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HOMOTOPY THEORIES FOR BRAIDS AND TOPOLOGICAL GENERALIZATIONS

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Abstract

We consider braid groups up to homotopy. Recently in [16] it was found a new presentation for homotopy braid groups, more symmetric than the classical one, and implying that $\tilde{P}_n$ is isomorphic to RP_n. This presentation was found using a topologic approach. In this work we find the same presentation by a combinatorial approach. Then we focus on surface braids and welded braids, giving an overview of the groups and of the results existing in the literature regarding their homotopy.

Sommario

Consideriamo i gruppi di trecce omotopiche. Recentemente, in [16], è stata trovata una nuova presentazione per i gruppi di trecce omotopiche, più simmetrica di quella classica, da cui si deduce che $\tilde{P}_n$ è isomorfo a RP_n. La presentazione è stata trovata usando un approccio topologico. In questa tesi ritroviamo la stessa presentazione mediante un approccio combinatorio. Poi trattiamo trecce su superfici e trecce welded, facendo una panoramica dei gruppi e dei risultati esistenti in letteratura riguardo ai loro quozienti omotopici.

Contents

Introduction

The study of braid groups started in 1891 with the work of Hurwitz, who considered them as fundamental groups of configuration spaces of n points in the complex plane. However, the first proper definition was given in 1925 by Artin [1]. He described a braid geometrically as the equivalence class, up to isotopy, of a collection of paths, called strands, in the cylinder $\mathbb{D}^2 \times [0,1]$, monotones with respect to the second coordinate, provided that the set of the initial points on the disc and the set of the endpoints on the disc coincide and different strands never intersect. He gave a presentation for braid groups, and also for their pure subgroups, which consist of the elements whose endpoints are not permuted.

Over the course of the century, braid groups were further defined in several different ways, e.g. as mapping class groups of a punctured disc or as subgroups of the group of automorphisms of the free group F_n.

Artin was also interested in studying homotopies between geometric braids. He defined two geometric braids isotopic when one can be deformed into the other by a continuous map such that at any time of the deformation it gives back a braid with the same endpoints. Then he defined two geometric braids homotopic if one can be deformed into the other by a continuous map that fixes the endpoints, and such that different strands never intersect, but allowing each strand to self-intersect itself during the deformation. In his paper of 1947 [2], he asked if the group of braids with n strands under the homotopy relation was the same as the group of braids with n strands under isotopy. The problem remained unsolved for years, until Goldsmith, in 1974, proved in her work [12] that isotopy and homotopy for braids are not the same, and gave a first presentation for the homotopy braid groups. Further studies on the subject followed; among the most recent, we want to recall the work of Liu [19] in 2015, and the one of Graff [16] in 2022. The latter gave a more symmetric presentation for the homotopy braid groups, using an unusual approach which involved claspers, and proved that there is an isomorphism between the group of pure braids under homotopy and the reduced pure braid group, which is the subgroup where every generator commutes with all its conjugates. His results will be extended in his PH.D. thesis.

Braids are versatile objects, which lead to several generalizations such as virtual braids, loop braids, welded braids or surface braids. From such generalizations arise other studies on homotopy.

Surface braid groups, first defined by Zariski during the 1930s, were studied throughout the century, and a first presentation was given by Scott [23] in 1970. Instead of considering a braid on the disc, as in the classical case, one can define it in the same geometric terms on a surface. Braid groups on a surface can also be considered from several points of view, e.g. as fundamental groups of configuration spaces of points on a surface, or as mapping class groups on a surface. The study of homotopy for surface braids is quite recent. Indeed, a first presentation of homotopy pure surface braid groups has been given by Yurasovskaya [24] in 2008.

Another example of a recent study on homotopy is the one of welded braids, which can be seen as a 4-dimensional generalization of classical braids. Welded braids can be described from several points of view and admit different names in each case. They have been considered topologically as ribbon braids, which are embeddings in $B^3 \times [0,1]$ of n disjoint copies of the oriented annulus $S^1 \times [0,1]$ with some specific properties. Ribbon braids can be described by three and two dimensional diagrams, namely braid broken diagrams and welded diagrams. They can be seen also as mapping class groups of a trivial link of unknotted circles in B^3 under the name of loop braids, as conjugating automorphisms of the free group, or as fundamental groups of the configuration space of a trivial link of unknotted circles. A survey on the equivalent definitions of welded braids can be found in [8].

The study of pure welded braids and pure surface braids up to homotopy requires the use of string links. A string link can be obtained by removing the monotony condition from the definition of pure braid. A classical result by Habegger and Lin [18] shows that every string link is link-homotopic to a pure braid. Therefore, it is equivalent to study pure braids up to homotopy or string links up to homotopy. In the same way, studying homotopy for surface and welded pure braids can be much easier if approached as surface and welded string links. This is the approach chosen in the surface context by Yurasovskaya [24], but also by Audoux, Bellingeri, Meilhan and Wagner, who studied homotopy for welded braids in [3] and [4]. In particular, they stated a result comparable to Habegger-Lin's classical one, which says that every welded string link is link-homotopic to a welded pure braid. Using this method, they managed to give an isomorphism between the group of welded pure braids up to homotopy and the group of conjugating automorphisms of the reduced free group. These results were extended by Darné [9], who gave a presentation of pure welded braid groups up to link-homotopy. However, many problems on the classification of welded string links up to homotopy are still open.

In this work, we consider homotopy for classical braids and we focus on finding the presentation given by Graff by using a combinatorial approach, with techniques similar to the ones used by Murasugi-Kurpita [20] and Liu [19]. It is still an open problem if this presentation has an equivalent in the surface and welded context; apparently, it is unlikely that there is an isomorphism between pure welded braids up to homotopy, pure surface braids up to homotopy and the reduced version of these groups up to isotopy.

The rest of this work is devolved to giving an overview of the above mentioned groups and on the results existing in the literature regarding their homotopy.

In Chapter 1 we introduce braids, providing several equivalent definitions, and we give a presentation for braid groups. Then we introduce homotopy of braids and we recall the presentations of the homotopy braid groups given by Goldsmith [12] and Liu [19]. Chapter 2 focuses on homotopy for surface braids. In Chapter 3 we introduce and present the state of the art of welded braids and welded string links up to link-homotopy. Finally, in Chapter 4, we prove two combinatorial lemmas before stating our main result on homotopy for classical braids.

Acknowledgements

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To the Laboratoire de Mathématiques Nicolas Oresme goes my gratitude for letting me witness the research world and allowing me to attend seminars and conferences which enriched my knowledge. Doing my internship in Caen was a great learning experience.

Lastly, I want to thank Emmanuel Graff for helping me understand his work on link-homotopy for braids and for the useful discussions.

Chapter 1

Braid groups

We start this thesis with an introduction to the braid groups. Braids were initially studied as fundamental groups of configuration spaces of n points in the complex plane, but the first proper definition is a geometric one which has been given by Artin in 1925 (see [1]). Then, other approaches led to several different definitions which has been proved to be equivalent over the years.

In this work, we have chosen to start by the geometric definition, which is of immediate comprehension and which is really useful to discuss homotopy. Afterwards, we state some other definitions, and we give partial proofs of their equivalence. The first one is in terms of configuration spaces, the second one involves mapping class groups, and the third one concerns automorphisms of the free group. The first part of this chapter ends with a presentation for the braid group given by Artin and some basic properties. Most of the results of this first part are taken from [10] and [15].
The second part of this chapter deals with the definition of homotopy for classical braids and Goldsmith's results on braids up to homotopy, and ends with a presentation for homotopy braid groups recently given by Liu.

1.1 Braids, Artin's classical definition

Let I denote, from now on, the unit interval $[0, 1]$. Let $p_1, \dots, p_n$ be distinguished points on the real axis inside the complex plane $\mathbb{C}$.

Definition 1.1.1. A n-component braid or n-braid $\beta = (\beta_1, \dots, \beta_n)$ is a collection of n paths

$$\beta_i : I \to \mathbb{C} \times I \qquad\qquad 1 \leq i \leq n$$

called strands, such that

1) the strands $\beta_i(I)$ are disjoint;

2) $\beta_i(t) \in \mathbb{C} \times \{t\}$;

3) there exists a permutation $\pi \in S_n$ such that $\beta_i(0) = (p_i, 0)$ and $\beta_i(1) = (p_{\pi(i)}, 1)$.

When π is the identity, the braid is called a pure braid.

Every braid can be represented by a 2-dimensional diagram, called braid diagram, which is a projection of the images of the β_i to the plane $\mathbb{R} \times I$, as we can see in Figure 1.1. By convention, $t = 0$ is usually drawn as top of the braid. In order to carry all the information, in each crossing we indicate graphically which strand goes over the other by erasing a small neighborhood of the underpassing strand.

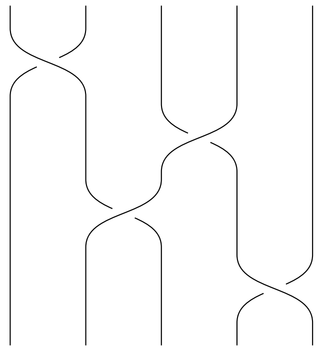

Figure 1.1: Braid diagram.

We define the product of two braids as a composition of paths, where the final endpoints of one braid are attached to the initial endpoints of the second one. More precisely, the product of the braid $(f_1(t), \ldots, f_n(t))$ with the braid $(g_1(t), \ldots, g_n(t))$ is the braid $(h_1(t), \ldots, h_n(t))$, where

$$h_i(t) = \begin{cases} f_i(2t) & 0 \leq t \leq 1/2 \\ g_{\pi_f(i)}(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

The heights of the two braids are scaled by 1/2 and then they are stacked one on top of the other, so the operation is usually called stacking and reparametrizing product. There is an example of product of two braids in Figure 1.2: we use the convention of not rescaling the heights, so that we can draw increasingly complicated braids.

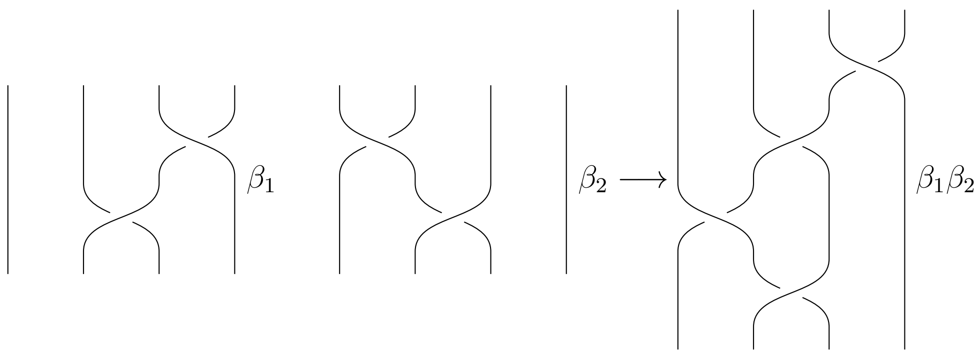

Figure 1.2: Product of braids.

Two braids are called isotopic if one can be deformed into the other by a continuous map such that at any time of the deformation we still have a braid with the same endpoints.

Definition 1.1.2. Let β and β' be two n-braids in $\mathbb{C} \times I$. We say that β is isotopic to β' if there exists a continuous map

$$H : (\mathbb{C} \times I) \times I \longrightarrow \mathbb{C} \times I$$

where $H(x,t,s) := H_s(x,t)$
such that:

1) $H_s(x,t)$ is an homeomorphism $\forall s \in I$;

2) $H_0 = \mathrm{id}$;

3) $(H_1 \circ \beta_1, \ldots, H_1 \circ \beta_n) = (\beta'_1, \ldots, \beta'_n)$;

4) $H_s|_{\mathbb{C}\times\{0,1\}} = \mathrm{id}$;

5) $H_s \circ \beta_i(t) \in \mathbb{C} \times \{t\}$ $\forall t \in I$.

Isotopy is an equivalence relation for braids. The set of isotopy equivalence classes of all n-braids with the product forms a group denoted by B_n. From now on, we will use the word braid to indicate an equivalence class of braids, and we will always omit the symbol [ ] for the classes: the abuse of notation is justified since we will always work up to isotopy. The identity element is the trivial braid, that is, $\beta = (\beta_1, \ldots, \beta_n)$ such that each strand β_i is the line segment connecting $p_i \times \{0\}$ and $p_i \times \{1\}$. The inverse of a given braid is obtained by taking its reflection through the plane $\mathbb{C} \times \{\frac{1}{2}\}$. Pure braids form a subgroup of B_n, denoted by P_n.

We denote by σ_i, for $1 \le i \le n-1$, the braid whose only crossing is the one created by the $(i+1)$-st strand passing in front of the i-th strand, as shown in Figure 1.3.

Remark 1.1.3. It is easy to understand that the group B_n is generated by $\sigma_1, \ldots, \sigma_n$. In fact, given any braid β we can find an element in the same isotopy class having its finitely many crossings at different horizontal levels. The braid β can be read from top to bottom as a product of the σ_i and their inverses.

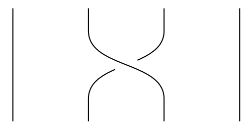

Figure 1.3: A generator σ_i for the braid group.

1.2 Braids as fundamental groups of configuration spaces

Let us consider the configuration space of n ordered distinct points in the complex plane $\mathbb{C}$, which is

$$M_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, \forall i \neq j\}.$$

Once we define the hyperplane $H_{ij} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i = z_j\} \in \mathbb{C}^n$ for $1 \le i < j \le n$, and the union of the hyperplanes $D = \bigcup_{1 \le i < j \le n} H_{ij}$, we can see M_n as the complement of D in $\mathbb{C}^n$:

$$M_n = \mathbb{C}^n \setminus D.$$

The union of hyperplanes D is usually called the big diagonal.
The symmetric group Σ_n acts on M_n by permuting coordinates. The quotient of M_n by this action is the configuration space of n unordered points in $\mathbb{C}$, which we can denote by

$$N_n = M_n/\Sigma_n.$$

The braid group on n strands B_n can be seen also as the fundamental group of N_n. Since the configuration space above defined is path connected, we will always omit the choice of a basepoint for the fundamental group.
Since a braid is a tuple $(f_1, \dots, f_n)$ where $f_i(t) \in \mathbb{C} \times \{t\}$, we have that the intersection with each plane $\mathbb{C} \times \{t\}$ is a point in the configuration space N_n. In this way, we can think of a braid $\beta = (\beta_1(t), \dots, \beta_n(t))$ as tracing out a loop of n-point configurations in $\mathbb{C}$ as t increases from 0 to 1. This identification gives the isomorphism. In this configuration space model, the generator σ_i of B_n described above corresponds to the element of $\pi_1(N_n)$ given by the loop of n-point configurations in $\mathbb{C}$ where the i-th and $(i+1)$-st points switch places by moving in a clockwise fashion, as indicated in Figure 1.4, and the others n-2 points remain fixed.

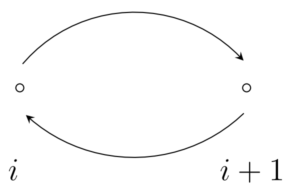

Figure 1.4: A standard generator of B_n in the configuration space model.

Remark 1.2.1. The pure braid group on n strands, P_n, is the fundamental group of M_n. That is $P_n \cong \pi_1(M_n)$.

1.3 Braids as mapping class groups of a punctured disc

The braid group B_n can be seen also as a mapping class group.

Definition 1.3.1. Let S be a compact, connected, orientable surface, with boundary ∂S, and for $n \geq 0$ let Q_n be a finite subset of $\mathrm{int}(S)$ consisting of n distinct points. Let $\mathrm{Homeo}^+(S, Q_n)$ denote the group of orientation-preserving homeomorphisms of S that leave Q_n setwise invariant and restrict to the identity on ∂S. The group is endowed with the compact-open topology.
The n-th mapping class group of S, denoted $MCG(S, n)$, is the group of isotopy classes of elements of $\mathrm{Homeo}^+(S, Q_n)$, where isotopies are required to fix the boundary pointwise and Q_n setwise.

$$MCG(S, n) = \pi_0(\mathrm{Homeo}^+(S, Q_n)).$$

So, it can be equivalently written as

$$MCG(S, n) = \mathrm{Homeo}^+(S, Q_n)/\,\mathrm{Homeo}_0(S, Q_n)$$

where $\mathrm{Homeo}_0(S, Q_n)$ is the connected component of the identity in $\mathrm{Homeo}^+(S, Q_n)$.

It is straightforward to check that $MCG(S, n)$ is indeed a group whose isomorphism class does not depend on the choice of Q_n. If $n = 0$ we write simply $\mathrm{Homeo}^+(S)$ and $MCG(S)$ for the corresponding groups.

Remark 1.3.2. In the literature there are sometimes slightly different definitions of $MCG(S)$, which take diffeomorphisms instead of homeomorphisms or homotopy classes instead of isotopy classes. The resulting groups are isomorphic.

We give one result on mapping class groups of the closed disc $\mathbb{D}^2$ that is used in the definition of braids as mapping class groups on the punctured disc. This result was proved by Alexander in 1923 and states that any homeomorphism ϕ of $\mathbb{D}^2$ that is the identity on the boundary $\partial\mathbb{D}^2$ is isotopic to the identity on $\mathbb{D}^2$, through homeomorphisms that are the identity on $\partial\mathbb{D}^2$.

Lemma 1.3.3. The group $MCG(\mathbb{D}^2)$ is trivial.

Proof. Identify $\mathbb{D}^2$ with the closed unit disc in $\mathbb{R}^2$. Let $\phi : \mathbb{D}^2 \to \mathbb{D}^2$ be a homeomorphism which is the identity on $\partial\mathbb{D}^2$. We define

$$F(x,t) = \begin{cases} (1-t)\phi(\frac{x}{1-t}) & 0 \le |x| < 1-t \\ x & 1-t \le |x| \le 1 \end{cases}$$

$F(x, 1)$ is the identity map of $\mathbb{D}^2$, so the result is an isotopy F from ϕ to the identity. $\square$

Let us consider the unitary closed disc $\mathbb{D}^2$ with n marked points. We can show that the mapping class group $MCG(\mathbb{D}^2, n)$ is isomorphic to the braid group B_n.

To show this we start by considering an homeomorphism ϕ of $\mathbb{D}^2$ that leaves invariant the set of marked points and that is the identity on $\partial\mathbb{D}^2$. Since ϕ is an homeomorphism of $\mathbb{D}^2$ that is the identity on the board, if we ignore the marked points we can use the Alexander lemma (see Lemma 1.3.3) to say that it is isotopic to the identity on $\mathbb{D}^2$. The isotopy F between ϕ and id moves the marked points around the interior of $\mathbb{D}^2$, which we can identify with $\mathbb{C}$, and takes them back to where they started, creating a loop in $\pi_1(N_n)$ which corresponds to a braid. It is not difficult to show that this association gives a well defined map from $MCG(\mathbb{D}^2, n)$ to B_n, which is a group isomorphism. A full proof, using short exact sequences, can be found in [10]. Hence braids can be seen as mapping classes of the punctured disc.

Remark 1.3.4. If we denote by $PMCG(\mathbb{D}^2, n)$ the subgroup of $MCG(\mathbb{D}^2, n)$ which elements are the isotopy classes of homeomorphisms that leave invariant the set of market points pointwise, we have $P_n \cong PMCG(\mathbb{D}^2, n)$.

Under the isomorphism between B_n and $MCG(\mathbb{D}^2, n)$, each generator σ_i corresponds to the isotopy class of a homeomorphism of the disc with n marked points that has support a twice-punctured disc and is described on this support by Figure 1.5.

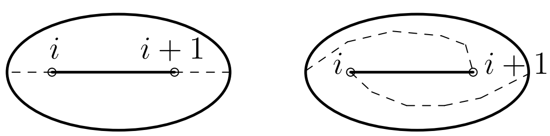

Figure 1.5: A half-twist.

1.4 Braids as automorphisms of the free group

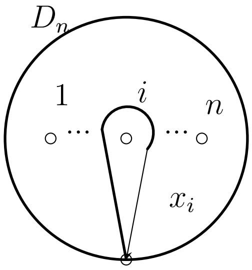

Figure 1.6: The loops $x_1, \dots, x_n$ are free generators of $\pi_1(D_n)$.

Let $F_n = F(x_1, x_2, \dots, x_n)$ be the free group of rank n. Artin in [1] gives also a characterization of the braid groups in terms of automorphisms of the free group.
Let us denote by D_n the disc $\mathbb{D}^2$ with n punctures. We remark that the fundamental group of the n times punctured disc D_n is the free group of rank n: $\pi_1(D_n) = \pi_1(D_n, *) = F_n$. If we fix the base point $*$, say in the boundary of D_n, we can take as free generators the loops $x_1, \dots, x_n$ depicted in Figure 1.6. If we consider braid $\beta \in B_n$ as a mapping class group, i.e. as an homeomorphism of D_n in itself, up to isotopy, then β acts on $\pi_1(D_n)$ as a isomorphism: it respects the concatenation of loops, and is bijective, as β^{-1} yields the inverse action. Hence β induces an automorphism of F_n, and this gives a representation:

$$\begin{array}{rccc} \rho: & B_n & \longrightarrow & \mathrm{Aut}(F_n) \\ & \beta & \longmapsto & \rho_\beta. \end{array}$$

The automorphism ρ_β can be easily described in the case $\beta = \sigma_i$, by giving the images of the generators of F_n under ρ_{σ_i}. We have that

$$\begin{array}{rcll} \rho_{\sigma_i}(x_i) & = & x_{i+1} & \\ \rho_{\sigma_i}(x_{i+1}) & = & x_{i+1}^{-1}x_i x_{i+1} & \\ \rho_{\sigma_i}(x_j) & = & x_j & \text{if } j \neq i, i+1. \end{array}$$

In Figure 1.7 and 1.8 the action of σ_i on the generators x_i and x_{i+1} is depicted.
The automorphism $\rho_{\sigma_i^{-1}}$ can be deduced from ρ_{σ_i}. For a general braid β written as a product of $\sigma_1, \dots, \sigma_{n-1}$ and their inverses, the automorphism ρ_β is the composition of the automorphisms corresponding to each letter.

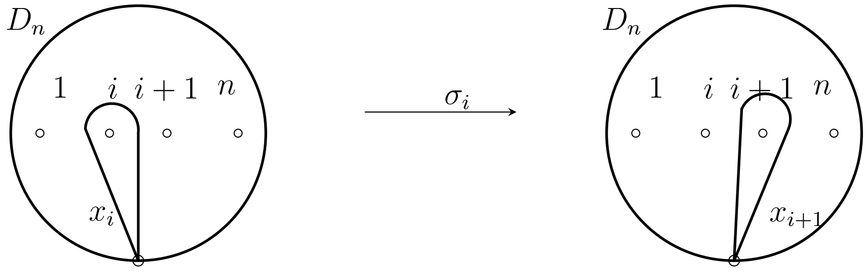

Figure 1.7: Action of σ_i on the generator x_i.

Artin in [2] showed that ρ is well defined by topological arguments, but we can see it coming easily from the presentation of B_n that is given in the next section.

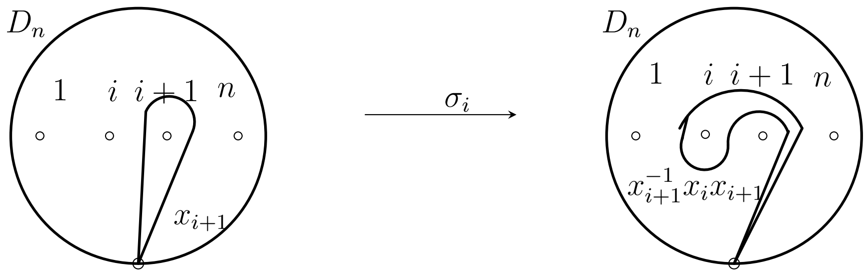

Figure 1.8: Action of σ_i on the generator x_{i+1}.

Remark 1.4.1. For every $\beta \in B_n$, the automorphism ρ_β sends each generator x_j to a conjugate of a generator.

Remark 1.4.2. For each $i = 1, \ldots, n-1$, we have $\rho_{\sigma_i}(x_1 \cdots x_n) = x_1 \cdots x_n$. In fact, $x_1 \cdots x_n$ corresponds to a loop that runs parallel to the boundary of D_n, enclosing the n points, so it's not deformed by any braid (up to isotopy), as we can see in Figure 1.9.

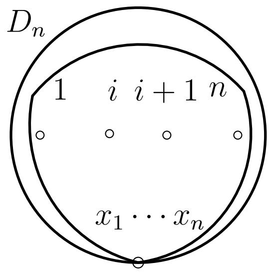

Figure 1.9: The loop $x_1 \cdots x_n$.

Artin in [1, 2] proved that these two conditions are not only necessary, but also sufficient for an element of $\mathrm{Aut}(F_n)$ to be induced by a braid, or equivalently to be in the subgroup $\rho(B_n)$ of $\mathrm{Aut}(F_n)$.

Theorem 1.4.3 ([1, 2])**.** *An automorphism $\beta \in \mathrm{Aut}(F_n)$ belongs to B_n if and only if β satisfies the following conditions:*

1) $\beta(x_i) = a_i x_{\pi(i)} a_i^{-1}$, $1 \leq i \leq n$;

2) $\beta(x_1 \cdots x_n) = x_1 \cdots x_n$

where $\pi \in S_n$ and $a_i \in F_n$.

1.5 Some properties of braid groups

We recall here some properties of braid groups, specifically the ones we are interested in for this work. For a more complete overview of braid groups' properties we suggest the readers to refer to [15].
We start by giving two short exact sequences which involve braid groups, already known by Artin [1].

The first one is really simple. To each braid in B_n we can associate the permutation it induces on its strands, that is, an element of the symmetric group Σ_n. This yields a well defined group homomorphism η from B_n to Σ_n, and the kernel of η is the subgroup of B_n formed by the braids inducing the trivial permutation, alias the pure braid group P_n. This led to the exact sequence

$$1 \to P_n \to B_n \xrightarrow{\eta} \Sigma_n \to 1. \tag{1.1}$$

The second exact sequence we consider, relates pure braid groups of distinct indices. Given a pure braid $\beta \in P_{n+1}$, by removing its last strand we obtain a pure braid $\tilde{\beta} \in P_n$. This yields a well defined homomorphism $\rho : P_{n+1} \to P_n$ which is clearly surjective. The kernel of this map consists of the braids in P_{n+1} whose first n strands form the trivial braid. Up to isotopy, every element of this subgroup can be seen with the first n strands vertical and the $n+1$-st which moves around them. If we look at this kind of elements as loops in the configuration space, they correspond to a motion of the $n+1$-st point, where the points $1, \dots, n$ do not move. This is equivalent to a motion of a point in the n-times punctured plane C_n. Hence, the kernel of the map ρ is isomorphic to $\pi_1(C_n)$, which is isomorphic to F_n. The resulting short exact sequence is:

$$1 \to F_n \xrightarrow{\iota} P_{n+1} \xrightarrow{\rho} P_n \to 1. \tag{1.2}$$

In this exact sequence, if F_n is freely generated by $x_1, \dots, x_n$, we can define

$$\iota(x_i) = (\sigma_n^{-1} \cdots \sigma_{i+1}^{-1})\sigma_i^2(\sigma_{i+1} \cdots \sigma_n)$$

for $i = 1, \dots, n$. This is the pure braid where the n-th strand passes behind the strands $n-1, \dots, i+1$, passes in front of the i-th strand, and again behind all the strands $i, \dots, n-1$, and all other strands are trivial.
There is a natural splitting $P_{n-1} \to P_n$ obtained by adding an extra strand, and so we see that $P_n \cong F_{n-1} \rtimes P_{n-1}$. A full proof of the correctness of this splitting, based on the approach with configuration spaces, can be found in [15].

Remark 1.5.1. Braid groups are torsion free: in fact, there is no way of obtaining the trivial braid by the product of a given braid β with itself finitely many times. There are several ways to prove it, according to the distinct approaches to the braid group; for a full exposition we suggest the reader to refer to [15].

1.6 Artin's presentation

The braid group on n strands B_n has been proved by Artin [1] to have a presentation which is stated in the next theorem. However, Artin did not give a detailed proof of the correctness of this presentation. There are in the literature various proofs of this result, most of which use the short exact sequences described in last section. One can use the fact that $P_2 \cong \mathbb{Z}$, and use the Reidemeister-Schreier method applied to Sequence 1.2 to construct a presentation of P_n, by induction on n. Then one can use Sequence 1.1 to deduce that the presentation given for B_n is correct.

We are going to give here a less technical proof, based on an argument by Zariski. We believe it is of more immediate comprehension, and it uses also some techniques similar to the ones that we use in the following.

Theorem 1.6.1. The braid group B_n has a presentation with generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and the following relations:

1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $i = 1, 2, \ldots, n-2$;

2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| > 1$ and $i, j = 1, 2, \ldots, n-1$.

Proof. It is easy to see, just by depicting the corresponding geometric braids, that relations 1) and 2) hold in B_n. Now, let $W = \sigma_{i_1}^{e_1} \cdots \sigma_{i_m}^{e_m}$ be a word in $\sigma_1, \ldots, \sigma_{n-1}$ and their inverses, and suppose that the braid determined by W is trivial. We must show that one can obtain the trivial word starting with W, and applying only the relations in the statement of the theorem, together with insertions or deletions of subwords of the form $(\sigma_i^{\mp 1} \sigma_i^{\pm 1})$. Since we already showed in Remark 1.1.3 that the elements σ_i generate the braid group, this proves the theorem.

For every $k = 0, \ldots, m$, let j_k be the position of the n-th puncture at the end of the motion represented by $\sigma_{i_1}^{e_1} \cdots \sigma_{i_k}^{e_k}$ (being the identity for $k = 0$). As W represents the trivial braid, it is clear than $j_0 = j_m = n$. Let us set also $\alpha_i = \sigma_i \sigma_{i+1} \cdots \sigma_{n-1}$, for $i = 1, \ldots, n-1$, and $\alpha_n = 1$. Then α_i represents a braid that sends the i-th puncture to the n-th position.

Using only permitted insertions, we can transform our word W into:

$$W = (\alpha_{j_0}^{-1} \sigma_{i_1}^{e_1} \alpha_{j_1})(\alpha_{j_1}^{-1} \sigma_{i_2}^{e_2} \alpha_{j_2}) \cdots (\alpha_{j_{m-1}}^{-1} \sigma_{i_m}^{e_m} \alpha_{j_m}).$$

This holds as $\alpha_{j_0} = \alpha_{j_m} = 1$.

Each parenthesized factor has one of the following forms:

1. $(\sigma_{n-1}^{-1} \cdots \sigma_i^{-1}) \sigma_i (\sigma_{i+1} \cdots \sigma_{n-1})$. This is clearly equivalent to the trivial word, so it can be removed.

2. $(\sigma_{n-1}^{-1} \cdots \sigma_i^{-1}) \sigma_i^{-1} (\sigma_{i+1} \cdots \sigma_{n-1})$. We will denote this word x_i^{-1}.

3. $(\sigma_{n-1}^{-1} \cdots \sigma_i^{-1}) \sigma_{i-1} (\sigma_{i-1} \cdots \sigma_{n-1})$. We will denote this word x_{i-1}.

4. $(\sigma_{n-1}^{-1} \cdots \sigma_i^{-1}) \sigma_{i-1}^{-1} (\sigma_{i-1} \cdots \sigma_{n-1})$. This is clearly equivalent to the trivial word, so it can be removed.

5. $(\sigma_{n-1}^{-1} \cdots \sigma_i^{-1}) \sigma_k^{\pm 1} (\sigma_i \cdots \sigma_{n-1})$ with $k < i - 1$. In this case, $\sigma_k^{\pm 1}$ commutes with the other letters, so by using permitted relations we can replace this word by the letter $\sigma_k^{\pm 1}$.

6. $(\sigma_{n-1}^{-1} \cdots \sigma_i^{-1}) \sigma_k^{\pm 1} (\sigma_i \cdots \sigma_{n-1})$ with $k > i$. Then, using the braid relations one has

$$\sigma_k (\sigma_i \cdots \sigma_{n-1}) = (\sigma_i \cdots \sigma_{n-1}) \sigma_{k-1}.$$

   Therefore the above word is equivalent to $\sigma_{k-1}^{\pm 1}$.

So, by the above procedure, we have replaced our original word W by a word in $\sigma_1, \ldots, \sigma_{n-2}, x_1, \ldots, x_{n-1}$ and their inverses. It is important that σ_{n-1} and σ_{n-1}^{-1} never appear in this writing alone, but always as parts of some word $x_i^{\pm 1}$.

Now, for $i = 1, \ldots, n-2$ and $j = 1, \ldots, n-1$, the word $\sigma_i^{-1} x_j \sigma_i$ can be written as a product of $x_1, \ldots, x_{n-1}$ and their inverses, by using only permitted relations. Notice that we use $\boxed{\quad\quad}$ to indicate that we manipulate the elements in the box.

1. If $i < j-1$ one can slide σ_i to the left and the resulting word is x_j.

2. If $i = j-1$, that is $j = i+1$, one has the following:

$$\begin{aligned}
\sigma_i^{-1}x_{i+1}\sigma_i &= \boxed{\sigma_i^{-1}(\sigma_{n-1}^{-1}\cdots\sigma_{i+2}^{-1})}\sigma_{i+1}^2\boxed{(\sigma_{i+2}\cdots\sigma_{n-1})\sigma_i} \\
&= (\sigma_{n-1}^{-1}\cdots\sigma_{i+2}^{-1})\sigma_i^{-1}\sigma_{i+1}^2\sigma_i(\sigma_{i+2}\cdots\sigma_{n-1}) \\
&= (\sigma_{n-1}^{-1}\cdots\sigma_{i+2}^{-1})\boxed{\sigma_i^{-1}\sigma_{i+1}\sigma_i}\boxed{\sigma_i^{-1}\sigma_{i+1}\sigma_i}(\sigma_{i+2}\cdots\sigma_{n-1}) \\
&= (\sigma_{n-1}^{-1}\cdots\sigma_{i+2}^{-1})\sigma_{i+1}\sigma_i\boxed{\sigma_{i+1}^{-1}\sigma_{i+1}}\sigma_i\sigma_{i+1}^{-1}(\sigma_{i+2}\cdots\sigma_{n-1}) \\
&= x_{i+1}x_ix_{i+1}^{-1},
\end{aligned}$$

   where the last equality is obtained just by permitted insertions.

3. If $i = j$ one has

$$\begin{aligned}
\sigma_i^{-1}x_i\sigma_i &= \sigma_i^{-1}(\sigma_{n-1}^{-1}\cdots\sigma_{i+2}^{-1}\sigma_{i+1}^{-1})\sigma_i^2(\sigma_{i+1}\boxed{\sigma_{i+2}\cdots\sigma_{n-1})\sigma_i} \\
&= \sigma_i^{-1}(\sigma_{n-1}^{-1}\cdots\sigma_{i+1}^{-1})\boxed{\sigma_i^2\sigma_{i+1}\sigma_i}(\sigma_{i+2}\cdots\sigma_{n-1}) \\
&= \sigma_i^{-1}(\sigma_{n-1}^{-1}\cdots\boxed{\sigma_{i+1}^{-1})\sigma_{i+1}}\sigma_i\sigma_{i+1}^2(\sigma_{i+2}\cdots\sigma_{n-1}) \\
&= \boxed{\sigma_i^{-1}(\sigma_{n-1}^{-1}\cdots\sigma_{i+2}^{-1})\sigma_i}\sigma_{i+1}^2(\sigma_{i+2}\cdots\sigma_{n-1}) \\
&= (\sigma_{n-1}^{-1}\cdots\sigma_{i+2}^{-1})\sigma_{i+1}^2(\sigma_{i+2}\cdots\sigma_{n-1}) \\
&= x_{i+1}.
\end{aligned}$$

4. If $i > j$, one has $\sigma_ix_j = x_j\sigma_i$, as one can see by sliding σ_i to the right, using the obvious relation at each time. Hence if $i > j$ one has $\sigma_i^{-1}x_j\sigma_i = x_j$.

It is clear that the above equations also imply that $\sigma_ix_j\sigma_i^{-1}$ can be written as a word in $x_1, \dots, x_{n-1}$ and their inverses. The resulting word is x_j if either $i < j-2$ or $i > j$, it is x_i if $i = j-1$, and it is $x_i^{-1}x_{i+1}x_i$ if $i = j$.

Therefore, starting with the word W, once we have rewritten it as a word in $\sigma_1, \dots, \sigma_{n-2}$, $x_1, \dots, x_{n-1}$ and their inverses, we can collect all the $\sigma_i^{\pm 1}$ on the right, so that we can write:

$$W = W_1W_2$$

where W_1 is a word in $x_1, \dots x_{n-1}$ and their inverses, and W_2 is a word in $\sigma_1, \dots, \sigma_{n-2}$ and their inverses.

Finally, we just need to notice that, as recalled above, by the split exact sequence (1.2), one has $P_n \cong F_{n-1} \rtimes P_{n-1}$, so every pure braid can be decomposed in a unique way as a product of a braid in $\iota(F_{n-1})$ and a braid in P_{n-1} (with the usual inclusion of P_{n-1} into P_n). We remark that $\iota(F_{n-1})$ is the free subgroup of P_n freely generated by $x_1, \dots, x_{n-1}$. Hence, as W is pure (W represents the trivial braid), the decomposition W_1W_2 is unique, meaning that W_1 represents the trivial element in F_{n-1} and W_2 represents the trivial element in P_{n-1}. As $x_1, \dots, x_n$ is a free set of generators of F_{n-1}, it follows that W_1 can be reduced to the trivial word by a sequence of permitted deletions. Therefore $W = W_2$, which is a word in $\sigma_1, \dots, \sigma_{n-2}$ and their inverses representing the trivial braid in B_{n-1}. The result then follows by induction on n. The base step with P_1 and F_1 is trivial. $\square$

Remark 1.6.2. It follows from the presentation of B_n that $B_2 \cong \mathbb{Z}$.

Remark 1.6.3. From the presentation of B_n it is easy to see that the abelianization of B_n is $\mathbb{Z}$ and that this $\mathbb{Z}$ is generated by the image of any σ_i under the abelianization map $B_n \to \mathbb{Z}$. In fact, if we take a commutator $[\sigma_i, \sigma_{i+1}]$, this is equivalent to

$$\sigma_i \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} = \sigma_{i+1}^{-1} \sigma_i.$$

If we quotient for all the commutators we have $\sigma_i = \sigma_{i+1}$, all the generators become the same, and the only thing that remains is the sign. The abelianization map $B_n \to \mathbb{Z}$ is the length homomorphism which counts the signed word length of elements of B_n in terms of the standard generators.

It was Artin again, in 1947, who gave the first presentation for the pure n-braid group. We state this result below.

Proposition 1.6.4. [2] The pure n-braid group P_n has a presentation with generators $A_{i,j}$ where $A_{i,j} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$ for $1 \leq i < j \leq n$ and the following relations:

1) $A_{r,s} \rightleftharpoons A_{i,j}$ if $1 \leq r < s < i < j \leq n$ or $1 \leq r < i < j < s \leq n$;
2) $A_{r,s}A_{r,j}A_{r,s}^{-1} = A_{s,j}^{-1}A_{r,j}A_{s,j}$ if $1 \leq r < s < j \leq n$;
3) $A_{r,s}A_{s,j}A_{r,s}^{-1} = A_{s,j}^{-1}A_{r,j}^{-1}A_{s,j}A_{r,j}A_{s,j}$ if $1 \leq r < s < j \leq n$;
4) $A_{i,j}^{-1}A_{s,j}A_{i,j} \rightleftharpoons A_{r,i}$ if $1 \leq r < s < i < j \leq n$.

Where the elements $A_{i,j}$ of the presentation represent the braid in Figure 1.10 and $\rightleftharpoons$ means that the two elements commute.

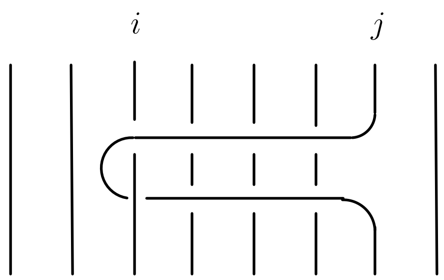

Figure 1.10: The generator $A_{i,j}$ of P_n.

Remark 1.6.5. Relation 4) is equivalent to

$$A_{r,i}A_{s,j}A_{r,i}^{-1} = [A_{i,j}^{-1}, A_{r,j}^{-1}]A_{s,j}[A_{i,j}^{-1}, A_{r,j}^{-1}]^{-1},$$

where $[a, b]$ is the commutator of the elements a and b.

Remark 1.6.6. Since all the defining relations for P_n are commutations, we have that the abelianization of P_n is a free abelian group with one generator for each generator of P_n. Thus

$$P_n^{ab} \cong \mathbb{Z}^{\binom{n}{2}}.$$

1.7 Homotopy of braids

Now we introduce the concept of homotopy of braids, which is essential for this work.

Definition 1.7.1. Let β and β' be two n-braids in $\mathbb{C} \times I$. We say that β is homotopic to β' if there exists a continuous map

$$H : (\mathbb{C} \times I) \times I \longrightarrow \mathbb{C} \times I$$

where $H(x, t, s) := H_s(x, t)$
such that:

1) $H_0 = \mathrm{id}$;
2) $(H_1 \circ \beta_1, \dots, H_1 \circ \beta_n) = (\beta'_1, \dots, \beta'_n)$;
3) $H_s|_{\mathbb{C}\times\{0,1\}} = \mathrm{id}$;
4) $H_s \circ \beta_i(I) \cap H_s \circ \beta_j(I) = \emptyset$ for $i \neq j$.

Basically, the difference between homotopy and isotopy is that during an homotopy deformation each strand is allowed to self-intersect itself, while during an isotopy deformation this is forbidden.

Definition 1.7.2. Homotopy is an equivalence relation for braids. Let us denote by $\tilde{B}_n$ the group of equivalence classes of n-braids under the homotopy relation, and by $\tilde{P}_n$ the group of equivalence classes of pure n-braids under the homotopy relation.

Two isotopic braids are also homotopic, but two homotopic braids are not necessarily isotopic. The question of whether homotopy and isotopy were the same for braids has long been unanswered. The answer, with an example of two homotopically equivalent but not isotopically equivalent braids, was given by Goldsmith in [12], together with a presentation for the homotopy braid groups. In Figure 1.11 we report Goldsmith's example, which shows an homotopy from an isotopically non trivial braid to the trivial braid. In the following we recall Goldsmith's results on homotopy.

Let $\Pi_n : B_n \to \tilde{B}_n$ be the map which sends every n-braid to its homotopy equivalence class. This map is clearly onto, therefore $B_n/\ker(\Pi_n)$ is isomorphic to $\tilde{B}_n$.

Proposition 1.7.3. [12] The kernel of Π_n is the normal subgroup of B_n generated by $[A_{i,j}, gA_{i,j}g^{-1}]$, where g $\in \langle A_{i,i+1}, A_{i,i+2}, \dots, A_{i,n}\rangle$ and $1 \leq i < j \leq n$.

From this result, Goldsmith obtains the first presentation for $\tilde{B}_n$ stated in the following theorem:

Theorem 1.7.4. [12] The set of equivalence classes of n-braids under homotopy forms a group, denoted by $\tilde{B}_n$, which has generators $\sigma_1, \dots, \sigma_{n-1}$ and the following relations:

1) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, for $i = 1, 2, \dots, n-2$;
2) $\sigma_i\sigma_j = \sigma_j\sigma_i$, for $|i - j| > 1$ and $j = 1, 2, \dots, n-1$;
3) $A_{j,k} \rightleftharpoons gA_{j,k}g^{-1}$, if $1 \leq j < k \leq n$, where g is an element of the subgroup of P_n generated by $A_{j,j+1}, A_{j,j+2}, \dots, A_{j,n}$.

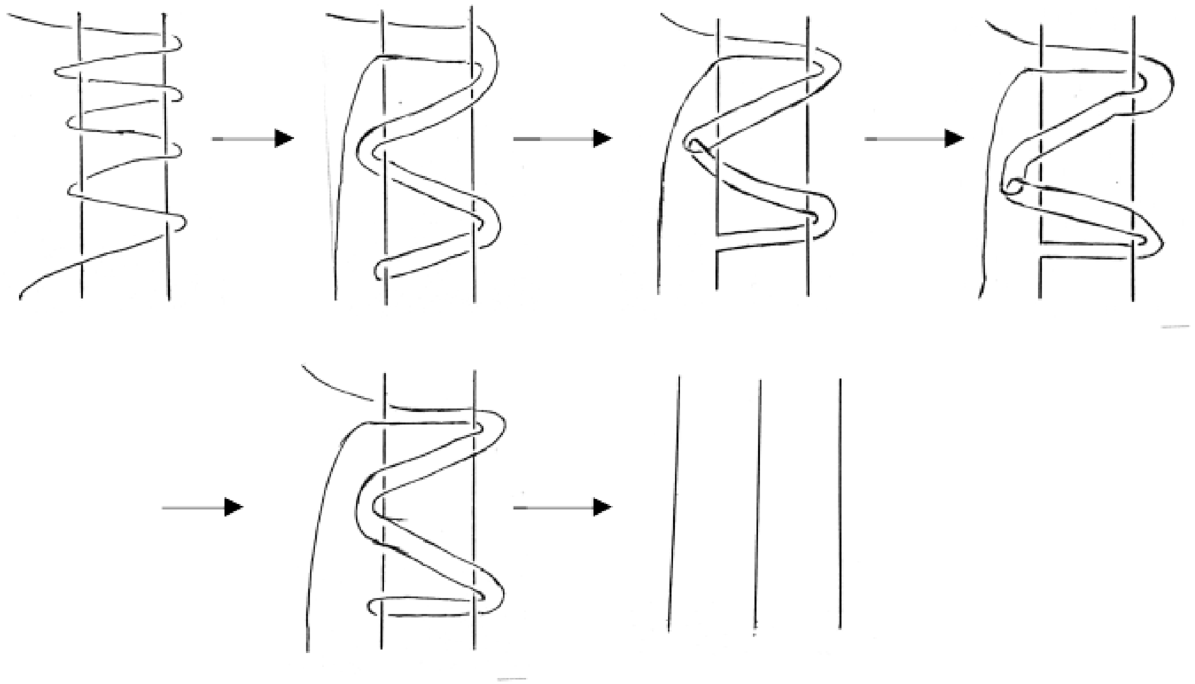

Figure 1.11: Goldsmith's example.

Another interesting presentation for $\tilde{B}_n$ is the one given by Liu in [19]. He first states the following result:

Lemma 1.7.5. [19] The set of every $[A_{i,j}, gA_{i,j}g^{-1}]$ and the set of every $[A_{i,j}, hA_{i,j}h^{-1}]$, where $g \in \langle A_{i,i+1}, A_{i,i+2}, ..., A_{i,n}\rangle$, $h \in \langle A_{1,j}, A_{2,j}, ..., A_{j-1,j}\rangle$ and $1 \leq i < j \leq n$, have the same normal closure.

With Lemma 1.7.5, Liu shows that

Proposition 1.7.6. [19] The presentation of $\tilde{B}_n$ can be stated either with the relation

$$A_{j,k} \rightleftharpoons gA_{j,k}g^{-1} \text{ where } 1 \leq j < k \leq n \text{ and } g \in \langle A_{j,j+1}, A_{j,j+2}, A_{j,n}\rangle,$$

or with the relation

$$A_{j,k} \rightleftharpoons gA_{j,k}g^{-1} \text{ where } 1 \leq j < k \leq n \text{ and } g \in \langle A_{1,k}, A_{2,k}, A_{k-1,k}\rangle.$$

1.8 String links

We end this chapter by introducing the notion of string links. If we take a pure braid and we eliminate the monotony condition we obtain a so called string-link.

Definition 1.8.1. Let $p_1, \ldots, p_n$ be distinguished points on the real axis inside the complex plane $\mathbb{C}$. A n-component string link $s = (s_1, \ldots, s_n)$ is a collection of n paths

$$s_i : I \to \mathbb{C} \times I \qquad\qquad 1 \leq i \leq n$$

called strands, such that

1) the strands $s_i(I)$ are disjoint;

2) $s_i(0) = (p_i, 0)$ and $s_i(1) = (p_i, 1)$.

String links on n strands, with the usual product given by stacking and reparametrizing and the trivial braid as the identity, form a monoid, denoted by SL_n. The pure braid group P_n is embedded into the monoid of string-links on n strands.

We are interested in string links because of their connection to pure braids up to homotopy.

Homotopy for string-links is defined in the same way it is defined for braids: two string-links are link-homotopic if there is a homotopy between them fixing the boundary, and such that the distinct components remain disjoint during the deformation. During the course of the deformation each strand is allowed to pass through itself but not through other strands.

A result by Habegger and Lin [18] in 1990 states that any string-link is link-homotopic to a pure braid. Thus, considering string-links up to link-homotopy is analogous to considering pure braids up to homotopy, and there is an isomorphism

$$SL_n/h \cong \tilde{P}_n,$$

where SL_n/h is the group of equivalent classes of n-string-links up to link homotopy.

Chapter 2

Surface braid groups

Surface braid groups are a natural generalization of the classical braid groups and of fundamental groups of surfaces. They were first defined by Zariski during the 1930's (braid groups on the sphere had been considered earlier by Hurwitz), were re-discovered by Fox during the 1960's, and were used subsequently in the study of mapping class groups. We are interested in giving an idea of what a surface braid is through some equivalent definitions, and to state a presentation for the group.

The first part of this chapter is taken from [17], the second one with the presentation from [14], the third one about homotopy from [24].

2.1 Surface braids as a collection of paths

Let M be a compact connected surface, not necessarily orientable, and let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a set of n distinct points of M.

Definition 2.1.1. A braid over M based at $\mathcal{P}$ is an n-tuple $\gamma = (\gamma_1, \dots, \gamma_n)$ of paths, $\gamma_i : I \longrightarrow M$, such that:

1) $\gamma_i(0) = P_i$ for $1 \leq i \leq n$;

2) $\gamma_i(1) \in \mathcal{P}$ for $1 \leq i \leq n$;

3) $\{\gamma_1(t), \dots, \gamma_n(t)\}$ are n distinct points of M, $\forall\ t \in I$.

In accordance with notation for classical braids, we will call γ_i the i-th strand of γ for $1 \leq i \leq n$.
If $\gamma_i(1) = P_i$ for all $i = 1, \dots, n$, the braid is said to be pure.

Two braids are said to be equivalent if there exists a homotopy which deforms one of them into the other, provided that at any time we have a geometric braid based at $\mathcal{P}$.

The product of two surface braids is defined by composing the strand of the first braid which ends at P_i with the i-th strand of the second braid. Equivalence classes of n-strands surface braids form a group with the product defined above, denoted by $B_n(M)$, and pure braids form a normal subgroup denoted by $PB_n(M)$.

2.2 Geometric braids on a surface

A closely related point of view on surface braids can be obtained defining them as geometric braids.

Let M be a connected surface, and let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a set of n distinct points of M.

Definition 2.2.1. A geometric n-braid in M is a collection $\beta = (\beta_1, \ldots, \beta_n)$ consisting of n arcs $\beta_i : I \to M \times I$, $i = 1, \ldots, n$, called strands, such that:

1. $\beta_i(0) = (P_i, 0)$ for $1 \leq i \leq n$;
2. $\beta_i(1) \in \mathcal{P} \times \{1\}$ for $1 \leq i \leq n$;
3. The strands are pairwise disjoint: $\beta_i(t) \neq \beta_j(t)$, $i \neq j$, $\forall t \in I$ and $\forall i, j \in \{1, \ldots, n\}$.
4. The strands are strictly monotone with respect to the t-coordinate: $\forall t \in I$ each strand meets the subset $M \times \{t\}$ in exactly one point.

Under this point of view it is easier to picture what a surface braid is, as we can see in Figure 2.1.

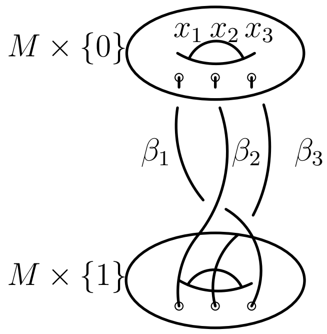

Figure 2.1: A geometric 3-braid with M equal to the 2-torus.

Two geometric braids are equivalent if there exist an isotopy, which keeps the endpoints of the strands fixed, from one to the other through n-braids (the strands remain pairwise disjoint during the isotopy). The set of equivalence classes of geometric surface braids, equipped with the usual stacking and reparametrizing product, form a group which is isomorphic to the surface braid group seen in the previous section.

In fact, given a surface braid $(\gamma_1(t), \ldots, \gamma_n(t))$, then $\beta = (\beta_1, \ldots, \beta_n)$, where $\beta_i(t) = (\gamma_i(t), t)$ for all $i = 1, \ldots, n$ and $t \in I$ is a geometric n-braid. Conversely, if we reparametrise each strand of a geometric n-braid β so that $\beta_i(t)$ is of the form $(\gamma_i(t), t)$, for $i = 1, \ldots, n$ and $t \in I$, where $\gamma = (\gamma_1(t), \ldots, \gamma_n(t))$ satisfies the opportune conditions for any $t \in I$, we obtain a surface braid seen as above as a collection of paths in M. The transition from a geometric n-braid to the collection of paths may be realized geometrically by projecting the strands lying in $M \times I$ onto the surface M.

Remark 2.2.2. Clearly if M is equal to $\mathbb{C}$ or to the 2-disc $\mathbb{D}^2$ then

$$B_n(M) \cong B_n \text{ and } PB_n(M) \cong P_n.$$

2.3 Surface braid groups as fundamental groups of configuration spaces

Definition 2.3.1. Let $F_n(M)$ denote the n-th configuration space of M defined by:

$$F_n(M) = \{(p_1, \ldots, p_n) \in M^n |\, p_i \neq p_j \text{ for all } i, j \in \{1, \ldots, n\}, i \neq j\}.$$

We equip $F_n(M)$ with the topology induced by the product topology on M^n. A transversality argument shows that $F_n(M)$ is a connected $2n$-dimensional open manifold. There is a natural free action of the symmetric group Σ_n on $F_n(M)$ by permutation of coordinates. The resulting orbit space $F_n(M)/\Sigma_n$, denoted by $D_n(M)$ is the n-th permuted configuration space of M, and may be thought of as the configuration space of n unordered points. Since it is a quotient of $F_n(M)$, $D_n(M)$ is connected too.

As we already pointed out in the previous chapter, $F_n(M)$ can be described also as $M^n \setminus D$, where D denotes the big diagonal of M^n:

$$D = \{(p_1, \dots, p_n) \in M^n | \, p_i = p_j \text{ for some } 1 \leq i < j \leq n\}.$$

It has been proved by Fox and Neuwirth that surface braids can be seen as fundamental groups of configuration spaces on a surface. Notice that the fact that $F_n(M)$ and $D_n(M)$ are connected implies that the isomorphism classes of $\pi_1(F_n(M))$ and $\pi_1(D_n(M))$ do not depend on the choice of basepoint.

Theorem 2.3.2. [11] Let $n \in \mathbb{N}$. Then $P_n(M) \cong \pi_1(F_n(M))$ and $B_n(M) \cong \pi_1(D_n(M))$.

This theorem leads to an immediate observation:

Remark 2.3.3. Since $F_1(M) = M$, we have that $B_1(M) \cong P_1(M) \cong \pi_1(M)$. The braid groups of M may be seen as generalizations of its fundamental group.

Also, we can notice that the natural inclusion $\iota : F_n(M) \hookrightarrow M^n$ induces a homomorphism of the corresponding fundamental groups (we know that $(\pi_1(M))^n = \pi_1(M^n)$):

$$\tilde{\iota} : P_n(M) \to (\pi_1(M))^n.$$

The inclusion $j : \mathbb{D}^2 \hookrightarrow \text{Int(M)}$ of a topological disc $\mathbb{D}^2$ in the interior of M induces a homomorphism $\tilde{j} : P_n \to P_n(M)$ that is injective for most surfaces. A well known result from Birman [6] is, in fact, that given M a compact, orientable surface different from $\mathbb{S}^2$, then the inclusion $j : \mathbb{D}^2 \hookrightarrow M$ induces an embedding $P_n \hookrightarrow P_n(M)$.

If M is different from $\mathbb{S}^2$ and $\mathbb{RP}^2$ then Goldberg showed that the following short sequence is exact:

$$1 \to \langle \text{Im}(\tilde{j}) \rangle^N \hookrightarrow P_n(M) \xrightarrow{\tilde{\iota}} (\pi_1(M))^n \to 1$$

where $\langle \text{Im}(\tilde{j}) \rangle^N$ denotes the normal closure of $\text{Im}(\tilde{j})$ in $P_n(M)$.

2.4 Surface braids as mapping class groups

Another well known interpretation of surface braid groups is the one which sees them as mapping class groups. Let M be a compact, connected surface. In particular, if we take a surface M different from $\mathbb{S}^2$, $\mathbb{RP}^2$, the torus or the Klein bottle, there is a short exact sequence:

$$1 \to B_n(M) \to MCG(M, n) \to MCG(M) \to 1.$$

For these surfaces the braid group $B_n(M)$ is thus isomorphic to the kernel of the homomorphism that corresponds geometrically to forgetting the marked points. It is easy to see that if $M = \mathbb{D}^2$, then $MCG(\mathbb{D}^2) = \{1\}$ for the Alexander's lemma (see Lemma 1.3.3) and we find again $B_n(\mathbb{D}^2) \cong MCG(\mathbb{D}^2, n)$.

If M is $\mathbb{S}^2$, $\mathbb{RP}^2$, the torus or the Klein bottle, one can find some similar sequences, a bit more complicated. A complete proof of these results can be found in [17].

2.5 Some properties of surface braid groups

Similarly to what happened for B_n, to each braid in $B_n(M)$ we can associate the permutation it induces on its strands, that is, an element of the symmetric group Σ_n. This yields a well defined group homomorphism η from $B_n(M)$ to Σ_n, and the kernel of η is the subgroup of $B_n(M)$ formed by the braids inducing the trivial permutation, alias $PB_n(M)$. This leads to the exact sequence

$$1 \to PB_n(M) \to B_n(M) \xrightarrow{\eta} \Sigma_n \to 1.$$

A lot of the properties of surface braid groups are specific to some surfaces, namely $\mathbb{S}^2$ and $\mathbb{RP}^2$. In fact, their braid groups are quite different from all the other cases: for example, they possess elements of finite order, while the braid groups $P_n(M)$ and $B_n(M)$ for all the other surfaces are torsion free.

We are not discussing here properties of the surface braid groups of the sphere and the projective plane, but we refer the reader to [17].

2.6 Geometrical representation of surface braids

We are interested in giving a geometrical representation of what a surface braid is. Since the discussion about homotopy will be set in the case in which M is a closed orientable surface of genus $g \geq 1$, from now on we focus on this case. We represent M as a polygon L of $4g$ sides, identified each with its opposite, as in Figure 2.2. Then a braid is represented in the cylinder $L \times I$ as on the left of Figure 2.3. In the same figure, on the right, we have a representation of the same braid as a loop in the configuration space of n points on the surface M represented as the polygon L.

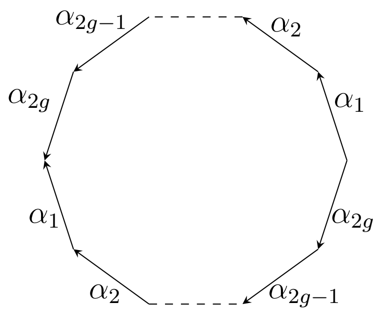

Figure 2.2: The polygon L representing M.

What is different from the disc case is that a strand can go through a wall of the cylinder and appear from the other side.

2.7 A presentation for surface braid groups

Many mathematicians computed presentations of surface braid groups. Among them, we recall Zariski [25], who gave a presentation in 1937, and Scott [23] in 1970. Over the last century, others gave presentations for specific surface braid groups such as the one over the sphere $\mathbb{S}^2$. More recently, there have been presentations of surface braid groups given by González-Meneses [14], Gonçalves and Guaschi [13], and Bellingeri [5].

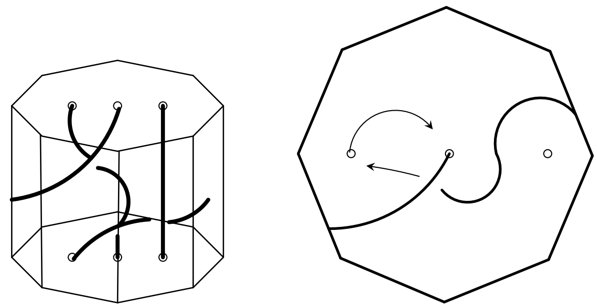

Figure 2.3: Two ways to see a braid on a surface.

We state here the presentation given by González-Meneses [14] in 2001, because it is the one used in the studies on homotopy of Yurasovskaya [24] and Lima [22] that we are going to present in the following. The results of this section are taken from [14].

Let us start by defining the generators of $B_n(M)$. We choose the n base points along the horizontal diameter of L, as shown in Figure 2.4. Now, given r, $1 \leq r \leq 2g$, we define the braid $a_{1,r}$ as follows: its only nontrivial strand is the first one, which goes through the r-th wall. Just for notation, the first strand will go upwards if r is odd, and downwards otherwise. We also define, for all $i = 1, \ldots, n-1$, the braid σ_i as in the same figure. Note that $\sigma_1, \ldots, \sigma_{n-1}$ are the classical generators of the braid group B_n. It has been proved that the set $\{a_{1,1}, \ldots, a_{1,2g}, \sigma_1, \ldots, \sigma_{n-1}\}$ is a set of generators of $B_n(M)$.

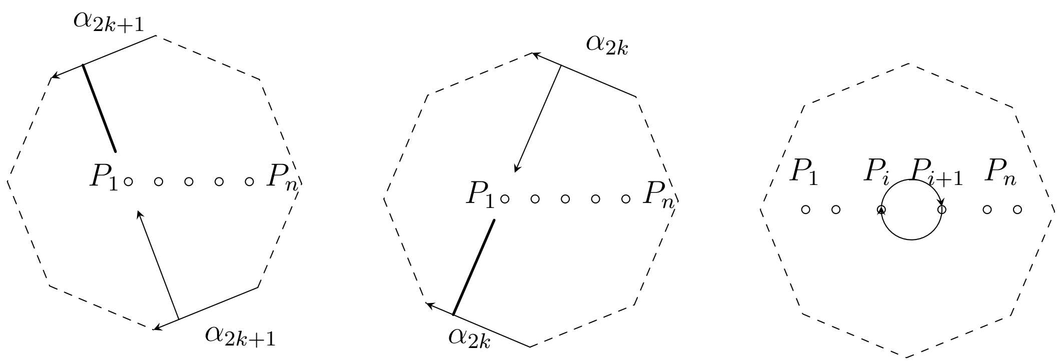

Figure 2.4: Elements of $B_n(M)$: from the left, $a_{1,2k+1}, a_{1,2k}$ and σ_i.

We observe that the classical relations in B_n:

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i = 1, 2, \ldots, n-2;$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1 \text{ and } i, j = 1, 2, \ldots, n-1,$$

still hold in $B_n(M)$. Also, if $i \in \{2, \ldots, n-1\}$ and $r \in \{1, \ldots, 2g\}$, then the non-trivial strands of σ_i and the one of $a_{1,r}$ may be taken to be disjoint, as pictured in Figure 2.5. This implies that these two braids commute. Hence we have:

$$a_{1,r}\sigma_i = \sigma_i a_{1,r}$$

for $1 \leq r \leq 2g$ and $i \geq 2$. Now, in order to find more relations between the set of generators, we do the following construction: denote by s_r the first strand of $a_{1,r}$, for all $r = 1, \ldots, 2g$, and consider all the paths $s_1, \ldots, s_{2g}$. We can cut the polygon L along

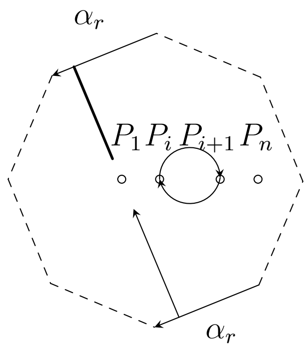

Figure 2.5: The braid $a_{1,r}\sigma_i$.

them, and glue the pieces along the paths $a_{1,1}, \ldots, a_{1,2g}$. We obtain another polygon of $4g$ sides which are labeled by $s_1, \ldots, s_{2g}$ (see in Figure 2.6 the case of a surface of genus 2, the general case is analogous). We will call this new polygon the P_1-polygon of M, since all of its vertices are identified to P_1, while L will be called the initial polygon. We obtain in this way a new representation of the surface M.

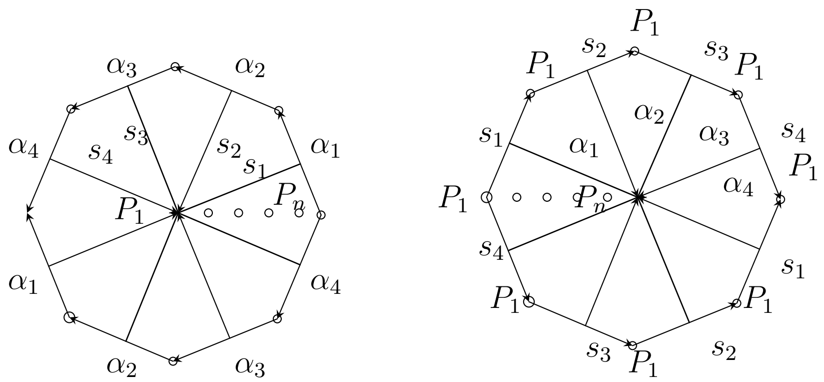

Figure 2.6: The initial and the P_1-polygons of a surface of genus 2.

We will use the P_1-polygon to show three more relations in $B_n(M)$. For instance, consider the product of braids $a_{1,1}\cdots a_{1,2g}a_{1,1}^{-1}\cdots a_{1,2g}^{-1}$. If we look at P_1-polygon, we see that it is equivalent to the braid on Figure 2.7. Also, this one can be seen in the initial polygon as a braid that does not go through the walls, namely, an element of B_n, the classical braid group. Then , we can easily see that it is equivalent to the braid $\sigma_1\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1$. So we have:

$$a_{1,1}\cdots a_{1,2g}a_{1,1}^{-1}\cdots a_{1,2g}^{-1} = \sigma_1\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1.$$

Now, for each $r = 1, \ldots, 2g-1$, we define the braid:

$$A_{2,r} = \sigma_1(a_{1,1}\cdots a_{1,r-1}a_{1,r+1}^{-1}\cdots a_{1,2g}^{-1})\sigma_1^{-1}.$$

We will use the P_1- polygon to see how it looks like. In the left hand side of Figure 2.8, we can see a braid which is equivalent to $A_{2,r}$ (if r is odd, the other case being analogous). If we cut and glue to see this braid in the P_1-polygon, we obtain the situation of the right hand side of Figure 2.8. That is, $A_{2,r}$ can be seen as a braid whose only nontrivial strand is the second one, which goes upwards and crosses once the r-th wall s_r. Note that, unlike

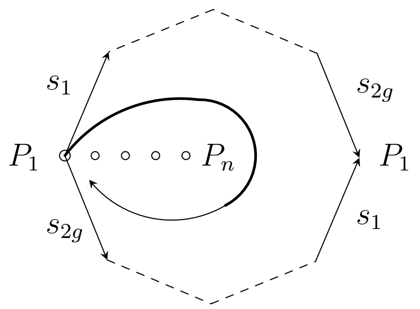

Figure 2.7: The braid $a_{1,1} \cdots a_{1,2g} a_{1,1}^{-1} \cdots a_{1,2g}^{-1}$.

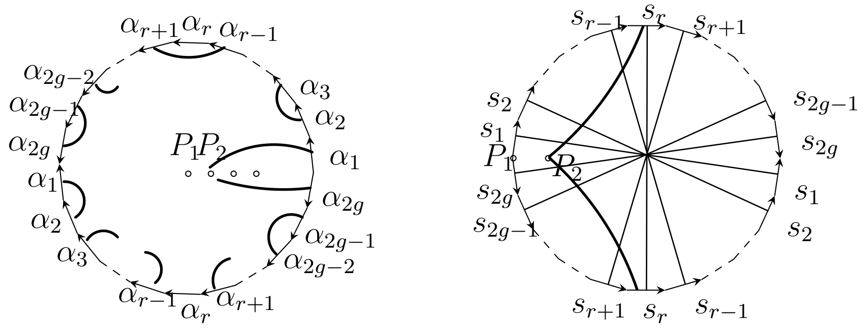

Figure 2.8: The braid $A_{2,r}$ in the initial polygon and in the P_1-polygon.

the case of $a_{1,r}$, the path $A_{2,r}$ always points upwards in the P_1-polygon, no matter the parity of r. Therefore we have seen that the braid $A_{2,r}$ can be represented by a geometric braid, whose only non trivial strand can be taken disjoint from all the paths s_t, $t \neq r$. This implies that

$$a_{1,t} A_{2,r} = A_{2,r} a_{1,t},\ 1 \leq t \leq 2g,\ 1 \leq r \leq 2g-1,\ t \neq r.$$

Now, we finish our set of relations by considering the commutator of the braids $(a_{1,1}, \cdots, a_{1,r})$ and $A_{2,r}$, for all $r = 1, \dots, 2g-1$. In Figure 2.9 we can see a sketch of the homotopy which starts with this commutator and deforms it to a braid equivalent to σ_1^2. Therefore, we obtain the relation:

$$(a_{1,1} \cdots a_{1,r}) A_{2,r} = \sigma_1^2 A_{2,r} (a_{1,1} \cdots a_{1,r}),\ 1 \leq r \leq 2g-1.$$

González-Meneses [14] proved that this are the only relations needed in the presentation of $B_n(M)$. We refer the reader to [14] for the proof. We give here his presentation of the n-braid group on a surface.

Theorem 2.7.1. [14] If M is a closed, orientable surface of genus $g \geq 1$, then $B_n(M)$ has a presentation with generators $\sigma_1, \dots \sigma_{n-1}, a_{1,1} \dots a_{1,2g}$, and the following relations:

1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, for $i = 1, 2, \dots, n-2$;

2) $\sigma_i \sigma_j = \sigma_j \sigma_i$, for $|i-j| > 1$ and $i, j = 1, 2, \dots, n-1$;

3) $a_{1,1} \cdots a_{1,2g} a_{1,1}^{-1} \cdots a_{1,2g}^{-1} = \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1$;

4) $a_{1,r} A_{2,s} = A_{2,s} a_{1,r}$, for $1 \leq r \leq 2g$, $1 \leq s \leq 2g-1$ and $r \neq s$;

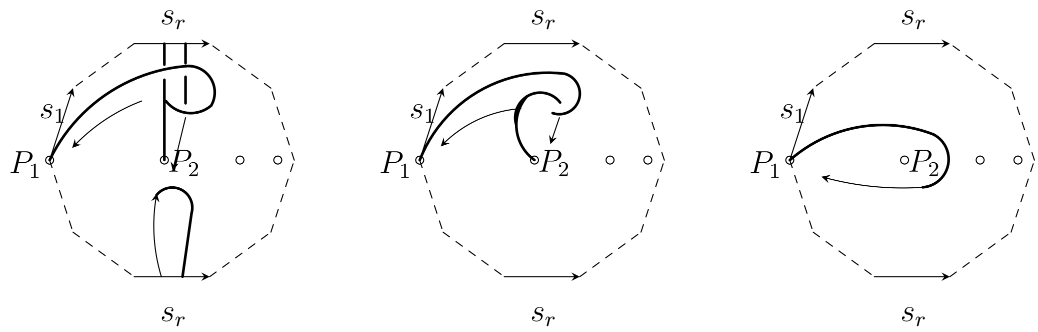

Figure 2.9: The braid $[a_{1,1} \cdots a_{1,r}, A_{2,r}]$.

5) $(a_{1,1} \cdots a_{1,r})A_{2,r} = \sigma_1^2 A_{2,r}(a_{1,1} \cdots a_{1,r})$, for $1 \leq r \leq 2g-1$;

6) $a_{1,r}\sigma_i = \sigma_i a_{1,r}$, for $1 \leq r \leq 2g$ and $i \geq 2$.

Before giving the presentation of $PB_n(M)$, let us define the generators for this group.

1. Let $a_{i,r}$ be the braid such that the i-th strand goes through the r-th wall. This strand will go upwards if r is odd, and downwards otherwise. The other strands are trivial.

2. Let $T_{i,j} = \sigma_i\sigma_{i+1} \cdots \sigma_{j-2}\sigma_{j-1}^2\sigma_{j-2} \cdots \sigma_{i+1}\sigma_i$ be the braid that starts in P_i, goes around P_j from the back and turns back to P_i passing in front of $P_j, \ldots, P_{i+1}$, and where all the other strands are trivial.

The braids defined previously are given in Figure 2.10.

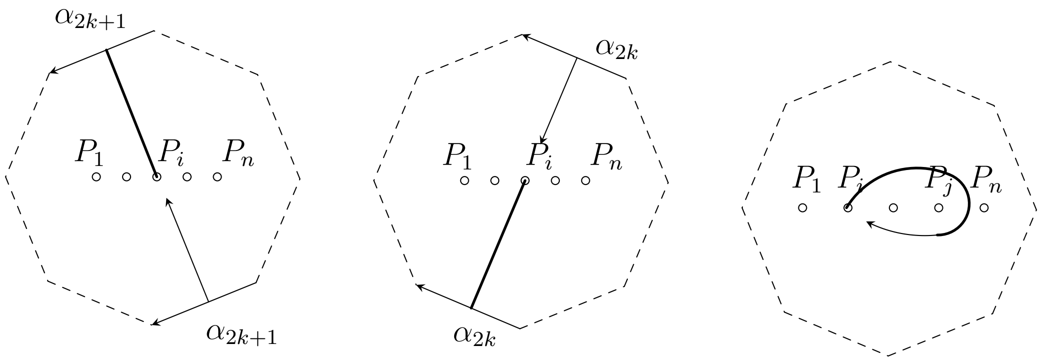

Figure 2.10: The braids $a_{i,r}$ and $T_{i,j}$.

In order to give the relations we need to define also

$$A_{j,r} = a_{j,1} \cdots a_{j,r-1}a_{j,r+1}^{-1} \cdots a_{j,2g}^{-1} \text{ for } 2 \leq j \leq n \text{ and } 1 \leq r \leq 2g-1.$$

If we denote by $s_{i,r}$ the i-th strand of $a_{i,r}$, for any $i \in \{2, \ldots, n\}$ we can define the P_i-polygon as we defined the P_1-polygon: we cut L along $s_{i,1}, \ldots, s_{i,2g}$ and glue along $a_{i,1}, \ldots, a_{i,2g}$.

Then, like in the representation of $A_{2,r}$ in the P_1-polygon, $A_{j,r}$ can be represented in the P_i-polygon (for $1 \leq i < j$), as the braid in Figure 2.11 whose only nontrivial strand is the j-th one, which goes upwards and crosses once the r-th wall $s_{i,r}$. Note that this representation does not depend on i, but it is only valid when $i < j$.

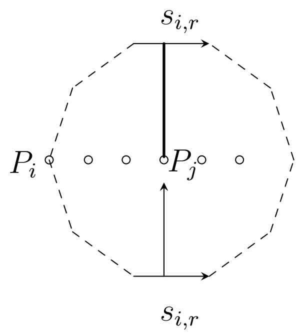

Figure 2.11: $A_{j,r}$ in the P_i-polygon.

Now, let us enunciate the presentation of $PB_n(M)$. For details about the relations and the proof, see [14].

Proposition 2.7.2. [14] The group $PB_n(M)$ admits a presentation with generators $a_{i,r}$, being $1 \leq i \leq n$ and $1 \leq r \leq 2g$, and $T_{j,k}$, $1 \leq j < k \leq n$, and relations:

1) $a_{n,1}^{-1}a_{n,2}^{-1}\cdots a_{n,2g}^{-1}a_{n,1}a_{n,2}\cdots a_{n,2g} = \prod_{i=1}^{n-1} T_{i,n-1}^{-1}T_{i,n}$;

2) $a_{i,r}A_{j,s} = A_{j,s}a_{i,r}$, for $1 \leq i < j \leq n$, $1 \leq r \leq 2g$, $1 \leq s \leq 2g-1$ and $r \neq s$;

3) $(a_{i,1}\cdots a_{i,r})A_{j,r}(a_{i,r}^{-1}\cdots a_{i,1}^{-1})A_{j,r}^{-1} = T_{i,j}T_{i,j-1}^{-1}$, for $1 \leq i < j \leq n$, $1 \leq r \leq 2g-1$;

4) $T_{i,j}T_{k,l} = T_{k,l}T_{i,j}$, for $1 \leq i < j < k < l \leq n$ or $1 \leq i < k < l \leq j \leq n$;

5) $T_{k,l}T_{i,j}T_{k,l}^{-1} = T_{i,k-1}T_{i,k}^{-1}T_{i,j}T_{i,l}^{-1}T_{i,k}T_{i,k-1}^{-1}T_{i,l}$, for $1 \leq i < k \leq j < l \leq n$;

6) $a_{i,r}T_{j,k} = T_{j,k}ai,r$, for $1 \leq i < j < k \leq n$ or $1 \leq j < k < i \leq n$ and $1 \leq r \leq 2g$;

7) $a_{i,r}(a_{j,2g}^{-1}\cdots a_{j,1}^{-1}T_{j,k}a_{j,2g}\cdots a_{j,1}) = (a_{j,2g}^{-1}\cdots a_{j,1}^{-1}T_{j,k}a_{j,2g}\cdots a_{j,1})a_{i,r}$,
for $1 \leq j < i \leq k \leq n$;

8) $T_{j,n} = \left(\prod_{i=1}^{j-1} a_{i,2g}^{-1}\cdots a_{i,1}^{-1}T_{i,j-1}T_{i,j}^{-1}a_{i,1}\cdots a_{i,2g}\right) a_{j,1}\cdots a_{j,2g}a_{j,1}^{-1}\cdots a_{j,2g}^{-1}$.

2.8 Homotopy for pure surface braid groups

Homotopy for surface braids is defined in accordance with the case of classical braids. Two braids γ and γ' on a surface M are homotopic if there is an homotopy of the strands in $M \times I$, fixing the endpoints, and deforming γ to γ' such that the images of different strands remain disjoint during the deformation.

In [24], Yurasovskaya studies homotopy for pure surface braids with the application of homotopy string links over a surface as an intermediate step due to their link-homotopical equivalence to pure surface braids. We recall here some of the main results of her work.

Definition 2.8.1. Let M be a closed, connected and orientable surface of genus $g \geq 1$. Let I be the unit interval I and let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a set of n distinct points of M. Let $I_1, \ldots, I_n$ be n copies of the interval I and denote with $\bigsqcup_{i=1}^n I_i$ the disjoint union of these intervals.

A string link σ on n strands over a surface M is a smooth or piecewise linear proper embedding

$$\sigma : \bigsqcup_{i=1}^{n} I_i \to M \times I$$

such that $\sigma_i(0) = (P_i, 0)$ and $\sigma_i(1) = (P_i, 1)$.

A string link over a surface can be seen as a pure braid on a surface without the monotony requirement. Every pure braid on a surface is a string link in itself.

Definition 2.8.2. We say that two string links σ and σ' are link-homotopic if there is a homotopy of the strands in $M \times I$, fixing $M \times \{0, 1\}$, and deforming σ to σ', such that the images of different strands remain disjoint during the deformation.

During the course of the deformation, each individual strand is allowed to pass through itself but not through other strands.

The following result is due to Fenn and Rolfsen:

Theorem 2.8.3. Every n-strand string link over a surface M is link-homotopic to a pure braid.

This theorem allows us to use the term link-homotopy pure braids instead of string links.

Define $H_n(M)$ the set of all pure braids in $PB_n(M)$ which are link-homotopic to the trivial braid. $H_n(M)$ is a normal subgroup of $PB_n(M)$. In fact, the product of two link-homotopically trivial braids produces a link homotopically trivial braid, and the inverse of a link-homotopically trivial braid is also link-homotopically trivial; also, if β is link-homotopically trivial, for any $x \in PB_n(M)$, $x\beta x^{-1}$ is clearly link-homotopically trivial.

Let us denote by $\widetilde{PB}_n(M)$ the set of link-homotopy classes of string links over the surface M, which will be called simply homotopy string links over surfaces. It has been proved that $\widetilde{PB}_n(M)$ is a group, isomorphic to $PB_n(M)/H_n(M)$.

Proposition 2.8.4. [24] The set $\widetilde{PB}_n(M)$ equipped with the usual product is a group isomorphic to the quotient of the pure braid group $PB_n(M)$ by the subgroup of link-homotopically trivial braids $H_n(M)$:

$$\widetilde{PB}_n(M) \cong PB_n(M)/H_n(M).$$

This result comes from the observations above: we know that each string link is link-homotopic to a pure braid, and that $\widetilde{PB}_n(M)$ can be seen as the quotient $PB_n(M)/H_n(M)$. Since $H_n(M)$ is normal, $\widetilde{PB}_n(M)$ is a group that inherits from $PB_n(M)$ the product and the inverse.

In [24], Yurasovskaya studies the case where S is a closed orientable surface of genus $g \geq 1$ with a single puncture, and finds a presentation for $\widetilde{PB_n}(S)$ using methods similar to the ones applied by González-Meneses to find Presentation 2.7.1. Then, considering M a closed orientable surface of genus $g \geq 1$ and S as the surface obtained by deleting a single point from M she finds a map between the groups of homotopy pure braids over S and those over M. She computes the kernel of the map and using the presentation of $\widetilde{PB}_n(S)$ she finds the presentation for $\widetilde{PB_n}(M)$ that is stated in the following.

The presentation of $\widetilde{PB}_n(M)$ has the same generators set as $PB_n(M)$ and the same relations with one more relation, defined by the commutator

$$[ft_{i,j}f^{-1}, gt_{i,j}g^{-1}] = 1,$$

where $t_{i,j} = \sigma_i\sigma_{i+1}\cdots\sigma_{j-2}\sigma_{j-1}^2\sigma_{j-2}^{-1}\cdots\sigma_{i+1}^{-1}\sigma_i^{-1}$ and $f, g \in F(2g+n-i)$, $i = 1, \dots, n$. We recall that $F(2g+n-i)$ is the notation for the free group $\pi_1(S \setminus \mathcal{P}_{n-i}, P_i)$ generated by

$$\{\{a_{i.r}\} \cup \{t_{i,j}\}; i+1 \leq j \leq n, 1 \leq r \leq 2g\}\},$$

where S is the surface M with a single point deleted and $\mathcal{P}_{n-i} = \{P_{i+1}, \dots P_n\}$.

In [24, Chapter 4] there is the following description of $\pi_1(S \setminus \mathcal{P}_{n-i}, P_i)$: it is seen as a free subgroup of $PB_n(S)$, which is denoted by $F(2g+n-i)$. The strands based at $\{P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_n\}$ are trivial and go vertically down. The P_i-based strand winds around the straight strands based at $\{P_{i+1}, \dots, P_n\}$ and through the walls of $M \times I$. Generators of the free subgroup $F(2g+n-i)$ correspond precisely to those of $\pi_1(S \setminus \mathcal{P}_{n-i}, P_i)$.

Proposition 2.8.5. [24] The group $\widetilde{PB}_n(M)$ admits a presentation with generators $a_{i,r}$, being $1 \leq i \leq n$ and $1 \leq r \leq 2g$, and $t_{j,k}$, $1 \leq j < k \leq n$, and relations:

1) $a_{n,1}^{-1}a_{n,2}^{-1}\cdots a_{n,2g}^{-1}a_{n,1}a_{n,2}\cdots a_{n,2g} = \prod_{i=1}^{n-1} T_{i,n-1}^{-1}T_{i,n}$;
2) $a_{i,r}A_{j,s} = A_{j,s}a_{i,r}$ for $1 \leq i < j \leq n$, $1 \leq r \leq 2g$, $1 \leq s \leq 2g-1$ and $r \neq s$;
3) $(a_{i,1}\cdots a_{i,r})A_{j,r}(a_{i,r}^{-1}\cdots a_{i,1}^{-1})A_{j,r}^{-1} = T_{i,j}T_{i,j-1}^{-1}$ for $1 \leq i < j \leq n$, $1 \leq r \leq 2g-1$;
4) $T_{i,j}T_{k,l} = T_{k,l}T_{i,j}$ for $1 \leq i < j < k < l \leq n$ or $1 \leq i < k < l \leq j \leq n$;
5) $T_{k,l}T_{i,j}T_{k,l}^{-1} = T_{i,k-1}T_{i,k}^{-1}T_{i,j}T_{i,l}^{-1}T_{i,k}T_{i,k-1}^{-1}T_{i,l}$ for $1 \leq i < k \leq j < l \leq n$;
6) $a_{i,r}T_{j,k} = T_{j,k}a_{i,r}$ for $1 \leq i < j < k \leq n$ or $1 \leq j < k < i \leq n$ and $1 \leq r \leq 2g$;
7) $a_{i,r}(a_{j,2g}^{-1}\cdots a_{j,1}^{-1}T_{j,k}a_{j,2g}\cdots a_{j,1}) = (a_{j,2g}^{-1}\cdots a_{j,1}^{-1}T_{j,k}a_{j,2g}\cdots a_{j,1})a_{i,r}$ for $1 \leq j < i \leq k \leq n$;
8) $T_{j,n} = \left(\prod_{i=1}^{j-1} a_{i,2g}^{-1}\cdots a_{i,1}^{-1}T_{i,j-1}T_{i,j}^{-1}a_{i,1}\cdots a_{i,2g}\right) a_{j,1}\cdots a_{j,2g}a_{j,1}^{-1}\cdots a_{j,2g}^{-1}$;
9) $[ft_{i,j}f^{-1}, gt_{i,j}g^{-1}] = 1$ where $1 \leq i < j \leq n$ and $f, g \in F(2g+n+i)$.

We have $T_{i,j} = t_{i,j}\cdots t_{i,i+1}$.

Remark 2.8.6. In [22, Remark 2.18], it is shown that the relation

$$[ft_{i,j}f^{-1}, gt_{i,j}g^{-1}] = 1,\ f, g \in F(2g+n+i)$$

can be replaced by the relation

$$[t_{i,j}, ht_{i,j}h^{-1}] = 1,\ h \in F(2g+n+i)$$

This will be useful for some considerations that we are going to do in Chapter 4. Next proposition is due to Yurasovskaya, but we recall it in the version of Lima, [22, Corollary 2.1.9].

Proposition 2.8.7. $H_n(M)$ is the smallest normal subgroup of $PB_n(M)$ generated by the commutators $[t_{i,j}, ht_{i,j}h^{-1}]$ as in the above relation.

$$H_n(M) = \langle\{[t_{i,j}, ht_{i,j}h^{-1}]\, 1 \leq i < j \leq n, h \in F(2g+n+i)\}\rangle^N,$$

where $\langle\ \ \rangle^N$ denotes the normal closure.

In the same work, Lemma 3.3.2, Lima proves also the following result.

Proposition 2.8.8. [22] It is possible to define an injective homomorphism

$$f_n : \tilde{P}_n \to \widetilde{PB}_n(M).$$

Chapter 3

Welded braid groups

Welded braids can be seen as a generalization in 4 dimensions of braids on the disc. Differently from the previous cases, they were studied under various points of view, each with a different name, and the proofs of the isomorphisms between all these groups were given at a later time. In this chapter, we follow the work of Damiani, who collected the different viewpoints in [8]. We start by defining ribbon braids, which are the topological counterparts of welded braids, and then we define welded diagrams and welded braid groups, stating the isomorphism between these groups and the groups of ribbon braids. Lately we give a definition of welded braids in terms of mapping class groups, in terms of fundamental groups of configuration spaces, and as automorphisms of F_n, which allows us to draw an easy comparison with the group of braids on the disc. We recall also the presentation given by Brendle and Hatcher in [7].

3.1 Ribbon braids

Before giving the definition we set some notation. Let us consider $B^4 \cong B^3 \times I$. For any submanifold $X \subset B^m \cong B^{m-1} \times I$, with $m = 3, 4$, we denote:

1) $\partial_\varepsilon X = X \cap (B^{m-1} \times \{\varepsilon\})$, with $\varepsilon \in \{0, 1\}$;

2) $\partial_* X = \partial X \setminus (\mathrm{int}(\partial_0 X) \cup \mathrm{int}(\partial_1 X))$;

3) $\overset{*}{X} = X \setminus \partial_* X$.

Definition 3.1.1. The image of an immersion $Y \subset X$ is said to be locally flat if and only if the couple (Y, X) is locally homeomorphic to the couple $(\mathbb{R}^k, \mathbb{R}^m)$ for some $k \leq m$, except on ∂X or ∂Y, where one of the $\mathbb{R}$ summands should be replaced by $\mathbb{R}_+$.

Definition 3.1.2. Given Y_1, Y_2 two submanifolds of B^m, their intersection $Y_1 \cap Y_2 \subset X$ is said to be flatly transverse if and only if it is locally homeomorphic to the transverse intersection of two linear subspaces $\mathbb{R}^{k_1}$ and $\mathbb{R}^{k_2}$ in $\mathbb{R}^m$, for some positive integers $k_1, k_2 \leq m$, except on ∂X, ∂Y_1 or ∂Y_2, where one of the $\mathbb{R}$ summands should be replaced by $\mathbb{R}_+$.

Definition 3.1.3. Given two submanifolds Y_1, Y_2, ribbon discs are intersections $D = Y_1 \cap Y_2 \subset \mathbb{R}^4$ that are isomorphic to the 2-dimensional disc, such that $D \subset \mathrm{int}(Y_1)$, $\mathrm{int}(D) \subset \mathrm{int}(Y_2)$ and ∂D is an essential (i.e. not homotopic to a point, a puncture, or a boundary component) curve in ∂Y_2.

We are now ready to give the definition of ribbon braids.

Definition 3.1.4. [8] Let $D_1, \dots, D_n$ be a collection of discs in B^2 and let $C_i = \partial D_i$ be the oriented boundary of D_i. Let $A_1, \dots, A_n$ be locally flat embeddings in $\overset{*}{B^4}$ of n disjoint copies of the oriented annulus $S^1 \times I$. We say that

$$b = \bigsqcup_{i \in \{1, \dots, n\}} A_i$$

is a geometric ribbon braid if:

1) the boundary of each annulus ∂A_i is a disjoint union $C_i \sqcup C_j$, for $C_i \in \partial_0 B^4$ and for some $C_j \in \partial_1 B^4$. The orientation induced by A_i on ∂A_i coincides with the orientation of the two boundary circles C_i and C_j;

2) the annuli A_i are fillable, in the sense that they bound immersed 3-balls $\subset \mathbb{R}^4$ whose singular points consist in a finite number of ribbon discs;

3) it is transverse to the lamination $\bigcup_{t \in I} B^3 \times \{t\}$ of B^4, which means that at each parameter t the intersection between b and $B^3 \times \{t\}$ is a collection of exactly n circles;

4) the orientations of the circles are concordant, at each parameter t, to the orientations of the circles that compose the boundary of the annulus.

If condition 1) is replaced by

1) $\partial A_i = C_i \times \{0, 1\}$ for all $i \in \{1, \dots, n\}$ and orientation induced by A_i on ∂A_i coincides with that of C_i,

we obtain a pure geometric ribbon braid.
Two ribbon braids are said to be equivalent if there is an isotopy between them, which is a continuous deformation that sends one to the other fixing the boundary circles. Damiani proved in [8] a theorem from which we deduce that equivalence for ribbon braids can be expressed, as for braids on the disc, with an ambient isotopy of $\mathbb{R}^4$ that brings one to the other.

Theorem 3.1.5. [8] Every isotopy of a geometric ribbon braid $F : b \times I \to B^3 \times I$ extends to an isotopy $G : (B^3 \times I) \times I \to B^3 \times I$ which is the identity on the boundary.

The proof of this theorem uses techniques which involve the approach as fundamental groups and mapping class groups explained in the next sections, together with sophisticated tools of algebraic topology. Product of ribbon braids is given by stacking and reparametrizing. The trivial ribbon braid $U = \bigsqcup_{i \in \{1, \dots, n\}} C_i \times I$ is the unit element for this product. Equivalence classes of geometric ribbon braids up to continuous deformations through the class of continuous ribbon braids fixing the boundary circles, equipped with the product above defined, form a group, that we denote by rB_n.

Definition 3.1.6. Pure ribbon braids form a subgroup of rB_n, denoted by PrB_n.

We have that PrB_n coincides with the kernel of the homomorphism from rB_n to the group of permutations Σ_n that associates to a ribbon braid the permutation induced on the boundary circles.
There are two ways to represent diagrammatically ribbon braids, respectively in 3 and 2 dimensions: in the former case via broken surface diagrams and in the latter via welded diagrams; let us see how they work.

3.2 Braid broken surface diagrams

As for classical braids, we can consider a projection in general position of a ribbon braid in the 3-dimensional space: in this case we obtain a braid broken surface diagram, which we define in the following.

Definition 3.2.1. Let $A_1, \dots, A_n$ be locally flat embeddings in $\overset{*}{B}{}^3$ of n disjoint copies of the oriented annulus $S^1 \times I$. We can say that

$$S = \bigcup_{i \in \{1,\dots,n\}} A_i$$

is a braid broken surface diagram if:

1) for each $i \in \{1, \dots, n\}$, the oriented boundary ∂A_i is the disjoint union $C_i \sqcup C_j$, for C_i in $\partial_0 B^3$ and for some C_j in $\partial_1 B^3$. The orientation induced by A_i on ∂A_i coincides with the orientation of one of the two boundary circles C_i and C_j;

2) it is transverse to the lamination $\bigcup_{t \in I} B^2 \times \{t\}$ of B^3, that is: at each parameter t, the intersection between S and $B^2 \times \{t\}$ is a collection of exactly n circles, not necessarily disjoint;

3) the set of connected components of singular points in S, denoted by $\Sigma(S)$, consists of flatly transverse disjoint circles in $(\bigcup_{i=1}^{n} \operatorname{int}(A_i))$.

For each element of $\Sigma(S)$, a local ordering is given on the two circle preimages. To specify the order we erase a small neighborhood of the lower preimage in the interior of the annulus it belongs to. We recall a lemma that allows us to state the correspondence between ribbon braids and braid broken surface diagrams:

Lemma 3.2.2. [8] Any generic projection of a ribbon braid from B^4 to B^3 is a braid broken surface diagram. Conversely any braid broken surface diagram is the projection of a ribbon braid.

Moreover, there is a stronger result which involves symmetric braid broken surface diagrams, defined below.

Definition 3.2.3. A braid broken surface diagram is said to be symmetric if for each pair of preimage circles the following properties are satisfied:

1) one of the preimage circles is essential in $\bigcup_{i=1}^{n} \operatorname{int}(A_i)$ and the other is not;

2) there is a pairing of the elements of $\Sigma(S) = \bigsqcup_r \{c_1^r, c_2^r\}$ such that, for each r, the essential preimages of c_1^r and c_2^r:
   (a) are respectively lower and higher than their non essential counterparts with respect to the associate order;
   (b) bound an annulus in $\bigcup_{i=1}^{n} \operatorname{int}(A_i)$;
   (c) this annulus avoids $\Sigma(S)$.

Lemma 3.2.4 ([3, 8])**.** Any ribbon braid can be represented by a symmetric braid broken surface diagram.

3.3 Welded diagrams

Broken surface diagrams can be associated also to 2-dimensional diagrams, called welded diagrams, whose definitions and properties we give in the following.

Definition 3.3.1. A strand diagram on n strands is a set of oriented arcs in $\mathbb{R}^2$, monotone with respect to the second coordinate, from the points $(0,1),\dots,(0,n)$ to $(1,1),\dots,(1,n)$. The arcs are allowed to have double points of three kinds, called classical positive, classical negative and welded (see Figure 3.1). We call σ_i the elementary diagram representing the $i+1$-th strand passing over the i-th strand, and ρ_i the welded crossing of the strands i and $i+1$. We will denote by D_n the set of strand diagrams on n strands.

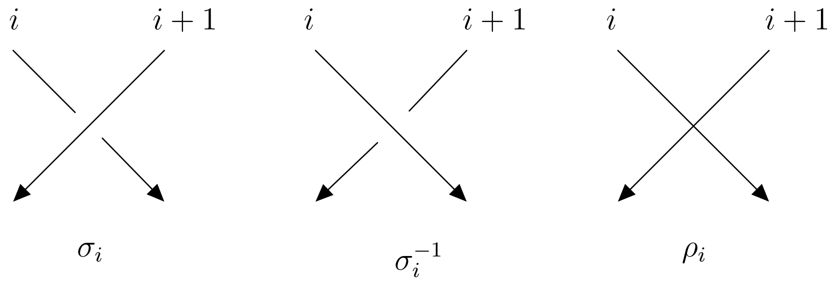

Figure 3.1: Crossings of a strand diagram.

Definition 3.3.2. A welded braid is an equivalence class of strand diagrams under the equivalence relation given by planar isotopy and by four kinds of moves that we will show graphically in Figures 3.2, 3.3, 3.4, 3.5 and which are called classical Reidemeister moves, virtual Reidemeister moves, mixed Reidemeister moves and welded Reidemeister moves. The equivalence is called welded Reidemeister equivalence.

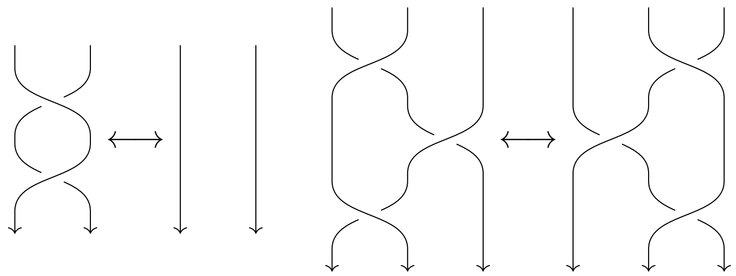

Figure 3.2: Classical Reidemeister moves.

Equivalence classes of n-strand diagrams by welded Reidemeister equivalence form a group, denoted by WB_n and called the welded braid group on n strands. The product is the usual one given by stacking and rescaling, mirror image is the inverse and the trivial diagram is the identity. As usual, the diagrams are read from top to bottom.

Passing through symmetric braid broken surfaces, 4-dimensional ribbon braids can be described using 2-dimensional welded braids. Let b be a welded braid. We associate to it a symmetric braid broken surface diagram in the following way. Consider B^2 and embed it as $B^2 \times \frac{1}{2}$ into B^3. Taken a tubular neighbourhood $N(b)$ of b, we have that $\partial_\varepsilon N(b) = \sqcup_{i\in\{1,\dots,n\}} D_i \times \varepsilon_i$, where $\varepsilon_i \in \{0,1\}$. Each crossing is sent to a 4-punctured

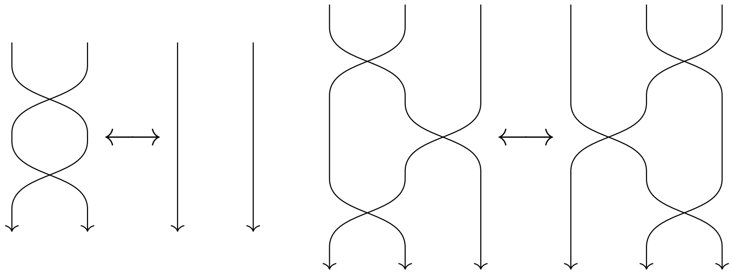

Figure 3.3: Virtual Reidemeister moves.

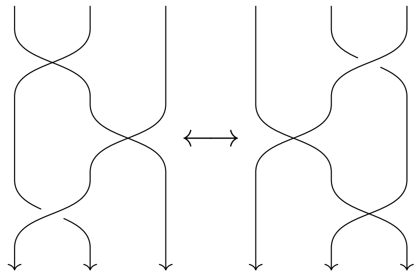

Figure 3.4: Mixed Reidemeister moves.

sphere. Then, according to the partial order defined on the double points of welded braid diagrams, we modify the sphere into the broken surfaces shown in Figure 3.6.

Given a welded braid b, being d the symmetric braid broken surface diagram associated to b, we can define a map $\mathrm{Tube} : WB_n \longmapsto rB_n$ that sends b to the ribbon braid associated to d.

Proposition 3.3.3. [8] The map $\mathrm{Tube} : WB_n \longmapsto rB_n$ is surjective.

3.4 Welded braid groups as mapping class groups.

Definition 3.4.1. In what follows we generalize to the 3-dimensional case the definition of mapping class group given in Definition 1.3.1. Let M be a compact, connected, orientable 3-manifold with boundary, and N an orientable submanifold contained in the interior of M, not necessarily connected or non-empty. Let $\mathrm{Homeo}^+(M, N)$ be the group of homeomorphisms $f : M \to M$ that fix ∂M pointwise, preserve orientation on both M and N, and globally fix N. The multiplication in this group is given by the usual composition. $\mathrm{Homeo}^+(M, N)$ is a topological group when equipped with the compact-open topology.

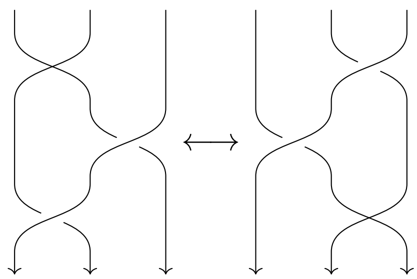

Figure 3.5: Welded Reidemeister moves.

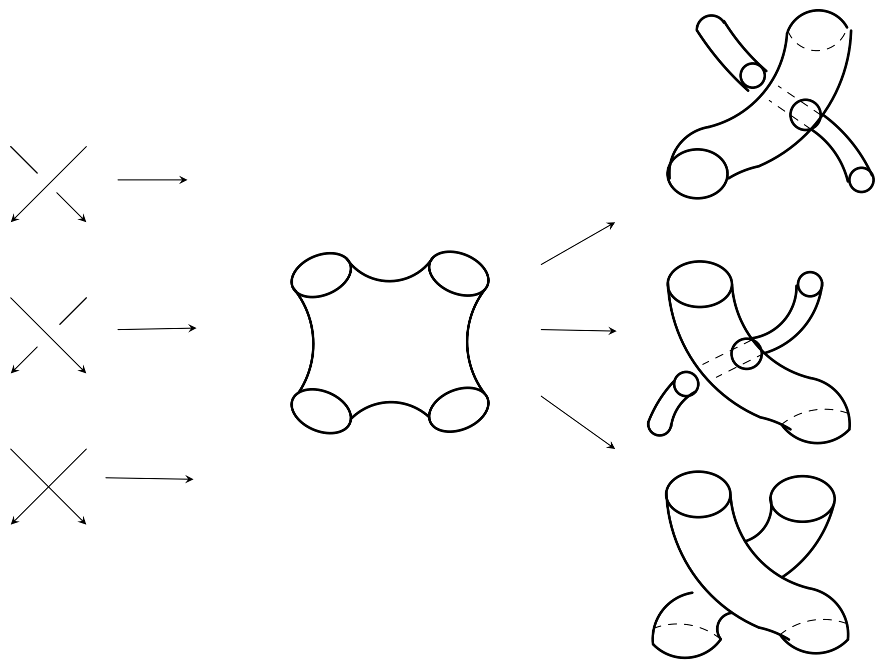

Figure 3.6: From welded diagrams to broken surface diagrams.

The mapping class group of a 3-manifold M with respect to a submanifold N, denoted by $MCG(M, N)$, is the group of isotopy classes of elements of $\mathrm{Homeo}^+(M, N)$, where isotopies are required to fix the boundary pointwise and N globally.

$$MCG(M, N) = \pi_0(\mathrm{Homeo}^+(M, N)).$$

Every homeomorphism of that kind induces a permutation on the connected components of N in the natural way. The pure mapping class group of a 3-manifold M with respect to a submanifold N, denoted by $PMCG(M, N)$, is the subgroup of elements of $MCG(M, N)$ that send each connected component of N to itself.

Definition 3.4.2. Let us fix $n \geq 1$, and let $C = C_1 \bigsqcup \cdots \bigsqcup C_n$ be a collection of n disjoint, unknotted, oriented circles, that form a trivial link of n components in $\mathbb{R}^3$. For our purposes, we can assume in the following that C is contained in the xy-disk in the 3-ball B^3. The loop braid group on n components, denoted by LB_n, is the mapping class group $MCG(B^3, C)$.

The pure loop braid group on n components, denoted by PLB_n, is the pure mapping class group $PMCG(B^3, C)$, that is the subgroup of $MCG(B^3, C)$ which elements are the isotopy classes of homeomorphisms that send each connected component of N to itself.

3.5 Welded braid groups as fundamental groups of configuration spaces

Definition 3.5.1. Let $n \geq 1$, and let $\mathcal{UR}_n$ be the space of configurations of n Euclidean, unordered, disjoint, unlinked circles in B^3 lying on planes parallel to a fixed one. The untwisted ring group UR_n is its fundamental group. Similarly, let $\mathcal{PUR}_n$ be the space of configurations of n Euclidean ordered, disjoint, unlinked circles lying on planes parallel to a fixed one. The pure untwisted ring group PUR_n is its fundamental group.

Let p be the orbit projection $\mathcal{PUR}_n \to \mathcal{UR}_n$ that forgets the order of the circles. In [8] it is shown that p is a regular $n!$-sheeted cover with Σ_n as group of deck transformations. From this follows that PUR_n is a subgroup of UR_n, and we have the short exact sequence

$$1 \longrightarrow PUR_n \longrightarrow UR_n \longrightarrow \Sigma_n \longrightarrow 1.$$

Damiani proved in [8, Proposition 3.12], the following statement.

Proposition 3.5.2. For $n \geq 1$, there are natural isomorphisms between pure untwisted ring group PUR_n and the pure loop braid group PLB_n, and between their respective unordered versions UR_n and LB_n.

Both these groups can be seen in an isomorphism with the group of ribbon braids on n strands.

Proposition 3.5.3. [8] For $n \geq 1$, there is an isomorphism between the pure ribbon braid group PrB_n and the pure loop braid group PLB_n.

Proof. Let us start with an observation. Taken b a geometric ribbon braid, the transversality forces $b \cap (B^3 \times \{t\})$ to be the disjoint union of n circles, for all $t \in I$. This allows us to think to a ribbon braid as a trajectory $\beta = (C(1), \dots, C_n(t))$ of circles in $B^3 \times I$. Let us define now

$$\phi(\beta) : I \to PLB_n$$

as the morphism defined by $t \to (C_1(t), \dots, C_n(t))$. By definition, $\phi(\beta)$ is a loop in the configuration space PUR_n, and corresponds to an element of PLB_n through the isomorphism in Proposition 3.5.2. This map induces a bijection

$$\phi_* : PrB_n \to PLB_n.$$

Indeed, two pure geometric braids β' and β'' are equivalent if and only if there is an ambient isotopy of $B^3 \times I$ from the identity map to an homeomorphism of $B^3 \times I$ in itself that maps β' to β''. That by construction would be an isotopy (so in particular a homotopy) between the two associated loops in PLB_n. Moreover products are preserved, so ϕ_* is a isomorphism. □

Proposition 3.5.4. [8] For $n \geq 1$, there is an isomorphism between the ribbon braid group rB_n and the loop braid group LB_n.

Proof. As in Proposition 3.5.3 we fix an element $\beta = (C_1(t), \dots, C_n(t))$ of rB_n and we define a map:

$$\hat{\phi}(\beta) : I \to LB_n$$

by $t \to (C_1(t), \dots, C_n(t))$. The element $\hat{\phi}(\beta)$ is a loop in the configuration space UR_n. This loop corresponds to an element of LB_n through the isomorphism from Proposition 3.5.2. Then $\hat{\phi}$ induces a homomorphism

$$\hat{\phi}_* : rB_n \to LB_n.$$

We consider the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & PrB_n \arrow[r] \arrow[d, "\phi_*", "\cong"'] & rB_n \arrow[r] \arrow[d, "\hat{\phi}_*"] & \Sigma_n \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & PLB_n \arrow[r] & LB_n \arrow[r] & \Sigma_n \arrow[r] & 1
\end{tikzcd}$$

It is commutative by construction of ϕ and $\hat{\phi}$. By applying the five lemma, the statement is proved. □

3.6 A presentation for welded braid groups

Brendle and Hatcher gave a presentation for the untwisted ring groups UR_n:

Proposition 3.6.1. [7] For $n \geq 1$ the group UR_n admits a presentation with generators $\{\sigma_i, \rho_i\}$, with $1 \leq i \leq n-1$, and the following relations:

1) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, for $i = 1, 2, \ldots, n-2$;

2) $\sigma_i\sigma_j = \sigma_j\sigma_i$, for $|i-j| > 1$ and $i, j = 1, 2, \ldots, n-1$;

3) $\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}$ for $i = 1, \ldots, n-2$;

4) $\rho_i\rho_j = \rho_j\rho_i$ for $|i-j| > 1$ and $i = 1, \ldots, n-1$;

5) $\rho_i^2 = 1$ for $i = 1, \cdots, n-1$;

6) $\rho_i\sigma_j = \sigma_j\rho_i$ for $|i-j| > 1$ and $i = 1, \ldots, n-1$;

7) $\rho_{i+1}\rho_i\sigma_{i+1} = \sigma_i\rho_{i+1}\rho_i$ for $i = 1, \ldots, n-2$;

8) $\sigma_{i+1}\sigma_i\rho_{i+1} = \rho_i\sigma_{i+1}\sigma_i$ for $i = 1, \ldots, n-2$.

The elements σ_i and ρ_i of the presentation represent the following loops in UR_n: we place the n rings in a standard position in the yz-plane with centers along the y-axis, then the σ_i is the loop that permutes the i-th and the $(i+1)$-st circles by passing the i-th circle through the $(i+1)$-st. The loop ρ_i permutes them passing the i-th around the $(i+1)$-st. We can see both in Figure 3.7.

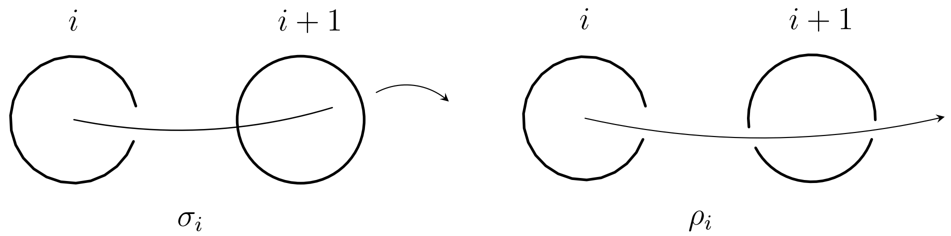

Figure 3.7: The elements σ_i and ρ_i.

We recall that welded braids were initially introduced as a graphic representation of the above presentation. Through this point of view, the generators σ_i and ρ_i can be drawn as in Figure 3.8.

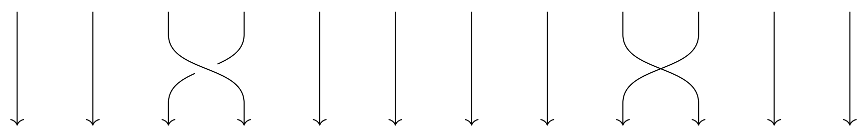

Figure 3.8: Generators σ_i and ρ_i.

There is also a presentation for PUR_n given by Brendle and Hatcher:

Proposition 3.6.2. For $n \geq 1$, the group PUR_n admits a presentation with generators $\alpha_{i,j}$ for $1 \leq i \neq j \leq n$ and relations:

1) $\alpha_{i,j}\alpha_{k,l} = \alpha_{k,l}\alpha_{i,j}$ if $\{i,j\} \cap \{k,l\} = \emptyset$;

2) $\alpha_{i,k}\alpha_{j,k} = \alpha_{j,k}\alpha_{i,k}$ for i,j,k pairwise distinct;

3) $\alpha_{i,j}(\alpha_{i,k}\alpha_{j,k}) = (\alpha_{i,k}\alpha_{j,k})\alpha_{i,j}$ for i,j,k pairwise distinct.

Each element $\alpha_{i,j}$ of the presentation represents the movement of the i-th circle passing through the j-th circle and going back to its position, as in Figure 3.9. The representation through welded diagrams can be seen in Figure 3.10.

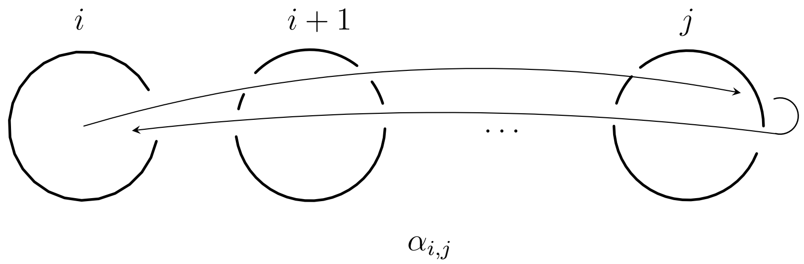

Figure 3.9: The element $\alpha_{i,j}$.

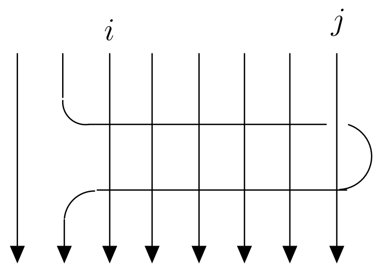

Figure 3.10: The element $\alpha_{i,j}$.

Remark 3.6.3. From the above result follows also the isomorphism between ribbon braids and welded braids.

Proposition 3.6.4. [8] The map Tube : $WB_n \longmapsto rB_n$ is an isomorphism.

Proof. We know from Proposition 3.5.4 that rB_n is isomorphic to LB_n, and from Proposition 3.5.2 that LB_n is isomorphic to UR_n. From the above presentation we know that UR_n is isomorphic to WB_n. The map Tube was already known to be surjective from Proposition 3.3.3. The result follows. □

3.7 Welded braids as automorphisms of the free group

A definition of welded braid groups can be given also in terms of automorphisms of F_n. We take the elements of $\mathrm{Aut}(F_n)$ of the following form:

$$\sigma_k : \begin{cases} x_k \longmapsto x_{k+1} & \\ x_{k+1} \longmapsto x_{k+1}^{-1} x_k x_{k+1} & \\ x_l \longmapsto x_l & \text{if } l \neq k, k+1 \end{cases}$$

The elements $\{\sigma_k\}$ for $1 \leq k \leq n-1$ generate the braid subgroup B_n of $\mathrm{Aut}(F_n)$ isomorphic to the braid group on n strands (see which we have seen in Section 1.4). We add another family of automorphisms:

$$\rho_i : \begin{cases} x_i \longmapsto x_{i+1} \\ x_{i+1} \longmapsto x_i \\ x_j \longmapsto x_j & \text{if } j \neq i, i+1 \end{cases}$$

The elements ρ_i generate the permutation subgroup of $\bar{S_n}$ of $\mathrm{Aut}(F_n)$. Both these set of generators, σ_k and ρ_i, generate the permutation-conjugacy subgroup of $\mathrm{Aut}(F_n)$, denoted by PC_n and containing the automorphisms that send each generator in the conjugate of some generator of F_n, in symbols:

$$\alpha : x_i \longmapsto w_i^{-1} x_{\pi(i)} w_i$$

where $\pi \in \Sigma_n$ and w_i is an element of F_n. As widely explained in [8], there is an isomorphism between the subgroup PC_n of $\mathrm{Aut}(F_n)$ and the group UR_n. Moreover, there is an isomorphism between the group of pure welded braids PUR_n and the subgroup of the automorphisms of F_n that send each generator to a conjugate of itself, in symbols:

$$\alpha : x_i \longmapsto w_i^{-1} x_i w_i$$

where w_i is an element of F_n. The subgroup is called basis-conjugation subgroup, denoted by $Aut_C(F_n)$, and it is generated by elements of the form:

$$\alpha_{i,j} : \begin{cases} x_i \longmapsto x_j^{-1} x_i x_j \\ x_k \longmapsto x_k & \text{if } k \neq i, j \end{cases}$$

The correspondence between the elements of UR_n that we called σ_i, ρ_i and $\alpha_{i,j}$ and the automorphisms of PC_n with the same name justifies the abuse of notation.

3.8 Ribbon tubes and welded string links

A similar notion can be given in order to study homotopy for welded braids. In the ribbon context, the natural analogues of string links are the so called ribbon tubes. They have been studied up to link-homotopy by Audoux, Bellingeri, Meilhan and Wagner in [3], where they showed that every ribbon tube is link-homotopic to a pure ribbon braid (result that can be considered the ribbon equivalent to Habegger-Lin's classical one).

This section is taken from [3], and we want to recall some of their main results. Furthermore, Darné [9] found a presentation for the group of pure welded braids up to homotopy that is partially a consequence of the above mentioned work. We are going to recall his presentation.

Definition 3.8.1. Let $A_1, \ldots, A_n$ be locally flat embeddings in $\overset{*}{B^4}$ of n disjoint copies of the oriented annulus $S^1 \times I$. We say that

$$T = \bigsqcup_{i \in \{1,\ldots,n\}} A_i$$

is a ribbon tube if:

1) $\partial A_i = C_i \times \{0,1\}$ for all $i \in \{1,\dots,n\}$ and orientation induced by A_i on ∂A_i coincides with that of C_i;

2) the annuli A_i are fillable, in the sense that they bound immersed 3-balls $\subset \mathbb{R}^4$ whose singular points consist in a finite number of ribbon discs.

The set of ribbon tubes up to isotopy fixing the boundary circles is denoted by rT_n. Equipped with the stacking and reparametrizing product, and with unit element the trivial ribbon tube $\bigsqcup_{i=1}^{n} C_i \times I$, rT_n is a monoid.

Definition 3.8.2. An element of rT_n is said to be monotone if it has a representative which is flatly transverse to the lamination $\bigcup_{t\in I} B^3 \times \{t\}$ of B^4.

It is proved in [3, Proposition 2.4], that the subset of rT_n whose elements are monotone is a group for the stacking product. Also, two monotone ribbon tubes which are equivalent in rT_n are always related by a monotone isotopy, which is an isotopy moving only through monotone objects ([3, Remark 2.26]).This implies that the group of monotone elements of rT_n is equal to the group of pure ribbon braids defined in Definition 3.1.6, so we are going to denote it with PrB_n.

Ribbon tubes can be described by broken surface diagrams, that we are going to define here. As we could expect, they are the analogous of braid broken surface diagrams defined in Definition 3.2.1, without the monotony requirement and where the associated permutation is the identity.

Definition 3.8.3. Let $A_1,\dots,A_n$ be locally flat embeddings in $\overset{*}{B}{}^3$ of n disjoint copies of the oriented annulus $S^1 \times I$. We can say that

$$S = \bigsqcup_{i\in\{1,\dots,n\}} A_i$$

is a broken surface diagram if:

1) $\partial A_i = C_i \times \{0,1\}$ for all $i \in \{1,\dots,n\}$ and orientation induced by A_i on ∂A_i coincides with that of C_i;

2) the set of connected components of singular points in S, denoted by $\Sigma(S)$, consists of flatly transvers disjoint circles in $(\bigcup_{i=1}^{n} \mathrm{int}(A_i))$.

There are two results that suggest, as for the case of ribbon braids, that broken surface diagrams can be thought of as 3-dimensional representations of ribbon tubes. These results are stated in [3, Lemma 2.13, Lemma 2.14], and they are:

Lemma 3.8.4. Any ribbon tube admits, up to isotopy, a projection to B^3 which is a broken surface diagram. Conversely any broken surface diagram is the projection of a unique ribbon tube.

Lemma 3.8.5. Any ribbon tube can be represented by a symmetric broken surface diagram.

Ribbon tubes can be represented also by welded string links diagrams.

Definition 3.8.6. A n-component welded string link diagram is a locally flat immersion L of n intervals $\bigsqcup_{i=1}^{n} I_i$ in $\overset{*}{B}{}^2$, called strands, such that:

1. each strand I_i has boundary $\partial I_i = \{p_i\} \times \{0, 1\}$ and is oriented from $\{p_i\} \times \{0\}$ to $\{p_i\} \times \{1\}$, where $1 \leq i \leq n$;
2. the singular set $\Sigma(L)$ of L is a finite set of flatly transverse points.

Moreover, for each element of $\Sigma(L)$, a partial ordering is given on the two preimages. We have three types of crossings, as for welded pure braids: positive, negative and welded (when the two preimages are not comparable).

Definition 3.8.7. A welded string link diagram is said to be monotone if it is flatly transverse to the lamination $\bigcup_{t \in I} I \times \{t\}$ of B^2.

The set of welded string links up to isotopy, quotiented by the Reidemeister moves defined in Figures 3.2, 3.3, 3.4, 3.5 and in Figure 3.11, forms a monoid with the stacking product and with unit element the trivial diagram $\bigcup_{i=1}^{n} p_i \times I$. The monoid will be denoted WSL_n. Monotone welded string links form a subset of WSL_n with a group structure,

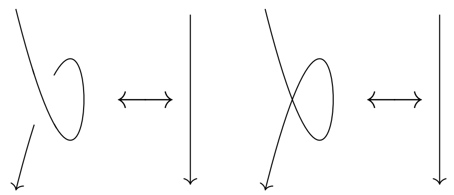

Figure 3.11: other Reidemeister moves for welded string links (self-crossings).

which can be proved to be equivalent to PWB_n. This allows us to denote also the subset of monotone welded string links with PWB_n, and to call his elements welded pure braids.

Similarly to the case of welded braids, we can define a map Tube: $WSL_n \longmapsto rT_n$ which associates to a welded string link diagram L a symmetric broken surface diagram, and a ribbon tube to the broken surface diagram. The entire procedure to define the map Tube is explained in [3]. Differently from ribbon braids, it is still unknown whether the correspondence between ribbon tubes and symmetric broken surface diagram up to Reidemeister moves is injective. That leads to the following proposition:

Proposition 3.8.8. The map Tube: $WSL_n \longmapsto rT_n$ is well defined and surjective.

It is still unknown whether the map is also injective, but if we restrict it to monotone objects on both sides we obtain an isomorphism: $PWB_n \cong PrB_n$.

3.9 Homotopy for pure welded braid groups

Definition 3.9.1. Two welded string link diagrams are related by a self-virtualization if one can be obtained from the other by turning a classical self-crossing (i.e. a classical crossing where the two preimages belong to the same components) into a welded one.

The equivalence relation on WSL_n generated by self-virtualization is called v-equivalence. The quotient of WSL_n under v-equivalence, which is compatible with the stacking product, is denoted by WSL_n^v. We denote by $PWB_n^v \subset WSL_n^v$ the subset of elements having a monotone representative.

Theorem 3.9.2. [3] Every welded string link is monotone up to self-virtualization.

We introduce now link-homotopy for ribbon tubes.

Definition 3.9.3. A singular ribbon tube is a locally flat immersion T of n annuli $\bigsqcup_{i=1}^{n} A_i$ in $\overset{*}{B}{}^4$ such that:

1) $\partial A_i = C_i \times \{0,1\}$ for all $1 \leq i \leq n$ and the orientation induced by A_i on ∂A_i coincides with that of C_i;

2) the singular set of T is a single flatly transverse circle, called singular loop, whose preimages are two circles embedded in $\bigcup_{i=1}^{n} \mathrm{int}(A_i)$, an essential and a non essential one;

3) there exist n locally flat immersed 3-balls $\bigcup_{i=1}^{n} B_i$ such that:

   (a) $\partial_* B_i = \mathrm{int}(A_i)$ and $\partial_\varepsilon B_i = D_i \times \{\varepsilon\}$ for all $1 \leq i \leq n$, $\varepsilon \in \{0,1\}$;

   (b) the singular set of $\bigcup_{i=1}^{n} B_i$ is a disjoint union of flatly transverse discs, all of them being ribbon singularities but one, whose preimages are two discs, bounded by the preimages of the singular loop, one in $\bigcup_{i=1}^{n} \partial_* B_i$ and the other with interior in $\bigcup_{i=1}^{n} \mathrm{int}(B_i)$.

We say that a singular ribbon tube is self-singular if and only if both preimages of the singular loop belong to the same tube component.

Definition 3.9.4. Two ribbon tubes T_1 and T_2 are said to be link-homotopic if and only if there is a 1-parameter family of regular and self-singular ribbon tubes from T_1 to T_2 passing through a finite number of self-singular ribbon tubes.

We denote by rT_n^h the quotient of rT_n by the link-homotopy equivalence, which is compatible with the monoidal structure of rT_n. Furthermore, we denote by PrB_n^h the image of PrB_n in rT_n^h.

Now we recall one of the main results of [3], which can be considered a higher-dimensional analogue of Habegger-Lin's statement about homotopy for string links:

Theorem 3.9.5. [3] Every ribbon tube is link-homotopic to a monotone ribbon tube.

Corollary 3.9.6. [3] The set rT_n^h is a group for the stacking product.

In [3], this group is described with automorphisms of the Reduced Free group.

Definition 3.9.7. Let F_n be the free group of rank n. We denote by RF_n the reduced free group of rank n, which is the smallest quotient where each generator commutes with all its conjugates.

$$RF_n := F_n/\{[x_i, x^{-1}x_ix] \mid 1 \leq i \leq n,\ x \in F_n\}$$

It has been proved in [3] that there is a group isomorphism between ribbon tubes up to link-homotopy and the group of basis-conjugating automorphisms of the reduced free group.

Definition 3.9.8. We define $\mathrm{Aut}_C(RF_n)$ the group of basis-conjugating automorphisms of RF_n, which are the automorphisms of RF_n which send a generator in a conjugate of itself.

$$\mathrm{Aut}_C(RF_n) := \{f \in \mathrm{Aut}(RF_n) \mid \forall\ 1 \leq i \leq n,\ \exists\ x \in RF_n,\ f(x_i) = x^{-1}x_ix\}$$

Theorem 3.9.9. [3] There is an isomorphism between rT_n^h and $\mathrm{Aut}_C(RF_n)$.

In the same work, it has been proved that welded string links up to self-virtualization are also isomorphic to basis-conjugating automorphisms of RF_n. We recall here two last results about welded string links up to self-virtualization.

Theorem 3.9.10. [3] The monoids WSL_n^v, PWB_n^v and $\mathrm{Aut}_C(RF_n)$ are isomorphic.

Proposition 3.9.11. [3] The map Tube:$WSL_n^v \longmapsto rT_n^h$ is a well defined group isomorphism.

So, link-homotopy for ribbon tubes corresponds to self-virtualization moves in welded diagrams.
Darné (see [9, Diagram 4.0.1]) showed that there is an embedding of the group of homotopy pure braids into the group of basis-conjugating automorphisms of RF_n.

Proposition 3.9.12. There is an embedding of $\tilde{P_n}$ into $\mathrm{Aut}_C(RF_n)$.

We have that the group of homotopy pure braids embeds into the group of homotopy pure welded braids. A presentation for the group of pure n-welded braids up to homotopy has been given by Darné in [9], using the isomorphism with $\mathrm{Aut}_C(RF_n)$.

Proposition 3.9.13. [9] For $n \geq 1$, the group of homotopy pure welded braids on n strands admits a presentation with generators $\alpha_{i,j}$ for $1 \leq i \neq j \leq n$ and relations:

1) $\alpha_{i,j}\alpha_{k,l} = \alpha_{k,l}\alpha_{i,j}$ if $\{i,j\} \cap \{k,l\} = \emptyset$;
2) $\alpha_{i,k}\alpha_{j,k} = \alpha_{j,k}\alpha_{i,k}$ for i,j,k pairwise distinct;
3) $\alpha_{i,j}(\alpha_{i,k}\alpha_{j,k}) = (\alpha_{i,k}\alpha_{j,k})\alpha_{i,j}$ for i,j,k pairwise distinct;
4) $[\alpha_{i,j}, w, \alpha_{i,j}] = 1$, for $j < i$ and $w \in \langle \alpha_{i,k} \rangle_{k<i}$;
5) $[\alpha_{j,i}, w, \alpha_{k,i}] = 1$, for $j,k < i$ and $w \in \langle \alpha_{i,h} \rangle_{h<i}$;
6) $[\alpha_{i,j}, w, \alpha_{j,i}] = 1$, for $i < j$ and $w \in \langle \alpha_{j,k} \rangle_{k<j,k\neq i}$,

where given three elements a,b,c, we have $[a,b,c] := [a,[b,c]]$.

Chapter 4

A new presentation for the homotopy braid groups

The goal of this thesis is to retrieve the more symmetric presentation for $\tilde{B}_n$ given by Graff [16] using an algebraic approach instead of the topological one used in [16]. First, we prove two lemmas.

Lemma 4.0.1. Let β be a pure n-braid of the form $A_{s,k}^{\varepsilon}A_{1,j}A_{s,k}^{-\varepsilon}$, with $\varepsilon = \pm 1$. Then, β can always be written in the form $gA_{1,j}g^{-1}$, where g is an element of the subgroup of P_n generated by $A_{1,2}, A_{1,3}, \ldots, A_{1,n}$.

Proof. Let us prove the result for every braid of the form $A_{s,k}A_{1,j}A_{s,k}^{-1}$, i.e. $\varepsilon = 1$. We are using the relations given in the presentation of P_n in Proposition 1.6.4. Also, we use $\boxed{}$ to indicate that we manipulate the elements in the box using the mentioned relations. For $1 < s < j \leq n$,

$$
\begin{aligned}
A_{s,j}A_{1,j}A_{s,j}^{-1} &= A_{1,s}^{-1}\boxed{A_{1,s}A_{s,j}A_{1,s}^{-1}}\boxed{A_{1,s}A_{1,j}A_{1,s}^{-1}}\boxed{A_{1,s}A_{s,j}^{-1}A_{1,s}^{-1}}A_{1,s} \\
&= A_{1,s}^{-1}A_{s,j}^{-1}A_{1,j}^{-1}A_{s,j}A_{1,j}A_{s,j}A_{s,j}^{-1}A_{1,j}A_{s,j}A_{s,j}^{-1}A_{1,j}^{-1}A_{s,j}^{-1}A_{1,j}A_{s,j}A_{1,s} \\
&= A_{1,s}^{-1}\boxed{A_{s,j}^{-1}A_{1,j}^{-1}A_{s,j}}A_{1,j}\boxed{A_{s,j}^{-1}A_{1,j}A_{s,j}}A_{1,s} \\
&= A_{1,s}^{-1}A_{1,s}A_{1,j}^{-1}A_{1,s}^{-1}A_{1,j}A_{1,s}A_{1,j}A_{1,s}^{-1}A_{1,s} = A_{1,j}^{-1}A_{1,s}^{-1}A_{1,j}A_{1,s}A_{1,j}.
\end{aligned}
$$

For $1 < s < k < j \leq n$ and $1 < j < s < k \leq n$, we have $A_{s,k}A_{1,j}A_{s,k}^{-1} = A_{1,j}$. For the case $1 < j < k \leq n$, it holds that

$$
\begin{aligned}
A_{j,k}A_{1,j}A_{j,k}^{-1} &= A_{j,k}\boxed{A_{1,j}A_{j,k}^{-1}A_{1,j}^{-1}}A_{1,j} \\
&= A_{j,k}A_{j,k}^{-1}A_{1,k}^{-1}A_{j,k}^{-1}A_{1,k}A_{j,k}A_{1,j} \\
&= A_{1,k}^{-1}\boxed{A_{j,k}^{-1}A_{1,k}A_{j,k}}A_{1,j} \\
&= A_{1,k}^{-1}A_{1,j}A_{1,k}A_{1,j}^{-1}A_{1,j} \\
&= A_{1,k}^{-1}A_{1,j}A_{1,k}.
\end{aligned}
$$

For $1 < s < j < k \leq n$, we want to prove that $A_{s,k}A_{1,j}A_{s,k}^{-1} = [A_{1,k}^{-1}, A_{1,s}^{-1}]A_{1,j}[A_{1,s}^{-1}, A_{1,k}^{-1}]$.

This is equivalent to show that $A_{1,j} = [A_{1,k}^{-1}, A_{1,s}^{-1}]^{-1} A_{s,k} A_{1,j} A_{s,k}^{-1} [A_{1,s}^{-1}, A_{1,k}^{-1}]^{-1}$. We have

$$
\begin{aligned}
&[A_{1,k}^{-1}, A_{1,s}^{-1}]^{-1} A_{s,k} A_{1,j} A_{s,k}^{-1} [A_{1,s}^{-1}, A_{1,k}^{-1}]^{-1} \\
&= A_{1,s}^{-1} A_{1,k}^{-1} A_{1,s} A_{1,k} A_{s,k} A_{1,j} A_{s,k}^{-1} A_{1,k}^{-1} A_{1,s}^{-1} A_{1,k} A_{1,s} \\
&= A_{1,s}^{-1} A_{1,k}^{-1} \boxed{A_{1,s} A_{1,k} A_{1,s}^{-1}} A_{1,s} A_{s,k} A_{1,j} A_{s,k}^{-1} A_{1,s}^{-1} \boxed{A_{1,s} A_{1,k}^{-1} A_{1,s}^{-1}} A_{1,k} A_{1,s} \\
&= A_{1,s}^{-1} A_{1,k}^{-1} A_{s,k}^{-1} A_{1,k} A_{s,k} A_{1,s} A_{s,k} A_{1,j} A_{s,k}^{-1} A_{1,s}^{-1} A_{s,k}^{-1} A_{1,k}^{-1} A_{s,k} A_{1,k} A_{1,s} \\
&= A_{1,s}^{-1} A_{s,k} \boxed{A_{s,k}^{-1} A_{1,k}^{-1} A_{s,k}^{-1} A_{1,k} A_{s,k}} A_{1,s} A_{s,k} A_{1,j} A_{s,k}^{-1} A_{1,s}^{-1} \boxed{A_{s,k}^{-1} A_{1,k}^{-1} A_{s,k} A_{1,k} A_{s,k}} A_{s,k}^{-1} A_{1,s} \\
&= A_{1,s}^{-1} A_{s,k} A_{1,s} A_{s,k}^{-1} A_{1,s}^{-1} A_{1,s} A_{s,k} A_{1,j} A_{s,k}^{-1} A_{1,s}^{-1} A_{1,s} A_{s,k} A_{1,s}^{-1} A_{s,k}^{-1} A_{1,s} \\
&= A_{1,s}^{-1} A_{s,k} \boxed{A_{1,s} A_{1,j} A_{1,s}^{-1}} A_{s,k}^{-1} A_{1,s} \\
&= A_{1,s}^{-1} A_{s,k} A_{s,j}^{-1} A_{1,j} A_{s,j} A_{s,k}^{-1} A_{1,s} \\
&= A_{1,s}^{-1} A_{s,j}^{-1} \boxed{A_{s,j} A_{s,k} A_{s,j}^{-1}} A_{1,j} \boxed{A_{s,j} A_{s,k}^{-1} A_{s,j}^{-1}} A_{s,j} A_{1,s} \\
&= A_{1,s}^{-1} A_{s,j}^{-1} A_{j,k}^{-1} A_{s,k} A_{j,k} \boxed{A_{1,j} A_{j,k}^{-1} A_{s,k}^{-1} A_{j,k}} A_{s,j} A_{1,s} \\
&= A_{1,s}^{-1} A_{s,j}^{-1} A_{j,k}^{-1} A_{s,k} A_{j,k} A_{j,k}^{-1} A_{s,k}^{-1} A_{j,k} A_{1,j} A_{s,j} A_{1,s} \\
&= A_{1,s}^{-1} \boxed{A_{s,j}^{-1} A_{1,j} A_{s,j}} A_{1,s} \\
&= A_{1,s}^{-1} A_{1,s} A_{1,j} A_{1,s}^{-1} A_{1,s} \\
&= A_{1,j}.
\end{aligned}
$$

Now, we prove the result for every braid of the form $A_{s,k}^{-1} A_{1,j} A_{s,k}$, that is $\varepsilon = -1$. For $1 < s < j \le n$, we have $A_{s,j}^{-1} A_{1,j} A_{s,j} = A_{1,s} A_{1,j} A_{1,s}^{-1}$. For $1 < s < k < j \le n$ and $1 < j < s < k \le n$, it holds that $A_{s,k}^{-1} A_{1,j} A_{s,k} = A_{1,j}$. Let $\Theta_n : B_n \to B_n$ be the strand-reversing automorphism of the n-braid group defined by $\Theta_n(\sigma_i) = \sigma_{n-i}^{-1}$ when $1 \le i \le n-1$. As shown in [19, Lemma 0.2.5], we have $\Theta_n(A_{i,j}) = A_{n-j+1,n-i+1}^{-1}$. Therefore, for $1 < j < k \le n$, the equality

$$
\begin{aligned}
\Theta_n(A_{j,k}^{-1} A_{1,j} A_{j,k}) &= A_{n-k+1,n-j+1} A_{n-j+1,n}^{-1} A_{n-k+1,n-j+1}^{-1} \\
&= A_{n-j+1,n}^{-1} A_{n-k+1,n}^{-1} A_{n-j+1,n}^{-1} A_{n-k+1,n} A_{n-j+1,n} \\
&= \Theta_n(A_{1,j} A_{1,k} A_{1,j} A_{1,k}^{-1} A_{1,j}^{-1})
\end{aligned}
$$

implies that $A_{j,k}^{-1} A_{1,j} A_{j,k} = A_{1,j} A_{1,k} A_{1,j} A_{1,k}^{-1} A_{1,j}^{-1}$. Moreover, for $1 < s < j < k \le n$, the relation

$$
\begin{aligned}
\Theta_n(A_{s,k}^{-1} A_{1,j} A_{s,k}) &= A_{n-k+1,n-s+1} A_{n-j+1,n}^{-1} A_{n-k+1,n-s+1}^{-1} \\
&= [A_{n-s+1,n}^{-1}, A_{n-k+1,n}^{-1}] A_{n-j+1,n}^{-1} [A_{n-s+1,n}^{-1}, A_{n-k+1,n}^{-1}]^{-1} \\
&= \Theta_n([A_{1,s}, A_{1,k}] A_{1,j} [A_{1,s}, A_{1,k}]^{-1})
\end{aligned}
$$

implies that $A_{s,k}^{-1} A_{1,j} A_{s,k} = [A_{1,s}, A_{1,k}] A_{1,j} [A_{1,s}, A_{1,k}]^{-1}$.
In conclusion, we showed that $A_{s,k}^{\varepsilon} A_{1,j} A_{s,k}^{-\varepsilon} = g A_{1,j} g^{-1}$ for $\varepsilon = \pm 1$. $\square$

Lemma 4.0.2. Let β be a pure n-braid of the form $h A_{1,j} h^{-1}$ with $h \in P_n$. Then, β can always be written in the form $g A_{1,j} g^{-1}$, where g is an element of the subgroup of P_n generated by $A_{1,2}, A_{1,3}, \ldots, A_{1,n}$.

Proof. We prove it by induction on the length of h. Suppose

$$h = A_{k_1,s_1}^{\varepsilon_1} A_{k_2,s_2}^{\varepsilon_2} \cdots A_{k_m,s_m}^{\varepsilon_m}$$

where $\varepsilon_i = \pm 1$. If $m = 1$, it is true due to Lemma 4.0.1. If it is true for $m-1$, then it can be written by inductive hypothesis as

$$\begin{aligned} hA_{1,j}h^{-1} &= A_{k_1,s_1}^{\varepsilon_1}(A_{k_2,s_2}^{\varepsilon_2} \cdots A_{k_m,s_m}^{\varepsilon_m} A_{1,j} A_{k_m,s_m}^{-\varepsilon_m} \cdots A_{k_2,s_2}^{-\varepsilon_2}) A_{k_1,s_1}^{-\varepsilon_1} \\ &= A_{k_1,s_1}^{\varepsilon_1}(\tilde{h} A_{1,j} \tilde{h}^{-1}) A_{k_1,s_1}^{-\varepsilon_1}, \text{ where } \tilde{h} \in \langle A_{1,2}, A_{1,3}, \ldots, A_{1,n} \rangle. \end{aligned}$$

Let us say $\tilde{h} = A_{1,t_1}^{l_1} A_{1,t_2}^{l_2} \cdots A_{1,t_p}^{l_p}$, with $l_i = \pm 1$. Then

$$\begin{aligned} A_{k_1,s_1}^{\varepsilon_1}(\tilde{h} A_{1,j} \tilde{h}^{-1}) A_{k_1,s_1}^{-\varepsilon_1} &= A_{k_1,s_1}^{\varepsilon_1}((\prod A_{1,t_i}^{l_i}) A_{1,j} (\prod A_{1,t_i}^{-l_i})) A_{k_1,s_1}^{-\varepsilon_1} \\ &= A_{k_1,s_1}^{\varepsilon_1}(\prod A_{1,t_i}^{l_i}) A_{k_1,s_1}^{-\varepsilon_1} A_{k_1,s_1}^{\varepsilon_1} A_{1,j} A_{k_1,s_1}^{-\varepsilon_1} A_{k_1,s_1}^{\varepsilon_1} (\prod A_{1,t_i}^{-l_i}) A_{k_1,s_1}^{-\varepsilon_1} \\ &= \prod (A_{k_1,s_1}^{\varepsilon_1} A_{1,t_i}^{l_i} A_{k_1,s_1}^{-\varepsilon_1}) A_{k_1,s_1}^{\varepsilon_1} A_{1,j} A_{k_1,s_1}^{-\varepsilon_1} \prod (A_{k_1,s_1}^{\varepsilon_1} A_{1,t_i}^{-l_i} A_{k_1,s_1}^{-\varepsilon_1}). \end{aligned}$$

According to Lemma 4.0.1, every element in the form $A_{k_1,s_1}^{\varepsilon_1} A_{1,t_i}^{\pm l_i} A_{k_1,s_1}^{-\varepsilon_1}$ can be written as $h' A_{1,t_i}^{\pm l_i} h'^{-1}$ and $A_{k_1,s_1}^{\varepsilon_1} A_{1,j} A_{k_1,s_1}^{-\varepsilon_1}$ can be written as $h'' A_{1,j} h''^{-1}$, where h', h'' are elements of $\langle A_{1,2}, A_{1,3}, \ldots, A_{1,n} \rangle$.
Then, β is in the form $gA_{1,j}g^{-1}$ where $g \in \langle A_{1,2},\ A_{1,3}, \ldots, A_{1,n} \rangle$. □

This lemma is a useful tool to prove a more interesting result, which is stated in next proposition: the commutator of a generator of P_n with the conjugate of the same generator for a generic element of P_n is always homotopically trivial.

Proposition 4.0.3. Let β be a pure n-braid of the form $\beta = [A_{j,k}, gA_{j,k}g^{-1}]$, g being an element of P_n. Then β is homotopically trivial.

Proof. Let us denote $\Sigma_j = \sigma_1^{-1}\sigma_2^{-1} \cdots \sigma_{j-1}^{-1}$. We know from [19, Lemma 0.2.4], that $\Sigma_j A_{j,k} \Sigma_j^{-1} = A_{1,k}$. In order to prove that $[A_{j,k}, gA_{j,k}g^{-1}] = 1$, we conjugate $[A_{j,k}, gA_{j,k}g^{-1}]$ by Σ_j and we obtain

$$\begin{aligned} \Sigma_j [A_{j,k}, gA_{j,k}g^{-1}] \Sigma_j^{-1} &= [\Sigma_j A_{j,k} \Sigma_j^{-1}, \Sigma_j g A_{j,k} g^{-1} \Sigma_j^{-1}] \\ &= [A_{1,k}, \Sigma_j g \Sigma_j^{-1} \Sigma_j A_{j,k} \Sigma_j^{-1} \Sigma_j g^{-1} \Sigma_j^{-1}] \\ &= [A_{1,k}, \tilde{g} A_{1,k} \tilde{g}^{-1}], \end{aligned}$$

where $\Sigma_j g \Sigma_j^{-1} = \tilde{g} \in P_n$. We know from Lemma 4.0.2 that $\tilde{g} A_{1,k} \tilde{g}^{-1}$ can be written as $hA_{1,k}h^{-1}$, for some $h \in \langle A_{1,2}, A_{1,3}, \ldots, A_{1,n} \rangle$. Our commutator becomes $[A_{1,k}, hA_{1,k}h^{-1}]$. From Goldsmith's presentation of $\tilde{B}^n$ (Theorem 1.7.4), we know that a commutator of this form is homotopically trivial. So,

$$\Sigma_j [A_{j,k}, gA_{j,k}g^{-1}] \Sigma_j^{-1} = 1 \text{ in } \tilde{B}^n,$$

which implies

$$[A_{j,k}, gA_{j,k}g^{-1}] = 1 \text{ in } \tilde{B}^n.$$

The proof is complete. □

This last proposition shows that we can deduce the relation $gA_{j,k}g^{-1} \rightleftharpoons A_{j,k}$ in $\tilde{B}_n$ with $g \in P_n$ directly from the relation $hA_{j,k}h^{-1} \rightleftharpoons A_{j,k}$ in $\tilde{B}_n$ with $h \in \langle A_{1,2}, A_{1,3}, \ldots, A_{1,n}\rangle$. The inverse is obvious. Thus we can replace the last relation in Goldsmith's presentation with this new one, finding a new presentation for $\tilde{B}_n$:

Proposition 4.0.4. The set of equivalence classes of n-braids under homotopy $\tilde{B}_n$ has generators $\sigma_1, \ldots, \sigma_{n-1}$ and the following relations:

1) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, 2, \ldots, n-2$;

2) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$ and $j = 1, 2, \ldots, n-1$;

3) $A_{j,k} \rightleftharpoons gA_{j,k}g^{-1}$ if $1 \le j < k \le n$ and g is an element of P_n.

To obtain a similar presentation for $\tilde{P}_n$, we recall here a Lemma proved in [16]:

Lemma 4.0.5. [16] The normal subgroup $J \triangleleft B_n$ generated in B_n by elements of the form $[A_{ij}, \lambda A_{ij}\lambda^{-1}]$ for $\lambda \in P_n$, seen as a subgroup of P_n, coincides with the normal subgroup of P_n generated by elements of the form $[A_{ij}, \lambda A_{ij}\lambda^{-1}]$, for $\lambda \in P_n$.

Proof. For $k \in \{1, \ldots, n-1\}$, $1 \le i < j \le n$ and $\lambda \in P_n$ we compute:

$$\sigma_k[A_{ij}, \lambda A_{ij}\lambda^{-1}]\sigma_k^{-1} = \begin{cases} [A_{i+1j}, \lambda_1 A_{i+1j}\lambda_1^{-1}] & \text{if } i = k \text{ and } j \neq k+1 \\ [A_{i+1j}, \lambda_2 A_{i+1j}\lambda_2^{-1}] & \text{if } j = k \\ A_{kk+1}[A_{i-1j}, \lambda_3 A_{i-1j}\lambda_3^{-1}]A_{kk+1}^{-1} & \text{if } i = k+1 \\ A_{kk+1}[A_{ij-1}, \lambda_4 A_{ij-1}\lambda_4^{-1}]A_{kk+1}^{-1} & \text{if } i \neq k \text{ and } j = k+1 \\ [A_{ij}, \lambda A_{ij}\lambda^{-1}] & \text{otherwise,} \end{cases}$$

with $\lambda_i \in P_n$ for $i \in \{1, 2, 3, 4\}$. Therefore the conjugates $\sigma_k[A_{ij}, \lambda A_{ij}\lambda^{-1}]\sigma_k^{-1}$ are always conjugates of $[A_{i'j'}, \lambda' A_{i'j'}(\lambda')^{-1}]$ in P_n for some $1 \le i' < j' \le n$ and $\lambda' \in P_n$ and the proof is done. ☐

The previous lemma allows us to explicitely write a presentation for the set of equivalence classes of pure n-braids under homotopy.

Corollary 4.0.6. The homotopy pure n-braid group $\tilde{P}_n$ has a presentation with generators $A_{i,j}$ and the following relations:

1) $A_{r,s} \rightleftharpoons A_{i,j}$ if $1 \le r < s < i < j \le n$ or $1 \le r < i < j < s \le n$;

2) $A_{r,s}A_{r,j}A_{r,s}^{-1} = A_{s,j}^{-1}A_{r,j}A_{s,j}$ if $1 \le r < s < j \le n$;

3) $A_{r,s}A_{s,j}A_{r,s}^{-1} = A_{s,j}^{-1}A_{r,j}^{-1}A_{s,j}A_{r,j}A_{s,j}$ if $1 \le r < s < j \le n$;

4) $A_{i,j}^{-1}A_{s,j}A_{i,j} \rightleftharpoons A_{r,i}$ if $1 \le r < s < i < j \le n$;

5) $A_{j,k} \rightleftharpoons gA_{j,k}g^{-1}$ where $1 \le j < k \le n$ and g is an element of P_n.

Through this presentation we can easily find a connection between $\tilde{P}_n$ and the reduced version of P_n.

Definition 4.0.7. Let G be a group normally generated by elements $g_1, \ldots, g_p$. We denote by

$$RG := G/\langle [g_i, hg_ih^{-1}] | 1 \le i \le p,\ h \in G\rangle^N$$

the reduced version of G, which is the smallest quotient where each generator commutes with all its conjugates.

Remark 4.0.8. The homotopy pure n-braid group is isomorphic to the reduced version of P_n:

$$\tilde{P}_n \cong RP_n.$$

Proof. Through the presentation in Corollary 4.0.6, $\tilde{P}_n$ is isomorphic to the quotient of P_n by the relation $A_{j,k} \rightleftharpoons gA_{j,k}g^{-1}$ where $1 \leq j < k \leq n$ and g is an element of P_n. That is by definition the reduced version of P_n. □

This leads us to question if we can find a more symmetric presentation for $\widetilde{PB}_n(M)$ and PrB_n^h as we did in Corollary 4.0.6 for $\tilde{P}_n$.

In the surface context, we think that a surjective map

$$g_n : \widetilde{PB}_n(M) \to RPB_n(M),$$

could be found, where $RPB_n(M)$ denotes the reduced version of $PB_n(M)$, obtained by quotienting out further relations. In order to do this, it is necessary to show that the relation $[t_{i,j}, ht_{i,j}h^{-1}] = 1$, $h \in F(2g+n+i)$ is valid in $RPB_n(M)$, which is not immediate, since the t_{ij} are not among the generators of $PB_n(M)$ given in Proposition 2.7.2. Also, we don't know if this map could be an isomorphism. In fact, we don't know if the relation $[T_{i,j}, gT_{i,j}g^{-1}] = 1$, $g \in PB_n(M)$ could be a direct consequence of the relation $[t_{i,j}, ht_{i,j}h^{-1}] = 1$, $h \in F(2g+n+i)$, and we don't know if $[a_{i,r}, ga_{i,r}g^{-1}] = 1$ with $g \in PB_n(M)$ is true in $\widetilde{PB}_n(M)$, so we are not sure that the isomorphism could be find, together with a more symmetric presentation.

In the welded context, it seems that we can not hope to apply a further symmetrization of the presentation as we did in Corollary 4.0.6. The problem is still open, but we think that through the presentation in 3.9.13 one should be able to show that the reduced pure welded group and the homotopy pure welded group are not isomorphic.

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