ALMA MATER STUDIORUM · UNIVERSITY OF BOLOGNA

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Master Degree in Physics

Black Holes as Quantum Bound States

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Academic Year 2020/2021

Abstract

We present a new quantum description for the Oppenheimer-Snyder model of gravitational collapse of a ball of dust. Starting from the geodesic equation for dust in spherical symmetry, we introduce a time-independent Schrödinger equation for the radius of the ball. The resulting spectrum is similar to that of the Hydrogen atom and Newtonian gravity. However, the non-linearity of General Relativity implies that the ground state is characterised by a principal quantum number proportional to the square of the ADM mass of the dust. For a ball with ADM mass much larger than the Planck scale, the collapse is therefore expected to end in a macroscopically large core and the singularity predicted by General Relativity is avoided. Mathematical properties of the spectrum are investigated and the ground state is found to have support essentially inside the gravitational radius, which makes it a quantum model for the matter core of Black Holes. In fact, the scaling of the ADM mass with the principal quantum number agrees with the Bekenstein area law and the corpuscular model of Black Holes. Finally, the uncertainty on the size of the ground state is interpreted within the framework of the *Uncertainty Principle*.

Contents

Introduction

Constructing a quantum theory of gravity is probably the most challenging problem in modern physics. The general path is based on finding the full quantum theory and then derive the predicted consequences. This approach is still far from its final success. As possible guide it is useful to have quantized models that already reproduce observational data. That was the first step in the derivation of *Quantum Electro Dynamics* that, without the already known results for the quantization of atomic spectrum, would have been impossible to obtain. In this spirit a new quantization model for Black Holes is proposed in this thesis.

It is widely expected that a quantum theory of gravity should cover the classical singularities that General Relativity predicts. Until such a theory is available we need to study how specific quantum models that describe compact objects cover singularities. Introducing a quantum way to describe gravitational collapse, and the possible Black Hole formation, is fundamental to understand the ability to cover the singularities of a quantum theory of gravity. After the classical study of a collapse for a ball of dust [1] many quantum approaches have been proposed. Almost all the quantum gravitational collapse models that we present in this thesis have one characteristic in common, they choose as quantum degree of freedom the areal radius of the ball. The reason for this choice is clear: it is a known observable that we already understand how to interpret. With this treatment the vast majority of the initial degrees of freedom are excluded. The hope is to introduce a notion of entropy that reproduces the grade of disorder due to the neglected degrees of freedom.

The thesis is organized as follows. In the first chapter the classical Oppenheimer-Snyder model for the collapse of a star is discussed, and the massive geodesic equation in the Schwarzschild space-time is obtained [1, 3, 4, 5].

In the second chapter two different semi-classical quantization models are presented. Both are based on the research for classical trajectories of the collapsing ball, and both reproduce the newtonian idea of a bounce for the collapsing matter. The first model [12] treats the collapse of massive particles from the point of view of two different observers, one stationary and one comoving with respect to the dust cloud. The second model instead studies the rather unrealistic case of a collapsing shell of photons [14]. These models have basically the same result: the wave function for the areal radius bounce back. Therefore, problems for a stable state that corresponds to a Black Hole arise.

The third chapter is dedicated to the corpuscular model of Black Holes [19]. In this picture Black Holes are seen as a Bose condensates, and all of their characteristics are

obtained in function of the occupation number. The condensate is composed of weakly interacting gravitons, and even if its structure is very clear and stable what it seems less obvious is the mechanism in which such a condensate is formed.
In the fourth chapter we develop a new quantization model for the collapse. Instead of looking for classical trajectories we search for quantum bound states. We used the geodesic equation as a time independent Schrödinger one, the final result is a spectrum similar to an Hydrogen-like atom. We will consider two different derivations, the first one keeping a generic orbital quantum number, the second setting it to zero. In our procedure the singularity is avoided in a fully Quantum Mechanics formalism. It is extremely unlikely to find matter in correspondence of the singularity, just like for the electron in the Bohr atom it is extremely unlikely to be found in the nucleus. Finally, uncertainty relations for our *Bound States Quantization Model* (BSQM) are obtained, and the *Heisenberg Uncertainty Principle* for the position and the momentum of the collapsing shell is derived. Also we obtain a preferred ratio between the mass of the source and the effective collapsing mass.
The final chapter is devoted to the description of a complete, and rather simplified, model for the Black Hole structure [26]. This model define both the wave function for the radial position of a particle and the wave function for the horizon. The two wave functions are combined to obtain the probability that a particle might form a Black Hole. The *Generalized Uncertainty Principle* is introduced in a fully form, therefore the concept of a minimal length arises.

1 Classical gravitational collapse

We start reviewing the Oppenheimer-Snyder model for the spherical collapse of a star that is approximated by a ball of incoherent mass [1, 3, 4].

1.1 Phases of a star

The destiny of a star is almost completely fixed by its characteristics like mass and density. After the formation of the stellar object, composed mainly of Hydrogen and dust, we have a stable state. The self attraction is compensated by pressure and outgoing radiation. The internal pressure is due to the fusion processes of the Hydrogen atoms that create Helium. Indeed, once the attraction has compressed enough matter to start the processes of transformation from Hydrogen to Helium those prevent the cooling of the star.
After all the Hydrogen of the star has been used, the production inside the star switches to heavier elements up to ^{56}Fe. This phase is much shorter than the previous, and if the object is sufficiently massive the heat is so high that eventually electrons will be emitted by atoms and a degenerate electron gas is formed. At this stage we have the White Dwarf. Using Quantum Mechanics *Chandrasekhar* [7] was able to demonstrate that if the initial mass exceeds 1.44 solar masses this White Dwarf is not a stable state. It must either expel the exceeding material, or continue to evolve towards a more compact object.
At these ranges of pressure the electrons and the protons of the nuclei turn into neutrons, that are the most closely packed nuclear matter. Even if this states, known as Neutron Stars, are really dense objects, if the initial mass is sufficiently big and not enough material has been radiated away they will not be the final state. At some time the pressure will no longer be able to balance the gravitational attraction, and the collapse will continue indefinitely beyond the gravitational radius R_H.
In the next sections we are going to present the only collapse model that is possible to solve analytically, the one for incoherent mass (dust). Dust has neither pressure nor any form of interaction except for gravitation. This rather unrealistic model does not predict any stable state, therefore it cannot fairly describe a celestial object. However, the exact solution of Einstein equations is an important feature of this kind of models, and it will play a fundamental role in the quantization process that is the core of this thesis.

1.2 Massive geodesics in spherical symmetry

The geodesic equation for a spherical symmetric space-time will be used as starting point in our new quantization process, we now recover it. We start from the Schwarzschild metric written using Schwarzschild coordinates [4]

$$ds^2 = -e^{\nu(R)}dt^2 + e^{-\nu(R)}dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.1}$$

The case of interest is the motion of a test particle m in the presence of an Arnowitt-Deser-Misner (ADM) mass M_0 [2]. We now apply the definition of a geodesic trajectory. We define the four-velocity of a massive particle

$$u^\mu = \frac{dx^\mu}{ds} \tag{1.2}$$

Geodesics are the trajectories along which this four-velocity is parallelly transported [6][1]

$$u^\mu \nabla_\mu u^\nu = \ddot{x}^\nu + \Gamma^\nu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0 \tag{1.3}$$

Two are the relevant integrals of motion: the energy per unit mass measured by an asymptotic observer and the angular momentum per unit mass (see Appendix A for the details). The energy is

$$u_t = \frac{E_m}{m} = const \tag{1.4}$$

while the angular momentum

$$u_\phi = \frac{L_m}{m} = const \tag{1.5}$$

The resulting geodesic equation can be expressed in the form:

$$\left(\frac{dR}{ds}\right)^2 - \frac{E_m^2}{m^2} + \left(1 - \frac{2G_N M_0}{R}\right) + \left(1 - \frac{2G_N M_0}{R}\right)\frac{L_m^2}{R^2 m^2} = 0 \tag{1.6}$$

Quantum effects will introduce uncertainties in the geometry of geodesics [8]. We are going to neglect these uncertainties and use the unperturbed geodesic equation. This approach seems equivalent to neglecting the effects that small quantum perturbations have on a classical background metric. This approximation leads to the ambiguity on the definition of the Hilbert states, different observers would define the vacuum differently. This consideration may be important for future developments.

1.3 Oppenheimer-Snyder model

We analyze the model of a collapsing ball of dust in the vacuum, the Oppenheimer-Snyder model [1]. We start with a form of the metric in spherical symmetry that uses a coordinates system comoving with the dust [3]

$$ds^2 = -d\tau^2 + e^{\nu(\tau,\eta)}d\eta^2 + R(\tau,\eta)^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.7}$$

[1] $\dot{x} = \dfrac{dx}{ds}$.

τ is the proper time of a particle of dust, η is the position in the comoving reference frame. It can be shown easily that dust always moves along geodesics. We start from the covariant conservation of the energy-momentum tensor

$$\nabla_\mu T^\mu_\nu = 0 \tag{1.8}$$

For dust ($p = 0$) the energy-momentum tensor of a perfect fluid reduces to

$$T^\mu_\nu = \rho u^\mu u_\nu \tag{1.9}$$

with $\rho(\tau, \eta)$ which is the proper energy density and u^μ is the four-velocity of dust. If we contract the covariant conservation with u^ν we obtain

$$u^\nu \nabla_\mu T^\mu_\nu = u^\nu \nabla_\mu(\rho u^\mu u_\nu) = u^\nu \rho u^\mu \nabla_\mu u_\nu + u^\nu u_\nu \nabla_\mu(\rho u^\mu) = 0 \tag{1.10}$$

Since we are considering timelike trajectories ($u^\nu u_\nu = -1$) we can write

$$\nabla_\mu(\rho u^\mu) = u^\nu \rho u^\mu \nabla_\mu u_\nu \tag{1.11}$$

The contracted conservation equation can also be written as

$$u^\nu \nabla_\mu T^\mu_\nu = u^\nu \nabla_\mu(\rho u^\mu u_\nu) = \nabla_\mu(u^\nu u_\nu \rho u^\mu) - u_\nu \rho u^\mu \nabla_\mu u^\nu = 0 \tag{1.12}$$

and rearranged into

$$\nabla_\mu(\rho u^\mu) = -u^\nu \rho u^\mu \nabla_\mu u_\nu \tag{1.13}$$

Combined with (1.11) this gives us $\nabla_\mu(\rho u^\mu) = 0$. Finally looking back at the conservation equation (1.10) and plugging this result

$$\nabla_\mu T^\mu_\nu = \nabla_\mu(\rho u^\mu u_\nu) = \rho u^\mu \nabla_\mu u_\nu + u_\nu \nabla_\mu(\rho u^\mu) = \rho u^\mu \nabla_\mu u_\nu = 0 \tag{1.14}$$

The geodesic equation has been obtained

$$u^\mu \nabla_\mu u^\nu = 0 \tag{1.15}$$

In the metric (1.7) we have used a reference frame comoving with dust. In such a frame the dust four-velocity is $u^\mu = (1, 0, 0, 0)$, therefore the only non vanishing components of the energy-momentum tensor (1.9) is

$$T^0_0 = -\rho(\tau, \eta) \tag{1.16}$$

Starting from (1.7) we can calculate the non-vanishing Christoffel symbols [2]

$$\Gamma^\tau_{\eta\eta} = \dot{\nu}\frac{e^\nu}{2} \qquad \Gamma^\tau_{\theta\theta} = R\dot{R} \qquad \Gamma^\tau_{\phi\phi} = R\dot{R}\sin^2\theta \tag{1.17}$$

$$\Gamma^R_{RR} = \frac{\nu'}{2} \qquad \Gamma^R_{\theta\theta} = -e^{-\nu}RR' \qquad \Gamma^R_{\phi\phi} = -e^{-\nu}RR'\sin^2\theta \qquad \Gamma^R_{R\tau} = \frac{\dot{\nu}}{2} \tag{1.18}$$

$$\Gamma^\theta_{R\theta} = \frac{R'}{R} \qquad \Gamma^\theta_{\theta\tau} = \frac{\dot{R}}{R} \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \tag{1.19}$$

$$\Gamma^\phi_{R\phi} = \frac{R'}{R} \qquad \Gamma^\phi_{\phi\tau} = \frac{\dot{R}}{R} \qquad \Gamma^\phi_{\theta\phi} = \cot\theta \tag{1.20}$$

[2] $x' = \dfrac{dx}{d\eta}$.

The equations of motion can be written as [3]

$$\mathbf{R}_\tau^\tau - \frac{\mathbf{R}}{2} = \left(\frac{2R''}{R} - \frac{\nu' R'}{R} + \frac{R'^2}{R^2}\right) e^{-\nu} - \frac{\dot{R}\dot{\lambda}}{R} - \frac{\dot{R}^2}{R^2} - \frac{1}{R^2} = -k\rho(\tau,\eta) \tag{1.21}$$

$$\mathbf{R}_R^R - \frac{\mathbf{R}}{2} = \frac{R'^2}{R^2} e^{-\nu} - \frac{2\ddot{R}}{R} - \frac{1}{R^2} = 0 \tag{1.22}$$

$$\mathbf{R}_\theta^\theta - \frac{\mathbf{R}}{2} = \mathbf{R}_\phi^\phi - \frac{\mathbf{R}}{2} = \left(\frac{R''}{R} - \frac{R'\nu'}{2R}\right) e^{-\nu} - \frac{\dot{r}\dot{\nu}}{2R} - \frac{\ddot{\nu}}{2} - \frac{\dot{\nu}^2}{4} - \frac{\ddot{R}}{R} = 0 \tag{1.23}$$

$$\mathbf{R}_{R\tau} = \frac{\dot{\nu} R'}{R} - \frac{2\dot{R}'}{R} = 0 \tag{1.24}$$

We take the last expression

$$\dot{\nu} = \frac{2\dot{R}'}{R'} = \frac{(\dot{R'^2})}{R'^2} \tag{1.25}$$

and integrate it to obtain

$$e^{\nu} = \frac{R'^2}{1 - \epsilon f^2(\eta)} \tag{1.26}$$

with $\epsilon = 0, \pm 1$ and $f(\eta)$ an arbitrary function. Substituting into (1.22)

$$2\ddot{R}R + \dot{R}^2 = -\epsilon f^2(\eta) \tag{1.27}$$

and defining $u = \dot{R}^2$ we can write

$$\frac{d(Ru)}{dR} = -\epsilon f^2(\eta) \tag{1.28}$$

This last expression is solved by

$$\dot{R}^2 = -\epsilon f^2(\eta) + \frac{F(\eta)}{R} \tag{1.29}$$

with an arbitrary function F. In (1.26) we can now eliminate f and find out that equation (1.23) is identically satisfied, while (1.21) gives us

$$k\rho = \frac{F'}{R'R^2} \tag{1.30}$$

Defining, with $\epsilon \neq 0$, a new time variable

$$d\lambda = f\frac{d\tau}{R} \tag{1.31}$$

equation (1.29) becomes

$$\left(\frac{\partial R}{\partial \lambda}\right)^2 = \frac{F}{f^2} R - \epsilon R^2 \tag{1.32}$$

This is solved by

$$R = \frac{F(\eta)}{2f^2(\eta)} h'_\epsilon(\lambda) \tag{1.33}$$

$$\tau - \tau_0 = \pm \frac{F(\eta)}{2f^3(\eta)} h_\epsilon(\lambda) \tag{1.34}$$

[3] $\mathbf{R}_{\mu\nu\lambda\sigma}$ indicates the Riemann tensor and its contractions.

where

$$h_\epsilon = \begin{cases} \lambda - \sin\lambda & \text{for } \epsilon = +1 \\ \sinh\lambda - \lambda & \text{for } \epsilon = -1 \end{cases} \tag{1.35}$$

This is the solution for $\epsilon \neq 0$. With $\epsilon = 0$ equation (1.29) is immediately solved by

$$\tau - \tau_0(\eta) = \pm\frac{2}{3}F^{-1/2}(\eta)R^{3/2} \tag{1.36}$$

From equations (1.33) and (1.36) we get $R(\tau, \eta)$. Now with (1.26) we can express the metric (1.7) using the *Tolman solution* [9]

$$ds^2 = -d\tau^2 + \left(\frac{\partial R}{\partial \eta}\right)^2 \frac{d\eta^2}{1 - \epsilon f^2(\eta)} + R^2(\tau, \eta)d\Omega^2 \tag{1.37}$$

This form is actually not fixed yet. We would like to choose a particular matter distribution ρ and then determinate the metric finding the expression for f, F, and τ_0. Unfortunately this cannot be done. We have to fix those functions and study a particular matter distribution.
Let us consider the case for a star of finite dimensions immersed in the vacuum. Inside we have $\rho \neq 0$ while outside $\rho = 0$. The two solutions for these two regions have to match on the surface of the star $\eta = \eta_0$.
For the interior region the most simple case is obtained when ρ does not depend on the position η. We have a uniform matter distribution within the star. Another great simplification comes from setting

$$R = K(\tau)\eta \tag{1.38}$$

and consequently obtaining from (1.30), a suggested form for F:

$$F = \frac{1}{3}k\hat{M}\eta^3 \tag{1.39}$$

where we have defined

$$\rho K^3(\tau) \equiv \hat{M} = const \tag{1.40}$$

We fix $f = \eta$ and $\tau_0 = 0$ and therefore the metric (1.37) takes the form

$$ds^2 = -d\tau^2 + K^2(\tau)\left(\frac{d\eta^2}{1 - \epsilon\eta^2} + R^2 d\Omega^2\right) \tag{1.41}$$

also from (1.33) and (1.34) we find

$$K(\lambda) = \frac{1}{6}k\hat{M}h'_\epsilon(\lambda) \tag{1.42}$$

$$\tau = -\frac{1}{6}k\hat{M}h_\epsilon(\lambda) \tag{1.43}$$

$$h_\epsilon(\lambda) = \begin{cases} \lambda - \sin\lambda & \text{for } \epsilon = +1 \\ \dfrac{\lambda^3}{6} & \text{for} \epsilon = 0 \\ \sinh\lambda - \lambda & \text{for } \epsilon = -1 \end{cases} \tag{1.44}$$

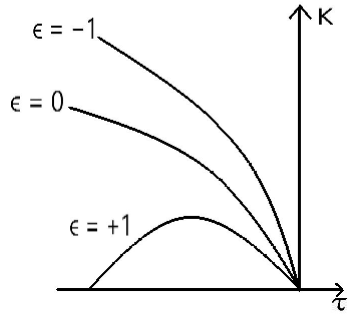

Fig. 1.1
Radius of a collapsing star

We have chosen the negative sign for the proper time τ. It is clear that the interior of the star is represented by the Friedman-Robertson-Walker-Lamaitre metric [4] and since the radius $R = K(\lambda)\eta$ depends on time, the star is either collapsing or expanding. We are considering dust and then we conclude that it is collapsing. In Fig.1.1 we can see that if $\epsilon = 0, -1$ the star radius decreases from an infinite value to zero at $\tau = 0$, while for $\epsilon = +1$ the star first expands until it reaches a maximal radius and then bounce back to a vanishing radius.
We now focus on the exterior region. Since we are in the vacuum with a spherically symmetric source, the Birkoff theorem tells us that the solution is the Schwarzschild one [5]. The Tolman solution (1.37) holds for arbitrary ρ. It must also cover the case with vanishing matter density that is the outer region of our spherical star. Recalling (1.30) with $\rho = 0$

$$k\rho = \frac{F'}{R'R^2} = 0 \tag{1.45}$$

We conclude that in the exterior region F is constant. In the previous section we have recovered the geodesic equation in Schwarzschild space time starting from a metric expressed in Schwarchild's coordinates. In the Tolman solution we have used different coordinates. However in both the descriptions dust particles follow geodesic trajectories, in particular particles on the surface $\eta = \eta_0$ of the star do. We take the geodesic equation (1.6) for radial motion ($L_m = 0$) in Schwarzschild

$$\left(\frac{dR}{ds}\right)^2 = \frac{E_m^2}{m^2} - 1 + \frac{2G_N M_0}{R} \tag{1.46}$$

We notice that in the reference frame comoving with dust the interval of proper time for a dust particle is equal to the line element: $ds^2 = d\tau^2$. Therefore equation (1.46) must coincide with (1.29)

$$\dot{R}^2 = -\epsilon f^2(\eta) + \frac{F(\eta)}{R} \tag{1.47}$$

Clearly we obtain for the exterior region

$$F = 2G_N M_0 \tag{1.48}$$

The exterior and interior solutions, have to match smoothly at the surface $\eta = \eta_0$. The necessary condition for this to happen is

$$R(\eta_0, \tau) = K(\tau)\eta_0 \tag{1.49}$$

where on the left hand side we have the radius in the exterior region while on the right hand side we have the radius for the interior region defined by (1.38). We stick with $\epsilon \neq 0$. Choosing also for the exterior section $\tau_0 = 0$, this condition can only be satisfied at all τ if both sides have the same functional dependence on τ. We now plug the new expression (1.48) for F in the exterior region into the expression (1.33) for the radius

$$R(\eta_0, \tau) = \frac{G_N M_0}{f^2(\eta_0)} h_\epsilon'(\lambda) \tag{1.50}$$

This is the outside radius R on the surface η_0. Now we do the same thing with the expression (1.34) for τ. Substituting the expression (1.48) for F outside and choosing again the minus sign

$$\tau = -\frac{G_N M_0}{f^3(\eta)} h_\epsilon(\lambda) \tag{1.51}$$

Now we match this exterior expression for τ with the interior one (1.43) on η_0. We get

$$\frac{G_N M_0}{f^3(\eta_0)} = \frac{1}{6} k \hat{M} \tag{1.52}$$

Now we combine the expression (1.50) for the external radius with the matching condition (1.49) and also with the expression (1.42) for the internal K

$$\frac{G_N M_0}{f^2(\eta_0)} h_\epsilon'(\lambda) = \frac{1}{6} k \hat{M} \eta_0 h_\epsilon'(\lambda) \tag{1.53}$$

Now considering (1.52) we write

$$\frac{G_N M_0}{f^2(\eta_0)} = \frac{G_N M_0}{f^3(\eta_0)} \eta_0 \tag{1.54}$$

We can conclude that $f(\eta_0) = \eta_0$. From (1.53) and considering the definition (1.40) of $\hat{M}$ we can finally write

$$6 G_N M_0 = k \rho K^3 \eta_0^3 \tag{1.55}$$

When this condition is satisfied we have a continuous metric on the stellar surface, which is exactly what we were looking for.

We call m the mass of the Schwarzschild source in the Newtonian theory. Thanks to the relation [3]

$$2 G_N M_0 = \frac{km}{4\pi} \tag{1.56}$$

we can write

$$\frac{4}{3} \pi \rho \eta_0 K^3 = m \tag{1.57}$$

A similar procedure can be performed for $\epsilon = 0$. This solution for the gravitational field of a collapsing star clearly shows that for the interior of the star there are no peculiarities even when the surface η_0 is inside the Schwarzschild radius $R = K\eta_0 = 2G_N M_0$. The singularity arises only when $K(\tau) = 0$.

2 Semi-classical models

All the Black Holes quantization models start by excluding the vast majority of the initial degrees of freedom. Indeed we know that Black Holes are characterised only by their mass, charge and angular momentum. The final result of a quantization procedure then should be a wave function that depends on these informations about the Black Hole and encodes the values (actually the average values) of some observables. Restricting to the Oppenheimer-Snyder case the wave equation should depend only on the mass of the dust cloud. The quantization models that we will present choose to focus on a wave function from which the only information that can be extrapolated is the areal radius of the collapsing ball. The grade of disorder due to all the possible degrees of freedom that were initially excluded should be represented by a suitable form of entropy.
The quantization models, up to now, tried to obtain wave packets that reproduce classical trajectories but inevitably ended with a bounce and, as we are going to show qualitatively in the next sections, did not reach a truly stable state for a Black Hole. Our *Bound States Quantization Model*, instead of looking for classical trajectories reinstall the bound states of canonical Quantum Mechanics basically reproducing an Hydrogen-like structure.

2.1 Collapse for different observers

We now present the result of a quantization process for an Oppenheimer-Snyder collapse viewed by a co-moving observer and by a stationary one [10, 11, 12]. It will be clear that while both of them will predict a bounce away from the horizon for the collapsing matter, the stationary one will also predict the re-collapse towards the horizon for matter emanating from it. Although this result seems encouraging for the formation of a Black Hole, it is important to underline that the bounce happens outside the photonic sphere[1] and the re-collapse starts between the horizon and the photonic radius, which means that nothing similar to a Black Hole comes out yet from this approach of a stationary observer.
The results for the co-moving observer might be obtained both from a canonical *Dirac's quantization* or from a quantization based on *affine coherent states*. Since we are only interested in a qualitative analysis the details of these procedures, that under some assumptions are completely equivalent, will not be presented. The trajectory of a collapsing

[1] For the Schwarzschild case the radius of the photonic sphere is $R_p = \dfrac{3}{2} R_H$.

observer in the quantized Oppenheimer-Snyder model is [12]

$$R(\tau) = \left[\frac{\hbar^2\delta}{G_N M} + \frac{9G_N M}{2}(\tau - \tau_0)^2\right]^{1/2} \tag{2.1}$$

For large R this reproduces classical trajectories of collapse and expansion connected by a bounce that basically is substituting the singularity. τ is the proper time of the observer, $R(\tau_0)$ is the minimum radius that depends on ambiguities of the quantization and M is the ADM mass of the source. Depending on the value of the parameter δ the dust cloud will fall to a stage within the horizon $R = 2G_N M$, then a state similar to a Black Hole is formed. The lifetime of such a state is

$$\Delta\tau = \tau_+ - \tau_- \tag{2.2}$$

$R(\tau_\pm) = 2G_N M$. This expression is evaluated by [12]

$$\Delta\tau = \frac{M}{3}\sqrt{1 - \left(\frac{R_0}{2G_N M}\right)^3} \tag{2.3}$$

If the minimal radius R_0 is sufficiently small the lifetime of a Black Hole, from the point of view of a co-moving observer, is proportional to the mass.
We now focus on the stationary observer. The Hamiltonian used in the quantization process is multivalued, it is divided in two branches. The internal branch corresponds to trajectories that classically remain inside the Black Hole horizon, the external branch corresponds to trajectories that classically are outside the horizon. The variables used in the Hamiltonian are: $A = (1/2)R^2$ proportional to the surface area of the collapsing cloud, and its canonical momentum P_A. The final form of the Hamiltonian operator, acting on the position space, makes the task to identify the wave function almost impossible. Still we can look into the quantum corrections to the dynamics. We present the final result of the quantization just to show the relation between the canonical variables [12]

$$M = -\frac{\Gamma(2\beta - 1)\Gamma(2\beta)}{\Gamma\left(2\beta + \frac{1}{2}\right)\Gamma\left(2\beta - \frac{3}{2}\right)} H_\pm(P_A, A) \tag{2.4}$$

β is a positive parameter, H_+ and H_- are the external and internal branches of the Hamiltonian. For $|P_A| \to +\infty$ we obtain $A = 2(G_N M)^2$ which corresponds to the classical horizon radius $R = 2G_N M$. This is the branching point for the Hamiltonian. In order to present the dynamical results we stick to value of M close to the Planck scale, since it is the only way to perform numerical analysis. It can be shown analytically that the obtained results are valid also at cosmological scale [12].
We start by looking at Fig.2.1 where the quantum corrected trajectories for the outside branch in the phase space is very close to the classical one. The system is divided into two parts one asymptotically collapsing towards the horizon, and one escaping from it. This behaviour is reproduced also solving the equation of motions Fig.2.2. The dust property of necessarily falling along geodesics allowed us to define a time T.

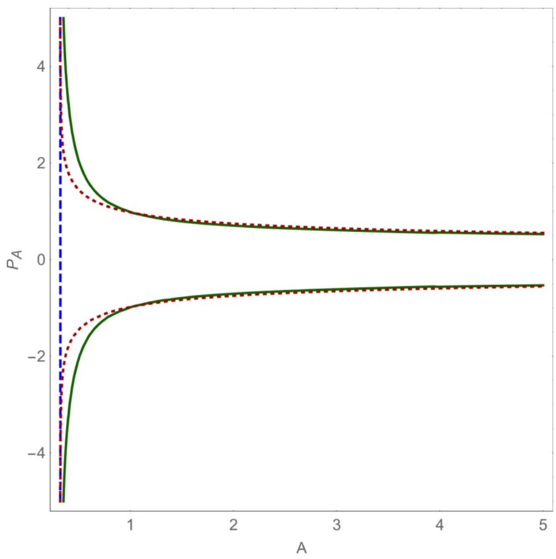

Fig. 2.1
$M = 0.4$ $\beta = 1$.
Quantum corrected phase space portraits for the outside branch of the Hamiltonian (green line), compared to the classical counterpart (red line) and to the horizon (blue line). Masses are given in Planck units [12].

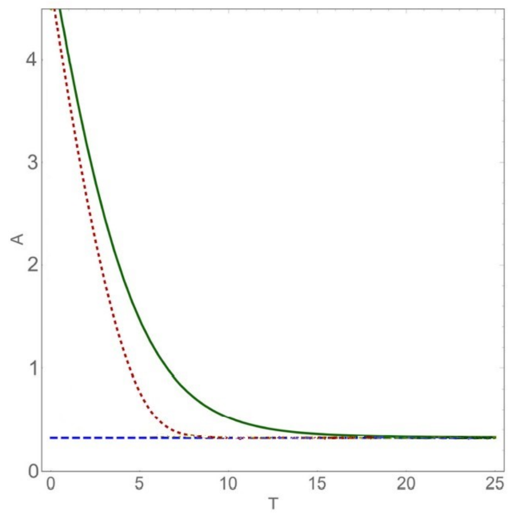

Fig. 2.2
$M = 0.4$ $\beta = 1$.
Quantum corrected trajectories for the outside branch of the Hamiltonian (green line) compared to the classical counterpart (red line) collapsing towards the horizon (blue line) from the outside. Masses are given in Planck units [12].

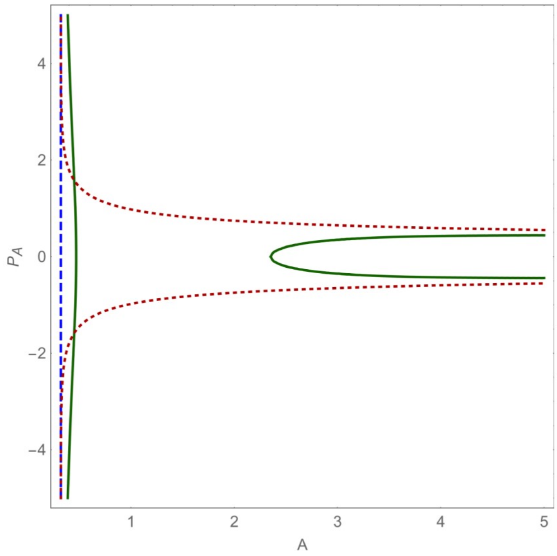

Fig. 2.3
$M = 0.4$ $\beta = 5$.
Quantum corrected phase space portraits for the outside branch of the Hamiltonian (green line), compared to the classical counterpart (red line) and to the horizon (blue line). Masses are given in Planck units [12].

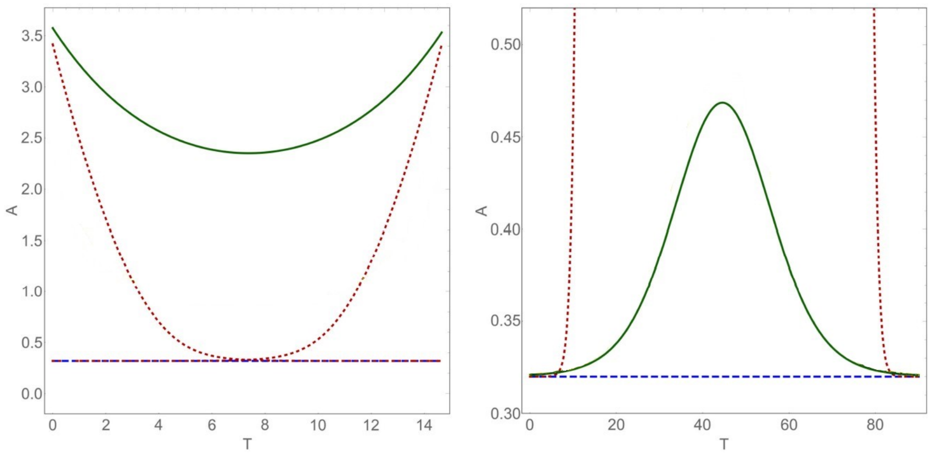

Fig. 2.4
$M = 0.4$ $\beta = 5$.
Quantum corrected trajectories for the outside branch of the Hamiltonian (green line) compared to the classical counterpart (red line) and to the horizon (blue line). On the left we have a collapse followed by a bounce, on the right we have an escaping trajectory followed by a recollapse. Masses are given in Planck units [12].

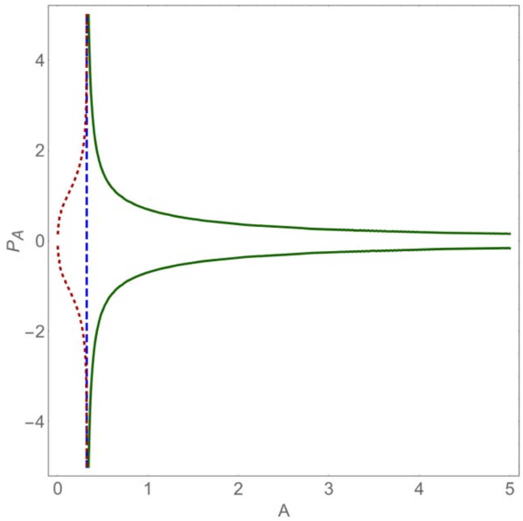

Fig. 2.5
$M = 0.4$ $\beta = 1$.
Quantum corrected phase space portraits for the inside branch of the Hamiltonian (green line), compared to the classical counterpart (red line) and to the horizon (blue line). Masses are given in Planck units [12].

When the value of β is increased, Fig.2.3, the situation is completely different. We observe the formation of a branch collapsing from and then expanding to infinity, and also of a branch near the horizon. In Fig.2.3 it is evident the presence of a bounce for matter coming from the infinity and a re-collapse for matter coming from the horizon. Again this result is confirmed from the solution of the equations of motion. In Fig.2.4 we have on the left the motion of a collapsing cloud that after the bounce escapes towards infinity, while on the right we have matter that after an expansion from the horizon re-collapses to it. It is clear that the re-collapse is a much slower process than the bounce. The inside branch is in great contrast with the expected classical trajectories. In Fig.2.5 it is shown that the classical trajectory ends in the singularity, while the quantum corrected one escapes at infinity. Increasing the value of β a bounce, near the classical singularity, happens and after that the cloud is able to escape at infinity Fig.2.6. It is worth notice that the bounce appears increasing β and keeping M fixed, but also keeping β fixed and increasing M. Indeed with $M = 1$ and $\beta = 1$ the exact same result of Fig.2.6, where $M = 0.4$ and $\beta = 5$, is obtained.

We underline that the point where the two branches will reunify, $P_A \to \pm\infty$, cannot be reached in a final amount of time. The two branches can be considered as separated theories, it is not really necessary to use a multivalued Hamiltonian.

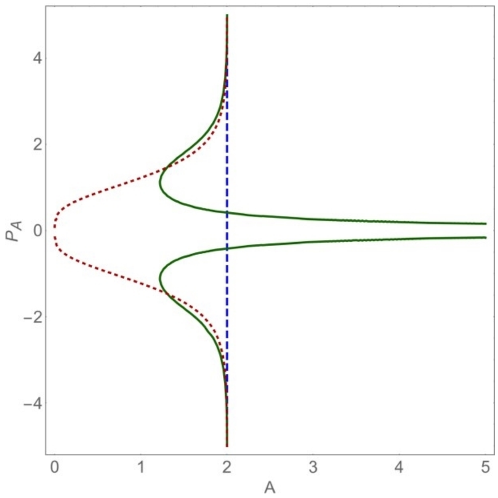

Fig. 2.6
$M = 0.4$ $\beta = 5$.
Quantum corrected phase space portraits for the inside branch of the Hamiltonian (green line), compared to the classical counterpart (red line) and to the horizon (blue line). Masses are given in Planck units [12].

We now look into the main problem with this approach: it does not predict a stable Black Hole state for a static observer. We are interested in how the picture changes by varying M and β. To study that we plot the values of A for different M and β, keeping $P_A = 0$. We start by considering the outside branch in Fig.2.7. The different green dotted lines indicate trajectories with different values of β. When on one of these lines we have two different values of A corresponding to the same value of M, it means that there are two different areas with zero momentum (zero velocity) with the same value of β and M. This can be verified only in the presence of a bounce (for the initially collapsing matter) or in the presence of a re-collapse (for the initially escaping matter). The values of the minimal area at which the bounce happens, for fixed M and β, grow a lot slower for the inside branch, Fig.2.8, with respect to the outside one Fig.2.7. This is the only difference between the two branches. For every β there is a critical mass after which we do not see a bounce or a re-collapse anymore. This critical values grow with β.
It seems that the bounce of the dust cloud happens outside the photon sphere, while the re-collapse is between the photon sphere and the horizon. Everything outside of the photon sphere is visible from the static observer. Therefore, for the dust cloud it is impossible to reproduce a Black Hole. In order to obtain the expected observational results, the bounce should have happened between the horizon and the re-collapse surface, with the latter within the photon sphere, as it is. In this way a stable Black Hole state would have been reached. In the astrophysical context if we want all the dust to bounce we need to push the critical mass almost to infinity. To do so we have to increase β at ranges where numerical analysis is impossible. We can avoid this difficulty thanks to an analytical procedure [12] which shows that, for $\beta \to +\infty$ the value of the minimal area for the bounce diverges to infinity while the maximal area for the re-collapse asymptotically

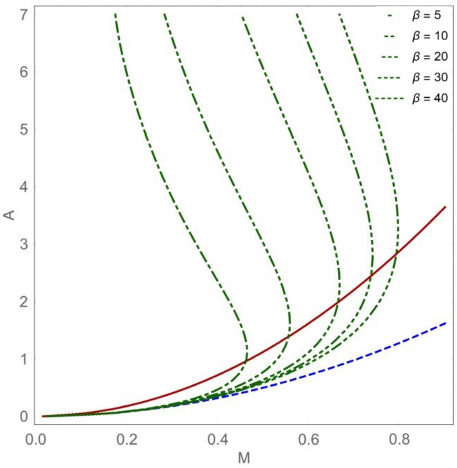

Fig. 2.7
Quantum corrected phase space portraits at $P_A = 0$ for the outside branch of the Hamiltonian for different β (green lines) compared to the photon sphere (red line) and to the horizon (blue line). Masses are given in Planck units [12].

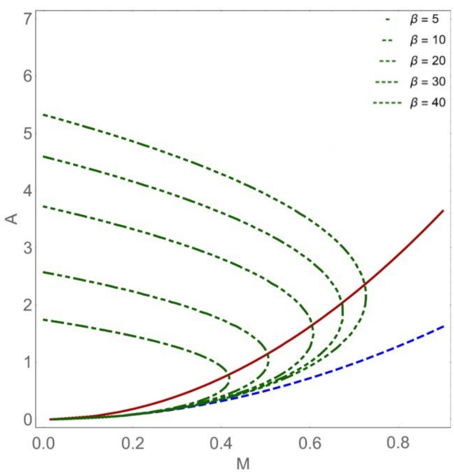

Fig. 2.8
Quantum corrected phase space portraits at $P_A = 0$ for the inside branch of the Hamiltonian for different β (green lines) compared to the photon sphere (red line) and to the horizon (blue line). Masses are given in Planck units [12].

approaches the horizon, which is exactly the starting classical result. It is then impossible to find a value of β such that the bounce always happens regardless of the mass, and this is a problematic quantization ambiguity.
For a static observer the horizon is never formed and then it is impossible even to define the notion of lifetime for the Black Hole. This clearly does not resemble the many observational evidences on Black Holes.

2.2 Null shells

We now present a model of collapse that takes into account spherically symmetric shells composed of zero rest mass particles, lightlike shells [14, 16, 17]. In General Relativity gravity affects light as well, such a collapse is predicted and might end up forming a Black Hole.
At the end of quantization process it is possible to identify a precise wave function for the collapsing null shells [14]

$$\Psi_{k\lambda}(t,r) = \frac{k!(2\lambda)^{k+1/2}}{\sqrt{2\pi(2k)!}}\left[\frac{i}{(\lambda+it+ir)^{k+1}} - \frac{i}{(\lambda+it-ir)^{k+1}}\right] \tag{2.5}$$

where k is a positive integer, λ is positive length and

$$t = \frac{u+v}{2} \tag{2.6}$$

$$r = \frac{-u+v}{2} \tag{2.7}$$

are the usual canonical transformation for the Eddington-Finkelstein coordinates [6].
The first important result comes from noticing that

$$\lim_{r\to 0} \Psi_{k\lambda}(t,r) = 0 \tag{2.8}$$

The probability to find the shell at vanishing radius is null. The singularity is covered in this quantization model. This is a directed consequence of unitary evolution rather than a boundary condition [14]. As we are going to see, this Quantum Mechanics interpretation of the cosmic censorship used to cover the singularity will be present also in our *Bound States Quantization Model.*
It follows from the wave equation that the shells will bounce and no event horizon is formed, although an object similar to a Black Hole is not excluded [14]. Indeed, if the expectation value for the energy evaluated with (2.5) is big enough, respect to the Planck energy, a great part of the wave packet can be squeezed within the Schwarzschild radius. Basically in the interior region the ingoing quantum shells develop, after the bounce, into a superposition of ingoing and outgoing shells, and their destructive interference avoids the singularity formation.

3 Black Holes as Bose-Einstein condensates

We present a model where Black Holes are seen as Bose-condensates that maximize N, the occupation number of the gravitons that constitute the Black Hole. A N-Bose condensate is a state of matter composed by very low energy bosons which mostly occupy the lowest quantum state. We are going to see that N is also the quantum measure of *classicality* [19, 20]. In this picture Black Holes are leaky bound states of weakly interacting particles that exist for an arbitrary N. Hawking radiation and Beckenstein entropy are reproduced in function of N without the introduction of any geometrical concept, not even an horizon. This is the corpuscular model of Black Holes.

3.1 Quantum Einstein gravity

Einstein's General Relativity, when introduced as a quantum theory, propagates a weakly coupled particle with spin 2, the graviton. At low energies a dimensionless self-coupling of gravitons can be written as [19]

$$\alpha_{gr} = \frac{\hbar G_N}{\lambda^2} = \frac{l_p^2}{\lambda^2} \tag{3.1}$$

where l_p is the Planck length

$$l_p \equiv \sqrt{G_N \hbar} \tag{3.2}$$

For wavelength $\lambda \ll l_p$ the coupling becomes strong and the theory cannot be treated perturbatively. Since we are dealing with a dimensionless coupling constant the hope is that introducing at some scale greater than l_p new terms, these will give us the possibility to evaluate gravitational amplitudes at arbitrarily short distances.

An alternative possibility, that avoids this Wilsonian UV-completion, predicts that Einstein's gravity is self complete. Therefore, it prevents us to investigate at arbitrarily short distances responding to any high energy scattering process by producing large occupation number N of wavelength $\lambda \gg l_p$ [21]. This non-Wilsonian approach is able to perfectly reproduce the Black Hole physics. We are going to show that these results can be restored in an Einstein's gravity quantum theory of gravitons.

In such a framework any classical object is described as a quantum bound state of high occupation number $N \gg 1$. N is maximized for $\lambda = R_H = 2G_N M$. Therefore, among all the possible sources with a fixed physical size, Black Holes are the ones that maximize N, they are the most classical objects among all the possibilities with a characteristic wavelength λ. It is important to remark that they exist for arbitrary N, which is the number of particles of the self imposed bound state. Those particles are the gravitons whose wavelength is [19]

$$\lambda = \sqrt{N} l_p \tag{3.3}$$

Those gravitons are weakly interacting via the coupling $1/N$. Their binding energy

$$V = \frac{\hbar}{\sqrt{N} l_p} \tag{3.4}$$

is just below the escaping energy of the condensate. Therefore, we will be able to reproduce the Hawking radiation without the need to introduce any geometrical principle.

3.2 Classicality

In classical physics the Schwarzschild radius $R_H \equiv 2G_N M$ is the most relevant quantity for the gravitational properties of the source. Its physical meaning is to set the distance at which the gravitational local effects of a specific source become strong. This is a classical length in the sense that it can be arbitrarily small, although not every form of energy describes a gravitational source. We are going to see that this is due to the fact that not all the sources are able to product at least one graviton's quantum.
To be classical a source needs to have a gravitational radius much bigger than the quantum length scale of the problem. The length at which the quantum fluctuations of the space time metric become important is the Planck length (3.2). We now define others important quantum quantities, like the Planck mass

$$m_p \equiv \sqrt{\frac{\hbar}{G_N}} \tag{3.5}$$

The Compton and de Broglie wavelengths of the source defined through its mass M and momentum P are

$$l_c \equiv \frac{\hbar}{M} \tag{3.6}$$

$$l_{db} \equiv \frac{\hbar}{P} \tag{3.7}$$

They set the scale at which the energy of the quantum fluctuations $E = \hbar/l_{c(db)}$ become comparable to the energy of the source.
All this quantum quantities vanish in the limit of $\hbar \to 0$ keeping M and G_N fixed. In General Relativity, that is a classical theory with $\hbar = 0$, it is possible to have Black Holes of microscopic sizes. We know that Quantum Mechanics describes reality better than classical physics, then we have to keep $\hbar$ and G_N fixed. We are able to vary only M and reach classicality when the mass reflects into a gravitational radius $R_H \gg l_p$. We have just introduced the idea to treat *classicality* in a quantum field theory. The

parameter that measures classicality is the gravitons occupation number.
Let us take a spherically symmetric source with uniform density, the actual composition is not important. The radius R_s of this source is well above its gravitational radius. For such a source a linear approximation of gravity, like the newtonian theory, is valid. Indeed, the newtonian potential that might play the role of the metric perturbation is

$$\phi(R) = -\frac{R_H}{R} \tag{3.8}$$

and falls of as R^2 for $R < R_s$. The linearized metric, obtained with the above perturbation, represents a superpositions of gravitons in a quantum field point of view. Classicality is measured by the occupation number of these gravitons, which form a Bose condensate that as long as $R_s > R_H$ cannot self-sustain. It requires some form of external source. The situation changes when the gravitational radius is crossed by the source radius: the condensate becomes self-sustained and the Black Hole classical state is reached.
We start by studying the region $R_s \gg R_H$. Through a Fourier analysis of the perturbation (3.8) the graviton's occupation number N is evaluated as

$$N = \frac{1}{\hbar} M R_H \tag{3.9}$$

Let us see how. The gravitational energy can be seen as the sum of the energies of each graviton with wavelength λ [19]

$$E_{gr} \sim \frac{M R_H}{R_s} \sim \sum_{\lambda} N_\lambda \hbar \lambda^{-1} \tag{3.10}$$

It is important to underline that this approximation as a simple sum of non-interacting particles is due to the distribution being peaked at $\lambda = R_s$. Shorter wavelengths are exponentially suppressed. Since we are at $R_s \gg R_H \gg l_p$ the gravitons that we are considering are at very long wavelength (weakly interacting) and their bound energy can be ignored. Actually with $R_s \gg R_H$ not only the individual interaction between gravitons can be ignored, but also the interaction between one single particle and the rest of the condensate. This is the reason why at this scale there is not any self-sustain for the Bose condensate (indeed the first expression in (3.10) is very small in this regime). If we now simply divide the total gravitational energy (3.10) for the typical quantum of energy $\hbar/\lambda$, recalling that $\lambda \sim R_s$, expression (3.9) is obtained.
This result remains valid also for $R_s \sim R_H$ where the interaction between the single graviton and the rest of the condensate is not negligible anymore. Indeed, as long $R_s \gg l_p$ the interactions between individual gravitons are still small, and the total gravitational energy can still be approximated as sum of single particle energies. From the expression (3.9) the criteria of classicality can be written as

$$N \gg 1 \tag{3.11}$$

A configuration is classical when it has many gravitons. We notice that based on (3.10), if $R_s < R_H$ the gravitational energy would exceed the total mass. In this setting the

physical size cannot be less than the gravitational radius. For a Black Hole we have $R_s = R_H$ and the gravitational energy is the whole mass of the object. Therefore, for fixed M and R_s the Black Hole is the source that maximizes N given by the (3.9). In this sense Black Holes are the most classical sources.
Given the above definitions and results we can write

$$N \sim \frac{R_H^2}{l_p^2} \sim \frac{l_p^2}{l_c^2} \sim \frac{M^2}{m_p^2} \tag{3.12}$$

In the classical limit we expect a divergent N because in classical physics the number of quanta for any field is infinite. In fact the limit $\hbar \to 0$ gives us vanishing values for l_p, l_c, m_p and consequently a divergent occupation number.
In the Black Hole case we have $\lambda \sim R_s = R_H$. Considering (3.2) and (3.1) the occupation number (3.9) becomes

$$N = \frac{\lambda^2}{l_p^2} \equiv \alpha_{gr}^{-1} \tag{3.13}$$

This result is quite remarkable: the measure of classicality is given by the inverse of the quantum coupling constant. Exactly as we expected, the weaker the quantum effects are the more the system has to be considered "classical".
Since a Black Hole maximizes the occupation number for a fixed size $R_s \sim R_H$, if we try to increase the N of a Black Hole we would result in increasing his size. The typical wavelength λ of the gravitons would increase by a factor $\sqrt{N}$, as it can be easily recovered by (3.13).
We conclude this section considering the gravitational field for a non relativistic electron. The occupation number is

$$N = \frac{m_e^2}{m_p^2} \sim 10^{-44} \tag{3.14}$$

This means that an electron does not contains any quanta of graviton and it cannot be considered as a gravitational source.

3.3 No-hair theorem

The occupation number N is independent from the composition of the source, it only depends on its mass. Equation (3.10) shows how gravitational self-energy decreases with the physical radius R_s. Therefore, for increasing values of the physical dimension the wavelength of the gravitons grows because their energy decreases. The effects of the gravitons on the dynamics become irrelevant at large scales. On the contrary when R_s approaches R_H gravitation becomes dominant. When a source approaches its gravitational radius it becomes an N-particle state condensate. N only depends on the energy of the center of mass, and it is the same for classical sources with many long wavelength quanta, $N_{source} \gg 1$, than for quantum sources with few short wavelength quanta, $N_{source} \sim 1$. If we consider as an example the scattering of two particles with very high center of mass energy M, the whole state is described by a many particles state with $N_{source} = 2$ and

N soft gravitons.
The idea is that the source can be quantum or classical but the Black Hole characteristics depend only on N. This idea does not apply to different bosonic fields that are not generated by energy. In the scattering of two charged particles the occupation number of photons is of the order of the fine structure (the coupling constant of the electromagnetic field $\alpha \sim 1/137$). Therefore, the elementary particles generated by the electromagnetic field (e.g. photons) do not represent classical states, their occupation number is less than one.
We know that all gravitational sources are composed of particles. We assume the contact interactions between these particles to be weak, but their summed effects are strong enough to keep a bound state for the source. The minimal energy for a single quantum of the source is given be the *Uncertainty Principle*

$$E = \frac{\hbar}{R_s} \tag{3.15}$$

Now it is trivial to say that the maximal occupation number of quanta for the source is its the total energy of the source M, over the energy of the single quantum (3.15). Therefore

$$N_{source}^{max} = M\frac{R_s}{\hbar} \tag{3.16}$$

It is worth notice that this number correspond to soft quanta of the source. As already mentioned, from the point of view of the Black Hole structure the situation would have been equivalent with fewer quanta for the source but with higher energies. Whatever the occupation number N_{source} is it will be below N_{source}^{max}. We can now re-elaborate one of the conclusions of the previous section. As long as $R_s \gg R_H$ the number of of quanta of the source overcomes the gravitons, as can be easily seen confronting (3.9) with (3.16). This is not true anymore when $R_s \sim R_H$, where the gravitons dominates over the quanta of the source. At this scales any source becomes classical, even if it was quantum originally. When $R_s \sim R_H$ the source is a gravitational self-sustained Bose condensate of N weakly interacting gravitons, from which we are able to extrapolate all quantum effects of Black Holes without any geometrical notion.

3.4 Quantum Black Holes

Recalling (3.13) we can write the wavelength of the gravitons that form the Black Hole

$$\lambda = \sqrt{N} l_p \tag{3.17}$$

and their coupling constant

$$\alpha_{gr} = \frac{1}{N} \tag{3.18}$$

Since they are weakly interacting we write the total mass of this bound state as

$$M = \sum_{\lambda} N_\lambda \hbar \lambda^{-1} = N\hbar\lambda^{-1} = \sqrt{N}\frac{\hbar}{l_p} \tag{3.19}$$

which is the sum of the energy of every single quanta. At divergent N the wavelength gives a good approximation of the dimension of the Black Hole, whose horizon scales as $R_H^2 \sim \lambda^2 = N l_p^2$. It seems that this surface is a collection of Planck cells. This should not be mistaken as a support for the Planck scale limit. As we have said no geometrical objects are introduced in this picture, not even an horizon. The Black Hole is simply a collection of N weakly interacting gravitons. This bound state, that exists for arbitrarily large N, is always characterized by the maximal possible occupation number. Therefore, this state is leaky since the escape energy is just above the energy of the quanta, and we can prove it defining an escape wavelength and show that is equal to the Black Hole one. The fact that the Black Hole state is leaky will play a fundamental role to recover the Hawking radiation.

We focus on objects with occupation number $N \gg 1$ composed by gravitons with large wavelength λ. We can write the interaction strength between a pair of gravitons using the (3.1) as

$$\hbar\alpha_{gr} = \hbar^2 \frac{G_N}{\lambda^2} \tag{3.20}$$

The effective potential of the collective attraction on each graviton is

$$V(r)|_{r \gtrsim \lambda} = \hbar\alpha_{gr} N \frac{1}{\lambda} \tag{3.21}$$

The wave equation for the condensate might be approximated, given the weak interaction, by

$$\Psi = \prod_i^N \psi_i \tag{3.22}$$

where ψ_i is the solution of a single graviton Schrödinger equation with the collective potential. The expression (3.21) reaches the maximal value for $r = \lambda$. Then we write the escape energy as

$$E_e = \frac{\hbar}{\lambda_e} = \hbar\alpha_{gr} N \frac{1}{\lambda} \tag{3.23}$$

If we now set the escape wavelength as the one that saturates the condensate, i.e. $\lambda = \lambda_e$ and we plug in the last expression the (3.20) we get

$$\frac{\hbar}{\lambda_e} = \hbar^2 \frac{G_N N}{\lambda_e^3} \tag{3.24}$$

which results into

$$\lambda = \lambda_e = \sqrt{N\hbar G_N} = \sqrt{N} l_p \tag{3.25}$$

The escape wavelength is equal to the Black Hole wavelength (3.17). Equivalently, substituting this expression for the wavelength in (3.20) and (3.23) we get the relations (3.18) and (3.19) for the Black Hole state.

N gravitons of wavelength $\lambda = \sqrt{N} l_p$ form a quantum leaky bound Black Hole state for arbitrarily large N.

3.5 Hawking radiation

The Hawking radiation of a Black Hole originates from an effect similar to the quantum depletion in Bose condensates. This phenomenon happens because even at zero temperature there are always some particles with energy above the ground state in a condensate. In the Black Hole, as we have seen, the escape energy is slightly above the energy of the weakly interacting gravitons. Since the only characteristic of the Black Hole is N, it is obvious that after the emission of a graviton the resulting condensate with occupation number $N-1$ is still a Black Hole. The process can be easily simplified considering the binding energy

$$E_e = \frac{\hbar}{\sqrt{N} l_p} \tag{3.26}$$

which is obtained from (3.23) substituting the expressions for a Black Hole (3.18) and (3.17). The escaping graviton gains enough energy to exceed this binding value thanks to scattering processes with the gravitational potential.
The most probable process is a $2 \to 2$ scattering where one acquires the sufficient energy level to escape. Since both of these gravitons have wavelength $\lambda = \sqrt{N} l_p$ their energy will be small and consequently also the transferred momentum will be small. The rate for this process in a Bose condensates is [19]

$$\Gamma = \frac{1}{N^2} N^2 \frac{\hbar}{\sqrt{N} l_p} \tag{3.27}$$

The first factor $1/N^2$ is the square of the interaction strength, the N^2 is a combinatoric term, and the final $\frac{\hbar}{\sqrt{N} l_p}$ is the energy involved in the process. The characteristic time of emission is

$$\Delta t = \frac{\hbar}{\Gamma} \tag{3.28}$$

and the energy emitted is

$$\Delta M = -\frac{\hbar}{\lambda} = -\frac{\hbar}{\sqrt{N}} \tag{3.29}$$

These two results can be combined to obtain

$$\frac{dM}{dt} = -\frac{\Gamma}{\sqrt{N} l_p} = -\frac{\hbar}{N l_p^2} \tag{3.30}$$

We now define the temperature for the Black Hole

$$T \equiv \frac{\hbar}{\sqrt{N} l_p} \tag{3.31}$$

The Hawking evaporation rate is easily recovered

$$\frac{dM}{dt} = -\frac{T^2}{\hbar} \tag{3.32}$$

We underline that the property of a negative heat-capacity is immediately integrated in our model given the dependence of T on N. Indeed as evaporation proceeds the value of N

decreases and T grows. This means that as the Black Hole looses energy its temperature increases.
The Hawking result has been restored without the use of any geometrical concept and without exceeding the energy limit of the theory going at trans-planckian values.

3.6 Entropy

The final aspect of the Black Hole theory that we want to re-introduce within this Bose condensate frame is the notion of entropy. We are going to see how to look at the occupation number N as the entropy of the Black Hole. The approach is limited to an estimates of the magnitude, the derivation of the exact coefficients is excluded.
The only possible interpretation of entropy that can be given in this condensate model is in a Boltzmann fashion. The entropy is linked to the number of possible quantum states in which N gravitons can exist. If the gravitons are non-interacting and indistinguishable the number of possible states would have gone with N^α where α is the number of states for a single graviton. However, we know that gravitons do interact, even if weakly, and the possible states exponentially grows with N. Because of this weak interaction the wave function of the whole condensate can be viewed as a product of wave functions of distinguishable *flavours*. The total number of possible states can be seen as the product of states for each flavour

$$n_{states} = \prod_j \xi_j \tag{3.33}$$

where $j = 1...N_{flavours}$ labels the flavours and ξ_j is the number of states for each of them. The possibility of having these different flavoured gravitons arises only if the condensate is a bound state of sub-condensates. Each of these sub-condensates has the same properties of the main one, they are a leaky Bose condensate. For any occupation number it is possible to adjust a wavelength in order to match the escape one. Then N_j constituents can form a union of wavelength $\sqrt{N_\alpha} l_p$ and energy $M_\alpha = \sqrt{N_\alpha}\hbar/l_p$. The flavour is a set of $\alpha = 1...n_j$ unions. These unions form a bound state of the mass of the Black Hole

$$M = \sum_{a=1} M_a \tag{3.34}$$

At the leading order the unions must also satisfy

$$N = \sum_{a=1} N_a \tag{3.35}$$

Here the index a runs from 1 to all the unions in all the flavours. The wave functions of these flavours are orthogonal (at the leading order of the interaction $1/N$), and then form eigenstates of the Hamiltonian with energy equal to the Black Hole mass. When the number of unions is of order one the number of their constituents clearly is of order $N_a \sim N$. This is the cut-off for the maximal number of flavours , i.e. for the maximal number of possible sets of unions, that grows with N.

The collective Black Hole wave function is given by the direct product of $N_{flavours} \sim N$ non-interacting and distinguishable flavours

$$\Psi_{BH} = \prod_{j}^{N} \psi_j \tag{3.36}$$

Each of the flavour wave function has degeneracy ξ_j. Since all the ψ_j have similar characteristics, they represents a soft bound state of soft sub-bound states with occupation number of order N, there must be a number of possible states ξ which is a good approximation for each ξ_j. Then we can write

$$n_{states} = \prod_{j}^{N} \xi_j \sim \xi^N \tag{3.37}$$

Now it is straightforward to evaluate a Boltzmann entropy as

$$s \sim \log(n_{states}) \sim N \tag{3.38}$$

The number N indicates the number of microstates of the Black Hole viewed as an N-Bose condensate. That is because the weakly coupled gravitons can form N flavours and each of them has a characteristic degeneracy $\sim \xi$. The number of possible configurations is then ξ^N, there is an exponential scaling with N.

4 Bound states quantization model

We now present in details a new quantization procedure for Black Holes. Instead of trying to reproduce the classical trajectories, as in the semi-classical cases, we look for the bound states of canonical Quantum Mechanics. The main difference with the other models lies in the absence of a bounce for the collapsing shell. As we are going to see the fundamental state is inside the Schwarzschild radius by far, and the formation of the event horizon in then restored. In order to start this quantization procedure we have to choose the variable that we want to quantize. The most logical choice is the areal radius of the ball because we already know the physical meaning of this variable, that gives us the area of the horizon. All the other degrees of freedom are basically excluded for the sake of simplicity. Indeed, this is the only way in which we are able to perform calculations effectively. The exclusion of the vast majority of the degrees of freedom should be reflected in the definition of an entropy for this new Black Hole structure.

4.1 Radius wave function

The radial motion ($L_m = 0$) in the Schwarzschild space is governed by the geodesic equation (1.6)

$$\left(\frac{dR}{ds}\right)^2 + 1 - \frac{2G_N M_0}{R} = \frac{E_m^2}{m^2} \tag{4.1}$$

We apply this equation to a collapsing ball of dust with total mass M, in order to study the motion of an external shell with $m = \epsilon M$. The ADM mass is $M_0 = (1-\epsilon)M$ [2]. Plugging the usual definition of the momentum

$$P_M = m\left(\frac{dR}{ds}\right) \tag{4.2}$$

the geodesic equation (4.1) takes the form

$$H_\epsilon \equiv \frac{P_M^2}{2\epsilon M} - \frac{G_N \epsilon(1-\epsilon)M^2}{R} = \frac{\epsilon M}{2}\left(\frac{E_m^2}{\epsilon^2 M^2} - 1\right) \tag{4.3}$$

We now switch to quantum physics defining

$$\widehat{P}_M = -i\hbar\nabla_R \tag{4.4}$$

which allows us to write the (4.3) as a Schrödinger equation $\widehat{H}_\epsilon \Phi = \mathcal{E}_\epsilon \Phi$. Indeed

$$\widehat{H}_\epsilon \Phi = -\frac{\hbar^2}{2\epsilon M}\nabla_R^2 \Phi - \frac{G_N \epsilon(1-\epsilon)M^2}{R}\Phi = \frac{\epsilon M}{2}\left(\frac{E_m^2}{\epsilon^2 M^2} - 1\right)\Phi = \mathcal{E}_\epsilon \Phi \tag{4.5}$$

Dividing the wave function Φ into a part dependent on R, one dependent on θ, and one dependent on ϕ (the usual spherical variables) we get

$$\Phi(R, \theta, \phi) = \Psi(R)\Omega(\theta)\Gamma(\phi) \tag{4.6}$$

We can recast the Schrödinger equation (4.5) as:

$$\frac{\partial^2 \Psi}{\partial R^2} + \frac{2}{R}\frac{\partial \Psi}{\partial R} + \frac{2M}{\hbar^2}\left(\epsilon \mathcal{E}_\epsilon + \frac{\epsilon^2(1-\epsilon)M^2 G_N}{R}\right)\Psi - \frac{l(l+1)}{R^2}\Psi = 0 \tag{4.7}$$

which is solved (see Appendix B for the details) by

$$\Psi_{\bar{n}}(R) = \sqrt{\frac{\epsilon^6(1-\epsilon)^3 M^9}{\pi \bar{n}^5 m_p^9 l_p^3}}\, e^{-\dfrac{\epsilon^2(1-\epsilon)RM^3}{\bar{n} m_p^3 l_p}} L^1_{\bar{n}-1}\left(\frac{2\epsilon^2(1-\epsilon)RM^3}{\bar{n} m_p^3 l_p}\right) \tag{4.8}$$

with an integer $\bar{n} > 1$.
L_q^p are the *Laguerre Polynomials*. This wave function describes, in a quantum mechanics formalism, the behaviour of the external shell of mass m during a gravitational collapse where the rest of the total mass is the ADM mass. In the solution of the Schrödinger equation we have followed a process similar to the quantization of an Hydrogen-like atom. Indeed, we have a principal quantum number $\bar{n} > 1$ and an orbital quantum number l. Since we have to represent a spherically symmetric situation it is reasonable to set $l = 0$ in order to have spherical orbitals for our shell. By looking at the (4.7), it is evident that the choice of setting $l = 0$ has been made only at the end of the procedure, in order to have a more flexible solution of the wave equation. We will show that if $l = 0$ is substituted in the Schrödinger from the beginning an equivalent result is obtained.
The principal quantum number can be written as (B)

$$\bar{n} = N_M + n \tag{4.9}$$

and for the fundamental state of the shell we have

$$\bar{n}_0 = N_M = \epsilon(1-\epsilon)\left(\frac{M}{m_p}\right)^2 \tag{4.10}$$

which is an extremely big number (order of 10^{76} if we take for M a solar mass). This is an important result because it implies that a large number of states are inaccessible for the collapsing shell. Among these states there is also the one with $\langle R \rangle = 0$. The singularity is then covered, as one should expect from an effective quantum gravity theory. The only classical quantity that we can extrapolate from the wave function is the areal radius R. Therefore, we do not identify the singularity through a principal quantum number $\bar{n}$, instead we use the average R corresponding to that $\bar{n}$. Strictly speaking a state with

$R = 0$ is accessible, but the minimum $\bar{n}$ is much bigger than the one corresponding to an average areal radius $\langle R \rangle = 0$. Consequently, the probability that our shell might be found in the singularity is almost zero. In this way we have somehow implemented the cosmic censorship: the singularity is covered since the probability of the shell to be found in it is essentially zero.
The expression of N_M exhibits a dependence on M^2, the same dependence is present for the area of the horizon that gives the famous Beckenstein entropy. The fundamental principal quantum number N_M, and the area of the horizon depend in the same way on the same quantized matter degree of freedom. This is a consistency condition of the Einstein theory, we have restored it in a complete different frame without requesting it a priori. Indeed, plugging the Einstein equation we require that the degrees of freedom of the matter part are also the degrees of freedom of the gravitational part. In our quantization model gravity is represented by the number N_M and it depends on the matter degrees of freedom.
The energy spectrum for this quantization process is (B)

$$E_m^n = \epsilon M \left[1 - \epsilon^2 (1-\epsilon)^2 \left(\epsilon(1-\epsilon) + n \left(\frac{m_p}{M} \right)^2 \right)^{-2} \right]^{1/2} \tag{4.11}$$

The role of the eigenvalues demands some careful considerations. From the classical description we expect E_m^n to be the energy measured by an asymptotic stationary observer. However, his role in the picture of a *Bound States Quantization Model* is far less clear, it seems hard even to give a specific definition to energy. The only well defined quantities are the mass M and the radius R thanks to their physical interpretation. The role of $\mathcal{E}_n$ might be linked to the energy measured by a comoving observer.
Speculations can be made on how Quantum Mechanics prevents what in classical physics would be unavoidable: the collapse into a singularity. Keeping in mind the equivalence with the first quantization of the Hydrogen atom, we recall the proposal by De Broglie: the permitted orbits are those whose length could be divided in an integer number of electron's wavelength [23]. Maybe here a similar argument might be carried. The role of the permitted orbits is taken by the permitted areas, while the role of the electron's wavelength is taken by a characteristic surface measure obtained from the wave function (4.8). It should be possible to divide the permitted areas into an integer number of characteristic area measures. The only information that we can extrapolate from the wave function is the value of the average areal radius of the 2-sphere that corresponds to each $\bar{n}$. Indeed using some results for the *Laguerre Polynomials* [22] we are able to evaluate the average radius corresponding to $\bar{n}$, starting from the (4.8):

$$\langle R_{\bar{n}} \rangle = \langle \Psi_n | R | \Psi_n \rangle = 4\pi \int_0^\infty R^3 |\Psi_n|^2 dR = \frac{3\bar{n}^2 m_p^3 l_p}{2\epsilon^2(1-\epsilon)M^3} \tag{4.12}$$

Dust is matter with no pressure or interactions of any kind except for gravitation. Classically there is no reason for a ball of dust to stop its contraction and maintain a static state. Instead, quantum theory is telling us that in order to have a well defined energy spectrum we will have a fundamental state with a non zero average radius. In particular substituting (4.10) in (4.12) we get

$$\langle R_{N_M}\rangle = \frac{3(1-\epsilon)Ml_p}{2m_p} = \frac{3}{4}(1-\epsilon)R_H \tag{4.13}$$

We have an upper limit for the compactness

$$\frac{R_H}{\langle R_{\bar{n}}\rangle} \leq \frac{4}{3(1-\epsilon)} \tag{4.14}$$

We now evaluate the quantum of the Hamiltonian. From the classical point of view they are the differences between energy levels measured by a comoving observer. In the quantization process (see Appendix B for the details) we have been able to show that

$$\mathcal{E}_{\bar{n}} = -\frac{M^5\epsilon^3(1-\epsilon)^2}{2m_p^4\bar{n}^2} \tag{4.15}$$

so we can write

$$\begin{aligned}\delta H = |\mathcal{E}_{\bar{n}+1} - \mathcal{E}_{\bar{n}}| = \left|-\frac{M^5\epsilon^3(1-\epsilon)^2}{2m_p^4}\left[\frac{1}{(N_M+n+1)^2} - \frac{1}{(N_M+n)^2}\right]\right| = \\ = \left|-\frac{M^5\epsilon^3(1-\epsilon)^2}{2m_p^4}\left[\frac{N_M^2+n^2+2nN_M-N_M^2-n^2-1-2N_Mn-2n-2N_M}{(N_M^2+n^2+1+2N_Mn+2n+2N_M)(N_M^2+n^2+2nN_M)}\right]\right|\end{aligned} \tag{4.16}$$

and if we now consider $0 \leq n \ll N_M$ and keep only the leading power of N_M we can approximate:

$$\delta H \simeq \frac{M^5\epsilon^3(1-\epsilon)^2}{2m_p^4}\left(\frac{2N_M}{N_M^4}\right) = \frac{M^5\epsilon^3(1-\epsilon)^2 m_p^6}{m_p^4\epsilon^3(1-\epsilon)^3M^6} \tag{4.17}$$

Finally we write

$$\delta H \simeq m_p\frac{m_p}{(1-\epsilon)M} \tag{4.18}$$

In this section we have presented a way to obtain a wave function for a dust shell starting from a classical geodesic equation used as Schrödinger equation. We have chosen a vanishing classical angular momentum L_m since the beginning. The expected spherically symmetric situation allowed us to set $l=0$. In order to perform a more general derivation, we have kept a generic l in the quantization and only at the end we make it to vanish. Here we want to show that if $l=0$ from the beginning the final result is equivalent.
We start from the Schrödinger equation (4.7) with $l=0$

$$\frac{\partial^2\Psi}{\partial R^2} + \frac{2}{R}\frac{\partial\Psi}{\partial R} + \frac{2M}{\hbar^2}\left(\epsilon\mathcal{E}_\epsilon + \frac{\epsilon^2(1-\epsilon)M^2G_N}{R}\right)\Psi = 0 \tag{4.19}$$

and after the definition

$$\Psi \equiv \frac{\psi}{R} \tag{4.20}$$

we obtain (see Appendix C for the details) as a solution

$$\psi_{\bar{n}}(x) = Ce^{-x/2}xF[1-\bar{n}, 2, x] \tag{4.21}$$

F is an *Hypergeometrical function*, C is an integration constant, while $\bar{n}$ and x have the same definitions of the case for a generic l: $\bar{n}$ is an integer bigger than 1, while

$$x \equiv 2R\frac{\epsilon^2(1-\epsilon)M^3}{m_p^3 l_p \bar{n}} \tag{4.22}$$

We are going to show, thanks to the relation between *Hypergeometrical function* and *Laguerre Polynomials* [23], that this solution is completely equivalent to the one obtained before. We have that

$$L_p^q(x) = \binom{p+q}{p} F[-p, q+1, x] \tag{4.23}$$

In our solution p and q take values

$$p = \bar{n} - 1 \tag{4.24}$$

$$q = 1 \tag{4.25}$$

and then we can simply substitute

$$\binom{\bar{n}}{\bar{n}-1} F[1-\bar{n}, 2, x] = L_{\bar{n}-1}^1(x) \tag{4.26}$$

The solution (4.21) can be written as

$$\psi_{\bar{n}}(x) = \frac{C}{(\bar{n})} e^{-x/2} x L_{\bar{n}-1}^1(x) \tag{4.27}$$

Now replacing x, returning to Ψ, and doing the same normalization process done in the previous case (see Appendix B for the details), we get exactly the result (4.8):

$$\Psi_{\bar{n}}(R) = \sqrt{\frac{\epsilon^6(1-\epsilon)^3 M^9}{\pi \bar{n}^5 m_p^9 l_p^3}}\, e^{-\frac{\epsilon^2(1-\epsilon)RM^3}{\bar{n} m_p^3 l_p}} L_{\bar{n}-1}^1\left(\frac{2\epsilon^2(1-\epsilon)RM^3}{\bar{n} m_p^3 l_p}\right) \tag{4.28}$$

We have shown that if we set $l = 0$ from the very beginning of the quantization process the final results for the wave function, and for all the observables do not change. The term containing l in (4.7) comes from solving the angular part of the Schrödinger equation. Since we wanted to study the radial motion of the shell we set $L_m = 0$. Classically we expect a spherically symmetric situation which is perfectly compatible with the choice of $l = 0$. If we want to analyze a more general case, where the spherical symmetry of the shell might be broken by a non-zero angular momentum, we should keep $l \neq 0$ in order to obtain non-spherical orbitals. To do so we have to stick with the first quantization approach.

4.2 Probability distribution

This quantization process produced as wave function the expression (4.8) that contains the *Laguerre Polymomials*

$$\Psi_{\bar{n}}(R) = \sqrt{\frac{\epsilon^6(1-\epsilon)^3 M^9}{\pi \bar{n}^5 m_p^9 l_p^3}}\, e^{-\frac{\epsilon^2(1-\epsilon)RM^3}{\bar{n} m_p^3 l_p}} L_{\bar{n}-1}^1\left(\frac{2\epsilon^2(1-\epsilon)RM^3}{\bar{n} m_p^3 l_p}\right) \tag{4.29}$$

where

$$L^1_{\bar{n}-1}(x) = \frac{e^x x^{-1}}{(\bar{n}-1)!}\frac{d^{\bar{n}-1}}{dx^{\bar{n}-1}}(e^{-x}x^{\bar{n}}) \tag{4.30}$$

From the wave function we are able to evaluate the probability that the dust shell after the collapse is inside the horizon. This probability is calculated via the integral

$$P(R < R_H) = \int_0^{R_H} \mathcal{P}_{\bar{n}}(R)dR = 4\pi \int_0^{R_H} |\Psi_{\bar{n}}(R)|^2 R^2 dR \tag{4.31}$$

which basically is the probability that $R < R_H$ where the latter is the Schwarzschild radius $2G_N M$. The shape for $\mathcal{P}_{\bar{n}}$ presents a number of zeros (and peaks) equals to $\bar{n}$ for the first values of R due to the presence of $L^1_{\bar{n}-1}$, and then an exponential decrease. The peak heights increase with R. Some examples of plots for the distribution $\mathcal{P}_{\bar{n}}$ are shown in Fig.4.1-4.3. When we try to plot the case of a collapsing ball of dust with mass similar to the Sun problems arise. In this case the smallest $\bar{n}$ is $N_M \sim 10^{76}$ and it is impossible to find a numerical result for such a large number. We now try to evaluate analytically how much the last peak for the fundamental state is close to $\langle R_{N_M} \rangle$, the closer they are the more relevant the last peak is compared to the previous ones. We expect to find the last peak near to the average value of R, but still on its right given the asymmetrical shape of the distribution density.
We know that when $n \to \infty$ the zeros of the *Laguerre Polynomials* L^α_n can be expressed in terms of the zeros for the *Bessel function* J_α. In particular [24]

$$\lim_{n\to\infty} \nu \lambda^\alpha_{(n,k)} = j^2_{(\alpha,k)} \tag{4.32}$$

$\lambda^\alpha_{(n,k)}$ is the zero of the *Laguerre* of order $k = 1, 2, ..., n$ while $j_{(\alpha,k)}$ is the zero of the *Bessel* of the same order. Also we have

$$\nu = 4n + 2\alpha + 2 \tag{4.33}$$

In our case

$$n = \bar{n} - 1 = N_M - 1 \tag{4.34}$$

$$\alpha = 1 \tag{4.35}$$

$$\nu = 4(\bar{n}-1) + 4 = 4\bar{n} = 4N_M \tag{4.36}$$

The *Bessel function* J_1 can be written as [23]

$$J_1 = \frac{1}{x^2}\sin x - \frac{1}{x}\cos x \tag{4.37}$$

J_1 vanishes when

$$f(x) = \tan x - x = 0 \tag{4.38}$$

We now look at the plot of $f(x)$ in Fig.4.4. The distance between two zeros of this function is always bigger than π. It can never be less or equal because the x-distance

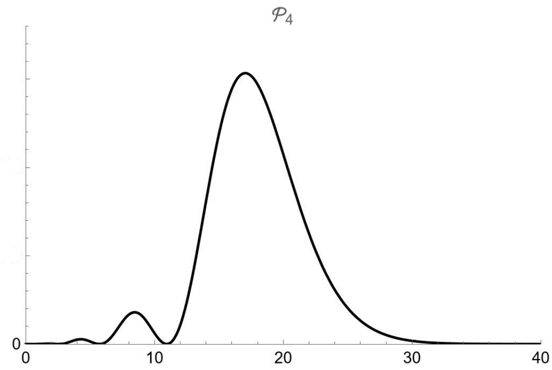

Fig. 4.1
Distribution $\mathcal{P}_{\bar{n}}$ with $\bar{n} = 4$. The first peak cannot be observed at this scale.

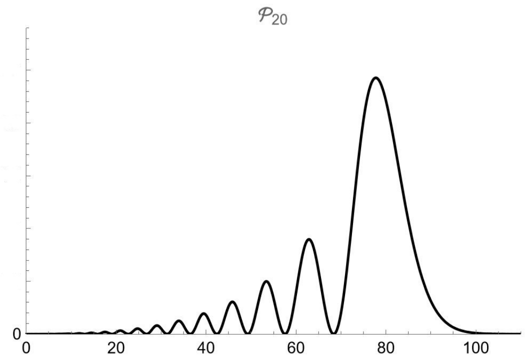

Fig. 4.2
Distribution $\mathcal{P}_{\bar{n}}$ with $\bar{n} = 20$. The first peaks cannot be observed at this scale.

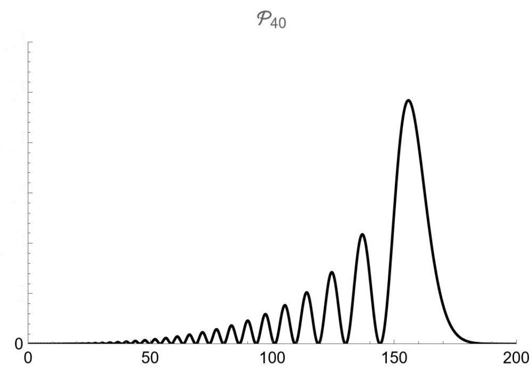

Fig. 4.3
Distribution $\mathcal{P}_{\bar{n}}$ with $\bar{n} = 40$. The first peaks cannot be observed at this scale.

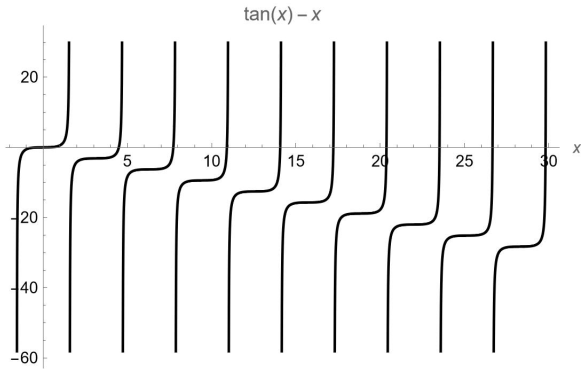

Fig. 4.4
Plot of the function $f(x) = \tan x - x$. The zeros tend to occupy positions at $N\dfrac{\pi}{2}$ where N is odd.

between the horizontal flexes is always π. Also it can never be bigger than $(3/2)\pi$. Then we can write

$$j_{(1,1)} - j_{(1,0)} = \pi + \sigma^1 \tag{4.39}$$
$$j_{(1,2)} - j_{(1,1)} = \pi + \sigma^2 \tag{4.40}$$
$$\dots \tag{4.41}$$
$$j_{(1,N_M)} - j_{(1,N_M-1)} = \pi + \sigma^{N_M} \tag{4.42}$$

We notice that the value of σ^k decreases with k

$$\frac{\pi}{2} > \sigma^1 > \dots\dots > \sigma^{N_M} \gtrsim 0 \tag{4.43}$$

Basically the distance between two zeros of the *Bessel function* tends to π. It seems fair to consider a finite convergence value for the series:

$$\beta^n = \frac{(\sigma^1 + \dots\dots + \sigma^n)}{n} \tag{4.44}$$

Now we can write (recalling that $j_{1,0} = 0$)

$$j_{(1,N_M-1)} = (j_{(1,N_M-1)} - j_{(1,N_M-2)}) + (j_{(1,N_M-2)} - j_{(1,N_M-3)}) + \dots\dots = (N_M - 1)(\pi + \beta^{N_M-1}) \tag{4.45}$$
$$j_{(1,N_M)} = (j_{(1,N_M)} - j_{(1,N_M-1)}) + (j_{(1,N_M-1)} - j_{(1,N_M-2)}) + \dots\dots = (N_M)(\pi + \beta^{N_M}) \tag{4.46}$$

and since σ^{N_M} is small we can say $\beta^{N_M-1} \approx \beta^{N_M}$. From these expressions it is possible to determine the last two zeros of $L^1_{N_M-1}(x)$ through (4.32)

$$\lambda^1_{(N_M-1,N_M-1)} = \frac{j^2_{(1,N_M-1)}}{4N_M} = \frac{(N_M-1)^2}{4N_M}(\pi + \beta^{N_M})^2 \tag{4.47}$$
$$\lambda^1_{(N_M-1,N_M)} = \frac{j^2_{(1,N_M)}}{4N_M} = \frac{N_M^2}{4N_M}(\pi + \beta^{N_M})^2 \tag{4.48}$$

We now want to evaluate the position of the last peak and see if our $\langle R_{N_M}\rangle$ brings us close to it. The last peak is to the right of the last zero. The peak widths tend to be equal asymptotically, we shall take the width of the next-to-last peak, divide it by two, and add this to the position of the last zero in order to find the last peak. We start by evaluating the width of the next-to-last peak

$$l_{N_M} = \lambda^1_{(N_M-1,N_M)} - \lambda^1_{(N_M-1,N_M-1)} = \frac{1}{2}(\pi + \beta^{N_M})^2 \tag{4.49}$$

now we divide this by two, and then we add the position of the last zero in order to find the last peak

$$\gamma_{N_M} = \frac{l_{N_M}}{2} + \lambda^1_{(N_M-1,N_M)} = (N_M+1)\frac{(\pi+\beta^{N_M})^2}{4} \tag{4.50}$$

Now we want to compare this to the value of the argument of $L^1_{N_M-1}$ evaluated with the (4.12)

$$\langle R_{N_M}\rangle = \frac{3m_p^3 l_p}{2\epsilon^2(1-\epsilon)M^3}N_M^2 \tag{4.51}$$

which, using (4.29), is

$$\langle x_{N_M}\rangle = \frac{2\epsilon^2(1-\epsilon)M^3}{m_p^3 l_p N_M}\langle R_{N_M}\rangle = 3N_M \tag{4.52}$$

We want γ_{N_M} as close as possible to $\langle x_{N_M}\rangle$ in order to have the value of the average radius $\langle R_{N_M}\rangle$ in the vicinity the radius for the last peak of the distribution.
Actually with

$$\gamma_{N_M} \gtrsim \langle x_{N_M}\rangle \;\;\Rightarrow\;\; \beta^{N_M} \gtrsim 2\sqrt{3} - \pi \approx 0.32 \tag{4.53}$$

we obtain that the radius of the last peak is slightly bigger than $\langle R_{N_M}\rangle$, which is exactly what we want.
A first numerical analysis of few values of the series σ^k clearly shows that $\beta^{N_M} \ll 1$ which allows us to say that the last peak is indeed "close" to $\langle R_{N_M}\rangle$.
Given the relation between zeros of the *Laguerre Polynomials*, and the zeros of the *Bessel functions* one may speculate over the conditions for a more extended equivalence between those expressions (see Appendix D for the details).

4.3 Available states

We now want to give a rough estimate for the number of possible states that can be occupied inside a Black Hole formed after the collapse of a ball of dust. We know, from the previous section, that the probability density $4\pi|\Psi_{\bar{n}}(R)|^2$ will be highly peaked on the value $\langle R_{\bar{n}}\rangle$. We start by evaluating the number of states $\bar{n}_H$ corresponding to an average radius that is the Schwarszchild radius R_H using (4.12)

$$\langle R_{\bar{n}_H}\rangle = 2G_N M = \frac{3\bar{n}_H^2 m_p^3 l_p}{2\epsilon^2(1-\epsilon)M^3} \tag{4.54}$$

Recalling

$$l_p = \sqrt{\hbar G_N} \tag{4.55}$$

$$m_p = \sqrt{\frac{\hbar}{G_N}} \tag{4.56}$$

$$N_M = \epsilon(1-\epsilon)\left(\frac{M}{m_p}\right)^2 \tag{4.57}$$

we obtain

$$2G_N M = \frac{3\bar{n}_H^2 \hbar^2}{2\epsilon^2(1-\epsilon)M^3 G_N} \tag{4.58}$$

$$\bar{n}_H^2 = \frac{4\epsilon^2(1-\epsilon)M^4 G_N^2}{3\hbar^2} \tag{4.59}$$

$$\bar{n}_H^2 = \frac{4}{3(1-\epsilon)} N_M^2 \tag{4.60}$$

We are able to evaluate the number of available states between the one that corresponds to the Schwarzschild radius $\bar{n}_H = n_H + N_M$, and the fundamental one $\bar{n}_0 = N_M$:

$$n_H = \bar{n}_H - \bar{n}_0 = \frac{2}{\sqrt{3(1-\epsilon)}} N_M - N_M = \left(\frac{2-\sqrt{3(1-\epsilon)}}{\sqrt{3(1-\epsilon)}}\right) N_M \tag{4.61}$$

We will show that exists a preferred ratio $\epsilon = 0.5$, for such a value we get an available number of states $n_H \simeq 0.15(M/m_p)^2$. This rough estimate could be used to evaluate the grade of disorder of the Black Hole and calculate its entropy. Indeed, all of these configurations correspond to the same macrostate observed outside: the Black Hole. This is the Boltzmann interpretation of the entropy. Problems might arise considering different forms of matter: dust collapses without losing energy, all the states correspond to the same total M. This allows us to say that the microstates correspond to the same macrostate. For different forms of matter that is not the case anymore, and this interpretation of entropy may not be valid.

4.4 Uncertainty relations

A result of any quantization model must be some sort of *Uncertainty Principle* that links the uncertainty for couples of observables, like energy and time or position and momentum. First we will derive the standard Heisenberg principle using the variance on the areal radius and on the momentum, then in the next section we will minimize the uncertainty for the areal radius to determine a preferred value for the ratio ϵ.
In this section we are going to apply the result of the previous ones to our case. In particular we are going to evaluate the uncertainties for the position and momentum of the collapsing shell.
We report the wave function (4.8), but this time plugging the definition of a particular

radius r_g similar to the *Bohr radius* for the Hydrogen atom

$$r_g \equiv \frac{m_p^3 l_p}{M^3} \tag{4.62}$$

obtaining

$$\Psi_{\bar{n}}(R) = \sqrt{\frac{\epsilon^6(1-\epsilon)^3}{\pi \bar{n}^5 r_g^3}} e^{-\frac{\epsilon^2(1-\epsilon)R}{r_g \bar{n}}} L^1_{\bar{n}-1}\left(\frac{2\epsilon^2(1-\epsilon)R}{r_g \bar{n}}\right) \tag{4.63}$$

In order to work in the momentum space we define the correspondent of r_g. The most natural choice seems to be:

$$\Delta_g \equiv m_p \frac{l_p}{r_g} \tag{4.64}$$

The wave function takes the form:

$$\Psi_{\bar{n}}(P) = \sqrt{\frac{\epsilon^6(1-\epsilon)^3}{\pi \bar{n}^5 \Delta_g^3}} e^{-\frac{\epsilon^2(1-\epsilon)P}{\Delta_g \bar{n}}} L^1_{\bar{n}-1}\left(\frac{2\epsilon^2(1-\epsilon)P}{\Delta_g \bar{n}}\right) \tag{4.65}$$

We use the result (4.12) for the average radius considering the definition (4.62):

$$\langle R_{\bar{n}} \rangle = \langle \Psi_{\bar{n}} | R | \Psi_{\bar{n}} \rangle = 4\pi \int_0^\infty R^3 |\Psi_{\bar{n}}(R)|^2 dR = \frac{r_g 3\bar{n}^2}{2\epsilon^2(1-\epsilon)} \tag{4.66}$$

Using the known results for the *Laguerre polynomials* [22] we can write

$$\langle R^2_{\bar{n}} \rangle = 4\pi \int_0^\infty R^4 |\Psi_n(R)|^2 dR = \frac{r_g^2 \bar{n}^2}{2\epsilon^4(1-\epsilon)^2}(5\bar{n}^2+1) \tag{4.67}$$

The variance takes the form

$$\langle \Delta R^2_{\bar{n}} \rangle = \langle R^2_{\bar{n}} \rangle - \langle R_{\bar{n}} \rangle^2 = \frac{r_g^2}{4\epsilon^4(1-\epsilon)^2}(10\bar{n}^4 + 2\bar{n}^2 - 9\bar{n}^4) = \frac{r_g^2}{4\epsilon^4(1-\epsilon)^2}(\bar{n}^4 + 2\bar{n}^2) \tag{4.68}$$

We now perform the same calculations in the momentum space:

$$\langle P_{\bar{n}} \rangle = \langle \Psi_{\bar{n}} | P | \Psi_{\bar{n}} \rangle = 4\pi \int_0^\infty P^3 |\Psi_{\bar{n}}(P)|^2 dP = \frac{\Delta_g 3\bar{n}^2}{2\epsilon^2(1-\epsilon)} \tag{4.69}$$

$$\langle P^2_{\bar{n}} \rangle = 4\pi \int_0^\infty P^4 |\Psi_{\bar{n}}(P)|^2 dP = \frac{\Delta_g^2 \bar{n}^2}{2\epsilon^4(1-\epsilon)^2}(5\bar{n}^2+1) \tag{4.70}$$

which result in:

$$\langle \Delta P^2_{\bar{n}} \rangle = \langle P^2_{\bar{n}} \rangle - \langle P_{\bar{n}} \rangle^2 = \frac{\Delta_g^2}{4\epsilon^4(1-\epsilon)^2}(10\bar{n}^4 + 2\bar{n}^2 - 9\bar{n}^4) = \frac{\Delta_g^2}{4\epsilon^4(1-\epsilon)^2}(\bar{n}^4 + 2\bar{n}^2) \tag{4.71}$$

Using the definition (4.64) we can invert this last expression

$$r_g^2 = \frac{(\bar{n}^4 + 2\bar{n}^2) m_p^2 l_p^2}{4\epsilon^4(1-\epsilon)^2 \langle \Delta P_n^2 \rangle} \tag{4.72}$$

and finally write the product of the standard variations. We substitute (4.72) in (4.68) and the product of the standard variations becomes [1]

$$\langle \Delta R_{\bar{n}} \rangle \langle \Delta P_{\bar{n}} \rangle = \sqrt{\langle \Delta R^2_{\bar{n}} \rangle \langle \Delta P^2_{\bar{n}} \rangle} = \frac{(\bar{n}^4 + 2\bar{n}^2)\hbar}{4\epsilon^4 (1-\epsilon)^2} \tag{4.73}$$

Evaluating the derivative for this expression with respect to ϵ, and keeping $\bar{n} = N_M = \epsilon(1-\epsilon)\left(\frac{M}{m_p}\right)$, we get:

$$\begin{aligned} &\frac{m_p l_p}{4} \frac{d}{d\epsilon} \left[\frac{\epsilon^4 (1-\epsilon)^4 \left(\frac{M}{m_p}\right)^8 + 2\epsilon^2 (1-\epsilon)^2 \left(\frac{M}{m_p}\right)^4}{\epsilon^4 (1-\epsilon)^2} \right] = \\ &= \frac{m_p l_p}{4} \frac{d}{d\epsilon} \left[(1-\epsilon)^2 \left(\frac{M}{m_p}\right)^8 + \frac{2}{\epsilon^2} \left(\frac{M}{m_p}\right)^4 \right] = \\ &= \frac{m_p l_p}{4} \left[-2(1-\epsilon) \left(\frac{M}{m_p}\right)^8 - \frac{4}{\epsilon^3} \left(\frac{M}{m_p}\right) \right] \end{aligned} \tag{4.74}$$

It is important to notice that this derivative is always negative for values of ϵ between 0 and 1, which is the interval of interest in our case.
The uncertainty (4.73) written as

$$\langle \Delta R_{N_M} \rangle \langle \Delta P_{N_M} \rangle = \frac{m_p l_p}{4} \left[(1-\epsilon)^2 \left(\frac{M}{m_p}\right)^8 + \frac{2}{\epsilon^2} \left(\frac{M}{m_p}\right)^4 \right] \tag{4.75}$$

diverges for $\epsilon \to 0$, while for $\epsilon = 1$ it becomes the standard *Heisenberg Principle*:

$$\langle \Delta R_{N_M} \rangle \langle \Delta P_{N_M} \rangle = \frac{m_p l_p}{2} \left(\frac{M}{m_p}\right)^4 \tag{4.76}$$

The negativity of the derivative and positive value for the uncertainty when $\epsilon = 1$ imply something very important: there are no real values of ϵ that make the expression (4.73) to vanish. This means that what we have found is a good manifestation of the *Uncertainty Principle*. The fact that this uncertainty diverges for $\epsilon \to 0$ suggests that if we want to quantize the collapse of null shells with mass $m = 0$, we have to consider a different starting point, we should not simply make the mass $m = \epsilon M$ to vanish in the Schrödinger equation (4.5).
We conclude this section with a careful look at the variances for R and P for the fundamental state:

$$\bar{n} = N_M = \epsilon(1-\epsilon) \left(\frac{M}{m_p}\right)^2 \tag{4.77}$$

Starting from the expressions (4.68) and (4.71) substituting the definitions (4.62) and (4.64) we get

$$\langle \Delta R_{N_M} \rangle = \sqrt{\langle \Delta R^2_{N_M} \rangle} \sim \frac{r_g}{2\epsilon^2 (1-\epsilon)} N^2_M = \frac{(1-\epsilon) l_p}{2 m_p} M \tag{4.78}$$

[1] Recall that $m_p l_p = \hbar$.

while

$$\langle \Delta P_{N_M} \rangle = \sqrt{\langle \Delta P_{N_M}^2 \rangle} \sim \frac{\Delta_g}{2\epsilon^2(1-\epsilon)} N_M^2 = \frac{(1-\epsilon)}{2m_p^6} M^7 \tag{4.79}$$

Looking to the expression of $\langle \Delta R_{N_M} \rangle$ we see that it scales with M that we can approximate with a solar mass. Therefore, this approach gives us a great value for the uncertainty. In section 4.2 we have shown the average radius is near the last peak of the *Laguerre Polynomials*, the width of the next-to-last peak (4.49) could be used as uncertainty on the position because the width of the peaks tend to be equal for $\bar{n} \gg 1$. The variance on the momentum grows much more rapidly: $\langle \Delta P \rangle \sim M^7$. We could interpret this as an effective quantum pressure that balances the gravitational collapse.

4.5 Effective collapsing mass

The product of the uncertainties $\langle \Delta R_{\bar{n}} \rangle$ and $\langle \Delta P_{\bar{n}} \rangle$ does not show a minimum for any value of ϵ. We are now going to show a procedure based on the minimization only of $\langle \Delta R_{\bar{n}} \rangle$, that might be used to choose a particular value of ϵ. Starting from (4.68)

$$\langle \Delta R_{\bar{n}} \rangle = \sqrt{\langle \Delta R_{\bar{n}}^2 \rangle} = \frac{r_g}{2\epsilon^2(1-\epsilon)} \bar{n} \sqrt{(\bar{n}^2+2)} \tag{4.80}$$

and using (4.66) we can write the expression:

$$\frac{\langle \Delta R_{\bar{n}} \rangle}{\langle R_{\bar{n}} \rangle} = \frac{r_g}{2\epsilon^2(1-\epsilon)} \bar{n} \sqrt{(\bar{n}^2+2)} \frac{2\epsilon^2(1-\epsilon)}{3r_g \bar{n}^2} = \frac{\sqrt{\bar{n}^2+2}}{3\bar{n}} \tag{4.81}$$

We recall that the fundamental state is characterized by

$$\bar{n} = N_M = \epsilon(1-\epsilon) \left(\frac{M}{m_p}\right)^2 = (\epsilon - \epsilon^2) a \tag{4.82}$$

where we have defined $a = \left(\frac{M}{m_p}\right)^2$. The uncertainty expression (4.81) for the fundamental state becomes

$$\frac{\langle \Delta R_{N_M} \rangle}{\langle R_{N_M} \rangle} = \frac{1}{3a} \left[\frac{\sqrt{(\epsilon-\epsilon^2)^2 a^2 + 2}}{\epsilon - \epsilon^2} \right] = \frac{1}{3a} \left[\frac{\sqrt{(\epsilon^2 - 2\epsilon^3 + \epsilon^4) a^2 + 2}}{\epsilon - \epsilon^2} \right] \tag{4.83}$$

We notice that this expression diverges for both $\epsilon = 0, 1$. Performing the derivative with respect to ϵ and setting the result to 0 in order to obtain the minimum we get

$$-\frac{1}{(\epsilon-\epsilon^2)^2}(1-2\epsilon)\sqrt{(\epsilon^2-2\epsilon^3+\epsilon^4)a^2+2} + \frac{(2\epsilon - 6\epsilon^2 + 4\epsilon^3)a^2}{(\epsilon-\epsilon^2)2\sqrt{(\epsilon^2-2\epsilon^3+\epsilon^4)a^2+2}} = 0 \tag{4.84}$$

$$(4\epsilon - 2)[(\epsilon^2 - 2\epsilon^3 + \epsilon^4)a^2 + 2] + \epsilon(1-\epsilon)(2\epsilon - 6\epsilon^2 + 4\epsilon^3)a^2 = 0 \tag{4.85}$$

$$8\epsilon - 4 = 0 \tag{4.86}$$

$$\epsilon = \frac{1}{2} \tag{4.87}$$

We have obtained that for $\epsilon = 0.5$ the uncertainty on the radius is minimized. We can say that the preferred ratio between m and M is 0.5.
It is interesting to notice that the value $\epsilon = 0.5$ minimize the uncertainty on $\langle R_{N_M} \rangle$ and at the same time it maximizes the value of N_M for a fixed M. The more $\langle R_{N_M} \rangle$ is "classical" the more the shell is far from the singularity $\bar{n} = 0$. That seems odd because the classical result predicts the collapse in the singularity.
The minimal uncertainty (4.81) for the fundamental state (4.82) is

$$\langle \Delta R_{N_M} \rangle = \frac{1}{3a} \left[\frac{\sqrt{\left(\frac{1}{2} - \frac{1}{4}\right)^2 a^2 + 2}}{\frac{1}{2} - \frac{1}{4}} \right] \langle R_{N_M} \rangle \simeq \frac{1}{3} \langle R_{N_M} \rangle \tag{4.88}$$

This value for the uncertainty might seems quite big but recalling (4.13)

$$\langle R_{N_M} \rangle = \frac{3}{4}(1 - \epsilon) R_H \tag{4.89}$$

and substituting $\epsilon = \frac{1}{2}$ we obtain

$$\langle R_{N_M} \rangle = \frac{3}{8} R_H \tag{4.90}$$

Now adding this expression for the radius with its uncertainty (4.88)

$$\langle R_{N_M} \rangle + \langle \Delta R_{N_M} \rangle = \frac{4}{3} \langle R_{N_M} \rangle = \frac{R_H}{2} \tag{4.91}$$

It is worth notice that what we are giving here is an overestimation of the possible maximum value for the areal radius R_{N_M} of the collapsing shell. That is because the probability distribution evaluated with $\Psi_{N_M}(R)$, is absolutely not symmetric: is a succession of higher and higher peaks with the last one, the more relevant by far, placed close to the value $\langle R_{N_M} \rangle$. Anyway, the result (4.91) is telling us that it is very unlikely to observe the shell of mass $m = (1/2)M$ outside the Scharzschild radius R_H.

5 Quantum horizon

We now introduce the wave function for an horizon associated to a generic and localized Quantum Mechanics particle. This new object will allow us to identify the uncertainties on the positions of the particle and of the horizon within a *Generalised Uncertainty Principle*. We will apply this construction to the case of a particle wave function described by a guassian wave packet at rest in the Minkowski space-time. The exact probability on the formation of a Black Hole is calculated in this simple model [26].

5.1 Horizon Quantum Mechanics

We start by looking at a generic spherically symmetric space-time

$$ds^2 = g_{ij}dx^i dx^j + R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.1}$$

R is the areal radius, $x^i = (x^1, x^2)$ are coordinates on a surface where θ and ϕ are constant. The horizon is a *most outer trapped surface*, and its definition is based on the escape velocity on it that must be equal to the speed of light. Its position is then determined via the equation [30]

$$g^{ij}\nabla_i R \nabla_j R = 0 \tag{5.2}$$

Once we fix the coordinates $x^1 = t$ and $x^2 = R$, we can write for a spherically symmetric source the condition to be an horizon

$$g^{RR} = 1 - \frac{R_H}{R} = 0 \tag{5.3}$$

where

$$R_H \equiv 2l_p \frac{M(t,R)}{m_p} = 2G_N M(t,R) \tag{5.4}$$

is the Horizon radius. The ADM mass $M(t,R)$ is the total energy enclosed in a two-sphere of radius R. It can be calculated as [2]

$$M(t,R) = \frac{4\pi}{3}\int_0^R \rho(t,\bar{R})\bar{R}^2 d\bar{R} \tag{5.5}$$

where ρ is the energy density. It is very difficult to verify the condition (5.2) in general, but we can say that an horizon exists if there are values of R such that

$$R_H \equiv 2G_N M(t,R) \geqslant R \tag{5.6}$$

When enough energy M is packed in a sufficiently small volume defined by R_H, the effects of gravity on the structure of space-time cannot be neglected, this is the *hoop conjecture* [31]. The energy M is not limited from below, and therefore the horizon radius can be arbitrarily small. The minimal value comes from the uncertainty on the position of a particle of Quantum Mechanics. As we have done in Section 3.2, we define the Compton wavelength as the scale at which quantum effects overcome gravitational ones. We define the Compton wavelength as in (3.6) and using (5.4)

$$l_c \equiv \frac{\hbar}{M} = \frac{l_p m_p}{M} \simeq \frac{l_p^2}{R_H} \tag{5.7}$$

The basic idea, enforced by common experience, that Quantum Mechanics describes the nature better than classical physics immediately brings us to require that

$$R_H \gtrsim l_c \tag{5.8}$$

Given the expressions (5.7), (5.4) and the last inequality we get

$$R_H \gtrsim l_p \tag{5.9}$$

$$M \gtrsim m_p \tag{5.10}$$

5.2 Horizon wave function

We now recover the definition of the horizon wave function. For simplicity we stick with spherically symmetric objects at rest in a chosen reference frame. Then the particle is described by spherical wave function $\psi_S \in L^2(\mathbb{R}^3)$ [26] [1]. We assume decomposition into energy eigenstates

$$|\psi_S\rangle = \sum_E C(E)|\psi_E\rangle \tag{5.11}$$

The sum is the spectral decomposition in Hamiltonian eigenstates

$$\hat{H}|\psi_E\rangle = E|\psi_E\rangle \tag{5.12}$$

We can invert the expression for the Schwarzschild radius (5.4) to express the energy

$$E = m_p \frac{R_H}{2l_p} \tag{5.13}$$

This is the energy corresponding to the horizon. Thus, we now define the un-normalized horizon wave function as [26]

$$\tilde{\psi}_H(R_H) = C\left(m_p \frac{R_H}{2l_p}\right) \tag{5.14}$$

that can be normalized via the scalar product

$$\langle\psi_H|\phi_H\rangle = 4\pi \int_0^\infty \psi_H^*(R_H)\phi_H(R_H)R_H^2 dR_H \tag{5.15}$$

[1] $L^2(\mathbb{R}^3)$ is vector space of all square integrable functions $f : \mathbb{R}^3 \to \mathbb{C}$.

Thanks to this definition we can interpret $|\psi_H\rangle$ as if it yields the probability that an horizon associated to a particle in the quantum state $|\psi_S\rangle$ is detected at R_H. This probability is evaluated as

$$P_H(R_H) = 4\pi R_H^2 |\psi_H(R_H)|^2 \tag{5.16}$$

This is the probability that a sphere with radius R_H is an horizon. Given this pure quantum mechanical definition of the horizon the latter obviously becomes "fuzzy", although it is not clear what does this means experimentally. Defining the probability to find a particle inside a sphere of radius R_H as

$$P_S(R < R_H) = 4\pi \int_0^{R_H} |\psi_S(R)|^2 R^2 dR \tag{5.17}$$

we can compute the probability that a particle with wave function ψ_S, and horizon wave function ψ_H, is actually a Black Hole. First we write the probability to find the particle inside its horizon

$$P_<(R < R_H) = P_S(R < R_H) P_H(R_H) \tag{5.18}$$

Then integrating over all the possible values for R_H we evaluate the probability that the particle is actually a Black Hole

$$P_{BH} = \int_0^\infty P_<(R < R_H) dR_H \tag{5.19}$$

We now analyse a particular example using a gaussian wave packet to exemplify this construction.

5.3 Gaussian particle

We choose a flat space-time with a massive particle described by a spherically symmetric gaussian wave function [26]

$$\psi_S(R) = \frac{e^{-\dfrac{R^2}{2l^2}}}{l^{3/2}\pi^{3/4}} \tag{5.20}$$

We fix the width l as the Compton wavelength (5.7)

$$l = l_c = \frac{l_p m_p}{M} \tag{5.21}$$

As we have already done in Section 4.4, we can switch to the momentum space through the definition of

$$\Delta = \frac{m_p l_p}{l} = M \tag{5.22}$$

and recast the wave equation

$$\psi_S(P) = \frac{e^{-\dfrac{P^2}{2\Delta^2}}}{\Delta^{3/2}\pi^{3/4}} \tag{5.23}$$

Given the choice of sticking with the Minkowski space-time we can use the usual mass-shell relation to define the energy

$$E^2 = P^2 + M^2 \tag{5.24}$$

If we now plug the expression (5.13) for the energy corresponding to the horizon we express the momentum P as

$$P^2 = E^2 - M^2 = m_p^2 \frac{R_H^2}{4l_p^2} - M^2 \tag{5.25}$$

Now substituting this expression in (5.23) we get the unnormalized horizon wave function

$$\tilde{\psi}_H(R_H) = \frac{l^{3/2} e^{-\frac{l^2 M^2}{2l_p^2 m_p^2}} e^{-\frac{l^2 R_H^2}{8l_p^4}}}{\pi^{3/4} l_p^{3/2} m_p^{3/2}} \tag{5.26}$$

Via the normalization defined by (5.15) we define the final horizon wave function [26]

$$\psi_H(R_H) = \frac{l^{3/2} e^{-\frac{l^2 R_H^2}{8l_p^4}}}{2^{3/2} \pi^{3/4} l_p^3} \tag{5.27}$$

From (5.20) the uncertainty for $\psi_S(R)$ is $\langle R^2 \rangle = l^2$ while from (5.27) the one for $\psi_H(R_H)$ is $\langle R_H^2 \rangle \simeq 4l_p^4/l^2$. Clearly in order to have the particle inside its horizon we have to require

$$\langle R_H^2 \rangle > \langle R^2 \rangle \tag{5.28}$$

$$l_p \gtrsim l \tag{5.29}$$

Given the definition (5.21) from the last inequality we get

$$M \gtrsim m_p \tag{5.30}$$

which is the (5.10) that has been derived in a completely quantum mechanical frame. We define the probability for the particle to be at a radius R

$$P_S(R) = 4\pi^2 R^2 |\psi_S(R)|^2 \tag{5.31}$$

We can now plot the probabilities $P_H(R_H)$ (of having an horizon at R_H) and $P_S(R)$ (for a particle to be at radius R) for this particular gaussian case. It seems from Fig.5.1 that for values of $M < m_p$ the most probable position for a massive particle is outside its horizon. The situation changes considering $M > m_p$ as it can be seen in Fig.5.2. In this latter case the greatest probability to find the particle is clearly inside its horizon. These two different behaviours will be reproduced in the probability for the Black Hole formation P_{BH}.

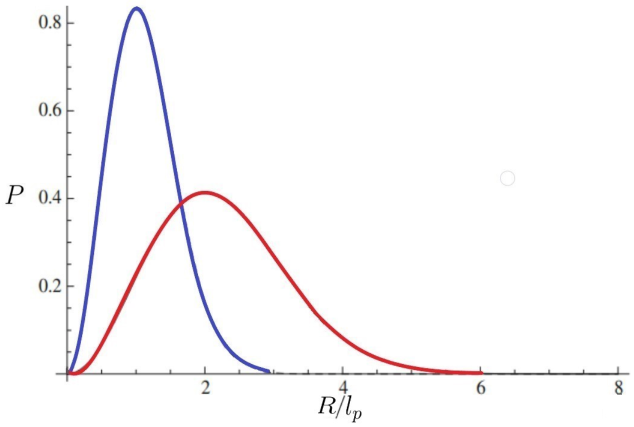

Fig.5.1
Probability $P_H(R_H)$ (blue line) to have an horizon in R_H, and probability $P_S(R)$ (red line) for a particle to be a R. $M = m_p/2$. [26]

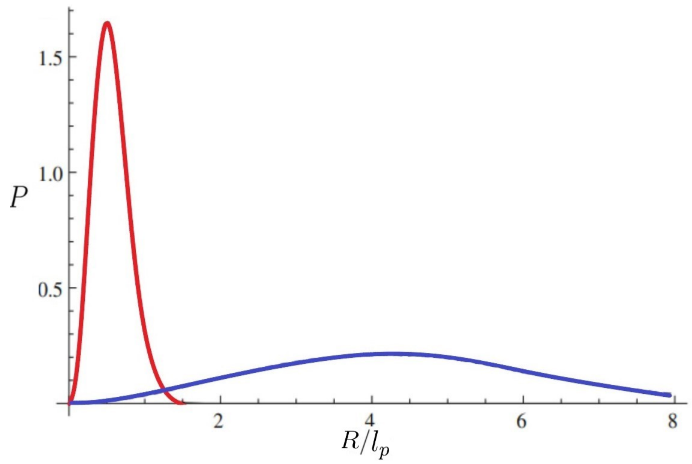

Fig.5.2
Probability $P_H(R_H)$ (blue line) to have an horizon in R_H, and probability $P_S(R)$ (red line) for a particle to be a R. $M = 2m_p$. [26]

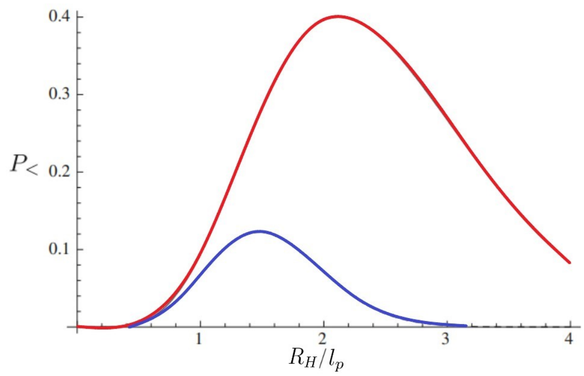

Fig.5.3
Probability $P_<(l)$ for a particle to be inside its own horizon R_H with $l = l_p$ (red line) and $l = 2l_p$ (blue line). [26]

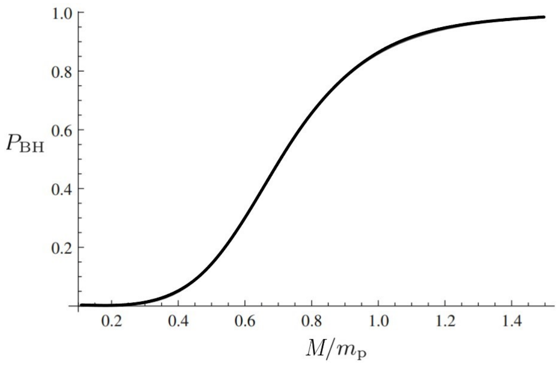

Fig.5.4
Probability $P_{BH}(M)$ that a particle with $l \sim M^{-1}$ is a Black Hole. [26]

We now evaluate the probability (5.18) to find the gaussian particle inside its own horizon [26]

$$P_<(l) = \frac{l^3 R_H^2 e^{-\frac{l^2 R_H^2}{4 l_p^4}}}{2\sqrt{\pi} l_p^6} \left[Erf\left(\frac{R_H}{l}\right) - \frac{2 R_H e^{-\frac{R_H^2}{l^2}}}{\sqrt{\pi} l} \right] \tag{5.32}$$

plotted in Fig.5.3 for two different values of the gaussian width l. It is clear that this probability decreases for increasing l. Given the (5.21) we have $l \sim M^{-1}$, therefore the probability $P_<$ increases for increasing M.
We can find the probability (5.19) of the Black Hole formation for a gaussian particle, in function of l

$$P_{BH}(l) = \frac{2}{\pi}\left[\arctan\left(2\frac{l_p^2}{l^2}\right) + 2\frac{l^2(4 - l^4/l_p^4)}{l_p^2(4 + l^4/l_p^4)^2}\right] \tag{5.33}$$

and in function of M

$$P_{BH}(M) = \frac{2}{\pi}\left[\arctan\left(2\frac{M^2}{m_p^2}\right) + 2\frac{m_p^2(4 - m_p^4/M^4)}{M^2(4 + m_p^4/M^4)^2}\right] \tag{5.34}$$

We conclude this section by describing Fig.5.4 that plots this last expression. We observe that the probability to be a Black Hole increases as the mass of the particle approaches the Planck mass m_p.
All this description has been performed in a flat space-time, therefore any self-gravity effect of the source has been excluded from this picture. This would clearly become a problem for objects with macroscopic masses that this procedure might fail to describe. A different energy relation for curved space-times and suitable modes, rather than gaussian plane waves, could be able to describe situations with $M \gg m_p$.

5.4 Deformed commutators

In this section the origins of a *Generalized Uncertainty Principle* are introduced [25, 26]. It is a well known fact of Quantum Mechanics that the uncertainties on the measurements of position and momentum must fulfill the *Heisenberg Uncertainty Principle*

$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{5.35}$$

which is generated by the Heisenberg algebra

$$[\hat{x}, \hat{p}] = i\hbar \tag{5.36}$$

This last expression is telling us that a generic quantum state $|\psi\rangle$ cannot be an eigenstate of position and momentum at the same time.
Introducing a deformed algebra we obtain a *Generalized Uncertainty Principle* (GUP), whose manifestation is relevant only at energy scales at least of order of the Planck mass m_p. A deformed algebra has a commutator which is not linear in its elements and it is defined through a deformation parameter, with the dimension of a mass, that in some limit will recover the usual Lie Algebra.
Some assumptions are in order [25][2]

1. The rotational group is undeformed:

$$[\vec{J}, J_i] = 0 \tag{5.37}$$
$$[J_i, x_j] = i\epsilon_{ijk}x_k \tag{5.38}$$
$$[J_i, p_j] = i\epsilon_{ijk}p_k \tag{5.39}$$
$$\tag{5.40}$$

2. The translation group is undeformed:

$$[p_i, p_j] = 0 \tag{5.41}$$

3. The commutator $[x, p]$ depends on the deformation parameter k with dimension of a mass.

The most general forms for the commutators are [25]

$$[x_i, x_j] = i\frac{\hbar^2}{k^2}\epsilon_{ijk}J_k a(E) \tag{5.42}$$
$$[x_i, p_j] = i\hbar\delta_{ij}f(E) \tag{5.43}$$

where the i factor is there to assure hermicity of x_i, p_i and J_i. $a(E)$ and $f(E)$ are dimensionless functions of E/k . In order to recover the canonical expression we set $f(0) = 1$. Then for $|k| \to +\infty$ the usual *Heisenberg Principle* is re-obtained. Given the usual expression for the relativistic energy:

$$E^2 = p^2 + m^2 \tag{5.44}$$

[2]Latin indices take values $1, 2, 3$.

it can be shown that [25]

$$[x_i, E] = i\hbar f(E)\frac{p_i}{E} \tag{5.45}$$

and consequently

$$[x_i, a(E)] = \frac{da}{dE}[x_i, E] = i\hbar f(E)\frac{p_i}{E}\frac{da}{dE} \tag{5.46}$$

The forms of a and f are restricted by the *Jacobi identity*:

$$[x_i, [x_j, x_k]] + [x_j, [x_k, x_i]] + [x_k, [x_i, x_j]] = 0 \tag{5.47}$$

Using relations (5.42) we have

$$[x_i, [x_j, x_k]] = i\frac{\hbar^2}{k^2}\epsilon_{jkp}[x_i, J_p a(E)] \tag{5.48}$$

After dropping the factor $i\frac{\hbar^2}{k^2}$ and develop this result into:

$$\begin{aligned} &\epsilon_{jkp}(x_i J_p a(E) - J_p a(E) x_i) = \epsilon_{jkp}(x_i J_p a(E) - J_p a(E) x_i + J_p x_i a(E) - J_p x_i a(E)) = \\ &= \epsilon_{jkp}([x_i, J_p] a(E) + J_p [x_i, a(E)]) = \epsilon_{jkp}\left(-i\epsilon_{pil} x_l a(E) + i\hbar J_p f(E)\frac{p_i}{E}\frac{da}{dE}\right) \end{aligned} \tag{5.49}$$

we notice that the first term vanishes. Indeed, since $i \neq j \neq k$ then $\epsilon_{jkp} = \epsilon_{jki}\delta_{ip}$ and we can write

$$\epsilon_{jkp}\epsilon_{pil} = \epsilon_{jki}\delta_{ip}\epsilon_{pil} = 0 \tag{5.50}$$

This result applied to (5.48) implies

$$[x_i, [x_j, x_k]] = -\frac{h^3 f(E)}{k^2 E}\frac{da}{dE}\epsilon_{jkp} J_p p_i = -\frac{h^3 f(E)}{k^2 E}\frac{da}{dE}\epsilon_{jki}\delta_{ip} J_p p_i \tag{5.51}$$

Now we can rewrite the *Jacobi Identity* (5.47) as

$$\frac{da}{dE}\delta_{ip} J_p p_i + \frac{da}{dE}\delta_{jp} J_p p_j + \frac{da}{dE}\delta_{kp} J_p p_k = \frac{da}{dE}\vec{J}\cdot\vec{p} = 0 \tag{5.52}$$

Since the *Jacobi identity* must be true either we choose a representation with $\vec{J}\cdot\vec{p} = 0$ or not, we conclude that $a(E) = const$. Through a suitable renormalization we can set $a = \pm 1$. We now analyze another form of the *Jacobi identity*

$$[x_i, [x_j, p_k]] + [x_j, [p_k, x_i]] + [p_k, [x_i, x_j]] = 0 \tag{5.53}$$

The terms must be evaluated one by one using (5.42-5.43)

$$[x_i, [x_j, p_k]] = i\hbar\delta_{jk}[x_i, f(E)] = -\hbar^2 p_i \delta_{jk}\frac{f(E)}{E}\frac{df}{dE} \tag{5.54}$$

$$[x_j, [p_k, x_i]] = -i\hbar\delta_{ki}[x_j, f(E)] = \hbar^2 p_j \delta_{ki}\frac{f(E)}{E}\frac{df}{dE} \tag{5.55}$$

$$[p_k, [x_i, x_j]] = \pm i\frac{\hbar^2}{k^2}\epsilon_{ijp}[p_k, J_p] = \pm\frac{h^2}{k^2}\epsilon_{ijp}\epsilon_{pkl} p_l = \pm\frac{h^2}{k^2}(-\delta_{jk} p_i + \delta_{ki} p_j) \tag{5.56}$$

and once we put all of these together we obtain for the *Jacobi identity* (5.53)

$$-h^2 p_i \left(\frac{f(E)}{E} \frac{df}{dE} \pm \frac{1}{k^2} \right) \delta_{jk} + h^2 p_j \left(\frac{f(E)}{E} \frac{df}{dE} \pm \frac{1}{k^2} \right) \delta_{ki} = 0 \tag{5.57}$$

which results into

$$\frac{f(E)}{E} \frac{df}{dE} = \mp \frac{1}{k^2} \tag{5.58}$$

Plugging the condition $f(0) = 1$ the last equation may be integrated to obtain

$$f(E) = \left(1 \mp \frac{E^2}{k^2} \right)^{1/2} \tag{5.59}$$

We choose to keep the plus sign. Substituting into (5.43) we get

$$[x_i, p_j] = i\hbar \delta_{ij} \left(1 + \frac{E^2}{k^2} \right)^{1/2} \tag{5.60}$$

which is the deformed algebra that generates the *Generalized Uncertainty Principle*

$$\Delta x_i \Delta p_j \geq \frac{\hbar}{2} \delta_{ij} \langle \left(1 + \frac{E^2}{k^2} \right)^{1/2} \rangle \tag{5.61}$$

This expression can be expanded in powers of $\frac{E^2}{k^2}$ defining

$$(\Delta p)^2 = \langle p^2 \rangle - p^2 \tag{5.62}$$

and recalling

$$E^2 = p^2 + m^2 \tag{5.63}$$

the final result is [25]

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left(1 + \frac{E^2 + (\Delta p)^2}{2k^2} \right) \tag{5.64}$$

Indeed, if we consider a deformation parameter $|k| \to +\infty$ we recover the usual *Heisenberg Principle*. For $E \ll k$ and $\Delta p \lesssim k$ instead we get

$$\Delta x \geq \frac{\hbar}{2\Delta p} + const \times \Delta p \tag{5.65}$$

This is the form for the GUP that we were looking for. In our case we can interpret this as the uncertainty for the formation of a Black Hole. The first term is the uncertainty on the position of a particle, or on the position of the shell for a ball of dust. The second term is the uncertainty on whether or not a particular surface is an horizon. In the next section we will formulate a GUP using a gaussian wave packet.
By looking at this expression it can already be said that for energies of order $\hbar/k$ exists a minimum spatial length. This is clearly a non-Lorentz invariant concept, we should always be able to perform a boost and reduce the length. We might conclude that Lorentz invariance will not be preserved by a quantum gravity theory.

5.5 GUP

We dedicate this final section to the formulation of a *Generalized Uncertainty Principle* for the gaussian Black Hole description. As we have done for the *Bound States Quantization Model*, we define the uncertainty on the radial position with the variance of R evaluated with (5.20) [26]. Starting from the average radius

$$\langle R \rangle = \langle \psi_S | R | \psi_S \rangle = 4\pi \int_0^\infty R^3 |\psi_S(R)|^2 dR \tag{5.66}$$

and the average square radius

$$\langle R^2 \rangle = 4\pi \int_0^\infty R^4 |\psi_S(R)|^2 dR \tag{5.67}$$

we can write the variance

$$\langle \Delta R^2 \rangle = \langle R^2 \rangle - \langle R \rangle^2 = \left(\frac{3\pi - 8}{2\pi} \right) l^2 \tag{5.68}$$

We can now do the same thing for the horizon wave function. We calculate the variance using (5.27)

$$\langle \Delta R_H^2 \rangle = \langle R_H^2 \rangle - \langle R_H \rangle^2 = 4\pi \int_0^\infty R_H^4 |\psi_H(R_H)|^2 dR_H - \left(4\pi \int_0^\infty R_H^3 |\psi_H(R_H)|^2 dR_H \right)^2 \tag{5.69}$$

that results in

$$\langle \Delta R_H^2 \rangle = 4 \left(\frac{3\pi - 8}{2\pi} \right) \frac{l_p^4}{l^2} \tag{5.70}$$

In order to recover a GUP as it is formulated in the Section 5.4 we need to define also the variance for the momentum of the particle. We can do this by means of the wave equation in the momentum space (5.23)

$$\Delta P^2 \equiv \langle \Delta P^2 \rangle = 4\pi \int_0^\infty P^4 |\psi_S(P)|^2 dP - \left(4\pi \int_0^\infty P^3 |\psi_S(P)|^2 dP \right)^2 \tag{5.71}$$

which is

$$\Delta P^2 = \left(\frac{3\pi - 8}{2\pi} \right) \frac{m_p^2 l_p^2}{l^2} \tag{5.72}$$

This expression can be inverted to write the gaussian width

$$l^2 = \left(\frac{3\pi - 8}{2\pi} \right) \frac{m_p^2 l_p^2}{\Delta P^2} \tag{5.73}$$

We introduce a parameter γ. This defines the relevance of the horizon uncertainty with respect to the one of the particle radial position, in order to define the uncertainty for the whole Black Hole. Indeed, we write the latter as

$$\Delta R \equiv \sqrt{\langle \Delta R^2 \rangle} + \gamma \sqrt{\langle \Delta R_H^2 \rangle} = \left(\frac{3\pi - 8}{2\pi} \right) l_p \frac{m_p}{\Delta P} + 2\gamma l_p \frac{\Delta P}{m_p} \tag{5.74}$$

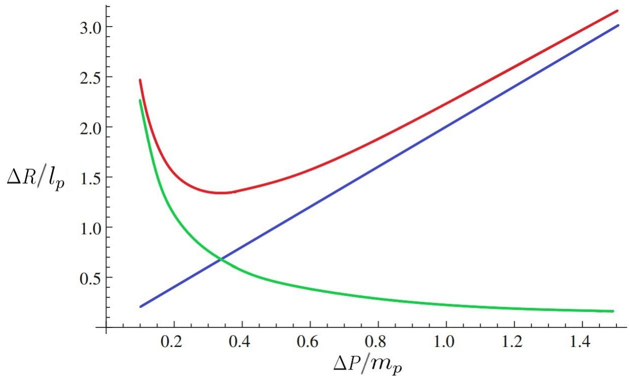

Fig.5.5
Black Hole uncertainty ΔR (red line) as a combination of the uncertainty on the particle position $\sqrt{\langle \Delta R^2 \rangle}$ (green line) and of the uncertainty on the horizon position $\sqrt{\langle \Delta R_H^2 \rangle}$ (blue line). $\gamma = 1$. [26]

This is the Black Hole uncertainty given by the uncertainty on the position of the particle combined with the uncertainty on the position of the horizon.
The expression (5.74) is exactly like the one expected in (5.65). In fact setting

$$\Delta P = \sqrt{\frac{3\pi - 8}{\pi\gamma}} \frac{m_p}{2} \tag{5.75}$$

we can define a minimum measured length

$$\Delta R \geq 2\sqrt{\gamma \frac{3\pi - 8}{\pi}} l_p \tag{5.76}$$

It is important to recall that the arise of a well defined GUP should not be viewed as fundamental principle. This is just a further proof that the quantization procedure employed is actually valid. The existence of a minimum measurable length is intrinsically linked to the Black Hole formation, if we imagine a single particle in the vacuum its energy is not limited, there is no minimal length. Only once a second particle is introduced and the two collide with enough energy a Black Hole might be formed. With a Black Hole also the minimal length is introduced.

Conclusions

The complete quantum theory of gravity is one of the ultimate goal of theoretical research in physics. The general way to tackle this problem is looking for a complete theory. We are still far from this result. In the spirit of the path followed for the birth of the *Quantum Electro Dynamics*, that is a complete theory, we can look for quantization models of compact objects in order to reproduce observed behaviours. Many proposals have been made in this way.

In this thesis we describe some of them that aim to quantize the Black Hole formation. The idea is to focus on the areal radius of the collapsing ball because it is a variable which we already know how to experimentally interpret. A final wave function for this radius, when obtained, is able to reproduce many of the known results of Black Holes. Up to now, the path has been to look for the classical trajectories, and all the results of these approaches have one peculiarity in common: they predict a bounce for the collapsing shell. The problem of these models is that they basically fail in describing the formation of a stable state corresponding to a Black Hole.

The new proposal in this thesis is based on the research for the bound states of Quantum Mechanics for a collapsing shell of incoherent matter. Starting from the geodesic equation for radial motion, we use it as a time independent Schrödinger equation. The procedure of the Bohr quantization for the Hydrogen atom has been followed as an inspiration. Part of the initial mass of the dust is considered as source, the rest is the collapsing shell. A first attempt keep a generic orbital number. This choice is motivated by the fact that in the future a possible new approach might be done considering a more realistic case, where the angular momentum of the ball is not set to zero. In order to do so we will need not spherically symmetric orbitals, and therefore a non vanishing orbital quantum number. This could be a way to produce a quantum model for Kerr Black Holes. Another procedure, done keeping a vanishing orbital number from the beginning, has been proven to be equivalent to the more general one. As expected for a complete quantum theory the singularity is covered.

The obtained wave function contains the *Laguerre polynomials* just like the radial part of the Hydrogen atom. A principal quantum number is obtained, and it scales with the square of the mass of the source. This is a consistency condition of the Einstein gravity restored here without requesting it a priori. Indeed, the classical equations of motion relate gravity on the left hand side with matter on the right hand side. In our case, gravity is represented by the principal quantum number introduced starting from the geodesic equation, while the matter part is obviously the mass of the ball. The interesting result is

that this principal quantum number scales as the area of the two spheres, and therefore it scales as the Beckenstein entropy.
The fundamental state is characterized by a principal quantum number of order 10^{76} when we consider a solar mass as a source. In order to give a rough estimate of the entropy in a Boltzmann way, we have evaluated the number of states between the fundamental state, and the one corresponding to the horizon. One of the problems is the high value for the principal quantum number that makes impracticable any numerical analysis. To partially solve this issue, an analytical procedure has been used to show that the average radius, corresponding to the fundamental state, is near to the last peak of the *Laguerre polynomials* in the wave function.
The uncertainty on the radial position of the shell is defined through the variance of the radial wave function. Switching to the momentum space the same thing has been done for the momentum of the collapsing matter. The product of these uncertainties is a new formulation of the *Heisenberg Principle*. Minimizing the variance on the radius we have evaluated the preferred ratio between the core mass and the collapsing shell.
In the last chapter a simplified, but more complete, description of the Black Hole structure is presented. The wave function for the radial position of a particle is taken to be gaussian wave packet. A new horizon wave function is introduced, and the Black Hole formation probability is calculated via the product of the particle probability distribution and the horizon probability distribution. In this model a full *Generalized Uncertainty Principle* is considered.
As future developments many questions remain open. A proper horizon wave function for the *Bound States Quantization Model* is still missing. Once this will be present a full *Generalized Uncertainty Principle* might be obtained also in this case, and with that comes the notion of a minimal length. A deeper analysis of the Laguerre polynomials with divergent principal quantum number might be used to better understand this model, and also to introduce the Kerr Black Holes. Also a proper definition for the entropy is needed, because the Boltzmann interpretation given in this thesis seems problematic given its dependence on the dust model for the source. The new entropy needs to recover the disorder due to all the degrees of freedom excluded once the choice of quantizing only the radius has been made.

A Schwarzschild geodesics

We are going to recover the geodesic equation used as starting point in our quantization procedure [5]. Starting from the Schwarzschild metric written as:

$$ds^2 = -e^{\nu(R)}dt^2 + e^{-\nu(R)}dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{A.1}$$

The geodesic equation is:

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0 \tag{A.2}$$

We recall that our solution is static, i.e. $g_{\alpha\beta,t} = 0$, and diagonal in order to simplify the evaluation of the Christoffel symbols

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\mu}(g_{\alpha\mu,\beta} + g_{\beta\mu,\alpha} - g_{\alpha\beta,\mu}) \tag{A.3}$$

The only non vanishing symbols are:

$$\Gamma^t_{tR} = \Gamma^t_{Rt} = \frac{\nu'}{2} \tag{A.4}$$

$$\Gamma^R_{tt} = \frac{1}{2}e^{2\nu}\nu' \tag{A.5}$$

$$\Gamma^R_{RR} = -\frac{\nu'}{2} \tag{A.6}$$

$$\Gamma^R_{\theta\theta} = -Re^\nu \tag{A.7}$$

$$\Gamma^R_{\phi\phi} = -Re^\nu\sin^2\theta \tag{A.8}$$

$$\Gamma^\theta_{\theta R} = \Gamma^\theta_{R\theta} = \frac{1}{R} \tag{A.9}$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \tag{A.10}$$

$$\Gamma^\phi_{\phi R} = \Gamma^\phi_{R\phi} = \frac{1}{R} \tag{A.11}$$

$$\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta \tag{A.12}$$

The geodesic equation becomes

$$\mu = t \qquad \ddot{t} + \Gamma^t_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \ddot{t} + \nu'\dot{R}\dot{t} = 0 \tag{A.13}$$

$$\mu = R \qquad \ddot{R} + \Gamma^R_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \ddot{R} + \frac{1}{2}e^{2\nu}\nu'\dot{t}^2 - \frac{\nu'}{2}\dot{R}^2 - Re^\nu\dot{\theta}^2 - Re^\nu(\sin^2\theta)\dot{\phi}^2 = 0 \tag{A.14}$$

$$\mu = \theta \qquad \ddot{\theta} + \Gamma^\theta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \ddot{\theta} + \frac{2}{R}\dot{R}\dot{\theta} - (\sin\theta\cos\theta)\dot{\phi}^2 = 0 \tag{A.15}$$

$$\mu = \phi \qquad \ddot{\phi} + \Gamma^\phi_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \ddot{\phi} + \frac{2}{R}\dot{R}\dot{\phi} + 2(\cot\theta)\dot{\theta}\dot{\phi} = 0 \tag{A.16}$$

From each one of these equations it is possible to recover $t(s)$, $R(s)$, $\theta(s)$, $\phi(s)$. In spherical symmetry it is always possible to restrict the motion on the equatorial plane, i.e. $\theta = \pi/2$ with $\dot{\theta} = 0$. I can remove the equation $\mu = \theta$.
The geodesic equation might be written as:

$$u^{\mu}_{;\beta} u^{\beta} = 0 \tag{A.17}$$

where as usual $u^{\mu} = \dfrac{dx^{\mu}}{ds}$. Considering

$$\begin{align}
g_{\alpha\mu} u^{\mu}_{;\beta} u^{\beta} &= u_{\alpha;\beta} u^{\beta} = 0 \tag{A.18}\\
u_{\alpha,\beta} u^{\beta} - \Gamma^{\lambda}_{\alpha\beta} u_{\lambda} u^{\beta} &= 0 \tag{A.19}\\
u_{\alpha,\beta} u^{\beta} &= \frac{1}{2} g^{\lambda\sigma} (g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma}) u_{\lambda} u^{\beta} \tag{A.20}\\
u_{\alpha,\beta} u^{\beta} &= \frac{1}{2} g_{\beta\sigma,\alpha} u^{\sigma} u^{\beta} \tag{A.21}\\
& \tag{A.22}
\end{align}$$

combined with

$$u_{\alpha,\beta} u^{\beta} = \frac{\partial u_{\alpha}}{\partial x^{\beta}} \frac{dx^{\beta}}{ds} = \frac{du_{\alpha}}{ds} \tag{A.23}$$

the geodesic equation can be expressed as

$$\frac{du_{\alpha}}{ds} = \frac{1}{2} g_{\beta\sigma,\alpha} u^{\sigma} u^{\beta} \tag{A.24}$$

Now it is possible to define two integrals of motion since

$$\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} = 0 \quad \Rightarrow \quad u_{\alpha} = const \tag{A.25}$$

The energy and the angular momentum per unit mass are

$$\begin{align}
u_t &= \frac{E}{m} = const \tag{A.26}\\
u_{\phi} &= \frac{L}{m} = const \tag{A.27}
\end{align}$$

because there are no components of the metric depending on t or ϕ

$$ds^2 = -\left(1 - \frac{2G_N M}{R}\right) dt^2 + \left(1 - \frac{2G_N M}{R}\right)^{-1} dR^2 + R^2 d\Omega^2 \tag{A.28}$$

Let us continue the derivation of the form (1.6) of the geodesic equation

$$\begin{cases}
u^t = \dfrac{dt}{ds} = g^{t\alpha} u_{\alpha} = g^{tt} u_t = -\dfrac{E}{m\left(1 - \dfrac{2G_N M}{R}\right)} \\[2ex]
u^3 = \dfrac{d\phi}{ds} = g^{\phi\alpha} u_{\alpha} = g^{\phi\phi} u_{\phi} = \dfrac{L}{mR^2} \\[2ex]
g_{\mu\nu} u^{\mu} u^{\nu} = -1
\end{cases} \tag{A.29}$$

These three equations combined, recalling $u^\theta = 0$, give

$$g_{tt}(u^t)^2 + g_{RR}(u^R)^2 + g_{\phi\phi}(u^\phi)^2 = -1 \tag{A.30}$$

$$-\left(1 - \frac{2G_N M}{R}\right) \frac{E^2}{m^2\left(1 - \frac{2G_N M}{R}\right)^2} + \frac{\dot{R}^2}{\left(1 - \frac{2G_N M}{R}\right)} + R^2 \frac{L^2}{m^2 R^4} = -1 \tag{A.31}$$

that can be finally written as

$$\dot{R}^2 - \frac{E^2}{m^2} + \left(1 - \frac{2G_N M_0}{R}\right) + \left(1 - \frac{2G_N M_0}{R}\right) \frac{L^2}{R^2 m^2} = 0 \tag{A.32}$$

which is exactly what we were looking for.

B BSQM states for $l \neq 0$

We start from the geodesics equation:

$$\left(\frac{dR}{ds}\right)^2 + 1 - \frac{2G_N M_0}{R} = \frac{E_m^2}{m^2} \tag{B.1}$$

with:

$$P_M = m\left(\frac{dR}{ds}\right) \tag{B.2}$$

We can rewrite the equation (B.1) as

$$H_\epsilon \equiv \frac{P_M^2}{2\epsilon M} - \frac{G_N \epsilon(1-\epsilon)M^2}{R} = \frac{\epsilon M}{2}\left(\frac{E_m^2}{\epsilon^2 M^2} - 1\right) \tag{B.3}$$

With the substitution

$$\widehat{P}_M = -i\hbar\nabla_R \tag{B.4}$$

in the geodesics we obtain the Schrödinger equation $\widehat{H}_\epsilon \Phi = \mathcal{E}_\epsilon \Phi$:

$$\widehat{H}_\epsilon \Phi = -\frac{\hbar^2}{2\epsilon M}\nabla^2_{\vec{R}}\Phi - \frac{G_N \epsilon(1-\epsilon)M^2}{R}\Phi = \frac{\epsilon M}{2}\left(\frac{E_m^2}{\epsilon^2 M^2} - 1\right)\Phi = \mathcal{E}_\epsilon \Phi \tag{B.5}$$

that can be written, after having expressed the laplacian in spherical coordinates, as:

$$\frac{\partial^2 \Phi}{\partial R^2} + \frac{2}{R}\frac{\partial \Phi}{\partial R} + \frac{1}{R^2 \sin^2\theta}\left[\sin\theta \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{\partial^2 \Phi}{\partial \phi^2}\right] + \frac{2M}{\hbar^2}\left(\epsilon \mathcal{E}_\epsilon + \frac{\epsilon^2(1-\epsilon)M^2 G_N}{R}\right)\Phi = 0 \tag{B.6}$$

Now writing

$$\Phi(R,\theta,\phi) = \Psi(R)\Omega(\theta)\Gamma(\phi) \tag{B.7}$$

We obtain, inserting the wave equation in this form:

$$\begin{aligned}\Omega\Gamma\frac{\partial^2 \Psi}{\partial R^2} + \Omega\Gamma\frac{2}{R}\frac{\partial \Psi}{\partial R} + \frac{1}{R^2\sin^2\theta}\left[\Psi\Gamma\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Omega}{\partial\theta}\right) + \Psi\Omega\frac{\partial^2\Gamma}{\partial\phi^2}\right] \\ + \frac{2M}{\hbar^2}\left(\epsilon\mathcal{E}_\epsilon + \frac{\epsilon^2(1-\epsilon)M^2 G_N}{R}\right)\Psi\Omega\Gamma = 0\end{aligned} \tag{B.8}$$

which is easily rearranged into:

$$\begin{aligned}R^2\frac{1}{\Psi}\left[\frac{\partial^2\Psi}{\partial R^2} + \frac{2}{R}\frac{\partial\Psi}{\partial R} + \frac{2M}{\hbar^2}\left(\epsilon\mathcal{E}_\epsilon + \frac{\epsilon^2(1-\epsilon)M^2 G_N}{R}\right)\Psi\right] \\ + \frac{1}{\sin^2\theta}\left[\frac{1}{\Omega}\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Omega}{\partial\theta}\right) + \frac{1}{\Gamma}\frac{\partial^2\Gamma}{\partial\phi^2}\right] = 0\end{aligned} \tag{B.9}$$

Now keeping constant R we find that the only way to satisfy this equation is having the angular part equal to a constant, that we write as:

$$\frac{1}{\sin^2\theta}\left[\frac{1}{\Omega}\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Omega}{\partial\theta}\right)+\frac{1}{\Gamma}\frac{\partial^2\Gamma}{\partial\phi^2}\right]=-l(l+1) \tag{B.10}$$

Substituting:

$$\frac{\partial^2\Psi}{\partial R^2}+\frac{2}{R}\frac{\partial\Psi}{\partial R}+\frac{2M}{\hbar^2}\left(\epsilon\mathcal{E}_\epsilon+\frac{\epsilon^2(1-\epsilon)M^2G_N}{R}\right)\Psi-\frac{l(l+1)}{R^2}\Psi=0 \tag{B.11}$$

We now simplify the problem through the definition of ψ:

$$\Psi\equiv\frac{\psi}{R} \tag{B.12}$$

and performing:

$$\frac{\partial^2\Psi}{\partial R^2}=\frac{2}{R^3}\psi-\frac{2}{R^2}\frac{\partial\psi}{\partial R}+\frac{1}{R}\frac{\partial^2\psi}{\partial R^2} \tag{B.13}$$

$$\frac{2}{R}\frac{\partial\Psi}{\partial R}=-\frac{2}{R^3}\psi+\frac{2}{R^2}\frac{\partial\psi}{\partial R} \tag{B.14}$$

Putting together these results with (B.11) we finally obtain

$$\frac{\partial^2\psi}{\partial R^2}+\left(\frac{2M}{\hbar^2}\epsilon\mathcal{E}_\epsilon-\frac{l(l+1)}{R^2}+2\frac{\epsilon^2(1-\epsilon)M^3G_N}{R\hbar^2}\right)\psi=0 \tag{B.15}$$

We now define:

$$K^2\equiv\frac{2M}{\hbar^2}\epsilon\mathcal{E}_\epsilon \tag{B.16}$$

Since we are considering bound states we expect $\mathcal{E}_\epsilon<0$ then:

$$K=i\tilde{k} \tag{B.17}$$

We define

$$r_g\equiv\frac{\hbar^2}{M^3G_N} \tag{B.18}$$

and also:

$$\tilde{k}\equiv\frac{\epsilon^2(1-\epsilon)}{r_g\bar{n}} \tag{B.19}$$

where $\bar{n}$ is a parameter. The Schrödinger equation becomes

$$\frac{\partial^2\psi}{\partial R^2}+\left(-\tilde{k}^2-\frac{l(l+1)}{R^2}+2\frac{\epsilon^2(1-\epsilon)}{Rr_g}\right)\psi=0 \tag{B.20}$$

The last definition is

$$x\equiv 2R\frac{\epsilon^2(1-\epsilon)}{r_g\bar{n}} \tag{B.21}$$

made in order to write

$$\frac{\partial^2\psi}{\partial R^2} = \frac{\partial^2 x}{\partial R^2}\frac{\partial\psi}{\partial x} + \left(\frac{\partial x}{\partial R}\right)^2\frac{\partial^2\psi}{\partial x^2} = \left(2\frac{\epsilon^2(1-\epsilon)}{r_g\bar{n}}\right)^2\frac{\partial^2\psi}{\partial x^2} \tag{B.22}$$

$$\tilde{k}^2 = \left(\frac{\epsilon^2(1-\epsilon)}{r_g\bar{n}}\right)^2 = \frac{1}{4}\left(2\frac{\epsilon^2(1-\epsilon)}{r_g\bar{n}}\right)^2 \tag{B.23}$$

$$\frac{l(l+1)}{R^2} = \left(2\frac{\epsilon^2(1-\epsilon)}{r_g\bar{n}}\right)^2\frac{l(l+1)}{x^2} \tag{B.24}$$

$$2\frac{\epsilon^2(1-\epsilon)}{r_g R} = \left(2\frac{\epsilon^2(1-\epsilon)}{r_g\bar{n}}\right)^2\frac{\bar{n}}{x} \tag{B.25}$$

The equation takes the form:

$$\frac{\partial^2\psi}{\partial x^2} + \left(-\frac{1}{4} - \frac{l(l+1)}{x^2} + \frac{\bar{n}}{x}\right)\psi = 0 \tag{B.26}$$

Now we consider the limit $x \to 0$:

$$\frac{\partial^2\psi}{\partial x^2} - \frac{l(l+1)}{x^2}\psi = 0 \tag{B.27}$$

which is solved by $\psi \propto x^{l+1},\ x^{-l}$. Requiring $\psi(0) = 0$ we choose $\psi \propto x^{l+1}$. Then considering $x \to +\infty$:

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{4}\psi = 0 \tag{B.28}$$

which is solved by $\psi \propto e^{\pm x/2}$. Requiring for the wave function to be bounded at infinity we restrict to $\psi \propto e^{-x/2}$. We have obtained a suggested form:

$$\psi(x) \propto x^{l+1}e^{-x/2}F(x) \tag{B.29}$$

Plugging this into the first term of the Schrödinger equation leads to

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial}{\partial x}\left[(l+1)x^l e^{-x/2}F(x) - \frac{1}{2}e^{-x/2}x^{l+1}F(x) + x^{l+1}e^{-x/2}\frac{\partial F}{\partial x}\right] \tag{B.30}$$

$$\begin{aligned}\frac{\partial^2\psi}{\partial x^2} = {}&(l+1)lx^{l-1}e^{-x/2}F - (l+1)x^l e^{-x/2}F + 2(l+1)x^l e^{-x/2}\frac{\partial F}{\partial x} + \frac{1}{4}e^{-x/2}x^{l+1}F + \\ &- e^{-x/2}x^{l+1}\frac{\partial F}{\partial x} + x^{l+1}e^{-x/2}\frac{\partial^2 F}{\partial x^2}\end{aligned} \tag{B.31}$$

while the second term becomes

$$\left[-\frac{1}{4} + \frac{l(l+1)}{x^2} + \frac{\bar{n}}{x}\right]\psi = -\frac{1}{4}e^{-x/2}x^{l+1}F - l(l+1)x^{l-1}e^{-x/2}F + \bar{n}x^l e^{-x/2}F \tag{B.32}$$

Combining the two parts we obtain:

$$x\frac{\partial^2 F}{\partial x^2} + (2l+2-x)\frac{\partial F}{\partial x} - (l+1-\bar{n})F = 0 \tag{B.33}$$

which is the Kummer equation solved using the *Laguerre Polynomials* $L_q^p(x)$. Then:

$$F(x) \propto L_{\bar{n}-l-1}^{2l+1}(x) \tag{B.34}$$

where $\bar{n}$ is an integer greater than 1. The suggested form for the wave function becomes:

$$\psi_{\bar{n}} \propto x^{l+1} e^{-x/2} L_{\bar{n}-l-1}^{2l+1}(x) \tag{B.35}$$

We now focus on the normalization conditions in order to determine the coefficient of $\psi_{\bar{n}}$. Starting from the request:

$$\int_0^\infty |\psi_{\bar{n}}(R)|^2 dR = \int_0^\infty \frac{r_g \bar{n}}{2\epsilon^2(1-\epsilon)} |\psi_{\bar{n}}(x)|^2 dx = 1 \tag{B.36}$$

The *Laguerre Polynomials* satisfy the orthogonality relation:

$$\int_0^\infty x^{p+1} e^{-x} (L_q^p(x))^2 dx = \frac{(p+q)!}{q!}(p+2q+1) \tag{B.37}$$

In our case we have $p = (2l+1)$ and $q = (\bar{n}-l-1)$ that give us:

$$\int_0^\infty x^{2l+2} e^{-x} (L_{\bar{n}-l-1}^{2l+1}(x))^2 dx = \frac{(\bar{n}+l)!}{(\bar{n}-l-1)!} 2\bar{n} \tag{B.38}$$

We notice that the expression inside this integral is exactly the suggested form (B.35) for $|\psi_{\bar{n}}|^2$. Plugging this result into the request (B.36) we obtain:

$$\psi_{\bar{n}}(R) = \sqrt{\frac{2\epsilon^2(1-\epsilon)(\bar{n}-l-1)!}{2r_g\bar{n}^2(\bar{n}+l)!}} x^{l+1} e^{-x/2} L_{\bar{n}-l-1}^{2l+1}(x) \tag{B.39}$$

Since we are considering the Schwarzschild case we would expect a spherical shape for the orbitals. In order to obtain that we set $l = 0$. Substituting $x = \dfrac{2R\epsilon^2(1-\epsilon)}{r_g\bar{n}}$ we finally arrive to:

$$\psi_{\bar{n}}(R) = \sqrt{\frac{4\epsilon^6(1-\epsilon)^3}{\bar{n}^5 r_g^3}} R\, e^{-\dfrac{\epsilon^2(1-\epsilon)R}{r_g\bar{n}}} L_{\bar{n}-1}^1\left(\frac{2\epsilon^2(1-\epsilon)R}{r_g\bar{n}}\right) \tag{B.40}$$

We now go back to the original wave equation $\Psi = \dfrac{\psi}{R}$ which has to be normalized in spherical coordinates:

$$\int_0^{2\pi} d\phi \int_{-1}^1 d(\cos\theta) \int_0^\infty R^2 |\Psi|^2 dR = 1 \tag{B.41}$$

then we obtain

$$\Psi_{\bar{n}}(R) = \sqrt{\frac{\epsilon^6(1-\epsilon)^3}{\pi\bar{n}^5 r_g^3}} e^{-\dfrac{\epsilon^2(1-\epsilon)R}{r_g\bar{n}}} L_{\bar{n}-1}^1\left(\frac{2\epsilon^2(1-\epsilon)R}{r_g\bar{n}}\right) \tag{B.42}$$

Now considering

$$l_p = \sqrt{G_N \hbar} \quad m_p = \sqrt{\frac{\hbar}{G_N}} \tag{B.43}$$

$$r_g = \frac{\hbar^2}{M^3 G_N} = \frac{m_p^3 l_p}{M^3} \tag{B.44}$$

we have

$$\Psi_{\bar{n}}(R) = \sqrt{\frac{M^9 \epsilon^6 (1-\epsilon)^3}{\pi \bar{n}^5 m_p^9 l_p^3}} e^{-\frac{M^3 \epsilon^2 (1-\epsilon) R}{m_p^3 l_p \bar{n}}} L^1_{\bar{n}-1} \left(\frac{2M^3 \epsilon^2 (1-\epsilon) R}{m_p^3 l_p \bar{n}} \right) \tag{B.45}$$

We recall the definition of $\tilde{k}$ keeping in mind that $\bar{n} > 1$ is an integer that we write as $\bar{n} = N_M + n$:

$$\tilde{k}^2 = \frac{\epsilon^4 (1-\epsilon)^2}{\bar{n}^2} \frac{M^6 G_N^2}{\hbar^4} = -2 \frac{M}{\hbar^2} \epsilon \mathcal{E}_\epsilon \tag{B.46}$$

. Combining the expressions (B.16), (B.19) and (B.18) we obtain

$$\mathcal{E}_{\epsilon \bar{n}} = \frac{\epsilon M}{2} \left(\frac{E_m^2}{\epsilon^2 M^2} - 1 \right) = -\frac{M^5 G_N^2}{\hbar^2} \frac{\epsilon^3 (1-\epsilon)^2}{2 \bar{n}^2} \tag{B.47}$$

considering

$$\frac{G_N^2}{\hbar^2} = \frac{1}{m_p^4} \tag{B.48}$$

we get

$$0 \leq \frac{E_m^2}{\epsilon^2 M^2} = 1 - \frac{\epsilon^2 (1-\epsilon)^2}{\bar{n}^2} \left(\frac{M}{m_p} \right)^4 \tag{B.49}$$

At the ground state we have $\bar{n} = N_M$ and in order to have $E_m = 0$ we write

$$N_M = \epsilon (1-\epsilon) \left(\frac{M}{m_p} \right)^2 \tag{B.50}$$

Performing some calculations:

$$\begin{aligned}
\mathcal{E}_{\epsilon \bar{n}} &= -\frac{M}{2} \left(\frac{M}{m_p} \right)^4 \frac{\epsilon^3 (1-\epsilon)^2}{\bar{n}^2} = \\
&= -\frac{M\epsilon}{2} \left(\frac{M}{m_p} \right)^4 \frac{\epsilon^2 (1-\epsilon)^2}{\left[\epsilon (1-\epsilon) \left(\frac{M}{m_p} \right)^2 + n^2 \right]^2} = \\
&= -\frac{M\epsilon}{2} \frac{\epsilon^2 (1-\epsilon)^2}{\left[\left[\epsilon (1-\epsilon) + n^2 \left(\frac{m_p}{M} \right)^2 \right] \right]^2} = \\
&= \frac{M\epsilon}{2} \left(\frac{E_m^2}{\epsilon^2 M^2} - 1 \right)
\end{aligned} \tag{B.51}$$

We finally compute the expression for the energy levels:

$$E_m^n = \epsilon M \left[1 - \epsilon^2 (1-\epsilon)^2 \left(\epsilon (1-\epsilon) + n \left(\frac{m_p}{M} \right)^2 \right)^{-2} \right]^{1/2} \tag{B.52}$$

C BSQM states for $l = 0$

We start from the Schrödinger equation (4.7) setting $l = 0$

$$\frac{\partial^2 \Psi}{\partial R^2} + \frac{2}{R}\frac{\partial \Psi}{\partial R} + \frac{2M}{\hbar^2}\left(\epsilon \mathcal{E}_\epsilon + \frac{\epsilon^2(1-\epsilon)M^2 G_N}{R}\right)\Psi = 0 \tag{C.1}$$

Through the definition of

$$\Psi = \frac{\psi}{R} \tag{C.2}$$

we can eliminate the first derivative

$$\frac{\partial^2 \psi}{\partial R^2} + \left(\frac{2M\epsilon \mathcal{E}_\epsilon}{\hbar^2} + \frac{2\epsilon^2(1-\epsilon)M^3 G_N}{\hbar^2 R}\right)\psi = 0 \tag{C.3}$$

Now we perform the same definitions of the previous case

$$r_g \equiv \frac{\hbar^2}{M^3 G_N} \tag{C.4}$$

$$K = i\tilde{k} \equiv \frac{\sqrt{-2M\epsilon \mathcal{E}_\epsilon}}{\hbar} \tag{C.5}$$

$$\tilde{k} \equiv \frac{\epsilon^2(1-\epsilon)}{r_g \bar{n}} \tag{C.6}$$

The equation takes the form

$$\frac{\partial^2 \psi}{\partial R^2} + \tilde{k}^2\left(-1 + \frac{2\epsilon^2(1-\epsilon)}{r_g R \tilde{k}^2}\right)\psi = 0 \tag{C.7}$$

We introduce new variables:

$$\rho_0 \equiv \frac{2\epsilon^2(1-\epsilon)}{r_g \tilde{k}} \tag{C.8}$$

$$\rho \equiv R\tilde{k} \tag{C.9}$$

$$\frac{\partial^2 \psi}{\partial R^2} = \tilde{k}^2 \frac{\partial^2 \psi}{\partial \rho^2} \tag{C.10}$$

The final form for the Schrödinger is

$$\frac{\partial^2 \psi}{\partial \rho^2} + \left(-1 + \frac{\rho_0}{\rho}\right)\psi = 0 \tag{C.11}$$

which is solved using the *Hypergeometrical functions* $F[a, b, x]$:

$$\psi(\rho) = Ce^{-\rho}\rho F\left[1 - \frac{\rho_0}{2}, 2, 2\rho\right] \tag{C.12}$$

The *Hypergeometrical function* $F[a, b, c]$ has to fulfil some conditions in order for this solution to have a finite polynomial form. In particular the first argument a needs to be a negative integer. The second argument b must be greater than a. In our case this condition is obviously satisfied. If we now plug the first condition into our solution

$$1 - \frac{\rho_0}{2} < 0 \tag{C.13}$$

$$1 - \frac{\epsilon^2(1-\epsilon)}{r_g\tilde{k}} < 0 \tag{C.14}$$

$$1 - \bar{n} < 0 \tag{C.15}$$

We have found that $\bar{n} > 1$ has to be an integer, which is the exact same condition obtained with the generic l procedure. Finally we define

$$x \equiv 2\rho = 2R\tilde{k} = 2R\frac{\epsilon^2(1-\epsilon)}{r_g\bar{n}} \tag{C.16}$$

Again this definition is exactly equal to one of the previous case. The final expression of the solution takes the form

$$\psi_{\bar{n}}(x) = Ce^{-x/2}xF[1 - \bar{n}, 2, x] \tag{C.17}$$

where both $\bar{n}$ and x represent the same quantities of the case where the value of l wasn't set to zero from the beginning.

D Bessel approximation

We are interested to find out under which conditions it is possible to treat *Laguerre polynomials* using *Bessel functions*. The latter are solutions of the equation

$$\frac{\partial^2 \psi(R)}{\partial R^2} + \left(\frac{V}{R} - \frac{l(l+1)}{R^2} \right) \psi(R) = 0 \tag{D.1}$$

We now start from expression (B.15) that we report

$$\frac{\partial^2 \psi}{\partial R^2} + \left(\frac{2M}{\hbar^2} \epsilon \mathcal{E}_\epsilon - \frac{l(l+1)}{R^2} + 2\frac{\epsilon^2(1-\epsilon)M^3 G_N}{R\hbar^2} \right) \psi = 0 \tag{D.2}$$

In order to re-obtain the Bessel equation (D.1) in this case we need

$$\frac{\epsilon(1-\epsilon)M^2 G_N}{R} \gg |\mathcal{E}_\epsilon| \tag{D.3}$$

At the end of the Appendix B we have recovered an expression for $\mathcal{E}_\epsilon$ that is

$$|\mathcal{E}_{\epsilon\bar{n}}| = \frac{M^5 \epsilon^3 (1-\epsilon)^2}{2m_p^4 \bar{n}^2} \tag{D.4}$$

The condition (D.3) for the *Bessel functions* becomes

$$\bar{n}^2 \gg \frac{RM^3\epsilon^2(1-\epsilon)}{2G_N m_p^4} \tag{D.5}$$

We now choose a radius near the horizon $R \sim 2G_N M$ and get

$$\bar{n}^2 \gg \frac{M^4 \epsilon^2 (1-\epsilon)}{m_p^4} \tag{D.6}$$

The minimum $\bar{n}$ exhibits the same relation on M of the fundamental quantum number N_M. Indeed plugging the values: $\epsilon = 0.5$, that is the evaluated preferred ratio, $M = 2 \times 10^{30} kg$ which a solar mass, and $m_p = 2.18 \times 10^{-8} kg$ we obtain

$$\bar{n}_0^2 \approx 8.80 \times 10^{150} \tag{D.7}$$

which gives us $\bar{n}_0 \approx 2.97 \times 10^{75}$ which perfectly reproduces the expected N_M for the *Laguerre polynomials*. It seems theoretically possible to recover the spectrum using *Bessel functions* when excited states are considered, however it is important to underline that numerous simulations have shown that this equivalence between *Bessel functions* and *Laguerre polynomials* it is not valid.

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