ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

SCUOLA DI SCIENZE
Corso di Laurea Magistrale in Matematica

Gromov-hyperbolicity of strictly pseudoconvex domains and applications

Supervisors:
Prof. Loredana Lanzani
Prof. Irina Markina

Author:
Giacomo Gavelli

Sessione 6
Anno Accademico 2021/2022

"Pure mathematics is the world's best game. It is more absorbing than chess, more of a gamble than poker, and lasts longer than Monopoly. It's free. It can be played anywhere. Archimedes did it in a bathtub."
Richard J. Trudeau

Introduction

The reason of interest for this thesis is to investigate bounded strictly pseudoconvex domains with C^2-smooth boundary, that is domains $\Omega \subset \mathbb{C}^n$ which admit a strictly plurisubharmonic defining function of class C^2. This condition of pseudoconvexity permits to show that the boundary of such sets is connected by "horizontal" paths, where the horizontal subbundle of the tangent bundle is an (n-1)-dimensional complex manifold. Being $\partial\Omega$ connected by horizontal paths, it turns out that the Carnot-Carathéodory metric is well defined.

Central part of the thesis is to show that bounded strictly pseudoconvex domains with C^2-smooth boundary, when endowed with the Kobayashi metric, are Gromov-hyperbolic. This result was proved in 2000 by Balogh and Bonk in [1]. Chapter 4 is entirely devoted to the proof of this Theorem.

Chapter 1, 2 and 3 introduce the terminology and the preliminary results needed to carry out the proof in Chapter 4.

Chapter 1 deals in detail with the geometric structure of strictly pseudoconvex domains and the construction of the Carnot-Carathéodory metric on the boundary.

In Chapter 2 we present the notion of Gromov-hyperbolicity. Such notion is a generalization of the "classical" notion of hyperbolicity, originally given for Riemannian manifolds with constant negative curvature, to general metric spaces. Gromov-hyperbolic spaces are of central interest because they provide a general framework for the classical regularity theory of extensions for biholomorphisms and proper holomorphic maps.

In Chapter 3 we introduce another notion of hyperbolicity, due to Kobayashi (1967), for complex spaces. It is based on the definition of an intrinsic semi-distance function on any complex space, and such space is said to be Kobayashi-hyperbolic if such semi-distance function is an actual distance function.

Notice that a Kobayashi-hyperbolic space is a metric space, hence it makes sense to investigate its Gromov-hyperbolicity. The domains of interest in this Thesis result to be Kobayashi-hyperbolic and Chapter 4, as previously mentioned, deals with the proof of their Gromov-hyperbolicity.

In the final part of the thesis we present a recent application of the results by Balogh and Bonk to the theory of functions with Bounded Mean Oscillation (BMO spaces, for short). In the setting of strictly pseudoconvex domains there are at least two notions of such spaces: BMO spaces defined via balls in the Kobayashi metric and dyadic BMO spaces. Hu, Huo, Lanzani, Palencia and Wagner recently proved in [7] that such notions are equivalent. A key role in the proof of such equivalence is indeed played by the results presented by Balogh and Bonk in [1].

Introduzione

Il motivo di interesse per questa tesi è di indagare domini strettamente pseudoconvessi, limitati, con bordo C^2, cioè domini $\Omega \subset \mathbb{C}^n$ che ammettono una funzione definente strettamente plurisubarmonica di classe C^2. Questa condizione di pseudoconvessità permette di mostrare che il bordo di tali insiemi è connesso per curve "orizzontali", dove il sottofibrato orizzontale del fibrato tangente è una (n-1)-varietà complessa. Essendo il bordo $\partial\Omega$ connesso per curve orizzontali, risulta ben definita la metrica di Carnot-Carathéodory.

Parte centrale della tesi è dimostrare che domini strettamente pseudoconvessi, limitati, con bordo $\mathbb{C}^2$, quando dotati della metrica di Kobayashi, siano iperbolici nel senso di Gromov. Questo risultato è stato provato nel 2000 da Balogh e Bonk in [1]. Il capitolo 4 è interamente dedicato alla prova di questo Teorema.

I capitoli 1, 2 e 3 introducono la terminologia e i risultati preliminari necessari per portare a termine la dimostrazione nel capitolo 4.

Il capitolo 1 affronta nel dettaglio la struttura geometrica di tali domini strettamente pseudoconvessi e la costruzione della metrica di Carnot-Carathéodory sul bordo.

Nel capitolo 2 viene introdotta la nozione di iperbolicità nel senso di Gromov. Tale nozione è una generalizzazione della nozione "classica" di iperbolicità, data originalmente per varietà Riemanniane con curvatura costante negativa, a generali spazi metrici. Gli spazi iperbolici nel senso di Gromov sono di speciale interesse poiché forniscono una cornice generale per la teoria classica per la regolarità di estensioni per biolomorfismi e mappe proprie.

Nel capitolo 3 viene introdotta un'ulteriore nozione di iperbolicità, dovuta a Kobayashi (1967), per spazi complessi. Si basa sulla definizione di una semi-distanza intrinseca ad ogni spazio complesso, e tale spazio viene detto iperbolico secondo Kobayashi se tale semi-distanza risulta essere un'effettiva distanza.

Si noti che uno spazio iperbolico secondo Kobayashi è uno spazio metrico, dunque è sensato interrogarsi sulla sua iperbolicità nel senso di Gromov. I domini di interesse in questa Tesi risultano essere sempre iperbolici secondo Kobayashi e il capitolo 4, come già menzionato, affronta la dimostrazione della loro iperbolicità nel senso di Gromov.

Nella parte finale della tesi viene presentata una recente applicazione dei risultati di Balogh e Bonk alla teoria delle funzioni a oscillazione media limitata (spazi BMO, in breve). Nell'ambito di domini strettamente pseudoconvessi ci sono almeno 2 nozioni di tali spazi: spazi BMO definiti via palle nella metrica di Kobayashi e spazi BMO diadici. Hu, Huo, Lanzani, Palencia e Wagner hanno recentemente dimostrato in [7] che queste nozioni sono equivalenti. Un ruolo chiave nella dimostrazione di tale equivalenza è giocato proprio dai risultati presentati da Balogh e Bonk in [1].

Contents

Chapter 1

Strictly pseudoconvex domains and the Carnot-Carathéodory metric

In this Chapter we deal in detail with the geometric structure of strictly pseudoconvex domains and the construction of the Carnot-Carathéodory metric on their boundary. A domain $\Omega \subset \mathbb{C}^n$ is strictly pseudoconvex if it admits a strictly positive definite Hermitian form in a neighborhood of the boundary, the Levi form. Such form is defined in terms of a defining function for Ω, which is required to be strictly plurisubharmonic.

On the other hand, it turns out that the Levi form associated to any defining function for such Ω (without requiring strict plurisubharmonicity) is strictly positive definite when restricted to the complex (or horizontal) tangent space at any point on the boundary.

In section 1.2 we exploit this fact and we choose one particular defining function for Ω (the so called "signed distance function") and we use it to define the Carnot-Carathéodory metric. This choice of defining function will play a crucial role in the proof of the main result (the Balogh-Bonk Theorem 4.1). In Section 1.3 we show how the Carnot-Carathéodory metric can be approximated by a class of Riemannian metrics on $\partial\Omega$.

1.1 Strictly pseudoconvex domains

Here we introduce the notion of Strict Pseudoconvexity and highlight some basic properties. We also give two examples of strictly pseudoconvex domains: the ball and the Siegel domain.

Def 1.1. ***(Plurisubharmonicity):*** *A C^2-smooth function $\varphi : U \subset \mathbb{C}^n \to \mathbb{R}$ is said to be* ***strictly plurisubharmonic*** *if its complex Hessian matrix*

$$L_\varphi(z) := \left(\frac{\partial^2 \varphi}{\partial z^j \partial \bar{z}^k}(z) \right)_{j,k=1,\cdots,n}$$

induces a strictly positive definite Hermitian form in $\mathbb{C}^n$, that is,

$$L_\varphi(z; Z) := \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z^j \partial \bar{z}^k}(z) Z_j \overline{Z_k} > 0 \quad \forall\ z \in U,\ Z \in \mathbb{C}^n \setminus \{0\}.$$

The Hermitian form

$$Z \mapsto L_\varphi(z; Z)$$

is the ***Levi form*** *of φ at $z \in U$.*

Def 1.2. ***(Defining function):*** *Let $\Omega \subset \mathbb{C}^n$ be an open set and k be an integer between 1 and ∞. We say that $\partial\Omega$ is of class C^k (or C^k-smooth) if there is an open neighborhood U of $\partial\Omega$ and a C^k-smooth function $\varphi : U \to \mathbb{R}$ with the following properties:*

- $U \cap \Omega = \{z \in U \ ; \ \varphi(z) < 0\}$,
- $d\varphi(z) \neq 0 \ \ \forall \ z \in U$.

φ is called a ***defining function*** *for Ω.*

We remark that any domain of class C^k admits infinitely many defining functions, see Lemma 1.2 below.

Def 1.3. ***(Pseudoconvexity):*** *A domain $\Omega \subset \mathbb{C}^n$ is said to be* ***strictly pseudoconvex*** *if Ω admits a strictly plurisubharmonic defining function. In particular, $\partial\Omega$ is of class $\mathbb{C}^k$ with $k \geq 2$.*

Proposition 1.1. *Let $\Omega_1,\ \Omega_2 \subseteq \mathbb{C}^n$ be open sets such that there exists a biholomorphism $\Phi : N_1 \to N_2$, where N_1 and N_2 are neighborhoods of $\bar{\Omega}_1$ and $\bar{\Omega}_2$ respectively, which is also a biholomorphism between Ω_1 and Ω_2. Then, Ω_1 is strictly pseudoconvex $\iff$ Ω_2 is strictly pseudoconvex.*

Proof. Let us assume that Ω_2 is strictly pseudoconvex and let φ_2 be a strictly plurisubharmonic defining function for Ω_2, i.e. there exists a neigborhood U_2 of $\partial\Omega_2$ such that:

1. $U_2 \cap \Omega_2 = \{z \in U_2 \ ; \ \varphi_2(z) < 0\}$,
2. $d\varphi_2(z) \neq 0 \quad \forall \ z \in U_2$,
3. $L_{\varphi_2}(z; Z) > 0 \quad \forall \ z \in U_2,\ Z \in \mathbb{C}^n \setminus \{0\}$.

The goal is now to show that $\varphi_1 := \varphi_2 \circ \Phi : \Phi^{-1}(U_2) \to \mathbb{R}$ is a strictly plurisubharmonic defining function for Ω_1.

First, $U_1 := \Phi^{-1}(U_2)$ is an open neighborhood of $\partial\Omega_1$, since Φ is a biholomorphism. We have to check:

1. $U_1 \cap \Omega_1 = \{w \in \mathbb{C}^n \ ; \ \varphi_1(w) < 0\}$,
2. $d\varphi_1(w) \neq 0 \quad \forall \ w \in U_1$,
3. $L_{\varphi_1}(w; W) > 0 \quad \forall \ w \in U_1,\ W \in \mathbb{C}^n \setminus \{0\}$.

1. Since Φ is a biholomorphism, $U_1 \cap \Omega_1 = \Phi^{-1}(U_2) \cap \Phi^{-1}(\Omega_2) = \Phi^{-1}(U_2 \cap \Omega_2)$. Moreover,

$$\varphi_1(w) < 0 \iff (\varphi_2(\Phi(w)) < 0 \iff$$

$$\Phi(w) \in \{U_2 \cap \Omega_2\} \iff w \in \Phi^{-1}(U_2 \cap \Omega_2) = U_1 \cap \Omega_1 \qquad \checkmark.$$

2. By the chain rule (see [16], Theorem 9.15),

$$0 = d\varphi_1(w) = d(\varphi_2 \circ \Phi)(w) = d\varphi_2(\Phi(w))(d\Phi(w)) \iff d\Phi(w) = 0.$$

It also follows from the chain rule that $(d\Phi^{-1})(\Phi(w))$ is the inverse of $d\Phi(w)$. In particular, $d\Phi(w) \neq 0 \quad \forall\, w \in U_1$. Summarizing,

$$d\varphi_1(w) \neq 0 \quad \forall\, w \in U_1 \qquad \checkmark.$$

3. According to (2.3.32) in [13], the complex Hessian matrix of $\varphi_1 = \varphi_2 \circ \Phi$ at $w \in U_1$ is given by

$$D_\Phi(w)^* L_{\varphi_2}(\Phi(w)) D_\Phi(w),$$

where

$$D_\Phi(w) = \begin{pmatrix} \frac{\partial \Phi_1}{\partial w_1}(w) & \cdots & \frac{\partial \Phi_1}{\partial w_n}(w) \\ \vdots & \ddots & \vdots \\ \frac{\partial \Phi_n}{\partial w_1}(w) & \cdots & \frac{\partial \Phi_n}{\partial w_n}(w) \end{pmatrix}$$

is the holomorphic derivative matrix of Φ and $D_\Phi(w)^*$ is its conjugate transpose. According to Sylvester's law of inertia (see [5], Theorem 9.13, page 313), the number of positive, negative and zero eigenvalues of $D_\Phi(w)^* L_{\varphi_2}(\Phi(w)) D_\Phi(w)$ is the same as for $L_{\varphi_2}(\Phi(w))$, i.e. it has n positive eigenvalues. Therefore, the Levi form $L_{\varphi_1}(w;\cdot)$ is strictly positive definite. □

Notice that Proposition 1.1 does not mean that strict pseudoconvexity is a biholomorphic invariant! It states that strict pseudoconvexity is preserved if a biholomorphism is defined in a *neighborhood* of the closure of a strictly pseudoconvex domain Ω.

Example 1.1. *The unit ball in $\mathbb{C}^n$ is strictly pseudoconvex.*

Indeed, if we consider $\mathbb{B}^{2n} := \{z = (z_1, \cdots, z_n) \in \mathbb{C}^n \ ; \ \sum_{i=1}^n |z_i|^2 < 1\}$ and

$$\varphi(z) := \sum_{i=1}^n |z_i|^2 - 1 = \sum_{i=1}^n z_i \bar{z}_i - 1,$$

then $\mathbb{B}^{2n} = \{z \in \mathbb{C}^n \ ; \varphi(z) < 0\}$, and

$$\frac{\partial^2 \varphi}{\partial z^j \partial \bar{z}^k}(z) = \frac{\partial}{\partial z^j}(z_k) = \delta_{jk}.$$

$$\Rightarrow L_\varphi(z) = Id_n.$$

Hence the Levi matrix of r is positive definite for all $z \in \mathbb{C}^n$, making $\mathbb{B}^{2n}$ strictly pseudoconvex.

Example 1.2. *The Siegel domain*

$$\mathscr{S} := \{w = (w_1 \cdots, w_n) \in \mathbb{C}^n \ ; \ Im(w_n) > \sum_{i=1}^{n-1} |w_i|^2\}$$

is strictly pseudoconvex.

To see this, according to Proposition 1.1, it is enough to check that it is biholomorphically equivalent to the unit ball $\mathbb{B}^{2n}$ through a biholomorphism defined in a neighborood of $\bar{\mathscr{S}}$. We consider the generalized Cayley transform

$$\Phi : \mathbb{C}^n \setminus \{-1\} \to \mathbb{C}^n \setminus \{-i\}$$

given by:

$$\Phi(z) := \left(\frac{z_1}{1+z_n}, \cdots, \frac{z_{n-1}}{1+z_n}, i\frac{1-z_n}{1+z_n} \right).$$

Φ is clearly olomorphic and we can check it is invertible with inverse

$$\Phi^{-1}(w) = \left(\frac{2iw_1}{i+w_n}, \cdots, \frac{2iw_{n-1}}{i+w_n}, \frac{i-w_n}{i+w_n} \right).$$

We claim that Φ maps the unit ball $\mathbb{B}^{2n}$ to $\mathscr{S}$. Indeed,

$$|z_n|^2 + \sum_{i=1}^{n-1} |z_i|^2 < 1 \iff \left| \frac{i-w_n}{i+w_n} \right|^2 + \sum_{i=1}^{n-1} \frac{|2iw_i|^2}{|i+w_n|^2} < 1$$

$$\iff 4\sum_{i=1}^{n-1} |w_i|^2 < |i+w_n|^2 - |i-w_n|^2| \iff 4\sum_{i=1}^{n-1} |w_i|^2 < 4Im(w_n).$$

Remark 1.1. *If $\Omega \subset \mathbb{C}^n$ is a strictly pseudoconvex domain, its boundary is a $(2n-1)$-dimensional real manifold, being the 0-set of a real valued function $\varphi : \partial\Omega \subset U \to \mathbb{R}$ with non-vanishing differential.*

This allows to study the tangent space $T_p\partial\Omega$ at $p \in \partial\Omega$, for which there is the following explicit expression in terms of the differential of any sufficiently regular defining function.

Lemma 1.1. *Let $\Omega \subset \mathbb{C}^n$ be a domain with boundary of class C^2 and let φ be a defining function for Ω. Then, for all $p \in \partial\Omega$,*

$$T_p\partial\Omega = \left\{ Z \in \mathbb{C}^n \ ; \ Re\langle \bar{\partial}\varphi(p), Z \rangle = 0 \right\}, \tag{1.1}$$

where

$$\bar{\partial}\varphi(p) = \left(\frac{\partial\varphi}{\partial\bar{z}^1}(p), \cdots, \frac{\partial\varphi}{\partial\bar{z}^n}(p) \right),$$

$$\frac{\partial\varphi}{\partial\bar{z}^k}(p) = \frac{1}{2}\left(\frac{\partial\varphi}{\partial x^k}(p) + i\frac{\partial\varphi}{\partial y^k}(p) \right) \quad \forall\ k = 1, \cdots, n$$

and $\langle W, Z \rangle := \sum_{i=1}^n W_i\bar{Z}_i$ is the standard Hermitian product of two vectors.

Proof. According to [11], Th. 3.4 page 154, the tangent space of an (embedded real) manifold M at a point p is characterized by a (or rather, any) defining function f as

$$T_pM = Ker\ df(p).$$

In our case, $M = \partial\Omega$ and $f = \varphi$. If we identify $\mathbb{C}^n = \mathbb{R}^{2n}$ and write in coordinates $(z_1, \cdots, z_n) = (x_1, y_1, \cdots, x_n, y_n)$, we get

$$\nabla\varphi(p) = \left(\frac{\partial\varphi}{\partial x^1}(p), \frac{\partial\varphi}{\partial y^1}(p), \cdots, \frac{\partial\varphi}{\partial x^n}(p), \frac{\partial\varphi}{\partial y^n}(p) \right).$$

On the other hand, if $Z = (a_1 + ib_1, \cdots, a_n + ib_n) \in \mathbb{C}^n$,

$$Re\langle \bar{\partial}_\varphi(p), Z\rangle = 0 \iff Re\left(\sum_{k=1}^{n} \frac{1}{2}\left(\frac{\partial\varphi}{\partial x^k}(p) + i\frac{\partial\varphi}{\partial y^k}(p)\right)(a_k - ib_k)\right) = 0$$

$$\iff Re\left[\sum_{k=1}^{n} \frac{\partial\varphi}{\partial x^k}(p)a_k + \frac{\partial\varphi}{\partial y^k}(p)b_k + i\left(\frac{\partial\varphi}{\partial y^k}(p)a_k - \frac{\partial\varphi}{\partial x^k}(p)b_k\right)\right] = 0$$

$$\iff \sum_{k=1}^{n} \frac{\partial\varphi}{\partial x^k}(p)a_k + \frac{\partial\varphi}{\partial y^k}(p)b_k = 0.$$

If we call $V := (a_1, b_1, \cdots, a_n, b_n) \in \mathbb{R}^{2n}$, this means that

$$d\varphi(p)(V) = \nabla\varphi(p) \cdot V = 0,$$

Where $W \cdot V = \sum_{k=1}^{2n} V_k W_k$ is the standard scalar product in $\mathbb{R}^{2n}$. Therefore,

$$\{Z \in \mathbb{C}^n \; ; \; Re\langle \bar{\partial}\varphi(p), Z\rangle = 0\} = \{V \in \mathbb{R}^{2n} \; ; \; d\varphi(p)(V) = 0\}$$

$$= Ker\ d\varphi(p) = T_p\partial\Omega.$$

□

Def 1.4. ***(Horizontal subspace):*** *Given Ω and φ as in Lemma 1.1 and $p \in \partial\Omega$, we call*

$$H_p\partial\Omega := \{Z \in \mathbb{C}^n \; ; \; \langle \bar{\partial}\varphi(p), Z\rangle = 0\} \tag{1.2}$$

the complex tangent space to Ω at p, also known as the ***horizontal subspace****.*

Note that $H_p\partial\Omega \subset T_p\partial\Omega$ (see (1.1) and (1.2)). It is important to observe that the horizontal subspace at $p \in \partial\Omega$ is independent of the choiche of the defining function φ. This is a consequence of the following result from [15].

Lemma 1.2. *Suppose $\Omega \subset \mathbb{C}^n$ is a domain with boundary of class C^2, and let φ_1, φ_2 be two defining functions for Ω in a neighborhood U of $\partial\Omega$. Then there exists a positive function $h \in C^1(U)$ such that*

- *$\varphi_1 = h \cdot \varphi_2$ on U,*
- *$d\varphi_1(p) = h(p) \cdot d\varphi_2(p)$ for $p \in \partial\Omega$.*

Proof. See [15], Lemma 2.5, page 51. □

Corollary 1.1. *Let $\Omega \subset \mathbb{C}^n$ be a domain with boundary of class C^2. Then the horizontal subspace $H_p\partial\Omega$ does not depend on the choice of defining function for Ω.*

Proof. We need to prove that if φ is a defining function for Ω, then

$$\{Z \in \mathbb{C}^n \; ; \; \langle \bar{\partial}\varphi(p), Z\rangle = 0\}$$

does not depend on φ.

Let $\varphi_1,\ \varphi_2$ be two distinct defining functions for Ω. By Lemma 1.2, there exists a positive function h defined in a neighborhood of $\partial\Omega$ such that $d\varphi_1(p) = h(p)\cdot d\varphi_2(p)$ for any $p\in\partial\Omega$. This implies that for all $k=1,\cdots,n$

$$\frac{\partial\varphi_1}{\partial\bar{z}^k}(p) = \frac{1}{2}\left(\frac{\partial\varphi_1}{\partial x^k}(p) + i\frac{\partial\varphi_1}{\partial y^k}(p)\right) =$$

$$\frac{h(p)}{2}\left(\frac{\partial\varphi_2}{\partial x^k}(p) + i\frac{\partial\varphi_2}{\partial y^k}(p)\right) = h(p)\frac{\partial\varphi_2}{\partial\bar{z}^k}(p).$$

Therefore, $\bar\partial\varphi_1(p) = h(p)\bar\partial\varphi_2(p)$ and, since h is positive,

$$\left\{Z\in\mathbb{C}^n\ ;\ \langle\bar\partial\varphi_1(p), Z\rangle = 0\right\} = \left\{Z\in\mathbb{C}^n\ ;\ \langle\bar\partial\varphi_2(p), Z\rangle = 0\right\}.$$

□

For any $p\in\partial\Omega$ we have that $H_p\partial\Omega$ is a $2(n-1)$-dimensional real vector space, or a $(n-1)$-dimensional complex vector space. Then, for each $p\in\partial\Omega$ we get the decomposition $\mathbb{C}^n = H_p\partial\Omega\oplus N_p\partial\Omega$, where $N_p\partial\Omega$ is the complex 1-dimensional subspace of $\mathbb{C}^n$ orthogonal to $H_p\partial\Omega$ (with respect to the standard Hermitian product).

Therefore, any vector $Z\in\mathbb{C}^n$ can be written uniquely as $Z = Z_H + Z_N$, with $Z_H\in H_p\partial\Omega,\ Z_N\in N_p\partial\Omega$. We can further decompose $Z_N = Z_{N,1} + Z_{N,2}$, where $Z_{N,1} = Z_N\cap T_p\partial\Omega$ and $Z_{N,2}\perp T_p\partial\Omega$. Notice that this decomposition is not trivial only if $n\geq 2$, as for $n=1$ we would have a $0-$dimensional horizontal subspace. For this reason, from this point on, we will always assume $n\geq 2$ when dealing with horizontal vectors.

The next result states that the Levi form associated to <u>any</u> C^2-smooth defining function of a strictly pseudoconvex domain is positive definite on the horizontal subspace.

Lemma 1.3. *Let $\Omega\subset\mathbb{C}^n$, $n\geq 2$, be a strictly pseudoconvex domain, and let φ be a C^2-smooth defining function for Ω. Then*

$$L_\varphi(p;Z) = \sum_{j,k=1}^{n}\frac{\partial^2\varphi}{\partial z^j\partial\bar{z}^k}(p)Z_j\overline{Z_k} > 0 \tag{1.3}$$

for all $p\in\partial\Omega$ and $Z\in\mathbb{C}^n\setminus\{0\}$ such that

$$\sum_{i=1}^{n}\frac{\partial\varphi}{\partial z^i}(p)Z_i = 0, \tag{1.4}$$

where $\frac{\partial\varphi}{\partial z^i}(p) = \frac{1}{2}\left(\frac{\partial\varphi}{\partial x^i}(p) - i\frac{\partial\varphi}{\partial y^i}(p)\right)$. (1.4) is equivalent to $\langle\bar\partial\varphi(p), Z\rangle = 0$ and it is just another way to express that Z is a horizontal vector at p.

Proof. Let φ_1,φ_2 be C^2-smooth defining functions for Ω and consider the function $h>0$ from Lemma 1.2. A computation based on Lemma 1.2 yields

$$L_{\varphi_1}(p;Z) = 2Re\left[\overline{(\nabla h(p)\cdot Z)}(\nabla\varphi_2(p)\cdot Z)\right] + h(p)L_{\varphi_2}(p;Z)\quad\forall\ Z\in\mathbb{C}^n. \tag{1.5}$$

For details of the computation see [15] page 56.

Now, if $Z \in H_p(\partial\Omega)$, i.e. (1.4) holds for φ_1 and φ_2, the first term in the RHS of (1.5) vanishes, and we are left with

$$L_{\varphi_1}(p; Z) = h(p) L_{\varphi_2}(p; Z) \quad \forall\, Z \in H_p(\partial\Omega). \tag{1.6}$$

If Ω is strictly pseudoconvex, we are granted that there exists at least one C^2-smooth defining function φ_0 such that the Levi form is strictly positive definite, i.e. (1.3) holds for all $Z \in \mathbb{C}^n \setminus \{0\}$. Now, if φ is another C^2-smooth defining function, (1.6) grants us that the result holds. □

1.2 The Carnot-Carathéodory metric

The goal of this section is to define the Carnot-Carathéodory metric on the boundary of a bounded strictly pseudoconvex domain with respect to the signed distance function. This metric will be crucial in the study of the Gromov-hyperbolicity for strictly pseudoconvex domains in Chapter 4.

Notice that the following result holds in general in $\mathbb{R}^n$ for $n \geq 2$, but since all the other results require pseudoconvexity, we will state it in the setting $\mathbb{C}^n$ to maintain the same notation troughout the Chapter.

Def 1.5. *Let $\Omega \subset \mathbb{C}^n = \mathbb{R}^{2n}$ be a bounded domain with C^2-smooth boundary. For all $x \in \mathbb{C}^n$, denote with $\delta(x) := d_E(x, \partial\Omega)$ the Euclidean distance of x to the boundary of Ω. The* ***signed distance function*** *is defined as*

$$\rho(x) := \begin{cases} -\delta(x) & x \in \Omega \\ \delta(x) & x \in \mathbb{C}^n \setminus \Omega. \end{cases}$$

Lemma 1.4. *Let $\Omega \subset \mathbb{C}^n$, be a bounded domain with C^2-smooth boundary. Then there exists $\varepsilon_0 > 0$ such that*

1. *$\forall\, x \in N_{\varepsilon_0}(\partial\Omega) := \{x \in \mathbb{C}^n \;;\; \delta(x) < \varepsilon_0\}$ there exists a unique point $\pi(x) \in \partial\Omega$ such that*
$$|x - \pi(x)| = \delta(x).$$
Henceforth, we refer to $\pi : N_{\varepsilon_0} \to \partial\Omega$ as the ***projection map****.*

2. *The signed distance $\rho : \mathbb{C}^n \to \mathbb{R}$ is C^2-smooth on $N_{\varepsilon_0}(\partial\Omega)$.*

3. *The fibers of the projection map $\pi : N_{\varepsilon_0} \to \partial\Omega$ are the Euclidean segments*
$$\pi^{-1}(p) = (p - \varepsilon_0 n(p), p + \varepsilon_0 n(p)), \tag{1.7}$$
where $n(p)$ is the outer unit normal vector of $\partial\Omega$ at $p \in \partial\Omega$.

4. *The gradient of ρ satisfies, for all $x \in N_{\varepsilon_0}(\partial\Omega)$,*
$$\nabla\rho(x) = n(\pi(x)). \tag{1.8}$$

5. *The projection map $\pi : N_{\varepsilon_0}(\partial\Omega) \to \partial\Omega$ is C^1-smooth.*

Proof. See [1], page 509. □

If we now consider $\Omega \subset \mathbb{C}^n$ to be a strictly pseudoconvex domain, the previous Lemma 1.4, in particular point *2.*, implies that the signed distance function is a C^2-smooth defining function. Consequently, Lemma 1.3 grants that the Levi form $L_\rho(p,\cdot)$ is strictly positive definite on $H_p\partial\Omega$ for all $p \in \partial\Omega$, where $H_p\partial\Omega$ is the horizontal subspace of the tangent space $T_p\partial\Omega$.

Def 1.6. ***(Horizontal curves):*** *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$ be a bounded strictly pseudoconvex domain. An absolutely continuous curve $\gamma : [0,1] \to \partial\Omega$ is called **horizontal** if*

$$\dot{\gamma}(t) \in H_{\gamma(t)}\partial\Omega \quad for\ a.e.\ t \in [0,1].$$

To define the Carnot-Carathéodory metric we need to see that any two points on $\partial\Omega$ can be connected by horizontal curves. For this, we need to introduce vector fields on $\partial\Omega$.

Def 1.7. ***(vector fields on $\partial\Omega$ and their brackets):***

- *A **vector field** on $\partial\Omega$ is an expression of the form*

$$X = \sum_{k=1}^{n} a_k \frac{\partial}{\partial z^k} + \bar{a}_k \frac{\partial}{\partial \bar{z}_k},$$

where $a_k \in C^2(\partial\Omega; \mathbb{C})$ and

$$Z := (a_1(p), \cdots, a_n(p)) \in T_p(\partial\Omega) \quad \forall p \in \partial\Omega.$$

- *We say that X is an **horizontal vector field** if*

$$Z := (a_1(p), \cdots, a_n(p)) \in H_p(\partial\Omega) \quad \forall p \in \partial\Omega.$$

We will denote with $\Gamma(T\partial\Omega)$ the collection of all vector fields on $\partial\Omega$ and with $\Gamma(H\partial\Omega)$ the collection of all horizontal vector fields on $\partial\Omega$.

- *Given any two vector fields $X, Y \in \Gamma(T\partial\Omega)$ acting on $C^2(\partial\Omega; \mathbb{C})$, we let $[X,Y]$ denote their bracket, which is a new vector field defined as follows:*

$$[X,Y](f) := X(Y(f)) - (Y(X(f)), \qquad \forall f \in C^2(\partial\Omega; \mathbb{C}).$$

In terms of differential forms,

$$X \in \Gamma(H\partial\Omega) \iff X \in Ker\ \theta, \quad with\ \theta := \sum_{k=1}^{n} \frac{\partial\varphi}{\partial \bar{z}^k} d\bar{z}_k. \tag{1.9}$$

Theorem 1.1. ***(Chow's Theorem):*** *Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain and consider $X_1, \cdots, X_{2n-2}$ horizontal vector fields on $\partial\Omega$. Assume that the following condition holds:*

$$\begin{gathered} X_1, \cdots, X_{2n-2}\ \text{and their iterated brackets}\ [X_i, X_j], [[X_i, X_j], X_k], \ldots \\ \text{span the real tangent space}\ T_p\partial\Omega\ \text{at every point}\ p \in \partial\Omega. \end{gathered} \tag{1.10}$$

Then any two points $p, q \in \partial\Omega$ can be connected by a horizontal path.

(1.10) is called Chow's condition.

Proof. For a proof in the more general setting of sub-Riemannian manifolds, see [2], page 15. □

If $X \in \Gamma(H\partial\Omega)$, we write

$$iX := \sum_{k=1}^{n} ia_k \frac{\partial}{\partial z^k} - i\bar{a}_k \frac{\partial}{\partial \bar{z}_k}.$$

We have $iX \in \Gamma(H\partial\Omega)$ as $H\partial\Omega$ is a complex vector space.

Remark 1.2. *In Theorem 1.1 we are not assuming that $X_1, \cdots, X_{2n-2}$ are linearly indipendent; in case they are linearly indipendent, then condition 1.10 takes the simpler form*

$$\begin{gathered} X_1, \cdots, X_{2n-2} \text{ and the bracket } [X_j, iX_j] \text{ for some } j \in \{1 \cdots, 2n-2\} \\ \text{span the real tangent space } T_p\partial\Omega \text{ at any point } p \in \partial\Omega. \end{gathered} \tag{1.11}$$

Proposition 1.2. *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a bounded strictly pseudoconvex domain. Then the simplified Chow's condition (1.11) is satisfied.*

Proof. We want to show that if we consider any vector field X in the horizontal bundle $H\partial\Omega$, the bracket $[X, iX]$ generates the "missing direction" in $T\partial\Omega$, which is given by $N\partial\Omega \cap T\partial\Omega$. We will see that

$$\theta([X, iX]) \neq 0, \quad i.e. \quad [X, iX] \notin Ker\ \theta.$$

Then, according to (1.9), $[X, iX] \notin \Gamma(H\partial\Omega)$ and we are done. From [17], Proposition 20.13 page 232, we get the formula

$$\theta([X,Y]) = X(\theta(Y)) - Y(\theta(X)) - d\theta(X,Y), \quad \forall\ X, Y \in \Gamma(T\partial\Omega).$$

Notice that if $X, Y \in \Gamma(H\partial\Omega)$, then $\theta(X) = \theta(Y) = 0$, hence

$$\theta([X,Y]) = -d\theta(X,Y). \tag{1.12}$$

Therefore, it suffices to show that $d\theta(X, iX) \neq 0\ \forall\ X \in \Gamma(H\partial\Omega)$. By antisimmetry of the wedge product,

$$d\theta = \sum_{j,k=1}^{n} \left(\frac{\partial^2 \varphi}{\partial z^j \partial \bar{z}^k} (dz_j \wedge d\bar{z}_k) + \frac{\partial^2 \varphi}{\partial \bar{z}^j \partial \bar{z}^k} (d\bar{z}_j \wedge d\bar{z}_k) \right) = \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z^j \partial \bar{z}^k} (dz_j \wedge d\bar{z}_k).$$

According to [17], Proposition 3.27 page 30, we have that, given 1-forms τ, ω and vector fields X, Y,

$$(\tau \wedge \omega)(X,Y) = det \begin{bmatrix} \tau(X) & \tau(Y) \\ \omega(X) & \omega(Y) \end{bmatrix}.$$

In our case, this gives

$$d\theta(X, iX) =$$

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z^j \partial \bar{z}^k}(dz_j \wedge d\bar{z}_k)\left(\sum_{\nu=1}^{n} a_\nu \frac{\partial}{\partial z^\nu} + \bar{a}_\nu \frac{\partial}{\partial \bar{z}^\nu}, \sum_{\beta=1}^{n} i a_\beta \frac{\partial}{\partial z^\beta} - i\bar{a}_\beta \frac{\partial}{\partial \bar{z}_\beta}\right) =$$

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z^j \partial \bar{z}^k} det \begin{bmatrix} a_j & i a_j \\ \bar{a}_k & -i\bar{a}_k \end{bmatrix} = -2i \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z^j \partial \bar{z}^k} a_j \bar{a}_k.$$

Hence, by (1.12)

$$\theta([X, iX]) = 2i \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z^j \partial \bar{z}^k} a_j \bar{a}_k > 0.$$

In the last step we used the fact that by strict pseudoconvexity of Ω the Levi form is strictly positive definite. In conclusion, $[X, iX] \notin \Gamma(H\partial\Omega)$, therefore the hypothesis of Chow's Theorem are satisfied. ☐

Corollary 1.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with C^2-smooth boundary. Then, for all $p, q \in \partial\Omega$ we have that*

$$\{\gamma : [0,1] \to \partial\Omega \ ; \ \gamma \text{ is a horizontal curve joining } p \text{ and } q\} \neq \emptyset.$$

Proof. Immediate from Proposition 1.2. ☐

Def 1.8. ***(Levi length and The Carnot-Carathéodory metric):***

- *The **Levi length** of a horizontal curve is defined as*

$$l_{L_\rho}(\gamma) := \int_0^1 \sqrt{L_\rho\left(\gamma(t); \dot{\gamma}(t)\right)} dt. \tag{1.13}$$

- *The **Carnot-Carathéodory metric** on $\partial\Omega$ corresponding to the signed distance function ρ is then defined as*

$$d_H(p, q) := \inf\{l_{L_\rho}(\gamma) \ ; \ \gamma \text{ is a horizontal curve joining } p \text{ and } q\}. \tag{1.14}$$

Notice that (1.13) is well defined as $L_\rho\left(\gamma(t); \dot{\gamma}(t)\right) \geq 0$ for a.e. $t \in [0,1]$ as previously stated. In particular, $L_\rho(\gamma(t); \dot{\gamma}(t)) = 0 \iff \dot{\gamma}(t) = 0$.

To check that d_H is indeed a distance function one can proceed exactly as in the proof that a Riemannian metric induces a distance function (see Proposition A.1).

Let us now take a moment to appreciate how horizontal vector fields and their brackets behave on a concrete example: the unit ball $\mathbb{B}^4 \subset \mathbb{C}^2$.

Example 1.3. *Computation of $[X, iX]$ for a choice of $X \in \Gamma(H\partial\mathbb{B}^4)$.*

In Example 1.1 we have seen that

$$\mathbb{B}^4 := \left\{z = (z_1, z_2) \in \mathbb{C}^2 \ ; \ |z_1|^2 + |z_2|^2 < 1\right\}$$

is a bounded strictly pseudoconvex domain with strictly plurisubharmonic defining function $\varphi(z) := |z_1|^2 + |z_2|^2 - 1 = z_1\bar{z}_1 + z_2\bar{z}_2 - 1$. The horizontal subspace at a boundary point $p = (z_1, z_2) \in \partial\mathbb{B}^4$ is

$$\begin{aligned} H_p\partial\mathbb{B}^4 = \left\{A = (a_1, a_2) \in \mathbb{C}^2 \ ; \ \langle\bar{\partial}\varphi(z), A\rangle = 0\right\} = \\ \left\{A = (a_1, a_2) \in \mathbb{C}^2 \ ; \ z_1\bar{a}_1 + z_2\bar{a}_2 = 0\right\}. \end{aligned} \tag{1.15}$$

Then, for each $p = (z_1, z_2) \in \partial\mathbb{B}^4$, a particular horizontal vector $A_p \in H_p\partial\mathbb{B}^4$ is $A_p := (\bar{z}_2, -\bar{z}_1)$, as

$$\langle \bar{\partial}\varphi(p), A_p \rangle = z_1\bar{a}_1 + z_2\bar{a}_2 = z_1z_2 - z_2z_1 = 0.$$

Let us denote $\partial_{z_j} := \frac{\partial}{\partial z_j}$, $\partial_{\bar{z}_j} := \frac{\partial}{\partial \bar{z}_j}$, $j = 1, 2$. Consider the horizontal vector field $X \in \Gamma(H\partial\mathbb{B}^4)$

$$X := \bar{z}_2\partial_{z_1} + z_2\partial_{\bar{z}_1} - \bar{z}_1\partial_{z_2} - z_1\partial_{\bar{z}_2} \tag{1.16}$$

which corresponds, at any point $p = (z_1, z_2) \in \partial\mathbb{B}^4$, to the tangent vector A_p (see Def 1.7). Then,

$$iX = i\bar{z}_2\partial_{z_1} - iz_2\partial_{\bar{z}_1} - i\bar{z}_1\partial_{z_2} + iz_1\partial_{\bar{z}_2}.$$

Let us now compute explicitly the bracket $[X, iX] := X(iX) - (iX)X$. We start by computing separately the two terms $X(iX)$ and $(iX)X$.

$$\begin{aligned}
\bullet\mathbf{X(iX)}= &(\bar{z}_2\partial_{z_1} + z_2\partial_{\bar{z}_1} - \bar{z}_1\partial_{z_2} - z_1\partial_{\bar{z}_2})(i\bar{z}_2\partial_{z_1} - iz_2\partial_{\bar{z}_1} - i\bar{z}_1\partial_{z_2} + iz_1\partial_{\bar{z}_2}) = \\
&i({\bar{z}_2}^2\partial_{z_1z_1} + |z_2|^2\partial_{\bar{z}_1z_1} - \bar{z}_1\bar{z}_2\partial_{z_2z_1} - z_1\bar{z}_2\partial_{\bar{z}_2z_1} - z_1\partial_{z_1} \\
&- |z_2|^2\partial_{z_1\bar{z}_1} - {z_2}^2\partial_{\bar{z}_1\bar{z}_1} + \bar{z}_1z_2\partial_{z_2\bar{z}_1} + \bar{z}_1\partial_{\bar{z}_1} + z_1z_2\partial_{\bar{z}_2\bar{z}_1} \\
&- \bar{z}_2\bar{z}_1\partial_{z_1z_2} - z_2\bar{z}_1\partial_{\bar{z}_1z_2} - z_2\partial_{z_2} + {\bar{z}_1}^2\partial_{z_2z_2} + |z_1|^2\partial_{\bar{z}_2z_2} \\
&+ \bar{z}_2z_1\partial_{z_1\bar{z}_2} + \bar{z}_2\partial_{\bar{z}_2} + z_2z_1\partial_{\bar{z}_1\bar{z}_2} - |z_1|^2\partial_{z_2\bar{z}_2} - {z_1}^2\partial_{\bar{z}_2\bar{z}_2}) = \\
&\quad i({\bar{z}_2}^2\partial_{z_1z_1} - {z_2}^2\partial_{\bar{z}_1\bar{z}_1} + {\bar{z}_1}^2\partial_{z_2z_2} - {z_1}^2\partial_{\bar{z}_2\bar{z}_2} - 2\bar{z}_1\bar{z}_2\partial_{z_1z_2} \\
&\qquad +2z_1z_2\partial_{\bar{z}_1\bar{z}_2} - z_1\partial_{z_1} + \bar{z}_1\partial_{\bar{z}_1} - z_2\partial_{z_2} + \bar{z}_2\partial_{\bar{z}_2}).
\end{aligned} \tag{1.17}$$

Similarly,

$$\begin{aligned}
\bullet\mathbf{(iX)X}= &(i\bar{z}_2\partial_{z_1} - iz_2\partial_{\bar{z}_1} - i\bar{z}_1\partial_{z_2} + iz_1\partial_{\bar{z}_2})(\bar{z}_2\partial_{z_1} + z_2\partial_{\bar{z}_1} - \bar{z}_1\partial_{z_2} - z_1\partial_{\bar{z}_2}) = \\
&i(- |z_2|^2\partial_{\bar{z}_1z_1} - \bar{z}_1\bar{z}_2\partial_{z_2z_1} + z_1\bar{z}_2\partial_{\bar{z}_2z_1} + z_1\partial_{z_1} \\
&+ |z_2|^2\partial_{z_1\bar{z}_1} - {z_2}^2\partial_{\bar{z}_1\bar{z}_1} - \bar{z}_1z_2\partial_{z_2\bar{z}_1} - \bar{z}_1\partial_{\bar{z}_1} + z_1z_2\partial_{\bar{z}_2\bar{z}_1} \\
&- \bar{z}_2\bar{z}_1\partial_{z_1z_2} + z_2\bar{z}_1\partial_{\bar{z}_1z_2} + z_2\partial_{z_2} + {\bar{z}_1}^2\partial_{z_2z_2} - |z_1|^2\partial_{\bar{z}_2z_2} \\
&- \bar{z}_2z_1\partial_{z_1\bar{z}_2} - \bar{z}_2\partial_{\bar{z}_2} + z_1z_2\partial_{\bar{z}_1\bar{z}_2} + |z_1|^2\partial_{z_2\bar{z}_2} - {z_1}^2\partial_{\bar{z}_2\bar{z}_2}) = \\
&\quad i({\bar{z}_2}^2\partial_{z_1z_1} - {z_2}^2\partial_{\bar{z}_1\bar{z}_1} + {\bar{z}_1}^2\partial_{z_2z_2} - {z_1}^2\partial_{\bar{z}_2\bar{z}_2} - 2\bar{z}_1\bar{z}_2\partial_{z_1z_2} \\
&\qquad +2z_1z_2\partial_{\bar{z}_1\bar{z}_2} + z_1\partial_{z_1} - \bar{z}_1\partial_{\bar{z}_1} + z_2\partial_{z_2} - \bar{z}_2\partial_{\bar{z}_2}).
\end{aligned} \tag{1.18}$$

From (1.17) and (1.18) we notice that when we compute $X(iX) - (iX)X$ all the derivatives of 2^{nd} order cancel out and we obtain

$$[X, iX] = -2i(z_1\partial_{z_1} - \bar{z}_1\partial_{\bar{z}_1} + z_2\partial_{z_2} - \bar{z}_2\partial_{\bar{z}_2}).$$

According to Def 1.7, the vector field $[X, iX]$ corresponds, at any point $p = (z_1, z_2) \in \partial\mathbb{B}^4$, to the tangent vector

$$V_p = (-2iz_1, -2iz_2) \in T_p\partial\mathbb{B}^4. \tag{1.19}$$

Notice that, as we expect from Proposition 1.2, V_p is <u>not</u> in the horizontal subspace $H_p\partial\mathbb{B}^4$, since

$$\langle \bar{\partial}\varphi(p), V_p \rangle = z_1\overline{(-2iz_1)} + z_2\overline{(-2iz_2)} = 2iz_1\bar{z}_1 + 2iz_2\bar{z}_2 =$$

$$2i(|z_1|^2 + |z_2|^2) = 2i \neq 0.$$

Let us now focus on the boundary point $p_0 := (1,0) \in \partial\mathbb{B}^4$ to get a better understanding of what exactly are the horizontal and the real tangent spaces.

Let us write the defining function φ in real coordinates:

$$\varphi(x_1, y_1, x_2, y_2) = {x_1}^2 + {y_1}^2 + {x_2}^2 + {y_2}^2.$$

Then the real tangent space at $p_0 = (1,0,0,0) \in \mathbb{R}^4$ is

$$T_{p_0}\partial\mathbb{B}^4 \left\{V \in \mathbb{R}^4 \ ; \ d\varphi(p_0)(V) = 0\right\} = \left\{V = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4 \ ; \ 2v_1 = 0\right\} =$$
$$Span_{\mathbb{R}} \left\{(0,1,0,0), (0,0,1,0), (0,0,0,1)\right\}.$$

On the other hand, according to (1.15), the horizontal subspace at p_0 is

$$H_{p_0}\partial\mathbb{B}^4 = \left\{A = (a_1, a_2) \in \mathbb{C}^2 \ ; \ a_1 = 0\right\} = Span_{\mathbb{R}} \left\{(0,0,1,0), (0,0,0,1)\right\}.$$

the first part of Example 1.3 says that, if we consider the horizontal vector field X as in (1.16), which at p_0 corresponds the horizontal vector $A_{p_0} = (0,0,-1,0)$, the bracket $[X, iX]$ generates the missing direction in the real tangent space. Indeed, according to (1.19), $[X, iX]$ corresponds at p_0 to the tangent vector

$$V_{p_0} = (-2i, 0) = (0, -2, 0, 0) \in T_{p_0}\partial\mathbb{B}^4 \setminus H_{p_0}\partial\mathbb{B}^4.$$

Lemma 1.5. *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a bounded strictly pseudoconvex domain. Then $\exists\ C \geq 1$ such that*

$$\frac{1}{C}|Z| \leq L_\rho(p; Z)^{\frac{1}{2}} \leq C|Z| \tag{1.20}$$

for each $p \in \partial\Omega$, $Z \in H_p\partial\Omega$. This means that if we replace the Levi length of a curve by its Euclidean length in the definition of the Carnot-Carathéodory metric (1.14) we get an equivalent metric.

Proof. See [1], page 506. □

1.3 The Approximation Lemma

Here we present an essential ingredient needed to study the Gromov-hyperbolicity for strictly pseudoconvex domains in Chapter 4 (see Section 4.4), namely Lemma 1.6 below: it states that the Carnot-Carathéodory metric can be approximated by a class of Riemannian metrics G_κ on $\partial\Omega$ in a specific quantitative sense.

In our setting the size of balls can be described quite explicitly by the following proposition stated in [1], page 513.

Proposition 1.3. ***(Box-Ball estimate).*** *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$ be a bounded strictly pseudoconvex domain with C^2-smooth boundary. Then there exists $\varepsilon_0 > 0$ and $C \geq 1$ such that for all $0 < \varepsilon \leq \varepsilon_0$ and $p \in \partial\Omega$*

$$Box\left(p, \frac{\varepsilon}{C}\right) \subseteq B_H(p, \varepsilon) \subseteq Box(p, C\varepsilon),$$

where $B_H(p,\varepsilon) := \{q \in \partial\Omega \ ; \ d_H(p,q) < \varepsilon\}$ and $Box(p,\varepsilon) := \{p+Z \in \partial\Omega \ ; \ |Z_H| < \varepsilon, \ |Z_{N,1}| < \varepsilon^2\}$. Here the decomposition $Z = Z_H + Z_N$ is taken at p and we remember $Z_{N,1} = Z_N \cap T_p\partial\Omega$.

Lemma 1.6. ***(The Approximation Lemma):*** *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a bounded strictly pseudoconvex domain. Then there exists a constant $C > 0$ such that for all $\kappa > 0$ the following holds: if $p, q \in \partial\Omega$, $d_H(p,q) \geq \frac{1}{\kappa}$, then*

$$\frac{1}{C} d_\kappa(p,q) \leq d_H(p,q) \leq C d_\kappa(p,q),$$

where d_κ is the distance function associated with the Riemannian metric

$$G_\kappa^2(p;Z) := \kappa^2 |Z_{N,1}|^2 + L_\rho(p;Z_H), \quad p \in \partial\Omega,\ Z \in T_p\partial\Omega.$$

Proof. We follow the proof given in [1], lemma 2.3 page 513. Let us fix $\kappa > 0$ and take G_κ as above. We divide the proof in two parts, in which we seek respectively for an upper and lower bound for d_κ in terms of d_H.

Upper bound: If $p, q \in \partial\Omega$ and $\alpha : [0,1] \to \partial\Omega$ is a (piecewise) C^1-smooth horizontal curve joining p and q, then $\dot{\alpha}_N \equiv 0$. Hence,

$$G_\kappa^2(\alpha(t);\dot{\alpha}(t)) = L_\rho(\alpha(t);\dot{\alpha}(t)) \quad for\ a.e.\ t \in [0,1].$$

Therefore, $l_{G_\kappa}(\alpha) = l_{L_\rho}(\alpha)$. Remember that for $d_\kappa(p,q)$ we minimize the G_κ-length over all possible C^1-smooth curves on $\partial\Omega$ joining p and q, not just the horizontal ones. Therefore,

$$d_\kappa(p,q) \leq d_H(p,q). \tag{1.21}$$

Lower bound: For the moment, assume that the following claim holds.

Claim 1.3.1. *There exists constants $\kappa_0 > 0$ and $C > 0$ such that we have the following implication:*

$$d_H(a,b) \geq \frac{1}{k} \Rightarrow d_\kappa(a,b) \geq \frac{C}{\kappa} \quad \forall\ a,b \in \partial\Omega,\ \kappa \geq \kappa_0. \tag{1.22}$$

Consider $p, q \in \partial\Omega$ with $d_H(p,q) \geq \frac{1}{\kappa}$.

• If $\kappa \geq \kappa_0$, let $\alpha : [0,1] \to \partial\Omega$ be a (piecewise) C^1-smooth curve joining p and q. There exists $N \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_N = 1$ s.t. for $x_j := \alpha(t_j)$ we have

$$\frac{1}{\kappa} \leq d_H(x_{j-1},x_j) \leq \frac{2}{\kappa}, \quad for\ j = 1, \cdots N. \tag{1.23}$$

Then, (1.22) applied to x_{j-1} and x_j leads to

$$l_{G_\kappa}(\alpha) = \sum_{j=1}^N l_{G_\kappa}\left(\alpha_{\restriction[t_{j-1},t_j]}\right) \geq \sum_{j=1}^N d_\kappa(x_{j-1},x_j) \underbrace{\geq}_{(1.22)}$$

$$\sum_{j=1}^N \frac{C}{\kappa} \underbrace{\geq}_{(1.23)} \frac{C}{2} \sum_{j=1}^N d_H(x_{j-1},x_j) \geq \frac{C}{2} d_H(p,q).$$

Taking the infimum over all admissible curves α we obtain

$$d_\kappa(p,q) \geq \frac{C}{2} d_H(p,q).$$

• If $0 < \kappa < \kappa_0$, let $\kappa_1 := \frac{1}{diam_H(\partial\Omega)}$. Notice that $\frac{1}{\kappa} \underbrace{\leq}_{hyp.} d_H(p,q) \leq diam_H(\partial\Omega) = \frac{1}{\kappa_1}$, so $\kappa_1 \leq \kappa < \kappa_0$ and

$$\kappa_1 \cdot d_H(p,q) \leq 1. \tag{1.24}$$

Moreover, $G_\kappa \geq \left(\frac{\kappa}{\kappa_0}\right) G_{\kappa_0}$ implies $d_\kappa \geq \left(\frac{\kappa}{\kappa_0}\right) d_{\kappa_0}$. Hence,

$$d_\kappa(p,q) \geq \left(\frac{\kappa}{\kappa_0}\right) d_{\kappa_0}(p,q) \geq \left(\frac{\kappa_1}{\kappa_0}\right) d_{\kappa_0}(p,q) \underbrace{\geq}_{(1.22)}$$

$$\left(\frac{\kappa_1}{\kappa_0}\right) \frac{C}{\kappa_0} \underbrace{\geq}_{(1.24)} C \left(\frac{\kappa_1}{\kappa_0}\right)^2 d_H(p,q).$$

Summarizing, provided the claim

$$\begin{cases} d_\kappa(p,q) \geq \frac{C}{2} d_H(p,q), & \kappa \geq \kappa_0 \\ d_\kappa(p,q) \geq C \left(\frac{\kappa_1}{\kappa_0}\right)^2 d_H(p,q), & 0 < \kappa < \kappa_0. \end{cases}$$

Up to renaming $C > 0$,

$$d_H(p,q) \leq C d_\kappa(p,q) \qquad \forall\ p,q \in \partial\Omega\ s.t.\ d_H(p,q) \geq \frac{1}{\kappa}. \tag{1.25}$$

(1.21) and (1.25) yields

$$d_\kappa(p,q) \leq d_H(p,q) \leq C d_\kappa(p,q) \qquad \forall\ p,q \in \partial\Omega\ s.t.\ d_H(p,q) \geq \frac{1}{\kappa}.$$

Therefore, we finish the proof of Approximation Lemma 1.6 provided the proof of claim 1.3.1.

Proof of Claim 1.3.1. *In the following, $C_1, C_2, \cdots$ will denote constants depending on Ω, but <u>not</u> on κ. According to (1.20), $\exists\ C_1 > 0$ s.t.*

$$C_1|Z|^2 \leq L_\rho(p; Z) \quad for\ p \in \partial\Omega,\ Z \in H_p\partial\Omega.$$

Let $\kappa_0 := \max\{\sqrt{2C_1}, \frac{1}{\varepsilon_0}\}$. Suppose $\kappa \geq \kappa_0$ and let $a, b \in \partial\Omega$ such that $d_H(a,b) \geq \frac{1}{\kappa}$. Since $\kappa \geq \kappa_0 \geq \frac{1}{\varepsilon_0}$, we have $\frac{1}{\kappa} \leq \varepsilon_0$, hence Proposition 1.3 yields

$$Box\left(a, \frac{C_2}{\kappa}\right) \subseteq B_H\left(a, \frac{1}{\kappa}\right)$$

for some $0 < C_2 \leq 1$. Since $d_H(a,b) \geq \frac{1}{\kappa}$,

$$b \notin Box\left(a, \frac{C_2}{\kappa}\right). \tag{1.26}$$

By points 2. and 4. of Lemma 1.4, the outer unit normal $n = \nabla\rho = 2\bar{\partial}_\rho$ is a C^1-smooth function on $\partial\Omega$. Hence, it is Lipschitz, i.e. $\exists\ C_3 > 0$ such that

$$|n(x) - n(y)| \leq C_3|x - y| \qquad \forall\ x, y \in \partial\Omega. \tag{1.27}$$

Now let $\alpha : [0,1] \to \partial\Omega$ be any (piecewise) C^1-smooth curve connecting a and b, and denote $A := l_E(\alpha)$ the Euclidean length of α. Since $\dot\alpha(t) \in T_{\alpha(t)}\partial\Omega$, we have

$$Re\langle \dot\alpha(t), n(\alpha(t))\rangle \underbrace{=}_{(1.1)} 0 \qquad for\ a.e.\ t \in [0,1].$$

Therefore,

$$\begin{gathered} |Re\langle \dot\alpha(t), n(a)\rangle| = |Re\langle \dot\alpha(t), n(\alpha(t)) + n(a) - n(\alpha(t))\rangle| = \\ |Re\left[\langle \dot\alpha(t), n(\alpha(t))\rangle + \langle \dot\alpha(t), n(a) - n(\alpha(t))\rangle\right]| = \\ |Re\langle \dot\alpha(t), n(a) - n(\alpha(t))\rangle| \le |\langle \dot\alpha(t), n(a) - n(\alpha(t))\rangle| \le \\ |\dot\alpha(t)||n(a) - n(\alpha(t))| \underbrace{\le}_{(1.27)} |\dot\alpha(t)|C_3|a - \alpha(t)| \le C_3 A|\dot\alpha(t)|. \end{gathered} \tag{1.28}$$

Integrating, we obtain

$$\begin{gathered} |Re\langle b - a, n(a)\rangle| = \left|Re\langle \int_0^1 \dot\alpha(t)dt, n(a)\rangle\right| = \left|\int_0^1 Re\langle \dot\alpha(t), n(a)\rangle dt\right| \le \\ \int_0^1 |Re\langle \dot\alpha(t), n(a)\rangle dt| \underbrace{\le}_{(1.28)} C_3 A^2. \end{gathered} \tag{1.29}$$

Now we notice that, for all $Z, W \in \mathbb{C}^n$,

$$|Z|^2 \ge \frac{|W|^2}{2} - |Z - W|^2. \tag{$\triangle$}$$

Indeed,

$$\begin{gathered} |Z|^2 + |Z - W|^2 - \frac{|W|^2}{2} \ge |Z|^2 + (|Z| - |W|)^2 - \frac{|W|^2}{2} = \\ 2|Z|^2 - 2|Z||W| + \frac{|W|^2}{2} = 2\left(|Z| + \frac{|W|}{2}\right)^2 \ge 0. \end{gathered}$$

Consequently,

$$|Z|^2 + |Z - W|^2 - \frac{|W|^2}{2} \ge 0 \iff |Z|^2 \ge \frac{|W|^2}{2} - |Z - W|^2.$$

From ($\triangle$) and (1.28) we get, for a.e. $t \in [0,1]$,

$$\begin{gathered} |\dot\alpha_N(t)|^2 = |\langle \dot\alpha(t), n(\alpha(t)\rangle|^2 = |Im\langle \dot\alpha(t), n(\alpha(t))\rangle|^2 \underbrace{\ge}_{(\triangle)} \\ \frac{|Im\langle \dot\alpha(t), n(a)\rangle|^2}{2} - |Im\langle \dot\alpha(t), n(\alpha(t))\rangle - Im\langle \dot\alpha(t), n(a)\rangle|^2 = \\ \frac{|Im\langle \dot\alpha(t), n(a)\rangle|^2}{2} - |Im\langle \dot\alpha(t), n(\alpha(t)), n(a)\rangle|^2 \underbrace{\ge}_{(1.28)} \\ \frac{|Im\langle \dot\alpha(t), n(a)\rangle|^2}{2} - C_3^2 A^2 |\dot\alpha(t)|^2. \end{gathered} \tag{1.30}$$

Now we denote

$$C_4 := \min\left\{\frac{\sqrt{2C_1}}{C_3}, \frac{C_2}{2\sqrt{C_3}}, C_2\right\}$$

and distinguish two cases:

$$\textit{(1)} \quad A \geq \frac{C_4}{\kappa},$$

$$\textit{(2)} \quad A < \frac{C_4}{\kappa}.$$

(1) *Recall that* $\kappa \geq \kappa_0 \geq \sqrt{2C_1}$. *For* $A \geq \frac{C_4}{\kappa}$, *we have*

$$l_{G_\kappa}(\alpha) = \int_0^1 \left(\kappa^2|\dot\alpha_N(t)|^2 + L_\rho(\alpha(t);\dot\alpha_H(t))\right)^{\frac{1}{2}} dt \geq$$

$$\int_0^1 \left(\kappa^2|\dot\alpha_N(t)|^2 + C_1|\dot\alpha_H(t)|^2\right)^{\frac{1}{2}} dt \geq \int_0^1 \left(2C_1|\dot\alpha_N(t)|^2 + C_1|\dot\alpha_H(t)|^2\right)^{\frac{1}{2}} dt \geq$$

$$\sqrt{C_1}\int_0^1 \left(|\dot\alpha_N(t)|^2 + |\dot\alpha_H(t)|^2\right)^{\frac{1}{2}} dt = \sqrt{C_1}A \geq \frac{\sqrt{C_1}C_4}{\kappa}. \tag{1.31}$$

(2) *In this case* $C_1 \geq \frac{\kappa^2}{2}C_3^2A^2$. *Indeed, this is equivalent to* $\frac{\sqrt{2C_1}}{C_3} \geq \kappa A$, *which is true as* $\frac{\sqrt{2C_1}}{C_3} \geq C_4 \underbrace{>}_{(2)} \kappa A$. *Moreover, since* $|\dot\alpha(t)| = |\dot\alpha_H(t)|^2 + |\dot\alpha_N(t)|^2$ *and* $\kappa \geq \kappa_0 \geq \sqrt{2C_1}$,

$$\kappa^2|\dot\alpha_N(t)|^2 + C_1|\dot\alpha_H(t)|^2 \geq \frac{\kappa^2}{2}|\dot\alpha_N(t)|^2 + C_1|\dot\alpha(t)|^2. \tag{1.32}$$

This gives

$$l_{G_\kappa}(\alpha) \geq \int_0^1 \left(\kappa^2|\dot\alpha_N(t)|^2 + C_1|\dot\alpha_H(t)|^2\right)^{\frac{1}{2}} dt \geq$$

$$\int_0^1 \left(\frac{\kappa^2}{2}|\dot\alpha_N(t)|^2 + C_1|\dot\alpha(t)|^2\right)^{\frac{1}{2}} dt \geq \int_0^1 \left(\frac{\kappa^2}{2}|\dot\alpha_N(t)|^2 + \frac{\kappa^2}{2}C_3^2A^2|\dot\alpha(t)|^2\right)^{\frac{1}{2}} dt =$$

$$\frac{\kappa}{\sqrt{2}}\int_0^1 \left(|\dot\alpha_N(t)|^2 + C_3^2A^2|\dot\alpha(t)|^2\right)^{\frac{1}{2}} dt \underbrace{\geq}_{(1.30)} \frac{\kappa}{\sqrt{2}}\int_0^1 \left(\frac{1}{2}|Im\langle\dot\alpha(t), n(a)\rangle|\right)^{\frac{1}{2}} dt \geq$$

$$\frac{\kappa}{2}|Im\langle b-a, n(a)\rangle|. \tag{1.33}$$

Now we denote $(b-a)_N := \langle b-a, n(a)\rangle$ *the projection of* $(b-a)$ *onto* $N_a\partial\Omega$. *From (1.29) and* $C_4 \leq \frac{C_2}{2\sqrt{C_3}}$, *i.e.* $C_3 \leq \frac{C_2^2}{4C_4^2}$, *we obtain*

$$|Im(b-a)_N| \geq |(b-a)_N| - |Re(b-a)_N| \underbrace{\geq}_{(1.29)} |(b-a)_N| - C_3A^2 \geq$$

$$|(b-a)_N| - \frac{C_2^2}{4C_4^2}A^2 \underbrace{\geq}_{(2)} |(b-a)_N| - \frac{C_2^2}{4\kappa^2}. \tag{1.34}$$

(1.33) and (1.34) yields

$$l_{G_\kappa}(\alpha) \geq \frac{\kappa}{2}|(b-a)_N| - \frac{C_2^2}{8\kappa}.$$

On the other hand, $C_4 \leq C_2$ *implies*

$$|(b-a)_H| \leq |b-a| \leq A \underbrace{<}_{2)} \frac{C_4}{\kappa} \leq \frac{C_2}{\kappa}. \tag{1.35}$$

In view of (1.26), which states that $b \notin Box\left(a, \frac{C_2}{\kappa}\right)$*, either* $(b-a)_H \geq \frac{C_2}{\kappa}$ *or* $(b-a)_{N,1} \geq \frac{C_2^2}{\kappa^2}$*. By (1.35), we must have* $|(b-a)_{N,1}| \geq \frac{C_2^2}{\kappa^2}$*. Hence,* $|(b-a)_N| \geq |(b-a)_{N,1}| \geq \frac{C_2^2}{\kappa^2}$*, and*

$$l_{G_\kappa}(\alpha) \geq \frac{\kappa}{2}\frac{C_2^2}{\kappa^2} - \frac{C_2^2}{8\kappa} = \frac{3C_2^2}{8\kappa}. \tag{1.36}$$

(1.31) and (1.36) show that

$$l_{G_\kappa}(\alpha) \geq \frac{C}{\kappa}$$

for some uniform $C > 0$*. Taking the infimum over all admissible paths* α *we obtain the result.* ✓

□

Chapter 2

Gromov hyperbolicity

The goal of this Chapter is to give a notion of hyperbolicity for general metric spaces, to then study some of their properties.

In section 2.1 we give a notion of hyperbolicity for geodesic metric spaces in terms of δ-slim triangles (this notion is due to Rips). We will show three equivalent conditions for a space to be Rips-hyperbolic, and this shall help us understand the "intuitive" geometric approach.

In section 2.2 we introduce the Gromov product for metric spaces which not need to be geodesics. This will allow us to introduce the Gromov condition for (δ)-hyperbolicity. This section ends with the proof that the Rips and Gromov conditions are compatible when dealing with metric spaces which are also geodesic. For Sections 2.1 and 2.2 we follow [4], part III, Chapter H, Section 1.

Section 2.3 is devoted to the proof that the classical hyperbolic spaces $\mathbb{H}^n$, $\mathbb{B}^n$ and $\mathbb{U}^n$ are Gromov-hyperbolic. Due to the fact that these are isometric as Riemannian manifolds, it is enough to show it for the Poincaré half-space $\mathbb{U}^n$.

2.1 Rips condition and δ-hyperbolicity

We start this section by recalling some basic definitions on geodesic metric spaces.

Def 2.1. ***(geodesics)*** *Let (X, d) be a metric space. A* ***geodesic path*** *joining two points $x, y \in X$ is a map $c : [0, l] \to X$ such that $c(0) = x$, $c(l) = y$ and*

$$d(c(t), c(t')) = |t - t'| \ \forall \ t, t' \in [0, l]. \tag{2.1}$$

$[x, y] := c([0, l])$ is called a ***geodesic segment*** *with endpoints x and y. (X, d) is a* ***(uniquely) geodesic metric space*** *if $\forall \ x, y \in X$, $\exists$ a (unique) geodesic path joining x and y.*

Def 2.2. ***(path length)*** *Let $[a, b]$ be a real interval. We call $P_{[a,b]}$ the set of the partitions of $[a, b]$:*

$$P_{[a,b]} := \{a = t_0 < t_1 \ldots < t_n = b; \ n \in \mathbb{N}\}.$$

The ***length*** *of a continuous path $c : [a, b] \to X$ is defined as*

$$l(c) := \sup_{\pi \in P_{[a,b]}} \sum_{i=1}^{n} d(c(t_i), c(t_{i-1})).$$

If $l(c) < \infty$, then c is said to be ***rectifiable****.*

Remark 2.1. *$l(c) \geq d(c(a), c(b))$, as $\{t_0 = a,\ t_1 = b\}$ is just one partition (the smallest) between all the possible ones for which we take the supremum.*

If c is a geodesic between two points $x, y \in X$, we get a telescopic sum which does not depend on the partition

$$l(c) = \sup_{\pi \in P_{[0,l]}} \sum_{i=1}^{n} d(c(t_i), c(t_{i-1})) \underbrace{=}_{(2.1)} \sup_{\pi \in P_{[0,l]}} \sum_{i=1}^{n} t_i - t_{i-1} =$$

$$\sup_{\pi \in P_{[0,l]}} t_n - t_0 = l - 0 = l = d(x, y).$$

Hence geodesics (when they exist) are paths with minimal lengths joining their ending points, and their length is exactly the distance between the endpoints. Here we used the fact that any rectifiable curve on $[a, b]$ can be reparametrized on $[0, l]$, where l is the distance between the endpoints, by $\tilde{c}(t) := c\left(\left(\frac{b-a}{l}\right) t + a\right)$.

Def 2.3. ***(geodesic triangles)*** *Let X be a geodesic metric space. A* ***geodesic triangle*** *with vertices $x, y, z \in X$ is a closed set $\Delta(x, y, z) \subset X$ bounded by three geodesic segments joining its vertices. If X is uniquely geodesic, then $\Delta(x, y, z)$ is unique.*

We are now ready to give the definition of Rips-hyperbolicity.

Def 2.4. ***(Rips-hyperbolicity)*** *Let $\delta \geq 0$. A geodesic triangle in a metric space X is said to be* ***δ-slim*** *if each of its sides is contained in the δ-neighborhood of the union of the other two sides (see figure 2.1). A geodesic metric space X is said to be* ***δ-hyperbolic*** *if every triangle in X is δ-slim. We say that X satisfies the* ***Rips condition*** *(or that X is* ***Rips-hyperbolic****) if $\exists\ \delta \geq 0$ s.t. X is δ-hyperbolic.*

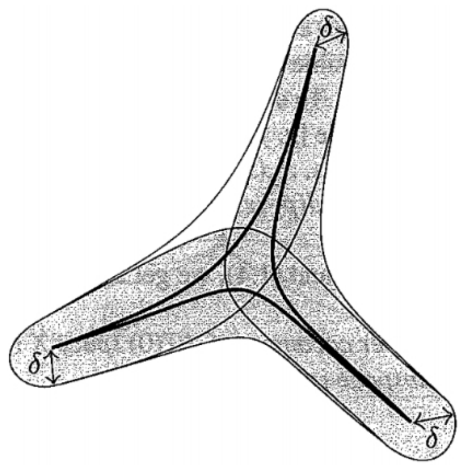

Figure 2.1: A δ-slim triangle

Sometimes we are interested in properties which depend only on the length of the sides of a geodesic triangle. In this case it can be useful to look at triangles in the Euclidean plane $\mathbb{E} := (\mathbb{R}^2, d_E)$, where d_E is the Euclidean distance, with the same sides length of the triangle we are dealing with.

Def 2.5. ***(Comparison triangles)*** *Let (X, d_X) be a metric space, and $\Delta := \Delta(x, y, z)$ be a geodesic triangle with vertices $x, y, z \in X$. A* ***comparison triangle*** *in $\mathbb{E}^2$ for Δ is a Euclidean triangle in $\mathbb{E}^2$ with vertices $\bar{x}, \bar{y}, \bar{z}$ such that*

$$d_X(x, y) = d_E(\bar{x}, \bar{y}), \quad d_X(y, z) = d_E(\bar{y}, \bar{z}), \quad d_X(x, z) = d_E(\bar{x}, \bar{z}).$$

Such a triangle exists thanks to the triangle inequality in X, and it is unique up to (Euclidean) isometry. We will denote such a comparison triangle $\Delta_E := \Delta_E(\bar{x}, \bar{y}, \bar{z})$.

We define $O_{\Delta_E} \subset \Delta_E$ the inscribed circle, and we denote the three meeting points between the comparison triangle Δ_E and its inscribed circle O_{Δ_E} as $\bar{i_x}, \bar{i_y}, \bar{i_z}$.

Moreover, we call $C_\Delta : \partial\Delta_E \to \partial\Delta$ the isometry between the boundaries of the triangles such that

$$C_\Delta(\bar{x}) = x, \; C_\Delta(\bar{y}) = y, \; C_\Delta(\bar{z}) = z,$$

and we get

$$i_x := C_\Delta(\bar{i_x}) \in [y, z], \; i_y := C_\Delta(\bar{i_y}) \in [x, z], \; i_z := C_\Delta(\bar{i_z}) \in [x, y].$$

i_x, i_y, i_z are called the ***internal points*** *of Δ.*

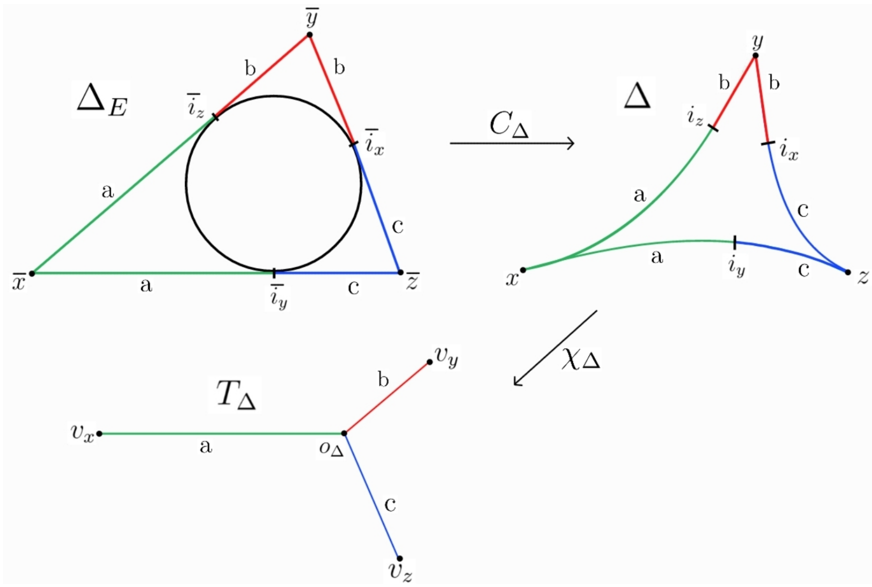

Figure 2.2: geometry of geodesic triangles

Now let (X, d) be a metric space and let $\Delta := \Delta(x, y, z)$ be a geodesic triangle in X. We define $T_\Delta := T(a, b, c)$ the metric tree that has three vertices v_x, v_y, v_z of valence 1, one vertex o_Δ of valence 3 and three edges, connecting v_x, v_y and v_z with o_Δ, of length a, b, c, s.t.

$$a + b = d(x, y); \quad a + c = d(x, z); \quad b + c = d(y, z).$$

Such a tree T_Δ (figure 2.2) is called a **tripod** . The map $\{x, y, z\} \to \{v_x, v_y, v_z\}$ extends uniquely to a map

$$\chi_\Delta : \partial\Delta \to T_\Delta$$

whose restriction to each side of Δ is an isometry. We note that the only pre-image that may have more than two points is $\chi_\Delta^{-1}(o_\Delta) = \{p \in \Delta \ ; \ \chi_\Delta(p) = o_\Delta\}$, which contains a point on each side of Δ. Moreover, this points are the images in Δ of the points at which the comparison triangle $\Delta_E \subset \mathbb{E}^2$ meets its inscribed circle. Then, $\chi_\Delta^{-1}(o_\Delta) = \{i_x, i_y, i_z\}$.

Def 2.6. ***(δ-thin triangles)*** *Let Δ be a geodesic triangle in a metric space (X, d_X), and consider the map $\chi_\Delta : \Delta \to T_\Delta$ defined above. Let $\delta \geq 0$. The triangle Δ is said to be* ***δ-thin*** *if $p, q \in \chi_\Delta^{-1}(t)$ implies $d_X(p,q) \leq \delta$ for all $t \in T_\Delta$. The diameter of $\chi_\Delta^{-1}(o_\Delta)$ is denoted* ***insize***Δ.

Let us now see how this definitions are related to the one of δ-slim triangles.

Proposition 2.1. ***(equivalent conditions for the Rips-hyperbolicity)*** *Let X be a geodesic space. The following conditions are equivalent:*

1. *$\exists\ \delta_0 \geq 0$ s.t. every geodesic triangle in X is δ_0-slim (X is Rips-hyperbolic).*
2. *$\exists\ \delta_1 \geq 0$ s.t. every geodesic triangle in X is δ_1-thin.*
3. *$\exists\ \delta_2 \geq 0$ s.t. insize$\Delta \leq \delta_2$ for every geodesic triangle Δ in X.*

Proof. Throughout the proof we consider a generic geodesic triangle $\Delta := \Delta(x, y, z)$ in X, and we denote its sides $[x, y], [y, z], [x, z]$. Figure 2.2 is useful to follow the arguments.

- **(2⇒1)** If $p \in [x, y]$, we want to show that $d(p, [x, z] \cup [y, z]) \leq \delta_0$, for some $\delta_0 \geq 0$ which does <u>not</u> depend on p. If $p = x$ or $p = y$ or the triangle is degenerate (two vertices coincide), any $\delta_0 \geq 0$ works. Otherwise, $card\left(\chi_\Delta^{-1}(\chi_\Delta(p))\right) = 2$. By hypothesis (2 holds) if $p' \in \chi_\Delta^{-1}(\chi_\Delta(p))$, then $d(p, p') \leq \delta_1$.

  If we repeat the argument for $p \in [x, z]$ or $p \in [y, z]$ we obtain the same result. Therefore, Δ is $\delta_1 - slim$ (we can then consider $\delta_0 = \delta_1$).

- **(1⇒3)** Now we consider the internal points i_x, i_y, i_z. Let us focus on $i_x \in [y, z]$. By hypothesis (1 holds) it lies in the δ_0-neighborhood of either $[x, y]$ or $[y, z]$. Let us suppose, without loss of generality, that it lies in the δ_0-neighborhood of $[x, y]$ and pick $p \in [x, y]$ such that $d(i_x, p) \leq \delta_0$.

  By the triangle inequality, $|d(y, p) - d(y, i_x)| \leq d(p, i_x) \leq \delta_0$. By construction, $d(y, i_x) = d(y, i_z)$, so $d(i_z, p) = |d(y, p) - d(y, i_z)| = |d(y, p) - d(y, i_x)| \leq \delta_0$.

  Hence $d(i_x, i_z) \leq d(i_x, p) + d(p, i_z) \leq 2\delta_0$.

  Rephrasing the same argument, we get

  $$d(i_y, \{i_x, i_z\}) \leq 2\delta_0 \quad and \quad d(i_z, \{i_x, i_y\}) \leq 2\delta_0.$$

  Therefore, $insize\Delta = diam\chi_\Delta^{-1}(o_\Delta) = \max\{d(i_x, i_y), d(i_x, i_z), d(i_y, i_z)\} \leq 4\delta_0 =: \delta_2$.

- **(3⇒2)** Let us now consider a point $p \in [y, z]$ such that $d(y, p) < d(y, i_x)$. Then $\chi_\Delta^{-1}(\chi_\Delta(p)) = \{p, q\}$, with $q \in [x, y]$, $d(y, q) = d(y, p)$. We now want to prove that $d(p, q) \leq \delta_1$ for some $\delta_1 \geq 0$ which does not depend on p, q. We do it by constructing a geodesic triangle of which p and q are internal points.

  Let $c : [0, 1] \to X$ be a monotone parametrization of $[y, z]$. For each $t \in [0, 1]$, consider a geodesic triangle $\Delta_t := \Delta(x, y, c(t))$, which has two sides that are $[y, x]$ and $c([0, t])$. We focus on the internal point of Δ_t on the side $c([0, t])$: for $t = 0$ it is y, and for $t = 1$ it is i_x. The internal point of Δ_t on the side $c([0, t])$ varies continuously as a function of t, hence is has to be p for some $t_0 \in [0, 1]$. Moreover, $d(y, p) = d(y, q)$, therefore q is also an internal point of Δ_{t_0}. Since Δ_{t_0} is a geodesic triangle, by assumption,

$$d(p, q) \leq insize\Delta_{t_0} \underbrace{\leq}_{(3)} \delta_2.$$

  We notice that the choice of p is made without loss of generality, as for all $p' \in \Delta$ it suffices to choose a vertex and an internal point of Δ such that p' belongs to the segment joining them. Then one can repeat verbatim the proof considering $q' \in \chi_\Delta^{-1}(\chi_\Delta(p'))$. Hence it suffices to pick $\delta_1 := \delta_2$.

□

2.2 Gromov product and (δ)-hyperbolicity

Def 2.7. ***(Gromov product)*** *Let (X, d) be a metric space (not necessarily geodesic). The **Gromov product** of two points $y, z \in X$ with respect to $x \in X$ is*

$$(y \cdot z)_x := \frac{d(y, x) + d(z, x) - d(y, z)}{2}.$$

Remark 2.2. *We notice that the Gromov product $(y \cdot z)_x$ measures exactly the length of the edge "a" of the tripod T_Δ associated to $\Delta := \Delta(x, y, z)$.*

Def 2.8. ***(Gromov-hyperbolicity)*** *Let $\delta \geq 0$. A metric space (X, d) is said to be **(δ)-hyperbolic** if*

$$(x \cdot y)_w \geq \min\{(x \cdot z)_w, (y \cdot z)_w\} - \delta \tag{2.2}$$

*for all $x, y, z, w \in X$. X is said to satisfy the **Gromov condition** (or to be **Gromov-hyperbolic**) if $\exists\ \delta \geq 0$ such that X is (δ)-hyperbolic.*

Henceforth, when we say that a metric space (X, d) is hyperbolic, we mean that it is Gromov-hyperbolic.

For fixed $w_0 \in X$, we omit the subscript in the Gromov product, i.e. we denote $(x \cdot y) := (x \cdot y)_{w_0}\ \forall\ x, y \in X$.

Lemma 2.1. *If $w_0 \in X$ and*

$$(x \cdot y) \geq \min\{(x \cdot z), (y \cdot z)\} - \delta \quad \forall\ x, y, z \in X,$$

then,

$$(x \cdot y) + (z \cdot w) - \min\{(x \cdot z) + (y \cdot w), (y \cdot z) + (x \cdot w)\} \geq -2\delta \tag{2.3}$$

$\forall\ x, y, z, w \in X$.

Proof. **Step 1:** Let us first suppose that $\max\{(x \cdot z), (x \cdot w), (y \cdot z)\} = (x \cdot z)$. Then, by hypothesis,

$$\begin{cases} (x \cdot y) \geq \min\{(x \cdot z), (y \cdot z)\} - \delta = (y \cdot z) - \delta \\ (z \cdot w) \geq \min\{(x \cdot z), (x \cdot w)\} - \delta = (x \cdot w) - \delta. \end{cases}$$

Summing term by term we obtain

$$(x \cdot y) + (z \cdot w) \geq (y \cdot z) + (x \cdot w) - 2\delta \geq$$

$$\min\{(x \cdot z) + (y \cdot w), (y \cdot z) + (x \cdot w)\} - 2\delta \quad \checkmark.$$

Step 2: If $\max\{(x \cdot z), (x \cdot w), (y \cdot z)\} = (x \cdot w)$, we permute z and w. By symmetry of Gromov product, this does not influence the quantity $(x \cdot y) + (z \cdot w)$, and brings us again in the situation of Step 1. We do the same for the case $\max\{(x \cdot z), (x \cdot w), (y \cdot z)\} = (y \cdot z)$, permuting x and y.

□

This Lemma yields the following result.

Proposition 2.2. *If condition (2.2) holds for all $x, y, z \in X$ and a fixed point $w_0 \in X$, then the space is (2δ)-hyperbolic.*

Proof. It suffices to show that the LHS of equation (2.3) is equal to

$$(x \cdot y)_w - \min\{(x \cdot z)_w, (y \cdot z)_w\},$$

as it would mean that

$$(x \cdot y)_w \geq \min\{(x \cdot z)_w, (y \cdot z)_w\} - 2\delta \quad \forall\ x, y, z, w \in X,$$

which is exactly the definition of (2δ)-hyperbolicity. By definition of Gromov product, the LHS of (2.3) is

$$\begin{aligned} (x \cdot y) + (z \cdot w) - \min\{(x \cdot z) + (y \cdot w), (x \cdot w) + (y \cdot z)\} = \\ \frac{1}{2}(d(x, w_0) + d(y, w_0) - d(x, y) + d(z, w_0) + d(w, w_0) - d(z, w) - \\ \min\{d(x, w_0) + d(z, w_0) - d(x, z) + d(y, w_0) + d(w, w_0) - d(y, w), \\ d(x, w_0) + d(w, w_0) - d(x, w) + d(y, w_0) + d(z, w_0) - d(y, z)\}) = \\ -\frac{1}{2}(d(x, y) + d(z, w) + \min\{-d(x, z) - d(y, w), -d(x, w) - d(y, z)\}). \end{aligned} \tag{2.4}$$

On the other hand,

$$\begin{aligned} (x \cdot y)_w - \min\{(x \cdot z)_w, (y \cdot z)_w\} = \frac{1}{2}(d(x, w) + d(y, w) - d(x, y) - \\ \min\{d(x, w) + d(z, w) - d(x, z), d(y, w) + d(z, w) - d(y, z)\}) \end{aligned} \tag{2.5}$$

The goal is to show that (2.4) is equal to (2.5), and this is true if and only if

$$-\min\{-d(x,z)-d(y,w),-d(x,w)-d(y,z)\}=$$
$$d(x,w)+d(y,w)-\min\{d(x,w)-d(x,z),d(y,w)-d(y,z)\}.$$

There are two options for the RHS of the last expression:

1. $RHS = d(y,w) + d(x,z)$.
   This happens if
   $$d(x,w)-d(x,z) \le d(y,w)-d(y,z) \iff$$
   $$-d(x,z)-d(y,w) \le -d(x,w)-d(y,z).$$
   Then $LHS = d(x,z) + d(y,w) = RHS$.
2. $RHS = d(x,w) + d(y,z)$.
   This happens if
   $$d(x,w)-d(x,z) \ge d(y,w)-d(y,z) \iff$$
   $$-d(x,z)-d(y,w) \ge -d(x,w)-d(y,z).$$
   Then $LHS = d(x,w) + d(y,z) = RHS$.

□

Remark 2.3. *(Geometry behind Gromov condition) By the definition of Gromov Product 2.7, one can rewrite (2.2) as a 4-points condition:*

$$d(x,w)+d(y,z) \le \max\{d(x,y)+d(z,w),\ d(x,z)+d(y,w)\}+2\delta, \tag{2.6}$$

for all $w,x,y,z \in X$.

Let us now think of w,x,y,z as the vertices of a tetrahedron. The expressions d(x,y)+d(z,w), d(x,z)+d(y,w) and d(x,w)+d(y,z) are called the ***pair sizes*** *of $\{w,x,y,z\}$.*

The reformulated inequality (2.6) states that if we list the pair sizes in increasing order, $S \le M \le L$, then

$$L - M \le 2\delta.$$

If we consider comparison triangles $\bar{\Delta}(x,w,y)$ and $\bar{\Delta}(x,w,z)$ in $\mathbb{E}^2$ and we glue them on the edge $[\bar{x},\bar{w}]$ we obtain the configuration in figure 2.3, where the pair sizes correspond to the sum of the lengths of the opposite sides and sum of the lengths of the diagonals.

We are now ready to show that for a geodesic metric space the two notions of Gromov and Rips hyperbolicity we have given coincide.

Theorem 2.1. *Let X be a geodesic space. Then X is Rips-hyperbolic $\iff$ X is Gromov-hyperbolic.*

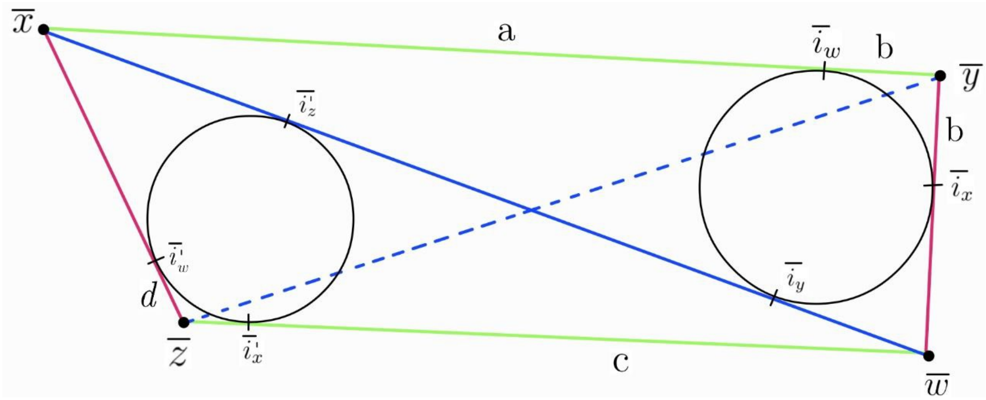

Figure 2.3: glued comparison triangles

Proof. By Prop. 2.1, X is Rips-hyperbolic $\iff \exists\, \delta \geq 0$ s.t. $insize\Delta \leq \delta$ for every geodesic triangle Δ in X. Therefore, it suffices to prove that this condition is satisfied $\iff$ X is Gromov-hyperbolic.

($\Rightarrow$) First we show that if $insize\Delta \leq \delta$ for all geodesic triangles $\Delta \subset X$, then X is (δ)-hyperbolic. Given $w, x, y, z \in X$ we can assume without loss of generality S, M, L as before. We must show that $L \leq M + 2\delta$.

Let $\Delta = \Delta(x, w, y)$ and $\Delta' = \Delta(x, z, w)$ be geodesic triangles, and denote their internal points with (i_x, i_w, i_y) and (i'_x, i'_z, i'_w). Consider now the path from y to z through i_x, i_y, i'_z, i'_w (see figure (2.4) for the picture for the comparison triangles $\Delta_E(\bar{x}, \bar{w}, \bar{y})$, $\Delta'_E(\bar{x}, \bar{z}, \bar{w})$).

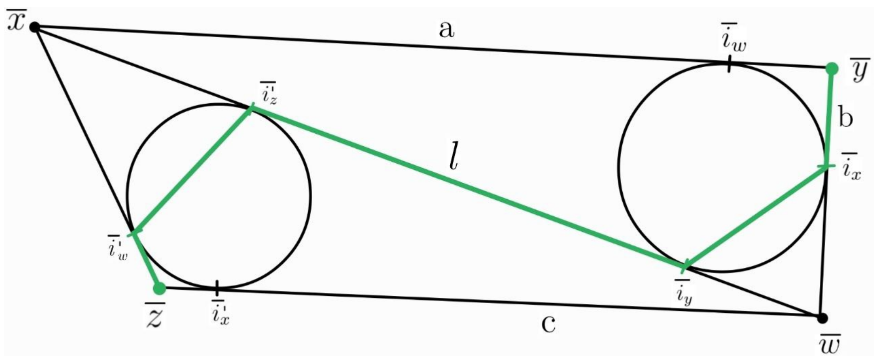

Figure 2.4:

Let us denote $l := d(i'_y, i'_z)$. By the triangle inequality we get

$$d(y, z) \leq b + \underbrace{d(i_x, i_y)}_{\leq \delta} + l + \underbrace{d(i'_z, i'_w)}_{\leq \delta} + d \leq b + l + d + 2\delta.$$

On the other hand, $d(x, w) = a + c - l$, thus

$$L = d(x, w) + d(y, z) \leq a + c - l + b + d + l + 2\delta =$$

$$\underbrace{a + b}_{d(x,y)} + \underbrace{c + d}_{d(z,w)} + 2\delta = M + 2\delta.$$

($\Leftarrow$) Now we assume that X is (δ)-hyperbolic, and we deduce that $insize\Delta \leq 6\delta$ for all geodesic triangles $\Delta = \Delta(x, y, z) \subset X$.

Let us consider the internal point $i_x \in [y, z]$. We want to apply the 4-points condition (2.6) to $\{x, y, z, i_x\}$.

Claim 2.2.1. *The biggest of the pair sizes for $\{x, y, z, i_x\}$ is $d(x, i_x) + d(y, z)$.*

Proof of Claim 2.2.1. *Since $d(y, z) = d(y, i_x)+d(i_x, z)$, by the triangle inequality*

$$2\,[d(x, i_x) + d(y, z)] = [d(x, i_x) + d(i_x, z)] + [d(x, i_x) + d(i_x, y)] + d(y, z)$$
$$\geq d(x, z) + d(x, y) + d(y, z) = P(\Delta),$$

where $P(\Delta)$ denotes the perimeter of Δ. On the other hand,

$$d(z, i_x) + d(x, y) = d(z, i_x) + d(x, i_z) + d(i_z, y) = \frac{P(\Delta)}{2},$$

$$d(y, i_x) + d(x, z) = d(y, i_x) + d(x, i_y) + d(i_y, z) = \frac{P(\Delta)}{2}. \qquad \checkmark$$

Therefore, (2.6) yields

$$d(x, i_x) + d(z, y) \leq \max\{d(z, i_x) + d(x, y),\ d(y, i_x) + d(x, z)\} + 2\delta \leq \frac{P(\Delta)}{2} + 2\delta.$$

We notice that even $d(y, z)$ and $d(x, i_z)$ sum up to $\frac{P(\Delta)}{2}$, hence

$$d(x, i_x) + d(z, y) \leq d(y, z) + d(x, i_z) + 2\delta \Longrightarrow d(x, i_x) - d(x, i_z) \leq 2\delta.$$

Following an argument analogous to the proof of Claim 2.2.1 for the points $\{x, y, z, i_z\}$, one can show that $d(z, i_z) + d(x, y) \leq \frac{P(\Delta)}{2} + 2\delta$. Notice that $d(y, x) + d(z, i_x) = \frac{P(\Delta)}{2}$, hence

$$d(z, i_z) + d(x, y) \leq d(y, x) + d(z, i_x) + 2\delta \iff d(z, i_z - d(z, i_x) \leq 2\delta.$$

Let us now consider the four points $\{x, z, i_x, i_z\}$. Now the pair sizes are: $d(x, i_z) + d(z, i_x) = d(x, z)$, $d(x, z) + d(i_x, i_z)$ and $d(x, i_x) + d(z, i_z)$. The last of these is such that

$$d(x, i_x) + d(z, i_z) = \underbrace{d(x, i_x) - d(x, i_z)}_{\leq 2\delta} + d(x, i_z) + d(z, i_x) \underbrace{-d(z, i_x) + d(z, i_z)}_{\leq 2\delta} \leq$$
$$d(x, i_z) + d(z, i_x) + 4\delta = d(x, z) + 4\delta. \tag{2.7}$$

Following the order in which we have written the pair sizes, the second is greater than the first, being the first itself plus a positive quantity. The third is greater than the first as well, and this can be seen as follows: If we call O_{xz} the intersection point between $[x, i_x]$ and $[z, i_z]$ (see figure 2.5), and we look at the geodesic triangles $\Delta(z, i_x, O_{xz})$ and $\Delta(x, i_z, O_{xz})$, we get

$$d(x, i_z) \leq d(x, O_{xz}) + d(O_{xz}, i_z),$$

$$d(z, i_x) \leq d(z, O_{xz}) + d(O_{xz}, i_x).$$

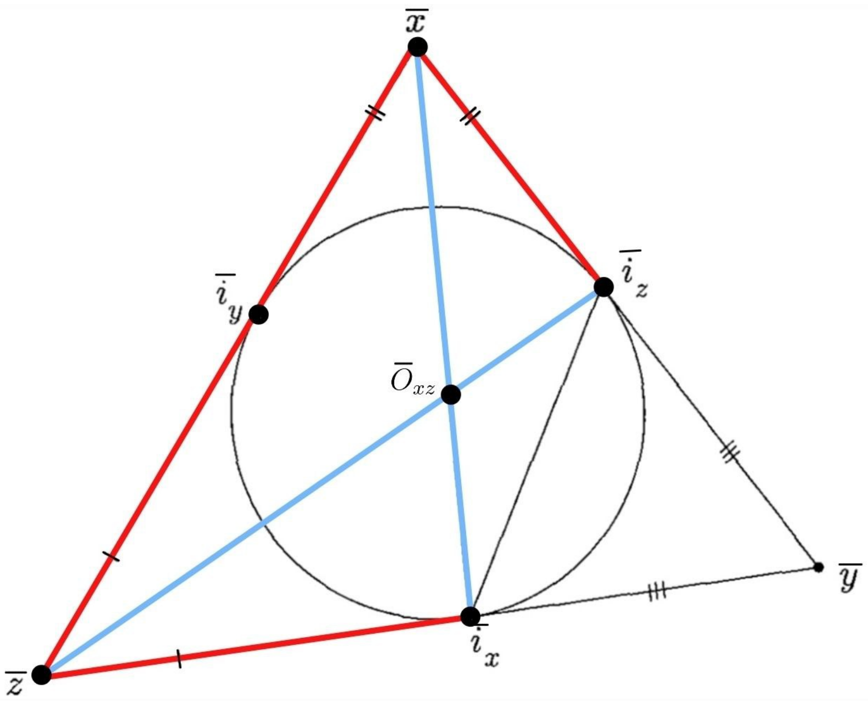

Figure 2.5: 4 points condition for $\{x, z, i_x, i_z\}$

The sum of the left hand sides give $d(x, z)$, i.e. the first pair size, while the sum of the right hand sides gives exactly $d(x, i_x) + d(z, i_z)$, i.e. the third pair size. Therefore, applying the 4 points condition (2.6) to $\{x, z, i_x, i_z\}$ we get

$$d(x, z) + d(i_x, i_z) - d(x, i_x) - d(z, i_z) \leq 2\delta.$$

Now we apply the condition (2.7) found above and we get

$$d(x, z) + d(i_x, i_z) - d(x, z) - 4\delta \leq 2\delta \iff d(i_x, i_z) \leq 6\delta.$$

The argument can be repeated starting with $i_y \in [x, z]$ and $i_z \in [x, y]$ to get also

$$d(i_y, i_x) \leq 6\delta, \qquad d(i_z, i_y) \leq 6\delta;$$
$$\implies insize\Delta \leq 6\delta.$$

□

2.3 Classical hyperbolic spaces are Gromov-hyperbolic

In this section we want to show that Gromov-hyperbolicity generalizes the notion of hyperbolic spaces as Riemannian manifolds with constant negative (sectional) curvature. In particular, we want to show that the "classical hyperbolic spaces" (hyperboloid, Poincaré ball and Poincaré half-space) are Gromov-hyperbolic when endowed with the distance functions induced by specific Riemannian metrics.

We start by introducing the classical models of hyperbolic geometry.

Def 2.9. ***(hyperboloid model)*** *$\mathbb{H}^n_R$ is the "upper sheet" of the two sheeted hyperboloid in $\mathbb{R}^{n+1}$, defined as*

$$\mathbb{H}^n_R := \{(\xi_1, \ldots, \xi_n, \tau) \in \mathbb{R}^{n+1} \ s.t.\ \tau^2 - |\xi|^2 = R^2,\ \tau > 0\},$$

endowed with the Riemannian metric

$$h^1_R := i^* m,$$

where $i : \mathbb{H}^n_R \hookrightarrow \mathbb{R}^{n+1}$ is the inclusion, and m is the Minkowski metric on $\mathbb{R}^{n+1}$:

$$m = (d\xi_1)^2 + \ldots + (d\xi_n)^2 - (d\tau)^2. \tag{2.8}$$

Def 2.10. ***(Poincaré ball model)*** *$\mathbb{B}^n_R$ is the ball of radius R in $\mathbb{R}^n$, with the metric given in coordinates $(u_1, \ldots, u_n)$ by*

$$h^2_R := 4R^2 \frac{(du_1)^2 + \ldots + (du_n)^2}{(R - |u|^2)^2}.$$

Def 2.11. ***(Poincaré half-space model)*** *$\mathbb{U}^n_R$ is the upper half space in $\mathbb{R}^n$, defined in coordinates $(x_1, \ldots, x_{n-1}, y)$ such that $y > 0$, with the metric*

$$h^3_R := R^2 \frac{(dx_1)^2 + \ldots + (dx_{n-1})^2 + dy^2}{y^2}.$$

Theorem 2.2. ***(Classical hyperbolic Spaces)*** *For all $R > 0$, the Riemannian manifolds $\mathbb{H}^n_R$, $\mathbb{B}^n_R$, $\mathbb{U}^n_R$ are isometric.*

Proof. See [14], Prop. 3.5 page 38. □

The goal now is to prove that the classical hyperbolic space with radius 1 (in any of its realization $\mathbb{H}^n$, $\mathbb{B}^n$, $\mathbb{U}^n$) is Gromov-hyperbolic. It suffices to prove it for the Poincaré half-space $\mathbb{U}^n$ for $n = 2$, as each geodesic triangle in $\mathbb{U}^n$ can be seen in an isometrically embedded copy of $\mathbb{U}^2$.

We will now give a proof of the Gromov-hyperbolicity of $\mathbb{U}^2$, following the proof of Theorem 7.2, in [3], page 285, which relies on a direct check of the 4-points condition (2.6). The idea of this proof will be used again in Chapter 4 to check the Gromov-hyperbolicity of strictly pseudoconvex domains with the "artificial" distance g in (4.3).

For this proof we will need an expression for the distance function d_{h^3}.

Lemma 2.2. *If $P, Q \in \mathbb{U}^2$, then*

$$d_{h^3}(P, Q) = 2 \ln \frac{|P - Q| + |P - \bar{Q}|}{2\sqrt{Im(P) Im(Q)}}. \tag{2.9}$$

Proof. See [10], Th. 1.2.6 page 6. □

Theorem 2.3. *$\mathbb{U}^2$ is Gromov-hyperbolic.*

Proof. Now we want to check that the 4-points condition (2.6) is satisfied, i.e. $\exists\, \delta \geq 0$ s.t. for all $P_j = x_j + iy_j,\ j = 1, \cdots 4$,

$$d_{h^3}(P_1, P_2) + d_{h^3}(P_3, P_4) \leq$$

$$\max\{d_{h^3}(P_1, P_3) + d_{h^3}(P_2, P_4), d_{h^3}(P_1, P_4) + d_{h^3}(P_2, P_3)\} + 2\delta.$$

Let us define $r_{jk} := |P_j - P_k| + |P_j - \bar{P}_k|$, for $j, k = 1, \cdots 4$, where $|P_j - P_k|$ is the Euclidean distance. Then

1. $r_{jk} = r_{kj}$, as $|P_j - \bar{P}_k| = |P_k - \bar{P}_j|$;
2. $r_{jk} \leq r_{jl} + r_{lk}$ for $l = 1, \cdots 4$, thanks to the triangle inequality.

Claim 2.3.1. $r_{12}r_{34} \leq 4\max\{r_{13}r_{24}, r_{14}r_{23}\}$.

Proof of Claim 2.3.1. *If we assume that* $\min\{r_{13}, r_{32}, r_{14}\} = r_{13}$*, then by the two properties above*

$$r_{12} \leq r_{13} + r_{32} \leq 2r_{23};$$

$$r_{34} \leq r_{31} + r_{14} \leq 2r_{14};$$

$$\Rightarrow r_{12}r_{34} \leq 2r_{23}2r_{14} \leq \max\{r_{23}r_{14}, r_{13}r_{24}\}.$$

If $\min\{r_{13}, r_{32}, r_{14}\} = r_{32}$*, we permute* P_2 *and* P_1*. This doesn't change the procedure above, by simmetry of* r_{jk}*.*

If $\min\{r_{13}, r_{32}, r_{14}\} = r_{14}$*, we argue the same, permuting* P_3 *and* P_4*.* ✓

If we divide each side of the above inequality by $4\sqrt{y_1y_2y_3y_4}$ we get, by monotonicity of the logaritm

$$\ln\left(\frac{r_{12}}{2\sqrt{y_1y_2}}\frac{r_{34}}{2\sqrt{y_3y_4}}\right) \leq \ln\left(4\max\left\{\frac{r_{23}r_{14}}{4\sqrt{y_1y_2y_3y_4}}, \frac{r_{13}r_{24}}{4\sqrt{y_1y_2y_3y_4}}\right\}\right).$$

We recall that the logaritm of a product is the sum of logaritms, and the formula for the hyperbolic distance (2.9), to get

$$d_{h^3}(P_1, P_2) + d_{h^3}(P_3, P_4) \leq$$

$$\max\{d_{h^3}(P_1, P_3) + d_{h^3}(P_2, P_4), d_{h^3}(P_1, P_4) + d_{h^3}(P_2, P_3)\} + 2\ln(4).$$

Therefore, $\mathbb{U}^2$ is $(\ln 4)$-hyperbolic and, as a consequence, Gromov-hyperbolic. □

Remark 2.4. ***(Extension to n>2):*** *One can prove that* $(\mathbb{U}^n, d_{h^3})$ *is (uniquely) geodesic (see [14] Proposition 5.14, page 83). Hence, by Theorem 2.1 and 2.3,* $\mathbb{U}^2$ *is Rips-hyperbolic. To prove Gromov-hyperbolicity of* $\mathbb{U}^n$ *it then suffices to check Rips-hyperbolicity, and this can be done by noticing that each geodesic triangle in* $\mathbb{U}^n$ *is contained in an isometrically embedded copy of* $\mathbb{U}^2$*.*

Chapter 3

Kobayashi hyperbolicity

As in Chapter 2 we defined a notion of hyperbolicity for general metric spaces, in Chapter 3 we shall give a definition of hyperbolicity for complex spaces, where a complex space is roughly speaking a closed subspace of $\mathbb{C}^n$ given as a zero set of some $k < n$ holomorphic functions. Particularly, complex manifolds, CR manifolds and some other objects are included into "complex spaces".

Kobayashi, in 1967, defined a natural semi-distance on any complex space. Instead of taking the distance between two points by linking them by Riemannian geodesics and computing the infimum of their length with respect to the Riemannian metric, he joins points by a chain of discs and takes their distance computing an infimum with respect to the hyperbolic metric. He calls a complex space **hyperbolic** when the semi-distance is a distance. Notice that the Kobayashi semi-distance may be identically zero. For example, it is identically zero on the complex plane $\mathbb{C}$. This occurs because $\mathbb{C}$ contains arbitrarily big discs.

We show that the Kobayashi semi-distance on the Poincaré disc is indeed equal to the Poincaré distance, making $\mathbb{D} := \mathbb{B}^2$ Kobayashi-hyperbolic. As a consequence, we will be able to give another example of Kobayashi-hyperbolic spaces: the polydiscs.

We also see that the bounded strictly pseudoconvex domains $\Omega \subset \mathbb{C}^n$ defined in Chapter 2 are Kobayashi-hyperbolic. As a consequence, we will be allowed to study (in Chapter 4) Gromov-hyperbolicity of such metric spaces when endowed with the Kobayashi distance. For Sections 3.1 and 3.2 we follow [12], Chapter I, Sections 1 and 2.

In Section 3.3 we give an equivalent definition of the Kobayashi semi- distance as the semi-distance function induced by a semi-Finsler metric

3.1 Holomorphic maps and the hyperbolic distance

The core of the concept of Kobayashi-hyperbolicity lies in Schwarz-Pick Lemma, which gives a bound for the derivative of any holomorphic map from the unit disc $\mathbb{D} := \mathbb{B}^2$ to itself, making it distance decreasing for the hyperbolic distance. On a complex manifold (or complex space) X, the Kobayashi semi-distance results to be the largest semi-distance d_X on X such that

$$d_X(f(x), f(y)) \leq d_{h^2}(x, y),$$

for all holomorphic maps f from the unit disc $\mathbb{D}$ to X. In a sense, this formula generalizes Schwarz-Pick lemma to all complex spaces.

Def 3.1. ***(semi-distance):*** *A* ***semi-distance function*** *over a set X is a non-negative function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in Z$:*

1. *$d(x, y) = d(y, x)$;*
2. *$d(x, y) \leq d(x, z) + d(z, y)$;*
3. *$x = y \quad \Rightarrow \quad d(x, y) = 0$.*

This definition differs from the one of "distance function" only for the last condition, which for a distance function is $x = y \iff d(x, y) = 0$.

Def 3.2. ***(distance decreasing maps):*** *Let (X, d), (X', d') be two spaces with semi-distance functions d, d'. A map $f : X \to X'$ is said to be* ***distance decreasing*** *if*

$$d'(f(x_1), f(x_2)) \leq d(x_1, x_2) \quad \forall\, x_1, x_2 \in X.$$

Proposition 3.1. ***(Schwarz-Pick Lemma):*** *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map of the disc into itself. Then for each $z \in \mathbb{D}$*

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}, \tag{3.1}$$

and equality holds if and only if f is an automorphism.

Proof. See [12], Prop. 1.1 □

Corollary 3.1. *A holomorphic map $f : \mathbb{D} \to \mathbb{D}$ is distance decreasing for the hyperbolic distance, i.e.*

$$d_{h^2}(f(z), f(w)) \leq d_{h^2}(z, w), \quad \forall\, z, w \in \mathbb{D}.$$

Proof. If we consider a (piecewise) differentiable path $\gamma : [0, 1] \to X$ joining z and w, then $f \circ \gamma : [0, 1] \to X$ is a (piecewise) differentiable path joining $f(z)$ and $f(w)$. Then

$$d_{h^2}(f(z), f(w)) \leq 2 \int_0^1 \frac{|(\dot{f \circ \gamma})(t)|}{1 - |f(\gamma(t))|^2} dt =$$

$$2 \int_0^1 \frac{|f'(\gamma(t))||\dot{\gamma}(t)|}{1 - |f(\gamma(t))|^2} dt \underbrace{\leq}_{(3.1)} 2 \int_0^1 \frac{|\dot{\gamma}(t)|}{1 - |\gamma(t)|^2} dt.$$

Taking the inf over all possible paths γ on the RHS, to obtain $d_{h^2}(z, w)$, we get the result. □

3.2 Kobayashi semi-distance

The Kobayashi semi-distance is defined on complex spaces, but since in this section we will be dealing mostly with complex manifolds, we recall the definition.

Def 3.3. *A* ***Complex manifold*** *is a topological manifold with an atlas of charts to the open unit disc in $\mathbb{C}^n$, such that the transition maps are holomorphic.*

Let now X be a complex manifold (space), and consider $x, y \in X$. We consdier a sequence of holomorphic embeddings

$$f_k : \mathbb{D} \to X, \qquad k = 1, \cdots, m, \quad m \in \mathbb{N},$$

such that $f_k(\mathbb{D}) \cap f_{k+1}(\mathbb{D}) \neq \emptyset$, $x \in f_1(\mathbb{D})$, $y \in f_m(\mathbb{D})$. We take $x =: x_0$, $y =: x_m$ and for each $k = 1, \cdots, m-1$ we pick $x_k \in f_k(\mathbb{D}) \cap f_{k+1}(\mathbb{D})$ and we call $p_k := f_k^{-1}(x_k)$, $p_0 = f_1^{-1}(x_0)$, $p_m = f_m^{-1}(x_m)$. We consider the geodesic joining p_{k-1} to p_k, and by its image we connect x_{k-1} to x_k. In this way we obtain a piecewise smooth curve $\gamma_{x,y}$ connecting x and y, called a **Kobayashi path** (see figure 3.1). Taking the infimum over all possible Kobayashi paths, i.e. all the possible choices of $\{f_k\}_{k=1,\cdots,m}$ and $\{p_k\}_{k=0,\cdots,m}$, we define the **Kobayashi semi-distance**

$$d_{K,X}(x, y) := \inf \sum_{k=1}^{m} d_{h^2}(p_{k-1}, p_k). \tag{3.2}$$

$\sum_{k=1}^{m} d_{h^2}(p_{k-1}, p_k)$ is called a **Kobayashi sum**.

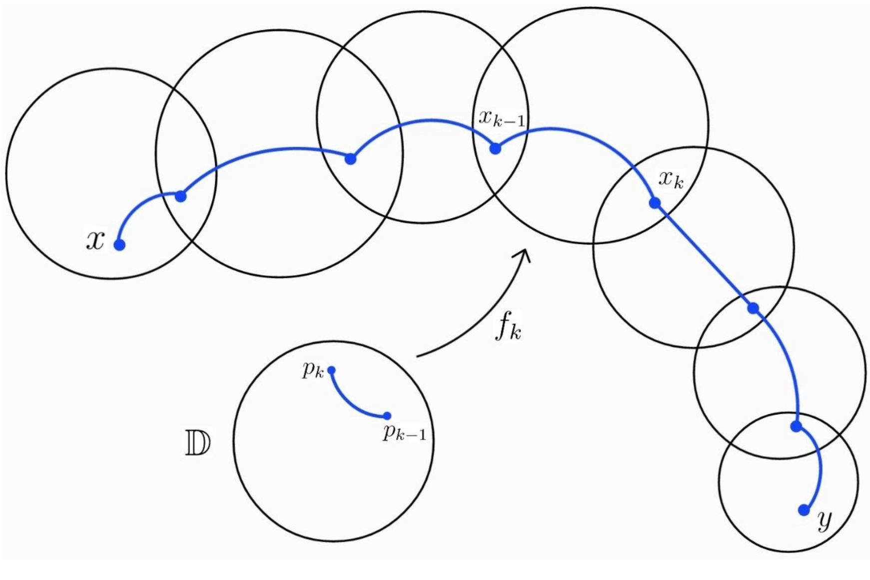

Figure 3.1: Kobayashi path

Remark 3.1. *$d_{K,X}$ is a semi-distance. Indeed,*

1. *$d_{K,X}(x, y) = d_{K,X}(y, x)$ by symmetry of d_{h^2};*

2. *$d_{K,X}(x, y) \leq d_{K,X}(x, z) + d_{K,X}(z, y)\ \forall\, z \in X$, as if we concatenate Kobayashi paths $\gamma_{x,z}$ and $\gamma_{z,y}$, we obtain a Kobayashi path $\gamma_{x,z,y}$ joining x and y via z. Such paths are only a subset of all possible Kobayashi paths joining x and y, hence $d_{K,X}(x, y)$ is lower than the infimum of the Kobayashy sums taken over the Kobayashi paths $\gamma_{x,z,y}$, which is exactly $d_{K,X}(x, z) + d_{K,X}(z, y)$.*

3. $d_{K,X}(x,y) \geq 0 \ \forall \ x,y \in X$, *as* d_{h^2} *is a distance, but* $d_{K,X}(x,y)$ *may be 0 even for* $x \neq y$.

Remark 3.2. *If X is not connected, then there is no Kobayashi path joining two points* x, y *in two distinct connected components. In such case we put* $d_{K,X}(x,y) = \infty$.

If X is connected, then there exists a chain of discs in X joining x to y, so $d_{K,X}(x,y) < \infty$. *For a proof of this last statement see [12], page 15.*

Example 3.1. *If* $X = \mathbb{D}$, *then* $d_{K,\mathbb{D}}$ *coincides with the hyperbolic distance.*

Proof. Let $z, w \in \mathbb{D}$. Since the geodesic joining z and w is the path of minimal length between them (2.1),

$$d_{h^2}(z,w) \leq d_{K,\mathbb{D}}(z,w).$$

On the other hand, if $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic embedding,

$$d_{K,\mathbb{D}}(z,w) \leq d_{h^2}(f^{-1}(z), f^{-1}(w)) \underbrace{\leq}_{3.1} d_{h^2}(z,w).$$

$$\Longrightarrow d_{K,\mathbb{D}}(z,w) = d_{h^2}(z,w) \quad \forall \ z,w \in \mathbb{D}.$$

□

Example 3.2. *If* $X = \mathbb{C}$ *(with the Euclidean metric), then* $d_{K,\mathbb{C}} \equiv 0$.

Proof. Let $z \neq w \in \mathbb{C}$. Consider a sequence of embeddings $f_k : \mathbb{D} \to \mathbb{C}$ such that $f_k(0) = z$ and $f_k(w_k) = w$, with $w_k \xrightarrow[k\to\infty]{h^2} 0$. One can do so by choosing w_k in the direction of $w - z$, applying a dilation λ_k such that $|\lambda_k(w_k)| = |w - z|$ and then applying the translation $0 \mapsto z$. This yields

$$d_{K,\mathbb{C}}(z,w) \leq \inf_{k\in\mathbb{N}} d_{h^2}(0, w_k) = 0.$$

□

Proposition 3.2. *Let* $f : X \to Y$ *be a holomorphic map of complex manifolds (spaces). Then f is distance decreasing for the Kobayashi semi-distance, i.e.* $\forall \ x, x' \in X$, *we have*

$$d_{K,Y}(f(x), f(x')) \leq d_{K,X}(x,x').$$

Proof. This is a direct consequence of Schwarz-Pick Lemma 3.1. □

Lemma 3.1. *Let* X, Y *be complex manifolds (spaces). Then, for* $x, x' \in X, \ y, y' \in Y$, *we have*

$$d_{K,X\times Y}((x,y),(x',y')) \geq \max[d_{K,X}(x,x'), d_{K,Y}(y,y')]. \tag{3.3}$$

Proof. Let us consider $\pi_X : X \times Y \to X, \ \pi_X(x,y) := x$. The projection π_X is holomorphic, hence by Prop. 3.2 is distance decreasing:

$$d_{K,X\times Y}((x,y),(x',y')) \geq d_{K,X}(\pi_X(x,y), \pi_X(x',y')) = d_{K,X}(x,x').$$

We can do the same with $\pi_Y : X \times Y \to Y, \ \pi_Y(x,y) := y$, and this concludes the proof. □

3.3 Kobayashi hyperbolicity

We are now ready to give the definition of Kobayashi-hyperbolicity.

Def 3.4. ***(Kobayashi hyperbolicity):*** *A complex manifold (space) is said to be* ***Kobayashi-hyperbolic*** *(or* ***K-hyperbolic****) if the Kobayashi semi-distance* $d_{K,X}$ *is distance, i.e.* $x \neq y \Rightarrow d_{K,X}(x,y) \neq 0$.

Remark 3.3. *By Examples 3.1 and 3.2 we can immediately conclude that* $\mathbb{D}$ *is K-hyperbolic, while* $\mathbb{C}$ *is not.*

Example 3.3. *Polydiscs are K-hyperbolic, where a polydisc is a cartesian product of discs,* $\mathbb{D}^N := \underbrace{\mathbb{D} \times \cdots \times \mathbb{D}}_{N \text{ times}}$. *In particular,*

$$d_{K,\mathbb{D}^N}((x_1, \cdots, x_N), (y_1, \cdots, y_N)) = \max_{i=1,\cdots,N} d_{K,\mathbb{D}}(x_i, y_i). \tag{3.4}$$

Proof. We check (3.4) for $N = 2$. Up to isometry we can assume $(x_1, x_2) = (0, 0)$ and $|y_1| \geq |y_2|$. Let us consider

$$f : \mathbb{D} \to \mathbb{D} \times \mathbb{D} \quad , \quad f(z) := \left(z, \frac{y_2}{y_1} z\right).$$

Then $f(0) = (0, 0)$ and $f(y_1) = (y_1, y_2)$. Moreover, f is distance decreasing (Prop. 3.2), therefore

$$d_{K,\mathbb{D}^2}((0, 0), (y_1, y_2)) = d_{K,\mathbb{D}}(f(0), f(y_1)) \leq d_{K,\mathbb{D}}(0, y_1).$$

Lemma 3.1 gives us the opposite inequality, $d_{K,\mathbb{D}^2}((0, 0), (y_1, y_2)) \geq d_{K,\mathbb{D}}(0, y_1)$, and this concludes the proof. □

Important properties of Kobayashi-hyperbolicity are that it is preserved by the product of K-hyperbolic manifolds and by subspaces.

Lemma 3.2. *Product of K-hyperbolic manifolds (spaces) is K-hyperbolic, i.e.* X, Y *are K-hyperbolic* $\Rightarrow$ $X \times Y$ *is K-hyperbolic.*

Proof. If $d_{K,X\times Y}((x, y), (x', y')) = 0$, then Lemma 3.1 yields

$$\max[d_{K,X}(x, x'), d_{K,Y}(y, y')] = 0.$$

The last expression implies that $x = x',\ y = y' \Rightarrow (x, y) = (x', y')$. □

Lemma 3.3. *Let* Y *be a (complex) subspace of* X*, or let* $F : Y \to X$ *be holomorphic and injective. If* X *is hyperbolic, then* Y *is hyperbolic.*

Proof. A holomorphic map $f : \mathbb{D} \to Y$, is also a holomorphic map from $\mathbb{D}$ to X. Hence, a Kobayashi path between two points x, y in Y is also a Kobayashi path in X. If $d_Y(x, y) = 0$, then $d_X(x, y) = 0$ as well. Therefore, if X is Kobayashi-hyperbolic, so is Y. □

Example 3.4. *As a consequence of Lemma 3.3, a bounded domain in* $\mathbb{C}^n$ *is K-hyperbolic, since it is a subset (considered as an injective holomorphic inclusion) into a polydisc (obtained as a product of discs with appropriate radius), which is K-hyperbolic by (3.4).*

Remark 3.4. *Any bounded strictly pseudoconvex domain* $\Omega \subset \mathbb{C}^n$ *is K-hyperbolic, being a bounded domain in* $\mathbb{C}^n$.

3.4 Kobayashi metric as a Finsler metric

In this section we want to give an equivalent definition of the Kobayashi semi-distance as the semi-distance function induced by a semi-Finsler metric (alternatively, a semi-length function). Specifically, the Royden function. For a detailed treatment of the topic see [12], Chapter 4.1.

Def 3.5. ***(Finsler metric):*** *A* ***Finsler metric****, or a* ***length function****, on the tangent bundle of a (real or complex) manifold* Ω *is a real-valued non-negative function* $F : TM \to \mathbb{R}$ *such that*

1. *F is continuous*

   and for each $x \in \Omega,\ Z \in T_x\Omega$,

2. $F(x; Z) = 0 \iff Z = 0$,

3. $F(x; cZ) = |c|F(x; Z)\ \forall\ c \in \mathbb{C}$.

Def 3.6. ***(induced distance function):*** *On any (real or complex) manifold* Ω*, a Finsler metric* F *defines a distance function by*

$$d_F(x, y) := \inf_{\substack{\gamma(0)=x,\\ \gamma(1)=y}} \int_0^1 F(\gamma(t); \dot\gamma(t))dt = \inf_{\substack{\gamma(0)=x,\\ \gamma(1)=y}} l_F(\gamma), \tag{3.5}$$

where $\gamma : [0, 1] \to \Omega$ *is a piecewise* C^1-*smooth path.*

Remark 3.5. *The difference between a Riemannian and a Finsler metric is that the second one defines a norm, for each tangent space* $T_p\Omega$*, that does not need to be induced by a scalar product.*

Def 3.7. ***(semi-Finsler metric):*** *A* ***semi-Finsler metric****, or a* ***semi-length function****, is a function* $F : T\Omega \to \mathbb{R}$ *as in 3.5 where instead of continuity the requirement is that* F *is upper-semicontinuous and point 2. becomes*

$$Z = 0 \Rightarrow F(x; Z) = 0.$$

Remark 3.6. *Notice that if* F *is a Finsler metric, then* d_F *is a distance function, while if* F *is a semi-Finsler metric then* d_F *is a semi-distance function.*

Def 3.8. ***(Royden function):*** *Let* Ω *be a complex manifold. For* $x \in \Omega,\ Z \in T_x\Omega$*. Define the* ***Royden function*** *as*

$$\begin{aligned} K(x; Z) := \inf\{|V|\ ;\ V \in \mathbb{C} \text{ and } \exists\ f : \mathbb{D} \to \Omega \text{ holomorphic s.t.} \\ f(0) = x,\ df(0)V = Z\}. \end{aligned} \tag{3.6}$$

In [12], Chapter 4.1, it is shown that K is a semi-length function, in particular

Proposition 3.3. *The Royden function (3.6) on* Ω *is the largest semi-length function* F *such that every holomorphic map* $f : \mathbb{D} \to \Omega$ *if* F*-decreasing.*

Proof. See [12], page 91. □

Moreover, it holds the following fundamental result.

Theorem 3.1. *The Royden semi-distance induced by the Royden function (3.6) is equal to the Kobayashi semi-distance.*

Proof. See [12], page 92. □

Remark 3.7. *On Kobayashi hyperbolic manifolds, the Kobayashi semi-distance is a distance function and the Royden function is a Finsler metric, which for obvious reason we will henceforth call the Kobayashi metric.*

Chapter 4

Bounded strictly pseudoconvex domains are Gromov-hyperbolic

In this Chapter we aim to investigate the relationship between Kobayashi and Gromov-hyperbolicity. We present a result, proven by Bonk and Balogh in 2000 in [1], that grants that bounded strictly pseudoconvex domains, when equipped with the Kobayashi distance, are Gromov-hyperbolic. Such result consists in showing that the Kobayashi distance on such a domain Ω is roughly-isometric to a distance function g which makes (Ω, g) Gromov-hyperbolic, and this rough-isometry implies Gromov-hyperbolicity with respect to the Kobayashi distance. Notice that, by Remark 3.4, bounded strictly pseudoconvex domains are $K-$*hyperbolic.*

4.1 Balogh-Bonk Theorem

Def 4.1. ***(rough-isometries):*** *Let $f : (X, d_X) \to (Y, d_Y)$ be a map between metric spaces, and let $k > 0$. We say that f is a k-**rough-isometry** if*

$$d_X(x, y) - k \leq d_Y(f(x), f(y)) \leq d_X(x, y) + k \quad \forall\ x, y \in X. \tag{4.1}$$

*If $d_1,\ d_2$ are distinct distance functions on the same set X, they are said to be **roughly-isometric** if there exists a constant $k > 0$ such that*

$$d_1(x, y) - k \leq d_2(x, y) \leq d_1(x, y) + k \quad \forall\ x, y \in X. \tag{4.2}$$

Def 4.2. *Let $\Omega \subset \mathbb{C}^n,\ n \geq 2$, be a bounded strictly pseudoconvex domain. By Lemma 1.4, there exists $N := N_{\varepsilon_0}(\partial\Omega) \cap \Omega$ such that for each $x \in N$ there is a unique projection $\pi(x) \in \partial\Omega$, with $d_E(x, \pi(x)) = \delta(x)$. In this environment, it is possible to choose an extension $\pi : \Omega \to \partial\Omega$ of this projection to the whole Ω. It suffices to choose, for each $x \in \Omega$, a point $\pi(x) \in \partial\Omega$ with $d_E(x, \pi(x)) = \delta(x)$. Denote $h(x) := \sqrt{\delta(x)}$ and $a \vee b := \max\{a, b\}$.*

We define

$$g(x, y) := 2 \ln \left(\frac{d_H(\pi(x), \pi(y)) + h(x) \vee h(y)}{\sqrt{h(x)h(y)}} \right). \tag{4.3}$$

Remark 4.1. *Notice that the extension of π from N to the whole domain Ω is not uniquely determined. Different choices of π lead to expressions of g in (4.3) that* *agree up to a bounded additive term. This does not affect the following results. From now on we fix a choice of such extension.*

Lemma 4.1. (Ω, g) *is Gromov-hyperbolic.*

Proof. Gromov-hyperbolicity of (Ω, g) can be shown following the same argument of Theorem 2.3, where the quantities r_{jk} are defined as

$$r_{jk} := d_H(\pi(x_j), \pi(x_k)) + h(x_j) \vee h(x_k).$$

The required properties of symmetry and triangle inequality continue to hold, as

- $r_{jk} = r_{kj} \ \forall \ k, j = 1, \cdots, 4$ by symmetry of d_H,
- $r_{jk} \le r_{jl} + r_{lk} \ \forall \ l = 1, \cdots, 4$ by triangle inequality for d_H and positiveness of h.

From here, the proof is exactly the same, dividing both sides of

$$r_{12}r_{34} \le 4 \max\{r_{13}r_{24}, r_{14}r_{23}\}$$

by $\sqrt{h(x_1)h(x_2)h(x_3)h(x_4)}$ and using properties of logarithm. □

The central result of the Chapter is the following.

Theorem 4.1. ***(Balogh, Bonk [1]):*** *Let $\Omega \subset \mathbb{C}^n$, $n \ge 2$ be a bounded strictly pseudoconvex domain, then $\exists \ C = C(\Omega) \ge 0$ such that, $\forall \ x, y \in \Omega$,*

$$g(x, y) - C \le d_{K,\Omega}(x, y) \le g(x, y) + C,$$

i.e. g and $d_{K,\Omega}$ are $C-$roughly-isometric.

Corollary 4.1. *Suppose Ω as in Th. 4.1. Then Ω, when equipped with the Kobayashi distance, is Gromov-hyperbolic.*

Proof. Let $x, y, z, w \in \Omega$. By Gromov-hyperbolicity of (Ω, g) (Lemma 4.1), the 4-points condition (2.6) is satisfied, i.e. $\exists \ \delta \ge 0$ s.t.

$$g(x, y) + g(z, w) \le \max\{g(x, z) + g(y, w), g(x, w) + g(y, z)\} + 2\delta.$$

By Th. 4.1, there exists $C = C(\Omega) > 0$ s.t. $\forall \ x, y, z, w \in \Omega$

$$d_{K,\Omega}(x, y) + d_{K,\Omega}(z, w) \le g(x, y) + g(z, w) + 2C \le$$

$$\max\{g(x, z) + g(y, w), g(x, w) + g(y, z)\} + 2\delta + 2C \le$$

$$\max\{d_{K,\Omega}(x, z) + d_{K,\Omega}(y, w) + 2C, d_{K,\Omega}(x, w) + d_{K,\Omega}(y, z) + 2C\} + 2\delta + 2C =$$

$$\max\{d_{K,\Omega}(x, z) + d_{K,\Omega}(y, w), d_{K,\Omega}(x, w) + d_{K,\Omega}(y, z)\} + 2\delta + 4C,$$

which is the condition for Gromov-hyperbolicity of $(\Omega, d_{K,\Omega})$. □

To prove Theorem 4.1 we will use a result showing how a local estimate for a Finsler metric on a bounded strictly pseudoconvex domain implies a global estimate for the induced distance function.

Henceforth, whenever we are dealing with a point $x \in \Omega$ and a vector $Z \in \mathbb{C}^n$, the splitting $Z = Z_H + Z_N$ will always be considered at $p = \pi(x)$ (see Remark 1.4).

Theorem 4.2. *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$ be a bounded strictly pseudoconvex domain. Suppose F is a Finsler metric on Ω with the following property: $\exists\ \varepsilon_0 > 0$, $s > 0$, $C_1 > 0$, $C_2 \geq 1$ such that for all $x \in N_{\varepsilon_0}(\partial\Omega) \cap \Omega$ and all $Z \in T_x\Omega$ we have*

$$\begin{aligned}(1 - C_1\delta(x)^s)\left(\frac{|Z_N|^2}{4\delta(x)^2} + \frac{1}{C_2}\frac{L_\rho(\pi(x); Z_H)}{\delta(x)}\right)^{\frac{1}{2}} &\leq F(x; Z) \\ \leq (1 + C_1\delta(x)^s)\left(\frac{|Z_N|^2}{4\delta(x)^2} + C_2\frac{L_\rho(\pi(x); Z_H)}{\delta(x)}\right)^{\frac{1}{2}}&.\end{aligned} \tag{4.4}$$

Then, if d_F is the distance function induced by F, $\exists\ C > 0$ such that

$$g(x, y) - C \leq d_F(x, y) \leq g(x, y) + C \quad \forall\ x, y \in \Omega.$$

Theorem 4.1 would be a corollary of Theorem 4.2 if we had an estimate as in (4.4) for the Kobayashi metric, as a Finsler metric, on Ω (see Remark 3.7). This is indeed the case, and the estimate is stated in the next result.

Proposition 4.1. *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a bounded strictly pseudoconvex domain. If K is the Kobayashi (Finsler) metric on Ω, then for every $\varepsilon > 0$ there exists $\varepsilon_0 > 0$, $C \geq 0$ such that for all $x \in N_{\varepsilon_0}(\partial\Omega) \cap \Omega$ and all $Z \in T_x\Omega$ we have*

$$\begin{aligned}\left(1 - C\delta(x)^{\frac{1}{2}}\right)\left(\frac{|Z_N|^2}{4\delta(x)^2} + (1-\varepsilon)\frac{L_\rho(\pi(x); Z_H)}{\delta(x)}\right)^{\frac{1}{2}} &\leq K(x; Z) \\ \leq \left(1 + C\delta(x)^{\frac{1}{2}}\right)\left(\frac{|Z_N|^2}{4\delta(x)^2} + (1+\varepsilon)\frac{L_\rho(\pi(x); Z_H)}{\delta(x)}\right)^{\frac{1}{2}}&.\end{aligned} \tag{4.5}$$

Proof. The proof of the estimate (4.5) is very technical and we are not going to include it here. A sketch of the proof is given in [1], the article By Balogh and Bonk where Theorems 4.1 and 4.2 are presented. □

The following sections (4.2, 4.3 and 4.4) are devoted to the proof of Theorem 4.2. In the last section we motivate how (4.5) works exactly as (4.4) even if slightly different. With these, we will be able to conclude that Theorem 4.1 and Corollary 4.1 hold.

- Every time we will need a constant $\varepsilon_0 > 0$, we will "update" it to be small enough to satisfy the hypothesis of all the results that require it (Lemma 1.4, Lemma 4.3, ...). Given such ε_0 we define $K := \{x \in \Omega \quad \delta(x) \geq \varepsilon_0\}$ and $N := \{x \in \Omega\ ;\ \delta(x) < \varepsilon_0\} = N_{\varepsilon_0}(\partial\Omega) \cap \Omega$, where we recall that $\delta(x)$ indicates the Euclidean distance of x to the boundary $\partial\Omega$. Then K is compact, N is open, $N \cap K = \emptyset$ and $N \cup\ K = \Omega$.

- Throughout the Chapter we will denote by C positive constants only depending on ε_0 and the other constants associated to Ω and F. The actual value of C may change even within the same line.

Structure of the proof of Theorem 4.2: The goal is to prove that there exists a constant $C > 0$ such that for all $x, y \in \Omega$

$$g(x, y) - C \leq d_F(x, y) \leq g(x, y) + C, \tag{4.6}$$

provided that (4.4) is true. Given $x, y \in \Omega$, denote by $p := \pi(x)$ and $q := \pi(y)$ the projections to the boundary. In order to prove (4.6) we will distinguish various cases depending on the relative positions of $x, y \in \Omega$:

Case 1: $x, y \in K$.

Case 2: $x \in N,\ y \in K$ (or viceversa).

Case 3: $x, y \in N,\ d_H(p,q) \le h(x) \vee h(y)$.

Case 4: $x, y \in N\ d_H(p,q) > h(x) \vee h(y)$.

Section 4.2 will deal with the proof of the cases 1 and 2. Cases 3 and 4 will be proven respectively in section 4.3 and 4.4. In each section, before the proof, all the needed results will be presented.

4.2 Proof of cases 1 and 2

Proof. **(Proof of case 1).** In this case

$$d_F(x,y) \le C \quad \forall\ x, y \in K$$

for some fixed $C > 0$, as d_F is continuous and K is compact. Moreover, g is bounded on K, as

$$g(x,y) = 2\ln\left(\frac{d_H(\pi(x),\pi(y)) + h(x) \vee h(y)}{\sqrt{h(x)h(y)}}\right) \le$$

$$2\ln\left(\frac{diam_H(\partial\Omega) + diam(\Omega)}{\sqrt{\varepsilon_0}}\right),$$

where

$$diam_H(\partial\Omega) := \max_{p,q\in\partial\Omega} d_H(p,q) < \infty,$$

$$diam(\Omega) := \max_{\xi,\eta\in\Omega} |\xi - \eta| < \infty$$

as $\partial\Omega$ and Ω are compact. Therefore

$$g(x,y) - C \le 0 \le d_F(x,y) \le g(x,y) + C.$$

□

To deal with the proof of case 2 we need the following result, that will be used repeatedly even for cases 3 and 4.

Lemma 4.2. *Let $\gamma : [0,1] \to \overline{N} \cap \Omega$ be a C^1-smooth curve joining two points $u, v \in \overline{N} \cap \Omega$. Then $\exists\ C > 0$ s.t.*

$$l_F(\gamma) \ge \left|\ln\frac{h(v)}{h(u)}\right| - C. \tag{4.7}$$

If $\gamma(t) = u + t(v-u)$ is a Euclidean segment contained in some fiber $\pi^{-1}(p)$, for $p \in \partial\Omega$, then

$$l_F(\gamma) \in \left[\left|\ln\frac{h(v)}{h(u)}\right| \pm C\right], \tag{4.8}$$

where the notation $A \in [B \pm C]$ means $A \in [B - C, B + C]$.

Proof. Notice that from the computations carried out in the proof of Lemma 1.1 and from Lemma 1.4, we can obtain $2\bar{\partial}\rho(p) = n(p)$, for $p \in \partial\Omega$. Remember that, for $x \in \Omega$ sufficiently close to the boundary, $\rho(x) = -\delta(x)$, hence $\bar{\partial}\delta(x) = -\bar{\partial}\rho(x) = -n(\pi(x))$. Then,

$$\left|\frac{d}{dt}h(\gamma(t))\right|_{t=t_0} = \left|\frac{d}{dt}\left[\sqrt{\delta(\gamma(t))}\right]\right|_{t=t_0} = \frac{1}{2\sqrt{\delta(\gamma(t_0))}}\left|\frac{d}{dt}\delta(\gamma(t))\right|_{t=t_0} =$$

$$\frac{|\nabla\delta(\gamma(t_0))\cdot\dot{\gamma}(t_0)|}{2h(\gamma(t_0))} = \frac{|Re\langle\bar{\partial}\delta(\gamma(t_0)),\dot{\gamma}(t_0)\rangle|}{h(\gamma(t_0))} = \frac{|Re\langle n(\pi(\gamma(t_0))),\dot{\gamma}(t_0)\rangle|}{2h(\gamma(t_0))} \leq$$

$$\frac{|\langle n(\pi(\gamma(t_0))),\dot{\gamma}(t_0)\rangle|}{2h(\gamma(t_0))} = \frac{|\dot{\gamma}_N(t_0)|}{2h(\gamma(t_0))}.$$

Now we use the LHS of the hypothesis (4.4) to get the first bound:

$$l_F(\gamma) = \int_0^1 F(\gamma(t);\dot{\gamma}(t))dt \underbrace{\geq}_{(4.4)}$$

$$\int_0^1 (1 - C_1\delta(\gamma(t))^s)\left(\frac{|\dot{\gamma}_N(t)|^2}{4\delta(\gamma(t))^2} + \frac{1}{C_2}\frac{L_\rho(\pi(\gamma(t));\dot{\gamma}_H(t))}{\delta(\gamma(t))}\right)^{\frac{1}{2}} dt \geq$$

$$\int_0^1 (1 - C_1\delta(\gamma(t))^s)\left(\frac{|\dot{\gamma}_N(t)|^2}{4\delta(\gamma(t))^2}\right)^{\frac{1}{2}} dt = \int_0^1 (1 - C_1\delta(\gamma(t))^s)\frac{|\dot{\gamma}_N(t)|}{2h(\gamma(t))^2}dt \geq$$

$$\int_0^1 \frac{1 - C_1 h(\gamma(t))^{2s}}{h(\gamma(t))}\left|\frac{d}{dt}h(\gamma(t))\right| dt = \int_0^1 \left|\frac{d}{dt}\left[\ln(h(\gamma(t)) - \frac{C_1}{2s}h(\gamma(t))^{2s}\right]\right| dt \geq$$

$$\left|\left[\ln(h(\gamma(t)) - \frac{C_1}{2s}h(\gamma(t))^{2s}\right]_0^1\right| = \left|\ln\frac{h(v)}{h(u)} - \frac{C_1}{2s}\left(h(v)^{2s} - h(u)^{2s}\right)\right| \geq$$

$$\left|\ln\frac{h(v)}{h(u)}\right| - \frac{C_1}{2s}\varepsilon_0^s.$$

Now, if $\gamma([0,1]) \subseteq \overline{N} \cap \pi^{-1}(p)$ is a straight line, by Lemma 1.4,

$$n(\pi(\gamma(t))) = n(p), \quad \dot{\gamma}(t) = \pm|u-v|n(p),$$

$$|\dot{\gamma}_N(t))| \equiv |u-v|, \quad |\dot{\gamma}_H(t)| \equiv 0.$$

In particular, for all $t \in [0,1]$, $\dot{\gamma}(t)$ is parallel to $n(p)$, hence

$$\left|\frac{d}{dt}h(\gamma(t))\right|_{t=t_0} = \frac{|\nabla\delta(\gamma(t_0))\cdot\dot{\gamma}(t_0)|}{2h(\gamma(t_0))} = \frac{|n(\pi(\gamma(t_0)))\cdot\dot{\gamma}(t_0)|}{2h(\gamma(t_0))} =$$

$$\frac{|n(p)\cdot\dot{\gamma}(t_0)|}{2h(\gamma(t_0))} = \frac{|\dot{\gamma}(t_0)|}{2h(\gamma(t_0))}.$$

Using this and the RHS of the hypothesis (4.4), with a computation similar to above one obtains that

$$l_F(\gamma) \leq \left|\ln\frac{h(v)}{h(u)}\right| + C,$$

and this concludes the proof. □

Proof. **(Proof of case 2).** Here, by assumption,

$$h(x) \vee h(y) = h(y) \geq \sqrt{\varepsilon_0},$$

hence

$$g(x,y) = 2\ln\left(\frac{d_H(\pi(x),\pi(y)) + h(y)}{\sqrt{h(x)h(y)}}\right) =$$

$$\ln\frac{1}{h(x)} + \ln\left(\frac{d_H(\pi(x),\pi(y)) + h(y)}{\sqrt{h(y)}}\right)^2.$$

Claim 4.2.1. $\exists\, C > 0$ *s.t.*

$$-C \leq \ln\left(\frac{d_H(\pi(x),\pi(y)) + h(y)}{\sqrt{h(y)}}\right)^2 \leq C. \tag{4.9}$$

Proof of Claim 4.2.1. *Lower bound:*

$$\ln\left(\frac{d_H(\pi(x),\pi(y)) + h(y)}{\sqrt{h(y)}}\right)^2 \geq \ln\left(\frac{h(y)}{\sqrt{h(y)}}\right)^2 = \ln(h(y)) \geq \ln\sqrt{\varepsilon_0}.$$

Upper bound:

$$\ln\left(\frac{d_H\pi(x),\pi(y)) + h(y)}{\sqrt{h(y)}}\right)^2 \leq \ln\frac{(d_H(\pi(x),\pi(y)) + diam(\Omega))^2}{\sqrt{\varepsilon_0}} \leq$$

$$\ln\frac{(diam_H(\partial\Omega) + diam(\Omega))^2}{\sqrt{\varepsilon_0}}.$$

If we pick $C \in \mathbb{R}^+$ *big enough such that*

$$-C < \ln\sqrt{\varepsilon_0} \qquad \text{and} \qquad \ln\frac{(diam_H(\partial\Omega) + diam(\Omega))^2}{\sqrt{\varepsilon_0}} < C$$

we get the result, i.e.

$$g(x,y) \in \left[\ln\frac{1}{h(x)} \pm C\right] \qquad \checkmark. \tag{4.10}$$

• Now we look for a <u>upper bound</u> for $d_F(x,y)$ in terms of $\ln\frac{1}{h(x)}$. Let us denote $x' := p - \varepsilon_0 n(p)$, where we recall that $n(p)$ is the outer unit normal vector of $\partial\Omega$ at $p \in \partial\Omega$. By Lemma 1.4, $x' \in \pi^{-1}(p)$, $\pi(x') = p$ and $x \in (x', p)$.

Consider the path $\gamma(t) := x + t(x' - x)$ for $t \in [0,1]$.

Claim 4.2.2. $d_F(x,x') \leq \ln\frac{1}{h(x)} + C$ *for some* $C > 0$.

Proof of Claim 4.2.2. *By (4.8) in Lemma 4.2 we have* $l_F(\gamma) \in \left[\ln\frac{h(x')}{h(x)} \pm C\right]$ *for some* $C > 0$. *Moreover,* $h(x') = \sqrt{\varepsilon_0}$, *hence (up to renaming* C*)*

$$l_F(\gamma) \in \left[\ln\frac{1}{h(x)} \pm C\right].$$

Then,

$$d_F(x,x') \leq l_F(\gamma) \leq \ln\frac{1}{h(x)} + C \qquad \checkmark.$$

The upper bound for $d_F(x,y)$ is obtained as follows:

$$d_F(x,y) \leq d_F(x,x') + d_F(x',y),$$

and by Case 1 $d_F(x',y) < C$ as $x', y \in K$. This remark and Claim 4.2.2 lead to

$$d_F(x,y) \leq \ln \frac{1}{h(x)} + C. \tag{4.11}$$

(4.11) together with (4.10) yields

$$g(x,y) + 2C \underbrace{\geq}_{(4.10)} \ln \frac{1}{h(x)} - C + 2C = \ln \frac{1}{h(x)} + C \underbrace{\geq}_{4.11} d_F(x,y). \tag{4.12}$$

• Now we look for a <u>lower bound</u> for $d_F(x,y)$ in terms of $\ln \frac{1}{h(x)}$. Let $\gamma : [0,1] \to \Omega$ be a (piecewise) C^1-smooth curve joining x and y. Consider y' to be the "first" point on the curve such that $y' \in K$, i.e. $h(y') = \sqrt{\varepsilon_0}$. Denote with $\tilde{\gamma}$ the subcurve of γ joining x and y'. By (4.7) in Lemma 4.2

$$l_F(\gamma) \geq l_F(\tilde{\gamma}) \underbrace{\geq}_{(4.7)} \ln \frac{h(y')}{h(x)} - C = \ln \frac{1}{h(x)} - C.$$

Therefore,

$$d_F(x,y) = \inf_{\substack{\gamma(0)=x \\ \gamma(1)=y}} l_F(\gamma) \geq \ln \frac{1}{h(x)} - C. \tag{4.13}$$

Similarly to (4.12), we have that (4.13) together with (4.10) yields

$$g(x,y) - 2C \underbrace{\leq}_{(4.10)} \ln \frac{1}{h(x)} + C - 2C = \ln \frac{1}{h(x)} - C \underbrace{\leq}_{(4.13)} d_F(x,y). \tag{4.14}$$

Up to renaming $2C$ with C in (4.12) and (4.14) we obtain the result. □

4.3 Proof of case 3

We recall that, for any $p \in \partial\Omega$, each vector $Z \in \mathbb{C}^n$ can be uniquely decomposed as $Z = Z_H + Z_N$ with $Z_H \in H_p\partial\Omega$ and $Z_N \in N_p\partial\Omega$. The following result gives a relation between a curve $\gamma : [0,1] \to N_{\varepsilon_0}(\partial\Omega) \cap \Omega$ and its projection on the boundary $\alpha := \pi \circ \gamma : [0,1] \to \partial\Omega$, for ε_0 sufficiently small. For the tangent vectors $\dot{\gamma}(t)$ and $\dot{\alpha}(t)$ we will consider the splitting into horizontal and normal components at $\alpha(t)$ and write

$$\dot{\gamma}(t) = \dot{\gamma}_H(t) + \dot{\gamma}_N(t),$$

$$\dot{\alpha}(t) = \dot{\alpha}_H(t) + \dot{\alpha}_N(t).$$

Lemma 4.3. *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$ be a bounded domain. Then $\exists\ \varepsilon_0$ and $C > 0$ such that if $\gamma : [0,1] \to N_{\varepsilon_0}(\partial\Omega) \cap \Omega$ is a C^1-smooth curve and $\alpha := \pi \circ \gamma$ its projection to the boundary, then the following estimates hold for all $t \in [0,1]$:*

1. $|\dot\gamma_H(t) - \dot\alpha_H(t)| \le C\delta(\gamma(t))|\dot\gamma(t)|$*;*

2. $|\dot\gamma_H(t) - \dot\alpha_H(t)| \le C\delta(\gamma(t))|\dot\alpha(t)|$*;*

3. $|\dot\alpha_N(t)| \le |\dot\gamma_N(t)| + C\delta(\gamma(t))|\dot\gamma(t)|$*;*

4. $|\dot\gamma_N(t)| \le |\dot\alpha_N(t)| + C\delta_0|\dot\alpha(t)|$*, if* $\delta(\gamma(t)) \equiv \delta_0$.

Proof. See [1] page 511. □

We are now ready to deal with the proof of case 3.

Proof. **(Proof of case 3).** Recall that now $x, y \in N$. We can assume, without loss of generality, $h(y) \ge h(x)$, so that the situation to deal with is the following:

$$d_H(p,q) \le h(y) \le \sqrt{\varepsilon_0}.$$

Claim 4.3.1. $\exists\, C > 0$*, indipendent on* x, y*, such that*

$$g(x,y) \in \left[\ln\frac{h(y)}{h(x)} \pm C\right]. \tag{4.15}$$

Proof of Claim 4.3.1. • *Lower bound:*

$$g(x,y) = 2\ln\left(\frac{d_H(p,q)+h(y)}{\sqrt{h(x)h(y)}}\right) \ge 2\ln\frac{h(y)}{\sqrt{h(x)h(y)}} = \ln\frac{h(y)}{h(x)}.$$

• *Upper bound:*

$$g(x,y) \le 2\ln\left(\frac{h(y)+h(y)}{\sqrt{h(x)h(y)}}\right) = 2\ln\frac{2\sqrt{h(y)}}{\sqrt{h(x)}} = \ln 4\frac{h(y)}{h(x)} = \ln\frac{h(y)}{h(x)} + \ln 4 \qquad \checkmark.$$

• Now we look for a <u>lower bound</u> for $d_F(x,y)$ in terms of $\ln\frac{h(y)}{h(x)}$. We notice that as done in case 2 we can obtain an estimate similar to (4.13) for $d_F(x,y)$.

Indeed, by (4.7) in Lemma 4.2, if γ is a C^1-smooth curve joining x and y, one gets

$$l_F(\gamma) \ge \ln\frac{h(y)}{h(x)} - C.$$

Hence,

$$d_F(x,y) = \inf_{\substack{\gamma(0)=x\\ \gamma(1)=y}} l_F(\gamma) \ge \ln\frac{h(y)}{h(x)} - C. \tag{4.16}$$

Again as in case 2, (4.16) together with (4.15) yields

$$g(x,y) - C \le d_F(x,y). \tag{4.17}$$

• Now we look for a <u>upper bound</u> for $d_F(x,y)$ in terms of $\ln\frac{h(y)}{h(x)}$.

Let $x' \in \pi^{-1}(p) \cap N$ be the <u>unique</u> point such that $\delta(x') = \delta(y)$, i.e. $x' := p - \delta(y)n(p)$. The curve $[x, x']$ is a straight line segment, then by (4.8) in Lemma 4.2 we get

$$d_F(x,x') \le \ln\frac{h(x')}{h(x)} + C = \ln\frac{h(y)}{h(x)} + C. \tag{4.18}$$

Now it only remains to find an upper bound for $d_F(x', y)$. By definition of the Carnot-Carathéodory metric (1.14), there exists a C^1-smooth horizontal path $\alpha : [0,1] \to \partial\Omega$ such that $\alpha(0) = p,\ \alpha(1) = q$ and

$$l_{L_\rho}(\alpha) \leq 2d_H(p,q). \tag{4.19}$$

We can lift this curve at height $\delta(y)$ with

$$\tilde{\gamma} : [0,1] \to \Omega \quad ; \quad \tilde{\gamma}(t) := \alpha(t) - \delta(y)n(\alpha(t)).$$

$\tilde{\gamma}$ is still C^1-smooth and $\tilde{\gamma}(0) = x',\ \tilde{\gamma}(1) = y,\ \delta(\tilde{\gamma}(t)) = \delta(y)\ \forall\ t \in [0,1]$.

Claim 4.3.2. $F(\tilde{\gamma}(t); \dot{\tilde{\gamma}}(t))^2 \leq C\left(\frac{|\dot{\tilde{\gamma}}_N(t)|^2}{4\delta(y)^2} + \frac{|\dot{\tilde{\gamma}}_H(t)|^2}{\delta(y)}\right) \quad \forall\ t \in [0,1]$.

Proof of Claim 4.3.2. *By the RHS of the hypothesis (4.4),*

$$F(\tilde{\gamma}(t); \dot{\tilde{\gamma}}(t))^2 \leq (1 + C_1\delta(\tilde{\gamma}(t))^s)^2 \left(\frac{|\dot{\tilde{\gamma}}_N(t)|^2}{4\delta(\tilde{\gamma}(t))^2} + C_2\frac{L_\rho(\pi(\tilde{\gamma}(t)); \dot{\tilde{\gamma}}_H(t))}{\delta(\tilde{\gamma}(t))}\right) \leq$$

$$\max_{x\in\overline{N}} (1 + C_1\delta(x)^s)^2 \left(\frac{|\dot{\tilde{\gamma}}_N(t)|^2}{4\delta(y)^2} + C_2\frac{L_\rho(\pi(\tilde{\gamma}(t)), \dot{\tilde{\gamma}}_H(t))}{\delta(y)}\right).$$

By the RHS of (1.20) in Lemma 1.5 and by renaming the constants,

$$F(\tilde{\gamma}(t); \dot{\tilde{\gamma}}(t))^2 \leq C\left(\frac{|\dot{\tilde{\gamma}}_N(t)|^2}{4\delta(y)^2} + \frac{|\dot{\tilde{\gamma}}_H(t)|^2}{\delta(y)}\right) \qquad \checkmark.$$

Now we recall that α is horizontal, i.e. $|\dot{\alpha}_N(t)| = 0$, and apply Lemma 4.3, points *2.* and *4.*, to get

$$F(\tilde{\gamma}(t); \dot{\tilde{\gamma}}(t))^2 \underbrace{\leq}_{4.} C\left(\frac{C^2\delta(y)^2|\dot{\alpha}(t)|^2}{4\delta(y)^2} + \frac{|\dot{\tilde{\gamma}}_H(t)|^2}{\delta(y)}\right) \leq$$

$$C\left(\frac{C^2|\dot{\alpha}(t)|^2}{4} + \frac{(|\dot{\tilde{\gamma}}_H(t) - \dot{\alpha}_H(t)| + |\dot{\alpha}_H(t)|)^2}{\delta(y)}\right) \underbrace{\leq}_{2.}$$

$$C\left(\frac{C^2|\dot{\alpha}(t)|^2}{4} + \frac{(C\delta(y)|\dot{\alpha}_H(t)| + |\dot{\alpha}_H(t)|)^2}{\delta(y)}\right) \leq$$

$$C\left(\frac{C^2}{4}|\dot{\alpha}_H(t)|^2 + \frac{(C\delta(y) + 1)^2}{\delta(y)}|\dot{\alpha}_H(t)|^2\right) =$$

$$\frac{|\dot{\alpha}_H(t)|^2}{\delta(y)} C\left(\frac{C^2\delta(y)}{4} + (C\delta(y) + 1)^2\right).$$

Since $\delta(y) \leq \varepsilon_0$, up to renaming the constant $C > 0$ we get

$$F(\gamma(t); \dot{\gamma}(t))^2 \leq C\frac{|\dot{\alpha}_H(t)|^2}{\delta(y)}.$$

Applying again Lemma 1.5, but this time focusing on the LHS of (1.20), we obtain

$$F(\tilde{\gamma}(t); \dot{\tilde{\gamma}}(t))^2 \leq C\frac{L_\rho(\alpha(t); \dot{\alpha}_H(t))}{\delta(y)}. \tag{4.20}$$

(4.20) leads to the following inequality for the integrated forms:

$$d_F(x', y) \leq l_F(\tilde{\gamma}) \leq \frac{C}{h(y)} l_{L_\rho}(\alpha) \underbrace{\leq}_{4.19} \frac{C}{h(y)} 2 d_H(p, q).$$

By hypothesis $d_H(p, q) \leq h(y)$, hence

$$d_F(x', y) \leq C. \tag{4.21}$$

The upper bound for $d_F(x, y)$ is now given by (4.18) and (4.21):

$$d_F(x, y) \leq \ln \frac{h(y)}{h(x)} + C. \tag{4.22}$$

Again as in case 2, (4.22) together with (4.15) yields

$$d_F(x, y) \leq g(x, y) + C. \tag{4.23}$$

□

4.4 Proof of case 4

The following result relates a curve $\gamma : [0, 1] \to N_{\varepsilon_0}(\partial\Omega) \cap \Omega$ to its projection to the boundary in a way that will facilitate the use of the Approximation Lemma 1.6 in the proof case 4.

Lemma 4.4. *Let $\Omega \in \mathbb{C}^n$, $n \geq 2$, be a bounded strictly pseudoconvex domain. Then $\exists\ \varepsilon_0 > 0$ and $C > 0$ such that if $\gamma : [0, 1] \to N_{\varepsilon_0}(\partial\Omega) \cap \Omega$ is C^1-smooth and $\alpha := \pi \circ \gamma$, then for $t \in [0, 1]$ it holds*

$$A_\gamma := \frac{|\dot{\gamma}_N(t)|^2}{4\delta(\gamma(t))^2} + \frac{L_\rho(\pi(\gamma(t)); \dot{\gamma}_H(t))}{\delta(\gamma(t))} \geq C \left[\frac{|\dot{\alpha}_N(t)|^2}{\delta(\gamma(t))^2} + \frac{L_\rho(\alpha(t); \dot{\alpha}_H(t))}{\delta(\gamma(t))} \right].$$

Proof. Let ε_0 be small enough to apply Lemmas 1.4 and 4.3. In this proof C_1, C_2, C_3, C_4 and C_5 will indicate positive constants independent of γ and t.

Remember (△) in the proof of the Approximation Lemma 1.6, i.e. for $Z, W \in \mathbb{C}^n$,

$$|Z - W|^2 \geq \frac{|W|^2}{2} - |Z|^2.$$

(△), together with *1.* from Lemma 4.3, yields

$$C_1 \delta(\gamma(t))^2 |\dot{\gamma}(t)|^2 \underbrace{\geq}_{1.} |\dot{\gamma}_H(t) - \dot{\alpha}_H(t)|^2 \underbrace{\geq}_{(\triangle)} \frac{1}{2} |\dot{\alpha}_H(t)|^2 - |\dot{\gamma}_H(t)|^2 \iff$$

$$|\dot{\gamma}_H(t)|^2 \geq \frac{1}{2} |\dot{\alpha}_H(t)|^2 - C_1 \delta(\gamma(t))^2 |\dot{\gamma}(t)|^2. \tag{$\flat$}$$

Similarly, (△) and *3.* from Lemma 4.3 yield

$$|\dot{\gamma}_N(t)|^2 \geq \frac{1}{2} |\dot{\alpha}_N(t)|^2 - C_2 \delta(\gamma(t))^2 |\dot{\gamma}(t)|^2. \tag{$\sharp$}$$

Now we apply (♭) and (♯), together with (1.20), and obtain

$$\frac{A_\gamma}{2} \geq \frac{|\dot{\gamma}_N(t)|^2}{8\delta(\gamma(t))^2} + \frac{|\dot{\gamma}_H(t)|^2}{2C_3\delta(\gamma(t))} \geq$$

$$\frac{\left(\frac{|\dot{\alpha}_N(t)|^2}{2} - C_2\delta(\gamma(t))^2|\dot{\gamma}(t)|^2\right)}{8\delta(\gamma(t))^2} + \frac{\left(\frac{|\dot{\alpha}_H(t)|^2}{2} - C_1\delta(\gamma(t))^2|\dot{\gamma}(t)|^2\right)}{2C_3\delta(\gamma(t))} \geq$$

$$C_4\left(\frac{|\dot{\alpha}_N(t)|^2}{\delta(\gamma(t))^2} + \frac{|\dot{\alpha}_H(t)|^2}{\delta(\gamma(t))}\right) - C_5|\dot{\gamma}(t)|^2.$$

Notice that, for ε_0 small enough, $\frac{A_\gamma}{2} \geq C_5|\dot{\gamma}(t)|^2$. Therefore,

$$A_\gamma \geq C_4\left(\frac{|\dot{\alpha}_N(t)|^2}{\delta(\gamma(t))^2} + \frac{|\dot{\alpha}_H(t)|^2}{\delta(\gamma(t))}\right) \underbrace{\geq}_{(1.20)} C\left(\frac{|\dot{\alpha}_N(t)|^2}{\delta(\gamma(t))^2} + \frac{L_\rho(\alpha(t);\dot{\alpha}_H(t))}{\delta(\gamma(t))}\right).$$

□

We now have everything we need to deal with the last part of the proof of Theorem 4.1.

Proof. **(Proof of case 4).** Recall that now

$$\begin{cases} x, y \in N, \\ d_H(p,q) > h(x) \vee h(y). \end{cases} \tag{4.24}$$

Claim 4.4.1. $g(x,y) \in \left[2\ln \frac{d_H(p,q)}{\sqrt{h(x)h(y)}} \pm C\right]$.

Proof of Claim 4.4.1. • *Lower bound:*

$$g(x,y) = 2\ln\left(\frac{d_H(p,q) + h(x) \vee h(y)}{\sqrt{h(x)h(y)}}\right) > 2\ln\frac{d_H(p,q)}{\sqrt{h(x)h(y)}},$$

as $h(x) \vee h(y) > 0$.
• *Upper bound:*

$$g(x,y) = 2\ln\left(\frac{d_H(p,q) + h(x) \vee h(y)}{\sqrt{h(x)h(y)}}\right) \underbrace{<}_{(4.24)} 2\ln\frac{2d_H(p,q)}{\sqrt{h(x)h(y)}} =$$

$$2\ln\frac{d_H(p,q)}{\sqrt{h(x)h(y)}} + 2\ln 2.$$

As in cases 2 and 3 we want to show that the same estimate holds for d_F, i.e.

$$d_F(x,y) \in \left[2\ln\frac{d_H(p,q)}{\sqrt{h(x)h(y)}} \pm C\right]. \tag{4.25}$$

If we manage to do so we have the result, following the same steps of (4.12) and (4.14).

•**Upper bound:** Introduce $h_0 := \min\{\sqrt{\varepsilon_0}, d_H(p,q)\}$ and denote

$$\begin{cases} x' := p - h_0^2 n(p) \\ y' := q - h_0^2 n(p). \end{cases}$$

Notice $h(x') = h(y') = h_0$. By (4.8) in Lemma 4.2, as in (4.18),

$$\begin{cases} d_F(x,x') \le \ln \frac{h(x')}{h(x)} + C = \ln \frac{h_0}{h(x)} + C \\ d_F(y,y') \le \ln \frac{h(y')}{h(y)} + C = \ln \frac{h_0}{h(y)} + C. \end{cases}$$

As in case 3 it can be seen that, being x' and y' at the same height,

$$d_F(x',y') \le C.$$

Indeed, following the same argument of case 3, one gets to

$$d_F(x',y') \le \frac{C}{h(y')} 2 d_H(p,q) = \frac{C}{h_0} d_H(p,q),$$

but here we are <u>not</u> in the situation $d_H(p,q) \le h_0$, so we have to distinguish two scenarios:

- $h_0 = d_H(p,q) \Rightarrow d_F(x',y') \le C.$
- $h_0 = \sqrt{\varepsilon_0} \Rightarrow d_F(x',y') \le \frac{C}{\sqrt{\varepsilon_0}} diam_H(\partial\Omega).$

We can then obtain the upper bound thanks to the triangle inequality for d_F :

$$d_F(x,y) \le d_F(x,x') + d_F(x',y') + d_F(y',y) \le$$

$$2 \ln \frac{h_0}{\sqrt{h(x)h(y)}} + C \le 2 \ln \frac{d_H(p,q)}{\sqrt{h(x)h(y)}}.$$

• **(Lower bound):** Let us now consider an arbitrary (piecewise) C^1-smooth curve $\gamma : [0,1] \to \Omega$ joining x and y. We want to find a proper lower bound for its $F-$length, as it was done in cases 2 and 3. To do so, let us define

$$H_\gamma := \max_{t\in[0,1]} h(\gamma(t)).$$

Notice that there exists $t_\gamma \in [0,1]$ such that $H_\gamma = h(\gamma(t_\gamma))$, and consider the two subcurves

$$\gamma_1 := \gamma_{\restriction_{[0,t_\gamma]}}, \qquad \gamma_2 := \gamma_{\restriction_{[t_\gamma,1]}}.$$

Now we distinguish two cases:

A) $H_\gamma \ge h_0$;

B) $H_\gamma < h_0$.

A) If $H_\gamma \leq \sqrt{\varepsilon_0}$, then $\gamma_1([0,t_\gamma]) \subseteq \overline{N} \cap \Omega$ and we can apply (4.7) in Lemma 4.2 to get

$$l_F(\gamma_1) \geq \ln \frac{h(\gamma(t_\gamma))}{h(x)} - C = \ln \frac{H_\gamma}{h(x)} - C \geq \ln \frac{h_0}{h(x)} - C.$$

If instead $H_\gamma > \sqrt{\varepsilon_0}$, $\gamma_1(t_\gamma) \in K$ and we can proceed as in case 2 to obtain

$$l_F(\gamma_1) \geq \ln \frac{\sqrt{\varepsilon_0}}{h(x)} - C \geq \ln \frac{h_0}{h(x)} - C.$$

For the same reason

$$l_F(\gamma_2) \geq \ln \frac{h_0}{h(y)} - C.$$

Now notice that, due to the boundedness of d_H on $\partial\Omega$, there is a constant $0 < C \leq 1$ such that

$$C d_H(p,q) \leq h_0 \leq d_H(p,q).$$

It suffices to pick $C = \frac{\sqrt{\varepsilon_0}}{diam_H(\partial\Omega)}$, where ε_0 can be adapted to obtain $C \leq 1$. Consequently,

$$\begin{aligned} l_F(\gamma) = l_F(\gamma_1) + l_F(\gamma_2) \geq \ln \frac{h_0^2}{h(x)h(y)} - C \geq \\ \ln \left(C^2 \frac{d_H(p,q)^2}{h(x)h(y)} \right) - C = 2 \ln \frac{d_H(p,q)}{\sqrt{h(x)h(y)}} - C. \end{aligned} \tag{4.26}$$

B) In this case we have $\gamma([0,1]) \subseteq N$. Let $\alpha := \pi \circ \gamma$ be the projection of γ to the boundary. Since $h(x) \leq H_\gamma$, there exists $m_\gamma \in \mathbb{N}$ such that

$$\frac{H_\gamma}{2^{m_\gamma}} \leq h(x) \leq \frac{H_\gamma}{2^{m_\gamma - 1}}.$$

Let us now define $s_0 := 0$,

$$s_i := \min \left\{ s \in [0,t_\gamma] \ ; \ h(\gamma(s)) = \frac{H_\gamma}{2^{m_\gamma - i}} \right\} \quad for \quad i = 1, \cdots m_\gamma$$

and put

$$x_i := \gamma(s_i), \qquad p_i := \pi(x_i) \qquad for \quad i = 0, \cdots m_\gamma.$$

Notice that $\frac{h(x_i)}{h(x_{i-1})} = 2$ for $i = 1, \cdots m_\gamma$. There are now two scenarios:

$$B_1) \ \exists \ l \in \{1, \cdots m_\gamma\} \quad s.t. \quad d_H(p_{l-1}, p_l) > \frac{d_H(p,q)}{8 \cdot 2^{m_\gamma - l}}; \tag{4.27}$$

$$B_2) \ d_H(p_{i-1}, p_i) \leq \frac{d_H(p,q)}{8 \cdot 2^{m_\gamma - i}} \qquad \forall \ i = 1, \cdots m_\gamma. \tag{4.28}$$

B_1) In this scenario, we define the constant $\kappa_\gamma > 0$ s.t.

$$\frac{1}{\kappa_\gamma} := \frac{d_H(p,q)}{8 \cdot 2^{m_\gamma - l}} < d_H(p_{l-1}, p_l),$$

i.e. the hypothesis of the Approximation Lemma 1.6 are satisfied by p_{l-1}, p_l and κ_γ. For $t \in [s_{l-1}, s_l]$ we have

$$h(\gamma(t)) \leq \frac{H_\gamma}{2^{m_\gamma - l}} < \frac{h_0}{2^{m_\gamma - l}} \leq \frac{d_H(p,q)}{2^{m_\gamma - l}} = \frac{8}{k_\gamma}.$$

Now, by the LHS of the hypothesis (4.4) and Lemma 4.4,

$$l_F\left(\gamma_{\restriction_{[s_{l-1},s_l]}}\right)=\int_{s_{l-1}}^{s_l}F(\gamma(t);\dot\gamma(t))dt\underbrace{\geq}_{(4.4)}$$

$$\int_{s_{l-1}}^{s_l}(1-C_1\delta(\gamma(t))^s)\left(\frac{|\dot\gamma_N(t)|^2}{4\delta(\gamma(t))^2}+\frac{1}{C_2}\frac{L_\rho(\pi(\gamma(t));\dot\gamma_H(t))}{\delta(\gamma(t))}\right)^{\frac12}dt\geq$$

$$\min_{x\in\overline{N}}\frac{1-C_1\delta(x)^s}{C_2}\int_{s_{l-1}}^{s_l}\left(\frac{|\dot\gamma_N(t)|^2}{4\delta(\gamma(t))^2}+\frac{L_\rho(\pi(\gamma(t));\dot\gamma_H(t))}{\delta(\gamma(t))}\right)^{\frac12}dt\underbrace{\geq}_{Lemma\ 4.4}$$

$$C\int_{s_{l-1}}^{s_l}\left(\frac{|\dot\alpha_N(t)|^2}{\delta(\gamma(t))^2}+\frac{L_\rho(\pi(\alpha(t));\dot\alpha_H(t))}{\delta(\gamma(t))}\right)^{\frac12}dt=$$

$$C\int_{s_{l-1}}^{s_l}\frac{1}{h(\gamma(t))}\left(\frac{|\dot\alpha_N(t)|^2}{h(\gamma(t))^2}+L_\rho(\pi(\alpha(t));\dot\alpha_H(t))\right)^{\frac12}dt\geq$$

$$C\frac{2^{m_\gamma-l}}{H_\gamma}\int_{s_{l-1}}^{s_l}\left(\left(\frac{\kappa_\gamma}{8}\right)^2|\dot\alpha_N(t)|^2+L_\rho(\pi(\alpha(t));\dot\alpha_H(t))\right)^{\frac12}dt\geq$$

$$C\frac{2^{m_\gamma-l}}{H_\gamma}\int_{s_{l-1}}^{s_l}\left((\kappa_\gamma^2|\dot\alpha_N(t)|^2+L_\rho(\pi(\alpha(t));\dot\alpha_H(t))\right)^{\frac12}dt\geq$$

$$C\frac{2^{m_\gamma-l}}{H_\gamma}d_{\kappa_\gamma}(p_{l-1},p_l),$$

where d_{κ_γ} is the distance function induced by the Riemannian metric in the Approximation Lemma 1.6:

$$G_{\kappa_\gamma}(p;Z):=\kappa_\gamma^2|Z_N|^2+L_\rho(p;Z_H),\qquad Z\in T_p\partial\Omega.$$

Claim 4.4.2.

$$l_F\left(\gamma_{\restriction_{[s_{l-1},s_l]}}\right)\geq C\frac{d_H(p,q)}{H_\gamma}.\tag{4.29}$$

Proof of Claim 4.4.2. *Since* $\frac{1}{\kappa_\gamma}\leq d_H(p_{l-1},p_l)$, *by the Approximation Lemma 1.6*

$$d_H(p_{l-1},p_l)\leq Cd_{\kappa_\gamma}(p_{l-1},p_l).$$

Then, up to renaming C,

$$l_F\left(\gamma_{\restriction_{[s_{l-1},s_l]}}\right)\geq C\frac{2^{m_\gamma-l}}{H_\gamma}d_{\kappa_\gamma}(p_{l-1},p_l)\geq C\frac{2^{m_\gamma-l}}{H_\gamma}d_H(p_{l-1},p_l)\underbrace{\geq}_{(4.27)}$$

$$C\frac{2^{m_\gamma-l}}{H_\gamma}\frac{d_H(p,q)}{8\cdot2^{m_\gamma-l}}=C\frac{d_H(p,q)}{H_\gamma}\qquad\checkmark.$$

Now let $t_1:=s_{m_\gamma}\leq t_\gamma$ (the first time that γ reaches heighth H_γ can be before t_γ). As a consequence of (4.29) and (4.7) in Lemma 4.2 we get

$$l_F\left(\gamma_{\restriction_{[0,t_1]}}\right)=l_F\left(\gamma_{\restriction_{[0,s_{l-1}]}}\right)+l_F\left(\gamma_{\restriction_{[s_{l-1},s_l]}}\right)+l_F\left(\gamma_{\restriction_{[s_l,t_1]}}\right)\geq$$

$$\ln \frac{h(x_{l-1})}{h(x_0)} + C\frac{d_H(p,q)}{H_\gamma} + \ln \frac{h(x_m)}{h(x_l)} - C =$$

$$\ln \left(\underbrace{\frac{h(x_{l-1})}{h(x_l)}}_{\frac{1}{2}} \frac{h(x_m)}{h(x_0)} \right) + C\frac{d_H(p,q)}{H_\gamma} - C = \ln \frac{H_\gamma}{h(x)} + C\frac{d_H(p,q)}{H_\gamma} - C.$$

Applying the same considerations to γ_2 instead of γ_1 we can find $t_2 \in [t_\gamma, 1]$ s.t.

$$l_F\left(\gamma_{\restriction_{[t_2,1]}}\right) \geq \ln \frac{H_\gamma}{h(y)} + C\frac{d_H(p,q)}{H_\gamma} - C.$$

B_2) Let us denote $p_1 := \pi(\gamma(t_1)),\ p_2 := \pi(\gamma(t_2))$. In this scenario,

$$d_H(p,p_1) \leq \sum_{i=1}^{m_\gamma} d_H(p_{i-1},p_i) \underbrace{\leq}_{(4.28)} \frac{d_H(p,q)}{8} \sum_{i=1}^{m_\gamma} \frac{1}{2^{m_\gamma - i}} \leq \frac{d_H(p,q)}{4}.$$

On the other hand, again by (4.7) in Lemma 4.2,

$$l_F(\gamma_{\restriction_{[0,t_1]}}) \geq \ln \frac{h(x_{m_\gamma})}{h(x_0)} - C = \ln \frac{H_\gamma}{h(x)} - C.$$

As in the previous scenario, the same considerations can be applied to γ_2 instead of γ_1.

Summarizing the results obtained in B_1) and B_2) we have the following possibilities:

$$l_F\left(\gamma_{\restriction_{[0,t_1]}}\right) \geq \ln \frac{H_\gamma}{h(x)} + C\frac{d_H(p,q)}{H_\gamma} - C. \qquad (\heartsuit)$$

$$\begin{cases} l_F\left(\gamma_{\restriction_{[0,t_1]}}\right) \geq \ln \frac{H_\gamma}{h(x)} - C \\ d_H(p,p_1) \leq \frac{d_H(p,q)}{4}. \end{cases} \qquad (\diamondsuit)$$

$$l_F\left(\gamma_{\restriction_{[t_2,1]}}\right) \geq \ln \frac{H_\gamma}{h(y)} + C\frac{d_H(p,q)}{H_\gamma} - C. \qquad (\clubsuit)$$

$$\begin{cases} l_F\left(\gamma_{\restriction_{[t_2,1]}}\right) \geq \ln \frac{H_\gamma}{h(y)} - C \\ d_H(q,p_2) \leq \frac{d_H(p,q)}{4}. \end{cases} \qquad (\spadesuit)$$

Claim 4.4.3. *Suppose that ($\diamondsuit$) and ($\spadesuit$) hold simultaneously. Then*

$$l_F\left(\gamma_{\restriction_{[t_1,t_2]}}\right) \geq C\frac{d_H(p,q)}{H_\gamma}.$$

Proof of Claim 4.4.3. *Start by noticing that*

$$d_H(p_1,p_2) \geq d_H(p,q) - d_H(p,p_1) - d_H(q,p_2) \underbrace{\geq}_{(\diamondsuit)+(\spadesuit)} \frac{d_H(p,q)}{2}. \qquad (\star)$$

Now we want to apply the Approximation Lemma 1.6 as in B_1), so we define the constant $\kappa > 0$ such that

$$\frac{1}{\kappa} := \frac{d_H(p,q)}{2} \leq d_H(p_1, p_2)$$

and we notice that

$$h(\gamma(t)) \leq H_\gamma < h_0 \leq d_H(p,q) = \frac{2}{\kappa}.$$

Following the same steps of B_1), we get

$$l_F\left(\gamma_{\restriction_{[t_1,t_2]}}\right) \geq C\frac{1}{H_\gamma}\int_{t_1}^{t_2}\left(\frac{\kappa^2}{4}|\dot{\alpha}_N(t)|^2 + L_\rho(\alpha(t);\dot{\alpha}_H(t))\right)^{\frac{1}{2}} dt \geq$$

$$\frac{C}{H_\gamma}\int_{t_1}^{t_2}\left(\kappa^2|\dot{\alpha}_N(t)|^2 + L_\rho(\alpha(t);\dot{\alpha}_H(t))\right)^{\frac{1}{2}} dt \geq \frac{C}{H_\gamma} d_k(p_1,p_2),$$

where d_K is the distance function induced by the Riemannian metric

$$G_\kappa(p;Z) := \kappa^2|Z_N|^2 + L_\rho(p;Z_H), \qquad Z \in T_p\partial\Omega.$$

Now we apply the Approximation Lemma 1.6 and conclude

$$l_F\left(\gamma_{\restriction_{[t_1,t_2]}}\right) \geq \frac{C}{H_\gamma} d_H(p_1,p_2) \underbrace{\geq}_{(\star)} \frac{C}{2}\frac{d_H(p,q)}{H_\gamma} \qquad \checkmark.$$

Consequently,

$$l_F(\gamma) = l_F\left(\gamma_{\restriction_{[0,t_1]}}\right) + l_F\left(\gamma_{\restriction_{[t_1,t_2]}}\right) + l_F\left(\gamma_{\restriction_{[t_1,1]}}\right) \geq$$

$$\ln\frac{H_\gamma}{h(x)} + C\frac{d_H(p,q)}{H_\gamma} + \ln\frac{H_\gamma}{h(y)} - C = 2\ln\frac{H_\gamma}{\sqrt{h(x)h(y)}} + C\frac{d_H(p,q)}{H_\gamma} - C.$$

Up to a modification of C, the last inequality holds for any combination of white-black seeds. In other words,

$$l_F(\gamma) \geq 2\ln\frac{H_\gamma}{\sqrt{h(x)h(y)}} + C\frac{d_H(p,q)}{H_\gamma} - C \tag{4.30}$$

for any $\gamma : [0,1] \to \Omega$ joining x, y, with $H_\gamma < h_0$.

The RHS of (4.30), as a function of H_γ, has a mininum for $H_\gamma = Cd_H(p,q)$. This yields the lower bound

$$l_F(\gamma) \geq 2\ln\frac{d_H(p,q)}{\sqrt{h(x)h(y)}} - C. \tag{4.31}$$

Notice that (4.26) and (4.31) are the same estimate respectively for the cases A) and B). We can therefore take the infimum over all admissible curves joining x and y and we obtain

$$d_F(x,y) \geq 2\ln\frac{d_H(p,q)}{\sqrt{h(x)h(y)}} - C.$$

In conclusion, (4.25) holds and this completes the proof. □

4.5 Estimate for the Kobayashi metric

Notice that the estimate (4.5) for the Kobayashi metric is slightly different from the estimate (4.4) in Theorem 4.2, as it is not possible to have $C_2 \geq 1$ and $\varepsilon > 0$ such that

$$C_2 = 1 + \varepsilon \quad and \quad \frac{1}{C_2} = 1 - \varepsilon.$$

Nonetheless, this does not affect the outcome

$$g(x,y) - C \leq d_{K,\Omega}(x,y) \leq g(x,y) + C.$$

Indeed, if in the proof of Theorem 4.2 we consider $F = K$, the estimate (4.5) gets the job done exactly as the estimate (4.4), as one can notice that the applications of the estimate all lie in the fact that $C_2 \geq 1$ and $\frac{1}{C_2} \leq 1$, it does not matter that $\frac{1}{C_2}$ is the actual reciprocal of C_2. Therefore, since $1 + \varepsilon \geq 1$ and $1 - \varepsilon \leq 1$, we get the same outcome. In particular, the LHS of the estimate is used in the first part of the proof of Lemma 4.2 and in the scenario B_1), in the proof of case 4. The RHS of the estimate is used in the second part of the proof of Lemma 4.2 and in the proof of Claim 4.3.2.

By this observations we can finally conclude that Theorem 4.1, and consequently Corollary 4.1, hold.

Chapter 5

An application of the Balogh-Bonk Theorem

In Chapter 5 we present a recent application of Balogh-Bonk Theorem 4.1 to the theory of functions with Bounded Mean Oscillation (BMO spaces, for short).

In section 5.1 we introduce two notions of BMO spaces on bounded strictly pseudoconvex domains:

- BMO spaces defined via balls in the Kobayashi metric (BMO^p_r);
- BMO spaces defined via dyadic tents ($BMO^p_{\mathcal{D}}$) .

In section 5.2 we show, following [7], that such BMO spaces are equivalent, in the sense that they are identical as sets and the corresponding norms are equivalent. A key role in the proof is indeed played by Theorem 4.1: the idea is that the dyadic cubes we consider are comparable to balls in the Carnot-Carathérodory metric (see Lemma 5.2, Lemma 5.3), and Theorem 4.1 gives an explicit relationship between the Carnot Carathéodory metric and the Kobayashi metric: we take advantage of this explicit relationship to show the inclusion $BMO^p_{\mathcal{D}} \subseteq BMO^p_r$, see Proposition 5.1.

5.1 Notions of BMO and strict pseudoconvexity

Let us fix, for the whole Chapter, a bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with C^2-smooth boundary. The first definition we give is the one of BMO spaces in terms of balls in the Kobayashi metric.

Def 5.1. *(BMO^p_r): Let $1 \leq p < \infty$, $r > 0$ and $b \in L^p_{loc}(\Omega)$. We say that $b \in BMO^p_r(\Omega)$ if*

$$||b||^p_{BMO^p_r} := \sup_{z\in\Omega} \fint_{E(z,r)} |b(w) - \langle b\rangle_{E(z,r)}|^p dw < \infty,$$

where $E(z,r) := \{w \in \Omega \ ; \ d_{K,\Omega}(w,z) < r\}$ denotes the Kobayashi ball centered at z with radius r and

$$\langle b\rangle_{E(z,r)} := \fint_{E(z,r)} b(w)dw = \frac{1}{|E(z,r)|}\int_{E(z,r)} b(w)dw$$

is the mean value of b on $E(z,r)$.

Lemma 5.1. *For $0 < r < r'$, $BMO^p_{r'}(\Omega) \subseteq BMO^P_r(\Omega)$ with bounded inclusion, i.e. there exists a constant $C = C(r, r') > 0$ such that, if $b \in BMO^p_{r'}(\Omega)$, then*

$$||b||_{BMO^p_r} \leq C(r, r')||b||_{BMO^p_{r'}}.$$

Proof. See [7], Remark 2.19, page 11. □

For the notion of dyadic BMO spaces, We need to introduce some notation.
Notation:

- If $A = A(\xi)$ and $B = B(\xi)$ are two real-valued positive functions of the same variable ξ, we write $A \approx B$ if there exist two constants $0 < C_1 \leq C_2 < \infty$ such that
$$C_1 B(\xi) \leq A(\xi) \leq C_2 B(\xi)$$
for each ξ in the domain of A and B. If not otherwise specified, the constants C_1 and C_2 will only depend on Ω.

- If $S \subseteq \partial\Omega$, we denote $\sigma(S)$ the volume of S, where σ is the induced Lebesgue measure on $\partial\Omega$.

Now we follow [7] for the construction of dyadic tents and "kubes" in the interior of Ω. We start by giving a precise construction of a system of dyadic cubes on the boundary of Ω.

Def 5.2. *Let (X, d) be a metric space. We say that (X, d) is a **geometrically doubling space** if there exists a positive integer M such that, for every $x \in X$ and $r > 0$, the ball $B_d(x, r) := \{y \in X \ ; \ d(x, y) < r\}$ can be covered by at most M balls $B_d\left(x_i, \frac{r}{2}\right), \quad i = 1, \cdots M$.*

Thanks to the Box-Ball estimate 1.3 one can notice that, for sufficiently small radius $r > 0$,

$$\sigma\left(B_{d_H}(x, r)\right) \approx r^{2n}. \tag{5.1}$$

As a consequence, $(\partial\Omega, d_H)$ is a geometrically doubling space (see [7], pages 4,5). Therefore, due to results in [7] and [9], the construction of dyadic systems is precise.

Lemma 5.2. *([7], lemma 2.2). Let $\delta > 0$ be sufficiently small and let $s > 1$ be a parameter. Then there exist reference points $\left\{p^{(k)}_j\right\}_{\substack{k\in\mathbb{N}\\ j\in I(k)}}$ on $\partial\Omega$ and an associated collection $\mathcal{Q} := \left\{Q^k_j\right\}_{\substack{k\in\mathbb{N}\\ j\in I(k)}}$ of subsets of $\partial\Omega$, with $p^{(k)}_J \in Q^k_j$, such that the following properties hold:*

1) For each fixed $k \in \mathbb{N}$, $\left\{p^{(k)}_j\right\}_{j\in I(k)}$ is a largest set of points in $\partial\Omega$ satisfying $d_H\left(p^{(k)}_j, p^{(k)}_i\right) > s^{-k}\delta$ for all $j, i \in I(k)$, meaning that if $p \in \partial\Omega$ is not in $\left\{p^{(k)}_j\right\}$, then there exists an index j_0 such that $d_H\left(p, p^{(k)}_{j_0}\right) \leq s^{-k}\delta$.

2) For each fixed $k \in \mathbb{N}$,

$$\bigcup_{j\in I(k)} Q^k_j = \partial\Omega \quad \textit{and} \quad Q^k_j \cap Q^k_i = \emptyset \quad \textit{when } i \neq j.$$

3) If $l > k$, then, $\forall\ i \in I(l),\ j \in I(k)$, either $Q_i^l \subset Q_j^k$ or $Q_i^l \cap Q_j^k = \emptyset$.

4) There exist positive constants c and C such that for all j and k

$$B_{d_H}(p_j^{(k)}, cs^{-k}\delta) \subseteq Q_j^k \subseteq B_{d_H}(p_j^{(k)}, Cs^{-k}\delta),$$

where $B_{d_h}(\cdot,\cdot)$ indicates a ball with respect to the Carnot-Carathéodory metric (1.14).

5) There exists $M \in \mathbb{N}$ such that each Q_j^k contains at most M sets Q_i^{k+1}.

The collection $\left\{p_j^{(k)}\right\}_{\substack{k\in\mathbb{N}\\ j\in I(k)}}$ is called a ***system of dyadic points****. The set Q_j^k is called a* ***dyadic cube of generation*** *k centered at $p_j^{(k)}$, and the collection $\mathcal{Q}$ is called a* ***collection of dyadic cubes****.*

Lemma 5.3. *([9], Theorem 4.1). Let δ and $\left\{p_j^{(k)}\right\}_{\substack{k\in\mathbb{N}\\ j\in I(k)}}$ be as in Lemma 5.2. Then there are finitely many collections $\{\mathcal{Q}_l\}_{l=1}^N$ such that the following conditions hold:*

1) Each collection $\mathcal{Q}_l$ is associated to the system of dyadic points $\left\{p_j^{(k)}\right\}$ and they satisfy all the properties in Lemma 5.2.

2) For any $p \in \partial\Omega$ and small $r > 0$, there exist $Q_{j_1}^{k_1} \in \mathcal{Q}_{l_1}$ and $Q_{j_2}^{k_2} \in \mathcal{Q}_{l_2}$ such that

$$Q_{j_1}^{k_1} \subseteq B_{d_H}(p,r) \subseteq Q_{j_2}^{k_2} \quad \text{and} \quad \sigma\left(B_{d_H}(p,r)\right) \approx \sigma\left(Q_{j_1}^{k_1}\right) \approx \sigma\left(Q_{j_2}^{k_2}\right).$$

Remark 5.1. *From Remark 4.13 in [9], the generation of the cube $Q_{l_2}^{k_2}$ above only depends on the radius r. In particular, k_2 is such that*

$$s^{-k_2-2} < r \leq s^{-k_2-1}.$$

Def 5.3. ***(Dyadic tents):*** *For a given dyadic system $\mathcal{Q}_l = \left(Q_j^k\right)_{\substack{k\in\mathbb{N}\\ j\in I(k)}}$ on $\partial\Omega$, the corresponding* ***dyadic tents*** *on the interior of Ω can now be constructed as follows:*

$$\hat{K}_j^k := \left\{z \in \Omega\ ;\ \pi(z) \in Q_j^k \text{ and } \delta(z) < s^{-2k}\delta^2\right\}.$$

We denote $\hat{K}^{-1} := \Omega$. The collection of dyadic tents of $\mathcal{Q}_l$ is denoted $\mathcal{T}_l := \left\{\hat{K}_j^k\right\}_{\substack{k\in\mathbb{N}\\ j\in I(k)}} \cup \hat{K}^{-1}$.

Notice that if $\hat{K}_{j_1}^{k_1}$ and $\hat{K}_{j_2}^{k_2}$ are two dyadic tents in $\mathcal{T}_l$ and $k_2 > k_1$, then either $\hat{K}_{j_2}^{k_2} \subset \hat{K}_{j_1}^{k_1}$ or $\hat{K}_{j_2}^{k_2} \cap \hat{K}_{j_1}^{k_1} = \emptyset$.

Def 5.4. ***(Dyadic "kubes"):*** *For the collection of dyadic tents $\mathcal{T}_l$ corresponding to the collection of dyadic cubes $\mathcal{Q}_l$, define the* ***center*** *of each tent $\hat{K}_j^k$ to be the unique point $c_j^{(k)} \in \Omega$ such that*

$$\pi\left(c_j^{(k)}\right) = p_j^{(k)};$$

$$\delta\left(c_j^{(k)}\right) = \frac{1}{2} \sup_{\substack{z \in K_j^k \\ \pi(z) = p_j^{(k)}}} \delta(z).$$

We denote $K^{-1} := \Omega \setminus \left(\bigcup_{j \in I(0)} \hat{K}_j^0\right)$ *and for each tent* $\hat{K}_j^k$ *we define the* ***dyadic "kube"*** *of* $\hat{K}_j^k$ *as*

$$K_j^k := \hat{K}_j^k \setminus \left(\bigcup_l \hat{K}_l^{k+1}\right),$$

where l *is any index in* $I(k+1)$ *such that* $p_l^{(k+1)} \in Q_j^k$*. We call* $\mathcal{K}_l := \left\{K_j^k\right\}_{\substack{k \in \mathbb{N} \\ j \in I(k)}} \cup K^{-1}$ *the collection of dyadic kubes corresponding to the collections of dyadic tents* $\mathcal{T}_l$*.*

The following lemmas collects properties of the dyadic tents and kubes.

Lemma 5.4. *([8], Lemma 3.11): Let* $\mathcal{T}_l = \left\{\hat{K}_j^k\right\}$ *be the collection of dyadic tents induced by the collection* $\mathcal{Q}_l$*. Let* K_j^k *be the kube of* $\hat{K}_j^k$*. Then*

1) K_j^k*'s are pairwise disjoint and*

$$\bigcup_{K_j^k \in \mathcal{K}_l} K_j^k = \Omega;$$

2) $|K_j^k| \approx |\hat{K}_j^k| \approx s^{-k(2n+2)}\delta^{2n+2}$*.*

This creates a tree structure on the kubes. We use the notation $K_j^k \preceq K_{j_0}^{k_0}$ *to indicate that* K_j^k *is a* ***descendant*** *of* $K_{j_0}^{k_0}$*, that is* $p_j^{(k)} \in Q_{j_0}^{k_0}$*. On the other hand, we say that* $K_{j_0}^{k_0}$ *is an* ***ancestor*** *of* K_j^k*. We use the same terminology of descendant and ancestor also for cubes and tents.*

Lemma 5.5. *([6], Th. 2.15) Each dyadic kube is comparable to a Kobayashi ball near the boundary in the following way. There exists a constant* $\beta > 0$ *such that for all* j, k, l *the following containment holds for* $K_j^k \in \mathcal{K}_l$ *:*

$$K_j^k \subset E\left(c_j^{(k)}, \beta\right).$$

Moreover, there are implicit constants $C_1, C_2 > 0$ *depending only on* Ω *and the parameters* δ, s *in the dyadic system such that*

$$C_1\left|E\left(c_j^{(k)}, \beta\right)\right| \leq \left|K_j^k\right| \leq C_2\left|E\left(c_j^{(k)}, \beta\right)\right|.$$

Lemma 5.6. *([7], Lemma 3.4) There exists a constant* $C_\beta > 0$ *depending only on* β *(from Lemma 5.5) such that if* $K_j^{k_0+1} \subset K_{j_0}^{k_0}$*, then*

$$\left|\langle b\rangle_{E\left(c_{j_0}^{(k_0)}, \beta\right)} - \langle b\rangle_{E\left(c_j^{(k_0+1)}, \beta\right)}\right| \leq C_\beta ||b||_{BMO_{3\beta}^p} \qquad \forall\, b \in BMO_{3\beta}^p(\Omega).$$

Finally, we are able to define the space $BMO_{\mathcal{D}}^p(\Omega)$.

Def 5.5. ***(***$BMO_{\mathcal{D}}^p$***):*** *Let* $1 \leq p < \infty$ *and* $b \in L^p(\Omega)$*. We say the* $b \in BMO_{\mathcal{D}}^p(\Omega)$ *if*

$$||b||_{BMO_{\mathcal{D}}^p}^p := \sup_{\substack{\hat{K}_l^k \in \mathcal{T}_l \\ l = 1, \cdots N}} \fint_{\hat{K}_j^k} |b(w) - \langle b\rangle_{\hat{K}_j^k}|^p dw < \infty.$$

5.2 Equivalence of the two BMO spaces

The goal of this section is to prove the following Theorem.

Theorem 5.1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with C^2-smooth boundary. Then*

$$BMO^p_r(\Omega) = BMO^p_{\mathcal{D}}(\Omega)$$

with equivalent norms for all $r \geq 3\beta$, where $\beta > 0$ was introduced in Lemma 5.5.

The proof of Theorem relies on the following results.

Proposition 5.1. *For any Kobayashi ball $E(z,r) \subset \Omega$ there exist $C_1(r), C_2(r) > 0$ and a tent $\hat{K}_z \in \mathcal{T}_l$, for some $l \in \{1, \cdots N\}$, such that $E(z,r) \subseteq \hat{K}_z$ with*

$$C_1(r)|\hat{K}_z| \leq |E(z,r)| \leq C_2(r)|\hat{K}_z|, \quad \textit{i.e.} \quad |E(z,r)| \approx |\hat{K}_z|.$$

Proof. • We can assume that z is sufficiently close to the boundary, i.e. $\delta(z) < \varepsilon_0$ for some appropriate $\varepsilon_0 > 0$. Otherwise we denote, as in Chapter 4,

$$K := \{x \in \Omega \ ; \ \delta(x) \geq \varepsilon_0\}.$$

For each $z \in K$, $r > 0$ we can then consider the tent $\hat{K}_z := \hat{K}^{-1} = \Omega$. Of course $E(z,r) \subseteq \hat{K}_z$ and $|E(z,r)| \leq |\hat{K}_z| = |\Omega|$. On the other hand, since K is compact,

$$\min_{x \in K} |E(x,r)| =: m(r) > 0.$$

Then $|E(z,r)| \geq \frac{m(r)}{|\Omega|} \left|\hat{K}_z\right|$, which yields $|E(z,r)| \approx \hat{K}_z$.

• Therefore, let $z \in \Omega \setminus K$ and $r > 0$. Here Balogh-Bonk Theorem 4.1 plays a central role. We recall from [7], Corollary 2.13 page 10, that $|E(z,r)| \approx \delta(z)^{n+1}$, where the implicit constants only depend on r. Thus, it suffices to find a tent $\hat{K}_z$ such that

$$E(z,r) \subseteq \hat{K}_z \quad \textit{and} \quad |\hat{K}_z| \approx \delta(z)^{n+1}. \tag{5.2}$$

Let us consider a point $w \in E(z,r)$. By Balogh-Bonk Theorem 4.1 we obtain that there is a constant $\tilde{C}(r) > 0$ such that $h(w) < \tilde{C}(r)h(z)$. Indeed, $w \in E(z,r) \iff d_{K,\Omega}(w,z) < r$ and by Theorem 4.1 $g(w,z) - C \leq d_{K,\Omega}(w,z) < r$, which means, by definition of g,

$$2\ln\left(\frac{d_H(\pi(w),\pi(z)) + h(w) \vee h(z)}{\sqrt{h(w)h(z)}}\right) - C < r.$$

Then

$$\begin{aligned} &2\ln\frac{h(w)}{\sqrt{h(w)h(z)}} - C \leq r \iff \ln\frac{\sqrt{h(w)}}{\sqrt{h(z)}} \leq \frac{r+C}{2} \\ &\iff \sqrt{h(w)} \leq e^{\frac{r+C}{2}}\sqrt{h(z)} \iff h(w) \leq e^{r+C}h(z). \end{aligned} \tag{5.3}$$

Then it suffices to pick $\tilde{C}(r) := e^{r+C}$. On the other hand, Theorem 4.1 also yields

$$\ln\frac{d_H(\pi(w),\pi(z))}{\sqrt{h(z)h(w)}} \leq \frac{r+C}{2} \iff$$

$$d_H(\pi(w),\pi(z)) \leq e^{\frac{r+C}{2}}\sqrt{h(z)h(w)} \underbrace{\leq}_{5.3} e^{r+C}h(z) = \tilde{C}(r)h(z).$$

Let us denote $t := \tilde{C}(r)h(z)$. As a consequence of Lemma 5.3, point *2)* (here we need z to be close to the boundary), we can find a dyadic cube $Q^{k_0}_{j_0}$ corresponding to a system of dyadic cubes $\mathcal{Q}_l$ such that

$$\pi(z) \in Q^{k_0}_{j_0}, \quad B_{d_H}(\pi(z),t) \subseteq Q^{k_0}_{j_0} \quad and \quad \sigma(Q^{k_0}_{j_0}) \approx \sigma\left(B_{d_H}(\pi(z),t)\right).$$

Since $d_H(\pi(w),\pi(z)) \leq t$, we have

$$\pi(w) \in B_{d_H}(\pi(z),t) \subseteq Q^{k_0}_{j_0}. \tag{5.4}$$

Now we consider the tent associated to the cube $Q^{k_0}_{j_0}$,

$$\hat{K}^{k_0}_{j_0} := \{w \in \Omega\ ;\ \pi(w) \in Q^{k_0}_{j_0}\ and\ \delta(w) < s^{-2k_0}\delta^2\}.$$

We can see that $|\hat{K}^{k_0}_{j_0}| \approx \delta(z)^{n+1}$, with implicit constants depending only on Ω and r. Notice that (5.1) gives $\sigma(B_{d_H}(z,t)) \approx t^{2n}$ and point *4)* of Lemma 5.2 gives $\sigma(Q^{k_0}_{j_0}) \approx (s^{-k_0}\delta)^{2n}$. Hence

$$(s^{-k_0}\delta)^{2n} \approx \sigma(Q^{k_0}_{j_0}) \approx \sigma(B_{d_H}(z,t)) \approx t^{2n}$$

$$\iff s^{-k_0}\delta \approx t. \tag{5.5}$$

Therefore, from Lemma 5.4, point *2)*, we obtain

$$|\hat{K}^{k_0}_{j_0}| \approx s^{-k_0(2n+2)}\delta^{2n+2} = \left(s^{-k_0}\delta\right)^{2n+2} \approx t^{2n+2} =$$

$$(\tilde{C}(r)h(z))^{2n+2} = \tilde{C}(r)^2\delta(z)^{n+1}.$$

In conclusion,

$$|\hat{K}^{k_0}_{j_0}| \approx \delta(z)^{n+1} \approx |E(z,r)|. \tag{5.6}$$

From (5.4) and (5.6), $\hat{K}^{k_0}_{j_0}$ is clearly candidate for being the tent $\hat{K}_z$, but unfortunately we are not granted that $E(z,r) \subseteq \hat{K}^{k_0}_{j_0}$ as we cannot prove the condition

$$\delta(w) < s^{-2k_0}\delta^2\ for\ all\ z \in E(z,r). \tag{5.7}$$

Then, we seek for an ancestor $\hat{K}^{k'}_{j'}$ of $\hat{K}^{k_0}_{j_0}$ with comparable volume such that condition (5.7) is satisfied with k' in place of k_0.

From (5.5) we know that there exists a constant $\hat{C} > 0$ only depending on Ω and r such that $t \leq \hat{C}s^{-k_0}\delta$. Thanks to Remark 5.1, we can assume that k_0 is big enough to satisfy $\hat{C} < s^{k_0}$, provided that z is sufficiently close to the boundary. Let us now pick

$$m := \min\{n \in \mathbb{N}\ ;\ \hat{C} < s^n\},$$

and denote $k' := k_0 - m$. It is important that m does not depend on z.

Consider $\hat{K}^{k'}_{j'}$ to be the ancestor of $\hat{K}^{k_0}_{j_0}$ of generation k'. Then $E(z,r) \subseteq \hat{K}^{k'}_{j'}$, because if $w \in E(z,r)$, then

$$\pi(w) \in Q^{k_0}_{j_0} \subseteq Q^{k'}_{j'}, \quad and$$

$$\delta(w) \underbrace{\leq}_{(5.3)} t^2 \leq \left(\hat{C}s^{-k_0}\delta\right)^2 = \left(\hat{C}s^{k'-k_0}s^{-k'}\delta\right)^2 =$$

$$\left(\frac{\hat{C}}{s^m}s^{-k'}\delta\right)^2 \leq \left(s^{-k'}\delta\right)^2.$$

Moreover, since m does not depend on z, the volumes of $\hat{K}^{k_0}_{j_0}$ and $\hat{K}^{k'}_{j'}$ are comparable with constans only depending on Ω, r and the coefficients chosen for the construction of the dyadic system:

$$|\hat{K}^{k'}_{j'}| \approx s^{-k'(2n+2)}\delta^{2n+2} = s^{m(2n+2)}s^{-k_0(2n+2)}\delta^{2n+2} \approx$$

$$|\hat{K}^{k_0}_{j_0}| \approx \delta(z)^{n+1} \approx |E(z,r)|.$$

In conclusion, we can choose $\hat{K}_z := \hat{K}^{k'}_{j'}$.

□

Lemma 5.7. *$BMO^p_{\mathcal{D}}(\Omega) \subseteq BMO^p_r(\Omega)$ for all $r > 0$ with bounded inclusion, i.e. for all $r > 0$ there exists $C(r) > 0$ such that*

$$||b||_{BMO^p_r} \leq C(r)||b||_{BMO^p_{\mathcal{D}}} \qquad \forall\ b \in BMO^p_{\mathcal{D}}(\Omega).$$

Proof. As a consequence of Proposition 5.1, for any $z \in \Omega,\ b \in BMO^p_{\mathcal{D}}$, it holds

$$\begin{aligned} ||b||_{BMO^p_{\mathcal{D}}} &\geq \left(\frac{1}{|\hat{K}_z|}\int_{\hat{K}_z}|b(w) - \langle b\rangle_{\hat{K}_z}|^p dw\right)^{\frac{1}{p}} \geq \\ &\left(\frac{C_1(r)}{|E(z,r)|}\int_{E(z,r)}|b(w) - \langle b\rangle_{\hat{K}_z}|^p dw\right)^{\frac{1}{p}}. \end{aligned} \tag{5.8}$$

Therefore, for all $z \in \Omega$ we have

$$\left(\fint_{E(z,r)}|b(w) - \langle b\rangle_{E(z,r)}|^p dw\right)^{\frac{1}{p}} \leq$$

$$\left(\fint_{E(z,r)}(|b(w) - \langle b\rangle_{\hat{K}_z}| + |\langle b\rangle_{\hat{K}_z} - \langle b\rangle_{E(z,r)}|)^p dw\right)^{\frac{1}{p}}$$

$$\left(\fint_{E(z,r)}|b(w) - \langle b\rangle_{\hat{K}_z}|^p dw\right)^{\frac{1}{p}} + \left(\fint_{E(z,r)}|\langle b\rangle_{\hat{K}_z} - \langle b\rangle_{E(z,r)}|^p dw\right)^{\frac{1}{p}} \overbrace{\leq}^{\textbf{(!!!)}}$$

$$2\left(\fint_{E(z,r)}|b(w) - \langle b\rangle_{\hat{K}_z}|^p dw\right)^{\frac{1}{p}} \underbrace{\leq}_{(5.8)} \frac{2}{C_1(r)^{\frac{1}{p}}}||b||_{BMO^p_{\mathcal{D}}}.$$

Taking the supremum over all $z \in \Omega$ we obtain, for all $r > 0$,

$$||b||_{BMO^p_r} \leq \frac{2}{C_1(r)^{\frac{1}{p}}}||b||_{BMO^p_{\mathcal{D}}}.$$

That is, $b \in BMO^p_{\mathcal{D}}(\Omega) \Rightarrow b \in BMO^p_r(\Omega) \quad \forall\ r > 0$. In the step marked by **(!!!)** we used the following fact:

$$\left(\fint_{E(z,r)} |\langle b\rangle_{\hat{K}_z} - \langle b\rangle_{E(z,r)}|^p dw\right)^{\frac{1}{p}} = |\langle b\rangle_{\hat{K}_z} - \langle b\rangle_{E(z,r)}| =$$

$$\left|\langle b\rangle_{\hat{K}_z} - \fint_{E(z,r)} b(w)dw\right| = \left|\fint_{E(z,r)} \langle b\rangle_{\hat{K}_z} - b(w)dw\right| \leq$$

$$\fint_{E(z,r)} |\langle b\rangle_{\hat{K}_z} - b(w)|dw \leq \left(\fint_{E(z,r)} |\langle b\rangle_{\hat{K}_z} - b(w)|^p dw\right)^{\frac{1}{p}}.$$

□

Proof. **(Proof of Theorem 5.1).** On account of Lemma 5.7 we only need to show that

$$BMO^p_r(\Omega) \subseteq BMO^p_{\mathcal{D}}(\Omega)$$

with bounded inclusion, for $r \geq 3\beta$.

Let us consider $r \geq 3\beta$ and $b \in BMO^p_r(\Omega)$. The goal is to show that there exists a constant $C > 0$ such that

$$||b||_{BMO^p_{\mathcal{D}}} \leq C||b||_{BMO^p_r}.$$

From Lemma 5.1 it is enough to show that

$$||b||^p_{BMO^p_{\mathcal{D}}} \leq C||b||^p_{BMO^p_{3\beta}}$$

for some constant $C > 0$. For any dyadic tent $\hat{K}^k_j$ there holds

$$\left(\fint_{\hat{K}^k_j} \left|b(w) - \langle b\rangle_{\hat{K}^k_j}\right|^p dw\right)^{\frac{1}{p}} \leq$$

$$\left(\fint_{\hat{K}^k_j} \left(\left|b(w) - \langle b\rangle_{E\left(c^{(k)}_j,\beta\right)}\right| + \left|\langle b\rangle_{E\left(c^{(k)}_j,\beta\right)} - \langle b\rangle_{\hat{K}^k_j}\right|\right)^p dw\right)^{\frac{1}{p}} \leq$$

$$\left(\fint_{\hat{K}^k_j} \left|b(w) - \langle b\rangle_{E\left(c^{(k)}_j,\beta\right)}\right|^p dw\right)^{\frac{1}{p}} + \left(\fint_{\hat{K}^k_j} \left|\langle b\rangle_{E\left(c^{(k)}_j,\beta\right)} - \langle b\rangle_{\hat{K}^k_j}\right|^p dw\right)^{\frac{1}{p}} \leq$$

$$2\left(\fint_{\hat{K}^k_j} \left|b(w) - \langle b\rangle_{E\left(c^{(k)}_j,\beta\right)}\right|^p dw\right)^{\frac{1}{p}}.$$

Then it suffices to show that

$$\sup_{\substack{\hat{K}^k_j \in \mathcal{T}_l \\ l=1,\cdots N}} \fint_{\hat{K}^k_j} \left|b(w) - \langle b\rangle_{E\left(c^{(k)}_j,\beta\right)}\right|^p dw \leq C||b||^p_{BMO^p_{3\beta}} \tag{5.9}$$

for some constant $C > 0$. Let us start by fixing a tent $\hat{K}^{k_0}_{j_0}$ and remember by Def 5.4 that

$$\hat{K}^{k_0}_{j_0} = \bigcup_{k \geq k_0} \bigcup_{\substack{j \in I(k) \\ K^k_j \preceq K^{k_0}_{j_0}}} K^k_j.$$

Moreover, from Lemma 5.5, there exists $C_1 = C_1(\Omega, \delta, s) > 0$ such that for all k, j

$$K_j^k \subset E\left(c_j^{(k)}, \beta\right), \quad C_1 \left|E\left(c_j^{(k)}, \beta\right)\right| \leq |K_j^k|.$$

Then,

$$\left(\fint_{\hat{K}_{j_0}^{k_0}} \left|b(w) - \langle b\rangle_{E\left(c_{j_0}^{(k_0)},\beta\right)}\right|^p dw\right)^{\frac{1}{p}} =$$

$$\left(\sum_{k\geq k_0} \sum_{\substack{j\in I(k)\\ K_j^k \preceq K_{j_0}^{k_0}}} \frac{1}{\left|\hat{K}_{j_0}^{k_0}\right|} \int_{K_j^k} \left|b(w) - \langle b\rangle_{E\left(c_{j_0}^{(k_0)},\beta\right)}\right|^p dw\right)^{\frac{1}{p}} =$$

$$\left(\sum_{k\geq k_0} \sum_{\substack{j\in I(k)\\ K_j^k \preceq K_{j_0}^{k_0}}} \frac{\left|K_j^k\right|}{\left|\hat{K}_{j_0}^{k_0}\right|} \fint_{K_j^k} \left|b(w) - \langle b\rangle_{E\left(c_{j_0}^{(k_0)},\beta\right)}\right|^p dw\right)^{\frac{1}{p}} \leq$$

$$\sum_{k\geq k_0} \sum_{\substack{j\in I(k)\\ K_j^k \preceq K_{j_0}^{k_0}}} \left(\frac{\left|K_j^k\right|}{\left|\hat{K}_{j_0}^{k_0}\right|} \fint_{K_j^k} \left|b(w) - \langle b\rangle_{E\left(c_{j_0}^{(k_0)},\beta\right)}\right|^p dw\right)^{\frac{1}{p}} \leq$$

$$\sum_{k\geq k_0} \sum_{\substack{j\in I(k)\\ K_j^k \preceq K_{j_0}^{k_0}}} \Bigg(\underbrace{\frac{\left|K_j^k\right|}{\left|\hat{K}_{j_0}^{k_0}\right|} \frac{1}{C_1\left|E\left(c_j^{(k)}, \beta\right)\right|} \int_{E\left(c_j^{(k}, \beta\right)} \left|b(w) - \langle b\rangle_{E\left(c_j^{(k)},\beta\right)}\right|^p dw}_{=:A_1(k,j)}\Bigg)^{\frac{1}{p}} +$$

$$\sum_{k\geq k_0} \sum_{\substack{j\in I(k)\\ K_j^k \preceq K_{j_0}^{k_0}}} \Bigg(\underbrace{\frac{\left|K_j^k\right|}{\left|\hat{K}_{j_0}^{k_0}\right|} \fint_{K_j^k} \left|\langle b\rangle_{E\left(c_j^{(k)},\beta\right)} - \langle b\rangle_{E\left(c_{j_0}^{(k_0)},\beta\right)}\right|^p dw}_{=:A_2(k,j)}\Bigg)^{\frac{1}{p}}.$$

For each $k \geq k_0$ we obtain, by applying Lemma 5.6 and the triangle inequality $(k - k_0)$ times,

$$\left|\langle b\rangle_{E\left(c_j^{(k)},\beta\right)} - \langle b\rangle_{E\left(c_{j_0}^{(k_0},\beta\right)}\right| \leq C_\beta(k - k_0)||b||_{BMO^p_{3\beta}}. \tag{5.10}$$

Thus, by (5.10) and point *2)* in Lemma 5.4, we get that there exists $\tilde{C} > 0$ such that

$$A_2(k, j) \leq \tilde{C} \frac{s^{-k(2n+2)}\delta^{2n+2}}{s^{-k_0(2n+2)}\delta^{2n+2}} C_\beta^p (k - k_0)^p ||b||^p_{BMO^p_{3\beta}} =$$

$$\tilde{C} s^{(k_0-k)(2n+2)} C_\beta^p (k-k_0)^p |b||^p_{BMO^p_{3\beta}}.$$

Moreover,

$$A_1(k,j) \le \frac{\tilde{C}}{C_1} \frac{s^{-k(2n+2)}\delta^{2n+2}}{s^{-k_0(2n+2)}\delta^{2n+2}} ||b||^p_{BMO^p_\delta} =$$
$$\frac{\tilde{C}}{C_1} s^{(k_0-k)(2n+2)} ||b||^p_{BMO^p_\beta}.$$

Therefore,

$$\left(\fint_{\hat{K}^{k_0}_{j_0}} \left| b(w) - \langle b \rangle_{E\left(c^{(k_0)}_{j_0},\beta\right)} \right|^p dw \right)^{\frac{1}{p}} \le \sum_{k\ge k_0} \sum_{\substack{j\in I(k)\\ K^k_j \preceq K^{k_0}_{j_0}}} A_1(k,j)^{\frac{1}{p}} + A_2(k,j)^{\frac{1}{p}} \le$$

$$\sum_{k\ge k_0} \sum_{\substack{j\in I(k)\\ K^k_j \preceq K^{k_0}_{j_0}}} \left(\frac{\tilde{C}}{C_1}\right)^{\frac{1}{p}} s^{\frac{(k_0-k)(2n+2)}{p}} ||b||^p_{BMO^p_\beta} + \tilde{C}^{\frac{1}{p}} s^{\frac{(k_0-k)(2n+2)}{p}} C_\beta^{\frac{1}{p}} (k-k_0) |b||_{BMO^p_{3\beta}}.$$

Up to renaming the constants,

$$\fint_{\hat{K}^{k_0}_{j_0}} \left| b(w) - \langle b \rangle_{E\left(c^{(k_0)}_{j_0},\beta\right)} \right|^p dw \le$$
$$C_{\beta,\delta,s,p} ||b||^p_{BMO^p_{3\beta}} \sum_{k\ge k_0} \sum_{\substack{j\in I(k)\\ K^k_j \preceq K^{k_0}_{j_0}}} s^{(k_0-k)(2n+2)} (1+k-k_0)^p \le$$
$$C_{\beta,\delta,s,p} ||b||^p_{BMO^p_{3\beta}} \sum_{k=1}^{\infty} \frac{1}{s^{k(2n+2)}} (1+k)^p.$$

Since for the construction of dyadic cubes we requested $s > 1$, the series

$$\sum_{k=1}^{\infty} \frac{1}{s^{k(2n+2)}} (1+k)^p$$

converges. Therefore,

$$\fint_{\hat{K}^{k_0}_{j_0}} \left| b(w) - \langle b \rangle_{E\left(c^{(k_0)}_{j_0},\beta\right)} \right|^p dw \le \tilde{C}_{\beta,\delta,s,p} ||b||^p_{BMO^p_{3\beta}} \qquad \textit{for all tents } \hat{K}^{k_0}_{j_0}.$$

By taking the supremum over all tents we obtain (5.9) with $C = \tilde{C}_{\beta,\delta,s,p}$, and this concludes the proof. □

Notice that this last proof allows us to conclude that any $b \in BMO^p_r(\Omega)$, which a priori is only locally $p-$integrable, in fact belongs to $L^p(\Omega)$. This is not true in general for Euclidean BMO spaces. Moreover,

$$BMO^p_r(\Omega) = BMO^p_{r'}(\Omega) \quad \forall\, r, r' \ge 3\beta.$$

In [7], page 32, it is shown that the technical assumption $r \ge 3\beta$ in Theorem 5.1 can be removed by further decomposing the dyadic structure. This allows us to generalize Theorem 5.1 as follows.

Theorem 5.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with C^2-smooth boundary. Then*

$$BMO_r^p(\Omega) = BMO_{\mathcal{D}}^p(\Omega)$$

with equivalent norms for all $r > 0$.

Appendix A

Appendix

A.1 Riemannian manifolds

Def A.1. *(**Riemannian manifolds**) A **Riemannian manifold** is a smooth manifold M together with the assignment of a scalar product to the tangent space T_pM at each point $p \in M$, such that these scalar products vary smoothly with p. Such an assignment of a scalar products is called a **Riemannian metric** on M (it can be seen also as a section of $T^*M \otimes T^*M$).*
If $(U, x^1, \ldots x^n)$ is a chart on M and $g_{ij}(x)$ are smooth functions of $x \in U$ for $i, j = 1 \ldots n$, the scalar product between two tangent vectors at x, $X := \sum_{i=1}^n X_i(x)\frac{\partial}{\partial x^i}$ and $Y := \sum_{i=1}^n Y_i(x)\frac{\partial}{\partial x^i}$, is

$$< X, Y >:= \sum_{i,j=1}^{n} g_{ij}(x)X_i(x)V_j(x).$$

We can denote the scalar product as $ds^2 = \sum_{i,j=1}^n g_{ij}(x)dx_i dx_j$.

Def A.2. *(**Riemannian isometry**) If (M_1, g_1) and (M_2, g_2) are Riemannian manifolds, a diffeomorphism φ from M_1 to M_2 is called an **isometry** if $\varphi^* g_2 = g_1$, i.e.*

$$\forall p \in M_1,\ V, W \in T_pM_1, \quad (\varphi^* g_2)(V, W) = g_1(V, W),$$

*where the **pullback metric** $\varphi^* g_2$ on M_1 is defined as*

$$(\varphi^* g_2)(V, W) = g_2(\varphi_* V, \varphi_* W). \tag{A.1}$$

Def A.3. *(**Riemannian length**) If (M, g) is a Riemannian manifold and $c : [a, b] \to M$ is a differentiable path in M, we denote its **velocity** vector at time t as $\dot{c}(t) \in T_{c(t)}M$, and $|\dot{c}(t)|_g$ is its norm with respect to the Riemannian metric (i.e. to the given scalar product). The **Riemannian length** $l_R(c)$ of c is defined as*

$$l_R(c) := \int_a^b |\dot{c}(t)|_g dt.$$

Proposition A.1. *(**Induced distance**) If (M,g) is a connected Riemannian manifold, the function $d_g : M \times M \to \mathbb{R}^+$, which to two points $x, y \in M$ associates the infimum of the Riemannian length of differentiable paths $c : [0, 1] \to M$ such that $c(0) = x,\ c(1) = y$, is a distance function.*

$$d_g(x, y) := \inf_{\substack{c(0)=x,\\ c(1)=y}} l_R(c).$$

This makes (M, d_g) a metric space.

Proof.

- **symmetry:** If $c(t)$ goes from x to y, then $c(1-t)$ goes from y to x and has the same length. Hence

$$d_g(x,y) = \inf_{\substack{c(0)=x \\ c(1)=y}} l_R(c) = \inf_{\substack{c(0)=y \\ c(1)=x}} l_R(c) = d_g(y,x).$$

- **triangle inequality:** If $x, y, z \in M$, then

$$d_g(x,y) + d_g(y,z) \geq d_g(x,z),$$

as if we concatenate a path from x to y and one from y to z, we obtain a path from x to z. Such a path is contained in the set of the paths which are considered to compute the distance between x and z.

- **positivity:** Fix $p \in M$, and consider a chart $(V, \underbrace{x^1, \ldots, x^n}_{\phi})$ around p, such that $\phi(p) = 0$ and $\phi(V)$ contains the closed unit ball in $\mathbb{R}^n$. Let us call $U := \{q \in V \ ; \ \sum_{i=1}^n x^i(q)^2 \leq 1\}$ the set of the points in V which are mapped into the unit ball. Suppose that in these local coordinates the Riemannian metric is given by the expression

$$ds^2 = \sum_{i,j=1}^n g_{ij}(q) dx_i dx_j.$$

We want to compare it to the Euclidean metric on U:

$$ds_E^2 = \sum_{i=1}^n dx_i^2.$$

Given $q \in U$ and $v \in T_qM$, we denote by $|v|_E$ (respectively $|v|$) the norm of v with respect to ds_E^2 (respectively ds^2). Let

$$m_1(q) := \inf_{\substack{v \in T_qM \\ |v|_E = 1}} |v| \qquad , \qquad m_2 := \sup_{\substack{v \in T_qM \\ |v|_E = 1}} |v|.$$

By compactness of $\{|v|_E = 1\}$, we have $0 < m_1(q) \leq m_2(q) < \infty$. Where $m_1(q) > 0$ because it is a mininum, and it being 0 would mean that there is a $v \in T_qM$ such that $|v|_E = 1$ and $|v| = 0$. But this is impossible, as the second condition would imply $v = 0$, which is not compatible with $|v|_E = 1$.

Moreover, for all $v \in T_q(U)$ we have

$$m_1(q)|v|_E \leq |v| \leq m_2(q)|v|_E.$$

Therefore, if we pick a (piecewise) differentiable path c in U, we have

$$m_1 l_E(c) \leq l_R(c) \leq M_2 l_E(c),$$

where $l_E(c)$ denotes the length with respect to the Euclidean metric ds_E^2 and $m_1 := (\min_{q \in U} m_1(q)) < M_2 := (\max_{q \in U} m_2(q))$ are finite as U is compact. In particular, this shows that the distances d_g and d_E are equivalent on U, hence d_g is positive definite.

□

Bibliography

[1] Zoltán M Balogh and Mario Bonk. Gromov hyperbolicity and the kobayashi metric on strictly pseudoconvex domains. *Commentarii Mathematici Helvetici*, 75(3):504–533, 2000.

[2] André Bellaıche and Jean-Jacques Risler. Sub-riemannian geometry, volume 144 of progress in mathematics, 1996.

[3] Mario Bonk and Oded Schramm. Embeddings of gromov hyperbolic spaces. In *Selected Works of Oded Schramm*, pages 243–284. Springer, 2011.

[4] Martin R Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319. Springer Science & Business Media, 2013.

[5] James B Carrell. Groups, matrices, and vector spaces. *A group theoretic approach to linear algebra. Springer, New York*, 2017.

[6] Bingyang Hu and Zhenghui Huo. Dyadic carleson embedding and sparse domination of weighted composition operators on strictly pseudoconvex domains. *Bulletin des Sciences Mathématiques*, 173:103067, 2021.

[7] Bingyang Hu, Zhenghui Huo, Loredana Lanzani, Kevin Palencia, and Nathan A Wagner. The commutator of the bergman projection on strongly pseudoconvex domains with minimal smoothness. *arXiv preprint arXiv:2210.10640*, 2022.

[8] Zhenghui Huo, Nathan A Wagner, and Brett D Wick. Bekollé-bonami estimates on some pseudoconvex domains. *Bulletin des Sciences Mathématiques*, 170:102993, 2021.

[9] Tuomas Hytönen and Anna Kairema. Systems of dyadic cubes in a doubling metric space. *arXiv preprint arXiv:1012.1985*, 2010.

[10] Svetlana Katok. *Fuchsian groups*. University of Chicago press, 1992.

[11] Ermanno Lanconelli. *Lezioni di analisi matematica 1*. Pitagora, 1994.

[12] Serge Lang. *Introduction to complex hyperbolic spaces*. Springer Science & Business Media, 2013.

[13] Jiri Lebl. *Tasty bits of several complex variables*. Lulu. com, 2019.

[14] John M Lee. *an Introduction to Curvature*, volume 176. Graduate text in mathematics, 1997.

[15] R Michael Range. *Holomorphic functions and integral representations in several complex variables*, volume 108. Springer Science & Business Media, 1998.

[16] Walter Rudin et al. *Principles of mathematical analysis*, volume 3. McGraw-hill New York, 1976.

[17] Loring W Tu. *An Introduction to Manifolds.* Springer, 2011.

Acknowledgments

At the end of this journey, I want to thank everyone who has been at my side during the past five years.

As far as it concerns this thesis, I want to express my deepest gratitude to my supervisors, Prof. Loredana Lanzani and Prof. Irina Markina. I had the privilege to work and study with two great mathematicians and at the same time fabulous human beings. They both have always been very helpful whenever I had math-related problems, and it was a pleasure to talk with them about my life both as a student and outside the university environment. In particular, Prof. Lanzani's methodical approach really helped me to keep track of all the topics we treated and to organize a well-ordered exposition. I am grateful to Prof. Markina for making me discover the world of hyperbolic geometry, which I had never approached before, and for giving me the possibility to present my first seminar in front of an audience of experts.

I also want to thank Prof. Sergio Venturini and Dr. Nathan Wagner who gave a crucial support for the drafting of the first and the last chapter, respectively.

A special mention goes to Prof. Bruno Franchi, who supervised me on my bachelor thesis, for everything he tought me and for lighting up my passion in Mathematical Analysis.

I am grateful to everyone in the university who makes the Erasmus project, and all the others projects abroad, possible. My experience in Norway made me grow significantly both in the mathematical and human side.

It is mandatory to thank my family for supporting me in all the choices I made, even if they had no actual idea of what I was really doing.

Finally, I would like to thank my girlfriend Anna and all the friends that shared an enormous amount of experiences with me in these 5 years, which I am positive to say they had been the best five years of my life. Please, help me making them worse than the next five.

To all of you, thank you from the bottom of my heart.