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Classification of Lagrangian planes in Kummer-type Hyperkähler Manifolds

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Abstract

We generalise a result from [Bak15] on K3 type hyperkähler manifolds proving that a line in a Lagrangian plane on an hyperkähler manifold X of Kummer type has Beauville-Bogomolov-Fujiki square $-\frac{n+1}{2}$ and order 2 in the discriminant group of $H^2(X,\mathbb{Z})$. Viceversa, an extremal primitive ray of the Mori cone verifying these conditions is in fact the class of a line in some Lagrangian plane. In doing so, we show, on moduli spaces of Bridgeland stable objects on an abelian surface, that Lagrangian planes on the fiber of the Albanese map correspond to sublattices of the Mukai lattice verifying some numerical condition.

Sommario

Generalizziamo un risultato da [Bak15] sulle hyperkäler di tipo $K3$ provando che la retta di un qualsiasi piano Lagrangiano su una varietà hyperkähler X di tipo Kummer ha quadrato $-\frac{n+1}{2}$ rispetto alla forma di Beauville-Bogomolov-Fujiki e che ha ordine due nel gruppo dicriminante di $H^2(X,\mathbb{Z})$. Viceversa un raggio estremale primitivo del cono di Mori che verifica questa uguaglianza è in effetti classe di una retta in qualche piano Lagrangiano. Nel provarlo, mostriamo che, negli spazi di moduli di oggetti stabili secondo Bridgeland su una superficie abeliana, i piani Lagrangiani in una fibra della mappa di Albanese corrispondono a sottoreticoli del reticolo di Mukai che verifichino alcune condizioni numeriche.

Contents

Introduction

The study of hyperkähler manifolds arises quite naturally from two different approaches: from one side, they appear as one of the "building blocks" of Ricci-flat Kahler manifolds (together with complex tori and Calabi-Yau manifolds) and from the other one they provide the higher dimensional analogues of K3 surfaces (for which a more classical theory is well known).

The first motivation is closer to riemaniann geometry and has been linked to problems from physics, in particular from string theory. The second point of view is more algebraic: on K3 surfaces, Torelli type results are well known (similar to the classical Torelli result for abelian varieties) giving a correspondence between isomorphisms of surfaces and isomorphisims of the Hodge structure on the second degree cohomology. Such results do have weaker analogues on hyperkähler manifolds (provided by Verbitsky in [Ver09] and refined by Marman in [Mar11]) that allow to study the geometry via a lattice structure on the H^2 of the manifold, whose bilinear pairing is called Beauville-Bogomolov-Fujiki form.

Because of these algebraic results, hyperkäler manifolds provide a testing ground for many more general conjectures in algebraic geometry. Nonetheless, one very central problem in the field is that (probably because of the extreme rigidity of these objects) examples are quite hard to find. At the moment, the only ones known are classified in two families (presented by Beauville in [Bea83]) called $K3$-type and Kummer type, each one giving a deformation class in every even dimension , with the exception of two sporadic examples from O'Grady (discovered in [O'G97],[O'G00]) that only exist in dimension 6 and 10.

It is in this perspective, as a generalisation of $K3$ surfaces, that our result has to be interpreted. On a $K3$ surface an explicit description of the cone of effective curves (the Mori cone) grants that extremal rays are those of square -2 for the intersection form.
Hasset and Tschinkel in [HT09, Thesis 1.1] proposed that something similar could happen in higher dimension: they conjectured that the square (with respect to the lattice structure of H^2) of the class of a line in a Lagrangian

plane (a Lagrangian plane is a subvariety isomorphic to a projective space of half the dimension of the manifold) is constant for every such plane and invariant in the deformation family of the hyperkähler. Morever they suggested that this square could be the minimal one achievable by the class of an effective curve.

Since then, results in this direction have been provided by the same two authors in [HT10] for a Kummer 4-fold, and from Bakker in [Bak15] for K3-type in any dimension. More work has been done by Song in his PHD thesis [Son], where he conjectured a formula for the class of a line that has been proved in the K3-type case under the additional hypothesis of primitivity by Oberdieck in [Obe22].

We adapt the techniques from the article of Bakker to extend his results to the Kummer type case. The main achievement of this thesis is the following theorem: (here $\varepsilon_X = 1$ for M of $K3$ type, $\varepsilon_X = 0$ for the Kummer case)

Theorem 0.0.1 (see Cor. 4.5.4 and Thm. 4.5.5)
Let (M, h) be an holomorphic symplectic variety of K3 or Kummer-type and dimension 2n, with $\mathbb{P}$ a Lagrangian plane, and let $R \in H_2(M, \mathbb{Z})$ be the class of the line.
Then

$$(R, R) = -\frac{n + 1 + 2\varepsilon_X}{2} \text{ and } 2R \in H^2(M, \mathbb{Z}) \tag{1}$$

Moreover for R a primitive generator of an extremal ray of the Mori cone, R is the class of a line in a Lagrangian plane if and only if R verifies the condition (1).

This theorem is obtained, modulo some general argument of deformation for hyperkähler manifolds, mainly using techniques relating the birational geometry of moduli spaces of Bridgeland stable objects on a $K3$ or abelian surface to sublattices of the Mukai lattice, which can be thought as the Groethendieck group of the surface equipped with the bilinear form defined by the relative Euler characteristic.

This notion of stability has been introduced by Bridgeland in [Bri02] and [Bri03] as a categorification of some properties of the more classical Gieseker stability. The mathematical need they solve is that, differently from Gieseker's, Bridgeland stability is stable under Fourier-Mukai transform. This is one of the reasons it gives rise to a very well behaved theory of moduli spaces, retaining most of the results that are known for Gieseker stability, while introducing more geometry in the relation between the space Stab(X) of such stability conditions (which can be realised as a complex manifold) on a $K3$ surface X and birational models of the moduli space: in particular there is a wall and chamber decomposition of Stab(X), such that conditions in the same chamber have the same (semi)stable objects, and in the case of a $K3$ or abelian surface each chamber corresponds to a K-trivial birational model of the same hyperkäler manifold.

This connection with the Minimal Model Program for hyperkäler manifolds have been developed first by Yoshioka, with Minamide and Yanagida, in the case of abelian surfaces (for example in [Yos12; MYY11b]) and by Bayer and Macrì in [BM14] for $K3$ type manifolds. In particular they found that every birational model of a moduli space can be interpreted in terms of wall crossing of the stability condition. Moreover, to each wall $\mathcal{W}$ is associated a sublattice of the Mukai lattice $\mathcal{H}_{\mathcal{W}}$ and the geometry of the contraction is encoded in the numerical properties of $\mathcal{H}_{\mathcal{W}}$.

All this will be made more precise in the corresponding chapters.

This perspective is the main point of view of our work, that makes use of this interpretation in terms of wall crossing and associated sublattices to achieve a classification of Lagrangian planes on moduli spaces: (for the notations see Sections 3.2.1 and 4.2)

Theorem 0.0.2 (see Prop. 4.4.2)
Let $v \in \tilde{H}_{alg}(X,\mathbb{Z})$ primitive and $v^2 > 0$, and let σ be a generic stability condition w.r.t. v.
If $\mathbb{P} \subset \mathbf{Y}_\sigma(v)$ is an extremal Lagrangian plane, then there is a $\mathbb{P}$ type sublattice $\mathcal{H} \subset \tilde{H}_{alg}(X,\mathbb{Z})$.
Moreover, for a generic stability condition $\sigma_0 \in \mathcal{W}_{\mathcal{H}}$, if $s \in \mathcal{H}$ is a minimal square class realising the minimum $(s,v) = \frac{v^2}{2}$ and $P := [v = s + (v-t)]$, there is an open dense set U of M_P s.t $\mathbb{P}$ is one of the connected components of $U \cap \mathbf{Y}_\sigma(v)$

This theorem is the key point that allows us to do an almost direct computation for the class of a line of a Lagrangian plane in a moduli space in terms of Mukai homomorphism. Then the proof of the main theorem 0.0.1 only relies on general deformation arguments.

Structure

We will now give an outline of the structure of this thesis.

1. Chapter 1 contains preliminary material from the general theory of hyperkählers: we make more precise the assertions at the beginning of this introduction motivating the study of such manifolds both from a riemaniann and an algebraic point of view.

We review the construction of Beauville's examples and we present the Markman Hodge theoretic Torelli theorem, defining the BBF form and the period map.

We conclude recalling the cone theorem, a classical and central result in the study of birational geometry, with its specialisation to the case of hyperkäler manifolds: in particular we will recall that contractible lines stay in the closure of the movable cone.

2. Chapter 2 is all about the definition and presentation of the main properties of Bridgeland stability. Since the subject is based in the theory of derived categories, we will recall some terminology from category theory and present the construction of the derived category of an abelian category.

   Then we will get into the core of the chapter, where we will give the definition of a stability condition and will present the main ideas around Bridgeland stability, with some special attention to the case of the bounded derived category of coherent sheaves on a $K3$ or abelian surface.

3. In Chapter 3 we approach the study of moduli spaces.
   We will define the Mukai vector to fix some numerical invariant, and recall results on the existence and projectivity of coarse moduli spaces for the stack of families of Bridgeland (semi)stable objects.
   This part is the most geometrical and is designed to provide an overview of the tools that we use to prove our result.

4. Finally in the last Chapter 4 we start tackling the proof of our generalisation. A precise plan is explained at the beginning of the chapter.

   A general idea, is that we will use and adapt results of [BM14] to classify extremal Lagrangian planes on moduli spaces of Bridgeland stable objects using sublattices of the Mukai lattice respecting some numerical conditions (that we call $\mathbb{P}$-type). From this we will be able to calculate the class of a line using the Mukai homorphism.

   Then to extend the result to the entire deformation type we will repeat almost the same argument of [Bak15] and use more general results about density of periods of moduli spaces in small deformation of hyperkähler manifolds to conclude.

Chapter 1

Preliminaries on Hyperkähler manifolds

In this chapter we will review some material on hyperkähler manifolds.

1.1 Hodge decomposition and Kähler condition

A complex manifold X of dimension n can also be seen as a real manifold of dimension $2n$. Clearly multiplication by i in each chart glue together to a map $I : X \to X$ such that $I^2 = -\operatorname{id}_X$, whose differential define at the level of tangent spaces a $\mathbb{R}$ linear map $d_xI : T_xX \to T_xX$.
This can be used to define a Hodge structure on each tangent space: let $T_xX_{\mathbb{C}} := T_xX \otimes \mathbb{C}$, considering the complexified map $I_{\mathbb{C}} : T_xX_{\mathbb{C}} \to T_xX_{\mathbb{C}}$ gives a decomposition $T_xX_{\mathbb{C}} = T_xX^{1,0} \oplus T_xX^{0,1}$ where $T_xX^{1,0}$ and $T_xX^{0,1}$ are the eigenspaces of d_xI respectively of eigenvalue i and $-i$.
This induces a similar decomposition on the cotangent space that in turn gives a bidegree on the exterior algebra $\bigwedge T_xX_{\mathbb{C}}$.
Globalising the construction, we get sheaves $\mathcal{A}^{p,q}$ as $\mathcal{C}^\infty_{\mathbb{C}}$-modules whose fibres are differential forms of bidegree p, q (so on coordinate open sets, sections are linear combination of forms of degree p, q with coefficients in the complex valued $\mathcal{C}^\infty$ functions on X). Similarly, indicating $\mathcal{O}_X$ the sheaf of holomorphic functions, we define $\Omega^{p,q}$ the $\mathcal{O}_X$-module having the same fibres.
Now, if $\mathcal{A}^k = \oplus_{p+q=k}\mathcal{A}^{p,q}$ is the sheaf of complex valued k-forms, the usual differential is a map $d : \mathcal{A}^k \to \mathcal{A}^{k+1}$.
We can decompose it defining two new operators: let $\Pi^{p,q} : \mathcal{A}^{p+q} \to \mathcal{A}^{p,q}$ the natural projection, then we define $\partial := \Pi^{p+1,q} \circ d : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}$ and $\bar{\partial} := \Pi^{p,q+1} \circ d : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$.

It can be verified that $\bar{\partial}^2 = \partial^2 = 0$, so that it makes sense to define:

Definition 1.1.1
The chain complexes $(\mathcal{A}^{p,\cdot}, \bar{\partial})$ are called Dolbeault complexes.
The (p,q)-Dolbeault cohomology is the cohomology

$$H^{p,q}(X) := H^q(\mathcal{A}^{p,\cdot}(X), \bar{\partial})$$

the key importance of these objects is made evident by the following fact:

Proposition 1.1.2 ([Huy05] Cor 2.6.21)
The Dolbeault cohmology of X computes the cohomology of the sheaf Ω^p, i.e. $H^{p,q} \cong H^q(X, \Omega^p)$

Note that in particular $H^{p,0}(X) = H^0(X, \Omega^p) = \Omega^p(X)$ are the holomorphic p-forms: in fact the kernel of $\bar{\partial} : \mathcal{A}^{p,0}(X) \to \mathcal{A}^{p,1}(X)$ is exactly the subspace of holomorphic global p-forms.
What we would like to do now is to introduce a Riemaniann metric on X. This is naturally associated to a connection on the tangent bundle, the Levi-Civita connection ∇ (the unique torsion free connection compatible with the metric).
Clearly there are some compatibility conditions we could ask between the metric and the complex structure.
For example, ∇ would induce a connection also on the holomorphic tangent bundle T_X (which is the $\mathcal{O}_X$-module having as section on coordinate open sets the linear combination of complexified tangent vectors, taken with holomorphic coefficients: it is therefore a subsheaf of TX). Recalling the decomposition of $TX = TX^{0,1} \oplus TX^{1,0}$, since $T_X \cong TX^{0,1}$ as vector spaces. it would make sense to ask for the restriction to the $0,1$ part of the connection $\nabla^{0,1}$ to coincide with the $\bar{\partial}$ operator (extended from $\mathcal{A}$ to bundles in the obvious way: the bundle can be seen as $TX \otimes \mathcal{A}$ and the operator acts on the second term).
Similarly, we could ask for the complex structure I to be parallel with respect to ∇.
All these geometric conditions do not hold in general, since they are equivalent to the following request:

Definition 1.1.3
A Riemann metric g on X is Kähler if the form $\omega = g(I\cdot, \cdot)$ is closed: $d\omega = 0$.
The form ω is called a Kähler form.
A complex manifold admitting a Kähler metric is said to be Kähler.

It is a theorem that this condition is equivalent to the desired property for ∇ (see [Huy05] Appendix 4.A).

One of the main consequences for a complex manifold of being Kähler is the existence of the Hodge decomposition:

Proposition 1.1.4 ([Huy05] Cor. 3.2.12)
Let X be a compact Kähler manifold. Then there exists a decomposition

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

.
Moreover, with respect to complex conjugation on $H^*(X,\mathbb{C})$ one has $\overline{H^{p,q}} = H^{q,p}$ and by Serre duality $H^{p,q} \cong H^{n-p,n-q}$.
This decomposition do not depend on the choice of the Kähler form.

It is a common notation to write $H^{p,q}(X,\mathbb{R})$ (or $H^{p,q}_{\mathbb{R}}$ when there is no ambiguity) for $H^{p,q} \cap H^{p+q}(X,\mathbb{R})$ and similarly for $\mathbb{Z}$ instead of $\mathbb{R}$.
As the Hodge decomposition do not depend on the Kähler form, the following makes sense:

Definition 1.1.5
The Kähler cone of a Kähler manifold is the cone $\mathcal{K}_X \subset H^{1,1}(X,\mathbb{R})$ of Kähler classes.

1.2 Motivation

Hyperkähler manifolds can be approached in different ways, both from a more algebraic and a more riemannian geometry point of view.
To build an intuition, the latter one is easier to grasp, so we will start from there. Anyway, for the main part on the work in chapter 4 we will use the algebraic one.

Let (X, g) be a Riemannian manifold. Fixed a point $x \in X$, for any closed loop based in x we can parallel transport (with respect to the Levi Civita connection ∇) the tangent space, defining an automorphism $T_x(X) \to T_x(X)$. This defines a subgroup of $GL(T_x(X))$ that we call holonomy group and we indicate $\mathrm{Hol_x}(\nabla)$.
For a connected manifold this doesn't depend on the point x: taking $x, y \in X$ composing with parallel trasport along a path γ connecting x to y gives an isomorphism $\mathrm{Hol_x}(\nabla) \cong \mathrm{Hol_y}(\nabla)$ (note that the isomorphism strongly depends on γ so it is far from beeing natural).
Therefore, when we are only interested in the isomorphism class of the group, we can drop the subscript and talk about the holonomy group $\mathrm{Hol}(\nabla)$ of the manifold (X, g).
A big part of the geometry of the manifold is encoded in the holonomy group:

Theorem 1.2.1
Let (M, g) a complete simply connected Riemannian manifold. Then M is isometric to a product $M_0 \times M_1 \times ... \times M_k$ where M_0 is a euclidean space and M_i are irreducible. This decomposition is unique up to permutation of the factors. Let $p = (p_0, ..., p_k) \in M$ be a point and $\mathrm{Hol}(M_i)$ the holonomy group of M_i in p_i. Then the holonomy group of M in p is the product $\prod \mathrm{Hol}(M_i)$ acting on $T_pM = \prod T_{p_i}M_i$ by the product representation. If M is Kähler, all M_i are Khäler and the isometry is biholomorphic.

This naturally asks for the study of irreducible representation, aiming to a classification of manifolds based on their holonomy. In the case of Ricci-flat varieties (equivalently, of trivial first Chern class), Beauville refined the theorem: he showed that only two cases are possible for $\mathrm{Hol}(M_i)$, since each one must be either the special group of unitary complex transformations $SU(m) \subset SO(2m)$ or the special group of unitary quaternionic transformations $Sp(m) \subset SO(4m)$.

Theorem 1.2.2 ([Bea83] Thm. 1)
Let X be a compact Kähler Ricci-flat manifold. The universal cover is isomorphic (as a Kähler manifold) to a product $\mathbb{C}^k \times \prod V_i \times \prod X_j$ where $\mathbb{C}^k$ are equipped with the standard metric, V_i and X_j are simply connected Kähler manifolds of holonomy group respectively $SU(m)$ and $Sp(m)$. This decomposition is unique up to permutation.
Moreover there exists a finite étale cover of X isomorphic (as a Kähler manifold) to a product $T \times \prod V_i \times \prod X_j$, where T is a complex torus and V_i, X_j are as above.

The pieces of type X_j are those that we call hyperkähler:

Definition 1.2.3
Let (X, g) a compact Riemann manifold of dimension $4n$. We say that it is Hyperkähler if $\mathrm{Hol}(\nabla) \cong Sp(n)$

At this point it could seem that these manifolds have nothing to do with algebraic geometry, being more related to differential-riemannian properties. Nonetheless, an equivalent definition can be given in a much more algebraic flavour:

Definition 1.2.4 (Irreducible holomorphic symplectic)
A compact Kähler complex manifold is said to be irreducible holomorphic symplectic (IHS) if:

1. *it is symply connected*

2. *there exist a non degenerate holomorphic symplectic form σ s.t.*

$$H^{2,0}(X, \mathbb{C}) = \mathbb{C}\sigma$$

Actually these two definitions happen to be exactly the same. In fact

Proposition 1.2.5 ([Bea83] Prop 4)
Let X be a compact Kähler complex manifold of dimension $\dim_{\mathbb{C}} X = 2r$. The following conditions are equivalent:

- *X admits a Kähler metric with holonomy group a subgroup of $Sp(r)$.*
- *X admits a symplectic structure*

Moreover, the following conditions are equivalent:

- *X admits a Kähler metric with holonomy group $Sp(r)$*
- *X is IHS*

1.3 Known types of hyperkählers

While finding examples of hyperkähler of higher dimension is an active research problem, the case of surfaces is much easier and reduce to surfaces known as $K3$:

Definition 1.3.1
A $K3$ surface is a simply connected compact complex manifold of dimension 2 admitting a non-degenerate holomorphic symplectic form.

The following is not completely trivial, as a priori $K3$ surfaces are not required to be Kähler, but in dimension two this follows automatically

Proposition 1.3.2
Hyperkähler manifolds of dimension 2 are the same as $K3$ surfaces

Examples of $K3$ surfaces are fairly easy to provide: the Fermat quartic (the zero locus of $x_0^4 + x_1^4 + x_2^4 + x_3^4$ in $\mathbb{P}^3$) is a $K3$ surface.
A different but very classical example is the Kummer surface.
Consider an abelian surface T, let $i : T \to T$ be the involution $x \mapsto -x$ and take the quotient by the action of i over T. Blowing up the 16 singular points, one gets a smooth manifold which turns out to be a $K3$ surface.

Before citing the known examples, a first observation is that deformations of hyperkähler manifolds stay hyperkähler:

Proposition 1.3.3 ([Bea83] Prop 9 Rmk.1)
Let $X \to B$ a smooth proper deformation of an IHS manifold X_0 over an analytic base B with Kähler fibres . Then every deformation $X_s, s \in B$ is IHS.

Therefore the interest is in finding non-deformation equivalent examples. At the moment, there are very few known deformation types. Apart from two sporadic cases (one in dimension 10 and one in dimension 6, called respectively $OG10, OG6$: the O'Grady examples), there are only two known deformation types for each even dimension: the $K3[n]$ type and the $Kum[n]$ type. The next subsection will give an idea of the construction.

1.3.1 Beauville examples

In [Bea83], Beauville used these low dimensional cases to generate in each even dimension two families of hyperkähler, providing two examples of non-deformation equivalent hyperkähler manifolds.
We will just sketch the main steps.
First start from a compact complex surface S. Let $S^{(r)}$ be the r-th symmetric power of S (alias the quotient of S^r by the symmetric group $\mathfrak{S}_r$), and let $\pi : S^r \to S^{(r)}$ be the projection map. Let $\Delta \subset S^r$ be the closed subset of r-uples having at least one repeated element. Then $S^{(r)}$ is singular on $\pi(\Delta)$.
The Hilbert scheme (or Douady space) of r points on S, has a projection $\epsilon : S^{[r]} \to S^{(r)}$ which is a resolution of singularities.
Denote Δ_* the open subset of Δ where there is exactly one repetition (all elements are different except two). Let $S^r_* = (S^r \setminus \Delta) \cup \Delta_*$, $S^{(r)}_* = \pi(S^r_*)$ its image on $S^{(r)}$ and $S^{[r]}_* = \epsilon^{-1}(S^{(r)}_*)$.
It can be deduced that $S^{[r]}_*$ can be identified with the quotient of the blow up $\mathrm{Bl}_\Delta(S^r_*)$ of the diagonal $\Delta \subset S^r_*$ by the action of the symmetric group (which extends from S^r to the entire blow up).
We get a commutative diagram:

$$\begin{array}{ccc} \mathrm{Bl}_\Delta(S^r_*) & \longrightarrow & S^{[r]}_* \\ \big\downarrow & & \big\downarrow \\ S^r_* & \longrightarrow & S^{(r)}_* \end{array}$$

If S admits a non-degenerate symplectic form σ, clearly one can take the sum of its pullback by each projection $S^r \to S$ to get a symplectic form on

S^r invariant under the action of $\mathfrak{S}_r$. Pulling back on $\mathrm{Bl}_\Delta(S^r_*)$ gives a non degenerate symplectic form still invariant under the action of the symmetric group. Therefore, it comes from a form on $S^{[r]}_*$. It is completely non trivial, but this non-degenerate symplectic form obtained on an open set of $S^{[r]}$ extends to a global non-degenerate holomorphic symplectic form.
Also, it can be proved that $S^{[r]}$ has the same fundamental group of S ([Bea83] Lemme 1), and there are explicit formulas for the cohomology.
There are two kind of surfaces we can input in this construction: $K3$ and abelian ones.
Starting from a $K3$ surface, for each r we get directly a $2r$ complex dimensional hyperkähler manifold. This is not the case for abelian surfaces: the fundamental group is non-trivial.
To solve the problem, Beauville shows that we can kill the fundamental group restricting to a fibre of the Albanese map. Explicitly he considers the map $S^{(r)} \to S$ sending the r unordered points to their sum: then the composition $\mathrm{alb} : S^{[r]} \to S^{(r)} \to S$ is the Albanese map of the Hilbert scheme and a fibre $K_r := \mathrm{alb}^{-1}(0)$ is a $2r$-dimensional hyperkähler manifold. We call K_r the generalised Kummer manifold of S of dimension $2r$ (notice that for $r = 1$ the construction gives exactly the Kummer surface).

The explicit formulas on cohomology allow to check these two examples are not deformation equivalent (they have different Betti numbers).
The deformation class of the Hilbert scheme of r points on a $K3$ surface is called $K3[r]$, the one of K_r is called $Kum[r]$.
With the exception of the two O'Grady's examples, any other known hyperkähler has been proved to be in one of these classes.

1.4 Torelli theorem

1.4.1 Beauville-Bogomolov-Fujiki form

On $K3$ surfaces the second order cohomology is Poincaré-dual to itself, therefore the intersection form of $H^*(X, \mathbb{Z})$ induces a perfect pairing on $H^2(X, \mathbb{Z})$, making it a lattice. Moreover, there is the additional data of the Hodge decomposition.
A big part of the geometry of $K3$ surfaces is encoded in its lattice and Hodge theoretic properties:

Theorem 1.4.1 ([Huy16] Thm.5.3)
Two complex $K3$ surfaces X, X' are isomorphic if and only if there exists an Hodge isometry $\psi : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$.

Moreover, any Hodge isometry $\psi : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ *with* $\psi(\mathcal{K}_X) \cap \mathcal{K}_{X'} \neq \emptyset$ *is induced (via pullback) by a unique isomorphism* $X \xrightarrow{\sim} X'$

It's kind of surprising that this central role of the second degree cohomology is somehow preserved in higher dimensions.
But clearly the intersection form is not any more a bilinear form on H^2.
The right generalisation is due to Beauville, Bogomolov and Fujiki :

Definition 1.4.2
Let X *be an* IHS *manifold of dimension* $2n$ *and* σ *a generator for* $H^{2,0}(X,\mathbb{C})$. *For* $\alpha \in H^2(X,\mathbb{C})$ *let* $\alpha = \lambda\sigma + \beta + \mu\bar{\sigma}$ *be its Hodge decomposition.*
Let:

$$q_X(\alpha) := \lambda\mu + \frac{n}{2}\int_X \beta^2(\sigma\bar{\sigma})^{n-1}$$

This defines a quadratic form q_X *on* $H^2(X,\mathbb{C})$ *and therefore a bilinear form. They are called Beaville-Bogomolov-Fujiki form (BBF form) resp. pairing.*

From the definition, it is obvious that, with respect to this form, the decomposition $H^2(X,\mathbb{R}) = (H^{0,2} \oplus H^{2,0})_{\mathbb{R}} \oplus H^{1,1}_{\mathbb{R}}$ is an orthogonal decomposition.
Notice that if $n = 1$ (which means X is a $K3$ surface) we clearly recover the intersection form. Another direct consequence of the definition is that the form is positive definite on $(H^{0,2} \oplus H^{2,0})_{\mathbb{R}}$ and a direct calculation shows that for a Kähler form ω, we also get $q_X(\omega) > 0$.
The complete statement is:

Proposition 1.4.3 ([Ell+12], Cor. 23.11)
The BBF form q_X *on* $H^2(X,\mathbb{R})$ *has index* $(3, b_2 - 3)$ *(where* b_2 *is the second Betti number). If* $[\omega] \in H^{1,1}_{\mathbb{R}}$ *is a Kähler class,* q_X *is positive definite on* $(H^{0,2} \oplus H^{2,0})_{\mathbb{R}} \oplus \mathbb{R}[\omega]$ *and negative definite on the primitive (1,1)-part* $H^{1,1}(X)_\omega$

Moreover it can be normalised to induce a primitive integral quadratic form on the integral cohomology:

Proposition 1.4.4 ([Ell+12] Prop.23.14)
Let X *be an irreducible holomorphic symplectic manifold. Then there exists a positive constant* $c \in \mathbb{R}$ *such that* $q_X(\alpha)^n = c\int_X \alpha^{2n}$ *for all* $\alpha \in H^2_{\mathbb{R}}(X)$. *In particular,* q_X *can be renormalised such that* q_X *is a primitive integral quadratic form on* $H^2(X,\mathbb{Z})$.

1.4.2 Period map

Consider a deformation like in 1.3.3 of a hyperkähler manifold X on an analytic space. We can assume it to be the universal deformation $f : \mathcal{X} \to \mathcal{M}$ family known as Kuranishi family (it is a universal deformation in the sense that every germ of a proper smooth family on an analytic base is pullback of this one). A result of Bogomolov (see [Bea83] Sec. 8) showed that up to restricting $\mathcal{M}$ we can assume it to be smooth and connected and that there exists a diffeomorphism $u : X \times \mathcal{M} \to \mathcal{X}$. For $s \in \mathcal{M}$ we denote $\mathcal{X}_s = f^{-1}(s)$, we write σ_s for the symplectic form on $\mathcal{X}_s$ and $u_s : X \to \mathcal{X}_s$ for the restriction of u.
We can define the period map as

Definition 1.4.5
The local period map is the application:

$$\begin{aligned} p : \mathcal{M} &\to \mathbb{P}(H^2(X, \mathbb{C})) \\ s &\mapsto u_s^*(\sigma_s) \end{aligned}$$

Theorem 1.4.6
Consider the open subset on a quadric:

$$\Omega := \{[\alpha] \in \mathbb{P}(H^2(X, \mathbb{C})), q_X(\alpha) = 0, q_X(\alpha + \bar{\alpha}) > 0\}$$

The image of p is contained in Ω and $p : \mathcal{M} \to \Omega$ is a local isomorphism.

Definition 1.4.7
The open set Ω on a quadric of the previous proposition is called period domain

In order to state a global version of this result, the first thing we notice is that there isn't any special fibre over zero. We need to introduce a somehow external reference, and this leads to the definition of a marking:

Definition 1.4.8
([Bea83] Thm. 5) A marking on a hyperkähler manifold X is the datum of an even lattice Λ of signature $(3, b_2 - 3)$ and an isometry $\eta : H^2(X, \mathbb{Z}) \to \Lambda$ where the bilinear form on $H^2(X, \mathbb{Z})$ is the BBF pairing.
A couple (X, η) of a hyperkähler manifold and a marking is called a marked hyperkähler.

In the local case, for each fiber $\mathcal{X}_s$ the marking is the monodromy operator induced by parallel transport $H^2(\mathcal{X}_s, \mathbb{Z}) \to H^2(X, \mathbb{Z})$.

To substitute $\mathcal{M}$ we introduce the coarse moduli space $\mathfrak{M}_\Lambda$ of marked hyperkähler (on the fixed lattice Λ). Unfortunately it is not a manifold: in particular it is not Hausdorff.
Then we can give a global definition:

Definition 1.4.9
The global period map is the application:

$$P : \mathfrak{M}_\Lambda \to \mathbb{P}(\Lambda \otimes \mathbb{C})$$
$$(X, \eta) \mapsto [\eta(H^{2,0}(X))]$$

and similarly define $\Omega_\Lambda := \{[a] \in \mathbb{P}(\Lambda \otimes \mathbb{C}), (a, a) = 0, (a, \bar{a}) > 0\}$.
To control non-separable points the following definition will be useful:

Definition 1.4.10
The positive cone $\mathcal{C}_X$ of a hyperkähler manifold X is the connected component of the cone:

$$\left\{\alpha \in H^{1,1}(X, \mathbb{R}), (\alpha, \alpha) > 0\right\}$$

containing the Kähler cone $\mathcal{K}_X$.

The global result due to Verbitsky (see [Ver09]) is:

Theorem 1.4.11 ([Mar11] Thm.2.2)
Fix a connected component $\mathfrak{M}^0_\Lambda \subset \mathfrak{M}_\Lambda$

1. *The period map P restricts to a surjective holomorphic map $P_0 : \mathfrak{M}^0_\Lambda \to \Omega_\Lambda$*
2. *The fiber $P_0^{-1}(p)$ consists of pairwise inseparable points for all $p \in \Omega_\Lambda$*
3. *Let (X_1, η_1) and (X_2, η_2) be two inseparable points of $\mathfrak{M}^0_\Lambda$, then X_1, X_2 are birationally equivalent*
4. *The marked pair $(X, \eta) \in \mathfrak{M}_\Lambda$ is a Hausdorff point if and only if $\mathcal{C}_X = \mathcal{K}_X$*
5. *The fiber $P_0^{-1}(p), p \in \Omega_\Lambda$, consists of a single Hausdorff point if $\Lambda^{1,1}(p)$ is trivial or if $\Lambda^{1,1}(p)$ is of rank 1, generated by a class α satisfying $(\alpha, \alpha) > 0$*

The last two conditions suggest that fixed a period (which means, fixed the Hodge structure on the H^2) the ambiguity on the birational model of X is encoded in a choice of the Kähler cone inside the positive cone. In fact, while $\mathcal{K}_X$ depends on X, the positive cone is defined lattice theoretically

(except for the connected component to be taken, but in [Mar11] Sec. 4 it is shown that this is determined by the connected component $\mathfrak{M}^0_\Lambda$) and is fixed on Λ once the Hodge structure has been chosen.
This is exactly the refinement obtained by Markman in its Hodge theoretic version of global Torelli.
For $(X, \eta) \in \mathfrak{M}^0_\Lambda$ let $\mathrm{Mon}^2_{Hdg}(X)$ be the group of monodromy operators (Hodge isometries induced on $H^2(X; \mathbb{Z})$ by parallel transport operators on X).

Definition 1.4.12
A Kähler-type chamber of $\mathcal{C}_X$ is a subset of the form $g[f^*\mathcal{K}_X]$ for $f : X \dashrightarrow Y$ a bimeromorphic map to an IHS manifold Y and $g \in \mathrm{Mon}^2_{Hdg}(X)$.

Fixing a period $p \in \mathbb{P}(\Lambda \otimes \mathbb{C})$ we can define the positive cone $\mathcal{C}_p \subset \Lambda$ and the Kähler-type chambers of $\mathcal{C}_p$ are defined as the subsets of the form $\eta(Ch)$, with (X, η) a marked pair and Ch a Kähler-type chamber on X. Denote $\mathcal{KT}(p)$ the set of Kähler-type chambers.
For any pair $(X, \eta) \in P_0^{-1}(p)$ we can inject $\mathrm{Mon}^2_{Hdg}(X) \hookrightarrow O(\Lambda)$ via $g \mapsto \eta g \eta^{-1}$. Denote $\mathrm{Mon}^2_{Hdg}(p)$ the image of this injection (which do not depend on the choice of the pair in the fiber). This group acts on elements of $\mathcal{KT}(p)$ permuting chambers.
Defining the action of $\mathrm{Mon}^2_{Hdg}(p)$ on the fiber $P_0^{-1}(p)$ as $g(\tilde{X}, \tilde{\eta}) = (\tilde{X}, g\tilde{\eta})$, he gets :

Theorem 1.4.13 ([Mar11] Thm. 5.16)
The map

$$\rho : P_0^{-1}(p) \to \mathcal{KT}(p)$$

given by $\rho(X, \eta) = \eta(\mathcal{K}_X)$ is a $\mathrm{Mon}^2_{Hdg}(p)$-equivariant bijection

Two main problems still persist in achieving a complete understanding of the moduli space of marked hyperkählers: a description of $\mathcal{KT}(p)$ only depending on p is still not known, and also the relation between different connected components is not clear.

1.5 Cone theorem

In the last paragraphs we have dealt with the birational models which stay hyperkäler.
Clearly this is not enough to run the Minimal Model Program, since contractions can introduce singularities. One of the central tools in MMP is the Cone theorem, which characterises classes of effective curves that can be

contracted (in the category of complex spaces). In the case of hyperkähler manifols, the classical result can be stated in the following way:

Theorem 1.5.1 ([HT07] Prop. 11)
Let X be a smooth projective variety with $K_X = 0$ and Δ an effective $\mathbb{Q}$-divisor on X. Then the closed cone of effective curves $\overline{\mathrm{NE}}_1(X)$ can be expressed:

$$\overline{\mathrm{NE}}_1(X) = \overline{\mathrm{NE}}_1(X)_{\Delta.C\geq 0} + \sum_j \mathbb{R}_{\geq 0}[C_j], \Delta.C_j < 0$$

where the C_j are extremal and represent rational curves collapsed by contractions of X. This is locally finite in the following sense: Given an ample divisor A and $\varepsilon > 0$, there are a finite number of C_j with $C_j.(\Delta + \varepsilon A) < 0$.

Remark. A remarkable difference between this and the classical formulation is that nothing is required on the singularities of Δ. This is due to the fact that (as explained in [HT07] in the remark following the theorem), it's always possible to find $\varepsilon > 0$ such that $(Y, \varepsilon\Delta)$ has the right type of singularities (Kawamata log terminal).

An interesting consequence of the fact that X is hyperkähler is that the divisors associated to contractions of these extremal rays C_j are contained in the closure of the movable cone.
This can be summarised as follows.
Take $C \in \mathrm{NE}_X$ having negative BBF square :$(C_j, C_j) < 0$. The orthogonal must contain some $D \in \mathcal{C}_X$ for signature reasons. Now take A an ample divisor on X.
Clearly we have

$$(D - \varepsilon A).C = -\varepsilon A.C < 0$$

Fix $\Delta_\varepsilon := D - \varepsilon A$ and apply the cone theorem 1.5.1 to get a contraction $\pi : X \to \bar{X}$. Assume that D has been chosen general enough such that C was the only class in the orthogonal. Because $\pi_*\Delta_\varepsilon$ is now nef and big, it is movable so the codimension of $|\Delta_\varepsilon|$ is at least 2.
Results from Kaledin (see [Kal06]) assure that the codimension of the singular locus is at least 4. Pulling it back on X, the base locus will have at least codim 2 so it is movable. Therefore since $\Delta_\varepsilon \xrightarrow[\varepsilon\to 0]{} D$ the divisor D must be in the closure of the movable cone.

Chapter 2

Bridgeland stability

Bridgeland stability conditions have been introduced in [Bri02] in an attempt to study Gieseker stability of sheaves.

The main idea is to give some kind of categorical version of the properties used in constructing moduli spaces for more classical stability. In this perspective, it is natural to try to recover as much as possible of the classical picture, in particular a wall and chamber decomposition of the space of stability conditions. This attempt actually results in a more formal but better behaved theory of moduli spaces, where the price to pay is shifted in the objects parametrised: a triangulated category is needed so instead of the usual category of coherent sheaves, the attention is brought to its bounded derived category.
Moreover Bridgeland stability makes sense for any triangulated category, so it can be used to look for new invariant of autoequivalences of triangulated categories and it is stable under Fourier-Mukai transform.
However, we will just provide definitions for the general case, while the main focus will be on the more concrete case of complexes of sheaves.

2.1 The large volume limit

This section serves more as a motivation to the study of Bridgeland stability. The facts reported here will not be used in the rest of the thesis and will be presented relying on definitions and notations that will be introduced in the rest of the chapter.

Fix X to be a $K3$ surface.
One of the problem of Gieseker stability is that it is not preserved by twisting with a line bundle (while it is the case for slope stability). A slight

generalisation of Gieseker stability that can take twists into account is the following:

Definition 2.1.1
Let $\beta, \omega \in \mathrm{NS}(X) \otimes \mathbb{R}$ *a pair of* $\mathbb{R}$*-divisors with* $\omega \in \mathrm{Amp}(X)$ *ample. Let* E *be a torsion-free sheaf on* X *with Mukai vector* $v(E) = (r(E), c_1(E), s(E))$ *and define*

$$\mu_{\beta,\omega}(E) := \frac{(c_1(E) - r(E)\beta) \cdot \omega}{r(E)} \qquad \text{and} \qquad \nu_{\beta,\omega}\frac{s(E) - c_1(E) \cdot \beta}{r(E)}$$

A torsion-free sheaf E *on* X *is said to be twisted semistable with respect to the pair* (β, ω) *if*

$$\mu_{\beta,\omega}(A) < \mu_{\beta,\omega}(E)$$

or

$$\mu_{\beta,\omega}(A) = \mu_{\beta,\omega}(E) \text{ and } \nu_{\beta,\omega}(A) \leq \nu_{\beta,\omega}(E)$$

for all subsheaves $0 \neq A \subset E$

Remark. Notice that for $\beta = 0$ this reduces to Gieseker stability

The relation with Bridgeland stability appears as follows: for every pair β, ω as before we can actually define a stability condition that has central charge dual to $\exp(\beta + i\omega)$ similar to those that we will present in 2.5.2. Taking large enough multiples of ω, the objects that stays semistable are the shifts of twisted semistable sheaves. In this sense we can describe twisted (and therefore Gieseker) stability as limits of Bridgeland stability for large volume, meaning for large enough multiples of the ample divisor. The precise statement is:

Theorem 2.1.2 ([Bri03] Prop. 14.2)
Fix a pair $\beta\omega \in \mathrm{NS}(X) \otimes \mathbb{Q}$ *with* $\omega \in \mathrm{Amp}(X)$ *ample.*
For integers $n >> 0$ *there is a unique stability condition* $\sigma_n \in U(X)$ *having central charge* $Z_n(-) := (\exp(\beta + in\omega), -)$.
Suppose $E \in D^b(X)$ *satisfies*

$$r(E) > 0 \text{ and } (c_1(E) - r(E)\beta) \cdot \omega > 0$$

Then E *is* σ_n*-semistable for* $n >> 0$ *precisely if* E *is a shift of a* (β, ω)*-twisted semistable sheaf on* X

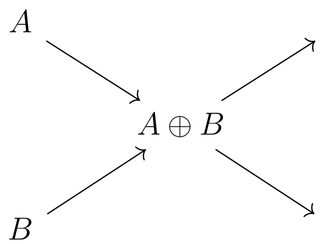
2.2 Some category theory language

A big part of the content of this chapter is purely categorical. Even if we are interested in the specific case of coherent sheaves on a complex variety, there is really no advantage in stating nor proving these results in the concrete context and it could actually make arguments less transparent.
For this reason, we introduce some of the needed language from the more abstract point of view.
For the reader not familiar with the basic language of category theory a reference could be [Lan13].

The categories we will be interested in usually exhibit additional structure.

Definition 2.2.1
Let R be a ring. An R-category is a category $\mathcal{A}$ such that for each couple $A, B \in \operatorname{Ob}\mathcal{A}$, the set $\operatorname{Hom}_{\mathcal{A}}(A, B)$ has an R-module structure such that

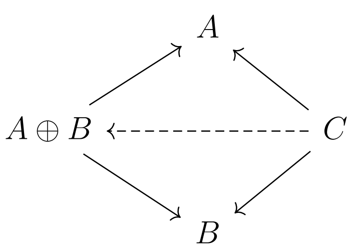
1. *The composition $\operatorname{Hom}_{\mathcal{A}}(A, B) \times \operatorname{Hom}_{\mathcal{A}}(B, C) \to \operatorname{Hom}_{\mathcal{A}}(A, C)$ is R-bilinear*
2. *There exists an object $0 \in \mathcal{A}$ such that $\operatorname{Hom}(0, 0)$ is trivial*
3. *For any couple $A, B \in \operatorname{Ob}(\mathcal{A})$ there exists an object $A \oplus B \in \operatorname{Ob}(\mathcal{A})$ and morphisms*



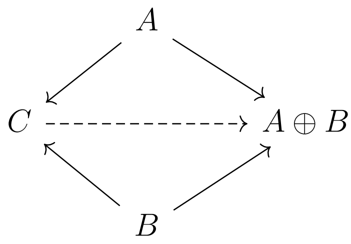
making $A \oplus B$ both the direct sum and the direct product of A, B

A $\mathbb{Z}$-category is called an additive category

We recall that direct product means that for all diagrams of the following form there exists a unique $C \dashrightarrow A \oplus B$

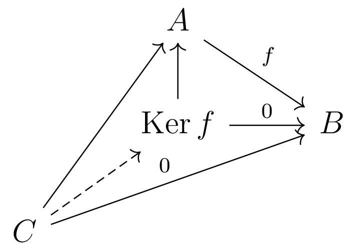


and similarly direct sum means that all diagrams of the following form have a unique $A \oplus B \dashrightarrow C$ fitting

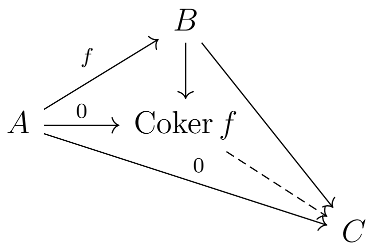


An additive category typically is not enough, in particular doesn't allow to define (co)-homology. In analogy to categories arising from algebraic structures, one is interested to restore the concept of Ker and Coker. The two notions can be categorified via universal properties.

Definition 2.2.2
For a morphism $f : A \to B$ *the kernel is the morphism* $\operatorname{Ker} f : \operatorname{Ker} f \to A$ *(abusing notation we indicate both the morphism and the object with the same symbol) such that commutative diagram of this form exist and can always be completed with a unique* $\dashrightarrow$



Dually the cokernel $\operatorname{Coker} f : B \to \operatorname{Coker} f$ *is the morphism such that commutative diagram of this form exist and can always be completed with a unique* $\dashrightarrow$



Then one can define image and coimage as

Definition 2.2.3
For a morphism $f : A \to B$, its image is defined

$$\operatorname{Im} f := \operatorname{Ker} \operatorname{Coker} f$$

and dually

$$\operatorname{Coimg} f := \operatorname{Coker} \operatorname{Ker} f$$

The nice categories on which this kind of operations are always allowed are the abelian ones

Definition 2.2.4
An abelian category is an additive category such that it admits kernel and cokernel for any morphism f and the natural map $\operatorname{Coimg} f \to \operatorname{Im} f$ is an isomorphism

The main reason to work in an abelian category is that here exact sequences make sense:

Definition 2.2.5
A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if $\operatorname{Ker} g = \operatorname{Im} f$

Unfortunately, asking for a category to be abelian is a very strong property, and we will see that introducing the derived category we will be forced to leave the abelian realm. So we will need some notion to get around exact sequences when they are not available, introducing some different structure. An adequate substitute in our case will be provided by distinguished triangles:

Definition 2.2.6
A triangulated category is an additive category $\mathcal{A}$ together with an additive equivalence

$$T : \mathcal{A} \to \mathcal{A}$$

called the shift functor, and a set of distinguished triangles

$$A \to B \to C \to T(A)$$

subject to the axioms TR1-TR4.
We denote $A[1] = T(A)$.

TR1 1. *Any triangle of the form*

$$A \xrightarrow{id_A} A \longrightarrow 0 \longrightarrow A[1]$$

is distinguished.

2. *Any triangle isomorphic to a distinguished triangle is distinguished*

3. *any morphism* $f : A \to B$ *can be completed to a distinguished triangle*

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1]$$

TR2 The triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is distinguished if and only if

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is distinguished

TR3 Suppose there is a commutative diagram of distinguished triangles

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A[1] \end{array}$$

then it can be completed by a (not necessarily unique) $h : C \to C'$

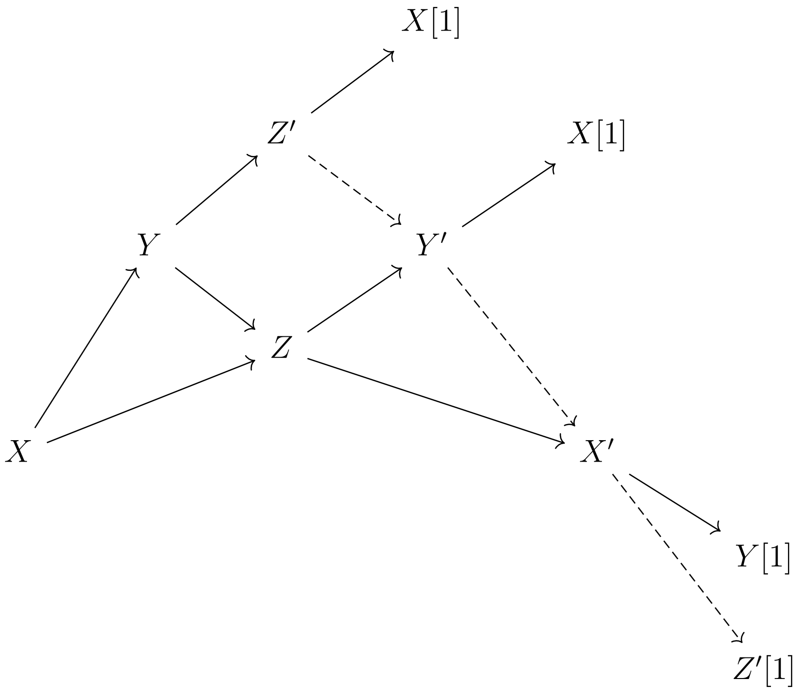
TR4 Given distinguished triangles

$$A \longrightarrow B \longrightarrow C' \longrightarrow A[1]$$

$$B \longrightarrow C \longrightarrow A' \longrightarrow B[1]$$

$$A \longrightarrow C \longrightarrow B' \longrightarrow A[1]$$

fitting in a commutative diagram



There exist a distinguished triangle

$$C' \dashrightarrow B' \dashrightarrow A' \dashrightarrow C[1]$$

completing the commutative diagram

Notice that for a category to be triangulated (meaning, to admit a triangulated structure) is not a weaker (nor stronger) condition than to be abelian. They are two different request and distinguished triangles are not in general an analogue to exact sequences: it is just in the context of the homotopy or derived category (which will be defined in the next section) that distinguished triangles can be chosen in such a way that they in some sense carry the same information. It is also important to stress that while abelianity is an intrinsic property of the category, the choices of the shift functor and the family of distinguished triangles are not canonical.

2.3 The derived category

References for this section can be found in [Huy06; Ver96]

This section is actually purely categorical, so it really makes no difference to present the general case of an abelian category $\mathcal{A}$.
Anyway to have a more concrete example, fix a complex variety X and let $\mathcal{A}$ be the category of coherent sheaves on X.
In this example, it is usually useful for studying homological properties in $\mathcal{A}$ to consider resolutions via locally free sheaves. Therefore it seems natural to introduce a new category having as objects such complexes. Anyway it is technically easier to allow any object of $\mathcal{A}$ to appear in the complex.
This leads to the definition:

Definition 2.3.1
The category of complexes $\mathrm{Kom}(\mathcal{A})$ *on* $\mathcal{A}$ *is the category having as objects diagrams in* $\mathcal{A}$ *of the form*

$$... \to A^{i-1} \xrightarrow{d_A^{i-1}} A^i \xrightarrow{d_A^i} A^{i+1} \to ...$$

where $A^i \in \mathrm{Ob}(\mathcal{A})$ *and* $d_A^i \circ d_A^{i-1} = 0$*. Such diagrams are called complexes and* A^i *are called the degree* i *component of* A*.*
Morphisms between complexes A, B *of* $\mathrm{Kom}(\mathcal{A})$ *are commutative diagrams of the form*

$$\begin{array}{ccccccccc} ... & \longrightarrow & A^{i-1} & \xrightarrow{d_A^{i-1}} & A^i & \xrightarrow{d_A^i} & A^{i+1} & \longrightarrow & ... \\ & & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & & \\ ... & \longrightarrow & B^{i-1} & \xrightarrow{d_B^{i-1}} & B^i & \xrightarrow{d_B^i} & B^{i+1} & \longrightarrow & ... \end{array}$$

This category is clearly abelian as $\mathcal{A}$ was.
Plus, it has a very remarkable functor, the shift functor:

Definition 2.3.2
The shift functor is the functor $[1] : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{A})$ *defined:*

- *On objects,* $A \in \mathrm{Ob}(\mathcal{A})$ *is mapped to the complex* $A[1]$ *such that*
  - $A[1]^i = A^{i+1}$
  - $d_A^i = d_A^{i+1}$
- *On morphisms,* $f \in \mathrm{Hom}(A, B)$ *is mapped to* $f[1]$ *such that* $f[1]_i = f_{i+1}$

Proposition 2.3.3 ([Huy06] Cor 2.5)
The shift functor is an equivalence of categories and its inverse is denoted $[-1]$*.*
The notation $[\pm n]$ *is used to indicate the composition* $[\pm 1]^n$ *for* $n \in \mathbb{N}$*.*

Still the shift doesn't make $\mathrm{Kom}(\mathcal{A})$ a triangulated category: the obvious choice of short exact sequence in $\mathcal{A}$ to define distinguished triangles doesn't respect the needed axioms.
For example, if we try to define distinguished triangle as those diagram $A \to C \to B \to A[1]$ for A, B, C fitting in a short exact sequence $0 \to A \to C \to B \to 0$ clearly it doesn't respect the axiom TR2.
Because we are mainly interested in cohomological property of the complex, we actually can condense some information: first of all homotopic equivalent complexes have the same homology, and moreover in $\mathrm{Kom}(\mathcal{A})$ there are morphisms that are not isomorphism and that induce isomorphisms in cohomology, called quasi-isomorphisms.
Fortunately this is possible in a universal way:

Theorem 2.3.4 ([Huy06] Thm 2.10)
There exists a category $\mathcal{D}(\mathcal{A})$ and a functor $Q : \mathrm{Kom}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ such that

1. *If $f : A \to B$ is a quasi-isomorphism in $\mathrm{Kom}(\mathcal{A})$, then $Q(f)$ is an isomorphism in $\mathcal{D}(\mathcal{A})$.*

2. *Any functor $\mathrm{Kom}(\mathcal{A}) \to \mathcal{F}$ satisfying the previous property factorise uniquely via Q*

Definition 2.3.5
The category $\mathcal{D}(\mathcal{A})$ of the previous theorem is called the derived category of $\mathcal{A}$

Remark. Clearly we have a functor $\mathcal{A} \to \mathrm{Kom}(\mathcal{A})$ sending an object to the corresponding complex concentrated in degree 0. This identification allows to think object of $\mathcal{A}$ as object of $\mathcal{D}(\mathcal{A})$ composing with the functor $\mathrm{Kom}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$.

This definition is fairly abstract and in this form can be seen as a special case of a more general construction of localisation of morphisms (see [Ver96, Ch. 2]). However, a direct proof of the existence gives also a description of the objects and morphisms of the category. The rest of the section will give a sketch of the main steps to prove the theorem. Details are in [Huy06]
It is useful to introduce an intermediate category:

Definition 2.3.6
The homotopic category of $\mathcal{A}$ is the category $\mathcal{K}(\mathcal{A})$ having

1. *as objects the same objects as $\mathrm{Kom}(\mathcal{A})$*

2. *as morphisms equivalence classes of homotopy equivalent morphisms*

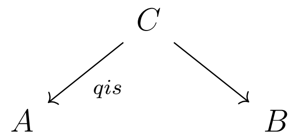
We recall what homotopic equivalent means from homological algebra:

Definition 2.3.7
Two morphisms $f, g \in \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A, B)$ are homotopy equivalent if there is a collection $h = (h^i)_{i\in\mathbb{N}}$ of morphisms $h^i : A^i \to B^{i+1}$ such that:

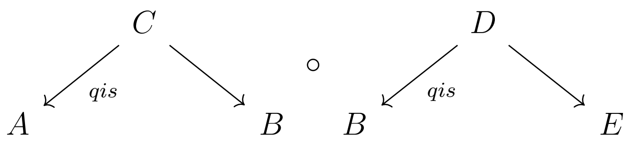
$$f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i$$

The collection h is called an homotopy.

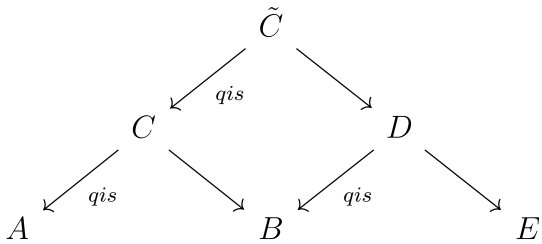
With the introduction of the homotopic category, the only morphism that we want to get rid of are the quasi-isomorphisms.
In $\mathcal{D}(\mathcal{A})$ they must became isomorphism, so for every diagram in $\mathcal{K}(\mathcal{A})$ of the form:



(where *qis* stands for quasi isomorphism) there must be a morphism $A \to B$ in $\mathcal{D}(\mathcal{A})$.
Therefore we take such diagrams as the definition of morphisms in $\mathcal{D}(\mathcal{A})$.
To define composition is not completely obvious: it would be natural to pose



as the diagram resulting as



But the existence of $\tilde{C}$ is not trivial. To solve this we introduce the cone of a morphism:

Definition 2.3.8
Let $f : A \to B$ be a morphism in Kom($\mathcal{A}$)*. Its mapping cone is the complex $C(f)$ defined as*

$$C(f)^i = A^{i+1} \oplus B^i \text{ and } d^i_{C(f)} = \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix}$$

Clearly the natural injection $B^i \to A^{i+1} \oplus B^i$ and projection $A^{i+1} \oplus B^i \to A^{i+1} = A[1]^i$ induce two natural morphisms $B \to C(f)$ and $C(f) \to A[1]$. This is in some sense the best avatar of short exact sequences that we have in this more general context:

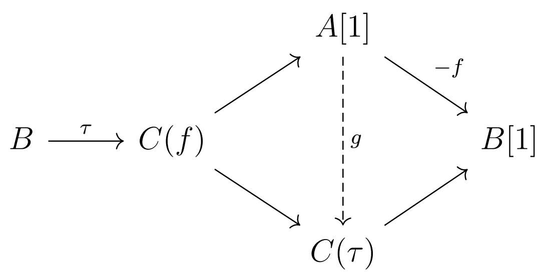
Proposition 2.3.9 ([Huy06] Ex. 2.27)
For every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ there is an isomorphism $C \to C(f)$ in $\mathcal{D}(\mathcal{A})$

Since as we said short exact sequences were not a suitable choice to define distinguished triangles, the next candidates are of the kind

$$A \to B \to C(f) \to A[1]$$

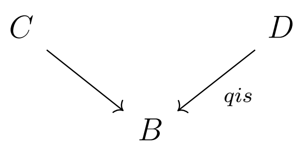
.
In this case the TR2 axiom holds in $\mathcal{K}(\mathcal{A})$

Proposition 2.3.10 ([Huy06] Prop. 2.16)
Let $f \in \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A, B)$ and $\tau : B \to C(f)$ the natural morphism, then there exists an isomorphism $g \in \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A[1], C(\tau))$ such that the following diagram commutes:

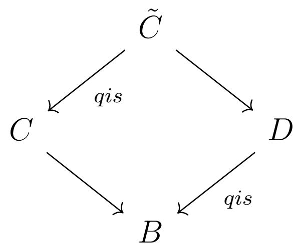


This is also the key ingredient of the following proposition, solving the problem of the composition of morphisms in $\mathcal{D}(\mathcal{A})$:

Proposition 2.3.11 ([Huy06] Prop 2.17)
Every diagram in $\mathcal{K}(\mathcal{A})$ of the form



can be completed in a commutative diagram



Finally, it is not trivial since all the other axiom must be tested, but it can be verified that

Proposition 2.3.12 ([Huy06] Prop 2.24)
The categories $\mathcal{K}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ equipped with the shift functor and with the family of distinguished triangles given by triangles isomorphic to some sequence $A \xrightarrow{f} B \to C(f) \to A[1]$, is a triangulated category

Notice that a direct consequence of 2.3.9 is

Corollary 2.3.13
Every short exact sequence induces a distinguished triangle.
More precisely, for any exact sequence $0 \to A \to B \to C \to 0$ there is a map $C \to A[1]$ making $A \to B \to C \to A[1]$ a distinguished triangle.

Some notable subcategories of $\mathrm{Kom}(\mathcal{A}), \mathcal{K}(\mathcal{A}), \mathcal{D}(\mathcal{A})$ are those composed by complexes having many zero components. It is possible to re-run the entire construction starting from the subcategory in $\mathrm{Kom}(\mathcal{A})$ to recover the corresponding ones in the other two.
Anyway, this provides essentially the same objects that one would get restricting $\mathcal{D}(\mathcal{A})$ to the full subcategory of object for which the corresponding cohomology groups vanish :

Definition 2.3.14
Let $* = +$(resp. $-$, resp b and let $\mathrm{Kom}(\mathcal{A})^*$ be the full subcategory of $\mathrm{Kom}(\mathcal{A})$ of complexes having $A^i = 0$ for $i << 0$ (resp. $i >> 0$ resp. $|i| >> 0$).
Let $\mathcal{D}(\mathcal{A})^+$ (resp. $-, b$) be the corresponding derived category.

Proposition 2.3.15 ([Huy06] Prop 2.30)
The natural functors $\mathcal{D}(\mathcal{A})^+ \to \mathcal{D}(\mathcal{A})$ (resp. $\mathcal{D}(\mathcal{A})^-, \mathcal{D}(\mathcal{A})^b$) define equivalences with the full triangulated subcategory of $\mathcal{D}(\mathcal{A})$ with $H^i(A) = 0$ for $i << 0$ (resp. $i >> 0$, $|i| >> 0$)

2.3.1 Grothendieck group

The classical definition of Grothendieck group can be generalised to abelian categories:

Definition 2.3.16
Let $\mathcal{A}$ be an abelian category. Its Grothendieck group $K(\mathcal{A})$ is the quotient of the free group on objects of $\mathcal{A}$ quotiented by the relations $A - B + C = 0$ for $A, B, C \in \operatorname{Ob}\mathcal{A}$ fitting in an exact sequence $0 \to A \to B \to C \to 0$

If the abelianity fails, but the triangulated structure is provided, we can define a similar notion :

Definition 2.3.17
Let $\mathcal{D}$ be a triangulated category. Its Grothendieck group $K(\mathcal{D})$ is the quotient of the free group on objects of $\mathcal{D}$ quotiented by the relations $A - B + C = 0$ for $A, B, C \in \operatorname{Ob}\mathcal{D}$ fitting in a distinguished triangle $A \to B \to C \to A[1]$

We said that the derived category of an abelian category is not abelian, but it is triangulated.
Fortunately, the following proposition shows the information in the Grothendieck group of the derived category is essentially the same:

Proposition 2.3.18 ([Vir] Prop. 0.3)
Let $\mathcal{A}$ be an abelian category and $\mathcal{D}(\mathcal{A})$ it's derived category.
Then

$$K(\mathcal{A}) \cong K(\mathcal{D}(\mathcal{A}))$$

Proof. The natural map on objects $\mathcal{A} \to \mathcal{D}(\mathcal{A})$ induces a map on Grothendieck groups by 2.3.13 that we will call β.
We need to provide an inverse.
Define

$$\alpha : K(\mathcal{D}(\mathcal{A})) \to K(\mathcal{A})$$
$$[A] \mapsto \sum_i (-1)^i [H^i(A)]$$

This is a well defined group homomorphism because given a triangle $A \to B \to C \to A[1]$ the long exact sequence induced in homology gives that $\alpha(B) = \alpha(A) + \alpha(C)$.
To prove this is an inverse for α, the composition $\alpha \circ \beta$ is easy to check (the complex associated to an object A has only one term, so cohomology is A in degree 0, 0 otherwise). For the composition $\beta \circ \alpha$ we need some more work.
Let $A \in \mathcal{D}(\mathcal{A})$ and A^n be the non vanishing term of highest degree.

Then we can obtain a complex isomorphic to A by writing the commutative diagram:

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & A^{n-2} & \longrightarrow & A^{n-1} & \xrightarrow{\phi} & A^n & \longrightarrow & 0 \\ & & = & & = & & \uparrow{\scriptstyle \operatorname{Im}\phi} & & \\ \ldots & \longrightarrow & A^{n-2} & \longrightarrow & A^{n-1} & \longrightarrow & \operatorname{Im}\phi & \longrightarrow & 0 \end{array}$$

This is a morphism in $\mathcal{D}(\mathcal{A})$ and its cone is isomorphic to $H^n(A)[-n]$ (in $\mathcal{D}(\mathcal{A})$) which has the same class in the Grothendieck group as $H^i(A)$. We iterate the procedure to finish the proof. □

2.4 Bridgeland stability

Let $\mathcal{D}$ be a triangulated category. We introduce the notation

$$\begin{array}{ccc} A & \longrightarrow & B \\ & \nwarrow \; \swarrow & \\ & C & \end{array}$$

meaning $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$ is an exact triangle.

Once again, if someone wants to avoid the abstract nonsense, one can think of $\mathcal{D}$ as the bounded derived category of coherent sheaves (to which we will reduce in the next section).

Mimicking the Harder-Narasimhan filtration of slope stability, Bridgeland gives the following definition

Definition 2.4.1

A slicing of $\mathcal{D}$ is a family $\mathcal{P} = (\mathcal{P}(\phi))_{\phi\in\mathbb{R}}$ of full additive subcategories of D such that

1. $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$
2. *for* $\phi_1, \phi_2 \in \mathbb{R}_{>0}, \phi_1 > \phi_2$ *then* $\forall E_1 \in \mathcal{P}(\phi_1), E_2 \in \mathcal{P}(\phi_2), \operatorname{Hom}(E_1, E_2) = 0$
3. *for* $E \in \mathcal{D}, E \neq 0$ *there exists an Harder-Narasimhan filtration a.k.a:* $\exists (E_i)_{i=0,\ldots,n} \in \mathcal{D}$ *s.t*

$$\begin{array}{ccccccccc} 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & \ldots & \longrightarrow & E_{n-1} & \longrightarrow & E_n = E \\ & \nwarrow \; \swarrow & & \nwarrow \; \swarrow & & \nwarrow \; \swarrow & & \nwarrow \; \swarrow & \\ & A_1 & & A_2 & & A_{n-1} & & A_n & \end{array}$$

with $A_i \in \mathcal{P}(\phi_i)$ and the sequence $(\phi_i)_i = 1...n$ strictly decreasing.
The A_i are called Jordan-Hölder factors of E, and are unique up to permutation.

Definition 2.4.2
A Bridgeland stability condition σ on $\mathcal{D}$ is the datum of a couple $(Z, \mathcal{P})$ where:

- *$\mathcal{P}$ is a slicing of $\mathcal{D}$*
- *$Z : K(X) \to \mathbb{C}$ is a group homomorphism from the Grothendieck group of X verifying*
$$Z(\mathcal{P}(\phi) \setminus \{0\}) \subseteq \mathbb{R}_{>0}e^{i\pi\phi}$$
for every $\phi \in \mathbb{R}$.

The morphism Z is called the central charge.

The existence of a stability condition forces the slicing to be in abelian subcategories

Lemma 2.4.3 ([Bri02] Lemma 5.2)
If $(Z, \mathcal{P})$ is a stability condition, then the $\mathcal{P}(\phi)$ are abelian

In order to be able to have decent results, it is necessary to restrict to some more hypothesis:

Definition 2.4.4
A slicing $\mathcal{P}$ is locally finite if there exists $\eta \in \mathbb{R}_{>0}$ such that for all $t \in \mathbb{R}$ the subcategory $\mathcal{P}(t - \eta, t + \eta)$ are of finite length.
A stability condition is locally finite if its slicing is locally finite.

These locally finite stability conditions are those we are mainly interested in, because their set admits a structure of complex manifold.
This is obtained by introducing a generalised metric on the set of locally finite slicings and for each stability condition a generalised norm.
Alternatively, the same topology can be induced by a generalised metric.
This boils down to the following result

Proposition 2.4.5 ([Bri02] Thm. 1.2)
Let $Stab(\mathcal{D})$ be the set of locally finite stability conditions.
It admits a topology such that for $\Sigma \subset Stab(\mathcal{D})$ a connected component, there is a linear subspace with a linear topology $V(\Sigma) \subset \mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$ such that the map $(Z, \mathcal{P}) \mapsto Z$ defines a local homeomorphism $\Sigma \to V(\Sigma)$

A very useful fact is that $Stab(\mathcal{D})$ has two groups acting on it

Proposition 2.4.6 ([Bri02] Lemma 8.2)
The space $\mathrm{Stab}(\mathcal{D})$ *carries:*

- *a right action of* $\widetilde{GL}(2,\mathbb{R})^+$ *(the universal cover of* $GL(2,\mathbb{R})$*)*
- *a left action by isometries of the group* $\mathrm{Aut}(\mathcal{D})$ *of autoequivalences of* $\mathcal{D}$

The two actions commute

The left action clearly makes $\psi \in \mathrm{Aut}(\mathcal{D})$ act by sending $(Z, \mathcal{P}) \mapsto (Z \circ \psi^{-1}, \psi(\mathcal{P}))$.
The right action is defined considering elements of $\widetilde{GL}(2,\mathbb{R})$ as couples (T, f), with $T \in GL(2,\mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}$ increasing, verifying $f(\phi+1) = f(\phi)+1$ and such that the induced map on $S^1 = \mathbb{R}/2\mathbb{Z} = \mathbb{R}^2 \setminus \{0\}/\mathbb{R}_{>0}$ are the same. Then (T, f) acts as $(Z, \mathcal{P}) \mapsto (T^{-1} \circ Z, \mathcal{P}')$ where $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$.

2.5 On K3 and abelian surfaces

We go back to the more geometric context: from now on, $\mathcal{D}$ will be the bounded derived category of coherent sheaves on a K3 or an abelian surface X.
In this case $\mathcal{D}$ is of finite type and it is possible to define a bilinear form χ on $K(X)$ (the Grothendieck group is the same for the two categories of coherent sheaves and its bounded derived one) as, for $E, F \in \mathcal{D}$

$$\chi(E, F) := \sum_{i \in \mathbb{N}} (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_X(E, F[i])$$

This allows one to define the numerical Grothendieck group

$$\mathcal{N}(X) := K(X)/K(X)^{\perp}$$

(notice that $K(X)^{\perp} = {}^{\perp}K(X)$) on which χ induces a non degenerate bilinear form.

Remark. The Chern character $\mathrm{ch} : K \to \tilde{H}_{alg}(X,\mathbb{Z})$ induces an isomorphism $\mathcal{N}(X) \cong \tilde{H}_{alg}(X,\mathbb{Z})$.
This could be used to define the Mukai product in the first place for a more abstract perspective than the one we will propose in the next chapter.

Definition 2.5.1
A stability condition $\sigma = (Z, \mathcal{P})$ is called numerical if Z factors trough $\mathcal{N}(X)$. Equivalently if there is $\pi(\sigma) \in \mathcal{N}(X) \otimes \mathbb{C}$ such that $Z(\cdot) = -\chi(\pi(\sigma), \cdot)$

From now on, $\mathrm{Stab}(X)$ will be the space of locally finite numerical stability conditions.
First of all, the problem of the existence of stability conditions should be addressed.
In particular it would be nice to construct stability conditions from an ample divisor and a twist, as it's done for Gieseker stability: E.Macrì and P.Stellari independently achieved exactly that.

Proposition 2.5.2 ([Bri03] Prop. 7.1)
Let $\omega, \beta \in \mathrm{NS}(X) \otimes (Q)$ with $\omega \in Amp(X)$ and define:

$$Z_{\beta,\omega}(E) := (\exp(\beta + i\omega), v(E))$$

Then there exist a stability condition σ having Z as central charge. Moreover, all skyscraper sheaves $\mathcal{O}_{X,x}$ are σ-stable.

and Bridgeland generalised the result :

Proposition 2.5.3 ([Bri03] Lemma 6.2,6.3)
Take $\beta, \omega \in \mathrm{NS}(X) \otimes \mathbb{R}$ with $\omega \in \mathrm{Amp}(X)$ and $\omega^2 > 2$. Then the function $Z_{\beta,\omega}$ (defined as in the previous proposition) is the central charge of a stability condition for which all skyscrapers sheaves are stable.

Remark. The condition $\omega^2 > 2$ is stronger than what is actually needed. In fact Bridgeland states that it is enough to ask for $Z_{\beta,\omega}(S) \notin \mathbb{R}_{\leq 0}$ for spherical objects.

These are not all the possible stability conditions. But they almost form a connected component of $\mathrm{Stab}(X)$. We will explain in what sense, but first we need to introduce the definition of good stability condition.

If $\mathrm{Stab}^*(X) \subset \mathrm{Stab}(X)$ is a connected component, the projection of 2.4.5 becomes $\pi : \mathrm{Stab}^*(X) \to \mathcal{N}(X) \otimes \mathbb{C}$ and also the space $V(\Sigma)$ in the proposition coincide with $\mathcal{N}(X) \otimes \mathbb{C}$.
Moreover, π is a covering map over the subset $\mathcal{P}_0(X) \subset \mathcal{N}(X) \otimes \mathbb{C}$ defined as follows:
Let $\mathcal{P}(X) \subset \mathcal{N}(X) \otimes \mathbb{C}$ the subset of vector whose imaginary and real part span a positive definite 2-plane in $\mathcal{N}(X) \otimes \mathbb{R}$. Then define

$$\mathcal{P}_0(X) := \mathcal{P}(X) \setminus \bigcup_{\substack{\delta \in \mathcal{N}(x) \\ (\delta,\delta) = -2}} \delta^{\perp}$$

Definition 2.5.4
A connected component $\Sigma \subset \mathrm{Stab}(X)$ *is said to be good if* $\pi(\mathrm{Stab}(X)) \cap \mathcal{P}_0(X) \neq \emptyset$.
A stability condition is said to be good if it lies in a good component.

Definition 2.5.5
We denote $\mathrm{Stab}^{\dagger}(X) \subset \mathrm{Stab}(X)$ *the good component containing stability conditions for which skyscraper sheaves* $\mathcal{O}_{X,x}$ *are stable of the same phase for every* $x \in X$

Remark. In the previous definition, it is not at all trivial that $\mathrm{Stab}^{\dagger}(X)$ is unique or that it even exists. This requires some work developed in [Bri03] Sect.11.

So by definition $\mathrm{Stab}^{\dagger}(X)$ contains all stability conditions arising from 2.5.3. The main reason to restrict to good components is that they allow a wall-chamber decomposition

Proposition 2.5.6 ([Bri03] Prop 9.3)
Let $B \subset \mathrm{Stab}^*(X)$ *be a compact subset and* $S \subset \mathcal{D}$ *a subset of bounded mass in a good component* $\mathrm{Stab}^*(X) \subset \mathrm{Stab}(X)$ *(i.e.* $\sup_{E \in S} m_{\sigma}(E) < \infty$ *for some* $\sigma \in \mathrm{Stab}^*(X)$*).*
Then there is a finite collection $\{\mathcal{W}_j, j \in \Gamma\}$ *of real codimension 1 submanifolds of* $\mathrm{Stab}^*(X)$ *such that for any connected componet*

$$C \subset B \setminus \bigcup_{\jmath \in \Gamma} \mathcal{W}_j$$

the set of semistable objects in S *stays constant for* σ *varying in* C.
Moreover if $E \in S$ *has primitive Mukai vector,* S *is stable for all* $\sigma \in C$.

Remark. In the previous theorem the fact that boundedness is tested for some $\sigma \in \mathrm{Stab}^*(X)$ is equivalent to asking that for any σ the sup is finite. This is a consequence of the definition of the topology on $\mathrm{Stab}(X)$ via the generalised metric.

Remark. Notice that the collection of objects having fixed Mukai vector is bounded

Still a priori $\mathrm{Stab}^{\dagger}(X)$ contains more stability conditions than those constructed before.
In the fist place, because the action of $\widetilde{GL}^+(2, \mathbb{R})$ on one of the stability conditions stays in the same connected component.
But this can be in some sense modded out : define

$$U(X) := \{\sigma \in \mathrm{Stab}(X) | \sigma \text{ is good and } \mathcal{O}_{X,x} \text{ is } \sigma\text{-stable for any } x \in X\}$$

By definition $\mathrm{Stab}^{\dagger}(X)$ is the component containing $U(X)$. Then

Proposition 2.5.7 ([Bri03] Prop. 10.3)
For each $\sigma \in U(X)$ there is a unique element $g \in \widetilde{GL}^{+}(2,\mathbb{R})$ that sends σ in one of the stability condition of 2.5.3

The boundary of $U(X)$ has been described in Sect 12 of [Bri03] as a locally finite union of codimension one real submanifolds, and for a stability condition to be on one of these corresponds to having a certain Jordan-Holder filtration for sheaves $\mathcal{O}_{X,x}$.
These filtrations involve spherical objects (for the definition of spherical object see the remark after definition 4.2) that do not exist on abelian sufaces (see [Bri03, Lemma 15.1]).
Because of that, in the abelian case there is no boundary. This means that $U(X) = \mathrm{Stab}^{\dagger}(X)$, which implies that $\mathrm{Stab}^{\dagger}(X) / \widetilde{GL}^{+}(2,\mathbb{R})$ can be identified with $NS(X)_{\mathbb{R}} \times \mathrm{Amp}(X)_{\mathbb{R}}$.

Chapter 3

Moduli spaces of Bridgeland semistable objects

As we have already said, one of the main reason to introduce Bridgeland stability to replace Gieseker stability, is a very well behaved theory of moduli spaces. In particular the case of a $K3$ and of an abelian suface have been studied in depth by various authors (the two main references will be the works of Bayer and Macrì for $K3$s, and those of Yoshioka for abelian surfaces). In particular for the next chapter the main result that we need to cite is that minimal model program for moduli spaces can be runned via wall-crossing: every birational model of a moduli space (for stability condition in a certain connected component) has an interpretation in terms of changing the chamber of the stability condition.

3.1 Mukai lattice

As for almost every moduli problem, we will need to fix some numerical invariant: in this case it will be the Mukai vector. We will state most results for vector bundles, but the same can be extended first to the category of coherent sheaves using a resolution via locally free sheaves, and then to the bounded derived category of coherent sheaves $D^b(X)$ using additivity.

A very pervasive and classical result in complex geometry is the Riemann-Roch theorem, that for our purpose can be stated as following: :

Theorem 3.1.1 ([Huy05] Thm. 5.1.1)
Let E be an holomorphic vector bundle on a compact complex manifold X.

Then its Euler characteristic is given by:

$$\chi(X, E) = \int_X \operatorname{ch}(E) \operatorname{Td}(X)$$

Remark. Clearly the integral is to be intended as the integration of the maximal degree component of the form.

With this in mind, the motivation behind the following definition will be clearer:

Definition 3.1.2
Let E be a vector bundle on a complex manifold X.
We define its Mukai vector $v(E)$ as:

$$v(E) := \operatorname{ch}(E)\sqrt{\operatorname{Td}(X)}$$

By Riemann-Roch it is clear that it is enough to use the Mukai vectors to calculate the relative Euler characteristic of two vector bundles.
To recall, we define

Definition 3.1.3
Let E, F two vector bundles on X. The relative Euler caracteristic is:

$$\chi(E, F) = \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{Ext}^i(E, F)$$

which gives the following identity :

Lemma 3.1.4
Let E, F be two vector bundles on X. Then:

$$\chi(E, F) = \chi(X, F \otimes E^*)$$

Therefore it's a very simple calculation that:

$$\begin{aligned}\chi(E, F) =& \chi(X, F \otimes E^*) = \int_X \operatorname{ch}(F \otimes E^*) \operatorname{Td}(X) = \int_X \operatorname{ch}(F) \operatorname{ch}(E^*) \operatorname{Td}(X) \\ &= \int_X v(F)v(E^*)\end{aligned}$$

In this way we can evaluate the Euler characteristic of two vector bundles as a function of their Mukai vectors very easily.
Let us fix X to be a complex surface. Using the explicit description of the Chern character we can observe that taking the dual inside the Mukai vector corresponds to a sign change in the H^2 component.
Therefore we give the following definitions:

Definition 3.1.5
Let X be a complex surface.
Consider the decomposition $H^*(X,\mathbb{Z}) = H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})$.
Define for $v = (a, B, c) \in H^*(X,\mathbb{Z})$ the dual

$$\check{v} := (a, -B, c)$$

Then define the product:

$$(a, B, c)(a', B', c') := BB' - ac' - a'c$$

or equivalently

$$< v, w >:= -\int_X v\check{w}$$

This defines a lattice structure on $H^*(X,\mathbb{Z})$. We will call this lattice Mukai lattice and we will indicate it with the notation $\tilde{H}(X,\mathbb{Z})$.

To sum up the discussion above the definition we can state:

Proposition 3.1.6
Let E, F holomorphic vector bundles on X. Then:

$$\chi(E, F) = -(v(E), v(F))$$

Proof. We continue the chain of equalities presented to motivate the definition of Mukai lattice:

$$\chi(E,F) = \chi(X, F \otimes E^*) = \int_X \operatorname{ch}(F \otimes E^*) \operatorname{Td}(X) = \int_X \operatorname{ch}(F) \operatorname{ch}(E^*) \operatorname{Td}(X) =$$
$$\int_X v(F)v(E^*) = \int_X v(F)v(E)\check{} = -(v(F), v(E))$$

□

Some more structure appears on the Mukai lattice once we have an Hodge decomposition for X.

Definition 3.1.7
Assume X is also Kähler (so that it admits a Hodge decomposition).
Then there is a natural pure weight 2 Hodge structure on the Mukai lattice posing:

$$\tilde{H}^{0,2} = H^{0,2}$$
$$\tilde{H}^{2,0} = H^{2,0}$$
$$\tilde{H}^{1,1} = H^0 \oplus H^{1,1} \oplus H^4$$

We denote $\tilde{H}_{alg}(X,\mathbb{Z}) = \tilde{H}^{1,1} \cap \tilde{H}(X,\mathbb{Z})$ and we call it the algebraic part of the Mukai lattice.

Such X for us will be a $K3$ or abelian surface.
Note that the first Chern class takes holomorphic vector bundles to (1,1) forms so the Chern character (and the Mukai vector) is in $\tilde{H}_{alg}(X,\mathbb{Z})$, and that the Todd class $\mathrm{Td}(X)$ is either 1 in the abelian case or in the $K3$ one: in both cases it is invertible.
Therefore the map $v : K(X) \to \tilde{H}_{alg}(X,\mathbb{Z})$ is an isomorphism if and only if the Chern character is one.
This has beeen proved in [AT14]

Proposition 3.1.8 ([AT14] Sec. 2.1,2.2)
The Chern character induces an isomorphism between the Grothendieck group of the derived category and the algebraic part of the Mukai lattice.

Moreover proposition 3.1.6 shows that it is an isometry of lattices. This can also be seen as an altenative way to define the Mukai product on the Mukai lattices using the identification with $K(X)$ (but this would be limited to these cases of the $K3$ and the abelian surfaces).

3.2 Projectivity and existence of moduli spaces

Let X be a $K3$ or abelian surface. We will use the notation:

$$\varepsilon_X := \begin{cases} 1 & \text{if } X \text{ is a } K3 \text{ surface} \\ 0 & \text{if } X \text{ is an abelian surface} \end{cases}$$

Let $v \in \tilde{H}_{alg}(X,\mathbb{Z})$ a Mukai vector and $\sigma \in \mathrm{Stab}^\dagger(X)$ a stability condition and $\phi \in (0,1]$ a phase.

Definition 3.2.1
A flat family of σ-(semi)stable objects of Mukai vector v and phase ϕ on an algebraic space S is an object $\mathcal{E} \in D^b(S \times X)$ such that for any closed point $s \in S$, denoting $i_s : X \hookrightarrow S \times X$ the inclusion $x \mapsto (s,x)$, then $i_s^*\mathcal{E} \in D^b(X)$ is a σ (semi)stable object of Mukai vector v and phase ϕ

We can indicate the corresponding stack of σ-semistable (resp. stable) objects of Mukai vector v and phase ϕ as $\mathfrak{M}_\sigma(v,\phi)$(resp. $\mathfrak{M}^s_\sigma(v,\phi)$).
First notice that the $\widetilde{\mathrm{GL}}(2,\mathbb{R})$ action gives isomorphisms $\mathfrak{M}_\sigma(v,\phi) \cong \mathfrak{M}_\sigma(v,\phi')$ for any $\phi,\phi' \in (0,1]$. So we can reduce to the case $\phi = 1$ and drop the phase from the notation.
A priori, it is not clear even when semistable objects exist, but it has been proved that this depends on the Mukai vector:

Lemma 3.2.2 ([BM12] Thm. 6.8, [Yos12])
Let $v \in H^*_{alg}(X, \mathbb{Z})$. Assume that $v = mv_0$ with $m \in \mathbb{Z}_{>0}$ and v_0 primitive with $v_0^2 \geq -2\varepsilon_X$.
Then $\mathfrak{M}_\sigma(v, \phi)(\mathbb{C})$ is non-empty for all $\sigma \in \mathrm{Stab}^\dagger(X)$ and $\phi \in \mathbb{R}$ such that $Z(v) \in \mathbb{R}_{>0}e^{i\phi\pi}$

Thanks to works from Toda, Yoshioka, Bayer and Macrì, under some assumptions on v and σ these stacks admit a coarse moduli space. We recall what this means:

Definition 3.2.3
Let $\mathcal{F}$ be a presheaf. A coarse moduli space is an algebraic space M together with a morphism $F \to h_M$ such that:

1. *for any algebrically closed field $\mathfrak{k}$, $F(\mathfrak{k}) \to h_M(\mathfrak{k}) = \mathrm{Hom}(\mathfrak{k}, M)$*
2. *has the following universal property: for each M' algebraic space and $F \to h'_M$ there exists a morphism $M \to M$ such that*

$$\begin{array}{ccc} F & \longrightarrow & h_{M'} \\ & \searrow \quad \nearrow & \\ & h_M & \end{array}$$

When it exists, we will denote $\mathbf{M}_\sigma(v)$ (resp. $\mathbf{M}^s_\sigma(v)$) the coarse moduli space for $\mathfrak{M}_\sigma(v)$(resp. $\mathfrak{M}^s_\sigma(v)$).
The previous definition for our case means the following: take an algebraic space S and $\mathcal{E} \in D^b(X \times S)$ a family of σ-(semi)stable objects of Mukai vector v (and phase 1). Then by definition $\mathcal{E} \in \mathfrak{M}_\sigma(v)(S) \to \mathbf{M}_\sigma(v)(S)$. So for each family $\mathcal{E}$ we get a morphism $S \to \mathbf{M}_\sigma(v)$. Notice that the viceversa is not true, alias not every morphism $S \to \mathbf{M}_\sigma(v)$ induces a family, except when $S = \mathfrak{k}$. The universal property states that $\mathbf{M}_\sigma(v)$ is minimal, in the sense that any other algebraic space doing so will factor through $\mathbf{M}_\sigma(v)$.

Theorem 3.2.4 ([BM12] Thm. 1.3,[MYY11b] Thm. 0.0.2)
Let X be a smooth projective $K3$ surface and let $v \in \tilde{H}_{alg}(X, \mathbb{Z})$.
If $\sigma \in \mathrm{Stab}^\dagger(X)$ is generic (does not lie on a wall with respect to v) then there is a coarse moduli space $\mathbf{M}_\sigma(v)$ for $\mathfrak{M}_\sigma(v)$ as a normal projective irreducible variety, parametrising the S-equivalence classes, with $\mathbb{Q}$-factorial singularities.

Remark. When we say that the coarse moduli space parametrises S-equivalence classes, we admit that we have been slightly imprecise: in order to get base

change (and having a stack more than a functor) we need to collapse some things. So the $\mathbb{C}$-points of the moduli space are actually S-equivalence classes which, we recall, means that objects with the same JH factors are collapsed together.
Notice that for stable objects the S-equivalence classes coincides with the object itself, so there is no problem. The precise definition of the stack and the moduli can be found in [AHLH18] and in more generality in [Bay+21].

At least for the moduli of stable objects the expected dimension can be computed checking that if $A \in D^b(X)$ is a stable object of Mukai vector v, it follows $v^2 = ext^1(A, A) - 2$ so $ext^1(A, A) = v^2 + 2$.
Notice that the moduli space of stable objects, if not empty, exists and is smooth of the expected dimension. This can be deduced from a classical result from Mukai and Artamkin [Art89; FMM12; IM19], which asserts that obstructions to deformations for a sheaf $\mathcal{F}$ are contained in the kernel of a surjective trace map $\mathrm{Ext}^2_X(\mathcal{F}, \mathcal{F}) \to H^2(X, \mathcal{O})$. But if $\mathcal{F}$ is simple and X is a surface, this is a surjective map $\mathbb{C} \to \mathbb{C}$, so the kernel is trivial and deformations are unobstructed.
Imposing numerical conditions on the Mukai vector also assures that the moduli space has the expected dimension (The following result is stated for twisted $K3$ surfaces, but Yoshioka and also Macrì prove the projectivity of moduli spaces constructing isomorphisms with such moduli, so it must hold in the same way for $K3$ and abelian surfaces)

Theorem 3.2.5 ([BM14] Thm. 2.15)
Let $v = mv_0 \in H^*(X, \mathbb{Z})$ be a Mukai vector, with v_0 primitive and $m > 0$ and let $\sigma \in \mathrm{Stab}^\dagger(X)$ a generic stability condition with respect to v. Then

1. *The coarse moduli space $\mathbf{M}_\sigma(v)$ is non-empty if and only if $v^2 + 2\varepsilon_X \geq 0$*

2. *$\mathbf{M}^s_\sigma(v) \neq \emptyset$ and $\dim \mathbf{M}_\sigma(v) = v^2 + 2$ if and only if $m = 1$ or $v_0^2 > 0$.*

Actually, something more can be said and Bayer and Macrì state this fact as part of a discussion just before the previous theorem: if v is primitive (and the condition is generic) the moduli spaces of stable and semistable objects coincides. In the non primitive case, if $v^2 > 0$, all that can be said is that the moduli of stable objects is a smooth open set of the moduli space of semistable objects.
Now, because of the first statement in the theorem, the non empty cases that do not fit in the situation of the second statement are those of non-primitive isotropic vector, which give the following

Proposition 3.2.6 (from [BM12] Lemma 6.6)
Let $v \in H^*(X,\mathbb{Z})$ of the form $v = mv_0$, with $v_0^2 = 0$ and $m > 1$. Then

$$\mathbf{M}_\sigma(v) \cong \mathrm{Sym}^m(\mathbf{M}_\sigma(v_0))$$

3.2.1 Hyperkählers in moduli

Fix $\sigma \in \mathrm{Stab}^\dagger(X)$ and $v \in H^*(X,\mathbb{Z})$ a primitive Mukai vector with $v^2 > 0$. We have seen that the moduli space $\mathbf{M}_\sigma(v)$ is an irreducible projective variety, smooth and with a symplectic form.
We can hope it is an hyperkähler, the only condition left to check is the triviality of the fundamental group. While this actually holds in the case of a $K3$ surface, for an abelian surface it is not as straightforward. Mimicking the classical construction by Beauville, we can find a generalised Kummer inside the moduli space of an abelian surface.
Let us fix T an abelian suface, then the Albanese map of $\mathbf{M}_\sigma(v)$ can be described as follow.
Fix an $E_0 \in \mathfrak{M}_\sigma(v)(\mathbb{C})$. Then define

$$\begin{aligned} \mathrm{alb} : &\mathbf{M}_\sigma(v) \to T \times T\check{} \\ &E \quad\to (\det \Phi_{\boldsymbol{P}}(E - E_0), \det(E - E_0)) \end{aligned}$$

where $\boldsymbol{P}$ is the Poincarè bundle on $T \times T\check{}$ defined trivially by the fact that $T\check{}$ parametrises line bundles (equivalent to zero) on T.

Definition 3.2.7
Let $v \in H^*(X,\mathbb{Z})$ a primitive Mukai vector with $v^2 \geq 6$ and $\sigma \in \mathrm{Stab}^\dagger(X)$ generic. We denote $\mathbf{K}_\sigma(v)$ the fiber of alb.

Theorem 3.2.8 ([Yos12] Thm 1.13)
For v, σ as in the previous definition, $\mathbf{K}_\sigma(v)$ is an irreducible symplectic manifold of $\dim \mathbf{K}_\sigma(v) = v^2 - 2$ which is deformation equivalent to a generalised Kummer variety.

In what follows, when it is needed to discriminate the two cases, we will denote $\mathbf{Y}_\sigma(v) \subset \mathbf{M}_\sigma(v)$ respectively

$$\begin{aligned} &\mathbf{M}_\sigma(v) \text{ if } X \text{ is a } K3 \text{ surface} \\ &\mathbf{K}_\sigma(v) \text{ if } X \text{ is an abelian surface} \end{aligned}$$

3.3 Some tools on moduli spaces

In this section we gather some useful tools that are available to work on moduli spaces. In particular we will recall some facts about HN filtrations in families, then we will proceed to define the Mukai homomorphism and identify a Nef divisor associated with a stability condition, which will be the main character in the next section linking the decomposition of the space of stability conditions, separating chambers of general stability conditions. with that of the Mori cone, separating the ample cones of different birational models.

3.3.1 Filtration in families

A very natural thing we would like to do is to decompose a (semi)stable object in its Jordan-Hölder factors, possibly with respect to different stability conditions. To do so in a geometric way on the moduli space, we need to be able to define in some way an Harder-Narasimhan filtration on a (quasi)universal family.
This can be done, under some hypotesis on the stability condition, and in the precise sense that will be stated at the end of this subsection. Now let Y be a smooth projective variety over $\mathbb{C}$ and $\sigma \in \mathrm{Stab}^{\dagger}(Y)$ a stability condition on $D^b(Y)$.

Definition 3.3.1
We say σ satisfies openness of stability if the following condition holds: for any scheme of finite type over $\mathbb{C}$ and for any $\mathcal{E} \in D^b(S \times Y)$ such that its derived restriction $\mathcal{E}_s$ is a σ-semistable object of $D^b(Y)$ for some $s \in S$, there exists an open neighborhood $U \subset S$ of s such that $\mathcal{E}_{s'}$ is σ-semistable for all $s' \in U$ (equivalently $\mathcal{E}_{|U}$ is a σ semistable family on U)

in our context this is always the case

Theorem 3.3.2 ([Tod07], Section 3)
Openness of stability holds on $K3$ (or abelian) surfaces and $\sigma \in \mathrm{Stab}^{\dagger}(X)$

Theorem 3.3.3
Let $\sigma \in \mathrm{Stab}^{\dagger}(Y)$ be an algebraic stability condition satisfying openness of stability. Let S be an irreducible variety over $\mathbb{C}$ and an object $\mathcal{E} \in D^b(S \times Y)$. Then there exists a system of maps:

$$0 = \mathcal{E}^0 \to \mathcal{E}^1 \to ... \to \mathcal{E}^m = \mathcal{E}$$

in $D^b(Y)$ and an open set $U \subset S$ with the following property: for any $s \in U$ the derived restriction of the system of maps

$$0 = \mathcal{E}_s^0 \to \mathcal{E}_s^1 \to \dots \to \mathcal{E}_s^m = \mathcal{E}_s$$

is the HN filtration of $\mathcal{E}_s$.

So using this result we can obtain the decomposition as follows. Let $v \in H^*_{alg}(X, \mathbb{Z})$ and $\sigma \in \mathrm{Stab}^\dagger(X)$ generic. Let E a σ stable object of Mukai vector v (which corresponds to a point in $\mathbf{M}_\sigma(v)$). Take γ a different stability condition. Let A_i be the γ-stable factors of E and denote $a_i := v(A_i)$ their Mukai vectors. Denote $M_{\gamma,P}$ the locally closed stratum of objects having JH partition which is a refinement of P.
Suppose that $\mathbf{M}_\sigma(v)$ admits a universal family $\mathcal{E}$. Then consider the restriction $\mathcal{E}_{|M_{\gamma,P}}$ and take the filtration in families $\mathcal{E}_i$. Setting $\mathcal{A}_i := {}^{\mathcal{E}_{i+1}}\!/_{\mathcal{E}_i}$ we have families of γ stable objects on an open set of $M_{\gamma,P}$ with generical fiber having Mukai vector a_i, which induce respectively rational maps $M_{\gamma,P} \dashrightarrow \mathbf{M}^s_\gamma(a_i)$. By universal property of fiber product we get a rational map $M_{\gamma,P} \dashrightarrow \bigtimes \mathbf{M}^s_\gamma(a_i)$.
If a universal family is not available, take an étale neighborhood $f : U \to \mathbf{M}_\sigma(v)$ of s admitting a universal family. The maps obtained by the previous construction from U will factor through f (see [BM14] end of the proof of Lemma 6.5).

3.3.2 The Mukai homomorphism

The first thing we are naturally interested into is the $H^2(M, \mathbb{Z})$ for $M = \mathbf{M}_\sigma(v)$ a moduli space.
The Mukai homomorphism will provide a connection between this and the cohomology of X.
Remember that taking Mukai vector is an isometry between the numerical Grothendieck group with the relative Euler characteristic and the Mukai lattice $v : (K_{num}(X), \chi) \cong (H^*_{alg}(X, \mathbb{Z}), (,))$.
We denote $v^\# = v^{-1}(v^\perp)$, equivalently the orthogonal to the preimage of the chosen Mukai vector.
A classical construction is that of the Donaldson morphism:

Definition 3.3.4
Let S be an algebraic space.
Let $\sigma \in \mathrm{Stab}^\dagger(X)$, $v \in H^*(X, \mathbb{Z})$ and $\mathcal{E} \in \mathfrak{M}_\sigma(v)(S)$.
We define the Donaldson morphism

$$\lambda_\mathcal{E} : v^\# \to N^1(X)$$

as the composition

$$v^{\#} \xrightarrow{p_X^*} K_{num}^{perf}(S \times X)_{\mathbb{R}} \xrightarrow{\cdot[\mathcal{E}]} K_{num}^{perf}(S \times X)_{\mathbb{R}} \xrightarrow{(p_S)_*} K_{num}^{perf}(S)_{\mathbb{R}} \xrightarrow{\det} N^1(S)$$

which explicitly means

$$\lambda([E]) := \det((p_S)_*([\mathcal{E}] \cdot p_X^*[E])) = \det(\Phi_{\mathcal{E}}(E))$$

The most natural family to choose would be a universal family.
But since the moduli spaces are semistable, we cannot hope to find a global universal family. Fortunately we have something quite close to it:

Definition 3.3.5
Let S be an algebraic space of finite type over $\mathbb{C}$. Fix $v \in H^*(X, \mathbb{Z})$ a Mukai vector and $\sigma \in \mathrm{Stab}^\dagger(X)$ a stability condition.

1. *A flat family $\mathcal{E} \in D^b(X \times S)$ is a quasi-family on S if for each $s \in S$ closed point there exists an integer $\rho > 0$ and a σ-semistable object E of Mukai vector v (said differently: $E \in \mathfrak{M}_\sigma(v)$) such that $\mathcal{E}_{|\{s\}\times X} \cong E^\rho$. If S is connected ρ doesn't depend on s and is called similitude of $\mathcal{E}$*

2. *two quasi-families $\mathcal{E}_1, \mathcal{E}_2$ on S are called equivalent if there are V_1, V_2 vector bundles on S such that $\mathcal{E} \otimes V_1 \cong E_2 \otimes V_2$*

3. *A quasi-universal family is a quasi-family $\mathcal{E}$ such that for any scheme S' and any quasi family $\mathcal{E}'$ on S' there exists a morphism $f : S' \to S$ such that $f^*\mathcal{E}$ and $\mathcal{E}'$ are equivalent.*

Thanks to a result of Mukai,

Theorem 3.3.6 ([BM12] Rmk 4.6)
If $\mathfrak{M}_\sigma(v) = \mathfrak{M}_\sigma^s(v)$ and admits a coarse moduli space $\mathbf{M}_\sigma(v)$, there exists a quasi-universal family on $\mathbf{M}_\sigma(v)$ which is unique up to equivalence.

Therefore we can define the Donaldson morphism choosing a quasi-universal family.
Its dual version is called the Mukai homomorphism, and can be defined as:

Definition 3.3.7
Fix a primitive Mukai vector $v \in H^*_{alg}(X, \mathbb{Z})$ and $\sigma \in \mathrm{Stab}^\dagger(X)$ generic with respect to v.
We define the Mukai homomorphism $\theta : v^\perp \to H^2(\mathbf{M}_\sigma(v))$ by the formula

$$\theta(w).C = \frac{1}{\rho}(w, \Phi_{\mathcal{E}}(\mathcal{O}_c))$$

where C is a curve in $\mathbf{M}_\sigma(v)$, $w \in v^\perp$ and $\mathcal{E}$ is a quasi-universal family of similitude ρ. The definition doesn't depend on $\mathcal{E}$.

It is just a calculation ([BM12] Rmk 5.5) that the following identity holds:

$$\theta(v(E)) = -\lambda(E^{\check{}})$$

for $E \in D^b(X)$.
Moreover, under positivity of the Mukai vector, θ identifies the Neron-Severi of $\mathbf{Y}_\sigma(v)$ with a sublattice of the Mukai lattice:

Theorem 3.3.8 ([BM12] Thm. 6.10, [Yos12] Thm.1.13)
Let $v \in H^*_{alg}(X, \mathbb{Z})$ be a primitive Mukai vector with $v^2 \geq 6(1 - \varepsilon_X)$, and let $\sigma \in \mathrm{Stab}^\dagger(X)$ be a generic stability condition.
Then $\mathbf{Y}_\sigma(v)$ is an irreducible symplectic projective manifold and the Mukai homomorphism induces an isomorphism:

$$\theta : v^\perp \xrightarrow{\sim} \mathrm{NS}(\mathbf{Y}_\sigma(v)) \quad \textit{if } v^2 > 0$$
$$\theta : {}^{v^\perp}\!/_{v} \xrightarrow{\sim} \mathrm{NS}(\mathbf{Y}_\sigma(v)) \ \textit{if } v^2 = 0$$

3.3.3 A Nef divisor

Recall that $\mathrm{Stab}^\dagger(X)$ projects very naturally on $\mathrm{Hom}(K_{num}(X), \mathbb{C})$ via the map $\sigma = (\mathbb{Z}, \mathcal{P}) \mapsto Z_\sigma := Z$ (cfr. 2.4.5).
Since the Euler characteristic is a non-degenerate bilinear form on $K_{num}(X)_\mathbb{R}$, we can assign to a stability condition σ the vector $w_\sigma \in K_{num}(X)$ dual to $\mathfrak{I} Z_\sigma$ and get via the Donaldson morphism λ an element of $N^1(X)$.
This subsection will summarise the main steps exposed in [BM12] to give an explicit description of such a divisor.
Firstly it is defined at the level of stacks:

Definition 3.3.9 ([BM12] Def. 3.2)
Let $C \to \mathfrak{M}_\sigma(v)$ be an integral projective curve over $\mathfrak{M}_\sigma(v)$ with induced universal family $\mathcal{E} \in D^b(C \times X)$ and $\Phi_\mathcal{E} : D^b(C) \to D^b(X)$ the corresponding Fourier-Mukai transform.
We define the following real number:

$$\mathfrak{L}_\sigma.C := \mathfrak{I} Z_\sigma(\Phi_\mathcal{E}(\mathcal{O}_C))$$

Then it is not modified by:

1. *tensoring $\mathcal{E}$ with the pull-back of a line bundle on C*
2. *replacing $\mathcal{O}_C$ with any line bundle on C*

We think of $\mathfrak{L}_\sigma$ as a divisor class in $N^1(\mathfrak{M}_\sigma(v))$.

Proposition 3.3.10 ([BM12] Lemma 3.3)
The divisor class $\mathfrak{L}_\sigma$ *is nef:* $\mathfrak{L}_\sigma.C \geq 0$.
Moreover, we have that $\mathfrak{L}_\sigma.C > 0$ *if and only if for two general closed points* $c, c' \in C$ *the corresponding objects* $\mathcal{E}_c, \mathcal{E}_{c'} \in D^b(X)$ *are not S-equivalent*

Taking a family $\mathcal{E} \in \mathfrak{M}_\sigma(v)(S)$ on some algebraic space S, it is possible to induce a divisor on S:

Theorem 3.3.11 ([BM12] Thm. 4.1)
The assignment $C \mapsto \mathfrak{L}_\sigma.C$ *only depends on the numerical class of the curve in* $N_1(S)$.
It defines a nef divisor class $\ell_{\sigma,\mathcal{E}} \in N^1(S)$ *invariant under tensoring* $\mathcal{E}$ *with a line bundle pulled back from* S.
Moreover for a curve $C \subset S$ *we have* $\ell_{\sigma,\mathcal{E}}.C > 0$ *if and only if for two general closed points* $c, c' \in C$*, the corresponding objects* $\mathcal{E}_c, \mathcal{E}_{c'} \in D^b(X)$ *are not S-equivalent*

To see that this divisor is the one described at the beginning of the subsection, denote $w_\sigma \in K_{num}(X)$ the only vector such that $\chi(w_\sigma, -) = \Im Z(-)$.

Proposition 3.3.12 ([BM12] Prop. 4.4)
For any integral curve $C \subset S$*, we have*

$$\lambda_{\mathcal{E}}(w_\sigma).C = \Im Z(\Phi_{\mathcal{E}}(\mathcal{O}_C)) = \ell_{\sigma,\mathcal{E}}.C$$

Taking a quasi-universal family $\mathcal{E}$ on a moduli space $\mathbf{M}_\sigma(v)$ defines $\ell_{\sigma,\mathcal{E}} \in N^1(\mathbf{M}_\sigma(v))$. To make the choice canonical, since changing the quasi-universal family multiplies the divisor by the similitude, we can normalise and define $\ell_\sigma := \frac{1}{\rho}\ell_{\sigma,\mathcal{E}}$ for any $\mathcal{E}$ quasi-universal family of similitude ρ.
Now, fix a chamber $\mathcal{C} \subset \mathrm{Stab}^\dagger(X)$. Since σ-semistable objects stay the same for $\sigma \in \mathcal{C}$ we write $\mathbf{M}_\mathcal{C}(v)$ to underline the independence on $\sigma \in \mathcal{C}$.
Then we can define a map:

$$\begin{aligned} \ell_\mathcal{C} : & \mathcal{C} \to N^1(\mathbf{M}_\mathcal{C}(v)) \\ & \sigma \mapsto \ell_\sigma \end{aligned}$$

Abusing the notation we will indicate $\ell_\mathcal{C}$ also the co-restriction to $N^1(\mathbf{Y}_\sigma(v))$. So we can map each chamber in the ample cone of the moduli space. What happens on the walls?

3.4 MMP for moduli spaces

3.4.1 Wall-crossing

Fix a primitive Mukai vector $v \in H^*(X,\mathbb{Z})$ with $v^2 \geq -2 + 8\varepsilon_X$, take $\mathcal{C}_\pm$ two adjacent chambers of $\mathrm{Stab}^\dagger(X)$ and $\mathcal{W} \subset \mathrm{Stab}^\dagger(X)$ to be the wall separating them. Clearly we have the two maps $\ell_{\mathcal{C}_\pm} : \mathcal{C}_\pm \to N^1(\mathbf{Y}_{\mathcal{C}_\pm}(v))$ induced by the quasi-universal families $\mathcal{E}_\pm$.
Fix a generic stability condition $\sigma_0 \in \mathcal{W}$ (meaning, that doesn't lie in the intersection with another wall) and let us call $\sigma_\pm \in \mathcal{C}_\pm$ two generic stability conditions on each side of the wall. Since $\sigma_\pm$ semistable objects stay semistable for σ_0, the two quasi-families $\mathcal{E}_\pm$ are also σ_0 quasi-families, so they induce two nef divisors $\ell_{0,\mathcal{E}_\pm} \in N^1(\mathbf{Y}_{\sigma_\pm}(v))$.

Theorem 3.4.1 ([BM12] Thm. 1.4,[MYY11b] Prop 3.29 (2))
The classes $\ell_{0,\pm}$ are big and nef and induce birational contraction morphisms

$$\pi_\pm : \mathbf{Y}_{\sigma_\pm}(v) \to Y_\pm$$

with $Y_\pm$ normal irreducible projective varieties. The curves contracted by $\pi_\pm$ are exactly those of objects S-equivalent with respect to σ_0

The idea is that since $\mathbf{Y}_{\sigma_\pm}(v)$ have trivial canonical bundles, the Base Point Free Theorem ([Kol+98] Theorem 3.3) shows $\ell_{0,\pm}$ are semi-ample and therefore they induce the contractions ([Laz04]). The second statement descends directly from 3.3.11.
To discriminate the behaviour around the wall, there are 4 possible situations

Definition 3.4.2
A wall $\mathcal{W}$ is called:

1. *a fake wall if there are no curves in $\mathbf{Y}_{\sigma_\pm}(v)$ S-equivalent to each other with respect to σ_0.*
2. *a totally semistable wall if $\mathbf{M}^s_{\sigma_0}(v) = \emptyset$*
3. *a flopping wall if we can identify $Y_+ = Y_-$ and the induced map $\mathbf{Y}_{\sigma_+}(v) \dashrightarrow \mathbf{Y}_{\sigma_-}(v)$ induces a flopping contraction*
4. *a divisorial wall if the morphisms $\pi^\pm$ are both divisorial contractions*

3.4.2 Lattice associated to a wall

A very useful tool to classify walls consist in associating to each wall a sublattice of the Mukai lattice.

Definition 3.4.3
Let $v \in H^*_{alg}(X, \mathbb{Z})$ a primitive Mukai vector with $v^2 > 0$ and $\mathcal{W} \subset \mathrm{Stab}^\dagger(X)$ a wall with respect to v. The lattice associated to $\mathcal{W}$ is

$$\mathcal{H}_\mathcal{W} := \left\{ w \in H^*_{alg}(X, \mathbb{Z}), \Im \frac{Z(w)}{Z(v)} = 0 \text{ for all } \sigma = (Z, \mathcal{P}) \in \mathcal{W} \right\}$$

sublattices arising in this way have the following properties

Proposition 3.4.4 ([BM14] Prop. 5.1)
In the same hypothesis of the previous definition, $\mathcal{H}_\mathcal{W}$ has the following properties:

1. *It is a primitive sublattice of rank two and of signature (-1,1) (with respecto to te restriction of the Mukai form)*
2. *Let σ_+, σ_- two sufficiently close and generic stability conditions on opposite sides of $\mathcal{W}$, and consider a σ_+stable object $E \in \mathbf{M}_{\sigma_+}(v)$. Then every factor A_i of the HN filtration of E has Mukai vector $v(A_i) \in \mathcal{H}_\mathcal{W}$*
3. *If $\sigma_0 \in \mathcal{W}$ is a generic stability condition on the wall, the previous property holds substituting σ_0 with σ_-*

Not all primitive sublattices of rank 2 and signature (-1,1) come from a wall. This leads to the definition of potential walls

Definition 3.4.5
Let $\mathcal{H} \subset H^*_{alg}(X, \mathbb{Z})$ a primitive sublattice of rank 2 and signature (-1,1). A potential wall associated to $\mathcal{H}$ is denoted $\mathcal{W}_\mathcal{H}$ and is a connected component of the real codimension 1 submanifold of stability conditions $\sigma = (Z, \mathcal{P})$ such that $Z(\mathcal{H})$ is contained in a line.

A useful fact is that proposition 3.4.4 still holds substituting to $\mathcal{W}, \mathcal{H}_\mathcal{W}$ the potential wall of a lattice $\mathcal{W}_\mathcal{H}, \mathcal{H}$.
All the geometry of the contraction at the wall is encoded in the numerical properties of the lattice $\mathcal{H}_\mathcal{W}$. The following result is specifically proved for $K3$ surfaces.

Theorem 3.4.6 ([BM14] Thm. 5.7)
Let X be a $K3$ surface, fix $v \in H^*_{alg}(X, \mathbb{Z})$ a primitive positive Mukai vector. Let $\mathcal{H} \subset H^*_{alg}(X, \mathbb{Z})$ be a primitive hyperbolic rank two sublattice containing v. Let $\mathcal{W} \subset \mathrm{Stab}^\dagger(X)$ be a potential wall associated to $\mathcal{H}$.
The set $\mathcal{W}$ is a totally semistable wall if and only if there exists either an isotropic class $w \in \mathcal{H}$ with $(v, w) = 1$, or an effective spherical class $s \in \mathcal{C}_\mathcal{W} \cap \mathcal{H}$ with $(s, v) < 0$. In addition:

1. *The set $\mathcal{W}$ is a wall inducing a divisorial contraction if one of the following three conditions holds:*
   (Brill-Noether): there exists a spherical class $s \in \mathcal{H}$ with $(s, v) = 0$
   (Hilbert-Chow): there exists an isotropic class $w \in \mathcal{H}$ with $(w, v) = 1$
   (Lie-Gieseker-Uhlenbeck):there exists an isotropic class $w \in \mathcal{H}$ with $(w, v) = 2$

2. *Otherwise v can be written as the sum $v = a + b$ of two positive classes, or if there exists a spherical class $s \in \mathcal{H}$ with $0 < (s, v) \leq \frac{v^2}{2}$ then $\mathcal{W}$ is a wall corresponding to a flopping contraction.*

3. *In all other cases $\mathcal{W}$ is either a fake wall (if it is totally semistable) or it is not a wall*

In the case of abelian surfaces the picture is similar. The only thing that is different is that Brill.Noether contraction do not occur (which agrees with the absence of spherical classes).

3.4.3 Birationality of moduli spaces

One consequence of the study of wall crossing is the following result:

Theorem 3.4.7 ([BM14] Thm. 1.1, [MYY11a] Cor. 3.3.9)
Let $\sigma, \tau \in \mathrm{Stab}^\dagger(X)$ be two generic stability condition with respect to $v \in H^*_{alg}(X, \mathbb{Z})$ a primitive Mukai vector with $v^2 > 0$.

1. *The two spaces $\mathbf{M}_\sigma(v)$ and $\mathbf{M}_\tau(v)$ are birational to each other*

2. *More precisely there is a birational map induced by a derived (anti-)equivalence Φ of $D^b(X)$ in the following sense: there exists a common open subset $U \subset \mathbf{M}_\sigma(v), U \subset \mathbf{M}_\tau(v)$, with complement of codimension at least two such that for any $u \in U$ the corresponding objects $\mathcal{E}_s \in M_\sigma V$ and $\mathcal{F}_u \in \mathbf{M}_\tau(v)$ are related via $\mathcal{F}_u = \Phi(\mathcal{E}_u)$.*

This leads to the following observation: we can improve the construction of $\ell_{\mathcal{C}}$ (see end of section 3.3.3). Since for any two generic stability condition we have a rational map between moduli spaces, we can identify the Neron-Severi $N^1(\mathbf{M}_\sigma(v))$ for any generic $\sigma \in \mathrm{Stab}^\dagger(X)$ and deduce a map $\mathrm{Stab}^\dagger(X) \setminus \bigcup_{\mathcal{W} \text{ walls}} \mathcal{W} \to N^1(\mathbf{M}_\sigma(v)) \to N^1(\mathbf{Y}_\sigma(v))$. Moreover the study of wall crossing also allows to prove that

Lemma 3.4.8 ([BM14] 10.1)
Let $C_+, C_- \subset \mathrm{Stab}^\dagger(X)$ adjacent chambers separeted by the wall $\mathcal{W}$. Then the two maps $\ell_{\mathcal{C}_+}, \ell_{\mathcal{C}_-}$ agree on $\mathcal{W}$ when extended by continuity.
Let $\pi^+ : \mathbf{Y}_{\mathcal{C}_+}(v) \to Y_+$ the contraction induced by $\mathcal{W}$, then

1. *When π^+ is an isomorphism or a small contraction then the maps $\ell_{\mathcal{C}_+}, \ell_{\mathcal{C}_-}$ are analytic continuations of each others*

2. *When π^+ is a divisorial contraction, then the analytic continuations of $\ell_{\mathcal{C}_+}, \ell_{\mathcal{C}_-}$ differ by the reflection at the divisor D contracted π^+.*

This allows to think of ℓ as a continuous map defined globally on $\mathrm{Stab}^\dagger(X)$. This result summarizes all the geometry encoded in ℓ and makes precise the statement about Minimal Model Program in the introduction to this section:

Theorem 3.4.9
Fix a base point $\sigma \in \mathrm{Stab}^\dagger(X)$ generic with respect to a primitive positive Mukai vector $v \in H^*_{alg}(X, \mathbb{Z})$.

1. *Under the identification of the Neron-Severi groups induces by the birational maps of 3.4.7 the maps $\ell_{\mathcal{C}}$ glue to a piece-wise analytic continuous map*
$$\ell : \mathrm{Stab}^\dagger(X) \to \mathrm{NS}(\mathbf{M}_\sigma(v))$$

2. *The image of ℓ is the intersection of the movable and big cone of $\mathbf{M}_\sigma(v)$*

3. *The map ℓ is compatible, in the sense that for generic $\sigma' \in \mathrm{Stab}^\dagger(X)$, the moduli space $\mathbf{M}_{\sigma'}(v)$ is the birational model corresponding to $\ell(\sigma')$. In particular every K-trivial smooth birational model of $\mathbf{M}_\sigma(v)$ appears as a moduli space $\mathbf{M}_{\mathcal{C}}(v)$ of Bridgeland stable objects for some chamber $\mathcal{C} \subset \mathrm{Stab}^\dagger(X)$*

4. *For a chamber $\mathcal{C} \subset \mathrm{Stab}^\dagger(X)$, we have $\ell_{\mathcal{C}} = \mathrm{Amp}(\mathbf{M}_{\mathcal{C}}(v))$*

Chapter 4

Main results

This chapter will present the original part of this work.
As anticipated in the introduction, our result is a generalisation to Kummer type hyperkählers of the one proved by Bakker on $K3$-type presented in [Bak15].
His strategy can be broken down to the following steps:

- prove the existence, on moduli spaces, of extremal Lagrangian planes associated to a sublattice with some numerical conditions
- check that all such planes verify the conditions on the line
- prove that all extremal Lagrangian planes on moduli spaces can be reached by the previous construction
- extend to the entire deformation type class via a density argument

Our generalisation follows the same frame and for most of the argument the two cases can be developed in parallel.
Because of this, to avoid redundant constructions, we will try as much as possible to condense analogue lemmas and propositions in one. We will try to empathise where the two cases diverge.

The first section 4.1 will be devoted to recall generalities about Lagrangian planes. We will then fix notations and provide analogues to some results of Bayer and Macrì on rank two sublattices of the Mukai lattice.
In Section 4.3 we will construct the planes and reproduce the calculation for the homology class of a line. Then the last part of the result on moduli spaces will be proved in 4.4 and we will need to point out some differences between the two deformation types, the K3-type and the generalised Kummer: the proofs are almost the same but they differ in some technical details.

Finally in 4.5 we will review how to extend the results to the entire deformation type class. Here the argument of Bakker stays practically unaltered because it relies on general results on hyperkähler manifolds.

One last note on the presentation order: the main results of sections 4.3 and 4.4 do not depend one on the other, and they could be exchanged. The choice of this particular order has been made because it is easier and maybe cleaner to present things this way. On the other hand, section 4.4 could be seen as the motivation to give definitions as we do, and it provides a better reason why we expect this machinery to work out. We will say more on this point of view in the concerned section. Anyway, if the numerical conditions on the sublattice seem a little bit unnatural, we suggest to give a look a few pages after to the characterisation to get a better idea of why this is not so ad hoc as it appears.

4.1 Lagrangian planes

Our main objects of study are Lagrangian planes. In this section we will recall the definition and some generalities.
Let V be a $\mathbb{R}$-vector space of dimension $2n$ and let σ be a symplectic bilinear form.

Definition 4.1.1
A Lagrangian subspace is a maximal isotropic subspace of V with respect to σ

It is very clear that Lagrangian subspaces must be of dimension n. The global analogue of this is a Lagrangian subvariety of a symplectic manifold

Definition 4.1.2
Let (X, σ) be a symplectic manifold
(a.k.a. X is a real manifold of dimension $2n$ and σ a 2-form such that the bilinear form σ_x induced on the tangent space T_xX is symplectic for all $x \in X$).
A Lagrangian subvariety is a submanifold $i : L \hookrightarrow X$ such that for all $l \in L, x = i(l)$ the inclusion $i_* : T_lL \hookrightarrow T_xX$ identifies T_lL to a Lagrangian subspace of T_xX with respect to σ_x

From now on, we will be working on $\mathbb{C}$ in the category of complex manifolds and the symplectic manifold will be a holomorphic symplectic manifold (see definition 1.2.4).
In particular it is interesting to reduce ourselves to grassmanians

Definition 4.1.3
Let X be a holomorphic symplectic manifold.
A Lagrangian grassmanian of X is a Lagrangian subvariety isomorphic to a grassmanian $Gr_{\mathbb{C}}(n,r)$ for some $n,r \in \mathbb{N}$.
A Lagrangian plane of X is a Lagrangian subvariety isomorphic to a projective space $\mathbb{P}^n_{\mathbb{C}}$ for some $n \in \mathbb{N}$.

Notice that a Lagrangian plane is nothing more than a half-dimensional projective space embededd in X.
In fact, all Lagrangian planes must be half dimensional because all Lagrangian subspaces of a $2n$ dimensional space are n dimensional, and at the same time any projective space $\mathbb{P}$ that embeds in X must be isotropic: the restriction of σ to $\mathbb{P}$ is a holomorphic form but on $\mathbb{P}^n_{\mathbb{C}}$ there are no non-trivial holomorphic 2-forms.

Why are grassmanians and projective space so special? One particular, and crucial in what will follow, property is that they are very rigid:

Proposition 4.1.4 ([Bak15] Lemma 11)
Let $G \subset X$ be a Lagrangian Grassmanian in a holomorphic symplectic variety. Then G does not deform as a subscheme. If $G \cong \mathbb{P}$ is a Lagrangian plane, then no curve $C \subset \mathbb{P}$ deforms out of $\mathbb{P}$

Proof. This proof is slightly different from the one of Bakker, and I have to thank Enrico Fatiguenti for sharing it with me.
First, observe that because G is Lagrangian, we get an isomorphism $\Omega_G \cong \mathcal{N}_{G|X}$ where $\mathcal{N}_{G|X}$ is the normal bundle of the immersion $G \hookrightarrow X$. Therefore it is clear that

$$H^0(\mathcal{N}_{G|X}) \cong H^0(\Omega_G) \cong H^{0,1}(G)$$

The Hodge numbers of the Grassmanian are known (they can be obtained by the Schubert cellular decomposition) and in particular the $h^{0,1}(G) = 0$ (more generally, the cohomology of a Grassmanian is non-zero only in even degree and it concentrates in the central $H^{p,p}$).
This prove the first assertion, since $H^0(\mathcal{N}_{G|X})$ implies that there are no deformation of G as a subscheme of X.
For the second part, we consider the normal sequence induced by the embedding ([Stacks, Tag 0473]):

$$0 \to \mathcal{N}_{C|\mathbb{P}} \to \mathcal{N}_{C|X} \to \mathcal{N}_{\mathbb{P}|X_{|C}} \to 0$$

To prove the assertion is reduced to prove that $H^0(N_{\mathbb{P}|X_{|C}}) \cong H^0(\Omega_{\mathbb{P}|C}) = 0$ since then one can deduce that $H^0(\mathcal{N}_{C|\mathbb{P}}) \to H^0(\mathcal{N}_{C|X})$ is an isomorphism.

Tensoring the sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{|C} \to 0$$

provided by the embedding of the curve in $\mathbb{P}$ by $\Omega_{\mathbb{P}}$ we get:

$$0 \to \Omega_{\mathbb{P}} \otimes \mathcal{I}_C \to \Omega_{\mathbb{P}} \to \Omega_{\mathbb{P}|C} \to 0$$

Since $H^0(\Omega_{\mathbb{P}}) = 0$, it is once again possible to move the problem on checking $H^1(\Omega_{\mathbb{P}} \otimes \mathcal{I}_C) = 0$.
Now we consider the Euler sequence:

$$0 \to \Omega_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}} \to 0$$

and we tensor for the ideal of the curve $\mathcal{I}_C$. We get

$$0 \to \Omega_{\mathbb{P}} \otimes \mathcal{I}_C \to \mathcal{I}_C(-1)^{n+1} \to \mathcal{I}_C \to 0$$

Then we have that

$$H^0(\mathcal{I}_C) \to H^1\Omega_{\mathbb{P}} \otimes \mathcal{I}_C \to H^1(\mathcal{I}_C(-1))^{n+1}$$

.
We can see that $H^0(\mathcal{I}_C) = 0$ since it is the kernel of the map $H^0(\mathcal{O}_{\mathbb{P}}) \to H^0(\mathcal{O}_C)$ which is an isomorphism (They both are $\mathbb{C}$ and the map is non-zero).
So in the end, the only thing we need is that $H^1(\mathcal{I}_C(-1)) = 0$. But again from the sequence of the curve we get that this is the case since $H^0(\mathcal{O}_C(-1)) = 0 = H^1(\mathcal{O}_{\mathbb{P}}(-1))$. □

Moreover Lagrangian planes are known to be contractible in the analytic (or even algebraic) category.
However, to assure the contraction can be realised inside the category of complex spaces, we need to ask something more:

Definition 4.1.5
A Lagrangian plane $\mathbb{P} \subset X$ is called extremal if the class of a line in $\mathbb{P}$ is an extremal ray of the Mori cone

As a consequence of the cone theorem, extremal Lagrangian planes can be contracted.

4.2 Notations and rank 2 sublattices

For the rest of this chapter we will refer to this section for notations. Let X be a $K3$ or an abelian surface. We will set

$$\varepsilon_X := \begin{cases} 1 & \text{for } X \text{ a } K3 \text{ surface} \\ 0 & \text{for } X \text{ an abelian surface} \end{cases}$$

As before $\tilde{H}(X,\mathbb{Z})$ will be the Mukai lattice. Fix a Mukai vector $v \in \tilde{H}_{alg}(X,\mathbb{Z})$, with $v^2 > 0$.
For $\sigma \in \mathrm{Stab}^{\dagger}(X)$ we recall that $\mathbf{M}_\sigma(v)$ is the moduli space of σ-semistable sheaves of Mukai vector v and phase 1. We denote

$$\mathbf{Y}_\sigma(v) := \begin{cases} \mathbf{M}_\sigma(v) & \text{for } X \text{ a } K3 \text{ surface} \\ \mathbf{K}_\sigma(v) & \text{for } X \text{ an abelian surface} \end{cases}$$

Definition 4.2.1
A pointed period is a pair $(\tilde{\Lambda}, v)$ where $\tilde{\Lambda}$ is a pure weight 2 Hodge structure on the Mukai lattice with $\dim \tilde{\Lambda}^{2,0} = 1$ and $v \in \tilde{\Lambda}_{alg}$

Definition 4.2.2
A pointed sublattice of a pointed period $(\tilde{\Lambda}, v)$ is a saturated sublattice $\mathcal{H} \subset \tilde{\Lambda}_{alg}$ containing v.

Not all elements of the Mukai lattice can be reached as Mukai vector of some sheaf

Definition 4.2.3
An element $a \in \tilde{H}_{alg}(X,\mathbb{Z})$ is called an effective class if there exist a complex $A \in \mathcal{D}^b(X)$ such that $v(A) = a$

There is a minimal square that can be achieved by effective classes of stable objects in the Mukai lattice:

Lemma 4.2.4
Let $s \in \tilde{H}_{alg}(X,\mathbb{Z})$ be an effective class, coming as Mukai vector of a stable object for some stability condition, then $(s,s) \geq -2\varepsilon_X$

Proof. this is almost trivial because for any stable object $S \in \mathcal{D}^b(X)$ we have

$$(v(S), v(S)) = -2\dim \mathrm{Hom}(S,S) + \dim \mathrm{Ext}^1(S,S) \geq 2\varepsilon_X$$

Definition 4.2.5
A class $s \in \tilde{H}_{alg}(X, \mathbb{Z})$ is said to be of minimal square if it is primitive and $s^2 = -2\varepsilon_X$.
An object $S \in D^b(X)$ is said to be of minimal square if $v(S)$ is a class of minimal square.

Remark. A simple object of minimal square $S \in D^b(X)$ is the same as an object verifying:

$$\begin{aligned} \hom(S,S) = \operatorname{ext}^0(S,S) &= 1 \\ \operatorname{ext}^1(S,S) &= 2(1-\varepsilon_X) \\ \operatorname{ext}^2(S,S) &= 1 \end{aligned}$$

Remark. In the case of $K3$ surfaces, classes of minimal square are spherical classes (alias, classes of square -2), and simple objects of minimal square are spherical objects. Moreover it is clear that spherical classes must be primitive: if $s = m\hat{s}$ then $-2 = s^2 = m^2\hat{s}^2$ which implies $m = 1$.

Definition 4.2.6
A pointed sublattice $\mathcal{H} \subset \tilde{H}_{alg}(X, \mathbb{Z})$ is said to be of $\mathbb{P}$ type if

$$\frac{v^2}{2} = \min_{\substack{s \in \mathcal{H}, \\ s \text{ minimal square}}} |(s, v)|$$

We will show in a few pages that to give a $\mathbb{P}$ type sublattice grants us a decomposition of the Mukai vector in two effective classes of minimal square. We need some preliminary work, which for the case of $K3$ surfaces is contained in [BM14].The proofs hold almost identical for the abelian case, so we group them up.

Proposition 4.2.7
Let $S \in D^b(X)$ σ-semistable of minimal square. Then the Jordan-Holder factors of S are simple objects of minimal square

Proof. Suppose S is not simple (otherwise there is nothing to prove). Then we could get its J-H factors $A_1, ..., A_m$ and the corresponding Mukai vectors $a_1, ..., a_m$. So there would be a map

$$\prod_i \mathbf{M}^s_\sigma(a_i) \to \mathbf{M}_\sigma(s)$$

which locally has a section. So the space on the left must have smaller dimension than the one on the right. This yields $\sum_i a_i^2 + 2m \leq s^2 + 2$.

Because $m \geq 2$ cleary this implies $s^2 \geq a_i^2$ for each i so by the minimality 4.2.4 we get $a_i^2 = -2\varepsilon_X$.
The primitivity of a_i is trivial in the $K3$ case, while in the abelian one we can observe that any primitive vector generating a_i must be isotropic and the proposition 3.2.6 applies.
So each stable factor must be of minimal square.

Remark. Notice that in abelian surfaces the same procedure gives an even stronger result: from $\sum_i a_i^2 + 2m \leq s^2 + 2$ we get $2 - 2\varepsilon_X \geq m(-2\varepsilon_X) + 2m$ and so $0 \geq 2(m-1)(1-\varepsilon_X)$ implying $m = 1$. So we find once again that for semistable objects having primitive isotropic Mukai vector is equivalent to being stable.
So notice that in fact on abelian surfaces S, being of minimal square implies being simple.

Once this preliminary lemma is available, we can recover also in the case of abelian surfaces the following

Proposition 4.2.8
Let $\mathcal{W}$ the potential wall associated to a primitive hyperbolic lattice $\mathcal{H}$, and $\sigma \in \mathcal{W}$ generic.
Then one of the following holds:

- *There are no minimal square classes in $\mathcal{H}$*
- *There is a unique minimal square class $s \in \mathcal{H}$ (up to sign), and there exists $S \in \mathcal{P}_\sigma(1)$ with $v(S) = s$*
- *There are infinitely many minimal square classes, and there are exactly two minimal square classes $s, t \in \mathcal{H}$ such that there exist $S, T \in \mathcal{P}_\sigma(1)$ with $v(S) = s, v(T) = t$.*
  Moreover on abelian surfaces they are the only two classes of minimal square.

Proof. If $s \in \mathcal{H}$ is a minimal square class, $(s, s) \geq -2$ so the moduli space $\mathbf{M}_\sigma(s)$ is non-empty. In other words there exists a σ-semistable object $S \in \mathcal{P}_\sigma(1)$ and $v(S) = s$.
If the minimal square class is unique, then S is simple: otherwise it would have different Jordan-Holder factors, which are of minimal square by 4.2.7, and their Mukai vectors , which are contained in $\mathcal{H}$ by [[BM14], 5.1], would contradict unicity.
For the same reasoning if there are two linearly independent classes of minimal square, then there must be two stable objects $S, T \in \mathcal{P}_\sigma(1)$ of minimal

square, that have independent Mukai vectors. Notice that for $K3$ surfaces a priori they are not the same as those we started with, while for the abelian case they must be the same because of the remark stated above.
Suppose now there are three stable objects $S_1, S_2, S_3 \in \mathcal{P}_\sigma(1)$ of minimal square and set $s_i = v(S_i)$.
Because they are of the same phase, $\hom(S_i, S_j) = \delta_{ij}$ where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$ and by Serre duality the same holds for $ext^2(S_i, S_j) = \delta_{ij}$.
So we can write down the following relations:

$$(s_i, s_j) = -2\delta_{ij} + ext^1(S_i, S_j)$$

which give

$$\begin{aligned}(s_i, s_j) \geq& 0 \qquad \text{for } i \neq j \\ (s_i, s_i) =& -2\varepsilon_X\end{aligned}$$

Now it is a lattice theoretic argument to prove that such s_i's do not exist.
The sublattice $\mathcal{H}$ is a rank 2 lattice of signature $(1, -1)$. We can assume s_1, s_2 to be linearly independent.
Writing $s_3 = xs_1 + ys_2$ and setting $m := (s_1, s_2) = \mathrm{ext}^1(S_1, S_2)$ we get

$$(s_1, s_3) \geq 0 \implies ym \geq 2x\varepsilon_X \tag{4.1}$$
$$(s_2, s_3) \geq 0 \implies xm \geq 2y\varepsilon_X \tag{4.2}$$
$$(s_3, s_3) = -2\varepsilon_X \implies 2xym + (-2\varepsilon_X)(x^2 + y^2 - 1) = 0 \tag{4.3}$$

from the last one we can conclude

$$2xym = 2\varepsilon_X(x^2 + y^2 - 1) \leq ymx + xmy - 2\varepsilon_X = 2xym - 2\varepsilon_X$$

and therefore

$$0 \leq -2\varepsilon_X$$

Therefore in the case of $K3$ surfaces we are done.
For abelian surfaces, it is enough to observe that $(s_3, s_3) = -2\varepsilon_X$ reads:

$$2xym = 0$$

therefore forcing either x or y to be 0.
But then s_3 is a multiple of s_2 or s_1, which contradicts primitivity. □

Now we prove the decomposition assuming a minimality condition on v. This will turn out not to be restrictive in any way: if the decomposition holds for such a minimal v then via reflections we can extend the result to any v. We will say that

Definition 4.2.9
The Mukai vector v is said to be minimal for $\mathcal{H}$ if

$$(v, s) \geq 0, \text{ for any effective class of minimal square } s \in \mathcal{H}$$

Proposition 4.2.10
Let $\mathcal{H}$ be a $\mathbb{P}$ type sublattice, and $\sigma_0 \in \mathcal{W}_{\mathcal{H}}$ a generic condition on the wall. If v is minimal for $\mathcal{H}$, then there exist two objects of minimal square $S, T \in \mathcal{P}_{\sigma_0}(1)$ such that $v(S), v(T) \in \mathcal{H}$ and $v = v(S) + v(T)$.

Proof. Let $s \in \mathcal{H}$ one of the minimal square class realising the minimum $(s, v) = \frac{v^2}{2}$ in the definition of $\mathbb{P}$ type.
Then $t := v - s$ is of minimal square:

$$t^2 = v^2 + s^2 - 2(v, s) = s^2 = -2\varepsilon_X$$

and t also realise the minimum

$$(t, v) = v^2 - (s, v) = \frac{v^2}{2}$$

This fact also grants us that t is primitive: otherwise $t = m\hat{t}$ would give $\frac{v^2}{2} = m(\hat{t}, v)$ and $(\hat{t}, v) < \frac{v^2}{2}$ that contradict the fact that $\frac{v^2}{2}$ is a minimum.
By 4.2.8 we get that there exist two simple objects $S, T \in \mathcal{P}_{\sigma_0}(1)$ of minimal square having linearly independent Mukai vectors s_0, t_0.
Writing $s = xs_0 + yt_0$ (with $x, y \in \mathbb{N}$) we apply the minimality and the definition of $\mathbb{P}$ type:

$$\frac{v^2}{2} = (s, v) = x(s_0, v) + y(t_0, v) \geq (x + y)\frac{v^2}{2}$$

therefore $x = 1$ and $y = 0$ or viceversa, and up to exchanging s and t we have $s = s_0$ and $t = t_0$. □

As anticipated before, the assumption of minimality is not required.

Proposition 4.2.11
Let $\mathcal{H}$ be a $\mathbb{P}$ type sublattice, and $\sigma \in \mathcal{W}_{\mathcal{H}}$ a generic condition on the wall. Then there exist two objects of minimal square $S, T \in \mathcal{P}_{\sigma_0}(1)$ such that $v(S), v(T) \in \mathcal{H}$ and $v = v(S) + v(T)$

Proof. The case of abelian surfaces is the easier one: notice that in the previous proof the minimality was used to prove that $s_0, t_0 = s, t$. This is automatic for abelian surfaces.
In the case of $K3$ something more must be said: it has been proved in [BM14,

Prop. 6.8] that there is a sequence of spherical reflections $R := R_{s_1} \circ ... \circ R_{s_m}$ taking v to a minimal vector. We can therefore get the decomposition after the reflections for $v_0 := R(v) \in R(\mathcal{H})$: for a generic $\sigma \in \mathrm{Stab}^\dagger(X)$ there exist $S_0, T_0 \in \mathcal{P}_{\sigma_0}(1)$ with $v(S_0) = s_0, v(T) = t_0$ classes of minimal square realising the minimum in the definition of $\mathbb{P}$ type such that $v_0 = s_0 + t_0$. To each spherical reflection corresponds a spherical twist $\mathcal{T}_i : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ which is an autoequivalence of categories inducing R_{s_i} in cohomology. Reversing this sequence we get $S = \mathcal{T}_m^{-1} \circ ... \circ \mathcal{T}_1^{-1}(S_0)$ and similarly for T. Their Mukai vector $s = R^{-1}(s_0), t = R^{-1}(t_0)$ are still of minimal square and realise the minimum. $\square$

Finally we want to point out that this decomposition is actually an equivalent definition of $\mathbb{P}$ type lattice:

Proposition 4.2.12
Let $\mathcal{H} \subset \tilde{H}_{alg}(X, \mathbb{Z})$ be a rank two pointed sublattice such that for $\sigma_0 \in \mathcal{W}_{\mathcal{H}}$ there exist two simple object of minimal square $S, T \in \mathcal{P}_{\sigma_0}(1)$ with Mukai vector $v(S) = s, v(T) = t$ giving a decomposition $v = s + t$, $s, t \in \mathcal{H}$ and $(s, v) = \frac{v^2}{2}$. Then $\mathcal{H}$ is of $\mathbb{P}$ type.

Proof. Clearly s, t are of minimal square (for the same reasoning used to prove t was of minimal square in 4.2.10) and by 4.2.8 they are the only two coming as Mukai vectors of simple objects of minimal square of phase 1.
So for any other effective class of minimal square α we can write $\alpha = xs + yt$ for $x, y \in \mathbb{N}$ not both zero. Therefore $|(\alpha, v)| = |(x + y)|\frac{v^2}{2} \geq \frac{v^2}{2}$

4.3 Construction of Lagrangian planes

We are now ready to construct the Lagrangian plane.
The idea is the following: we consider the objects granted by the proposition of last section, then we take extensions of the two. This is a projective space of the right dimension. The rest of the proposition serves to give a canonical way to embed it in the moduli space.
There is only one caveat: while for $K3$ surfaces there is no choice for the two stable objects and therefore we get immediatly a single projective space, for abelian surfaces we don't really have a way to canonically choose the two objects, so we get a projective bundle over a 4-fold parametrising the possible choices.

Firstly, we review an application of the filtration in families presented in [BM14]: take the moduli space $\mathbf{M}_\sigma(v)$ a stability condition $\gamma \in \mathrm{Stab}^\dagger(X)$

and a partition $P = [v = \sum_i a_i]$.
Consider $M_{\gamma,P}$ the closed stratum of points parametrising S-equivalence classes of objects which J-H partition with respect to γ is P.
Then suppose for a moment $\mathbf{M}_\sigma(v)$ admits a universal family $\mathcal{E}$.
Taking the filtration in families with respect to γ of $\mathcal{E}_{|M_{\gamma,P}}$ we get a sequence $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to ... \to \mathcal{E}_n$. The quotient of a family of γ (semi)stable objects is still a family of γ (semi)stable objects. So the quotients $\mathcal{A}_i := {}^{\mathcal{E}_{i+1}}/_{\mathcal{E}_i}$ are families of σ stable modules and on an open subset $U \subset M_{\gamma,P}$, the fibers $(\mathcal{A}_i)_{|x}$ for $x \in U$ are the Jordan Holder factors of $\mathcal{E}_x$ with respect to γ.
But given the description of $M_{\gamma,P}$, it is clear that $\mathcal{A}_i$ are families of γ-stable objects of Mukai vector a_i.
Therefore we get a rational map

$$JH : M_{\gamma,P} \dashrightarrow \mathbf{M}^s_\gamma(a_1) \times ... \times \mathbf{M}^s_\gamma(a_n)$$

that to a point (in the open set of definition) parametrising a complex associates the tuple of its JH factors.

If a universal family doesn't exist, we can still define the map on an étale neighbourhood, where a universal family exists, and the defined map will factor trough the moduli space (see [BM14] in the proof of Lemma 6.5).
Moreover it is clear that we have a section associating to the Jordan-Holder factor the trivial extension.
What are the fibres of this map?
They correspond to possible extensions of the Jordan-Holder factors.
The following proposition constructs explicitly the fibers for partition of lenght 2:

Proposition 4.3.1
Let σ be a (generic) stability condition with respect to v, $\gamma \in Stab(X)$.
Given a partition $P = [v = s + t]$ with $(s,t) \neq 0$ let JH be defined as above on $M_{\gamma,P}$.
Let $S, T \in \mathcal{P}_\sigma(1)$ be simple objects such that $v(S) = s, v(T) = t$, then

$$JH^{-1}((S,T)) \cong \mathbb{P}(\mathrm{Ext}^1(S,T))$$

Proof. Fix $\mathbb{P} := \mathbb{P}(\mathrm{Ext}^1(S,T))$ and $p_X : \mathbb{P} \times X \to X$ the projection.
Define $E \in D^b(X)$ as the cone:

$$(p_X^* S)(1) \to E \to p_X^* T \to (p_X^* S)(1)[1]$$

Reducing to the fiber on $l \in \mathbb{P}$ we get a triangle

$$S \to E_l \to T \to S[1] \tag{4.4}$$

so $v(E_l) = v(S) + v(T) = v$.
It remains to prove E_l is γ-semistable.
But we can observe that taking $\mathrm{Hom}(T, -)$ on the last triangle we get an exact sequence

$$\begin{array}{ccccccc} \mathrm{Hom}(T,S) & \longrightarrow & \mathrm{Hom}(T,E_l) & \longrightarrow & \mathrm{Hom}(T,T) & \longrightarrow & \mathrm{Hom}(T,S[1]) \\ \cong & & & & \cong & & = \\ 0 & & & & \mathbb{C} & & \mathrm{Ext}^1(T,S) \neq 0 \end{array}$$

the last map in the sequence acts as multiplication by a scalar, so it is injective if and only if the last term is non zero. But this clearly happens, because

$$\dim \mathrm{Hom}(T, S[1]) = ext^1(T,S) = (v(T), v(S)) \neq 0$$

Using the fact that $\mathrm{Hom}(T, E_l) = 0$ we can invoke the following category theory lemma:

Lemma 4.3.2
Let A, B simple object of an abelian category and E an object fitting into a sequence:

$$A \hookrightarrow E \twoheadrightarrow B^y$$

such that $\mathrm{Hom}(B, E) = 0$.
Then every proper quotient of E is of the form B^z for some $z \in \mathbb{N}$

Therefore every proper quotient of E_l is of the form T^z for some $z \in \mathbb{N}$ and so it has the same phase as T with respect to γ. So E_l cannot be destabilised, in other words it is γ-semistable and E induces a morphism $\mathbb{P} \to \mathbf{M}_\sigma(v)$. To see it realises an isomorphism with the fibre, it is obvious that the image is contained in the fibre (in 4.4 we see that S, T are the JH factors of E_l) and because of the dimension it must be an isomorphism. □

Our Lagrangian planes arise simply as fibres of bundles like those in the last proposition starting from the decomposition of 4.2.10.
Here the case of abelian surfaces is slightly more subtle: we need to intersect the bundle with the Kummer $\mathbf{K}_\sigma(v)$.
To state the two cases as one, we say:

Proposition 4.3.3
Let $\mathcal{H}$ be a $\mathbb{P}$ type sublattice, then there are lagragian planes in $\mathbf{Y}_\sigma(v)$.
In particular for generic $\sigma_0 \in \mathcal{W}_\mathcal{H}$ and $P = [v = s + t]$ the corresponding partition, there is an open set $U \subset M_P$ such that $Y_\sigma(v) \cap U$ is a union of

isolated Lagrangian planes.
In the case of a $K3$ surface, U can be taken to be M_P and there is only one component (only one plane appears in the union)

Proof. By 4.2.10 we have for generic $\sigma_0 \in \mathcal{W}_{\mathcal{H}}$ the decomposition P and two minimal square objects $S, T \in \mathcal{P}_{\sigma_0}(1)$ with Mukai vecors respectively s, t.
We can apply 4.3.1 to get an open dense set $U \subset M_P$ with a map $U \to \mathbf{M}^s_{\sigma_0}(s) \times \mathbf{M}^s_{\sigma_0}(t)$, which is a fiber bundle with fiber $\mathbb{P}^n$, where $n+1 = (s,t)$.
It is easy to verify that $\dim \mathbf{Y}_\sigma(v) = 2n$:

$$2n = 2(s,t) - 2 = 2(s,v) - 2(s,s) - 2 = v^2 + 4\varepsilon_X - 2 = \dim Y$$

In the case of $K3$ surfaces, there is nothing more to prove: the base $\mathbf{M}^s_{\sigma_0}(s) \times \mathbf{M}^s_{\sigma_0}(t)$ is a single point.
Hence U is the fibre, it is isomorphic to $\mathbb{P}^n$ and therefore $M_P = U$ by density.
The Kummer case is slightly more subtle.
We need to prove that:

1. Fibres are contained in $\mathbf{K}_\sigma(v)$

2. The intersection of $\mathbf{K}_\sigma(v)$ with the base is transverse (this only makes sense locally using a section)

1:
Take a fibre $\mathbb{P}$, which we know to be isomorphic to $\mathbb{P}^n$.
Then it would induce a rational curve via the Albanese map:

$$\mathbb{P} \subset \mathbf{M}_\sigma(v) \to X \times X\check{}$$

But there are no rational curves on an abelian variety.
So $\mathbb{P}$ is contained in a fibre of the Albanese map, which by definition is $\mathbf{K}_\sigma(v)$
2:
By the previous point we know that each fiber is contained in one of the fibers of the Albanese map, so fixed $\mathbf{K}_\sigma(v)$ one specific fiber is either contained in it or disjoint.
If the planes were not isolated then we would get a deformation of the Lagrangian plane inside the Kummer. But this is not possible because of the rigidity (see lemma 4.1.4) □

Now, observe that on planes arising from this construction we can calculate the class of a line.

Proposition 4.3.4
Let $R \in H_2(X, \mathbb{Z})$ be the class of a line on the Lagrangian plane $\mathbb{P}$ obtained by the spherical lattice $\mathcal{H}$. Let $s, t \in \mathcal{H}$ be the two classes of minimal square realising the minimum, as in the construction of proposition 4.3.3.
Then $R = \pm\theta\check{}(s)$, where θ is the Mukai homomorphism defined in 3.3.7.
Moreover, crossing $\mathcal{W}_{\mathcal{H}}$, R changes the sign: so $\mathbb{P}$ is extremal.

Proof. Up to exchanging names we can suppose $\phi(S) < \phi(T)$.
Take $\mathcal{C}$ a curve in $\mathbf{M}_\sigma(v)$.
Then

$$\theta(w).\mathcal{C} = (w, v(\Phi_E((O)_C)))$$

To prove the thesis it is enough to assume $w \in v^\perp$.
So in our case, if R is the class of a certain $\mathbb{P}^1 \subset \mathbf{M}_\sigma(v)$ then

$$\theta(w).R = (w, v(p_{X*}E_{|\mathbb{P}^1\times X}))$$

To calculate the last part we can use the K-theory and this very useful projection formula:

Proposition 4.3.5 ([Har13], ch III ex 8.3)
Let $X \to Y$ a morphism of ringed spaces, let $\mathcal{F}$ an $\mathcal{O}_X$-module and $\mathcal{E}$ a locally free $\mathcal{O}_Y$-module of finite rank.
Then for each $i \in \mathbb{N}$

$$\mathrm{R}^i\, f_*(\mathcal{F} \otimes f^*\mathcal{E}) \cong \mathrm{R}^i\, f^*(\mathcal{F}) \otimes \mathcal{E}$$

so in K-theory we can write:

$$p_{X*}E_{|\mathbb{P}^1\times X} = p_{X*}((p_X^*S)(1)_{|\mathbb{P}^1\times X}) + p_{X*}p_X^*T_{|\mathbb{P}^1\times X} = S \cdot p_{X*}p_{\mathbb{P}^1}^*\mathcal{O}_{\mathbb{P}^1}(1) + T$$

We can calculate $p_{X*}p_{\mathbb{P}^1}^*\mathcal{O}_{\mathbb{P}^1}(1)$ as:

Lemma 4.3.6
Let X be a projective variety, let $p_X, p_{\mathbb{P}^n}$ be the projection from $X \times \mathbb{P}^n$.
Then for $k \geq 0$:

$$p_{X*}p_{\mathbb{P}^n}^*\mathcal{O}_{\mathbb{P}^n}(k) \cong \mathcal{O}_X^{\oplus\binom{n+k}{n}}$$

Proof. It follows by a direct computation using proposition 4.3.5:
take an open set $U \subset X$. Then

$$p_{X*}p_{\mathbb{P}^n}^*\mathcal{O}_{\mathbb{P}^n}(k)(U) = p_{\mathbb{P}^n}^*\mathcal{O}_{\mathbb{P}^n}(k)(p_X^{-1}(U)) = p_{\mathbb{P}^n}^*\mathcal{O}_{\mathbb{P}^n}(k)(\mathbb{P}^n \times X)$$

The pull back is defined with the inverse limit, but the only open subset containing $p_{\mathbb{P}^n}(\mathbb{P}^n \times U) = \mathbb{P}^n$ is the entire $\mathbb{P}^n$ so we continue:

$$\begin{aligned} p_{\mathbb{P}^n}^* \mathcal{O}_{\mathbb{P}^n}(k)(\mathbb{P}^n \times U) &= \mathcal{O}_{\mathbb{P}^n}(k)(\mathbb{P}^n) \otimes_{\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n)} \mathcal{O}_{X\times\mathbb{P}^n}(U \times \mathbb{P}^n) \\ &= \mathbb{C}^{\binom{n+k}{n}} \otimes_{\mathbb{C}} \mathcal{O}_X(U) = \mathcal{O}_X(U)^{\oplus\binom{n+k}{n}} \end{aligned}$$

□

so in K theory $p_{X*}p_{\mathbb{P}^1}^*\mathcal{O}_{\mathbb{P}^1}(1) = 2$ and therefore

$$\theta(w).R = (w, v(2S + T)) = (w, 2s + t) = (w, v + s) = (w, s)$$

On the other side of the wall, we have that $\phi(S) > \phi(T)$, so we get

$$R = \check{\theta}(t) = \check{\theta}(-s) = -\check{\theta}(s)$$

. □

4.4 Characterisation of extremal Lagrangian planes

In this section we will see that actually every extremal Lagrangian plane can be obtained by the construction of 4.3.3.
As we already said, the order of these two sections could be exchanged. While the chosen one is more direct, this part is more geometrical and could be seen as the motivation to look for $\mathbb{P}$ type sublattices in the first place.
We will summarise the ideas before entering the proper proof.
It is a general fact descending from the cone theorem that extremal Lagrangian planes can be contracted to a point: it is enough to observe that, as the class of a line is extremal, we can contract it and in doing so we contract all the the span of the line, which surely includes the plane. A priori we could be contracting much more than the Lagrangian plane.
Bayer and Macrì [BM14] provided that minimal models on moduli spaces can be runned via wall crossing, so the contraction of the Lagrangian plane must be a map between moduli spaces obtained by crossing some wall.
Therefore it is natural to study the properties of the corresponding sublattice and we will see that it is of $\mathbb{P}$ type.
Moreover the contraction sends the plane on a point in a moduli space, alias a S-equivalence class. This naturally gives a stratum associated with the partition corresponding to the equivalence class, and in the proof we will see that (for $K3$ surfaces) this is exactly the plane that we started with.

All this gives a strong motivation for all the previous section and I think very much clarifies the geometric picture.
Before getting into the main result of this section, notice that

Lemma 4.4.1
Fix a Mukai vector v and a generic stability condition σ. Let γ another stability condition and A_i be γ-stable objects such that $a_i = v(A_i)$.
Let $P = [v = \sum a_i]$and $M_{\gamma,P} \subset \mathbf{M}_\sigma(v)$ the corresponding locally closed stratum. Let $C \subset M_{\gamma,P}$ a subvariety and $c \in C$ a point whose J-H factor with respect to γ are the A_i.
Then if the A_i deform on some base S, C deforms as well locally around c in $M_{\sigma,P}$ on the same base S .

Proof. Consider the rational map $M_{\gamma,P} \dashrightarrow \bigtimes_i \mathbf{M}_\gamma(a_i)$. The argument is local so we will abuse the notation writing morphisms defined locally as morphisms. Notice that at least locally it has a section (taking direct sum of universal families)
Let c_i the image of c via $C \to M_P \to \bigtimes_i \mathbf{M}^s_\gamma(a_i) \to \mathbf{M}^s_\gamma(a_i)$.
Since deformations of stable objects on a surface are not obstructed the deformation of A_i over S gives a deformation of c_i in $\mathbf{M}^s_\gamma(a_i)$ over S as $\mathbf{M}^s_\gamma(a_i) \supset \mathcal{C}_i \to S$.
Taking the fiber product over S we get a deformation $\mathcal{C} = \bigtimes_S \mathcal{C}_i$ in the product of moduli spaces:

$$\begin{array}{ccccccc} & & & & M_{\gamma,P} & \subset & \mathbf{M}_\sigma(v) \\ & \nearrow & & & \downarrow & & \\ C & \longrightarrow & \mathcal{C} & \longrightarrow & \bigtimes \mathbf{M}^s_\gamma(a_i) & & \\ \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & S & & & & \end{array} \tag{4.5}$$

Using the local section $\bigtimes \mathbf{M}^s_\gamma(a_i) \to M_{\gamma,P}$ gives the desired deformation. □

Proposition 4.4.2
Let $v \in \tilde{H}_{alg}(X, \mathbb{Z})$ primitive and $v^2 > 0$, and let σ be a generic stability condition w.r.t. v.
If $\mathbb{P} \subset \mathbf{Y}_\sigma(v)$ is an extremal Lagrangian plane, then there is a $\mathbb{P}$ type sublattice $\mathcal{H} \subset \tilde{H}_{alg}(X, \mathbb{Z})$.
Moreover, for a generic stability condition $\sigma_0 \in \mathcal{W}_\mathcal{H}$, if $s \in \mathcal{H}$ is a minimal square class realising the minimum $(s, v) = \frac{v^2}{2}$ and $P := [v = s + (v - t)]$,

there is an open dense set U of M_P s.t $\mathbb{P}$ is one of the connected components of $U \cap \mathbf{Y}_\sigma(v)$

Proof. The proofs diverge a little in the two cases, even if the structure is really similar.
Since $\mathbb{P} \subset \mathbf{Y}_\sigma(v)$ is extremal we get a contraction of the Lagrangian plane $\pi : \mathbf{Y}_\sigma(v) \to M$.
But from 3.4.9, this contraction must be obtained crossing a wall. Therefore we know that π contracts objects of an S equivalence class. Let $A_1, ..., A_m$ be the Jordan Holder factors of the S-equivalence class contracted by π, let $a_i := v(A_i)$ and let $P := [v = \sum_i a_i]$ the associated partition.
Now the key observation is that

Lemma 4.4.3
*The A_i's are **rigid** on $\mathbf{Y}_\sigma(v)$ so they are simple object of minimal square Moreover*

$$\sum ext^1(A_i, A_i) \leq \operatorname{codim}_{\mathbf{M}_\sigma(v)} \mathbf{Y}_\sigma(v)$$

Proof. Suppose A_i deform on a base S_i. Then composing with $S_i \to S := \bigtimes_i S_i$ gives a deformation over S for each A_i.
Let $C \subset \mathbb{P}$ be a curve. By 4.4.1 locally around a point C deforms as well over S. But the deformations of C would not contract via π: locally A_i stay stable for σ_0 so the Jordan Holder factors of a point in the deformation of $\mathcal{C}$ over a point of $(b_1, ..., b_k) \in S$ are the corresponding deformation of A_i over $b_i \in S_i$.
This would imply that $\mathcal{C}$ has deformed outside $\mathbb{P}$, but this is not possible by 4.1.4.
The last part is easy to deduce: it suffices to observe that $\dim S = \sum_i ext^1(A_i, A_i)$ and that the previous reasoning actually gives that $\mathbf{Y}_\sigma(v)$ intersect B transversally (locally there is a section at least as a differentiable bundle, but differentiability is enough to compare dimensions). □

Here the two cases really diverge.

- **abelian surface**: In the abelian case, we know that $\operatorname{codim}_{\mathbf{M}_\sigma(v)} \mathbf{Y}_\sigma(v) = 4$ and the A_i must have $ext^1(A_i, A_i) \geq 2$. Therefore we get that if $\#A_i$ is the number of factors:

$$4 \geq \sum ext^1(A_i, A_i) \geq 2\#A_i$$

  Clearly $\#A_i$ can't be 1, because otherwise the S-equivalence class for σ_0 would have been composed of one single element, that would have

been part of only one S-equivalent class for σ. This would have implied that only one point of $\mathbf{M}_\sigma(v)$ would have contracted via π to $\pi(\mathbb{P})$, which is false.
So $\#A_i$ must be 2 and consequently $ext^1(A_i, A_i) = 2$.
This in addition to the fact that the A_i are simple gives us that there are only two a_i and that they are primitive isotropic vectors.
We conclude that $\mathcal{H}$ is of $\mathbb{P}$ type by 4.2.12.

- **K3 surface**: For $K3$ surfaces the bound with the codimension is not very interesting: $\mathbf{Y}_\sigma(v) = \mathbf{M}_\sigma(v)$ so $\operatorname{codim} \mathbf{Y}_\sigma(v) = 0$, and rigidity of A_i is global, so the inequality reduces to $0 \geq 0$.
But in this case the fact that the two classes s, t are unique in $\mathcal{H}$ (proved in 4.2.8) allows us to write $P = [xs + yt]$ because they are the only two classes available (in other words each a_i must be s or t).
Up to exchanging names we can suppose $\phi(S) < \phi(T)$ with respect to σ. Due to the fact that S and T are rigid (so they do not self extend) we can write the Harder Narasimhan filtration for every E represented by a point in $\mathbb{P}$ as:
$$S^x \to E \to T^y \to S^x[1] \tag{4.6}$$
Using this fact we can prove that

Claim. The locally closed stratum M_P is a subscheme of the scheme of maps $\mathbb{P}^{x-1} \times \mathbb{P}^{y-1} \to \mathbb{P}Ext^1(S, T)$ quotiented by the action of $PGL(x) \times PGL(y)$

(For the proof of this claim, see Appendix A) Therefore we get a bound on the dimension of M_P:

$$\dim M_P \leq xy(s,t) - (x^2y^2) = \frac{v^2}{2} + x^2 + y^2 - x^2y^2 = n - (x^2-1)(y^2-1)$$

But at the same time, $\mathbb{P} \subset M_P$ so $n = \dim \mathbb{P} \leq \dim M_P$.
This forces either $x = 1$ or $y = 1$ (because they are positive by definition).
If $y = 1$ (and similarly otherwise) then $M_P = Gr(x, (s,t))$, but once again $\mathbb{P} \subset M_P$ implies $x = 1$ and $\mathbb{P} = M_P$.

□

4.5 Extension to deformation type

Once the result is proved for moduli spaces the procedure to extend it to the entire deformation type is exactly the same in the cases of $K3$ type and generalised Kummer: Bakker uses results that do not depend on the assumption of $K3$ type, referring to arguments that hold for general hyperkähler.
The idea is simply that given a general hyperkähler of $K3$ type or generalised Kummer the moduli spaces are dense in the Kuranishi family (the universal family of deformation for hyperkähler), so we can find a general deformation which is a moduli space. Then if we have a Lagrangian plane on the original manifold this deforms to the moduli space, where we can use the result just proven to get the $\mathbb{P}$ type sublattice and deform it back. For primitive generators of extremal rays, we can also do the process in the other direction: we start from the homology class verifying the numerical conditions, we get a $\mathbb{P}$ type sublattice, we deform it to a moduli space to obtain a Lagrangian plane that we deform back.

Let M be a $K3$ type manifold of dimension $2n$. There is a monodromy invariant extension ([Mar11] Corollary 9.5):

$$0 \to H^2(M, \mathbb{Z}) \to \tilde{\Lambda}(M) \to \mathcal{Q} \to 0$$

where $\tilde{\Lambda}$ is a pure weight-2 Hodge structure on the Mukai lattice of M polarised by the intersection form and $\mathcal{Q}$ is of rank 1.
We will write $v(M)$ for a primitive generator of $H^2(X, \mathbb{Z})^{\perp} \subset \tilde{\Lambda}(M)$ and $\theta\check{} : \tilde{\Lambda} \to H_2(M, \mathbb{Z})$ the dual of the embedding.

Almost the same happens for the abelian case: see [Wie18, Thm. 4.9]
These notations coincide with those we gave more explicitly on moduli spaces and allow us to state a characterisation of the Mori cone:

Theorem 4.5.1 ([BHT13] Thm. 1, [KLCM15] Thm.2.9)
Let (M, h) be a polarised holomorphic symplectic variety of $K3$ type or of generalised Kummer type. The Mori cone of M is generated by the positive cone and classes of the form

$$\left\{ \theta\check{}(a) | a \in \tilde{\Lambda}(M)_{alg}, a^2 \geq -2\varepsilon_X, |(a, v)| \leq \frac{v^2}{2}, h.\theta(a) > 0 \right\}$$

From results of Voisin [Voi92] and Ran [Ran95] we have, respectively, that deformations of the manifold for which a Lagrangian subvariety deform staying Lagrangian are those preserving some Hodge structure and that they

are unobstructed . More precisely if $i : L \hookrightarrow M$ is a Lagrangian subvariety of a symlpectic manifold M, the deformation are those that preserve the Hodge structure $\ker i^* \subset H^*(M, \mathbb{Z})$.
This allows Bakker to prove the following lemma for the $K3$ type case, but the same proof holds for Kummer type:

Theorem 4.5.2
Let (M, h) be an holomorphic symplectic variety of $K3$ (resp. Kummer) type and dimension $2n$, with $\mathbb{P}$ a Lagrangian plane, and let $R \in H_2(M, \mathbb{Z})$ be the class of the line. Then $\tilde{\Lambda}(M)$ admits a $\mathbb{P}$ type sublattice $\mathcal{H}$ and $R = \theta\check{}(s)$ with $s \in \mathcal{H}$ a class of minimal square with $|(v, s)| = \frac{v^2}{2}$

Proof. Because $R^2 < 0$ the argument in ([BHT13] Prop.3) provides a smooth proper family $\mathcal{M} \to B$ over an irreductible analytic base such that on some $0 \in B$ where it specialises to M the local system $\mathrm{R}^{2n-2} f_* \mathbb{Z}$ admists an algebraic section specialising to R. This means that paths in B induce via parallel transport Hodge isometries in cohomology.
Therefore the Lagrangian plane $\mathbb{P}$ deforms to the general fiber of this deformation.
Periods of moduli spaces are dense in the Kuranishi family, so we can always find a specialisation of the deformation to a moduli space for which the plane $\mathbb{P}$ does not degenerate and $\mathbb{P}$ is extremal (this two conditions are open, see [BHT13]) and a path in the base connecting the period of (M, h) and of the moduli space.
So the plane deforms to an extremal Lagrangian plane on a moduli space where we have a $\mathbb{P}$ type lattice $\mathcal{H}$ and the class of a line is as wanted.
Transporting back, via parallel transport along the path, $\mathcal{H}$ to M gives the $\mathbb{P}$ type sublattice, as monodromy operators are isometries.

Recall that the discriminant group of a lattice Λ is the quotient Λ^*/Λ where $\Lambda^* := \mathrm{Hom}(\Lambda, \mathbb{Z})$ is the dual lattice.
Now we can just do the following computation:

Lemma 4.5.3
If $\tilde{\Lambda}(M)$ admits a class of minimal square $a \in \tilde{\Lambda}(M)$ such that $(a, v) = \frac{v^2}{2}$, then $R = \theta\check{}(a)$ has $(R, R) = -\frac{n+1+2\varepsilon_X}{2}$ and order 2 in the discriminant group of $H^2(M, \mathbb{Z})$

Proof. Remember that $\theta\check{}$ is the orthogonal projection onto $v^\perp$, composed with the inclusion $H^2(M, \mathbb{Z}) \to H_2(X, \mathbb{Z})$ we have

$$(R, R) = (a - \frac{v}{2})^2 = a^2 - (a, v) + \frac{v^2}{4} = -2\varepsilon_X - \frac{v^2}{4}$$

On the other hand we know that

$$\frac{v^2}{2} = n + 1 - 2\varepsilon_X$$

so we get

$$(R, R) = -2\varepsilon_X - \frac{v^2}{4} = -\frac{4\varepsilon_X + n + 1 - 2\varepsilon_X}{2} = -\frac{n + 1 + 2\varepsilon_X}{2}$$

Clearly $2a - v \in v^{\perp}$ so R is 2-torsion ($2R \in H^2(X, \mathbb{Z}) = v^{\perp} \subset \tilde{\Lambda}(M)$), but $R \neq 0$ in the discriminant group, so R has order 2.

to conclude:

Corollary 4.5.4
Let (M, h) be as above and let R be the class of a line of a Lagrangian plane. Then $(R, R) = -\frac{n+1+2\varepsilon_X}{2}$ and $2R \in H^2(M, \mathbb{Z})$.

Remark. Our result is in alignment to that of Hasset and Tshinkel [HT10] on Kummer fourfolds: in that case $n = 2$ and so $(R, R) = -\frac{3}{2}$

The partial converse Thm 25 in [Bak15], holds unaltered:

Theorem 4.5.5
Let (M, h) as above, and let $R \in H_2(M, \mathbb{Z})$ be a primitive generator of an extremal ray of the Mori cone. Then R is the class of a line in a Lagrangian plane if and only if $(R, R) = -\frac{n+1+2\varepsilon_X}{2}$ and $2R \in H^2(M, \mathbb{Z})$.

Proof. One direction of the implication is a direct consequence of the previous theorem.
The extremality of R implies that because of the description of the Mori cone , $R = m\theta\check{}(a)$ for some a as in 4.5.1, and primality forces $m = 1$.
To construct the $\mathbb{P}$ type lattice $\mathcal{H}$ one can use 4.2.12: defining $b := v - a$ gives the decomposition that proves $\mathcal{H} = \mathrm{Span}_{\mathbb{Z}}(a, b)$ is of $\mathbb{P}$ type. The only thing to verify is that $(a, v) = \frac{v^2}{2}$, but this is a consequence of the numerical condition:

$$-\frac{n + 1 + 2\varepsilon_X}{2} = (R, R) = (a - \frac{(a, v)}{v^2} v)^2 = a^2 - \frac{(a, v)^2}{v^2}$$

implies

$$\frac{(a, v)^2}{v^2} = a^2 + \frac{n + 1 + 2\varepsilon_X}{2} = -2\varepsilon_X + \frac{n + 1 + 2\varepsilon_X}{2} = \frac{n + 1 - 2\varepsilon_X}{2} = \frac{v^2 + 2 - 2}{4} = \frac{v^2}{4}$$

so $(a, v) = \frac{v^2}{2}$, and therefore $\mathcal{H}$ is of $\mathbb{P}$-type.
Now, as in the previous theorem, we have a smooth proper family along which R stays algebraic that specialises to a moduli space on which the image of R is extremal. So we have an extremal Lagrangian plane on the moduli space. Deforming it back to M gives the desired Lagrangian plane $\mathbb{P}$ having class of a line R. It doesn't degenerate because of the extremality and primitivity conditions on R. □

Appendix A

Proof of Claim in 4.4.2

Here we will give some more details about the proof of the claim in Thm 4.4.2. We have decided to isolate this part of the proof and relegate it in the appendix because it is fairly involved, using some results and calculations on triangulated categories, while not adding much to the geometric picture. It would have moved the focus from the main objective of Section 4.4.
First we need to derive a classical consequence in the context of derived category of the Five Lemma. First observe that :

Proposition A.0.1
Let $\mathcal{D}$ be a triangulated category. Let $A \in \mathcal{D}$. Then any distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow X[1]$$

induces an exact sequence

$$\mathrm{Hom}(A, X) \xrightarrow{f_*} \mathrm{Hom}(A, Y) \xrightarrow{g_*} \mathrm{Hom}(A, Z)$$

Proof. Clearly the composition $g_* \circ f_*$ is 0 by functoriality.
To check exactness, take some $\alpha : A \to Y$ such that $g_*(\alpha) = 0$. Then we have a diagram:

$$\begin{array}{ccccccc} A & \longrightarrow & 0 & \longrightarrow & A[1] & \xrightarrow{\mathrm{id}_{A[1]}} & A[1] \\ \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow{\scriptstyle h[1]} & & \downarrow{\scriptstyle \alpha[1]} \\ Y & \xrightarrow{g} & Z & \longrightarrow & X[1] & \xrightarrow{f[1]} & Y[1] \end{array}$$

where the last row is a distinguished triangle by $TR2$, the first by $TR1$ (combined with $TR2$) and the arrow $h[1]$ exists by $TR3$.
The commutativity of the right square gives $\alpha[1] = f[1] \circ h[1] = (f \circ h)[1]$ and therefore $\alpha = f_*(h)$. □

So using $TR2$ we can get an infinite exact sequence. This allows to convert the two Four Lemmas (and therefore the Five Lemma) in the Two-out-of-three property:

Lemma A.0.2
Given the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & A[1] \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\alpha[1]} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X[1] \end{array}$$

if α is an isomorphism and γ (resp. β) is a monomorphism (resp. epimorphism) then the β (resp. γ) is also a monomorphism (resp. epimorphism).

Proof. For any $A \in \mathcal{D}$ we have the morphism of long exact sequences :

$$\begin{array}{ccccccc} \mathrm{Hom}(A,X) & \xrightarrow{u_*} & \mathrm{Hom}(A,Y) & \xrightarrow{v_*} & \mathrm{Hom}(A,Z) & \xrightarrow{w_*} & \mathrm{Hom}(A,X[1]) \\ \downarrow{\scriptstyle\alpha_*} & & \downarrow{\scriptstyle\beta_*} & & \downarrow{\scriptstyle\gamma_*} & & \downarrow{\scriptstyle\alpha[1]_*} \\ \mathrm{Hom}(A,X') & \xrightarrow{u'_*} & \mathrm{Hom}(A,Y') & \xrightarrow{v'_*} & \mathrm{Hom}(A,Z') & \xrightarrow{w'_*} & X[1] \end{array}$$

So we by the classical Four Lemma we conclude that being α_* bijective and γ injective, β_* must be injective. The other case is similar. □

We are now ready to prove the claim:

Claim. In the same hypothesis and notation of the 4.4.2, in the case of a $K3$ surface, the locally closed stratum M_P is a subscheme of the scheme of maps $\mathbb{P}^{x-1} \times \mathbb{P}^{y-1} \to \mathbb{P}Ext^1(T,S)$ quotiented by the action of $PGL(x) \times PGL(y)$.

Proof. The idea is that we can use vector spaces U, V of dimension x, y to parametrise different injections $S \hookrightarrow S^x$ and $T \hookrightarrow T^y$, then we will get a map $U \otimes V \to \mathrm{Ext}^1$ which assign to each choice of these injections how the copy of S extends over the copy of T.
The picture to have in mind is that we want to complete the diagram:

$$\begin{array}{ccccccc} T[-1] & \dashrightarrow & S & \dashrightarrow & X_{\psi,\phi} & \dashrightarrow & T \\ \downarrow{\scriptstyle\phi[-1]} & & \downarrow{\scriptstyle\psi} & & \downarrow & & \downarrow{\scriptstyle\phi} \\ T^y[-1] & \xrightarrow{\alpha} & S^x & \xrightarrow{f} & E & \xrightarrow{g} & T^y \end{array}$$

To make this more clear, first choose U, V vector spaces of dimension x, y. We fix an Hermitian product on each of them (equivalently, we choose a basis)

Clearly, fixing the basis we fixed isomorphisims $S^x \cong S \otimes U$ and $T^y \cong T \otimes V$. Moreover, for each non-zero $u \in U$ and $v \in V$ we have injections

$$\begin{aligned} \psi_u \colon S &\to S \otimes U \quad \text{and} \quad & \phi_v \colon T &\to T \otimes V \\ s &\mapsto s \otimes u & t &\mapsto t \otimes v \end{aligned}$$

We can now consider the diagram

$$\begin{array}{ccc} T[-1] & \overset{\hat{\alpha}_{u,v}}{\dashrightarrow} & S \\ \big\downarrow{\scriptstyle \phi_v[-1]} & & \big\downarrow{\scriptstyle \psi_u} \\ T \otimes V[-1] & \xrightarrow{\alpha} & S \otimes U \end{array}$$

this is easy to complete since we have a section for ψ_u given by orthogonal projection: since we have fixed the product we have a projection $U \to \mathbb{C}u$ that induces $S \otimes U \to S \otimes \mathbb{C}u \cong S$.
This defines a map

$$\begin{aligned} \hat{\alpha} \colon U \otimes V &\to \operatorname{Ext}^1(T, S) \\ u \otimes v &\mapsto \hat{\alpha}_{u,v} \end{aligned}$$

We can now construct the concrete extension as a cone over $\hat{\alpha}_{u,v}$.
Defining $X_{u,v}$ as the cone of $\hat{\alpha}_{u,v}$ (see definition **??**), we have that the part of degree i is $X_{u,v}^{(i)} = T[-1]^{(i+1)} \oplus S^{(i)} = T^{(i)} \oplus S^{(i)}$.
Since E fits into a distinguished triangle, we can assume it to be the cone over α which means $E^{(i)} = T^{(i)} \otimes V \oplus S^{(i) \otimes U}$. It is just a calculation to verify that the maps $h_{u,v}^{(i)} := \phi_v^i \oplus \psi_u^i$ define a morphism $h_{u,v} : X_{u,v} \to E$ of complexes and complete the diagram:

$$\begin{array}{ccccccc} T[-1] & \xrightarrow{\hat{\alpha}_{u,v}} & S & \longrightarrow & X_{u,v} & \longrightarrow & T \\ \big\downarrow{\scriptstyle \phi_v[-1]} & & \big\downarrow{\scriptstyle \psi_u} & & \big\downarrow{\scriptstyle h_{u,v}} & & \big\downarrow{\scriptstyle \phi_u} \\ T \otimes V[-1] & \xrightarrow{\alpha} & S \otimes U & \xrightarrow{f} & E & \longrightarrow & T \otimes V \end{array}$$

Now we want to take into account stability: in particular we will see that for E to be stable implies that the two maps $U \to \operatorname{Hom}(V, \operatorname{Ext}^1(T, S))$ and $V \to \operatorname{Hom}(U, \operatorname{Ext}^1(T, S))$ are injective.
Suppose the first one is not. Then take a non zero $u \in \operatorname{Ker}$. Clearly, this is the same as saying that $\hat{\alpha}_{u,v} = 0$ for all $v \in V$.
It is a direct consequence of the definition of cone that the cone over the zero morphism is the direct sum. So $X_{u,v} = S \oplus T$ and we have an arrow $\gamma_{u,v} : T \to X_{u,v}$ such that $T \to X_{u,v} \to T$ is the identity.
Notice that from the analogue of the Four Lemma, h is a monomorphism.

The idea now is to glue together the $X_{u,v}$ into $S \oplus (T \otimes V)$ preserving injectivity (in the sense of monomorphism).
We already have $h' : S \to E$ as the composition $f \circ \psi_u$. We can define $h'' : T \otimes V \to E$ by the formula:

$$h''(t \otimes v) := h_{u,v}(\gamma_{u,v}(t))$$

. This induces a map $h : S \oplus (T \otimes V) \to E$, by universal property.
So finally we need to check injectivity: using the section of ψ_u from before, we have $T \otimes V \to S$, so by construction we have:

$$\begin{array}{ccccccc}
T \otimes V[-1] & \xrightarrow{\psi_u^{-1} \circ \alpha_{u,v}} & S & \longrightarrow & S \oplus (T \otimes V) & \longrightarrow & T \otimes V \\
= & & \downarrow \psi_u & & \downarrow h_{u,v} & & = \\
T \otimes V[-1] & \xrightarrow{\alpha} & S \otimes U & \xrightarrow{f} & E & \longrightarrow & T \otimes V
\end{array}$$

The first row is a distinguished triangle if and only if $\psi_u^{-1} \circ \alpha_{u,v} = 0$. But this is clearly the case: we can show it is zero on a set of generators of $T \otimes V[-1]$, so in particular it is enough to test it on elements of the form $t \otimes v$. But then we can use the fact that they are in the image of ϕ_v, so it's easy to see that:

$$\psi_u^{-1} \circ \alpha_{u,v}(t \otimes v) = \psi_u^{-1} \circ \alpha_{u,v}(\phi_v(t)) = \hat{\alpha}_{u,v}(t) = 0$$

so because of the Five lemma $S \oplus (T \otimes V) \hookrightarrow E$.
But this destabilises E since the phase of T is bigger than the one of S so $S \oplus (T \otimes V)$ has bigger phase than E (which have the same phase as $S^x \oplus T^y$).
To conclude, the map $\hat{\alpha} : U \otimes V \to \mathrm{Ext}^1(T, S)$ induces a map $\mathbb{P}^x \times \mathbb{P}^y \to \mathbb{P}(\mathrm{Ext}^1(T, S))$. There is still a problem: this map has been produced by choosing a basis. We can now show that it is actually independent of that choice.
But this is easy to show: if D, C are the corresponding automorphisisms of U, V it is clear by the construction that this commutes:

$$\begin{array}{ccc}
T[-1] & \xrightarrow{\hat{\alpha}_{u,v}} & S \\
\downarrow \phi_v[-1] & & \downarrow \psi_u \\
T \otimes V[-1] & \xrightarrow{\alpha} & S \otimes U \\
\downarrow id_T \otimes D & & \downarrow id_S \otimes C \\
T \otimes V[-1] & \xrightarrow{\beta} & S \otimes U \\
\phi_{D(v)}[-1] \uparrow & & \psi_{C(u)} \uparrow \\
T[-1] & \xrightarrow{\hat{\beta}_{C(u),D(v)}} & S
\end{array}$$

were the first and last row are actually the same (non so c'è una notazione per questo? intendo che posso mettere l'identità tra T[-1] sopra e sotto, e uguale con S) So the map is $\hat{\alpha}$ is $\mathrm{GL}(x) \times \mathrm{GL}(y)$ invariant and we can map M_P to the quotient of the scheme of bidegree (1,1) morphisms $\mathbb{P}^x \times \mathbb{P}^y \to \mathbb{P}(\mathrm{Ext}^1(T, S))$. Now we can prove it is injective and we are done: for this observe that if we take $\alpha, \beta : T \otimes V[-1] \to S \otimes U$, if we suppose that the corresponding corresponding maps are the same $\hat{\alpha} = \hat{\beta}$, then for a generator of the form $t \otimes v$ we can write:

$$\alpha(t \otimes v) = \alpha(\phi_v(t)) = \psi_u(\alpha_{u,v}(t)) = \psi_u(\beta_{u,v}(t)) = \beta(t \otimes v)$$

And now we are done. □

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