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Thermodynamic Bethe Ansatz for a Family of Scattering Theories with $\mathfrak{U}_q(\mathfrak{sl}(2))$ Symmetry

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Abstract

In this thesis work we analyze a wide class of 1+1 dimensional integrable scattering theories with $\mathfrak{U}_q(\mathfrak{sl}(2))$ quantum group symmetry, whose asymptotic states are multiplets of particles with iso-spin k/2. Their two-body S-matrices have been recently found in terms of the R matrix of the quantum group. Since they satisfy Yang-Baxter equation, unitarity and crossing symmetry, they represent a consistent integrable factorized scattering theory. The question of finding the corresponding underlying QFT can be addressed once the Thermodynamic Bethe Ansatz (TBA) is obtained. In this work we get the TBA equations and we compare them to previous known results of S. R. Aladim e M. J. Martins for the particular case when $q \to 1$.

Sommario

In questo lavoro di tesi analizziamo una vasta classe di teorie integrabili di scattering con simmetria di quantum group $\mathfrak{U}_q(\mathfrak{sl}(2))$, i cui stati asintotici sono multipletti di particelle con iso-spin k/2. Le loro matrici S a due corpi sono state recentemente trovate in termini della matrice R del quantum group. Dato che soddisfano l'equazione di Yang-Baxter, unitarietà e crossing symmetry, esse descrivono una teoria di scattering consistente. Il problema di trovare la corrispondente teoria di campo quantistica può essere investigato una volta noto il Thermodynamic Bethe Ansatz (TBA). In questo lavoro le equazioni di TBA sono ricavate e vengono confrontate con precedenti risultati di S. R. Aladim e M. J. Martins per il particolare caso in cui $q \to 1$.

Contents

Introduction

"*The field of integrable systems is born together with classical mechanics*" O. Babelon, D. Bernard and M. Talon write in [1] and it is probably true. It is believed that the main aim of Physics has always been that of describing with more or less elegant mathematical formulas the behaviour of Nature, but this is not enough. Once the fundamental law is found, one has to look for its solutions and then compare this solution with empirical data, *i.e.* the experiments.

One of the first examples in the history of physics of the application of this process is given by gravitation: in 1687 Sir I. Newton understood the law that describes the mutual gravitational attraction between two massive bodies, which can be exactly solved and these solutions were in perfect accordance with the Kepler's observations done in the beginning of the same century. One could try to do the same thing with other theories but unfortunately it is not easy to find the solution of a given equation that describes some phenomena.

To make matter worse it was later understood, thanks to more precise measurements and the advent of modern physics such as quantum mechanics and special relativity, that generally Nature works in a more complicated way than expected. For these reasons many theories, such as Maxwell electromagnetism, were discovered to be special cases of a more general theory called quantum field theory, or to go back to our example, the Newton theory of gravitation was corrected and improved by A. Einstein in 1915 with his General Theory of Relativity.

These laws that describe the behaviour of Nature turn out to be mathematically very complicated and they are almost impossible to solve exactly. However, in many cases, the parameters that describe those theories are small enough such that one can safely perform a perturbative expansion of the interesting quantities and therefore an approximate solution can be computed. Given the stunning results obtained, for a large part of the last century this method had been the cutting edge of theoretical physics. Just to give an insight of the precision of this idea, in the framework of Quantum Electrodynamics (QED), whose the coupling constant is the fine structure constant $\alpha \simeq 1/137 = 0.007$, the anomalous magnetic moment of the electron was measured with a precision of 12 significant digits, making this result one of the most accurately verified predictions in the history of physics (see e.g. [2]).

However, physicists soon understood that there are phenomena that cannot be described in terms of a perturbative theory, e.g. the problem of quarks confinement, strongly correlated systems, string theory and AdS/CFT: this reignited the interest in the study of exact solutions, *i.e.* the property of integrability.

Meanwhile from the front of Statistical Mechanics, after the outstanding ideas of H. Bethe to exactly solve the Heisenberg model, scientists like R. Baxter, B. Sutherland and M. Takahashi just to name a few, started to analyze a huge number of spin chain models, making progress towards a new way to intend integrability in the quantum realm. Another step was done in the late '80s by brothers Zamolodchikov, who were able to find the exact expression for the S-matrix of the sine-Gordon model (SGM), together with other theories with $O(n)$ symmetry [3].

In particular it was understood that in the SGM there is a precise value of the coupling constant, namely $\beta = \sqrt{8\pi}$, where the underlying symmetry is $SU(2)$, making the model equivalent to the chiral Gross-Neveu model, instead of just $U(1)$, which holds for all the other values of β.

D. Bernard and A. LeClair then showed in [4] that considering non-local currents it was possible to extend the $SU(2)$ symmetry to every value of β introducing a quantum deformation of its universal enveloping algebra: with this seminal work, the concept of *quantum group* made his entrance in the integrability world (see e.g. [5] for a review). This led in 1995 V. A. Fateev, E. Onofri and Al. B. Zamolodchikov to study the $O(3) \simeq SU(2)/\mathbb{Z}_2$ sigma model [6] and in particular its quantum group deformation, the so-called *sausage model*, the name coming from the shape of the target space once the deformation is considered.

The natural question is whether this construction holds also at higher spin representation, namely $3/2$, 2, *etc.* A first attempt was was made by S. R. Aladim and M. J. Martins [7], who constructed a class of rational S-matrices based on $SU(2)_k$ symmetry (where k is the order of the representation of the algebra, *i.e.* $k = 2s$

where s is the iso-spin) and performed their TBA. Recently C. Ahn and F. Ravanini [8] were able to obtain the exact S-matrix for the quantum deformation of the rational case just mentioned.

The aim of this thesis is to understand how TBA equations work in the rational case and then generalize this result in the deformed one in order to have an insight on the critical behaviour of these theories.

Having outlined the framework, we have organized the work into the following chapters:

- in Chapter 1 we are going to introduce the concept of integrability starting from the classical point of view, which can be precisely defined via Liouville theorem, until we arrive to the quantum world. In this case there is not a unique definition of integrable model, except for $1+1$ integrable models. In these theories, the S-matrix is severely constrained by some basic requirement and can be therefore exactly computed. We know that any physical QFT theory admits a scattering operator but unfortunately, knowing an S-matrix does not mean that it describes a real physical theory. The powerful tool used to verify whether it is true or not is the Thermodynamic Bethe Ansatz (TBA).

- In Chapter 2 we are going to describe the Thermodynamic Bethe Ansatz method. In particular, in the first part we are going to analyze the TBA for diagonal scattering theories, where an elegant structure related to Lie algebras appear. In the second part we are going to study the case of non-diagonal scattering theories and in particular we are going to focus on the sine-Gordon model as an example. In this case the TBA calculations become more complicated than in the previous one, since new degrees of freedom appear, related to the appearance of new non-physical particles called magnons.

- In Chapter 3 we are going to analyze with the TBA method two classes of scattering theories. In the first part we are going to focus on theories which describe particles of arbitrary spin $k/2$ with $SU(2)$ symmetry, introduced in [7]. These particular models are very useful in the study of integrable models at particular values of the coupling constant. It is then possible to generalize these theories by deforming the underlying Lie algebra into the so called quantum group $\mathfrak{U}_q(\mathfrak{sl}(2))$. In the last part of this chapter we are going to derive the TBA equations of these models, in order to compare them with the undeformed case.

1

Classical And Quantum Integrability Theory

In this chapter we are going to introduce the concept of integrability in physics. In the first part we are going to introduce the formalism of classical mechanics and then we will focus on the most important result for our purposes, which is the Liouville theorem. Then a brief overview on classical field theories will be presented, in order to better understand the more complex topic of quantum field integrability, which is described in the second part of the chapter.

In particular, in the last section, we are going to describe the object of biggest interest in the quantum integrability realm, which is the S-matrix. Indeed, it can be shown that in $1+1$ dimensions, by only exploiting its defining properties, it is possible to define the scattering matrix without knowing anything of the related QFT: this method is known as ***bootstrap*** program.

1.1 Classical Integrability

In classical mechanics, the state of a system with n degrees of freedom is uniquely determined by the position q_i and its canonically conjugated momentum p_i where $i = 1, \ldots, n$. The evolution in time is given by Hamilton equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{1.1}$$

where $H(q_i, p_i)$ is the Hamiltonian of the system. For two arbitrary dynamical functions F and G one can define the Poisson brackets

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right) \tag{1.2}$$

such that the evolution of any function defined on the phase space becomes

$$\dot{F} = \{H, F\}. \tag{1.3}$$

Thanks to this definition it is clear that H doesn't change in time, *i.e.* $\{H, H\} = 0$.

We say that a dynamical system is Liouville integrable if it possesses n independent conserved quantities $Q_1, \ldots, Q_n$ which are in mutual involution, namely

$$\{H, Q_i\} = 0 = \{Q_i, Q_j\} \qquad \text{for all } i, j = 1, \ldots, n. \tag{1.4}$$

If these requirements are fulfilled, the equation of motion of the system can be suitably solved by "quadratures", *i.e.* by a series of nested replacements.

A key concept in the modern approach to integrability is that of the ***Lax pairs***. Suppose that it is possible to find two matrices $M(q_i, p_i)$ and $L(q_i, p_i)$ such that the equations of motion can be written in the following form

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [M, L]. \tag{1.5}$$

In this case it is possible to define

$$I_k = \operatorname{Tr} L^k, \qquad k \in \mathbb{N}, \tag{1.6}$$

which is easy to prove to be conserved in time for every k, $\dot{I}_k = 0$. It is even possible to formulate the same problem in terms of Lax pairs that depend on a non-dynamical parameter, $L(\lambda)$, $M(\lambda)$: this becomes very useful in the study of integrability in classical field theories, since it gives an infinite number of pairs parametrized by λ.

However, this method has some intrinsic problem: first of all, it is not guaranteed that the set of conserved quantities is linearly independent and in involution, therefore the Liouville theorem is not automatically satisfied; secondly there is not a standard procedure to find the Lax pairs able to recast the equation of motion in the form (1.5) so, even if very powerful, this method is difficult to apply.

It is possible to show (see [1] for the proof) that the involution between the conserved charges (1.6) is equivalent to the existence of a matrix r such that

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2], \tag{1.7}$$

where $L_1 = L \otimes \mathbb{1}$ and $L_2 = \mathbb{1} \otimes L$. Moreover, since the Poisson brackets (1.7) have to satisfy the Jacobi identity, the form of the r matrix is constrained, namely

$$[L_1, [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] + \{L_2, r_{13}\} - \{L_3, r_{12}\}] + \text{cyclic perm.} = 0. \tag{1.8}$$

In particular, if r is independent of the dynamic variables and it is anti-symmetric, equation (1.8) is trivially satisfied when

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{1.9}$$

Equation (1.9) is called ***classical Yang-Baxter equation***, and r is the ***classical*** R-***matrix***, in analogy to what happens in the quantum spin chain case.

A more interesting situation is given by classical field theories[1]. In this case the system has infinite degrees of freedom, therefore, referring to the Liouville theorem, one should find infinitely many conserved charges in involution to ensure the integrability of the system, which is clearly a difficult task. For this reason the Lax formalism introduced above turns out to be a powerful tool.

As an example let us analyze the Korteweg-de Vries (KdV) equation, which is the perfect prototype of integrable field theory, introduced at the end of 19th century to describe the motion of water waves and then borrowed by the most diverse fields of physics. The equation reads

$$\partial_t u(t, x) = 6u(t, x)\partial_x u(t, x) - \partial_x^3 u(t, x). \tag{1.10}$$

[1] We will restrict to the case of 1+1 dimensional theories, meaning that one dimension represent the time t and the other the space x.

It admits a Lax pair representation, namely

$$L = -\partial_x^2 + u(t,x), \tag{1.11a}$$
$$M = 4\partial_x^3 - 3[u(t,x)\partial_x + \partial_x u(t,x)]. \tag{1.11b}$$

Thanks to equation (1.6) it is possible to find an infinite amount of conserved charges (see e.g. [9]). For example the first three read

$$I_1 = \int dx\, u(t,x), \tag{1.12a}$$
$$I_2 = \int dx\, u(t,x)^2, \tag{1.12b}$$
$$I_3 = \int dx \left(u(t,x)^3 + \frac{(\partial_x u(t,x))^2}{2} \right). \tag{1.12c}$$

An interesting feature of many integrable field theories, including the discussed KdV equation, is that they admit solutions of the equation of motion that are well-localized in space and which preserve their shape in the time evolution, called *solitons*. In the scattering processes between these solutions, the fact that the theory admits infinite conserved quantities, constraints the number and the nature of such particles to be conserved. Therefore the collision is always of elastic nature and, additionally, it turns out that the scattering of more than two solitons can always be factorized in a sequence of pairwise interaction. This feature is of crucial importance in the quantum world, that we are going to analyze in the next section.

1.2 Quantum Integrability

Giving a precise definition of quantum integrability is a difficult task, given the wide variety of cases that it is possible to encounter. Usually one starts by considering spin chains, or the quantum counterpart of some classical integrable systems described above, but for our purposes we are going to describe the situation for a very specific case: quantum field theories in 1+1 dimensions.

The reasons why we focus on 2-dimensional systems are multiple, but one hint comes from the renown *Coleman-Mandula* theorem [10]. This fundamental result, which is based on several assumptions[2], states that the most general symmetry underlying any massive QFT in 3+1 dimensions, can be expressed as the direct product of the Lorentz-Poincaré group $\mathcal{A}$ and a group of internal symmetry $\mathcal{I}$, namely

$$\mathcal{A} \otimes \mathcal{I}. \tag{1.13}$$

In particular the authors showed that any theory in $1 + d$ dimensions ($d > 1$) which admits a conserved charge with spin higher that 1 have a trivial S-matrix, *i.e.* there is no interaction between particles. But these arguments surprisingly do not apply in the (1+1)-dimensional case: here space-time and internal symmetries can be combined in a non trivial way and in particular conserved charges of arbitrary spin do not affect the nature of the S-matrix.

It becomes clear that the S-matrix has a relevant role in the study of integrable systems and for this reasons we are going to review some theory about the scattering processes.

1.2.1 The S-Matrix

Let us consider a theory defined in generic space-time dimensions with a short range interaction: this always allows to consider the particles in an asymptotic state, *i.e.* far enough from the other particles for which the interaction can be neglected. Since in scattering processes the particles involved are only of physical nature with momentum p_μ, they must be on-shell, namely

$$p_\mu p^\mu = m^2. \tag{1.14}$$

The scattering matrix is defined as the unitary and invariant operator that transforms an incoming state into an outgoing state, namely

$$|f\rangle = S\,|i\rangle\,. \tag{1.15}$$

[2] In the years the assumptions of the theorem have been relaxed, for example if one allows also anticommutation relations between the generators, will obtain the supersymmetry algebra, see [11].

To prove unitarity, one can consider a superposition of basis vectors $|n\rangle$ of the Hilbert space,

$$|\psi\rangle = \sum\nolimits_n \alpha_n |n\rangle \,, \tag{1.16}$$

then the conservation of total probability leads to the following relation

$$1 = |\, \langle\psi'|\, S\, |\psi\rangle\, |^2 = \sum\nolimits_{m,n} \alpha_m^* \alpha_n \langle m|\, S^\dagger S\, |n\rangle \,, \tag{1.17}$$

which holds true only if the S-matrix is unitary:

$$S^\dagger S = \mathbb{1}. \tag{1.18}$$

Since it is reasonable to think that transition amplitudes do not change under a generic Lorentz transformation, we expect the S-matrix to be an invariant itself, which means that it is a function of Lorentz scalars only. These scalars are nothing but the Mandelstam variables s, t and u.

In the 2-dimensional case, it is convenient to describe the energy and momentum of particles (p^0, p^1) in terms of rapidities,

$$p_i^0 = m_i \cosh\theta, \qquad p_i^1 = m_i \sinh\theta_i, \tag{1.19}$$

and for the Lorentz invariance property, the S-matrix will be a function of the difference $\theta_i - \theta_j$ only.

The incoming and outgoing states are generated by the iterative action of some vertex operator $V_a(\theta_a)$ on the vacuum,

$$|V_{a_1}(\theta_1)\dots V_{a_n}(\theta_n)\rangle = V_{a_1}^\dagger(\theta)\dots V_{a_n}^\dagger(\theta_n)\,|0\rangle \,. \tag{1.20}$$

These operators are generalization of the usual creation/destruction operator which satisfy the so called *Faddeev-Zamolodchikov algebra*:

$$V_a(\theta_1)V_b(\theta_2) = \sum\nolimits_{cd} S_{ab}^{cd}(\theta_{12})V_d(\theta_2)V_c(\theta_1), \tag{1.21a}$$

$$V_a^\dagger(\theta_1)V_b^\dagger(\theta_2) = \sum\nolimits_{cd} S_{ab}^{cd}(\theta_{12})V_d^\dagger(\theta_2)V_c^\dagger(\theta_1), \tag{1.21b}$$

$$V_a(\theta_1)V_b^\dagger(\theta_2) = \sum\nolimits_{cd} S_{ab}^{cd}(-\theta_{12})V_d(\theta_2)V_c(\theta_1) + 2\pi\delta_{ab}\delta(\theta_{12}). \tag{1.21c}$$

In particular for the incoming particles $\theta_1 > \theta_2 > \cdots > \theta_n$ while for the outgoing ones $\theta_1 < \theta_2 < \cdots < \theta_n$. This prescription on the rapidities can be understood looking at figure 1.1.

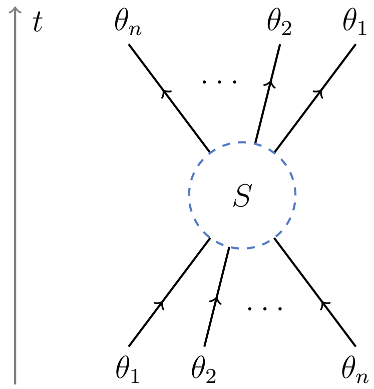


Figure 1.1: n-body collision event with rapidity ordering prescription.

If the considered theory is also integrable, and therefore there exist infinite conserved charges $\mathcal{Q}_i$, several constraints are induced on the form of the S-matrix.

In particular in 1980 S. J. Parke [12] proved the so-called ***Parke theorem***, which states that the presence of at least two higher-spin charges ensures two fundamental properties of the scattering process:

- Absence of particle production, meaning that the masses m_i and rapidities θ_i are conserved before and after the collision process;
- Factorization of the S-matrix, ***i.e.*** the property that a n particle collision can be completely factorized in $n(n-1)/2$ subsequent 2-particles scattering events.

These results can be written in a single condition as

$$S^{b_1 b_2 \dots b_m}_{a_1 a_2 \dots a_n}(\theta_{a_1}, \dots, \theta_{a_n}; \theta_{b_1}, \dots, \theta_{b_n}) = \delta_{nm} \times \prod_{i=1}^{n} \delta(\theta_{a_i} - \theta_{b_i}) \prod_{\substack{i,j,k,l=1 \\ i<j,k<l}}^{n} S^{b_l b_k}_{a_i a_j}(\theta_{a_i} - \theta_{a_j}). \tag{1.22}$$

The proof of Parke theorem is particularly long and cumbersome, and we are not going to present it here, but the important fact on which it is based is that the absence of particle production allows to formulate the problem in a "classical" way, *i.e.* it is possible to introduce a wave function to describe the set of asymptotic particles: in this way it is for example possible to prove the equivalence between the two amplitudes of the $3 \to 3$ processes depicted in figure 1.2. This equivalence leads to the renown Yang-Baxter equation

$$S_{a_1 a_2}^{\alpha\beta}(\theta_{12}) S_{\alpha a_3}^{b_1 \gamma}(\theta_{13}) S_{\beta\gamma}^{b_2 b_3}(\theta_{23}) = S_{a_1 \alpha}^{\gamma b_3}(\theta_{12}) S_{a_2 a_3}^{\beta\alpha}(\theta_{12}) S_{\gamma\alpha}^{b_1 b_2}(\theta_{12}), \tag{1.23}$$

where $\theta_{ij} = \theta_i - \theta_j$.

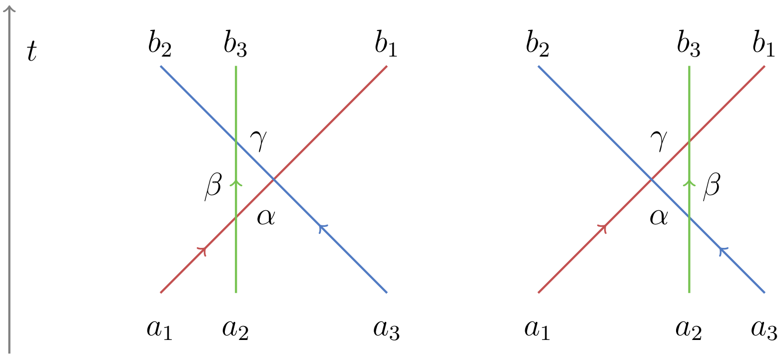


Figure 1.2: Two different but equivalent amplitudes of the $3 \to 3$ scattering process: their equivalence leads to the Yang-Baxter equation.

1.2.2 2-Particle S-Matrix

Since in every integrable QFT the factorizability of the S-matrix is always guaranteed by Parke theorem, it is important to understand better the general properties of the 2-particles S-matrix. For what said above,

$$|V_a(\theta_1)V_b(\theta_2)\rangle_{in} = S_{ab}^{cd}(\theta_{12})\, |V_c(\theta_1)V_d(\theta_2)\rangle_{out}\,, \quad \theta_1 > \theta_2. \tag{1.24}$$

Since momenta are conserved in 1+1 dimensional scattering theories, the Mandelstam variable $u = 0$, while the other two are linearly dependent $t(\theta_{12}) = s(i\pi - \theta_{12})$, therefore the S-matrix elements are function of s (or t) only.

The S-matrix should be invariant under discrete symmetries C, P and T, leading to the following relations:

- charge conjugation C:

$$S_{ab}^{cd}(\theta) = S_{\bar{a}\bar{b}}^{\bar{c}\bar{d}}(\theta), \tag{1.25}$$

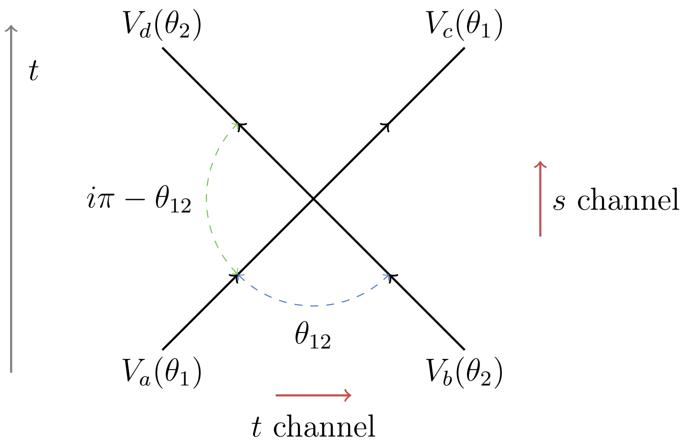


Figure 1.3: Situation in a two-particle scattering in 1+1 dimensions. The two available channels are related by a rotation of an angle π.

i.e. the scattering amplitude does not change by replacing every particle with its own antiparticle;

- parity P:

$$S_{ab}^{cd}(\theta) = S_{ba}^{dc}(\theta); \tag{1.26}$$

- time reversal T:

$$S_{ab}^{cd}(\theta) = S_{dc}^{ba}(\theta). \tag{1.27}$$

The unitarity condition (1.18) still holds, and in this case can be rewritten in terms of the components of S, namely

$$\sum\nolimits_{cd} S_{ab}^{cd}(\theta) S_{cd}^{ef}(-\theta) = \delta_a^e \delta_b^f. \tag{1.28}$$

Another property called ***crossing symmetry*** comes from the equivalence that can be established between the two possible scattering channel t and s, as can be seen in figure 1.3, namely

$$S_{ab}^{cd}(\theta) = S_{\bar{d}a}^{\bar{b}c}(i\pi - \theta). \tag{1.29}$$

Finally one is interested in the analiticy properties of the S-matrix, which can be studied by writing the Mandelstam variable s in terms of rapidities

$$s = (p_a + p_b)^2 = m_a^2 + m_b^2 + 2m_1 m_b \cosh\theta_{12}. \tag{1.30}$$

If one performs an analytic continuation on the real variable s, two branch cuts will arise for $s > (m_a + m_b)^2$ and $s < (m_a - m_b)^2$. The physical region is represented by the values of s that lies above the branch cut on the right, *i.e.* $s^+ = s + i0$ with $s > (m_a + m_b)^2$. Since S is a real analytic function, one can assume complex conjugate values at complex conjugate points,

$$S_{ab}^{cd}(s^*) = (S_{ab}^{cd}(s))^*. \tag{1.31}$$

Imposing unitarity condition (1.18) for this physical sheet, one obtain

$$S_{ab}^{cd}(s^+)(S_{dc}^{ef}(s^+))^* = \delta_a^e \delta_b^f, \tag{1.32}$$

and employing the real analicity property one finds that

$$S_{ab}^{cd}(s^+) S_{cd}^{ef}(s^-) = \delta_a^e \delta_b^f. \tag{1.33}$$

where $s^- = s - i0$. It is now possible to translate this condition for the θ variable. Inverting equation (1.30) one finds

$$\theta_{12} = \log \frac{s - m_a^2 - m_b^2 + \sqrt{(s - (m_a + m_b)^2)(s - (m_a - m_b)^2)}}{2 m_a m_b}. \tag{1.34}$$

This relation maps the physical sheet of the s-plane into the physical strip $0 \leq \mathrm{Im}\{\theta_{12}\} \leq \pi$. Other strips are mapped periodically on the strips $n\pi \leq \mathrm{Im}\,\theta \leq (n+1)\pi$ where $n = \ldots, -1, 0, 1, \ldots$. The integrability of the theory implies that $S(\theta)$ is a meromorfic function and it assumes real values only on the imaginary θ-axis.

So far we have analysed the property of the S-matrix, without talking about the particle content of the theory. It turns out that the presence of a simple pole in the physical strip is the signal of the presence of a bound state in the particle spectrum of the theory. Let us suppose that the S-matrix present a pole of the form iu_{ab}^k. The amplitude corresponding to this pole is

$$S_{ab}^{cd} \simeq i \frac{\mathbb{P}_{ab}^k R^k \mathbb{P}_k^{cd}}{\theta - iu_{ab}^k}, \tag{1.35}$$

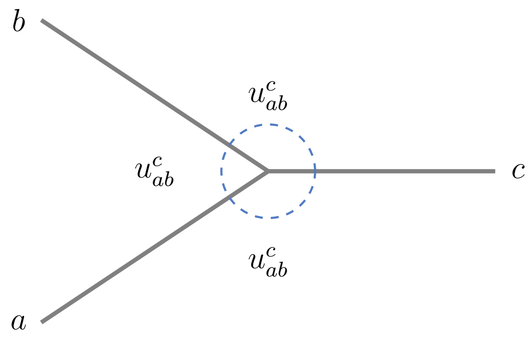


Figure 1.4: Graphical representation of the fusing angles.

where R^k is the residue which has been projected through the operator $\mathbb{P}$ from the 1-particle space onto the bound state space. From the definition of s, the mass of the bound state reads

$$m_k^2 = m_a^2 + m_b^2 + 2m_a m_b \cos u^k_{ab}. \tag{1.36}$$

The quantity u^c_{ab} is called ***fusing angle*** and employing the crossing symmetry property it is possible to find the other angles obtained from different channels.

Looking at figure 1.4, the relation that connects these three numbers is clearly

$$u^a_{bc} + u^b_{ac} + u^c_{ab} = 2\pi. \tag{1.37}$$

If these bound states are considered on the same footing of the other particles, then the amplitudes for bound states can be written in terms of those of the particles, and vice versa, namely

$$S_{cd}(\theta) = S_{ad}(\theta + i\bar{u}^b_{a\bar{c}})S_{bd}(\theta - i\bar{u}^a_{b\bar{c}}), \tag{1.38}$$

where $\bar{u} = \pi - u$. This relation can be graphically visualised by the equivalence of the two processes represented in figure 1.5.

This is the renown ***bootstrap equation***, and it represent one of the most powerful tool in the integrability realm. One can start by finding all the poles of an S-matrix, which means finding all the bound states of the theory. Then, using equation (1.38) and equation (2.36) one can evaluate all the scattering amplitudes of the theory.

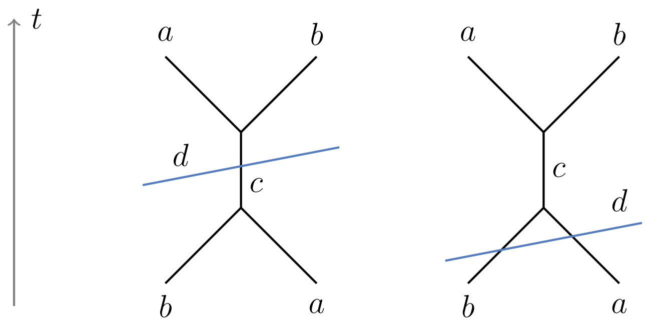


Figure 1.5: The two scattering processes described by the bootstrap equation (2.38).

Finally, one has to check if the theory has been correctly "bootstrapped", *i.e.* if the particle spectrum just found is the one which reproduces correctly the set of poles of the starting S-matrix. This procedure has been widely used in the '80s to find the exact S-matrices of numerous models, see e.g. [13], [14], [15], [3].

2

Thermodynamic Bethe Ansatz

In this chapter we are going to discuss the Thermodynamic Bethe Ansatz method (TBA), introduced for the first time in 1969 by C. N. Yang and C. P. Yang [16] to study the thermodynamic of a one-dimensional system of bosons which interact through a δ-potential and then generalized to the relativistic case in the early '90s by the work of Al. Zamolodchikov [17, 18] and R. Klassen and E. Melzer [19, 20].

The power of TBA consists in the fact that knowing an S matrix, which can be built up in two dimensions using few general assumptions, namely unitarity (1.28), crossing symmetry (1.29) and the fulfillment of Yang-Baxter equation (1.23), one can inspect the thermodynamic of a theory in an infinite volume ***i.e.*** computing quantities such as energy or entropy. Finally, considering integrable QFTs as particular perturbations of CFTs with some relevant fields preserving integrability as shown in [13], one can see the CFT as the high-energy (or UV) limit of the corresponding QFT. We will analyze two different situations, first when the S-matrix of the theory is diagonal ***i.e.*** it describes purely elastic scatterings and then when it is not diagonal.

In the former case, as it was noticed by Al. Zamolodchikov in [21] and then formalized in [22], there is a close connection between these theories and A, D, E Lie algebras: in particular it can be shown that the structure of the TBA equations emerges directly from the Dynkin diagram of an underlying algebra.

In the latter case instead, there is no such a clear connection between the two and one has to find the TBA equations of the problem using more subtle methods.

2.1 Diagonal S-matrix Theories

To begin with, let us consider a ***purely elastic scattering theory***, ***i.e.*** a QFT in $1+1$ dimensions described by a factorizable diagonal S-matrix. In particular, one can consider N particles[1], distributed in the positions $x_1 < x_2 < \cdots < x_N$ on a circle of length L large enough so that they can be considered in an asymptotic state, which means that their mutual interaction is neglectable. For this reason, it is possible to describe the whole system with a wave function which is the sum of plane waves with unknown amplitudes, also known as Bethe ansatz wave function, which was introduced by H. Bethe in [23]. Defining the N momenta that still have to be determined with p_j, one can explicitly write

$$\Psi(x_1, \ldots, x_N; \mathcal{Q}) = \sum_{\mathcal{P}} A_{\mathcal{P}}(\mathcal{Q}) e^{i \sum_j p_{\mathcal{P}_j} x_j}, \tag{2.1}$$

where $\mathcal{P}$ is a permutation of the quasi-momenta, $\mathcal{Q}$ is the permutation that specifies the particle order, also called ***simplex*** and $A_{\mathcal{P}}(\mathcal{Q})$ is the unknown amplitude.

If one tries to interchange two of the particles on the circle $\{\ldots, i, j, \ldots\} \to \{\ldots, j, i, \ldots\}$, ***i.e.*** changes the simplex $\mathcal{Q} \to \mathcal{Q}'$, the wave function construction would be broken, since the interaction becomes significant; however in this situation all the informations needed to describe the scattering event are encoded in the well known scattering matrix $S_{ij}(p_i, p_j)$.

Therefore for consistency one gets

$$A_{\mathcal{P}}(\mathcal{Q}') = S_{ij}(p_i, p_j) A_{\mathcal{P}'}(\mathcal{Q}), \tag{2.2}$$

which means that the amplitude obtained by exchanging two particles is equivalent to the amplitude relative to the initial simplex but with the momenta exchanged and multiplied by the corresponding scattering element. As discussed in the previous chapter, it is convenient to introduce the rapidities θ_i, which are a useful parametrization for particles' momenta in $1+1$ dimensions

$$(p_i^0, p_i^1) = (m_i \cosh \theta_i, m_i \sinh \theta_i), \tag{2.3}$$

[1] The generalization to the case of different types of particles is straightforward.

such that S_{ij} is a function of the difference $\theta_i - \theta_j$ only. Since the spacial variable is compactified on the circle, periodic (anti-periodic) boundary conditions can be imposed if the particles are bosons (fermions):

$$\Psi(x_1, \ldots, x_i, \ldots, x_N; \mathcal{Q}) = \pm\Psi(x_1, \ldots, x_i + L, \ldots, x_N; \mathcal{Q}), \tag{2.4}$$

for every $i = 1, \ldots, N$ and every simplex $\mathcal{Q}$.

However, bringing the particle from the initial position x_i to the final $x_i + L$ on the circle, one has to take into account all the scattering processes that take place, as shown in figure 2.1 , thus

$$\Psi(x_1, \ldots, x_i + L, \ldots, x_N; \mathcal{Q}) = e^{ip_iL} \prod_{j\neq i} S_{ij}(\theta_{ij})\Psi(x_1, \ldots, x_i, \ldots, x_N; \mathcal{Q}), \tag{2.5}$$

where, for the sake of brevity, $\theta_{ij} = \theta_i - \theta_j$.

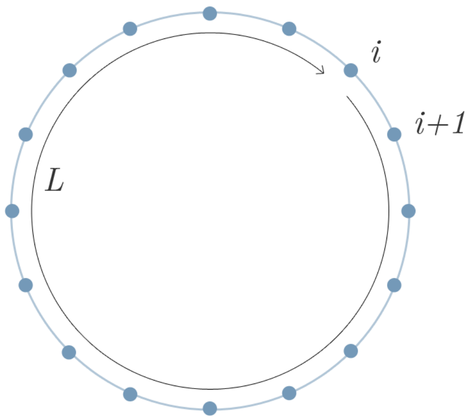


Figure 2.1: Imposing (anti)periodic boundary conditions the the i-th particle scatters with the other particles on the circle and the wave function doesn't change.

Therefore, considering equations (2.3), (2.4) and (2.5) one finally gets the quantization condition for the rapidities:

$$e^{iLm_i \sinh\theta_i} \prod_{j\neq i} S_{ij}(\theta_{ij}) = \pm 1 \qquad \text{for,} \quad i = 1, \ldots, N \tag{2.6}$$

also known as Bethe Equations, whose solutions are the admissible rapidities of the particles, also called ***roots***. Since the theory is relativistic, once equation (2.6) is solved the energy and the spatial momentum of the system are given by:

$$\mathcal{E} = \sum_{i=1}^{N} m_i \cosh\theta_i, \qquad \mathcal{P} = \sum_{i=1}^{N} m_i \sinh\theta_i. \tag{2.7}$$

Taking the logarithm on both sides of equation (2.6) one gets

$$L m_i \sinh\theta_i + \sum_{j\neq i} \delta_{ij}(\theta_i - \theta_j) = 2\pi n_i \tag{2.8}$$

where $\delta_{ij}(\theta_i - \theta_j) = -i \ln S_{ij}(\theta_i - \theta_j)$ represent the phase shift of the wave function. The numbers $\{n_i\}_{i=1}^{N}$ can be understood as occupation numbers denoting the accessible states and they are integers in the bosonic case or half-integers in the fermionic one. Together with the rapidities, they denote each possible state of the system:

$$|n_1, \theta_i; \ldots; n_N, \theta_N\rangle \tag{2.9}$$

Furthermore, if particles are identical, one has to take into account additional section rules on the set of rapidities $\{\theta_1, \ldots, \theta_N\}$ since if particle are bosons, the Bethe wave function must be symmetric under the exchange of two of them, while if they are fermions, the function must be anti-symmetric. Since the S-matrix has to be unitary, see equation (1.28), one has two possible cases:

- In the first case

  $$S_{ij}(0) = -1, \tag{2.10}$$

  which means that the wave function becomes anti-symmetric under exchange of two particles. This is clearly not compatible with Bose-Einstein statistic: this means that if one is dealing with bosons, each value of the rapidity can be taken by one particle only (in some sense it is possible to say that they have a ***fermion-like*** behaviour), thus all integers n_i are different. Otherwise, if one is dealing with fermions this case is in perfect agreement with Fermi-Dirac statistic and there are no restrictions on the values of the n_i (that is a ***boson-like*** behaviour).

- In the second case

$$S_{ij}(0) = 1, \tag{2.11}$$

which is the opposite of the previous situation: bosons behave like bosons and fermions behave like fermions.

2.1.1 Thermodynamics and Finite Size Effects

The next step to fully develop the TBA, is to deform the cylindrical geometry of the two-dimensional space-time by imposing periodic boundary conditions also on the time dimension. Since the space dimension was already compactified, one is now dealing with the topology of a torus generated by two geodesic of circumference R and L respectively, as shown in figure 2.2.

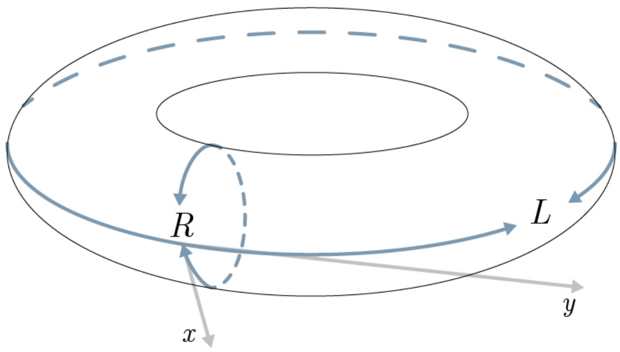


Figure 2.2: Graphical representation of the torus generated by the two circumferences $\mathcal{C}_R$ and $\mathcal{C}_L$.

It is easy to see that the old geometry can be re-obtained by considering the cylinder as a limiting case of the torus, where the $\mathcal{C}_L$ circumference is sent to infinity.

There are two possible topologically equivalent ways to quantize such a theory, based on the choice one takes for the axes:

- x as space dimension and y as time dimension. In this case quantum states which live in the Hilbert space $\mathcal{H}_R$ are evolved by the Hamiltonian

$$H_R = \frac{1}{2\pi} \int_{\mathcal{C}_R} dx\, T_{yy}. \tag{2.12}$$

Considering the partition function, in the $L \to \infty$ limit, the main contribution is given by the lowest value of the energy $\mathcal{E}_0(R)$,

$$\mathcal{Z}(R, L) \simeq \lim_{L\to\infty} \mathrm{Tr}_{\mathcal{H}_R} e^{-H_R L} \simeq e^{-\mathcal{E}_0(R)L} \tag{2.13}$$

- y as space dimension and $-x$ as time dimension[2]. In this case quantum states which live in the Hilbert space $\mathcal{H}_L$ are evolved by the Hamiltonian

$$H_L = \frac{1}{2\pi} \int_{\mathcal{C}_R} dy\, T_{xx}. \tag{2.14}$$

Taking now the same $L \to \infty$ limit, the length R can be understood as the inverse of a temperature and for this reason the leading term of the partition function is

$$\mathcal{Z}(R, L) \simeq \lim_{L\to\infty} \mathrm{Tr}_{\mathcal{H}_L} e^{-H_L R} \simeq e^{-Lf(R)R}, \tag{2.15}$$

where $f(R)$ is the free energy per unit length at temperature $1/R$.

Comparing the two results (2.13) and (2.15) one finds the fundamental relation

$$\mathcal{E}_0(R) = Rf(R), \tag{2.16}$$

which shows the link between the vacuum energy of the theory and the thermodynamics of the system.

It is possible to parametrize the ground state energy as (see [24])

$$\mathcal{E}_0(R) = -\frac{\pi \widetilde{c}(r)}{6R}, \tag{2.17}$$

where $r = mR$ is a dimensionless parameter and $\widetilde{c}(r)$ is a scaling function that can be directly computed via TBA. The most important property of this function is its ultraviolet limit, *i.e.* $r \to 0$, for which the behaviour of (2.17) is controlled by the underlying conformal field theory. Indeed, it is possible to show that

$$\mathcal{E}_0(R) = \frac{2\pi}{R}\left(\Delta_{\mathsf{min}} + \bar{\Delta}_{\mathsf{min}} - \frac{c}{12}\right), \tag{2.18}$$

where Δ_{min} is the scaling dimension of the lowest operator. If then $\Delta_{\mathsf{min}} = \bar{\Delta}_{\mathsf{min}}$,

[2] The negative sign is to preserve the frame orientation

the function $\widetilde{c}(r)$ tends to the effective central charge of the theory

$$c_{\mathrm{eff}} = c - 24\Delta_{\mathrm{min}}. \tag{2.19}$$

$\mathcal{E}_0$ represents the Casimir energy of the model, which is described in more detail in Appendix A.

2.1.2 Thermodynamic Limit

Since the system of transcendental equations (2.8) is in general difficult to solve, one can try to study them in the thermodynamic limit[3], *i.e.*

$$L \to \infty \qquad N \to \infty \qquad \text{with} \quad N/L = \text{constant}. \tag{2.20}$$

In this situation the number of particles grows $\sim L$ as $L \to \infty$ and for this reason the spectrum of the rapidities tends to become dense, the distance between two consecutive levels being of order $|\theta_{i+1} - \theta_i| \sim 1/mL$. Therefore it is convenient to introduce a continuous rapidity density of particles and of available states

$$\sigma(\theta) = \lim_{TD} \frac{m_{i+1} - m_i}{\theta_{i+1} - \theta_i}, \qquad \sigma_{\mathrm{tot}}(\theta) = \sigma(\theta) + \widetilde{\sigma}(\theta) = \lim_{TD} \frac{n_{i+1} - n_i}{\theta_{i+1} - \theta_i} \tag{2.21}$$

respectively, where $m_{i+1} - m_i$ is the number of particles in a given interval of rapidities $\theta_{i+1} - \theta_i$, while $n_{i+1} - n_i$ is the number of allowed states, not necessarily filled with particles, in the same interval. In equation (2.21) the holes density $\widetilde{\sigma}$ has been introduced for later purposes.

In fact, in the thermodynamic limit, one can estimate discrete sums over rapidities as integrals:

$$\sum_i f(\theta_i) \xrightarrow{\mathrm{TD}} \int d\theta\, f(\theta)\sigma(\theta) \tag{2.22}$$

For example, thanks to this fact, one can compute the energy defined in (2.7) as

$$\mathcal{E}[\sigma(\theta)] = \int_{-\infty}^{+\infty} d\theta\, m \cosh\theta\sigma(\theta). \tag{2.23}$$

It is now possible to study the thermodynamic limit of equation (2.8), replacing

[3] In the following, we denote by TD this limiting procedure.

the sum with the integral as shown in (2.22)

$$mL\sinh\theta_i + \int_{-\infty}^{+\infty} d\theta'\,\sigma(\theta')\delta(\theta_i - \theta') = 2\pi n_i. \tag{2.24}$$

Subtracting this result by the same equation obtained for $i-1$ and dividing by $\theta_i - \theta_{i-1}$, together with equation (2.21), in the continuum limit one finds

$$mL\cosh\theta + (\varphi * \sigma)(\theta) = 2\pi\sigma_{\text{tot}}(\theta), \tag{2.25}$$

where the kernel has been introduced

$$\varphi(x) = \frac{\mathrm{d}\delta(x)}{\mathrm{d}x} = -i\frac{\mathrm{d}\ln S(x)}{\mathrm{d}x}. \tag{2.26}$$

The convolution of two functions is defined in the usual way

$$(f * g)(\theta) = \int_{-\infty}^{+\infty} d\theta'\, f(\theta - \theta')g(\theta') \tag{2.27}$$

To study the thermodynamic at the equilibrium one has to minimize the free Helmholtz energy

$$\mathcal{F}[\sigma,\widetilde{\sigma}] = \mathcal{E}[\sigma] - T\mathcal{S}[\sigma,\widetilde{\sigma}]. \tag{2.28}$$

where the energy has been already defined in (2.23) and, as mentioned before, since the theory is defined on a compactified time direction of circumference R, one can identify $1/R$ as the temperature T of the system. The last ingredient is the entropy $\mathcal{S}$, which can be calculated starting from the number of ways of distributing m_i particles among n_i levels:

$$\Omega_F = \frac{m_i!}{n_i!(m_i - n_i)!} \tag{2.29}$$

if the particles are fermions or

$$\Omega_B = \frac{(n_i + m_i + 1)!}{n_i!(n_i - 1)!} \tag{2.30}$$

if they are bosons. In the thermodynamic limit the number of levels can be approximated with $n_i \sim L\rho(\theta_i)\Delta\theta$ and that of particles with $m_i \sim L\sigma(\theta_i)\Delta\theta$, therefore, since $\mathcal{S} = \ln \Omega_{B/F}$, the entropies in the two cases are

$$\mathcal{S}_F[\sigma, \widetilde{\sigma}] = \int d\theta \, [(\sigma + \widetilde{\sigma}) \ln(\sigma + \widetilde{\sigma}) - \sigma \ln \sigma - \widetilde{\sigma} \ln(\widetilde{\sigma})], \tag{2.31}$$

$$\mathcal{S}_B[\sigma, \widetilde{\sigma}] = \int d\theta \, [(\sigma_{\text{tot}} + \sigma) \ln(\sigma_{\text{tot}} + \sigma) - \sigma_{\text{tot}} \ln \sigma_{\text{tot}} - \sigma \ln \sigma]. \tag{2.32}$$

To minimize the free energy (2.28) one can introduce the Lagrange multiplier $\xi(\theta)$ and define the functional

$$\Phi[\sigma, \widetilde{\sigma}, \xi] = R\mathcal{E}[\sigma] - \mathcal{S}[\sigma, \widetilde{\sigma}] + \xi(\theta)\mathcal{V}[\sigma, \widetilde{\sigma}], \tag{2.33}$$

where $\mathcal{V}[\sigma, \widetilde{\sigma}]$ is the constraint equation (2.25). In the fermionic case one will find

$$\begin{aligned}\Phi[\sigma, \widetilde{\sigma}, \xi] = \int d\theta \, [&mR \cosh \theta \sigma - (\sigma + \widetilde{\sigma}) \ln(\sigma + \widetilde{\sigma}) + \sigma \ln \sigma + \widetilde{\sigma} \ln(\widetilde{\sigma}) \\ &+ \xi(2\pi(\sigma + \widetilde{\sigma}) - mL \cosh \theta - (\varphi * \sigma)].\end{aligned} \tag{2.34}$$

Taking functional derivatives with respect to σ, $\widetilde{\sigma}$ and ξ and setting them equal to zero one gets the extremal conditions

$$\frac{\delta \Phi}{\delta \sigma} = mR \cosh \theta - \ln(\sigma + \widetilde{\sigma}) + \log \sigma + 2\pi\xi - (\varphi * \xi) = 0, \tag{2.35a}$$

$$\frac{\delta \Phi}{\delta \widetilde{\sigma}} = - \ln(\sigma + \widetilde{\sigma}) + \ln \widetilde{\sigma} + 2\pi\xi = 0, \tag{2.35b}$$

$$\frac{\delta \Phi}{\delta \xi} = 2\pi(\sigma + \widetilde{\sigma}) - mL \cosh \theta - \varphi * \sigma = 0. \tag{2.35c}$$

Using equation (2.35b), one can solve for ξ and then substituting into (2.35a) one finds

$$mR \cosh \theta - \ln \frac{\widetilde{\sigma}}{\sigma} - \left(\varphi * \frac{1}{2\pi} \ln \frac{\sigma + \widetilde{\sigma}}{\widetilde{\sigma}} \right) = 0. \tag{2.36}$$

This equation can be written in a more elegant way introducing the ***pseudoenergy*** defined as follows

$$\epsilon(\theta) = \log \frac{\widetilde{\sigma}(\theta)}{\sigma(\theta)}, \tag{2.37}$$

obtaining

$$\epsilon(\theta) = mR \cosh \theta - \frac{1}{2\pi} \Big(\varphi * \ln \big(1 + e^{-\epsilon(\theta)} \big) \Big). \tag{2.38}$$

The calculations in the bosonic follow the same steps, introducing another pseudoenergy

$$\epsilon(\theta) = \log \frac{\sigma_{\text{tot}} + \sigma}{\sigma}. \tag{2.39}$$

It is possible to collect the two result in one equation, keeping in mind that the upper sign refers to fermion and the lower to bosons:

$$\epsilon(\theta) = mR\cosh\theta \mp \frac{1}{2\pi}\Big(\varphi * \ln\big(1 \pm e^{-\epsilon(\theta)}\big)\Big), \tag{2.40}$$

which is a non linear integral equation called ***Thermodynamic Bethe Ansatz Equation*** whose solutions are the pseudoenergies of the particles. For example, starting from (2.16) and (2.28) and using (2.23) and (2.32) or (2.33), it is possible to calculate the Casimir energy $\mathcal{E}_0$ in the fermionic case in terms of the pseudoenergies:

$$\begin{aligned}
\mathcal{E}_0(R) &= \frac{R\mathcal{F}(R)}{L} = \frac{1}{L}\int d\theta\, R\mathcal{E}(R) - \mathcal{S}_F \\
&= \frac{1}{L}\int d\theta \Big(mR\sigma\cosh\theta - (\sigma + \widetilde{\sigma})\ln(\sigma + \widetilde{\sigma}) + \sigma\ln\sigma + \widetilde{\sigma}\ln(\widetilde{\sigma})\Big) \\
&= \frac{1}{L}\int d\theta \left[\sigma\left(mR\cosh\theta + \ln\left(\frac{\sigma}{\sigma + \widetilde{\sigma}}\right)\right) + \widetilde{\sigma}\ln\left(\frac{\widetilde{\sigma}}{\sigma + \widetilde{\sigma}}\right)\right] \\
&= \frac{1}{2\pi L}\int d\theta\, \ln\left(\frac{\widetilde{\sigma}}{\sigma + \widetilde{\sigma}}\right)\big(2\pi(\sigma + \widetilde{\sigma}) + (\varphi * \sigma)\big) \\
&= -\frac{m}{2\pi}\int d\theta\, \cosh\theta \ln\left(1 + \frac{\sigma}{\widetilde{\sigma}}\right) = \frac{m}{2\pi}\int d\theta\, \cosh\theta\ln\left(1 + e^{-\epsilon(\theta)}\right),
\end{aligned} \tag{2.41}$$

where equations (2.35a), (2.35c) have been used. In the second last line the order of convolution has been changed[4]. Performing similar calculations one finds the result for the bosonic case:

$$\mathcal{E}_0(R) = \frac{m}{2\pi}\int d\theta\, \cosh\theta\ln\left(1 - e^{-\epsilon(\theta)}\right). \tag{2.42}$$

At this point it becomes clear that once one has found the solutions of TBA equation (2.40), one can potentially solve exactly the thermodynamic of the problem, in the sense that no perturbative expansions are needed (see e.g. [16]). However, the main problem that arise is that equation (2.40) in a non-linear integral equation and for this reason, difficult to solve, but apart from that it still gives a lot of informations

[4]Namely, $\int h(x)(f*g)(x) = g(x)(f*h)(x)$, which holds whether $f(x) = f(-x)$, as in this case.

about the considered theory as it will be shown in the next sections.

2.1.3 Ultraviolet and Infrared Regimes

The first attempt of solution that can be done of equation (2.40), is by studying its two asymptotic regimes, namely the high-energy limit, or UV limit, or even conformal limit, and the low energy limit, or IR limit. Let us firstly analyze the ultraviolet regime[5]. The TBA equation is

$$\epsilon(\theta) = r\cosh\theta - \frac{1}{2\pi}(\varphi * L)(\theta), \tag{2.43}$$

where $r = mR$ is the dimensionless scaling length, while

$$L(\theta) = \ln\big(1 + e^{-\epsilon(\theta)}\big) \tag{2.44}$$

Note ***en passant*** that both $\epsilon(\theta)$ and $L(\theta)$ are even function. To study the UV regime, where the mass scale goes to zero, or equivalently the correlation length tends to be infinite, one can send $r \to 0$.

In this case it is possible to notice that the solutions tend to flatten in a central region, known as ***plateau region*** as shown in figure 2.3.

In this interval, which can be roughly estimated to be $[-\ln(2/r), \ln(2/r)]$, the function becomes constant and this constant value is known as ***plateau value***, given by the transcendental equation

$$\epsilon_a = \sum_{b=1}^{n} N_{ab}\ln\big(1 + e^{-\epsilon_b}\big), \tag{2.45}$$

where

$$N_{ab} = -\frac{1}{2\pi}\int d\theta\, \varphi_{ab}. \tag{2.46}$$

To study the conformal limit one can compare equations (2.17) and (2.41) to define

[5] We will now focus on particles of fermionic type, *i.e.* with $S(0) = -1$. This because there is strong reason to think these interacting theories are the only fully consistent, since in the bosonic case negative pseudoenergies, that is to say complex rapidities, can appear and we can't give any physical interpretation

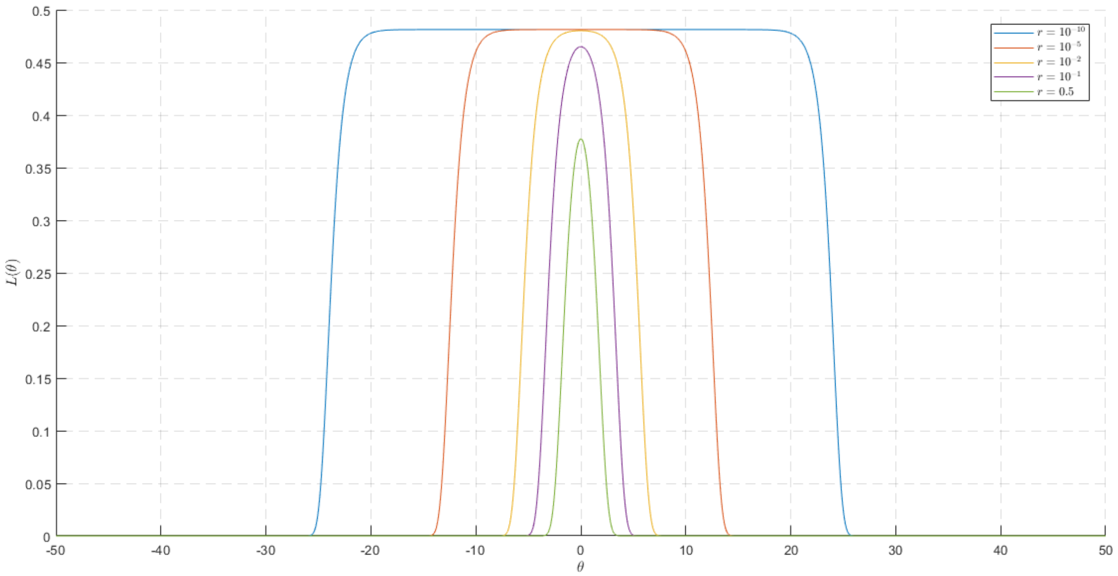


Figure 2.3: Plots of $L(\theta)$ for different values of r for the 3-Potts model, computed numerically with MATLAB. One can clearly see that while r becomes small, the plateau region starts to form.

the scaling function

$$\widetilde{c}(r) = \frac{3r}{\pi^2} \int d\theta \, \cosh\theta L(\theta). \tag{2.47}$$

Looking at the behaviour of this function in the $r \to 0$, one can extract informations about the underlying conformal field theory, *i.e.* it is possible to find the value of the effective central charge, thanks to equation (2.18). For example in Figure 2.4 the scaling function for the 3-Potts model has been plotted, where it is possible to see that for $r = 0$, the function seems to approach the expected value $\widetilde{c}(0) = 2/5 = 0.4$.

Since, as mentioned before, $\epsilon(\theta)$ and $L(\theta)$ are even function in θ, it is possible to replace the lower boundary of the integral in equation (2.43) with zero, multiplying by 2. Furthermore, it is easy to see that when $r \to 0$, the r dependence is encoded in a displacement of the falloff of the plateau. For this reason, when studying the UV limit, one can shift the θ variable to the edge and approximate $\cosh\theta$ with $\frac{e^\theta}{2}$[6], finding a new TBA equation

$$\epsilon_{\mathsf{kink}}(\theta) = \frac{1}{2} r e^\theta - \frac{1}{2\pi}(\varphi * L_{\mathsf{kink}}(\theta)), \tag{2.48}$$

[6] This is valid only in the $\theta \gg 0$ limit, that is outside the central plateau region.

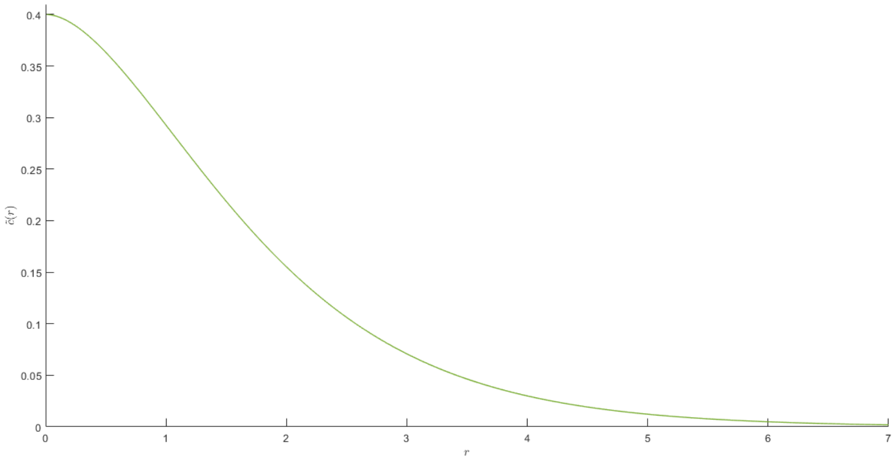


Figure 2.4: Behaviour of $\widetilde{c}(r)$ in the 3-Potts model, obtained by iterative method on TBA equation (2.40).

where we defined the shifted quantities, or ***kink*** quantities as

$$\epsilon_{\mathsf{kink}}(\theta) = \epsilon(\theta + \ln(2/r)), \tag{2.49a}$$

$$L_{\mathsf{kink}}(\theta) = L(\theta + \ln(2/r)). \tag{2.49b}$$

The scaling function (2.47) in the deep UV regime reads

$$\widetilde{c}(0) = \frac{6}{\pi^2} \lim_{r\to 0} \int_0^\infty d\theta \, L_{\mathsf{kink}}(\theta) r \frac{e^\theta}{2}. \tag{2.50}$$

One can take the derivative with respect to θ of equation (2.48) to rewrite the last factor of the previous equation, finding

$$\widetilde{c}(0) = \frac{6}{\pi^2} \lim_{r\to 0} \int_0^\infty d\theta \, L_{\mathsf{kink}}(\theta) \left[\frac{\mathrm{d}\epsilon_{\mathsf{kink}}(\theta)}{\mathrm{d}\theta} + \left(\varphi * \frac{e^{\epsilon_{\text{kink}}}}{1+e^{\epsilon_{\text{kink}}}} \frac{\mathrm{d}\epsilon_{\mathsf{kink}}}{\mathrm{d}\theta} \right) \right]. \tag{2.51}$$

After a series of integrations by parts and using again equation (2.48) one finds

$$\widetilde{c}(0) = \frac{6}{\pi^2} \mathcal{L} \left(\frac{1}{1+e^{\epsilon_a}} \right) = \frac{6}{\pi^2} \int_0^\infty dx \, \frac{x + \epsilon_P/2}{e^{x+\epsilon_P}+1}, \tag{2.52}$$

where $\mathcal{L}(x)$ is the Rogers dilogaritmic function (see Appendix B):

$$\mathcal{L}(x) = -\frac{1}{2} \int_0^x dx \left[\frac{\ln(1-t)}{t} + \frac{\ln t}{1-t} \right]. \tag{2.53}$$

To end this section let us mention what happens reaching the IR regime, $r \to \infty$. In this case it is easy to see that equation (2.43) reduces to $\epsilon(\theta) \simeq r\cosh\theta$ and consequently one can approximate $L(\theta) \simeq e^{-r\cosh\theta}$. Substituting into (2.47) one finds

$$\widetilde{c}(r) \simeq \frac{6r}{\pi^2}\int_0^\infty d\theta\,\cosh\theta e^{-r\cosh\theta} = \frac{6r}{\pi^2}K_1(r), \tag{2.54}$$

where $K_1(x)$ is the modified Bessel function of the second kind (see e.g. [25], §**8.432**).

2.1.4 Universal TBA and Y-System

Since for many integrable systems the S-matrix is known, it is possible to study their thermodynamic properties through the TBA analysis and try to understand some general features of the underlying theories. Indeed, it was noticed [21, 20] that there is a class of systems, that has a common and elegant structure for their TBA.

Such kind of systems, called ADE theories, are described by a set of non linear integral equation, as shown before, for unknown pseudoenergies $\epsilon_\alpha(\theta)$ where $a = 1, 2, \ldots, n$ is the number of species of particles.

This number turns out to be equal to the rank of some Lie algebra $\mathcal{A}$ of the class A_n[7], D_n and E_n (see Appendix C for more details); in particular each particle can be graphically represented by a node of the Dynkin diagram of $\mathcal{A}$ to which is associated the so called ***driving term*** $\nu_a = m_a R\cosh\theta$[8].

Let us now consider the TBA equation

$$\epsilon_\alpha(\theta) = \nu_\alpha(\theta) - \frac{1}{2\pi}\sum_b(\varphi_{ab} * L_b)(\theta), \tag{2.55}$$

where φ_{ab} is related through (2.26) to the S-matrix $S_{ab}(\theta_a - \theta_b)$ which describe the scattering of particle a with particle b and L_b has the usual definition (2.44). It is possible to use the fundamental matrix identity (see e.g. [22] for the proof)

$$\left(\delta_{ab} - \frac{1}{2\pi}\hat{\varphi}_{ab}(\omega)\right)^{-1} = \delta_{ab} - \frac{1}{2\cosh(\omega/h)}I_{ab}. \tag{2.56}$$

[7] There is a special reduction of the class A_{2n}, namely the $T_n = A_{2n}/Z_2$ algebras, that should be treated separately. A complete discussion can be found in [22].

[8] The vector that collect all the mass terms is called the Perron-Frobenius vector, which is the eigenvector of the algebra's incidence matrix with all positive entries: it correspond to the highest eigenvalue.

where h is the dual Coxeter number, I is the incidence matrix of some Dynkin diagram[9] and

$$\hat{\varphi}(\omega) = \int d\theta \, e^{i\omega\theta} \varphi_{ab}(\theta) \tag{2.57}$$

is the Fourier transform of the scattering kernel (2.26). Computing this identity for $\omega = 0$, one recovers the known matrix relation

$$N = C(2 - C)^{-1}, \tag{2.58}$$

where N is the matrix defined in equation (2.46) and C is the Cartan matrix of $\mathcal{A}$. Thanks to this identity and defining $\epsilon_a = \ln y_a$, (2.45) can be rewritten as

$$y_a^2 = \prod_b (1 + y_b)^{I_{ab}}. \tag{2.59}$$

Taking the Fourier transform of equation (2.55) and multiplying it by $\delta_{ab} - I_{ab}\widetilde{R}(\theta)$, with $\widetilde{R}(\theta) = (2\cosh(\pi\theta/h))^{-1}$ one gets

$$\epsilon_a(\theta) = \nu_a(\theta) - \frac{1}{2\pi} \sum_b I_{ab}(\varphi_h * (\nu_b - \log\left(1 + e^{-\epsilon_b}\right)))(\theta), \tag{2.60}$$

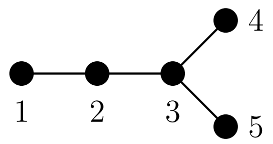
where

$$\varphi_h(\theta) = \frac{h}{2\cosh(h\theta/2)}. \tag{2.61}$$

The fundamental fact about equation (2.60) is that the TBA equation follows once one fixes the Dynkin diagram of an algebra and its incidence matrix. In particular, it turns out that the kernel (2.61), which is the anti-Fourier transform of $\widetilde{R}$, is ***universal***, ***i.e.*** it does not depend on the specific system one is considering, but just on the Coxeter number h.

For example let us consider the Lie algebra $\mathcal{A} = D_5$. Its Dynkin diagram and incidence matrix read

[9] The incidence matrix element I_{ab} is 1 whether node a is connected to node b, 0 if they are not linked.



$$I_{D_5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \quad (2.62)$$

Therefore, putting this matrix into equation (2.60), one will directly find the TBA equation that can be solved in the two regimes with the methods presented in the previous section.

Another way to approach the problem is to recast (2.55) into a series of functional equation, performing an analytical continuation on the variable $\theta \to \theta \pm i\pi/h$[10]. After some manipulation, one can rewrite the equation in the following form

$$Y_a\left(\theta + i\frac{\pi}{h}\right) Y_a\left(\theta - i\frac{\pi}{h}\right) = \prod_{b=1}^{r} [1 + Y_b(\theta)]^{I_{ab}}, \quad (2.63)$$

where

$$Y_a(\theta) = e^{\epsilon_a(\theta)}, \quad (2.64)$$

which is called Y-system. One important aspect of this equation is that its stationary solutions, *i.e.* θ-independent, are the y_a that appear in equation (2.59): for this reason it is believed that this formulation encodes more information than the usual TBA. Moreover, Y-functions fulfill certain periodicity conditions

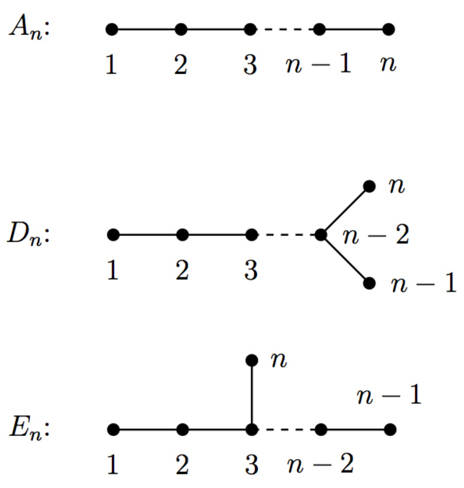
$$Y_a(\theta + i\pi P) = Y_{\bar{a}}(\theta), \qquad P = \frac{h+2}{h} \quad (2.65)$$

where $\bar{a}$ represents the antiparticle of a, as shown in figure 2.5.

Moreover, as noticed in [21], the periodicity P is strictly related to the conformal dimension of the perturbing field in the various cases

$$\Delta = 1 - 1/P. \quad (2.66)$$

[10] This is highly nontrivial, since one has to know the position of the poles of the pseudoenergies in the complex plane. However calculations become easier when the ground state is considered because it is know that it posses a strip where no poles or zeros are present.



$$\bar{a} = n + 1 - a, \quad a = 1, \dots, n$$

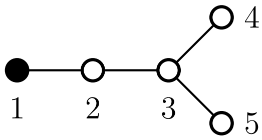
$$\begin{cases} \text{for } n \text{ even:} \quad \bar{a} = a, \quad a = 1, \dots, n \\ \text{for } n \text{ odd:} \begin{cases} \bar{a} = a, \quad a = 1, \dots, n-2 \\ \bar{n} = n - 1 \end{cases} \end{cases}$$

$$\begin{cases} \text{for } n = 6\text{:} \quad \bar{1} = 5, \quad \bar{2} = 4, \quad \bar{3} = 3, \quad \bar{6} = 6 \\ \text{for } n = 7, 8\text{:} \quad \bar{a} = a, \quad a = 1, \dots, n \\ n \leq 8 \end{cases}$$

Figure 2.5: On the left the Dynkin diagrams for ADE Lie algebras. On the right the particle-antiparticle relations described by equation (2.66).

However the classification of these systems is independent of h: this means that it is possible to choose other values for it, loosing the property of being the Coxeter number of some algebra.

This fact was employed by Al. B. Zamolodchikov in the beginning of the '90s to describe the renormalization group flow between minimal models [26, 27]. He noticed that the TBA on these systems can be written in a simple and elegant way considering also massless Dynkin nodes, *i.e.* introducing a new definition of the driving term $\nu_a(\theta) = \delta_{ak} mR \cosh\theta$. These massless nodes are relative to fictitious particles called magnons, which have to be introduced in the process of diagonalization of the S-matrix (for more details see section § 2.2.1). The most relevant fact is that even though one is dealing with a completely different problem, the ADE classification of Y-systems still holds also in the magnonic case. Therefore, as an example, it is possible consider the theory $\mathcal{A} = (D_5)_1$ where the label 1 indicates the position of the massive node. The corresponding "Dynkin diagram" and incidence matrix are



$$I_{(D_5)_4} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.67}$$

Even though the incidence matrices is the same of (2.62), the nature of the nodes give rise to completely different TBA results.

2.2 Non-Diagonal S-Matrix Theories

So far, the TBA techniques for theories which are described by a diagonal S-matrix have been presented. The next step is to discuss what happens when the scattering matrix of a given theory is non-diagonal. In this case the TBA equations for rapidities –the analog of equation (2.40)– are much more difficult to derive, since every time a given particle make a trip around the circle, its state can change. Therefore, one has to deal with the diagonalization of a certain number of non-diagonal S-matrices, a problem that can be solved by the method that goes under the name of ***Algebraic Bethe Ansatz***, discussed briefly in the next section.

Then, as representative of non-diagonal theories, the sine-Gordon model will be presented and the computation of the TBA will be developed.

2.2.1 Algebraic Bethe Ansatz

The Algebraic Bethe Ansatz (ABA) is a very useful tool in the study of spin chains integrability and it can be thought as the second quantization version of the Coordinate Bethe Ansatz, which was essentially the version used by Bethe in [23] to solve the isotropic Heisenberg chain.

The main idea of this method is enlarging the space one is working with, in order to "decouple" the interaction between physical degrees of freedom and leaving just that with the auxiliary ones. One can think of this auxiliary degree of freedom as a new particle, a sort of a probe, that propagates through the chain, which lives in a Hilbert space $\mathcal{V}_a$, where a labels different auxiliary particle's spaces, if needed.

First of all, given an operator $\mathcal{L}_{ij}$, called Lax operator, one can build up the so called monodromy matrix

$$\mathcal{M}_a(\lambda|\boldsymbol{\theta}) = \mathcal{L}_{N,a}(\lambda - \theta_N)\mathcal{L}_{a,N-1}(\lambda - \theta_{N-1})\dots\mathcal{L}_{a,1}(\lambda - \theta_1), \tag{2.68}$$

where λ is the rapidity of the auxiliary particle and $\boldsymbol{\theta} = \{\theta_1, \dots, \theta_N\}$. It is easy to see that equation (2.68) basically represent the scattering of the auxiliary particle with all the particles of the chain. Now one has to trace over the auxiliary space in order to get the so called ***colour transfer matrix***

$$\mathcal{T}(\lambda|\boldsymbol{\theta}) = \mathrm{Tr}_a\, \mathcal{M}_a(\lambda|\boldsymbol{\theta}) \tag{2.69}$$

which is equivalent to impose periodic boundary conditions. Since the monodromy matrix is an operator $\mathcal{M}_a : \mathcal{V}_a \times \mathcal{H} \to \mathcal{V}_a \times \mathcal{H}$, it can be represented in a matrix form[11]

$$\mathcal{T}(\lambda|\boldsymbol{\theta}) = \begin{pmatrix} A(\lambda|\boldsymbol{\theta}) & B(\lambda|\boldsymbol{\theta}) \\ C(\lambda|\boldsymbol{\theta}) & D(\lambda|\boldsymbol{\theta}) \end{pmatrix}, \tag{2.70}$$

where every entry of the matrix is an operator acting on the $\mathcal{H}$ space. In this formulation,

$$\mathcal{T}(\lambda|\boldsymbol{\theta}) = A(\lambda|\boldsymbol{\theta}) + D(\lambda|\boldsymbol{\theta}). \tag{2.71}$$

One can now introduce a ***pseudo-vacuum*** state $|\Omega\rangle$ which hosts no particles excitation, which is annihilated by the $C(\lambda|\boldsymbol{\theta})$ operator and from which one can generate a generic state by the application of $B(\lambda|\boldsymbol{\theta})$:

$$C(\lambda|\boldsymbol{\theta})\,|\Omega\rangle = 0, \qquad \prod_{j=1}^{M} B(\lambda_j|\boldsymbol{\theta})\,|\Omega\rangle = |\Psi\rangle\,. \tag{2.72}$$

At the end, having defined all these quantities one is interested in solving the following eigenvalue equation

$$\mathcal{T}(\lambda|\boldsymbol{\theta})\,|\Psi\rangle = [A(\lambda|\boldsymbol{\theta}) + D(\lambda|\boldsymbol{\theta})] = \Lambda(\lambda|\boldsymbol{\theta})\,|\Psi\rangle\,, \tag{2.73}$$

which can be worked out using the commutation relation between operators A, B, C and D coming from the Yang-Baxter equation, satisfied by the S-matrix itself.

In particular, one will find that developing the calculations, non-diagonal terms will appear in equation (2.73), which shouldn't be present in an eigenvalue equation.

Setting them to zero, will produce a series of equations, called the ***Bethe equations***, which represent a sort of constraints on the auxiliary rapidities λ_i.

In the TBA framework, the introduction of such an equation, necessary for the diagonalization of the S-matrix, bring the presence of additional massless excitations, called ***magnons***.

[11] For simplicity we now take $\dim \mathcal{V}_a = 2$

2.2.2 Sine-Gordon Model

The sine-Gordon model is described by the following action

$$\mathcal{S} = \int \mathrm{d}^2 x \left[\frac{1}{2} (\partial_\mu \phi(x))^2 + \frac{\mu^2}{\beta^2} \cos\left(\beta \phi(x)\right) \right], \tag{2.74}$$

and is widely used in the most various fields of physics such as the study of crystals, light and superconductivity. Thanks to Euler-Lagrange equation one can find the equation of motion,

$$\partial_t^2 \phi(x) - \partial_x^2 \phi(x) - \frac{\mu^2}{\beta} \sin\left(\beta \phi(x)\right) = 0, \tag{2.75}$$

from which it becomes clear where the name of the model come from.

From the point of view of this thesis the most important feature of the sine-Gordon model is the fact that it possesses an infinite number of integrals of motion in involution, and this property ensures both its classical and quantum integrability.

If one looks at the potential, $V[\phi] = \mu^2/\beta^2 \cos(\beta\phi)$ it is easy to understand that this model presents an infinite series of degenerate minima, each placed at $\phi = 2\pi n_i/\beta$, where $i = 1, 2, \ldots$. The quantum interpretation of this feature is that they correspond to an infinite family of vacua, whose excitation can be thought as particles of mass μ. Together with them, there are also topological excitations, associated to the field configuration between two vacua. Without going into details, the solution with topological charge ± 1 are called ***solitons*** and ***antisoliton*** respectively, their name coming from the fact that they are well-localized functions without dispersion. Using the method of the ***Bäklund transformations***, it is possible to generate multiparticle solutions[12]: one of these is the soliton-antisoliton solution, also called ***breather***.

2.2.3 S-Matrix And Bethe-Yang Equations

The S-matrix of the sine-Gordon model was first calculated by brothers Zamolodchikov in 1979; here I will briefly summarize the main results, the explicit calculations can be found in [3].

[12] It is not possible to simply create a linear combination of solutions since the sine-Gordon model is non-linear.

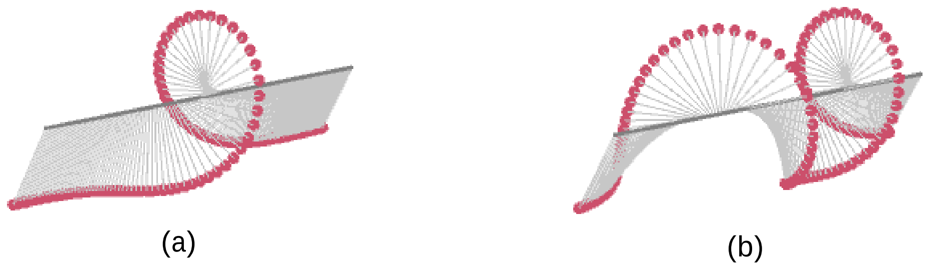


Figure 2.6: A graphical representation of a soliton solution 2.6a, and a breather bound state 2.6b, considering a system of pendula, which represent the discrete version of the sine-Gordon theory. Simulated with ***Mathematica***, adapted from [28].

In the following, it is useful to introduce a renormalized coupling constant:

$$\xi = \frac{\beta^2/8\pi}{1-\beta^2/8\pi}. \tag{2.76}$$

One can define the two particles of the $O(2)$ model as A_1 and A_2, but to reveal the structure of the scattering theory it is more useful to define the complex linear combinations

$$A(\theta) = A_1(\theta) + iA_2(\theta), \qquad \bar{A}(\theta) = A_1(\theta) - iA_2(\theta). \tag{2.77}$$

In terms of these excitations, the scattering equations become

$$\begin{aligned} A(\theta_1)\bar{A}(\theta_2) &= S_T(\theta)\bar{A}(\theta_2)A(\theta_1) + S_R(\theta)A(\theta_2)\bar{A}(\theta_1), \\ A(\theta_1)A(\theta_2) &= S_0(\theta)A(\theta_2)A(\theta_1), \\ \bar{A}(\theta_1)\bar{A}(\theta_2) &= S_0(\theta)\bar{A}(\theta_2)\bar{A}(\theta_1). \end{aligned} \tag{2.78}$$

and the amplitudes can be collected into the 4×4 matrix

$$\boldsymbol{S}(\theta) = \begin{pmatrix} S_0(\theta) & & & \\ & S_T(\theta) & S_R(\theta) & \\ & S_R(\theta) & S_T(\theta) & \\ & & & S_0(\theta) \end{pmatrix} = \frac{S_0(\theta)}{a(\theta)} \begin{pmatrix} a(\theta) & & & \\ & b(\theta) & c(\theta) & \\ & c(\theta) & b(\theta) & \\ & & & a(\theta) \end{pmatrix}, \tag{2.79}$$

where S_T and S_R are the transmission and reflection amplitudes in the soliton-antisoliton scattering process, while S_0 is the transmission amplitude in the soliton-

soliton interaction (or the antisoliton-antisoliton one via charge conjugation). In the second equality the matrix $\boldsymbol{R}(\theta)$ has been highlighted since it correspond to the spin 1/2 R-matrix coming from the study of the quantum group $\mathfrak{U}_q(sl(2))$ while the prefactor is due only to the property of unitarity and crossing symmetry satisfied by the S matrix itself.

Using the the Yang-Baxter equation (1.23), it is possible to find the explicit expression for the different elements:

$$S_0(\theta) = e^{-i\chi(\theta)}, \qquad \chi(\theta) = \int_0^{+\infty} dk \, \frac{\sin(k\theta)}{k} \frac{\sinh(k(\pi - \xi)/2)}{\sinh(\xi k/2)\cosh(\pi k/2)}, \tag{2.80}$$

$$a(\theta) = \sinh \frac{\pi}{\xi}(\theta - i\pi), \qquad b(\theta) = -\sinh \frac{\pi\theta}{\xi}, \qquad c(\theta) = -\sinh \frac{i\pi^2}{\xi}. \tag{2.81}$$

The particle spectrum of this model can be organized in two regimes:

1. $\xi > \pi$. In this regime no poles of the S-matrix fall into the physical strip and soliton-antisoliton bound states cannot be created, therefore it is sometimes called the ***repulsive*** regime of the theory and the only possible particles are

$$s^+, s^- \qquad \text{of mass } \mu.$$

2. $\xi < \pi$. In this regime, the $b_a = (s^+ s^-)$ bound states, the breathers, can appear and for this reason it is called the attractive regime. The possible particles are

$$b_a, \quad a = 1, \ldots, \lfloor \pi/\xi \rfloor^{13} \qquad \text{of mass } m_a = 2\mu \sin \frac{a\xi}{2}.$$

Together with the soliton-soliton scattering amplitude, using the bootstrap equations one can calculate the S-matrix relative to the scattering of a soliton and a breather

$$S_{aA}^{aA}(\theta) = \frac{\sinh\theta + i\cos(a\xi/2)}{\sinh\theta - i\cos(a\xi/2)} \prod_{l=1}^{a-1} \frac{\sin^2(\frac{a-2l}{4}\xi - \frac{\pi}{4} + i\frac{\theta}{2})}{\sin^2(\frac{a-2l}{4}\xi - \frac{\pi}{4} - i\frac{\theta}{2})}, \tag{2.82}$$

[13] The symbol $\lfloor x \rfloor$ represent the integer part of x.

and between two breathers

$$
\begin{aligned}
S_{ab}^{ab}(\theta) =& \frac{\sinh\theta + i\sin\left(\frac{a+b}{2}\xi\right)}{\sinh\theta - i\sin\left(\frac{a+b}{2}\xi\right)} \frac{\sinh\theta + i\sin\left(\frac{a-b}{2}\xi\right)}{\sinh\theta + i\sin\left(\frac{a-b}{2}\xi\right)} \\
&\times \prod_{l=1}^{\min(a,b)-1} \frac{\sin^2\left(\frac{b-a-2l}{4}\xi + i\frac{\theta}{2}\right)\cos^2\left(\frac{b+a-2l}{4}\xi + i\frac{\theta}{2}\right)}{\sin^2\left(\frac{b-a-2l}{4}\xi - i\frac{\theta}{2}\right)\cos^2\left(\frac{b+a-2l}{4}\xi - i\frac{\theta}{2}\right)}.
\end{aligned}
\tag{2.83}
$$

The procedure to get TBA equation from these amplitudes is very similar to the one described in section 2.1. Following the ABA method discussed in section 2.2.1, without repeating the calculations, the diagonalization of the colour transfer matrix leads to the following Bethe equation for magnon rapidities

$$
\prod_{l=1}^{N} \frac{\sinh\frac{\pi}{\xi}(\theta_l - \lambda_j - i\pi/2)}{\sinh\frac{\pi}{\xi}(\theta_l - \lambda_j + i\pi/2)} = \prod_{l=1}^{M} \frac{\sinh\frac{\pi}{\xi}(\lambda_l - \lambda_j - i\pi)}{\sinh\frac{\pi}{\xi}(\lambda_l - \lambda_j + i\pi)}. \tag{2.84}
$$

It is straightforward to check that both members of equation (2.84) are periodic in the imaginary part of λ, with period $p_0 = \xi$, for this reason it is convenient to consider $-p_0 < \operatorname{Im}\lambda_j \leq p_0$. This value will play a central role in the study of theories with a trigonometric S-matrix such as sine-Gordon. Then, denoting with N_s the number of solitons and with N_a the number of breathers on the line, such that $N = N_s + \sum_a N_a$, one can send a particle of rapidity θ_j through the chain making it scatter with the other particles, distinguishing the case in which a soliton takes the trip

$$
e^{i\mu L\sinh\theta_j} \prod_{a=1}^{\lfloor\pi/\xi\rfloor} \prod_{l=1}^{N_b} S_a(\theta_j - \theta_l)\, \mathrm{Tr}_j \prod_{l=1\neq i}^{N_s} \boldsymbol{S}(\theta_j - \theta_l) = 1, \tag{2.85}
$$

from the case when a breather does

$$
e^{im_a L\sinh\theta_j} \prod_{b=1}^{\lfloor\pi/\xi\rfloor} \prod_{j=1\neq i}^{N_b} S_{ab}(\theta_j - \theta_l) \prod_{l=1\neq i}^{N_s} S_a(\theta_j - \theta_l) = 1, \tag{2.86}
$$

where m_a is the mass of the bound state and $i = 1, \ldots, N$. Once the trace is computed via ABA, one finally gets the quantization equations for solitons and breathers rapidities:

$$e^{-i\mu L\sinh\theta_j} = \prod_{a=1}^{\lfloor\pi/\xi\rfloor}\prod_{l=1}^{N_b} S_a(\theta_j-\theta_l)\prod_{l=1}^{N_s} S_0(\theta_j-\theta_l)\prod_{k=1\neq l}^{M}\frac{\sinh\pi/\xi(\theta_l-\lambda_k-i\pi/2)}{\sinh\pi/\xi(\theta_l-\lambda_k+i\pi/2)}, \quad (2.87)$$

$$e^{im_a L\sinh\theta_j}\prod_{b=1}^{\lfloor\pi/\xi\rfloor}\prod_{l=1}^{N_b} S_{ab}(\theta_j-\theta_l)\prod_{l=1}^{N_s} S_a(\theta_j-\theta_l) = 1, \quad (2.88)$$

2.2.4 Thermodynamic Limit And Bethe Strings

The thermodynamics of the model has been studied in [29], making use of the duality between the sine-Gordon model and the massive Thirring model (MT), demonstrated by Coleman in [30]. It has also been shown in [31] that under a precise scaling limit, the sine-Gordon model is equivalent to the XYZ anisotropic spin $1/2$ quantum chain analysed by Takahashi and Suzuki in [32].

As example we will develop the calculations in the repulsive regime ($\xi > \pi$), where the particle spectrum is purely solitonic. Proceeding with the analysis similarly to section 2.1.2, one can take the thermodynamic limit, which in this case means

$$N_s, M, L \to \infty \quad \text{with} \quad N_s/L = \text{const} = M/L. \quad (2.89)$$

Sending the number of magnonic excitations to infinity, a very particular phenomenon happens to the solutions of Bethe equation (2.88): they start to organize themselves in the complex plane into bound states called ***strings***, defined as follows

$$\lambda_\alpha^{(m)} = \lambda^{(m)} + \frac{i\pi}{2}(m+1-2\alpha), \quad (2.90)$$

where $\alpha = 1, 2\ldots, m$ and the real part $\lambda^{(m)}$ is called the center of the string. For this reason, every time one comes across the product over all magnons, it is possible to write

$$\prod_{l=1}^{M} \longrightarrow \prod_{m\in\mathfrak{M}}\prod_{l=1}^{\xi_m}\prod_{\alpha=1}^{m}, \quad (2.91)$$

where $\mathfrak{M}$ represent the set of all types of strings, ξ_m is the number of strings of type m and α runs over all the elements of the given string.

Remembering that these solutions have a periodicity p_0, different situations can occur: one is when p_0 is an irrational number, another when it is a rational number or finally when it is an integer. The first two situations are, not surprisingly, more complicated to study and they were attacked by Takahashi and Suzuki in [32], using the notion of continued fraction. Their results were then applied to solve quantum system via TBA: for example Tateo in [33] solved the sine-Gordon model in the rational p_0 case. As shown below, in the rational and especially in the integer limit, things becomes particularly simpler since the number of TBA non linear coupled equations for the pseudoenergies ϵ_a becomes finite, with $a = 0, \ldots, N$. For these reasons, from now on, we will consider $p_0 = N$, where $N \in \mathbb{N}$.

At this point it is possible to repeat the same calculation as in section 2.1.2, with some minor modifications. First of all, since in the thermodynamic limit the rapidities become dense, it is necessary to introduce the density of roots and the density of strings:

$$\bar{\sigma}_0(\theta) = \frac{n_i - n_{i-1}}{\theta_i - \theta_{i-1}} \qquad \bar{\sigma}_n(\theta) = \frac{m_i - m_{i-1}}{\lambda_i - \lambda_{i-1}}. \tag{2.92}$$

where the n_i and m_i are sorts of quantum numbers that come from the logarithm of equations (2.87) and (2.84), a similar procedure of (2.20). Here the bar indicate that one is dealing with the total density, *i.e.* the sum of the particle density and the hole density, since it is impossible to say whether a level n_i for particles or m_i for magnons is occupied. Taking the logarithm, the thermodynamic limit and the derivative of (2.87) and (2.84) one finds

$$\frac{\mu L}{2\pi} \cosh\theta + (\phi * \sigma_0)(\theta) - \sum_{n=1}^{N} (\psi_n * \sigma_n)(\theta) = \sigma_0(\theta) + \widetilde{\sigma}_0(\theta), \tag{2.93}$$

$$(\psi_m * \sigma_0)(\lambda) - \sum_{n=1}^{N} (A_{nm} * \sigma_n)(\lambda) = \widetilde{\sigma}_m(\lambda). \tag{2.94}$$

where the convolution is defined in the usual way as $(f*g)(x) = \int dy\, f(x-y)g(y)$. It becomes useful to introduce, following the notation introduced in [19], the functions

$$f_\alpha(x) = \frac{\sinh\Big(\frac{\pi}{\xi}(x + \frac{i\pi}{2}\alpha)\Big)}{\sinh\Big(\frac{\pi}{\xi}(x - \frac{i\pi}{2}\alpha)\Big)}. \tag{2.95}$$

The integral kernels read

$$\phi(\theta) = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\theta}\log S_0(\theta), \qquad \psi_n(\theta) = \frac{1}{2\pi i}\sum_{\alpha=1}^{n}\frac{\mathrm{d}}{\mathrm{d}\theta}\log f_1(\theta - \lambda_\alpha^{(n)}), \tag{2.96a}$$

$$A_{nm}(\lambda - \lambda') = \delta(\lambda')\delta_{nm} + \frac{1}{2\pi i}\sum_{\beta=1}^{m}\sum_{\alpha=1}^{n}\frac{\mathrm{d}}{\mathrm{d}\lambda}\log f_2(\lambda_\beta^{(m)} - \lambda_\alpha^{\prime(n)}) \tag{2.96b}$$

where the dependence on the string internal quantum numbers α or β has been highlighted since thanks to equation (2.90) the sums can be drastically simplified.

At this point, as discussed in the previous section, one has to define the energy and the entropy of the system and then define the free energy. Varying the latter with respect to the various densities of particles and holes (both in the real and magnon case) one finally gets the TBA equations. They can be further simplified employing the relation between the kernels in the Fourier space, and eventually come to the final Y-system.

In particular, in the special case when the coupling constant is a multiple of an integer number, the model becomes reflectionless and therefore the S-matrix becomes diagonal. Therefore the TBA equations can be recast into a Y-system and one finds that the related Dynkin diagram is of the D_n type, ash shown in figure 2.7.

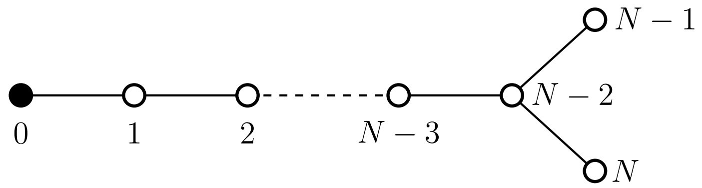


Figure 2.7: Graphical representation of TBA equations for the sine-Gordon model.

3

TBA Analysis

In this chapter we are going to analyze a wide class of scattering theories: in particular in the first section we will focus on those which can be built up exploiting the $SU(2)_k$ symmetry of the system, where k is the order of the considered representation. This construction is of great interest since many models, such as sine-Gordon or the O(3) non-linear sigma model turn out to develop such symmetry for some specific value of the coupling constant.

In the second part of the chapter we are going to study a new class of scattering theories, namely those obtained by requiring the $\mathfrak{U}_q(\mathfrak{sl}(2))$ symmetry. This idea was first introduced by D. Bernard and A. LeClair in [4]: they showed that in $1+1$D integrable QFT it is possible to find certain non-local currents, whose existence is due by some underlying q-deformed algebra. As it happens in the "underformed" case, one can then exploit this symmetry to build from scratch the S-matrix. Using this new scattering matrices we are going to derive the TBA equations and we will try to understand the relation between the two cases.

3.1 $SU(2)_k$ Factorizable S-Matrix Theories

These theories were first analyzed in a paper by S. R. Aladim and M. J. Martins [7]; here we are going to report the result with some minor correction.

In order to construct the S-matrix one uses the standard procedure of studying the corresponding lattice model, which in this case is represented by the spin $k/2$ Heisenberg chain, whose Boltzmann weights and relative R-matrix were already computed in [34]. Imposing then unitarity (see equation (1.28)) and crossing symmetry (see equation (1.29)), one finds the exact form of the S-matrix:

$$\begin{aligned} S(\theta) &= \prod_{l=\text{even}}^{k} \frac{\theta + il\pi}{\theta - il\pi} R(\theta,\pi) \qquad \text{for } k \text{ even}, \\ &= S_0(\theta) \prod_{l=\text{odd}}^{k} \frac{\theta + il\pi}{\theta - il\pi} R(\theta,\pi) \qquad \text{for } k \text{ odd}, \end{aligned} \tag{3.1}$$

where

$$S_0(\theta) = \frac{\Gamma(1/2 - i\theta/2\pi)\Gamma(i\theta/2\pi)}{\Gamma(1/2 + i\theta/2\pi)\Gamma(-i\theta/2\pi)} \tag{3.2}$$

is the prefactor coming from imposing unitarity and crossing symmetry, while

$$R(\theta,\pi) = \mathbb{P}^0 + \sum_{\alpha=1}^{k} \prod_{l=1}^{\alpha} \frac{\theta - il\pi}{\theta + il\pi} \mathbb{P}^\alpha, \tag{3.3}$$

is the R-matrix obtained from the spin chain model, $\mathbb{P}^j$ being the projectors over the j state of the Hilbert space. Without going into further details, using the ABA method described in § 2.2.1, one can diagonalize the inhomogeneous transfer matrix $\mathcal{T}(\theta_1, \ldots, \theta_L)$ whose elements are defined as the product of S-matrices (the computation in the homogeneous case has been done in [35], the generalization is straightforward):

$$\mathcal{T}_j(\theta_!, \ldots, \theta_L)^{\alpha'_1 \ldots \alpha'_L}_{\alpha_1 \ldots \alpha_L} = \sum_{\gamma,s} S^{\alpha_{j+1}\alpha'_{j+1}}_{\alpha_j \gamma_j}(\theta_j - \theta_{j+1}) \ldots S^{\alpha_{j+1}\alpha'_{j+1}}_{\alpha_j \gamma_j}(\theta_j - \theta_{j-1}). \tag{3.4}$$

As described in the previous chapter, the first step to get informations about the thermodynamics of the system, is to take a physical particle and perform a complete trip on a circle of length L. Every time the particle scatters with another particle, the wave function acquire an additional term, which is nothing but the eigenvalue of the inhomogeneous transfer matrix discussed earlier. Therefore in this situation, the analogous of equation (2.6) reads

$$e^{imL\sinh\theta_j} \prod_{l=1}^{L} \widetilde{S}_0(\theta_j - \theta_l) \prod_{l=1}^{M} \frac{\theta_j - \lambda_l + i\pi k/2}{\theta_j - \lambda_l - i\pi k/2} = 1. \tag{3.5}$$

where $\widetilde{S}_0$ is a function that depends on the spin representation, namely

$$\begin{aligned} \widetilde{S}_0(\theta) &= \prod_{l=\text{odd}}^{k} \frac{\theta - il\pi}{\theta + il\pi} \qquad \text{for } k \text{ even}, \\ &= S_0(\theta) \prod_{l=\text{even}}^{k} \frac{\theta - il\pi}{\theta + il\pi} \qquad \text{for } k \text{ odd}. \end{aligned} \tag{3.6}$$

Together with this equation one must consider the defining equation for magnon rapidities λ_i, obtained during the ABA diagonalization

$$\prod_{l=1}^{M} \frac{\lambda_j - \lambda_l - i\pi}{\lambda_j - \lambda_l + i\pi} = -\prod_{l=1}^{L} \frac{\lambda_j - \theta_l - i\pi k/2}{\lambda_j + \theta_l + i\pi k/2}. \tag{3.7}$$

To investigate the thermodynamics of the models, one has to take the thermodynamic limit, $N, L \to \infty$ with $M/L = \text{const}$. In this regime, it is known that the solutions of equation (3.7) organize themselves into strings in the complex λ plane, following equation (2.90) and as already pointed out one can express the product over magnon rapidities as in formula (2.91).

Considering first equation (3.7) and defining the functions

$$f_\alpha(x) = \frac{x - i\pi\alpha/2}{x + i\pi\alpha/2}, \tag{3.8}$$

one gets

$$-\prod_{n=1}^{\infty} \prod_{l=1}^{\xi_n} \prod_{\alpha=1}^{n} f_2(\lambda_{\beta,j}^{(m)} - \lambda_{\alpha,l}^{(n)}) \prod_{l=1}^{L} f_{-k}(\lambda_{\beta,j}^{(m)} - \theta_l) = 1. \tag{3.9}$$

The surprising fact is that it is always possible to get rid of the α and β dependencies, *i.e.* it is always possible to express the equations only in terms of the centers of strings. To see this fact let us take the product over $\beta = 1, \ldots, m$,

$$-\prod_{n=1}^{\infty} \prod_{l=1}^{\xi_n} \prod_{\beta=1}^{m} \prod_{\alpha=1}^{n} f_2(\lambda_{\beta,j}^{(m)} - \lambda_{\alpha,l}^{(n)}) \prod_{l=1}^{L} \prod_{\beta=1}^{m} f_{-k}(\lambda_{\beta,j}^{(m)} - \theta_l) = 1. \tag{3.10}$$

Then, expressing $\lambda_\alpha^{(n)}$ using its string definition (2.90), one can simplify the products that appear in (3.10), namely

$$\prod_{\beta=1}^{m} f_{\pm k}(\lambda_{\beta,j}^{(m)} - \theta_l) = \prod_{i=1}^{\min(k,m)} f_{\pm(m+k-2i+1)}(\lambda_j^{(m)} - \theta_l), \tag{3.11a}$$

$$\begin{aligned}\prod_{\beta=1}^{m}\prod_{\alpha=1}^{n} f_2(\lambda_{\beta,j}^{(m)} - \lambda_{\alpha,l}^{(n)}) = f_{|m-n|}(\lambda_{jl}^{(m,n)}) \\ \times \prod_{i=1}^{\min(m,n)-1} f_{|m-n|+2i}^2(\lambda_{jl}^{(m,n)}) f_{|m-n|+2\min(m,n)}(\lambda_{jl}^{(m,n)}),\end{aligned} \tag{3.11b}$$

where in the last equation, for sake of brevity, we put $\lambda_{jl}^{(m,n)} = \lambda_j^{(m)} - \lambda_l^{(n)}$. At this point, using these results and introducing the particles and holes densities for physical particles (σ_0 and $\widetilde{\sigma}_0$) and magnons (σ_n and $\widetilde{\sigma}_n$) as defined in equation (2.92), one can compute the logarithmic derivative, obtaining

$$-\sum_{n=1}^{\infty} \int d\lambda'\, K_{nm}(\lambda - \lambda')\sigma_n(\lambda') + \int d\theta\, \psi_m(\lambda - \theta)\sigma_0(\theta) = \widetilde{\sigma}_m(\lambda). \tag{3.12}$$

where we have introduced the integral kernels

$$\begin{aligned}K_{nm}(\lambda - \lambda') = \delta(\lambda - \lambda')\delta_{mn} + \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\lambda}\log\Big[f_{|m-n|}(\lambda - \lambda') \\ \times \prod_{i=1}^{\min(m,n)-1} f_{|m-n|+2i}^2(\lambda - \lambda') f_{|m-n|+2\min(m,n)}(\lambda - \lambda')\Big],\end{aligned} \tag{3.13a}$$

$$\psi_m(\lambda - \theta) = -\frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\lambda}\log\prod_{j=1}^{\min(k,m)} f_{-(m+k-2j+1)}(\lambda - \theta), \tag{3.13b}$$

The explicit expressions of these kernels is quite complicated, however things become simpler when one takes their Fourier transform.

From definition (3.8) one finds

$$\mathscr{F}\left(\frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}x}\log f_{\pm\alpha}(x)\right)(\omega) = \pm e^{-\pi\omega\alpha/2}, \tag{3.14}$$

it is possible to show after cumbersome calculations that

$$\widetilde{K}_{mn}(\omega) = \coth\left(\frac{\pi|\omega|}{2}\right)\left[e^{-\pi|\omega||m-n|/2} - e^{-\pi|\omega|(m+n)/2}\right], \tag{3.15a}$$

$$\widetilde{\psi}_m(\omega) = \frac{1}{2\cosh(\pi\omega/2)} \coth\left(\frac{\pi|\omega|}{2}\right) \left[e^{-\pi|\omega||m-k|/2} - e^{-\pi|\omega|(m+k)/2}\right]. \tag{3.15b}$$

In particular it is important to notice that the $\widetilde{\psi}_m$ kernel can be expressed in term of the $\widetilde{K}_{mn}$ one, namely

$$\widetilde{\psi}_m(\omega) = \widetilde{p}(\omega)\widetilde{K}_{mk}(\omega), \tag{3.16}$$

where we have introduced the universal kernel

$$\widetilde{p}(\omega) = (2\cosh(\pi\omega/2))^{-1}. \tag{3.17}$$

This fact, which will will be true also for the $\widetilde{\phi}$ kernel (coming from the other equation), is of fundamental importance to obtain the ***universal form*** of the TBA. Using the standard definition of convolution (2.27), the equation takes the simple form

$$\widetilde{\sigma}_m(\lambda) = (\widetilde{\psi}_m * \sigma_0)(\lambda) - \sum_{n=1}^{\infty}(\widetilde{K}_{mn} * \sigma_n)(\lambda) \qquad m = 1, 2, \dots \tag{3.18}$$

Considering now equation (3.5) and repeating similar calculations, one finds

$$\frac{Lm}{2\pi}\cosh\theta + \int d\theta'\, \phi(\theta-\theta')\sigma_0(\theta') + \sum_{n=1}^{\infty}\int d\lambda\, \psi_n(\theta-\lambda)\sigma_n(\lambda) = \sigma_0(\theta) + \widetilde{\sigma}_0(\theta), \tag{3.19}$$

where the kernel ψ_n is the same of equation (3.15b) and

$$\phi(\theta-\theta') = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\theta}\log \widetilde{S}_0(\theta-\theta'), \tag{3.20}$$

which in Fourier transform reads

$$\phi(\omega) = \frac{1}{4\cosh^2(\pi\omega/2)}(1 - e^{\pi|\omega|k}) = \widetilde{p}^2(\omega)\widetilde{K}_{kk}(\omega). \tag{3.21}$$

both in even and odd k cases (see Appendix D for more details), therefore one finally gets

$$\sigma_0(\theta) + \widetilde{\sigma}_0(\theta) = \frac{mL\cosh(\theta)}{2\pi} + (\phi * \sigma_0)(\theta) - \sum_{n=1}^{\infty}(\psi_n * \sigma_n)(\theta). \tag{3.22}$$

It is possible to compute the equilibrium condition for the free energy defining the function

$$\Phi[\sigma, \widetilde{\sigma}, \xi, \mu_i] = \mathcal{E} - T\mathcal{S} + \xi\mathcal{V}^{(1)} + \sum_{j=1}^{\infty} \mu_i \mathcal{V}_j^{(2)} \tag{3.23}$$

where $\mathcal{E}$ and $\mathcal{S}$ are the energy and the entropy of the model (see equations (2.23) and (2.31)), wile $\mathcal{V}^{(1)}$ and $\mathcal{V}^{(2)}$ are equations (3.18) and (3.22) interpreted as constraints, $\xi(\theta)$ and $\mu_j(\lambda)$ being the Lagrange multipliers. Varying the result obtain with respect to the various densities and multipliers and imposing the the equilibrium condition, *i.e.* setting the resulting equations equal zero, one finally gets the TBA equations

$$\epsilon_0(\theta) = mR\cosh\theta - (\phi * L_0)(\theta) - \sum_{m=1}^{\infty}(\psi_m * (\log(1 + e^{\epsilon_m})))(\theta), \tag{3.24a}$$

$$\log\left(1 + e^{-\epsilon_m(\lambda)}\right) = (\psi_m * L_0)(\lambda) + \sum_{n=1}^{\infty}(K_{mn} * \log(1 + e^{\epsilon_n}))(\lambda), \tag{3.24b}$$

where we have introduced

$$\epsilon_0(\theta) = \frac{\widetilde{\sigma}_0(\theta)}{\sigma_0(\theta)}, \qquad \epsilon_n(\lambda) = \frac{\sigma_n(\lambda)}{\widetilde{\sigma}_n(\lambda)} \quad n = 1, 2, \dots \tag{3.25}$$

which represent the pseudoenergies of particles and magnons, respectively.

In order to obtain the universal form of the TBA, which encode the Dynkin structure of the model, one has to invert the K_{mn} kernel (3.15a), defining a matrix K_{mn}^{-1} such that

$$\sum\nolimits_{n'}(K_{mn'}^{-1} * K_{n'n})(\lambda) = \delta(\lambda)\delta_{mn}. \tag{3.26}$$

This matrix is given by

$$K_{mn}^{-1}(\lambda) = \delta(\lambda)\delta_{mn} - p(\lambda)(\delta_{m,n+1} + \delta_{m,n-1}). \tag{3.27}$$

Taking the convolution of $K_{im}^{-1}(\lambda)$ with equation (3.24b) and then summing over the m index, one finds

$$\epsilon_i(\lambda) = \delta_{ik}(p * L_0)(\lambda) + (p * ((1 - \delta_{1i})L_{i-1} + L_{i+1})) \qquad i = 1, 2, \dots, \tag{3.28}$$

where the Kronecker delta has been introduced to take into account that the first node, $i = 1$, cannot be linked to the zeroth magnonic node, since there is no zero-magnonic node.

Considering now (3.24b) for $m = k$ one finds

$$\epsilon_k(\lambda) + L_k(\lambda) = (\psi_k * L_0)(\lambda) + \sum_{n=1}^{\infty}(K_{k,n} * L_n)(\lambda). \tag{3.29}$$

Taking the convolution of $p(x)$ with this result and exploiting properties (3.16) and (3.21), one finds the relation

$$-(\phi * L_0)(\lambda) - \sum_{n=1}^{\infty}(\psi_m * \log(1 + e^{\epsilon_m}))(\lambda) = -(p * L_k)(\lambda). \tag{3.30}$$

Using this relation it is possible to replace the last two terms in equation (3.24a), obtaining

$$\epsilon_0(\theta) = mR\cosh\theta - (p * L_k)(\theta). \tag{3.31}$$

As explained in the previous chapter, these universal equation can be graphically represented with a Dynkin diagram, as shown in figure 3.1.

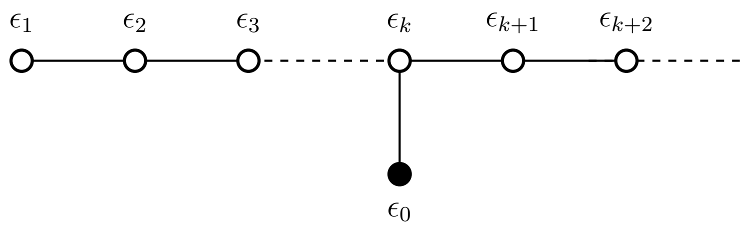


Figure 3.1: Graphical representation of the TBA equations for an arbitrary spin $s = k/2$ scattering theory with rational S-matrix.

The result we just derived correctly predict that massless magnonic nodes with pseudoenergy ϵ_i (equation (3.28)) are physically coupled only to the preceding and following nodes and also that the k-th of them (where $k = 2s$) is coupled with the massive term L_0. On the other side (see equation (3.31)), the massive node turns out to be coupled only to the k-th node. This is exactly what we expected from the already known cases, namely the $k = 1$ and $k = 2$ models [36, 6].

3.2 $\mathfrak{U}_q(\mathfrak{sl}(2))$ Factorizable S-Matrix Theories

In this section we are going to derive TBA equations for the q-deformed S-matrices at arbitrary spin computed in [8], exploiting the $\mathfrak{U}_q(\mathfrak{sl}(2))$ symmetry.

Diagonalizing these matrices using the ABA method presented in § 2.2.1 we have the following equations

$$e^{imL\sinh\theta_j}\prod_{l=1}^{L}\widetilde{S}_0(\theta_j-\theta_l)\prod_{l=1}^{M}\frac{\sinh\gamma(\theta_j-\lambda_l+i\pi k/2)}{\sinh\gamma(\theta_j-\lambda_l-i\pi k/2)}=1, \tag{3.32}$$

which describes the situation in which a particle of rapidity θ_j and spin $s=k/2$ is scattered with all the particles of the chain and brought back to the starting position. The parameter γ is the deformation parameter, which is related to the quantum group parameter q by the relation $q=e^{2i\pi\gamma}$.

The prefactor $\widetilde{S}_0$ appearing in (3.32) is the factor due to the request of unitarity, crossing symmetry and the fulfillment of Yang-Baxter equation. It can be shown that the prefactor is given defined as

$$\begin{aligned}\widetilde{S}_0(u) &= \prod_{l=\text{odd}}^{k}\frac{\sinh\gamma(\theta-il\pi)}{\sinh\gamma(\theta+il\pi)} \qquad \text{for } k \text{ even},\\ &= S_0(\theta)\prod_{l=\text{even}}^{k}\frac{\sinh\gamma(\theta-il\pi)}{\sinh\gamma(\theta+il\pi)} \qquad \text{for } k \text{ odd}.\end{aligned} \tag{3.33}$$

It is easy to see that the $k=2$ case correctly reproduces the known results on the sausage model [6, 37]. We also assume that the term $S_0(\theta)$ can written in an integral form, such that when $k=1$ one obtain the renown sine-Gordon prefactor obtained by brothers Zamolodchikov in [3], reproduced in (2.80).

As in the undeformed case, the diagonalization of the S-matrix via ABA leads to the presence of a number of M magnonic rapidities λ_l, which can be thought as auxiliary non-physical particles. These rapidities turn out to be solutions of a series of Bethe constraint equations

$$\prod_{l=1}^{M}\frac{\sinh\gamma(\lambda_j-\lambda_l-i\pi)}{\sinh\gamma(\lambda_j-\lambda_l+i\pi)}=-\prod_{l=1}^{L}\frac{\sinh\gamma(\lambda_j-\theta_l-i\pi k/2)}{\sinh\gamma(\lambda_j+\theta_l+i\pi k/2)}. \tag{3.34}$$

As already mentioned in § 2.2.4, the periodicity $p_0 = \pi/\gamma$ of equation (3.34) in the imaginary λ direction, can impose several constraint on the length of strings as explained in the most general case in [32].

It is useful to introduce the functions f_α,

$$f_\alpha(x) = \frac{\sinh\gamma(x - i\alpha\pi/2)}{\sinh\gamma(x + i\alpha\pi/2)}, \tag{3.35}$$

which are the "trigonometrized" version of definition (3.8), with the property that $f_{-\alpha}(x) = f_\alpha(-x)$. Following the usual steps, one can define

$$P_\alpha(x) = \frac{\mathrm{d}}{\mathrm{d}x}\log f_\alpha(x) = 2i\gamma\frac{\sin(\gamma\pi\alpha)}{\cosh(2\gamma x) - \cos(\gamma\pi\alpha)}, \tag{3.36}$$

whose Fourier transform can be easily computed (see e.g. [38], §**7.7**).,

$$\widetilde{P}_\alpha(\omega) = \int_{-\infty}^{+\infty} dx\, e^{i\omega x} P_\alpha(x) = i\pi\frac{\sinh\left(\frac{\pi\omega}{2\gamma}(1-\gamma\alpha)\right)}{\sinh\left(\frac{\pi\omega}{2\gamma}\right)}. \tag{3.37}$$

This Fourier transform holds only when $0 < \pi(1-\gamma\alpha) < \pi$, namely when $\alpha < 1/\gamma$: this means that this analysis is valid only in the regime in which the periodicity p_0 is larger than the length of the string. At this point, following the usual procedure, we can take the logarithm of equation (3.32) while performing the thermodynamic limit,

$$\begin{aligned} imL\sinh\theta_j &+ \int_{-\infty}^{+\infty} d\theta' \log\widetilde{S}_0(\theta_j - \theta')\sigma_0(\theta') \\ &+ \sum_{n=1}^{N-1}\int_{-\infty}^{+\infty} d\lambda \sum_{\alpha=1}^{n} \log f_k(\theta_j - \lambda_{\alpha l}^{(n)})\sigma_n(\lambda) = 2\pi i n_j. \end{aligned} \tag{3.38}$$

Subtracting this result to the same equation substituting j with $j-1$ and dividing by $2\pi i(\theta_j - \theta_{j-1})$ we get

$$\begin{aligned} \frac{mL}{2\pi}\cosh\theta &+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty} d\theta' \frac{\mathrm{d}}{\mathrm{d}\theta}\log\widetilde{S}_0(\theta - \theta')\sigma_0(\theta') \\ &+ \frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{-\infty}^{+\infty} d\lambda \sum_{\alpha=1}^{n}\frac{\mathrm{d}}{\mathrm{d}\theta}\log f_k(\theta - \lambda_\alpha^{(n)})\sigma_n(\lambda) = \bar{\sigma}_0(\theta), \end{aligned} \tag{3.39}$$

where $\bar{\sigma}_0(\theta) = \sigma_0(\theta) + \widetilde{\sigma}_0(\theta)$ is the total density, expressed as the sum of the particles density and the holes one, as defined in (2.92).

Using definition (2.27) we can write the previous equation in a more compact form

$$\sigma_0(\theta) + \widetilde{\sigma}_0(\theta) = \frac{mL}{2\pi}\cosh\theta + (\phi * \sigma_0)(\theta) - \sum_{n=1}^{\infty}(\psi_n * \sigma_n)(\theta), \tag{3.40}$$

where the integral kernels have been introduced, namely

$$\phi(x) = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}x}\log\widetilde{S}_0(x), \tag{3.41a}$$

$$\psi_n(x) = -\frac{1}{2\pi i}\sum_{\alpha=1}^{n}\frac{\mathrm{d}}{\mathrm{d}x^{(n)}}\log f_k(x^{(n)}_{-\alpha}) = \sum_{i=1}^{\min(n,k)} P_{-(n+k-2i+1)}(x^{(n)}). \tag{3.41b}$$

where in the last equality we have used the property (3.11a) (which is surprisingly valid also when f is defined as in equation (3.35)). Considering now equation (3.34) summed over $\beta = 1, \ldots, m$ and repeating the same steps, we obtain

$$\widetilde{\sigma}_m(\lambda) = -\sum_{n=1}^{\infty}(K_{mn} * \sigma_n)(\lambda) + (\psi_m * \sigma_0)(\lambda), \tag{3.42}$$

where ψ_m is defined as in (3.41b) while K_{mn} is

$$K_{mn}(x) = \delta(x)\delta_{mn} + \frac{1}{2\pi i}\sum_{\beta=1}^{m}\sum_{\alpha=1}^{n}\frac{\mathrm{d}}{\mathrm{d}x^{(m)}}\log f_2(x^{(m)}_\beta - x^{(n)}_\alpha), \tag{3.43}$$

$$= \delta(x)\delta_{mn} + P_{|m-n|}(x^{(m,n)}) + 2\sum_{i=1}^{\min(m,n)-1} P_{|m-n|+2i}(x^{(m,n)}) \tag{3.44}$$

$$+ P_{|m-n|+2\min(m,n)}(x^{(m,n)}),$$

where again in the last equality equation (3.11b) in its trigonometric version has been used.

Computing explicitly this last kernel in the Fourier space one finds

$$\widetilde{K}_{mn}(\omega) = \coth\left(\frac{\pi\omega}{2}\right)\frac{\sinh\left(\frac{\pi\omega}{2\gamma}(1-\max(m,n)\gamma)\right)}{\sinh\left(\frac{\pi\omega}{2\gamma}\right)}\sinh\left(\frac{\pi\omega}{2}\min(m,n)\right). \tag{3.45a}$$

It is clear that even in this case it is possible to express every kernel in term of the

K_{mn} one: again this will ensure the possibility of recasting the equations in their universal form. Indeed,

$$\widetilde{\psi}_m(\omega) = \widetilde{p}(\omega)\widetilde{K}_{mk}(\omega), \tag{3.46a}$$

$$\widetilde{\phi}(\omega) = \widetilde{p}^2(\omega)\widetilde{K}_{kk}(\omega). \tag{3.46b}$$

Moreover, even if the kernels look more complicated than those obtained in the previous case, it can be shown that in the $\gamma \to 0$ limit they correctly reproduce their undeformed counterparts (3.15a), (3.15b) and (3.21), indeed in this limit there is a term in the numerator of the three equations that never vanishes and that once combined with the ending hyperbolic sine gives Aladim and Martins result.

The next step is to define the energy as we did in the diagonal case (see equation (2.23)),

$$\mathcal{E}[\sigma_0] = \int_{-\infty}^{+\infty} d\theta\, \cosh\theta\sigma_0(\theta), \tag{3.47}$$

and the entropy, which this time will contain also the contribution of the magnon densities:

$$\mathcal{S}[\sigma_n, \widetilde{\sigma}_n] = \sum_{n=0}^{\infty}\int_{-\infty}^{+\infty} dx\, [\sigma_n(x) + \widetilde{\sigma}_n(x)]\log[\sigma_n(x) + \widetilde{\sigma}_n(x)] \\ - \sigma_n(x)\log\sigma_n(x) - \widetilde{\sigma}_n(x)\log\widetilde{\sigma}_n(x). \tag{3.48}$$

so that we can build up the free energy, defined as in equation (2.28). To get informations about the equilibrium configuration, we need to vary this function with respect to the various densities, keeping equations (3.40) and (3.42) as constraints, which means that we need to introduce $\xi(\theta)$ and $\mu_j(\lambda)$ as Lagrange multipliers, as done in the undeformed case. The calculation at this point are straightforward and they basically reproduce the steps done in the previous section, with the only notable modification being the explicit expression of the integral kernels.

Finally we end up with the TBA equations

$$\epsilon_0(\theta) = r\cosh\theta - (\phi * L_0)(\theta) - \sum_{m=1}^{\infty}(\psi_m * \log(1+e^{\epsilon_m}))(\theta), \tag{3.49a}$$

$$L_n(\lambda) = (\psi_n * L_0)(\lambda) + \sum_{m=1}^{\infty}(K_{mn} * \log(1+e^{\epsilon_m}))(\lambda), \tag{3.49b}$$

where $r = mR$ is the relevant dimensionless parameter, ϵ_i are the pseudoenergies defined as in (3.25) (the index 0 is relative to the physical particle while n refers to the magnons) and the integral kernel L is defined as in (2.44).

Even in this case one can find a relation in order to invert the K_{mn} kernel (3.45a) and as in the trigonometric case, it is given by

$$K_{mn}^{-1}(\lambda) = \delta(\lambda)\delta_{mn} - p(\lambda)(\delta_{m,n+1} + \delta_{m,n-1}). \tag{3.50}$$

where $p(\lambda)$ has the usual Fourier transform (3.17).

Taking the convolution of $K_{im}^{-1}(\lambda)$ with equation (3.49b) and then summing over the m index, one finds

$$\epsilon_i(\lambda) = \delta_{ik}(p * L_0)(\lambda) + (p * (L_{i-1} + L_{i+1})) \qquad i = 1, 2, \ldots, \tag{3.51}$$

and repeating the same reasoning of the previous section, the other equation reads

$$\epsilon_0(\theta) = mR \cosh\theta - (p * L_k)(\theta). \tag{3.52}$$

Therefore we have just shown that in the trigonometric case the structure of the Dynkin diagram given by these TBA equations is identical to the rational one.

However, as already mentioned, the periodicity $p_0 = \pi/\gamma$ of equations (3.32) and (3.34) in the imaginary λ direction plays a crucial role. Indeed, it can be shown that the irreducible representations of a quantum group are in 1-1 correspondence with the Bethe strings forming in the thermodynamic limit. In fact, when the deformation parameter q is not a root of unity, *i.e.* when γ is an irrational number, the infinitely many representations of the quantum group are those of the group and therefore we get a situation similar to the undeformed case, with an infinite number of magnonic nodes. However, when one considers γ to be a rational multiple of π, which correspond to the case when q is a root of unity, it is possible to show that among all the possible representation of the quantum group only a finite number is irreducible and therefore we expect a finite amount of allowed strings in the TBA analysis. This will lead to a closure of the diagram, meaning that it will end after a certain node. In particular, when $1/\gamma$ is π divided by some integer N, the number of allowed strings is N. In the latter case one possibility for the final diagram could be the one represented in figure 3.2, where the magnonic part of it becomes a D_N-type Dynkin diagram .

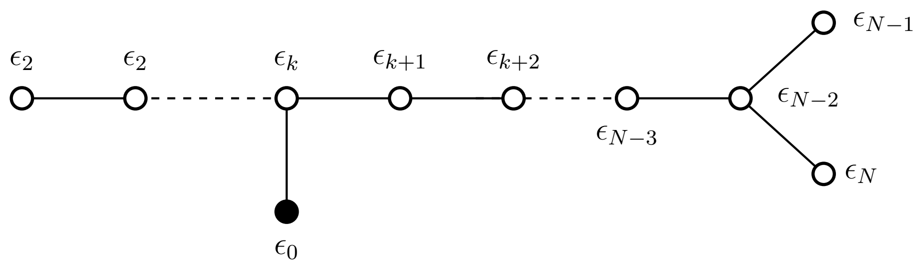


Figure 3.2: Potential graphical representation of TBA equations for the deformed scattering theories. We assume that $\gamma = \pi/N$.

In particular we note that this is the same result that we obtain in the two known cases namely $k = 1$, which is the sine-Gordon model (see figure 2.7) and $k = 2$, which is the sausage model [37].

Conclusions and Outlooks

In this thesis we investigated two different families of scattering theories at arbitrary spin by means of the Thermodynamic Bethe Ansatz method.

At first we considered a set of models introduced by S.R Aladim and M. J. Martins [7], described by non diagonal S-matrices which are invariant under $SU(2)$ symmetry acting on the multiplets of particles that lie in the $s = k/2$ representation. These theories are of great importance as many quantum field theories present this type of symmetry (e.g. the sine-Gordon model at $\beta = \sqrt{8\pi}$ where it becomes a $SU(2)$ Gross-Neveu model or the $O(3)$ sigma model). Since these models are described by non-diagonal S-matrices, we had to take into account the presence of magnons rapidities, which can be considered as a mathematical artefact of fictitious massless particles, allowing to diagonalize the transfer matrix of the model: these additional objects make the analysis more complex with respect to the diagonal case. However, in the thermodynamic limit, the magnonic solutions tend to organize into Bethe strings and it is then possible to simplify the calculations.

We then obtained the "raw" TBA equations of these models, which turned out to be defined through three different integral kernels: one describing the magnon-magnon scattering, one the magnon-particle scattering and one the particle-particle one. Surprisingly these three object are all related, and thanks to this relation it has been possible to further simplify the system, resulting in a universal form of the TBA equations.

This result is of great importance since it is the starting point to analyze the critical behaviour of these theories, for example by computing their central charge, which we know to be wrong in [7].

It is known that it is possible to deform the $SU(2)_k$ symmetry algebra underlying these theories preserving integrability in order to obtain a new set of scattering matrices with a quantum group symmetry $\mathfrak{U}_q(\mathfrak{sl}(2))$: these exact S-matrices has been recently found in [8]. At this point the natural question to be answered was how to describe the TBA of these new models and whether they are related to their undeformed counterparts. Analyzing these new scattering theories we understood that the structure of the equations remains basically the same with new definitions of the three integral kernels previously mentioned. Additionally, taking the $q \to 1$ limit, *i.e.* the undeformed limit, the two results are equal.

Finally, we computed the TBA equations for this class of theories: as expected, it reproduces the same result as the undeformed case. However in this situation the number of total magnons may be finite, depending on the periodicity of the TBA equations, leading to different results.

This is just a first step in the analysis of these new theories, and there are still things to do to fully understand their physical meaning:

- first of all, once the universal form is derived, one should compute the central charge, in order to have a first insight on the critical behaviour of these models in their UV limit.
- Once it is understood what kind of CFT these models are describing, one should perform a conformal perturbation preserving integrability, by some relevant field, as explained in [13], in order to obtain the massive QFTs that would be equivalent to the studied scattering models.
- If this analysis is able to put in relation the deformed S-matrices with a physical QFT, it is then possible to try to establish a duality between these theories and some sigma models, as proposed in recent papers by V. A. Fateev *et al.*

A

The Casimir Effect

Let us briefly describe how to derive the expression of the Casimir energy in terms of conformal parameters, such as the central charge and the scaling dimensions of fields.

First of all we consider an euclidean theory defined on a cylinder $\mathfrak{C}$ of width R, parametrized by the coordinates $\tau = (-\infty, \infty)$ and σ, such that $\sigma = \sigma + 2\pi$. Every point of this space can be written as

$$\zeta = \tau + i\sigma. \tag{A.1}$$

Next one can introduce the radial conformal mapping

$$\begin{aligned} \rho : \mathfrak{C} &\longrightarrow \mathbb{C} \\ \zeta &\longmapsto z = e^{2\pi(\tau + i\sigma)/R} \end{aligned} \tag{A.2}$$

which maps slices at equal τ of $\mathfrak{C}$ into concentric circumferences in $\mathbb{C}$, in particular $\tau = -\infty$ to mapped into the origin and $\tau = \infty$ in the point at infinity of the complex plane. The stress-energy tensor of the theory changes under local conformal mappings $z \to \eta$,

$$T(z) = T(\eta) \left(\frac{\mathrm{d}\eta}{\mathrm{d}z}\right)^2 + \frac{c}{12}\{\eta, z\} \tag{A.3}$$

where $\{\cdot, \cdot\}$ is the Schwartz derivative (for more detail see e.g. [39]).

In the case we are analysing we have

$$T_{\mathfrak{C}}(\zeta) = \left(\frac{2\pi}{R}\right)^2 \left[T_{\mathbb{C}}(z)z^2 - \frac{c}{24}\right]. \tag{A.4}$$

and a similar expression for the anti-holomorfic component $\bar{T}$. Combining equation (A.4) and the definition of the Virasoro generators

$$L_0 = \frac{1}{2\pi i}\oint dz\, zT(z), \qquad \bar{L}_0 = -\frac{1}{2\pi i}\oint d\bar{z}\, \bar{z}\bar{T}(z). \tag{A.5}$$

it is possible to calculate the Hamiltonian of the system

$$H = \frac{1}{2\pi}\int_0^R d\sigma\, (T(\sigma) - \bar{T}(\sigma)) = \frac{2\pi}{R}(L_0 + \bar{L}_0) - \frac{\pi c}{6R}. \tag{A.6}$$

The minimum value of the energy is thus

$$\mathcal{E}_0 = -\frac{\pi c_{\text{eff}}}{6R} \tag{A.7}$$

with

$$c_{\text{eff}} = c - 24\Delta_{\text{min}}, \tag{A.8}$$

c being the central charge of the theory and Δ_{min} the lowest eigenvalue of L_0. Equation (A.7) is a particular case of (2.17) and it represents the definition of Casimir energy: it is inversely proportional to the width of the cylinder and vanishes when $R \to \infty$, *i.e.* when it reduces to a plane, therefore it is a purely topological quantity that depends on the geometry of the system.

For unitary theories it turns out that $c_{\text{eff}} = c$ and $c > 0$, while for non-unitary ones Δ and c can be both negative. However it can be shown that the effective central charge of every minimal model, unitary or not, is always positive.

B

Rogers Dilogarithms

In this appendix we will briefly summarize some of the properties of the Rogers dilogarithm function, which is ubiquitous in the TBA calculations and in general in the study of integrable systems, see e.g. [40].

B.1 Definition

The Rogers dilogarithm function can be defined via its integral representation

$$\mathcal{L}(x) = -\frac{1}{2}\int_0^x dt \left(\frac{\log(1-t)}{t} + \frac{\log t}{1-t}\right). \tag{B.1}$$

where $0 \leq x \leq 1$. It turns out that it is deeply connected to the Euler dilogarithm,

$$\mathrm{Li}_2(x) = -\frac{1}{2}\int_0^x \frac{\log(1-t)}{t} = \sum_{n=1}^{\infty} \frac{x^2}{n^2}, \tag{B.2}$$

since it is possible to show that

$$\mathcal{L}(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log x \log(1-x). \tag{B.3}$$

B.2 Properties

Among a huge number of identities (which can be found e.g. in [41]), we have that

$$\mathcal{L}(x) + \mathcal{L}(1-x) = \frac{\pi^2}{6}, \tag{B.4}$$

and

$$\mathcal{L}(x) + \mathcal{L}(y) = \mathcal{L}(xy) + \mathcal{L}\left(\frac{x(1-y)}{1-xy}\right) + \mathcal{L}\left(\frac{y(1-x)}{1-xy}\right), \tag{B.5}$$

where $x > 0$ and $y < 1$. This last identity, calculated for $x = y$ leads to the Abel duplication formula

$$\mathcal{L}(x^2) = 2\mathcal{L}(x) - 2\mathcal{L}\left(\frac{1}{1+x}\right). \tag{B.6}$$

Using these properties, it is possible to evaluate the dilogarithm for some special values, which are very useful in TBA analysis:

$$\mathcal{L}(0) = 0, \tag{B.7}$$

$$\mathcal{L}(1) = \frac{\pi^2}{6}, \tag{B.8}$$

$$\mathcal{L}(-1) = -\frac{\pi^2}{12}, \tag{B.9}$$

$$\mathcal{L}\left(\frac{1}{2}\right) = \frac{\pi^2}{12}, \tag{B.10}$$

$$\mathcal{L}\left(\frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{10}, \tag{B.11}$$

$$\mathcal{L}\left(\frac{3-\sqrt{5}}{2}\right) = \frac{\pi^2}{15}. \tag{B.12}$$

C

Lie Algebras

A Lie algebra $\mathcal{A}$ of dimension n is a vector space equipped with a binary anti-symmetric operation $[\cdot,\cdot] : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$, called Lie bracket, that must satisfy the Jacobi identity;

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0, \qquad x,y,z \in \mathcal{A}. \tag{C.1}$$

It is possible to relate every element of a Lie algebra with one element of the connected part of a Lie group containing the unit element via the exponential map:

$$\begin{aligned} \exp : \mathcal{A} &\longrightarrow G \\ x &\longmapsto g = e^{iax} \end{aligned} \tag{C.2}$$

for some parameter a. A Lie algebra can be defined by specifying a set of generators $\{J^a\}_{a=1}^n$, such that

$$[J^a, J^b] = \sum_c i f_c^{ab} J^c. \tag{C.3}$$

The number n of these generators is called the dimension of the algebra, while the constants f_c^{ab} are the structure constants, which turns out to be real if the generators are self-adjoint. A map that associate every element of $\mathcal{A}$ to a linear operator acting on some vector space V is called a ***representation*** of the algebra, the dimension of V being the dimension of the representation. Among all the J^a is always possible to find r generators[1] H^i, $i = 1, \dots, r$, such that

$$[H^i, H^j] = 0. \tag{C.4}$$

This set of generators forms the ***Cartan subalgebra*** $\mathcal{H}$. The other generators can always be chosen as particular combination of the starting J^a that satisfy

$$[H^i, E^\alpha] = \alpha^j E^\alpha. \tag{C.5}$$

Therefore for a given E^α it is possible to define an object $\alpha = (\alpha^1, \dots, \alpha^r)$ which naturally maps every $H^i \in \mathcal{H}$ to the relative number α^i of equation (C.5) by $\alpha(H^i) = \alpha^i$. These objects are called ***roots*** and by virtue of what said above they are element of the dual Cartan subalgebra H^*. Since equation (C.5) is invariant under Hermitian conjugation, one can infer that if α is a root, $-\alpha$ is a root too. Moreover the roots are generally linearly dependent. Therefore one can introduce a basis for H^*, and express every root as

$$\beta = \sum_{i=1}^{r} n_i \alpha^{i}{}^{2}, \tag{C.6}$$

where α_i are called simple roots. These roots are positive roots[3] that cannot be expressed as the sum of two positive roots.

It is possible to define a Killing form in the algebra $\mathcal{A}$, by

$$K(x, y) = \mathrm{Tr}(\mathrm{ad}(x)\mathrm{ad}(y)), \tag{C.7}$$

where $\mathrm{ad}(\cdot)$ represents the adjoint representation, ***i.e.*** the representation that maps the elements of the algebra into itself and where the action of the generator x on y is represented by $\mathrm{ad}(x)y = [x, y]$.

The fundamental role of the Killing form is that it induces a Killing form in the dual Cartan subalgebra (for a detailed discussion see e.g. [39] ch. 13), ***i.e.*** a scalar product which allows to compute lengths and angles between simple roots, defined

[1] r is the rank of the algebra.

[2] From now on, the superscript does not refer to the i-th component of a root, but it indicates the i-th simple root.

[3] A positive root is a root for which its first element is positive.

as

$$(\alpha, \beta) = K(H^\alpha, H^\beta), \qquad \text{such that } (\alpha, \alpha) = |\alpha|^2. \tag{C.8}$$

It is possible to define the Cartan matrix, whose elements are defined as

$$C_{ij} = \frac{2(\alpha^i, \alpha^j)}{|\alpha^j|^2}. \tag{C.9}$$

It can be shown that the diagonal elements of this matrix are equal to 2, while the non diagonal ones are necessarily negative integers. Moreover the Schwarz inequality implies that the product $C_{ij}C_{ji} < 4$, thus the elements can take the values 0, -1, -2 or -3 only.

From these considerations it is possible to show that there are only two possible root lengths (long and short) and the angles between them can be d 90°, 120°, 130° and 150°. Since it is possible to uniquely define an algebra only knowing the Cartan matrix, *i.e.* the relations between the various roots, it is possible to introduce a diagrammatic notation, known as Dynkin diagram, which encodes all the informations about the system of roots of a given algebra. The rules are simple:

- A root of long length is represented by an empty circle ○, while a short root by a filled circle ●;
- Two circles are connected by 1, 2 or 3 lines whether the angle between them is 120°, 130° and 150° respectively. If they are not connected, the two roots are orthogonal.

Therefore, all the possible simple Lie algebras can be classified as in table C.1.

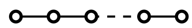
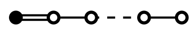
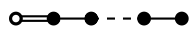
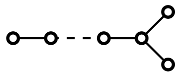
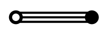
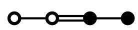
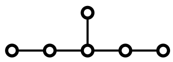
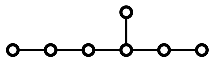
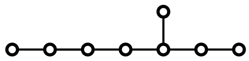
Algebra	**Group**	**Dimension**	**Dynkin diagram**	$\boldsymbol{h}$
A_r	$SU(r+1)$	$r(r+2)$, $r \geq 1$		r+1
B_r	$O(2r+1)$	$r(2r+1)$, $r \geq 2$		2r
C_r	$Sp(2r)$	$r(2r+1)$, $r \geq 2$		2r
D_r	$O(2r)$	$r(2r-1)$, $r \geq 3$		2r-2
G_2		14		6
F_4		52		12
E_6		78		12
E_7		133		18
E_8		248		30

Table C.1: Classification of simple Lie algebras.

D

Computation of ϕ Kernel

As mentioned in Chapter 3, the ϕ kernel is the one which comes from the prefactor of the S-matrix, obtained from unitarity, crossing symmetry and Yang-Baxter equation. It is the one of biggest interest, since it is directly related to the scattering properties of the theory. We are going to prove here the fact that the ϕ kernel is identical both in the even and odd k case.

The prefactor is defined as follows

$$\widetilde{S}_0(\theta) = \prod_{l=\text{odd}}^{k} \frac{\theta - il\pi}{\theta + il\pi} \qquad \text{for } k \text{ even,} \tag{D.1a}$$

$$= S_0(\theta) \prod_{l=\text{even}}^{k} \frac{\theta - il\pi}{\theta + il\pi} \qquad \text{for } k \text{ odd,} \tag{D.1b}$$

where

$$S_0(\theta) = \frac{\Gamma(1/2 - i\theta/2\pi)\Gamma(i\theta/2\pi)}{\Gamma(1/2 + i\theta/2\pi)\Gamma(-i\theta/2\pi)}. \tag{D.2}$$

D.1 Even Case

The calculation in the even case are rather simple. Indeed, taking the logarithmic derivative of the right hand side of equation (D.1a) one gets

$$\sum_{l=0}^{k/2-1} \frac{\mathrm{d}}{\mathrm{d}x} \log f_{2(2l+1)}(x) = \sum_{l=0}^{k/2-1} P_{2(2l+1)}(x), \tag{D.3}$$

where $f_\alpha(x)$ is the function defined in equation (3.8). To explicitly compute the sum, it is useful to perform the calculation in momentum space. The Fourier transform of $P_\alpha(x)$ reads

$$\widetilde{P}_\alpha(\omega) = -2\pi i e^{\pi|\omega|\alpha/2}. \tag{D.4}$$

Therefore equation (D.3) becomes

$$\begin{aligned}\mathscr{F}\left(\sum_{l=0}^{k/2-1}\frac{\mathrm{d}}{\mathrm{d}x}\log f_{2(2l+1)}(x)\right)(\omega) &= -2\pi i\sum_{l=0}^{k/2-1} e^{\pi|\omega|(2l+1)} \\ &= 2\pi i\left[\frac{1}{4\cosh^2(\pi\omega/2)}\coth\left(\frac{\pi|\omega|}{2}\right)\left(1-e^{\pi|\omega|k}\right)\right].\end{aligned} \tag{D.5}$$

This is exactly the result of equation (3.21). The factor $2\pi i$ is simplified by the right hand side of equation (3.5) once the logarithm is taken.

D.2 Odd Case

In this case calculations are a bit more cumbersome, due to the presence of the prefactor (D.2). The logarithm of the right hand side of equation (D.1b) is

$$\log S_0(\theta) + \sum_{l=1}^{(k-1)/2}\log f_{4l}(x). \tag{D.6}$$

Let us now focus on the first term of this sum, indeed using the well-known relation $\Gamma(1+z) = z\Gamma(z)$ one can rewrite it as

$$-\log\frac{\Gamma\left[1+\left(\frac{i\theta}{2\pi}-\frac{1}{2}\right)\right]}{\Gamma\left[1+\left(\frac{i\theta}{2\pi}-\frac{1}{2}\right)+\frac{1}{2}\right]} + \log\frac{\Gamma\left[1+\left(-\frac{i\theta}{2\pi}-\frac{1}{2}\right)\right]}{\Gamma\left[1+\left(-\frac{i\theta}{2\pi}-\frac{1}{2}\right)+\frac{1}{2}\right]} \tag{D.7}$$

Now it is possible to use the following integral representation (see e.g. [25], §**3.427, 10**)

$$\log\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+\gamma)} = \int_0^\infty\frac{dx}{x}\left[e^{-\alpha x}\frac{(1-e^{-\gamma x})}{e^x-1} - \gamma e^{-x}\right], \tag{D.8}$$

which holds whether $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\gamma > 0$.

Therefore after changing the integration variable $x \to 2\pi\omega$, it becomes

$$-\int_0^\infty \frac{d\omega}{\omega}\left(\frac{1-e^{-\pi\omega}}{e^{\pi\omega}-e^{-\pi\omega}}\right)e^{-i\omega x}+\int_0^\infty \frac{d\omega}{\omega}\left(\frac{1-e^{\pi\omega}}{e^{\pi\omega}-e^{-\pi\omega}}\right)e^{i\omega x}. \tag{D.9}$$

Using some trigonometric manipulations and a change of integration variable in the second integral $\omega \to -\omega$ one finds

$$-\int_0^\infty \frac{d\omega}{2\omega}\frac{e^{-\pi\omega/2}}{\cosh(\pi\omega/2)}e^{-i\omega x}-\int_{-\infty}^0 \frac{d\omega}{2\omega}\frac{e^{\pi\omega/2}}{\cosh(\pi\omega/2)}e^{-i\omega x}, \tag{D.10}$$

which can be rewritten in a more compact form as

$$-\int_{-\infty}^\infty \frac{d\omega}{2\omega}\frac{e^{-\pi|\omega|/2}}{\cosh(\pi\omega/2)}e^{-\pi\omega x}. \tag{D.11}$$

Taking the derivative of this result, as customary in this kind of calculations, one gets

$$\frac{\mathrm{d}}{\mathrm{d}x}\log S_0(x)=\int_{-\infty}^\infty \frac{d\omega}{2\pi}\left[i\pi\frac{e^{-\pi|\omega|/2}}{\cosh(\pi\omega/2)}\right]e^{-i\omega x}, \tag{D.12}$$

where we have highlighted in square brackets the expression of the Fourier transform of the term under analysis.

Let us now compute the x-derivative of the second term of equation (D.6), namely

$$\sum_{l=1}^{(k-1)/2}\log\frac{\mathrm{d}}{\mathrm{d}x}f_{4l}(x)=\sum_{l=1}^{(k-1)/2}P_{4l}(x). \tag{D.13}$$

Using the Fourier expression of $P_\alpha(x)$ given in (D.4), one gets

$$\begin{aligned}\mathscr{F}\left(\sum_{l=1}^{(k-1)/2}\log\frac{\mathrm{d}}{\mathrm{d}x}f_{4l}(x)\right)(\omega)&=2\pi i\sum_{l=1}^{(k-1)/2}e^{-2\pi|\omega|l}\\&=\pi i\left[\frac{e^{-\pi|\omega|}-e^{-\pi|\omega|k}}{\sinh(\pi|\omega|)}\right].\end{aligned} \tag{D.14}$$

Therefore, combining the two results, one finds

$$\int_{-\infty}^\infty \frac{d\omega}{2\pi}e^{-i\omega x}2\pi i\left[\frac{e^{-\pi|\omega|/2}}{2\cosh(\pi\omega/2)}+\frac{e^{-\pi|\omega|}-e^{-\pi|\omega|k}}{2\sinh(\pi|\omega|)}\right]. \tag{D.15}$$

Again, manipulating this result with some simple trigonometric identities, it follows that

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega x} 2\pi i \left[\frac{1}{4\cosh^2(\pi\omega/2)} \coth\left(\frac{\pi|\omega|}{2}\right) (1 - e^{\pi|\omega|k}) \right] \tag{D.16}$$

As already pointed out in the even case, the factor $2\pi i$ will cancel out once this expression is put into the equation.

As claimed, the results in the even and odd case are equal: this will lead to a unique TBA analysis for both cases.

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