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An Introduction to Sheaf Theory and the Foundations of Condensed Mathematics

Tesi di Laurea in Logica Matematica

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Introduction

Sheaves as mathematical structures first appeared in the context of algebraic geometry, where they were first employed in the study of homology and cohomology of topological spaces. The axiomatization of the concept of sheaf was introduced years after their first uses as computational tools in this framework, and only in the 1950s, with the works of A.Grothendieck and J.P.Serre among others, a categorical definition of sheaves was produced. Category theory was then starting to become an important independent branch of mathematical logic, first recognized as a foundational theory of different aspects of mathematics by S.Eilenberg and S.Mac Lane with their pioneering works in the early 1960s. This theory found immediate application in the expansion of sheaf theory, which was rendered an object of study and not merely a tool; in fact, today sheaf theory has important connections with separate branches of mathematics, from analysis to algebraic geometry to logic. In this thesis, we will also be able to comment on some of the advances made by A.Grothendieck in 1961 with the definition of Grothendieck topology (see Chapter 3), which included among the possible domains of sheaves not only categories of strict topological derivation such as the category of open sets of a topological space, but also any abstract category which could be endowed with a (Grothendieck) topology.

The first chapter of this thesis will provide some basic notions and lemmas of category theory which will be employed in the following sections. Category theory is a discipline which arose from the observation that various branches of mathematics have the common nature of studying specific mathematical objects and morphisms between them which preserve their structures. This observation led to the identification and formalization of constructions such as limits or pullbacks which could then be applied in contexts where they had never appeared before. The last section of this chapter is dedicated to the definition of abelian category, whose aim is to single out the categories which have the algebraic structure that is necessary in order to employ them in algebraic geometry. As a consequence, some of the most common properties of abelian groups are generalized; the importance of these is such that, as will be shown in the last chapter, the attempt to have new objects with those characteristics has lately led to the development of a new branch of mathematical logic: condensed mathematics.

The second chapter will mainly follow **[6]** to define the notion of sheaf of sets and examine their structure. Sheaves of sets are defined as functors from the opposite category $\mathcal{O}(X)^{op}$ of the category of open subsets of a topological space X to the category **Sets** of all sets, which verify a requirement, the sheaf condition. This condition encapsulates in a categorical manner two common properties which, for example, are fundamental in continuous maps: when given a map $f : X \to Y$, we can restrict it on a subset $U \subseteq X$ in a unique way to yield a continuous map; conversely, several continuous maps defined on subsets $U_i \subseteq X$ can be collated to yield a unique map defined on the whole space, provided that they coincide on the intersections $U_i \cap U_j$ for every i, j. We will also define a bundle over a topological space X, which is merely a continuous function p between topological spaces with codomain X, and associate each bundle to a sheaf, named its sheaf of cross-sections. This independent construction is then shown to be very important: we will prove that every sheaf of sets has the common structure of being a sheaf of cross-sections of a bundle; moreover, we will also prove a categorical adjunction which will identify a process called sheafification, which is capable of yielding a "best approximating" sheaf when applied to any functor F between the categories above, even when F is not a sheaf. A concrete example will also be computed. In the end, an equivalence of categories will be shown to closely relate sheaves and certain bundles called ètale bundles.

In the third chapter we will enlarge our setting from sheaves on the category $\mathcal{O}(X)^{op}$ to sheaves on any site (i.e. any category which is equipped with a Grothendieck topology). This is a generalization of the concept of topology which is defined on an arbitrary category $\mathbf{C}$ in terms of its morphisms. Namely, such a topology assigns to every object of $\mathbf{C}$ a collection of families of morphisms, called covering sieves, which verify some fundamental axioms. As happens in topology with the notion of basis of a topology, we will define some concepts such as precoverage, coverage and pretopology which will allow us to generate a topology starting from families of morphisms which verify weaker conditions than the axioms for a topology. The last section of the chapter will define the notion of sheaf on any topologized category, by "translating" the sheaf condition in terms of covering sieves and diagrams of morphisms.

The closing chapter will study sheaves on a specific site, the site of profinite topological spaces. These sheaves, also called condensed sets, are the starting point of a new theory conceptualized by P.Scholze in the last decade: condensed mathematics. The theory aims to provide a better framework to study algebraic objects in a way that avoids forgetting about their topological structures, when present. For example, it will be shown that the category of topological abelian groups is not an abelian category. However, we will construct a functor which assigns to any topological space X a condensed set $\underline{X}$ which embodies it, in the sense that $\underline{X}$ is a functor that maps any profinite set S onto the set of continuous maps $S \to X$. As a consequence, we can prove that the category of condensed abelian groups is abelian and conclude that, in certain conditions, it can be very useful to think in terms of the corresponding condensed sets rather than in terms of the topological spaces one is interested in. It was precisely through arguments of which ours is a small example that condensed mathematics have seen a vibrant evolution in the last years, both leading to the introduction of new mathematical objects and to the discovery of new proofs for already existing theorems: for example, some classical theorems of complex geometry whose proofs were based on analytic arguments were proven in a more algebraic way.

Contents

Chapter 1

Notions of Category Theory

In this chapter we seek to provide some basic notions of category theory which will be applied in the context of sheaf theory. We will start by defining the concept of category, together with transformations within and between categories; then we will examine special relationships and properties that relate such morphisms (functors), such as being adjoints, pullbacks, and so forth. Finally, we will prove some useful lemmas which will be applied in the forecoming chapters. The references for the results contained in this chapter are [**5**] and [**6**].

1.1 Categories and morphisms

Definition 1.1. *A category* $\boldsymbol{C}$ *is a collection of objects (A, B, C,...) and morphisms (f, g, h,...), together with the following four operations:*

- *dom associates to each morphism* f *an object of* $\boldsymbol{C}$*,* $dom(f)$*, called its domain;*
- *cod analogously pairs a morphism* f *and its codomain* $cod(f)$*, an object of* $\boldsymbol{C}$*;*
- *The third operation defines for each object* C *a morphism* 1_C *(or* id_C*), called the identity morphism;*
- *The operation of composition associates to a pair of morphisms* (f, g) *such that* $dom(f) = cod(g)$ *a morphism* $f \circ g$*, their composite.*

Additionally, four axioms must hold, namely:

For all objects C*,* D *of* $\boldsymbol{C}$ *and for all morphisms* f*,* g*,* h*, composable when needed:*

(i) $dom(1_C) = C = cod(1_C)$*;*

(ii) $dom(f \circ g) = dom(g)$ *and* $cod(f \circ g) = cod(f)$*;*

(iii) $1_D \circ f = f$ *and* $f \circ 1_C = f$*;*

(iv) $(f \circ g) \circ h = f \circ (g \circ h)$*.*

We will denote the collection of objects of $\boldsymbol{C}$ *as* $Ob(\boldsymbol{C})$ *and the collection of morphisms in* $\boldsymbol{C}$ *between two objects* C*,* D *as* $Hom_{\boldsymbol{C}}(C, D)$*.*

Example 1.2. Some examples of categories, some of which will be employed later, are:

- The category **Sets**, whose objects are (small) sets (i.e. sets which all belong to the same universal set U), together with functions as morphisms;
- **Top**, with topological spaces as objects and continuous maps as morphisms;
- Once we fix a topological space X, we can consider $\mathcal{O}(X)$, the category whose objects are open subspaces of X, where the only morphism $U \to V$ is the inclusion of U in V, if present;
- Let X be a fixed topological space. The category **Top/X** has morphisms with codomain X as objects, and morphisms between these objects are given by commutative triangles, as shown below: given $f : Y \to X$ and $g : Z \to X$, there is a morphism $h : Y \to Z \in Hom_{\mathbf{Top/X}}(f, g)$ if and only if

$$\begin{tikzcd} Y \arrow[rr, "h"] \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & X & \end{tikzcd} \tag{1.1}$$

  commutes. The composition is given by attaching two triangles by their common side.
- Given a category $\mathbf{C}$, we can define its opposite category $\mathbf{C}^{op}$ as the category with $Ob(\mathbf{C}^{op}) = Ob(\mathbf{C})$ and morphisms given by $Hom_{\mathbf{C}^{op}}(C, D) = Hom_{\mathbf{C}}(D, C)$, i.e. every morphism $f : C \to D$ in $\mathbf{C}$ identifies a morphism $f^{op} : D \to C$ in $\mathbf{C}^{op}$. Note that $f^{op} \circ g^{op} = (g \circ f)^{op}$.

Let's continue by defining some basic properties of morphisms within a category:

Definition 1.3. *Let* $\boldsymbol{C}$ *be a category. A morphism* $f \in Hom_{\boldsymbol{C}}(C, D)$ *is called:*

- *a monomorphism if for every* $B \in Ob(\boldsymbol{C})$ *and for every* $g, h : B \to C$ *we have that* $fg = fh$ *implies* $g = h$*. We also say that* f *is monic;*

- *an epimorphism if for every object E and for every $g, h \in Hom_{\boldsymbol{C}}(D, E)$ such that $gf = hf$ it follows that $g = h$;*
- *an isomorphism if there exists $g : D \to C$ such that $f \circ g = 1_D$ and $g \circ f = 1_C$.*

Next, we recall the notion of a morphism of categories, also called a functor.

Definition 1.4. *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be categories. A functor $T : \boldsymbol{C} \to \boldsymbol{D}$ is an operation which assigns to each object C of $\boldsymbol{C}$ an object TC of $\boldsymbol{D}$, and to each arrow $f : A \to C$ of $\boldsymbol{C}$ an arrow $T(f) : TA \to TC$ of $\boldsymbol{D}$, in such a way that*

$$T(1_C) = 1_{TC}, \qquad T(f \circ g) = Tf \circ Tg,$$

the latter holding for any pair of composable morphisms f, g.

Relations between different functors will become paramount in the following chapters, hence we shall now investigate some of the most important ones.

Definition 1.5. *Let $F, G : \boldsymbol{C} \to \boldsymbol{D}$ be functors. A natural transformation $\alpha : F \to G$ is a function which assigns to each object C of $\boldsymbol{C}$ a morphism $\alpha_C : FC \to GC$ of $\boldsymbol{D}$ such that every arrow $f : C \to C'$ in $\boldsymbol{C}$ yields a commutative diagram*

$$\begin{tikzcd} FC \arrow[r, "\alpha_C"] \arrow[d, "F(f)"'] & GC \arrow[d, "G(f)"] \\ FC' \arrow[r, "\alpha_{C'}"] & GC'. \end{tikzcd} \tag{1.2}$$

In this case, we say that α_C is natural in C. Moreover, if every component of α (i.e. every α_C, for $C \in Ob(\boldsymbol{C})$) is an isomorphism, then α is called a natural isomorphism between $\boldsymbol{C}$ and $\boldsymbol{D}$.

Example 1.6. Let **CRng** be the category of commutative (small) rings, and **Grp** the category of (small) groups. Consider a commutative ring K and let K^* be the group of its invertible elements. Let M be a $n \times n$ matrix with entries in K; M is therefore invertible if and only if $det_K M \in K^*$. Moreover, $det_K : GL_n(K) \to K^*$ is a morphism of groups. Finally, since the determinant of a $n \times n$ matrix is given by the same formula in all rings, det_K is natural: the diagram

$$\begin{tikzcd} GL_n(K) \arrow[r, "det_K"] \arrow[d, "GL_n(f)"'] & K^* \arrow[d, "f^*"] \\ GL_n(K') \arrow[r, "det_{K'}"] & (K')^* \end{tikzcd} \tag{1.3}$$

commutes for every choice of morphism $f : K \to K'$ of commutative rings. As a consequence, the transformation $det : GL_n(-) \to (-)^*$ is natural between two functors $\mathbf{CRng} \to \mathbf{Grp}$.

Now consider two categories $\mathbf{C}$, $\mathbf{D}$. Notice that every category has an identity functor and that given three functors $F, G, H : \mathbf{C} \to \mathbf{D}$ together with two natural transformations $\alpha : F \to G$, $\beta : G \to H$ one can define the composite natural transformation $\beta \circ \alpha$ by $(\beta \circ \alpha)_C = \beta_{G(C)} \circ \alpha_C$. This considerations lead us to the formulation of the functor category $\mathbf{D}^{\mathbf{C}}$, whose objects are functors from $\mathbf{C}$ to $\mathbf{D}$, with natural transformations between them as morphisms. We will now investigate some basic properties of functors.

Definition 1.7. *A functor $F : \boldsymbol{C}^{op} \to \boldsymbol{D}$ is called a contravariant functor, whereas a functor $F : \boldsymbol{C} \to \boldsymbol{D}$ is sometimes called a covariant functor.*

A functor $F : \boldsymbol{C} \to \boldsymbol{D}$ is said to be:

- *full (respectively faithful), if for every pair of objects C, C' of $\boldsymbol{C}$, the operation*

$$\begin{gathered} Hom_{\boldsymbol{C}}(C, C') \to Hom_{\boldsymbol{D}}(FC, FC') \\ f \mapsto F(f) \end{gathered}$$

  is surjective (respectively injective);

- *an equivalence of categories, if F is full, faithful and every object of $\boldsymbol{D}$ is isomorphic to an object in the image of F.*

Definition 1.8. *Let $\boldsymbol{C}$ be a small category. A subfunctor Q of a functor $P : \boldsymbol{C}^{op} \to \boldsymbol{Sets}$ is a functor $\boldsymbol{C}^{op} \to \boldsymbol{Sets}$ such that each QC is a subset of PC and each $Qf : QD \to QC$ is the restriction of $Pf : PD \to PC$.*

1.2 Limits and adjunctions

In this section we examine the most common universal constructions in a category. The term universal is used in this context to highlight that a certain object is fully determined by a construction up to isomorphism.

One of the most important notions is that of limit.

Definition 1.9. *Let $\boldsymbol{C}$, $\boldsymbol{J}$ be two categories, with $\boldsymbol{J}$ a small "indexing" category. Con-*

sider the functor category $\boldsymbol{C}^{\boldsymbol{J}}$, together with the diagonal functor

$$\begin{aligned} \Delta : \boldsymbol{C} &\to \boldsymbol{C}^{\boldsymbol{J}} \\ C &\mapsto \Delta_{\boldsymbol{J}}(C) : \boldsymbol{J} \to \boldsymbol{C} \\ j &\mapsto C \\ f : j \to j' &\mapsto 1_C. \end{aligned}$$

A natural transformation $\pi : \Delta_{\boldsymbol{J}}(C) \to A$, where A is another functor in $\boldsymbol{C}^{\boldsymbol{J}}$, is determined by maps $\pi_j : C \to A(j)$, with $j \in \boldsymbol{J}$, such that for every $u : j \to k$ the diagram

$$\begin{tikzcd} & C \arrow[dl, "\pi_j"'] \arrow[dr, "\pi_k"] & \\ A(j) \arrow[rr, "A(u)"] & & A(k) \end{tikzcd} \tag{1.4}$$

commutes (since $C = \Delta_{\boldsymbol{J}}(C)(j) = \Delta_{\boldsymbol{J}}(C)(k)$). Hence such a natural transformation π is called a cone $\pi : C \to A$ on the diagram A, with vertex C.

A limiting cone of $A : \boldsymbol{J} \to \boldsymbol{C}$ consists of an object $\varprojlim A \in \boldsymbol{C}$ and a universal cone π, with vertex $\varprojlim A$: that is, a cone such that for every other cone $f : C \to A$ there exists a unique arrow $g : C \to \varprojlim A$ in $\boldsymbol{C}$ such that for each $u : j \to k \in \boldsymbol{J}$ the following diagram commutes:

$$\begin{tikzcd} C \arrow[rr, dashed, "g"] \arrow[dr, "f_j"] \arrow[ddr, bend right, "f_k"'] & & \varprojlim A \arrow[dl, "\pi_j"'] \arrow[ddl, bend left, "\pi_k"] \\ & A(j) \arrow[d, "A(u)"] & \\ & A(k). & \end{tikzcd} \tag{1.5}$$

The limit is said to be finite if $\boldsymbol{J}$ has a finite number of elements.

Observation 1.10. Most of the definitions stated in this section have a dual notion, that is to say an analogous concept where all of the arrows in the defining diagrams are reversed: for example, the dual of the notion of limit is that of a colimit. A cocone with vertex C on $A : \mathbf{J} \to \mathbf{C}$ will be a natural transformation $A \to \Delta_{\mathbf{J}}(\mathbf{C})$; if it exists, the universal cocone on A is called the colimit of A, with vertex $\varinjlim A$.

Some special instances of limits are so recurring that they deserve special consideration: we will now examine three of these structures, namely those of terminal object, of pullback and of equalizer. The formal duals of these constructions are called initial objects, pushforwards and coequalizers.

Definition 1.11. *A terminal object of a category $\boldsymbol{C}$ is an object $\{*\}$ such that any other object D of $\boldsymbol{C}$ has a unique morphism $D \to \{*\}$. Such an object can be considered a limit by taking $\boldsymbol{J}$ to be the empty category.*

Definition 1.12. *Consider $\boldsymbol{J} = \bullet \to \bullet \leftarrow \bullet$, $F : \boldsymbol{J} \to \boldsymbol{C}$ a functor, in fact a pair of arrows $B \xrightarrow{f} A \xleftarrow{g} D$ in $\boldsymbol{C}$. A cone over such a functor is composed of a vertex C together with a pair of arrows k, h such that this diagram commutes:*

$$\begin{tikzcd} C \arrow[r, "k"] \arrow[d, "h"'] & D \arrow[d, "g"] \\ B \arrow[r, "f"] & A. \end{tikzcd} \tag{1.6}$$

Such a cone, if it is universal, is called a pullback square and its vertex C is named $B \times_A D$ (sometimes called the fibre product of B and D over A). If A is the terminal object $\{*\}$, then $B \times_{\{*\}} D = B \times D$ is the product between B and D.

Definition 1.13. *Let $\boldsymbol{J} = \bullet \rightrightarrows \bullet$. A functor $F : \boldsymbol{J} \to \boldsymbol{C}$ is then a pair of parallel arrows $f, g : A \to B$ in $\boldsymbol{C}$. A cone over such a functor is a morphism $e : E \to A$ such that $fe = ge$. The equalizer of f and g is then defined as the limiting cone over this diagram; in other words e is the equalizer of f and g if $fe = ge$ and for any other morphism $u : X \to A$ in $\boldsymbol{C}$ with this property there exists a unique $v : X \to E$ such that $ev = u$, as in*

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ X. \arrow[u, dashed, "v"] \arrow[ur, "u"'] & & \end{tikzcd} \tag{1.7}$$

*For example, in **Sets** the equalizer between two arbitrary functions $A \to B$ is given by $E = \{a \in A \mid f(a) = g(a)\}$, with e the set inclusion. However, in a generic category $\boldsymbol{C}$ two parallel arrows need not have an equalizer.*

Now we will demonstrate a short lemma which will be useful later.

Lemma 1.14. *Let $e : E \to A$ be the equalizer of two arrows $f, g : A \to B$. Then e is monic.*

Proof. Consider two parallel arrows $j, l : C \to E$ such that $ej = el$. It follows that $f(ej) = (fe)j = (ge)j = g(ej)$, thus by the universal property of equalizers we have that there is a unique $k : C \to E$ such that $ek = ej$, as in the diagram below:

$$\begin{tikzcd} C \arrow[r, shift left, "j"] \arrow[r, shift right, "l"'] \arrow[r, bend left=60, dashed, "k"] & E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B. \end{tikzcd} \tag{1.8}$$

By uniqueness of k, since $ek = ej = el$, it follows that $k = j = l$. Since j and l were arbitrary, e is monic. ■

Lastly, we need to introduce another kind of relationship between two functors such that the domain of one of them is the codomain of the other; it can be thought of as a weaker form of categorical equivalence, going under the name of adjunction.

Definition 1.15. *Let $\boldsymbol{A}$, $\boldsymbol{X}$ be two categories, together with two functors $F : \boldsymbol{X} \to \boldsymbol{A}$, $G : \boldsymbol{A} \to \boldsymbol{X}$. G is a right adjoint for F (and conversely F is a left adjoint for G) if for all objects X of $\boldsymbol{X}$, A of $\boldsymbol{A}$ there exists a bijection $\theta : Hom_{\boldsymbol{X}}(X, GA) \cong Hom_{\boldsymbol{A}}(FX, A)$ which is natural in X and in A.*

This means that given any pair of morphisms $\alpha : A \to A'$ of $\boldsymbol{A}$, $\xi : X' \to X$ of $\boldsymbol{X}$, θ is such that both of the following diagrams commute:

$$\begin{tikzcd} Hom_{\boldsymbol{X}}(X, GA) \arrow[r, "\theta"] \arrow[d, "(G\alpha)_*"] & Hom_{\boldsymbol{A}}(FX, A) \arrow[d, "\alpha_*"] \\ Hom_{\boldsymbol{X}}(X, GA') \arrow[r, "\theta"] & Hom_{\boldsymbol{A}}(FX, A') \end{tikzcd} \qquad \begin{tikzcd} Hom_{\boldsymbol{X}}(X, GA) \arrow[r, "\theta"] \arrow[d, "\xi^*"] & Hom_{\boldsymbol{A}}(FX, A) \arrow[d, "F\xi^*"] \\ Hom_{\boldsymbol{X}}(X', GA) \arrow[r, "\theta"] & Hom_{\boldsymbol{A}}(FX', A) \end{tikzcd} \tag{1.9}$$

where α_* is the (right) composition with α and ξ^* the (left) composition with ξ.

Definition 1.16. *Let $\begin{tikzcd} \boldsymbol{X} \arrow[r, shift left, "F"] & \boldsymbol{A} \arrow[l, shift left, "G"] \end{tikzcd}$ be a pair of adjoint functors. The unit of the adjunction is the unique map yielded by the choice $A = FX$, i.e. $\eta_X : X \to GFX$ such that $\theta(\eta_X) = 1_{FX}$. Moreover, η_X is universal among arrows from X to an object of the form GA, in the sense that each $f : X \to GA$ uniquely determines another $h : FX \to A$ such that the triangle*

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & GFX \arrow[d, dashed, "Gh"] \\ & GA \end{tikzcd} \tag{1.10}$$

commutes. The morphisms η_X, grouped together, constitute a natural transformation $\eta : 1_{\boldsymbol{X}} \to GF$.

Analogously, choosing $X = GA$, $f = 1_{GA}$ yields the counit $\epsilon_A : FGA \to A$, which is universal among arrows from an object of the form FX to A. Once again, $\epsilon : FG \to 1_{\boldsymbol{A}}$ is a natural transformation.

Finally, it can be proven by taking $f = 1_{GA} : X = GA \to GA$ in (1.10) *(respectively $h = 1_{FX}$ in the dual triangle) that there are two commutative triangles*

$$\begin{tikzcd} F \arrow[d, "F\eta"] \arrow[dr, "1_F"] & \\ FGF \arrow[r, "\epsilon F"'] & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G. \end{tikzcd} \tag{1.11}$$

The fact that these triangles commute is actually equivalent to the fact that F and G form an adjunction.

1.3 Abelian categories

The last notion we will make use of is that of abelian category. Intuitively, these are categories in which morphisms can have desirable properties such as having kernels and cokernels, being exact, forming short exact sequences, and so on. The typical example of abelian category is the category **Ab** of abelian groups: by considering how many of the properties stated are fundamental in the development of homology theory, one can understand by analogy the importance of the structure of abelian category.

Definition 1.17. *A category $\boldsymbol{C}$ is called Ab-category or preadditive category if for each pair of objects A, B the hom-set $Hom_{\boldsymbol{C}}(A, B)$ is an additive abelian group, and if the composition of morphisms in $\boldsymbol{C}$ is bilinear: that is, for any arrows $f, f' : A \to B$, $g, g' : B \to C$ it follows that*

$$(g + g') \circ (f + f') = g \circ f + g \circ f' + g' \circ f + g' \circ f'. \tag{1.12}$$

*Examples of Ab-categories include the category **Ab** of (small) abelian groups and the categories **R-Mod**, **Mod-R** of (small) right and left modules on a ring R.*

Observation 1.18. Every object A of an Ab-category **C** has a unique morphism $\mathbb{Z} \to Hom_{\mathbf{C}}(A, A)$, completely determined by the image of $1 \in \mathbb{Z}$, which will be identified as $1_A \in Hom_{\mathbf{C}}(A, A)$.

Now, we will give a few preliminary definitions. These define structures which will be needed to make an Ab-category abelian.

Definition 1.19. *Let $\boldsymbol{C}$ be a (small) category. An object A is called a zero object (or a null object) if it is both terminal and initial, i.e. if for every other object B of $\boldsymbol{C}$ there is a unique morphism $B \to A$ as well as a unique morphism $A \to B$.*

Definition 1.20. *Two objects A, B of a category $\boldsymbol{C}$ have a biproduct diagram if there exist an object C, together with arrows p_1, p_2, i_1, i_2 such that the diagram*

$$\begin{tikzcd} A \arrow[r, shift right, "i_1"'] & C \arrow[l, shift right, "p_1"'] \arrow[r, shift left, "p_2"] & B \arrow[l, shift left, "i_2"] \end{tikzcd} \tag{1.13}$$

is such that

$$p_1 i_1 = 1_A, \quad p_2 i_2 = 1_B, \quad i_1 p_1 + i_2 p_2 = 1_C.$$

Definition 1.21. *Let $f : A \to B$ be a morphism in a category $\boldsymbol{C}$ equipped with a null object Z. Consider the composite of the unique arrows $A \to Z$, $Z \to B$, which will be named $0 : A \to B$. A kernel k of f is an equalizer of the arrows $f, 0 : A \to B$: an arrow $S \to A$ such that $fk = 0$ and for every $h : C \to B$ such that $fh = 0$, there is a unique $h' : C \to S$ that makes the diagram*

$$\begin{tikzcd} S \arrow[dr, "k"] \arrow[drr, "0"] & & \\ & A \arrow[r, "f"] & B \\ C \arrow[uu, dashed, "h'"] \arrow[ur, "h"] \arrow[urr, "0"'] & & \end{tikzcd} \tag{1.14}$$

commute.

Definition 1.22. *An Ab-category $\boldsymbol{C}$ is an abelian category if the following are true:*

(i) *$\boldsymbol{C}$ has a zero object;*

(ii) *$\boldsymbol{C}$ has binary biproducts (i.e. all pairs of objects have a biproduct);*

(iii) *every arrow in $\boldsymbol{C}$ has both a kernel and a cokernel;*

(iv) *every monic morphism is a kernel, and every epic arrow is a cokernel.*

Observation 1.23. The category **Ab** of abelian groups, as well as the categories **R-Mod**, **Mod-R** of right and left modules on a ring R, is abelian. The most important consequence in this context is that since every arrow has a kernel and a cokernel, exact sequences can be defined in these categories; therefore, homology and cohomology can be developed.

Chapter 2

Sheaves of Sets

The aim of this chapter is to introduce the notion of a sheaf of sets on a given topological space. This concept generalizes the idea of locally grouping all functions with a certain common property (e.g. continuity); a sheaf is in fact a collection of sets composed by the restrictions of such functions on open subsets of the space X. Some further properties must be verified, which encapsulate the common understood behaviour of (e.g. continuous) functions when they interact with each other on intersections. We then go on to describe the internal structure of a sheaf and how it can be viewed as a special kind of bundle. Most of the content of this chapter refers to **[6]**.

2.1 Sheaves of sets on a topological space

In this section we define sheaves and presheaves, we consider their fundamental properties and we then focus on the task of collating together different sheaves to make a bigger one.

Definition 2.1. *A presheaf of sets P on a topological space X is a functor $P : \mathcal{O}(X)^{op} \to$ **Sets**.*

Definition 2.2. *Let X be a topological space. A sheaf of sets F on X is a functor $F : \mathcal{O}(X)^{op} \to$ **Sets** such that every open covering $U = \bigcup_{i \in I} U_i$ yields an equalizer diagram*

$$FU \overset{e}{\dashrightarrow} \prod_{i \in I} FU_i \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j \in I} F(U_i \cap U_j) \tag{2.1}$$

where for $t \in FU$, $e(t) = \{t|_{U_i} \mid i \in I\}$; for a family $t_i \in FU_i$, $p\{t_i\} = \{t_i|_{U_i \cap U_j} \mid i, j \in I\}$ and $q\{t_i\} = \{t_j|_{U_i \cap U_j} \mid i, j \in I\}$.

Definition 2.3. *$Sh(X)$ is the category of sheaves of sets on X, whose morphisms are natural transformations of functors $F \to G$. By definition, $Sh(X)$ is a full subcategory of $\boldsymbol{Sets}^{\mathcal{O}(X)^{op}}$.*

Some examples of sheaves are:

Example 2.4.

- D such that for an open subset $U \subseteq X$, $DU = \{f \mid f : U \to \mathbb{R}$ function$\}$, and for $V \subseteq U$, the function $DU \to DV$ is the embedding $DU \hookrightarrow DV$;

- I, defined as $IU = \{f \mid f : U \to [0,1]$ is continuous$\}$;

- $\mathcal{C}^k$ such that $\mathcal{C}^k U = \{$functions $U \to \mathbb{R}$ of class $\mathcal{C}^k\}$.

Observation 2.5. The equalizer diagram in the definition of sheaf is the formalization of the familiar concept of collation: for example, in the case of the sheaf of continuous functions $f : U \subseteq \mathbb{R} \to \mathbb{R}$, it states that given open subsets U_i, U_j and functions $f_i : U_i \to \mathbb{R}$, $f_j : U_j \to \mathbb{R}$ such that $f_i|_{U_i} = f_j|_{U_j}$, there is a unique continuous function $f : U_i \cup U_j \to \mathbb{R}$ such that $f|_{U_i} = f_i$ and $f|_{U_j} = f_j$.

On the other hand, the functor B such that BU is the set of bounded functions from U to $\mathbb{R}$ is not a sheaf, since the collation of bounded functions may yield an unbounded function, therefore in the equalizer diagram the choice of FU is not unique in the sense given by the definition of equalizer (see Definition 1.13). In fact B is an example of a presheaf that is not a sheaf.

Observation 2.6. For the universal properties of (small) products in a category, the diagram

$$\begin{tikzcd}
 & FU_i \arrow[r] & F(U_i \cap U_j) \\
FU \arrow[ur] \arrow[r, dashed, "e"] \arrow[dr] & \prod_{i \in I} FU_i \arrow[u] \arrow[d] \arrow[r, dashed, shift left, "p"] \arrow[r, dashed, shift right, "q"'] & \prod_{i,j \in I} F(U_i \cap U_j) \arrow[u] \arrow[d] \\
 & FU_j \arrow[r] & F(U_i \cap U_j)
\end{tikzcd} \tag{2.2}$$

in which the vertical arrows are projections, only commutes for unique choices of e, p, q; therefore any small category with products $\mathbf{C}$ could replace $\mathbf{Sets}$ as the codomain of the sheaf F. For example, by replacing $\mathbf{C}$ with $\mathbf{Ring}$ one can define sheaves of rings on X, and so on with (abelian) groups, R-modules, R-algebras. In this section, however, we will focus on sheaves of sets.

Observation 2.7. Let $\mathbf{C}$ be a category. Any object C of $\mathbf{C}$ yields a presheaf

$$\begin{aligned}\mathbf{y}(C) : \mathbf{C}^{op} &\to \mathbf{Sets}\\ D &\mapsto Hom_{\mathbf{C}}(D, C)\\ \alpha : D' \to D &\mapsto \mathbf{y}(C)(\alpha),\end{aligned}$$

where $\mathbf{y}(C)(\alpha)$ maps $u : D \to C$ onto $u \circ \alpha : D' \to C$. Moreover, it can be proven that any morphism $f : C \to D$ in $\mathbf{C}$ induces a natural transformation $\mathbf{y}(C) \to \mathbf{y}(D)$; therefore, $\mathbf{y}$ is a functor $\mathbf{C} \to \mathbf{Sets}^{\mathbf{C}^{op}}$, called the Yoneda embedding. This embedding proves to be a full and faithful functor, hence $\mathbf{C}$ can be seen as a full subcategory of the category formed by its presheaves.

The Yoneda embedding is a special case of the following lemma, whose proof can be found in [**5**], p.61.

Lemma 2.8 (Yoneda). *Let $\boldsymbol{C}$ be a category, and consider a functor $P : \boldsymbol{C} \to \boldsymbol{Sets}$. For any object C of $\boldsymbol{C}$, there is a bijection $\boldsymbol{y} : Nat(\boldsymbol{y}(C), P) \cong PC$, sending every natural transformation $\alpha : \boldsymbol{y}(C) \to P$ to $\alpha_C(1_C)$, the image of the identity of C.*

Definition 2.9. *A presheaf P on a category $\boldsymbol{C}$ that is isomorphic to a presheaf of the form $\boldsymbol{y}(C)$, with C an object of $\boldsymbol{C}$, is called a representable presheaf.*

Definition 2.10. *A continuous function $f : X \to Y$ between topological spaces induces a functor $f_* : Sh(X) \to Sh(Y)$ that maps a sheaf F on X onto a sheaf f_*F on Y given by $\mathcal{O}(Y)^{op} \xrightarrow{f^{-1}} \mathcal{O}(X)^{op} \xrightarrow{F} \boldsymbol{Sets}$. f_*F is called the direct image of F under f.*

Note that the assignment $Sh(f) = f_*$ gives a functor $Sh : \mathbf{Top} \to \mathbf{Top}$; as a consequence, for example, two homeomorphic topological spaces X and Y have isomorphic categories of sheaves.

We now present a result which allows us to construct sheaves on a topological space with a different method.

Observation 2.11. If F is a sheaf on X, and $U \subseteq X$ is an open subset of X, then the sheaf $F|_U$ is a sheaf on U (as a topological space with subspace topology).

Theorem 2.12. *Let X be a topological space, covered by open sets $X = \bigcup_{k \in I} W_k$. If, for each k, l in I, there exist sheaves F_k, F_l such that*

$$F_k|_{(W_k \cap W_l)} = F_l|_{(W_k \cap W_l)}, \tag{2.3}$$

then there exists a sheaf F on X such that $F|_{W_k} \cong F_k$ for every k. Moreover, such sheaf F is unique up to isomorphism.

Proof. Let F_{kl} be the sheaf in (2.3), defined on $W_k \cap W_l$. We shall define the desired sheaf F on a set U as FU such that

$$FU \longrightarrow \prod_{k \in I} F_k(U \cap W_k) \rightrightarrows \prod_{k,l \in I} F_{kl}(U \cap W_k \cap W_l) \tag{2.4}$$

is an equalizer. If $V \subseteq U$, by the universal property of equalizer diagrams there is a unique map $FU \to FV$, which we'll take as the morphism given by F on the inclusion map $V \hookrightarrow U$. Therefore, such F is a presheaf. In order to prove it a sheaf, we shall consider an open covering $\{U_i\}$ of U, together with the following diagram:

$$\begin{tikzcd}
FU \arrow[r, "\varphi_3"] \arrow[d, "\varphi_1"] & \prod\limits_{k \in I} F_k(U \cap W_k) \arrow[r, shift left, "e"] \arrow[r, shift right, "f"'] \arrow[d, "\varphi_4"] & \prod\limits_{k,l \in I} F_{kl}(U \cap W_k \cap W_l) \arrow[d, "\varphi_6"] \\
\prod\limits_{i \in I} FU_i \arrow[r, "\varphi_2"] \arrow[d, shift right, "a"'] \arrow[d, shift left, "b"] & \prod\limits_{i,k \in I} F_k(U_i \cap W_k) \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[d, shift right, "c"'] \arrow[d, shift left, "d"] & \prod\limits_{k,i,l \in I} F_{kl}(U_i \cap W_k \cap W_l) \arrow[d, shift right] \arrow[d, shift left] \\
\prod\limits_{i,j \in I} F(U_i \cap U_j) \arrow[r, "\varphi_5"] & \prod\limits_{i,j,k \in I} F_k(U_i \cap U_j \cap W_k) \arrow[r, shift left] \arrow[r, shift right] & \prod\limits_{i,j,k,l \in I} F_{kl}(U_i \cap U_j \cap W_k \cap W_l).
\end{tikzcd}$$

By definition, all of the rows are equalizers. Moreover, since F_k and F_{kl} are sheaves, the last two columns are equalizers as well. The diagram chase below shows that the first column is an equalizer, and therefore that F is a sheaf. First of all, we note that since the lower left square commutes, we have that $c\varphi_2\varphi_1 = \varphi_5 a\varphi_1 = \varphi_5 b\varphi_1$; since φ_5 is monic (by Lemma 1.14) we have that $a\varphi_1 = b\varphi_1$. Now, let $\psi : X \to \prod\limits_{i \in I} FU_i$ be an arrow such that $a\psi = b\psi$, as in the diagram below.

$$\begin{tikzcd}
 & FU \arrow[r, "\varphi_3"] \arrow[d, "\varphi_1"] & \prod\limits_{k \in I} F_k(U \cap W_k) \arrow[r, shift left, "e"] \arrow[r, shift right, "f"'] \arrow[d, "\varphi_4"] & \prod\limits_{k,l \in I} F_{kl}(U \cap W_k \cap W_l) \arrow[d, "\varphi_6"'] \\
X \arrow[ur, dashed, bend left, "w"] \arrow[ur, dashed, "v"'] \arrow[r, "\psi"] & \prod\limits_{i \in I} FU_i \arrow[r, "\varphi_2"] \arrow[ur, dashed, "\lambda"] \arrow[d, shift right, "a"'] \arrow[d, shift left, "b"] & \prod\limits_{i,k \in I} F_k(U_i \cap W_k) \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[d, shift right, "c"'] \arrow[d, shift left, "d"] & \prod\limits_{k,i,l \in I} F_{kl}(U_i \cap W_k \cap W_l) \arrow[d, shift right] \arrow[d, shift left] \\
 & \prod\limits_{i,j \in I} F(U_i \cap U_j) \arrow[r, "\varphi_5"] & \prod\limits_{i,j,k \in I} F_k(U_i \cap U_j \cap W_k) \arrow[r, shift left] \arrow[r, shift right] & \prod\limits_{i,j,k,l \in I} F_{kl}(U_i \cap U_j \cap W_k \cap W_l).
\end{tikzcd}$$

We then have that

$$\begin{aligned}
\varphi_5 a\psi &= \varphi_5 b\psi \\
c\varphi_2\psi &= d\varphi_2\psi \quad \text{by commutativity of the bottom left square.}
\end{aligned} \tag{2.5}$$

Since the central column is an equalizer, by the universal property there is a unique λ such that $\varphi_2 = \varphi_4\lambda$, as in the diagram. Note that λ is monic: if x, y are such that $\lambda x = \lambda y$, it follows that $\varphi_4\lambda x = \varphi_4\lambda y$, so $\varphi_2 x = \varphi_2 y$ which means $x = y$ since φ_2 is monic (as an equalizer arrow). Now

$$\begin{aligned} g\varphi_2\psi &= h\varphi_2\psi \\ g\varphi_4\lambda\psi &= h\varphi_4\lambda\psi \\ \varphi_6 e\lambda\psi &= \varphi_6 f\lambda\psi \quad \text{by commutativity of the top right square} \\ e\lambda\psi &= f\lambda\psi \qquad \text{since } \varphi_6 \text{ is monic,} \end{aligned} \tag{2.6}$$

so given that the top row is an equalizer, there is a unique $v : X \to FU$ such that $\lambda\psi = \varphi_3 v$. Consider the composition $\varphi_2\varphi_1 v$. We have that $\varphi_4\lambda\varphi_1 v = \varphi_2\varphi_1 v = \varphi_4\varphi_3 v = \varphi_4\lambda\psi$. Since φ_4 and λ are monic, their composition is monic, so $\varphi_1 v = \psi$. It only remains to show that the morphism v is unique. Let $w : X \to FU$ be another arrow such that $\varphi_1 w = \psi = \varphi_1 v$. Note that φ_1 is monic: for any x, y such that $\varphi_1 x = \varphi_1 y$ it follows that $\varphi_4\varphi_3 x = \varphi_2\varphi_1 x = \varphi_2\varphi_1 y = \varphi_4\varphi_3 y$, so $x = y$ since both φ_3 and φ_4 are monic. In particular, considering v, w as x and y, we have that $v = w$. Therefore the left-hand column is an equalizer; this means that F is a sheaf. The uniqueness of such F is readily proven: consider another sheaf G for which GU can substitute FU in the diagram above so that the left-hand column is an equalizer diagram as well. By the properties of the two equalizers, there are two unique maps $x : GU \to FU$, $y : FU \to GU$ as in the following commutative figure:

$$\begin{tikzcd} FU \arrow[r, "\varphi_3"] \arrow[d, dashed, shift left, "x"] \arrow[dr, "\psi" near start] & \prod_{k\in I} F_k(U\cap W_k) \\ GU \arrow[u, dashed, shift left, "y"] \arrow[r, "\widetilde{\varphi_3}"'] \arrow[ur, "\widetilde{\psi}"' near end] & \prod_{k\in I} G_k(U\cap W_k). \end{tikzcd} \tag{2.7}$$

Consider the composition yx: we have that

$$\begin{aligned} \varphi_3 1_{FU} &= \varphi_3 = \psi x = \varphi_3 yx \\ \widetilde{\varphi_3} 1_{GU} &= \widetilde{\varphi_3} = \widetilde{\psi} y = \widetilde{\varphi_3} xy. \end{aligned} \tag{2.8}$$

Since φ_3 and $\widetilde{\varphi_3}$ are monic, we have that $yx = 1_{FU}$ and $xy = 1_{GU}$, hence FU and GU are isomorphic. By repeating the same argument on the equalizers of the left-hand columns, we deduce that the isomorphisms $FU \to GU$, $FV \to GV$ commute with the inclusion $V \subseteq U$, so there is in fact an isomorphism of sheaves between F and G. ∎

2.2 Bundles

We present a different mathematical structure, that of a bundle over a topological space X, which is *a priori* unrelated to the notion of sheaf, but will turn out to be closely adjacent.

Definition 2.13. *Let X be a topological space. A continuous map $p : Y \to X$, or equivalently an object of **Top/X**, is called a bundle over X.*
*A morphism of bundles $p : Y \to X$, $p' : Y' \to X$ is a morphism in **Top/X**: a continuous map $f : Y \to Y'$ such that $p'f = p$.*

Definition 2.14. *The fiber of a bundle $p : Y \to X$ on a point $x \in X$ is the set $p^{-1}\{x\}$.*

Definition 2.15. *A cross-section of a bundle $p : Y \to X$ is a continuous map $s : X \to Y$ such that $ps = 1_X$ (therefore, an arrow $id_X \to p$ in **Top/X**).*

Observation 2.16. Let $p : Y \to X$ be a bundle on X. If U is an open subset of X, then p restricts to the bundle $p_U : p^{-1}U \to U$ over U, and the diagram

$$\begin{tikzcd} p^{-1}U \arrow[r, hook] \arrow[d, "p_U"] & Y \arrow[d, "p"] \\ U \arrow[r, hook] \arrow[ur, dashed, "s"] & X \end{tikzcd} \tag{2.9}$$

is a pullback diagram in **Top**. A cross-section s of p_U is a continuous map $s : U \to Y$ such that $ps = i : U \hookrightarrow X$ is the inclusion.

We shall give some example of bundles to make the definition clearer.

Example 2.17.
- A covering map $p : \widetilde{X} \to X$ is a bundle over X, and a cross-section of p on an open evenly covered subset U of X is the lift of the inclusion $i : U \hookrightarrow X$ itself.

- If X is a topological space, and L is a real vector space, regarded as a topological space, the projection $X \times L \to X$ is a bundle over X whose cross-sections are continuous maps $X \to L$.

- A real vector bundle Y over X is a bundle $p : Y \to X$ such that:

  (i) For each $x \in X$, $p^{-1}x$ is a real vector space;

(ii) For each point $x \in X$, there is an open neighbourhood V equipped with a real vector space L and an isomorphism Φ, linear on each fiber, such that

$$\begin{tikzcd} p^{-1}V \arrow[r, "\Phi"] \arrow[d, "p_V"] & V \times L \arrow[d] \\ V \arrow[r, equal] & V \end{tikzcd} \tag{2.10}$$

For example, the tangent bundle on a smooth manifold M is a real vector bundle in this sense.

Definition 2.18. *Let $p : Y \to X$ be a bundle on X. The sheaf of cross-sections of p is the sheaf $\Gamma_p : \mathcal{O}(X)^{op} \to$ **Sets** such that for a subset $U \subseteq X$,*

$$\Gamma_p U = \{s \mid s : U \to Y \text{ and } ps = i : U \hookrightarrow X\}$$

and the inclusion $V \subseteq U$ induces the restriction $\Gamma_p U \to \Gamma_p V$.

Therefore, every bundle on X induces a sheaf on X; additionally, a map of bundles $f : p \to p'$ induces a morphism $\Gamma_p \to \Gamma_{p'}$ of sheaves on X, where $\Gamma_p U \to \Gamma_{p'} U$ is given by $s \mapsto fs$. In this way, Γ is a functor from bundles to sheaves.
In the following subsections, we aim to show that every sheaf on a topological space can be seen as a sheaf of cross-sections of bundles on that space.

2.3 Germs and stalks

Definition 2.19. *Let $x \in X$ be a point of a topological space. Let $P : \mathcal{O}(X)^{op} \to$ **Sets** be a presheaf on X. If U, V are open neighbourhoods of x, two elements $s \in PU$, $t \in PV$ are said to* have the same germ at x *if there is an open set $W \subseteq (U \cap V)$ such that $x \in W$ and $s|_W = t|_W \in PW$.*
The relation "has the same germ at x as" *is an equivalence relation, and the equivalence class of s is the germ of s at x, denoted $germ_x s$.*

In order to define stalks, we must consider

$$P_x = \{germ_x s \mid s \in PU,\, x \in U,\, U \text{ open subset of } X\}$$

the set of all germs at x. Let $P^{(x)}$ be the restriction of the presheaf P to open neighbourhoods of x. Since the inclusion $x \in W \subseteq U$ means that for $s \in PU$, $germ_x s =$

$germ_x(s|_W)$, we have a diagram

$$\begin{tikzcd}
& PU \arrow[ddl, "\tau_U"'] \arrow[d] \arrow[ddr, "germ_x"] & \\
& PW \arrow[dl] \arrow[dr] & \\
L & & P_x \arrow[ll, dashed, "t"]
\end{tikzcd} \tag{2.11}$$

in which P_x is the colimit and $germ_x$ the colimiting cone: $P_x = \varinjlim_{x \in U} PU$. By the definition of "having the same germ", moreover, if $\{\tau_U : PU \to L\}_{x \in U}$ is any other cone over $P^{(x)}$, then there is a unique function $t : P_x \to L$ such that $t \circ germ_x = \tau$.

Definition 2.20. *The set P_x of all germs at x is called the stalk of P at x.*

Observation 2.21. If $x \in U \subseteq X$, with U an open subset, a morphism of presheaves $h : P \to Q$ induces in x a unique function $h_x : P_x \to Q_x$ such that the diagram

$$\begin{tikzcd}
PU \arrow[r, "h_U"] \arrow[d, "germ_x"'] & QU \arrow[d, "germ_x"] \\
P_x \arrow[r, dashed, "h_x"'] & Q_x
\end{tikzcd} \tag{2.12}$$

commutes. Therefore, there are functors

$$\boldsymbol{Sets}^{\mathcal{O}(X)^{op}} \to \boldsymbol{Sets}$$
$$h \mapsto h_x,$$

"take the germ at x".

We shall now consider a specific bundle on X. Define

$$\Lambda_P = \coprod_{x \in X} P_x = \{germ_x s \mid x \in X, s \in PU\}, \tag{2.13}$$

and $p : \Lambda_P \to X$ as the map $germ_x s \mapsto x$. We can choose a topology on Λ_P that makes p continuous, and therefore a bundle: it is sufficient to associate to $s \in PU$ a function

$$\dot{s} : U \to \Lambda_P$$
$$x \mapsto germ_x s.$$

In fact, $\dot{s}$ is a section of p. We then define a basis of open subsets of Λ_P as $\{\dot{s}(U) \mid s \in PU,\ U \subseteq X \text{ open}\}$, so that an open set in Λ_P is a union of images of open sets $U \subseteq X$ through the sections $\dot{s}$.

Proposition 2.22. *The map $P \mapsto (p : \Lambda_P \to X)$ is a functor from presheaves to bundles. Moreover, this bundle $p : \Lambda_P \to X$ is a local homeomorphism.*

Proof. Firstly, we consider a morphism of presheaves $h : P \to Q$. We have seen that h induces a unique map on stalks at every x, $h_x : P_x \to Q_x$. We shall then define the image of h as the morphism of bundles $\alpha : \Lambda_P \to \Lambda_Q$ given by the disjoint union of these maps h_x. We can check that α is continuous on a basis of the topology of Λ_Q: let $\dot{t}(U) = \{germ_x^Q t \mid x \in U\}$ be an open subset of Λ_Q. Then

$$\begin{aligned}\alpha^{-1}[\dot{t}(U)] &= \coprod_{x\in U} h_x^{-1}[\dot{t}(U)] \\ &= \coprod_{x\in U} \{germ_x^P s \mid h_x(germ_x^P s) = germ_x^Q t\} \\ &\overset{(2.12)}{=} \coprod_{x\in U} \{germ_x^P s \mid germ_x^Q(h_U s) = germ_x^Q t\}.\end{aligned} \tag{2.14}$$

Therefore, on a stalk P_x we have that the germs in $\alpha^{-1}[\dot{t}(U)]$ are exactly those of the functions s such that $germ_x^Q(h_U s) = germ_x^Q t$, i.e. such that there exists a neighbourhood $W \subseteq U$ of x such that $h_U s|_W = t|_W$. We can then show that $\alpha^{-1}[\dot{t}(U)]$ is open by using this condition: consider $germ_x^P s \in \alpha^{-1}[\dot{t}(U)]$, and consider its neighbourhood $\dot{s}(W)$. Let x' be a point in W: we have that $germ_{x'}^P s$ and $germ_x^P s$ are in the same open subset $\dot{s}(U)$ of Λ_P. We now compute $\alpha(germ_{x'}^P s)$:

$$\alpha(germ_{x'}^P s) = h_{x'}(germ_{x'}^P s) = germ_{x'}^Q(h_U s). \tag{2.15}$$

But since $x' \in W$, and since we had that $h_U s|_W = t|_W$, we conclude that $germ_{x'}^Q(h_U s) = germ_{x'}^Q t = \dot{t}(x') \in \dot{t}(U)$, hence $germ_{x'}^P s \in \alpha^{-1}[\dot{t}(U)]$. Given that $germ_{x'}^P s$ was arbitrary, it follows that $\alpha^{-1}[\dot{t}(U)]$ is open, and therefore α is continuous.

In order to show that p is a local homeomorphism, we will first show that the functions $\dot{s}$ are homeomorphisms: if $s \in PU$ and $t \in PV$ are such that their sections $\dot{s}$, $\dot{t}$ agree at $x \in U \cap V$, then by definition of germ it follows that $\{y \in U \cap V \mid \dot{s}y = \dot{t}y\}$ is an open set $W \subseteq U \cap V$, and $\dot{s}|_W = \dot{t}|_W$, thus any $\dot{s}$ is continuous. Moreover, the $\dot{s}$ maps are open and injections trivially, therefore they are in fact homeomorphisms when restricted to their image.

Finally, any point $germ_x s$ of Λ_P has an open neighbourhood $\dot{s}(U)$ such that $p|_{\dot{s}(U)}$ has $\dot{s} : U \to \dot{s}(U)$ as a two-sided inverse, hence p is a local homeomorphism. ∎

2.4 Cross-sections and the sheafification functor

In this section, we aim to show that the construction of sheaves of sections shown before gives us information about the structure of any sheaf, not only sheaves of sections.

More precisely, we will prove a theorem which states that every sheaf is in fact a sheaf of cross-sections. Later on, we will define a functor which assigns to every presheaf its "best approximating" sheaf.

At first, given a presheaf P on X, we shall consider the bundle $\Lambda_P \to X$ defined above. Define, for each open subset $U \subseteq X$,

$$\begin{aligned} \eta_U : PU &\to \Gamma\Lambda_P(U) \\ s &\mapsto \dot{s}. \end{aligned} \tag{2.16}$$

Note that the diagram

$$\begin{array}{ccc} PU & \longrightarrow & PV \\ {\scriptstyle \eta_U}\big\downarrow & & \big\downarrow{\scriptstyle \eta_V} \\ \Gamma\Lambda_P(U) & \longrightarrow & \Gamma\Lambda_P(V) \end{array} \tag{2.17}$$

commutes, therefore

$$\eta : P \to \Gamma \circ \Lambda_P \tag{2.18}$$

is a natural transformation of functors. The construction of the sheaf $\Gamma\Lambda_P$ from a presheaf P is called sheafification.

Now we have all the requisites to prove the following theorem:

Theorem 2.23. *Let P be a sheaf of sets on a topological space X. Then $\eta : P \to \Gamma \circ \Lambda_P$ as above defined is an isomorphism of sheaves $P \cong \Gamma\Lambda_P$; in other words, every sheaf is a sheaf of cross-sections.*

Proof. We will start by showing that η_U is injective: take $s, t \in PU$ such that $\dot{s} = \dot{t}$. This means that for each $x \in U$, $germ_x s = germ_x t$; therefore for each x there is an open set $V_x \subseteq U$ such that $s|_{V_x} = t|_{V_x}$. Since $U \subseteq \bigcup_{x\in U} V_x$, this inclusion induces a map $PU \to \prod_{x\in U} PV_x$ with regard to which s and t have the same image; hence $s = t$ by uniqueness of the collation of maps given by the definition of sheaf (see Observation 2.5). Now let $h : U \to \Lambda_P$, $h \in \Gamma\Lambda_P(U)$ be a cross-section of the bundle $p : \Lambda_P \to X$ on U. Then for each point $x \in U$ we have an open set U_x and an element $s_x \in PU_x$ such that $hx = germ_x s_x$. By definition, $\dot{s}_x U_x$ is an open subset of Λ_P; therefore by continuity of h we must have an open set $V_x \subseteq U$ with $x \in V_x \subseteq U_x$ such that $hV_x \subseteq \dot{s}_x U_x$: equivalently, $h = \dot{s}_x$ on V_x. Thus, we can cover U with open sets V_x and consider an element $s_x|_{V_x}$ in each PV_x. On each intersection $V_x \cap V_y$ we have $\dot{s}_x = h = \dot{s}_y$; therefore for $z \in V_x \cap V_y$ we have $germ_z s_x = germ_z s_y$, hence $s_x|_{V_x\cap V_y} = s_y|_{V_x\cap V_y}$ by injectivity of η_U. Therefore the family $\{s_x\}_{x\in U}$ has the same image under both of the standard maps $\prod_{x\in U} PV_x \rightrightarrows \prod_{x,y\in U} P(V_x \cap V_y)$. In conclusion, again by collation we have that there exists $s \in PU$ such that $s|_{V_x} = s_x$. Then at each x, $hx = germ_x s_x = germ_x s$, so $h = \dot{s}$. Finally, we have proved that η is surjective, therefore an isomorphism. ■

In fact, more can be said about the process of sheafification. The intuitive fact that the sheafification $\Gamma\Lambda_P$ of a presheaf P is the sheaf that best approximates P can be expressed more precisely by the following theorem.

Theorem 2.24. *Let X be a topological space. The category $Sh(X)$ of sheaves of sets on X is reflective in the category $\boldsymbol{Sets}^{\mathcal{O}(X)^{op}}$ of presheaves on X: that is to say, the inclusion functor*

$$Sh(X) \rightarrowtail \boldsymbol{Sets}^{\mathcal{O}(X)^{op}}$$

has a left adjoint.

Proof. We will show that the composition $\Gamma \circ \Lambda$ is precisely that left adjoint, with η as the unit. First of all, we must prove that the morphism of presheaves $\eta : P \to \Gamma\Lambda_P$ is universal from P to sheaves. Let F be a sheaf and $\theta : P \to F$ be a map of presheaves, as in the following diagram:

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[dr, "\theta"'] & \Gamma\Lambda_P \arrow[d, dashed, "\sigma"] \\ & F. \end{tikzcd} \tag{2.19}$$

We aim to give a unique σ such that the diagram commutes. Since Theorem 2.23 states that η is an isomorphism, let $\sigma = \eta^{-1}\Gamma\Lambda_\theta$. Therefore, in the diagram below

$$\begin{tikzcd} P \arrow[r, "\eta"] \arrow[d, "\theta"'] & \Gamma\Lambda_P \arrow[d, "\Gamma\Lambda_\theta"] \arrow[dl, "\sigma"] \\ F \arrow[r, "\eta"'] & \Gamma\Lambda_F \end{tikzcd} \tag{2.20}$$

the bottom triangle commutes by definition of σ, and the outer square commutes by naturality of η; this means that $\theta = (\eta^{-1} \circ \Gamma\Lambda_\theta) \circ \eta \overset{def}{=} \sigma \circ \eta$.

Lastly, we show the unicity of σ. Let σ,$\tau : \Gamma\Lambda_P \to F$ be maps such that $\sigma \circ \eta = \tau \circ \eta : P \to F$. Consider an open set U and a section $h \in \Gamma\Lambda_P(U)$. For each $x \in U$ there exists an open neighbourhood V_x and an element $s_x \in PV_x$ such that $hx = germ_x s_x$. As in the proof of Theorem 2.23, we can restrict V_x so that $h|_{V_x} = \dot{s}_x = \eta_{V_x}(s_x)$. Therefore, $\sigma(h)|_{V_x} = \sigma(h|_{V_x}) = \sigma\eta(s_x) = \tau\eta(s_x) = \tau(h|_{V_x}) = \tau(h)|_{V_x}$. This means that $\sigma(h)$ and $\tau(h)$ agree on an open cover $\bigcup_{x\in U} V_x = U$; then, by collation, it follows that $\sigma(h) = \tau(h)$. Since h was an arbitrary section of Λ_P, we have that $\sigma = \tau$. ■

2.4.1 An example of sheafification

In this subsection we will consider a presheaf that is not a sheaf and, by applying the functors defined above, turn it into a sheaf. This example expands the one presented in [**2**], pp.50-51.

Let $I = [0, 1] \subseteq \mathbb{R}$ be the unit interval, equipped with the Euclidean topology of $\mathbb{R}$; let A be a set of cardinality greater than one and define

$$\begin{aligned} F : \mathcal{O}(I)^{op} &\to \mathbf{Sets} \\ \emptyset &\mapsto \{*\} \\ \emptyset \neq U &\mapsto A \\ (U \hookrightarrow V) &\mapsto id_A. \end{aligned}$$

Note that in order for a functor to be a sheaf, it must map the empty set to a one-point set (the equalizer diagram requires so, since a product over an empty index set I is the one-point set $\{*\}$). Therefore, we might as well solve this problem right away and focus on the actual reason why F is not a sheaf. Two open disconnected subsets of I, such as $B = [0, \frac{1}{3})$ and $C = (\frac{2}{3}, 1]$, would need to be mapped to two different copies of A in order for the collation property to be valid, otherwise when we consider a section of F on B, i.e. an element $a_1 \in A$, and a different section on C, i.e. another element $a_2 \neq a_1$ of A, since a_1 and a_2 restrict to the same element $\{*\}$ on $B \cap C = \emptyset$, we should have a unique section s on $F(B \cup C)$ such that $s|_B = a_1$ and $s|_C = a_2$. But since the restriction maps from $B \cup C$ to B and C are identity maps, it follows that $a_1 = s = a_2$, which is a contradiction.

We now compute the sheafification $\Gamma\Lambda_F$. By definition, an element of $\Gamma\Lambda_F$ is a section $\dot{s} : U \to \Lambda_P = \coprod_{x \in U} F_x$ such that $p\dot{s} = i : U \hookrightarrow I$. In this argument, however, we will use an equivalent definition, stating that

$$F^+ = \left\{ \dot{s} : U \to \coprod_{x \in U} F_x \;\middle|\; \begin{array}{l} \forall x \in U \;\; \dot{s}(x) \in F_x \text{ and} \\ \forall x \in U \; \exists \; V \text{ with } x \in V \subseteq U, \; \exists \; t_V \in FV \text{ s.t.} \\ \forall y \in V, \;\; \dot{s}(y) = germ_t y \end{array} \right\}.$$

We shall then prove the following proposition:

Proposition 2.25. *Let F be a presheaf of sets on X and consider the sheafification $\Gamma\Lambda_F$ defined in Section 2.4. There is a bijection between $\Gamma\Lambda_F$ and the set F^+ defined above.*

Proof. Let $U \subseteq X$ be any open subset of X. Consider a function $\dot{s} : U \to \Lambda_F = \coprod_{x \in U} F_x$; we shall prove that $\dot{s} \in \Gamma\Lambda_F$ if and only if $\dot{s} \in F^+$.

Let $\dot{s}$ be continuous and such that $p\dot{s} = id_U$. Then, for every $x \in U$, $\dot{s}x = germ_x s \in F_x$ by definition of the stalk F_x; moreover, since by Proposition 2.22 p is a local homeomorphism, we have that for $x \in U$, we can take $V = U$, $t = s$ and it follows that for each $y \in X$, $\dot{s}y = germ_y s$ by definition of $\dot{s}$.

On the other hand, we can verify that an element $\dot{s}$ of F^+ is continuous: for $germ_x s = \dot{s}x \in \Lambda F$, by definition we have an open neighbourhood V of x and an element $t \in FV$ such that $germ_y s = germ_y t$ for each $y \in V$. This means that $\dot{s}x$ has an open neighbourhood $\dot{t}(V)$ such that its counterimage V is an open neighbourhood of x, hence $\dot{s}$ is continuous. Note that V is the proper counterimage, and not merely a subset of the counterimage, since $\dot{t}$ being a section of p means that $p\dot{t} = id_V$, and $p|_{\dot{t}(V)}$ is a local homeomorphism. Moreover, $\dot{s}$ is a section since for each $x \in U$ there is a local section t such that $\dot{s}x = germ_x t$, so $p\dot{s}x = p(germ_x t) = p(\dot{t}x) = x$ since $p\dot{t} = id_V$. ∎

We now compute the stalks F_x:

$$\begin{aligned} F_x &= \{(U, s) \mid U \text{ open and } s \in FU\}_{/\sim} \\ &= \{(U, a) \mid U \text{ open and } a \in A\}_{/\sim} \end{aligned}$$

where $(U, a) \sim (V, b)$ if and only if there exist an open neighbourhood $W \subseteq U \cap V$ such that $F(U \to W)(a) = F(V \to W)(b)$. These morphisms are by definition the identity, so by taking W to be $U \cap V$ we have that $(U, a) \sim (V, b)$ if and only if $a = b$. This yields $F_x \cong A \ \forall x \in I$.

Now we compute the induced maps $FU \to F_x$: they are given by

$$\begin{array}{ccccc} A = FU & \longrightarrow & F_x & \xleftrightarrow{\cong} & A \\ \\ a & \longmapsto & [(U, a)]_{/\sim} & \xleftrightarrow{\cong} & a \end{array}$$

and so they are in fact the identity map on A. Finally, we can conclude that

$$\begin{aligned} \Gamma\Lambda_F(U) &= \left\{ \dot{s} : U \to A \,\middle|\, \begin{array}{l} \forall x \in U \ \exists V \text{ with } x \in V \subseteq U, \\ \exists t \in FV = A \text{ s.t. } \forall y \in V, \ \dot{s}(y) = germ_y t = t \end{array} \right\} \\ &= \{\dot{s} : U \to A \mid \forall x \in U \ \exists V \text{ with } x \in V \subseteq U \text{ such that } \dot{s}|_V \text{ is constant}\} \\ &= \{\dot{s} : U \to A \mid \dot{s} \text{ is locally constant}\}. \end{aligned}$$

2.5 Sheaves and ètale spaces

We have seen in the previous section that every sheaf is a sheaf of cross-sections of an appropriate bundle p, which was shown to be a local homeomorphism. Inspired by this fact, we now take a further look into the exact relationship between bundles and sheaves. To begin with, we give a more precise definition of the stated condition of local homeomorphism.

Definition 2.26. *A bundle $p : E \to X$ is ètale over X if every $e \in E$ is equipped with an open neighbourhood $V \subseteq E$ such that pV is an open subset of X and the restriction $p|_V : V \to pV$ is a homeomorphism.*

For example, all covering spaces $\tilde{X} \to X$ are ètale; however many ètale bundles are not covering maps. Note that for each open subset $U \subseteq X$ we have the pullback square

$$\begin{tikzcd} E_U \arrow[r, hook] \arrow[d, "p|_{E_U}"'] & E \arrow[d, "p"] \\ U \arrow[r, hook] & X \end{tikzcd} \tag{2.21}$$

which means that the map $E_U \to U$ is also ètale over U. We may then define a section of E to be a continuous map $s : U \to E$ such that $ps = i : U \hookrightarrow X$.

Observation 2.27. Any ètale bundle $p : E \to X$, and any of its sections, is open; moreover, since every point $e \in E$ is in the image of at least a section $p^{-1}|_{pV} : pV \to V \subseteq E$ as in Definition 2.26, the images sU of all open subsets of X through all sections s form a base for the topology of E.

Now that we have identified the class of bundles which most concerns us, namely ètale bundles, we can prove the following meaningful result:

Theorem 2.28. *Let X be a topological space. There is an adjunction between bundles on X and presheaves on X given by the functors*

$$\boldsymbol{Top/X} = \boldsymbol{Bund}X \underset{\Lambda}{\overset{\Gamma}{\rightleftarrows}} \boldsymbol{Sets}^{\mathcal{O}(X)^{op}}, \tag{2.22}$$

which map a bundle $p : Y \to X$ into the sheaf of cross-sections of Y, Γ_p and a presheaf P on X into the bundle $\Lambda P = \Lambda_P$ of germs of P. The unit and the counit, respectively, of this adjunction are the natural transformations

$$\eta_P : P \to \Gamma\Lambda P, \ \ \epsilon_Y : \Gamma\Lambda Y \to Y. \tag{2.23}$$

We have shown in Theorem 2.23 that for any sheaf P, η_P is an isomorphism; moreover, if Y is ètale over X, ϵ_Y is an isomorphism. Thus the restriction of the functors Γ and Λ respectively on ètale bundles and on sheaves gives an equivalence of categories

$$\boldsymbol{\grave{E}tale}X \rightleftarrows Sh(X), \tag{2.24}$$

which also means that $\textbf{Ètale}X$ *is a coreflective full subcategory of* $\textbf{Bund}X$*.*

Proof. The transformation η_P has been the object of Theorem 2.23, where we defined it as $\eta_P(s) = \dot{s} \in \Gamma\Lambda P$ and showed it an isomorphism for P a sheaf. This proof shall then focus on ϵ_Y. Let $p : Y \to X$ be a bundle; then we have that

$$\begin{aligned} \Lambda\Gamma Y &= \coprod_{x \in X} (\Gamma Y)_x = \{germ_x s \mid x \in X, s \in \Gamma Y(U)\} \\ &= \{\dot{s}x \mid x \in X, s : U \to Y \text{ section}\}. \end{aligned} \tag{2.25}$$

We then define $\epsilon_Y(\dot{s}x) = sx \in Y$. Note that if $s : U \to Y$, $t : V \to Y$ are sections which have the same germ $\dot{s}x = \dot{t}x$ at x, then they must agree on an open neighbourhood of x, which implies that their image under ϵ_Y is the same, $sx = tx$. We can prove that ϵ_Y is continuous by testing it on an element of the base $\{sU \mid U \subseteq X \text{ open}, s : U \to Y \text{ section}\}$ of the topology on Y (see Observation 2.27): evidently, $\epsilon_Y^{-1}(sU) = \dot{s}(U)$ is open in $\Lambda\Gamma Y$, equipped with the topology defined in Section 2.3. We also have that

$$\begin{tikzcd} \Lambda\Gamma Y \arrow[r, "\epsilon_Y"] \arrow[dr, "q"'] & Y \arrow[d, "p"] \\ & X \end{tikzcd} \tag{2.26}$$

where we named q the map $germ_x s = \dot{s}x \mapsto x$ introduced in Section 2.3. The diagram commutes since $p \circ \epsilon_Y(\dot{s}x) = p(sx) = x$, s being a section of p. This means that ϵ_Y is in fact a map of bundles. Moreover, ϵ_Y is also natural in Y: consider a morphism $\gamma : Y \to Z$ of bundles on X. We have the diagram

$$\begin{tikzcd} \Lambda\Gamma Y \arrow[r, "\epsilon_Y"] \arrow[d, "\Lambda\Gamma\gamma"'] & \Lambda\Gamma Z \arrow[d, "\epsilon_Z"] \\ Y \arrow[r, "\gamma"'] & Z \end{tikzcd} \qquad \begin{tikzcd} \dot{s}x \arrow[r, mapsto] \arrow[d, mapsto] & sx \arrow[d, mapsto] \\ (\dot{\gamma s})x \arrow[r, mapsto] & \gamma sx \end{tikzcd} \tag{2.27}$$

which commutes. As a consequence, ϵ is natural, and therefore a natural transformation $\Gamma\Lambda \to 1_{\textbf{Bund}X}$.

Now, let Y be ètale over X. Consider $y \in Y$ such that $py = x$, together with an open neighbourhood U of x equipped with a cross-section $s : U \to Y$ which maps x in y. We shall define an inverse for ϵ_Y, $\theta_Y : Y \to \Lambda\Gamma Y$, as $\theta_Y(y) = \dot{s}x$. We can easily verify the independence of this definition from the choice of the cross-section s, since we have noted before that if $\dot{s}x = \dot{t}x$, then $y = sx = tx$. Moreover, θ_Y is continuous because the counterimage an element of the basis of the topology on the codomain, $\dot{s}(U)$, is $\theta_Y^{-1}(\dot{s}(U)) = p^{-1}(U)$ which is open by continuity of p. Lastly,

$$\begin{aligned} \theta_Y(\epsilon_Y(\dot{s}x)) &= \dot{s}x \\ \epsilon_Y(\theta_Y(sx)) &= sx. \end{aligned} \tag{2.28}$$

We have then found an inverse for ϵ_Y. Next, we must prove that both composites of η and ϵ are identities. Indeed, we have that

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\eta_\Gamma} & \Gamma\Lambda\Gamma & \xrightarrow{\Gamma_\epsilon} & \Gamma \\ \\ \Gamma_Y(U) \ni s & \longmapsto & \dot{s} & \longmapsto & s \end{array} \tag{2.29}$$

and

$$\begin{array}{ccccc} \Lambda & \xrightarrow{\Lambda_\eta} & \Lambda\Gamma\Lambda & \xrightarrow{\epsilon_\Lambda} & \Lambda \\ \\ \Lambda_Y \ni germ_x s & \longmapsto & germ_x \dot{s} & \longmapsto & \dot{s}x = germ_x s. \end{array} \tag{2.30}$$

This completes the proof that Λ and Γ form an adjunction, with unit η and counit ϵ.

Next, we will use the following characterizations of $Sh(X)$ and $\mathbf{\grave{E}tale}X$:

$$\begin{aligned} Sh(X) &= \{\Gamma B \mid B \text{ bundle}\} \\ \mathbf{\grave{E}tale}X &= \{\Lambda P \mid P \text{ presheaf}\}. \end{aligned}$$

Note that by Theorem 2.23 and by this proof, $\eta_{\Gamma B} : \Gamma B \to \Gamma\Lambda\Gamma B$ and $\epsilon_{\Lambda P} : \Lambda\Gamma\Lambda P \to \Lambda P$ are isomorphisms. Consider the diagram

$$\begin{tikzcd} Sh(X) \arrow[r, shift left, "\Lambda_0"] \arrow[d, hook, "i"] & \mathbf{\grave{E}tale}X \arrow[l, shift left, "\Gamma_0"] \arrow[d, hook, "j"'] \\ \mathbf{Sets}^{\mathcal{O}(X)^{op}} \arrow[r, shift left, "\Lambda"] & \mathbf{Bund}X \arrow[l, shift left, "\Gamma"] \end{tikzcd} \tag{2.31}$$

where i and j are the inclusions of the full subcategories, and Λ_0, Γ_0 are the restrictions of Γ and Λ. Since the restrictions of the unit and counit are isomorphisms which satisfy the triangular identities, we have the stated equivalence between $Sh(X)$ and $\mathbf{\grave{E}tale}X$. It now remains to be shown that $\mathbf{\grave{E}tale}X$ is coreflective in $\mathbf{Bund}X$, i.e. that the inclusion functor $\mathbf{\grave{E}tale}X \rightarrowtail \mathbf{Bund}X$ has a right adjoint. Since the image of Γ is in $Sh(X)$, Γ restricts to a functor $\Gamma' : \mathbf{Bund}X \to Sh(X)$ such that $j \circ \Gamma' = \Gamma$. Therefore, the original adjunction restricts to an adjunction

$$\Gamma' : \mathbf{Bund}X \rightleftarrows Sh(X) : \Lambda \circ i. \tag{2.32}$$

By composing this last adjunction with the equivalence in the top row of (2.31), we have that $\Lambda_0\Gamma'$ is right adjoint to $\Lambda i \Gamma_0$; but $i\Gamma_0 = \Gamma j$ since (2.31) commutes, and $\Lambda\Gamma j \cong j$ since $\epsilon_{\Lambda P}$ is an isomorphism, so $\Lambda_0\Gamma'$ is right adjoint to $j : \textbf{Ètale}X \to \textbf{Bund}X$, therefore making the former a coreflective full subcategory of the latter. ■

Chapter 3

Grothendieck Topologies and Sites

In this chapter, we aim to define an entity which plays a role analogous to that of a topology on a space X, but on a category $\mathbf{C}$ with pullbacks: such structure is called a Grothendieck topology. These topologies provide us with the adequate environment in which we can move our first steps towards defining sheaves on mathematical objects which are more general than topological spaces.

3.1 Sieves and Grothendieck topologies

Definition 3.1. *A sieve S on C is a family of morphisms in $\boldsymbol{C}$ with codomain C such that for any A, B and for any morphisms $f : A \to C, g : B \to A$ it follows that*

$$f \in S \Rightarrow f \circ g \in S. \tag{3.1}$$

The idea behind this definition is that if a function f "goes through" the sieve, so does anything smaller: for example, a sieve on an object c of a partially ordered set regarded as a category (i.e. with its elements as objects and a morphism $a \to b$ if and only if $a \leq b$) is a set S of elements $b \leq c$ such that $a \leq b \in S$ implies $a \in S$.

Observation 3.2. Sieves on an object C of a category $\mathbf{C}$ can alternatively be defined as subfunctors of $\mathbf{y}(C)$, where $\mathbf{y}$ is the Yoneda embedding described in Observation 2.7. Indeed, a sieve S identifies the subfunctor $Q \subseteq \mathbf{y}(C)$ such that $QA = \{f \mid f : A \to C,\ f \in S\} \subseteq Hom_{\mathbf{C}}(A, C)$; conversely, a subfunctor Q yields a sieve $S = \{f \mid \exists A \in Ob(\mathbf{C}),\ f : A \to C,\ f \in QA\}$. Each operation is the inverse of the other, hence the characterization.

A common operation on sieves is that of pulling them back along an arrow: consider a sieve S on C and an arrow $h : D \to C$; the set

$$h^*S = \{g \mid cod(g) = D,\ hg \in S\} \tag{3.2}$$

is a sieve on D. We now seek to generalize the concept of sheaf of sets on a topological space by considering sheaves on "topologized" categories.

Definition 3.3. *Let $\boldsymbol{C}$ be a category. A Grothendieck topology on $\boldsymbol{C}$ is a function J such that for each object C of $\boldsymbol{C}$, $J(C)$ is a collection of sieves on C which respect the following conditions:*

(i) the maximal sieve $t_C = \{f \mid cod(f) = C\}$ is in $J(C)$;

(ii) (stability axiom) if $S \in J(C)$ and $h : D \to C$ is any morphism, then $h^*S \in J(D)$;

(iii) (transitivity axiom) if $S \in J(C)$ and R is a sieve on C such that for all $h : D \to C$ in S, $h^*R \in J(D)$, then $R \in J(C)$.

An element $S \in J(C)$ is called a covering sieve *on C. If $\boldsymbol{C}$ is a small category and J is a Grothendieck topology on $\boldsymbol{C}$, we say that the pair $(\boldsymbol{C}, J)$ is a site.*

Observation 3.4. If $R, S \in J(C)$, we can consider an element $f : D \to C$ of R. By (ii), it follows that $f^*S \in J(D)$. Note that since $f \in R$, $f^*S = f^*(R \cap S)$; moreover, $R \cap S$ is obviously a sieve. Therefore, by applying (iii) to R and $R \cap S$, it follows that $(R \cap S) \in J(C)$, hence any two covers R, S in $J(C)$ have a common refinement.

An example of a site can be given on the category $\mathcal{O}(X)$ which we already introduced, where X is a topological space. One can define a sieve S covering an open subset $U \subseteq X$ to be a family of subsets such that $V' \subseteq V$, $V \in S$ implies $V' \in S$. Call $J(U)$ the collection of such sieves. The maximal sieve on an object U will simply be the collection of all open subsets $V \subseteq U$, since an arrow in $\mathcal{O}(X)$ is the inclusion $V \subseteq U$; therefore $t_U \in J(U)$. The stability axiom holds: consider $S \in J(U)$, together with an inclusion $h : V \hookrightarrow U$; then h^*S is precisely the family of subsets of V, which is a sieve in $J(V)$ as a consequence of the definition of J (if $V' \in h^*S$, $W \subseteq V' \in h^*S$, then $W \subseteq V' \subseteq V \subseteq U$, which means $W \in h^*S$). Finally, if R is any other sieve on U, the condition that for any arrow $i : V \to U \in S$, $i^*R \in J(V)$ means that (since i can only be the inclusion) i^*R is precisely the collection of all subsets of V. Therefore, the definition of i^*R can be read as "$V' \subseteq V \subseteq U$, $V \subseteq U \in R$ implies $V' \in R$", i.e. $R \in J(U)$.

As with topologies on topological spaces, a Grothendieck topology can be described by specifying the elements of a basis, as follows:

Definition 3.5. *Let $\boldsymbol{C}$ be a category with pullbacks (a modified definition works without pullbacks). A basis for a Grothendieck topology on $\boldsymbol{C}$ is a function which assigns to each object C a collection $K(C)$ of families of morphisms with codomain C such that:*

(i') for any isomorphism $f : D \to C$, $\{f : D \to C\} \in K(C)$;

(ii') (stability axiom) if $\{f_i : C_i \to C \mid i \in I\} \in K(C)$, then for any morphism $g : D \to C$ the family of pullbacks $\{\pi_i : C_i \times_C D \to D \mid i \in I\} \in K(D)$;

(iii') (transitivity axiom) if $\{f_i : C_i \to C \mid i \in I\} \in K(C)$, and if there is for every $i \in I$ a family of maps $\{g_{ij} : D_{ij} \to C_i \mid j \in J_i\} \in K(C_i)$, then the family of composites $\{f_i \circ g_{ij} : D_{ij} \to C \mid i \in I, j \in J_i\} \in K(C)$.

To justify the introduction of this notion, we should prove the following result:

Proposition 3.6. *A basis K of a Grothendieck topology on $\boldsymbol{C}$ identifies a topology J on $\boldsymbol{C}$ through*

$$S \in J(C) \text{ if and only if } \exists\, R \in K(C) \text{ such that } R \subseteq S. \tag{3.3}$$

Proof. Consider an object C of $\mathbf{C}$. We will show that such $J(C)$ verifies the three properties of a J-cover:

(i) the maximal sieve $t_C = \{f \mid cod(f) = C\}$ is in $J(C)$ since it contains $\{f : C' \to C \mid f \text{ isomorphism}\} \in K(C)$ (see (i'));

(ii) (stability) consider $S \in J(C)$, together with a morphism $g : D \to C$. Choose any $R \subseteq S$, $R \in K(C)$ and let $T \in K(D)$ be the family of pullbacks described in (ii'). Therefore, the elements of T are morphisms $h : D \times_C C' \to D$ such that the pullback diagram

$$\begin{tikzcd} D \times_C C' \arrow[r] \arrow[d, "h"'] & C' \arrow[d, "f"] \\ D \arrow[r, "g"] & C \end{tikzcd} \tag{3.4}$$

commutes for some $f \in R$. It follows that $T \subseteq g^*S$, so $g^*S \in J(D)$;

(iii) (transitivity) let $S \in J(C)$. By (3.3) there exists $A \subseteq S$ with $A \in K(C)$. Consider $A = \{h_i : C_i \to C \mid i \in I\}$. For any sieve R on C such that $f^*R \in J(D_f)$ for all $f : D_f \to C$ in S, in particular we have that $h_i^*R \in J(C_i)$ for all $h_i \in A$, $i \in I$. Therefore every h_i^*R must have a subset $B = \{g_{ij} \mid j \in J_i,\ dom(g_{ij}) = D_{ij},\ h_i \circ g_{ij} \in R\}$ with $B \in K(C_i)$. By (iii'), the family $N = \{h_i \circ g_{ij} \mid i \in I,\ j \in J_i\} \in K(C)$; since $N \subseteq R$, we have that $R \in J(C)$.

Finally, we can conclude that J is a topology on $\mathbf{C}$. ∎

Example 3.7. A few examples of topologies on different categories may be:

- the trivial topology on a category $\mathbf{C}$, in which for all objects C, $J(C)$ only contains the maximal sieve t_C;
- consider a small subcategory $\mathbf{T}$ of $\mathbf{Top}$ which is closed under finite limits and under taking open subspaces. The open cover topology on T is the topology generated by the basis K such that $K(X) = \{f_i : Y_i \hookrightarrow X \mid i \in I\}$, where each Y_i is an open subset of X, f_i is the corresponding embedding and $\bigcup_{i\in I} Y_i = X$;
- a topology which will be useful later, on the same category $\mathbf{T}$ as above, is the one generated by the basis K' such that $\{f_i : Y_i \to X \mid i \in I\} \in K'(X)$ if and only if there is an open surjection $f : \coprod_{i\in I} Y_i \to X$ which restricts to f_i on every Y_i.

The following few pages will be dedicated to the introduction of a technical alternative to the concept of basis: the notion of precoverage, an entity capable of generating a Grothendieck topology. One can think of these as a weaker alternative to a proper basis, and in fact bases are precoverages. Some among the following results will be crucial in the forecoming sections. The definition of (pre)coverage and the subsequent remarks and lemmas follow the presentation of [**1**].

Definition 3.8. *Let $\boldsymbol{C}$ be a category. A coverage τ on $\boldsymbol{C}$ is the collection of a set $Cov_\tau(X)$ of sieves for each object X of $\boldsymbol{C}$, such that for every $S \in Cov_\tau(X)$ and for every $f : Y \to X$ there is a sieve $R \subseteq f^*S$ such that $R \in Cov_\tau(Y)$.*

Definition 3.9. *Let τ be a coverage on a category $\boldsymbol{C}$. The Grothendieck topology generated by τ is the topology J such that for any object C of $\boldsymbol{C}$, $J(C)$ is the intersection of all $J'(C)$, where J' ranges over all Grothendieck topologies such that every covering sieve of $Cov_\tau(C)$ is in $J'(C)$. In other words, J is the intersection of all Grothendieck topologies containing τ.*

Observation 3.10. A Grothendieck topology is clearly a coverage, by the stability axiom. As a coverage, it generates itself in the way above described.

Definition 3.11. *A family of morphisms $F = \{f_i : X_i \to X\}_{i\in I}$ generates a sieve S such that $S = \{f : dom f \to X \mid f \text{ factors through some } f_i\}$.*

Definition 3.12. *Let $\boldsymbol{C}$ be a category. A precoverage π on $\boldsymbol{C}$ is a law that assigns to every object X of $\boldsymbol{C}$ a set $Cov_\pi(X)$ of covering families of morphisms with target X, such that for every $\{f_i : X_i \to X\}_{i\in I} \in Cov_\pi(X)$ and for every morphism $g : Y \to X$, there exists a family $\{h_j : Y_j \to Y\}_{j\in J} \in Cov_\pi(Y)$ such that each $g \circ h_j$ factors through* *an f_i, as exemplified by the diagram below:*

$$\begin{tikzcd} Y_j \arrow[r, dashed] \arrow[d, "h_j"'] & X_i \arrow[d, "f_i"] \\ Y \arrow[r, "g"] & X. \end{tikzcd} \tag{3.5}$$

The coverage generated by a precoverage π is defined as that whose covering sieves are precisely those generated by the covering families of π in the sense of Definition 3.11. The Grothendieck topology generated by π will then be the one generated by the coverage it generates.

Observation 3.13. A basis is itself a precoverage, as follows immediately by the stability axiom for bases. In fact, the topology generated by a basis K as a basis (see Proposition 3.6) coincides with the topology generated as a precoverage: we will prove this result immediately.

Proof. Consider a basis K, together with the Grothendieck topology J it generates as a basis. Let β be the coverage generated by K, and J' the topology generated by β. For an object X, consider $S \in J(X)$ and take $F \in K(X)$ such that $F \subseteq S$. Let S' be the sieve generated by F; then $S' \in Cov_\beta(X)$, hence $S' \in J'(X)$. Now, every $f \in S' \subseteq S$ is such that $f^*S = Hom(-, domf)$ is in $J'(domf)$, by the stability axiom. Therefore $S \in J'(X)$, hence $J(X) \subseteq J'(X)$ for every X. Conversely, $\beta \subseteq J$, and therefore $J' \subseteq J$. ■

3.2 Sheaves on a site

We now seek to define sheaves of sets on a given site $(\mathbf{C}, J)$. The notion is totally analogous, but we have to specify how the definition of sheaf matches with the covers given by sieves rather than open subsets.

Definition 3.14. *Let $\boldsymbol{C}$ be a small category, equipped with a Grothendieck topology J. A presheaf of sets on $\boldsymbol{C}$ is, as before, a functor $P : \boldsymbol{C}^{op} \to \boldsymbol{Sets}$.*

Definition 3.15. *Let $(\boldsymbol{C}, J)$ be a site, and consider a presheaf P on $\boldsymbol{C}$, together with a sieve $S \in J(C)$, with C object of $\boldsymbol{C}$. A matching family for S of elements of P is a function which assigns to every arrow $f : D \to C$ of S an element $x_f \in P(D)$, in such a way that*

$$P(g)(x_f) \overset{def}{=} x_f \cdot g = x_{fg} \quad \text{for all } g : E \to D \text{ in } \boldsymbol{C}. \tag{3.6}$$

Note that since S is a sieve, $fg \in S$. Another perspective is given by considering S as a subfunctor of $\boldsymbol{y}(C) = Hom(-, C)$: therefore, a matching family is exactly a natural transformation $S \to P$, where condition (3.6) *is precisely the naturality of the functor.*

If for all $f \in S$ there exists a single element $x \in P(C)$ such that $x \cdot f = x_f$, we say that x is an amalgamation of the matching family.

Definition 3.16. *A presheaf $P : \boldsymbol{C}^{op} \to \boldsymbol{Sets}$ is a sheaf if every matching family for any cover of any object C of $\boldsymbol{C}$ has a unique amalgamation.*

Diagrammatically, this definition can be expressed by the request that for each object C of $\mathbf{C}$ and for each cover $S \in J(C)$, the diagram

$$P(C) \xrightarrow{e} \prod_{f \in S} P(domf) \overset{p}{\underset{a}{\rightrightarrows}} \prod_{\substack{f \in S \\ domf = codg}} P(domg) \tag{3.7}$$

is an equalizer. In this case, e is given by $e(x) = (x \cdot f)_{f \in S}$, p and a act on $\mathbf{x} = (x_f)_{f \in S} \in \prod_{f \in S} P(domf)$ as $(p(\mathbf{x}))_{f,g} = x_{fg}$, $(a(\mathbf{x}))_{f,g} = x_f \cdot g$.

This is equivalent to the request that for all sieves S on any object C, any natural transformation $f : S \to P$ must extend to $Hom(-, C)$ in a unique way, i.e. the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, tail] & P \\ Hom(-, C) \arrow[ur, dashed] & \end{tikzcd} \tag{3.8}$$

hence for every covering sieve S, the inclusion $S \hookrightarrow \mathbf{y}(C)$ induces an isomorphism $Hom(S, P) \cong Hom(\mathbf{y}(C), P)$.

The introduction of the notions of basis, coverage and precoverage were useful in the following sense: it almost always is much easier to verify the sheaf condition on the covering families of a precoverage or of a basis, than on the covering sieves of a Grothendieck topology. Moreover, once established that it holds on any of these two cases, it must hold on the generated topology as well. We will not prove this fact, but rather redirect the reader to [**1**], p.11 and [**6**], pp.123-124.

The last proposition of this chapter will show that the category of sheaves on a site is closed under taking limits.

Proposition 3.17. *Consider a site $(\boldsymbol{C}, J)$, together with a diagram of presheaves $\boldsymbol{J} \to \boldsymbol{Sets}^{\boldsymbol{C}^{op}}$. If all presheaves P_j are sheaves, their limit $\varprojlim P_j$ in the category of presheaves is a sheaf.*

Proof. Consider an object C of $\mathbf{C}$ and a covering sieve S for C. By hypothesis, for every $j \in \mathbf{J}$ we have an equalizer diagram

$$P_j(C) \longrightarrow \prod_{f \in S} P_j(domf) \rightrightarrows \prod_{f,g} P_j(domg). \tag{3.9}$$

Since limits commute with limits (see for example **[5]**, p.231), the limit of these equalizer diagrams is the equalizer of the limit presheaf, therefore

$$(\varprojlim P_j)(C) \longrightarrow \prod_{f \in S} (\varprojlim P_j)(domf) \rightrightarrows \prod_{f,g} (\varprojlim P_j)(domg) \tag{3.10}$$

is an equalizer, hence the sheaf condition holds for $\varprojlim P_j$. ■

Chapter 4

Condensed Sets

The following chapter will introduce a concept theorized by D. Clausen and P. Scholze in order to provide coherence between the topologies on sets/groups/rings and the algebraic nature of morphisms between them. For example, the category **TopAb** of topological abelian groups is not itself an abelian category, as one may wish: but the category of condensed abelian groups will be. As a consequence, even though outside of the scope of this thesis, homology and cohomology theories can be derived for condensed abelian groups. The content of this section is mostly derived from [**1**], [**4**] and [**7**].

4.1 Profinite spaces and condensed sets

As we pointed out, the crucial motivation that prompted the definition of condensed sets is that of providing the context of an abelian category for the study of the (co)homology of topological spaces which have an algebraic structure. This was necessary since, for example, often the topological and the algebraic structures of abelian groups interact in such a way that the category **TopAb** itself does not verify some important properties.

Concretely, we will now see that the category of topological abelian groups is not abelian. It suffices to consider the "identity" map $f : (\mathbb{R}, \tau_D) \to (\mathbb{R}, \tau_\epsilon)$, where the topologies are, respectively, the discrete and the Euclidean. This morphism is not an isomorphism in **TopAb** because the unique map $g : (\mathbb{R}, \tau_\epsilon) \to (\mathbb{R}, \tau_D)$ such that $fg = 1_{(\mathbb{R}, \tau_\epsilon)}$ is not continuous, hence not a morphism in **TopAb**. If **TopAb** were to be abelian, this should be explained by the existence of a nontrivial kernel or cokernel. However, the zero map $0 : (\mathbb{R}, \tau_D) \to (\mathbb{R}, \tau_\epsilon)$ is trivial, hence the kernels and cokernels of f are themselves trivial.

We shall now pave our way towards the definition of condensed set. These will prove to be the adequate structures with which we will be able to solve this problem: in the end, we will conclude that the category **Cond**(**Ab**) of condensed abelian groups will be an abelian category, and we will identify a cokernel for the morphism proposed above.

The following definition can be found in [**8**] as Definition 4.20.1.

Definition 4.1. *A diagram (i.e. a functor) $M : \boldsymbol{J} \to \boldsymbol{C}$ is cofiltered if the following conditions are true:*

(i) $\boldsymbol{J}$ is non-empty;

(ii) for every pair of objects X, Y of $\boldsymbol{J}$ there exists an object Z together with two morphisms $Z \to X$, $Z \to Y$;

(iii) for every pair of objects X, Y of $\boldsymbol{J}$, and for every pair of morphisms $f, g : X \to Y$ there exists an arrow $h : W \to X$ of $\boldsymbol{J}$ such that $M(f \circ h) = M(g \circ h)$ (as morphisms in $\boldsymbol{C}$).

The category $\boldsymbol{J}$ is cofiltered if $id_{\boldsymbol{J}}$ is cofiltered. In this case, any functor from $\boldsymbol{J}$ to any category will be cofiltered.

A cofiltered limit is the limiting cone (or less appropriately its vertex) over a cofiltered diagram.

Observation 4.2. The notions of filtered diagram, filtered (co)limit and filtered category are the formal duals of those above.

Example 4.3. A profinite set is a cofiltered limit of finite sets, viewed as discrete topological spaces in the category **Top**. Profinite sets form the category **Prof**, whose morphisms are continuous maps.

Observation 4.4. It might well be clearer to imagine profinite sets through the characterization given by this lemma (Lemma 5.22.2 in [**8**]), whose proof can be found there.

Lemma 4.5. *Let X be a topological space. Then X is a profinite set if and only if it is compact, Hausdorff and totally disconnected (i.e. its only connected subspaces are singletons).*

Profinite sets are the mathematical objects we will work on. Since the category **Prof** is large, we must settle the possible set-theoretic problems that could arise by fixing an uncountable strong limit cardinal κ (i.e. such that for every cardinal $\lambda < \kappa$ it follows that $2^{\lambda} < \kappa$) and considering only profinite sets of cardinality less than κ. We will still write **Prof** to denote the category formed by these profinite sets.

Observation 4.6. The category **Prof** has pullbacks: i.e. for any two maps $f : Y \to X$, $g : Z \to X$ with common target, the fibre product $Y \times_X Z$ is in **Prof**. A proof is given in [**1**], p.13.

Prof is made a site by considering the Grothendieck topology τ generated by the precoverage of finite jointly surjective families: let S be a profinite set; then a finite collection of morphisms $\{f_i : S_i \to S\}_{i\in I}$ is a covering family if and only if the induced arrow (resulting from the universal property of coproducts) $\coprod_{i\in I} S_i \to S$ is surjective. A proof that these families form a precoverage, and hence that $(\mathbf{Prof}, \tau)$ is a site, can be found in [**1**], p.14.

Definition 4.7. *Let* $\boldsymbol{C}$ *be a category. The category* $\boldsymbol{Cond}(\boldsymbol{C})$ *of condensed objects of* $\boldsymbol{C}$ *is the category of* $\boldsymbol{C}$*-valued sheaves on the site* $(\boldsymbol{Prof}, \tau)$*. For example, a condensed group is a contravariant functor* $T : \boldsymbol{Prof}^{op} \to \boldsymbol{Grp}$ *such that the sheaf condition holds: for every profinite set* S *and for every jointly surjective family* $\mathcal{F} = \{f_i : S_i \to S\}_{i\in I}$ *the diagram*

$$T(S) \xrightarrow{e} \prod_{\substack{f_i\in\mathcal{F}\\ i\in I}} T(S_i) \overset{p}{\underset{a}{\rightrightarrows}} \prod_{\substack{f_i\in\mathcal{F}\\ codg=S_i}} T(domg) \tag{4.1}$$

is an equalizer. In general, given a condensed set T*, the set* $T(*)$ *is called its underlying set.*

We shall now provide two requisites which are equivalent to this sheaf condition; but first consider the following lemmas:

Lemma 4.8. *Let* $\Sigma = \{f_i : S_i \to S \mid i \in I\}$ *be a sieve formed by a finite number of jointly surjective maps on an object* S *of* $\boldsymbol{Prof}$*. Consider a presheaf* T *on* $\boldsymbol{Prof}$*, together with a family* $F = \{(x_{f_i})_{i\in I}\}$*, where* $x_{f_i} \in T(S_i)$ *for every* $i \in I$*.*

Such a family is (the image of) a matching family for Σ *if and only if for any commutative diagram*

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "g"'] & S_j \arrow[d, "f_j"] \\ S_i \arrow[r, "f_i"] & S \end{tikzcd} \tag{4.2}$$

it follows that $x_{f_i} \cdot g = x_{f_j} \cdot h$*.*

Proof. ($\Rightarrow$) Suppose F is a matching family in the sense of Definition 3.15. Consider any commutative diagram like that of Equation (4.2). We immediately have that $x_{f_i} \cdot g = x_{f_i g} = x_{f_j h} = x_{f_j} \cdot h$, where the paracentral equalities are true by applying the matching condition.

($\Leftarrow$) Suppose F verifies the condition stated above for any commutative diagram. We must prove that for every $i \in I$ and for every $g : Y \to S_i$ it follows that $x_{f_i} \cdot g = x_{f_i g}$. In our hypotheses I is finite; therefore we can order it and prove the thesis by induction. First of all, we can assume Σ contains the empty function, since the empty set is profinite (it is compact, Hausdorff and totally disconnected) and Σ is still finite and jointly surjective. Obviously the empty function verifies the matching condition trivially, since it cannot be composed from the left. Now suppose the thesis be true for all $j < i$, and consider f_i together with any $g : Y \to S_i$. Since the category **Prof** has pullbacks, we can form the following commutative diagram, where $j < i$:

$$\begin{tikzcd}
Y \arrow[dr, dashed, "\delta"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "g"'] & & \\
& S_i \times_S S_j \arrow[r, "q"] \arrow[d, "p"'] & S_j \arrow[d, "f_j"] \\
& S_i \arrow[r, "f_i"] & S
\end{tikzcd} \tag{4.3}$$

where δ is the unique function such that $g = p\delta$ and $h = q\delta$ (in fact, we will define h as $q\delta$). Finally, we have that

$$\begin{aligned}
x_{f_i} \cdot g &\overset{def}{=} T(g)(x_{f_i}) && \\
&= T(\delta)T(p)(x_{f_i}) && \text{by functoriality of } T \\
&\overset{def}{=} T(\delta)(x_{f_i} \cdot p) && \\
&= T(\delta)(x_{f_j} \cdot q) && \text{by hypothesis, since } f_i p = f_j q \\
&= T(\delta)(x_{f_j q}) && \text{by inductive hypothesis} \\
&= x_{f_j q \delta} && \text{id.} \\
&= x_{f_i p \delta} && \\
&= x_{f_i g}. &&
\end{aligned}$$

■

Lemma 4.9. *Let $T : \boldsymbol{C}^{op} \to \boldsymbol{Sets}$ be a presheaf. Consider an object X of $\boldsymbol{C}$, together with a family of morphisms $F = \{f_i : X_i \to X\}_{i \in I}$. Suppose the fibre products $X_i \times_X X_j$ exist for all $i, j \in I$. Then a family $(x_{f_i})_{i \in I}$, with each $x_{f_i} \in T(X_i)$, is (the image of) a matching family for F if and only if*

$$x_{f_i} \cdot \pi_{ij,1} = x_{f_j} \cdot \pi_{ij,2}$$

for all $i, j \in I$, where $\pi_{ij,1} : X_i \times_X X_j \to X_i$, $\pi_{ij,2} : X_i \times_X X_j \to X_j$ are the two projections from the fibre product.

Proof. ($\Rightarrow$) Let $(x_{f_i})_{i\in I}$ be (the image of) a matching family. Then $\pi_{ij,1} : X_i \times_X X_j \to X_i$ and $\pi_{ij,2} : X_i \times_X X_j \to X_j$ can be placed in the commutative pullback square and thus $x_{f_i} \cdot \pi_{ij,1} = x_{f_j} \cdot \pi_{ij,2}$ for all $i, j \in I$ by the lemma immediately above.

($\Leftarrow$) Let $(x_{f_i})_{i\in I}$ be a family, with $x_{f_i} \in T(X_i)$ such that $x_{f_i} \cdot \pi_{ij,1} = x_{f_j} \cdot \pi_{ij,2}$ for all $i, j \in I$. Consider $g : Y \to X_i$, $h : Y \to X_j$ such that $f_i \circ g = f_j \circ h$. By the universal property of pullbacks, we have the diagram

$$\begin{tikzcd} Y \arrow[dr, dashed, "l"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "g"'] & & \\ & X_i \times_X X_j \arrow[r, "\pi_{ij,2}"] \arrow[d, "\pi_{ij,1}"'] & X_j \arrow[d, "f_j"] \\ & X_i \arrow[r, "f_i"] & X, \end{tikzcd} \tag{4.4}$$

so $x_{f_i} \cdot g = x_{f_i} \cdot \pi_{ij,1} \cdot l = x_{f_j} \cdot \pi_{ij,2} \cdot l = x_{f_j} \cdot h$, hence the matching condition is fulfilled. ∎

Observation 4.10. In case all relevant fibre products exist, the sheaf condition for a presheaf T (i.e. that every matching family have a unique amalgamation) is equivalent to the requisite that

$$T(X) \xrightarrow{e} \prod_{i\in I} T(X_i) \overset{p_1}{\underset{p_2}{\rightrightarrows}} \prod_{(i,j)\in I\times I} T(X_i \times_X X_j) \tag{4.5}$$

be an equalizer diagram. Here the maps are $e(x) = (x \cdot f_i)_{i\in I}$, and for $\mathbf{x} = (x_{f_i})_{i\in I} \in \prod_{i\in I} T(X_i)$, $(p_1(\mathbf{x}))_{i,j} = x_{f_i} \cdot \pi_{ij,1}$, $(p_2(\mathbf{x}))_{i,j} = x_{f_j} \cdot \pi_{ij,2}$.

Proposition 4.11. *The covering families of morphisms of* ***Prof*** *of the types:*

(i) $\{f_i : S_i \to S\}_{i\in I}$, where I is finite and $\coprod_{i\in I} S_i \to S$ is an isomorphism;

(ii) singleton families $\{p : S' \to S\}$, with p a surjective morphism;

form a precoverage on ***Prof****. Moreover, this precoverage generates the same topology τ generated by finite jointly surjective families.*

Proof. Let $\{f_i : S_i \to S\}$ be a family of morphisms of type (i). Consider any morphism of profinite sets $g : R \to S$, and define $R_i = g^{-1}(f_i(S_i))$. Since each f_i is an omeomorphism when restricted to its image, we have that $R = \bigcup_{i\in I} R_i$; hence the inclusions $R_i \hookrightarrow R$

are a finite jointly surjective family. We now have to show that each restriction $g|_{R_i}$ factors through f_i, for every $i \in I$. We can write $g|_{R_i} = f_i \circ (f_i|_{f(S_i)})^{-1} \circ g|_{R_i}$, hence the precoverage condition is verified.

On the other hand, consider a surjective morphism $p : S' \to S$, i.e. a family of type (ii). Given $g : R \to S$, since the category **Prof** has pullbacks, we can pull back p along g to obtain a morphism $q : R \times_S S' \to R$, which is surjective: consider $z \in R$; we have that $g(z) \in S$ is, by surjectivity of p, equal to $p(w)$ for some $w \in S'$. This means that the element $(z, w) \in R \times_S S'$ since $g(z) = p(w)$, and $q((z, w)) = z$, i.e. z is in the image of q. Moreover, by definition of fibre product, the composition of this morphism with g factors through f. Thus, the precoverage condition is valid with respect to any morphism of type (i) or (ii). Denote the precoverage constituted by the families of either type by π_1, and the topology it generates by τ_1. Let π_2 be the precoverage formed by finite jointly surjective families, which generates the topology τ. Since $\pi_1 \subseteq \pi_2$, we have that $\tau_1 \subseteq \tau$. We need to show that the converse also holds. Let $Z \in \tau(S)$ be a sieve on a profinite set S generated by a finite jointly surjective family $\{f_i : S_i \to S\}_{i \in I}$. Call $f : \coprod_{i \in I} S_i \to S$ the surjective map induced by the f_i. For each $j \in I$, let $\phi_j : S_j \hookrightarrow \coprod_{i \in I} S_i$ be the inclusion. As for each i, $f \circ \phi_i = f_i \in Z$, it follows $\phi_i \in f^*Z$; so $f^*Z \in \tau_1(\coprod_{i \in I} S_i)$ because it contains the family $\{\phi_i\}_{i \in I}$, which is of type (i). Moreover, for any map h with target $\coprod_{i \in I} S_i$, by the transitivity axiom we have that $h^*(f^*Z) \in \tau_1(\coprod_{i \in I} S_i)$. Let Z_f be the sieve generated by $\{f\}$: $Z_f \in \tau_1(S)$ since $\{f\}$ is a family of type (ii). For any $g : R \to S \in Z_f$, we can write $g = f \circ h$ (by Definition 3.11). Then

$$g^*Z = (f \circ h)^*Z = h^*(f^*Z) \in \tau_1(\coprod_{i \in I} S_i). \tag{4.6}$$

By the transitivity axiom of Grothendieck topologies, this implies $Z \in \tau_1(S)$, so $\tau \subseteq \tau_1$. The two are then equal by double inclusion. ■

Theorem 4.12. *Consider the site $(\textbf{\textit{Prof}}, \tau)$. A presheaf T on* ***Prof*** *is a sheaf if and only if $T(\emptyset) = \{*\}$ and it satisfies two conditions:*

1. *for any finite collection $(S_i)_{i \in I}$ of profinite sets, the natural map*

$$T(\coprod_{i \in I} S_i) \to \prod_{i \in I} T(S_i) \tag{4.7}$$

is a bijection;

2. *for any surjection $S' \to S$ of profinite sets, together with the fibre product $S' \times_S S'$ and the two projections p_1, p_2 to S, the map*

$$T(S) \to \{x \in T(S') \mid x \cdot p_1 = x \cdot p_2 \in T(S' \times_S S')\} \tag{4.8}$$

is a bijection.

Proof. Let us call a sieve of type (i) or (ii) according to the proposition above. Recall that our site is equivalently generated by sieves of those types. We will show that T satisfies the sheaf condition with respect to every sieve of type (i) (respectively, (ii)) if and only if it satisfies 1 (respectively, 2).

Consider a finite family $\{f_i : S_i \to S\}_{i\in I}$ of type (i). We can assume that $S = \coprod_{i\in I} S_i$ and that each f_i is the inclusion $S_i \hookrightarrow \coprod_{i\in I} S_i$. Then the fibre products $S_i \times_S S_j$ are empty for $i \neq j$ and equal to S_i otherwise. By Lemma 4.9, since in the nonempty case the two maps $\pi_{ij,1}$ and $\pi_{ij,2}$ are both equal to the identity on S_i, it follows that the sheaf condition is equivalent to the existence of a unique equalizer map e for the two identical maps $\pi_{ij,1}$ and $\pi_{ij,2}$. This implies that e itself, if and only if it exists, is an isomorphism for a more general reason, as proven below. Consider $1_{\prod_{i\in I} T(S_i)}$, the identity: in particular, it obviously verifies $1_{\prod_{i\in I} T(S_i)} \circ \pi_{ij,1} = 1_{\prod_{i\in I} T(S_i)} \circ \pi_{ij,2}$ just like any map. By the universal property of the equalizer, there must be a unique map $k : \prod_{i\in I} T(S_i) \to T(\coprod_{i\in I} S_i)$ as in this diagram:

$$\begin{tikzcd} T(\coprod_{i\in I} S_i) \arrow[r, "e"] & \prod_{i\in I} T(S_i) \arrow[r, shift left, "\pi_{ij,1}"] \arrow[r, shift right, "\pi_{ij,2}"'] & \prod_{(i,j)\in I\times I} T(S_i \times_S S_j) \\ \prod_{i\in I} T(S_i). \arrow[u, dashed, "k"] \arrow[ur, "1_{\prod_{i\in I} T(S_i)}"'] & & \end{tikzcd} \tag{4.9}$$

Moreover, $ek = 1_{\prod_{i\in I} T(S_i)}$. As an equalizer, e is monic; from $eke = 1_{\prod_{i\in I} T(S_i)} e = e$ with a left cancellation we have $ke = 1_{T(\coprod_{i\in I} S_i)}$, hence e is precisely the isomorphism required by condition 1.

In the case of sieves of type (ii), the sheaf condition on the generating singleton family is equivalent to the fact that the diagram

$$T(S) \xrightarrow{e} T(S') \underset{p_2}{\overset{p_1}{\rightrightarrows}} T(S' \times_S S') \tag{4.10}$$

is an equalizer. Now, for any object in the image of $T(p)$, i.e. for any $y \in T(S')$ such that $y = x \cdot p$ for some $x \in T(S)$, we have that $y \cdot p_1 = y \cdot p_2$ since $x \cdot p \cdot p_1 = x \cdot (p \circ p_1) = x \cdot (p \circ p_2) = x \cdot p \cdot p_2$. This means that $T(p)$ is a morphism from $T(S)$ to

$E = \{y \in T(S') \mid y \cdot p_1 = y \cdot p_2\}$. We then find ourselves in this situation:

$$\begin{tikzcd} T(S) \arrow[r, "e"] \arrow[d, "T(p)"', shift right] & T(S') \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & T(S' \times_S S') \\ E. \arrow[u, dashed, "k"', shift right] \arrow[ur, hook, "i"'] & & \end{tikzcd} \tag{4.11}$$

Here k is the morphism induced by the universal property and i is the inclusion (hence it is monic). Finally, we have that

$$i = ek = iT(p)k \tag{4.12}$$

$$e = iT(p) = ekT(p). \tag{4.13}$$

From these equation, by left cancellation (since both e and i are monic) we conclude that $T(p)$ and k are inverses, hence the required bijection holds. ■

4.2 Relationship with topological spaces

Now that we have a more precise description of condensed sets, we must clarify how exactly these functors are related to the usual topological spaces (or rings, groups,...).

Definition 4.13. *A topological space X is compactly generated if for every map $f : X \to Y$, the fact that, for all compact Hausdorff spaces S with a map $S \to X$, the composite $S \to X \to Y$ is continuous implies that f itself is continuous. The inclusion of compactly generated spaces into topological spaces is a forgetful functor that admits a right adjoint $X \mapsto X^{cg}$, where X^{cg} is a topological space with underlying set X, equipped with the quotient topology for the map $\coprod S \to X$, where the disjoint union admits every compact Hausdorff space S with a map to X.*

Observation 4.14. Note that any compact Hausdorff space X has a surjection from a profinite set S, for example by considering the Stone-Čech compactification of X as a discrete set. We will not discuss this topic in detail, since it would distract us from the main focus: it will be sufficient for us to know that such a map exists. Moreover, since any Stone-Čech compactification is compact, this surjection will automatically be closed, hence a quotient map. This enables us to replace compact Hausdorff spaces with profinite spaces in the definition above, leaving the definition itself and the functor $X \mapsto X^{cg}$ unaltered.

Once again, to settle the set-theoretical problems, we will adopt the following definition:

Definition 4.15. *A topological space X is called κ-compactly generated if it is equipped with the quotient topology for the map $\coprod S \to X$, where S now ranges over compact Hausdorff spaces of cardinality less than κ (with κ being the strong limit cardinal introduced above). Such compact Hausdorff spaces admit a surjection from their compactification S' as discrete sets; furthermore S' will be a subset of $\mathcal{P}(\mathcal{P}(S))$, hence $|S'| \leq 2^{2^{|S|}} < \kappa$. We will therefore substitute S with S' in the definition, and write $X^{\kappa-cg}$ for the topological space given by the underlying set of X, equipped with the quotient topology for the map $\coprod S \to X$, where S ranges over κ-small compact Hausdorff spaces. $X \mapsto X^{\kappa-cg}$ will be the right adjoint of the forgetful functor from κ-small compact Hausdorff spaces to topological spaces.*

Theorem 4.16. *The correspondence*

$$G : \boldsymbol{Top} \to \boldsymbol{Cond}(\boldsymbol{Top})$$
$$T \mapsto \underline{T}$$

from topological spaces to κ-small condensed sets, where

$$\underline{T} : \boldsymbol{Prof}^{op} \to \boldsymbol{Sets}$$
$$S \mapsto \{f : S \to T \mid f \text{ is continuous}\}$$
$$g : R \to S \mapsto \underline{T}(g)(f : S \to T) = g \circ f : R \to T,$$

is a faithful functor, and its restriction to the full subcategory of κ-compactly generated topological spaces is fully faithful. We define the action of G on a continuous function $h : A \to B$ between topological spaces as the natural transformation $\underline{h} : \underline{A} \to \underline{B}$ such that for any profinite set S, the morphism $\underline{h}_S : \underline{A}(S) \to \underline{B}(S)$ assigns to $f : S \to A$ its composite $hf : S \to B$. The facts that $\underline{T}$ is a condensed set and that G is a functor will be proven below.

Moreover, G admits a left adjoint F, sending $P \to P(*)_{top}$, where P is a condensed set and $P(*)_{top}$ is its underlying set $P(*)$, equipped with the quotient topology for the map $\coprod_{S \to P} S \to P(*)$. Here the disjoint union ranges over all κ-small profinite sets S with a map to P.

Lastly, the counit of the adjunction $\underline{X}(*)_{top} \to X$ agrees with the counit $X^{\kappa-cg} \to X$ of the adjunction between κ-compactly generated topological spaces and all topological spaces; as a consequence, $\underline{X}(*)_{top} \cong X^{\kappa-cg}$.

Proof. First of all, we must show that G is a functor. If we suppose that $\underline{T}$ is in fact a condensed set, as we will prove immediately after, it only remains to show that G maps a morphism $h : A \to B$ of topological spaces, i.e. a continuous function, onto a morphism $\underline{h}$ of sheaves. Consider any profinite space S, together with any morphism $f : S \to R$ of profinite spaces. Define

$$\begin{aligned} \underline{h}_S : \underline{A} &\to \underline{B} \\ \gamma : S \to A &\mapsto h\gamma : S \to B. \end{aligned}$$

Then the following square commutes by associativity of composition:

$$\begin{array}{ccc} \underline{A}(R) & \xrightarrow{\underline{h}_R} & \underline{B}(R) \\ \downarrow{\scriptstyle \underline{A}(f)} & & \downarrow{\scriptstyle \underline{B}(f)} \\ \underline{A}(S) & \xrightarrow{\underline{h}_S} & \underline{B}(S). \end{array} \tag{4.14}$$

Therefore $\underline{h}$ is a natural transformation, and G is a functor.

In order to prove that $\underline{T}$ is a condensed set, we will note that $\underline{T}(\emptyset) = \{*\}$ and verify the conditions 1 and 2 of Theorem 4.12 (observing that Condition 1 holds when verified on two arbitrary profinite sets, by iteration). The bijection between $\underline{T}(S_1 \coprod S_2)$ and $\underline{T}(S_1) \times \underline{T}(S_2)$ is evident by sending each continuous function $f : S_1 \coprod S_2 \to T$ to the pair $(f|_{S_1}, f|_{S_2})$, and vice versa. Consider a surjection $p : S' \to S$ of profinite sets. In particular, p is a surjection of compact Hausdorff spaces, therefore it is a quotient map. It is therefore closed, so that any map $S \to T$ for which $S' \to S \to T$ is continuous must itself be continuous. The bijection of Condition 2 is then given by $\underline{T}(S) \ni f \mapsto f \circ p \stackrel{def}{=} g \in \underline{T}(S')$: it is easily shown that $g \cdot p_1 = \underline{T}(p_1)(g) \stackrel{def}{=} gp_1 = fpp_1 = fpp_2 = gp_2 \stackrel{def}{=} \underline{T}(p_2)(g) = g \cdot p_2 \in \underline{T}(S' \times_S S')$. Conversely, any $g : S' \to T$ can be written as psg, where s is a section of p (i.e. a map $s : S \to S'$ such that $ps = 1_{S'}$, which exists since p is surjective); therefore $g = p \circ f$ for $f = sg$.

Before showing the (full) faithfulness of G, we will begin by proving that the adjunction holds; we will see that the latter implies the former. This part of the proof follows [**10**]. We want to show that for every κ-condensed set T and for every topological space X there is a bijection

$$\Phi : Hom_{\mathbf{Cond(Top)}}(T, \underline{X}) \cong Hom_{\mathbf{Top}}(T(*)_{top}, X) \tag{4.15}$$

that is natural in T and X. Recall that the topology on $T(*)_{top}$ is the quotient topology for $\pi : \coprod_{(S, f \in T(S))} S \to T(*)$. By the Yoneda lemma (Lemma 2.8), every $f \in T(S)$ is in correspondence with a morphism of condensed sets from the representable sheaf $\mathbf{y}(s)$ to T, which we will denote itself as f. This means that f induces the map $\mathbf{y}(s)(*) = S \to T(*)$ given by $f_{\{*\}}$. Write $\iota_f : S \to \coprod_{(S',f')} S'$ for the insertion of the coordinate corresponding to (S, f) in the domain of π. The unit $\eta : 1_{\mathbf{Cond(Top)}} \to GF$ is the natural transformation given by

$$\begin{aligned}(\eta_T)_S : T(S) &\to Hom_{\mathbf{Top}}(S, T(*)_{top}) \\ f &\mapsto \pi \circ \iota_f.\end{aligned}$$

We therefore have the maps

$$\begin{aligned}\Phi : Hom_{\mathbf{Cond(Top)}}(T, \underline{X}) &\leftrightarrow Hom_{\mathbf{Top}}(T(*)_{top}, X) : \Psi \\ f &\mapsto f_{\{*\}} \\ g \circ \eta_T &\leftarrow\!\shortmid g.\end{aligned}$$

We only need to check that Φ and Ψ are inverses. Consider a continuous function $g : T(*)_{top} \to X$; then, $\Phi(\Psi(g)) : T(*)_{top} \to X$ is the map $(\Psi(g))_{\{*\}} : T(*)_{top} \to \underline{X}(*)_{top}$. For any $t \in T(*)_{top}$, we have that $(\Psi(g))_{\{*\}}(t) = (g \circ \pi \circ \iota_f) : \{*\} \to X = g(t)$, since $\pi \circ \iota_f : \{*\} \to X$ is the constant map with value t. We therefore conclude that $\Phi(\Psi(g)) = g$ as they are pointwise equal. Conversely, consider a natural transformation $f : T \to \underline{X}$. The natural transformation $\Psi(\Phi(f)) = \Psi(f_{\{*\}})$ is such that for any profinite set S with a map $h \in T(S)$, $\Psi(f_{\{*\}})_S(h) = f_{\{*\}} \circ \pi \circ \iota_h : S \to X$. But by definition, $\pi \circ \iota_h = h_{\{*\}} : h_S(*)_{top} \to T(*)_{top}$; thus $\Psi(f_{\{*\}})_S(h) = f_{\{*\}} \circ h_{\{*\}} = (fh)_{\{*\}}$. This is the continuous map $f_S(h) : S \to X$, hence $\Psi(\Phi(f)) = f$ by arbitrariness of S and h. The bijection formed by Φ and Ψ is natural in X and in T. Indeed, consider a continuous function $h : X_0 \to X_1$. The diagram

$$\begin{tikzcd} Hom_{\mathbf{Cond(Top)}}(T, \underline{X_0}) \arrow[r, "\Phi"] \arrow[d, "\underline{h}_*"'] & Hom_{\mathbf{Top}}(T(*)_{top}, X_0) \arrow[d, "h_*"'] \\ Hom_{\mathbf{Cond(Top)}}(T, \underline{X_1}) \arrow[r, "\Phi"] & Hom_{\mathbf{Top}}(T(*)_{top}, X_1) \end{tikzcd} \tag{4.16}$$

commutes since $(\underline{h} \circ f)_{\{*\}} = \underline{h}_{\{*\}} \circ f_{\{*\}} = h \circ f_{\{*\}}$. The naturality in T is completely analogous.

By Observation 4.14, every compact Hausdorff space is the quotient of a profinite space. As a consequence, the quotient topologies τ_1, τ_2 with which we equipped X to define $\underline{X}(*)_{top}$ and $X^{\kappa-cg}$ are in fact the same: they are defined to be the finest topologies for which all maps $S \to X$ are continuous, where S is respectively a κ-profinite set and a κ-compact Hausdorff space. Hence $\tau_2 \subseteq \tau_1$ as all profinite sets are compact Hausdorff. Conversely, consider a compact Hausdorff space T with a map $f : T \to \underline{X}(*)_{top}$; let S be the profinite space of which T is a quotient via $p : S \to T$. By hypothesis, τ_1 makes $f \circ p$ continuous. If $U \subseteq \underline{X}(*)_{top}$ is open for τ_1, we have that $(f \circ p)^{-1}(U) = p^{-1}(f^{-1}(U))$ is open in S. Since p is a quotient map, this is equivalent to $f^{-1}(U)$ being open, hence f must itself be continuous and by definition we have the other inclusion $\tau_1 \subseteq \tau_2$. As a consequence, $\underline{X}(*)_{top} \cong X^{\kappa-cg}$. Finally, we can conclude that

$$\begin{aligned} Hom_{\mathbf{Cond(Top)}}(\underline{X}, \underline{Y}) &\cong Hom_{\mathbf{Top}}(\underline{X}(*)_{top}, Y) \\ &= Hom_{\mathbf{Top}}(X^{\kappa-cg}, Y) \\ &\hookrightarrow Hom_{\mathbf{Top}}(X, Y), \end{aligned}$$

where the last arrow is an isomorphism for a κ-compactly generated space $X = X^{\kappa-cg}$. Therefore the restriction of G to κ-compactly generated spaces is fully faithful. ■

The last theorem that we will introduce explains us that a wide number of categorical properties of **Ab** are also verified in the condensed category **Cond**(**Ab**); indeed many of Grothendieck's axioms for abelian categories, denoted $(AB-)$, are shown to hold. While some also hold in any category of sheaves of abelian groups, axioms $(AB4*)$ and $(AB6)$ hardly ever do generally.

Theorem 4.17. *The category **Cond**(**Ab**) of κ-condensed abelian groups is abelian. Moreover, the following properties are verified:*

- $(AB3)-(AB3*)$ *all colimits and limits exist;*
- $(AB4)-(AB4*)$ *direct sums (coproducts) and arbitrary products are exact (i.e. if for every index j there are short exact sequences of condensed abelian groups $0 \longrightarrow A_j \longrightarrow B_j \longrightarrow C_j \longrightarrow 0$, then $0 \longrightarrow \prod_j A_j \longrightarrow \prod_j B_j \longrightarrow \prod_j C_j \longrightarrow 0$ is also exact; the same for coproducts);*
- $(AB5)$ *filtered colimits are exact;*
- $(AB6)$ *given any family of filtered categories $(I_j)_{j\in J}$ (see Observation 4.2) with functors $i \mapsto M_i$ to κ-condensed abelian groups, the map*

$$\varinjlim_{(i_j \in I_j)_j} \prod_{j\in J} M_{i_j} \to \prod_{j\in J} \varinjlim_{i_j \in I_j} M_{i_j} \tag{4.17}$$

is an isomorphism of condensed abelian groups.

This theorem is more easily shown by making use of the following notions.

Definition 4.18. *A compact Hausdorff space S is extremally disconnected if every surjection $f : S' \to S$ from any other compact Hausdorff space splits, i.e. there exists a section $g : S \to S'$ such that the composition fg is the identity on S.*

Observation 4.19. The notion of extremally disconnected space is closely entwined with that of Stone-Čech compactification: for example, every compactification of a discrete set is extremally disconnected, and every extremally disconnected space is a retract of a Stone-Čech compactification. Moreover, extremally disconnected spaces can be characterized by the property that the closure of any open subset be open. For more information, see [**3**].

Lemma 4.20. *There is an equivalence of categories between **Cond**(**Top**), the category of κ-condensed sets, and the category of sheaves on the site of κ-small extremally disconnected compact Hausdorff spaces, whose covering families are finite jointly surjective families.*

Proof. Note that any compact Hausdorff extremally disconnected set S' is also totally disconnected: consider $x \in S'$ and its connected component $C(x)$. For any other $y \in S'$ we want to show that $y \notin C(x)$. Since S' is Hausdorff, there is an open neighbourhood U of x with $y \notin \overline{U}$. But $\overline{U} \cap C(x)$ is both open and closed, thus $\overline{U} \cap C(x) = C(x)$, meaning that $y \notin C(x)$.

As a consequence, any sheaf of (κ-small) extremally disconnected spaces is also a sheaf on **Prof** (recall Lemma 4.5). Given such a sheaf T, we will need to extend its domain on all profinite sets in an unique way. This is easily accomplished: for any profinite set S, we shall choose its Stone-Čech compactification S' together with the unique surjection $p : S' \to S$. By pulling back p along itself we obtain the diagram

$$\begin{tikzcd} S' \times_S S' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & S' \arrow[d, "p"] \\ S' \arrow[r, "p"] & S. \end{tikzcd} \tag{4.18}$$

Since **Prof** is stable under pullbacks, the profinite set $S' \times_S S'$ also has a unique surjection $q : S'' \to S'$ from its compactification. We can construct $T(S)$ as the equalizer of the two maps $T(p\pi_1 q)$ and $T(p\pi_2 q)$. This choice is unique because of the uniqueness of p, q, S', S''. The equivalence between the categories is then easily verified by considering, in the other direction, the restriction of any condensed set to extremally disconnected spaces. ∎

Theorem 4.17. By the equivalence just established, we shall prove the results by considering the category of sheaves of abelian groups on the site of κ-small extremally disconnected spaces: i.e. of functors

$$M : \{\kappa - \text{small extremally disconnected spaces}\}^{op} \to \mathbf{Ab}$$

verifying the sheaf condition. In this case, the sheaf condition is equivalent to the requirement regarding the empty set and to point 1 alone, since condition 2 is automatically fulfilled by the fact that any surjective map p of extremally disconnected sets splits (for more details, see [**9**]).

The fact that $\mathbf{Cond}(\mathbf{Ab})$ is an abelian category depends on the more general fact that for any abelian category $\mathbf{C}$, the category $\mathbf{C}^{\mathbf{D}}$ of functors from an arbitrary category $\mathbf{D}$ is abelian. Indeed, for any two functors $S, T : \mathbf{D} \to \mathbf{C}$, the termwise addition on $Hom_{\mathbf{C}^{\mathbf{D}}}(S, T)$ defined by $(\alpha+\beta)_C = \alpha_C+\beta_C : SC \to TC$ is commutative for every object C of $\mathbf{C}$. The null object of $\mathbf{C}^{\mathbf{D}}$ is the functor $\mathbf{D} \to \mathbf{C}$ with constant value the null object of $\mathbf{C}$; the biproduct $S \oplus T$ of any pair S, T of functors is given by $(S \oplus T)C = SC \oplus TC$; the kernel K of a natural transformation α is defined as $K_C = ker(\alpha_C)$.

Moving on to the axioms, the key concept is that the limits and colimits of such sheaves M can be formed pointwise. For any category $\mathbf{J}$ together with a functor to abelian sheaves of extremally disconnected spaces $j \mapsto M_j$, we have that $(\varprojlim M_j)(S) \overset{def}{=} \varprojlim(M_j(S))$ is the limit, where the limit on the right hand side is a limit of abelian groups. Analogously, the colimit will be $(\varinjlim M_j)(S) \overset{def}{=} \varinjlim(M_j(S))$. These pointwise limit and colimit are certainly presheaves, but the fact that they are sheaves is not banal. In the case of the limit, it follows from the more general Proposition 3.17; however, since limits and colimits do not commute in general, we must straightforwardly prove that such $\varinjlim M_j$ is a sheaf in this case. Consider two extremally disconnected sets S_1, S_2. We have that

$$\begin{aligned}
(\varinjlim M_j)(S_1 \amalg S_2) &\overset{def}{=} \varinjlim(M_j(S_1 \amalg S_2)) \\
&\cong \varinjlim(M_j(S_1) \times M_j(S_2)) \\
&\cong \varinjlim(M_j(S_1) \amalg M_j(S_2)) \\
&\cong \varinjlim(M_j(S_1)) \amalg \varinjlim(M_j(S_2)) \\
&\cong \varinjlim(M_j(S_1)) \times \varinjlim(M_j(S_2)) \\
&\overset{def}{=} (\varinjlim M_j)(S_1) \times (\varinjlim M_j)(S_2),
\end{aligned}$$

where we used the sheaf properties of the sheaves M_j, the fact that in $\mathbf{Ab}$ finite products and coproducts coincide, and the commutativity of colimits. Therefore, the sheaf condition also holds in this case.

The fact that both limits and colimits are pointwise means that the axioms are true because they hold in **Ab**. For example, $(AB4*)$ is verified in this way:

Consider an indexing category $\mathbf{J}$, together with three functors

$$F_1, F_2, F_3 : \mathbf{J} \to \{\text{abelian sheaves of } \kappa\text{-small extremally disconnected sets}\}$$

such that for each object j the sequence

$$0 \longrightarrow F_1(j) \xrightarrow{\alpha_j} F_2(j) \xrightarrow{\beta_j} F_3(j) \longrightarrow 0, \tag{4.19}$$

where α_j and β_j are morphisms of sheaves, is exact (i.e. $Im(\alpha_j) = ker(\beta_j)$)[1]. We want to show that

$$0 \longrightarrow (\prod_{j\in\mathbf{J}} F_1)(j) \xrightarrow{\prod \alpha_j} (\prod_{j\in\mathbf{J}} F_2)(j) \xrightarrow{\prod \beta_j} (\prod_{j\in\mathbf{J}} F_3)(j) \longrightarrow 0 \tag{4.20}$$

is also exact. However, since products are in particular limits, we have that $(\prod_{j\in\mathbf{J}} F_i)(j) = \prod_{j\in\mathbf{J}} (F_i(j))$ for every $i = 1, 2, 3$. As a consequence, for any extremally disconnected set S,

$$(Im(\prod \alpha_j))_S = Im((\prod \alpha_j)_S) = ker((\prod \beta_j)_S) = (ker(\prod \beta_j))_S, \tag{4.21}$$

where the middle equality holds because it holds for every j, since axiom $(AB4*)$ is true for abelian groups. As a consequence, $Im(\prod \alpha_j) = ker(\prod \beta_j)$, hence the axiom is true in $\mathbf{Cond}(\mathbf{Ab})$. ■

Observation 4.21. Let us now reconsider the example presented at the beginning of this chapter: that of the map $f : (\mathbb{R}, \tau_D) \to (\mathbb{R}, \tau_\epsilon)$. We had seen that in an abelian category the failure of this map to be an isomorphism ought to be explained by the presence of a nontrivial kernel or cokernel. This problem is now solved in the category $\mathbf{Cond}(\mathbf{Ab})$: the corresponding map $\underline{f} : \underline{(\mathbb{R}, \tau_D)} \to \underline{(\mathbb{R}, \tau_\epsilon)}$ is a morphism of which we can evaluate the kernel and cokernel. The kernel of a morphism of sheaves of abelian condensed sets Φ is given by the sheaf $(ker\Phi)(S) = ker(\Phi(S))$ for every profinite set S. The sheaf condition is verified by noticing that a kernel is an instance of equalizer (therefore, a limit) and by

[1] Generally, one might have to consider the sheafification of $Im(\alpha_j)$, which has a natural injection onto $F_2(j)$.

applying Proposition 3.17. In this case,

$$\begin{aligned}(ker\underline{f})(S) &= \underline{(\mathbb{R},\tau_D)}(S) \big/ \underline{(\mathbb{R},\tau_\epsilon)}(S) \\ &= \{\text{continuous maps } f : S \to (\mathbb{R},\tau_D)\} \big/ \{\text{continuous maps } f : S \to (\mathbb{R},\tau_\epsilon)\} \\ &= \{[0]\},\end{aligned}$$

i.e. the kernel of $\underline{f}$ is banal. Conversely, the cokernel of $\underline{f}$ is the condensed abelian group given by

$$\begin{aligned}(coker\underline{f})(S) &= coker(\underline{f}(S)) \\ &= \{\text{continuous maps } f : S \to (\mathbb{R},\tau_\epsilon)\} \big/ \{\text{continuous maps } f : S \to (\mathbb{R},\tau_D)\}.\end{aligned}$$

This construction does not always verify the sheaf condition: in most cases, the cokernel presheaf must undergo sheafification in order to obtain a sheaf. However, we can straightforwardly check that the sheaf condition holds in this case. By Proposition 4.11, we can consider only covering families of types (i) and (ii). Let $\{f_i : S_i \to S\}_{i\in I}$ be a matching family of type (i). Consider the assignment $f_i \mapsto x_{f_i} \in coker(\underline{f}(S_i))$. By the isomorphism $\coprod_{i\in I} S_i \cong S$, we can view the sets S_i as forming a partition of S, and the functions x_{f_i}, considered as equivalence classes in the quotient, as defined on the elements of the partition. Since $S_i \cap S_j$ is empty for all $i, j \in I$, it follows that the unique amalgamation of the matching family is given by considering the function $S \to \mathbb{R}$ given by the disjoint union of the functions f_i, as no gluing constraint is posed on intersections. Analogously, the sheaf condition must also hold for families of type (ii), hence $coker\underline{f}$ is a sheaf.

Finally, the desired nontriviality of the cokernel is shown, for example, by considering the profinite set $\mathbb{N} \cup \{\infty\}$ (which is profinite since it is compact as the Alexandrov compactification of $\mathbb{N}$, Hausdorff and totally disconnected). Indeed, $(coker\underline{f})(\mathbb{N} \cup \{\infty\}) \neq \{[0]\}$ since there exist non-locally constant convergent real number sequences which are not eventually constant.

Bibliography

[1] D. Ásgeirsson: *The Foundations of Condensed Mathematics*, 21 ECTS thesis, University of Paris, July 2021. Available at url `https://dagur.sites.ku.dk/files/2022/01/condensed-foundations.pdf`

[2] A. Brown, S. Cultrera di Montesano, O. Draganov, C. Fillmore, T. Heiss: *Sheaf Theory and Applications*, Lecture notes, Spring semester of 2020, IST Austria. Available at url `https://sebastianocultrera.com/wp-content/uploads/2022/10/Sheaf_Theory_and_Applications.pdf`

[3] A. M. Gleason: *Projective topological spaces*, Illinois J. Math. 2,(4A) 482 - 489, November 1958. Available at url `https://doi.org/10.1215/ijm/1255454110`

[4] B. Le Stum: *An Introduction to Condensed Mathematics*, Lecture notes, University of Rennes, Spring semester of 2024. Available at url `https://perso.univ-rennes1.fr/bernard.le-stum/bernard.le-stum/Enseignement_files/CondensedBook.pdf`

[5] S. Mac Lane: *Categories for the Working Mathematician.* Springer-Verlag: New York, NY, 1971

[6] S. Mac Lane, I. Moerdijk: *Sheaves in Geometry and Logic.* First edition. Springer-Verlag: New York, NY, 1992

[7] P. Scholze: *Lectures on Condensed Mathematics*, Lecture notes, University of Bonn, Summer semester of 2019. Available at url `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`

[8] The Stacks Project authors: *The Stacks Project*, 2024.
Definition 4.20.1 Available at url `https://stacks.math.columbia.edu/tag/04AY`
Lemma 5.22.2 Available at url `https://stacks.math.columbia.edu/tag/08ZY`

[9] A.Topaz, Condensed mathematics, MathOverflow answer consulted on 09/09/2024. Available at url `https://mathoverflow.net/q/429839`

[10] R. van Dobben de Bruyn, Adjunction between topological spaces and condensed sets, MathOverflow answer consulted on 19/08/2024. Available at url https://mathoverflow.net/q/420373

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