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Differential calculi on quantum principal bundles in the Đurđević approach

M.Sc. Thesis in Geometry

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Abstract

In this thesis we study the Đurđević theory of differential calculi on quantum principal bundles within the domain of noncommutative geometry. Throughout the exposition, an algebraic approach based on Hopf algebras is employed. We begin by briefly recalling the foundational concepts of Hopf algebras and comodule algebras. Hopf-Galois extensions are introduced, along with an important correspondence result which relates crossed product algebras and cleft extensions. Differential calculus over algebras is reviewed, with a particular focus on the covariant case. The Đurđević theory is presented in a modern language. We compare this theory to existing literature on the topic. We extend the Đurđević theory and provide explicit realisations of quantum principal bundles and complete differential calculi on the noncommutative algebraic 2-torus, the quantum Hopf fibration and on crossed product algebras, resulting in an original contribution within this thesis.

Sommario

In questa tesi presentiamo la teoria di Đurđević dei calcoli differenziali su fibrati principali quantistici nel dominio della geometria noncommutativa. Durante l'esposizione, adottiamo un approccio algebraico basato sulle algebre di Hopf. Iniziamo brevemente ricordando i concetti fondamentali delle algebre di Hopf e delle comodule algebras. Introduciamo le estensioni di Hopf-Galois, insieme ad un importante risultato di corrispondenza che collega le crossed product algebras e le estensioni cleft. Esaminiamo il calcolo differenziale sulle algebre, con particolare attenzione al caso covariante. La teoria di Đurđević viene presentata adottando un linguaggio moderno. Confrontiamo questa teoria con la letteratura esistente sull'argomento. Estendiamo la teoria di Đurđević, e provvediamo alcune realizzazioni esplicite di fibrati principali quantistici e calcoli differenziali completi sul 2-toro noncommutativo, sulla fibrazione di Hopf quantistica e sulle crossed product algebras, risultando in un contributo originale all'interno di questa tesi.

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Contents

Introduction

To do geometry, instead of working with points of a manifold we may consider the commutative algebra of C^∞ functions on such. Noncommutative geometry extends this idea considering such algebras to be possibly noncommutative [6]. The corresponding underlying "space" is called a "quantum space", borrowing the terminology from physics, where the process of quantisation consists in turning the commutative algebra of observables on the phase space into a noncommutative one. Here we explore aspects of noncommutative geometry in which a fundamental role is played by "quantum groups", i.e. the deformed function algebras on Lie groups [25, 12], and more in general Hopf algebras, commonly accepted as the natural analogue of a group in this noncommutative setting. These structures originally appeared as generalisation of symmetry groups for certain physical integrable systems [13]. Moreover they appear in the noncommutative generalisation of principal bundles of classical differential geometry, known as Hopf-Galois extensions, or principal comodule algebras, originally in [5, 14] and more recently in [1, 18]. The total and base space are understood as algebras on which the quantum group acts in a precise sense. Our main motivation to explore this subject is to extend the differential geometry of homogeneous spaces to the noncommutative case. Within this noncommutative setting the mathematical theory of quantum groups its undoubtedly interesting on its own, but since geometry originally emerged as a practical subject we would also like to provide some motivation coming from physics. The quantum theory of fields gives a description of fundamental particles and interactions in terms of quantum fields defined on a suitable Lorentzian manifold. This theory culminates in the "Standard Model", a gauge theory which gives a quantum description of almost all the known fundamental particles. Interactions are given by gauge fields (or gauge mediators) that mathematically are understood as connections on certain principal bundles. A quantum description of gravity is ruled out in this framework owing to non-renormalisability of the latter. Efforts to reconcile gravity with quantum field theory have led to the search for a quantum theory of gravity, aiming to unify general relativity with quantum field theory. One notable approach of investigation lies in the concept of gravity as a gauge theory within the Cartan moving frame setting, where connections on certain principal bundles play a central role [24]. The commonly accepted general indication is that the small-scale structure of space-time could not be reasonably modelled according to usual continuum geometry. Geometry at the Planck scale may be modified in order to account for the presence of quantum effects, and noncommutative geometry could provide a possible realisation of the latter. In fact, within this approach, geometry can be specialised to deal with discrete spaces or finite dimensional algebras, with the possibility of recovering key insights on physics beyond the Planck scale. The classical *emergent* geometry of smooth (possibly Riemannian) manifolds is recovered as a *low energy* approximation via a suitable limit process [16].

In this thesis we study Đurđević's theory of quantum principal bundles, originally presented in [8, 10]. Đurđević introduces a comprehensive and natural theory of principal bundles within the framework of noncommutative geometry. In this theory, quantum groups act as the structure groups, while general quantum spaces serve as total space and base manifolds. The study focuses on developing a differential calculus specifically tailored for quantum principal bundles. This includes the introduction and examination of algebras of horizontal and vertical differential forms on the bundle. A natural "braiding" emerges, from which the formalism of connections is elaborated upon, with attention given to operators such as bimodule connections, metrics, Levi-Civita connections, covariant derivative, and curvature. A quantum counterparts of infinitesimal gauge transformations is given. Our aim is to present this theory in a modern light, offering three significant contributions. First, we translate the original papers into a more modern mathematical language, making the approach accessible to a wider audience. Second, we establish connections between Đurđević's work and existing literature on quantum principal bundles, particularly engaging with the influential research of [5]. This highlights how the Đurđević approach builds upon, rather than competes with, established frameworks, extending them to higher-order forms. We introduce new examples of Đurđević's quantum principal bundles, including bicovariant calculi, crossed product calculi, and the fundamental Hopf fibration on the quantum sphere. This integration with established methodologies, along with the examples provided, is indicative of the significance of Đurđević's framework in noncommutative differential geometry. Furthermore, we clarify and expand Đurđević's findings, particularly those presented in the work [10]. Doing so, we adopt an algebraic perspective, focusing on principal comodule algebras and Hopf algebras, without presuming the existence of compact quantum spaces or groups. While we omit certain technical details, such as considerations related to $*$-involutions, these aspects remain ripe for future investigation. Moreover, exploration into the braiding, connections, and metrics associated with Đurđević's work has also been postponed to upcoming inquiries.

Outline

In Chapter 1 we briefly introduce the main algebraic structures we deal with. We consider Hopf algebras H, and $H-$comodule algebras together with the corresponding subalgebra of (co)invariant elements. These algebras respectively play the role of structure group, total and base space of a quantum principal bundle provided some auxiliary conditions hold. We provide some explicit realisations of Hopf algebras and comodule algebras.

In Chapter 2 we define Hopf-Galois, cleft and trivial extensions. Such extensions are of the form $B \subseteq A$, where A is a (right) $H-$comodule algebra and B the subalgebra of $H-$coinvariant elements [17]. We discuss some properties of convolution invertible morphisms (cleaving maps) from Hopf algebras to comodule algebras. It is proven that trivial extensions are cleft, and that cleft extensions are Hopf-Galois. Some explicit realisations of trivial, cleft and Hopf-Galois extensions are discussed. We introduce the translation map as an "inverse" of the canonical Hopf-Galois map of a Hopf-Galois extension. Some properties of the translation map are discussed. Crossed product algebras $B\sharp_\sigma H$ are defined starting from an algebra B, satisfying some additional assumptions, and a Hopf algebra H. We provide a correspondence result between cross product algebras $B\sharp_\sigma H$ and cleft extensions $B \subseteq B\sharp_\sigma H$ [7].

In Chapter 3 we begin discussing differential geometry in the noncommutative setting. We define the notion of first order differential calculus over an algebra and provide some explicit realisations. Such structures are non-unique, in the sense that many different first order differential calculi can be defined on the same algebra. We introduce the first order universal differential calculus and provide a theorem [25] stating how every first order calculus over an algebra can be induced as a quotient of the latter. The notion of first order covariant calculi over algebras is introduced [6]. We provide a classification result of those first order differential calculi which are covariant [25], i.e. compatible with a coaction on the algebra. We consider the extension of the first order theory to higher order differential calculi. The universal differential calculus is introduced. Higher order covariant differential calculi are discussed. We consider the maximal prolongation [8], being the biggest differential calculus that can be defined starting from a given first order differential calculus. We prove how the maximal prolongation of the universal first order differential calculus is its tensor algebra, i.e. the universal differential calculus. Moreover, every differential calculus turns out to be induced as a quotient of the universal.

In Chapter 4 we delve into Đurđević's theory of quantum principal bundles. We start with a

bicovariant first order differential calculus Γ over a Hopf algebra H. The corresponding maximal prolongation $\Gamma^\wedge$ is considered. The (differential graded) subalgebra of coinvariant elements $\Lambda^\bullet$ is introduced. In Đurđević's framework the definition of quantum principal bundle is that of faithfully flat Hopf-Galois extensions $B \subseteq A$. Vertical forms $\mathfrak{ver}^\bullet(A)$ over the total space A of the bundle are introduced. We show that these form a differential calculus. The notion of complete differential calculus $\Omega^\bullet(A)$ over the total space is introduced. In such scenario we provide a surjective morphism $\pi_{\mathfrak{ver}} : \Omega^\bullet(A) \twoheadrightarrow \mathfrak{ver}^\bullet(A)$ projecting a total space form onto the space of vertical forms. We show that the differential calculus of vertical forms is complete if the calculus on the total space is. The noncommutative analogue of horizontal forms $\mathfrak{hor}^\bullet(A)$ is introduced. It is shown that these form a graded subalgebra of $\Omega^\bullet(A)$. In particular $\mathfrak{hor}^\bullet(A)$ is a right $H-$comodule algebra. It is shown how first order horizontal, vertical and total space constitute a natural short exact sequence of $A-$modules $0 \to \mathfrak{hor}^1(A) \hookrightarrow \Omega^1(A) \twoheadrightarrow \mathfrak{ver}^1(A) \to 0$. We observe that within other approaches exactness of this sequence is not guaranteed, see for example [1]. It is discussed how the same sequence fails to be exact for higher order forms. The differential calculus over the base space $B := A^{coH}$ is introduced. It is shown how differential forms $\Omega^\bullet(B)$ over the base space coincides with the intersection of right $H-$coinvariant and horizontal forms. We argue that $\Omega^\bullet(B)$ is not a differential calculus in general. However, in all explicit examples we consider the base calculus will be a differential calculus generated by B. We compare Đurđević's theory of quantum principal bundles with the one in [5, 2]. Explicit realisations of complete differential calculi and quantum principal bundles are presented. It is shown how every Hopf algebra H over a field $\Bbbk$ naturally forms a quantum principal bundle $\Bbbk \subseteq H$. A complete differential calculus is constructed along the "group algebra" $\mathbb{C}(\mathbb{Z})$. We show that the noncommutative algebraic 2-torus forms a cleft extension, and so a quantum principal bundle, under the right $H-$coaction of the Hopf algebra $\mathcal{O}(\mathrm{U}(1))$. We construct a complete differential calculus over the total space of such bundle. We show how the space of right $H-$coinvariant forms 1−forms of the bundle is generated by the base space itself, and so forms a differential calculus over the base. We consider the quantum Hopf fibration with total space $A = \mathcal{O}_q(\mathrm{SU}(2))$ under the action of the group algebra $H = \mathcal{O}_{q^2}(\mathrm{U}(1))$. There is a complete differential calculus over the total space exists and that the corresponding forms over the base space are a differential calculus. Finally, we define differential calculi over crossed product algebras $B\sharp_\sigma H$. According to Chapter 2 it is clear that every crossed product algebra $B\sharp_\sigma H$ is a quantum principal bundle and that every cleft extension is of this form. We show that the differential calculus over the total space $B\sharp_\sigma H$ is complete given the differential calculus on H is complete. Moreover the corresponding base forms are a differential calculus.

We point out that we make a new contribution by establishing completeness of the differential calculi on total spaces of the examples discussed. Moreover, within the same examples, we are able to recover differential calculi on the corresponding base spaces. The results of this thesis will be summarised in a forthcoming publication.

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Notation

Symbol	Description	First time appearing
A	An algebra/H-comodule algebra	Page 1
Δ	The coproduct on a coalgebra/bialgebra/Hopf algebra	Page 1
ϵ	The counit on a coalgebra/bialgebra/Hopf algebra	Page 1
μ	The product on an algebra/bialgebra/Hopf algebra	Page 1
η	The unit on an algebra/bialgebra/Hopf algebra	Page 1
$\Bbbk$	A field	Page 1
H	A Hopf algebra	Page 3
S	The antipode on a Hopf algebra	Page 3
j	The cleaving map	Page 5
$*$	The convolution product	Page 5
$\triangleright$	Right H-module action on a vector space/algebra	Page 7
$\triangleleft$	Left H-action on a vector space/algebra	Page 7
${}_V\Delta$	Left H-coaction on a vector space/algebra	Page 7
Δ_V	Right H-coaction on a vector space/algebra	Page 7
${}^{coH}A$	The subalgebra of left H-coinvariant elements	Page 8
A^{coH}	The subalgebra of right H-coinvariant elements	Page 8
$\otimes_B$	The balanced tensor product over B	Page 13
$B \subseteq A$	A trivial/cleft/Hopf-Galois extension	Page 13
$\cdot$ (usually omitted)	Left/right A-module action	page 24

Chapter 1

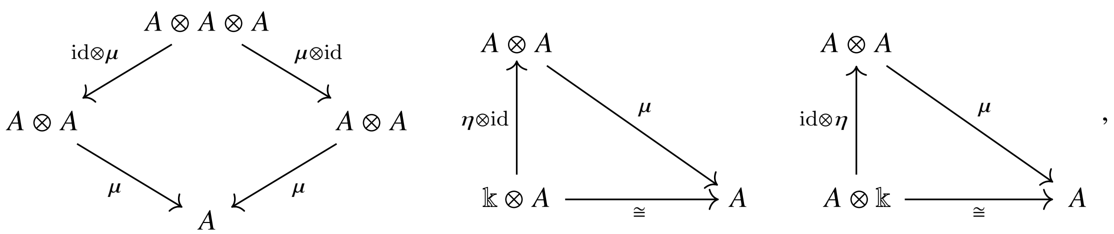
Hopf algebras

This chapter provides a brief introduction to the subject of Hopf algebras, being the fundamental "group" structures upon which we develop the theory of quantum principal bundles presented in this thesis. We start by discussing concepts of algebra and coalgebras in 1.1. We define bialgebras and Hopf algebras in 1.2, the latter being bialgebras equipped with an "antipode" which we show to be unique and satisfies the property of being an antibialgebra map. In section **??** we introduce the notion of $H-$comodule algebras, playing the role of total spaces in the theory of quantum principal bundles. The main references we will be using are [15, 2, 21, 17].

1.1 Algebras and coalgebras

We provide the definition of algebra, expressing all structures as linear maps.

Definition 1.1.1. An algebra A over a field $\mathbb{k}$ is a $\mathbb{k}-$vector space together with a product map $\mu : A \otimes A \to A$ and a unit element 1_A which can be equivalently written as a map $\eta : \mathbb{k} \to A$ by $\eta(1) = 1_A$ such that the following diagrams commute

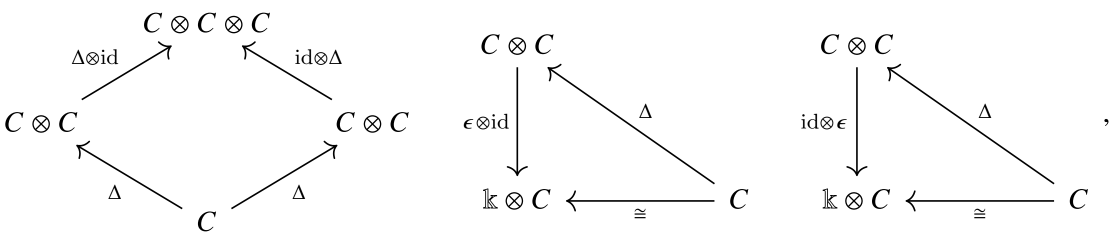


that is

$$\begin{aligned} \mu \circ (\mathrm{id} \otimes \mu) &= \mu \circ (\mu \otimes \mathrm{id}), \\ \mu \circ (\eta \otimes \mathrm{id}) &= \mathrm{id}, \\ \mu \circ (\mathrm{id} \otimes \eta) &= \mathrm{id}\,. \end{aligned} \tag{1.1}$$

Commutativity of these diagrams are representative of associativity and unitality, respectively. The notion of coalgebra is obtained via such diagrammatic approach by reversing arrows.

Definition 1.1.2. A coalgebra C over a field $\mathbb{k}$ is a $\mathbb{k}-$vector space togehter with a coproduct map $\Delta : C \to C \otimes C$ and a counit map $\epsilon : C \to \mathbb{k}$ such that the following diagrams commute



that is

$$\begin{aligned}(\mathrm{id}\otimes\Delta)\circ\Delta &= (\Delta\otimes\mathrm{id})\circ\Delta,\\ (\epsilon\otimes\mathrm{id})\circ\Delta &= \mathrm{id},\\ (\mathrm{id}\otimes\epsilon)\circ\Delta &= \mathrm{id}\,.\end{aligned} \tag{1.2}$$

Commutativity of these diagrams are representative of *coassociativity* and *counitality* of the coalgebra C.

Notation 1.1.3. When dealing with such algebraic structures it is customary to adopt the "Sweedler notation" We denote

$$\Delta(c) := c_1 \otimes c_2, \quad \text{for all c} \in C, \tag{1.3}$$

which is a concise way of keeping track of the numbering of tensor factors. This is particularly useful to write down in a more explicit way the axiom of a coalgebra (an even more). □

Coassociativity reads

$$\begin{aligned}(\Delta\otimes\mathrm{id})\circ\Delta(c) &= (\Delta\otimes\mathrm{id})\circ(c_1\otimes c_2)\\ &= (\Delta(c_1)\otimes c_2)\\ &= c_{11}\otimes c_{12}\otimes c_2;\\ (\mathrm{id}\otimes\Delta)\circ\Delta(c) &= (\mathrm{id}\otimes\Delta)\circ(c_1\otimes c_2)\\ &= c_1\otimes\Delta(c_2)\\ &= c_1\otimes c_{21}\otimes c_{22},\end{aligned} \tag{1.4}$$

or

$$c_{11}\otimes c_{12}\otimes c_{22} = c_1\otimes c_2\otimes c_3 = c_1\otimes c_{21}\otimes c_{22}. \tag{1.5}$$

The convention is to relable indexes from the lowest to the highest[1] keeping track of the order of factors. Unitality reads

$$\begin{aligned}(\epsilon\otimes\mathrm{id})\circ\Delta(c) &= (\epsilon\otimes\mathrm{id})\circ(c_1\otimes c_2)\\ &= (\epsilon(c_1)\otimes c_2)\\ &= \epsilon(c_1)c_2\\ &= c\\ &= c_1\epsilon(c_2)\\ &= (\mathrm{id}\otimes\epsilon)\circ\Delta(c),\end{aligned} \tag{1.6}$$

or

$$\epsilon(c_1)c_2 = c = c_1\epsilon(c_2). \tag{1.7}$$

Example 1.1.4. We provide a short example to explain the Sweedler notation. Let H be a Hopf algebra. We consider the map $(\Delta\otimes\Delta)\circ\Delta : H \to H\otimes H\otimes H\otimes H$. For $h \in H$ we find

$$\begin{aligned}(\Delta\otimes\Delta)\circ\Delta(h) &= (\Delta\otimes\Delta)(h_1\otimes h_2)\\ &= \Delta(h_1)\otimes\Delta(h_2)\\ &= h_{11}\otimes h_{12}\otimes h_{21}\otimes h_{22},\end{aligned} \tag{1.8}$$

and relabel according to Notation 1.1.3 as $(\Delta\otimes\Delta)\circ\Delta(h) = h_1\otimes h_2\otimes h_3\otimes h_4$.

[1]Here indexes 11 and 12 are of course relative to tensor product component preceeding the index 2. Similarly for 21 and 22.

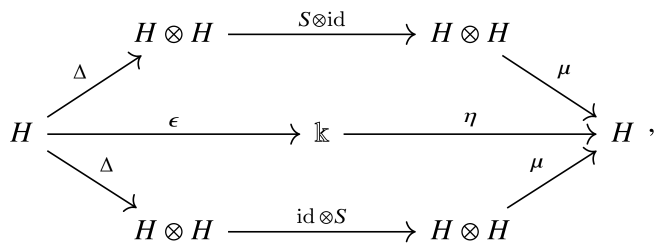
1.2 Bialgebras and Hopf algebras

The next definition is the one of bialgebra, featuring simultaneously the algebraic structure of an algebra and a coalgebra.

Definition 1.2.1. A bialgebra H over $\mathbb{k}$ is a $\mathbb{k}$–vector space equipped with linear maps $\mu : H\otimes H \to H$, $\eta : \mathbb{k} \to H$, and algebra maps $\Delta : H \to H \otimes H$, $\epsilon : H \to \mathbb{k}$ such that (H, μ, η) is an algebra and (H, Δ, ϵ) is a coalgebra.

We now define a Hopf algebra starting with a bialgebra with an additional requirement, that is a map playing the role of "group inversion".

Definition 1.2.2. A Hopf algebra is a bialgebra $(H, \Delta, \epsilon, \mu, \eta)$ together with an "antipode" map $S : H \to H$ such that the following diagram commutes



that is to say

$$\begin{aligned}\mu \circ (S \otimes \mathrm{id}) \circ \Delta(h) &= \mu \circ (S(h_1) \otimes h_2) \\ &= S(h_1)h_2 \\ &= \eta \circ \epsilon(h) \\ &= \epsilon(h) \\ &= h_1 S(h_2) \\ &= \mu \circ (\mathrm{id} \otimes S) \circ \Delta(h),\end{aligned} \tag{1.9}$$

or

$$S(h_1)h_2 = \epsilon(h) = h_1 S(h_2). \tag{1.10}$$

Definition 1.2.3. A Hopf algebra is commutative if the underlying algebra structure is. Moreover, defining flip : $H \otimes H \to H \otimes H$ as flip$(a \otimes b) = b \otimes a$ we say a Hopf algebra is cocommutative if flip $\circ\, \Delta = \Delta$, i.e.

$$\begin{aligned}\text{flip} \circ \Delta(h) &= \text{flip}(h_1 \otimes h_2) \\ &= h_2 \otimes h_1 \\ &= h_1 \otimes h_2,\end{aligned} \tag{1.11}$$

or $h_1 \otimes h_2 = h_2 \otimes h_1$ for every $h \in H$.

We provide a few explicit realisations of Hopf algebras by the following examples.

Example 1.2.4. Let $(G, \cdot, e)$ be a group and let $\mathbb{k}$ be a field. We define the group algebra $H = \mathbb{k}G$ as the $\mathbb{k}$–vector space generated by elements of G. Accordingly every element in H is of the form $\sum_{g\in G} k_g g$ with finitely many $k_g \in \mathbb{k}$ being non-zero. The algebra structure on $\mathbb{k}G$ is given by the associative product

$$\sum_{g\in G} k_g g \cdot \sum_{h\in G} k'_h h = \sum_{g,h\in G} k_g k'_h g \cdot h. \tag{1.12}$$

The unit element is e. The coalgebra structure is extended linearly by the following maps acting on elements of G

$$\begin{aligned}&\Delta : G \to G \otimes G, \qquad g \mapsto g \otimes g, \\ &\epsilon : G \to \mathbb{k}, \qquad g \mapsto 1\end{aligned} \tag{1.13}$$

as

$$\Delta\left(\sum_{g\in G} k_g g\right) = \sum_{g\in G} k_g g \otimes g, \quad \epsilon\left(\sum_{g\in G} k_g g\right) = \sum_{g\in G} k_g. \tag{1.14}$$

Finally, the Hopf algebra structure follows defining a morphism $S : H \to H$ defined as $S(g) = g^{-1}$ on elements of G and linearly extending to elements of H as $S(\sum_{g\in G} k_g g) = \sum_{g\in G} k_g g^{-1}$. It is an easy check that $(H, \Delta, \epsilon, \mu, \eta, S)$ is a Hopf algebra by those assignments. □

Example 1.2.5. Let V be a $\Bbbk$–vector space. Recall the tensor algebra $\mathcal{T}V$ over V is defined as

$$\mathcal{T}(V) := \Bbbk \oplus V \oplus (V \otimes V) \oplus \dots .$$

We define

$$\begin{aligned} &\Delta : \mathcal{T}V \to \mathcal{T}V \otimes \mathcal{T}V, && x \mapsto x \otimes 1 + 1 \otimes x \\ &\epsilon : \mathcal{T}V \to \Bbbk, && x \mapsto 0 \\ &S : \mathcal{T}V \to \mathcal{T}V, && x \mapsto -x \end{aligned} \tag{1.15}$$

for every $x \in V$. ExtendingΔ, ϵ to algebra maps and S as an antibialgebra map over $\mathcal{T}V$ realises a Hopf algebra space. □

Example 1.2.6. Let $\Bbbk = \mathbb{C}$ and let $q \in \mathbb{C}$ be non-zero. We define the "quantum group" $H = \mathrm{SL}_q(2)$ as the free algebra generated as $H = \mathrm{span}_{\mathbb{C}}\{\alpha, \beta, \gamma, \delta\}$ modulo relations

$$\begin{gathered} \alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma, \\ \beta\gamma = \gamma\delta, \quad \alpha\delta - \delta\alpha = (q - q^{-1})\beta\gamma, \quad \alpha\delta - q\beta\gamma = 1. \end{gathered} \tag{1.16}$$

Coproduct, counit and antipode are provided as

$$\begin{aligned} &\Delta : H \to H \otimes H, && \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha\otimes\alpha + \beta\otimes\gamma & \alpha\otimes\beta + \beta\otimes\delta \\ \gamma\otimes\alpha + \delta\otimes\gamma & \gamma\otimes\beta + \delta\otimes\delta \end{pmatrix}, \\ &\epsilon : H \to \Bbbk, && \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ &S : H \to H, && \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \delta & q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix}. \end{aligned} \tag{1.17}$$

□

Proposition 1.2.7. *Let H be a Hopf algebra. Then its antipode $S : H \to H$ is unique.*

Proof. Let $h \in H$ and let S_1, S_2 be two antipode maps. Then we find

$$\begin{aligned} S_1(h) &= S_1(h_1\epsilon(h_2)) \\ &= S_1(h_1)\epsilon(h_2) \\ &= S_1(h_1)h_{21}S_2(h_{22}) \\ &= S_1(h_1)h_2S_2(h_3) \\ &= \epsilon(h_1)S_2(h_2) \\ &= S_2(\epsilon(h_1)h_2) \\ &= S_2(h). \end{aligned} \tag{1.18}$$

□

Next we introduce the convolution algebra in order to prove that the antipode map $S : H \to H$ is a morphism of antibialgebras. The first definition is the one of convolution invertible morphism between algebras.

Definition 1.2.8. A $\mathbb{k}$–linear map $j : H \longrightarrow A$ is said to be convolution invertible if we can find another $\mathbb{k}$–linear map $i : H \longrightarrow A$ such that $j(h_1)i(h_2) = \epsilon(h)1_A = i(h_1)j(h_2)$, for every element $h \in H$. In this case one usually writes $i = j^{-1}$ or vice-versa.

Proposition 1.2.9 (Convolution algebra)**.** *Given an associative unital algebra (A, μ, η) and a a coassociative counital algebra (C, Δ, ϵ) consider the vector space $Hom_{\mathbb{k}}(C, A)$ of $\mathbb{k}$–linear maps $C \to A$. Define a $\mathbb{k}$–linear map*

$$\begin{aligned} * : Hom_{\mathbb{k}}(C, A) \otimes Hom_{\mathbb{k}}(C, A) &\longrightarrow Hom_{\mathbb{k}}(C, A) \\ (f, f') &\mapsto f * f' := \mu \circ (f \otimes f') \circ \Delta. \end{aligned} \tag{1.19}$$

We call $(Hom_{\mathbb{k}}(C, A), *)$ the convolution algebra. It is s an associative unital algebra with unit $\eta \circ \epsilon$.

Proof. We already know $\mathrm{Hom}_{\mathbb{k}}(C, A)$ is a vector space. Given an element $x \in C$ we find

1. $*$ is bilinear:

$$\begin{aligned} &(kf + hg) * (k'f' + h'g')(x) \\ &\quad= \mu \circ ((kf + hg) \otimes (k'f' + h'g')) \otimes \Delta(x) \\ &\quad= \mu \circ (kf \otimes k'f' + kf \otimes h'g' + hg \otimes k'f + hg \otimes h'g') \circ (x_1 \otimes x_2) \\ &\quad= \mu \circ (kf(x_1) \otimes k'f'(x_2) + kf(x_1) \otimes h'g'(x_2) \\ &\quad+ hg(x_1) \otimes k'f'(x_2) + hg(x_1) \otimes h'g'(x_2)) \\ &\quad= kk'f(x_1)f'(x_2) + kh'f(x_1)g'(x_2) + hk'g(x_1)f'(x_2) + hh'g(x_1)g'(x_2) \\ &\quad= kk'(f * f')(x) + kh'(f * g')(x) + hk'(f * g)(x) + hh'(g * g')(x) \end{aligned}$$

   for every $f, g, f', g' \in \mathrm{Hom}_{\mathbb{k}}(C, A)$ and $k, k', h, h' \in \mathbb{k}$.

2. $*$ is associative:

$$\begin{aligned} ((f * g) * h)(x) &= \mu \circ ((f * g) \otimes h)\Delta(x)) \\ &= \mu \circ ((f * g)(x_1) \otimes h(x_2)) \\ &= \mu \circ (\mu \circ (f \otimes g)(x_{11} \otimes x_{12}) \otimes h(x_2)) \\ &= \mu \circ (f(x_{11})g(x_{12}) \otimes h(x_2)) \\ &= f(x_1)g(x_2)h(x_3) \\ &= (f * (g * h))(x) \end{aligned} \tag{1.20}$$

3. $\eta \circ \epsilon$ is a unit:

$$\begin{aligned} ((\eta \circ \epsilon) * f)(x) &= \mu \circ ((\eta \circ \epsilon) \otimes f)(x_1 \otimes x_2) \\ &= \mu \circ (\eta(\epsilon(x_1)) \otimes f(x_2)) \\ &= \eta(\epsilon(x_1))f(x_2) \\ &= f(x), \end{aligned} \tag{1.21}$$

   and similarly for the other component. □

In view of Proposition 1.2.9 the notion of convolution invertibility amounts to invertibility in the convolution algebra.

Proposition 1.2.10. *Let H be a Hopf algebra.*

1. *if $f : C \to H$ is a coalgebra map then $S \circ f$ is the convolution inverse of f in $Hom_{\mathbb{k}}(C, H)$;*

2. *if $f : H \to A$ is an algebra map then $f \circ S$ is the convolution inverse of f in $Hom_{\mathbb{k}}(H, A)$.*

Proof. It follows easily by a direct computation

$$\begin{aligned}(S\circ f * f)(x) &= \mu\circ(S\circ f\otimes f)(x_1\otimes x_2)\\ &= \mu\circ(S(f(x_1))\otimes f(x_2))\\ &= S(f(x_1))f(x_2)\\ &= S(f(x)_1)f(x)_2\\ &= \epsilon(f(x))1.\end{aligned} \tag{1.22}$$

Similarly the other point follows. □

In view of the last two proposition we have the following.

Proposition 1.2.11. *Let H be a Hopf algebra. Then its antipode $S : H \to H$ satisfies*

1. $S(hg) = S(g)S(h)$, *for every* $h, g \in H$;
2. $S(1) = 1$;
3. $(S\otimes S)\circ\Delta(h) = \mathit{flip}\circ\Delta\circ S(h)$, *for every* $h \in H$;
4. $\epsilon\circ S(h) = \epsilon(h)$, *for every* $h \in H$.

Proof. In order:

1. We define $f := S\circ\mu : H\otimes H \to H$ and $f' := \mu\circ\text{flip}\circ(S\otimes S)$. By Proposition 1.2.10 we have that f is the convolution inverse of μ in the convolution algebra $\text{Hom}_{\mathbb{k}}(H\otimes H, H)$. Moreover f' is also a left inverse of $\mu : H\otimes H \to H$, indeed

$$\begin{aligned}(f' * \mu)(h\otimes h') &= f'(h_1\otimes h'_1)\mu(h_2\otimes h'_2)\\ &= [\mu\circ\text{flip}\circ(S\otimes S)(h_1\otimes h'_1)]h_2h'_2\\ &= S(h'_1)S(h_1)h_2h'_2\\ &= \epsilon(h)\epsilon(h')\\ &= \epsilon(hh').\end{aligned} \tag{1.23}$$

Therefore, by uniqueness, $f = f'$ and we have the first property.

2. Since $\eta : \mathbb{k} \to H$ is a coalgebra map we know $S\circ\eta$ is its convolution inverse. Moreover η is the unit of the convolution algebra $\text{Hom}_{\mathbb{k}}(\mathbb{k}, H)$ and so the convolution inverse of itself. Accordingly $S\circ\eta = \eta$, or $S(1) = 1$.

3. Let $g := \Delta\circ S : H \to H\otimes H$ and $g' := (S\otimes S)\circ\text{flip}\circ\Delta : H \to H\otimes H$. We have g, g' are both left inverses of $\Delta : H \to H\otimes H$ in the convolution algebra $\text{Hom}_{\mathbb{k}}(H, H\otimes H)$, indeed, since Δ is an algebra map g is the convolution inverse of Δ by Proposition 1.2.10, moreover for every $h \in H$ we get

$$\begin{aligned}(g' * \Delta)(h) &= g'(h_1)\Delta(h_2)\\ &= (S(h_1)_2\otimes S(h_1)_1)(h_{21}\otimes h_{22})\\ &= (S(h_2)\otimes S(h_1))(h_3\otimes h_4)\\ &= S(h_2)h_3\otimes S(h_1)h_4\\ &= 1\otimes S(h_1)h_2\\ &= 1\otimes\epsilon(h).\end{aligned} \tag{1.24}$$

Therefore $g = g'$ by uniqueness proving point 3.

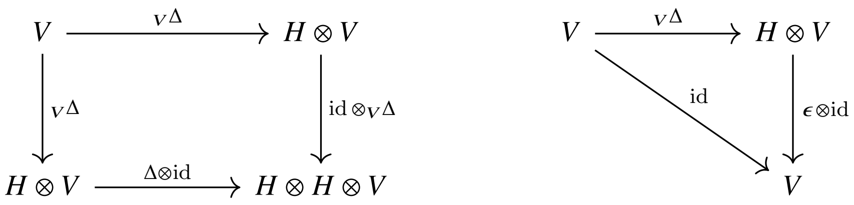
4. Finally, since $\epsilon : H \to \mathbb{k}$ is an algebra map $\epsilon\circ S$ must be its convolution inverse. On the other hand ϵ is the unit of the convolution algebra $\text{Hom}_{\mathbb{k}}(H, \mathbb{k})$ and so its the convolution inverse of itself. Therefore $\epsilon\circ S = \epsilon$. □

1.2.1 Comodule and comodule algebras

Let H be a Hopf algebra.

Definition 1.2.12. A left $H-$module algebra is a vector space V which is a left module over H, such that $h \triangleright 1_A = \epsilon(h)1_A$ and $h \triangleright (vw) = (h_1 \triangleright v)(h_2 \triangleright w)$ whenever $v, w \in V$ and $h \in H$. Similarly, A right $H-$module algebra is a vector space V which is a right module over H, such that $1_A \triangleleft h = 1_A\epsilon(h)$ and $(vw) \triangleleft h = (v \triangleleft h_1)(w \triangleleft h_2)$, whenever $v, w \in V$ and $h \in H$.

Definition 1.2.13. A left $H-$comodule is a vector space V together with a map ${}_V\Delta : V \longrightarrow H \otimes V$ such that the diagrams

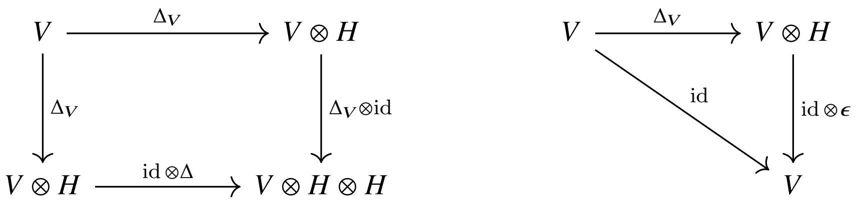


commute, i.e.

$$\begin{aligned}(\mathrm{id} \otimes {}_V\Delta) \circ {}_V\Delta &= (\Delta \otimes \mathrm{id}) \circ {}_V\Delta, \\ (\epsilon \otimes \mathrm{id}) \circ {}_V\Delta &= \mathrm{id}\,.\end{aligned} \tag{1.25}$$

The map ${}_V\Delta$ is called a left $H-$coaction on V.

A right $H-$comodule on V is a vector space V together with a map $\Delta_V : V \longrightarrow V \otimes H$ such that the diagrams

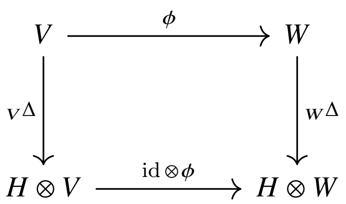


commute, i.e.

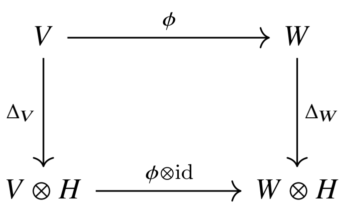
$$\begin{aligned}(\mathrm{id} \otimes \Delta) \circ \Delta_V &= (\Delta_V \otimes \mathrm{id}) \circ \Delta_V, \\ (\mathrm{id} \otimes \epsilon) \circ \Delta_V &= \mathrm{id}\,.\end{aligned} \tag{1.26}$$

The map Δ_V is called a right $H-$coaction on V.

Definition 1.2.14. Let $(V, {}_V\Delta)$ and $(W, {}_W\Delta)$ be left H-comodules. A morphism of left H-comodules, or a left $H-$colinear map, is a $\mathbb{k}$-linear map $\phi : V \longrightarrow W$ such that the diagram



commutes, i.e. ${}_W\Delta \circ \phi = (\mathrm{id} \otimes \phi) \circ {}_V\Delta$. Let (V, Δ_V) and (W, Δ_W) be right H-comodules. A morphism of right H-comodules, or a right $H-$colinear map, is a $\mathbb{k}$-linear map $\phi : V \longrightarrow W$ such that the diagram



commutes, i.e. i.e. $\Delta_W \circ \phi = (\phi \otimes \mathrm{id}) \circ \Delta_V$.

Notation 1.2.15. The Sweedler notation for right and left H−coactions on V is the following:

$$_V\Delta v =: \sum v_{-1} \otimes v_0 =: v_{-1} \otimes v_0, \quad \Delta_V v =: \sum v_0 \otimes v_1 =: v_0 \otimes v_1, \tag{1.27}$$

for all $v \in V$.

Definition 1.2.16. A right H−comodule algebra is a right H−comodule A such that $\mu : A \otimes A \to A$ and $\eta : \Bbbk \to A$ are right H−colinear maps, namely

$$\Delta_A(ab) = \Delta_A(a)\Delta_A(b), \quad \Delta_A(1_A) = 1_A \otimes 1_H. \tag{1.28}$$

Similarly, a left H−comodule algebra is a left H−comodule A such that $\mu : A \otimes A \to A$ and $\eta : \Bbbk \to A$ are left H−colinear maps, namely

$$_A\Delta(ab) = {_A\Delta(a)}{_A\Delta(b)}, \quad {_A\Delta(1_A)} = 1_H \otimes 1_A. \tag{1.29}$$

Definition 1.2.17. Let $(V, {_V\Delta})$ and $(W, {_W\Delta})$ be left H−comodule algebras. A morphism of left H−comodule algebras is a morphism $\phi : V \to W$ of left H−comodules that is also an algebra map, namely $\phi(vv') = \phi(v)\phi(v')$ for every $v, v' \in V$. Similarly for right H−comodule algebras.

Definition 1.2.18. Let A be a left H−comodule algebra. We define the subalgebra ^{coH}A of left-coinvariant elements under the left H−coaction $\lambda_A : A \to H \otimes A$ as

$$^{coH}A := \{a \in A : \ {_A\Delta(a)} = 1 \otimes a\}. \tag{1.30}$$

Similarly, if A is a right H−comodule algebra, we define the subalgebra of right-coinvariant elements under the right H−coaction $\Delta_A : A \to A \otimes H$ as

$$A^{coH} := \{a \in A : \ \Delta_A(a) = a \otimes 1\}. \tag{1.31}$$

Example 1.2.19. Let us define the adjoint right H−coaction as

$$\mathrm{ad}_R : H \to H \otimes H, \quad h \mapsto h_2 \otimes S(h_1)h_3. \tag{1.32}$$

We have that (H, ad_R) is a right H−comodule. Indeed

$$\begin{aligned}(\mathrm{id} \otimes \Delta) \circ \mathrm{ad}_R(h) &= (\mathrm{id} \otimes \Delta) \circ (h_2 \otimes S(h_1)h_3) \\ &= h_2 \otimes \Delta(S(h_1)h_3) \\ &= h_2 \otimes \Delta(S(h_1))\Delta(h_3) \\ &= h_2 \otimes (S(h_1)_1 \otimes S(h_1)_2)(h_{31} \otimes h_{32}) \\ &= h_2 \otimes (S(h_{12}) \otimes S(h_{11}))(h_{31} \otimes h_{32}) \\ &= h_2 \otimes S(h_{12})h_{31} \otimes S(h_{11})h_{32} \\ &= h_3 \otimes S(h_2)h_4 \otimes S(h_1)h_5.\end{aligned} \tag{1.33}$$

On the other hand

$$\begin{aligned}(\mathrm{ad}_R \otimes \mathrm{id}) \circ \mathrm{ad}_R(h) &= (\mathrm{ad}_R \otimes \mathrm{id}) \circ (h_2 \otimes S(h_1)h_3) \\ &= \mathrm{ad}_R(h_2) \otimes S(h_1)h_3 \\ &= h_{22} \otimes S(h_{21})h_{23} \otimes S(h_1)h_3 \\ &= h_3 \otimes S(h_2)h_4 \otimes S(h_1)h_5.\end{aligned} \tag{1.34}$$

□

Example 1.2.20. Every Hopf algebra H can be seen as a right $H-$comodule algebra via the coproduct $\Delta : H \to H \otimes H$. Indeed, let $h \in H$. We have that $(\Delta \otimes \mathrm{id}) \circ \Delta(h) = (\mathrm{id} \otimes \Delta) \circ \Delta(h)$, since it is the coassociativity axiom for a Hopf algebra. Moreover $(\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id}$ is counitality. Finally $\Delta(ab) = \Delta(a)\Delta(b)$ and $\Delta(1_H) = 1_H \otimes 1_H$. □

Definition 1.2.21. Let H be a Hopf algebra over a field $\Bbbk$. An $H-$Hopf module is a vector space V being both an $H-$module and an $H-$comodule and for which ${}_V\Delta : V \to H \otimes V$ is a left $H-$module map, i.e. ${}_V\Delta(h \cdot v) = (\Delta h).({}_V\Delta v)$, for all $h \in H$ and $v \in V$. The dot represents the action of H on V.

Lemma 1.2.22 ([2],Lemma 2.17)**.** *Let V be a left $H-$Hopf module. Then $V \simeq H \otimes ({}^{coH}V)$, where the right hand side is a Hopf module by the Hopf module structure of H. Conversely, every vector space defines a Hopf module by $H \otimes ({}^{coH}V)$, giving an equivalence between $H-$Hopf modules and vector spaces.* □

Chapter 2

Hopf-Galois extensions

In this section H is a Hopf algebra, A is a right $H-$comodule algebra with right $H-$coaction $\Delta_A : A \to A \otimes H$. In 2.1 we give some basic results about convolution invertible morphisms, we provide the notions of trivial, cleft and Hopf-Galois extensions and we discuss some explicit realisations.

In 2.1.2 we introduce the adjoint coaction on the tensor product between a right $H-$comodule algebra and a Hopf algebra; we discuss some identities featuring such coaction.

In 2.1.3 we introduce the translation map, which is the inverse of the canonical map of a Hopf-Galois extension, and discuss some properties.

We introduce crossed product algebras in 2.2 and discuss the proof of a theorem due to Doi-Takeuchi providing one-to-one correspondence between cross product algebras and cleft extensions, reducing to a correspondence with smash product algebras if the extension is trivial.

Content of Sections 2.1 and 2.2 are essential for what we discuss in 4.2.

A full understanding of the theory of principal bundle presented in [8, 10, 11, 9] requires also sections 2.1.2 and 2.1.3.

The main works to which we refer are [3, 4, 17]

2.1 Hopf-Galois, cleft and trivial extensions

Let $B := A^{coH}$ be the subalgebra of coinvariant elements under Δ_A. We start with a lemma.

Lemma 2.1.1. *Let $j : H \to A$ be a right $H-$colinear convolution invertible map; denote by $j^{-1} : H \to A$ the convolution inverse of j. Then:*

1. *the convolution inverse satisfies $\Delta_A(j^{-1}(h)) = j^{-1}(h_2) \otimes S(h_1)$ for all $h \in H$;*

2. *there is a map $A \to B$ assigning $a_0 j^{-1}(a_1)$ to every element $a \in A$;*

3. *$j(1) \in A$ is invertible with inverse $j^{-1}(1)$;*

4. *there is a unital right $H-$colinear convolution invertible map $j' : H \to A$ assigning $j(1)^{-1}j(h)$ to every element $h \in H$, with convolution inverse $j'^{-1}(h) = j^{-1}(h)j(1)$;*

5. *if j is an algebra morphism then j^{-1} is anti-algebra morphism, that is $j^{-1}(hh') = j^{-1}(h')j^{-1}(h)$ and $j^{-1}(1) = 1$ for every $h, h' \in H$.*

Proof. In order:

1. Notice that by colinearity and properties of Δ_A the following equivalence holds:

$$\begin{aligned}\epsilon(h)1_A \otimes 1_H = \Delta_A(\epsilon(h)1_A) &= \Delta_A(j(h_1)j^{-1}(h_2)) \\ &= \Delta_A(j(h_1))\Delta_A(j^{-1}(h_2)) \\ &= (j(h_1)_0 \otimes j(h_1)_1)(j^{-1}(h_2)_0 \otimes j^{-1}(h_2)_1) \\ &= j(h_1)j^{-1}(h_3)_0 \otimes h_2 j^{-1}(h_3)_1,\end{aligned} \tag{2.1}$$

   for every $h \in H$. If we define $f := (j \otimes \mathrm{id})\Delta : H \to A \otimes H$ we have

$$\begin{aligned}\left(f * (\Delta_A \circ j^{-1})\right)(x) &= \mu \circ \left(f \otimes (\Delta_A \circ j^{-1})\right) \circ \Delta(x) \\ &= \mu \circ \left(f(x_1) \otimes \Delta_A \circ j^{-1}(x_2)\right) \\ &= \mu \circ \left((j \otimes \mathrm{id})\Delta(x_1) \otimes \Delta_A \circ j^{-1}(x_2)\right) \\ &= \mu \circ \left((j(x_{11}) \otimes x_{12}) \otimes \Delta_A(j^{-1}(x_2))\right) \\ &= \mu \circ \left(j(x_1)j^{-1}(x_3)_0 \otimes x_2 j^{-1}(x_3)_1\right) \\ &= \epsilon(x)1_A,\end{aligned} \tag{2.2}$$

   so that f is a convolution inverse of $\Delta_A \circ j^{-1}$ in the convolution algebra $\mathrm{Hom}_{\Bbbk}((H, A \otimes H), *)$. Another convolution inverse of f is provided by $j^{-1}(h_2) \otimes S(h_1)$. Therefore the claim follows from uniqueness of the inverse.

2. By a direct computation we find

$$\begin{aligned}\Delta_A(a_0 j^{-1}(a_1)) &= \Delta_A(a_0)\Delta_A(j^{-1}(a_1)) \\ &= (a_{00} \otimes a_{01})(j^{-1}(a_{12}) \otimes S(a_{11})) \\ &= a_{00} j^{-1}(a_{12}) \otimes a_{01} S(a_{11}) \\ &= a_0 j^{-1}(a_3) \otimes a_1 S(a_2) \\ &= a_0 j^{-1}(a_1) \otimes 1,\end{aligned} \tag{2.3}$$

   and therefore we have a coinvariant element in $B := A^{coH}$.

3. As j^{-1} is the convolution inverse of j we find

$$\begin{aligned}(j * j^{-1})(1) &= j^{-1}(1)j(1) \\ &= \epsilon(1)1 \\ &= 1 \\ &= j(1)j^{-1}(1) \\ &= (j^{-1} * j)(1),\end{aligned} \tag{2.4}$$

   so $j^{-1}(1)$ is the inverse of $j(1)$.

4. Follows from the previous point.

5. By proposition 1.2.10 we have $j^{-1} = j \circ S$. In particular

$$\begin{aligned}j^{-1}(hh') &= j(S(hh')) \\ &= j(S(h')S(h)) \\ &= j^{-1}(h')j^{-1}(h),\end{aligned} \tag{2.5}$$

   for every elements in $h, h' \in H$. Accordingly $j^{-1}(1) = 1$. □

This lemma contains several important features of the so called *cleaving map*, which we define in the following. More specifically we define the notion of *extension* in the setting of Hopf algebra coactions on comodule algebras. The following definition is of special interest for the development of the theory presented in 4.2.

Definition 2.1.2. We call $B \subseteq A$

1. a *trivial extension* if there is a convolution invertible morphism $j : H \to A$ of right $H-$comodule algebras;

2. a *cleft extension* if there is a convolution invertible morphism $j : H \to A$ of right H-comodules, to which we will refer to as *cleaving map*;

3. a *Hopf-Galois extension* if the canonical map[1]

$$\chi : A \otimes_B A \to A \otimes H, \quad a \otimes_B a' \mapsto aa'_0 \otimes a'_1$$

is invertible.

In virtue of Lemma 2.1.1 we assume $j, j^{-1} : H \to A$ to be unital maps.

The next result is presented to compare trivial, cleft and Hopf-Galois extensions. In particular We show trivial extensions to be cleft, and cleft extensions to be Hopf-Galois.

Proposition 2.1.3. *Every trivial extension is also cleft. Every cleft extension is also Hopf-Galois.*

Proof. The first statement is obviously satisfied. Indeed, the difference is that in trivial extensions we require $j : H \to A$ to be an $H-$comodule algebra map, whereas in cleft extension $j : H \to A$ is an $H-$comodule map.

Let $B \subseteq A$ be a cleft extension and let $j : H \to A$ be the cleaving map with convolution inverse $j^{-1} : H \to A$. Define a map $\chi^{-1} : A \otimes H \to A \otimes_B A$ by

$$\chi^{-1}(a \otimes h) := aj^{-1}(h_1) \otimes_B j(h_2),$$

where $a \in A$ and $h \in H$. From Lemma 2.1.1 we have

$$\begin{aligned} \chi(\chi^{-1}(a \otimes h)) &= \chi(aj^{-1}(h_1) \otimes_B j(h_2)) \\ &= aj^{-1}(h_1)j(h_2)_0 \otimes j(h_2)_1 \\ &= aj^{-1}(h_1)j(h_2) \otimes h_3 \\ &= a \otimes h. \end{aligned} \tag{2.6}$$

On the other hand, given $a, a' \in A$, we have

$$\begin{aligned} \chi^{-1}(\chi(a \otimes_B a')) &= \chi^{-1}(aa'_0 \otimes a'_1) \\ &= aa'_0 j^{-1}(a'_1) \otimes_B j(a'_2) \\ &= a \otimes_B a'_0 j^{-1}(a'_1) j(a'_2) \\ &= a \otimes_B a', \end{aligned} \tag{2.7}$$

where we exploited the very definition of balanced tensor product over B to move elements of B along the tensor product. □

In virtue of this Proposition, and that $a'_0 j^{-1}(a'_1) \in B$ by Lemma 2.1.1, we will generally refer to trivial, cleft and Hopf-Galois extensions as Hopf-Galois extension, specifying if the extension is trivial or cleft according to the special case.

[1] Notice the map $\chi : A \otimes_B A \to A \otimes H$ is well defined over $A \otimes_B A$, since $ab_0 a'_0 \otimes b_1 a'_1 = aba'_0 \otimes a'_1$.

2.1.1 Examples

We discuss some explicit realisations of Hopf-Galois extension.

Example 2.1.4. Right $H-$coinvariant elements of any bialgebra are simply scalars $\Bbbk \cong H^{coH}$. Indeed $\Delta(h) = h \otimes 1$ for any $h \in H$ implies $h = \epsilon(h)1_H$, i.e. that h must be a scalar multiple of the unit. On the other hand any such multiple is clearly right $H-$coinvariant.

Proposition 2.1.5. *Let H be a bialgebra for the moment. The canonical map $\chi : H \otimes H \to H \otimes H$ sending $h \otimes h' \mapsto hh'_1 \otimes h'_2$ is invertible if and only if H is a Hopf algebra.*

Proof. If S is an antipode for H then $h \otimes h' \mapsto hS(h'_1) \otimes h'_2$ is the inverse of the canonical map. If, on the other hand, χ is invertible, we have an antipode via $S : H \to H$, where

$$S(h) := (\mathrm{id} \otimes \epsilon) \circ \chi^{-1} \circ (1 \otimes h).$$

The map S satisfies the antipode axiom:

$$\begin{aligned} h_1 S(h_2) &= h_1 (\mathrm{id} \otimes \epsilon)(\chi^{-1}(1 \otimes h_2)) \\ &= (\mathrm{id} \otimes \epsilon)(\chi^{-1}(h_1 \otimes h_2)) \\ &= (\mathrm{id} \otimes \epsilon)(\chi^{-1}\chi(1 \otimes h)) \\ &= 1 \otimes \epsilon(h). \end{aligned} \tag{2.8}$$

Similarly $S(h_1)h_2 = \epsilon(h) \otimes 1$. Therefore any Hopf algebra is in particular a Hopf-Galois extension $\Bbbk \subseteq H$. Invertibility of the canonical map χ is equivalent to the existence of an antipode. □

Example 2.1.6. Let F and E be fields. We say that E is a field extension of F if $F \subseteq E$ is a subfield. This means E can be considered as a vector space over F. We call the $F-$dimension of E the *degree of the extension*. Let now G be a finite group acting by $\Bbbk-$automorphisms $\Phi_g : E \to E$ on the field E; denote by

$$F = E^G = \{x \in E \ : \ \Phi_g(x) = x, \text{ for all g} \in G\}$$

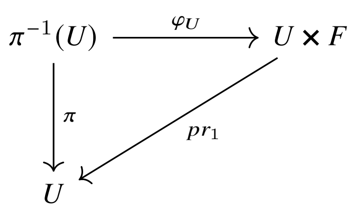
the set of *fixed points* of E under the action of G. The field extension $F \subseteq E$ is called *Galois* if and only if the action of G is faithful (that is, $\Phi_g(x) = x$ for every $x \in E$ means $g = e$, the identity element of G) on E if and only if the degree of the extension is equal to the cardinality of G.

Proposition 2.1.7. *Let $F \subseteq E$ be a Galois extension of degree $n = |G|$. Such field extension always corresponds to a Hopf-Galois extension with $F = B$, $E = A$ and $H = Hom_{\Bbbk}(\Bbbk G, \Bbbk)$.*

Proof. Consider $G = \{x_1, \ldots, x_n\}$ and let $\{b_1, \ldots, b_n\}$ a basis of the quotient field E/F. Define a Hopf algebra H as the dual of the group algebra $\Bbbk G$ introduced as in Example 1.2.4. H has basis $\{x^1, \ldots, x^n\}$ dual to the basis of G, namely $x^i(x_j) = \delta^i_j$. This gives a right $H-$coaction $\Delta_E : E \to E \otimes H$ explicitly presented as $\Delta_E(x) = \sum_{i=1}^n \Phi_{x_i}(x) \otimes x^i$. The canonical map reads $\chi(x \otimes_F x') = x \sum_{i=1}^n \Phi_{x_i}(x') \otimes x^i$ for all $x, x' \in E$.

Faithfulness of the $G-$action follows since we are dealing with Galois field extensions. Accordingly the canonical map χ is a bijection. □

Example 2.1.8. In differential geometry a fiber bundle is the datum of a smooth map $\pi : P \to M$ between smooth manifolds P, M and another smooth manifold F, such that for every $x \in M$ there is an open neighbourhood $U \subseteq M$ of x such that $\varphi_U : \pi^{-1}(U) \to U \times F$ are diffeomorphic and the diagram



,

where $pr_1 : U \times F \to U$ is to projection to the first factor, commutes. One calls P the total space, M the base space and F the fiber. The commutative diagram above is called a *local trivialisation* of the bundle. Here π must be a surjective summersion.

Let G be a Lie group acting freely and trasitively from the right on P, such that the fibers are preserved, in the sense that

$$\pi(p \cdot g) = \pi(p) \text{ for all } p \in P \text{ and } \; g \in G.$$

We call $\pi : P \to M$ a principal G−bundle. In particular the canonical map

$$P \times_M G \to P \times_M P, \quad (p, g) \mapsto (p, p \cdot g)$$

into the fibered product $P \times_M P = \{(p, p') \in P \times P' : \pi(p) = \pi(p')\}$ is a diffeomorphism.

Denote $A = C^\infty(P)$ and $B = C^\infty(M), H = C^\infty(G)$. The pull-back of the right action $P \times G \to P$ determines a right H−coaction $\Delta_A : A \to A \otimes H$ such that $B = A^{coH}$. Moreover the canonical map $\chi : A \otimes_B A \to A \otimes H$ is the pull-back of the canonical map of the principal bundle, and therefore it a bijection. Therefore $B \subseteq A$ forms a Hopf-Galois extension. □

2.1.2 Adjoint coactions

Recall the adjoint right H−coaction $\text{ad}_R : H \to H$ of Example 1.2.19, with $\text{ad}_R(h) = h_2 \otimes S(h_1)h_3$ on H. Consider the corresponding diagonal right H−coaction on the tensor product $A \otimes H$ defined as

$$\Delta^{\text{ad}}_{A\otimes H} : A \otimes H \to A \otimes H \otimes H, \quad a \otimes h \mapsto a_0 \otimes h_2 \otimes a_1 S(h_1)h_3.$$

Define two auxiliary maps

$$\begin{aligned} \chi' : A \otimes A &\longrightarrow A \otimes H, & a \otimes a' &\mapsto aa'_0 \otimes a'_1, \\ \nu : A \otimes H &\longrightarrow H \otimes A \otimes H, & a \otimes h &\mapsto a_1 S(h_1) \otimes a_0 \otimes h_2. \end{aligned} \tag{2.9}$$

We discuss some properties of those maps when $B \subseteq A$ is a Hopf-Galois extension. Let

$$\text{flip}_{A,H} : A \otimes H \to H \otimes A, \quad a \otimes h \mapsto h \otimes a.$$

Lemma 2.1.9. *Let A be a right H−comodule algebra. The following equations hold*

1. $(\text{id} \otimes \chi') \circ ((\textit{flip}_{A,H} \circ \Delta_A) \otimes \text{id}) = \nu \circ \chi'$;
2. $(\chi' \otimes \text{id}) \circ (\text{id} \otimes \Delta_A) = (\text{id} \otimes \Delta) \circ \chi'$;
3. $(\chi' \otimes \text{id}) \circ \Delta_{A\otimes A} = A^{ad}_{A\otimes H} \circ \chi'$.

Proof. We proceed in both cases by a direct calculation.

1. We find

$$\begin{aligned} (\text{id} \otimes \chi') \circ ((\text{flip}_{A,H} \circ \Delta_A) \otimes \text{id})(a \otimes a') &= (\text{id} \otimes \chi') \circ (\text{flip}_{A,H}(a_0 \otimes a_1) \otimes a') \\ &= (\text{id} \otimes \chi') \circ (a_1 \otimes a_0 \otimes a') \\ &= a_1 \otimes \chi'(a_0 \otimes a') \\ &= a_1 \otimes a_0 a'_0 \otimes a'_1. \end{aligned} \tag{2.10}$$

Similarly

$$\begin{aligned} (\nu \circ \chi')(a \otimes a') &= \nu(\chi'(a \otimes a')) \\ &= \nu(aa'_0 \otimes a'_1) \\ &= (aa'_0)_1 S(a'_{11}) \otimes (aa'_0)_0 \otimes a'_{12} \\ &= a_1 a'_1 S(a'_2) \otimes a_0 a'_0 \otimes a'_3 \\ &= a_1 \otimes a_0 a'_0 \otimes a'_1. \end{aligned} \tag{2.11}$$

2. Next we have

$$\begin{aligned}(\chi' \otimes \mathrm{id}) \circ (a \otimes \Delta_A(a')) &= (\chi' \otimes \mathrm{id}) \circ (a \otimes (a'_0 \otimes a'_1)) \\ &= \chi'(a \otimes a'_0) \otimes a'_1 \\ &= aa'_{00} \otimes a'_{01} \otimes a'_1 \\ &= aa'_0 \otimes a'_1 \otimes a'_2;\end{aligned} \tag{2.12}$$

$$\begin{aligned}(\mathrm{id} \circ \Delta) \circ \chi'(a \otimes a') &= (\mathrm{id} \otimes \Delta)(aa'_0 \otimes a'_1) \\ &= aa'_0 \otimes a'_1 \otimes a'_2.\end{aligned}$$

3. Finally

$$\begin{aligned}(\chi' \otimes \mathrm{id}) \circ \Delta_{A\otimes A}(a \otimes a') &= (\chi' \otimes \mathrm{id})(a_0 \otimes a'_0 \otimes a_1 a'_1) \\ &= \chi'(a_0 \otimes a'_0) \otimes a_1 a'_1 \\ &= a_0 a'_{00} \otimes a'_{01} \otimes a_1 a'_1 \\ &= a_0 a'_0 \otimes a'_1 \otimes a_1 a'_2,\end{aligned} \tag{2.13}$$

$$\begin{aligned}\Delta^{\mathrm{ad}}_{A\otimes H} \circ \chi'(a \otimes a') &= A^{\mathrm{ad}}_{A\otimes H}(aa'_0 \otimes a'_1) \\ &= (aa')_0 \otimes a'_{12} \otimes (aa'_0)_1 S(a'_{11}) a'_{13} \\ &= a_0 a'_0 \otimes a'_3 \otimes a_1 a'_1 S(a'_2) a'_4 \\ &= a_0 a'_0 \otimes a'_1 \otimes a_1 a'_2.\end{aligned}$$

□

2.1.3 The translation map

Let us consider the quotient map $\pi : A \otimes A \to A \otimes_B A'$. By the very definition of the Hopf-Galois canonical map $\chi : A \otimes_B A \to A \otimes H$ the composition $\chi \circ \pi : A \otimes A \to A \otimes H$ is exactly $\chi' : A \otimes A \to A \otimes H$ defined in Equation (2.9). Consequently relations derived in the Lemma 2.1.9 descend to the quotient.

Let χ be the Hopf-Galois canonical map of a Hopf-Galois extension then its inverse

$$\chi^{-1} : A \otimes H \to A \otimes_B A$$

is well defined.

Definition 2.1.10. The *translation map* is defined as $\kappa = \chi^{-1}|_{1_A \otimes H} : H \to A \otimes_B A$.

Notation 2.1.11. We use the Sweedler-like shorthand notation $\kappa(h) = h^{\langle 1 \rangle} \otimes h^{\langle 2 \rangle}$ to denote the action for the translation map, given any $h \in H$.

In the next proposition we exploit some important features of the translation map.

Proposition 2.1.12. *Let $B \subseteq A$ be a Hopf-Galois extension. For any $h, h' \in H$ and $a \in A$, we have:*

1. $h^{\langle 1 \rangle}(h^{\langle 2 \rangle})_0 \otimes (h^{\langle 2 \rangle})_1 = 1_A \otimes h$;
2. $a_0 (a_1)^{\langle 1 \rangle} \otimes_B (a_1)^{\langle 2 \rangle} = 1_A \otimes_B a$;
3. $\kappa(hh') = h'^{\langle 1 \rangle} h^{\langle 1 \rangle} \otimes_B h^{\langle 2 \rangle} h'^{\langle 2 \rangle}$;
4. $h^{\langle 1 \rangle} h^{\langle 2 \rangle} = \epsilon(h) 1_A$;
5. $h^{\langle 1 \rangle} \otimes_B (h^{\langle 2 \rangle})_0 \otimes (h^{\langle 2 \rangle})_1 = (h_1)^{\langle 1 \rangle} \otimes_B (h_1)^{\langle 2 \rangle} \otimes h_2$;

Proof. In order:

1. by a direct calculation

$$\begin{aligned} h^{\langle 1\rangle}(h^{\langle 2\rangle})_0 \otimes (h^{\langle 2\rangle})_1 &= \chi(h^{\langle 1\rangle} \otimes h^{\langle 2\rangle}) \\ &= \chi(k(h)) \\ &= \chi(\chi^{-1}(1_A \otimes h)) = 1_A \otimes h. \end{aligned} \tag{2.14}$$

2.

$$\begin{aligned} a_0(a_1)^{\langle 1\rangle} \otimes_B (a_1)^{\langle 2\rangle} &= a_0 k(a_1) = a_0\chi^{-1}(1 \otimes a_1) \\ &= \chi^{-1}(a_0 \otimes a_1) \\ &= \chi(1_A \otimes_B a) \\ &= 1_A \otimes_B a, \end{aligned} \tag{2.15}$$

where we used that $\chi : A \otimes_B A \to A \otimes H$ and $\chi^{-1} : A \otimes H \to A \otimes_B A$ are left A−linear.

3. Applying two times the first property:

$$\begin{aligned} \chi(h'^{\langle 1\rangle} h^{\langle 1\rangle} \otimes_B h^{\langle 2\rangle} h'^{\langle 2\rangle}) &= h'^{\langle 1\rangle} h^{\langle 1\rangle} (h^{\langle 2\rangle})_0 (h'^{\langle 2\rangle})_0 \otimes (h^{\langle 2\rangle})_1 (h'^{\langle 2\rangle})_1 \\ &= h'^{\langle 1\rangle} (h'^{\langle 2\rangle})_0 \otimes h (h'^{\langle 2\rangle})_1 \\ &= 1_A \otimes hh', \end{aligned} \tag{2.16}$$

so we find $k(hh') = \chi(h'^{\langle 1\rangle} h^{\langle 1\rangle} \otimes_B h^{\langle 2\rangle} h'^{\langle 2\rangle})$.

4. We consider a concatenation between the relations in Equations (2.9) descended to the quotient.

$$\begin{aligned} (\mathrm{id} \otimes \chi) \circ (\tau_{A,H} \circ \Delta_A \otimes_B \mathrm{id}) \circ \kappa(h) &= \nu \circ \chi \circ \kappa(h) \\ (\mathrm{id} \otimes \chi) \circ ((h^{\langle 1\rangle})_1 \otimes (h^{\langle 1\rangle})_0 \otimes_B h^{\langle 2\rangle}) &= S(h_1) \otimes 1_A \otimes h_2 \\ (h^{\langle 1\rangle})_1 \otimes (h^{\langle 1\rangle})_0 (h^{\langle 2\rangle})_0 \otimes (h^{\langle 2\rangle})_1 &= S(h_1) \otimes 1_A \otimes h_2, \end{aligned} \tag{2.17}$$

giving the claim after applying $(\epsilon \otimes \mathrm{id} \otimes \epsilon)$ to both sides.

5. We perform a calculation similar to the previous point. This time we concatenate the second equation of Equations (2.9) with $(\chi^{-1} \otimes \mathrm{id})$ on the left, and with κ on the right.

$$\begin{aligned} (\chi^{-1} \otimes \mathrm{id})(\chi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes_B \Delta_A)\kappa(h) &= (\chi^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\chi(\kappa(h)) \\ (\chi^{-1} \otimes \mathrm{id})(\chi(h^{\langle 1\rangle} \otimes_B h_0^{\langle 2\rangle}) \otimes (h^{\langle 2\rangle})_1) &= (\chi^{-1} \otimes \mathrm{id})(1_A \otimes h_1 \otimes h_2) \\ h^{\langle 1\rangle} \otimes_B (h^{\langle 2\rangle})_0 \otimes (h^{\langle 2\rangle})_1 &= \chi^{-1}(1_A \otimes h_1) \otimes h_2 \\ &= \kappa(h_1) \otimes h_2 \\ &= (h^{\langle 1\rangle})_1 \otimes_B (h^{\langle 2\rangle})_1 \otimes h_2. \end{aligned} \tag{2.18}$$

□

2.2 Crossed product algebras

In this section we define *crossed product algebras* and *smashed product algebras*. The Doi-Takeuchi correspondence between cleft extension and crossed product algebras (trivial extensions and smash product algebras) of [7] is proven.

Let B an algebra.

Definition 2.2.1. We say that H measures B if there is a $\Bbbk$−linear map

$$\cdot : H \otimes B \to B, \quad h \otimes b \mapsto h \cdot b,$$

such that

$$\begin{aligned} h \cdot 1_B &= \epsilon(h) 1_B, \\ h \cdot (bb') &= (h_1 \cdot b)(h_2 \cdot b'). \end{aligned}$$

We stress that $\cdot : H \otimes B \to B$ is not assumed to be an action.

Definition 2.2.2. Assume H measures B. Consider a map $\sigma : H \otimes H \to B$.

1. We call $\sigma : H \otimes H \to B$ a 2-*cocyle with values in* B if it is a convolution invertible morphism such that
$$\begin{aligned} &\sigma(h \otimes 1) = \epsilon(h)1_B = \sigma(1 \otimes h), \\ &(h_1 \cdot \sigma(h'_1 \otimes h''_1))\sigma(h_2 \otimes h'_2h''_2) = \sigma(h_1 \otimes h'_1)\sigma(h_2h'_2 \otimes h''), \end{aligned} \tag{2.19}$$
for all $h, h', h'' \in H$ and $b \in B$.

2. We call B a $\sigma-$*twisted left* $H-$*module* if there is a 2-cocycle $\sigma : H \otimes H \to B$ with values in B such that
$$\begin{aligned} &1 \cdot b = b, \\ &h \cdot (h' \cdot b) = \sigma(h_1 \otimes h'_1)((h_2h'_2) \cdot b)\sigma^{-1}(h_3 \otimes h'_3), \end{aligned} \tag{2.20}$$
for all $h, h' \in H$ and $b \in B$.

Lemma 2.2.3. *Given* B *a* $\sigma-$*twisted left* $H-$*module, we define a* $\Bbbk-$*linear map*

$$\mu_{\sharp_\sigma} : (B \otimes H) \otimes (B \otimes H) \to B \otimes H, \quad (b \otimes h) \otimes (b' \otimes h') \mapsto b(h_1 \cdot b')\sigma(h_2 \otimes h'_1) \otimes h_3h'_2, \tag{2.21}$$

for every $h, h' \in H$ *and* $b \in B$*, providing an associative unital product* $\mu_{\sharp_\sigma}$ *on* $B \otimes H$*, with unit* $1_B \otimes 1_H$.

Proof. First notice the $\Bbbk-$linear map $\mu_{\sharp_\sigma}$ is well defined. To prove associativity we proceed considering

$$\begin{aligned}
\mu_{\sharp_\sigma}(\mu_{\sharp_\sigma}((b \otimes h) \otimes (b' \otimes h')) \otimes (b'' \otimes h'')) &= \mu_{\sharp_\sigma}((b(h_1 \cdot b')\sigma(h_2 \otimes h'_1) \otimes h_3h'_2 \otimes b'' \otimes h'') \\
&= b(h_1 \cdot b')\sigma(h_2 \otimes h'_1)(h_3h'_2)_1 \cdot b''_1 \cdot \sigma((h_3h'_2)_2 \otimes h''_1) \otimes (h_3h'_2)_3h''_2 \\
\textbf{relabeling} \quad &= b(h_1 \cdot b')\sigma(h_2 \otimes h'_1)(h_{31}h'_{21} \cdot b''_1) \cdot \sigma(h_{32}h'_{22} \otimes h''_1) \otimes h_{33}h'_{23}h''_2 \\
\textbf{Equation (2.20)} \quad &= b(h_1 \cdot b')\sigma(h_2 \otimes h'_1)\sigma^{-1}(h_3 \otimes h'_2)(h_4 \cdot h'_3 \cdot b'')\sigma(h_5 \otimes h'_4)\sigma(h_6h'_5 \otimes h''_1) \otimes h_7h'_6h''_2 \\
\textbf{simplifying} \quad &= b(h_1 \cdot b')(h_2 \cdot h'_1 \cdot b'')\sigma(h_3 \otimes h'_2)\sigma(h_4h'_3 \otimes h''_1) \otimes h_5h'_4h''_2 \\
\textbf{measurability} \quad &= b(h_1 \cdot (b'(h'_1 \cdot b'')))\sigma(h_2 \otimes h'_2)\sigma(h_3h'_3 \otimes h''_1) \otimes h_4h'_4h''_2 \\
\textbf{Equation (2.19)} \quad &= b(h_1 \cdot (b'(h'_1 \cdot b'')\sigma(h_2 \otimes h''_1)))\sigma(h_2 \otimes h'_3h''_2) \otimes h_3h'_4h''_3 \\
\textbf{measurability} \quad &= b(h_1 \cdot (b'(h'_1 \cdot b'')\sigma(h_2 \otimes h''_1)))\sigma(h_2 \otimes h'_3h''_2) \otimes h_3h'_4h''_3 \\
&= \mu_{\sharp_\sigma}((b \otimes h) \otimes (b'(h'_1 \cdot b'') \cdot \sigma(h'_2 \otimes h''_1) \otimes h'_3h''_2) \\
&= \mu_{\sharp_\sigma}(b \otimes h \otimes \mu_{\sharp_\sigma}((b' \otimes h') \otimes (b'' \otimes h''))),
\end{aligned} \tag{2.22}$$

where we used property of B being a $\sigma-$twisted left H-module, so that

$$h(h' \cdot b) = \sigma^{-1}(h_1 \otimes h'_1)((h_2h'_2 \cdot b)(\sigma(h_3 \otimes h'_3)),$$

and the property of 2-cocycle with values in B. The last part of the proof is to show that $1_B \otimes 1_H$ is a unit, but this follows immediately. ◻

$B\sharp_\sigma H$ is called a *cross product algebra*. We define on $B\sharp_\sigma H$ the structure of a right $H-$comodule algebra with respect to the right $H-$coaction $\Delta_{\sharp_\sigma} := \mathrm{id}_B \otimes \Delta : B\sharp_\sigma H \to (B\sharp_\sigma H) \otimes H$. Accordingly, the subalgebra of coinvariant elements in $B\sharp_\sigma H$ is

$$(B\sharp_\sigma H)^{coH} = (B \otimes 1) \cong B.$$

Remark 2.2.4. Every left $H-$module algebra B is in particular a $\sigma-$twisted left $H-$module with respect to the trivial $2-$cocycle

$$\sigma : H \otimes H \to B, \quad h \otimes h' \mapsto \epsilon(hh')1_B.$$

The corresponding product

$$\mu_\sharp : (B \otimes H) \otimes (B \otimes H) \to B \otimes H, \quad (b \otimes h) \otimes (b' \otimes h') \mapsto b(h_1 \cdot b') \otimes h_2 h'$$

makes $B\sharp H := (B \otimes H, \mu_\sharp)$ an associative unital algebra, the *smash product algebra*.

The next theorem was proven in [7]. It shows that crossed product algebras are in $1:1$ correspondence with cleft extensions. Moreover, trivial extensions are in $1:1$ correspondence with smash product algebras.

Theorem 2.2.5. *Any crossed product algebra $B\sharp_\sigma H$ is a cleft extension $B \subseteq B\sharp_\sigma H$ with cleaving map $j : H \to B\sharp_\sigma H$ assigning $h \mapsto 1\sharp_\sigma h$. Conversely, giving a cleft extension $B \subseteq A$ with cleaving map $j : H \to A$ we define a $\sigma-$twisted left $H-$module action*

$$\cdot : H \otimes B \to B, \qquad h \otimes b \mapsto h \cdot b := j(h_1) b j^{-1}(h_2)$$

on B, and a 2-cocycle

$$\sigma : H \otimes H \to B, \qquad h \otimes h' \mapsto \sigma(h \otimes h') := j(h_1) j(h'_1) j^{-1}(h_2 h'_2)$$

with values in B. Then $A \cong B\sharp_\sigma H$ are isomorphic as right $H-$comodule algebras. A cleft extension is a trivial extension if and only if the corresponding crossed product algebra is a smash product algebra.

Proof. Given a crossed product algebra $B\sharp_\sigma H$ we consider a right $H-$colinear map

$$j : H \to B\sharp_\sigma H, \quad h \in H \mapsto 1\sharp_\sigma h. \tag{2.23}$$

This map is convolution invertible with inverse

$$j^{-1} : H \to B\sharp_\sigma H, \quad h \mapsto \sigma^{-1}(S(h_2) \otimes h_3)\sharp_\sigma S(h_1). \tag{2.24}$$

Indeed

$$\begin{aligned} j^{-1}(h_1) j(h_2) &= (\sigma^{-1}(S(h_2) \otimes h_3)\sharp_\sigma S(h_1))(1\sharp_\sigma h_4) \\ &= \mu_{\sharp_\sigma} \left[(\sigma^{-1}(S(h_2) \otimes h_3) \otimes S(h_1)) \otimes (1 \otimes h_4)\right] \\ &= \sigma^{-1}(S(h_2) \otimes h_3)(S(h_1)_1 \cdot 1)\sigma(S(h_1)_2 \otimes h_4) \otimes S(h_1)_3 h_5 \\ &= \sigma^{-1}(S(h_3) \otimes h_4)\sigma(S(h_2) \otimes h_5) \otimes S(h_1) h_6 \\ &= \sigma^{-1}(S(h_2)_2 \otimes h_3)\sigma(S(h_2)_2 \otimes h_4) \otimes S(h_1) h_5 \\ &= 1\sharp_\sigma S(h_1) h_2 \\ &= \epsilon(h) 1\sharp_\sigma 1, \end{aligned} \tag{2.25}$$

and similarly evaluating in the other order, proving $B \subseteq B\sharp_\sigma H$ is a cleft extension.

For the other implication, given a cleft extension $B \subseteq A$ with cleaving map $j : H \to A$ we prove $\sigma : H \otimes H \to B$ and $\cdot : H \otimes B \to B$ provide a $\sigma-$twisted left $H-$module structure on B. We have that $\cdot : H \otimes B \to B$ is a $H-$measure on B, indeed

$$\begin{aligned} (h \cdot 1_B) &= j(h_1) j^{-1}(h_2) \\ &= \epsilon(h) 1_B; \end{aligned} \tag{2.26}$$

moreover

$$\begin{aligned} (h_1 \cdot b)(h_2 \cdot b') &= j(h_1) b j^{-1}(h_2) j(h_3) b' j(h_4) \\ &= j(h_1) b b' j(h_2) \\ &= h \cdot (bb'). \end{aligned} \tag{2.27}$$

Moreover, $\sigma : H \otimes H \to B$ is a **2-cocycle** with values in B:

1. The image of σ is contained in B, indeed

$$\begin{aligned}\Delta_A(j(h_1)j(h_2)j^{-1}(h_2h_2')) &= \Delta_A(j(h_1))\Delta_A(j(h_1'))\Delta_A(j^{-1}(h_2h_2'))\\ &= (j(h_1)_0\otimes j(h_1)_1)(j(h_1')_0\otimes j(h_1')_1)(j^{-1}(h_2h_2')_2\otimes S(h_2h_2')_1)\\ &= j(h_1)j(h_1')j(h_4h_4')\otimes h_2h_2'S(h_3h_3')\\ &= j(h_1)j(h_1')j(h_2h_2')\otimes 1,\end{aligned} \tag{2.28}$$

where we used Lemma 2.2.5. The convolution inverse of σ is

$$\sigma^{-1}(h\otimes h') = j(h_1'h_1)j^{-1}(h_2)j^{-1}(h_2').$$

2. $\sigma(1\otimes h) = \sigma(h\otimes 1) = \epsilon(h)1$.

3. Finally

$$\begin{aligned}&(h_1\cdot\sigma(h_1'\otimes h_1''))(\sigma(h_2\otimes h_2'h_2''))\\ &\quad= j(h_{11})\sigma(h_1'\otimes h_1'')j^{-1}(h_{12})j(h_{21})j(h_{21}'h_{21}'')j^{-1}(h_{22}h_{22}'h_{22}'')\\ &\quad= j(h_{11})j(h_{11}')j(h_{11}'')j^{-1}(h_{12}'h_{12}'')j^{-1}(h_{12})j(h_{21})j(h_{21}'h_{21}'')j^{-1}(h_{22}h_{22}'h_{22}'')\\ &\quad= j(h_1)j(h_1')j(h_1'')j^{-1}(h_2'h_2'')j^{-1}(h_2)j(h_3)j(h_3'h_3'')j^{-1}(h_4h_4'h_4'')\\ &\quad= j(h_1)j(h_1')j(h_1'')j^{-1}(h_2h_2'h_2'')\\ &\quad= j(h_1)j(h_1')j^{-1}(h_2h_2')j(h_3h_3')j(h_1'')j^{-1}(h_4h_4'h_2'')\\ &\quad= \sigma(h_1\otimes h_1')\sigma(h_2h_2'\otimes h'').\end{aligned} \tag{2.29}$$

The measure $\cdot : H\otimes B\to B$ is a σ–twisted left H–module action, since

$$\begin{aligned}1\cdot b &= j(1)bj^{-1}(1) = b;\\ \sigma(h_1\otimes h_1')((h_2h_2')\cdot b)\sigma^{-1}(h_3\otimes h_3') &= \sigma(h_1\otimes h_1')(j(h_{21}h_{21}')bj^{-1}(h_{22}h_{22}'))\sigma^{-1}(\sigma_3\otimes\sigma_3')\\ &= j(h_1)j(h_1')j^{-1}(h_2h_2')j(h_3h_3')bj^{-1}(h_4h_4')j(h_5h_5')j^{-1}(h_6')j(h_6)\\ &= j(h_1)j(h_1')bj^{-1}(h_2')j^{-1}(h_2)\\ &= h\cdot(h'\cdot b),\end{aligned}$$

for every $h, h'\in H$ and $b\in B$. Consequently we can construct the crossed product algebra $B\sharp_\sigma H$.

The only thing left to show it that $B\sharp_\sigma H$ is isomorphic to A as a right H–comodule algebra, the explicit isomorphism is provided by

$$\theta : A\to B\sharp_\sigma H,\quad a\mapsto a_0j^{-1}(a_1)\otimes a_2.$$

This map is a right H–comodule morphism and is well defined according to Lemma 2.1.1. It is left to prove it is also an algebra morphism. We have

$$\begin{aligned}\theta(a)\theta(a') &= (a_0j^{-1}(a_1)\sharp_\sigma a_2)(a_0'j^{-1}(a_1)\sharp_\sigma a_2')\\ &= \mu_{\sharp_\sigma}(a_0(j^{-1}(a_1))\otimes a_2)\otimes(a_0'j^{-1}(a_1)\otimes a_2')\\ &= a_0j^{-1}(a_1)(a_{21}\cdot(a_0'(j^{-1}(a_1'))))\sigma(a_{22}\otimes a_{21}')\otimes a_{23}a_{22}'\\ &= a_0j^{-1}(a_1)j(a_{21})a_0'j^{-1}(a_1')j^{-1}(a_{22})j(a_{31})j(a_{21}')j^{-1}(a_{32}a_{22}')\otimes a_{33}a_{22}'\\ &= a_0j^{-1}(a_1)j(a_2)a_0'j^{-1}(a_1')j^{-1}(a_3)j(a_4)j(a_2')j^{-1}(a_5a_3')\otimes a_6a_4'\\ &= a_0a_0'j^{-1}(a_1a_1')\otimes a_2a_2'\\ &= \theta(aa').\end{aligned} \tag{2.30}$$

In the context of a trivial extension, then $j : H \to A$ is an algebra morphism. The induced 2–cocycle becomes trivial

$$\begin{aligned} \sigma(h \otimes h') &= j(h_1) j(h'_1) j^{-1}(h_2 h'_2) \\ &= j(h_1) j(h'_1) j^{-1}(h'_2) j^{-1}(h_2) \\ &= \epsilon(hh')1. \end{aligned} \tag{2.31}$$

Thus, $B\sharp H$ is a smash product algebra.

Conversely if $(B\sharp H, \mu_\sharp)$ is a smash product algebra, the induced cleaving map is an algebra map. □

Chapter 3

Differential calculus over algebras

In this chapter we explore the topic of differentials over algebras. In the context of (classical) differential geometry we usually considers a n−dimensional manifold M with a differentiable structure, that is, every open set of M is identified with a open set of $\mathbb{R}^n$ in "some" smooth way (smooth atlas). These *local patches* fit in such way that we can talk of a global structure on M. Tangent and cotangent bundles to M are defined as dual structures. They are respectively obtained by gluing the spaces of tangent vectors and spaces of covectors for every point in the *manifold* M. Sections of the tangent bundle act as derivations of the (commutative) algebra of C^∞ functions on M. Dually, we have a map $\mathrm{d} : C^\infty(M) \to \Gamma(T^*M)$ turning smooth functions into covectors. Differential forms are obtained by subsequent applications of the exterior derivative $\mathrm{d} : \Gamma(\Lambda^\bullet(T^*M)) \to \Gamma(\Lambda^\bullet(T^*M))$ on sections of the exterior powers of the cotangent bundle.

This construction can be generalised to the setting of *noncommutative* algebras. In this picture no actual topological space is required; we actually consider differential geometry over algebras in this sense.

In 3.1 we introduce first order differential calculi over $\mathbb{k}$−algebras, we provide some examples and prove a theorem by Woronowicz stating that every first order differential calculus on an algebra is induced as a quotient of the so called *universal* differential calculus. Then we introduce the notion of *covariant* calculus.

In 3.1.1 we prove another important classification result due to Woronowicz, providing a 1 to 1 correspondence between covariant calculi and certain ideals defined on the underlying algebra.

In 3.2 we generalise the first order theory to higher order differential forms. In 3.2.1 we discuss higher order covariant differential calculi and provide an explicit formula for the H−coaction on higher order differential forms.

In 3.3 we introduce the maximal prolongation, an higher order differential calculus defined just by the datum of a first order calculus over a $\mathbb{k}$−algebra. This is of particular interest for the theory developed in 4.2.

The main references we follow here are [2, 25, 19]

3.1 First order differential calculi

Definition 3.1.1. A first order differential calculus (Γ, d) over a $\mathbb{k}$−algebra A is the datum of

1. an A−bimodule Γ;
2. a linear map $\mathrm{d} : A \to \Gamma$ satisfying the Leibniz rule $\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b$ for every $a, b \in A$;
3. a surjectivity condition $\Gamma = A\mathrm{d}A$, i.e. $\Gamma = \mathrm{span}\{a\mathrm{d}b : a, b \in A\}$.

In classical differential geometry $A = C^\infty(M)$, and left and right module structure on Γ always coincide, meaning that for every $a, b \in A$ we have $a\mathrm{d}b = \mathrm{d}ba$. This is not true in the general noncommutative setting.

Remark 3.1.2 (Surjectivity from right). The surjectivity condition $\Gamma = A\mathrm{d}A$ of a differential calculus (Γ, d) on A is equivalent to surjectivity from the right $\Gamma = \mathrm{d}(A)\, A$. In fact, for every $\omega \in \Gamma$ there are $a^i, b^i \in A$, with $1 \leq i \leq n$ such that $\omega = a^i \mathrm{d}b^i$. We define $e^j, f^j \in A$ for $1 \leq j \leq 2n$ by $e^j = a^j b^j$, with $f^j = 1$ for $1 \leq j \leq n$ and $e^j = a^{j-n}$, $f^j = b^{j-n}$ for $n < j \leq 2n$. Then

$$\omega = a^i \mathrm{d}b^i = \mathrm{d}\big(a^i b^i\big) - \mathrm{d}\big(a^i\big)\, b^i = \mathrm{d}\big(e^j\big)\, f^j,$$

by the Leibniz rule. Similarly the other implication is proven.

Remark 3.1.3 (Connected calculus). From the Leibniz rule it immediately follows that every first order differential calculus satisfies $\mathrm{d}(1) = \mathrm{d}(1 \cdot 1) = 2\,\mathrm{d}(1)$, from which we find $\mathbb{k} \cdot 1 \subseteq \ker \mathrm{d}$. If the last is an equality we say the calculus is *connected*.

We provide a few explicit realisations of first order differential calculi.

Example 3.1.4 (q-differential calculus on the circle)**.** Let $q \in \mathbb{C}$ not a root of unity. Consider the algebra $A = \mathbb{S}^1_q = \mathbb{C}[t, t^{-1}]$ of rational polynomials in one variable t. Define Γ as the free left A−module generated by $\mathrm{d}t$. Every element of Γ will be of the form $f(t) \cdot \mathrm{d}t$. The free left A−module Γ becomes an A−bimodule via

$$f(t) \cdot \mathrm{d}t \cdot g(t) = f(t)g(qt) \cdot \mathrm{d}t$$

for $f, g \in A$. We define a $\mathbb{C}$−linear map $\mathrm{d} : A \longrightarrow \Gamma$, the exterior derivative, by

$$\mathrm{d}(f(t)) = \frac{f(qt) - f(t)}{t(q-1)} \cdot \mathrm{d}t,$$

for every $f \in A$. The exterior derivative satisfyies the Leibniz rule.

$$\begin{aligned} \mathrm{d}(f(t)) \cdot g(t) + f(t) \cdot \mathrm{d}(g(t)) &= \frac{f(qt) - f(t)}{t(q-1)} \cdot \mathrm{d}t \cdot g(t) + f(t)\frac{g(qt) - g(t)}{t(q-1)} \cdot \mathrm{d}t \\ &= \frac{f(qt)g(qt) - f(t)g(qt) + f(t)g(qt) - f(t)g(t)}{t(q-1)} \cdot \mathrm{d}t \\ &= \frac{f(qt)g(qt) - f(t)g(t)}{t(q-1)} \cdot \mathrm{d}t \\ &= \mathrm{d}(f \cdot g)\,(t), \end{aligned} \tag{3.1}$$

for all $f, g \in A$.

By construction $\Gamma = A \cdot \mathrm{d}t = A\mathrm{d}A$. Thus (Γ, d) is a first order differential calculus over A.

Example 3.1.5 (Quotient differential calculus)**.** Given a surjective algebra map $\pi : A \longrightarrow H$, and a first order differential calculus (Γ, d) on A, we define $I := \ker \pi$ and $N_I := I\mathrm{d}A + A\mathrm{d}I$. Clearly $I \subseteq A$ is an algebra ideal such that $H \cong A/I$.

Using the Leibniz rule one shows that $N_I \subseteq \Gamma$ is an A−bimodule. The induced calculus (Γ_H, d_H) is the quotient calculus on H. Namely, we set $\Gamma_H := \Gamma/N_I$ and $\mathrm{d}_H = \pi_\Gamma \circ \mathrm{d}$, where $\pi_\Gamma : \Gamma \longrightarrow \Gamma_H$ is the quotient map.

Definition 3.1.6. Let (Γ, d) on A and (Γ', d') on A' be first order differential calculi. A morphism of differential calculi is a tuple (Φ, ϕ), where $\Phi : \Gamma \longrightarrow \Gamma'$ is a $\mathbb{k}$−linear map, and $\phi : A \longrightarrow A'$ is an algebra morphism, such that

$$\Phi(a \cdot \omega \cdot b) = \phi(a) \cdot' \Phi \cdot' \phi(b), \tag{3.2}$$

for all $a, b \in A$ and $\omega \in \Gamma$, and such that the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\ \Phi\ } & \Gamma' \\ {\scriptstyle \mathrm{d}}\uparrow & & \uparrow{\scriptstyle \mathrm{d}'} \\ A & \xrightarrow{\ \phi\ } & A' \end{array}$$

commutes. In this case we say ϕ is *differentiable*. We write $\mathrm{d}\phi = \Phi$.

Proposition 3.1.7 (Universal differential calculus)**.** *For every algebra A there is a first order differential calculus (Γ_u, d_u) defined by*

$$\Gamma_u := \ker \mu_A = \left\{ \sum_i a^i \otimes b^i \in A \otimes A : \sum_i a^i b^i = 0 \right\},$$

with differential $\mathrm{d}_u(a) := 1 \otimes a - a \otimes 1$ for all $a \in A$.

Proof. The A–bimodule structure on Γ_u is induced from $A \otimes A$, i.e. the multiplication on the first tensor factor from the left and the second tensor factor from the right, respectively.

The map d_u maps into Γ_u and satisfies the Leibniz rule

$$\begin{aligned} \mathrm{d}_u(a)b + a\mathrm{d}_u(b) &= 1 \otimes ab - a \otimes b + a \otimes b - ab \otimes 1 \\ &= \mathrm{d}_u(ab), \end{aligned} \tag{3.3}$$

for all $a, b \in A$.

Let $a^i \otimes b^i \in \Gamma_u := \ker \mu_A$ be arbitrary. Then

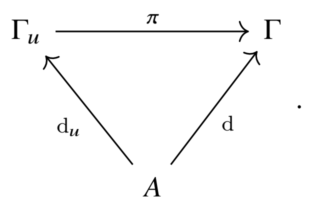
$$\begin{aligned} \sum_i a^i \mathrm{d}_u(b^i) &= \sum_i a^i \otimes b^i - \sum_i a^i b^i \otimes 1 \\ &= \sum_i a^i \otimes b^i, \end{aligned} \tag{3.4}$$

which proves surjectivity. Therefore (Γ_u, d_u) is a first order differential calculus over A.

□

The next theorem was first proven by Woronowicz in [25]. It states that every first order differential calculus is induced as quotient of the *universal* differential calculus in Proposition 3.1.7.

Theorem 3.1.8 (Woronowicz)**.** *Let A be a unital $\mathbb{k}$–algebra. Every first order differential calculus (Γ, d) over A can be obtained as a quotient of the universal differential calculus (Γ_u, d_u), meaning there exists an A–subbimodule $N \subseteq \Gamma_u$ such that $\Gamma \cong \Gamma_u/N$ as A–bimodules, and $\mathrm{d} = \pi \circ \mathrm{d}_u$, where $\pi : \Gamma_u \to \Gamma_u/N \cong \Gamma$ is the quotient map. By construction*



The denomination "universal" is used since for every calculus Γ on A there is a unique surjective morphism π such that the diagram shown commutes.

Proof. Let us consider any $\mathbb{k}$–algebra A and the first order differential calculus (Γ_u, d_u) of Proposition 3.1.7. Let (Γ, d) be any other differential calculus on A. Consider the A–bilinear map $\pi : \Gamma_u \to \Gamma$ assigning $a^i \otimes b^i \mapsto a^i \mathrm{d} b^i$.

The map π is surjective. Indeed, for any $\omega = a^i \mathrm{d} b^i \in \Gamma$ we have

$$\begin{aligned} \pi(\sum_i a^i \otimes b^i - \sum_i a^i b^i \otimes 1) &= \sum_i a^i \mathrm{d} b^i - \sum_i a^i b^i \, \mathrm{d}(1) \\ &= \sum_i a^i \mathrm{d} b^i. \end{aligned} \tag{3.5}$$

Considering the quotient of Γ_u by the subbimodule $N = \ker \pi$ we provide the isomorphism $\Gamma \cong \Gamma_u/N$.

Moreover

$$\begin{aligned} \pi(\mathrm{d}_u(a)) &= \pi(1 \otimes a - a \otimes 1) \\ &= 1 \cdot \mathrm{d}(a) \\ &= \mathrm{d}(a)\,, \end{aligned} \tag{3.6}$$

for every $a \in A$. □

3.1.1 The classification of first order H−covariant calculi

Here we introduce the notion of H−covariant differential calculi over A, where H is a Hopf algebra and A is an H−comodule algebra.

Definition 3.1.9. A first order differential calculus $(\Gamma), \mathrm{d})$ on a right H−comodule algebra (A, Δ_A) is called *right H−covariant* if Γ is a right H−covariant A−bimodule with coaction

$$\Delta_\Gamma : \Gamma \to \Gamma \otimes H$$

such that the differential $\mathrm{d} : A \to \Gamma$ is right H−colinear.

Similarly, it is called *left H−covariant* if Γ is a left H−covariant A−bimodule with coaction

$${}_\Gamma\Delta : \Gamma \to H \otimes \Gamma,$$

such that the differential $\mathrm{d} : A \to \Gamma$ is left H−colinear.

It is called *H−bicovariant* if its both left and right H−covariant.

Let us fix (Γ, d) a left covariant first order differential calculus over a Hopf algebra H. We denote the module of left coinvariant forms by

$$\Lambda^1 := \{\omega \in \Gamma : {}_\Gamma\Delta(\omega) = 1 \otimes \omega\}. \tag{3.7}$$

Definition 3.1.10. We define the *quantum Maurer-Cartan form* as the canonical $\mathbb{k}$-linear map from the kernel $H^+ = \ker \epsilon$ of the counit $\epsilon : H \to \mathbb{k}$ to the coinvariant forms as

$$\varpi : H^+ \to \Lambda^1, \quad h \mapsto S(h_1)\mathrm{d}h_2. \tag{3.8}$$

Lemma 3.1.11. *The quantum Maurer-Cartan form $\varpi : H^+ \to \Lambda^1$ is a surjective morphism and moreover $I = \ker \varpi \subseteq H$ is a right ideal.*

We will use this Lemma in the proof of the classification theorem due to Woronowicz, originally proven in [25]. This result provides a correspondence between right ideals $I \subseteq H$ that are in H^+ and left covariant first order differential calculi Γ over H. In particular explicit forms of the bimodule structure, the differential and left H−coaction are provided.

Proof. Let $\triangleright : H \otimes \Gamma \to \Gamma$ be the left module action of H on Γ. Considering a left-coinvariant form $\omega = a^i \mathrm{d}b^i$ we find

$$\begin{aligned} \triangleright \circ (S \otimes \mathrm{id}) \circ {}_\Gamma\Delta(\omega) &= \triangleright \circ (S \otimes \mathrm{id})(1 \otimes \omega) \\ &= \omega \end{aligned} \tag{3.9}$$

by left H−coinvariance.

Moreover

$$\begin{aligned} \triangleright \circ (S \otimes \mathrm{id}) \circ {}_\Gamma\Delta(\omega) &= \triangleright \circ (S \otimes \mathrm{id})(a^i_0 b^i_0 \otimes a^i_1 \mathrm{d}b^i_1) \\ &= \triangleright \circ (S(a^i_0 b^i_0) \otimes a^i_1 \mathrm{d}b^i_1) \\ &= S(b^i_0) S(a^i_0) a^i_1 \mathrm{d}b^i_1 \\ &= \epsilon(a^i) S(b^i_0) \mathrm{d}b^i_1 \\ &= \varpi(\epsilon(a^i) b^i - \epsilon(a^i b^i) 1_H), \end{aligned} \tag{3.10}$$

where we used that $\varpi(1_H) = 0$. Notice that every expression is well defined as $\epsilon(a^i)b^i - \epsilon(a^i b^i)1_H$ is in the kernel of ϵ, since ϵ is an algebra map. Accordingly ϖ maps onto left coinvariant forms. To show that $I = \ker\varpi$ is a right ideal we simply consider $\eta \in I$ and $h \in H$, to obtain

$$\begin{aligned} \varpi(\eta h) &= S(\eta_1 h_1)\,\mathrm{d}(\eta_2 h_2) \\ &= S(h_1)S(\eta_1)(\mathrm{d}\eta_2 h_2 + \eta_2 \mathrm{d}h_1) \\ &= S(h_1)\varpi(\eta)h_2 + S(h_1)\epsilon(\eta)\mathrm{d}h_2 \\ &= 0, \end{aligned} \tag{3.11}$$

therefore ηh is in I. □

Denote the quotient map by

$$\pi : H^+ \to H^+/I, \quad h \mapsto \pi(h) = [h]. \tag{3.12}$$

Theorem 3.1.12. *For any right ideal $I \subseteq H$ with $I \subseteq H^+$ we have a left covariant first order differential calculus (Γ, d) on H, where*

$$\Gamma = H \otimes (H^+/I), \quad \mathrm{d}h = (\mathrm{id}\otimes\pi)(\Delta(h) - h \otimes 1).$$

Γ is an $H-$bimodule via

$$h \cdot (h' \otimes [g]) = hh' \otimes [g], \quad (h \otimes [g]) \cdot h' = hh'_1 \otimes [gh'_2], \tag{3.13}$$

for $h, h' \in H$ and $g \in H^+$. The left $H-$coaction on Γ is ${}_\Gamma\Delta = \Delta \otimes \mathrm{id}_{H/I}$.

If $ad_R(I) \subseteq I \otimes H$, (Γ, d) is bicovariant with right $H-$coaction

$$\Delta_\Gamma(h \otimes [g]) = (h_1 \otimes [g_2]) \otimes h_2 S(g_1)g_3, \tag{3.14}$$

for $h \in H$ and $g \in H^+$. Moreover, every $H-$covariant first order differential calculus is of this form.

Proof. The bimodule relations do not depend upon the choice of a representative for a class under the projection map. Indeed

$$h \cdot (h' \otimes [g]) = h \cdot (h' \otimes [g']),$$

so we define the same element. The same for the other relation. Therefore Γ is an $H-$bimodule;in particular a right $H-$module.

Consider the given definition of the $\Bbbk-$linear map $\mathrm{d} : H \to \Gamma$. We have

$$\begin{aligned} \mathrm{d}(h)\,h' + h\,\mathrm{d}(h') &= (\mathrm{id}\otimes\pi)((h_1 \otimes h_2 - h \otimes 1) \cdot h' + h \cdot (h'_1 \otimes [h'_2] - h' \otimes 1) \\ &= hh'_1 \otimes [h_2 h'_2] - hh'_1 \otimes [h'_2] + hh'_1 \otimes [h'_2] - hh' \otimes [1] \\ &= hh'_1 \otimes [h_2 h'_2] - hh' \otimes [1] \\ &= \mathrm{d}(hh') \end{aligned} \tag{3.15}$$

for all $h, h' \in H$.

Let $h^i \otimes [g^i] \in \Gamma$ be arbitrary, with $h^i \in H$ and $g^i \in H^+$. Consider the combination $h^i S(g^i_1)\,\mathrm{d}(g^i_2)$, defining an element of the form $H\mathrm{d}H$. We find

$$\begin{aligned} h^i S(g^i_1)\,\mathrm{d}(g^i_2) &= h^i S(g^i_1)g^i_2 \otimes [g^i_3] - h^i S(g^i_1)g^i_2 \otimes [1] \\ &= h^i \otimes [g^i] - h^i \epsilon(g^i) \otimes [1] \\ &= h^i \otimes [g^i]. \end{aligned} \tag{3.16}$$

Surjectivity follows.

The map ${}_\Gamma\Delta : \Gamma \to H \otimes \Gamma$ is a left H−coaction, as it is defined by Δ and the identity; the diagram

$$\begin{array}{ccc} H \otimes \Gamma & \xrightarrow{\Delta \otimes \mathrm{id}_\Gamma} & H \otimes H \otimes \Gamma \\ \uparrow {\scriptstyle {}_\Gamma\Delta} & & \uparrow {\scriptstyle \mathrm{id}_H \otimes {}_\Gamma\Delta} \\ \Gamma & \xrightarrow{{}_\Gamma\Delta} & H \otimes \Gamma \end{array}$$

commutes as

$$\begin{aligned} (\mathrm{id}_H \otimes {}_\Gamma\Delta)(\Delta \otimes \mathrm{id}_\Gamma)(h \otimes [g]) &= (\mathrm{id}_H \otimes {}_\Gamma\Delta)(h_1 \otimes h_2 \otimes [g]) \\ &= (\mathrm{id}_H \otimes \Delta \otimes \mathrm{id}_\Gamma)(h_1 \otimes h_2 \otimes [g]) \\ &= h_1 \otimes h_2 \otimes h_3 \otimes [g] \\ &= (\Delta \otimes \mathrm{id}_\Gamma)(h_1 \otimes h_2 \otimes [g]) \\ &= (\Delta \otimes \mathrm{id}_\Gamma){}_\Gamma\Delta(h \otimes [g]). \end{aligned} \tag{3.17}$$

Compatibility with the H−bimodule structure follows

$$\begin{aligned} {}_\Gamma\Delta(h \cdot (h' \otimes [g]) \cdot h'') &= {}_\Gamma\Delta(hh'h''_1 \otimes [gh''_2]) \\ &= h_1h'_1h''_1 \otimes h_2h'_2h''_2 \otimes [gh''_3] \\ &= \Delta(h) \cdot {}_\Gamma\Delta(h' \otimes [g]) \cdot \Delta(h''), \end{aligned} \tag{3.18}$$

for all $h, h', h'' \in H$ and $g \in H^+$.

Left H−colinearity of the differential follows as

$$\begin{aligned} (\mathrm{id} \otimes \mathrm{d})\Delta(h) &= h_1 \otimes \mathrm{d}(h_2) \\ &= h_1 \otimes h_2 \otimes [h_3] - h_1 \otimes h_2 \otimes [1] \\ &= {}_\Gamma\Delta(\mathrm{d}(h)). \end{aligned} \tag{3.19}$$

Accordingly, (Γ, d) is a left covariant first order differential calculus.

If in addition $\mathrm{ad}_R(I) \subseteq I \otimes H$, the map $\Delta_\Gamma : \Gamma \to \Gamma \otimes H$ is well defined and moreover a right H−coaction:

$$\begin{aligned} (\Delta_\Gamma \otimes \mathrm{id})(\Delta_\Gamma(h \otimes [g]) &= \Delta_\Gamma(h_1 \otimes [g_2]) \otimes h_2S(g_1)g_3 \\ &= h_1 \otimes [g_3] \otimes h_2S(g_2)g_4 \otimes h_3S(g_1)g_5 \\ &= h_1 \otimes [g_2] \otimes (h_2S(g_1)g_3)_1 \otimes (h_2S(g_1)g_3)_2 \\ &= (\mathrm{id} \otimes \Delta)(\Delta_\Gamma(h \otimes [g])), \end{aligned} \tag{3.20}$$

for all $h \in H$ and $g \in H^+$. Proofs of compatibility of Δ_Γ with the bimodule structure and right H−colinearity of the differential are similar. We have shown that given a right ideal $I \subseteq H$ with $I \subseteq H^+$ we can construct (Γ, d) a bicovariant first order calculus on H.

On the other hand, given a first order differential calculus we construct a right ideal $I \subseteq H$ as the kernel of ϖ according to Lemma 3.1.11

For the $1:1$−correspondence we prove that, given a left covariant first order differential calculus Γ, we obtain $\Gamma \cong H \otimes H^+/I$ as left covariant first order differential calculi.

The first thing we prove is the isomorphism $\Gamma \cong H \otimes H^+/I$ as left covariant H−bimodules. Consider

$$\phi : \Gamma \to H \otimes H^+/I, \quad \omega \mapsto \omega_{-2} \otimes \varpi^{-1}(S(\omega_{-1})\omega_0). \tag{3.21}$$

Here ϖ^{-1} is the inverse of the quantum Maurer-Cartan form restricted as $\varpi : H^+/\ker\varpi \to \Lambda^1$, which is an isomorphism by Lemma 3.1.11. Notice that $S(\omega_{-1})\omega_0$ is a left coinvariant form.

We provide the inverse of ϕ as

$$\phi^{-1} : H \otimes H^+/I \to \Gamma, \quad h \otimes [g] \mapsto h\varpi(g). \tag{3.22}$$

This map is H−bilinear and left H−colinear, indeed

$$\begin{aligned}\phi^{-1}(h \cdot (h' \otimes [g]) \cdot h'') &= \phi^{-1}(hh'h''_1 \otimes [gh''_2]) \\ &= hh'h''_1 S(g_1 h''_2)\,\mathrm{d}\big(g_2 h''_3\big) \\ &= hh'h''_1 S(h''_2) S(g_1)\,\mathrm{d}\big(g_2 h''_3\big) \\ &= hh'S(g_1)(\mathrm{d}g_2 h'' + g_2 \mathrm{d}h'') \\ &= hh'\varpi(h)h'' + hh'\mathrm{d}h''\epsilon(g) \\ &= h \cdot \phi^{-1}(h' \otimes [g]) \cdot h'',\end{aligned} \tag{3.23}$$

for all $h, h', h'' \in H$ and $g \in H^+$.

Furthermore

$$\begin{aligned}(h \otimes [g])_{-1} \otimes \phi^{-1}((h \otimes [g])_0) &= h_1 \otimes \phi^{-1}(h_2 \otimes [g]) \\ &= h_1 \otimes h_2\varpi(g) \\ &= (h\varpi(g))_{-1} \otimes (h\varpi(g))_0 \\ &= \phi^{-1}(h \otimes [g])_{-1} \otimes \phi^{-1}(h \otimes [g])_0.\end{aligned} \tag{3.24}$$

The last thing to show is the relation between the differential on Γ and the one on $H \otimes H^+/I$. We have

$$\begin{aligned}\phi^{-1}(\mathrm{d}_I(h)) &= \phi^{-1}(h_1 \otimes [h_2] - h \otimes [1]) \\ &= h_1\varpi(h_2) - h\varpi(1) \\ &= h_1 S(h_2)\,\mathrm{d}(h_3) - h\,\mathrm{d}(1) \\ &= \mathrm{d}h,\end{aligned} \tag{3.25}$$

proving that ϕ is an isomorphism of left covariant first order differential calculi.

If we furthermore assume the calculus $(H \otimes H^+/I, \mathrm{d}_I)$ to be bicovariant, the ideal I satisfies $\Delta_\Gamma(H \otimes I) \subseteq H \otimes I \otimes H$, in particular

$$\Delta_\Gamma(1 \otimes g) = 1 \otimes g_2 \otimes S(g_3)g_1 \in H \otimes I \otimes H,$$

for all $g \in I$, namely $\mathrm{ad}_R(I) \subseteq I \otimes H$. In this case the isomorphism is a right H−covariant map, since

$$\begin{aligned}\phi^{-1}((h \otimes [g]_0 \otimes (h \otimes [g])_1) &= \phi^{-1}(h_1 \otimes [g_2]) \otimes h_2 S(g_1)g_3 \\ &= h_1\varpi(g_2) \otimes h_2 S(g_1)g_3 \\ &= h_1 S(g_2)\,\mathrm{d}(g_3) \otimes h_2 S(g_1)g_4 \\ &= (hS(g_1)\,\mathrm{d}(g_2))_0 \otimes (hS(g_1)\,\mathrm{d}(g_2))_1 \\ &= \phi^{-1}(h \otimes [g])_0 \otimes \phi^{-1}(h \otimes [g])_1,\end{aligned} \tag{3.26}$$

for every $h \in H$ and every $g \in H^+$ by the right H−covariance of d. □

3.2 Higher order differential calculi

In this section we introduce higher order differential calculi as differential graded algebras with a surjectivity requirement.

Definition 3.2.1 (Differential graded algebra)**.** A differential $\mathbb{N}_0$−graded algebra $(\Omega^\bullet, \wedge, \mathrm{d})$ is a $\mathbb{N}_0$−graded vector space $\Omega^\bullet = \bigoplus_{k \geq 0} \Omega^k$ endowed with

1. a map $\wedge : \Omega^\bullet \otimes \Omega^\bullet \to \Omega^\bullet$ of degree zero, i.e. $\Omega^k \wedge \Omega^\ell \subseteq \Omega^{k+\ell}$,
2. a map $\mathrm{d} : \Omega^\bullet \to \Omega^{\bullet+1}$ of degree 1, meaning $\mathrm{d}(\Omega^k) \subseteq \Omega^{k+1}$,

such that the *wedge product* $\wedge$ is associative and unital and the differential satisfies $\mathrm{d}^2 = 0$ and the graded Leibniz rule

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^{|\omega|}\omega \wedge \mathrm{d}\eta$$

for all $\eta \in \Omega^\bullet$ and homogeneous[1] elements ω.

Definition 3.2.2. Let $(\Omega^\bullet(A), \mathrm{d}_A, \wedge_A)$ over A and $(\Omega^\bullet(B), \mathrm{d}_B, \wedge_B)$ over B be differential graded algebras. A morphism of differential graded algebras is a morphism of graded algebras $\Phi : \Omega^\bullet(A) \to \Omega^\bullet(B)$ that respects the differential.

We say that a morphism of differential graded algebras $\Phi : \Omega^\bullet(A) \to \Omega^\bullet(B)$ is an extension of a morphism of algebras $\phi : A \to B$ if $\Phi|_A = \phi$.

Proposition 3.2.3. *Let $(\Omega^\bullet(A), \mathrm{d}_A, \wedge_A)$ and $(\Omega^\bullet(B), \mathrm{d}_B, \wedge_B)$ be differential graded algebras. Let $\Phi : \Omega^\bullet(A) \to \Omega^\bullet(B)$ be a morphism of differential graded algebras such that $\Phi|_A = \phi : A \to B$. We have that Φ is unique as an extension of ϕ. Moreover if ϕ is surjective then Φ is.*

Proof. Since the involved operations are linear it is enough to consider a homogeneous element $\omega = a^0\mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k \in \Omega^k(A)$, for $a^0, \ldots, a^k \in A$. We find

$$\begin{aligned} \Phi(a^0\mathrm{d}_A a^1 \wedge_A \cdots \wedge_A \mathrm{d}_A a^k) &= \Phi(a^0)\Phi(\mathrm{d}_A a^1) \wedge_B \cdots \wedge_B \Phi(\mathrm{d}_A a^k) \\ &= \phi(a^0)\mathrm{d}_B(\phi(a^1)) \wedge_B \cdots \wedge_B \mathrm{d}_B(\phi(a^k)), \end{aligned} \tag{3.27}$$

where we only used that Φ is a morphism of differential graded algebra. Accordingly we find that Φ is the unique extension of ϕ. Moreover, let ϕ be surjective and let $b^0 = \phi(a^0), \ldots, b^k = \phi(a^k) \in B$ for some $a^0, \ldots, a^k \in A$. Then by equation (3.27) we have that Φ is surjective, since

$$\begin{aligned} b^0\mathrm{d}_B b^1 \wedge_B \cdots \wedge_B \mathrm{d}_B b^k &= \phi(a^0)\mathrm{d}_B(\phi(a^1)) \wedge_B \cdots \wedge_B \mathrm{d}_B\phi(a^k) \\ &= \Phi(a^0)\Phi(\mathrm{d}_A a^1) \wedge_B \cdots \wedge_B \Phi(\mathrm{d}_A a^k) \\ &= \Phi(a^0\mathrm{d}_A a^1 \wedge_A \cdots \wedge_A \mathrm{d}_A a^k). \end{aligned} \tag{3.28}$$

□

Definition 3.2.4 (Differential calculus)**.** A differential calculus on a $\mathbb{k}$–algebra A is a differential $\mathbb{N}_0$–graded algebra $(\Omega^\bullet, \wedge, \mathrm{d})$ which is generated in degree zero and such that $\Omega^0 = A$. The former means that

$$\Omega^k = \mathrm{span}_{\mathbb{k}}\{a^0\mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k : a^0, \ldots, a^k \in A\}.$$

We call elements of Ω^k differential k–forms.

Lemma 3.2.5. *Let $(\Omega^\bullet(A), \mathrm{d}_A, \wedge_A)$ be a differential calculus over A and let $(\Omega^\bullet(B), \mathrm{d}_B, \wedge_B)$ be a differential graded algebra. A map $\Phi : \Omega^\bullet(A) \to \Omega^\bullet(B)$ is a morphism of differential graded algebras if and only if*

$$\Phi(a^0\mathrm{d}_A a^1 \wedge_A \cdots \wedge_A \mathrm{d}_A a^k) = \Phi(a^0) \wedge_B \mathrm{d}_B \circ \Phi(a^1) \wedge_B \cdots \wedge_B \mathrm{d}_B \circ \Phi(a^k), \tag{3.29}$$

for all $a^0, ..., a^k \in A$, with $k \geq 0$.

[1] With homogeneous we simply mean an element that belongs to a fixed vector space in the grading. $|\omega|$ is the degree of ω.

Proof. If Φ is a morphism of differential graded algebras then equation (3.29) is clear. On the other hand assume equation (3.29) to be true. Let $a^0 \mathrm{d}_A a^1 \wedge_A \cdots \wedge_A \mathrm{d}a^k \in \Omega^k(A)$ and $b^0 \mathrm{d}_A b^1 \wedge_A \cdots \wedge_A \mathrm{d}_A b^\ell \in \Omega^\ell(A)$. Then

$$\begin{aligned}\Phi[(a^0 \mathrm{d}_A a^1 \wedge_A \cdots \wedge_A \mathrm{d}a^k) \wedge_A (b^0 \mathrm{d}_A b^1 \wedge_A \cdots \wedge_A \mathrm{d}_A b^\ell)]&\\ = [\Phi(a^0)\mathrm{d}_B \Phi(a^1) \wedge_B \cdots \wedge_B \mathrm{d}_B \Phi(a^k)] \wedge_B [\Phi(b^0)\mathrm{d}_B \Phi(b^1) \wedge_B \cdots \wedge_B \mathrm{d}_B \Phi(b^\ell)]&\\ = \Phi(a^1 \mathrm{d}_A a^1 \wedge_A \cdots \wedge_A \mathrm{d}_A a^k) \wedge_B \Phi(b^1 \mathrm{d}_A b^1 \wedge_A \cdots \wedge_A \mathrm{d}_A b^\ell).&\end{aligned}$$

Moreover, Φ is compatible with the differential, as

$$\begin{aligned}\Phi(\mathrm{d}_A \omega) &= \Phi(\mathrm{d}_A(a^0 \mathrm{d}_A a^1 \wedge \cdots \mathrm{d}_A a^k)\\ &= \Phi(\mathrm{d}_A a^0 \wedge \cdots \mathrm{d}_A a^k)\\ &= \mathrm{d}_B \phi(a^0) \wedge_B \cdots \wedge_B \mathrm{d}_B \phi(a^k)\\ &= \mathrm{d}_B b^0 \wedge_B \cdots \wedge_B \mathrm{d}_B b^k\end{aligned} \tag{3.30}$$

equals

$$\begin{aligned}\mathrm{d}_B \Phi(\omega) &= \mathrm{d}_B \Phi(a^0 \mathrm{d}_A a^1 \wedge_A \cdots \wedge_A \mathrm{d}_A a^k)\\ &= \mathrm{d}_B(b^0 \mathrm{d}_B b^1 \wedge_B \cdots \wedge_B \mathrm{d}_B b^k)\\ &= \mathrm{d}b^0 \wedge_B \cdots \wedge_B \mathrm{d}_B b^k.\end{aligned} \tag{3.31}$$

□

Example 3.2.6 (Trivial DC)**.** On every algebra A there is a trivial DC $\Omega^k = 0$ for all $k > 0$ with the zero differential.

Example 3.2.7. The universal first order differential calculus of Example 3.1.7 can be extended to higher order forms, defining $\Omega_u^n := \cap_{k=1}^n \ker \mu_k$, where $\mu_k := \mathrm{id}_{A^{\otimes(k-1)}} \otimes \mu \otimes \mathrm{id}_{A^{\otimes(n-k)}} : A^{\otimes(n+1)} \to A^{\otimes n}$. The differential is

$$\mathrm{d}_u : \Omega_u^k \to \Omega_u^{k+1}, \quad a^0 \otimes \cdots \otimes a^k \mapsto \sum_{i=1}^{k+1} a^0 \otimes \cdots \otimes a^{i-1} \otimes 1 \otimes a^{i+1} \otimes \cdots \otimes a^k. \tag{3.32}$$

The wedge product $\wedge_u : \Omega_u^k \otimes \Omega_u^\ell \to \Omega^{k+\ell}$ is defined on a k–form ω and a ℓ–form η as

$$\omega \wedge_u \eta = a^0 \otimes \cdots \otimes a^{k-1} \otimes a^k b^0 \otimes b^2 \cdots \otimes b^\ell, \tag{3.33}$$

i.e. as the concatenation of forms.

Lemma 3.2.8. *The universal differential calculus* $(\Omega_u^\bullet, \wedge_u, \mathrm{d}_u)$ *is a differential calculus on* A.

Proof. During the proof we will denote d_u as d and $\wedge_u$ as $\wedge$ for simplicity. The A–bimodule structure on $\Omega_u^\bullet$ is the one induced from $A \otimes A$, i.e. the multiplication on the first tensor factor from the left and the second tensor factor from the right, respectively. The squared differential vanishes, as on the order 0 we find

$$\begin{aligned}\mathrm{d}^2(a) &= \mathrm{d}(1 \otimes a - a \otimes 1)\\ &= \mathrm{d}(1 \otimes a) - \mathrm{d}(a \otimes 1)\\ &= 1 \otimes 1 \otimes a - 1 \otimes 1 \otimes a + 1 \otimes a \otimes 1\\ &\quad - (1 \otimes a \otimes 1 - a \otimes 1 \otimes 1 + a \otimes 1 \otimes 1)\\ &= 0,\end{aligned} \tag{3.34}$$

and moreover at higher orders we find

$$\begin{aligned}
\mathrm{d}^2(a^0 \otimes \cdots \otimes a^k) &= \mathrm{d}\left(\sum_{i=0}^{k+1} a^0 \otimes \cdots \otimes a^{i-1} \otimes 1 \otimes a^i \otimes \cdots \otimes a^k\right) \\
&= \mathrm{d}\Big(1 \otimes a^0 \otimes \cdots \otimes a^k\Big) - \mathrm{d}\Big(a^0 \otimes 1 \otimes \cdots \otimes a^k\Big) + \cdots + (-1)^k \,\mathrm{d}\Big(a^0 \otimes \cdots \otimes a^k \otimes 1\Big) \\
&= 1 \otimes 1 \otimes a^0 \otimes \cdots \otimes a^k - 1 \otimes 1 \otimes a^0 \otimes \cdots \otimes a^k + 1 \otimes a^0 \otimes 1 \otimes \cdots \otimes a^k + \ldots \\
&- 1 \otimes a^0 \otimes 1 \otimes \cdots \otimes a^k + a^0 \otimes 1 \otimes 1 \otimes \cdots \otimes a^k + \ldots \\
&= 0.
\end{aligned} \tag{3.35}$$

We already know the Leibniz rule holds at order zero, namely on elements of A. For elements in Γ_u we find

$$\begin{aligned}
\mathrm{d}(\omega \wedge \gamma) &= \mathrm{d}\Big((a^0 \otimes a^1) \wedge (b^0 \otimes b^1)\Big) \\
&= \mathrm{d}\Big(a^0 \otimes a^1 b^0 \otimes b^1\Big) \\
&= 1 \otimes a^0 \otimes a^1 b^0 \otimes b^1 - a^0 \otimes 1 \otimes a^1 b^0 \otimes b^1 \\
&\quad + a^0 \otimes a^1 b^0 \otimes 1 \otimes b^1 - a^0 \otimes a^1 b^0 \otimes b^1 \otimes 1 \\
&= (1 \otimes a^0 \otimes a^1 - a^0 \otimes 1 \otimes a^1 + a^0 \otimes a^1 \otimes 1) \wedge (b^0 \otimes b^1) \\
&- (a^0 \otimes a^1) \wedge (1 \otimes b^0 \otimes b^1 - b^0 \otimes 1 \otimes b^1 + b^0 \otimes b^1 \otimes 1) \\
&= \mathrm{d}\omega \wedge \gamma - \omega \wedge \mathrm{d}\gamma,
\end{aligned} \tag{3.36}$$

and we can infer that the rule continues to hold at higher orders. Finally surjectivity of the calculus follows as any element $a^0 \otimes \cdots \otimes a^n$ can be obtained as

$$\begin{aligned}
a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^n &= a^0(1 \otimes a^1 - a^1 \otimes 1) \wedge \cdots \wedge (1 \otimes a^n - a^n \otimes 1) \\
&= (a^0 \otimes a^1) \wedge (1 \otimes a^2 - a^2 \otimes 1) \wedge \cdots \wedge (1 \otimes a^n - a^n \otimes 1) \\
&= (a^0 \otimes a^1 \otimes a^2)(1 \otimes a^3 - a^3 \otimes 1) \wedge \cdots \wedge (1 \otimes a^n - a^n \otimes 1) \\
&= \cdots \\
&= a^0 \otimes \cdots \otimes a^n.
\end{aligned} \tag{3.37}$$

Therefore we have $(\Omega^\bullet_u, \wedge_u, \mathrm{d}_u)$ is a differential calculus on A. □

Definition 3.2.9. A morphism from a differential calulus $(\Omega^\bullet, \wedge, \mathrm{d})$ over an algebra A to another differential calculus $(\Omega'^\bullet, \wedge', \mathrm{d}')$ over an algebra A' is a morphism of the underlying differential $\mathbb{N}_0$−graded algebras. Explicitly, it is a graded map $\Phi : \Omega^\bullet \to \Omega'^\bullet$ of degree 0, such that

$$\Phi(\omega \wedge \eta) = \Phi(\omega) \wedge \Phi(\eta), \quad \Phi \circ \mathrm{d} = \mathrm{d}' \circ \Phi, \tag{3.38}$$

for all $\omega, \eta \in \Omega^\bullet$. An isomorphism of differential calculi is a morphism of differential calculi which is also invertible.

3.2.1 Higher order H−covariant calculi

In the following we discuss a special class differential calculi. Those are differential calculi on H−comodule algebras such that the coaction *lifts* to graded algebra of forms and the differential is colinear, and are called differential covariant calculi. A $\mathbb{N}_0$−graded $\Bbbk$−vector space $V^\bullet = \bigoplus_{k \geq 0} V^k$ is a graded right H−comodule if there is a right H−coaction $\Delta^\bullet : V^\bullet \to V^\bullet \otimes H$ which is graded of degree 0[2]. It follows that for any $k \geq 0$ the vector space V^k is a right H−comodule via

$$\Delta^k := \Delta^\bullet|_{V^k} : V^k \to V^k \otimes H;$$

[2]Meaning that $\Delta^\bullet(V^k) \subseteq V^k \otimes H$ for all positive k's.

accordingly we write $\Delta^\bullet = \oplus_{k\geq 0}\Delta^k$.

A graded map $\phi : V^\bullet \to W^{\bullet+\ell}$ of degree ℓ between graded right H–comodules $(V^\bullet, \Delta_V^\bullet)$ and $(W^\bullet, \Delta_W^\bullet)$ is right H–colinear if the diagram

$$\begin{array}{ccc} V^k & \xrightarrow{\ \phi^k\ } & W^{k+\ell} \\ {\scriptstyle \Delta_V^k}\downarrow & & \downarrow{\scriptstyle \Delta_W^{k+\ell}} \\ V^k \otimes H & \xrightarrow[\ \phi^k\otimes \mathrm{id}_H\ ]{} & W^{k+\ell} \otimes H \end{array}$$

commutes for all $k \geq 0$, where here $\phi^k = \phi|_{V^k} : V^k \to W^{k+\ell}$.

In case $V^\bullet$ is a graded algebra we call it a *graded right H–comodule algebra* if it is a graded right H–comodule and the algebra structure is compatible with the right H–coaction in the usual sense.

Definition 3.2.10. A differential calculus over a right H–comodule algebra (A, Δ_A) is called *right H-covariant* if $(\Omega^\bullet, \wedge)$ is a graded right H–comodule algebra with graded coaction $\Delta^\bullet : \Omega^\bullet \to \Omega^\bullet \otimes H$ such that $\Delta^0 = \Delta_A : A \to A \otimes H$ and the differential $\mathrm{d} : \Omega^\bullet \to \Omega^{\bullet+1}$ is right H–colinear. Similarly, left covariant and bicovariant calculi are defined.

If we consider H as an H–comodule algebra with corresponding coaction given by the coproduct, we call the corresponding H–covariant calculus *covariant*.

Proposition 3.2.11. *A differential calculus $(\Omega^\bullet, \wedge, \mathrm{d})$ on a right H–comodule algebra A is right H–covariant if and only if one of the following conditions holds:*

1. *for all $k > 0$ and $a^{i,0}, \ldots, a^{i,k} \in A$ such that $\sum_i a^{i,0}\mathrm{d}a^{i,1} \wedge \cdots \wedge \mathrm{d}a^{i,k} = 0$ we have*
$$\sum_i a_0^{i,0}\mathrm{d}a_0^{i,1} \wedge \cdots \wedge \mathrm{d}a_0^{i,k} \otimes a_1^{i,0}a_1^{i,1} \ldots a_1^{i,k} = 0;$$

2. *the right H–coaction $\Delta_A : A \to A \otimes H$ is k–times differentiable for all $k > 0$, namely*
$$\begin{array}{ccccccc} A \otimes H & \xrightarrow{\ \mathrm{d}_\otimes\ } & \Omega^1 \otimes H & \xrightarrow{\ \mathrm{d}_\otimes\ } & \ldots & \xrightarrow{\ \mathrm{d}_\otimes\ } & \Omega^k \otimes H \\ {\scriptstyle \Delta_A}\uparrow & & \uparrow{\scriptstyle \Delta_{\Omega^1(A)}} & & & & \uparrow{\scriptstyle \Delta_{\Omega^k(A)}} \\ A & \xrightarrow{\ \mathrm{d}\ } & \Omega^1 & \xrightarrow{\ \mathrm{d}\ } & \ldots & \xrightarrow{\ \mathrm{d}\ } & \Omega^k \end{array}$$
commute, and for all $k > 0$ the map $\Delta_{\Omega^k(A)} : \Omega^k \to \Omega^k \otimes H$ is left A–linear, in the sense that $\Delta_{\Omega^k(A)}(a \cdot \omega) = \Delta_A(a)\Delta_{\Omega^k(A)}(\omega)$ for all $a \in A$ and $\omega \in \Omega^k$.

In this case the right H–coactions $\Delta_{\Omega^k(A)} : \Omega^k \to \Omega^k \otimes H$, $k > 0$ are explicitely given as

$$\Delta_{\Omega^k(A)}\left(\sum_i a^{i,0}\mathrm{d}a^{i,1} \wedge \cdots \wedge \mathrm{d}a^{i,k}\right) = \sum_i a_0^{i,0}\mathrm{d}a_0^{i,1} \wedge \cdots \wedge \mathrm{d}a_0^{i,k} \otimes a_1^{i,0}a_1^{i,1} \ldots a_1^{i,k}$$

for all $a^{i,0}, \ldots, a^{i,k} \in A$.

Proof. Let $(\Omega^\bullet, \wedge, \mathrm{d})$ be a differential calculus on a right H–comodule algebra (A, Δ_A). First we prove that 1. implies right H–covariance.

For $k > 0$ define a $\Bbbk$–linear map

$$\Delta_{\Omega^k(A)} : \Omega^k(A) \to \Omega^k(A) \otimes H, \quad \sum_i a^{i,0}\mathrm{d}a^{i,1} \wedge \cdots \wedge \mathrm{d}a^{i,k} \mapsto \sum_i a_0^{i,0}\mathrm{d}a_0^{i,1} \wedge \cdots \wedge \mathrm{d}a_0^{i,k} \otimes a_1^{i,0}a_1^{i,1} \ldots a_1^{i,k}.$$

The assumption of point 1 ensures this expression is well-defined, since it is zero whenever the input object is zero.

Moreover $\Delta_{\Omega^k(A)} : \Omega^k(A) \to \Omega^k(A) \otimes H$ is a right H−coaction:

$$\begin{aligned}(\Delta_{\Omega^k(A)} \otimes \mathrm{id})\Delta_{\Omega^k(A)}(\omega) &= \sum_i (\Delta_{\Omega^k(A)} \otimes \mathrm{id})(a_0^{i,0}\mathrm{d}a_0^{i,1} \wedge \cdots \wedge \mathrm{d}a_0^{i,k} \otimes a_1^{i,0} \ldots a_1^{i,k}) \\ &= \sum_i a_0^{i,0}\mathrm{d}a_0^{i,1} \wedge \cdots \wedge \mathrm{d}a_0^{i,k} \otimes a_1^{i,0} \ldots a_1^{i,k} \otimes a_2^{i,0} \ldots a_2^{i,k} \\ &= \sum_k (\mathrm{id} \otimes \Delta)(a_0^{i,0}\mathrm{d}a_0^{i,1} \wedge \cdots \wedge \mathrm{d}a_0^{i,k} \otimes a_1^{i,0} \ldots a_1^{i,k}) \\ &= (\mathrm{id} \otimes \Delta)\Delta_{\Omega^k(A)}(\omega),\end{aligned} \tag{3.39}$$

for every $\omega \in \Omega^k$.

Consequently, $\Omega^\bullet$ is a graded right H−comodule with respect to $\Delta_{\Omega^\bullet(A)} = \Delta_A \oplus \bigoplus_{k>0} \Delta_{\Omega^k(A)}$.

The differential $\mathrm{d}|_{\Omega^k} : \Omega^k(A) \to \Omega^{k+1}(A)$ is right H−colinear for every $k \geq 0$. Indeed, let $\omega \in \Omega^k$. By the Leibniz rule and $\mathrm{d}^2 = 0$ we have

$$\mathrm{d}(\omega) = \sum_i \mathrm{d}a^{i,0} \wedge \mathrm{d}a_0^{i,1} \wedge \cdots \wedge \mathrm{d}a_0^{i,k}.$$

Therefore

$$\begin{aligned}\Delta_{\Omega^{k+1}(A)} \circ \mathrm{d}(\omega) &= \sum_i \Delta^{k+1}(\mathrm{d}a^{i,0} \wedge \cdots \wedge \mathrm{d}a^{i,k}) \\ &= \sum_i \mathrm{d}a_0^{i,0} \wedge \cdots \wedge \mathrm{d}a_0^{i,k} \otimes a_1^{i,0} \ldots a_1^{i,k} \\ &= \sum_i (\mathrm{d} \otimes \mathrm{id}_H)(a_0^{i,0}\mathrm{d}a_0^{i,0} \wedge \cdots \wedge \mathrm{d}a_0^{i,k} \otimes a_1^{i,0} \ldots a_1^{i,k}) \\ &= (\mathrm{d} \otimes \mathrm{id}_H) \circ \Delta_{\Omega^k(A)}(\omega).\end{aligned} \tag{3.40}$$

To prove $\Delta_{\Omega^\bullet(A)}$ is a graded algebra map we show $\Delta_{\Omega^\bullet(A)}(\omega \wedge \eta) = \Delta_{\Omega^\bullet(A)}(\omega)(\wedge \otimes \mu)\Delta_{\Omega^\bullet(A)}(\eta)$ for all $\omega \in \Omega^k(A)$ and $\eta \in \Omega^\ell(A)$. For $k = 0$ this is automatic, since Δ_A is an algebra map. For $k = 1$, using the Leibniz rule, we find

$$\begin{aligned}&\Delta_{\Omega^{1+\ell}(A)}\left((\sum_i a^{i,0}\mathrm{d}a^{i,1}) \wedge (\sum_j b^{j,0}\mathrm{d}b^{j,1} \wedge \cdots \wedge \mathrm{d}b^{j,\ell})\right) \\ &= \sum_{i,j} \Delta^{1+\ell}\left(a^{i,0}\,\mathrm{d}\left(a^{i,1}b^{j,0}\right) \wedge \mathrm{d}b^{j,1} \wedge \cdots \wedge \mathrm{d}b^{j,\ell}\right) - \sum_{i,j} \Delta^{1+\ell}\left(a^{i,0}a^{i,1}\mathrm{d}b^{j,0} \wedge \mathrm{d}b^{j,1} \wedge \cdots \wedge \mathrm{d}b^{j,\ell}\right) \\ &= \sum_{i,j}\left(a_0^{i,0}\,\mathrm{d}\left(a_0^{i,1}b^{j,0}\right) \wedge \mathrm{d}b_0^{j,1} \wedge \cdots \wedge \mathrm{d}b_0^{j,1} - a_0^{i,0}a_0^{i,1}\mathrm{d}b^{j,0} \wedge \cdots \wedge \mathrm{d}b^{j,\ell}\right) \otimes a_1^{i,0}a_1^{i,1}b_1^{j,0}b_1^{j,1} \ldots b_1^{j,\ell} \\ &= \sum_{i,j} a_0^{i,0}\mathrm{d}a_0^{i,1} \wedge (b_0^{j,0}\mathrm{d}b_0^{j,1}) \wedge \cdots \wedge \mathrm{d}b_0^{j,1} \otimes a_1^{i,0}a_1^{i,1}b_1^{j,0}b_1^{j,1} \ldots b_1^{j,\ell} \\ &= \Delta_{\Omega^1(A)}\left(\sum_i a^{i,0}\mathrm{d}a^{i,1}\right)\Delta_{\Omega^\ell(A)}\left(\sum_j b^{j,0}\mathrm{d}b^{j,1} \wedge \cdots \wedge \mathrm{d}b^{j,\ell}\right)\end{aligned} \tag{3.41}$$

holds.

The cases $k > 1$ are proven by the same argumentation above by rewriting $\omega \wedge \eta$ in the form $\sum_i c^{i,0}\mathrm{d}c^{i,1} \wedge \cdots \wedge \mathrm{d}c^{i,k+\ell}$ for some $c^{i,0}, \ldots, c^{i,k+\ell}$s in A. Thus, we have $(\Omega^\bullet, \wedge, \mathrm{d})$ is a right H−covariant differential calculus.

On the other hand assuming the differential calculus to be right H−covariant we want to show the assumption 1. holds. Consider $\omega \in \Omega^k$ and apply $\Delta_{\Omega^k(A)}$ to obtain

$$\begin{aligned}\Delta_{\Omega^k(A)}\left(\sum_i a^{i,0}\mathrm{d}a^{i,1}\wedge\cdots\wedge\mathrm{d}a^{i,k}\right) &= \sum_i \Delta_A(a^{i,0})\Delta_{\Omega^1(A)}(\mathrm{d}a^{i,1})\cdots\Delta_{\Omega^1(A)}(\mathrm{d}a^{i,k})\\ &= \sum_i \Delta_A(a^{i,0})(\mathrm{d}\otimes\mathrm{id})\Delta_A(a^{i,1})\cdots(\mathrm{d}\otimes\mathrm{id})\Delta_A(a^{i,k}) \qquad (3.42)\\ &= \sum_i a_0^{i,0}\mathrm{d}a_0^{i,1}\wedge\cdots\wedge\mathrm{d}a_0^{i,k}\otimes a_1^{i,0}a_1^{i,1}\dots a_1^{i,k},\end{aligned}$$

since $\Delta_{\Omega^\bullet(A)}$ is an algebra map and d is colinear. In particular $\omega = 0$ implies $\Delta^k(\omega) = 0$ and thus we have the equivalence of point 1. with right $H-$covariance of the differential calculus.

For the point 2., if the differential calculus is right $H-$covariant with respect to the graded right $H-$coaction $\Delta_{\Omega^\bullet(A)} : \Omega^\bullet \to \Omega^\bullet \otimes H$ we have $\Delta_{\Omega^k(A)}$ is differentiable with $\mathrm{d}\Delta_{\Omega^k(A)} = \Delta_{\Omega^{k+1}(A)}$. By assumption $\Delta_{\Omega^\bullet(A)}$ is an algebra map and thus $\Delta_{\Omega^k(A)}(a\cdot\omega) = \Delta_A(a)\Delta_{\Omega^k(A)}(\omega)$ for every $a \in A$ and $\omega \in \Omega^k$, proving the first implication.

On the other hand by left linearity it is straightforward to prove $\Delta_{\Omega^k(A)}$ is of the required form for all positive values of k. Therefore, as the assumption of 1. are satisfied we have right $H-$covariance of the calculus. □

3.3 The maximal prolongation

Let us consider an associative unital algebra A over a field $\Bbbk$. Starting from (Γ, d) a first order differential calculus over A, we want to construct a differential calculus over A as a prolongation of the first order.

Let us consider the tensor bundle algebra

$$\Gamma^{\otimes_A} = \bigoplus_{i=0}^{\infty}\Gamma^{\otimes_A^n} = A\oplus\Gamma\oplus(\Gamma\otimes_A\Gamma)\oplus\dots, \qquad (3.43)$$

that is a graded, associative, unital algebra with product $\otimes_A$ and unit $1 \in A$. Consider the graded ideal

$$S^\wedge = \bigoplus_{n=0}^{\infty} S^\wedge \cap \Gamma^{\otimes_A^n} \subseteq \Gamma^{\otimes_A},$$

generated by elements $\sum_i \mathrm{d}a^i \otimes_A \mathrm{d}b^i$ where $a^i, b^i \in A$, such that $\sum_i a^i\mathrm{d}b^i = 0$.

We define accordingly the graded associative unital algebra $\Gamma^\wedge := \Gamma^{\otimes_A}/S^\wedge$, with induced product $\wedge$. The quotient algebra $\Gamma^\wedge$ is graded, indeed as

$$\begin{aligned}\Gamma^\wedge &= \frac{\bigoplus_{n=0}^{\infty}\Gamma^{\otimes_A^n}}{\bigoplus_{n=0}^{\infty}S^\wedge\cap\Gamma^{\otimes_A^n}}\\ &= \bigoplus_{n=0}^{\infty} x\frac{\Gamma^{\otimes_A^n}}{S^\wedge\cap\Gamma^{\otimes_A^n}} \qquad (3.44)\\ &\cong \bigoplus_{n=0}^{\infty}\frac{\Gamma^{\otimes_A^n}+S^\wedge}{S^\wedge},\end{aligned}$$

where in the last row we exploited the second isomorphism theorem. Accordingly the product $\wedge : \Gamma^\wedge \otimes \Gamma^\wedge \to \Gamma^\wedge$ is induced as

$$\begin{aligned}(\Gamma^{\wedge^j})\wedge(\Gamma^{\wedge^k}) &= \frac{(\Gamma^{\otimes_A^j}+S^\wedge)}{S^\wedge}\otimes_A\frac{(\Gamma^{\otimes_A^k}+S^\wedge)}{S^\wedge}\\ &\subseteq \frac{(\Gamma^{\otimes_A^{k+j}}+S^\wedge)}{S^\wedge}.\end{aligned} \qquad (3.45)$$

We report the following important result.

Theorem 3.3.1 ([8],Appendix B)**.** *The differential* $\mathrm{d}: A \to \Gamma$ *uniquely extends to a differential on* $\Gamma^\wedge$ *such that* $(\Gamma^\wedge, \wedge, \mathrm{d})$ *is a differential calculus on* A.

Proof. To show existence, we notice that the assignment

$$\mathrm{d}\left(\sum_i a_i \mathrm{d}b_i\right) = \sum_i \mathrm{d}a_i \wedge \mathrm{d}b_i$$

defines a linear map $\mathrm{d}: \Gamma \to \Gamma^\wedge$. This map is well defined since whenever $\sum_i a_i \mathrm{d}b_i = 0$ we have $\sum_i \mathrm{d}a_i \wedge \mathrm{d}b_i = 0$ as we quotiented out the ideal $S^\wedge$.

The $\mathrm{d}^2(a) = 0$ property for each $a \in A$ is easily checked as

$$\mathrm{d}^2 a = \mathrm{d}(\mathrm{d}a) = \mathrm{d}(1) \wedge \mathrm{d}a = 0. \tag{3.46}$$

Moreover, let $a \in A$ and let $\theta = \sum_i a_i \mathrm{d}b_i \in \Gamma$. We find

$$\begin{aligned} \mathrm{d}(a\theta) &= \mathrm{d}\left(\sum_i a a_i \mathrm{d}b_i\right) \\ &= \sum_i \mathrm{d}(a a_i)\, b_i \\ &= \sum_i \mathrm{d}(a)\, a_i \wedge \mathrm{d}b_i + a\, \mathrm{d}(a_i) \wedge \mathrm{d}b_i \\ &= \mathrm{d}(a) \wedge \sum_i a_i \mathrm{d}b_i + a \sum_i \mathrm{d}a \wedge \mathrm{d}b_i \\ &= \mathrm{d}(a) \wedge \theta + a\, \mathrm{d}(\theta)\,; \end{aligned} \tag{3.47}$$

and

$$\begin{aligned} \mathrm{d}(\theta a) &= \mathrm{d}\left(\sum_i a^i \mathrm{d}b^i a\right) \\ &= \mathrm{d} \sum_i (a_i\, \mathrm{d}(b_i a) - a_i b_i \mathrm{d}a) \\ &= \sum_i (\mathrm{d}a_i \wedge \mathrm{d}(b_i a) - \mathrm{d}(a_i b_i) \wedge \mathrm{d}a) \\ &= \sum_i \mathrm{d}a_i \wedge \mathrm{d}b_i\ a + \mathrm{d}a_i b_i \wedge \mathrm{d}a - \mathrm{d}a_i b_i \wedge \mathrm{d}a - a_i \mathrm{d}b_i \wedge \mathrm{d}a \\ &= \sum_i (\mathrm{d}a_i \wedge \mathrm{d}b_i) a - \sum_i (a_i \mathrm{d}b_i) \wedge \mathrm{d}a \\ &= \mathrm{d}(\theta)\, a - \theta \wedge \mathrm{d}a. \end{aligned} \tag{3.48}$$

From the last two properties we have the map d has the unique extension $\mathrm{d}^\wedge : \Gamma^{\otimes_A^2} \to \Gamma^\wedge$, satisfying

$$\mathrm{d}^\wedge(w \otimes_A u) = \mathrm{d}(w) \wedge \pi(u) + (-1)^{|w|} \pi(w) \wedge \mathrm{d}(u)\,, \tag{3.49}$$

where $\pi : \Gamma^{\otimes_A} \to \Gamma^\wedge$ is the projection map.

Equations (3.46) and (3.49) imply elements of $S^\wedge$ are in the kernel of the differential $\mathrm{d}^\wedge$, since

$$\mathrm{d}^\wedge \left(\sum_i \mathrm{d}a_i \otimes_A \mathrm{d}b_i\right) = \sum_i \mathrm{d}^2 a \wedge \pi(\mathrm{d}b_i) + (-1)^{|\mathrm{d}a|}\ \pi(\mathrm{d}a) \wedge \mathrm{d}^2 b = 0. \tag{3.50}$$

This extension can be generalised inductively to higher order forms. Consequently there exists a unique map $\mathrm{d} : \Gamma^\wedge \to \Gamma^\wedge$ defined as a factorisation of the map $\mathrm{d}^\wedge : \Gamma^{\otimes_A} \to \Gamma^\wedge$ through π as

$$\begin{array}{ccc} \Gamma^{\otimes_A} & \xrightarrow{\ \mathrm{d}^\wedge\ } & \Gamma^\wedge \\ {\scriptstyle \pi}\downarrow & \nearrow_{\mathrm{d}} & \\ \Gamma^\wedge & & \end{array} .$$

This map features all the required properties. □

In example 3.1.7 we introduced the universal first order differential calculus. Then in Theorem 3.1.8 we showed how any first order differential calculus can be induced as the quotient of the universal calculus. One may ask if this is also true when extended to higher order forms.

We recall the universal differential calculus $(\Omega_u^\bullet, \wedge_u, \mathrm{d}_u)$ of Example 3.2.7. Here the space of forms is $\Omega_u^\bullet := \cap_{k=1}^n \ker \mu_k$, where $\mu_k := \mathrm{id}_{A^{\otimes(k-1)}} \otimes \mu \otimes \mathrm{id}_{A^{\otimes(n-k)}} : A^{\otimes(n+1)} \to A^{\otimes n}$ denotes the multiplication of two subsequent tensor factors in the position k, and where the differential is given by $\mathrm{d}_u(a^0 \otimes \cdots \otimes a^n) := \sum_{k=0}^{n+1} (-1)^k a^0 \otimes \cdots \otimes a^{k-1} \otimes 1 \otimes a^k \otimes \cdots \otimes a^n$. The wedge product of forms is $(a^0 \otimes \cdots \otimes a^k) \wedge_u (b^0 \otimes \cdots \otimes b^\ell) = a^0 \otimes \cdots \otimes a^k b^0 \otimes \ldots b^\ell$, for all $a^0, \ldots, a^k, b^0, \ldots, b^\ell \in A$ and the bimodule structure on $\Omega_u^\bullet$ is given as left multiplication on the first tensor factor and right multiplication on the last tensor factor.

Lemma 3.3.2. *Let $(\Omega^\bullet, \wedge, \mathrm{d})$ be a differential calculus on A. Then $(\Omega^\bullet, \wedge, \mathrm{d})$ is induced by a surjective morphism of differential graded algebras $\Omega_u^\bullet \to \Omega^\bullet$. In other words, any differential calculus over an algebra is a quotient of the universal differential calculus.*

Proof. Following the lines in the proof of Theorem 3.1.8 we define a map

$$\Phi^\bullet : \Omega_u^\bullet \to \Omega^\bullet, \quad a^0 \otimes \cdots \otimes a^k \mapsto a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k. \tag{3.51}$$

For the action on the space of n–forms we write

$$\Phi^n : \Omega_u^n \to \Omega^n, \quad \omega \mapsto \cdot \circ (\mathrm{id}_A \wedge \mathrm{d} \wedge \cdots \wedge \mathrm{d}) \circ \omega, \tag{3.52}$$

where
$cdotA : A \otimes \Omega_u^\bullet \to \Omega_u^\bullet$ is the left A–action on $\Omega_u^\bullet$.

We notice that

$$\begin{aligned} \Phi^n(0 \otimes \cdots \otimes 0) &= \cdot \circ (0 \otimes \cdots \otimes 0) \\ &= 0 \wedge \cdots \wedge 0, \end{aligned} \tag{3.53}$$

therefore Φ^n and thus $\Phi := \bigoplus_{n \in \mathbb{N}} \Phi^n$ are well defined maps.

Define the projection map

$$\pi : A \otimes A \to \ker \mu, \quad a \otimes b \mapsto a \otimes b - ab \otimes 1. \tag{3.54}$$

Accordingly, we define the k–th projection between two successive adjacent tensor factors as $\pi^k := \mathrm{id}_{A^{\otimes(k-1)}} \otimes \pi \otimes \mathrm{id}_{A^{\otimes(n-k-1)}}$. Let also $\xi_n = \pi_n \cdots \pi_1(a^0 \otimes \cdots \otimes a^n) \in A^{\otimes(n+1)}$. As the various projections map to kernel of multiplications, we can infer that ξ_n is an element of Ω_u^n. We find

$$\begin{aligned} \Phi^n(\xi_n) &= \Phi^n(\pi_1 \ldots \pi_n(a^0 \otimes \cdots \otimes a^n)) \\ &= \cdot \circ (\mathrm{id}_A \otimes \mathrm{d} \otimes \cdots \otimes \mathrm{d})(\pi_n \ldots \pi_1(a^0 \otimes \cdots \otimes a^n)) \\ &= \cdot \circ (\mathrm{id}_A \otimes \mathrm{d} \otimes \cdots \otimes \mathrm{d}) \circ (a^0 \otimes a^1 - a^0 a^1 \otimes 1) \otimes \cdots \otimes (a^{n-1} \otimes a^n - a^{n-1} a^n \otimes 1) \\ &= \cdot \circ (\mathrm{id}_A \otimes \mathrm{d} \otimes \cdots \otimes \mathrm{d})(a^0 \otimes \cdots \otimes a^n) = \\ &= a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^n, \end{aligned} \tag{3.55}$$

showing the surjectivity of Φ^n and thus of $\Phi^\bullet$.

The map $\Phi^\bullet$ is left A−linear, since

$$\begin{aligned}\Phi^n(a(a^0 \otimes \cdots \otimes a^n)) &= aa^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^n \\ &= a\Phi^n(a^0 \otimes \cdots \otimes a^n).\end{aligned} \tag{3.56}$$

For right A−linearity consider

$$\begin{aligned}\Phi^n((a^0 \otimes \cdots \otimes a^n)) &= a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}(a^n a) \\ &= (a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^n)a + a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^{n-1} a^n \wedge \mathrm{d}a \\ &= \Phi^n(a^0 \otimes \cdots \otimes a^n)a + \omega \wedge \mathrm{d}(a)\,,\end{aligned} \tag{3.57}$$

and so right A−linearity follows if and only if the form ω vanishes, which is the case since

$$\begin{aligned}\omega &= a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}\left(a^{n-1}\right) a^n \\ &= a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}\left(a^{n-1} a^n\right) - a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}\left(a^{n-2}\right) a^{n-1} \wedge \mathrm{d}a^n \\ &= -a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^{n-2} a^{n-1} \wedge \mathrm{d}a^n \\ &= (-1)^2 a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}\left(a^{n-3}\right) a^{n-2} \wedge \mathrm{d}a^{n-1} \wedge \mathrm{d}a^n \\ &= (-1)^{n-1} a^0 a^1 \mathrm{d}a^2 \wedge \cdots \wedge \mathrm{d}a^n \\ &= 0,\end{aligned} \tag{3.58}$$

where we used that elements of $\Omega_u^\bullet$ are in the intersection of the kernel of all adjacent multiplication maps.

The map Φ is a morphism of graded algebra since, given $a^0 \otimes \cdots \otimes a^n \in \Omega_u^n$ and $b^0 \otimes \cdots \otimes b^m \in \Omega^m$ we find

$$\begin{aligned}\Phi(a^0 \otimes \cdots \otimes a^n) \wedge \Phi(b^0 \otimes \cdots \otimes b^m) &= a^0 \mathrm{d}a^1 \wedge \mathrm{d}a^n b^0 \wedge \mathrm{d}b^1 \wedge \cdots \wedge b^m \\ &= a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}\left(a^n b^0\right) \wedge \cdots \wedge \mathrm{d}b^m \\ &\quad - a^0 \mathrm{d}a^1 \wedge \mathrm{d}\left(a^{n-1}\right) a^n \wedge \mathrm{d}b^0 \wedge \ldots \mathrm{d}b^m \\ &= \Phi\left((a^0 \otimes \cdots \otimes a^n) \wedge_u (b^0 \otimes \cdots \otimes b^m)\right) \\ &\quad - \omega \wedge \mathrm{d}b^0 \wedge \cdots \wedge \mathrm{d}b^n \\ &= \Phi\left((a^0 \otimes \cdots \otimes a^n) \wedge_u (b^0 \otimes \cdots \otimes b^m)\right).\end{aligned} \tag{3.59}$$

Finally we have

$$\begin{aligned}\Phi^{n+1} \circ \mathrm{d}_u(a^0 \otimes \cdots \otimes a^n) &= \Phi^{n+1} \circ \left(\sum_{k=0}^{n+1} (-1)^k a^0 \otimes \ldots a^{k-1} \otimes 1 \otimes a^k \otimes \cdots \otimes a^n\right) \\ &= \Phi^{n+1}\left(1 \otimes a^0 \otimes \cdots \otimes a^n\right) \\ &\quad + \Phi^{n+1}\left(\sum_{k=1}^{n+1} (-1)^k a^0 \otimes \cdots \otimes a^{k-1} \otimes 1 \otimes a^k \otimes \cdots \otimes a^n\right) \\ &= \mathrm{d}a^0 \wedge \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^n \\ &= \mathrm{d} \circ \Phi^n(a^0 \otimes \cdots \otimes a^n),\end{aligned} \tag{3.60}$$

where we used that $1 \in \ker \mathrm{d}$. This proves that Φ respects the differentials. Therefore we proven that there exists a surjective morphism of differential graded algebra $\Phi : \Omega_u^\bullet \to \Omega^\bullet$. It follows that $(\Omega_u^\bullet / \ker \Phi, \wedge_u, \mathrm{d}_u) \cong (\Omega^\bullet, \wedge, \mathrm{d})$. □

Proposition 3.3.3. *The maximal prolongation of the first order universal differential calculus* (Γ_u, d_u) *is its tensor algebra, in other words the higher order universal differential calculus* $(\Omega_u^\bullet, \wedge_u, \mathrm{d}_u)$.

Proof. Let $\mathcal{T}A$ be the tensor algebra built from the first order universal differential calculus Γ_u on A, i.e.

$$\mathcal{T}A := \bigoplus_{n\in\mathbb{N}} \mathcal{T}^n A = A \oplus \Gamma_u \oplus \Gamma_u \otimes_A \Gamma_u \oplus \cdots .$$

Define $S^\wedge$ as the ideal of 2–forms generated by elements $\sum_i \mathrm{d}_u a^i \otimes_A \mathrm{d}_u b^i$ such that $\sum_i a^i \mathrm{d}_u b^i = 0$, where $\mathrm{d}_u : A \to \Gamma$ is the usual universal differential.

To construct the maximal prolongation of (Γ_u, d_u) we quotient out the ideal $S^\wedge$. We now show $\mathcal{T}A/S^\wedge$ is isomorphic to $\Omega_u^\bullet$.

From the Leibniz rule we see

$$\mathrm{d}\left(\sum_i a^i b^i\right) = \sum_i \mathrm{d}_u a^i b^i + \sum_i a^i \mathrm{d}_u b^i, \tag{3.61}$$

which equivalently tells us that $\sum_i \mathrm{d}_u a^i b^i$ is equal to zero since $\sum_i a^i b^i = 0$ in the very definition of Γ_u. Accordingly we find

$$\begin{aligned} \sum_i \mathrm{d}_u a^i b^i &= \sum_i a^i \otimes b^i - 1 \otimes a^i b^i = \sum_i a^i \otimes b^i = 0, \\ \sum_i a^i \mathrm{d}_u b^i &= \sum_i a^i b^i \otimes 1 - a^i \otimes b^i = -\sum_i a^i \otimes b^i = 0. \end{aligned} \tag{3.62}$$

Therefore, the ideal $S^\wedge$ is generated by 0, and the quotient $\mathcal{T}A/S^\wedge$ is naturally isomorphic to $\Omega_u^\bullet$ itself. Moreover, by theorem 3.3.1 we know that the differential $\mathrm{d}_u : \Omega^\bullet \to \Omega^\bullet$ is the unique extension of the differential $\mathrm{d} : A \to \Gamma$. □

Theorem 3.3.4 ([8],Appendix B)**.** *Let* $(\Omega^\bullet, \widetilde{\wedge}, \widetilde{\mathrm{d}})$ *be any differential calculus on* A *such that* $\Omega^1 = \Gamma$ *and* $\widetilde{d}|_A = \mathrm{d}$. *There exists a surjective morphism* $\Gamma^\wedge \to \Omega^\bullet$ *of differential graded algebras. In particular,* $(\Omega^\bullet, \widetilde{\wedge}, \widetilde{\mathrm{d}})$ *is a quotient of* $(\Gamma^\wedge, \wedge, \mathrm{d})$. □

Chapter 4

Quantum principal bundles

This chapter represents the core of this thesis work. The Đurđević theory of quantum principal bundles introduced in [10, 8] is presented adopting a modern language and notation. Essentially, a quantum principal bundle, or principal comodule algebra, is understood as a faithfully flat Hopf-Galois extension $B \subseteq A$, where A is a right $H-$comodule algebra, H is a Hopf algebra and B is the subalgebra of coinvariant elements in A under the coaction of H. In section 4.1 we introduce a left covariant first order differential calculus on the Hopf algebra H, the corresponding subalgebra of left invariant forms and the corresponding right $H-$module structure. We extend to the maximal prolongation of this first order calculus providing a differential calculus over H. We discuss the space of left invariant forms of this calculus. We then fix a bicovariant first order differential calculus over H.

In section 4.2 we provide the definition of quantum principal bundle as a faithfully flat Hopf-Galois extension. We discuss the space of vertical forms, which turns out to be a differential calculus under a suitable choice of wedge product and differential.

In section 4.2.2 we introduce the notion of complete calculus over a quantum principal bundle, being a differential calculus over the totale space such that the right $H-$coaction lifts to a morphism of differential graded algebras. We develop a theory for complete differential calculi over right H-comodule algebras. A definition of horizontal forms is provided; it is shown that for first order differential calculi there is a short exact sequence of $A-$modules involving horizontal, vertical and total space forms. We show how in general such exact sequence fails for higher order calculi. In subsection 4.2.3 we discuss differential calculus over the base space, i.e. over the right $H-$coinvariant elements. We show how the pull-back calculus by [2] and the one by Đurđević are different in general. However, we want to stress that in all explicit examples in this thesis the pull-back calculus coincides with the base space calculus. In subsection 4.2.4 we investigate the relation between quantum principal bundle of [5, 2] and Đurđević, providing a condition under which the two definition coincide. In particular, every quantum principal bundle in the sense of [5] on a faithfully flat Hopf-Galois extension is automatically first order complete in the sense of Đurđević.

In section 4.3 we provide some non-trivial examples of quantum principal bundles in the Đurđević theory. The non-commutative torus, the quantum Hopf fibration and crossed product calculi turn out to be complete differential calculi in our previous definition. Moreover, in these three examples, the base space calculus is generated by the base itself.

4.1 Preliminary notions

Let H be a Hopf algebra $(H, \mu, \eta, \Delta, \epsilon, S)$. Recall the maximal prolongation $\Gamma^\wedge$ introduced in section 3.3 obtained as a quotient of the tensor bundle algebra on a first order calculus Γ over H by the ideal $S^\wedge$. For now let Γ be a left-covariant first order differential calculus over H; let ${}_\Gamma\Delta : \Gamma \to H \otimes \Gamma$ be the corresponding left coaction of H. Denote by Λ^1 the space of left-invariant elements of Γ, namely

$$\Lambda^1 = \{\theta \in \Gamma : {}_\Gamma\Delta(\theta) = 1 \otimes \theta\}.$$

Consider the quantum Maurer-Cartan form $\varpi : H^+ \to \Lambda^1$ given by $\varpi(a) = S(a_1)\mathrm{d}a_2$ where $H^+ = \ker \epsilon$, and let furthermore $I \subseteq \ker(\epsilon) = H^+$ be the right $H-$ideal which corresponds canonically to Γ under the identification provided by Theorem 3.1.1. We have $\ker \varpi = I$, and because of this there exists a natural isomorphism $\Lambda^1 \cong H^+/I$. This isomorphism induces a right $H-$module structure on Λ^1 which will be denoted as $\leftharpoondown$. Explicitely, the action is given as

$$\begin{aligned} \varpi(a) \leftharpoondown b &= \varpi(ab - \epsilon(a)b) \\ &= \varpi(ab) - \epsilon(a)\varpi(b) \\ &= S(a_1b_1)\,\mathrm{d}(a_2b_2) - \epsilon(a)S(b_1)\mathrm{d}b_2 \\ &= S(b_1)S(a_1)(\mathrm{d}(a_2)\,b_2 + a_2\mathrm{d}b_2) - \epsilon(a)S(b_1)\mathrm{d}b_2 \\ &= S(b_1)\varpi(a)b_2 + S(b_1)\epsilon(a)\mathrm{d}b_2 - \epsilon(a)S(b_1)\mathrm{d}b_2 \\ &= S(b_1)\varpi(a)b_2, \end{aligned} \tag{4.1}$$

for each $a, b \in H$.

Remark 4.1.1. The Maurer-Cartan form satisfies the following relation among product of elements in H^+.

$$\begin{aligned} \varpi(ab) &= S(a_1b_1)\,\mathrm{d}(a_2b_2) \\ &= S(b_1)S(a_1)(\mathrm{d}a_2b_2 + a_2\mathrm{d}b_2) \\ &= S(b_1)\varpi(a)b_2 + S(b_1)\epsilon(a)\mathrm{d}b_2 \\ &= \varpi(a) \leftharpoondown b + \epsilon(a)\varpi(b), \end{aligned} \tag{4.2}$$

for every $a, b \in H^+$. □

Remark 4.1.2. By left covariance of the calculus the maps Δ and ${}_\Gamma\Delta$ admit common extensions to homomorphisms

$${}_{\Gamma^\wedge}\Delta : \Gamma^\wedge \to H \otimes \Gamma^\wedge \quad \text{and} \quad {}_{\Gamma^\otimes}\Delta : \Gamma^\otimes \to H \otimes \Gamma^\otimes$$

according to proposition 3.2.11. □

Let $\Lambda^\bullet$ be the differential graded subalgebra of left-invariant elements of $\Gamma^\wedge$, namely

$$\Lambda^\bullet = \bigoplus_{k \geq 0} \left\{\omega \in \Omega^k(H) \ : {}_{\Gamma^\wedge}\Delta(\omega) = 1 \otimes \omega\right\}. \tag{4.3}$$

Proposition 4.1.3. *The map $\pi_{inv} : \Gamma^\wedge \to \Gamma^\wedge$ defined by $\omega \mapsto S(\omega_{-1})\omega_0$ maps onto the space of invariant forms $\Lambda^\bullet$.*

Proof. Consider $\omega \in \Gamma^\wedge$, which can be written as $\omega = a^0\mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k$. We find that

$$\begin{aligned} \pi_{inv}(\omega) &= S(\omega_{-1})\omega_0 \\ &= \mu \circ (S \otimes \mathrm{id}) \circ {}_{\Gamma^\wedge}\Delta(\omega) \\ &= \mu \circ (S \otimes \mathrm{id}) \circ {}_{\Gamma^\wedge}\Delta(a^0\mathrm{d}a^1 \wedge \cdots \mathrm{d}a^k). \end{aligned} \tag{4.4}$$

and so

$$\begin{aligned} \pi_{inv}(\omega) &= \mu \circ (S \otimes \mathrm{id}) \circ (a^0_1 \otimes a^0_2)(\wedge \otimes \wedge)(a^1_1 \otimes \mathrm{d}a^1_2)(\wedge \otimes \wedge) \cdots (\wedge \otimes \wedge)(a^k_1 \otimes \mathrm{d}a^k_2) \\ &= \epsilon(a^0)S(a^1_1 \cdots a^k_1)\mathrm{d}a^1_2 \wedge \cdots \wedge \mathrm{d}a^k_2. \end{aligned} \tag{4.5}$$

Applying ${}_{\Gamma^\wedge}\Delta$ we obtain

$$\begin{aligned} {}_{\Gamma^\wedge}\Delta(\pi_{inv}(\omega)) &= {}_{\Gamma^\wedge}\Delta(\epsilon(a^0)S(a^1_1 \cdots a^k_1)\mathrm{d}a^1_2 \wedge \cdots \wedge \mathrm{d}a^k_2) \\ &= \epsilon(a^0)\Delta(S(a^1_1 \cdots a^k_1))(\mathrm{id} \otimes \mathrm{d})\Delta(a^2_2)(\wedge \otimes \wedge) \cdots (\wedge \otimes \wedge)(\mathrm{id} \otimes \mathrm{d})\Delta(a^k_2) \\ &= \epsilon(a^0)(S(a^1_2 \cdots a^k_2)a^1_3 \cdots a^k_3 \otimes S(a^1_1 \cdots a^k_1)\mathrm{d}a^1_4 \wedge \cdots \wedge \mathrm{d}a^k_4) \\ &= 1 \otimes \epsilon(a^0)S(a^1_1 \cdots a^k_1)\mathrm{d}a^1_2 \wedge \cdots \wedge \mathrm{d}a^k_2 \\ &= 1 \otimes S(a^0_1 \cdots a^k_1)a^0_2\mathrm{d}a^1_2 \wedge \cdots \wedge \mathrm{d}a^k_2. \end{aligned} \tag{4.6}$$

Clearly, π_{inv} is surjective, since for $\omega \in \Lambda^\bullet$ we obtain $\pi_{inv}(\omega) = \omega$. □

Proposition 4.1.4 ([2],Proposition 2.31)**.** *Let (Γ, d) be a left covariant first order differential calculus on a Hopf algebra H. Higher order coinvariant forms correspond to tensor products of coinvariant 1-forms quotiented by some relations. Precisely*

$$\Lambda^\bullet = (\Lambda^1)^\otimes / S^\wedge_{inv}, \tag{4.7}$$

where $S^\wedge_{inv} := \langle \varpi(\pi_\epsilon(a_1)) \otimes \varpi(\pi_\epsilon(a_2)) \mid a \in A \rangle$, where $\pi_\epsilon : H \to H^+$ is defined by $\pi_\epsilon(h) = h - \epsilon(h)1$. □

Proposition 4.1.5. *The right $H-$module structure $\leftharpoonup$ on Λ^1 can be extended uniquely to $\Lambda^\bullet$, i.e. $\Lambda^\bullet$ is a graded right $H-$module algebra, with*

$$\begin{aligned} 1 \leftharpoonup a &= \epsilon(a)1, \\ (\theta \wedge \eta) \leftharpoonup a &= (\theta \leftharpoonup a_1) \wedge (\eta \leftharpoonup a_2), \end{aligned} \tag{4.8}$$

for each θ, η either in $\Lambda^\bullet$, and every $a \in H$. Explicitly, $\theta \leftharpoonup a = S(a_1)\theta a_2$.

Proof. First of all, for any $\theta \in \Lambda^\bullet$ and $a, b \in H$, we have

$$\begin{aligned} (\theta \leftharpoonup a) \leftharpoonup b &= S(b_1)(\theta \leftharpoonup a)b_2 \\ &= S(b_1)S(a_1)\theta a_2 b_2 \\ &= S(a_1 b_1)\theta a_2 b_2 \\ &= \theta \leftharpoonup ab, \end{aligned} \tag{4.9}$$

i.e. $\leftharpoonup$ is a right $H-$action.

Moreover, let $\omega, \eta \in \Lambda^\bullet$. Then

$$\begin{aligned} (\theta \leftharpoonup h_1) \wedge (\eta \leftharpoonup h_2) &= (S(h_{11})\theta h_{12}) \wedge (S(h_{21})\eta h_{22}) \\ &= S(h_1)\theta h_2 \wedge S(h_3)\eta h_4 \\ &= S(h_1)\theta \wedge h_2 S(h_3)\eta h_4 \\ &= S(h_1)\theta \wedge \eta h_2 \\ &= (\theta \wedge \eta) \leftharpoonup h. \end{aligned} \tag{4.10}$$

□

Proposition 4.1.6. *The algebra $\Lambda^\bullet \subseteq \Gamma^\wedge$ is $\mathrm{d}-$invariant.*

Proof. Let $\mathrm{d} : \Gamma^\wedge \to \Gamma^\wedge$ be the differential extending $\mathrm{d} : H \to \Gamma$ constructed in 3.3.1. Let $\omega \in \Lambda^\bullet$ be a coinvariant form. By definition ${}_{\Gamma^\wedge}\Delta(\omega) = 1 \otimes \omega$. This means $a^0_{-1} \cdots a^k_{-1} = 1$. Since d is left $H-$colinear, it follows that

$$\begin{aligned} {}_{\Gamma^\wedge} \circ \Delta(\mathrm{d}\omega) &= (\mathrm{id} \otimes \mathrm{d}) \circ {}_{\Gamma^\wedge}\Delta(\omega) \\ &= 1 \otimes \mathrm{d}\omega. \end{aligned} \tag{4.11}$$

Proposition 4.1.7. *The "Maurer-Cartan equation" $\mathrm{d}\varpi(\pi_\epsilon(a)) = -\varpi(\pi_\epsilon(a_1))\varpi(\pi_\epsilon(a_2))$ holds.*

Proof. By a direct calculation we find

$$\begin{aligned}
\varpi(\pi_\epsilon(a_1)) \wedge \varpi(\pi_\epsilon(a_2)) &= S(\pi_\epsilon(a_{11}))\mathrm{d}\pi_\epsilon(a_{12}) \wedge S(\pi_\epsilon(a_{21}))\mathrm{d}\pi_\epsilon(a_{22}) \\
&= S(\pi_\epsilon(a_1))\mathrm{d}\pi_\epsilon(a_2) \wedge S(\pi_\epsilon(a_3))\mathrm{d}\pi_\epsilon(a_4) \\
&= S(\pi_\epsilon(a_1))\mathrm{d}\pi_\epsilon(a_2)S(\pi_\epsilon(a_3)) \wedge \mathrm{d}\pi_\epsilon(a_4) \\
&= S(\pi_\epsilon(a_1))(\mathrm{d}(\pi_\epsilon(a_2)S(\pi_\epsilon(a_3))) - \pi_\epsilon(a_2)\mathrm{d}S(\pi_\epsilon(a_3))) \wedge \mathrm{d}\pi_\epsilon(a_4) \\
&= S(\pi_\epsilon(a_1))\,\mathrm{d}(1) \wedge \mathrm{d}\pi_\epsilon(a_4) - S(\pi_\epsilon(a_1))\pi_\epsilon(a_2)\mathrm{d}S(\pi_\epsilon(a_3)) \wedge \mathrm{d}\pi_\epsilon(a_4) \\
&= -S(\pi_\epsilon(a_1))\pi_\epsilon(a_2)\,\mathrm{d}(S(\pi_\epsilon(a_3))) \wedge \mathrm{d}\pi_\epsilon(a_4) \\
&= -\epsilon(a_1)\,\mathrm{d}(S(\pi_\epsilon(a_{11}))) \wedge \mathrm{d}\pi_\epsilon(a_{12}) \\
&= -\,\mathrm{d}(S(\epsilon(a_1)a_2))) \wedge \mathrm{d}\pi_\epsilon(a_3) \\
&= -\,\mathrm{d}(S(\pi_\epsilon(a_1))) \wedge \mathrm{d}\pi_\epsilon(a_2) \\
&= -\,\mathrm{d}(\varpi(\pi_\epsilon(a)))\,,
\end{aligned} \tag{4.12}$$

since $\mathrm{d}(\varpi(\pi_\epsilon(a))) = \mathrm{d}(S(\pi_\epsilon(a_1))\mathrm{d}\pi_\epsilon(a_2)) = \mathrm{d}(S(\pi_\epsilon(a_1))) \wedge \mathrm{d}\pi_\epsilon(a_2)$. □

Lemma 4.1.8. *The differential of the right $H-$module structure on coinvariant differential forms reads*

$$\mathrm{d}(\theta \leftharpoondown a) = (\mathrm{d}(\theta) \leftharpoondown a) - \varpi(\pi_\epsilon(a_1))(\theta \leftharpoondown a_2) + (-1)^{|\theta|}(\theta \leftharpoondown a_1)\varpi(\pi_\epsilon(a_2)), \tag{4.13}$$

for $a \in H$ and

Proof. The left hand side reads

$$\begin{aligned}
\mathrm{d}(\theta \leftharpoondown a) &= \mathrm{d}(S(a_1)\theta a_2) \\
&= \mathrm{d}(S(a_1))\,\theta a_2 + S(a_1)\mathrm{d}\theta a_2 + (-1)^{|\theta|}S(a_1)\theta\mathrm{d}a_2.
\end{aligned} \tag{4.14}$$

Terms on the right hand side read

$$\begin{aligned}
\mathrm{d}\theta \leftharpoondown a &= S(a_1)\mathrm{d}\theta a_2; \\
\varpi(\pi_\epsilon(a_1))(\theta \leftharpoondown a_2) &= S(\pi_\epsilon(a_{11}))\,\mathrm{d}(\pi_\epsilon(a_{12}))\,S(a_{21})\theta a_{22} \\
&= S(\pi_\epsilon(a_1))\mathrm{d}\pi_\epsilon(a_2)S(a_3)\theta a_4 \\
&= S(\pi_\epsilon(a_1))(\mathrm{d}(\pi_\epsilon(a_2)S(a_3)) - \pi_\epsilon(a_2)\mathrm{d}S(a_3))\theta a_4 \\
&= -S(\pi_\epsilon(a_1))\pi_\epsilon(a_2)\mathrm{d}S(a_3)\theta a_4 \\
&= -\mathrm{d}S(\pi_\epsilon(a_1))\theta a_2; \\
(-1)^{|\theta|}(\theta \leftharpoondown a_1)\varpi(\pi_\epsilon(a_2)) &= (-1)^{|\theta|}S(a_{11})\theta a_{12}S(\pi_\epsilon(a_{21}))\mathrm{d}\pi_\epsilon(a_{22}) \\
&= (-1)^{|\theta|}S(a_1)\theta a_2 S(\pi_\epsilon(a_3))\mathrm{d}\pi_\epsilon(a_4) \\
&= (-1)^{|\theta|}S(a_1)\theta\mathrm{d}\pi_\epsilon(a_2),
\end{aligned} \tag{4.15}$$

therefore

$$\begin{aligned}
\mathrm{d}(\theta) \leftharpoondown a - \varpi(\pi_\epsilon(a_1))(\theta \leftharpoondown a_2) + (-1)^{|\theta|}(\theta \leftharpoondown a_1)\varpi(\pi_\epsilon(a_2)) = S(\pi_\epsilon(a_1))\mathrm{d}\theta a_2 \\
+ \mathrm{d}S(a_1)\theta a_2 \\
+ (-1)^{|\theta|}S(a_1)\theta\mathrm{d}\pi_\epsilon(a_2).
\end{aligned} \tag{4.16}$$

□

In the following we fix Γ a bicovariant first order differential calculus over H and consider its maximal prolongation to a higher order calculus $\Gamma^\wedge$.

Proposition 4.1.9. *Let H be a Hopf algebra and let $(\Omega^\bullet(H), \wedge, \mathrm{d})$ be a bicovariant differential calculus on H. Then $\Omega^\bullet(H)$ is a graded Hopf algebra with coproduct $\Delta^\bullet : \Omega^k(H) \to \Omega^m(H) \otimes \Omega^n(H)$, where $m+n=k$, sending $\omega \mapsto \omega_{(1)} \otimes \omega_{(2)} =: \omega_{-1} \otimes \omega_0 + \omega_0 \otimes \omega_1 = ({}_{\Gamma^\wedge}\Delta + \Delta_{\Gamma^\wedge})(\omega)$, counit $\epsilon^\bullet : \Omega^\bullet(H) \to \Bbbk$, with $\epsilon(\omega) = 0$ for any ω such that $|\omega| > 1$, and antipode $S^\bullet : \Omega^\bullet \to \Omega^\bullet$ defined as $S^\bullet(\omega) = -S(\omega_{-1})\omega_0 S(\omega_1)$.*

Proof. We start by checking coassociativity of $\Delta^\bullet$. Let $\omega \in \Omega^k(H)$. We have

$$\begin{aligned}(\Delta^\bullet \otimes \mathrm{id}) \circ \Delta^\bullet(\omega) &= (\Delta^\bullet \otimes \mathrm{id})(\omega_{-1} \otimes \omega_0 + \omega_0 \otimes \omega_1)\\ &= (\omega_{-1})_1 \otimes (\omega_{-1})_2 \otimes \omega_0\\ &+ (\omega_0)_{-1} \otimes (\omega_0)_0 \otimes \omega_1\\ &+ (\omega_0)_0 \otimes (\omega_0)_1 \otimes \omega_1\\ &= \omega_{-2} \otimes \omega_{-1} \otimes \omega_0 + \omega_{-1} \otimes \omega_0 \otimes \omega_1 + \omega_0 \otimes \omega_1 \otimes \omega_2\\ &= (\mathrm{id} \otimes \Delta^\bullet) \circ \Delta^\bullet(\omega).\end{aligned} \tag{4.17}$$

Next we have

$$\begin{aligned}(\epsilon \otimes \mathrm{id}) \circ \Delta^\bullet(\omega) &= (\epsilon \otimes \mathrm{id})(\omega_{-1} \otimes \omega_0 + \omega_0 \otimes \omega_{-1})\\ &= \epsilon(\omega_{-1}) \otimes \omega_0 + \epsilon(\omega_0) \otimes \omega_1\\ &= \epsilon(\omega_{-1})\omega_0 \otimes 1\\ &= \omega \otimes 1\\ &= \omega.\end{aligned} \tag{4.18}$$

Finally

$$\begin{aligned}\omega_{(1)} S(\omega_{(2)}) &= \omega_{-1} S(\omega_0) + \omega_0 S(\omega_1)\\ &= -\omega_{-1} S((\omega_0)_{-1})(\omega_0)_0 S((\omega_0)_1) + \omega_0 S(\omega_1)\\ &= -\omega_{-2} S(\omega_{-1})\omega_0 S(\omega_1) + \omega_0 S(\omega_1)\\ &= -\omega_0 S(\omega_1) + \omega_0 S(\omega_1)\\ &= 0\\ &= \epsilon(\omega)1.\end{aligned} \tag{4.19}$$

Similarly for $S(\omega_{(1)})\omega_{(2)}$. □

Remark 4.1.10. In [10] it is stated that the lifted graded coproduct $\Delta^\bullet = {}_{\Gamma^\wedge}\Delta + \Delta_{\Gamma^\wedge} : \Gamma^\wedge \to \Gamma^\wedge \otimes \Gamma^\wedge$ is a morphism of differential graded algebras. However, it turns out that $\Delta^\bullet$ is a morphism of graded algebras, but is not a morphism of differential graded algebras, since it is not compatible with the differential in general. Consider for instance a 2−form $\omega \in \Omega^2(H)$. Then, if we assume $\Delta^\bullet$ to be a morphism of differential graded algebras we get

$$\begin{aligned}\Delta^\bullet(\omega) &= \Delta^\bullet(a\mathrm{d}b \wedge \mathrm{d}c)\\ &= \Delta(a)\Delta^1(\mathrm{d}a)\Delta^1(\mathrm{d}b)\\ &= (a_1 \otimes a_2)(\mathrm{d} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{d})(b_1 \otimes b_2)(\mathrm{d} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{d})(c_1 \otimes c_2)\\ &= (a_1 \otimes a_2)(-b_1\mathrm{d}c_1 \otimes \mathrm{d}b_2 c_2 + \mathrm{d}b_1 \wedge \mathrm{d}c_1 \otimes b_2 c_2 + \mathrm{d}b_1 c_1 \otimes b_2 \mathrm{d}c_2 + b_1 c_1 \otimes \mathrm{d}b_2 \wedge \mathrm{d}c_2).\end{aligned}$$

On the other hand, according to Đurđević's definition we would get

$$\begin{aligned}\Delta^\bullet(\mathrm{d}\omega) &= ({}_{\Gamma^\wedge}\Delta + \Delta_{\Gamma^\wedge})(a\mathrm{d}b \wedge \mathrm{d}b)\\ &= {}_{\Gamma^\wedge}\Delta(a\mathrm{d}b \wedge \mathrm{d}c) + \Delta_{\Gamma^\wedge}(a\mathrm{d}b \wedge \mathrm{d}c)\\ &= a_1 b_1 c_1 \otimes a_2 \mathrm{d}b_2 \wedge \mathrm{d}c_2 + a_1\mathrm{d}b_1 \wedge \mathrm{d}c_1 \otimes a_2 b_2 c_2.\end{aligned} \tag{4.20}$$

Since two terms are missing, we conclude $\Delta^\bullet = {}_{\Gamma^\wedge}\Delta + \Delta_{\Gamma^\wedge}$ is not a morphism of differential graded algebras. □

Thus, we choose to restrict our attention to bicovariant differential calculi over Hopf algebras such that the coproduct lifts uniquely to a morphism $\Delta^\bullet : \Gamma^\wedge \to \Gamma^\wedge \otimes \Gamma^\wedge$ of differential graded algebras.

Notation 4.1.11. The action of $\Delta^\bullet : \Gamma^\wedge \to \Gamma^\wedge \otimes \Gamma^\wedge$ on a given form $\omega \in \Gamma^\wedge$ is denoted as $\Delta^\bullet(\omega) = \omega_{[1]} \otimes \omega_{[2]}$.

4.2 Quantum principal bundles

We now introduce the definition of quantum principal bundle in Đurđević' theory as a faithfully flat Hopf-Galois extension.

Definition 4.2.1. Let B be a ring and let X, Y, Z be $B-$modules. A $B-$module A such that

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is short exact if and only if

$$0 \longrightarrow X \otimes_B A \longrightarrow Y \otimes_B A \longrightarrow Z \otimes_B A \longrightarrow 0$$

is short exact, is called a faithfully flat left $B-$module.

Definition 4.2.2. A quantum principal bundle, or principal comodule algebra, is a faithfully flat Hopf-Galois extension $B \subseteq A$, where A is a right $H-$comodule algebra and $B := A^{coH}$ is the space of coinvariant forms in A under the coaction of H.

We consider differential calculi over such quantum principal bundles.

Recall that in classical differential geometry the vertical bundle and the horizontal bundle are vector bundles associated with a smooth fiber bundle. Specifically, given a smooth fiber bundle $\pi : E \to B$, the vertical bundle VE and horizontal bundle HE are subbundles of the tangent bundle satisfying $VE \oplus HE \cong TE$, the horizontal bundle choice being fixed by the connection. This implies that, over each point $e \in E$, the fibers V_eE and H_eE form complementary subspaces of the tangent space T_eE. The vertical bundle comprises all vectors tangent to the fibers.

4.2.1 Vertical forms

The following definition provides the non-commutative analogue of vertical forms of classical differential geometry.

Definition 4.2.3. Let $B \subseteq A$ be a quantum principal bundle. Let Γ be a bicovariant first order differential calculus over H, $\Gamma^\wedge$ the maximal prolongation and $\Lambda^\bullet$ the space of left coinvariant forms on H. The graded vector space $\mathfrak{ver}^\bullet(A) = A \otimes \Lambda^\bullet$ defines the space of vertical forms.

The space of vertical forms is a differential calculus thanks to the following lemma.

Lemma 4.2.4. *The space of vertical forms* $(\mathfrak{ver}(A), \wedge_{\mathfrak{ver}}, \mathrm{d}_{\mathfrak{ver}})$ *is differential calculus over* A *by the assignments*

$$(a \otimes \theta) \wedge_{\mathfrak{ver}} (b \otimes \eta) = ab_0 \otimes (\theta \leftharpoonup b_1) \wedge \eta, \tag{4.21}$$

$$\mathrm{d}_{\mathfrak{ver}}(a \otimes \theta) = a \otimes \mathrm{d}\theta + a_0 \otimes \varpi(\pi_\epsilon(a_1)) \wedge \theta, \tag{4.22}$$

for every $\theta, \eta \in \Lambda^\bullet$ *and* $a, b \in A$.

Proof. During the proof we will call write $(\wedge, \mathrm{d})$ for $(\wedge_{\mathfrak{ver}}, \mathrm{d}_{\mathfrak{ver}})$ to shorten the notation. We shall prove that $\wedge$ is an associative product that is unital, and moreover the differential satisfies the graded Leibniz rule and squares to zero. The product of vertical forms is associative, indeed

$$\begin{aligned} ((a\otimes\theta)\wedge(b\otimes\eta))\wedge(c\otimes\omega) &= (a_0b_0\otimes(\theta\leftharpoondown b_1)\wedge\eta)\wedge\omega \\ &= ab_0c_0\otimes((\theta\leftharpoondown b_1)\wedge\eta)\leftharpoondown c_1)\wedge\omega \\ &= ab_0c_0\otimes((\theta\leftharpoondown b_1)\leftharpoondown c_{11})\wedge(\eta\leftharpoondown c_{12})\wedge\omega \\ &= ab_0c_0\wedge(\theta\leftharpoondown b_1c_1)\wedge(\eta\leftharpoondown c_2)\wedge\omega \\ &= ab_0c_{00}\otimes(\omega\leftharpoondown b_1c_{01})\wedge(\eta\leftharpoondown c_1)\wedge\omega \\ &= (a\otimes\theta)\wedge(bc_0\otimes(\eta\leftharpoondown c_1)\wedge\omega \\ &= (a\otimes\theta)\wedge((b\otimes\eta)\wedge(c\otimes\omega)). \end{aligned} \tag{4.23}$$

Next we have $\mathfrak{ver}(P)$ is a unital algebra with unit $1\otimes 1$, indeed

$$\begin{aligned} (1\otimes 1)\wedge(a\otimes\theta) &= a_0\otimes(1\leftharpoondown a_1)\wedge\theta \\ &= a_0\otimes\epsilon(a_1)\theta \\ &= a_0\epsilon(a_1)\otimes\theta \\ &= a\otimes\theta \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} (a\otimes\theta)\wedge(1\otimes 1) &= a\otimes(\theta\leftharpoondown 1) \\ &= a\otimes S(1)\theta \\ &= a\otimes\theta. \end{aligned} \tag{4.25}$$

Now we use the Maurer-Cartan equation $\mathrm{d}\varpi(\pi_\epsilon(h)) = -\varpi(\pi_\epsilon(h_1))\wedge\varpi(\pi_\epsilon(h_2))$ and the property

$$\mathrm{d}(\theta\leftharpoondown h) = (\mathrm{d}(\theta)\leftharpoondown h) - \varpi(\pi_\epsilon(h_1))\wedge(\theta\leftharpoondown h_2) + (-1)^{|\theta|}(\theta\leftharpoondown h_1)\wedge\varpi(\pi_\epsilon(h_2))$$

to show we have a differential on the algebra $\mathfrak{ver}(A)$ defined by (4.22). We find

$$\begin{aligned} \mathrm{d}((a\otimes\theta)\wedge(b\otimes\eta)) &= \mathrm{d}(ab_0\otimes(\theta\leftharpoondown b_1)\wedge\eta) \\ &= ab_0\otimes\mathrm{d}((\theta\leftharpoondown b_1)\wedge\eta) \\ &+ a_0b_0\otimes\varpi(\pi_\epsilon(a_1b_1))\wedge(\theta\leftharpoondown b_2)\wedge\eta \\ &= ab_0\otimes(\mathrm{d}(\theta\leftharpoondown b_1)\wedge\eta) \\ &+ (-1)^{|\theta|}ab_0\otimes[(\theta\leftharpoondown b_1)\wedge\mathrm{d}\eta] \\ &+ a_0b_0\otimes\varpi(\pi_\epsilon(a_1b_1))\wedge(\theta\leftharpoondown b_2)\wedge\eta \\ &= ab_0\otimes(\mathrm{d}\theta\leftharpoondown b_1)\wedge\eta \\ &- ab_0\otimes\varpi(\pi_\epsilon(b_1))\wedge(\theta\leftharpoondown b_2)\wedge\eta \\ &+ (-1)^{|\theta|}ab_0\otimes(\theta\leftharpoondown b_1)\wedge\varpi(\pi_\epsilon(b_2))\wedge\eta \\ &+ (-1)^{|\theta|}ab_0\otimes(\theta\leftharpoondown b_1)\wedge\mathrm{d}\eta \\ &+ a_0b_0\otimes\varpi(\pi_\epsilon(a_1b_1))\wedge(\theta\leftharpoondown b_2)\wedge\eta. \end{aligned} \tag{4.26}$$

On the other hand

$$\begin{aligned} &\mathrm{d}(a\otimes\theta)\wedge(b\otimes\eta) + (-1)^{|\theta|}(a\otimes\theta)\wedge\mathrm{d}(b\otimes\eta) \\ &= (a\otimes\mathrm{d}\theta + a_0\otimes\varpi(\pi_\epsilon(a_1))\wedge\theta)\wedge(b\otimes\eta) \\ &+ (-1)^{|\theta|}(a\otimes\theta)\wedge(b\otimes\mathrm{d}\eta + b_0\otimes\varpi(\pi_\epsilon(b_1)\wedge\eta)) \\ &= ab_0\otimes(\mathrm{d}\theta\leftharpoondown b_2)\wedge\eta \\ &+ (-1)^{|\theta|}ab_0\otimes(\theta\leftharpoondown b_1)\wedge(\varpi(\pi_\epsilon(b_2))\wedge\eta) \\ &+ (-1)^{|\theta|}ab_0\otimes(\theta\leftharpoondown b_1)\wedge\mathrm{d}\eta \\ &+ a_0b_0\otimes((\varpi(\pi_\epsilon(a_1))\wedge\theta)\leftharpoondown b_1)\wedge\eta. \end{aligned} \tag{4.27}$$

We work out the last term of this expression as following.

$$
\begin{aligned}
&a_0b_0 \otimes ((\varpi(\pi_\epsilon(a_1)) \wedge \theta) \leftharpoonup b_1) \wedge \eta \\
\text{Equation (4.8)} &= a_0b_0[(\varpi(\pi_\epsilon(a_1)) \leftharpoonup b_1) \wedge (\theta \leftharpoonup b_2)] \wedge \eta \\
&= a_0b_0 \otimes [\varpi(\pi_\epsilon(a_1b_1)) \wedge (\theta \leftharpoonup b_2) - \epsilon(a_1)\varpi(\pi_\epsilon(b_1)) \wedge (\theta \leftharpoonup b_2)] \wedge \eta \\
\text{Equation (4.2)} &= a_0b_0 \otimes \varpi(\pi_\epsilon(a_1b_1)) \wedge (\theta \leftharpoonup b_2) \wedge \eta \\
&\quad - ab_0 \otimes \varpi(\pi_\epsilon(b_1)) \wedge (\theta \leftharpoonup b_2) \wedge \eta.
\end{aligned} \tag{4.28}
$$

Therefore we have

$$
\begin{aligned}
\mathrm{d}(a \otimes \theta) \wedge (b \otimes \eta) + (-1)^{|\theta|}(a \otimes \theta) \wedge \mathrm{d}(b \otimes \eta)& \\
= ab_0 \otimes (\mathrm{d}\theta \leftharpoonup b_1) \wedge \eta& \\
- ab_0 \otimes \varpi(\pi_\epsilon(b_1)) \wedge (\theta \leftharpoonup b_2) \wedge \eta& \\
+ (-1)^{|\theta|} ab_0 \otimes (\theta \leftharpoonup b_1) \wedge \varpi(\pi_\epsilon(b_2)) \wedge \eta& \\
+ (-1)^{|\theta|} ab_0 \otimes (\theta \leftharpoonup b_1) \wedge \mathrm{d}\eta& \\
+ a_0b_0 \otimes \varpi(\pi_\epsilon(a_1b_1)) \wedge (\theta \leftharpoonup b_2) \wedge \eta,&
\end{aligned} \tag{4.29}
$$

coinciding with Equation (4.26).

Moreover

$$
\begin{aligned}
\mathrm{d}^2(a \otimes \theta) &= \mathrm{d}(a \otimes \mathrm{d}\theta + a_0 \otimes \varpi(a_1) \wedge \theta) \\
&= \mathrm{d}(a \otimes \mathrm{d}\theta) + \mathrm{d}(a_0 \otimes \varpi(a_1) \wedge \theta) \\
&= a \otimes \mathrm{d}^2\theta + a_0 \otimes \varpi(a_1) \wedge \mathrm{d}\theta + a_0 \otimes \mathrm{d}(\varpi(a_1) \wedge \theta) + a_{00} \otimes \varpi(a_{01}) \wedge (\varpi(a_1) \wedge \theta) \\
&= 0,
\end{aligned} \tag{4.30}
$$

by associativity of the $\wedge$ product and the Maurer-Cartan equation.

To show that the surjectivity axiom holds start by considering $\mathfrak{ver}^1(A) = A \otimes \Lambda^1$. It follows that $\mathfrak{ver}^1(A) = A\mathrm{d}_{\mathfrak{ver}}A$, indeed

$$
\mathrm{d}(a \otimes 1) = a \otimes \mathrm{d}(1) + a_0 \otimes \varpi(\pi_\epsilon(a_1)) = a_0 \otimes \varpi(\pi_\epsilon(a_1)), \tag{4.31}
$$

but $\varpi(\pi_\epsilon(a_1))$ maps surjectively onto Λ^1. Thus $a_0 \otimes \varpi(a_1)$ is an element of $A \otimes \Lambda^1$, and by a suitable multiplication from left by elements of A we span $\mathfrak{ver}^1(A)$.

Next we show that $\mathfrak{ver}^{i+1}(A)$ is linearly generated by $\mathfrak{ver}^i(A)$. Let $(a \otimes \theta) \in \mathfrak{ver}^i(A)$. Differentiating we get

$$
\mathrm{d}(a \otimes \theta) = a \otimes \mathrm{d}\theta + a_0 \otimes \varpi(a_1) \wedge \theta
$$

by definition of $\mathrm{d}_{\mathfrak{ver}}$. Since the algebra $\Lambda^\bullet$ is d-invariant according to proposition 4.1.6 we conclude $\mathrm{d}\theta \in \Lambda^{k+1}$. □

The next lemma characterises the lift of the right $H-$coaction on A to the space of vertical forms. We remind $\Gamma^\wedge$ is a graded Hopf algebra with graded coproduct $\Delta^\bullet : \Gamma^\wedge \to \Gamma^\wedge \otimes \Gamma^\wedge$, and that we denote $\Delta^\bullet(\omega) = \omega_{[1]} \otimes \omega_{[2]}$.

4.2.2 Total space calculus

Now we introduce the notion of total calculus on a quantum principal bundle. Fix the usual bicoviariant first order differential calculus Γ on H. Denote the corresponding maximal prolongation by $\Omega^\bullet(H)$, for convenience.

Definition 4.2.5. A complete differential calculus over the quantum principal bundle $B = A^{coH} \subseteq A$ is a differential calculus $\Omega^\bullet(A)$ such that the coaction $\Delta_A : A \to A \otimes H$ extends to a morphism

$$\Delta_A^\wedge : \Omega^\bullet(A) \to \Omega_A^\bullet \otimes \Omega^\bullet(H)$$

of differential graded algebras.

From now on we will focus on complete differential calculi. We still continue to restrict our attention to those calculi where $\Delta^\bullet : \Omega^\bullet(H) \to \Omega^\bullet(H) \otimes \Omega^\bullet(H)$ is a morphism of differential graded algebras. The next result shows that $\Delta_A^\wedge$ is understood as a graded coaction itself.

Lemma 4.2.6. *We have*

$$(\Delta_A^\wedge \otimes \mathrm{id}) \circ \Delta_A^\wedge = (\mathrm{id} \otimes \Delta^\bullet) \circ \Delta_A^\wedge. \tag{4.32}$$

Moreover $\Delta_A^\wedge$ *is uniquely determined as extension of* Δ_A.

Proof. For $a \in \Omega^0(A) = A$ everything follows as $\Delta_A^\wedge$ extends Δ_A. Let $a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^{k+1}$ be in $\Omega^{k+1}(A)$ and assume by induction that equation (4.32) holds on $\Omega^k(A)$. We have

$$\begin{aligned}(\Delta_A^\wedge \otimes \mathrm{id})\Delta_A^\wedge(a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^{k+1}) &= (\Delta_A^\wedge \otimes \mathrm{id})\Delta_A^\wedge(a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k)(\wedge_A \otimes \wedge_H)\Delta_A^\wedge(\mathrm{d}a^{k+1}) \\ &= (\mathrm{id} \otimes \Delta^\bullet)\Delta_A^\wedge(a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k)(\wedge_A \otimes \wedge_H)\Delta_A^\wedge(\mathrm{d}a^{k+1}) \\ &= (\mathrm{id} \otimes \Delta^\bullet)\Delta_A^\wedge(a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k \wedge \mathrm{d}a^{k+1}).\end{aligned}$$

□

Lemma 4.2.7. $\Omega^\bullet(A)$ *is right* $H-$*covariant differential calculus with respect to the coaction*

$$\Delta_{\Omega^\bullet(A)} = (\mathrm{id} \otimes \pi_0)\Delta_A^\wedge : \Omega^\bullet(A) \to \Omega^\bullet(A) \otimes H.$$

In other words the following identities hold:

$$\begin{aligned}&(\Delta_{\Omega^\bullet(A)} \otimes \mathrm{id}) \circ \Delta_{\Omega^\bullet(A)} = (\mathrm{id} \otimes \Delta) \circ \Delta_{\Omega^\bullet(A)}; \\ &(\mathrm{id} \otimes \epsilon) \circ \Delta_{\Omega^\bullet(A)} = \mathrm{id}; \\ &(\mathrm{d} \otimes \mathrm{id}) \circ \Delta_{\Omega^\bullet(A)} = \Delta_{\Omega^\bullet(A)} \circ \mathrm{d}.\end{aligned} \tag{4.33}$$

Proof. Follows from theorem 3.2.11. □

The next proposition is used to construct a non-commutative analogue of the verticalising homeomorphism of classical differential geometry.

Proposition 4.2.8. *Let* $\Omega^\bullet(A)$ *be a complete calculus on* A. *Then there is a surjective morphism of differential graded algebras*

$$\pi_{\mathfrak{ver}} : \Omega^\bullet(A) \to \mathfrak{ver}^\bullet(A)$$

extending the identity $\mathrm{id}\colon A \to A$. *It is determined on homogeneous elements by*

$$\pi_{\mathfrak{ver}}|_{\Omega^k(A)} := (\mathrm{id} \otimes (\pi_{inv} \circ \pi_k)) \circ \Delta_A^\wedge|_{\Omega^k(A)} : \Omega^k(A) \to \mathfrak{ver}^k(A) \tag{4.34}$$

for all $k > 0$, *where* $\pi_k : \Omega^\bullet(H) \to \Omega^k(H)$ *is the obvious projection. Explicitly, we have*

$$\pi_{\mathfrak{ver}}(a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k) = a_0^0 a_0^1 \ldots a_0^k \otimes S(a_1^0 a_1^1 \ldots a_1^k) a_2^0 \mathrm{d}a_2^1 \wedge \cdots \wedge \mathrm{d}a_2^k \tag{4.35}$$

for all $a^0, \ldots, a^k \in A$.

Proof. Consider $\pi_{inv} : \Omega^\bullet(H) \to \Lambda^\bullet$ defined by $\pi_{inv}(\omega) = S(\omega_{-1})\omega_1$. We may equivalently write

$$\pi_{inv}(\omega) = \triangleright \circ (S \otimes \mathrm{id}) \circ {}_{\Gamma^\wedge}\Delta(\omega).$$

Let $\omega = a \in H$. We find

$$\begin{aligned}\pi_{inv}(\omega) &= \pi_{inv}(a)\\ &= \triangleright \circ (S \otimes \mathrm{id}) \circ {}_{\Gamma^\wedge}\Delta(a)\\ &= \triangleright \circ (S \otimes \mathrm{id}) \circ \Delta(a)\\ &= \triangleright \circ (S \otimes \mathrm{id}) \circ (a_1 \otimes a_2)\\ &= S(a_1)a_2\\ &= \epsilon(a)\end{aligned} \tag{4.36}$$

for any $\omega \in \Omega^\bullet(A)$. Let $\omega = \mathrm{d}b \in \Omega^1(H) = \Gamma$ for some $b \in H$. We have

$$\begin{aligned}\pi_{inv}(\omega) &= \pi_{inv}(\mathrm{d}b)\\ &= \triangleright \circ (S \otimes \mathrm{id}) \circ {}_{\Gamma^\wedge}\Delta(\mathrm{d}b)\\ &= \triangleright \circ (S \otimes \mathrm{id}) \circ {}_{\Gamma^\wedge}\Delta(\mathrm{d}b)\\ &= \triangleright \circ (S \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{d})(b_1 \otimes b_2)\\ &= \triangleright \circ (b_1 \otimes \mathrm{d}b_2)\\ &= S(b_1)\mathrm{d}b_2\\ &= S(b_1)\mathrm{d}b_2\\ &= \varpi(b).\end{aligned} \tag{4.37}$$

Let now $\omega \in \Omega^\bullet(H)$. Then $\omega = a^0\mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k$ for some $a^0, \ldots, a^k \in A$. We find

$$\begin{aligned}\pi_{inv}(\omega) &= \pi_{inv}(a^0\mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k)\\ &= \triangleright \circ (S \otimes \mathrm{id}) \circ {}_{\Gamma^\wedge}\Delta(a^0\mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k)\\ &= \triangleright \circ (S \otimes \mathrm{id}) \circ \Delta(a^0)(\wedge \otimes \wedge)(\mathrm{id} \otimes \mathrm{d})\Delta(a^1)(\wedge \otimes \wedge) \cdots (\wedge \otimes \wedge)(\mathrm{id} \otimes \mathrm{d})\Delta(a^k)\\ &= \triangleright \circ (S \otimes \mathrm{id}) \circ (a^0_1 \otimes a^0_2)(\wedge \otimes \wedge)(a^1_1 \otimes \mathrm{d}a^1_2)(\wedge \otimes \wedge) \cdots (\wedge \otimes \wedge)(a^k_1 \otimes \mathrm{d}a^k_2)\\ &= S(a^0_1 a^1_1 \cdots a^k_1)a^0_2\mathrm{d}a^1_2 \wedge \cdots \wedge \mathrm{d}a^k_2\\ &= \epsilon(a^0)S(a^1_1 a^2_1 \cdots a^k_1)\mathrm{d}a^1_2 \wedge \cdots \wedge \mathrm{d}a^k_2.\end{aligned} \tag{4.38}$$

Let us now consider a morphism $\pi_{\mathfrak{ver}} : \Omega^\bullet(A) \to \mathfrak{ver}^\bullet(A)$ by

$$\pi_{\mathfrak{ver}}(\omega) = (\mathrm{id} \otimes (\pi_{inv} \circ \pi^k)) \circ \Delta^\wedge_A(\omega). \tag{4.39}$$

$\pi_{\mathfrak{ver}}$ is obviously well defined. Moreover, given $a \in A$ we find

$$\pi_{\mathfrak{ver}}(a) = (\mathrm{id} \otimes \pi_{inv} \circ \pi_0)\Delta_A(a) = (a_1 \otimes \pi_{inv} \circ \pi_0(a_2)) = a \otimes 1 = a,$$

and so $\pi_{\mathfrak{ver}}|_A = \mathrm{id}_A$.

The morphism $\pi_{\mathfrak{ver}}$ is degree preserving. Indeed, given a k-form $\omega = a^0\mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k \in \Omega^k(A)$, we find

$$\begin{aligned}&(\mathrm{id} \otimes \pi_k)\Delta^\wedge_A(a^0\mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k)\\ &\quad= (\mathrm{id} \otimes \pi_k)\Delta_A(a^0)(\mathrm{id} \otimes \mathrm{d} + \mathrm{d} \otimes \mathrm{id})\Delta_A(a^1)(\wedge \otimes \wedge) \cdots (\wedge \otimes \wedge)(\mathrm{id} \otimes \mathrm{d} + \mathrm{d} \otimes \mathrm{id})\Delta_A(a^k)\\ &\quad= a^0_1 \cdots a^k_1 \otimes a^0_2\mathrm{d}a^1_2 \wedge \cdots \wedge \mathrm{d}a^k_2.\end{aligned} \tag{4.40}$$

Moreover

$$\begin{aligned}\pi_{\mathfrak{ver}}(\omega) &= a^0_1 \cdots a^k_1 \otimes \pi_{inv}(a^0_2\mathrm{d}a^1_2 \wedge \cdots \wedge \mathrm{d}a^k_2)\\ &= a^0_1 \cdots a^k_1 \otimes \epsilon(a^0_2)S(a^1_{2_1} a^2_{2_1} \cdots a^k_{2_1})\mathrm{d}a^1_{2_2} \wedge \cdots \wedge \mathrm{d}a^k_{2_2}\\ &= a^0_1\epsilon(a^0_2) \cdots a^k_1 \otimes S(a^1_1 \cdots a^k_1)\mathrm{d}a^1_2 \wedge \cdots \wedge \mathrm{d}a^k_2\\ &= a^0 a^1_0 \cdots a^k_0 \otimes S(a^1_1 \cdots a^k_1)\mathrm{d}a^1_2 \wedge \cdots \wedge \mathrm{d}a^k_2,\end{aligned} \tag{4.41}$$

providing an explicit formula for $\pi_{\mathfrak{ver}}$.

The second tensor factor in the above formula defines indeed a coinvariant form in $\Lambda^\bullet$. By induction:

$$\begin{aligned}
&{}_{\Gamma^\wedge}\Delta[S(a_1^1 \cdots a_1^{k+1})(\mathrm{d}a_2^1 \wedge \cdots \wedge \mathrm{d}a_2^{k+1})] \\
&\qquad = {}_{\Gamma^\wedge}\Delta[S(a_1^{k+1})S(a_1^1 \cdots a_1^k)(\mathrm{d}a_2^1 \wedge \cdots \wedge \mathrm{d}a_2^{k+1})] \\
&\qquad = {}_{\Gamma^\wedge}\Delta(S(a_1^{k+1})){}_{\Gamma^\wedge}\Delta[S(a_1^1 \cdots a_1^k)(\mathrm{d}a_2^1 \wedge \cdots \wedge \mathrm{d}a_2^k)]{}_{\Gamma}\Delta(\mathrm{d}a_2^{k+1}) \\
&\qquad = \Delta(S(a_1^{k+1}))[1 \otimes S(a_1^1 \cdots a_1^k)\mathrm{d}a_2^1 \wedge \cdots \wedge \mathrm{d}a_2^k](\mathrm{id} \otimes \mathrm{d})(a_0^{k+1} \otimes a_1^{k+1}) \\
&\qquad = (S(a_2^{k+1}) \otimes S(a_1^{k+1}))[1 \otimes S(a_1^1 \cdots a_1^k)\mathrm{d}a_2^1 \wedge \cdots \wedge \mathrm{d}a_2^k](\mathrm{id} \otimes \mathrm{d})(a_3^{k+1} \otimes a_4^{k+1}) \\
&\qquad = 1 \otimes S(a_1^1 \cdots a_1^{k+1})\mathrm{d}a_2^1 \wedge \cdots \wedge \mathrm{d}a_2^{k+1}.
\end{aligned}$$

We now show that $\pi_{\mathfrak{ver}}$ is a morphism of differential graded algebras. Thanks to lemma 3.29 this amounts to show

$$\begin{aligned}
\pi_{\mathfrak{ver}}(a^0\mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k) &= \pi_{\mathfrak{ver}}(a^0) \wedge_{\mathfrak{ver}} \mathrm{d}_{\mathfrak{ver}} \circ \pi_{\mathfrak{ver}}(a^1) \wedge_{\mathfrak{ver}} \cdots \wedge_{\mathfrak{ver}} \mathrm{d}_{\mathfrak{ver}} \circ \pi_{\mathfrak{ver}}(a^k) \\
&= (a^0 \otimes 1) \wedge_{\mathfrak{ver}} (a_0^1 \otimes \varpi(a_1^1)) \wedge_{\mathfrak{ver}} \cdots \wedge_{\mathfrak{ver}} (a_0^k \otimes \varpi(a_1^k)).
\end{aligned} \tag{4.42}$$

We proceed by induction. For a 1–form $\omega = a\mathrm{d}b \in \Omega^1(A)$ we have $\pi_{\mathfrak{ver}}(a\mathrm{d}b) = ab_0 \otimes \varpi(b_1)$, whereas

$$\begin{aligned}
\pi_{\mathfrak{ver}}(a) \wedge_{\mathfrak{ver}} \mathrm{d}_{\mathfrak{ver}} \circ \pi_{\mathfrak{ver}}(b_1) &= (a \otimes 1) \wedge_{\mathfrak{ver}} (b_0 \otimes \varpi(b_1)) \\
&= ab_0 \otimes (1 \leftharpoondown b_1) \wedge \varpi(b_2) \\
&= ab_0 \otimes S(b_1)b_2 \wedge \varpi(b_3) \\
&= ab_0 \otimes \varpi(b_1).
\end{aligned} \tag{4.43}$$

Assuming (4.42) hold for k–forms we have

$$\begin{aligned}
&\pi_{\mathfrak{ver}}(a^0) \wedge_{\mathfrak{ver}} \mathrm{d}_{\mathfrak{ver}} \circ \pi_{\mathfrak{ver}}(a^k) \wedge_{\mathfrak{ver}} \cdots \wedge_{\mathfrak{ver}} \mathrm{d}_{\mathfrak{ver}} \circ \pi_{\mathfrak{ver}}(a^{k+1}) \\
&\quad = \pi_{\mathfrak{ver}}(a^0\mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k) \wedge_{\mathfrak{ver}} \mathrm{d}_{\mathfrak{ver}} \circ \pi_{\mathfrak{ver}}(a^{k+1}) \\
&\quad = (a^0a_0^1 \cdots a_0^k) \otimes S(a_1^1 \cdots a_1^k)\mathrm{d}a_2^1 \wedge \cdots \wedge \mathrm{d}a_2^k \wedge_{\mathfrak{ver}} (a_0^{k+1} \otimes \varpi(a_1^{k+1})) \\
&\quad = (a^0a_0^1 \cdots a_0^{k+1}) \otimes (S(a_1^1 \cdots a_1^k)\mathrm{d}a_2^1 \wedge \cdots \wedge \mathrm{d}a_2^k \leftharpoondown a_1^{k+1}) \wedge \varpi(a_2^{k+1}) \\
&\quad = a^0a_0^1 \cdots a_0^{k+1} \otimes S(a_1^{k+1})S(a_1^1 \cdots a_1^k)\mathrm{d}a_2^1 \wedge \cdots \wedge \mathrm{d}a_2^k a_2^{k+1} \wedge S(a_3^{k+1})\mathrm{d}a_4^{k+1} \\
&\quad = a^0a_0^1 \cdots a_0^{k+1} \otimes S(a_1^{k+1})S(a_1^1 \cdots a_1^k)\mathrm{d}a_2^1 \wedge \cdots \wedge \mathrm{d}a_2^k \wedge a_2^{k+1}S(a_3^{k+1})\mathrm{d}a_4^{k+1} \\
&\quad = a^0a_0^1 \cdots a_0^{k+1} \otimes S(a_1^1 \cdots a_1^{k+1})\mathrm{d}a_2^1 \wedge \cdots \wedge \mathrm{d}a_2^{k+1}.
\end{aligned} \tag{4.44}$$

By proposition 3.2.3 we now know that $\pi_{\mathfrak{ver}} : \Omega^\bullet(A) \to \mathfrak{ver}^\bullet(A)$ is unique as extension of $\mathrm{id}_A : A \to A$. Moreover $\pi_{\mathfrak{ver}}$ is also surjective as id_A is.

□

Proposition 4.2.9. *The vertical forms $\mathfrak{ver}^\bullet(A)$ are complete, i.e. the right H-coaction $\Delta_A \colon A \to A \otimes H$ extends to a morphism $\Delta_{\mathfrak{ver}}^\wedge \colon \mathfrak{ver}^\bullet(A) \to \mathfrak{ver}^\bullet(A) \otimes \Omega^\bullet(H)$ of differential graded algebras. Moreover, the diagram*

$$\begin{array}{ccc}
\Omega^\bullet(A) & \xrightarrow{\ \Delta_A^\wedge\ } & \Omega^\bullet(A) \otimes \Omega^\bullet(H) \\
\downarrow{\scriptstyle \pi_{\mathfrak{ver}}} & & \downarrow{\scriptstyle (\pi_{\mathfrak{ver}} \otimes \mathrm{id})} \\
\mathfrak{ver}^\bullet(A) & \xrightarrow{\ \Delta_{\mathfrak{ver}}^\wedge\ } & \mathfrak{ver}^\bullet(A) \otimes \Omega^\bullet(H)
\end{array}$$

commutes, i.e. $\Delta_{\mathfrak{ver}}^\wedge \circ \pi_{\mathfrak{ver}} = (\pi_{\mathfrak{ver}} \otimes \mathrm{id})\Delta_A^\wedge$.

Proof. We want to define $\Delta^\wedge_{\mathfrak{ver}}$ by commutativity of the above diagram. To that purpose, let us consider a morphism

$$\widetilde{\Delta}^\wedge_{\mathfrak{ver}} : \Omega^\bullet(A) \to \mathfrak{ver}^\bullet \otimes \Omega^\bullet(H), \quad \omega \mapsto (\pi_{\mathfrak{ver}} \otimes \mathrm{id}) \circ \Delta^\wedge_A(\omega). \tag{4.45}$$

Let $\omega = a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k \in \Omega^k(A)$. If in particular $\omega \in \ker \pi_{\mathfrak{ver}}$ we have

$$\pi_{\mathfrak{ver}}(\omega) = a^0_0 a^1_0 \dots a^k_0 \otimes S(a^0_1 \dots a^k_1) a^0_2 \mathrm{d}a^1_2 \wedge \cdots \wedge \mathrm{d}a^k_2 = 0. \tag{4.46}$$

We now show that even $\widetilde{\Delta}^\wedge_{\mathfrak{ver}}(\omega) = 0$. Explicitly it reads

$$\begin{aligned}
\widetilde{\Delta}^\wedge_{\mathfrak{ver}}(\omega) &= (\pi_{\mathfrak{ver}} \otimes \mathrm{id}) \circ \Delta^\wedge_A(a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k) \\
&= \pi_{\mathfrak{ver}}(a^0_0 \mathrm{d}a^1_0 \wedge \cdots \wedge \mathrm{d}a^k_0) \otimes a^0_1 \cdots a^k_1 \\
&+ \pi_{\mathfrak{ver}}(a^0_0 a^1_0 \mathrm{d}a^2_0 \wedge \cdots \wedge \mathrm{d}a^k_0) \otimes a^0_1 \mathrm{d}a^1_1 \cdots a^k_0 \\
&+ \cdots \\
&+ \pi_{\mathfrak{ver}}(a^0_0 \cdots a^{k-1}_0 \mathrm{d}a^k_0) \otimes a^0_1 \mathrm{d}a^1_1 \wedge \cdots \wedge \mathrm{d}a^1_{k-1} a^k_0 \\
&+ \pi_{\mathfrak{ver}}(a^0_0 \cdots a^k_0) \otimes a^0_1 \mathrm{d}a^1_1 \wedge \cdots \wedge \mathrm{d}a^k_1,
\end{aligned} \tag{4.47}$$

where the dots stand for all the other possible combinations in $\Omega^r(A) \otimes \Omega^s(H)$ such that $s + r = k$.

Considering the first tensor factor of the summand in the above expression we find

$$\begin{aligned}
&(\Delta_A \otimes \Delta_{\Omega^k(A)}) \circ \pi_{\mathfrak{ver}}(a^0_0 \mathrm{d}a^1_0 \wedge \cdots \wedge \mathrm{d}a^k_0) \\
&= (\Delta_A \otimes \Delta_{\Omega^k(A)}) \circ (a^0_0 \cdots a^k_0 \otimes S(a^0_1 \cdots a^k_1) a^0_2 \mathrm{d}a^1_2 \wedge \cdots \wedge \mathrm{d}a^k_2) \\
&= a^0_0 \cdots a^k_0 \otimes a^0_1 \cdots a^k_1 \otimes S(a^0_1 \cdots a^k_1)_0 a^0_{20} \mathrm{d}a^1_{20} \wedge \cdots \mathrm{d}a^k_{20} \otimes S(a^0_1 \cdots a^k_1)_1 a^0_{21} \cdots a^k_{21} \\
&= a^0_0 \cdots a^k_0 \otimes a^0_1 \cdots a^k_1 \otimes S(a^0_{11} \cdots a^k_{11}) a^0_{20} \mathrm{d}a^1_{20} \wedge \cdots \wedge a^k_{20} \otimes S(a^0_{10} \cdots a^k_{10}) a^0_{21} \cdots a^k_{21} \\
&= a^0_0 \cdots a^k_0 \otimes a^0_1 \cdots a^k_1 \otimes S(a^0_3 \cdots a^k_3) a^0_4 \mathrm{d}a^1_4 \wedge \cdots \wedge \mathrm{d}a^k_4 \otimes S(a^0_2 \cdots a^k_2) a^0_5 \cdots a^k_5.
\end{aligned}$$

Multiplying the second and last tensor factors in the result of the last expression we have

$$a^0_0 \cdots a^k_0 \otimes S(a^0_1 \cdots a^k_1) a^0_2 \mathrm{d}a^1_2 \wedge \cdots \wedge \mathrm{d}a^k_2 \otimes a^0_3 \cdots a^k_3 = 0, \tag{4.48}$$

according to Equation (4.46). Therefore

$$a^0_0 \cdots a^k_0 \otimes S(a^0_1 \cdots a^k_1) a^0_2 \mathrm{d}a^1_2 \wedge \cdots \wedge \mathrm{d}a^k_2 \otimes a^0_3 \cdots a^k_3 \otimes a^0_4 \cdots a^k_4 = 0. \tag{4.49}$$

The above reasoning can be repeated for every summand in Equation (4.47). Therefore we conclude $\widetilde{\Delta}^\wedge_{\mathfrak{ver}}(\omega) = 0$. Thus, $\widetilde{\Delta}^\wedge_{\mathfrak{ver}} : \Omega^\bullet(A) \to \mathfrak{ver}^\bullet(A) \otimes \Omega^\bullet(H)$ descends to the morphism

$$\Delta^\wedge_{\mathfrak{ver}} : \mathfrak{ver}^\bullet(A) \cong \Omega^\bullet(A)/\ker \pi_{\mathfrak{ver}} \to \Omega^\bullet(A) \otimes \Omega^\bullet(H), \tag{4.50}$$

which makes the diagram commute. Since every other map in the diagram is a morphism of differential graded algebras also $\Delta^\wedge_{\mathfrak{ver}}$ is. □

Example 4.2.10. We consider a 2-form $\omega = a\mathrm{d}b \wedge \mathrm{d}c \in \Omega^2(A)$ in the kernel of $\pi_{\mathfrak{ver}}$, i.e.

$$a_0 b_0 c_0 \otimes S(a_1 b_1 c_1) a_2 \mathrm{d}b_2 \wedge \mathrm{d}c_2 = 0, \tag{4.51}$$

and show that $\widetilde{\Delta}^\wedge_{\mathfrak{ver}}(\omega) = 0$. We have

$$\begin{aligned}
\Delta^\wedge_A(a\mathrm{d}b \wedge \mathrm{d}c) &= a_0 b_0 c_0 \otimes a_1 \mathrm{d}b_1 \wedge \mathrm{d}c_1 \\
&+ a_0 \mathrm{d}b_0 \wedge \mathrm{d}c_0 \otimes a_1 b_1 c_1 \\
&+ a_0 \mathrm{d}b_0 c_0 \otimes a_1 b_1 \mathrm{d}c_1 \\
&- a_0 b_0 \mathrm{d}c_0 \otimes a_1 \mathrm{d}b_1 c_1.
\end{aligned} \tag{4.52}$$

Accordingly

$$\begin{aligned}\widetilde{\Delta}^{\wedge}_{\mathfrak{ver}}(\omega) &= (\pi_{\mathfrak{ver}} \otimes \mathrm{id}) \circ \Delta^{\wedge}_{A}(a\mathrm{d}b \wedge \mathrm{d}c)\\ (i) \quad &= a_0 b_0 c_0 \otimes a_1 b_1 c_1 \otimes a_2 \mathrm{d}b_2 \wedge \mathrm{d}c_2\\ (ii) \quad &+ a_0 b_0 c_0 \otimes S(a_1 b_1 c_1) a_2 \mathrm{d}b_2 \wedge \mathrm{d}c_2 \otimes a_3 b_3 c_3\\ (iii) \quad &+ a_0 b_0 c_0 \otimes S(a_1 b_1 c_1) a_2 \mathrm{d}b_2 c_2 \otimes a_3 b_3 \mathrm{d}c_3\\ (iv) \quad &- a_0 b_0 c_0 \otimes S(a_1 b_1 c_1) a_2 b_2 \mathrm{d}c_2 \otimes a_3 \mathrm{d}b_3 c_3.\end{aligned} \tag{4.53}$$

Considering Equation (4.51) we coact on the first tensor factor to obtain

$$\begin{aligned}(\Delta_A \otimes \mathrm{id})(a_0 b_0 c_0 \otimes S(a_1 b_1 c_1) a_2 \mathrm{d}b_2 \wedge \mathrm{d}c_2) &= 0\\ &= a_0 b_0 c_0 \otimes a_1 b_1 c_1 \otimes S(a_2 b_2 c_2) a_3 \mathrm{d}b_3 \wedge \mathrm{d}c_3.\end{aligned} \tag{4.54}$$

Multiplying the first and last tensor factor we have

$$a_1 b_1 c_1 \otimes a_2 \mathrm{d}b_2 \wedge \mathrm{d}c_2 = 0. \tag{4.55}$$

Therefore term (i) of equation (4.53) vanishes. Term (ii) is zero by the derivation in Proposition (4.2.9). Finally, considering again Equation (4.51) we have

$$\begin{aligned}(\Delta_A \otimes \pi_1 \otimes \pi_1) \circ (\mathrm{id} \otimes \Delta^{\bullet})\Delta_A(a_0 b_0 c_0) \otimes S(a_1 b_1 c_1) a_2 \mathrm{d}b_2 \wedge \mathrm{d}c_3) &= 0\\ &= \Delta_A(a_0 b_0 c_0) \otimes (\pi_1 \otimes \pi_1)\Delta^{\bullet}(S(a_1 b_1 c_1) a_2 \mathrm{d}b_2 \wedge \mathrm{d}c_2)\\ &= \Delta_A(a_0 b_0 c_0) \otimes S(a_1 b_1 c_1)_1 a_2 \mathrm{d}b_2 c_2 \otimes S(a_1 b_1 c_1)_2 a_3 b_3 \mathrm{d}c_3\\ &- \Delta_A(a_0 b_0 c_0) \otimes S(a_1 b_1 c_1)_1 a_2 b_2 \mathrm{d}c_2 \otimes S(a_1 b_1 c_1)_2 a_3 \mathrm{d}b_3 c_3\\ &= a_0 b_0 c_0 \otimes a_1 b_1 c_1 \otimes S(a_{22} b_{22} c_{22}) a_3 \mathrm{d}b_3 c_3 \otimes S(a_{21} b_{21} c_{21}) a_4 b_4 \mathrm{d}c_4\\ &- a_0 b_0 c_0 \otimes a_1 b_1 c_1 \otimes S(a_{22} b_{22} c_{22}) a_3 b_3 \mathrm{d}c_3 \otimes S(a_{21} b_{21} c_{21}) a_4 \mathrm{d}b_4 c_4\\ &= a_0 b_0 c_0 \otimes a_1 b_1 c_1 \otimes S(a_3 b_3 c_3) a_4 \mathrm{d}b_4 c_4 \otimes S(a_2 b_2 c_2) a_5 b_5 \mathrm{d}c_5\\ &- a_0 b_0 c_0 \otimes a_1 b_1 c_1 \otimes S(a_3 b_3 c_3) a_4 b_4 \mathrm{d}c_4 \otimes S(a_2 b_2 c_2) a_5 \mathrm{d}b_5 c_5.\end{aligned} \tag{4.56}$$

Multiplying the second and last tensor factors in the last expression's result we have

$$a_0 b_0 c_0 \otimes S(a_1 b_1 c_1) a_2 \mathrm{d}b_2 c_2 \otimes a_3 b_3 \mathrm{d}c_3 - a_0 b_0 c_0 \otimes S(a_1 b_1 c_1) a_2 b_2 \mathrm{d}c_2 \otimes a_3 \mathrm{d}b_3 c_3 = 0, \tag{4.57}$$

and this holds if and only both summands are zero. Therefore terms (iii) and (iv) of Equations (4.53) vanish. Accordingly $\widetilde{\Delta}^{\wedge}_{\mathfrak{ver}}(\omega) = 0$. □

Remark 4.2.11. It is also possible to work out the explicit form for $\pi_{\mathfrak{ver}}$ by assuming it to be the unique morphism of differential graded algebras extending the identity $\mathrm{id} : A \to A$. Let $a\mathrm{d}b \in \Omega^1(A)$. We find

$$\begin{aligned}\pi_{\mathfrak{ver}}(a\mathrm{d}b) &= a\pi_{\mathfrak{ver}}(\mathrm{d}b)\\ &= a\mathrm{d}_{\mathfrak{ver}}(b \otimes 1)\\ &= a(b_0 \otimes \varpi(b_1))\\ &= ab_0 \otimes S(b_{11})\,\mathrm{d}(b_{12})\\ &= ab_0 \otimes S(b_1)\mathrm{d}b_2.\end{aligned} \tag{4.58}$$

Let now $a\mathrm{d}b \wedge \mathrm{d}c \in \Omega^2(A)$. We have

$$\begin{aligned}\pi_{\mathfrak{ver}}(a\mathrm{d}b \wedge \mathrm{d}c) &= (ab_0 \otimes S(b_1)\mathrm{d}b_2) \wedge_{\mathfrak{ver}} (c_0 \otimes \varpi(c_1))\\ &= ab_0 c_0 \otimes (\varpi(b_1) \leftharpoonup c_1) \wedge \varpi(c_2)\\ &= ab_0 c_0 \otimes S(c_{10})\varpi(b_1)c_{11} \wedge \varpi(c_2)\\ &= ab_0 c_0 \otimes S(c_1)S(b_1)\mathrm{d}b_2 c_2 \wedge S(c_3)\mathrm{d}c_4\\ &= ab_0 c_0 \otimes S(b_1 c_1)\mathrm{d}b_2 \wedge c_2 S(c_3)\mathrm{d}c_4\\ &= ab_0 c_0 \otimes S(b_1 c_1)(\mathrm{d}b_2 \wedge \mathrm{d}c_2).\end{aligned} \tag{4.59}$$

Generalising for $\omega = a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k \in \Omega^k(A)$ it is easy to see

$$\pi_{\mathfrak{ver}}(a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k) = a^0 a^1_0 \cdots a^k_0 \otimes S(a^1_1 \cdots a^k_1)(\mathrm{d}a^1_2 \wedge \cdots \wedge \mathrm{d}a^k_2). \tag{4.60}$$

□

We now introduce the non-commutative analogue of horizontal forms of classical geometry.

Definition 4.2.12. Let $\Omega^\bullet(A)$ be a complete differential calculus on the quantum principal bundle $B := A^{coH} \subseteq A$. The horizontal forms of the bundle are defined as elements of the $\Omega^\bullet(A)$−subalgebra

$$\mathfrak{hor}^\bullet(A) := (\Delta^\wedge_A)^{-1}(\Omega^\bullet(A) \otimes H). \tag{4.61}$$

Notice that since since $\Omega^0(A) = A$ we have $(\Delta^\wedge_A)^{-1}(A \otimes H) = (\Delta_A)^{-1}(A \otimes H) = A$, i.e. $\mathfrak{hor}^0(A) = A$. This also means $\Omega^0(A) = \mathfrak{ver}^0(A) = \mathfrak{hor}^0(A)$.

Proposition 4.2.13. $\mathfrak{hor}^\bullet(A)$ *is a graded algebra. In other words, given* $\omega \in \mathfrak{hor}^k(A)$ *and* $\gamma \in \mathfrak{hor}^\ell(A)$ *we have* $\omega \wedge \gamma \in \mathfrak{hor}^{k+\ell}(A)$.

Proof. Given ω and γ as above we find

$$\Delta^\wedge_A(\omega \wedge \gamma) = \Delta^\wedge_A(\omega)(\wedge \otimes \mathrm{id})\Delta^\wedge_A(\gamma) \in (\Omega^k(A) \wedge \Omega^\ell(A)) \otimes H = \Omega^{k+\ell}(A) \otimes H, \tag{4.62}$$

therefore the wedge product of horizontal forms gives an horizontal forms. □

Lemma 4.2.14. *The graded algebra* $\mathfrak{hor}^\bullet(A)$ *is* $\Delta^\wedge_A$*−invariant, namely*

$$\Delta^\wedge_A(\mathfrak{hor}^\bullet(A)) \subseteq \mathfrak{hor}^\bullet(A) \otimes H. \tag{4.63}$$

In other words, $\mathfrak{hor}^\bullet(A)$ *is a right* H*−comodule algebra.*

Proof. Let $\varphi \in \mathfrak{hor}^\bullet(A)$. By the definition of horizontal forms we have

$$\begin{aligned}(\Delta^\wedge_A \otimes \mathrm{id}) \circ \Delta^\wedge_A(\varphi) &= (\mathrm{id} \otimes \Delta^\bullet) \circ \Delta^\wedge_A(\varphi) \\ &= (\mathrm{id} \otimes \Delta) \circ \Delta_{\Omega^\bullet_A}(\varphi),\end{aligned} \tag{4.64}$$

which is an element in $\Omega^\bullet(A) \otimes H \otimes H$; accordingly also $(\Delta^\wedge_A \otimes \mathrm{id}) \circ \Delta^\wedge_A(\varphi)$ is an element of the same space. It follows that $\Delta^\wedge_A(\varphi) \in \mathfrak{hor}^\bullet(A) \otimes H$, since then

$$\Delta^\wedge_A(\mathfrak{hor}^\bullet(A)) \otimes H = \Delta^\wedge_A((\Delta^\wedge_A)^{-1}(\Omega^\bullet(A) \otimes H)) \otimes H = \Omega^\bullet(A) \otimes H \otimes H.$$

Accordingly $\mathfrak{hor}^\bullet(A)$ is a right H−comodule algebra by the coaction

$$\Delta_{\mathfrak{hor}} := \Delta^\wedge_A|_{\mathfrak{hor}^\bullet(A)} : \mathfrak{hor}^\bullet(A) \to \mathfrak{hor}^\bullet(A) \otimes H.$$

□

Remark 4.2.15. Horizontal forms are not closed under the differential. This is true already for horizontal 1−forms. Indeed

$$\begin{aligned}\Delta^\wedge_A(\mathrm{d}\omega) &= (\mathrm{id} \otimes \mathrm{d} + \mathrm{d} \otimes \mathrm{id})\Delta^\wedge_A(\omega) \\ &= (\mathrm{id} \otimes \mathrm{d} + \mathrm{d} \otimes \mathrm{id})\Delta^\wedge_A(a\mathrm{d}b) \\ &= (\mathrm{id} \otimes \mathrm{d} + \mathrm{d} \otimes \mathrm{id})(a_0 \mathrm{d}b_0 \otimes a_1 b_1) \\ &= -a_0 \mathrm{d}b_0 \otimes \mathrm{d}(a_1 b_1) + \mathrm{d}a_0 \wedge \mathrm{d}b_0 \otimes a_1 b_1,\end{aligned} \tag{4.65}$$

so $\Delta^\wedge_A(\mathrm{d}\omega) \in (\Omega^1(A) \otimes \Omega^1(H)) \oplus (\Omega^2(A) \otimes H)$. □

The next lemma provides an exact sequence of A–modules involving vertical, horizontal and total space forms. The result we provide holds only for 1–forms and is false for higher orders. This, as we will see, happens precisely because forms in the kernel of $\pi_{\mathfrak{ver}}$ are not necessarily horizontal for $k > 1$.

Lemma 4.2.16. *There is a short-exact sequence of A–modules given by*

$$0 \longrightarrow \mathfrak{hor}^1(A) \xhookrightarrow{\iota} \Omega^1(A) \xrightarrow{\pi_{\mathfrak{ver}}} \mathfrak{ver}^1(A) \longrightarrow 0.$$

Proof. Injectivity of ι is obvious and surjectivity of $\pi_{\mathfrak{ver}}$ follows from proposition 4.2.8. Therefore we only need to prove that the kernel of $\pi_{\mathfrak{ver}}$ equals the image of ι, i.e. that $\ker(\pi_{\mathfrak{ver}}) = \mathfrak{hor}^\bullet(A)$.

Let $\varphi \in \mathfrak{hor}^k(A)$. We find

$$\begin{aligned} \pi_{\mathfrak{ver}}(\varphi) &= (\mathrm{id} \otimes \pi_{inv})(\mathrm{id} \otimes \pi_k)\Delta_A^\wedge(\varphi) \\ &= (\mathrm{id} \otimes \pi_{inv})(\mathrm{id} \otimes \pi_k)(\theta \otimes h) \\ &= \theta \otimes 0 = 0, \end{aligned} \tag{4.66}$$

so $\mathfrak{hor}^\bullet(A) \subseteq \ker(\pi_{\mathfrak{ver}})$, and in particular $\mathfrak{hor}^1(A) \subseteq \ker(\pi_{\mathfrak{ver}})$.

Let $\omega \in \ker(\pi_{\mathfrak{ver}}) \cap \Omega^1(A)$. We have

$$\pi_{\mathfrak{ver}}(\omega) = (\mathrm{id} \otimes \pi_{inv})(\mathrm{id} \otimes \pi_1)\Delta_A^\wedge(\omega) = 0, \tag{4.67}$$

which explicitely reads

$$ab_0 \otimes S(b_1)\mathrm{d}b_2 = 0. \tag{4.68}$$

Applying $(\mathrm{id} \otimes \triangleright) \circ (\mathrm{id} \otimes \Delta_A)$ to $\pi_{\mathfrak{ver}}(\omega)$ we find

$$\begin{aligned} (\mathrm{id} \otimes \triangleright) \circ (\mathrm{id} \otimes \Delta_A) \circ (ab_0 \otimes S(b_1)\mathrm{d}b_2) &= (\mathrm{id} \otimes \triangleright) \circ ((a_0 \otimes a_1)(b_0 \otimes b_1) \otimes S(b_2)\mathrm{d}b_3) \\ &= a_0 b_0 \otimes a_1 \mathrm{d}b_1 = 0. \end{aligned} \tag{4.69}$$

But then

$$\Delta_A^\wedge(a\mathrm{d}b) = a_0 b_0 \otimes a_1 \mathrm{d}b_1 + a_0 \mathrm{d}b_0 \otimes a_1 b_1 = a_0 \mathrm{d}b_0 \otimes a_1 b_1, \tag{4.70}$$

which means $\Delta_A^\wedge(\omega) \in \Omega^1(A) \otimes H$, i.e. $\omega \in \mathfrak{hor}^1(A)$. □

Remark 4.2.17. As we discussed this exact sequence fails on $\Omega^k(A)$ for $k > 1$. Indeed, given $\omega \in \ker(\pi_{\mathfrak{ver}}) \cap \Omega^k(A)$ we have

$$\Delta_A^\wedge(\omega) \in (A \otimes \Omega^k(H)) \oplus (\Omega^1(A) \otimes \Omega^{k-1}(H)) \oplus \cdots \oplus (\Omega^k(A) \otimes H),$$

and the fact that ω is also in the kernel of $\pi_{\mathfrak{ver}}$ is sufficient to obtain

$$\pi_{\mathfrak{ver}}(a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k) = a^0 a_0^1 \cdots a_0^k \otimes S(a_1^1 \cdots a_1^k)\mathrm{d}a_2^1 \wedge \cdots \wedge \mathrm{d}a_2^k = 0, \tag{4.71}$$

from which

$$(\Delta_A \otimes \mathrm{id}) \circ \pi_{\mathfrak{ver}}(\omega) = a_0^0 \cdots a_0^k \otimes a_1^0 \cdots a_1^k \otimes S(a_2^0 \cdots a_1^k a_2^0) a_3^0 \mathrm{d}a_3^1 \wedge \cdots \wedge \mathrm{d}a_3^k = 0. \tag{4.72}$$

Multiplying the second and last tensor factor we obtain

$$a_0^0 \cdots a_0^k \otimes a_1^0 \mathrm{d}a_1^1 \wedge \cdots \wedge \mathrm{d}a_1^k = 0, \tag{4.73}$$

which is not enough to conclude that ω is also horizontal. □

4.2.3 Base space calculus

Definition 4.2.18. Given a complete calculus $\Omega^\bullet(A)$ on a quantum principal bundle $B := A^{coH} \subseteq A$ we define the base space forms as

$$\Omega^\bullet(B) := \left\{\omega \in \Omega^\bullet(A) \ : \ \Delta_A^\wedge(\omega) = \omega \otimes 1\right\}. \tag{4.74}$$

The differential on $\Omega^\bullet(B)$ is the restriction of the differential on $\Omega^\bullet(A)$.

Recall that a first order differential calculus is called *connected* if the kernel of the differential is solely made by scalars, i.e. $\ker \mathrm{d} = \mathbb{k}$. The next proposition gives a characterisation of the base space forms under connectedness assumption on the total space calculus.

Proposition 4.2.19. *Given a connected, bicovariant, first order differential calculus (Γ, d) on H we have the identification $\Omega^\bullet(B) = \mathfrak{hor}^\bullet(A) \cap \mathrm{d}^{-1}(\mathfrak{hor}^\bullet(A))$, in other words the base space calculus consists of horizontal forms which are mapped to horizontal forms by the differential.*

Proof. Let $\omega \in \Omega^k(B)$. Then $\Delta_A^\wedge(\omega) = \omega \otimes 1$, i.e. $\Delta_A^\wedge(\omega) \in \Omega^k(A) \otimes H$, that implies $\omega \in \mathfrak{hor}^k(A)$. Moreover

$$\begin{aligned}\Delta_A^\wedge(\omega) &= (\mathrm{d} \otimes \mathrm{id} + (-1)^k \mathrm{id} \otimes \mathrm{d})\Delta_A^\wedge(\omega)\\ &= (\mathrm{d} \otimes \mathrm{id} + (-1)^k \mathrm{id} \otimes \mathrm{d})(\omega \otimes 1)\\ &= \mathrm{d}\omega \otimes 1,\end{aligned} \tag{4.75}$$

and so $\mathrm{d}\omega \in \mathfrak{hor}^{k+1}(A)$ and accordingly $\omega \in \mathfrak{hor}^k(A) \cap \mathrm{d}^{-1}(\mathfrak{hor}^{k+1}(A))$.

Let now $\omega \in \mathfrak{hor}^k(A) \cap \mathrm{d}^{-1}(\mathfrak{hor}^{k+1}(A))$. Then

$$\Delta_A^\wedge(a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k) = a_0^0 \mathrm{d}a_0^1 \wedge \cdots \wedge \mathrm{d}a_0^k \otimes a_1^0 \cdots a_1^k, \tag{4.76}$$

and moreover

$$\begin{aligned}\Delta_A^\wedge(\mathrm{d}\omega) &= (\mathrm{d} \otimes \mathrm{id} + (-1)^k \mathrm{id} \otimes \mathrm{d})\Delta_A^\wedge(\omega)\\ &= (\mathrm{d} \otimes \mathrm{id} + (-1)^k \mathrm{id} \otimes \mathrm{d})(a_0^0 \mathrm{d}a_0^1 \wedge \cdots \wedge \mathrm{d}a_0^k \otimes a_1^0 \cdots a_1^k)\\ &= \mathrm{d}a_0^0 \wedge \cdots \wedge \mathrm{d}a_0^k \otimes a_1^0 \cdots a_1^k\\ &+ (-1)^k a_0^0 \mathrm{d}a_0^1 \wedge \cdots \wedge \mathrm{d}a_0^k \otimes \mathrm{d}\left(a_1^0 \cdots a_1^k\right)\\ &= \mathrm{d}a_0^0 \wedge \cdots \wedge \mathrm{d}a_0^k \otimes a_1^0 \cdots a_1^k,\end{aligned} \tag{4.77}$$

and therefore $a_0^0 \mathrm{d}a_0^1 \wedge \cdots \wedge \mathrm{d}a_0^k \otimes \mathrm{d}(a_1^0 \cdots a_1^k) = 0$. By connectedness assumption on the first order differential calculus (Γ, d) on H we have $a_1^0 \cdots a_1^k \in \mathbb{k}$. Accordingly

$$\Delta_A^\wedge(\omega) \in \Omega^k(A) \otimes \mathbb{k} \quad \Rightarrow \quad \Delta_A^\wedge(\omega) = \omega \otimes 1 \quad \Rightarrow \quad \omega \in \Omega^k(B).$$

□

Proposition 4.2.20. *Elements of the form $B\mathrm{d}B$ are contained in $\Omega^1(B)$.*

Proof. Let $b, b' \in B$ and consider $b\mathrm{d}b'$. As $B := \{a \in A \ : \ \Delta_A(a) = a \otimes 1\}$, we find

$$\begin{aligned}\Delta_A^\wedge(b\mathrm{d}b') &= \Delta_A(b)(\wedge \otimes \wedge)\Delta_A(\mathrm{d}b')\\ &= (b \otimes 1)(\wedge \otimes \wedge)(\mathrm{d} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{d})\Delta_A(b')\\ &= (b \otimes 1)(\wedge \otimes \wedge)(\mathrm{d}b' \otimes 1)\\ &= (b\mathrm{d}b' \otimes 1),\end{aligned} \tag{4.78}$$

i.e. $b\mathrm{d}b' \in \Omega^1(B)$. This reasoning can be carried for higher order forms as follows. Consider $b^0\mathrm{d}b^1 \wedge \cdots \wedge \mathrm{d}b^k$, where $b^0, \dots, b^k \in B$. Then

$$\begin{aligned}\Delta_A^\wedge(b^0\mathrm{d}b^1 \wedge \cdots \wedge \mathrm{d}b^k) &= \Delta_A(b^0)(\wedge \otimes \wedge)(\mathrm{d} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{d})\Delta_A(b^1)(\wedge \otimes \wedge)\cdots \\ &\cdots(\wedge \otimes \wedge)(\mathrm{d} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{d})\Delta_A(b^k) \\ &= b^0\mathrm{d}b^1 \wedge \cdots \mathrm{d}b^k \otimes 1,\end{aligned} \tag{4.79}$$

and so $B\mathrm{d}B \wedge \cdots \wedge \mathrm{d}B \subseteq \Omega^k(B)$. □

Remark 4.2.21. The other inclusion, $\Omega^1(B) \subseteq B\mathrm{d}B$ does not hold in general. Indeed, let $\omega \in \Omega^1(A)$ be such that $\Delta_A^\wedge(\omega) = \Delta_A^\wedge(a\mathrm{d}a') = a\mathrm{d}a' \otimes 1$. We have

$$\begin{aligned}\Delta_A^\wedge(a\mathrm{d}a') &= \Delta_A(a)(\wedge \otimes \wedge)(\mathrm{d} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{d})\Delta_A(a') \\ &= a_0\mathrm{d}a'_0 \otimes a_1a'_1 + a_0a'_0 \otimes a_1\mathrm{d}a'_1 \\ &= a\mathrm{d}a' \otimes 1\end{aligned} \tag{4.80}$$

if and only if $a_0\mathrm{d}a'_0 \otimes a_1a'_1 = a\mathrm{d}a' \otimes 1$ and $a_0a'_0 \otimes a_1\mathrm{d}a'_1 = 0$.

Proposition 4.2.22. *Differential forms over the base space B are exactly the horizontal, right H−coinvariant forms; in other words $\Omega^\bullet(B) = \mathfrak{hor}^\bullet(A) \cap \Omega^\bullet(A)^{coH}$.*

Proof. Let $\omega \in \Omega^k(B)$ be a differential k−form on the base space. By definition $\omega = a^0\mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k$ such that $\Delta_A^\wedge(\omega) = \omega \otimes 1$. Explicitly we have

$$\begin{aligned}\Delta_A^\wedge(a^0\mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k) &= a_0^0\mathrm{d}a_0^1 \wedge \cdots \wedge \mathrm{d}a_0^k \otimes a_1^0 \cdots a_1^k \\ &+ (-1)^{k-1}a_0^0\mathrm{d}a_0^1 \wedge \cdots \wedge \mathrm{d}a_0^{k-1}a_0^k \otimes a_1^0 \cdots a^{k-1}\mathrm{d}a_1^k \\ &+ \dots \\ &- a_0^0\mathrm{d}a_0^1a_0^2 \cdots a_0^k \otimes a_1^0a_1^1\mathrm{d}a_1^2 \wedge \cdots \wedge \mathrm{d}a_1^k \\ &+ a_0^0 \cdots a_0^k \otimes a_1^0\mathrm{d}a_1^1 \wedge \cdots \wedge \mathrm{d}a_1^k,\end{aligned} \tag{4.81}$$

and since

$$\Delta_A^\wedge(a^0\mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k) = a^0\mathrm{d}a^1 \wedge \cdots \mathrm{d}a^k \otimes 1,$$

every summand in the above equation, except the first, vanish. This implies $\Delta_A^\wedge(a^0\mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^k) \in \Omega^k \otimes H$, and so ω is horizontal and right H−coinvariant. By the same calculation we find that every form ω that is both horizontal and right H−coinvariant, is in $\Omega^\bullet(B)$. □

4.2.4 Comparison with the Brzeziński-Majid approach

The following section supplies a comparison between the Brzeziński-Majid and Đurđević approaches to quantum principal bundles.

Definition 4.2.23 ([5, 2])**.** Let (Γ_A, d_A) be a right H−covariant first order differential calculus over a right H−comodule algebra A, and let (Γ_H, d_H) be a bicoviariant first order differential calculus over a Hopf algebra H. We have a Brzeziński-Majid quantum principal bundle if the map

$$\mathfrak{ver}_{BM} : \Gamma_A \to A \otimes \Lambda^1, \quad \mathfrak{ver}_{BM}(a\mathrm{d}_Aa') = aa'_0 \otimes \varpi(a'_1) = aa'_0 \otimes S(a'_1)\mathrm{d}_H(a'_2) \tag{4.82}$$

is well defined, and the sequence

$$0 \to A\mathrm{d}_A(B)A \hookrightarrow \Gamma_A \overset{\mathfrak{ver}_{BM}}{\longrightarrow} A \otimes \Lambda^1 \to 0 \tag{4.83}$$

is exact. We call $A\mathrm{d}_A(B)A$ the Brzeziński-Majid horizontal forms.

We talk of a quantum principal bundles referring to Đurđević, and of a Brzeziński-Majid quantum principal bundle referring to the last definition.

Definition 4.2.24. A differential calculus on a quantum principal bundle is called first order complete if the right $H-$coaction $\Delta_A : A \to A \otimes H$ is $1-$differentiable, i.e. if there is a morphism of first order differential calculi

$$\Delta_A^1 : \Omega^1(A \otimes H) \to (\Omega^1(A) \otimes H) \oplus (A \otimes \Omega^1(H)) \tag{4.84}$$

extending Δ_A:

$$\begin{array}{ccc} \Omega^1(A) & \xrightarrow{\Delta_A^1} & \Omega^1(A \otimes H) \\ \uparrow{\scriptstyle \mathrm{d}} & & \uparrow{\scriptstyle \mathrm{d}_\otimes} \\ A & \xrightarrow{\Delta_A} & A \otimes H \end{array}$$

We want to clarify the relation between the definition of quantum principal bundle in the Brzeziński-Majid and Đurđević approaches. As the Brzeziński-Majid approach only assumes a right $H-$comodule algebra, whereas the Đurđević's requires a faithfully flat Hopf-Galois extension, for a fair comparison we need to consider faithfully flat Hopf-Galois extensions in both scenarios.

Proposition 4.2.25. *Let $B = A^{coH} \subseteq A$ be a faithfully flat Hopf-Galois extension.*

1. *If there is a Brzeziński-Majid quantum principal bundle on A, then the maximal prolongation of Γ_A is first order complete.*

2. *If there is a first order complete differential calculus $\Omega^\bullet(A)$ on A with corresponding calculus $\Omega^\bullet(H)$ on the structure Hopf algebra, then the first order truncation $(\Omega^1(A), \Omega^1(H))$ is a Brzeziński-Majid quantum pricipal bundle if and only if the horizontal $1-$forms of Brzeziński-Majid and Đurđević' coincide.*

Proof. Given a Brzeziński-Majid quantum principal bundle (Γ_A, Γ_H) we show that the maps

$$\Delta_{\Gamma_A} : \Gamma_A \to \Gamma_A \otimes H, \quad a\mathrm{d}_A a' \mapsto a_0 \mathrm{d}_A a'_0 \otimes a_1 a'_1, \tag{4.85}$$

$$\mathfrak{ver} : \Gamma_A \to A \otimes \Gamma_H, \quad a\mathrm{d}_A a' \mapsto a_0 a'_0 \otimes a_1 \mathrm{d}_H a'_1 \tag{4.86}$$

are well defined. This gives exactly first order completeness of the maximal prolongation of Γ_A. The map Δ_{Γ_A} is well defined, since Γ_A is a right $H-$covariant first order differential calculus. Moreover, by assumption the map

$$\mathfrak{ver}_{BM} : \Gamma_A \to A \otimes \Gamma_H, \quad a\mathrm{d}_A(a') \mapsto aa'_0 \otimes S(a'_1)\mathrm{d}_H a'_2 = a_0 a'_0 \otimes S(a_1 a'_1) a_2 \mathrm{d}_H a'_2$$

is well defined, and thus

$$(\mathrm{id} \otimes \triangleright) \circ (\Delta_A \otimes \mathrm{id}) \circ \mathfrak{ver}_{BM} : \Gamma_A \to H \otimes \Gamma_H$$

is well defined. We have that the map $\mathfrak{ver}$ is well defined, since

$$\begin{aligned} (\mathrm{id} \otimes \triangleright) \circ (\Delta_A \otimes \mathrm{id}) \circ \mathfrak{ver}_{BM}(a\mathrm{d}_A a') &= (\mathrm{id} \otimes \triangleright) \circ (\Delta_A \otimes \mathrm{id})(a_0 a'_0 \otimes S(a_1 a'_1) a_2 \mathrm{d}_H a'_2) \\ &= (\mathrm{id} \otimes \triangleright)(a_0 a'_0 \otimes a_1 a'_1 \otimes S(a_2 a'_2) a_3 \mathrm{d}_h a'_3) \\ &= a_0 a'_0 \otimes \triangleright(a_1 a'_1 \otimes S(a_2 a'_2) a_3 \mathrm{d}_H a'_3) \\ &= a_0 a'_0 \otimes a_1 \mathrm{d}_H a'_1 \\ &= \mathfrak{ver}(a\mathrm{d}_A a'). \end{aligned} \tag{4.87}$$

For the other point, let us consider a first order complete differential calculus $\Omega^\bullet(A)$, and let $\Omega^\bullet(H)$ be the corresponding differential calculus on H. In particular $\Omega^\bullet(A)$ is right $H-$covariant and $\Omega^\bullet(H)$ is bicovariant. The vertical map $\mathfrak{ver} : \Omega^1(A) \to A \otimes \Omega^1(A)$ is well defined, and so

$$\mathfrak{ver}_{BM} = (\mathrm{id} \otimes \triangleright) \circ (\mathrm{id} \otimes S \otimes \mathrm{id}) \circ (\Delta_A \otimes \mathrm{id}) \circ \mathfrak{ver} \tag{4.88}$$

is well defined. Moreover, we have that $\pi_{\mathfrak{ver}}|_{\Omega^1(A)} : \Omega^1(A) \to A \otimes \Lambda^1$ explicitly reads

$$\pi_{\mathfrak{ver}}(a\mathrm{d}_A a') = a_0 a_1 \otimes S(a_1 a'_1) a_2 \mathrm{d}a'_2,$$

i.e. it coincides with $\mathfrak{ver}_{BM}(a\mathrm{d}_A a')$. This means $\mathfrak{ver}_{BM}$ is surjective, since $\pi_{\mathfrak{ver}}$ is. The morphism sending horizontal forms into total space form is automatically injective, therefore it is only left to show that $\ker \mathfrak{ver}_{BM} = A\mathrm{d}_A(B)A$. Since $\mathfrak{hor}^1(A) = \ker \pi_{\mathfrak{ver}} = \ker \mathfrak{ver}_{BM}$ this follows if and only if horizontal 1-forms of the Đurđević' and Brzeziński-Majid approaches are equal. □

4.3 Examples

In this section we provide some non trivial examples of quantum principal bundles and complete differential calculi. Let $\Omega^\bullet(A)$ be a complete differential calculus over A. The morphism $\Delta_A^\wedge : \Omega^\bullet(A) \to \Omega^\bullet(A) \otimes \Omega^\bullet(H)$ maps elements of fixed degree k in the direct sum decomposition of $\Omega^\bullet(A)$ as $\Delta_A^\wedge : \Omega^k(A) \to \Omega^s(A) \otimes \Omega^r(H)$, with $r + s = k$.

Proposition 4.3.1. *Let $\Omega^\bullet(A)$ be a complete calculus. Let us write*

$$\mathfrak{ver}^{k,\ell} := (\pi_A^k \otimes \pi_H^\ell) \circ \Delta_A^\wedge|_{\Omega^{k+\ell}(A)} : \Omega^{k+\ell}(A) \to \Omega^k(A) \otimes \Omega^\ell(H)$$

for the graded components of $\Delta_A^\wedge$. Then

$$\mathfrak{ver}^{k,\ell}(\omega \wedge \eta) = \sum_{m=0,\dots,|\omega|} \mathfrak{ver}^{m,|\omega|-m}(\omega)\mathfrak{ver}^{k-m,|\eta|-k+m}(\eta)$$

for all $\omega, \eta \in \Omega^\bullet(A)$ such that $|\omega| + |\eta| = k + \ell$.

Proof. Let us consider $\omega = a^0 \mathrm{d}a^1 \wedge \cdots \wedge \mathrm{d}a^r$ and $\eta = b^0 \mathrm{d}b^1 \wedge \cdots \wedge \mathrm{d}b^s$ in $\Omega^\bullet(A)$. We write $\mathrm{d}_\otimes = (\mathrm{id} \otimes \mathrm{d} + \mathrm{d} \otimes \mathrm{id})$ as a shorthand notation for this proof. On the left hand side we have

$$\begin{aligned} \mathfrak{ver}^{k,\ell}(\omega \wedge \eta) &= (\pi_k \otimes \pi_\ell)\Delta_A^\wedge(\omega \wedge \eta) \\ &= (\pi_k \otimes \pi_\ell)(\Delta_A(a^0)\mathrm{d}_\otimes \Delta_A(a^1) \wedge \cdots \wedge \mathrm{d}_\otimes \Delta_A(a^r) \\ &\quad \wedge \Delta_A(b^0)\mathrm{d}_\otimes \Delta_A(b^1) \wedge \cdots \wedge \mathrm{d}_\otimes \Delta_A(b^r)). \end{aligned} \tag{4.89}$$

On the right hand side we have

$$\begin{aligned} &\sum_{m=0,\dots,|\omega|} \mathfrak{ver}^{m,|\omega|-m}(\omega)\mathfrak{ver}^{k-m,|\eta|-k+m}(\eta) \\ &\qquad = \sum_{m=0,\dots,|\omega|} (\pi^m \otimes \pi^{|\omega|-m})(\Delta_A(a^0)\mathrm{d}_\otimes \Delta_A(a^1) \wedge \cdots \mathrm{d}_\otimes \Delta_A(a^r)) \\ &\qquad\qquad (\pi^{k-m} \otimes \pi^{|\eta|-k+m})(\Delta_A(b^0)\mathrm{d}_\otimes \Delta_A(b^1) \wedge \cdots \mathrm{d}_\otimes \Delta_A(b^s)). \end{aligned} \tag{4.90}$$

Since the projection maps fix the degree of left and right tensor factors of the above formulas we have the claim.

□

Example 4.3.2. Let H be a Hopf algebra and let (Γ, d) be a first order differential calculus on H. Consider the maximal prolongation of (Γ, d). The subalgebra of coinvariant elements $B = H^{coH}$ is made of elements that are invariant under the coaction of H on itself, i.e. the coproduct $\Delta : H \longrightarrow H \otimes H$. This subalgebra is naturally isomorphic to $\mathbb{k}$. Therefore we consider the Hopf-Galois map $\chi : H \otimes H \longrightarrow H \otimes H$ sending $h \otimes g \mapsto hg_1 \otimes g_2$, where no balanced tensor product is required after the established isomorphism $B \cong \mathbb{k}$. This map is a bijection and in fact we can provide explicitely the inverse $\chi^{-1} : H \otimes H \longrightarrow H \otimes H$ sending $h \otimes g \mapsto hS(g_1) \otimes h_2$, indeed

$$\chi \circ \chi^{-1}(h \otimes g) = \chi(hS(g_1) \otimes g_2) = hS(g_1)g_2 \otimes g_3 = h \otimes g, \tag{4.91}$$

$$\chi^{-1} \circ \chi(h \otimes g) = \chi^{-1}(hg_1 \otimes g_2) = hg_1S(g_2) \otimes g_3 = h \otimes g. \tag{4.92}$$

Therefore we have a quantum principal bundle. □

Example 4.3.3. Consider the group algebra $H = \mathbb{C}[\mathbb{Z}]$. H is generated by an element g, i.e. every element of H can be written as $\alpha_k g^k$, where $\alpha_k \in \mathbb{C}$ and $k \in \mathbb{Z}$. The coproduct, counit and antipode read

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1}. \tag{4.93}$$

Consider the first order differential calculus Γ over H defined as $\Gamma = \mathrm{span}_H\{\mathrm{d}g\} = H\mathrm{d}g$, with $(g^n\mathrm{d}g)g := qg^{n+1}\mathrm{d}g$, where q is not a root of unity. The differential $\mathrm{d} : H \longrightarrow \Gamma$ maps $\mathrm{d}(g) := \mathrm{d}g$ and in general

$$\begin{aligned} &\mathrm{d}(g^n) := (1 + q + \cdots + q^{n-1})g^{n-1}\mathrm{d}g, && \text{if } n \geq 0, \\ &\mathrm{d}(g^n) := -(q^n + \cdots + q^{-1})g^{n-1}\mathrm{d}g, && \text{if } n < 0, \\ &\mathrm{d}f(t) := \frac{(f(qt) - f(t))}{(t(q-1))}, && \text{for a rational function.} \end{aligned} \tag{4.94}$$

In particular $\mathrm{d}(g^{-1}) = -g^{-1}(\mathrm{d}g)g^{-1}$ much like a chain rule. We define right and left H−coactions on Γ as

$$\begin{aligned} &\Delta_\Gamma : \Gamma \longrightarrow \Gamma \otimes H, && {}_\Gamma\Delta : \Gamma \longrightarrow H \otimes \Gamma. \\ &g^n\mathrm{d}g \mapsto g^n\mathrm{d}g \otimes g^{n+1}, && g^n\mathrm{d}g \mapsto g^{n+1} \otimes g^n\mathrm{d}g. \end{aligned} \tag{4.95}$$

Since Δ extends to the first order calculus we have that (Γ, d) is a bicovariant first order differential calculus over H.

Moreover, since $\mathrm{d}(g^{-1}) + q^{-1}g^{-2}\mathrm{d}g = 0$, we find

$$\begin{aligned} (\mathrm{d}g^{-1} + q^{-1}g^{-2}\mathrm{d}g) \otimes \mathrm{d}g &= q^{-1}(\mathrm{d}g^{-1}g^{-1} + g^{-1}\mathrm{d}g^{-1}) \otimes \mathrm{d}g \\ &= -q^{-1}(g^{-1}\mathrm{d}gg^{-2} + g^{-2}\mathrm{d}gg^{-1}) \otimes \mathrm{d}g \\ &= -q^{-2}(1 + q^{-1})g^{-3}\mathrm{d}g \otimes \mathrm{d}g. \end{aligned} \tag{4.96}$$

The last expression vanishes on the quotient defined by the maximal prolongation. Therefore on this quotient we have $[\mathrm{d}g \otimes \mathrm{d}g] = 0$, as we assumed q not a root of unity. Accordingly we have no non-zero k−forms for $k > 1$, and thus a complete differential calculus. □

4.3.1 The noncommutative algebraic 2-torus

In the following example we will discuss the quantum principal bundle given by the Hopf algebra $H = \mathcal{O}(\mathrm{U}(1)) = \mathbb{C}[t, t^{-1}]$ and right H−comodule algebra $A = \mathcal{O}_\theta(\mathbb{T}^2) = \mathbb{C}[u, u^{-1}, v, v^{-1}]/\langle uv - e^{i\theta}vu\rangle$, where $\theta \in \mathbb{R}$. This algebra A is known as the non-commutative algebraic 2-torus. The corresponding Hopf algebra structure of H is given by coproduct $\Delta(t) = t \otimes t$, counit $\epsilon(t) = 1$ and antipode $S(t) = t^{-1}$. Let $\Delta_A : A \longrightarrow A \otimes H$ be the map assigning

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix} \otimes \begin{pmatrix} t \\ t^{-1} \end{pmatrix}. \tag{4.97}$$

This is a right H–coaction on A, indeed

$$(\Delta_A \otimes \mathrm{id}) \circ \Delta_A \begin{pmatrix} u \\ v \end{pmatrix} = (\Delta_A \otimes \mathrm{id}) \begin{pmatrix} u \\ v \end{pmatrix} \otimes \begin{pmatrix} t \\ t^{-1} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \otimes \begin{pmatrix} t \\ t^{-1} \end{pmatrix} \otimes \begin{pmatrix} t \\ t^{-1} \end{pmatrix}, \tag{4.98}$$

whereas

$$(\mathrm{id} \otimes \Delta) \circ \Delta_A \begin{pmatrix} u \\ v \end{pmatrix} = (\mathrm{id} \otimes \Delta) \begin{pmatrix} u \\ v \end{pmatrix} \otimes \begin{pmatrix} t \\ t^{-1} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \otimes \begin{pmatrix} t \\ t^{-1} \end{pmatrix} \otimes \begin{pmatrix} t \\ t^{-1} \end{pmatrix}. \tag{4.99}$$

Moreover

$$(\mathrm{id} \otimes \epsilon) \circ \Delta_A \begin{pmatrix} u \\ v \end{pmatrix} = (\mathrm{id} \otimes \epsilon) \begin{pmatrix} u \\ v \end{pmatrix} \otimes \begin{pmatrix} t \\ t^{-1} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \otimes 1 \cong \begin{pmatrix} u \\ v \end{pmatrix}. \tag{4.100}$$

The space of right H–coinvariant forms is given by

$$B := A^{coH} = \mathrm{span}_{\mathbb{C}}\{(uv)^k,\ k \in \mathbb{Z}\}.$$

Proposition 4.3.4. *The noncommutative algebraic 2-torus under the coaction of $O(U(1))$ is a cleft extension $B \subseteq A$. In particular it is a quantum principal bundle.*

Proof. Let us consider $j : H \to A$ sending

$$\begin{pmatrix} t^k \\ t^{-k} \end{pmatrix} \mapsto \begin{pmatrix} u^k \\ v^k \end{pmatrix},$$

for all $k \geq 0$. We show that $j : H \to A$ is a convolution invertible morphism of right H–comodules.

$$\begin{aligned} \Delta_A(j(t)) &= \Delta_A(u) = u \otimes t = (j \otimes \mathrm{id})\Delta(t), \\ \Delta_A(j(t^{-1})) &= \Delta_A(v) = v \otimes t^{-1} = (j \otimes \mathrm{id})\Delta(t^{-1}). \end{aligned} \tag{4.101}$$

We provide an explicit inverse for the map $j : H \to A$, that is $j^{-1} : H \to A$ sending

$$\begin{pmatrix} t^k \\ t^{-k} \end{pmatrix} \mapsto \begin{pmatrix} u^{-k} \\ v^{-k} \end{pmatrix};$$

indeed

$$\begin{aligned} j * j^{-1}(t^k) &= \mu \circ (j \otimes j^{-1}) \circ \Delta(t^k) = j(t^k) j^{-1}(t^k) = u^k u^{-k} = 1, \\ j * j^{-1}(t^{-k}) &= \mu \circ (j \otimes j^{-1}) \circ \Delta(t^{-k}) = j(t^{-k}) j^{-1}(t^{-k}) = v^k v^{-k} = 1, \end{aligned} \tag{4.102}$$

and similarly $j^{-1} * j = 1$ is easily verified.

Since every cleft extension is in particular a faithfully flat Hopf-Galois extension [4] we have the claim. □

We now investigate in depth the differential calculi over $H = O(\mathrm{U}(1))$ and $A = O_\theta(\mathbb{T}^2)$. For the differential structure over H, we consider a bicovariant first order differential calculus $(\Gamma = \mathrm{span}_H\{\mathrm{d}t\}, \mathrm{d})$ with bimodule relations

$$\begin{aligned} t\mathrm{d}t &= q^{\alpha} \mathrm{d}t t, \\ t^{-1}\mathrm{d}t &= q^{-\alpha} \mathrm{d}t t^{-1}. \end{aligned} \tag{4.103}$$

Here q is a understood as a deformation parameter and $\alpha \in \mathbb{R}$. Differentiating these relations we find

$$\mathrm{d}t \wedge \mathrm{d}t = -q^{\alpha} \mathrm{d}t \wedge \mathrm{d}t. \tag{4.104}$$

Accordingly there are no non-zero higher order forms.

Proceeding with the calculus on A, we define a bicovariant first order differential calculus $(\Omega^1(A) = \mathrm{span}_A\{\mathrm{d}u, \mathrm{d}v\}, \mathrm{d})$ via relations

$$\mathrm{d}u u = u \mathrm{d}u, \quad \mathrm{d}v v = v \mathrm{d}v, \quad \mathrm{d}u v = e^{-i\theta} v \mathrm{d}u, \quad \mathrm{d}v u = e^{i\theta} u \mathrm{d}v. \tag{4.105}$$

Differentiating, we find

$$\begin{aligned} &\mathrm{d}u \wedge \mathrm{d}v + e^{-i\theta}\mathrm{d}v \wedge \mathrm{d}u = 0, \\ &\mathrm{d}u \wedge \mathrm{d}u = 0, \\ &\mathrm{d}v \wedge \mathrm{d}v = 0, \end{aligned} \tag{4.106}$$

and clearly higher order forms vanish.

Proposition 4.3.5. *$\Omega^1(A) := \mathrm{span}_A\{\mathrm{d}u, \mathrm{d}v\}$ is a right H−covariant differential calculus on the noncommutative algebraic 2-torus. Moreover the map $\mathfrak{ver} := \mathfrak{ver}^{0,1}$ is well defined.*

Proof. We show the coaction $\Delta_A : A \to A \otimes H$ lifts to a morphism $\Delta^1_A : \Omega^1(A) \to \Omega^1(A) \otimes \Omega^1(H)$. We have

$$\Delta^1_A = \Delta_{\Omega^1(A)} + \mathfrak{ver} : \Omega^1(A) \to (A \otimes \Gamma) \oplus (\Omega^1 \otimes H).$$

To properly define this map we need

$$\begin{aligned} \Delta_{\Omega^1(A)}(\mathrm{d}u) &= (\mathrm{d} \otimes \mathrm{id})\Delta_A(u) = \mathrm{d}u \otimes t, \\ \Delta_{\Omega^1(A)}(\mathrm{d}v) &= (\mathrm{d} \otimes \mathrm{id})\Delta_A(v) = \mathrm{d}v \otimes t^{-1}, \end{aligned} \tag{4.107}$$

and then we extend $\Delta_{\Omega^1(A)}$ to $\Omega^1(A)$ by

$$\Delta_{\Omega^1(A)}(a\mathrm{d}u + a'\mathrm{d}v) := \Delta_A(a)\Delta_{\Omega^1(A)}(\mathrm{d}u) + \Delta_A(a')\Delta_{\Omega^1(A)}(\mathrm{d}v). \tag{4.108}$$

Moreover $\Delta_{\Omega^1(A)}$ is well defined, since

$$\begin{aligned} \Delta_{\Omega^1(A)}(\mathrm{d}u.v) &= \Delta_{\Omega^1(A)}(\mathrm{d}u)\Delta_A(v) \\ &= (\mathrm{d}u \otimes t)(v \otimes t^{-1}) \\ &= \mathrm{d}uv \otimes 1, \end{aligned} \tag{4.109}$$

$$\begin{aligned} \Delta_{\Omega^1(A)}(e^{-i\theta}v\mathrm{d}u) &= e^{-i\theta}\Delta_A(v)\Delta_{\Omega^1(A)}(\mathrm{d}u) \\ &= e^{-i\theta}(v \otimes t^{-1})(\mathrm{d}u \otimes t) \\ &= e^{-i\theta}v\mathrm{d}u \otimes 1 \\ &= \Delta_{\Omega^1(A)}(\mathrm{d}uv) \end{aligned} \tag{4.110}$$

and similarly $\Delta_{\Omega^1(A)}(\mathrm{d}v.u) = \Delta_{\Omega^1(A)}(e^{i\theta}u.\mathrm{d}v)$. To read more explicitly $\mathfrak{ver} : \Omega^1(A) \to A \otimes \Gamma$ we consider $a, a' \in A$ and develop

$$\begin{aligned} \Delta^\wedge_A(a\mathrm{d}a') &= \Delta_A(a)(\mathrm{d} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{d})\Delta_A(a') \\ &= a_0\mathrm{d}a'_0 \otimes a_1a'_1 + a_0a'_0 \otimes a_1\mathrm{d}a'_1. \end{aligned} \tag{4.111}$$

Accordingly we define on the basis elements

$$\mathfrak{ver}(\mathrm{d}u) = u \otimes \mathrm{d}t, \quad \mathfrak{ver}(\mathrm{d}v) = v \otimes \mathrm{d}t^{-1}. \tag{4.112}$$

Therefore we have

$$\begin{aligned} \mathfrak{ver}(\mathrm{d}uv) &= \mathfrak{ver}(\mathrm{d}u)\Delta_A(v) \\ &= (u \otimes \mathrm{d}t)(v \otimes t^{-1}) \\ &= uv \otimes \mathrm{d}tt^{-1} \end{aligned} \tag{4.113}$$

$$\begin{aligned} \mathfrak{ver}(e^{-i\theta}v\mathrm{d}u) &= e^{-i\theta}\Delta_A(v)\mathfrak{ver}(\mathrm{d}u) \\ &= e^{-i\theta}(v \otimes t^{-1})(u \otimes \mathrm{d}t) \\ &= e^{-i\theta}vu \otimes t^{-1}\mathrm{d}t \\ &= e^{-i\theta}vu \otimes q^{-\alpha}\mathrm{d}tt^{-1}. \end{aligned} \tag{4.114}$$

Similarly

$$\begin{aligned}\mathfrak{ver}(\mathrm{d}vu) &= \mathfrak{ver}(\mathrm{d}v)\Delta_A(u)\\ &= (v\otimes \mathrm{d}t^{-1})(u\otimes t)\\ &= vu\otimes \mathrm{d}t^{-1}t\\ &= -vu\otimes t^{-2}\mathrm{d}tt\\ &= -vu\otimes t^{-2}t\mathrm{d}t\\ &= -vu\otimes t^{-1}\mathrm{d}t,\end{aligned} \tag{4.115}$$

$$\begin{aligned}\mathfrak{ver}(e^{i\theta}u\mathrm{d}v) &= e^{i\theta}\Delta_A(u)\mathfrak{ver}(\mathrm{d}v)\\ &= e^{i\theta}(u\otimes t)(v\otimes \mathrm{d}t^{-1})\\ &= e^{i\theta}uv\otimes t\mathrm{d}t^{-1}\\ &= -e^{i\theta}uv\otimes tt^{-2}\mathrm{d}t\\ &= -e^{i\theta}uv\otimes t^{-1}\mathrm{d}t,\end{aligned} \tag{4.116}$$

where we assumed $\alpha = 0$. The map $\mathfrak{ver} : \Omega^1(A) \to A\otimes\Gamma$ is thus well defined along the relations between the generators of the first order differential calculus over A. Accordingly we define

$$\mathfrak{ver}(a\mathrm{d}a') := a_0a'_0\otimes a_1\mathrm{d}a'_1. \tag{4.117}$$

□

Proposition 4.3.6. $\Omega^2(A) := span_A\{\mathrm{d}u\wedge\mathrm{d}v\}$ *is a right H−covariant differential calculus on the noncommutative algebraic 2-torus. Moreover maps $\mathfrak{ver}^{1,1}$ and $\mathfrak{ver}^{0,2}$ are well defined.*

Proof. We show that $\Delta_A : A \to A\otimes H$ extends to a morphism

$$\Delta^2_A : \Omega^2(A) \to \underbrace{(\Omega^2(A)\otimes H)}_{\Delta_{\Omega^2(A)}} \oplus \underbrace{(\Omega^1(A)\otimes\Omega^1(H))}_{\mathfrak{ver}^{1,1}} \oplus \underbrace{(A\otimes\Omega^2(H))}_{\mathfrak{ver}^{0,2}}.$$

Let us consider $\omega = a\mathrm{d}a'\wedge\mathrm{d}a'' \in \Omega^2(A)$ for $a, a', a'' \in A$. We have

$$\begin{aligned}\Delta^\wedge_A(a\mathrm{d}a'\wedge\mathrm{d}a'') &= \Delta_A(a)(\mathrm{d}\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{d})\Delta_A(a')(\mathrm{id}\otimes\mathrm{d}+\mathrm{d}\otimes\mathrm{id})\Delta_A(a'')\\ &= (a_0\otimes a_1)(\mathrm{d}a'_0\otimes a_1 + a'_0\otimes\mathrm{d}a'_1)(\mathrm{d}a''_0\otimes a''_1 + a''_0\otimes\mathrm{d}a''_1)\\ &= (a_0\otimes a_1)(\mathrm{d}a'_0\wedge\mathrm{d}a''_0\otimes a'_1a''_1 + \mathrm{d}a'_0a''_0\otimes a'_1\mathrm{d}a''_1 - a'_0\mathrm{d}a''_0\otimes\mathrm{d}a'_1a''_1),\end{aligned} \tag{4.118}$$

where we exploited that there are no non-zero two forms on H. Accordingly we consider

$$\begin{aligned}\Delta_{\Omega^2(A)}(a\mathrm{d}a'\wedge\mathrm{d}a'') &:= a_0\mathrm{d}a'_0\wedge\mathrm{d}a''_0\otimes a_1a'_1a''_1,\\ \mathfrak{ver}^{1,1}(a\mathrm{d}a'\wedge\mathrm{d}a'') &:= a_0\mathrm{d}a'_0a''_0\otimes a_1a'_1\mathrm{d}a''_1 - a_0a'_0\mathrm{d}a''_0\otimes a_1\mathrm{d}a''_1a''_1,\end{aligned} \tag{4.119}$$

and $\mathfrak{ver}^{0,2}$ to be the zero map. These maps are well defined, since

$$\begin{aligned}\Delta_{\Omega^2(A)}(\mathrm{d}u\wedge\mathrm{d}v) &= \Delta_{\Omega^1(A)}(\mathrm{d}u)\Delta_{\Omega^1(A)}(\mathrm{d}v)\\ &= (\mathrm{d}u\otimes t)(\mathrm{d}v\otimes t^{-1})\\ &= (\mathrm{d}u\wedge\mathrm{d}v)\otimes 1.\end{aligned} \tag{4.120}$$

$$\begin{aligned}\mathfrak{ver}^{1,1}(\mathrm{d}u\wedge\mathrm{d}v) &= \mathrm{d}uv\otimes t\mathrm{d}t^{-1} - u\mathrm{d}v\otimes\mathrm{d}tt^{-1}\\ &= -\mathrm{d}uv\otimes t^{-1}\mathrm{d}t - u\mathrm{d}v\otimes\mathrm{d}tt^{-1}\\ &= -(\mathrm{d}uv + u\mathrm{d}v)\otimes\mathrm{d}tt^{-1}\\ &= -\,\mathrm{d}(uv)\otimes\mathrm{d}tt^{-1};\end{aligned} \tag{4.121}$$

$$\begin{aligned}\mathfrak{ver}^{1,1}(\mathrm{d}v\wedge\mathrm{d}u) &= \mathrm{d}vu\otimes t^{-1}\mathrm{d}t - v\mathrm{d}u\otimes \mathrm{d}t^{-1}t\\ &= \mathrm{d}vu\otimes t^{-1}\mathrm{d}t + v\mathrm{d}u\otimes t^{-2}\mathrm{d}tt\\ &= \mathrm{d}vu\otimes \mathrm{d}tt^{-1} + v\mathrm{d}u\otimes t^{-1}\mathrm{d}t\\ &= \mathrm{d}vu\otimes \mathrm{d}tt^{-1} + v\mathrm{d}u\otimes \mathrm{d}tt^{-1}\\ &= \mathrm{d}(vu)\otimes \mathrm{d}tt^{-1},\end{aligned}\tag{4.122}$$

and so

$$\begin{aligned}\mathfrak{ver}^{1,1}(\mathrm{d}u\wedge\mathrm{d}v) + e^{-i\theta}\mathfrak{ver}^{1,1}(\mathrm{d}v\wedge\mathrm{d}u) &= -\mathrm{d}(uv)\otimes\mathrm{d}tt^{-1} + e^{-i\theta}\,\mathrm{d}(vu)\otimes\mathrm{d}tt^{-1}\\ &\quad -\mathrm{d}(uv)\otimes\mathrm{d}tt^{-1} + \mathrm{d}\Big(e^{-i\theta}vu\Big)\otimes\mathrm{d}tt^{-1}\\ &= -\mathrm{d}(uv)\otimes\mathrm{d}tt^{-1} + \mathrm{d}(uv)\otimes\mathrm{d}tt^{-1}.\end{aligned}\tag{4.123}$$

□

Theorem 4.3.7. *The differential calculus $\Omega^\bullet(A)$ on the noncommutative algebraic 2-torus is complete.*

Proof. This follows directly from Proposition 4.3.5-4.3.6 □

Proposition 4.3.8. *The base space 1-forms are generated by B. In other words $B\mathrm{d}B = \Omega^1(B)$.*

Proof. According to lemma 4.2.20 the non trivial statement is that $B\mathrm{d}B \supseteq \Omega^1(B)$. A generic element in $\omega \in \Omega^1(B)$ can be written as $\omega = \alpha_{k\ell}u^kv^\ell\mathrm{d}u + \beta_{mn}u^mv^n\mathrm{d}v$, for $\alpha_{k\ell},\beta_{mn}\in\mathbb{C}$. We have

$$\begin{aligned}\Delta_A^\wedge(\alpha_{k\ell}u^kv^\ell + \beta_{mn}u^mv^n) &= \alpha_{k\ell}\Delta_A(u^kv^\ell)\Delta_A^1(\mathrm{d}u) + \beta_{mn}\Delta_A(u^mv^n)\Delta_A^1(\mathrm{d}v)\\ &= \alpha_{k\ell}(u^kv^\ell)\otimes t^{k-\ell}(\mathrm{d}\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{d})\Delta_A(u) + \beta_{mn}(u^mv^n)\otimes t^{m-n}(\mathrm{d}\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{d})\Delta_A(v)\\ &= \alpha_{k\ell}u^kv^\ell(\mathrm{d}u\otimes t^{k-\ell+1} + u\otimes t^{k-\ell}\mathrm{d}t) + \beta_{mn}u^mv^n(\mathrm{d}v\otimes t^{m-n-1} + v\otimes t^{m-n}\mathrm{d}t^{-1})\\ &= \alpha_{k\ell}u^kv^\ell\mathrm{d}u\otimes t^{k-\ell+1} + \beta_{mn}u^mv^n\mathrm{d}v\otimes t^{m-n-1}\\ &\quad + \alpha_{k\ell}u^kv^\ell u\otimes t^{k-\ell}\mathrm{d}t - \beta_{mn}u^mv^nv\otimes t^{m-n-2}\mathrm{d}t\\ &= (\alpha_{k\ell}u^kv^\ell\mathrm{d}u + \beta_{mn}u^mv^n\mathrm{d}v)\otimes 1\end{aligned}$$

if and only if

$$\begin{aligned}&\alpha_{k\ell}u^kv^\ell\mathrm{d}u\otimes t^{k-\ell+1} + \beta_{mn}u^mv^n\mathrm{d}v\otimes t^{m-n-1} = (\alpha_{k\ell}u^kv^\ell\mathrm{d}u + \beta_{mn}u^mv^n\mathrm{d}v)\otimes 1,\\ &\alpha_{k\ell}u^kv^\ell u\otimes t^{k-\ell}\mathrm{d}t - \beta_{mn}u^mv^nv\otimes t^{m-n-2}\mathrm{d}t = 0.\end{aligned}\tag{4.124}$$

The first equation tells we must have $k-\ell+1=0$ (or $\beta_{mn}=0$) and $m-n-1=0$ (or $\alpha_{k\ell}=0$), which leads to

$$\alpha_{k\ell}u^kv^\ell\mathrm{d}u\otimes t^{k-\ell+1} + \beta_{mn}u^mv^n\mathrm{d}v\otimes t^{m-n-1} = \alpha_{k,k+1}u^kv^{k+1}\mathrm{d}u\otimes 1 + \beta_{n,n+1}u^{n+1}v^n\mathrm{d}v\otimes 1.\tag{4.125}$$

The second equation reads

$$\begin{aligned}&\alpha_{k\ell}u^kv^\ell u\otimes t^{k-\ell}\mathrm{d}t - \beta_{mn}u^mv^nv\otimes t^{m-n-2}\mathrm{d}t\\ &\qquad = \alpha_{k,k+1}u^kv^{k+1}u\otimes t^{-1}\mathrm{d}t - \beta_{n+1,n}u^{n+1}v^nv\otimes t^{-1}\mathrm{d}t\\ &\qquad = (\alpha_{k,k+1}e^{i(k+1)\theta}u^{k+1}v^{k+1} - \beta_{n,n+1}u^{n+1}v^{n+1})\otimes t^{-1}\mathrm{d}t,\end{aligned}$$

which is zero if and only if

$$\alpha_{k,k+1}e^{i(k+1)\theta}u^{k+1}v^{k+1} - \beta_{n,n+1}u^{n+1}v^{n+1} = 0,\tag{4.126}$$

so we must have $n = k$, and accordingly $\alpha_{k,k+1}e^{i(k+1)\theta} = \beta_{k,k+1}$. Therefore a general element $\omega \in \Omega^1(B)$ reads

$$\begin{aligned}\alpha_{k\ell}u^k v^\ell \mathrm{d}u + \beta_{mn}u^m v^n \mathrm{d}v &= \alpha_{k,k+1}u^k v^{k+1}\mathrm{d}u + \beta_{k,k+1}u^{k+1}v^k \mathrm{d}v \\ &= \alpha_{k,k+1}(u^k v^k v\mathrm{d}u + e^{i(k+1)\theta}u^k u v^k \mathrm{d}v) \\ &= \alpha_{k,k+1}u^k v^k (v\mathrm{d}u + e^{i(k+1)\theta}u^k v^k \mathrm{d}v u e^{-i(k+1)\theta}) \\ &= \alpha_{k,k+1}u^k v^k (v\mathrm{d}u + \mathrm{d}v u) \\ &= \alpha_{k,k+1}(uv)^k \,\mathrm{d}(uv)\,,\end{aligned} \tag{4.127}$$

for any $k \in \mathbb{Z}$. Since $B := A^{coH} = \mathrm{span}_{\mathbb{C}}\{(uv)^k |\ k \in \mathbb{Z}\}$ we have the thesis. □

4.3.2 Quantum Hopf fibration and the Podleś sphere

Let us consider the Hopf algebra $H = \mathrm{O}(\mathrm{U}(1))$. We introduce $O_q(\mathrm{SU}(2))$ as the free algebra generated by elements $\alpha, \beta, \gamma, \delta$ modulo relations

$$\begin{aligned}&\beta\alpha = q\alpha\beta, \quad \gamma\alpha = q\alpha\gamma, \quad \delta\beta = q\beta\delta, \quad \delta\gamma = q\gamma\delta \\ &\gamma\beta = \beta\gamma, \quad \delta\alpha - \alpha\delta = (q - q^{-1})\beta\gamma, \quad \alpha\delta - q^{-1}\beta\gamma = 1.\end{aligned} \tag{4.128}$$

Proposition 4.3.9. *$A = O_q(SU(2))$ is a right $H-$comodule algebra under the right $H-$coaction*

$$\Delta_A : A \to A \otimes H, \quad \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \mapsto \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \otimes \begin{pmatrix}t & 0 \\ 0 & t^{-1}\end{pmatrix} = \begin{pmatrix}\alpha\otimes t & \beta\otimes t^{-1} \\ \gamma\otimes t & \delta\otimes t^{-1}\end{pmatrix}. \tag{4.129}$$

Proof. We check that Δ_A satisfies the axioms of a right $H-$coaction on A, indeed

$$\begin{aligned}(\Delta_A \otimes \mathrm{id}) \circ \Delta_A \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} &= \Delta_A \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \otimes \begin{pmatrix}t & 0 \\ 0 & t^{-1}\end{pmatrix} \\ &= \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \otimes \begin{pmatrix}t & 0 \\ 0 & t^{-1}\end{pmatrix} \otimes \begin{pmatrix}t & 0 \\ 0 & t^{-1}\end{pmatrix} \\ &= \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \otimes \Delta \begin{pmatrix}t & 0 \\ 0 & t^{-1}\end{pmatrix} \\ &= (\mathrm{id} \otimes \Delta) \circ \Delta_A \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix};\end{aligned} \tag{4.130}$$

moreover

$$(\mathrm{id} \otimes \epsilon) \circ \Delta_A \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} = (\mathrm{id} \otimes \epsilon) \circ \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \otimes \begin{pmatrix}t & 0 \\ 0 & t^{-1}\end{pmatrix} = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}. \tag{4.131}$$

□

Notation 4.3.10. With $|\cdot|$ we indicate the degree of an element in A; $|f|$ is explicitly defind by the relation $\Delta_A(f) = f \otimes t^{|f|}$, for any $f \in \{\alpha, \beta, \gamma, \delta\}$.

Remark 4.3.11. The subalgebra $B := A^{coH}$ of coinvariant elements under the right $H-$coaction Δ_A is known as the Podleś sphere [23, 22]. By a quick calculation it is easy to deduce generators of B must be of the form fg, where $f, g \in \{\alpha, \beta, \gamma, \delta\}$ and either $|f| = 1$ and $|g| = -1$ or the opposite. Modulo commutation relations we have 3 non equivalent generators for B and we shall fix

$$B_+ = \alpha\beta, \quad B_- = \gamma\delta, \quad B_0 = \gamma\beta. \tag{4.132}$$

One finds

$$\begin{aligned}B_- B_0 &= \gamma\delta\gamma\beta = q\gamma\gamma\delta\beta \\ &= q^2\gamma\gamma\beta\delta = q^2\gamma\beta\gamma\delta \\ &= q^2 B_0 B_-,\end{aligned} \tag{4.133}$$

and similarly

$$B_-B_+ = q^2B_0(1-q^2B_0), \quad B_+B_- = B_0(1-B_0). \tag{4.134}$$

Proposition 4.3.12 ([4]). *$B \subseteq A$ is a quantum principal bundle.* □

We define a first order differential calculus $(\Omega^1(A), \mathrm{d})$ over A as the free left A−module generated by

$$e^+ = q^{-1}\alpha\mathrm{d}\gamma - q^{-2}\gamma\mathrm{d}\alpha, \quad e^- = \delta\mathrm{d}b - q\beta\mathrm{d}\delta, \quad e^0 = \delta\mathrm{d}\alpha - q\beta\mathrm{d}\gamma, \tag{4.135}$$

with commutation relations

$$e^\pm f = q^{|f|} f e^\pm, \quad e^0 f = q^{2|f|} f e^0, \tag{4.136}$$

where $f \in \{\alpha,\beta,\gamma,\delta\}$ and $|\alpha| = |\gamma| = 1$, whereas $|\beta| = |\delta| = -1$.

Proposition 4.3.13. *$\Omega^1(A)$ is a right H−covariant differential calculus. Moreover the map $\mathfrak{ver} = \mathfrak{ver}^{0,1}$ is well-defined.*

Proof. We need to show that $\Delta_A : A \to A\otimes H$ extends to a morphism

$$\Delta^1_A = \Delta_{\Omega^1(A)} + \mathfrak{ver} : \Omega^1(A) \to (\Omega^1(A)\otimes H)\oplus(A\otimes\Omega^1(H))$$

restricting to the usual coaction on A. Such extension is completely determined by the generators. We want

$$\begin{aligned}\Delta_{\Omega^1(A)}(e^+) &= \Delta_{\Omega^1(A)}(q^{-1}\alpha\mathrm{d}\gamma - q^{-2}\gamma\mathrm{d}\alpha)\\ &= q^{-1}\Delta_A(\alpha)\Delta_{\Omega^1(A)}(\mathrm{d}\gamma) - q^{-2}\Delta_A(\gamma)\Delta_{\Omega^1(A)}(\mathrm{d}\alpha)\\ &= q^{-1}(\alpha\otimes t)(\mathrm{d}\otimes\mathrm{id})(\gamma\otimes t) - q^{-2}(\gamma\otimes t)(\mathrm{d}\alpha\otimes t)\\ &= e^+\otimes t^2\end{aligned} \tag{4.137}$$

and similarly $\Delta_{\Omega^1(A)}(e^-) = e^-\otimes t^{-2}$. Moreover

$$\begin{aligned}\Delta_{\Omega^1(A)}(e^0) &= \Delta_{\Omega^1(A)}(\delta\mathrm{d}\alpha - q\beta\mathrm{d}\gamma)\\ &= \Delta_A(\delta)(\mathrm{d}\otimes\mathrm{id})\Delta_A(\alpha) - q\Delta_A(\beta)(\mathrm{d}\otimes\mathrm{id})\Delta_A(\gamma)\\ &= (\delta\otimes t^{-1})(\mathrm{d}\alpha\otimes t) - q(\beta\otimes t^{-1})(\mathrm{d}\gamma\otimes t)\\ &= (\delta\mathrm{d}\alpha - q\beta\mathrm{d}\gamma)\otimes 1\\ &= e^0\otimes 1,\end{aligned} \tag{4.138}$$

and so we define

$$\Delta_{\Omega^1(A)}(e^\pm) := e^\pm\otimes t^{\pm2}, \quad \text{and} \quad \Delta_{\Omega^1(A)}(e^0) := e^0\otimes 1.$$

Proceeding in a similar fashion for $\mathfrak{ver} : \Omega^1(A) \to A\otimes\Omega^1(H)$ we need

$$\begin{aligned}\mathfrak{ver}(e^0) &= \mathfrak{ver}(\delta\mathrm{d}\alpha - q\beta\mathrm{d}\gamma)\\ &= \delta\alpha\otimes t^{-1}\mathrm{d}t - q\beta\gamma\otimes t^{-1}\mathrm{d}t\\ &= (\delta\alpha - q\beta\gamma)\otimes t^{-1}\mathrm{d}t;\end{aligned} \tag{4.139}$$

$$\begin{aligned}\mathfrak{ver}(e^+) &= \mathfrak{ver}(q^{-1}\alpha\mathrm{d}\gamma - q^{-2}\gamma\mathrm{d}\alpha)\\ &= q^{-1}\alpha\gamma\otimes t\mathrm{d}t - q^{-2}\gamma\alpha\otimes t\mathrm{d}t\\ &= (q^{-1}\alpha\gamma - q^{-2}\gamma\alpha)\otimes t\mathrm{d}t\\ &= q^{-1}(\alpha\gamma - q^{-1}\gamma\alpha)\otimes t\mathrm{d}t\\ &= 0\end{aligned} \tag{4.140}$$

$$\begin{aligned}\mathfrak{ver}(e^-) &= \mathfrak{ver}(\delta \mathrm{d}\beta - q\beta \mathrm{d}\delta)\\ &= \delta\beta \otimes t^{-1}\mathrm{d}t^{-1} - q\beta\delta \otimes t^{-1}\mathrm{d}t^{-1}\\ &= (q\beta\delta - \delta\beta)\otimes t^{-1}\mathrm{d}t^{-1}\\ &= 0,\end{aligned} \tag{4.141}$$

where for $\mathfrak{ver}(e^\pm)$ we exploited the commutation relations. Accordingly we define

$$\mathfrak{ver}(e^0) := 1 \otimes t^{-1}\mathrm{d}t, \quad \mathfrak{ver}(e^\pm) = 0. \tag{4.142}$$

With these definitions the map $\Delta^1_A := \Delta_{\Omega^1(A)} + \mathfrak{ver}$ extends correctly to $(\Omega^1(A), \mathrm{d})$ provided $\Delta_{\Omega^1(A)}$ and $\mathfrak{ver}$ satisfy the commutation relations among A and $\Omega^1(A)$. We have

$$\begin{aligned}\Delta_{\Omega^1(A)}(e^\pm f) &= \Delta_{\Omega^1(A)}(e^\pm)\Delta_A(f)\\ &= (e^\pm \otimes t^{\pm 2})(f \otimes t^{|f|})\\ &= e^\pm f \otimes t^{\pm 2} t^{|f|}\\ &= q^{|f|} f e^\pm \otimes t^{\pm 2} t^{|f|}\\ &= q^{|f|}\Delta_A(f)\Delta_{\Omega^1(A)}(e^\pm)\\ &= \Delta_{\Omega^1(A)}(q^f f e^\pm),\end{aligned} \tag{4.143}$$

$$\begin{aligned}\Delta_{\Omega^1(A)}(e^0 f) &= \Delta_{\Omega^1(A)}(e^0)\Delta_A(f)\\ &= (e^0 \otimes 1)\ (f \otimes t^{|f|})\\ &= e^0 f \otimes t^{|f|}\\ &= q^{2|f|} f e^0 \otimes t^{|f|}\\ &= q^{2|f|}\Delta_A(f)\Delta_{\Omega^1(A)}(e^0)\\ &= \Delta_{\Omega^1(A)}(q^{2|f|} f e^0),\end{aligned} \tag{4.144}$$

$$\begin{aligned}\mathfrak{ver}(e^\pm f) &= \mathfrak{ver}(e^\pm)\Delta_A(f)\\ &= 0\\ &= q^{|f|}\Delta_A(f)\mathfrak{ver}(e^\pm)\\ &= \mathfrak{ver}(q^{|f|} f e^\pm).\end{aligned} \tag{4.145}$$

$$\begin{aligned}\mathfrak{ver}(e^0 f) &= \mathfrak{ver}(e^0)\Delta_A(f)\\ &= (1 \otimes t^{-1}\mathrm{d}t)(f \otimes t^{|f|})\\ &= f \otimes t^{-1}\mathrm{d}t\ t^{|f|}\\ &= f \otimes t^{-1}\mathrm{d}t\ t^{|f|}\\ &= q^{2|f|} f \otimes t^{|f|} t^{-1}\mathrm{d}t,\end{aligned} \tag{4.146}$$

and

$$\mathfrak{ver}(q^{2|f|} f e^0) = q^{2|f|} f \otimes t^{|f|} t^{-1}\mathrm{d}t. \tag{4.147}$$

□

We define a second order differential calculus $(\Omega^2(A), \wedge, \mathrm{d})$ over A as the free left A–module generated as $\Omega^2(A) = \mathrm{span}_A\{e^\pm \wedge e^0, e^+ \wedge e^-\}$ with commutation relations [2]

$$\begin{gathered} e^+ \wedge e^- = -q^{-2} e^- \wedge e^+, \quad e^\pm \wedge e^0 = -q^{\mp 4} e^0 \wedge e^\pm, \quad \mathrm{d}e^0 = q^3 e^+ \wedge e^-,\\ \mathrm{d}e^\pm = \mp q^{\pm 2}[2]_{q^{-2}} e^\pm \wedge e^0, \quad e^\pm \wedge e^\pm = e^0 \wedge e^0 = 0,\end{gathered} \tag{4.148}$$

where $[2]_q = (1 - q^2)/(1 - q)$.

Proposition 4.3.14. *$\Omega^2(A)$ is right $H-$covariant. Moreover, maps*

$$\begin{aligned} \mathfrak{ver}^{1,1} &: \Omega^2(A) \to \Omega^1(A) \otimes \Omega^1(H), \\ \mathfrak{ver}^{0,2} &: \Omega^2(A) \to A \otimes \Omega^2(H) \end{aligned} \tag{4.149}$$

are well defined.

Proof. The morphism $\mathfrak{ver}^{0,2}$ is trivial since no-nonzero 2-forms occur on H. We define

$$\Delta_{\Omega^2(A)}(e^+ \wedge e^-) := e^+ \wedge e^- \otimes 1, \quad \Delta_{\Omega^2(A)}(e^+ \wedge e^0) := e^+ \wedge e^0 \otimes t^2, \\ \Delta_{\Omega^2(A)}(e^- \wedge e^0) := e^- \wedge e^0 \otimes t^{-2} \tag{4.150}$$

This is well-defined because it is compatible with all relations of 2-forms. In fact

$$\begin{aligned} \Delta_{\Omega^2(A)}(e^+ \wedge e^-) &= \Delta_{\Omega^1(A)}(e^+)\Delta_{\Omega^1(A)}(e^-) \\ &= (e^+ \otimes t^2)(e^- \otimes t^{-2}) \\ &= e^+ \wedge e^- \otimes 1; \end{aligned} \tag{4.151}$$

$$\begin{aligned} \Delta_{\Omega^2(A)}(e^\pm \wedge e^0) &= \Delta_{\Omega^1(A)}(e^\pm)\Delta_{\Omega^1(A)}(e^0) \\ &= (e^\pm \otimes t^{\pm 2})(e^0 \otimes 1) \\ &= e^\pm \wedge e^0 \otimes t^{\pm 2}, \end{aligned} \tag{4.152}$$

$$\begin{aligned} \Delta_{\Omega^2(A)}(-q^2 e^- \wedge e^+) &= -q^2 \Delta_{\Omega^2(A)}(e^- \wedge e^+) \\ &= -q^2 \Delta_{\Omega^1(A)}(e^-)\Delta_{\Omega^1(A)}(e^+) \\ &= -q^2 (e^- \otimes t^{-2})(e^+ \otimes t^2) \\ &= -q^2 e^- \wedge e^+ \otimes 1 \\ &= e^+ \otimes e^- \otimes 1 \\ &= \Delta_{\Omega^2(A)}(e^+ \wedge e^-); \end{aligned} \tag{4.153}$$

$$\begin{aligned} \Delta_{\Omega^2(A)}(-q^4 e^0 \wedge e^\pm) &= -q^4 \Delta_{\Omega^1(A)}(e^0)\Delta_{\Omega^1(A)}(e^\pm) \\ &= -q^4 (e^0 \otimes 1)(e^\pm \otimes t^{\pm 2}) \\ &= -q^4 e^0 \wedge e^\pm \otimes t^{\pm 2} \\ &= e^\pm \wedge e^0 \otimes t^\pm \\ &= \Delta_{\Omega^2(A)}(e^\pm \wedge e^0). \end{aligned} \tag{4.154}$$

$$\begin{aligned} \Delta_{\Omega^2(A)}(\mathrm{d}e^0) &= (\mathrm{d} \otimes \mathrm{id})\Delta_{\Omega^1(A)}(e^0) \\ &= \mathrm{d}e^0 \otimes 1 \\ &= e^+ \wedge e^- \otimes 1, \end{aligned} \tag{4.155}$$

$$\begin{aligned} \Delta_{\Omega^2(A)}(\mathrm{d}e^+) &= (\mathrm{d} \otimes \mathrm{id})\Delta_{\Omega^1(A)}(e^+) \\ &= \mathrm{d}e^+ \otimes t^2 \\ &= -q^2 [2]_{q^{-2}} e^+ \wedge e^0 \otimes t^2 \\ &= -q^2 [2]_{q^{-2}} (e^+ \otimes t^2)(e^0 \otimes 1) \\ &= -q^2 [2]_{q^{-2}} \Delta_{\Omega^1(A)}(e^+)\Delta_{\Omega^1(A)}(e^0) \\ &= \Delta_{\Omega^2(A)}(-q^2 [2]_{q^{-2}} e^+ \wedge e^0). \end{aligned} \tag{4.156}$$

$$\begin{aligned}\Delta_{\Omega^2(A)}(\mathrm{d}e^-) &= (\mathrm{d}\otimes\mathrm{id})\Delta_{\Omega^1(A)}(e^-)\\ &= \mathrm{d}e^-\otimes t^{-2}\\ &= q^{-2}[2]_{q^{-2}}e^-\wedge e^0\otimes t^{-2}\\ &= q^{-2}[2]_{q^{-2}}(e^-\otimes t^{-2})(e^0\otimes 1)\\ &= q^{-2}[2]_{q^{-2}}\Delta_{\Omega^1(A)}(e^-)\Delta_{\Omega^1(A)}(e^0)\\ &= \Delta_{\Omega^2(A)}(q^{-2}[2]_{q^{-2}}e^-\wedge e^0).\end{aligned} \tag{4.157}$$

and so we extend $\Delta_{\Omega^2(A)}$ to a map on $\Omega^2(A)$ by

$$\begin{aligned}\Delta_{\Omega^2(A)}(ae^+\wedge e^- + be^+\wedge e^0 + ce^-\wedge e^0) &= \Delta_A(a)\Delta_{\Omega^2(A)}(e^+\wedge e^-)\\ &\quad + \Delta_A(b)\Delta_{\Omega^2(A)}(e^+\wedge e^-) + \Delta_A(c)\Delta_{\Omega^2(A)}(e^-\wedge e^0).\end{aligned} \tag{4.158}$$

Moving to the vertical map $\mathfrak{ver}^{1,1}$ we have

$$\begin{aligned}\mathfrak{ver}^{1,1}(e^+\wedge e^-) &= \Delta_{\Omega^1(A)}(e^+)\mathfrak{ver}(e^-) + \mathfrak{ver}(e^+)\Delta_{\Omega^1(A)}(e^-)\\ &= 0\\ &= \mathfrak{ver}^{1,1}(\mathrm{d}e^0),\end{aligned} \tag{4.159}$$

$$\begin{aligned}\mathfrak{ver}^{1,1}(e^\pm\wedge e^0) &= \Delta_{\Omega^1(A)}(e^\pm)\mathfrak{ver}(e^0) + \mathfrak{ver}(e^\pm)\Delta_{\Omega^1(A)}(e^0)\\ &= \Delta_{\Omega^1(A)}(e^\pm)\mathfrak{ver}(e^0)\\ &= (e^\pm\otimes t^{\pm 2})(1\otimes t^{-1}\mathrm{d}t)\\ &= e^\pm\otimes t^{\pm 2-1}\mathrm{d}t,\end{aligned} \tag{4.160}$$

and so we define

$$\mathfrak{ver}^{1,1}(e^+\wedge e^-) := 0, \quad \mathfrak{ver}^{1,1}(e^\pm\wedge e^0) := e^\pm\otimes t^{\pm 2-1}\mathrm{d}t. \tag{4.161}$$

With these definitions on the generators the map $\mathfrak{ver}^{1,1}$ extends correctly to $\Omega^2(A)$ provided the commutation relations among generators of $\Omega^2(A)$ are preserved. The relation $\mathfrak{ver}^{1,1}(-q^2e^-\wedge e^+) = \mathfrak{ver}^{1,1}(e^+\wedge e^-) = 0$ is obvious.

In [2] (page 109) the following relations are provided:

$$\mathrm{d}\alpha = \alpha e^0 + q\beta e^+, \quad \mathrm{d}\beta = \alpha e^- - q^{-2}\beta e^0, \quad \mathrm{d}\gamma = \gamma e^0 + q\delta e^+, \quad \mathrm{d}\delta = \gamma e^- - q^{-2}\delta e^0. \tag{4.162}$$

Therefore

$$\begin{aligned}\mathfrak{ver}^{1,1}\circ\mathrm{d}(e^+) &= \mathfrak{ver}^{1,1}\circ\mathrm{d}\Big(q^{-1}\alpha\wedge\mathrm{d}\gamma - q^{-2}\gamma\mathrm{d}\alpha\Big)\\ &= \mathfrak{ver}^{1,1}(q^{-1}\mathrm{d}\alpha\wedge\mathrm{d}\gamma - q^{-2}\mathrm{d}\gamma\wedge\mathrm{d}\alpha)\\ &= q^{-1}\mathfrak{ver}^{1,1}(\mathrm{d}\alpha\wedge\mathrm{d}\gamma) - q^{-2}\mathfrak{ver}^{1,1}(\mathrm{d}\gamma\wedge\mathrm{d}\alpha)\\ &= q^{-1}\mathfrak{ver}^{1,1}[(\alpha e^0 + q\beta e^+)\wedge(\gamma e^0 + q\delta e^+)] - q^{-2}\mathfrak{ver}^{1,1}[(\gamma e^0 + q\delta e^+)\wedge(\alpha e^0 + q\beta e^+)]\\ &= \mathfrak{ver}^{1,1}(\alpha e^0\wedge\delta e^+) + \mathfrak{ver}^{1,1}(\beta e^+\wedge\gamma e^0) - q^{-1}\mathfrak{ver}^{1,1}(\gamma e^0\wedge\beta e^+) - q^{-1}\mathfrak{ver}^{1,1}(\delta e^+\wedge\alpha e^0)\\ &= \Delta_A(\alpha)\mathfrak{ver}(e^0)\Delta_A(\delta)\Delta_{\Omega^1(A)}(e^+) + \Delta_A(\beta)\Delta_{\Omega^1(A)}(e^+)\Delta_A(\gamma)\mathfrak{ver}(e^0)\\ &\quad - q^{-1}\Delta_A(\gamma)\mathfrak{ver}(e^0)\Delta_A(\beta)\Delta_{\Omega^1(A)}(e^+) - q^{-1}\Delta_A(\delta)\Delta_{\Omega^1(A)}(e^+)\Delta_A(\alpha)\mathfrak{ver}(e^0)\\ &= (\alpha\otimes t)(1\otimes t^{-1}\mathrm{d}t)(\delta\otimes t^{-1})(e^+\otimes t^2) + (\beta\otimes t^{-1})(e^+\otimes t^2)(\gamma\otimes t)(1\otimes t^{-1}\mathrm{d}t)\\ &\quad - q^{-1}(\gamma\otimes t)(1\otimes t^{-1}\mathrm{d}t)(\beta\otimes t^{-1})(e^+\otimes t^2) - q^{-1}(\delta\otimes t^{-1})(e^+\otimes t^2)(\alpha\otimes t)(1\otimes t^{-1}\mathrm{d}t)\\ &= -\alpha\delta e^+\otimes\mathrm{d}tt + \beta e^+\gamma\otimes t\mathrm{d}t + q^{-1}\gamma\beta e^+\otimes\mathrm{d}tt - q^{-1}\delta e^+\alpha\otimes t\mathrm{d}t\\ &= (-q^2\alpha\delta e^+ + q\beta\gamma e^+ + q\gamma\beta e^+ - \delta\alpha e^+)\otimes t\mathrm{d}t\\ &= -(q^2(\alpha\delta - q^{-1}\gamma\beta) + (\delta\alpha - q\gamma\beta))e^+\otimes t\mathrm{d}t\\ &= -(q^2+1)e^+\otimes t\mathrm{d}t,\end{aligned}$$

which equals

$$\mathfrak{ver}^{1,1}(-q^2[2]_{q^{-2}}e^+ \wedge e^0) = -(1+q^2)e^+ \otimes t\mathrm{d}t. \tag{4.163}$$

Moreover

$$\begin{aligned}
\mathfrak{ver}^{1,1} \circ \mathrm{d}\Big(e^0\Big) &= \mathfrak{ver}^{1,1} \circ \mathrm{d}(\delta \mathrm{d}\alpha - q\beta \mathrm{d}\gamma)\\
&= \mathfrak{ver}^{1,1}(\mathrm{d}\delta \wedge \mathrm{d}\alpha - q\mathrm{d}\beta \wedge \mathrm{d}\gamma)\\
&= \mathfrak{ver}^{1,1}[(\gamma e^- - q^{-2}\delta e^0) \wedge (\alpha e^0 + q\beta e^+) - q(\alpha e^- - q^{-2}\beta e^0) \wedge (\gamma e^0 + q\delta e^+)]\\
&= \mathfrak{ver}^{1,1}(\gamma e^- \wedge \alpha e^0 - q^{-1}\delta e^0 \wedge \beta e^+ - q\alpha e^- \wedge \gamma e^0 + q^{-1}\beta e^0 \wedge \delta e^+)\\
&= \Delta_A(\gamma)\Delta_{\Omega^1(A)}(e^-)\Delta_A(\alpha)\mathfrak{ver}(e^0) - q^{-1}\Delta_A(\delta)\mathfrak{ver}(e^0)\Delta_A(\beta)\Delta_{\Omega^1(A)}(e^+)\\
&- q\Delta_A(\alpha)\Delta_{\Omega^1(A)}(e^-)\Delta_A(\gamma)\mathfrak{ver}(e^0) + q^{-1}\Delta_A(\beta)\mathfrak{ver}(e^0)\Delta_A(\delta)\Delta_{\Omega^1(A)}(e^+)\\
&= (\gamma \otimes t)(e^- \otimes t^{-2})(\alpha \otimes t)(1 \otimes t^{-1}\mathrm{d}t) - q^{-1}(\delta \otimes t^{-1})(1 \otimes t^{-1}\mathrm{d}t)(\beta \otimes t^{-1})(e^+ \otimes t^2)\\
&- q(\alpha \otimes t)(e^- \otimes t^{-2})(\gamma \otimes t)(1 \otimes t^{-1}\mathrm{d}t) + q^{-1}(\beta \otimes t^{-1})(1 \otimes t^{-1}\mathrm{d}t)(\delta \otimes t^{-1})(e^+ \otimes t^2)\\
&= \gamma e^- \alpha \otimes t^{-1}\mathrm{d}t + q\delta\beta e^+ \otimes t^{-1}\mathrm{d}t - q\alpha e^- \gamma \otimes t^{-1}\mathrm{d}t + q\beta\delta e^+ \otimes t^{-1}\mathrm{d}t\\
&= (q\gamma\alpha + q\delta\beta - q^2\alpha\gamma + q^2\beta\delta)e^+ \otimes t^{-1}\mathrm{d}t\\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\mathfrak{ver}^{1,1} \circ \mathrm{d}(e^-) &= \mathfrak{ver}^{1,1} \circ \mathrm{d}(\delta \mathrm{d}\beta - q\beta \mathrm{d}\delta)\\
&= \mathfrak{ver}^{1,1}(\mathrm{d}\delta \wedge \mathrm{d}\beta - q\mathrm{d}\beta \wedge \mathrm{d}\delta)\\
&= \mathfrak{ver}^{1,1}[(\gamma e^- - q^{-2}\delta e^0) \wedge (\alpha e^- - q^{-2}\beta e^0) - q(\alpha e^- - q^{-2}\beta e^0) \wedge (\gamma e^- - q^{-2}\delta e^0)]\\
&= \mathfrak{ver}^{1,1}(-q^{-2}\gamma e^- \wedge \beta e^0 - q^{-2}\delta e^0 \wedge \alpha e^- + q^{-1}\alpha e^- \wedge \delta e^0 + q^{-1}\beta e^0 \wedge \gamma e^-)\\
&= -q^{-2}\Delta_A(\gamma)\Delta_{\Omega^1(A)}(e^-)\Delta_A(\beta)\mathfrak{ver}(e^0) - q^{-2}\Delta_A(\delta)\mathfrak{ver}(e^0)\Delta_A(\alpha)\Delta_{\Omega^1(A)}(e^-)\\
&+ q^{-1}\Delta_A(\alpha)\Delta_{\Omega^1(A)}(e^-)\Delta_A(\delta)\mathfrak{ver}(e^0) + q^{-1}\Delta_A(\beta)\mathfrak{ver}(e^0)\Delta_A(\gamma)\Delta_{\Omega^1(A)}(e^-)\\
&= -q^{-2}(\gamma \otimes t)(e^- \otimes t^{-2})(\beta \otimes t^{-1})(1 \otimes t^{-1}\mathrm{d}t) - q^{-2}(\delta \otimes t^{-1})(1 \otimes t^{-1}\mathrm{d}t)(\alpha \otimes t)(e^- \otimes t^{-2})\\
&+ q^{-1}(\alpha \otimes t)(e^- \otimes t^{-2})(\delta \otimes t^{-1})(1 \otimes t^{-1}\mathrm{d}t) + q^{-1}(\beta \otimes t^{-1})(1 \otimes t^{-1}\mathrm{d}t)(\gamma \otimes t)(e^- \otimes t^{-2})\\
&= -q^{-2}\gamma e^- \beta \otimes t^{-3}\mathrm{d}t + q^{-2}\delta\alpha e^- \otimes t^{-2}\mathrm{d}t t^{-1} + q^{-1}\alpha e^- \delta \otimes t^{-3}\mathrm{d}t + q^{-1}\beta\gamma e^- t^{-2}\mathrm{d}t t^{-1}\\
&= -q^{-3}\gamma\beta e^- \otimes t^{-3}\mathrm{d}t + q^{-4}\delta\alpha e^- \otimes t^{-3}\mathrm{d}t + q^{-2}\alpha\delta e^- \otimes t^{-3}\mathrm{d}t - q^{-3}\beta\gamma e^- t^{-3}\mathrm{d}t\\
&= q^{-2}(-q^{-1}\gamma\beta + q^{-2}\delta\alpha + \alpha\delta - q^{-1}\beta\gamma)e^- \otimes t^{-3}\mathrm{d}t\\
&= q^{-2}(-q^{-1}\gamma\beta + q^{-2}(\alpha\delta + (q - q^{-1})\beta\gamma) + \alpha\delta - q^{-1}\beta\gamma)e^- \otimes t^{-3}\mathrm{d}t\\
&= q^{-2}(1 + q^{-2})e^- \otimes t^{-3}\mathrm{d}t,
\end{aligned}$$

which equals

$$\begin{aligned}
\mathfrak{ver}^{1,1}(q^{-2}[2]_{q^{-2}}e^- \wedge e^0) &= q^{-2}(1+q^{-2})\mathfrak{ver}^{1,1}(e^- \wedge e^0)\\
&= q^{-2}(1+q^{-2})e^- \otimes t^{-3}\mathrm{d}t.
\end{aligned} \tag{4.164}$$

Finally

$$\begin{aligned}
\mathfrak{ver}^{1,1}(-q^{\mp 4}e^0 \wedge e^\pm) &= -q^{\mp 4}(\Delta_{\Omega^1(A)}(e^0)\mathfrak{ver}(e^\pm) + \mathfrak{ver}(e^0)\Delta_{\Omega^1(A)}(e^\pm))\\
&= -q^{\mp 4}\mathfrak{ver}(e^0)\Delta_{\Omega^1(A)}(e^\pm)\\
&= -q^{\mp 4}(1 \otimes t^{-1}\mathrm{d}t)(e^\pm \otimes t^{\pm 2})\\
&= -q^{\mp 4}e^\pm \otimes t^{-1}\mathrm{d}t\ t^{\pm 2}\\
&= -q^{\mp 4}e^\pm \otimes t^{-1}\mathrm{d}t\ t^{\pm 2}\\
&= e^\pm \otimes t^{\pm 2-1}\mathrm{d}t\\
&= \mathfrak{ver}^{1,1}(e^\pm \wedge e^0).
\end{aligned} \tag{4.165}$$

In the various calculations we exploited

$$\mathrm{d}tt^2 = q^4 t^2 \mathrm{d}t, \quad \mathrm{d}tt^{-2} = q^{-4} t^{-2} \mathrm{d}t$$

beside the various commutation relations between the generators of the algebra and Proposition (4.3.1). □

We define a third order differential calculus $(\Omega^3(A), \wedge, \mathrm{d})$ over A as the free left A–module generated as $\Omega^3(A) = \mathrm{span}_A\{e^+ \wedge e^- \wedge e^0\}$, with commutation relations

$$\mathrm{d}(e^+ \wedge e^-) = -q^{-2}\, \mathrm{d}(e^- \wedge e^+), \quad \mathrm{d}\Big(e^\pm \wedge e^0\Big) = -q^{\mp 4}\, \mathrm{d}\Big(e^0 \wedge e^\pm\Big). \tag{4.166}$$

Proposition 4.3.15. $\Omega^3(A)$ *is right H–covariant. Moreover the vertical maps*

$$\begin{aligned} \mathfrak{ver}^{2,1} &: \Omega^3(A) \to \Omega^2(A) \otimes \Omega^1(H), \\ \mathfrak{ver}^{1,2} &: \Omega^3(A) \to \Omega^1(A) \otimes \Omega^2(H), \\ \mathfrak{ver}^{0,3} &: \Omega^3(A) \to A \otimes \Omega^3(H), \end{aligned} \tag{4.167}$$

are well defined.

Proof. We start noticing $\mathfrak{ver}^{0,3}$ and $\mathfrak{ver}^{1,2}$ are trivial since $\Omega^k(H) = 0$ for $k > 1$. We need to show the coaction $\Delta_A : A \to A \otimes H$ extends to $\Omega^3(A)$ in the correct way. On the generator we have:

$$\begin{aligned} \Delta_{\Omega^3(A)}(e^+ \wedge e^- \wedge e^0) &= \Delta_{\Omega^1(A)}(e^+)\Delta_{\Omega^1(A)}(e^-)\Delta_{\Omega^1(A)}(e^0) \\ &= (e^+ \otimes t^2)(e^- \otimes t^{-2})(e^0 \otimes 1) \\ &= e^+ \wedge e^- \wedge e^0 \otimes 1; \end{aligned} \tag{4.168}$$

$$\begin{aligned} \mathfrak{ver}^{2,1}((e^+ \wedge e^-) \wedge e^0) &= \Delta_{\Omega^2(A)}(e^+ \wedge e^-)\mathfrak{ver}(e^0) + \mathfrak{ver}^{1,1}(e^+ \wedge e^-)\Delta_{\Omega^1(A)}(e^0) \\ &= (e^+ \wedge e^- \otimes 1)[1 \otimes t^{-1}\mathrm{d}t] \\ &= (e^+ \wedge e^-) \otimes t^{-1}\mathrm{d}t \\ &= \mathfrak{ver}^{2,1}(e^+ \wedge (e^- \wedge e^0)). \end{aligned} \tag{4.169}$$

Accordingly we define

$$\begin{aligned} \Delta_{\Omega^3(A)}(e^+ \wedge e^- \wedge e^0) &:= e^+ \wedge e^- \wedge e^0 \otimes 1, \\ \mathfrak{ver}^{2,1}(e^+ \wedge e^- \wedge e^0) &:= (e^+ \wedge e^-) \otimes t^{-1}\mathrm{d}t. \end{aligned} \tag{4.170}$$

With these definitions the map $\Delta^3_A := \Delta_{\Omega^3(A)} + \mathfrak{ver}^{2,1}$ provides a complete third order differential calculus over A provided the commutation relations between elements of the calculus are preserved. We have

$$\begin{aligned} \Delta_{\Omega^3(A)} \circ \mathrm{d}\Big(e^\pm \wedge e^0\Big) &= \Delta_{\Omega^3(A)}(\mathrm{d}e^\pm \wedge e^0 - e^\pm \wedge \mathrm{d}e^0) \\ &= \Delta_{\Omega^2(A)}(\mathrm{d}e^\pm)\Delta_{\Omega^1(A)}(e^0) - \Delta_{\Omega^1(A)}(e^\pm)\Delta_{\Omega^2(A)}(\mathrm{d}e^0) \\ &= (\mathrm{d}e^\pm \otimes t^{\pm 2})(e^0 \otimes 1) - (e^\pm \otimes t^\pm)(\mathrm{d}e^0 \otimes 1) \\ &= (\mathrm{d}e^\pm \wedge e^0 - e^\pm \wedge \mathrm{d}e^0) \otimes t^\pm \\ &= \mathrm{d}\Big(e^\pm \wedge e^0\Big) \otimes t^2 \\ &= -q^{\mp 4}\, \mathrm{d}\Big(e^0 \wedge e^\pm\Big) \otimes t^\pm \\ &= \Delta_{\Omega^3(A)}\Big(-q^{\mp 4}\, \mathrm{d}\Big(e^0 \wedge e^\pm\Big)\Big), \end{aligned} \tag{4.171}$$

and

$$\begin{aligned}\Delta_{\Omega^3(A)} \circ \mathrm{d}\big(e^+ \wedge e^-\big) &= \Delta_{\Omega^3(A)}(\mathrm{d}e^+ \wedge e^- - e^+ \wedge \mathrm{d}e^-)\\ &= \Delta_{\Omega^2(A)}(\mathrm{d}e^+)\Delta_{\Omega^1(A)}(e^-) - \Delta_{\Omega^1(A)}(e^+)\Delta_{\Omega^2(A)}(\mathrm{d}e^-)\\ &= (\mathrm{d}e^+ \otimes t^2)(e^- \otimes t^{-2}) - (e^+ \otimes t^2)(\mathrm{d}e^- \otimes t^{-2})\\ &= (\mathrm{d}e^+ \wedge e^- - e^+ \wedge \mathrm{d}e^-) \otimes 1\\ &= \mathrm{d}\big(e^+ \wedge e^-\big) \otimes 1\\ &= -q^{-2}\,\mathrm{d}\big(e^- \wedge e^+\big) \otimes 1\\ &= \Delta_{\Omega^3(A)}(-q^{-2}\,\mathrm{d}\big(e^- \wedge e^+\big)).\end{aligned} \tag{4.172}$$

Moving to the vertical maps we find

$$\begin{aligned}\mathfrak{ver}^{2,1} \circ \mathrm{d}\big(e^+ \wedge e^-\big) &= \mathfrak{ver}^{2,1}(\mathrm{d}e^+ \wedge e^- - e^+ \wedge \mathrm{d}e^-)\\ &= \Delta_{\Omega^2(A)}(\mathrm{d}e^+)\mathfrak{ver}(e^-) + \mathfrak{ver}^{1,1}(\mathrm{d}e^+)\Delta_{\Omega^1(A)}(e^-)\\ &\quad - \Delta_{\Omega^1(A)}(e^+)\mathfrak{ver}^{1,1}(\mathrm{d}e^-) - \mathfrak{ver}(e^+)\Delta_{\Omega^2(A)}(\mathrm{d}e^-)\\ &= -[(q^2+1)e^+ \otimes t\mathrm{d}t](e^- \otimes t^{-2}) - (e^+ \otimes t^2)(q^{-2}(1+q^{-2})e^- \otimes t^{-3}\mathrm{d}t)\\ &= q^{-4}(q^2+1)e^+ \wedge e^- \otimes t^{-1}\mathrm{d}t - q^{-2}(1+q^{-2})e^+ \wedge e^- \otimes t^{-1}\mathrm{d}t\\ &= -(q^{-2}+q^{-4}-q^{-2}-q^{-4})e^+ \wedge e^- \otimes t^{-1}\mathrm{d}t\\ &= 0,\end{aligned} \tag{4.173}$$

which equals.

$$\begin{aligned}-q^{-2}\mathfrak{ver}^{2,1} \circ \mathrm{d}\big(e^- \wedge e^+\big) &= -q^{-2}\mathfrak{ver}^{2,1}(\mathrm{d}e^- \wedge e^+ - e^- \wedge \mathrm{d}e^+)\\ &\quad - q^{-2}\mathfrak{ver}^{1,1}(\mathrm{d}e^-)\Delta_{\Omega^1)(A)}(e^+) - q^{-2}\Delta_{\Omega^1(A)}(e^-)\mathfrak{ver}^{1,1}(\mathrm{d}e^+)\\ &= -q^{-2}(q^{-2}(1+q^{-2})e^- \otimes t^{-3}\mathrm{d}t)(e^+ \otimes t^2) - q^{-2}(e^- \otimes t^{-2})((1+q^2)e^+ \otimes t\mathrm{d}t)\\ &= q^{-4}(1+q^{-2})e^- \wedge e^+ \otimes t^{-3}\mathrm{d}t t^2 - q^{-2}(1+q^2)e^- \wedge e^+ \otimes t^{-1}\mathrm{d}t\\ &= (1+q^{-2})e^- \wedge e^+ \otimes t^{-1}\mathrm{d}t - q^{-2}(1+q^2)e^- \wedge e^+ \otimes t^{-1}\mathrm{d}t\\ &= [1+q^{-2}-q^{-2}-1]e^- \wedge e^+ \otimes t^{-1}\mathrm{d}t\\ &= -q^2[1+q^{-2}-q^{-2}-1]e^+ \wedge e^- \otimes t^{-1}\mathrm{d}t\\ &= 0.\end{aligned}$$

Moreover

$$\begin{aligned}-\mathfrak{ver}^{2,1}(e^0 \wedge e^+) &= -\mathfrak{ver}^{2,1}(\mathrm{d}e^0 \wedge e^+ - e^0 \wedge \mathrm{d}e^+)\\ &= -(\mathfrak{ver}^{1,1}(\mathrm{d}e^0)\Delta_{\Omega^1(A)}(e^+) - \Delta_{\Omega^1(A)}(e^0)\mathfrak{ver}^{1,1}(\mathrm{d}e^+) - \mathfrak{ver}(e^0)\Delta_{\Omega^2(A)}(\mathrm{d}e^+))\\ &= \Delta_{\Omega^1(A)}(e^0)\mathfrak{ver}^{1,1}(\mathrm{d}e^+) + \mathfrak{ver}(e^0)\Delta_{\Omega^2(A)}(\mathrm{d}e^+)\\ &= (e^0 \otimes 1)(-(q^2+1)e^+ \otimes t^{-1}\mathrm{d}t) + (1 \otimes t^{-1}\mathrm{d}t)(\mathrm{d}e^+ \otimes t^2)\\ &= -(q^2+1)e^0 \wedge e^+ \otimes t^{-1}\mathrm{d}t - q^4\mathrm{d}e^+ \otimes t\mathrm{d}t,\end{aligned} \tag{4.174}$$

which equals

$$\begin{aligned}q^4\mathfrak{ver}^{2,1} \circ \mathrm{d}\Big(e^+ \wedge e^0\Big) &= q^4\mathfrak{ver}^{2,1}(\mathrm{d}e^+ \wedge e^0 - e^+ \wedge \mathrm{d}e^0)\\ &= q^4(\mathfrak{ver}^{1,1}(\mathrm{d}e^+)\Delta_{\Omega^1(A)}(e^0) - \Delta_{\Omega^1(A)}(\mathrm{d}e^+)\mathfrak{ver}(e^0))\\ &= q^4[(-(q^2+1)e^+ \otimes t^{-1}\mathrm{d}t)(e^0 \otimes 1) - (\mathrm{d}e^+ \otimes t^2)(1 \otimes t^{-1}\mathrm{d}t)]\\ &= q^4(q^2+1)e^+ \wedge e^0 \otimes t^{-1}\mathrm{d}t - q^4\mathrm{d}e^+ \otimes t\mathrm{d}t\\ &= -(q^2+1)e^0 \wedge e^+ \otimes t^{-1}\mathrm{d}t - q^4\mathrm{d}e^+ \otimes t\mathrm{d}t.\end{aligned} \tag{4.175}$$

Finally

$$\begin{aligned}
-\mathfrak{ver}^{2,1}(e^0 \wedge e^-) &= -\mathfrak{ver}^{2,1}(\mathrm{d}e^0 \wedge e^- - e^0 \wedge \mathrm{d}e^-)\\
&= -(\mathfrak{ver}^{1,1}(\mathrm{d}e^0)\Delta_{\Omega^1(A)}(e^-) - \Delta_{\Omega^1(A)}(e^0)\mathfrak{ver}^{1,1}(\mathrm{d}e^-) - \mathfrak{ver}(e^0)\Delta_{\Omega^2(A)}(\mathrm{d}e^-))\\
&= \Delta_{\Omega^1(A)}(e^0)\mathfrak{ver}^{1,1}(\mathrm{d}e^-) + \mathfrak{ver}(e^0)\Delta_{\Omega^2(A)}(\mathrm{d}e^-)\\
&= (e^0 \otimes 1)(q^{-2}(1+q^{-2})e^- \otimes t^{-3}\mathrm{d}t) + (1 \otimes t^{-1}\mathrm{d}t)(\mathrm{d}e^- \otimes t^{-2})\\
&= q^{-2}(1+q^{-2})e^0 \wedge e^- \otimes t^{-3}\mathrm{d}t - q^{-4}\mathrm{d}e^- \otimes t^{-3}\mathrm{d}t\\
&= (q^{-2}(1+q^{-2})e^0 \wedge e^- - q^{-4}\mathrm{d}e^-) \otimes t^{-3}\mathrm{d}t,
\end{aligned} \tag{4.176}$$

that is

$$\begin{aligned}
q^{-4}\mathfrak{ver}^{2,1} \circ \mathrm{d}\Big(e^- \wedge e^0\Big) &= q^{-4}\mathfrak{ver}^{2,1}(\mathrm{d}e^- \wedge e^0 - e^- \wedge \mathrm{d}e^0)\\
&= q^{-4}(\mathfrak{ver}^{1,1}(\mathrm{d}e^-)\Delta_{\Omega^1(A)}(e^0) - \Delta_{\Omega^1(A)}(\mathrm{d}e^-)\mathfrak{ver}(e^0))\\
&= q^{-4}[(q^{-2}(q^{-2}+1)e^- \otimes t^{-3}\mathrm{d}t)(e^0 \otimes 1) - (\mathrm{d}e^- \otimes t^{-2})(1 \otimes t^{-1}\mathrm{d}t)]\\
&= -q^{-4}q^{-2}(q^{-2}+1)e^- \wedge e^0 \otimes t^{-3}\mathrm{d}t - q^{-4}\mathrm{d}e^- \otimes t^{-3}\mathrm{d}t\\
&= (-q^{-4}q^{-2}(q^{-2}+1)e^- \wedge e^0 - q^{-4}\mathrm{d}e^-) \otimes t^{-3}\mathrm{d}t\\
&= [q^{-2}(1+q^{-2})e^0 \wedge e^- - q^{-4}\mathrm{d}e^-] \otimes t^{-3}\mathrm{d}t.
\end{aligned} \tag{4.177}$$

□

Theorem 4.3.16. $\Omega^\bullet(A)$ *is a complete differential calculus.*

Proof. According to the last propositions we have the existence of vertical maps and right H–covariance up to the third order calculus. Moreover $\Omega^k(A) = 0$ for $k > 3$ since $e^+ \wedge e^+ = e^- \wedge e^- = e^0 \wedge e^0 = 0$. We conclude $\Omega^\bullet(A) = A \oplus \Omega^1(A) \oplus \Omega^2(A) \oplus \Omega^3(A)$ is a complete differential calculus over A. □

Remark 4.3.17. In [2] (Proposition 2.35, page 111) it is stated that $\Omega^1(B)$ is spanned by

$$\begin{aligned}
\mathrm{d}x &= -1\beta\delta e^+ - \alpha\gamma e^-,\\
\mathrm{d}z &= \mathrm{d}(\gamma\delta) = \delta^2 e^+ + \gamma^2 e^-,\\
\mathrm{d}\bar{z} &= -q\,\mathrm{d}(\alpha\beta) = -q\beta^2 e^+ - q\alpha^2 e^-.
\end{aligned} \tag{4.178}$$

Still in [2] (page 112) the following relations are provided.

$$\begin{aligned}
\delta e^+ &= \alpha\mathrm{d}z + q^{-1}\gamma\mathrm{d}x,\\
\beta e^+ &= q^{-2}\gamma\mathrm{d}z - q\alpha\mathrm{d}x,\\
\alpha e^- &= q^2\beta\mathrm{d}x - q^{-1}\delta\mathrm{d}\bar{z},\\
\gamma e^- &= -\delta\mathrm{d}x - q\beta\mathrm{d}z.
\end{aligned} \tag{4.179}$$

These are of particular interest in the proof of the next proposition. □

Proposition 4.3.18. $\Omega^\bullet(B)$ *is a differential calculus.*

Proof. let us consider a generic element $\omega \in \Omega^1(B)$. We may write

$$\omega = ae^+ + be^- + ce^0, \quad a, b, c \in A, \tag{4.180}$$

and since we must have $\Delta^1_A(\omega) = \omega \otimes 1$ we find

$$\begin{aligned}\Delta^1_A(ae^+ + be^- + ce^0) &= \Delta_A(a)\Delta^\wedge_A(e^+) + \Delta_A(b)\Delta^\wedge_A(e^-) + \Delta_A(c)\Delta^\wedge_A(e^0) \\ &= (a \otimes t^{|a|})(e^+ \otimes t^2) + (b \otimes t^{|b|})(e^- \otimes t^{-2}) \\ &+ (c \otimes t^{|c|})[e^0 \otimes 1 + 1 \otimes t^{-1}\mathrm{d}t] \\ &= ae^+ \otimes t^{|a|}t^2 + be^- \otimes t^{-2}t^{|b|} + ce^+ \otimes t^{|c|} + c \otimes t^{|c|}t^{-1}\mathrm{d}t \\ &= (ae^+ + be^- + ce^0) \otimes 1\end{aligned} \tag{4.181}$$

if and only if we have one the following combinations

$$|a| = -2, |b| = 2, c = 0, \quad a = c = 0, |b| = 2, \quad b = c = 0, |a| = -2, \quad a = b = c = 0. \tag{4.182}$$

Accordingly every element in $\Omega^1(B)$ must be of the form $\omega = ae^+ + be^-$ with the stated prescriptions. We write down the possible elements of degree $2, -2$ in terms of generators:

$$\text{degree 2} : \alpha^2, \gamma^2, \alpha\gamma; \quad \text{degree -2} : \delta^2, \beta^2, \beta\gamma.$$

We now show that every possible element of the form $ae^+ + be^-$ with a of degree -2 and b of degree 2 gives rise to an element of the form $B\mathrm{d}B$. For elements of the form be^- we find

$$\begin{aligned}\alpha^2 e^- &= \alpha(q^2\beta\mathrm{d}x - q^{-1}\delta\mathrm{d}z) \\ &= q^2\alpha\beta\mathrm{d}x - q^{-2}\alpha\delta\mathrm{d}z; \\ \gamma^2 e^- &= \gamma(-\delta\mathrm{d}x - q\beta\mathrm{d}z) \\ &= -\gamma\delta\mathrm{d}x - q\gamma\beta\mathrm{d}z; \\ \alpha\gamma e^- &= \alpha(-\delta\mathrm{d}x - q\beta\mathrm{d}z) \\ &= -\alpha\delta\mathrm{d}x - q\alpha\beta\mathrm{d}z.\end{aligned} \tag{4.183}$$

Each of the above expressions define elements of the form $B\mathrm{d}B$ as we are always pairing degree 1 and degree -1 elements. By the same reasoning we exploit elements of the form ae^+, for which

$$\begin{aligned}\beta^2 e^+ &= \beta(q^{-2}\gamma\mathrm{d}z - q\alpha\mathrm{d}x) \\ &= q^{-2}\beta\gamma\mathrm{d}z - q\beta\alpha\mathrm{d}x; \\ \delta^2 e^+ &= \delta(\alpha\mathrm{d}z + q^{-1}\gamma\mathrm{d}x) \\ &= \delta\alpha\mathrm{d}z + q^{-1}\delta\gamma\mathrm{d}x; \\ \delta\beta e^+ &= \delta(q^{-2}\gamma\mathrm{d}z - q\alpha\mathrm{d}x) \\ &= q^{-2}\delta\gamma\mathrm{d}z - q\delta\alpha\mathrm{d}x.\end{aligned} \tag{4.184}$$

Moreover, in [2] (Proposition 2.35, page 113) it is stated that the volume form can be expressed in terms of elements in B and $\Omega^1(B)$. Therefore $\Omega^\bullet(B)$ is a differential calculus. □

4.3.3 Crossed product calculus

In this section we study the relevant example of crossed product calculi. We recall that Theorem 2.2.5 provides a 1 to 1 correspondence between crossed product algebras and cleft extensions $B \subseteq B\sharp_\sigma H$. Since every cleft extension is also a faithfully flat Hopf-Galois extension, we conclude every extension $B \subseteq B\sharp_\sigma H$ is a quantum principal bundle. Classically a cleft extension can be thought as a trivialisation of the bundle.

In section 2.2 we considered B a $\sigma-$twisted left $H-$module algebra, and a Hopf algebra H. We introduced the crossed product algebra as the tensor product $B \otimes H$ equipped with an associative unital product $\mu_{\sharp_\sigma} : (B \otimes H) \otimes (B \otimes H) \to (B \otimes H)$. For details we remind to Lemma 2.2.3.

Definition 4.3.19 ([20], Definition 3.3)**.** Let $(\Omega^1(B), \mathrm{d}_B)$ be a first order differential calculus on a $\sigma-$twisted $H-$module algebra B with measure $\cdot : H \otimes B \to B$ and 2-cocycle $\sigma : H \otimes H \to B$. We say that $(\Omega^1(B), \mathrm{d}_B)$ is a $\sigma-$twisted $H-$module differential calculus if there exists a linear map $\cdot : H \otimes \Omega^1(B) \to \Omega^1(B)$ such that

$$\begin{aligned} h \cdot (b\mathrm{d}_B b') &= (h_1 \cdot b)(h_2 \cdot \mathrm{d}_B b') \\ \mathrm{d}_B(h \cdot b) &= h \cdot \mathrm{d}_B b \\ \mathrm{d}_B \circ \sigma &= 0, \end{aligned} \tag{4.185}$$

for every $h \in H$ and $b, b' \in B$.

Theorem 4.3.20 ([20],Theorem 3.7)**.** *Given $(\Omega^1(H), \mathrm{d}_h)$ a bicovariant first order differential calculus on H and $(\Omega^1(B), \mathrm{d}_B)$ a $\sigma-$twisted $H-$module first order differential calculus on B, we obtain a right $H-$covariant first order differential calculus $(\Omega^1(B\sharp_\sigma H), \mathrm{d}_{\sharp_\sigma})$ on $B\sharp_\sigma H$, where*

$$\begin{aligned} &\Omega^1(B\sharp_\sigma H) := (\Omega^1(B) \otimes H) \oplus (B \otimes \Omega^1(H)), \\ &\mathrm{d}_{\sharp_\sigma} : B\sharp_\sigma H \to \Omega^1(B\sharp_\sigma H), \quad b \otimes h \mapsto \mathrm{d}_B b \otimes h + b \otimes \mathrm{d}_H h. \end{aligned} \tag{4.186}$$

□

Remark 4.3.21. In the proof of this Theorem an explicit form for the right $H-$coaction on $\Omega^1(B\sharp_\sigma H)$ is provided. Since $B\sharp_\sigma H$ is a right $H-$comodule algebra with

$$\Delta_{B\sharp_\sigma H} : B\sharp_\sigma H \to B\sharp_\sigma H \otimes H, \quad b \otimes h \mapsto b \otimes h_1 \otimes h_2, \tag{4.187}$$

we define

$$\begin{aligned} \Delta^1_{B\sharp_\sigma H} &: \Omega^1(B\sharp_\sigma H) \to \Omega^1((B\sharp_\sigma H) \otimes H), \\ &\omega \otimes h + b \otimes \eta \mapsto \omega \otimes h_1 \otimes h_2 + b \otimes \eta_0 \otimes \eta_1 + b \otimes \eta_{-1} \otimes \eta_0. \end{aligned} \tag{4.188}$$

This map is differentiable, indeed

$$\begin{aligned} \Delta^1_{B\sharp_\sigma H} \circ \mathrm{d}_{\sharp_\sigma}(b \otimes h) &= \Delta^1_{B\sharp_\sigma H}(\mathrm{d}_B b \otimes h + b \otimes \mathrm{d}_H h) \\ &= \Delta^1_{B\sharp_\sigma H}(\mathrm{d}_B b \otimes h) + \Delta^1_{B\sharp_\sigma H}(b \otimes \mathrm{d}_H h) \\ &= \mathrm{d}_B b \otimes h_1 \otimes h_2 + b \otimes (\mathrm{d}_H h)_0 \otimes (\mathrm{d}_H h)_1 + b \otimes (\mathrm{d}_H h)_{-1} \otimes (\mathrm{d}_H h)_0 \\ &= \mathrm{d}_B b \otimes h_1 \otimes h_2 + b \otimes \mathrm{d}_H h_1 \otimes h_2 + b \otimes h_1 \otimes \mathrm{d}_H h_2 \\ &= \mathrm{d}_{\sharp_\sigma}(b \otimes h_1) \otimes h_2 + (b \otimes h_1) \otimes \mathrm{d}_H h_2 \\ &= \mathrm{d}_{\sharp_\sigma} \circ \Delta_{B\sharp_\sigma H}(b \otimes h), \end{aligned} \tag{4.189}$$

where we exploited left/right colinearity of differential on H. Therefore $\Omega^1(B\sharp_\sigma H)$ is a right $H-$covariant first order differential calculus over $B\sharp_\sigma H$, and the vertical map $\mathfrak{ver} = \mathfrak{ver}^{0,1}$ is well defined. □

We generalise Definition 4.3.19 to higher order forms on the crossed product algebra $B\sharp_\sigma H$.

Definition 4.3.22 ([20],Definition 3.13)**.** Let $(\Omega^\bullet(B), \mathrm{d}_B)$ be a differential calculus on B. We say that it is a $\sigma-$twisted $H-$module differential calculus if there exists linear maps $\cdot : H \otimes \Omega^k(B) \to \Omega^k(B)$, for all $k \geq 1$, such that

$$\begin{aligned} h \cdot (b^0 \mathrm{d}_B b^1 \wedge \cdots \wedge \mathrm{d}_B b^k) &= (h_1 \cdot b^0)(h_2 \cdot \mathrm{d}_B b^1) \wedge \cdots \wedge (h_{k+1} \cdot \mathrm{d}_B b^k), \\ h \cdot \mathrm{d}_B b &= \mathrm{d}_B(h \cdot b), \\ \mathrm{d}_B \circ \sigma &= 0, \end{aligned} \tag{4.190}$$

for all $b, b^0, \cdots, b^k \in B$ and $h \in H$.

Theorem 4.3.23. *[[20],Theorem 3.15] Let $(\Omega^\bullet(B), \mathrm{d}_B)$ be a σ−twisted H−module differential calculus over B, and let $(\Omega^\bullet(H), \mathrm{d}_H)$ be a bicovariant differential calculus over H. Let us define*

$$\Omega^n(B\sharp_\sigma H) := \bigoplus_{i=0}^{n} \Omega^{n-i}(B) \otimes \Omega^i(H), \tag{4.191}$$

for all $n \geq 0$, and let

$$\begin{aligned} (\omega \otimes \eta) \wedge (\omega' \otimes \eta') &:= (-1)^{jk}(\omega \wedge (\eta_{-2} \cdot \omega')\sigma(\eta_{-1} \otimes \eta'_{-1})) \otimes (\eta_0 \wedge \eta'_0), \\ \mathrm{d}_{\sharp_\sigma}(\omega \otimes \eta) &:= \mathrm{d}_B\omega \otimes \eta + (-1)^i \omega \otimes \mathrm{d}_H \eta, \end{aligned} \tag{4.192}$$

for $\omega \in \Omega^i(B)$, $\eta \in \Omega^j(H)$ and $\omega' \in \Omega^k(B)$, $\eta' \in \Omega^\ell(H)$. Then $(\Omega^\bullet(B\sharp_\sigma H, \mathrm{d}_{\sharp_\sigma})$ is a right H−covariant differential calculus on $B\sharp_\sigma H$ with respect to which the right H−coaction $\Delta_{B\sharp_\sigma H}$ is differentiable. □

Proposition 4.3.24. *Let $\Omega^\bullet(H)$ be a complete differential calculus and let $\Omega^\bullet(B\sharp_\sigma H)$ be the corresponding (complete) crossed product calculus. Differential forms over the base space are the ones $\Omega^\bullet(B) \otimes 1 \cong \Omega^\bullet(B)$.*

Proof. Every differential calculus defined as in Theorem 4.3.23 is complete since the right H−coaction extends to higher order forms as a morphism $\Delta^\wedge_{B\sharp_\sigma H} : \Omega^\bullet(B\sharp_\sigma H) \to \Omega^\bullet(B\sharp_\sigma H) \otimes \Omega^\bullet(H)$. Let $\beta \otimes \gamma$ be a coinvariant form in $\Omega^\bullet(B\sharp_\sigma H)$. We have

$$\begin{aligned} \Delta^\bullet_{B\sharp_\sigma H}(\beta \otimes \gamma) &= \beta \otimes \gamma_{[1]} \otimes \gamma_{[2]} \\ &= \beta \otimes \gamma \otimes 1, \end{aligned} \tag{4.193}$$

if and only if $\Delta^\bullet(\gamma) = \gamma \otimes 1$. If $\gamma \in \Omega^0(H) = H$, than γ must be a scalar multiple of the unit.

On the other hand, given a k−form $\gamma = h^0 \mathrm{d}h^1 \wedge \cdots \wedge \mathrm{d}h^k$ on $\Omega^\bullet(H)$, we have

$$h^0_1 \ldots h^k_1 \otimes h^0_2 \mathrm{d}h^1_2 \wedge \cdots \wedge \mathrm{d}h^k_2 = 0. \tag{4.194}$$

Applying $(\epsilon \otimes \mathrm{id})$ to the above equation gives

$$\gamma = h^0 \mathrm{d}h^1 \wedge \cdots \wedge \mathrm{d}h^k = 0. \tag{4.195}$$

□

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