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MODELLING COMPETENCY: A COMPARISON BETWEEN ITALIAN AND DANISH APPROACH

Tesi magistrale in Didattica della Matematica

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Introduction

The European Union's definition of mathematical competence, given in 2006 says that it is "*the ability to develop and apply mathematical thinking in order to solve a range of problems in everyday situations*". This formulation underscores the importance of the process leading to the solution, in addition to the knowledge itself. Being mathematical competent entails more than just memorizing a series of theorems and definitions; it involves understanding and utilizing them in various contexts. Towards the end of the last century and particularly at the beginning of the current one, there was a shift towards competency-based learning, rather than solely focusing on theoretical knowledge.

Italy, with its strong tradition of textbook-centred education, encountered challenges in implementing significant reforms in the school system until 2010. However, during this period, a robust community of teacher-researchers emerged, striving to innovate teaching methodologies.

In Denmark, in 2000, the education minister established a commission in order to develop a novel approach to mathematical learning; these efforts culminated in the creation of the KOM framework ("*Competencies and mathematical learning*"), which delineates what it truly means to successfully learn and master mathematics. In particular, it defines what it's called as *competency flower*, a representation identifying eight distinct mathematical competencies, partly overlapped but independent of each other.

The present project primarily focus on two of these eight competencies: the *modelling competency* and the *problem handling competency*, considered as a significant component of the modelling cycle. The former encompasses aspects such as analyzing the fundamentals and properties of existing models and actively engaging in modelling within a given context; the latter can be summarised as identifying and formulating various types of mathematical problems and solving them both independently or posed by others, in the most appropriate manner.

The research aim of the current study can be summarised as follows:

> Investigate Italian upper secondary school students' potential difficulties with mathematical modelling, according to the KOM framework, and compare

them with Danish upper secondary school students' potential difficulties.

In essence, the objective is to conduct a deep examination of the Danish approach of developing and assessing competencies in high school students, with a particular focus on creating and designing tasks that are aligned with the specific competencies mentioned above. Then, a replication study is conducted, considering a Danish research made by Jankvist and Niss in 2019 ([14]) involving a mathematics test submitted to upper secondary school Danish students. By submitting the same tasks to upper secondary school Italian students we would like to identify differences and similarities in the difficulties encountered by the students.

The first chapter provides a brief overview of the Italian educational system, tracing its evolution from traditional practices of the last century to a more detailed description of the current national guidelines. The second chapter delves into the explanation of the KOM framework, outlining its structure and key characteristics. The third chapter conducts a deeper investigation into the two main competencies under study: *modelling* and *problem handing*, focusing mostly on the modelling cycle and the role that problem handling plays in it. In the fourth chapter, the focus shifts to the process of assessing mathematical competencies, analysing the Italian and the Danish systems and showing the example of PISA assessment related to the modelling cycle. The fifth chapter is a short one that tries to define exactly the research questions that will be addressed in the final part of the project.

Then, the practical part follows. The sixth chapter is about the methodologies used in submitting the tasks to Italian students and provides a brief presentation of the questions involved in the test. The seventh and final chapter presents the obtained results, followed by a subsequent analysis and discussion. Finally, a conclusion is drawn based on the findings.

Chapter 1

Italian curriculum

1.1 Introduction to Italian Didactic Tradition

The tradition of Mathematical Education in Italy traces back to the last century, beginning with Italy's unification and through the following times. After the Second World War, particularly during the 1950s-1970s, the world was characterised by what can be called a "*New Math* revolution". This period saw the development of new mathematics curricula, primarily in America but to a lesser extent in other European countries. These curricula were based on the recognition that children's learning of mathematics should not rely solely on memorization and practice but should also incorporate teaching for comprehension.

To better understand the meaning of teaching for comprehension, it's helpful to distinguish between two types of understanding: *instrumental understanding* and *relational understanding*. Instrumental understanding, as described by Skemp (2006), refers to "rules without reasons", indicating the misconception that merely knowing a rule and being able to apply it is sufficient to demonstrate understanding. Conversely, *relational understanding* entails a deeper awareness of the topic, "knowing both what to do and why" (Skemp, 2006 [28]). An illustrative example that clarifies this concept is given by the following analogy:

> *Imagine that two groups of children are taught music as a pencil and-paper subject. They are all shown the five-line stave, with the curly treble sign at the beginning; and taught that marks on the lines are called E, G, B, D, F. Marks between the lines are called F, A, C, E. They learn that a line with an open oval is called a minim, and is worth two with blacked-in ovals which are called crotchets, or four with blacked-in ovals and a tail which are called quavers, and so on – musical multiplication tables if you like. For one group of children, all their learning is of this kind and nothing beyond. [...] They would find it boring, and*

the rules to be memorised would be so numerous that problems like 'Write a simple accompaniment for this melody' would be too difficult for most. They would give up the subject as soon as possible, and remember it with dislike. The other group is taught to associate certain sounds with these marks on paper. For the first few years these are audible sounds, which they make themselves on simple instruments. After a time they can still imagine the sounds whenever they see or write the marks on paper. Associated with every sequence of marks is a melody, and with every vertical set a harmony. Much less memory work is involved [...]. These children would also find their learning intrinsically pleasurable, and many would continue it voluntarily. (Skemp, 1987, p. 157 [29])

This analogy, provided by the researcher Skemp, readily illustrates a common issue in mathematics. It's undeniable that many young students, and even adults, harbor a dislike, or sometimes even hatred, towards mathematics. There are numerous factors contributing to this negative attitude, but one significant reason is the inability to see the practical utility of what they're learning. They perceive mathematical rules just as mere mechanisms that are difficult to remember due to their senseless meaning, and that have no application in real life. This outcome underscores the grave danger that is hidden behind an instrumental understanding, and the urgent need to find an efficient way to promote relational understanding.

During a period when efforts were being made to effect these changes, the Italian education system remained largely unaffected by such innovations, remaining entrenched in a well-established textbook tradition. This stagnation hindered opportunities for improvement in school teaching methods. However, it's important to note that the Italian mathematical community did not stagnate. In fact, active discussions on these topics began, influenced by international developments. Thus, while the education system remained static, new research initiatives emerged in the field. An interesting aspect of the Italian tradition is that, despite the lack of curriculum reforms, teaching in schools evolved due to the behavior of teachers. The curricula adopted during that time, which persisted until 2010, did not impose strict rules for teachers to adhere to. This allowed them a certain degree of freedom in interpretation and planning of their activities. Consequently, teachers who were open to innovation had the opportunity to experiment and develop their ideas, leading to new approaches in math teaching and the emergence of a more robust research field in mathematics education (Blum et al. , 2019 [6]).

In the early 1970s, Italian research in mathematics education saw the initiation of a collaboration between universities and schools. This collaboration involved conducting research with a mixed group of teachers and academics, resulting in a fusion of various forms of knowledge. In developing new theories and teaching methods, both the practical needs of classrooms and academic traditions were taken into account. Over the subsequent years, these collaborative groups focused on innovating the Italian educational tradition, drawing insights from direct classroom experiences, teaching experiments, and laboratory observations. These approaches provided opportunities to assess the effectiveness of methods in real-time and identify any weaknesses. Despite these efforts, there remained some disparities between Italian and international research in didactics. This discrepancy stemmed from the Italian tradition's emphasis on epistemological and historical analysis of content, rather than on cognitive studies (Blum et al., 2019 [6]).

Significant changes in the Italian education system did not occur until the beginning of the new millennium. During this time, a delegation of mathematicians and teacher-researchers developed new guidelines that contributed to the formulation of updated Italian curricula for primary, secondary, and high schools. These developments will be further explored in the subsequent sections (Blum et al., 2019 [6]).

1.2 National guidelines for specific learning objectives

Before entering into the details of the Italian didactic guidelines is appropriate to provide a general overview of the Italian school system. In Italy there are ten years of mandatory school, covering students aged six to sixteen. This educational journey comprises five years of primary school, three years of secondary school, and eventually, five years of high school, which students can choose to conclude at the age of sixteen. While primary and secondary schooling are standardized for all students, upon completion of secondary school, students must select a high school that aligns with their interests and goals. Three main types of high schools exist:

- High school (*Liceo*);
- Technical school (*Istituto tecnico*);
- Vocational school (*Istituto professionale*).

All of these options provide the compulsory teaching of mathematics until graduation's year, but at different levels. The school called *liceo* includes different curricula, such as *classico* (focused on ancient languages), *scientifico* (scientific), *linguistico* (linguistic), *delle scienze umane* (focused on human sciences), *musicale e coreutico* (focused on music), *artistico* (focused on arts). Each high school type requires distinct guidelines tailored to its specific subjects and areas of interest. These guidelines can be categorized into four main types:

- National guidelines for the curriculum of primary and secondary school;
- National guidelines for specific learning objectives for *licei*;

- Guidelines for Technical schools;
- Guidelines for Vocational schools.

In accordance with the Italian tradition outlined in 1.1, these guidelines represent just a generic path to be followed, allowing for personal interpretation and granting teachers the freedom to adapt different methods to suit individual class and student needs. This research will deal mostly with aspects related to *Liceo scientifico*; hence, we will now analyse the national guidelines specific to *licei*, starting with the common elements among different types of high schools. Translating from the Italian national guidelines we can see that the overarching objective of high school education is defined as to "*provide the student with the cultural and methodological tools for a thorough understanding of reality, so that he puts himself, with a rational, creative, planning and critical attitude, in front of situations, phenomena and problems, and acquires knowledge, skills and competences both appropriate to the continuation of higher education, integration into social life and the world of work, and consistent with personal skills and choices*" (*Il profilo culturale, educativo e professionale dei Licei*, 2010, p.1 [26]). To achieve this objective, learning outcomes have been established across five different areas, which are general in nature and applicable to all high school specializations:

1. *Methodological area*:
    - the acquisition of an independent study method that could be used in different contexts and also in the academic world to continue the personal formation of the student;
    - the consciousness of the variety of methods used and the ability to evaluate and connect all of them to the topic studied.
2. *Logical-argumentative area*, which concerns:
    - the ability of arguing his opinion and of critically evaluating that one of the others;
    - the ability of reasoning in a logical way, identifying problems and looking for possible clever solutions.
3. *Linguistic and communicative area*, which concerns:
    - being able of reading and understanding a complex test in italian language, of writing and speaking precisely and accurately, by applying to various contexts;
    - foreign language skills at B2 level for at least one language (mandatory English);
    - the ability of using technological tools to study and for research.

4. *Historical-humanistic area*, which concerns:
    - a deep knowledge of the historical, political and geographical scenario of the world, with a particular focus on Italy and Europe;
    - the consciousness of the literary, philosophical, religious and artistic Italian and European tradition;
    - being able to understand the development of scientific thinking and to collocate in the historic framework the most important scientific and technological discoveries.
5. *Scientific, mathematical and technological area*, which concerns:
    - the comprehension of the specific formal language of mathematics and of the mathematical theories that are the basis of real life, and the ability of thinking mathematically;
    - the knowledge of fundamental aspects of physics and natural sciences, in addiction to the ability of understanding and using the scientific method of research;
    - the ability of using, in an appropriate way, technological tools to study and to solve problems. [26]

In addition to the overarching goals shared among all types of high schools, further differentiation occurs based on the specific focus of each type of school and subjects within them. Each subject follows a similar structure within the guidelines. It typically begins with a section on "*General guidelines and competencies*", followed by "*Specific learning objectives*", which is further divided into objectives for the first two-year period, the subsequent two-year period, and the final year of high school. The same structure is evident in the *National guidelines for the curriculum of primary and secondary school*, ensuring a continuity in these important documents.

The government emphasizes that, consistent with Italian tradition, significant autonomy is granted to each institution. Schools are encouraged to modify and adapt teaching methods to suit their specific contexts, considering factors such as the school's history and geographical location, available teachers and instruments, and connections with local resources. Moreover, a lot of independence is given also to singular subjects: there is a belief that certain abilities and competencies can be developed if the student engage with a curriculum that integrates various subjects while preserving the epistemic identities of individual disciplines. In addition to subject-specific competencies, there are also general competencies that are not explicitly specified but are implicit and shared across all subjects. An example could be the digital skills, that are said to be "result of field work of all disciplines", while competencies related to citizenship can be cultivated within the broader school environment.[27]

1.3 On what are these guidelines based?

Firstly, it's important to note that the guidelines are grounded in certain principles. Two key principles include the recognition of essential knowledge around which educational programming should be structured and the assertion of a unified knowledge framework that doesn't distinguish between the notion itself and the relevant abilities. These guidelines were initially established in 2010 and, while they remain largely unchanged, it's specified that modifications and adjustments may have occurred in subsequent years. They were developed by a commission comprising individuals from academic, cultural, and educational backgrounds, with input from students, parents, and teachers; these guidelines were published for public review and debate on a public platform for a month.

In terms of educational objectives, the guidelines were crafted in alignment with the strategies suggested by Europe for fostering a "knowledge society", as well as various national and international reference frameworks. These frameworks includes assessment such as OCSE PISA, which evaluates reading and mathematics skills at the beginning of high school, IEA TIMSS ADVANCED, which assesses mathematics and science competencies at the conclusion of high school, and INVALSI, which are Italian assessments that proves students' competencies at various educational stages. [27] At the basis of mathematics' INVALSI assessments, there's the idea that what has to be tested is not just the knowledge, but more the ability to apply it, and it is specified that "*the domains of reading, mathematical and scientific literacy are covered not merely in terms of mastery of the school curriculum, but in terms of important knowledge and skills needed in adult life*"[33].

It has to be said that both INVALSI testing and National Guidelines are built on studies conducted in the early 21st century, when a collaboration between academics and teachers started. This initiative led to the development of what is called "*Matematica per il cittadino*", literally translated as "Mathematics for the citizen". This project began in 2000, when the Italian Mathematics Union (UMI) established a specific commission in order to elaborate a new mathematical curriculum aligned with the needs of the modern Italian school system. That commission was headed by Ferdinando Arzarello, one of the best known figures in Italian Mathematics Education's scenario.

From this commission's work, three distinct curricula emerged: one for primary and secondary schools, another for the first four years of high school (with no distinction between different types of high schools), and the third for the fifth and final year of high school. More precisely this last curriculum involves a division between a path of deepening and one of consolidation. The former is intended for high schools where mathematics is a dominant subject, while the latter is for schools where mathematics is not as prominent, so that the notions acquired in the previous four years may be enough and it's just necessary to consolidate that knowledge and relate it to real life, in order to make it significant for adult life. In general, these curricula, aim mainly to be a guide for school teachers: they present a series of knowledge and abilities essential for every individual that is going to live in today's society, to be acquired through an organic pathway along the different school levels (Arzarello et al., 2003 [2]).

We're going to focus on the curricula related to high school.

The first one, known as "Matematica 2003", due to its publication in 2003, is structured similarly to its predecessor. It identifies four thematic cores aimed at continuing the development of knowledge initiated in primary and secondary school:

- *Number and algorithms*;
- *Space and figures*;
- *Relations and functions*;
- *Data and forecasts*.

Each core is divided into objectives for the first two-year period and the subsequent period. For each core, a table delineates the abilities students should possess and the knowledge, theories, and theorems they should be familiar with. After this presentation of topics, a series of suggestions is provided, in order to complete the guide and make it really helpful for a teacher, such as historical insights, main conceptual difficulties that could be found, and attitudes that are not recommended.

Furthermore, three transversal cores focus on students' mental processes:

- *Argumentation, conjecture, and demonstration*, a mixture core characterised by logical thinking;
- *Measurement*, related to direct experience with phenomena and quantities;
- *Problem posing and solving*, in order to develop student abilities and make them able to use their knowledge to build mathematical models.

Then, the project ends with some considerations about mathematics laboratory, which has been part of Italian tradition since the 1970s, and methodological indications. Concrete examples of classroom activities are provided to help teachers achieve the intended goals (Arzarello et al., 2003 [2]).

The third curriculum, called "Matematica 2004", focuses on the final year of high school. It consists of a first section regarding schools strictly connected with mathematics, following the same structure as "Matematica 2003". This section is followed by a consolidation part structured differently, consisting mainly of potential activities and examples for assessing students' levels (Arzarello et al., 2004 [2]).

Even though the national guidelines were developed more than five years later than this project, it has been the main guide for the outline that is still in place in Italian schools. It represents the culmination of the efforts of some of the key figures in Italian Mathematics Education.

1.4 Scientific High School: a specific case

To achieve the goal of this research, we will specifically focus on the guidelines for the scientific high school related to the subject of mathematics. The "*Liceo scientifico*" curriculum is organised so that, at the end of this path, the student "*will know the basic concepts and methods of mathematics, both internal to the discipline itself, and relevant for the description and prediction of phenomena, particularly related to the physical world*" (*Indicazioni nazionali riguardanti gli obiettivi specifici di apprendimento concernenti le attività e gli insegnamenti compresi nei piani degli studi previsti per i percorsi liceali [...]*, 2010, p.337 [27]). This first objective presented at the beginning of the guidelines focus on knowing mathematical concepts, without any reference to the applying of the methods studied. However, this sentence is placed at the beginning of the section regarding *general guidelines and competencies*. Indeed, it is followed by a list of the main concepts and methods that are central to the study path.

The eight concepts presented provide a generic overview on the mathematics topics treated and that should be mostly developed during the scientific high school. Most of them relate on knowledge, such as knowledge of the Euclidean geometry or knowledge of the basic mathematical tools for the study of physical phenomena, but there are also some references to the practical use of mathematics elements.

Given that the objects of this research are modelling competency and problem-solving skills, we are going to focus on the parts of Italian guidelines that explicitly refers to, or seem to recall, these two competencies, considered according with the definition given by the KOM framework.

In this first section about general competencies two points can be considered related to the modelling competency. Here the English translation is presented:

- *The concept of a mathematical model and a clear idea of the distinction between physical mathematisation and modelling mathematisation*;
- *Construction and analysis of simple mathematical models deriving from physical phenomena* (*Indicazioni nazionali riguardanti gli obiettivi specifici di apprendimento concernenti le attività e gli insegnamenti compresi nei piani degli studi previsti per i percorsi liceali [...]*, 2010, p.337 [27]).

In addition to the list, there is also a reference about the connection between mathematical models and problem solving. In fact, it is said that the student will know "*the basic methods for the construction of a mathematical model [...] and how to apply them in order to solve problems*" (*Indicazioni nazionali riguardanti gli obiettivi specifici di apprendimento concernenti le attività e gli insegnamenti compresi nei piani degli studi previsti per i percorsi liceali [...]*, 2010, p.337 [27]). This is to notice because underline the link between the two competencies which is also discussed a lot in the KOM framework and represent a significant aspect of this research.

A closer examination of the subsequent parts of the guidelines reveals that the development of modelling competency is expected to progress over the years: at the end of the first two-year period the student is expected to "*obtain information and derive the solutions of a mathematical model of phenomena, even in operational research or decision theory contexts*" (*Indicazioni nazionali riguardanti gli obiettivi specifici di apprendimento concernenti le attività e gli insegnamenti compresi nei piani degli studi previsti per i percorsi liceali [...]*, 2010, p.339 [27]). Additionally, a starting knowledge of informatics is required, including familiarity with the concept of algorithms and the ability to develop simple algorithms to solve basic mathematical models.

As students progress into the second two-year period, their modelling competency advances further, they "*will be able to build simple models of exponential growth or decrease, as well as periodic trends, also in relation to the study of other disciplines; both in a discreet and continuous context* (*Indicazioni nazionali riguardanti gli obiettivi specifici di apprendimento concernenti le attività e gli insegnamenti compresi nei piani degli studi previsti per i percorsi liceali [...]*, 2010, p.340 [27]).

Finally, by the end of the fifth year, students are expected to incorporate infinitesimal calculus into their modelling competency. They should deeply comprehend the role and significance of infinitesimal calculus in describing and modelling physical or other phenomena.

The problem handling competency is addressed in a less explicit manner in the Italian national guidelines. Apart from the connection with mathematical models already cited, there is no explicit reference to this kind of ability in the first section of the paper.

Analysing the subsequent sections it can be noticed that expressions that seem to relate to problem handling are primarily found in the objectives outlined for the first two-year period. Here it is said that the student "*will acquire the ability to perform calculations with literal expressions both to represent and to solve problems*" and "*will study functions both in a mathematical context and to represent the solution of practical problems*" (*Indicazioni nazionali riguardanti gli obiettivi specifici di apprendimento concernenti le attività e gli insegnamenti compresi nei piani degli studi previsti per i percorsi liceali [...]*, 2010, p.338 [27]).

However, in the subsequent sections of the guidelines, mentions of problem handling are less prominent. Particularly there are no explicit references to this competency considered as in the KOM framework.

A valuable aspect, related to problems and mentioned in the guidelines, concerns the technical proficiency. It is emphasized that while technical proficiency is important, mathematics should not solely focus on repeating fixed patterns but should involve a deep relational understanding of the problem itself. The main indication states: "*few fundamental concepts and methods, deeply acquired*" (*Indicazioni nazionali riguardanti gli obiettivi specifici di apprendimento concernenti le attività e gli insegnamenti compresi nei piani degli studi previsti per i percorsi liceali [...]*, 2010, p.338 [27]).

Chapter 2

The KOM framework

2.1 Introduction to Danish Didactic Tradition

The Didactics of Mathematics emerged as an independent scientific discipline more than fifty years ago, with Denmark playing a central role in its development. At the onset of the twentieth century, there was a growing interest in the Scandinavian countries regarding issues pertaining to the teaching of mathematics in schools. This interest may be the consequence of previous changes within the educational systems of these nations. It's important to note that the expression *Scandinavian countries* refers only to Denmark, Norway, and Sweden. These countries share similar traditions in didactics, owing to various resemblances such as the organization of their school systems, teacher education methods, language, and core values.

Regarding changes in the educational systems, in the latter half of the nineteenth century, Scandinavian countries established state school systems, that led to the necessity of training teachers. Consequently, separate teacher education programs were developed for compulsory and post-compulsory education levels. Furthermore, during the twentieth century, compulsory schooling was extended from seven to nine years, a structure that remains in place still today. (Blum et al., 2019 [6]).

The Danish school system is formed by one year of kindergarten plus nine years of mandatory high school, encompassing primary and lower secondary education, collectively known as "*Folkeskole*", with grades that go from zero to nine. Following this, there are three additional years (grades ten to twelve), which constitute upper secondary education. These years offer various paths, including scientific and linguistic options in *Gymnasium*, as well as *Vocational schools* and *Business schools*, allowing students to choose according to their interests and career goals.

In the latter half of the twentieth century, the *New Math* phenomenon, discussed in the previous chapter, also exerted its influence on Scandinavian countries, particularly Denmark. This prompted the establishment of a Nordic committee for the modernization of mathematics education between 1960 and 1967, involving Denmark, Norway, Sweden, and Finland. Consequently, due to the efforts of this committee and the involvement of numerous Danish researchers in the modernization process, New Math textbooks began to dominate the Danish school system. (Blum et al., 2019 [6]).

However, despite these changes in concrete mathematics education, it wasn't until around 1970 that the didactics of mathematics emerged as a distinct research field in Scandinavian countries. Initially, this research was primarily conducted by pedagogical figures who also carried out experiments and laboratories directly within classrooms; mathematics teachers were not actively involved in research studies. Instead, they underwent training courses during their academic preparation focused on teaching methods, which was considered an innovative aspect of the Danish educational system at that time.

In this period the *Royal Danish School of Educational Studies* was established, becoming one of Denmark's most significant educational centers. Today, it continues to play a fundamental role in Danish education and is now part of Aarhus University, although it remains located in Copenhagen. A notable research centre is situated in Roskilde University, which would later play a crucial role in the development of the KOM project (Blum et al., 2019 [6]). Another Danish active research group is located at the Copenhagen University, who is mainly directed to research in teaching and learning of mathematics in higher education, with a strong connection with the Theory of Didactical Situations. Since then, Denmark has increasingly focused on enhancing its mathematics education system, with particular attention to topics such as competencies, modelling, and real-life problem-solving, making it an ideal resource for the purpose of this research.

2.2 What is the KOM framework?

2.2.1 The Danish historical and educational context

On early 2000s, the Danish minister of Education established a commission that gave rise to one of the most renowned mathematics' framework: the KOM framework. The ministry aimed to revolutionize the way of teaching mathematics in Denmark at that time. The focus of the project was on *mathematical competency*, rather than rote memorization of definitions, theorems and methods. The urgent need for innovation in mathematics education system stemmed from various reasons, leading to the inception of this project.

One significant reason is the internationally recognized *justification problem*. This refers to the challenge of providing a concrete reason for justifying the fact that mathematics education is essential for students in a particular society. This problem is not to be underestimated, especially because with no doubt it's difficult to find an answer that is satisfactory for everyone. Ernest (1998 [8]) claims that there are "*a number of myths and fallacies about mathematics and its teaching which act as obstacles to the clarification of the aims and the justification of mathematics teaching*". These myths and fallacies generates a natural negative attitude towards mathematics itself making it really difficult to change public perception. Another complication in addressing the justification problem is that the goals that should be reached through mathematics are usually not made explicit. What it's easy to observe is just the reality of mathematics education, the tasks and the methods that are used and the school objectives, but these are not enough: there are underlying reasons and driving forces behind them, that are not observable (Niss, 1996 [18]).

The consequences of the justification problem were evident in Denmark at that time, prompting the need for reform in mathematics education. Niss (2003, p.1 [20]) describes this situation as follows: "*society needs a well educated population, to actively contribute to the shaping of society, and a broadly qualified work force, all of whom are able to activate mathematical knowledge, insights, and skills in a variety of situations and contexts*", but at the same time "*an increasing extent of young people opt away from educational programs with a strong component of mathematics*".

Furthermore, several other challenges can be identified, a few of which are outlined below:

- The ongoing debate around the concept *mathematics for all*: after the conclusion of the Second World War, there was a growing belief that mathematics education should be accessible to all. However, the outcomes often fell short of expectations. Numerous failures in achieving this goal led to the thinking that rigorous mathematics education should be reserved for those who can truly benefit from it, while the common citizen would only experience difficulty from excessively deep mathematical learning. It's evident that this issue is closely related with the previous one.

- *Implementation problems*: these are related to the people responsible for mathematics teaching. Throughout a student's academic journey, the role of teachers is essential. In Denmark, and in Italy too, it appears that primary and secondary school teachers often have a background primarily focused on pedagogical content rather than mathematical content. However, this changes in the subsequent phase, as high school professors possess a preparation centered solely on mathematical concepts, often neglecting didactic and pedagogical aspects.

- *Transition problems*, that Niss (2003, p.3 [20]) describes as problems "*about the identity and coherence of mathematics as a subject across the levels*". This refers to the fact that Danish mathematics education consists of distinct blocks at various levels without explicit connections between them, leading to confusion regarding what is essential to learn and which path to follow.

- *Assessing problems*, concerning the evaluation of student progress. These problems are particularly relevant to the objectives of this research, as subsequent sections will delve into how to assess and design accurate tasks. It is closely linked to the aforementioned challenges since assessing students' improvements becomes difficult when the goals they are striving to achieve are unclear.

These are just few of the difficulties encountered within the Danish mathematics education system at that time, culminating in the establishment of the KOM project (Niss, 2003 [20]). They are presented in this research in order to create a concrete historical context for the inception of the KOM framework. It should be noticed that the problems and situations described pertain to the time when the KOM project commission was established in the early 2000s, and subsequent changes may have occurred in the following years, influenced in part by KOM's innovations.

One of Danish Ministry's objectives in establishing the commission was to produce a fundamental guideline for the Danish school curriculum and to implement concrete reforms in the educational system. The KOM project commission, where "KOM" stands for *Competencies and the learning of mathematics* in Danish, comprised twelve members selected from mathematicians, mathematics teachers, and researchers in mathematics education. In addition to these figures there were also some individuals unrelated to the world of mathematics education and some *sparring partners*, whose job was to evaluate critically the work of the commission along the way.

An important aspect was the extensive open debates with mathematics teachers and schools before finalizing the project. This highlights how much was important to make a significant reform: the involvement of the main figures interested by the reform, precisely the teachers, was made to prevent that, after the closure of such a big project, teachers would have refused to apply what expected by the reform, making every researchers' efforts useless.

Regarding the main sources that helped the development of this framework, Niss and Højgaard (2019, [24]) identify two interconnected sources. Firstly, the personal experience of Morgan Niss himself, the head of the commission and a mathematics educator, provided valuable insights into the practice and application of mathematics. Secondly, the mathematics and science study program established at Roskilde University played a significant role in shaping an innovative framework. It's noteworthy that KOM was not directly influenced by existing frameworks but could rely on the great wealth of knowledge of his creators. However, it should be mentioned that between 1990s' and 00's there was a significant change in the learning world, that led to the creation of a direct link between the notion of academic achievement and that one of competency. Two notable projects concerning the role and the definition of competencies are (Geraniou and Jankvist, 2019 [9]):

- the OECD's DeSeCo, a project from the UCL Institute of Education (University College of London), in 1997. Its aim was to put together the opinions of various experts in order to identify and define the key competencies for dealing with the real world;
- the Programme for International Student Assessment (PISA), in 2006. It is an assessment project focused on the development of competencies at different levels. Due to its close relation with the topic treated in this research, it will be analysed deeper later.

The final version of KOM project was published in 2002, in Danish language. However, it soon became evident that an English translation was necessary, firstly to position the framework in the international context to widen the possibilities of improvements, but later due to the growing international interest in the project, so that it was essential to make it accessible to a wider audience. Despite this, the first official translation was not published until 2011, (Niss & Højgaard, 2011 [23]). This translation consist of just a portion of the original KOM, omitting the sections considered by the authors concerning specific issues for Danish school system.

Subsequently, the discourse prompted by the KOM framework evolved, leading to a more recent reexamination by the authors Niss and Højgaard. They define it as "*the first systematic journal account of the project*" (Niss & Højgaard, 2019, p.10 [24]), providing an updated conceptual framework based on research advancements made in the past two decades.

In this revision of the KOM framework, the authors have included a section addressing the unique definition of competency. There are certain concepts that may be mistaken for competencies but have distinct differences that should not be overlooked. This highlights the urgent need, especially at the start of the century, for a clear definition of competency.

The authors identify three concepts that are similar yet different from competency: procedural skill, factual knowledge, and understanding. They describe these to clarify how they differ from the concept of competency.

Niss and Højgaard (2019, p.20 [24]) say that a procedural skill is "*someone's ability to perform, with accuracy and certainty, a particular, methodologically well-defined, -oftentimes algorithmic— goal-oriented type of undertaking*". These procedural skills are not completely different from competencies, but they are much more simpler. We can said that in activating a competency, lots of procedural skills are needed, but they are not enough. They explain this fact through an analogy with the chemical molecule. A molecule is composed by many atoms, but has characteristics that are not exactly given by the sum of the single characteristic of each atoms it is composed by. The same happens for a competency: it is composed by many procedural skills but in its existence something else is involved, in addition to the details of the procedural skills needed.

Factual knowledge is closely related to competency, yet it's not sufficient on its own. Factual knowledge involves understanding mathematical concepts, while competency encompasses the ability to apply that knowledge effectively. However, it's important to note that factual knowledge serves as a prerequisite for activating competency: one can't act properly in a certain situation without knowing what to do. By the way, this is not sufficient, the ability to transform this knowledge into action is also necessary. Mathematically speaking, we could say that specific factual knowledge is a necessary but not sufficient condition for activating a competency, and vice versa: mathematical competency is necessary but not sufficient to achieve factual knowledge in a particular field.

A similar perspective applies to understanding. Sometimes, a student may possess a profound understanding of mathematical concepts, such as theorems and definitions, but in a passive manner, lacking awareness of their utility and proper application. This means having understanding without having mathematical competency. On the other hand, if one is able to activate a specific competency it should imply a certain amount of understanding. Ultimately, understanding is considered a subset of mathematical competency.

2.2.2 The task of the KOM project

As mentioned earlier, the primary objective of the KOM project was to introduce innovation into the Danish mathematics education system. Specifically, there were several key questions that needed to be addressed. The primary research question was as follows:

What does it mean to master mathematics? (Niss, 2003, p.5 [20])

Answering this question would provide valuable insights sought by the researchers. Here are a few of the questions, most relevant to the research aim:

- *Which mathematical competencies need to be developed with students at different stages of the education system?*

  This question involves the importance of competencies, that will be deepened in the next section, in order to provide an useful and effective mathematics learning.

- *How do we measure mathematical competence?*

  In this question lies the assessment problem, strictly related to this study and the objective of creating tasks that could assess a specific students' competency.

- *How can we, in Denmark, make use of international experiences with mathematics teaching?*

This question emphasizes the importance of international collaboration, implicit in the current research aim, which is to evaluate the Italian system while using a Danish approach to competency assessment.

- *What does society demand and expect of mathematics teaching and learning?*

  This final question addresses a theme discussed earlier, the justification problem, and the pressing need for concrete and explicit reasons to study mathematics.

The existence of several questions to be answered implies that the KOM project could not provide exact answers to all of them. Instead, the commission aimed to conduct a thorough analysis of the issues raised by these questions and undertake an analytical development project. Therefore, the expectation was not to find solutions to all the presented problems but rather to gain an understanding of what needed to be changed to initiate improvements in the Danish mathematics education system (Niss, 2003 [20]).

To fully comprehend the researchers' intentions, it is helpful to consider the specific task they described themselves as attempting to solve.

Their idea encompassed:

- *identifying and characterising, without going round in circles, what it means to master (i.e. know, understand, do and use) mathematics in itself and in different contexts irrespective of what specific mathematical content or syllabus is involved*;

- *describing the development and progression in mathematics teaching and learning both within and between different curricula*;

- *characterising different levels of mastery so as to describe the development and progression in the individual student's mathematical competence*;

- *comparing different mathematics curricula and different kinds of mathematics education in parallel or different stages of education in a way that goes beyond a mere comparison of curricula.* (Niss & Højgaard, 2011, p.47 [23])

What's crucial to emphasize is their belief that this task could be addressed through the definition and proper utilization of mathematical competence, a concept we'll elaborate on in the next section. They viewed it as a mean for describing mathematics curricula.

For the moment we're going to focus on the section of KOM project that directly involves competencies.

2.3 Mathematical competence

2.3.1 Competence and competency

The literal definition of competence is "*the ability to do something successfully or efficiently*". In simpler terms, having competence in a particular field means mastering the essential aspects of that field. More precisely, according to Niss (2003, p.6 [20]) "*to possess a competence in some domain of personal, professional or social life is to master essential aspects of life in that domain.* But what does it mean in the context of mathematics?

In the realm of mathematics, competence entails the ability to understand, utilize, and apply mathematical principles in various contexts, whether directly related to mathematics or not. This includes recognizing mathematical concepts in real-life situations and employing mathematical knowledge to describe or solve problems. It's clear that while knowledge and technical skills are necessary components of mathematical competence, they alone are not sufficient. There's something deeper, something inherent in the complex meaning of mathematical competence.

This aligns closely with what researchers in the KOM project were seeking: mathematical competence as the key to mastering mathematics. Consequently, it could serve as a framework for designing mathematics curricula, with a focus on fostering mathematical competence to empower students to master the subject.

To better understand how the notion of competence has been used in the KOM project, it's beneficial to examine the definition proposed by Niss and Højgaard (2019, p.12 [24]): "*competence is someone's insightful readiness to act appropriately in response to the challenges of given situations*". You may notice the use of the terms *readiness* and *appropriately*, rather than ability and successfully; this is in line with the objectives of the KOM project and their interpretation of competence.

The two experts explain that readiness doesn't mean just promptness to act in certain situations; it encompasses the ability to assess whether and how to act, including the decision not to act, which requires insight. This highlights one of the many facets of mathematical competence: its orientation towards action, whether physical or mental.

Another key word of the aforementioned definition is the term *challenge.* The authors (Niss & Højgaard, 2019 [24]) emphasize that what is challenging and at which level of challenge depends on who is facing a certain situation. Thus, the difficulty concerning a challenge may be different in dependence on the person experiencing it: what is challenging for one could be really easy for someone else; this happens frequently in mathematics education.

Furthermore, challenges can arise in various forms: it could be on personal level or on moral one, scientific or professional; there is a variety of challenges that an individual can meet. For this reason, the authors also offer a clear explanation of this concept, underlying that "*the degree to which certain actions meet the challenges is always a question of who the judges are that bring meaning and legitimacy to the actions*"(Niss & Højgaard, 2019, p.12 [24]).

Starting from the broad definition of competence, we can now move to define what is *mathematical competence.* In their 2019 revisitation, Niss and Højgaard revised the definition from their English translation of 2011. In the translation, in fact, it is said that mathematical competence "*comprises having knowledge of, understanding, doing, using and having an opinion about mathematics and mathematical activity in a variety of contexts where mathematics plays or can play a role*" (Niss & Højgaard, 2011, p.49 [23]). However, this definition was more general and did not address all the nuances of the terms used. In 2019, the authors opted for a more coherent definition in line with the new definition of competence. It reads: "*mathematical competence is someone's insightful readiness to act appropriately in response to all kinds of mathematical challenges pertaining to given situations.*" (Niss & Højgaard, 2019, p.12 [24]). This new definition keep it clear what we have defined before about the meaning of readiness and challenge, while specifying that mathematics challenges are taken into account.

In the 2019 revisitation, Niss and Højgaard also introduced a distinction between *competence* and *competency*, which was already used in the original translation but not clearly delineated. This distinction is crucial for understanding the subsequent development of the KOM project. While we have referred to mathematical competenc*e* in a general sense, the world competenc*y* refers to the various components of the competenc*e*, which will be elaborated and detailed in the following section. "*A mathematical competency is someone's insightful readiness to act appropriately in response* ***to a specific sort*** *of mathematical challenge in given situations*" (Niss & Højgaard, 2019, p.14 [24]). This definition encompasses all the features of the general definition, but specifies the type of mathematical challenge being addressed. An analogy can be drawn, considering mathematical competence as a building and mathematical competencies as its windows. This analogy suggests that certain aspects of challenging situations require specific competencies, or "*a specific kind of activation of mathematics*" (Niss & Højgaard, 2019, p.14 [24]).

Identifying and studying these competencies were two primary objectives of the KOM project. However, this task had lots of challenges. The number of competencies identified needed to strike a balance: not too large to avoid excessive detail, making each competency overly restrictive, yet not too small to be overly general and therefore less helpful. The KOM framework distinguish eight different competencies, each one with a proper identity but at the same time all connected with each other. These competencies will be detailed in the following section.

2.3.2 The eight mathematical competencies

Since a revisitation of the KOM project has been published we will consider both the original translation and the subsequent update in this study. We will present the eight mathematical competencies as originally conceived in 2002, noting any significant changes made in the revisitation.

The competencies have been divided into two groups based on the type of activity they are associated with:

- the ability to ask and answer questions in and with mathematics;
- the ability to deal with mathematical language and tools.

It's important to clarify, as the KOM researchers themselves do, that the term *ability* in this context does not refer to personal faculties possessed by an individual, but it is just a substantivation of *being able to*, aligning with the definition based on *readiness*.

The eight competencies can be represented as a flower, as done in the original project, shown in 2.1, with each petal representing a specific competency. The overlapping of all petals symbolizes the inevitable interconnection between them. Additionally, there is a non-empty intersection of all petals at the center of the flower. The intensity of colors varies, with maximum intensity in the central part of each petal, gradually becoming lighter towards both the center and the border of each petal.

It's essential to note that the division into two groups mentioned above does not imply that competencies within the same group are more interconnected with each other than with those in the other group. The relationships and connections are equal among all competencies.

The first group mentioned, concerning asking and answering questions, includes:

- *mathematical thinking competency*;
- *problem handling competency*;
- *modelling competency*;
- *reasoning competency.*

The second group, explicitly related to language and tools, comprises:

- *representing competency*;
- *symbol and formalism competency*;
- *communicating competency*;

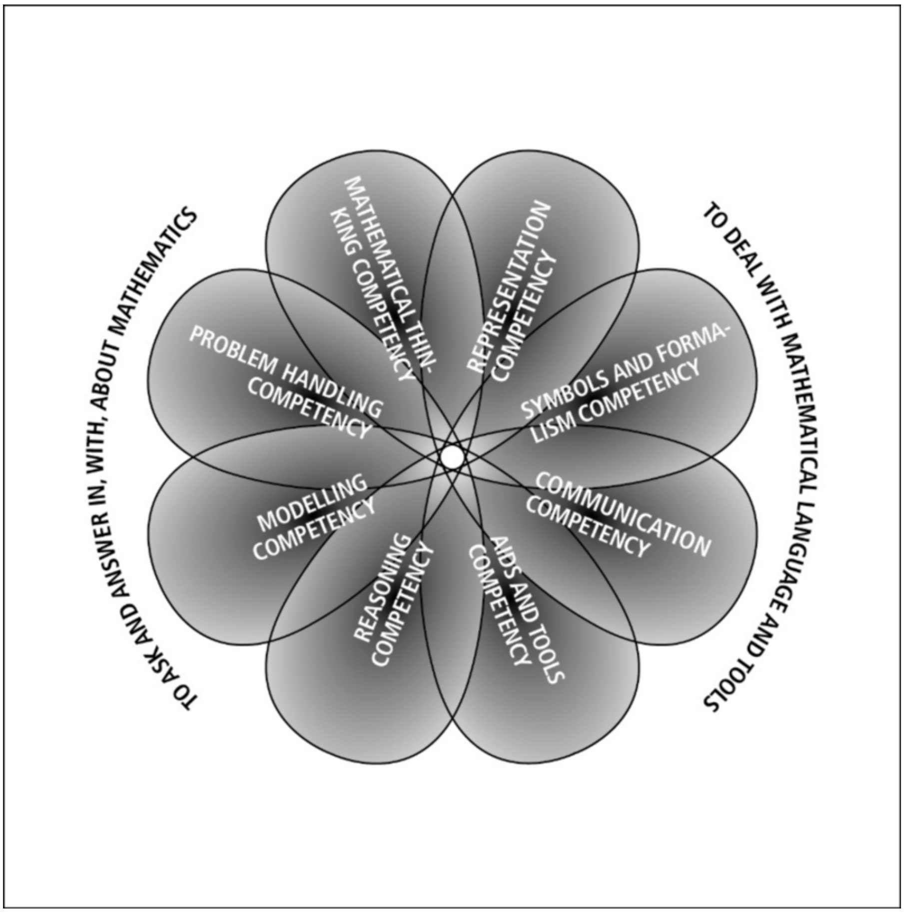

Figure 2.1: A visual representation of the eight competencies: the competencies flower

- *aids and tools competency.*

Before going into details of the eight competencies is important to talk about their dual nature. In the original translation the authors identified two parts of competencies' nature as "*investigative*" side and "*productive*"side (Niss & Højgaard, 2011 [23]). In the revisitation, the authors themselves renamed them, respectively, "*receptive facet*" and "*constructive facet*" (Niss & Højgaard, 2019 [24]). Despite the name changed the meaning has remained unchanged: the former refers to "*the individual's ability to relate to and navigate with respect to considerations and processes which have already been introduced (typically by others) into a given context or situation*"; the latter focus on "*the individual's ability to independently invoke and activate the competency to put it to use for constructive purposes in given contexts and situations*" (Niss & Højgaard, 2019, p.19 [24]).

Moreover, each competency possess a behavioral component. At the end of the KOM framework authors claimed that both parts of competencies involved activities, on mental or physical level, that are of behavioural nature. This underlines that the attention was directed to the individual being able to do a certain activity while using his competencies, but at the same time they exclude the possibility to consider them just as an “*individual’s response to given stimuli*” (Niss & Højgaard, 2011, p.71 [23]).

However, it’s noteworthy that in the revisitation, the authors chose to focus solely on the cognitive aspects of competencies, excluding affective, dispositional, and volitional characteristics. This difference is underlined in reference to their definition of mathematical competence which has been reported previously: they use the term readiness instead of disposition precisely for underlying that they’re talking about individual’s cognitive prerequisites to do a certain activity, without considering any other personal aspects.

Mathematical thinking competency

The mathematical thinking competency concerns the ability to pose mathematical questions and comprehend the various types of potential answers. Additionally, it entails an awareness of the range of possible mathematical questions.

According to Niss and Højgaard (2011, p.53 [23]) it also includes “*being able to recognise, understand and deal with the scope of given mathematical concepts (as well as their limitations) and their roots in different domains; extend the scope of a concept by abstracting some of its properties; understand the implications of generalising results; be able to generalise such results to larger classes of objects*”.

Central to this competency is the focus on mathematical concepts, questions, and answers, which one must understand in order to generate and respond to questions effectively.

Another essential aspect of this competency involves familiarity with different types of mathematical statements and assertions, such as theorems, conjectures, or conditional statements. One should be able to differentiate between them and recognize the role and significance of the quantifiers used in these statements.

To give some examples of possible mathematical questions and answers we can mention: “if so, under what conditions?”, “how many ... ?”, and “yes, because ...”, “no, because ...”, “it depends on the situation since ... ”.

The receptive facet of mathematical thinking competency comprehends the ability to understand and recognize mathematical questions or sentences formulated by others; its productive facet entails the ability to generate questions and answers independently.

It must be specified that, related to this competency, the focus is solely on the mathematical nature of the questions, nor about their theoretical content nor the correctness of the answers. These aspects will be addressed in subsequent competencies.

Problem handling competency

The problem handling competency, that was originally defined as problem tackling and successively changed in problem handling, comprises "*being able to put forward, i.e. detect, formulate, delimitate and specify different kinds of mathematical problems, [...]and partly being able to solve such mathematical problems in their already formulated form*" (Niss & Højgaard, 2011, p.55 [23]). This implies the capability to both formulate and solve mathematical problems, whether they are posed by oneself or by others, using various approaches.

It's essential to clarify the concept of a mathematical problem. Similar to the definition of a challenging task, the definition of a mathematical problem is subjective. In the KOM project, a problem that can be solved using routine skills and mechanical procedures is not considered a problem. Rather, we are referring to problems that require non-routine approaches for resolution.

The investigative facet of the problem handling competency involves critically evaluating and understanding solutions proposed by others, while its productive facet entails implementing problem-solving methods independently.

In the revisitation proposed in 2019 (Niss & Højgaard, 2019 [24]), a clearer distinction is made between solving or dealing with a mathematical problem and formulating a mathematical problem based on a non-mathematical context. The latter is primarily associated with the subsequent modelling competency.

Since this competency is of great interest for the purpose of this research, we will delve into further details in the next chapter.

Modelling competency

The modelling competency comprises lots of different abilities. Firstly, it involves the capacity to engage with existing models, which includes evaluating them and identifying their key properties. Additionally, it requires the ability to "de-mathematize" a model, extracting it from its mathematical form and deducing the real-world context it represents. These constitute the investigative facet of the modelling competency.

Another crucial aspect concerned by this competency is certainly the ability of creating a model from scratch, within a given context. This involves the process of mathematising a situation and identifying the characteristics that can be translated into a mathematical model. Subsequently, the competency involves working concretely with the mathematical problem, finding a solution, and evaluating its correctness. These are the main aspects of its productive facet.

Lastly, the modeling competency entails critically analyzing the model and effectively communicating with others about the process that led to its development and its results.

As for the problem handling competency, the modelling competency is going to be deepened in the next chapter, due to its important relevance for the aim of this study.

Mathematical reasoning competency

In the original translation, mathematical reasoning competency is defined as " *the ability to follow and assess mathematical reasoning, i.e. a chain of argument put forward by others, in writing or orally, in support of a claim*" (Niss & Højgaard, 2011, p.60 [23]). Central to this competency is the clear understanding of mathematical proof, to be able to understand the process that it involves and recognise the basic ideas that are fundamental in proving the thesis. Additionally, it involves the capacity to construct personal proofs of mathematical claims.

The core of this competency lies in the assessment of mathematical claims, identifying patterns of arguments and operations that validate the claim. It requires the knowledge of specific logical elements, such as counter example or heuristic based proof, and the ability to discern a concrete proof from mere intuition-based assertions.

It's important to note that when referring to proofs, we are not just referring to mathematical theorems but it also encompass problems and answers, anything that requires justification. This highlights the close relationship between reasoning competency, modelling competency and problem handling competency, as the former involves evaluating the results of the latter two competencies' work.

Here we can notice that the productive facet involves the ability of creating proofs on your own, while the investigative facet entails the comprehension and the critical analysis of proofs given by others.

Representing competency

The representing competency is defined as the ability to "*understand (i.e. decode, interpret, distinguish between) and utilise different kinds of representations of mathematical objects, phenomena, problems or situations [...] and being able to understand the reciprocal relations between different representational forms of the same entity*" (Niss & Højgaard, 2011, p.63 [23]). The aspect about understanding different kind of representations constitute the investigative facet of the representing competency, while the utilising them belongs to the productive facet.

This competency is based on the awareness of the existence of many different types of representation of the same mathematical object and requires the ability to identify and interpret them effectively.

The symbolic representation is a very important aspect of mathematics and requires an appropriate use of its potentiality. A deep knowledge of all kinds of possible representations is required, as well as of their strength and weaknesses, in order to be able to choose the representation that fit best with the purpose of the task. This implies a strong ability in choosing and switching between different kind of representations.

Each type of representation possesses distinct characteristics, and transitioning between them inevitably involves either a loss or an increase in information. Individuals with this competency should be capable of recognizing such changes and be fully aware of their implications.

It has to be specified that when talking about representations we are not just talking about that one of mathematical objects, but more generally about every kind of phenomena, problems or models.

Some examples of tasks that requires use of representation competency could be: the representation of numbers through the use of dots or bricks, the many different representations of number π, or illustrating the concept of linear function, which can be represented in an algebraic way, as a set of points, through a graph, and so forth.

Symbol and formalism competency

This competency, differently from the previous one, doesn't talk about symbols in the sense of switching from a representation to another, but it's more about the rules that should be followed in the use of mathematical symbols. It is known that mathematics require a certain level of formalism and understanding the rules mentioned above is a very important aspect of this competency; a student should have an insight into the nature of formal mathematical systems.

The definition given by the authors claims that symbol and formalism competency is "*being able to decode symbol and formal language; being able to translate back and forth between the mathematic symbol language and natural language; and being able to treat and utilise symbolic statements and expressions, including formulas*" (Niss & Højgaard, 2011, p.65 [23]). The ability of decoding is a component of the investigative facet, while the ability of treating and utilising symbolic statements constitute the productive facet.

When referring to rules we mean an understanding of the meaning of symbols and how they are employed. The term "symbols" encompasses not only those related to algebraic manipulations and calculus but also basic arithmetic. Very significant is the word *formalism.* It's important to understand that we're talking about using mathematical symbols according to the formal use, with a deep awareness of their role and meaning.

An example of something that is involved in this competency is the "*understanding that 406 stands for four hundreds, no tens and 6 ones*" (Niss & Højgaard, 2011, p.66 [23]). Or, moving to an higher level, being able to translate, from natural language to symbol language, the content of a specific set.

Communicating competency

The communicating competency encompasses both sides of the communication process. This means it comprises both being the sender of the communication or the receiver of some information. These two situations, even thought both characterizing the same act of communication, requires different abilities and respectively constitute the productive and the investigative facet of communicating competency.

The authors summarised them as it follows: "*being able to study and interpret others' written, oral or visual mathematical expressions or "texts", and, on the other hand, being able to express oneself in different ways and with different levels of theoretical or technical precision about mathematical matters*" (Niss & Højgaard, 2001, p.67 [23]). Here are references to many possible forms of mathematics communication, all of which are necessary for this competency.

The different forms of expressing mathematics—oral, written, or visual—refer to diverse representations of the same mathematical object, linking this competency with the previous two. In fact, communication often comprehends the use of mathematical symbols in order to make a topic understandable to everyone; but that's not the only thing. In addition many other factors have to be considered in communication, such as the knowledge and the background of the two entities involved in the communication, and the purpose of the communication itself.

The entities involved in the communication process can be diverse. We refer to communication both if considering the interpretation and decoding of a mathematics textbook or the dialogue between two students about mathematical topics.

Aids and tools competency

The aids and tools competency encompasses the understanding and utilization of various instruments that aid in comprehending and teaching mathematics. This competency involves both sides of the coins regarding tools: "*having knowledge of the existence and properties of the diverse forms of relevant tools used in mathematics and having an insight into their possibilities and limitations in different sorts of contexts, and, on the other hand, being able to reflectively use such aids*" (Niss & Højgaard, 2011, p. 68-69 [23]). The first side constitute the investigative facet of this competency, the second one constitute the productive facet.

This competency requires a deep awareness of the potential benefits of additional tools in learning mathematics. Mathematics has historically relied on various aids, ranging from ancient tools like the abacus to modern computer software. Given the multitude of available instruments, it is crucial to understand the utility and strengths of each in order to select the most suitable for a given purpose.

These aids expand the possibilities of representing mathematical objects and must adhere to formal rules. This highlights a deep connection between this competency, the representing competency, and the symbol and formalism competency. Each of these competencies contributes to the effective utilization of aids and tools in mathematical learning.

2.4 General comments about competencies

2.4.1 The three dimensions of possessing a competency

After a general overview of the competencies identified by the KOM framework, some comments are provided, in order to make the reader better understand the utility of these instrument.

First of all, as mentioned above, we should remember that the eight different competencies described are distinct but overlapped, in the sense that each of them has its own identity but they are strongly interconnected between each other.

Moreover, two aspects that seems not to have been discussed in the competencies system are mathematical intuition and mathematical creativity. Despite this, they are implicitly involved in several competencies. For example characteristics of mathematics intuition contribute to thinking, problem handling, reasoning and representation competencies, while creativity can be regarded as "*the essence of all the productive sides of competencies*"(Niss & Højgaard, 2011, p.71 [23]).

What is very relevant, especially for the purpose of this research, is a discussion, conducted both in the original and in the revisited version, about the three dimensions of possessing a competency. Three dimensions of people's mastery of a competency has been identified, named as follows: *degree of coverage*, *radius of action*, *technical level*.

The individual degree coverage of a person means "*the extent to which the person masters those aspects which characterise the competency*" (Niss & Højgaard, 2011, p.72 [23]). In other words is the amount of aspects within the competency that an individual can activate in certain circumstances, and to what extent they can do so independently. For instance, someone who can understand solutions to problems proposed by others but cannot propose his own solutions has a smaller degree of coverage in the problem handling competency compared to someone who can do both.

The radius of action "*represents the range and variety of different contexts and situations in which the individual can successfully activate the competency*" (Niss & Højgaard, 2019, p.21 [24]). This refers both to mathematical contexts and non-mathematical contexts. For example, a person who can activate problem handling competency for both geometry and algebra problems has a broader radius of action than someone who cannot do so for geometry problems. This is the most difficult dimension to assess.

The last dimension, the technical level, "*is determined by how conceptually and technically advanced the entities and tools are that can be activated in the relevant competency*" (Niss & Højgaard, 2011, p.73 [23]). This dimension refers to the theoretical background, including theories, theorems and results that an individual can draw upon while attempting to activate a certain competency. For instance, someone who can graphically represent functions with one or two variables has a higher technical level than someone who cannot represent functions with two variables.

These three dimensions, in a certain sense, allows us to measure the individual possessing of a competency, but not at all. It should be clear that it's not possible to compare two versions of the same competency following this idea: it makes no sense to compare the technical level of a person who is dealing with linear calculus problems to the one of someone who is dealing with probability problems. There are too many things that would need to be taken into account.

2.4.2 Their relationship with the subject of mathematics

After this description of competencies we should come back to the aim of KOM project, which was to start a development in Danish mathematics education system. To achieve this, the theory of mathematical competencies needed to be applied to the mathematics curriculum. One potential application was to use this study as a guide for defining the Danish mathematics curricula. By considering these competencies and the level of mastery required for each at different educational stages, it could help in designing the goals, structure, and organization of school curricula.

In this regard, an additional section is proposed regarding the relationship between competencies and mathematical subject matter areas. According to Niss and Højgaard (2019, p.22 [24]), they are seen as "*two independent but interacting dimensions of mastery of mathematics*". They cannot be derived from one another, but must be selected based on other criteria. The authors suggested a representation of this relationship as a matrix where each column represents a different subject matter area, going from 1 to n, while the rows refers to the eight competencies previously discussed. The matrix is illustrated in figure 2.2.

Each cell of the matrix should answer question such as "*what is the specific role of competency i (i= 1,2,...,8) in dealing with subject matter area j (j= 1,2,...,n) at this level?*", or equivalently, "*What is the specific role of subject matter area j (j= 1,2,...,n) in the activation of competency i (i= 1,2,...,8), at the level under consideration?*" (Niss & Højgaard, 2019, p.22 [24]). This matrix serves as a useful tool because it clarifies the role of a specific competency at a certain educational level by considering an entire row. Simultaneously, by examining an entire column, it elucidates which competencies are primarily involved in learning a specific subject matter area.

Competency \ Subj. matter area	Area 1	Area 2	···	Area n
Mathematical thinking				
Problem handling				
Modelling				
Reasoning				
Representation				
Symbols and formalism				
Communication				
Aids and tools				

Figure 2.2: A matrix structuring of the competencies × subject matter area of mathematics education

The matrix introduced the risk of teachers perceiving it as overly specific and feeling compelled to fill each cell with restrictive content. To address this concern, an alternative representation was suggested, providing clear space and inviting teachers to structure the representation according to their educational level and context. It is presented in figure 2.3.

This alternative matrix serves not only as a tool for describing competencies but also as meta-cognitive support for teachers. It guides teacher activity planning and facilitates dialogues and discussions among teachers to identify activities that align best with defined goals.

Having identified such competencies is a crucial point that make necessary to decide which thematic area of the subject is more adapt to reach the goal of activating a specific competency in the students. At the same time it can be also an instrument for the assessment part, making it clearer what has to be assessed while creating tasks for a certain matter area.

Moreover, the description of competencies also can benefit students directly. It can be an instrument that helps them in checking their improvements and their learning outcomes, while monitoring the level that they have reached for each competency (Niss & Højgaard, 2019 [24]).

It's evident that the description of competencies is not merely a summary of innate abilities developed along the school curriculum; rather, it requires active participation

Competency \ Subj. matter area	Area 1	Area 2	···	Area n
Mathematical thinking				
Problem handling				
Modelling				
Reasoning				
Representation				
Symbols and formalism				
Communication				
Aids and tools				

Figure 2.3: An open two-dimensional structuring of the competencies × subject matter area of mathematics education

from both students and teachers. Complete awareness of their potentiality is essential to achieve this goal.

2.5 Overview and judgement

In addition to defining the eight mathematical competencies, the researchers of KOM project also included a section addressing mathematics as a discipline. While the competencies focus on individual abilities in challenging situations, it's crucial to recognize that mathematics is a scientific discipline within the education system. Therefore, having a clear understanding of mathematics as a discipline is essential. This understanding doesn't solely come from activating mathematical competencies. Conversely, knowledge of mathematics' structure as an educational subject is not sufficient to master mathematics itself. These two aspects are deeply interconnected and mutually necessary.

For this reason, the KOM project offers insights into the nature and role of mathematics in the world, aiming to provide individuals with "*overview and judgement of the relations between mathematics and in conditions and chances in nature, society and culture*" (Niss & Højgaard, 2011, p.74 [23]). It's important to note that this definition doesn't focus on specific aspects of mathematics, as seen with competencies. Instead, it considers mathematics as a whole, having an overview and a sense of judgment about the whole discipline.

The Danish researchers identified three main types of overview and judgment:

- the actual application of mathematics in other subject and practice areas;
- the historical development of mathematics, both internally and from a social point of view;
- the nature of mathematics as a subject area.

The actual application of mathematics in other subject and practice areas

This first type of overview and judgement concerns the application of mathematics in everyday areas. Mathematics plays a critical role in numerous external contexts in our daily lives, and it's essential to fully recognize its significance.

This aspect have been extensively discussed in the previous description of competencies, due to the fact that is strictly related with modelling competency. Utilizing mathematical models is how external situations are translated into the realm of mathematics. However, having a high level of modelling competency alone is not sufficient.

When discussing overview and judgment related to the actual application of mathematics, we are referring to a more generalized form that also encompasses sociological and science philosophical aspects. This can be explained using some questions as examples, such as: "*Who, outside mathematics itself, actually use it for anything?*", "*On what conditions?*", "*With what consequences?*" (Niss & Højgaard, 2011, p.75 [23]).

The historical development of mathematics, both internally and from a social point of view

The second type of overview and judgement is about the development of mathematics as a discipline for millennia, in culture and society. This development has sometimes been driven by external needs from other fields of practice and at other times has pursued its own goals independently.

Having overview and judgement about its historical development entails more than just knowing the history of mathematics; it involves understanding its socially related development and discerning the mechanisms responsible for its evolution. Of course, a knowledge of the main events in the history of mathematics is necessary to reach this objective.

Unlike the previous type, there isn't a specific competency directly associated with this dimension of overview and judgment. This is because no historical mathematical competency has been identified.

Example of questions one should be able to answer if possess this dimension of overview and judgement are: "*How has mathematics developed through the ages?*", "*What types of actors were involved in the development?*", "*In which social situations did it take place?*" (Niss & Højgaard, 2011, p.76 [23]).

The nature of mathematics as a subject area

This final form of overview and judgment is shaped by the contributions of all eight competencies. It pertains to mathematics as a discipline with specific properties that are unique, distinguishing it from any other subject, yet also shares certain commonalities with other educational fields.

An example of a characteristic that defines mathematics is its method of deriving results and presenting clear proofs of them. While all competencies contribute to this aspect, the ones most closely associated with this dimension of overview and judgment are mathematical thinking, mathematical reasoning, and symbol and formalism competencies, as they emphasize these specific characteristics of mathematics.

Examples of questions related to the nature of mathematics as a subject area are: "*What is characteristic of mathematical problem formulation, thought and methods?*", "*What types of results are produced and what are they used for?*", "*What is its connection with other disciplines?*" (Niss & Højgaard, 2011, p.77 [23]).

2.6 Didactic and pedagogical competencies for teacher

In addition to the eight mathematical competencies described, the KOM framework also discusses the didactic and pedagogical competencies that a mathematics teacher should possess. These should be the main goals of mathematics teacher training. These competencies are presented in a more general and simpler manner than the mathematical competencies, as they were not the primary focus of the KOM project.

The authors identified the following six didactic and pedagogical competencies (Niss & Højgaard, 2011 [23]):

- Curriculum competency;
- Teaching competency;
- Competency of revealing learning;
- Assessment competency;
- Cooperation competency;
- Professional development competency.

We're going to give a short overview of them.

Curriculum competency

This competency pertains to mathematics curricula across various stages and levels. It involves the capacity to analyze, evaluate, and adapt teaching frameworks for mathematics, as well as the ability to implement diverse course plans based on existing frameworks.

Teaching competency

The teaching competency encompasses "*the creation of an abundant spectrum of teaching and learning situations*" (Niss & Højgaard, 2011, p. 86 [23]). It includes numerous activities that a teacher undertakes both in the classroom and while preparing lessons or assignments. This competency involves preparing materials and lessons tailored to students' needs, identifying various methods for teaching mathematics, motivating students to engage in learning mathematics, and effectively explaining the content and purpose of lessons to students.

Competency of revealing learning

This important competency involves the ability to identify and thoroughly understand the level of mathematical learning among students. It is crucial for teachers to analyze students' mastery of the eight competencies and to understand how their understanding develops over time.

Assessment competency

TThe assessment competency involves creating materials and tasks to evaluate students' progress both throughout and at the end of a course. It goes beyond merely identifying or constructing exercises; it includes critically evaluating students' responses. Additionally, it encompasses "*the ability to characterise the individual student's yield and competencies and the ability to be able to communicate with the students about observations and interpretations made*" (Niss & Højgaard, 2011, p. 87 [23]). Therefore, teachers should be able to assist students in overcoming obstacles identified during assessments.

Cooperation competency

The cooperation competency involves collaborating with colleagues, whether they are subject-specific or from other disciplines. It is crucial to engage with them regarding the importance of mathematics teaching and to work together on interdisciplinary projects. Additionally, this competency extends to engaging with external figures such as parents, authorities, or other stakeholders beyond the staff room. It is important to communicate with them about the parameters and constraints of teaching mathematics effectively.

Professional development competency

The professional development competency is described as "*a kind of a meta-competency*" (Niss & Højgaard, 2011, p. 88 [23]). It involves the ability to reflect on one's teaching practices and to enhance competencies in both mathematical and pedagogical content. This includes participating in new training courses, critically discussing ideas with colleagues, and staying updated with the latest trends and educational materials.

Chapter 3

Modelling and problem handling

3.1 Modelling competency

3.1.1 Different definitions of modelling competency

As modelling competency is a key focus of this research we will now delve deeper into its history and the processes involved. In addition to the definition of modelling competency provided by the KOM framework, various definitions and perspectives on modelling competency have emerged in educational history. First of all, we would like to focus on the meaning of the term “model”.

Initially, during the mid-nineteenth century, a model was perceived as a tangible object used to aid mathematics instruction, representing something else. An example of such an object could be stellated polyhedra, physical entities that could be touched and studied.

Over time, the meaning of the term “model” underwent a significant shift, becoming more theoretical. It came to denote “*a description of a system using mathematical concepts and language*”, that can be used to “*explain a system and to study the effects of different components, and to make predictions about behaviour*” (Stacey & Turner, 2015, p. 59-60 [30]).

The term “model” encompasses a myriad of system descriptions, ranging from simple to sophisticated, and finds application in various real-life scenarios.

A clear illustration of real-life modelling is provided by Stacey in the book “*Assessing mathematical literacy*”, related to PISA assessment experience. She explains how she makes tea, according to the rule that one spoonful of tea is required for each person and an additional one for the teapot. This can be seen as a linear model based on assumptions of dimensions of the teapot and the most preferred strength in tea’s taste, and validated by her grandmother's years of experience. It underscores how frequently we employ modelling in a mathematical sense without explicit recognition.

The process of creating a model primarily involves three key actions, which will be deepened in the next section when talking about the modelling cycle, but can be summarised as follows:

- identifying the significant variables related to the problem under consideration and gathering pertinent information about them;
- examining the relationships between the identified variables;
- creating a formula that facilitates finding the answer to the question (Stacey & Turner, 2015 [30]).

The modelling competency involves both creating your own model from real-life contexts and understanding or validating models proposed by others. There are two main ways of interpreting this competency: the "top-down" definition, as suggested by the KOM framework, which views *the* modelling competency as a primary object that could consist of secondary objects called sub-competencies, and the "bottom-up" definition, which considers a set of sub-competencies as primary, collectively constituting modelling competency.

To give a clear explanation of this distinction, "*the bottom-up view cannot admit that the whole is more than the sum of its parts*" (Niss & Blum, 2020, p. 80-83 [22]), since the modelling competency is exactly given by the sum of its sub-competencies, without acknowledging any emergent properties beyond these parts. Conversely, the top-down perspective implies that "*there is more to modelling competency than the set of sub-competencies it entails*" (Niss & Blum, 2020, p. 80-83 [22]).

In other words, in the top-down definition we have a central notion that involves the ability to construct and analyse models starting from an extra-mathematical context; this implies countless smaller components depending on the situation and the perspective adopted. On the other hand, the bottom-up definition focuses primarily on a set of sub-competencies that someone can partly possess or not; you may talk about the general modelling competency only when all sub-competencies have been achieved.

A term that has often been associated with "modelling" is "mathematisation", closely tied to the initial phase of the modelling process, where real-world situations are translated into mathematical contexts. Thus, modelling requires a phase of *mathematising*, which has been studied for long. For this reason many researchers concentrate on this aspect, treating the terms "modelling" and "mathematisation" interchangeably.

Concerning this topic it is interesting to consider the table proposed by Stacey (2015, p. 95 [30]), illustrating the evolution of the definition of mathematisation.

Mathematising (originally labelled 'modelling')	
2005	Mathematising, interpreting, validating
2006	Mathematising an extra-mathematical situation, or making use of a given or constructed model by interpreting or validating it in relation to the context
2007	Mathematising an extra-mathematical situation (which includes structuring, idealising, making assumptions, building a model), or making use of a given or constructed model by interpreting or validating it in relation to the context
2013	Translating an extra-mathematical situation into a mathematical model, interpreting outcomes from using a model in relation to the problem situation, or validating the adequacy of the model in relation to the problem situation

Figure 3.1: Development of the *mathematising* definition

It shows how, over the years, the definition has become more comprehensive, encompassing all aspects of modelling competency, with particular emphasis on real-world situations and, finally, also including the validation process of the model in relation with the practical initial situation.

In the next section we will delve into further details of these phases.

3.1.2 The modelling cycle

At this point it should be clear that modelling competency is deeply connected with extra-mathematical contexts. This connection has been explored in numerous research studies and illustrated through various representations, often depicted as a "*modelling cycle*".

All conceivable representations of this cycle encompass the three fundamental entities defining a mathematical model: (D, f, M), where D represents the extra-mathematical domain, M denotes the mathematical domain, and f is the mapping of selected objects in D to specific objects in M and vice-versa.

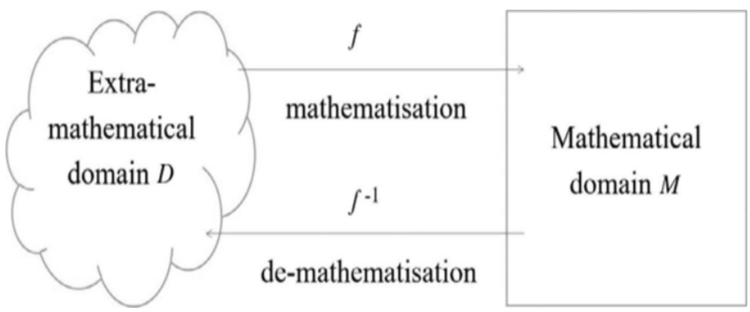

Figure 3.2: Minimal modelling cycle

The basic relationship between these elements can be depicted as shown in Figure 3.2. The process of transitioning from the extra-mathematical domain to the mathematical domain is called "*mathematisation*", while its inverse path, from mathematical domain M to extra-mathematical domain D is called "*de-mathematisation*".

An illustrative example of this one-to-one correspondence between the two different worlds could be the practice adopted by many countries of assigning each individual an identity code composed of numbers generated by specific algorithms. Transitioning from a person to their corresponding identity code represents movement through the function f, whereas reverting from the identity code to the individual involves the inverse function f^{-1}. However, the process of activating modelling competency can be far more intricate than this simplified example.

A more detailed representation, proposed by Niss (2010, [21]), is presented in Figure 3.3.

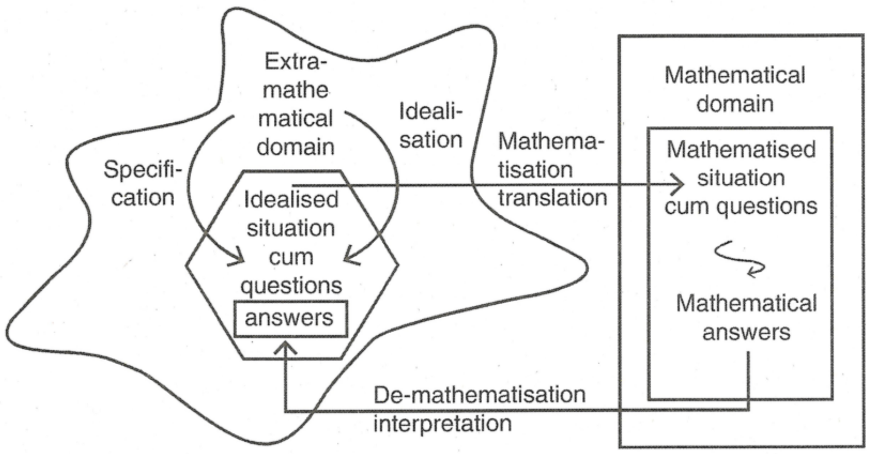

Figure 3.3: The modelling cycle

This picture, even if more articulated than the previous one, shares with it the three main components of modelling competency activation and the two acts of mathematisation and de-mathematisation.

In this representation, the extra-mathematical domain D is portrayed as an "amoeba", highlighting its often blurred and ill-defined nature. In this domain we can find the two phases of *specification* and *idealisation*, constituting a segment of the process that is called *pre-mathematisation*.

This phase involves identifying the main elements to consider, their roles, and their key features. Its aim is to simplify external situations, making them clearer before transitioning into the mathematical world. The result of this phase is a tailored and structured extra-mathematical situation that is called *idealised situation cum questions*, which precisely includes the elements, particularly the questions, that need to be addressed in the mathematical domain.

With the situation now refined, the mathematisation process begins, transforming it into a *mathematised situation cum questions*.

The subsequent step entails bringing the situation into the mathematical domain. Inside this domain M there is a wriggly arrow, signifying that challenges may arise even within the mathematical realm. This wriggly arrow refers to the problem handling competency, underlying the important role that it plays along the modelling cycle and that will be deepened at the end of this chapter.

Once mathematical answers are obtained, they are ready for the de-mathematisation process, converting them into answers applicable to the external world. These answers are depicted within the hexagon of the idealised situation cum questions, emphasizing the importance of evaluating them in comparison with the initial situation.

This phase is called *validation* and consists of "*the confrontation of the mathematical model and its outcomes with the properties of and goals concerning the extra-mathematical situation that the model was constructed to capture*" (Jankvist & Niss, 2020, p. 470 [14]). In other words, it comprises the critical analysis of the answer, to understand whether the solution aligns with the external situation and provides an acceptable resolution.

It's worth noting that the modelling cycle is not a precise depiction of subsequent steps in applying modelling competency but rather a reconstruction of the primary phases typically involved in the modelling process, which may occasionally overlap or omit certain steps.

Furthermore, each arrow in the cycle demands specific abilities and may pose challenges for students attempting to navigate it. Jankvist and Niss define these difficulties as "*stumbling blocks*" (Jankvist & Niss, 2020, p. 470 [14]), meaning any of the processes that could represent a big difficulty for the student and make him unable to activate the modelling competency. Studies have shown that these stumbling blocks are often encountered by students with a strong mathematical background but lacking proficiency in handling extra-mathematical situations and the associated competencies.

An additional characteristic of the modelling cycle, introduced by Niss (Niss, 2010 [21]) is the *implemented anticipation*, which we will examine in detail in the following subsection.

Implemented anticipation

Implemented anticipation, as described by Niss, is a crucial aspect of completing a successful mathematisation of an extra-mathematical situation. Niss identifies three points along the modelling cycle where anticipation by the modeller is essential for efficient completion.

- Firstly, when considering the extra-mathematical situation, anticipation is necessary to identify relevant objects and features aligned with the intended goal of the future model, and posing questions "*in light of their anticipated usefulness in mathematising*" (Stillman & Brown, 2014, p. 681 [31]).

- In a second moment, during the process of mathematisation, the modeller should anticipate the most suitable representations for capturing the situation. When doing this, not only mathematical knowledge is needed for a complete implemented anticipation, but also some past experience in mathematising similar situations that have been successful posed in a certain specific way.

- Finally, implemented anticipation is required regarding problem-solving strategies to tackle the mathematised situation. It involves predicting how the results of mathematisation can aid in finding answers to the original questions.

To sum up, implemented anticipation implies the awareness of what should occur within the mathematical context and making decisions based on these predictions. Niss (2010, [21]), identifies four main prerequisites needed in order to have a correct implemented anticipation by the students. They are:

- mathematical knowledge relevant to the situation;

- putting this knowledge to work for modelling;

- application-oriented beliefs about mathematics on the part of the student;

- "mathematical" self-confidence and perseverance.

These factors underscore a paradox in the development of modelling competency: students are expected to anticipate mathematical knowledge in modelling even before knowing how to model. This paradox is still under discussion, with efforts aimed at overcoming it and establishing an efficient approach to help students develop their modelling competency.

3.1.3 The three dimensions of possessing modelling competency

In the previous chapter we have discussed three dimensions of possessing a competency: the degree of coverage, the radius of action, and the technical level. Introduced by researchers in the KOM project, these dimensions serve as a qualitative means to assess an individual's competency.

Let's delve into what each of these dimensions means in the context of modelling competency:

- *Degree of coverage*: it concerns the extent to which the individual master the key aspects of mathematical modelling competency. It is strictly connected with all the steps forming the modelling cycle: an individual who is able to manage all the steps expected by the modelling cycle has an higher degree of coverage than someone who is able to carry on just the first step. Additionally, degree of coverage is associated with both the ability to construct models and evaluate them. For example, an individual that has a strong ability in analysing models proposed by others but is not able to create one on his own, has a lower degree of coverage than someone who is able to do both.

- *Radius of action*: it refers to the range of different contexts in which the individual can activate the modelling competency. This is deeply connected with non-mathematical context. Consequently, someone capable to activate modelling competency in various external situations, such as economics, household practices, carpentry and so on, has a larger radius of action than someone who can activate his modelling competency just in relation to economic problems.

- *Technical level*: it involves the baggage of mathematical notions, methods and concepts that one can count on while activating the modelling competency. When evaluating technical level in relation to modelling competency, consider, for instance, an individual proficient in creating and analyzing models involving various function types—polynomial, logarithmic, trigonometric, etc. Such an individual demonstrates a higher technical level of modelling competency compared to one proficient only in linear functions. (Niss & Blum, 2020 [22]).

While these dimensions serve as valuable tools for assessing competency possession, it's essential to exercise caution. They may seem to provide an ordered hierarchy of competency possession levels, but in reality, numerous factors must be considered. For instance, one individual may possess modelling competency at a lower degree of coverage but at a higher technical level than another individual. Additionally, competency coverage may vary depending on the context considered.

Nevertheless, these tools can be partly useful in evaluating the possession of the modelling competency, providing insight into the domains of mathematics most relevant to modelling competency and the extent to which it is situated within those domains.

3.1.4 Its relationship with the other competencies

In order to give a clear explanation of the relationship of modelling competencies with the other seven competencies defined by KOM framework, let's walk through an example of activating modelling competency, highlighting the roles played by other competencies during the modelling cycle process.

Consider the following situation: we want to model the cost of a taxi ride from S (starting point) to A (arrival point) in a certain city C (Niss & Blum, 2020, p. 10-11 [22]).

This is a general task that comprehends many different factors. In fact, it is known that many variables play a role in establishing the cost of a taxi ride, such as the number of passengers, the distance, the time of the day chosen for the ride, if it has been booked in advance, the number of luggages taken, the company chosen and many others.

For simplicity, let's focus on a basic scenario, without taking into consideration additional factors mentioned above, but just analysing a classic taxi ride situation, that could also be generalised in future.

> *Let's consider a taxi ride for one person without luggage in a car from a particular company during working hours (9am-5pm). What does such a ride cost?* (Niss & Blum, 2020, p. 10-11 [22]).

Let's make also the assumptions that the ride is on a precise route and that the cost depends only on an initial fixed basic cost, that we will indicate with letter b, the variable which represents distance travelled, x, and the cost per unit distance, which we indicate with d. Any other factors are not taken into considerations for the upcoming model.

The model that comes out of this situation can be expressed through the following linear function:

$$c(x) = dx + b, \tag{3.1}$$

where d, x and b are non-negative real numbers.

At this point the mathematisation phase is completed: we are now in the mathematical domain, with a mathematised situations with questions, and should proceed through the wriggly arrow of the modelling cycle to find the mathematical answer.

The task proposed requires the activation of the modelling competency and specific levels of the three dimensions of possessing it. Firstly, the student needs to possess a technical understanding of linear functions. Considering additional factors previously mentioned may necessitate an even higher technical level. Secondly, the task demands a high degree of coverage since the model is not provided by the exercise but must be created by the student. Lastly, the solver's radius of action should be extensive enough to encompass real-life situations, such as those involving money and taxi rides.

Going on with the resolution process we can say that, if we find out on the company's website that the rate per km is 2€ and the basic rate is 5€, the function becomes:

$$c(x) = 2x + 5, \tag{3.2}$$

So, let's suppose that the ride is 12.9 km long, we're looking for the value $c(12.9)$. Through making the calculation $2{\cdot}12.9{+}5 = 30.8$ we find the result. This is the mathematics solution. At this point the de-mathematisation of the result is needed: let's say that the taxi ride costs 30.8€.

The following step is the validation process: by considering the place in which we are and the average of taxi ride's prices we should evaluate if 30.8€ is a realistic amount to pay for such taxi ride.

This is a clear and realistic example of activating process of modelling competency. When did the other competencies play a role in this process?

Starting from the beginning of the process, when analysing the situation from the extra-mathematical domain in order to create an idealised situation cum questions, we may see as it was necessary to identify the variety of factors that played a role in this situation; this requires the activation of the mathematical thinking competency.

Then, we moved on defining a strategy to solve the situation, while first focusing on classic general situation and subsequently moving to the most specific one. In this phase the problem tackling competency was involved.

The development of the function, during the mathematisation phase, needed the collaboration of symbol and formalism competency, in order to identify the variables necessary for the calculation, and representation competency related to the algebraic representation of a linear function.

Going from the mathematised situation cum question to the mathematical answer required calculation: depending on the person who is trying to make it, aids and tools competency may be needed.

Finally, the reasoning competency is needed in the de-mathematisation phase, to critically analyse if the result obtained could be the answer of the initial problem, and the communication competency may be useful in a second moment, depending on the reason why the individual was trying to model this situation.

Thus, we have seen that the modelling process involves in different ways all the others competencies, and consequently give a help in their development. However, it's necessary to underline that the involvement of a competency in the activation of another one it's not enough for its own development: each competency need to be developed independently, and then used in others process. The same is true for the modelling competency.

Additionally, it's important to note that the modelling competency uniquely encompasses all the other competencies. In fact, we will see in the next section that it is not the same for problem handling competency. This distinction has led to some debate regarding the modelling competency and its role within the competencies framework.

3.2 Problem handling competency

3.2.1 What is a mathematical problem?

According to the route followed for the deepening of modelling competency, we are now going into further details of problem handling competency. Let's begin by examining the meaning of the term "*problem*".

A "problem" can be described as a "*situation which carries with it certain open questions that challenge somebody intellectually who is not in immediate possession of direct methods/procedures/algorithms etc. sufficient to answer the questions*" (Blum & Niss, 1991, p. 37 [5]). Essentially, a problem entails a situation distinct from routine tasks, demanding the activation of specialized competencies for resolution.

It's important to note that the distinction between what can be considered a problem and what is an ordinary task, is highly subjective. It depends on the individual who is trying to face the situation, his knowledge and his abilities: what represents a problem for some person could be an ordinary task for someone else; a variety of factors contribute in determining it.

Moving on to the concept of a "*mathematical* problem", we can identify two distinct types. Typically, we refer to *applied* mathematical problem considering real word's scenarios requiring the application of mathematical concepts. These problems are integrated into the realm of mathematics through the process of mathematisation within the modelling cycle, as discussed earlier.

Additionally, we define *purely* mathematical problem, a situation that is totally embedded in the mathematical realm. This definition, given by Blum and Niss, (1991, p.38 [5]), is accompanied by the clarification that "*this does not prevent pure problems from arising from applied ones, but as soon as they are lifted out of the extra-mathematical context which generated them they are no longer applied*". In essence, a purely mathematical problem can evolve from an applied problem transitioned into the mathematical domain.

According to Hilbert (1900), the first source of mathematical problems is the world of external phenomena, which prompts humans to pose questions that can be answered through the development of mathematical methods. Subsequently, much of the work occurs within the mathematical realm itself, driven by what Hilbert describes as "*the creative power of pure reason*" (Hilbert, 1900). This internal work enables the discovery of new fields of mathematics, which can lead to solutions for many real-world problems. The development of mathematics originates from "*this ever-recurring interplay between thought and experience*" (Hilbert, 1900).

Hilbert explains this with an analogy, stating:

> *A new problem, especially when it comes from the world of outer experience, is like a young twig, which thrives and bears fruit only when it is grafted carefully and in accordance with strict horticultural rules upon the old stem, the established achievements of our mathematical science* (Hilbert, 1900).

In other words, the domains of mathematics and the external world are interconnected and mutually beneficial for each other's development.

The distinction between purely mathematical problems and applied mathematical problems underscores the significance of the process of mathematisation, which has been described as an essential component of the modelling cycle. It's worth noting that the definitions mentioned previously originated at the end of the nineteenth century, predating the development of the KOM project.

These definitions align with the initial definition of problem handling competency outlined in the first version of KOM framework. Indeed, this claimed that problem handling was about "*being able to put forward, i.e. detect, formulate, delimitate and specify different kinds of mathematical problems, pure as well as applied*" (Niss & Højgaard, 2011, p. 55). The inclusion of applied problems directly related to the process of mathematising extra-mathematical situations, which is closely associated with activating modelling competency.

The authors themselves acknowledge that "*the boundary between dealing with applied mathematical problems and active mathematical model building is fluid*" (Niss & Højgaard, 2012, p. 56). To avoid this ambiguity, in the revised version of 2019, Niss and Højgaard (2019) refined the definition of problem handling competency, restricting it to problems entirely within the mathematical domain. The resolution of problems arising from extra-mathematical domains, but whose transition into the mathematical realm is complete, is termed *applied mathematical problem solving*.

Moreover, in the revisited definition of problem handling competency, there is an emphasis on the strategic aspects of problem solving. But what exactly is problem solving? Thus far, we have discussed posing a problem without considering the solving aspect. The expression *problem solving* refers to the process of dealing with a problem while trying to solve it. In other words, it concerns "*the ability to devise and implement strategies to solve mathematical problems*" (Niss & Højgaard, 2019, p. 15).

The focus of strategies for solving problems is a primary aspect of problem tackling competency, and as illustrated by the table proposed by Stacey and Turner (2016, p. 95), its interpretation has evolved over time.

The evolution of the definition of problem solving, here referred to as "*devising strategies for solving problems*", begins with a general consideration of the solution process and gradually shifts towards awareness of the strategies employed and control over the entire implementation process.

Devising strategies for solving problems (originally labelled 'Problem solving')	
2005	The planning, or strategic controlling, and implementation of mathematical solution processes, largely within the mathematical world
2006	Selecting or creating a mathematical strategy to solve problems arising from the task or context; successfully implementing the strategy
2007	Selecting or devising, as well as implementing, a mathematical strategy to solve problems arising from the task or context
2013	Selecting or devising a mathematical strategy to solve a problem as well as monitoring and controlling implementation of the strategy

Figure 3.4: Development of the *problem solving* definition (Stacey & Turner, 2016, p.95)

In the subsequent section, we will delve deeper into the main aspects of problem handling competency, concentrating on problem-solving strategies and the associated challenges.

3.2.2 Devising strategies for problem solving

Problem solving is often discussed without distinguishing between its creative and conceptual aspects. Early studies have revealed that every task, which is not routine but poses a challenge for the individual attempting to tackle it, entails two distinct challenges: *creative* and *conceptual*.

When making a distinction between *creative challenges* and *conceptual challenges*, it's important to note that these definitions are not mutually exclusive. While they represent distinct aspects, they are often intertwined in students' difficulties.

The term "*creative challenge* refers to a *discrepancy between what the student has experienced previously (and in some way has access to) and what is required by a conceivable method*" (Jäder, 2022, p.52). Essentially, creative challenges involve the reasoning about the correct strategy for solving a given problem. Due to the fact that it is not an ordinary task, the student may struggle to identify an adequate strategy due to lack of prior experience in a similar situation. This challenge may lead to the creation of a new solution method or significant modifications to an existing one to adapt it to the new scenario.

To better understand the definition of the conceptual challenge, it's necessary to understand additional definitions regarding a distinction between formal mathematical concepts and the cognitive process that conceived them.

To explain this, we are going to refer to the work of Tall and Vinner (1981), regarding *concept image* and *concept definition*, which pertain to the various mental processes that occur, consciously or unconsciously, during the understanding of mathematical concepts.

A *concept image* is a "*mental picture of the cognitive structure associated with a specific content*" (Jäder, 2022, p.51). It encompasses all the properties linked to a particular definition, shaped by years of experience. Therefore, an individual's concept image can evolve and change over time. For instance, initially, subtraction is presented as an operation to be performed only with natural numbers, leading to the concept image that subtraction involves taking the smaller number from the larger one to get a smaller result.

A concept image is said to be *evoked* when it is activated in relation to a specific problem. It is influenced by both the student's mathematical knowledge and his previous experience. Sometimes, a student may evoke conflicting concept images at different times: this can occasionally lead to the correct solution and at other times to an incorrect one.

An *adequate concept image* is the mental picture necessary to solve a specific task, the one the task aims to evoke during the resolution process.

Conversely, a *concept definition*, is "*a form of words used to specify that concept*" (Tall & Vinner, 1981, p.152). It is a personal definition that may differ from the formal one and is expressed in the individual's own words to explain what they have learned about a specific concept. Essentially, the concept definition reflects what the student has understood and, subsequently, evokes his own concept image.

We're now ready to give the definition of a *conceptual challenge*. A *conceptual challenge* is a "*discrepancy between a student's evoked concept image and an adequate concept image* (Jäder, 2022, p.52). Essentially, it represents a difficulty in understanding and interpreting the task correctly in order to solve it. Conceptual comprehension is crucial for utilizing mathematical concepts effectively and finding the solution.

These challenges manifest at various stages of the problem solving process, which has been described as formed by several steps. These steps can be summarised as it follows, according to Lithner (2007, p. 257):

- Encounter with a problematic situation (task).
- Selection of a solving strategy, which may involve deliberation on why a particular strategy should lead to the solution; through the term strategy here we could mean both a general approach or a local procedure.
- Implementation of the chosen strategy, requiring full awareness of the process and verification of correctness.
- Attainment of a conclusion.

In this framework, creative challenges primarily occur in the initial phases, when encountering a new situation and seeking the correct approach to solve it.

Developing a strategy to solve a problem is a crucial stage of the problem solving process. Undoubtedly, past experience in solving problems plays a significant role in formulating new ideas and methods to overcome any obstacles encountered. Therefore, mastering this phase requires ample opportunities for imitation and practice. While a problem is a new situation for the individual tackling it, it can be studied and analyzed with reference to previously solved problems, both personal and others', in order to adapt solving procedures to the new situation.

Pólya (1945, p.4) asserts that "*solving problem is a practical skill, like swimming. We acquire any practical skill by imitation and practice. [...] Trying to solve problems, you have to observe and to imitate what other people do when solving problems and, finally, you learn to do problems by doing them*".

In searching for the right strategy to solve a problem, we may need to change our perspective multiple times. Initially, we have a general overview of the problem, but as progress is made, our understanding deepens, offering new insights.

Pólya also (1945, p.5) identifies four distinct phases in the problem solving process. The first phase, encountering a problem, is described by Pólya as the need to understand the problem, finding out what is required. This is a critical point in problem solving. It is not uncommon for students to attempt to solve exercises without fully understanding them. There should be not only a deep comprehension of the problem but also a genuine desire to solve it. This desire often stems from a well-presented problem, which motivates the student to find a solution. In this phase specific questions can be helpful, such as *What is the unknown? What are the data?*

Subsequently, Pólya defines the step of searching for the correct strategy as *making a plan*: "*we have to see how the various items are connected, how the unknown is linked to the data*" (Pólya, 1945, p.5). At this point, the conceptual challenge can be decisive. When considering the data and their relationships, it is essential to understand their original meaning; having an inadequate concept image can lead to incorrect conclusions. Formulating a plan may present obstacles and requires creative ideas. The correct strategy might emerge after many unsuccessful attempts, with past experiences with other types of problems being very helpful.

The third step involves carrying out the previously made plan. This means implementing the strategy and attempting to find the solution. This is a fundamental step in the solving process because it generates the solution to the problem. However, it is important to note that this phase is worthless if the previous steps have not been completed successfully. At the same time, it can be easier than the earlier steps: if the plan has been well-conceived, the path will lead to the solution, requiring only time and patience to follow all the steps outlined in the plan.

The final step concerns the analysis and discussion of the outcome. But what exactly constitutes the solution to a problem? How should it be defined to be considered satisfactory? Hilbert (1900) states that, to define something as the solution of a mathematical problem, "*it shall be possible to establish the correctness of the solution by means of a finite number of steps based upon a finite number of hypotheses which are implied in the statement of the problem and which must always be exactly formulated*". In other words, a clear solution requires rigor in reasoning and logical deduction to be mathematically consistent.

Reflecting on the solution after reaching it is extremely important, even though it is often overlooked by students. Pólya (1945, p.14) suggests that, "*by reconsidering and reexamining the result and the path that led to it, they could consolidate their knowledge and develop their ability to solve problems*". The idea is that a problem should never be considered completely exhausted; there is always something left to explore or improve. Additionally, it is crucial to notice similarities and differences between various kind of problems, in order to understand their connections and enhance one's ability to develop new solving methods.

3.2.3 The three dimensions of possessing problem handling competency

Now, let's delve into a detailed analysis of the three dimensions of problem handling competency. These dimensions serve as a crucial tool for assessing a student's proficiency in problem handling.

Let's explore what each dimension entails when referring to problem handling competency:

- *Degree of coverage*: this dimension measures the extent to which a student masters the key aspects of problem handling competency. It assesses how many of these aspects a person can activate and to what degree independently. For instance, if an individual can implement strategies proposed by others but struggles to devise a strategy for a specific task, their degree of coverage is lower compared to someone who can actively perform both tasks.

- *Radius of action*: it represents the range of situations and contexts in which an individual can activate problem handling competency. If someone can activate problem handling competency across various mathematical domains such as algebra, geometry, probability, etc., they have a broader radius of action compared to someone who can only apply problem handling just to geometry problems, for example.

- *Technical level*: this dimension pertains to the mathematical knowledge and technical tools available to an individual when activating the problem handling competency. For instance, a person with extensive knowledge of problem solving strategies and significant experience in creating new solving methods possesses a higher technical level than someone lacking knowledge of potential solving methods and lacking personal experience in problem solving.

It's important to note that if any one of these three dimensions is absent in an individual, they do not possess problem handling competency. Each dimension plays a critical role in evaluating and understanding an individual's ability to handle problems effectively.

3.2.4 Its relationship with the other competencies

The problem tackling competency belongs to the group of competencies identified by the KOM framework as about *the ability to ask and answer questions in and with mathematics.*

While mathematical problems stem from non-routine mathematical questions, it's also true that not all mathematical questions present a problem. It's crucial for an individual to discern the distinction between problems and mathematical questions; this differentiation delineates problem-solving competency from mathematical thinking competency.

Being able to solve mathematical problems requires the mathematical thinking competency but these two competencies are not the same, indeed problem handling does not involve definitions and theorems to the same extent as mathematical thinking (Niss & Højgaard, 2012).

This is a clear example of the correlation that exists between problem handling competency and another competency defined by the KOM framework.

It's noteworthy that, unlike modelling competency, the activation process of problem handling competency doesn't involve all other competencies. This characteristic is unique to modelling competency and isn't shared with problem handling. However, there are certainly relationships between problem handling competency and some other competencies, and occasionally, they may influence its activation process.

For a more detailed understanding, let's consider an example, proposed in the original translation of the KOM framework (Niss & Højgaard, 2012, p.57), which manifests itself precisely as a mathematical question:

Can you make a triangle out of three sides of arbitrary length?

We can observe that the question is already within the mathematical domain. This highlights an important aspect, which we will delve into further in the next section, regarding the role of problem handling competency in relation to modelling: problem handling competency deals exclusively with problems within the mathematical realm.

Let's start with phase one of problem solving: the encountered problematic situation revolves around the possibility of constructing a triangle using any set of three numbers representing side lengths.

The subsequent step involves devising an appropriate strategy to address the question and making a plan. In this scenario, two primary strategies can be pursued: attempting to prove that the assertion is true or seeking a counterexample to demonstrate that it is false. Here, mathematical thinking competency is essential, as it entails understanding a mathematical question and discerning the various potential answers.

Let's consider the latter strategy and decide to construct a counterexample to prove that the claim is not universally true. At this juncture, mathematical reasoning competency comes into play. This involves leveraging knowledge and awareness of mathematical proofs and understanding the significance of the counterexample process in proving a statement. The counterexample is one of the most common types of mathematical proofs, but it is often quite difficult to understand. The reason lies in the nature of proof: to prove that a statement is true, you must demonstrate its validity in all possible situations. However, to prove that a statement is not true, you only need to find a single situation where it fails. So, in this specific example, the reasoning competency play a fundamental role in the activation process of the problem handling competency.

Representation competency may also be necessary during the second and the third step of problem solving process, to visually depict the situation through the drawing of a triangle.

Upon reaching a conclusion, we can examine the one proposed by the authors (Niss & Højgaard, 2012, p.57):

> *No. If we have e.g. side lengths 3, 5, and 10, and start by placing the two short sides each at an endpoint of the long side, the two short sides will not reach each other. Therefore no triangle will be formed.*

The solution provided incorporates elements of communication competency, as it endeavors to elucidate clearly why it is not feasible to construct a triangle with such lengths. This competency could be really useful in the last phase of the process, while reexamining the solution, gaining conclusion from it and comparing the problem to other ones.

Furthermore, aids and tools competency may be activated if practical instruments, such as rulers, are employed to aid in reasoning about side lengths, in phase two and three.

In summary, while problem handling competency may draw upon influences from other competencies in solving a problem, the process does not involve all of them to the same extent as modelling. Therefore, it does not contribute to their development at the same level.

3.3 Differences between modelling and problem handling

Modelling competency and problem handling competency share some similarities that could lead to confusion between the two of them. Here, we aim to highlight the differences between these competencies to clarify their relationship and to underline the role that problem handling plays along the modelling cycle, which is of our main interest.

First of all, it is necessary to specify that two different approaches regarding the relationship between these two competencies can be considered. In this project, we will focus on the Danish approach.

In the idea of the KOM framework modelling and problem handling are seen as two distinct competencies. As previously mentioned, modelling competency plays a slightly different role compared to other competencies within the KOM framework. In its activation, all other competencies can be utilized. Conversely, this is not the case for problem handling competency, although it plays a significant role throughout the modelling cycle.

To better understand the difference between these two competencies let's consider two examples of tasks proposed by Højgaard (2010, p. 255).

1. What is the relation between one's income and the tax paid?
2. How does the tax one pays depend on the income tax and the VAT?

The first example clearly invites the activation of a mathematical modelling process. There are numerous choices to be made regarding the task before applying mathematical concepts and techniques to solve it: including specification and idealisation of the situation, and the subsequent mathematisation. Consequently, all phases of the modelling cycle are required to find a solution. The feeling that come with this kind of task has been expressed as "*"perplexity due to too many roads to take and no compass given*" (Blomhøj & Jensen, 2003). This question addresses the general relationship between two real-life economic factors, without a predefined mathematical model, thus necessitating assumptions and decisions related to non-mathematical facts.

Conversely, the second task is about working within the model. Here there is a clearer reference to the mathematical world. Højgaard (2010) describes it as if involving just some phases of the modelling cycle, specifically: mathematisation, mathematical analysis and evaluation of model results. This task is accompanied by a feeling of "*knowing what the goal is without knowing how to achieve it*" (Blomhøj & Jensen, 2003).

To give a clear explication of the relationship between these two competencies, according the KOM framework, a new representation of the modelling cycle has been created, showed in Figure 3.5.

The idea coming from the KOM framework concerns two distinct competencies with a great relationship. As previously mentioned, problem handling mainly focuses on solving problems within the mathematical domain, involving the process of mathematisation and the transition from a *mathematised situation cum questions* to *mathematical answers*.

The figure illustrates the distinction between an *applied mathematical problem*, which is the one that arises from an extra-mathematical situation and has been mathematised,

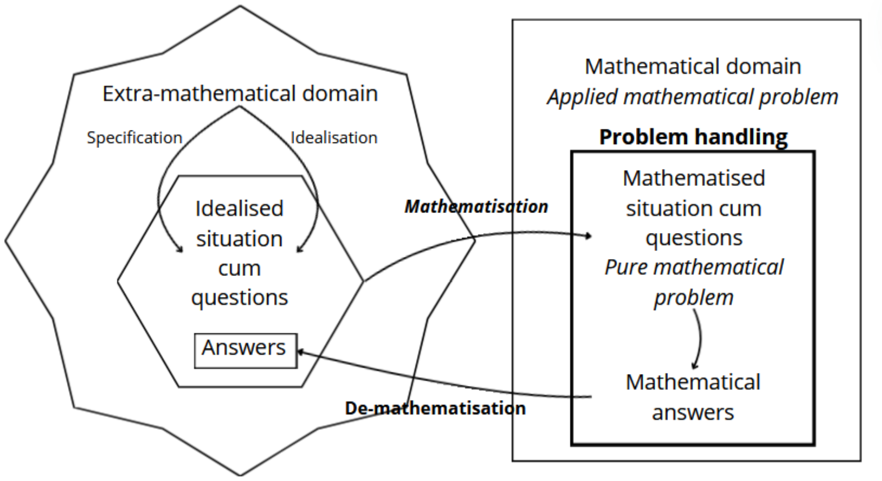

Figure 3.5: Modelling cycle and relationship with problem handling

and a *pure mathematical problem*, which follows the applied one but exists solely within the mathematical realm, without any reference to the real world.

It is important to note that problem handling can be seen as an internal component of the modeling cycle. However, like all phases of the cycle, it does not play a role every time the modelling competency is activated. It can happen that the *idealised situation cum answers*, when taken into the mathematical domain, represents a routine mathematical task, which may not be considered an effective problem for the individual facing it. In such cases, we are not dealing with problem handling, but rather with solving a routine task.

To conclude, the Danish approach to modelling and problem handling entails "*two competencies that do overlap, but have completely different cruxes*" (Højgaard, 2010, p.256).

In this research we will focus mainly on modelling competency, underlying, when needed, the role that problem handling plays along the modelling cycle.

Chapter 4

Assessment of mathematical competencies

4.1 Assessment in Mathematics

Assessment of students' level has always been a central topic of educational word, encompassing various subjects. Literally, *to assess* someone means to evaluate or estimate its ability or quality. Towards the end of the previous century, the international mathematics education community began paying increasing attention to assessment, recognizing that while many innovations were being made in the teaching of mathematics, assessment practices remained stagnant (Niss, 1993 [19]).

Firstly, it is essential to discuss the goals of assessment in Mathematics Education, which are largely shared with other school subjects. Since mathematics is taught in schools and students are engaged in learning it, a means to evaluate their progress is necessary.

Niss (1993, [19]) identifies three primary purposes of mathematics assessment:

- Provision of information;
- Base for decisions or actions;
- Shaping of social reality.

Provision of information

There are many reasons why information about a student's assessment may be necessary, and various individuals can find it useful.

Firstly, assessing a student's mathematical performance provides information to the student themselves, both regarding their standing in mathematics absolutely and relative to their peers.

Secondly, it is an invaluable tool for teachers. Assessment data can assist teachers in understanding each student's strengths and weaknesses, aiding in tailoring instruction to support their development. Additionally, it provides insight into the effectiveness of the teacher's instructional methods. Assessments conducted on a class or group level can help teachers gauge the efficacy of their teaching strategies and identify areas for improvement.

Thirdly, assessment information can inform society at large. It can be used for individual selection and placement purposes, as well as to evaluate the overall performance of students and the effectiveness of the educational system and teachers' efforts.

Base for decisions or actions

When discussing *taking decisions and actions*, Niss (1993, [19]) refers to the selections made in societies to identify individuals best suited for specific jobs, positions, or opportunities.

This selection process occurs at various levels, ranging from within institutional systems to a national scale, and relies on criteria established to rank candidates. These criteria often derive from the outcomes of assessment phases.

In today's society, many positions require knowledge and competencies related to mathematics. Therefore, it is crucial to develop efficient assessment methods to identify individuals best suited for these roles.

Shaping of social reality

This last reason presented by Niss is less explicit than the two mentioned before. With this expression he refers to the implicit societal desire for a disciplined world comprising individuals, educators, and institutions dedicated to hard work.

The emphasis placed on this aim varies depending on the society and its ideological system. Nonetheless, as Niss asserts (1993, p.12 [19]) "*any society holds it and has to hold it, to some extent, in order to have its citizens subject themselves to the often difficult, stressing and frustrating conditions inherent in the competition for education, job, position and material wealth*".

Having outlined the main goals of assessment, we can now discuss how assessment should be conducted and its key characteristics. An *assessment mode* is defined as "*the full spectrum of components in an assessment model that is or could be implemented*" (Niss, 1993, p.12 [19]).

It can be explained through an analogy with a vector that has the following components:

- *The subject of assessment*: this component considers whether the assessment is aimed at an individual student, a class group, or the entire institution.
- *The objects of assessment*: These are the mathematical knowledge content and competencies that are central to the assessment and are to be evaluated.
- *The items of assessment*: These are the tasks that constitute the assessment, designed to produce the desired outcomes.
- *The occasion of assessment*: This component accounts for differences between assessments designed for the end of a course and those meant to be completed during the term.
- *The procedures and the circumstances*: This involves the methods of assessment, the answer to the question "*What happens and who is expected to do what?*"(Niss, 1993 p.14 [19]) It includes factors such as the format of the assessment (written or oral tasks), the role of the teacher during the test, and other similar factors.
- *The judging and recording in assessment*: This refers to what aspects should be emphasized when evaluating the results of the assessment. This may include considerations such as the correctness, completeness, and explanation of the methods utilized.
- *The reporting of assessment outcomes*: This final component pertains to the aspects derived from the assessment results that are crucial for providing information to external parties and aiding in taking decisions and actions.

In the next chapter of this study, we will define the methodology for preparing and submitting tasks to Italian students, analyzing the characteristics of the assessment according to the components discussed here.

Since Italy and Denmark are the two countries involved in this research, we're delving into details of their assessment systems.

4.1.1 The assessment system in Italy

The Italian school system places significant emphasis on the connection between teaching and testing, where they evolve together throughout the academic year, with testing serving as a means for teaching across all subjects.

Specifically, mathematics is viewed as "*a systematic and progressive activity which starts in the earliest grades and proceeds in a spiral or fan-shaped development*" (Bazzini, 1993 p.101 [4]). This implies that the learning of mathematics is continuous over the years, necessitating constant evaluation of its progress. Therefore, assessment holds a central and crucial role in the student's educational journey also within the Italian school system.

The effectiveness of assessment is notably enhanced when the same teacher instructs a specific class for multiple years, allowing them to gain a comprehensive understanding of their students' progress. Throughout the school year, mathematics assessments in Italian schools mainly consist of written tests or oral exercises at the board.

During the year there are many different assessment methods that can be used at schools, in accordance with the variety of mathematical activities that could be performed. What is clear in the Italian tradition is that there is no much interest in standardized tests. The Italian traditional approach to assessment prefers written open tasks and problems, rather than multiple-choice. This is a consequence of the deep-routed idea that standardized tests cannot assess the students' knowledge in an effective way (Bazzini, 1993 [4]).

In more recent times some changes have been made, particularly concerning large-scale assessments. Due to the disappointing results that Italian students had in mathematics in PISA assessment in 2007, in 2008 Italy established the Italian Assessment System, in order to improve the Italian methods of assessment and consequently the outcomes of Italian students.

This led to the creation of a new evaluation system called INVALSI, which is based on standardised national tests to assess students' reading comprehension, grammatical knowledge and mathematics competency. These tests are submitted to all the italian schools in the same period of the year and to students who are attending Grades 2 and 5 of the primary school, Grades 6 and 8 of the lower secondary school, and Grade 10 of the upper secondary school (Arzarello et al., 2015, [1]).

The introduction of these standardised tests, really different from the kind of assessment to which Italian students were used until that time, actually led to some improvements, not only in students' results but also in mathematical methods of teaching. While the INVALSI tests were not the sole catalyst for innovation in the Italian education system, they certainly stimulated teacher interest in advancing mathematical activities (Arzarello et al., 2015, [1]).

Lastly, From a formal and national perspective, there are two main assessment events during an Italian student's academic journey: one at the end of lower secondary school and the other at the end of upper secondary school. Both assessments include written and oral components comprising open tasks, rather than standardized tests. However, while the former is administered by the school's teachers, the latter is a national exam evaluated by external assessors. Additionally, both assessments include a written mathematical test. (Bazzini, 1993 [4]).

4.1.2 The assessment system in Denmark

The Danish school system has a tradition in assessment that differs somewhat from the Italian system, although they share some similarities. One commonality is the recognition of mathematical knowledge as dynamic, requiring continuous assessment throughout the academic year.

Talking about this kind of assessment it has to be said that Denmark has a great tradition in considering written homework assignments, that are not considered just exercises to evaluate students' level but as "*part of the learning and training process*" (Niss & Højgaard, 2011, p. 146 [23]).

On the other hand, also oral presentations play an important role in Danish way of assessment. It's common to give students the opportunity to present some written material prepared at home to the rest of the class. In recent times, more and more attention is being paid to communication of mathematical contents.

Another way of assessment that has been increasing in Denmark tradition involves working in group in class, so that the teacher can be a supervisor of the activity and can find an unusual way to assess the students' level (Niss & Højgaard, 2011, [23]).

In the original text of the KOM project (Niss & Højgaard, 2011 [23]), the authors also say something about assessment. They underline that significant changes are being made in assessment in Danish schools. The classical forms of assessment used to involve individual oral and written tasks, but in recent years they have been relaxed in many ways. It means that these kind of tests have been partially replaced by "take home exam", given assignment that students must do at home and later present in class or give to the teacher. They explicitly say that "*the intention with this modification of written examination is mainly to reduce the distorting effect of a restrictive time factor on the quality of the answers*" (Niss & Højgaard, 2011, p. 143 [23]). It is desirable that, in this way, students won't be stressed by lack of time and we'll focus more on the solving of the tasks. Sometimes this approach can include working in groups.

About Danish formal way of assessment in upper secondary school we need, firstly, to distinguish two different levels in mathematics teaching: B-level is achieved at the end of the first two years, while A-level at the end of the third year; each year involves five hours of mathematics' lessons a week. For both of them there is a final examination which is prepared by national authorities and concerns a written examination paper. The written test involves both purely or applied mathematical problems. In addition, every year a number of selected classes are chosen by the minister to take also an oral examination, which is based on mathematical questions posed previously by the teacher and additional ones related to the topic that can be asked during the student's presentation (Niss, 1993 [17]).

4.2 How to assess mathematical competencies

The innovative work of the KOM project led also to the need of new ways of assessment, specifically related to mathematical competencies. It is now clear that not just mathematical knowledge has to be assessed in students', but also the development of each one of the eight mathematical competencies, taking into consideration the three levels of possessing a competency.

When talking about assessment of mathematical competencies we are not referring to something that can be showed just through a correct answer: it is related to identifying the presence and to what extent these competencies are used by the student during the carrying out of a mathematical activity.

Niss and Højgaard (2011, [23]), describe a mathematical activity as "*a set of conscious and goal oriented mathematical actions in a situation*". A mathematical activity involves different competencies: the central aim of assessment is to identify which competencies are needed by a specific mathematical activity and to what extent they play a role in an individual performance, in order to judge the person's possessing of competencies.

At the same time, it's important to notice that a mathematical activity only comprehends a subset of the eight mathematical competencies. This means that, in order to have a complete framework of individual's competencies is better to pose many different mathematical activities that will require in different times all the competencies (Niss & Højgaard, 2011 [23]).

For this reason, past studies mainly focused on one competency at a time, in order to analyse deeper its manifestation through student's activity. The authors of the KOM project define this assessment problem as "*finding a way of evaluating the individual person's mastery of a given mathematical competency*" (Niss & Højgaard, 2011, p.138 [23]). This can be expanded to identify many mathematical activities that together can give the student's mathematical competencies profile.

To find the correct solution for this, many obstacles must be overcome:

- Finding types of mathematical activities that demonstrate in a clear way the presence of a mathematical competency and that allow teachers and researchers to judge the level of possession of that specific competency.
- Finding a set of mathematical activities that can involve all the competencies in order to provide a complete mathematical competencies profile of the student.
- Finding a way to judge progressive improvements of the individual mastery of one or more competencies.

The last point underlines the importance of paying attention to the whole process of development. It means not just having assessment at the end of a specific topic or term, but also focusing on continuous assessment during teaching, that can be helpful both for the teacher and the student himself.

An essential aspect of assessment regards the fact of being goal-oriented, in the sense that the aim of the assessment must be clear to the person who is being evaluated. Concerning to this, Højgaard (2009, p.4 [12]) propose the analogy with the question *How are you?*, that should be followed by the specific *Regarding what*? "*If that is not clear (possibly implicitly) to the persons involved it will be an unfocused discussion with nothing common to refer to*". The following step can be looking for the level of possessing a competency.

According to him, there are three sub-processes that must be taken into consideration when preparing for assessment (Højgaard 2009, p.4):

- *Characterising* what you are looking for.
- *Identifying* the extent to which what you are looking for is present in the situations involved in the assessment.
- *Judging* the identified.

Following this path, one could be completely aware of the goal of assessment phase, knowing what to look for and able to judge the mastery of it.

Let's focus on the first obstacle to overcome in mathematical assessment, the identification of mathematical activities. In creating the tasks it's essential to consider two quality criteria (Niss, 1993):

- *Validity*: Does the assessment highlight what is supposed to highlight?
- *Reliability*: Is the assessment reliable?

The former refers mostly to initial phase of assessment, when deciding the tasks to submit to the students: it's necessary to pose great attention to their characterisation, in order to collect effective results. The latter involves the final part, about judging the result while being aware of the reliability of the assumptions made in this part of the process.

When these tasks should be related to assessment of competencies things become even more complicated. The correct way to design task in assessment of competencies is to find a perfect balance between high validity and high reliability, but at the beginning of the process a person cannot look for both simultaneously. The solution that Højgaard suggests is "*making a demand for high validity the point of departure when attempting to raise the reliability as much as possible, or vice versa*" (2009, p.5).

It means focusing on one of these two criteria and trying to involve the other one step by step. Between these two opposite strategies, the one suggested by Højgaard in his Danish studies is starting with high validity, because regardless of the outcomes of the assessment, tasks with high validity will keep the learning process on the right track, while tasks with low validity are pretty useless (Højgaard, 2009,).

In designing tasks related to competencies' assessment and starting from high validity, a procedural version of the process is proposed (Højgaard, 2009), consisting in answering the following questions:

1. *Which (competency) learning aims exist for the unit of teaching I am about to assess?*
2. *How do I understand these aims – especially if they are not initially chosen and formulated by me?*
3. *Which kind of presentations to guide student activity can I find or construct that I believe will be well suited to help the students in developing toward the established aims?*
4. *What signs in terms of certain kinds of student activity should I pay special attention to in order to identify the extent to which our aims are present in the situations assessed?*
5. *How do I judge what I have identified?*

The path presented begin with an initial attention to the competency or the competencies that are the main objects of the assessment and that requires complete awareness by the person who wants to assess them on other people.

Secondly, it's important to focus on the best way to assess students', thinking about the most appropriate tasks to present them and the indications they will need, in order to provide them with all the necessary instruments to complete the assessment in an effective way.

Thirdly, one should understand which aspects of the tasks presented are more significant in order to identify the level of possessing a competency, and focus on these factors while checking the outcomes of the students. Then, the judgement phase comes.

Another important aspect in assessing methods introduced by the KOM project, regards the three dimensions of possessing a competency: the degree of coverage, the radius of action and the technical level. It's important to notice that is extremely difficult to assess the three of them at the same time. Past discussions show that, in the mathematical assessment tradition, it's pretty common to pay a greater attention to the technical level of the students, rather than the degree of coverage and the radius of action (Højgaard, 2007).

The KOM project suggests to use these three dimensions of possessing a competency as a tool for mathematics assessment. They assume that, since the master of mathematics is a continuous process, there can't be reduction of these levels, but just growth or stagnation (Niss & Højgaard, 2011).

Since we are talking about three dimensions that can grow independently, the researchers of the KOM project decided to represent the progression in an individual's mastery of a mathematical competency as a cube, with its three dimensions that explicitly refer to degree of coverage, radius of action and technical level. The representation is showed in Figure 4.1.

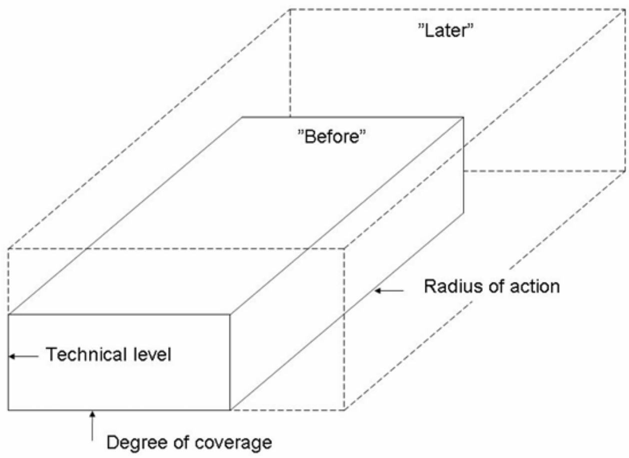

Figure 4.1: Visual presentation of progression in a person's mastery of a mathematical competency (Niss & Højgaard, 2011, p.141)

The entire cube represents a competency, meaning that the possessing of a competency can be increased in different ways, depending on which dimensions is interested by the development. Thus means that the mastery of a competency can be considered as the volume of a cube, "*the product of the fulfilments of the three dimensions*" (Niss & Højgaard, 2011, p. 159 [23]): it increases when one of the three dimensions increases.

It follows that if just one of the three dimensions of possessing a competency is equal to zero, then the student doesn't possess the competency at all. On the other hand, it means that different students can have the same level of mastery of a competency but with very different values of each dimension.

In the next section we're going to analyse deeper the aspects concerning task design and the three dimensions related to assessment for the specific case of the modelling competency, whose assessment is the main focus of this project.

4.3 Tasks related to modelling competency

4.3.1 Main challenges in modelling assessment

6 We have previously discussed that when talking about competencies the assessment process becomes complicated. Since the modelling competency has a more complex structure compared with the other seven competencies, when assessing it things become even more complicated and many variables have to be taken into account.

Studies of the last decades show that modelling competencies are usually weakly developed among students, and many of them present lots of difficulties in completing them, even if only some basic knowledge of mathematics and calculation is required (Niss & Blum, 2020 [22]).

The reason behind this outcome lies in the many challenges that modelling competency concerns.

First of all, a modelling task involves both the mathematical world and the extra-mathematical world. This means that to succeed in it, a student must possess both mathematical knowledge and some extra-mathematical knowledge. The former is essential for what concerns mathematisation and the resolution of the mathematised situation cum answers; the latter is needed for the idealisation of the initial situation, so in the pre-mathematisation phase, and for final validation and evaluation (Niss & Blum, 2020 [22]).

The involvement of the extra-mathematical world often generates negative perceptions in the students, that are more used to considering mathematics as a pure abstract discipline which has nothing to do with the rest of the world. Usually, students, are used to mathematics lessons that focus on rules, theorems and theoretical knowledge, and expect tasks which clearly refer to the topic dealt during the lesson, so that they know what to expect and can prepare themselves for the test. For this reason, when they find an open modelling environment, many of them get stuck and simply refuse to deal with it because they can't see the relationship between a real life situation and mathematics.

Additionally, very often, when solving tasks, students just try to understand the mathematical actions required by the task, and apply it without completely understanding what they are doing and the connection with the real world situation (Niss & Blum, 2020 [22]).

The dealing with recently introduced concepts can't go together with modelling tasks. In designing modelling tasks for students is necessary to take into account the "few years gap"; it comes from the fact that "*it may take a long time, and several experiences with such uses, before students can independently apply the mathematics they have learned to new and open modelling tasks*" (Niss & Blum, 2020, p.93 [22]). According to this way of thinking, there should always be a temporal distance of few years between the time a student learn a mathematical concept and the time he is able to use it to solve a modelling task.

The teacher play a significant role in developing the students' modelling competency. First, to change students' attitude towards pure mathematics is necessary to provide lessons that include correlations between it and other subjects, or between it and the external world, in order to overcome the deep-rooted idea that mathematics is for its own sake. Secondly, when designing tasks for modelling assessment a big teacher's effort is needed: it is essential to choose tasks that can be interesting for the students, nor too difficult nor too easy, and affordable with their level of mathematical knowledge.

To do this in a correct way the teacher must possess at least the same level of modelling competency that is required by the task; this means that development of mathematical competencies should be central also in teacher's educational path (Niss & Blum, 2020 [22]).

A great job that the teacher should do in order to prepare students for modelling tasks concerns the *didactical contract*. This concept, designed by Brousseau, refers to "*the set of habits, rules and expectations that are, explicitly or implicitly but mostly in a tacit manner, established between students and their teacher concerning their interaction and division of labour*" (Niss & Blum, 2020, p. 96 [22]). This is something that is often established unconsciously, and can be really dangerous if there is no complete awareness of it.

A well-know example of didactical contract proposed by Brousseau himself is called "the age of the captain". It is an experiment conducted in a primary school involving the following problem:

> *On a ship there are 26 sheep and 10 goats. What is the age of the captain?* (Brousseau, 2002, p. 263 [7])

The experiment showed direct consequences of the didactical contract: 78% of the primary students investigated gave n answer involving some operations between the numbers 26 and 10 (Brousseau, 2002, [7]). Such behaviors stem from the unwritten rules established between the teacher and the students, which lead the students to believe that a question posed by the teacher must have a reasonable answer, even if they cannot discern its purpose.

The consequences of the didactical contract that may represent a problem for modelling tasks are many, regarding students' expectations about the tasks given by the teacher. Here are some examples (Niss & Blum, 2020, [22]):

- The conviction that the task contains exactly all the data needed to complete it, which means that it is not necessary to make personal assumptions or to look for additional data hidden in the text.

- The idea, mentioned before, that the task must refer to a specific topic that has been introduced lately by the teacher.
- The idea that, if the task is set in a real-world context, the context is irrelevant and the only important thing is to identify the mathematical problem and carry on the calculations in the right way.
- The certainty that just a unique correct answer exists.

It clearly follows that a student with these expectations will be pretty much surprised in encountering an open modelling task, and this increases his probability of failure.

The last challenge to be presented in this section regards assessment in general. Not everything that is taught at school can or should be assessed, especially when talking about development of competencies and how difficult it is to really identify their possession level; but, conversely, it is well known that students put more effort in studying something that will be assessed (Niss & Blum, 2020, [22]). For this reason it's extremely important to find the correct way to make them aware of mathematical competencies and to judge their level of possession.

4.3.2 Holistic approach vs Atomistic approach

One of the main challenge in modelling assessment is caused by the complexity of this specific competency and the process of the modelling cycle that it involves. Due to the many phases composing it, it is extremely important to establish, before designing tasks, the aim of it, especially if it should assess all the modelling cycle or just some of its phases.

Concerning this aspect, two different approaches have been developed: the *holistic* approach and the *atomistic* approach.

The former "*seek to assess the entire modelling work or process at the same time in one shot*" (Niss & Blum, 2020, p. 134 [22]). It means that we are considering tasks that involve the entire modelling cycle and we want to assess the student's complete mastery of it.

Conversely, the latter "*refrain from taking all aspects of modelling work and processes into account at the same time but seek to assess only one or a few aspects at a time, typically by zooming in one or two steps in the ideal-typical modelling cycle*" (Niss & Blum, 2020, p. 134 [22]). This idea is based on the conviction that focusing on a single phase at a time it would be possible to have a clearer overview on the student performance in it.

While the holistic approach make it possible to consider the entire process of modelling activation in one time, the atomistic approach need to collect more combined assessments in order to give a comprehensive framework of the student's possessing of the modelling competency.

Depending on the strategy that a person decides to adopt, different assessment modes can be available for testing modelling competency. Frejd (2013) studies showed that there are five major modes of assessment for mathematical modelling, which are: written tests, projects, hands-on tests, portfolios, and contents.

The atomistic approach usually favors written tests, while assessment modes such as projects and portfolios are favoured in the holistic approach, being also long term tasks that can leave the student enough time to work along the whole modelling cycle (Niss & Blum, 2020, [22]).

We're now going into details of each phase of the modelling cycle. Many studies, as the one conducted by Jankvist and Niss (2019) that is of primary interest for this research, show that each phase of the modelling cycle can represent an obstacle for students, for different reasons.

That is because "*mathematical modelling is a cognitively demanding activity since several competencies must be activated along the road*" (Niss & Blum, 2020, p.115 [22]). For this reason, every step has its own difficulties and could block students.

The first step is "understanding the given situation and constructing a mental model of it". This phase conceals many challenging factors, such as the external context and the understanding of the written task. In understanding a situation coming from the outside world it is clear that the past experience of the individual and his familiarity with that specific context play a significant role. At the same time student's language proficiency should not be underestimated. In fact, from the didactical contract discussed above follows that "*in this subject you don't have to care about meaning but just about tackling tasks by applying a recently learnt method*" (Niss & Blum, 2020, p.116 [22]). This leads to the fact that many students simply don't pay attention to the careful reading of the part of the text regarding the extra-mathematical situation.

While assessing this phase of the modelling cycle, the teacher should focus on looking for some indications that show the complete understanding of the situation by the student. This is one of the most difficult phases to assess through written tasks, because usually the process of understanding takes place in the head of the modeller (Niss & Blum, 2020, [22]).

The second phase is the one about idealisation, "simplifying and structuring the situation". In this phase implemented anticipation by the students is extremely important in order to succeed. At this moment students have to identify the variables and think about a way to solve the situation presented. This way should of course involve mathematical actions that they are able to carry on, but without a correct implemented anticipation they may define variables leading to complex calculations or models that they are not able to solve. Additionally, sometimes the students don't pay much effort in this phase, again because too focused on the mathematical actions that will follow.

In assessing this phase, the teacher should pay attention to the variables identified and their relations between each other, in order to find out if there is awareness of the process that has to come and if there has been successful implemented anticipation (Niss & Blum, 2020, [22]).

The third step is mathematisation, which leads to the creation of a mathematised situation cum questions. In this phase mathematical knowledge plays a central role. The student should write a function or a mathematical model that will lead to mathematical answers. Both implemented anticipation and a deep understanding of the mathematical entities in use are needed and may represent difficulties for the individual.

This phase is probably easier to assess because concerns mathematical elements and their relations. The teacher should pay attention on the correctness of the mathematical product presented (Niss & Blum, 2020, [22]).

The fourth step is the solving of the mathematised situation. At this point, the mathematised situation obtained could be a routine task or a problem. In the first case the student should carry on a mathematical strategy to which he's used to, so the difficulties are mostly related to mathematical process and calculations involved in it. In the second case the activation of problem handling competency is required, thus a whole range of new difficulties appears, starting from the search for a new strategy to implement until the checking of results' validity.

At this point the teacher must be aware of the difference between the situation that requires problem handling and the one that does not; then he can assess the student's outcome by evaluating the mathematical calculations and the correctness of the obtained result (Niss & Blum, 2020 [22]).

The fifth step corresponds to the de-mathematisation phase, the "interpreting the mathematical results". A very common students' behaviour at this point is to consider the task almost finished, after having obtained a mathematical result. The reason of this is partly to attribute to the didactical contract and the separation between mathematical and extra-mathematical world. Moreover, the extra-mathematical situation comes into play and this generates an additional difficulty to the task, due to having to relate with two realms at the same time.

In this phase the teacher should pay attention on the correctness of the interpretation of the result in the extra-mathematical context (Niss & Blum, 2020 [22]).

The sixth step is about "validating and evaluating". This is a very tricky phase because sometimes students just don't do it. It should involves some reasoning about the sense of the result obtained and taken in the extra-mathematical world, to check if it can effectively be the answer of the initial question. It's pretty much common that students don't pay attention to the realistic sense and the appropriateness of the obtained result.

The role of the teacher in this phase is to check the correctness of the presented result and especially to look for indications that validating and evaluating really happened. As in the first phase, this is extremely difficult to assess because, if not explicitly written by the student, these actions take place in his own mind (Niss & Blum, 2020 [22]).

4.3.3 Assessing the three dimensions

As previously said, there are many factors to consider when designing task for competencies assessment. Regarding modelling competency we have to take into consideration all the phases of the modelling cycle and, at the same time, the three dimensions of possessing a competency. We're now going into details of what each of the three dimensions, degree of coverage, radius of action, and technical level, aims related to modelling competency.

Degree of coverage

To assess the degree of coverage of an individual regarding the modelling competency it's important to focus on how much independently the person is able to deal with models. To reach this goal it's useful to analyse the students' behaviour both with pre-existing models and with models that they must create. It follows that a person who is able to mathematise situation on his own has an higher degree of coverage than a person who can just carry on the process of models created by others.

To assess this dimension the holistic approach can be considered, because it is effective to consider the modelling cycle in its entirety. Indeed, "*someone who can carry through the various sub-processes in the mathematical modelling process, but only when prompted to do so, is more competent than someone who can not enter these processes at all, but less competent than the one who autonomously initiates the work*" (Højgaard, 2007, p. 144, [10]).

Radius of action

The radius of action concerns the context in which a person is able to activate a specific competency. It follows that, related to modelling competency, there are countless contexts coming from the outside world that can be taken into consideration.

Testing students with tasks referring to different extra-mathematical situations can be a way to estimate their radius of action of modelling competency. It is clear that the individual's radius of action is influenced by his personal experience, in the sense that someone who is an expert in everyday shopping situations may not be good enough in creating models coming from the field of architecture (Højgaard, 2007, [10]).

Technical level

This dimension involves "*the size and the content of the mathematical toolbox*" of the student (Højgaard, 2007, p.144 [10]). Talking about models it refers to the part of the modelling cycle that begin with mathematisation. The personal mathematical knowledge of an individual is extremely important in this phase because it makes difference if the model created can count on a functional expression or just on a simple equation.

Thus, to evaluate this dimension of possessing modelling competency is important to focus on the mathematised situation proposed, and the mathematical tools that are involved in it.

4.4 The PISA experience: an example of mathematical assessment

4.4.1 Pisa surveys

Mathematical assessment has always been a problem shared among all kind of schools and nationalities. From the need for a way to test educational systems all around the world, the project PISA came out, opening to a new kind of mathematical assessment. The name PISA stands for Programme for International Student Assessment, guided by the Organisation for Economic Co-operation and Development (OECD). It is a project that focuses on testing if educational systems are effectively preparing students "*for the challenges that they are likely to face in their futures*" (Stacey & Turner, 2015, p.6 [30]).

Thus, the focus is not on questions inside the school world, but on problems related to real life. The aspects that have been considered fundamental for young people are reading, writing and arithmetic, which is now better defined as mathematical literacy, involving not only calculations' ability but all branches of mathematical sciences.

PISA surveys are conducted every three years, on sample groups of fifteen-year-old students, chosen randomly in OECD and partner countries. The tasks and the results are chosen in a way that is possible to compare results of the same country taken in different years. The same is not possible for the ranking, because the number of countries that participate in the survey may differ from year to year.

We will focus on the PISA experience about the assessment of mathematical literacy. The mathematical literacy, defined initially in 2000, and then revisited in 2006 and 2012, is said to be: "*an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain, and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded* *judgments and decisions needed by constructive, engaged and reflective citizens*" (OECD 2013a, p.25).

It's important to underline how it explicitly refers to the connection between mathematical and extra-mathematical worlds, expressing the important role that this subject plays in everyday life. So, PISA's goal was to create a survey composed by mathematical tasks that could give a general overview of the mathematical literacy of students from all over the world.

We are presenting this example of mathematical assessment because its approach to assessment of mathematical literacy is similar to the one adopted by Danish researchers on mathematical competencies. Indeed, it is no coincidence that some of the tasks that we are going to use in this research project and that have been used in the Danish study taken into consideration (Jankvist & Niss, 2019 [14]) are taken from PISA past surveys. Additionally, it has to be said that, at the beginning, the PISA assessment was based on the KOM framework, involving seven of the eight competencies defined by Danish researcher. The mathematical thinking competency was left behind because considered too difficult to assess.

A Mathematics Expert Group is responsible for design and definition of PISA tasks. Each PISA item is classified according to six factors:

- *Assessment mode*: which could be paper-based or computer-based.
- *Process category*:it refers to the kind of mathematical actions required by the task, such as formulating situations mathematically, employing mathematical concepts, interpreting and evaluating mathematical outcomes.
- *Content category*: this one is about the topic treated in the item; the different possibilities are quantity, uncertainty and data, change and relationships, space and shape.
- *Context categories*: it refers to four real-world contexts that have been identified by PISA researchers, which are personal, societal, occupational, ans scientific. The personal items arise from daily life, societal ones from being an active citizen in the society, occupational problems refer to the world of work, and scientific ones are referred to science and technology.
- *Response type*: it refers to the structure of the answer, that can be selected response, such as multiple choice, or constructed response, such as manual calculations.
- *Cognitive demand*: it's a number that classifies items' difficulty compared to the general average. An average item has difficulty 0, while items with positive scores have difficulty over the average.

4.4.2 PISA items and the modelling cycle

The aspect of PISA surveys that are of main interest for the aim of this research are the ones concerning modelling competency. Since connection between the real world and the mathematical world is a central point of PISA assessment, the modelling competency play a significant role in this project.

The representation of the modelling cycle provided by PISA is showed in Figure 4.2.

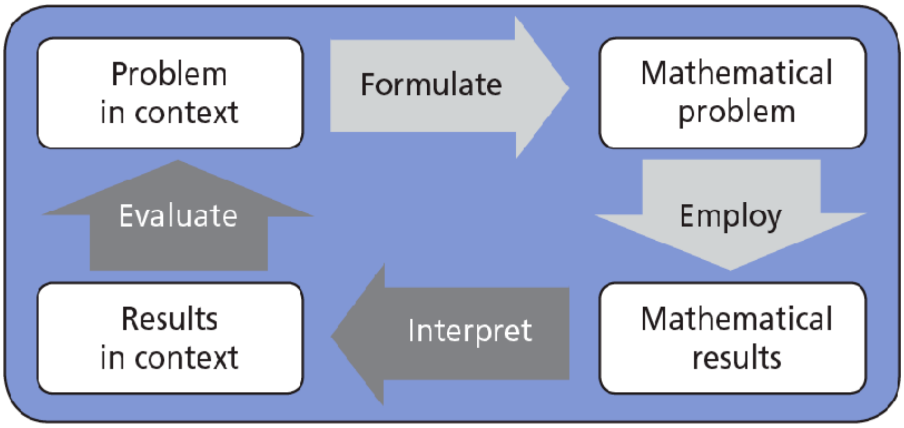

Figure 4.2: PISA 2012 model of mathematical modelling cycle (OECD 2013a)

We can notice that this representation is less detailed than the one from the KOM project analysed in the previous chapter, but they still share the most significant points, including the co-existence of external and mathematical worlds.

PISA representation presents four different stages of the modelling cycle:

- Formulate;
- Employ;
- Interpret;
- Evaluate.

At the beginning, a *problem in context* is identified, followed by the formulation of a mathematical problem, which corresponds to the act of mathematisation. The next step, employ, leads to mathematical results, and involves the resolution of the mathematical problem. It is followed by the interpretation of the mathematical result, which corresponds to de-mathematisation and leads to the result in an external context. Then, there is the evaluation of the result: at this point, if the found solution is satisfactory, the modelling ends, otherwise a modified problem in context has been established and the modelling cycle continues (Stacey & Turner, 2015 [30]).

The phases showed are the same used by PISA system to define process categories. So, PISA items primarily focus on these four steps of the modelling cycle because they reflect much use of mathematics in real life and involves the central aspects of modelling competency.

The strategy adopted by PISA items is to assess one of these four phases at a time, due to the fact that the tasks are designed to be solved by students in a short time and without additional resources, so it would be too difficult to handle an entire modelling cycle in each task (Stacey & Turner, 2015 [30]). PISA items follows the athomistic approach previously described.

Past studies show that "*student's genuine interest in a real-world context such as sustainability issue or the direct relevance of a context to students' lives can be harnessed to increase motivation*" (Stacey & Turner, 2015, p.78-79, [30]). Due to this, PISA put a lot of effort in identifying contexts that are attractive to students, considering the four contexts categories presented in the previous section and studying the factors of most familiarity and interest for fifteen-year-old students.

The students' outcomes from PISA surveys are then analysed in many different ways, to identify the main difficulties and draw conclusions about the educational system, to promote future improvements.

In the next chapter, some examples of PISA tasks will be provided, given that they have been used for the current project.

Chapter 5

Research questions

At this point of the research, after having introduced the main theoretical aspects that concern the object of this study, we can define more precisely the research questions to which the current research wants to find answers.

In the introduction, the following research aim has been presented:

> *Investigate Italian upper secondary school students' potential difficulties with mathematical modelling, according to the KOM framework, and compare them with Danish upper secondary school students' potential difficulties (Jankvist & Niss, 2019).*

From the beginning of this project it has been clear that the focus is on the Danish way to consider, teach and assess mathematical competencies. A deep analysis of the KOM Framework and the definition of the eight mathematical competencies has been provided. Among these eight competencies, two have resulted of primary interest for this research: the modelling competency and the problem handling competency, the latter considered as a potential component of the former.

The literature presented in Chapter 1 showed us how much Italian and Danish tradition in Mathematics Education have different approaches in the teaching of mathematics in the school. These differences are the point of departure of the present research that is trying to take aspects of the Danish approach into the Italian school system.

The first goal of the project can be presented through the following research question:

What aspects of modelling competency are developed in Italian students?

This research question is motivated by the difference between the two didactical approaches. Since Italian tradition has been presented as more related with textbooks and theoretical knowledge, while Danish tradition presents the development of competencies, such as mathematical modelling, as a main goal of the school curricula, this project wants to check whether the Italian approach, despite not being focused on it, provide students with opportunities to develop their mathematical modelling competencies.

The analysis that leads to the answer of this research question has been conducted considering two of the three dimensions of possessing a competency: the degree of coverage and the technical level. The reason of this decision lies in the difficulty of assessing radius of action, that may vary a lot also between students coming from different countries due to life experiences surely not very similar.

The reason of the focus on modelling competency and, when it plays a significant role during the modelling cycle, on the problem handling competency, lies in the important role that these abilities have in everyday life. Indeed, the central aspect of modelling competency is the transition from an extra-mathematical context to the mathematical world, the phase of the modelling cycle called mathematisation. This means that such a competency is essential for facing situations from the real world in the more effective way. Thus, assessment of modelling competency on Italian students is made in order to evaluate how much the mathematics taught in the school is preparing them for the challenges of the future real world in which they will live; we have seen in Chapter 4 that this was also the primary goal of the PISA assessment at the time it was developed.

In other words, what we want to analyse is at which level the Italian school system contribute to the development of the modelling competency even if not explicitly treated as a main component of the school curricula. Does it evolve indirectly in the students thanks to the teaching approach used or should we put more attention on it?

These questions relate directly with the next research question, that is:

> *What are their main difficulties and what are the possible reasons for these difficulties?*

This research question refers exactly to the "stumbling blocks" defined by Jankvist and Niss (2019, [14]), which play a significant role in the development of this research. We aims to identify the main difficulties of Italian students, analyse and evaluate them, in order to discuss the aspects that could lead to their generation and the ways to overcome them.

They can be caused by some mistakes related to the organisation of the mathematics curriculum, by a modelling competency not so much developed in the students, or maybe by an individual lack of mathematical knowledge. We want to analyse the difficulties and find out the reasons behind them.

This question is going to be answered following the path of analysing the results designed by Jankvist and Niss in 2019. The reason for this similarity is due to the third and last research question, whose aim is to create a comparison between the results of the two countries protagonists of the research. It claims:

What are the significant differences compared to the Danish upper secondary school students and why?

Unfortunately, there was no opportunity to evaluate students at the same time in the two countries, but results pertaining the same tasks have been collected and are used to establish a significant comparison. The goal behind this research question is to evaluate if two different didactical approaches can lead to significant opposite results in students' difficulties or can generate similar outcomes. To reach this goal we are going to evaluate the stumbling blocks of Italian students and compare them with the ones identified by Jankvist and Niss (2019, [14]).

Finally, the general purpose is to analyse the consequence of the didactical approach used in a country and try to gain enhancements by considering aspects coming from different countries; as decades of research in Didactics of Mathematics show, combine different cultures can lead to new methods and improvements.

Chapter 6

Submitting modelling tasks to Italian students: methodology and presentation of the tasks

6.1 Methodology and data collection

The crucial point of the current project is to submit modelling tasks to Italian students, in order to assess their modelling competency.

As discussed previously, Italian tradition has been strictly related to textbooks and memorisation for several years, and it doesn't possess a focus on mathematical competencies. On the other hand, Danish tradition provides mathematical modelling as a compulsory component of the curriculum of the upper secondary mathematics programme (Jankvist & Niss, 2019 [14]).

Due to differences between the two countries, the idea behind this research is to evaluate how much Italian students develop modelling competency, even if the school system is not focused on it. The goal is to provide them with some tasks coming from the real world, which comprehends the whole modelling cycle or just some phases of it, and analyse their ways to create and solve models coming from real life.

The students that take part in this project are attending the third year of upper secondary school, which corresponds to Grade 11, at a Scientific high school in the city of Bologna. Despite coming from the same Scientific high school, the total of forty-one students is composed by two classes that presents small differences in the main address of the school: one of the two is Scientific high school with one hour per week of mathematics enhancement, the other one is a Scientific high school focused on applied science. The hour of enhancement of the former is usually used to develop additional mathematical abilities, through programming or laboratories' activities. Apart from this, the two classes have exactly the same hours of mathematics per week and share the same curriculum.

The test submitted is composed by six tasks, taken from a Danish study (Jankvist & Niss, 2019 [14]) that will be detailed later in this section. One hour of time to solve it is given to the students. The tasks are administered by the students' teacher in whole class setting and during ordinary school hours.

A difference is made between the submission to the two classes. Discussions about didactical contract show that students' habits to not consider mathematics as related to the real world could make less effective the submission of the test, especially considering the Italian assessment tradition. This project provide Italian students, who are mostly used to problems posed in the mathematical world and focused on a specific topic, with open tasks that sometimes don't show explicitly a connection with mathematics. For this reason, the idea is to provide an introduction to modelling competency to only one of the two classes, in order to analyse if there is a significant difference between the results of the two classes. The choice of the class that should be informed about modelling competency is left to the teacher.

The students' introduction to modelling competency comprises a definition of the competency and a general overview of the modelling cycle, focusing on the initial phases, especially mathematisation, which are the most treated by the presented tasks. The hope is that, in this way, they won't be totally surprised by the open tasks composing the test and will be able to collocate themselves along the modelling cycle.

The objects of the discussion are the answers provided by the students to tasks that present written open questions. Unfortunately there is no opportunity to get in touch with the students and have a deeper understanding of their responses, thus the written answers represent the crucial part of the collected data.

The decision to submit exactly the six tasks provided in the study of Jankvist and Niss (2019, [14]) is due to another goal of this project: the comparison between the development of modelling competency in Italian and Danish students. Indeed, a comparison between the results showed in the Danish paper and the results obtained in Italian school is conducted, in order to identify the outcomes coming from two different educational traditions related to the role of mathematics in the outside world.

For this reason, we're now delving into details of the project carried on by Jankvist and Niss in 2019. The aim of their project was to analyse the main difficulties of Danish students with modelling competency, by providing them with a *detection test*.

A detection test is focused on a specific mathematical *problematique* and "*is a set of mathematical questions related to this problematique to be answered within a time frame that is sufficient for time pressure not to be an issue*" (Jankvist & Niss, 2019, p. 473 [14]).

The aim of a detection test is to detect students with some difficulties in the learning and understanding of mathematical entities. For this reason it consists of tasks that are not routine tasks but "*require students to think and act independently*" (Jankvist & Niss, 2019, p. 473 [14]).

This research belongs to a larger Danish project that aims to create figures called "math counsellors", whose role is precisely to identify students with particular difficulties in mathematical learning, in order to carry out specific interventions to reduce or totally overcome these difficulties.

The detection test used in the considered project contained thirteen questions, focusing especially on the early phases of the modelling process, because the main goal was to identify difficulties in getting started on a modelling problem. The students had to solve those questions in an hour and a half, during ordinary schools' hours and under their own teacher's supervision. 315 students coming from upper secondary school joined that research; they were from Grades 10, 11 and 12.

In the next sections we are providing a deeper analysis of six from the thirteen tasks used in the detection test, that are composing the test submitted to Italian students.

6.2 Presentation of the tasks

The six chosen tasks share the same format "written open response" and focus on different phases of it. Due to the fact that the research is not centered to mathematical knowledge but to mathematical modelling competency, the technical mathematical concepts required are not high level, in order to focus on the non-technical aspects of mathematical modelling. As we have discussed before, there's a great time gap between the acquirement of a certain mathematical knowledge and the ability to use it in a modelling context; thus, tasks requiring topics treated in the classroom recently would not have been in line with the purpose of the research.

It has to be underlined that the tasks presented are described by the authors as "*not full-fledged, extended modelling tasks*, but rather "*stylised, distilled modelling tasks* (Jankvist & Niss, 2019, p. 475, [14]). This means that they don't require students to engage a lot with extra-mathematical information, but they already provide some kind of idealisation. However, it is necessary that the students completely understand the sense of the statement and make decisions on his own, between many different paths for mathematisation. In conclusion, it can be said that "*each of them represents significant aspects of genuine modelling*" (Jankvist & Niss, 2019, p. 476, [14]).

The tasks are intended to assess degree of coverage and technical level of students' modelling competency, rather than radius of action. Indeed, to assess students' radius of action it would be necessary to provide them with many exercises related to various extra-mathematical contexts. Since six tasks are not enough to evaluate students' radius of action we will focus on the other two dimensions of possessing modelling competency.

Some of the tasks have been partially adapted to be closer to Italian students' reality.

Let's start the analysis of each task.

Task 1 - A walk in Rome

> *Gianni can walk from The Trevi Fountain to the Pantheon in 6 minutes. Greta needs 8 minutes. How long does it take if they walk together? Justify your answer.*

> **Solution:** Since Greta needs at least 8 minutes to complete the walk, while Gianni needs at least 6 minutes, the common constraint when they walk together is represented by the longer time, which is 8 minutes. Therefore, to complete the walk together, they need at least 8 minutes, which is also sufficient for both.

The original version of this task was about a walk in the Danish city of Roskilde. The location has been changed in order to make it more familiar to Italian students.

Such a question is totally located in the extra-mathematical world and the students are required to understand the correct mathematical way to answer it. It focuses on mathematisation rather than the phase of resolution involving problem handling competency. Indeed, calculation is not strictly necessary, some reasoning could lead to the correct solution.

The phase of the modelling cycle most relevant to this task is the mathematisation phase. Students may find this particularly challenging because it is uncommon to encounter such situations in traditional mathematical exercises. This could elicit two different reactions from students.

According to the discussion about the didactical contract, one of the most common reactions to this type of question is to apply a calculation without considering the meaning of the result obtained. For some students, it may seem logical to simply calculate the average of the two times presented and assume that together they will need 7 minutes to complete the walk. This reflects students' tendency to disregard the extra-mathematical context and focus solely on performing calculations with the given data, assuming that is what the teacher expects.

Additionally, due to the ambiguous connection between the question and the mathematical world, which is not as clear-cut as in exercises students are accustomed to, another common reaction is to refuse to complete the exercise because it seems absurd that the teacher would pose such a strange question. This is an important aspect that could be central in the difference between the students who have been previously prepared through a presentation of the modelling cycle, and the ones who are totally unaware of it.

Implemented anticipation is essential to enable students to create an efficient model, particularly by focusing on the key aspects necessary for solving the task.

The degree of coverage and the required technical level are not very high, aligning with the idea that the focus is on practical aspects of the modelling competency. The mathematical knowledge needed involves only the correct methods to solve a system of inequalities, which can be formulated using basic knowledge of constraints. If solving a system of inequalities is not a routine task for the student, problem-handling competency will need to be activated.

Task 2 - Height of building

Look at the photograph (Figure 6.1). How tall is the building in front? Justify your answer.

Figure 6.1: Illustration from task 2 (Jankvist & Niss, 2019, p. 480 [14])

Solution: There are two reasonable answers to this question:

- One could focus on the composition of the building. There are four upper floors and a ground floor, plus four decks and the roof deck. Using some real-life knowledge, one can estimate that the height of each upper floor is approximately 2.5 meters, the height of the ground floor is 3 meters, and each deck has a height of 0.5 meters. The total height can thus be estimated to be close to 15.5 meters.

- One could use the person with the red jumper who is near the building for scale. By estimating the average height of the individual (let's say 1.80 meters) and measuring his height in centimeters in the picture, one can calculate how many times the person's height fits into the height of the building. This calculation leads to a total height of approximately 17 meters.

This question can easily mislead students since it apparently provides no numerical data. Starting from just a picture, the student is asked to calculate the height of the illustrated building, but no explicit numerical data is given.

This is an example of a task that follows the holistic approach, activating the entire modelling cycle.

In the beginning, the process of mathematisation plays a significant role: an extra-mathematical situation without explicit connection to the mathematical world is presented, and the student must create a model to answer the question meaningfully. Implemented anticipation is crucial: identifying data that could be useful in the mathematical context is extremely important.

The degree of coverage needed to solve the task is not very high: in both cases, linear equations are sufficient to find the solution. The same can be said for the technical level.

Once the model has been created, in one of the two ways presented before, we have a mathematised situation cum questions. At this point, problem handling competency may be needed. This is subjective and depends on the individual student's ability to solve linear equations.

After the solution is found, de-mathematisation is required. This is followed by a significant phase of validation: it is essential, in this specific task, that the students realize that the result found mathematically represents the height of a building. They should critically analyze the number obtained to decide if it is realistic.

It can be noted that the didactical contract could be an obstacle for this task as well. Due to the apparent absurdity of the question and its unclear connection with the subject of mathematics, it is not unlikely that students will simply refuse to answer it, especially if they are not familiar with the modelling cycle process.

Additionally, the task does not refer to a specific mathematical topic but gives students complete freedom to choose their resolution process. This can be extremely confusing for students who are accustomed to the idea that in mathematical exercises, it is not necessary to make personal assumptions.

Task 3 - Oil field

> *An oil field contains 100 million barrels of oil. Alice says that if you extract 1 million barrels of oil every year, the oil field will be exhausted after 100 years. Anna says that if each year you extract 1% of the remaining oil, then the oil field never gets exhausted. Who is right and why?*

> **Solution:** Alice and Anna statements are based on different assumptions. Indeed, they are both right. If there are 100 million barrels of oil and you extract 1 million barrels every year, it will be exhausted after $100/1 = 100$ years; so, Alice is right. Anna suggests to extract 1% of the remaining oil every year. It means that, after n years, there is an amount of oil left equal to $100 \cdot 0.99^n > 0$. It means that the oil will never get exhausted, so Anna is right.

This task presents a more evident connection to the mathematical world, as students will likely realize immediately that mathematics is needed to solve it. However, pre-mathematisation and mathematisation are the phases most involved in this question, with a significant role played by implemented anticipation.

A mathematical model should be created to perform calculations and evaluate which of the two statements is true. The tricky part is that both statements are actually true because they are based on different assumptions, so one does not exclude the other. This can be an obstacle for students who are influenced by a strong didactical contract: it is difficult for them to accept that the answer to a question asking them to choose between two statements is not unique. This aspect can lead to different approaches among students. Some will likely not consider that both statements could be true, instead trying to prove the one they find more convincing and simply excluding the other.

At the beginning, students should be able to create a model for both statements. Alice's assumption requires an easier model and a lower degree of coverage than Anna's assumption. To prove the second statement, students need a higher degree of coverage that involves understanding exponential functions. The technical level required should be manageable for Grade 11 students but is higher than that required by the two previous tasks.

After creating the two models, problem handling competency comes into play. The calculation required by Alice's assumption should not be problematic for Grade 11 students, but the one proposed by Anna's assumption may require activating problem handling competency.

After the mathematical answer is found it is necessary to identify its realistic meaning. The validation phase can be tricky especially for the second statement: students may find it confusing that a source of oil will never get exhausted.

In this case, knowing about the process of the modelling cycle may not be very helpful: the conviction generated by the didactical contract that there must be a unique correct answer predominates over other considerations.

Task 4 - A hill in Athens

> *On a steep hill in Athens there is a 4 km long path leading to the top. Riccardo, who is in good shape, can climb the hill with an average speed of 3 km per hour and walk down again twice as fast. What is the average speed for the full walk? Justify your answer.*
>
> **Solution:** The length of the total walk is 8 km. The first step to find the answer is calculating the total time needed for the walk. He needs 80 minutes to go up the hill and 40 minutes to go down; these times are calculated based on the average speed given for the two different walks. By considering the total amount of time needed (120 minutes or 2 hours) and the length of the entire walk (8 km), the total average speed can be calculated. It is equal to 4 km/h.

This question provides clear references to mathematics and physics concepts. It is evident from the outset that some calculation is needed to find the solution, but the student must complete the mathematisation phase and translate the data into a mathematical expression. As seen in the previous questions, this phase can hide many obstacles that might complicate the resolution process.

In this specific case, physics knowledge is required: the student needs to remember the definition of average speed and the formula to calculate it; otherwise, it would be impossible to reach a correct result. This means that the technical level required is higher than in previous tasks. However, the degree of coverage required should be entirely manageable for Grade 11 students.

While converting the data into mathematical expressions during the mathematisation phase, some implemented anticipation is needed to create an efficient model that will easily lead to the solution. This could represent a significant obstacle for students due to the many difficulties associated with implemented anticipation, as discussed in Chapter 3.

Once the mathematical model has been created, some calculations are necessary. Depending on the student's ability and knowledge, these calculations could require the activation of the problem-handling competency.

Finally, after obtaining a result, the de-mathematisation phase is needed. In this case, it involves evaluating whether the number obtained at the end of the calculation process can represent an average speed and if it is realistic in relation to the described situation.

Task 5 - Pizza

A pizzeria serves two round pizzas of the same kind and thickness in two different sizes. The smaller one has a diameter of 30 cm and costs 30 zeds. The larger one has a diameter of 40 cm and costs 40 zeds. Which pizza is better value for money? Show your reasoning.

Solution: It should be clear that the amount of pizza is determine by the volume of the pizza, which is defined by the ratio area/price. The value for money of the smallest pizza is equal to $(15^2\pi)/30$, while the value for money of the biggest pizza is equal to $(20^2\pi)/40$. Since $(15^2\pi)/30 < (20^2\pi)/40$, the larger pizza gives the better value for money.

This task, which is actually a PISA item, involves prices expressed using zeds, an international currency invented by PISA.

This is a clear example of a question where some idealization has already been made. The situation is presented in a way that highlights its connection to the mathematical realm, which may help students identify the correct path for mathematisation. However, implemented anticipation is needed, especially in the first phase, to understand which aspects of the pizza are most significant.

The mathematisation phase is undoubtedly the one most addressed by this task: students should create a model to relate the diameter and price of the pizzas, in order to obtain numbers that can be compared to discuss value for money.

The researcher wants the student to recognize that value for money, considering that the pizzas have the same thickness and topping, depends only on the area-to-price ratio. The main difficulties in this task lie in the mathematisation phase. Students may struggle to understand the concept of value for money and to create a mathematical expression to calculate it.

Regarding mathematical knowledge, students need to know the formula for the area of a circle and the concept of a ratio. This should be manageable for Grade 11 students, but they must have a technical level high enough to know the formula for the area of a circle. Since the required calculations are not highly complex, a very high degree of coverage is not needed.

Once the model is created, students should be able to work with the given numbers and compare the results for the two pizzas. The activation of problem-handling competency may be needed in this phase.

The validation phase is not explicitly addressed by this task.

Task 6 - Wooden cube

> *A wooden cube, with all edges of length 2 cm, weighs 4.8 grams. What is the weight of a wooden cube with all edges of length 4 cm? Justify your answer.*
>
> **Solution:** Due to he fact that the two cubes are made of the same material, they should have the same densities, given by the ratio weight/volume. Since the volume of the larger cube is 8 times the volume of the smaller cube, the larger cube should weighs 8 times as much as the smaller cube. So, it weighs 38.4 grams.

This question focuses heavily on the mathematisation phase. It requires the student to identify the relationships between the given data, particularly understanding that density is determined by the ratio of weight to volume. One potential obstacle could be fixating on the lengths of edges rather than considering the cubes as solids. Effective implemented anticipation is crucial to create the correct model and avoid wasting time on irrelevant aspects of the data.

This task likely has a high technical level since it involves scientific knowledge related to density. The mathematical model is based on the assumption that both cubes are homogeneous and made of the same material, resulting in identical densities. This assumption could pose a challenge for Grade 11 students in correctly mathematising the situation.

Later, with the mathematised situation cum questions, it may happens that students don't know exactly how to solve it. In this case they are facing a mathematical problem and the activation of the problem handling competency is needed. However, the calculations required do not demand a high degree of coverage.

During the de-mathematisation phase, which follows the calculation phase, it is crucial to evaluate whether the result obtained makes sense. In this case, it's important to assess whether the student understands the concept of density and recognizes that the obtained result could effectively represent the weight of the larger wooden cube.

Chapter 7

Analysis and discussion of the collected data

7.1 Analysis of the results

In this chapter, we will present and discuss the data collected from Italian students and provide answers to the research questions. The analysis is conducted individually for each task and is organized according to the methodology used by Jankvist and Niss (2019 [14]).

Firstly, a distinction between satisfactory and unsatisfactory answers is made. The unsatisfactory answers are then categorized based on the stumbling blocks they appear to present. We will consider the same categories identified by the authors (2019 [14]), and add new categories if necessary.

We will identify Class A as the group that has not been informed about the modelling cycle, which consists of twenty students. Class B will be the group that has been briefly introduced to the modelling cycle and consists of twenty-one students.

7.1.1 Task 1 - A walk in Rome

Task 1 is an example of a question posed entirely within an extra-mathematical context, significantly different from the tasks to which students are accustomed in Italian upper secondary schools. Jankvist and Niss (2019 [14]) identified three types of stumbling blocks for this task. The answers from the Italian students have been satisfactorily divided into these same categories, without the need for additional groups.

The three types of stumbling blocks identified are:

- Type 0: The student refuses to deal with the question in a mathematical way. This type includes answers that do not refer to the mathematical context at all but instead evoke everyday considerations.
- Type 1: The student focuses on the mathematical aspect of the task without considering the extra-mathematical context. They proceed to make some calculations with the given numbers (usually calculating the average), without concretely thinking about the meaning of the numbers.
- Type 2: The student does not take the mathematical conditions seriously. This type of unsatisfactory answers includes suggestions that the two friends could find a compromise.

Table 7.1 shows the results of class A.

Total Class A	Missing	Satisfactory	Unsatisfactory	Type 0	Type 1	Type 2
20	0	10	10	3	2	5
%		50%	50%	30%	20%	50%

Tabella 7.1: Task 1 - Class A

We can see that there is a perfect balance between satisfactory and unsatisfactory answers. The predominant stumbling block in Class A is Type 2: many of the students suggest that the answer depends on the compromise chosen by the two friends, implying that Gianni could go a little slower to wait for Greta and Greta could try to go a little faster to prevent Gianni from waiting.

Total Class B	Missing	Satisfactory	Unsatisfactory	Type 0	Type 1	Type 2
21	1	12	8	1	3	4
%	4.76%	57.14%	38.09%	12.5%	37.5%	50%

Tabella 7.2: Task 1 - Class B

Table 7.2 shows a predominance of satisfactory answers. Among the unsatisfactory answers, the most common stumbling block is again Type 2, related to the non-acceptance of the mathematical conditions given by the task.

This stumbling block highlights difficulties of the students regarding the pre-mathematisation phase of the modelling cycle. They are not able to interpret the data given by the task in a mathematical context to create a useful model.

It should be noted that, even in the satisfactory answers, there is no evidence of a concrete mathematical model created by the students, such as involving inequalities. The students who gave satisfactory answers tend to explain their reasoning with logical statements, suggesting that since Greta can't go faster, Gianni has to wait for her. For this reason, there is no evidence of problem handling competency or of difficulties in activating it.

The group of unsatisfactory answers of Type 1 indicates difficulties with implemented anticipation. A Type 1 stumbling block involves not being able to identify the role of the given data and the correct way in which they should be used in the mathematical context. These difficulties lead to an inadequate mathematical model, such as calculating the average of the two times.

Notably, in Class A there is a higher percentage of Type 0 stumbling blocks (30%) than in Class B (12.5%). Since Type 0 involves the refusal to engage with this kind of task, we can assume that this difference is the result of the introduction to mathematical modelling presented to Class B. In fact, Class A students were not prepared at all for this kind of task. This can lead to greater difficulty in accepting a question entirely situated in the extra-mathematical world, compared to Class B students who were aware of the goal of the test.

Finally, to subsequently address the research questions, a table summarizing the results of the two classes is presented below:

Total	Missing	Satisfactory	Unsatisfactory	Type 0	Type 1	Type 2
41	1	22	18	4	5	9
%	2.44%	53.66%	43.9%	22.22%	27.78%	50%

Tabella 7.3: Task 1

7.1.2 Task 2 - Height of the building

Task number 2 is composed of a picture and the question to guess the height of the building shown. It differs from the mathematical exercises usually presented in Italian classes, providing interesting results.

As with the previous task, the categories of stumbling blocks identified by Jankvist and Niss (2019, [14]) have been sufficient to analyze the responses of Italian students. In fact, only two of the four groups of reasons for unsatisfactory answers identified by the authors (2019 [14]) have been used. These are:

- Type 0: This category involves students not accepting the task provided. It includes all the answers providing funny comments or random numbers, where the students, due to the apparent absurdity of the question, simply refuse to think about it in a mathematical way.

- Type 2: This category includes all the answers that provide a mathematical model but with unreasonable assumptions. Some examples may include unreasonable estimates of the height of the person or the floor, mistakes in the scale chosen between the picture and reality, and inappropriate choices of the person to compare.

Total Class A	Missing	Satisfactory	Unsatisfactory	Type 0	Type 2
20	0	16	4	1	3
%		80%	20%	25%	75%

Tabella 7.4: Task 2 - Class A

We can notice that the majority of the students answered correctly. This is surprising given the considerations previously made about the apparent absurdity of the task, especially considering that Class A knew nothing about the modelling cycle.

Since this task involves the whole modelling cycle, it is a very significant result.

Among the unsatisfactory answers, only one belongs to Type 0, as it gives a random number as result. Most of the unsatisfactory answers belong to Type 2: students who tried to reason by considering the number of floors failed because they did not consider the separation decks. Thus, the model proposed by them was partly unreasonable.

All students who considered the height of the person provided a reasonable model.

Total Class B	Missing	Satisfactory	Unsatisfactory	Type 0	Type 2
21	0	14	7	1.5	5.5
%		66.7%	33.3%	21.43%	78.57%

Tabella 7.5: Task 2 - Class B

Table 7.5 shows the results of Class B. We immediately notice that there is a higher percentage of unsatisfactory answers in Class B compared to Class A, which is surprising given that this class should have been more prepared for tasks like this.

Further details of the data collected show that it was easier for students who reasoned about the number of floors to fail. As in Class A, the main reason for their unreasonable models was not taking into consideration the separation decks or the roof.

Differently from Class A, there are also people in Class B who failed while considering the height of a person: one student chose a figure too far from the building, leading to mistakes in proportionality caused by perspective, and another gave an unreasonable estimation of the average height of a person.

Additionally, it can be noticed that in Class B, which was informed about the modelling cycle, there are two examples of satisfactory answers including an evident validation phase. Two students justified their answers by stating that it made sense that a building has such a height. This aspect is extremely important because it shows awareness of both the mathematical model created and the extra-mathematical meaning of the result obtained.

Total	Missing	Satisfactory	Unsatisfactory	Type 0	Type 2
41	0	30	11	2.5	8.5
%		73.17%	26.83%	22.73%	77.27%

Tabella 7.6: Task 2

The following table shows a summary of the results of the two classes:

Overall, the results of this task are satisfying, with a significant portion of students from both classes providing correct answers. The major stumbling block regards the mathematisation phase, while pre-mathematisation, problem handling and validation did not seem to be a problem.

7.1.3 Task 3 - Oil field

Task 3 presents a scenario where students must evaluate two different statements and determine their truthfulness. The biggest challenge here is that both statements are true but are based on different assumptions.

This task has a more evident connection to the mathematical world, yet it poses significant challenges in the modelling phase, especially for Grade 10 students. The data collected confirm this assumption.

For the analysis of this exercise we have used all the categories identified by Jankvist and Niss (2019 [14]). They are:

- Type 0: Students refuse to engage with the task or do not understand it due to their real-world knowledge.
- Type 1: Students try to solve the task but do not take it seriously, often assuming that only one of the statements can be correct.
- Type 2: Students realize that both statements could be correct but fail to accurately mathematise one or both situations.
- Type 3: Students create a mathematical model for the statements but fail to correctly solve or understand the mathematical treatment needed, indicating difficulties in activating problem handling competency.

Total Class A	Missing	Satisfactory	Unsatisfactory	Type 0	Type 1	Type 2	Type 3
20	1	1	18	5.5	4.5	4	4
%	5%	5%	90%	30.56%	25%	22.22%	22.22%

Tabella 7.7: Task 3 - Class A

Table 7.7 shows the results of Class A.

In analyzing the results of Class A, we observe that only one student answered satisfactorily, while all the others encountered issues in finding the solution. The highest percentage of stumbling blocks belongs to Type 0. Some students provided random answers without any explanation, while others simply refused to solve the problem because they did not agree with the real-world assumptions made. In particular, some students could not accept the idea that an oil field would never be exhausted without considering the mathematical aspects that could lead to this result.

However, we can see that the distribution of the stumbling blocks is fairly homogeneous. Type 1 involves students who assumed that both statements could not be correct, so they focused on proving one of the statements and consequently assumed the other was false. This reflects a consequence of the didactical contract.

Type 2 indicates difficulties with the mathematisation phase, while type 3 reflects challenges with the activation of problem handling competency. It should be noted that Alice's statement posed fewer problems for most students compared to Anna's statement. The majority of the class struggled with correctly modelling or solving the mathematical aspects of Anna's statement.

Total Class B	Missing	Satisfactory	Unsatisfactory	Type 0	Type 1	Type 2	Type 3
21	0	3	18	3.5	6.5	2	6
%		14.28%	85.71%	19.44%	30.95%	9.52%	28.57%

Tabella 7.8: Task 3 - Class B

Table 7.8 shows the results of Class B.

We can notice that Class B has a higher percentage of satisfactory answers. Importantly, compared to the correct answers from Class A, these answers present a clear exposition of a mathematical model. This could be a consequence of Class B students' awareness of the modelling cycle: they knew that a model should be created, so they tried to provide a clear mathematical explanation of their reasoning.

In Class B, there is a less homogeneous distribution of the stumbling blocks. The main difficulties were of types 1 and 3.

Type 1 occurred because many students considered only the first statement, proved it true, and simply ignored the other one. As assumed during the presentation of the task, even though Class B was informed about the modelling cycle, the didactical contract, along with the conviction that both statements cannot be true, predominates.

Type 3 indicates significant difficulties in activating the problem handling competency. It was common for students to not correctly understand or solve the mathematical model created for Anna's statement or to incorrectly interpret its result.

The summary of the results for this task follows:

Total	Missing	Satisfactory	Unsatisfactory	Type 0	Type 1	Type 2	Type 3
41	1	4	36	9	11	6	8.5
%	2.44%	9.76%	87.8%	25%	30.56%	16.67%	23.61%

Tabella 7.9: Task 3

7.1.4 Task 4 - A hill in Athens

Task 4, unlike the previous three tasks analyzed, presents a more evident connection with the mathematical context and requires a higher technical level.

In this case, Jankvist and Niss (2019 [14]) identified four categories of stumbling blocks, which are:

- Type 1: The student does not try to solve the task due to a lack of deep understanding of the question or lack of knowledge about average speed.
- Type 2: The student directly operates with the given numbers and calculates the average of the two average speeds.
- Type 3: The student understands that a mathematical model is needed but is unable to convert real-world information into an adequate mathematical model.
- Type 4: The student completes the mathematisation phase but is unable to operate correctly with the mathematical model due to mistakes in units or calculations.

Total Class A	Missing	Satisfactory	Unsatisfactory	Type 1	Type 2	Type 3	Type 4
20	1	7	12	1	8	2	1
%	5%	35%	60%	8.33%	66.67%	16.67%	8.33%

Tabella 7.10: Task 4 - Class A

Table 7.10 shows the results of Class A. We can notice that there is a predominance of unsatisfactory answers, likely due to the higher technical level required by this task.

Among the unsatisfactory answers, the most common stumbling block is Type 2. Many students simply calculated the average of the two average speeds given by the text without providing any reasons. This can be considered a consequence of the didactical contract and a manifest lack of reasoning about the question. Thus, the main difficulties of the students concern the mathematisation phase.

One student exhibited a Type 4 stumbling block, indicating difficulty in solving the mathematical problem created. This can be a subjective aspect, depending on the technical level of the individual.

An interesting answer falls into the Type 3 category. One student created an incorrect mathematical model, which led to the solution of 0 km/h. The student simply accepted this result without questioning it. This clearly demonstrates the absence of a validation phase: the student found a mathematical answer to his model but failed to contextualize it in the real world. If they had, they would have realized that it is impossible for Riccardo to go up and down the hill with an average speed of 0 km/h.

Total Class B	Missing	Satisfactory	Unsatisfactory	Type 1	Type 2	Type 3	Type 4
21	0	13	8	0	4	2	2
%		61.9%	38.1%		50%	25%	25%

Tabella 7.11: Task 4 - Class B

Table 7.11 shows the results of Class B.

It is evident that Class B achieved a predominance of satisfactory answers. However, among the unsatisfactory responses, there is a notable trend similar to Class A, with a higher percentage falling under Type 2.

Interestingly, in Class B, we observe instances of Type 3 stumbling blocks where students failed to validate their solutions. Specifically, two students created erroneous mathematical models that led them to conclude with an average speed of 0 km/h. This oversight is unexpected given that Class B students should have had more awareness of the modelling cycle, including the de-mathematisation phase, which should have facilitated their validation process.

Furthermore, it is worth noting that among the satisfactory answers, some students demonstrated a higher technical level by employing the harmonic mean. They were able to directly calculate the solution. Others succeeded in creating and solving a mathematical model through the activation of problem handling competency. Only two students encountered difficulties during the solving phase.

Here is a summary of the results of Task 4:

Total	Missing	Satisfactory	Unsatisfactory	Type 1	Type 2	Type 3	Type 4
41	1	20	20	1	12	4	3
%	2.44%	48.78%	48.78%	5%	60%	20%	15%

Tabella 7.12: Task 4

7.1.5 Task 5 - Pizza

Task 5, similar to Task 4, has a clearer connection to the mathematical world. Jankvist and Niss (2019 [14]) identified three types of stumbling blocks among unsatisfactory answers, which were also adequate to categorize the results from Italian students:

- Type 1: The student provides an answer without relevant reasoning or considerations, indicating a lack of understanding of the concept of value for money.
- Type 2: The student fails to correctly mathematise the real-world situation, commonly making the mistake of assuming a direct proportionality between the diameter and the price of the pizza.
- Type 3: The student successfully mathematises the situation but is unable to solve the mathematical problem correctly, reflecting difficulties in activating problem handling competency.

Total Class A	Missing	Satisfactory	Unsatisfactory	Type 1	Type 2	Type 3
20	1	5	14	8	3	3
%	5%	25%	70%	57.14%	21.43%	21.43%

Tabella 7.13: Task 5 - Class A

Table 7.13 shows that the majority of students provided unsatisfactory answers for this question. There are several reasons for this, but the highest percentage of issues falls under type 1 stumbling blocks.

Many students in Class A answered the question with real-life considerations or arguments that seemed logical to them. This stumbling block concerns the pre-mathematisation phase, where students are unable to conceptualize the real situation and identify the idealized situation necessary for a mathematical approach.

Additionally, we see that 4 students exhibited type 2 stumbling blocks. These students understood that a mathematical model was needed but faced difficulties during the mathematisation phase. The most common mistake was creating a model based on a proportional relationship between the diameter and the price of the pizza. This is an example of incorrect implemented anticipation, where students focused on the wrong mathematical concepts.

A small percentage of students also had difficulties with the activation of problem handling competency. These students were able to create a correct model but made errors in interpreting the results obtained.

Total Class B	Missing	Satisfactory	Unsatisfactory	Type 1	Type 2	Type 3
21	0	14	7	1	3	3
%		66.67%	33.33%	14.28%	42.86%	42.86%

Tabella 7.14: Task 5 - Class B

Table 7.14 shows the results of Class B.

This indicates a difference in the technical level of the two classes: Class B showed more awareness of the concept of the area/price or volume/price ratio.

Among the unsatisfactory answers, the highest percentage belongs to stumbling blocks of type 2 and type 3, while just one answer falls into type 1. This can be explained by the introduction to the modelling cycle proposed to Class B: they were aware of the necessity of a mathematical model, making them less likely to provide answers based on real life considerations.

Among the satisfactory answers, we can highlight a good example of the activation of problem handling competency. Instead of calculating the area/price or volume/price ratio of the two pizzas and comparing them, some students calculated the price per square centimeter of the first pizza and then determined what the price of the area of the second one should be if priced similarly. This shows a smart approach to find the solution, even if the students did not have a clear understanding of the exact concept of the area/price ratio to calculate the best value for money.

The summary of the results of the two classes is as it follows:

Total	Missing	Satisfactory	Unsatisfactory	Type 1	Type 2	Type 3
41	1	19	21	9	6	6
%	2.44%	46.34%	51.22%	42.86%	28.57%	28.57%

Tabella 7.15: Task 5

7.1.6 Task 6 - A wooden cube

This task requires a technical level high enough to involve the concept of density. Thus, it can be very subjective, depending on the mathematical knowledge of the individual. Jankvist and Niss (2019 [14]) identified four categories of stumbling blocks. We have used only two of them, which are:

- Type 1: The student does not provide any reasonable mathematisation.
- Type 2: The student completes an unsatisfactory mathematisation, considering proportionality not between the volume and the weight of the cube, but between edges or areas and weight.

Total Class A	Missing	Satisfactory	Unsatisfactory	Type 1	Type 2
20	0	13	7	4	3
%		65%	42.86%	35%	57.14%

Tabella 7.16: Task 6 - Class A

It is surprising to note that, despite the technical level required by the task, most of the class succeeded in solving it. This indicates that the concept of density was generally well understood by the students. Those who couldn't use the concept of density effectively found alternative methods. For example, some students demonstrated problem handling competency by reasoning that the larger cube contains eight smaller cubes, and thus its weight should be eight times than the weight of the smaller cube.

Among the unsatisfactory answers, the highest percentage falls into the stumbling block of type 1. Many students constructed a mathematical model based on direct proportionality between the length of an edge and the weight of the cube. These students had difficulties with implemented anticipation, focusing on incorrect data during the mathematisation phase.

Type 2 answers showed that some students understood that there was not a direct proportionality between the edge and the weight but they were not able to successfully complete the mathematisation, misunderstanding the meaning of cube.

Total Class B	Missing	Satisfactory	Unsatisfactory	Type 1	Type 2
21	0	17	4	3	1
%		80.95%	19.05%	75%	25%

Tabella 7.17: Task 6 - Class B

Table 7.17 shows the results of Class B.

In Class B, the percentage of satisfactory answers is even higher than in Class A, potentially confirming the higher technical level of Class B as suggested by the results of the previous task.

Among the unsatisfactory answers, the predominant stumbling block is type 1, where students made incorrect mathematical models, indicating difficulties in the mathematisation phase.

The summary of the results of the two classes for task 6 follows:

Total	Missing	Satisfactory	Unsatisfactory	Type 1	Type 2
41	0	30	11	7	4
%		73.17%	26.83%	63.64%	36.36%

Tabella 7.18: Task 6

7.2 Discussion

7.2.1 What aspects of modelling competencies are developed in Italian students?

The aim of this research is to analyse the possession of the modelling competency in upper secondary school Italian students, considering that it is not a main component of the school framework as it is in Denmark. We might expect this to be a disadvantage for Italian students, but it could also be possible that the Italian educational system indirectly contributes to the development of this competency.

To answer this question, we will consider the data previously analysed.

Question	Missing	Satisfactory	Unsatisfactory	Satisfactory %	Unsatisfactory %
1 (A walk)	1	22	18	55%	45%
2 (Building)	0	29	12	70.73%	29.27%
3 (Oil field)	1	4	36	10%	90%
4 (A hill)	1	20	20	50%	50%
5 (Pizza)	1	19	21	47.5%	52.5%
6 (Cube)	0	30	11	73.17%	26.83%

Tabella 7.19: Summary of the distribution of satisfactory and unsatisfactory answers for Italian students

The question with the highest percentage of satisfactory answers is Task 6, the one about the wooden cube. This result is quite surprising, given that it is likely the task requiring the highest technical level among the six proposed.

What is particularly interesting about the answers to this task is that some students used the concept of density, drawing on their mathematical knowledge, while others, perhaps not recognizing or not remembering the definition of density, activated their problem handling competency in a very clever way. Most of the students demonstrated a strong ability in mathematising the situation and solving not just a mathematical exercise but a mathematical problem.

We can notice that Task 2 has the second highest percentage of satisfactory answers. This is surprising given that Task 2 involves the entire modelling cycle and appears to be the furthest from the mathematical world.

Initially, we assumed it would be difficult for Italian students to deal with a question posed so differently from their usual tasks. However, they have shown remarkable ability in identifying the hidden data in the picture and estimating them based on their real-life knowledge, such as the average height of a person or a floor. They were able to determine which elements were significant for solving the task, demonstrating successful implemented anticipation, followed by a good mathematisation phase. The activation of problem handling competency was also evident in the majority of the students.

The third task in terms of higher percentage of satisfactory answers is Task 1. Similar to Task 2, it does not have a strong connection with the mathematical world but still achieved good results. It is noteworthy that none of the students presented a complete mathematical model involving mathematical elements; most of the satisfactory answers were generated through logical reasoning about the practical situation. For this reason, we probably cannot conclude that the results of this task demonstrate strong abilities in the mathematisation phase or problem handling, as there is no evidence of the use of mathematical elements to find the solution.

Then, we notice that Task 4 and Task 5 present a similar percentage of satisfactory answers: just under half of the students succeeded in solving these tasks. This is reasonable considering that they share some similarities. Both tasks have a more evident connection with the mathematical world and require a specific technical level, as they necessitate knowledge of exact formulas. Some students showed great abilities in creating the mathematical model, but many had difficulties with the initial phases of the modelling cycle, which we will analyze later to address the second research question. What seems to be consistently good in both satisfactory and unsatisfactory answers is the solving phase of the mathematical model.

The last task to be discussed is Task 3, which has a very low percentage of satisfactory answers. We will analyze the difficulties generated by this task later. For now, we can note that the students who answered it satisfactorily provided a clear and correct mathematical model, thus demonstrating a good level of modelling competency, especially in the mathematisation phase.

Therefore, we have surprisingly discovered that Italian students performed well in tasks that did not seem to be directly related to the mathematical world, but also in those requiring specific mathematical knowledge. However, the difficulties encountered did not pertain to the mathematical concepts themselves. In fact, the results show that the activation of the problem handling competency was successful for most students, and only a few had difficulties in solving the mathematical model.

In some cases, we have also seen that Italian students were successful in mathematisation and implemented anticipation. It seems that they were able to easily handle these phases of the modelling cycle when the tasks did not demand a high technical level or degree of coverage.

7.2.2 Main difficulties and reasons

The second research question aims to identify the main difficulties of Italian students related to solving modelling tasks. For this reason we have considered for each task various types of stumbling blocks and we will analyse them and discuss the possible causes.

Referring to Table 7.19 it is evident that Task 3 posed the most significant challenge for Italian students, with only 10% providing satisfactory answers.

Looking at Table 7.9 we can see that the answers are evenly distributed among the 4 types of stumbling blocks identified by Jankvist and Niss (2019 [14]).

The majority of students encountered a type 1 stumbling block, which stemmed from the belief that only one statement could be true. This stumbling block was due to pre-mathematisation: the students focused on incorrect aspects of the task, leading to an unsuccessful implemented anticipation.

Almost the same percentage of students (27.78%) exhibited type 0 stumbling blocks, indicating they either refused to solve the tasks or provided random answers. This could be due to the task's real-world context: the topic of oil fields may not be familiar to upper secondary school students, leading to misconceptions or disengagement.

We observed that 23.61% of students faced stumbling blocks related to working mathematically, while 15.28% struggled with the mathematisation phase. Thus, the main problems during the modelling cycle are concentrated around the pre-mathematisation phase and problem handling competency.

Next, in terms of difficulty, comes Task 5, with a total of 52.5% of unsatisfactory answers. The predominant stumbling block in this task was pre-mathematisation phase, including implemented anticipation. The second most common stumbling block, with a percentage close to the first, also involved pre-mathematisation: many students focused on the diameter to estimate the concept of the best value for money. Instead, fewer students manifested difficulties of type 3, hence regarding the mathematical treatment and the activation of problem handling competency.

Task 4 follows closely in terms of unsatisfactory answers. Here, results show a predominance of type 2 stumbling blocks, again involving the pre-mathematisation phase. Many students struggled to perform implemented anticipation correctly. Smaller percentages of students had difficulties with mathematisation and problem solving, and only one student did not attempt to solve the task.

Task 1 showed that half of the students' answers involved negotiation and compromise between two friends, indicating they did not take the mathematical conditions seriously and failed to create a correct mathematical model. The other half was almost equally divided between students who had problems with the pre-mathematisation phase and implemented anticipation, and those who did not accept the task because it was too unfamiliar or inconsistent with their didactical contract.

The last two tasks in terms of the number of unsatisfactory answers are Task 2 and Task 6.

Task 2 reveals that most students had problems in the mathematisation phase, with only a few not attempting the task. No students demonstrated difficulties in activating problem handling competency.

Task 6 had just 26.83% unsatisfactory answers. Among these, most were due to stumbling blocks related to pre-mathematisation and implemented anticipation. These students often focused on incorrect elements, typically establishing a proportionality between the length of the edge and the weight of the cube. Only a few students failed to mathematise the situation entirely.

From this analysis, it emerges that the main difficulties for Italian students are related to the pre-mathematisation phase and implemented anticipation. We observed that problem-handling competency was not an issue for the majority of students when needed, nor was the mathematisation phase. Although the mathematisation phase presented a stumbling block for some students, it generally followed quite easily if the pre-mathematisation phase was completed successfully.

Common across all six tasks are difficulties with the pre-mathematisation phase and implemented anticipation. Niss (2010 [21]) describes implemented anticipation as a crucial point in the modelling cycle, necessary for its successful completion. In this context, we are referring to the implemented anticipation needed for the pre-mathematisation phase, which involves identifying the elements that are mathematically relevant for creating the model.

The implemented anticipation is as important as it is complex. Niss (2010 [21]) claims that developing this factor is challenging because it requires prerequisites that can be well developed while using and succeeding with the modeling cycle. On the other hand, implemented anticipation is necessary to achieve success.

Among these prerequisites, it appears that relevant mathematical knowledge was not an issue for Italian students. When a higher technical level was required, they demonstrated sufficient understanding of the necessary mathematical elements. This is not surprising, as the tasks were designed to include mathematical concepts familiar to upper secondary school students, allowing the focus to remain on modelling aspects.

Italian students seemed to find it more challenging to apply their knowledge to modelling tasks, likely because they are not accustomed to such tasks. Modelling tasks are not common in Italian schools, and the Italian curriculum considers modelling competency more related to mathematical elements, making it unusual for students to apply their knowledge in an extra-mathematical context. This is also connected to another prerequisite described by Niss (2010 [21]), which is the “application-oriented beliefs about mathematics on the part of the student”.

We have previously discussed the importance of students' attitudes towards mathematics, a key reason for the birth of the KOM project. The deeply rooted belief that mathematics is self-contained prevents students from seeking mathematical solutions to presented tasks. This affects their performance throughout the modelling cycle, as successful implemented anticipation is difficult without awareness of mathematics' role in the specific task.

Supporting this argument, we note that Task 1, which was not evidently connected to the mathematical realm, was resolved by all students who answered satisfactorily using real-life considerations, without providing an exact mathematical model. Additionally, Tasks 1, 2, and 3 show type 0 stumbling blocks, while the others do not. A type 0 stumbling block indicates the student's refusal to engage with the task. Tasks 4, 5, and 6 did not require this category because there were no such answers. This might be because Tasks 4, 5, and 6 had a more explicit mathematical connection, making students recognize them as similar to exercises they are accustomed to and prompting them to seek a mathematical solution. Tasks 1, 2, and 3, lacking this explicit connection, led some students to find the tasks absurd or provide irrelevant answers.

Consequences of incorrect implemented anticipation included focusing on wrong elements of the text and incorrect pre-mathematisation, resulting in inadequate models. This was evident in Task 1, where many students calculated the average time taken by two friends without considering the real meaning. Similarly, in Task 4, many students calculated the average speed of the entire walk by averaging the two speeds, and in Task 5, some students assumed proportionality between diameter and price without considering the implications of increasing a circle's diameter.

The mathematisation phase did not appear to be a significant stumbling block for Italian students. Most were able to translate concepts into mathematical terms once pre-mathematisation was completed satisfactorily. However, it sometimes posed significant difficulties. For example, in Task 3, very few students succeeded in modelling Anna's statement. Possible reasons include the task's real-life context being distant from students' knowledge and the mathematical model requiring a higher degree of coverage than other tasks.

Therefore, we can conclude that mathematisation did not represent a significant stumbling block for Italian students if it involved a degree of coverage and technical level within their reach.

Task 3 also provides another point of discussion. Many students did not consider the possibility that both statements could be true, focusing only on the easier one. This may result from the didactical contract, as Italian students are not typically required to solve tricky tasks; they are more accustomed to exercises with precise questions presented in a certain way. This didactical contract may also explain the type 0 stumbling blocks in Tasks 1 and 2. We observe a slight difference in the percentages of these stumbling blocks between the class aware of the modeling cycle and the one that was not. The latter may have found the task formats more confusing.

Regarding problem handling competency, it does not seem to represent a significant stumbling block for Italian students. Few students encountered difficulties in the solving phase of the modelling cycle. This may be because the tasks involved calculations not particularly challenging for upper secondary school students, but it also suggests a certain level of preparation among Italian students.

Based on these findings, we can infer that Italian students develop problem handling competency more than modelling competency. The major stumbling blocks for Italian students concern the pre-mathematisation phase and implemented anticipation before mathematisation, indicating difficulties in dealing with extra-mathematical contexts. Conversely, solving problems within the mathematical realm did not create significant difficulties for most students. This outcome may result from the Italian didactic tradition, which emphasizes textbooks and problem-solving exercises, helping students develop their problem handling competency. However, they seem to lack the ability to take elements from the real world and translate them into a mathematical context.

In the next section, we will address the third research question, focusing on comparing the results of Italian students with those of Danish students as provided by Jankvist and Niss (2019 [14]).

7.2.3 Comparison with Danish students

The Danish results that we are going to consider are the ones discussed by Jankvist and Niss (2019 [14]). For completeness the table of Danish results is provided.

Question	Missing	Satisfactory	Unsatisfactory	Satisfactory %	Unsatisfactory %
1 (A walk)	13	165	137	54.6%	45.4%
2 (Building)	12	164	139	54.1%	45.9%
3 (Oil field)	17	149	149	50.0%	50.0%
4 (A hill)	21	43	251	14.6%	85.4%
5 (Pizza)	4	108	203	34.7%	65.3%
6 (Cube)	18	118	179	39.7%	60.3%

Tabella 7.20: Summary of the distribution of satisfactory and unsatisfactory answers for Danish students

A sample of 315 Danish students was involved in this research. Looking at 7.20 we notice that the task with the highest percentage of unsatisfactory answers is Task 4, with 85.4% compared to 50% for Italian students. Among the unsatisfactory answers, the most common stumbling block was type 2 for both Italian and Danish students, with a higher percentage among Danish students: unsuccessful pre-mathematisation and implemented anticipation are the main causes. However, there is a difference: while many Danish students (20.3%) presented difficulties in making sense of the situation, only one Italian student showed type 1 stumbling block; most Italian students had a high enough technical level to understand the mathematical elements involved in the task and the concept of average speed, enabling them to start with the pre-mathematisation phase.

The second task with the highest percentage of unsatisfactory answers is Task 5, as it is for Italian students, but the Danish percentage is higher (65.3% compared to 52.2% for Italian students). The most common stumbling block among Danish students is type 2, concerning pre-mathematisation and mathematisation phases, involving the choice of wrong elements to focus on and creating a model that did not reflect the concept of value for money. For Italian students, the most common stumbling block involves pre-mathematisation and implemented anticipation: many students had problems identifying the mathematical elements of the task and just focused on the real-life context, providing non-significant considerations or answers without explanation. Hence, in this case, Italian students seemed more focused on the real situation described, while Danish students had a deeper awareness of the need for mathematisation, even if they did not complete this phase satisfactorily.

The third task is Task 6, with 60.3% unsatisfactory answers. This result is surprising compared to the Italian one, where only 26.83% of students gave unsatisfactory answers. The main stumbling blocks for Italian students were the pre-mathematisation phase and the implemented anticipation required. The few students who did not succeed in this question made the mistake of considering proportionality between the length of the edge and the weight of the cube. The same happens for Danish students, with a percentage of 70% compared to the Italian 63.63%.

We can notice that Task 6 has the highest technical level. At the same time, it has a more evident reference to the mathematical world. This shows that Italian students are more confident with tasks pertaining to the mathematical world, rather than tasks situated in an extra-mathematical context with no evident data.

Considering Task 3, the percentage of Danish unsatisfactory answers is 50%, compared to the Italian 90%. For Danish students, the main stumbling block is type 1, involving problems in pre-mathematisation, while for Italian students there is a even distribution of unsatisfactory answers over the four types of stumbling blocks identified.

Then, Task 2 and Task 1 follow, with percentages very close to each other.

Task 2 has a high percentage of unsatisfactory answers (45.9%) compared to the Italian percentage (26.83%). Both Danish and Italian students exhibited the same major stumbling blocks, type 0 and type 2. Notably, Italian students did not find any difficulties in the phases of mathematisation, problem handling, and validation, whereas some Danish students did.

The percentages of unsatisfactory answers for Task 1 are nearly identical for Danish and

Italian students, as is the distribution of types of stumbling blocks. The main difficulties were in taking the mathematical conditions seriously; both groups reasoned about the realistic aspects of the question, suggesting compromises between the two friends rather than considering a mathematical solution.

Overall, the results for the Italian students are good, with higher percentages of satisfactory answers than those collected from the Danish students. In discussing these results, it is important to remember two aspects of the current research:

- The modelling cycle is a mandatory part of the Danish curriculum, while it is not in Italy. Hence, Italian students do not have a curriculum that focuses on the study of the modelling process.

- The number of Danish students involved in the study by Jankvist and Niss (2019 [14]) is significantly higher than the number of Italian students involved in this research. This can impact the results, as the Danish results may include a greater variety of cases.

Considering that Italian students' preparation does not focus on modelling exercises, the results achieved are satisfactory compared to the Danish ones. However, many aspects should be improved.

In general, the stumbling blocks encountered by Italian students have a distribution similar to those presented by the Danish results, but there are some differences that may provide insights into the success of the Italian curriculum.

It is noteworthy that not many Italian students presented difficulties with the activation of problem-handling competency; they show lower percentages of stumbling blocks related to problem handling, compared to Danish students. This result may suggest that the concentration of the Italian curriculum on exercises and problem-solving leads to a significant development of problem handling competency in Italian students. They seem able to handle a mathematical problem and successfully complete the required calculations.

Moreover, Italian students showed satisfactory results in tasks requiring a higher technical level, demonstrating knowledge of mathematical elements such as the concept of density or the area of a circle. As previously mentioned, these concepts were specifically within the level of upper secondary school students; despite this, it can happen that even well-known mathematical concepts, when placed in an unusual context, can confuse students. This reaction was not seen in the results obtained by Italian students; they reacted well to tasks explicitly involving mathematical elements, completing them more successfully than tasks with a less explicit connection to mathematics. This is likely due to their familiarity with exercises pertaining to the mathematical realm, making those tasks more familiar and easier to deal with.

In certain cases, Danish students demonstrated a deeper awareness of the need for mathematisation of the presented situation, while Italian students dealt with it using logical reasoning without providing a mathematical model. This may be caused by the difference in the use of the modelling cycle between the two groups of students. However, while mathematisation often represented a stumbling block for Danish students, Italian students proved that if they recognized the necessity of a mathematical model, they could construct it quite successfully.

The main difficulties encountered by Italian students were in pre-mathematisation and implemented anticipation, similar to Danish students. The pre-mathematisation phase is crucial for successfully creating the mathematical model, but it is also extremely difficult as it requires a deep understanding of the extra-mathematical situation presented. It is not surprising that Italian students found it difficult to deal with this early phase of the modelling cycle, suggesting that the Italian curriculum may not sufficiently consider the relationship between mathematics and the real world. This also indicates a strong influence of the didactical contract. However, since these difficulties are approximately at the same level as those of Danish students, we can assume that the indirect way of developing modelling and problem handling competencies in the Italian curriculum yields positive results. Increasing attention to the applications of mathematics in the real world and directly explaining the modelling cycle may help fill the gaps encountered.

Conclusion

The aim of this project was to investigate the potential difficulties faced by Italian upper secondary school students and compare their results with those of Danish students. We formulated three research questions, all of which were satisfactorily addressed.

Firstly, we examined the aspects of modelling competency developed by Italian students. The results from the solved tasks demonstrated that Italian students have successfully developed problem-solving skills and the ability to create mathematical models, once the key mathematical aspects were correctly identified. Notably, if students possessed the necessary mathematical knowledge, they were able to apply it effectively in a mathematical context.

Secondly, we analyzed the main stumbling blocks faced by Italian students, using the framework established by Jankvist and Niss (2019). We observed that the types of challenges encountered by Italian students were similar to those faced by Danish students, although some categories were absent due to the smaller sample size of Italian participants compared to the Danish cohort studied by Jankvist and Niss.

The primary difficulties for Italian students related to pre-mathematisation and implemented anticipation, indicating that the transition from the extra-mathematical world to the mathematical one is one of the biggest obstacles in developing modelling competency.

We then compared the Italian and Danish results. Interestingly, despite differences in the Italian and Danish curricula, both groups struggled with pre-mathematisation and implemented anticipation. This finding suggests the presence of universal mathematical obstacles that persist across different educational approaches.

In some cases, Danish students demonstrated a deeper understanding of the processes needed to solve the tasks and were able to create more effective models, while Italian students did not. This highlights the importance of being aware of the modelling cycle. This was also evident, for some tasks, in the differing results of Class A and Class B students.

However, the fact that the unsatisfactory performance of Italian students was similar to that of Danish students suggests that even an educational approach not directly focused on developing modelling competency can still foster its growth.

It is also worth noting that Italian students performed well on tasks with a higher technical level and, in some cases, even outperformed Danish students. This reinforces the idea that the Italian educational system, which emphasizes theoretical aspects, is effective in certain contexts.

This study involved a small group of Italian students, which may have influenced the range of stumbling blocks observed. Nonetheless, the results provide insight into how Italian students approach real-life problems requiring mathematical solutions.

The key takeaway from this study is that a balanced approach, combining Italy's theoretical emphasis with Denmark's focus on competencies, could enhance the development of modelling competency. Such an approach would involve both the awareness of the process that lead from the real world to the mathematical realm and the mathematical knowledge and ability to solve pure mathematical problems.

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