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Fuzzy Dark Matter from String Axions

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When it comes to the world around us,
is there any choice but to explore?

Lisa Randall [1]

Abstract

In this thesis we study the viability for ultra-light axions coming from moduli stabilisation in the Large volume scenario to fit recent observations about the dark matter presence in several galaxies and clusters. After a historical introduction on the dark matter problem and a review of the candidates proposed as dark matter constituents through the years, in Chapter 1 we present WIMPs and ALPs, which are nowadays believed to be the most likely constituents. In Chapter 2 we give the theoretical basis to analyse these particles, that is Supersymmetry and String theory, focusing especially on dimensional reduction and string compactifications in order to derive an effective theory. Then, in Chapter 3 we discuss the issue of moduli stabilisation and we deal with it in the Large volume scenario, examining the examples of Swiss cheese and Fibred Calabi-Yau manifolds. Finally, in Chapter 4 we present the original results of this thesis. Starting from recent observations claiming the existence of a preferred range of masses for the ultra-light axions constituting dark matter, we provide a theoretical explanation in the Large volume scenario. We demonstrate that a preferable mass exists for these axions and that axions having this mass could account for the total observed dark matter abundance in a natural way. In addition to this, we examine how to provide several axions with different masses by imposing reasonable cosmological hierarchies. Finally, we show how to generalise these results to Calabi-Yau manifolds with any number of axions, while in Chapter 5 some interesting outlooks of the present work are discussed.

Sommario

In questa tesi studiamo gli assioni ultra leggeri che provengono dalla stabilizzazione dei moduli nel regime di Large volume e la loro possibilità di riprodurre recenti osservazioni astrofisiche riguardanti la presenza di materia oscura in diverse galassie e ammassi. Dopo un'introduzione storica riguardante il problema della materia oscura e un riassunto delle varie proposte che si sono succedute nel corso degli anni per risolvere tale problema, nel Capitolo 1 presentiamo le WIMPs e le ALPs, che attualmente si ritiene siano i migliori candidati per costituire la materia oscura. Nel Capitolo 2 vengono fornite le basi teoriche a supporto dell'analisi di tali particelle: introduciamo la Supersimmetria e la Teoria delle Stringhe e, nello specifico, ci concentriamo sulla riduzione dimensionale e sulla compattificazione al fine di ottenere una teoria effettiva. Poi, nel Capitolo 3 discutiamo il problema della stabilizzazione dei moduli e lo risolviamo utilizzando il limite di Large volume, presentando come esempi due casi di topologie diverse di varietà di Calabi-Yau, cio Swiss cheese e Fibred Calabi-Yaus. Infine, nel Capitolo 4 discutiamo i risultati originali di questa tesi. Partendo da recenti osservazioni sperimentali che riportano uno specifico range di masse per gli assioni ultra-leggeri che costituirebbero la materia oscura, forniamo ad essi una spiegazione teorica nello scenario di Large volume. Inoltre, dimostriamo che esiste un preciso valore della massa che si ricava in un modo molto naturale, e che una popolazione assionica con tale massa potrebbe costituire tutta la densità di materia oscura osservata. Oltre a questo, esaminiamo come produrre vari assioni con masse diverse tra loro imponendo ragionevoli gerarchie cosmologiche. Chiudiamo la discussione mostrando come generalizzare questi risultati ad una varietà di Calabi-Yau con un qualsiasi numero di assioni. Nel Capitolo 5, per concludere, vengono presentate alcune interessanti prospettive per questi risultati.

Contents

Chapter 1

Dark Matter: A Brief Introduction

What is the Dark Matter?

This is probably one of the most important and difficult questions in physics nowadays. Since its presence was accepted around the 80s after a long debate, the scientific community began to work intensively on Dark Matter (DM), trying to answer the question at the beginning of this chapter. However, we still do not have any robust answer. Of course, during these decades, many and many theories have been proposed, and only a few survived tests and experiments. As the years go by, our knowledge about what DM should be increased, primarily thanks to the advent of the so-called precision cosmology. At the present, we are in possession of many data coming from both astrophysics and particle physics, that give us constraints on DM properties, behaviour, dynamics and so on. Unfortunately, these data are not enough to state which theory or proposal is the most accurate in describing DM, so we are still not able to prefer one theory over the other.

For these reasons, after a short historical introduction based on [2, 3], we are going to review briefly the most prominent proposals following [4, 5, 6]. After that, we will choose one of them to work with in this thesis.

1.1 Genesis of the problem

The first papers showing that something was missing in our comprehension of the universe came out in the 30s, and by around the 80s the need for new physical and cosmological features was widely recognized by the scientific community. The problem arose as a wide discrepancy between what was theoretically expected and what experimental data reported.

In 1933, the astronomer Fritz Zwicky published a pioneering paper [7] where he studied the mass of Coma cluster. Zwicky found that the velocity dispersion of galaxies was so high that, to keep the system stable, the average mass density of the cluster had to be much higher than the one deduced from the observable matter. Therefore, he first proposed the existence of unseen, dark matter. It was not until the 50s that this idea took hold in the astronomical debate, also due to the many more clusters that were found to have the same mass discrepancy as Coma cluster. However, Zwicky proposal was only one possibility among many others and by the end of the 60s and the beginning of the 70s, a large number of solutions to the cluster problem had been discussed.

Another important milestone was laid in 1970 by Vera Rubin and Kent Ford in the celebrated paper [8]. In their study of the rotation curve of Andromeda Nebula, they showed that the mass profile calculated from both the stars distribution in the galactic spiral and the mass-to-light ratio in the stellar disk did not match with the masses derived from the observed rotation curves. Form Kepler's Laws, the rotation velocities were expected to decrease with the distance from the galactic centre. Instead, it was found that rotation curves remained flat as the galactic radius increased. In short, it seemed that the galaxy had much more mass than the visible one, and this mass was gravitationally contributing to rotation curves. Otherwise, the gravity description used was not correct. Nevertheless, based on their measurement, Rubin and Ford drew no conclusion on whether DM existed or not.

The presence of unseen matter was a possible solution to two different issues that arose independently. The connection between the problems of flat rotation curves and mass discrepancy in the clusters became evident only when these issues were considered under a new prospective. Between the 60s and the 70s, the disciplinary boundaries between physics and astronomy faded into cosmology and many breakthroughs followed. A major subject of research was to understand whether the universe was flat, open or closed. In turn, this corresponded to finding the mass density of the universe Ω, which started to be considered a fundamental parameter in cosmology. At the time, it was still believed that the universe had to be close and physicists and astronomers were trying to motivate this statement through observations.

This is the background where in 1974 the work [9] by the physicist James Peebles and the astronomers Jeremiah Ostriker and Amos Yahil developed. In their search for Ω, for the first time they synthesized the two problems of galaxies and clusters into a single framework, stating that they were both due to a lack of mass in the universe. In particular, they found that the galactic masses had been underestimated by a factor of 10 or more. So, if the universe was close or at least flat, the amount of matter had to be way more than what was observed.

By the 80s, it was widely recognized that these two problems were in fact the *evidence* that something important was missing. The possible solutions proposed to solve the puzzle followed two distinct paths. The hypothesis that the missing mass consisted of one or more subatomic particle species (probably still unknown) gained enough support to become the leading paradigm. This is the particle physics way, and we will proceed in this direction. Nevertheless, a brief mention to the other main proposal is in place.

A relevant branch of research approaches the missing mass problem by assuming that our understanding of gravity could be either wrong or incomplete, in such a way that no mass may be properly missing. This perspective led to the so-called theories of modified gravity. Their first instances were formulated way before the 70s, but they really took off over the past decades, also thanks to the trigger given by the work of Rubin and Ford. We will not delve further into describing the theories of modified gravity; a satisfying review can be found in [10].

Instead, we will proceed by trusting General Relativity and by considering a new, unknown and unseen form of matter as responsible for the discrepancies between theory and experiments. Therefore, in the next sections we present a review of the most suitable DM candidates which have been proposed through the years.

Finally, it is interesting to notice that Zwicky used the term *dark matter* in German in 1933 and in English in 1937. However, apparently it had not been used again in the literature until 1979, when it appeared in a review paper [11] about Zwicky's work and the issues described above. Before that, the phrases *missing mass* or *missing matter* were used. In particular, at the early stages the word *dark* was only an adjective and the whole phrase referred to both particles and all the varieties of astrophysical material that happened to be too faint to be detected by telescopes. In the next sections, we will see that the meaning of this expression has considerably changed. Nowadays, with the phrase *dark matter* we address to any form of energy density which makes its presence felt only by its gravitational effects [4].

1.2 Review of dark matter candidates

It is now widely recognized that finding a suitable candidate for DM must be a crucial subject of research. This became even more pressing when the first estimates of the percentage of DM abundance present in the universe were made, showing that the DM contribution to the total mass of the universe was way more significant than ordinary matter. The last estimate made by Planck and WMAP [12] reported that the matter content in the universe is $\Omega_m \simeq 0.317$, but the contribution of the visible matter is only $\Omega_{m,v} \simeq 0.049$.

For this reason, the scientific community began a sort of golden rush to find appropriate DM constituents, in both a theoretical and experimental way. Before proceeding, we want to underline that there is no *a priori* reason for DM to be made of only one constituent [4]. Moreover, we are still not sure about which kind of particles can make up the DM, or even what mass these particles might be expected to have. Indeed, as we will see, the proposals span from 1 TeV down to 10^{-22} eV. Consequently, we think that the fairest approach to follow is to keep an open mind about particle physics while trying to place constraints using astrophysical, cosmological and theoretical considerations.

We can divide the DM candidates into three major groups on the basis of their physical features:

- *baryonic* and *non-baryonic* DM. By baryonic DM we refer to objects that are made of (more or less) known constituents, such as the Standard Model (SM) particles and the ordinary matter in general, but that are not visible either because their light is too faint or because their presence can only be inferred by their effects on the surrounding objects. On the contrary, non-baryonic DM is made of brand new particles described by theories beyond the SM, and their interaction with photons is so weak that they are actually not visible.

- *thermal* and *non-thermal* DM. This classification is based on the way DM constituents are produced. If a DM particle is in thermal equilibrium with the other components of the universe but suddenly it decouples, then this particle is said to be produced in a thermal way. Instead, a non-thermal particle is a particle that is produced via a dynamical mechanism, in such a way that it may not be in thermal equilibrium with the universe at the moment of its production.

- *hot* and *cold* DM. This distinction lies on the velocity the particles have when they decouple from the primordial plasma. A particle species is said to be hot DM (HDM) if it is relativistic when it decouples or, more precisely, if the velocity dispersion of these particles is relativistic. Instead, cold DM (CDM) is made of particles having a very slow velocity dispersion compared to the speed of light. Usually, in the term CDM also non-thermal DM is included. The discrimination between hot and cold is of extreme importance because HDM and CDM decouple from radiation before ordinary matter, and in turn they give rise to two different scenarios of structure formation in the early universe.

The first DM candidates proposed were of course baryonic. The astrophysical landscape numbers several examples of objects that are not visible. These are collectively referred to as Massive Astrophysical Compact Objects (MACHOs), and they include among others: white dwarfs, red and brown dwarfs, neutron star and astrophysical black holes. These objects can be detected mainly via gravitational lensing. However, statistical analysis of such events in our galaxy revealed that they occur in a very small number, implying that there are too few MACHOs to account for a significant fraction of the DM mass, in both our galaxy and the universe.

It is worth mentioning a particular class of MACHOs, that is the so-called Primordial Black Holes (PBHs). PBHs were first proposed by Stephen Hawing in 1971 in [13], and they have been studied ever since. The idea that they could represent a promising DM candidate got revived interest after the recent detection of gravitational waves by LIGO and Virgo and the advent of inflationary cosmology. Although PBHs could play an important cosmological role, it is still unclear if they can account for the total mass of the DM in the universe [14]. The main theoretical issue is the difficulty to generate PBHs which constitute 100% of DM in a natural way.

Another baryonic candidate proposed at the early stages is the massive neutrino, which can be simply described by making a little extension to the SM. Neutrinos are the typical HDM proponents, since they are very light (~ 1 eV) and interact only weakly and gravitationally with the other particles. However, they are now believed to represent a minor fraction of DM thanks to the recent developments in the theory of structure formation.

As it is well known, DM has a very important role in structure formation. Since it decouples from the primordial plasma before matter, DM creates the conditions for the perturbations to collapse, to grow and form astrophysical structures such as galaxies and clusters. Depending on whether the DM is hot or cold, two different scenarios can arise, and they are denoted respectively *top-down* and *bottom-up*. In a top-down scenario, the first structures to show up are the most massive ones, then the smaller structures are formed through fragmentation. Instead, in a bottom-up scenario the first structures to appear are the smallest ones, which subsequently converge and form bigger structures. These two trends follow from the different values of the Jeans mass M_J for the HDM and the CDM. M_J is a reference mass indicating whether a perturbation of matter can grow (if its mass is bigger than the M_J of the DM) or must fade away (if on the contrary its mass is smaller than the M_J of the DM). M_J scales with the cube of the free-streaming velocity so that the Jeans mass for HDM is around ten order bigger than for CDM. Because HDM has such a big value for M_J, the only perturbations to survive and then grow are the most massive ones.

Data from the cosmic microwave background (CMB) radiation as measured by the COBE satellite, showed that the primordial structures at high redshift prefer by far the bottom-up scenario, and only a minimal fraction of the structures are formed according to the top-down model. Indeed, since HDM has relativistic velocities, it is reasonable that it has difficulties in clumping together gravitationally. This is the reason why HDM has long been ruled out from being a possible constituent of the total amount of DM in the universe.

Therefore, all that remains is to consider only CDM candidates. Nowadays, the Standard Model of cosmology is also called ΛCDM from its main constituents, CDM and the dark energy Λ. We shall dedicate the next section to a review of the most prominent CDM candidates.

1.3 Two possible solutions: WIMPs and axions

In the previous section, we have seen that MACHOs are good CDM candidates, but they can account only for a minimal fraction of the total DM. Therefore, we have to move further towards new proposals coming from theories beyond the SM, since the SM does not provide a suitable DM candidate.

As well known, the most promising theories extending the SM are *Supersymmetry* and *String Theory*. These theories are not competitors. Actually, supersymmetry is required for consistency in string theory and it characterises the effective field theory of string theory, as we shall discuss in the next chapter. By the way, both of these theories include classes of particles that are very promising CDM candidates, and now most of the research efforts are spent on them. For this reason, a single treatment for each proposal is necessary. In doing this review, we will mainly follow [3, 15, 16, 17, 18, 19, 20, 21, 22, 23].

1.3.1 WIMPs from thermal freezeout

In view of what was said in this chapter, the fundamental criteria DM candidates must satisfy are basically two: they must interact in a very feeble way with SM particles and they must be sufficiently cold. A good realisation of these two features are weakly interacting massive particles (WIMPs). These particles have large mass scales ($m_{WIMP} \sim 10 - 10^3$ Gev) and this ensures that their velocities are non-relativistic. Moreover, due to the large mass of their mediator particles (such as W or Z bosons), they have very short-range interactions, thus making the interactions with other particles quite rare.

However, their large mass can also be a problem. Large masses mean large phase spaces, implying that they are unstable on cosmological time-scales. They could decay into lighter, SM particles and would not contribute significantly to the DM abundance we observe today. To ensure the stability of such heavy particles, one must introduce symmetries that preserve the number of WIMPs. It is not easy to motivate theoretically these symmetries, because a global symmetry may be broken by quantum gravity when the universe cools down whereas a local symmetry may lead to additional interactions.

Nevertheless, few cases exist where stable WIMPs are doable. The most famous and important example is the lightest supersymmetric partner (LSP) coming from the Minimal Supersymmetric Standard Model (MSSM) with R-parity. The MSSM is the simplest theory extending the SM. It consist in adding superpartners for all the observed SM particles. This leads to the following new particle spectrum: fermionic superpartners (gauginos) for gauge bosons to promote them to vector multiplets, bosonic superpartners (squarks and sleptons) for quarks and leptons to promote them to chiral multiplets, fermionic superpartners (Higgsinos) for the Higgs scalars. Hence, interactions are governed by gauge symmetry and supersymmetry.

Usually, the DM candidate taken into account from the MSSM is the neutralino, a Majorana fermion with spin $s = \frac{1}{2}$ that comes form the mixing of a neutral higgsino and a neutral gaugino. We will delve deeper into the MSSM and Supersymmetry in the next chapter. For now, let us introduce only a few properties.

R-parity is the symmetry that one must consider for the stability of the WIMPs and it is defined as

$$R \equiv (-1)^{3(B-L)+2s} = \begin{cases} +1 & \text{SM particles} \\ -1 & \text{superpartners} \end{cases}, \tag{1.1}$$

where s is the spin of the particle, while B and L are the baryon and lepton number of the particle respectively. As a consequence, R-parity ensures that superpartners can only be created or destroyed in pairs. Moreover, heavy superpartners can decay into lighter superpartners along with a number of SM particles, but the LSP cannot decay. All decay chains end up with a LSP in the final state. Therefore, R-parity, if preserved, stabilises LSPs.

Now, it is interesting to understand how these LSPs could have made up all of the DM we observe today. WIMPs are produced in a thermal way. In the early universe, when the temperature was bigger than m_{WIMP}, supersymmetric particles were in thermal equilibrium and nearly as abundant as SM particles. This equilibrium abundance was maintained by supersymmetric particles annihilations into SM particles and antiparticles. As the universe cooled down and expanded, the heavier supersymmetric particles could no longer survive, thus they eventually annihilated or decayed into neutralino or LSPs. Furthermore, some of the LSPs pair-annihilated into final states not containing supersymmetric particles but only SM particles. As the density decreased, the annihilation rate became small compared to the cosmological expansion, and the neutrinos experienced the so-called *freezeout*. Therefore, their density today is determined by this small rate of pair-annihilation and by the subsequent dilution due to the expansion of the universe. Note that neutralinos are predicted to freeze out at a temperature below their mass, thus they are non-relativistic throughout the history of the universe.

In order to obtain the present observed DM density via thermal production, the particle at hand must have an effective annihilation cross-section times its relative speed v of about $\langle \sigma v \rangle \simeq \alpha^2/(150\,\mathrm{GeV})^2$, α being the fine structure constant. This is roughly what is expected for a particle with interactions and mass range in the electroweak regime. Therefore, it seems that neutralinos, and WIMPs in general emerge naturally as perfect DM candidates. The fact that the perfect DM particle was found simply by having new physics above the SM electroweak scale was called a *striking coincidence*, or the *WIMPs miracle*. This observation, combined with theoretical arguments in favour of the existence of new physics at or around the electroweak scale elevated WIMPs to the class of most appealing DM candidates.

However, WIMPs as DM particles yield a serious phenomenological problem. In N-body gravitational simulations based on a DM WIMPs scenario, DM halos surrounding galaxies always present steep power-law (*cuspy*) mass density distributions at their centre. This is in contrast with all the observations concerning less-massive galaxies close to us and low-surface-brightness galaxies. Studies of the rotation curves of such galaxies revealed that they have a central DM density profile that is nearly flat. This discrepancy goes under the name of *cusp-core problem*.

During the last decade, some solutions have been proposed. However, the most promising and natural way to deal with the cusp-core problem seems to be the change of the nature of the particles that constitute the DM.

Note that the ability of Supersymmetry to provide a viable DM candidate is seen as a bonus, rather than as the main motivation to study such theories. As a matter of fact, Supersymmetry was first proposed to solve other problems of the SM, different from the DM one. This is the reason why the study of Supersymmetry will continue even if, one day, it will be find out that the DM is not made of supersymmetric particles. We will make a review of the most prominent features of Supersymmetry in the next chapter.

1.3.2 QCD Axions, ALPs and Fuzzy dark matter

The other leading DM candidate is a very light and very stable particle named *axion* by Wilczek [24] after a famous American laundry detergent. The axion is a pseudo-Nambu-Goldstone boson arising from spontaneously broken global chiral symmetries. It was first postulated in 1977 to solve the strong-charge-parity (CP) problem, but then came to describe a class of different particles that have an ubiquitous presence both in the SM and, above all, in the theories beyond the SM.

The theory of quantum chromodynamics (QCD) suffers from the so-called strong-CP problem. This comes from the fact that the QCD Lagrangian allows a total derivative term of the form

$$\mathcal{L}_{QCD} \supset \frac{g^2}{32\pi^2}\,\theta\,\mathrm{tr}\Big(G_{\mu\nu}\tilde{G}^{\mu\nu}\Big)\,, \tag{1.2}$$

where $G_{\mu\nu}$ is the gluonic field strength, $\tilde{G}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}G_{\alpha\beta}$ is its dual, g is the strong coupling constant and θ is a parameter arising from the study of the QCD vacuum structure which can span from 0 to 2π. The trace runs over the colour SU(3) indices.

The problem lies in the value of θ. The above term is odd under CP since it violates parity but conserves charge symmetry, and so it produces CP-violating effects, such as a non-vanishing electric dipole moment d_n of the neutron. If $\theta \sim \mathcal{O}(1)$, as one would naturally expect, the value of d_n produced by the above Lagrangian term is $\sim 10^{10}$ times larger than experimental upper bounds. Therefore, in order to be consistent with observations, θ must be smaller than 10^{-10}. However, if there were only CP-conserving strong interactions, then θ could be simply set to zero by symmetry.

The actual issue arises when we consider also the electroweak sector, since weak interactions generally violate CP. A generic Lagrangian term for weak interactions takes the form $\mathcal{L}_{\mathrm{w}} = \bar{q}_{i,R}M_{ij}q_{j,L} + (h.c.)$. In order to diagonalise the mass matrix M, one has to perform a chiral rotation of the quark fields, but this rotation does affect the QCD vacuum as well. The effect of this transformation is to change the measurable parameter in front of $G\tilde{G}$, which becomes $\theta = \tilde{\theta} +$ arg det M, where $\tilde{\theta}$ is the bare parameter. It is now clear that a value of $\theta < 10^{-10}$ is unlikely. Therefore, we are dealing with a true fine-tuning problem.

The solution to the strong-CP problem was proposed in 1977 by Roberto Peccei and Helen Quinn [25] and developed the year after by Steven Weinberg [26] and Frank Wilczek [24]. In their work, Peccei and Quinn showed that, by introducing a new global $U(1)_{PQ}$ symmetry that is spontaneously broken, θ can be *dynamically* set to zero, explaining naturally the light value observed. Then Weinberg and Wilczek independently pointed out that such a global symmetry also implies the presence of a Nambu-Goldstone boson, *i.e.* the axion a. Therefore, the introduction of the $U(1)_{PQ}$ symmetry into the theory replaces the static CP-violating angle θ with a dynamical CP-conserving field, that is a. This procedure is also called misalignment mechanism.

The solution relies basically on two ingredients: the Goldstone theorem and the presence of instantons in the QCD vacuum. The procedure can be summarized in three steps, as shown in figure 1.1. First of all, after the spontaneous breaking of the $U(1)_{PQ}$ symmetry, θ is promoted to a dynamical field $\theta(x)$ with a kinetic term in the Lagrangian of the form $\mathcal{L}_{\theta,kin} = \frac{1}{2}f^2\partial_\mu\theta(x)\partial^\mu\theta(x)$, where f has the dimension of mass and it is known as the axion decay constant. Since conventionally a scalar has mass dimension one, we can redefine $a(x) \equiv \theta(x)/f$. The contribution to the total SM Lagrangian given by $a(x)$ reads

$$\mathcal{L} = \mathcal{L}_{SM} + \frac{1}{2}\partial_\mu a(x)\,\partial^\mu a(x) + \frac{g^2}{32\pi^2}\frac{a(x)}{f}\,\mathrm{tr}\Big(G_{\mu\nu}\tilde{G}^{\mu\nu}\Big)\,. \tag{1.3}$$

Remarkably, (1.3) exhibits a shift symmetry for the axion under $a(x) \to a(x)+fc$, c being a constant. This symmetry is preserved at perturbative level and for $\Lambda_{QCD} < E < f$.

However, when non-perturbative (QCD instanton) effects switch on at the particular scale Λ_{QCD}, the symmetry is broken explicitly and a potential for a is generated and it takes the general form

$$V_{\text{eff}} \sim \Lambda^4_{QCD}\left[1 - \cos\frac{a(x)}{f}\right]. \tag{1.4}$$

Minimising the potential with respect to a gives the Peccei-Quinn solution: at $\langle a(x)\rangle = 0$ θ is set to zero dynamically. The proof of the CP-conservation of the instanton-corrected action is known as the Vafa-Witten theorem [27], which guarantees that the instanton potential is minimised at the CP-conserving value of a.

Moreover, thanks to these instanton effects, the axion gains a mass with scaling law

$$m_a \sim \frac{\Lambda^2_{QCD}}{f}. \tag{1.5}$$

A combination of collider and astrophysical experiments has given some rigid constraints to the value of f. The value of f is typically rather high,

$$10^9 \lesssim \frac{f}{\text{GeV}} \leq 10^{12}, \tag{1.6}$$

while the non-perturbative scale Λ_{QCD} is much lower, namely $\Lambda_{QCD} \sim 200$ MeV. In the relation (1.6), the lower bound is given by astrophysical observations whereas the upper

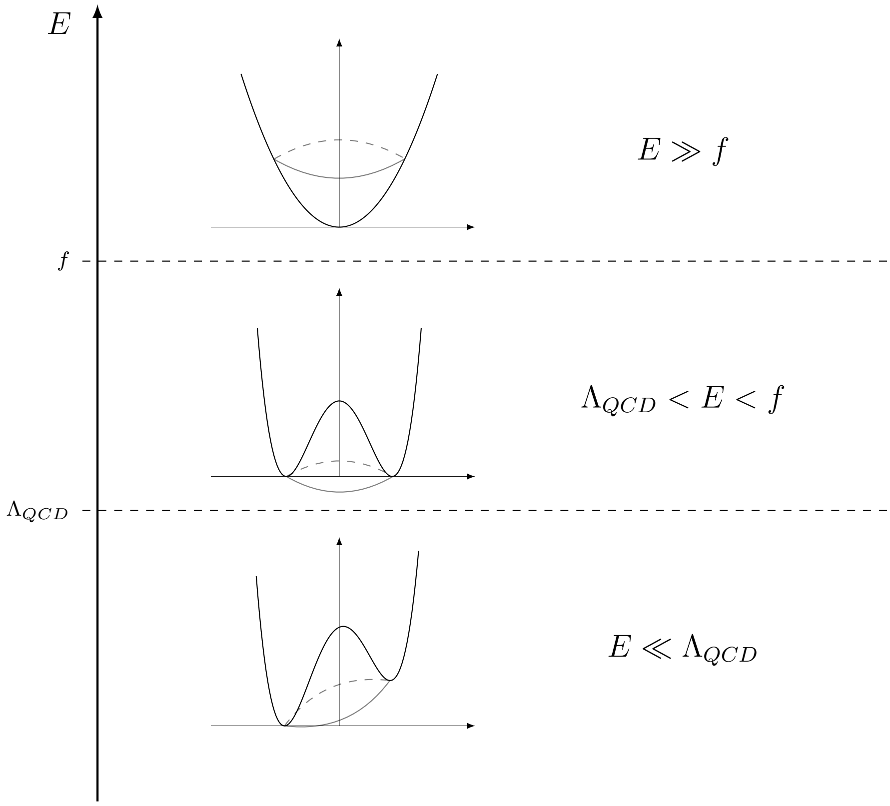

Figure 1.1: The three steps of the misalignment mechanism are displayed. First, when the energy scale E is much bigger than f, the $U(1)_{PQ}$ symmetry is preserved. Then, when E falls below f, the symmetry is broken spontaneously and a flat direction arise, indicating the presence of a Nambu-Goldstone boson, the axion. Finally, when E become smaller than Λ_{QCD}, instanton effects switch on and the potential is *tilted*. This gives a mass to the axion.

bound comes from the DM overproduction for $\theta \sim \mathcal{O}(1)$. This in turn means that the upper bound could be relaxed if $\theta \ll 1$. Therefore, from equation (1.5) with the values in (1.6) we can see that the axion has a parametrically small mass and from the last term in the Lagrangian (1.3) it follows that the axion has very feeble interactions.

Such a light and invisible particle as the axion can have very interesting implications in cosmology. Since it is a very stable particle on cosmological timescales, if an axionic population had been produced in the early universe, it would have survived until nowadays. Therefore, if this population is sufficiently abundant, it can constitute a viable DM candidate. It is worth noting here, for clarity, that actually the word *axion* can take on a variety of meanings, when it is used in other models different from QCD. This is why, from now on, we will generally make use of the phrase *Axion-Like Particles* (ALPs) to address every pseudo-scalar particle enjoying a shift-symmetry. Notice that, for a generic ALP the relation (1.5) is no longer valid, that is m_a and f are independent.

In the history of the universe, ALPs could have been produced in a thermal way via the decay of the so-called topological defects. However, for ALPs light enough to obey to the experimental constrains, this population would only be able to account for a very small fraction of the DM density we observe today.

However, another production mechanism is possible, and it turns out to be very close to the misalignment mechanism of the QCD axion reviewed above and displayed in figure 1.1, which is therefore a non-thermal production. It is based on the assumption that in the early universe the field $\theta(x)$ can acquire a random non-vanishing initial value, as one would expect from quantum fluctuations at this epoch. If this field has a mass m, after a time $t \sim m^{-1}$, the field will start oscillating around the minimum tending to minimise the potential.

Let us consider a real scalar field $\phi(t)$ playing the role of the ALP, with corresponding Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m_\phi^2\phi^2 + \mathcal{L}_I, \tag{1.7}$$

where $\mathcal{L}_I$ includes both ϕ self-interactions and interactions with other particles present in the primordial universe. Let us also assume that the universe undergoes a period of inflation at a certain value of the Hubble expansion parameter $H = \mathrm{d}\log a/\mathrm{d}t$.

If we consider a Friedman-Robertson-Walker universe, the equation of motion for ϕ is given by

$$\ddot{\phi} + 3H\dot{\phi} + m_\phi^2\phi = 0. \tag{1.8}$$

In the early universe, when $H \gg m_\phi$, the equation (1.8) describes an over-damped harmonic oscillator. Thus, at this epoch we can approximate $\dot{\phi} = 0$ and so $\phi = const$, since the field hardly moves due to Hubble friction. The equation of state of ϕ at early times implies $\omega_\phi = -1$, therefore the ALP behaves as a contribution to the vacuum energy. This is also why in some theories ALPs are used as models for Dark Energy and inflation.

At a later time, when $H \ll m_\phi$, the damping becomes under-critical and the field can start oscillating. Thus, the relation (1.8) describes an under-dumped harmonic regime. When the mass term represents the leading scale in the equation, the solution can be found with the form

$$\phi \simeq a^{-3/2}\cos m_\phi t. \tag{1.9}$$

This solution corresponds to fast oscillations with a slow amplitude decay. The equation of state during this epoch oscillates around the value $\omega_\phi = 0$, which is the same value as ordinary matter. Moreover, since ALPs are produced in a non-thermal way, they are non-relativistic at a very early stage of the universe, when the temperature is so high that other particles of the same mass are still thermally coupled. This is why ALPs produced via this sort of misalignment mechanisms are good CDM candidates.

The cosmological misalignment mechanism can take place in two different scenarios, implying different assumptions and different consequences. First of all, we must notice that we are left with three free parameters, *i.e.* m, f and θ. Instead, the classical PQ mechanism for the QCD axion involved only two free parameter, m and f. From this cosmological perspective, the value of θ has become a free parameter since its initial value is determined only by quantum fluctuations, and it is allowed to span freely from 0 to 2π. The initial value of θ has serious consequences on the DM abundance made of ALPs. As a matter of fact, $\Omega_{DM} \sim \sqrt{m} f^2 \theta^2$, approximatively.

The fact that we can put by hand the value of θ, together with the cosmological principle[1] represent the two motivations for the presence of the two different scenarios mentioned above. These scenarios can be summarised as follows:

- *Scenario A*: here $H_{inf} > f$, which means that inflation occurs before the breaking of the PQ symmetry. The present-day universe is then very likely to be composed by causally disconnected macro-regions, all presenting a different initial values of θ because of the quantum fluctuations. Therefore, in order to calculate the ALP density, it is necessary to perform an average over all the possible values of θ, giving $\bar{\theta} \simeq \pi/\sqrt{3}$. Hence, the initial misalignment angle becomes fixed and the prediction one can perform on the DM abundance is unique. It is worth noting that in this scenario a maximum possible temperature of the universe is present. Thus, there is a constraint on the maximum value that H_{inf} and consequently f can acquire. From the information given by Planck observations on the CMB, it was calculated that only ALPs with $f < 10^{14}$ GeV are allowed in the Scenario A. If an ALP has a value of f bigger than 10^{14} GeV, then it can only exist in the Scenario B.

- *Scenario B*: $H_{inf} < f$, *i.e.* inflation occurs after the breaking of the $U(1)_{PQ}$ symmetry. In this case it is the pre-inflation universe that is made of causally disconnected patches: a sort of multiverse is present initially, where each patch is a distinct universe. Then, each universe gains a different value of θ after the breaking of the PQ symmetry. The existence of many pre-inflation patches implies that values of θ arbitrarily close to 0 or to 2π cannot be excluded, even if they seem unnatural. Subsequently, every universe undergoes a period of inflation, at the end of which the Big Bang nucleosynthesis occurs. Since $\Omega_{DM} \sim \sqrt{m} f^2 \theta^2$ and since we have a lower limit imposed on f, if θ acquires a very high value in one of the universes, then that universe is very likely to collapse gravitationally because it produced too much matter. Therefore, if we found by chance that in the universe we live in θ has a very low, unnatural value, then probably this is the reason why

[1] As well known, the cosmological principle states that the universe must be homogeneous and isotropic at sufficiently large scales. This principle is indeed supported by the theory of inflation, which smooths every inhomogeneity that could have occurred before the inflationary epoch thanks to an exponential acceleration of the expansion rate.

our universe survived. Moreover, from $\Omega_{DM} \sim \sqrt{m} f^2 \theta^2$, we can find the right DM abundance playing with the value of f by taking it much bigger than 10^{14} GeV.

It is immediate to understand that Scenario A and Scenario B take into account different kinds of ALPs. While the QCD axion is more likely to be produced in Scenario A, Scenario B gives the phenomenological grounds for the existence of a family of ALPs with solid theoretical basis, *i.e.* string axions.

In string theory and supergravity, the term *axion* refers to pseudo-scalar fields enjoying PQ shift symmetries that are associated to the geometry of compact spatial dimensions. Axions arise in the compactification of extra-dimensions from the integration of p-form gauge potentials over p-cycles of the compact space. Furthermore, axions defined in this way are massless to all orders in perturbation theory thanks to the higher-dimensional gauge invariance. The axions then obtain mass by non-perturbative effects, just like the QCD axion. We will discuss string axions in a more appropriate way in the next chapter. Let us disclose for now only the aspects that are important for the present purposes.

String axions are strictly related to the geometry of the compact dimensions. Hence, their presence does not need to be postulated as for the QCD axion in the S. The number of axions depends on the topology of the compact manifold, and since many different kinds of compact manifolds are known to exist, the resulting average number of axions per manifold is large. Axions are a ubiquitous presence in any type of string theory compactification. In addiction, it turns out that also string axions are parametrically light and that the value of their masses can span many orders of magnitude. Most importantly, they can be lighter than QCD axions and acquire very high values of the decay constant f.

To be more quantitative, axions with masses of 10^{-18} eV $< m < 10^{-33}$ eV are allowed to exist. Such axions are usually referred to as *Ultra-Light Axions* (ULAs). Models involving ULAs as CDM are called *Fuzzy Dark Matter* (FDM) models. This setup is the one we will work on in this thesis.

A variety of reasons why FDM is a very promising DM candidate exist. The most prominent ones rely on two main phenomenological results. It was shown (*e.g.* in [28], where the term FDM was first used) that ULAs DM can solve the cusp-core problem affecting WIMPs in DM galactic halos. For this purpose, a ULA population with mass $m \sim 10^{-22}$ eV was considered, so that their wave-like nature was manifest on astrophysical scales and can prevent the formation of the kpc scale cusps.

Nevertheless, this mass scale turned out to be rather interesting. In the recent work [29], the authors showed that an attractive feature of FDM with a mass in the range $m \sim 10^{-22} - 10^{-21}$ eV is that a cosmic DM abundance $\Omega_{DM} \sim 1$ can arise naturally. Notice that these mass scales correspond to values of the decay constant of the order $f \sim 10^{16} - 10^{17}$ eV. Consequently, such ULAs would be produced in the Scenario B, *i.e.* before inflation occurred.

For all the reasons exposed in this section, FDM models are very promising and recently have become a main field of research. The purpose of the next chapters is to motivate with appropriate models a particle with such a light mass. Finally, in Chapter 4, we will study the possibility for these models to produce enough ULAs to account for the DM abundance we observe today.

Chapter 2

Beyond The Standard Model

The theory of the Standard Model (SM) represents one of the highest levels ever reached by the human mind and it is one of the most important breakthroughs in physics so far. Its predictive capabilities have been tested many times ad its remarkable success is unparalleled.

However, the beauty of the SM is corrupted by a number of theoretical and phenomenological issues which the SM fails to give an adequate answer. For example, the SM does not provide a viable DM candidate, as described in the previous chapter. This shows that the SM cannot be the ultimate theory and that we have to move further and look for new theories able to solve the SM issues. Nevertheless, since the SM undoubtedly works well below a certain energy scale, the new theory must be an ultra-violet completion of the SM. That is, the low energy limit of the new theory must recover the SM.

The aim of this chapter is to make a review of the two most prominent theories beyond the SM, *i.e.* Supersymmetry and String theory. Besides describing the main features of both the theories, we will especially delve into the aspects that are useful for the next chapters.

The following discussion is based on [20, 30, 31, 32, 33, 34].

2.1 Supersymmetry

Supersymmetry plays a crucial role in the structure of String Theory. It first appeared as a symmetry in the attempt of extending the bosonic string to include also fermions. Moreover, the simplest version of supersymmetry in 4D provide a viable solution to the electroweak hierarchy problem present in the SM. The electroweak hierarchy problem refers to the fact that no symmetries are present to prevent the masses of scalar particles (as the Higgs boson) from getting large loop corrections, and so to become too large compared to the observed value. Since the supersymmetry relates scalars to fermions, the chiral symmetries protecting the masses of the fermions also protect the masses of the scalars from acquiring quadratic divergences. More precisely, the mass of a chiral fermion is forced to be zero by chirality, consequently the mass of a scalar is protected against getting loop corrections, because the corrections are cancelled order by order in perturbation theory. Also, these cancellations are not only possible but also unavoidable if we assume the existence of supersymmetry.

Supersymmetry is a symmetry which relates bosons and fermions, combining them into the same multiplets, the supermultiplets. A supersymmetric transformation turns a bosonic state into a fermionic state, and vice versa. This requires an equal number of fermionic and bosonic degrees of freedom. Such a transformation is operated by the supersymmetry generators Q_α and $\bar{Q}_{\dot{\alpha}}$, which are the conserved supercharges from Noether theorem. Q_α and $\bar{Q}_{\dot{\alpha}}$ are fermionic operators that must satisfy the superalgebra given by the anticommutation relation

$$\left\{Q^A_\alpha, \bar{Q}_{\dot{\alpha}B}\right\} = 2\,\sigma^\mu_{\alpha\dot{\alpha}} P_\mu\, \delta^A_B, \tag{2.1}$$

all other relations being vanishing. The indices $A, B = 1, ..., \mathcal{N}$, where $\mathcal{N}$ is the number of supersymmetries present. It is demonstrable that $\mathcal{N} > 1$ supersymmetries are non-chiral. Since the SM is known to be chiral, it seems that only the $\mathcal{N} = 1$ supersymmetry will lead to a realistic symmetry broken at low energies. Therefore, we will specialise our discussion to $\mathcal{N} = 1$ supersymmetry.

Now, let us introduce the basic definitions of *superspace* and *superfield*. The superspace comes from the generalisation of the 4-dimensional Minkowski space made by including additional anticommutating spinorial coordinates θ_α and $\bar{\theta}_{\dot{\alpha}}$ on which supersymmetric transformation acts. As a result, the superspace is defined by the classical four dimensions plus extra fermionic dimensions, and it is parametrised by the set of coordinates $(x, \theta, \bar{\theta})$.

The spinors θ_α and $\bar{\theta}_{\dot{\alpha}}$ are Grassman variables and consequently they obey the Grassman algebra, *i.e.*

$$\begin{gathered} \theta^2 = 0, \quad \theta_1\theta_2 = -\theta_2\theta_1, \\ \int \mathrm{d}\theta = 0, \quad \int \mathrm{d}\theta\, \theta = \frac{\partial}{\partial\theta}\theta = 1. \end{gathered} \tag{2.2}$$

Superfields are defined as the fields which depend on the superspace coordinates $(x, \theta, \bar{\theta})$. From the properties of the Grassman variables follows that a general superfield S has a finite power-expansion in the spinorial coordinates, thus also a finite number of ordinary fields making up the supermultiplets. The most general superfield one can write in an expansion of the spinorial fields takes the form

$$\begin{aligned} S\left(x^\mu, \theta, \bar{\theta}\right) =& \varphi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta M(x) + \bar{\theta}\bar{\theta}N(x) + \left(\theta\sigma^\mu\bar{\theta}\right) A_\mu(x) \\ &+ (\theta\theta)\,\bar{\theta}\bar{\lambda}(x) + \left(\bar{\theta}\bar{\theta}\right)\theta\rho(x) + (\theta\theta)\left(\bar{\theta}\bar{\theta}\right) D(x). \end{aligned} \tag{2.3}$$

Supersymmetry acts on the superfield S as

$$\delta S = i\left(\epsilon\,\mathcal{Q} + \bar{\epsilon}\,\bar{\mathcal{Q}}\right) S, \tag{2.4}$$

where ϵ is a parameter and $\mathcal{Q}$ is a representation of the spinorial generators Q_α acting on the spinorial coordinates. The transformation for each component of S are

$$\begin{aligned}
\delta\varphi &= \epsilon\psi + \bar{\epsilon}\bar{\chi} \\
\delta\psi &= 2\epsilon M + \sigma^\mu\bar{\epsilon}\left(i\partial_\mu\varphi + A_\mu\right) \\
\delta\bar{\chi} &= 2\bar{\epsilon}N - \epsilon\sigma^\mu\left(i\partial_\mu\varphi - A_\mu\right) \\
\delta M &= \bar{\epsilon}\bar{\lambda} - \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\epsilon} \\
\delta N &= \epsilon\rho + \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\bar{\chi} \\
\delta A_\mu &= \epsilon\sigma_\mu\bar{\lambda} + \rho\sigma_\mu\bar{\epsilon} + \frac{i}{2}\left(\partial^\nu\psi\sigma_\mu\bar{\sigma}_\nu\epsilon - \bar{\epsilon}\bar{\sigma}_\nu\sigma_\mu\partial^\nu\bar{\chi}\right) \\
\delta\bar{\lambda} &= 2\bar{\epsilon}D + \frac{i}{2}\left(\bar{\sigma}^\nu\sigma^\mu\bar{\epsilon}\right)\partial_\mu A_\nu + i\bar{\sigma}^\mu\epsilon\partial_\mu M \\
\delta\rho &= 2\epsilon D - \frac{i}{2}\left(\sigma^\nu\bar{\sigma}^\mu\epsilon\right)\partial_\mu A_\nu + i\sigma^\mu\bar{\epsilon}\partial_\mu N \\
\delta D &= \frac{i}{2}\partial_\mu\left(\epsilon\sigma^\mu\bar{\lambda} - \rho\sigma^\mu\bar{\epsilon}\right).
\end{aligned} \tag{2.5}$$

Notice that δD is a total derivative.

If we perform the derivative $\partial_\mu S$, we find that this is still a superfield. Instead, if we perform the derivative on spinorial coordinates $\partial_\alpha S$, the result is no longer a superfield. As a consequence, we have to define a covariant derivative

$$\begin{aligned}
\mathcal{D}_\alpha &= \partial_\alpha + i\left(\sigma^\mu\right)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \\
\bar{\mathcal{D}}_{\dot{\alpha}} &= -\bar{\partial}_{\dot{\alpha}} - i\,\theta^\beta\left(\sigma^\mu\right)_{\beta\dot{\alpha}}\partial_\mu,
\end{aligned} \tag{2.6}$$

so that $\mathcal{D}_\alpha S$ is a superfield. The covariant derivative is very useful also for the following reason. Since S is not an irreducible representation of the supersymmetry algebra, we can eliminate some of its components, keeping it still a superfield. In order to extract its irreducible components, we have to impose consistent constraints on S. The most relevant superfields can be found imposing the following constraints:

- chiral superfield Φ such that $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi$=0,
- vector superfield V such that $V = V^\dagger$.

2.1.1 Chiral superfield

The chiral field Φ is the simplest irreducible superfield and it is characterised by the condition $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi$=0. The components of Φ satisfying that condition can be found in the following way. If we define $y^\mu \equiv x^\mu + i\theta\sigma^\mu\bar{\theta}$, we can perform the covariant derivative $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi\left(y,\theta,\bar{\theta}\right)$ and equal it to zero. In components, we find

$$\Phi\left(y,\theta\right) = \varphi(y^\mu) + \sqrt{2}\,\theta\psi(y^\mu) + \theta\theta F(y^\mu), \tag{2.7}$$

where the explicit $\bar{\theta}$-dependence is absent. Then, we can recast Φ in terms of x^μ as

$$\begin{aligned}\Phi\left(x,\theta,\bar{\theta}\right) =& \varphi(x) + \sqrt{2}\,\theta\psi + \theta\theta F(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\varphi(x)\\ &- \frac{i}{\sqrt{2}}\left(\theta\theta\right)\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}\left(\theta\theta\right)\left(\bar{\theta}\bar{\theta}\right)\partial_\mu\partial^\mu\varphi(x).\end{aligned} \tag{2.8}$$

Therefore, we see that the physical components of a chiral superfield are a scalar particle ϕ (such as a squark, a sleptons or the Higgs), a spin s=1/2 particle ψ (like quarks, leptons or Higgsinos) and an auxiliary field F in a way to be defined later.

Under supersymmetry, Φ transforms as

$$\delta\Phi = i\left(\epsilon Q + \bar{\epsilon}\bar{Q}\right)\delta\Phi, \tag{2.9}$$

and the transformations of its components read

$$\begin{aligned}\delta\varphi &= \sqrt{2}\,\epsilon\psi\\ \delta\psi &= i\sqrt{2}\,\left(\partial_\mu\varphi\sigma^\mu\bar{\epsilon}\right) + \sqrt{2}\,\epsilon F\\ \delta F &= i\sqrt{2}\,\left(\bar{\epsilon}\sigma^\mu\partial_\mu\varphi\right).\end{aligned} \tag{2.10}$$

Notice that the supersymmetric transformation of the auxiliary field F is a total derivative, just like the transformation of D in the general superfield S. Also, the product of superfields with the same chirality yields another chiral superfield. A consequence of these features is that the 4D space-time integral of the F term of an arbitrary polynomial of chiral superfields is invariant under supersymmetry. In particular, the most general renormalizable form of supersymmetric couplings involving chiral superfields reads

$$\mathcal{L}_W = \left(\left.W\left(\Phi\right)\right|_F + h.c.\right) \equiv \int \mathrm{d}^2\theta\,\left(\alpha + \lambda\Phi + \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3\right), \tag{2.11}$$

where the integration over $\mathrm{d}^2\theta$ selects the F terms of Φ. $W\left(\Phi\right)$ is called *superpotential* and is an holomorphic function, hence a chiral superfield itself.

The canonical kinetic terms for Φ are described by a real function K called *Kähler potential*, and they takes the form

$$\mathcal{L}_K = \left.K\left(\Phi,\Phi^\dagger\right)\right|_D \equiv \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,\Phi\Phi^\dagger, \tag{2.12}$$

where the integration over $\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}$ selects the D term of the polynomial $\Phi\Phi^\dagger$, that is a real superfield. We saw previously that the supersymmetric variation of the D terms is a total derivative. Thus, this leads to a supersymmetry-invariant term upon integration over the 4-dimensional space-time. It is noteworthy that the Lagrangian (2.12) provides a supersymmetric version of the kinetic terms of the form $K_{i\bar{j}}\,\partial_\mu\varphi^i\partial^\mu\varphi^{\bar{j}*}$, where

$$\frac{\partial^2 K}{\partial\varphi^i\,\partial\varphi^{\bar{j}*}} = K_{i\bar{j}}. \tag{2.13}$$

$K_{i\bar{j}}$ is a metric known as the *Kähler metric*, and a manifold endowed with the *Kähler metric* is called *Kähler manifold*. In differential geometry, Kähler manifolds enjoy many remarkable properties and they play a very important role in string theory, as we will describe in due course.

The most general Lagrangian for a chiral superfield can be written as

$$\mathcal{L} = K\left(\Phi, \Phi^\dagger\right)\Big|_D + \left(W\left(\Phi\right)|_F + h.c.\right). \tag{2.14}$$

By solving explicitly the integration, we come to the following Lagrangian

$$\mathcal{L} = \partial^\mu\varphi^*\partial_\mu\varphi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + FF^* + \left(\frac{\partial W}{\partial\varphi}F + h.c.\right) - \frac{1}{2}\left(\frac{\partial^2 W}{\partial\varphi^2}\psi\psi + h.c.\right), \tag{2.15}$$

which is known as *Wess-Zumino model.* Therefore, the interactions for chiral superfields give rise to the masses for the scalar field φ as well as for the spinor ψ, in such a way that the masses are equals. Moreover, the Yukawa coupling $g(\varphi\bar{\psi}\psi)$ arises, and its coupling constant g determines the scalar self-coupling $g^2\varphi^4$ as well.

It may happen that a supersymmetric Lagrangian is invariant under $U(1)$ global symmetries acting in a different way on the fermionic and scalar components of the same chiral multiplet. Such symmetries are called R-symmetries and denoted by $U(1)_R$. Their presence can be encoded as a $U(1)$ charge assignment to the superspace coordinates θ and $\bar{\theta}$. Thus, R-symmetry constrains the superpotential to couplings satisfying $\sum_i a_i = 2$, a being the value of the charge given to a certain coordinate.

Now, notice that the F-term Lagrangian

$$\mathcal{L}_F = FF^* + \frac{\partial W}{\partial\varphi}F + \frac{\partial W^*}{\partial\varphi^*}F^* \tag{2.16}$$

is quadratic in F and does not contain derivatives, *i.e.* the auxiliary field F does not propagate. For this reason, we can easily rule F out using the field equations

$$\begin{aligned} F^* + \frac{\partial W}{\partial\varphi} &= 0, \\ F + \frac{\partial W^*}{\partial\varphi^*} &= 0, \end{aligned} \tag{2.17}$$

and substitute the results back into the Lagrangian $\mathcal{L}_F$. This defines the so-called F-term scalar potential, namely

$$V_F(\varphi) \equiv \left|\frac{\partial W}{\partial \varphi}\right|^2 . \tag{2.18}$$

2.1.2 Vector superfield

The vector multiplet is defined by a superfield satisfying the constraint $V = V^\dagger$. Vector multiplets introduce in supersymmetric theories the gauge interactions, since they contain gauge bosons, that is $V = V(\lambda_\alpha, A_\mu, D; C, \chi_\alpha, N)$. The superfield V is subjected to a generalised gauge invariance. However, it is possible to gauge away the fields C, χ_α and N thanks to the so-called Wess-Zumino gauge fixing. Thus, the vector superfield in the Wess-Zumino gauge reads

$$V_{WZ}\left(x,\theta,\bar{\theta}\right) = \left(\theta\sigma^\mu\bar{\theta}\right) A_\mu(x) + (\theta\theta)\left(\bar{\theta}\bar{\lambda}(x)\right) + \left(\bar{\theta}\bar{\theta}\right)(\theta\lambda(x)) + \frac{1}{2}(\theta\theta)\left(\bar{\theta}\bar{\theta}\right) D(x), \tag{2.19}$$

where the remaining fields are the gauge boson A^μ, a Weyl spinor in the adjoint representation λ_α (that is a gaugino) and a real scalar auxiliary field D in a way to be defined later.

Let us consider only the abelian case. In this case, we can impose on Φ and V the transformations $\Phi \to \exp(iq\Lambda)\,\Phi$ and $V \to V - \frac{i}{2}\left(\Lambda - \Lambda^\dagger\right)$, where Λ is a chiral superfield defining the generalised gauge transformations, then $i\left(\Lambda - \Lambda^\dagger\right)$ is a vector superfield. Thanks to these relations, we can construct a gauge invariant quantity out of Φ and V, namely $\Phi^\dagger \exp(2qV)\,\Phi$. As a matter of fact, without this imposition, the Kähler potential would not be invariant under gauge transformations. Instead, $K = \Phi^\dagger \exp(2qV)\,\Phi$ is gauge invariant and it describes the interactions between V and Φ, plus the kinetic terms for both. Noticeably, it produces also a linear term in the auxiliary field D.

The abelian field-strength is defined as

$$W_\alpha = -\frac{1}{4}\left(\bar{\mathcal{D}}\bar{\mathcal{D}}\right)\mathcal{D}_\alpha V \tag{2.20}$$

and is both chiral and invariant under generalized gauge transformations. Hence, the kinetic Lagrangian for V reads

$$\mathcal{L}_{V,kin} = f(\Phi)\left.(W^\alpha W_\alpha)\right|_F + h.c., \tag{2.21}$$

where $f(\Phi)$ is the *gauge kinetic function*, it is itself a chiral superfield and is dimensionless. This function encodes the non-renormalizable couplings of the gauge supermultiplets to the chiral supermultiplets. Hence, $\mathcal{L}_{V,kin}$ is renormalizable if and only if f =const= τ.

An additional term can be added to the Lagrangian for the vector superfield, that is the *Fayet-Iliopoulos* (FI) term, which reads

$$\mathcal{L}_{FI} = \xi \left. V \right|_D = \frac{1}{2}\xi D, \tag{2.22}$$

ξ being a constant. The FI term is only present in the abelian case. Indeed, D is not charged under $U(1)$, hence making the FI term gauge-invariant under $U(1)$. Instead, for a non-abelian gauge theory, the D terms would transform under the gauge group. Consequently, the FI term in the non-abelian case is forbidden.

Now, we are able to write the renormalizable Lagrangian of the super QED with $f = \tau = 1/4$, which reads

$$\mathcal{L}_{sQED} = \left.\left(\Phi^\dagger e^{2qV}\Phi\right)\right|_D + \left(\left.W\left(\Phi\right)\right|_F + h.c.\right) + \left(\frac{1}{4}\left. W^\alpha W_\alpha\right|_F + h.c.\right) + \xi \left. V\right|_D, \tag{2.23}$$

where $V = V_{WZ}$ and

$$\left. W^\alpha W_\alpha \right|_F = D^2 - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2i\lambda\sigma^\mu\partial_\mu\bar{\lambda}. \tag{2.24}$$

Let us consider only the Lagrangian for the D terms, that is

$$\mathcal{L}_D = qD\left|\varphi\right|^2 + \frac{1}{2}D^2 + \frac{1}{2}\xi D, \tag{2.25}$$

which yields to the field equations

$$D + \frac{\xi}{2} + q\left|\varphi\right|^2 = 0. \tag{2.26}$$

When we substitute (2.26) back into $\mathcal{L}_D$, we get a semi-definite positive scalar potential for the D terms, which is given by

$$V_D\left(\varphi\right) \equiv \frac{1}{8}\left(\xi + 2q\left|\varphi\right|^2\right)^2. \tag{2.27}$$

Finally, combining (2.27) with (2.18), we find the expression for the total scalar potential, which reads

$$V\left(\varphi\right) = V_F\left(\varphi\right) + V_D\left(\varphi\right) = \left|\frac{\partial W}{\partial\varphi}\right|^2 + \frac{1}{8}\left(\xi + 2q\left|\varphi\right|^2\right)^2. \tag{2.28}$$

The non-abelian generalisation of the vector superfield V can be performed by introducing a different covariant derivative from (2.20) and placing a trace in front of the $W_\alpha W^\alpha$ term to keep it gauge invariant.

2.1.3 Supersymmetric action

It is well known that a generic action $\mathcal{S}$ is defined by the relation $\mathcal{S} = \int \mathrm{d}^4x\,\mathcal{L}$. Before writing down the supersymmetric action, it is worth remembering that the integral $\int \mathrm{d}^2\theta$ selects the F terms and the integral $\int \mathrm{d}^4\theta$ selects the D terms. Thus, the Lagrangian for the super QED can be recast in the following form

$$\mathcal{L} = \int \mathrm{d}^4\theta\, K + \left(\int \mathrm{d}^2\theta\, W + h.c.\right) + \left(\int \mathrm{d}^2\theta\, W_\alpha W^\alpha + h.c.\right). \tag{2.29}$$

In turn, the supersymmetric action reads

$$\begin{aligned} \mathcal{S}\left[K\left(\Phi^\dagger, e^{2qV}, \Phi\right), W\left(\Phi\right), f\left(\Phi\right), \xi\right] = &\int \mathrm{d}^4x \int \mathrm{d}^4\theta\, \left(K + \xi V\right) \\ &+ \int \mathrm{d}^2x \int \mathrm{d}^2\theta\, \left(W + f W_\alpha W^\alpha + h.c.\right). \end{aligned} \tag{2.30}$$

From the action in (2.30), we can see that in general the functions K, W, f and the FI constant ξ determine the structure of $\mathcal{N} = 1$ supersymmetric theories. There exist some important theorems, the so-called *non-renormalisation theorems*, which state how such functions behave under quantum corrections. These theorems can be formulated as follows

- K gets corrections order by order in perturbation theory;
- only one loop - corrections exist for $f\left(\Phi\right)$;
- $W\left(\Phi\right)$ and ξ are not renormalized in perturbation theory.

2.1.4 Supersymmetry breaking

Since no supersymmetric partner of any ordinary particle has been observed yet, supersymmetry should be broken at an energy above the electroweak scale or higher. Therefore, we expect supersymmetry to be broken spontaneously, meaning that the Lagrangian is invariant under supersymmetry, but the vacuum state is not. We speak of broken supersymmetry if the vacuum state satisfies $Q_\alpha \left|0\right\rangle \neq 0$.

Supersymmetry can be broken in two ways, that is by F terms and by D terms independently. Consider the transformation laws for the components of the chiral superfield in (2.10). If one of $\delta\varphi$, $\delta\psi$ or δF is non-vanishing, then the supersymmetry is broken. However, to preserve Lorentz-invariance we need $\langle\psi\rangle = \langle\partial_\mu\varphi\rangle = 0$. Thus, the condition for supersymmetry breaking simplifies to $\langle F\rangle \neq 0$. From the relations (2.10), we see that only the fermionic component is non-vanishing since $\delta\psi = \sqrt{2}\,\epsilon\langle F\rangle$. This means that ψ is a Goldstone fermion, or a *goldstino*. Moreover, recalling the F-term scalar potential in

(2.18), we see that supersymmetry is broken if the potential acquires a positive vacuum expectation value (vev), and vice versa.

Notice that, as a consequence of the non-renormalisation theorem, the superpotential W is not renormalised to all orders in perturbation theory. This means that, is supersymmetry is unbroken at tree-level, then it is also unbroken to all orders in perturbation theory. Therefore, in order to break the supersymmetry, we mus consider non-perturbative effects for W, *i.e.* supersymmetry can also be broken non-perturbatively.

Supersymmetry can be broken also by non-vanishing D terms. Consider the vector superfield $V(\lambda, A_\mu, D)$ with transformation laws under supersymmetry given by the relations $\delta\lambda$, δA_μ and δD in (2.5) imposing the Wess-Zumino gauge. Thus, requiring the vanishing of $\delta\lambda$, δA_μ and δD but keeping the Lorentz-invariance, we see that $\delta\lambda \sim \epsilon D$. This means that $\langle D \rangle \neq 0$, hence supersymmetry is broken. Supersymmetry breaking with a non-zero vev for the D term can occur in the abelian case through the FI term. We saw that the FI term is a linear term in the auxiliary field (see (2.22)) that can be added in the Lagrangian if we are in presence of a $U(1)$ gauge symmetry. The inclusion of this term may force the D-term scalar potential to get a non-vanishing vev. In (2.25), we presented all the D terms coming from the Lagrangian of the super QED. Then, deriving the equation of motion for D (2.26) and plugging them into $\mathcal{L}_D$, we found the D-term scalar potential as (2.27). Now, even if we require $\langle\varphi\rangle = 0$, from the equation of motion (2.26) we see that D is clearly non-vanishing thanks to the term coming from the FI contribution, *i.e.* supersymmetry must be broken. Moreover, substituting $D = -\frac{\xi}{2} - q\,|\varphi|^2$ into (2.27), the D-term scalar potential becomes

$$V_D = \frac{1}{8}\,\xi^2 + \frac{q\xi}{2}\,|\varphi|^2 + \frac{q^2}{2}\,|\varphi|^4\,. \tag{2.31}$$

If $\langle\varphi\rangle = 0$, then the mass for φ is given by the constant ξ coming from the FI term, namely $m_\varphi^2 = q\xi$. Notice that we must make the additional requirement of ξ and q to take the same sign.

2.2 Supergravity

We saw in the previous sections that supersymmetry is a global space-time symmetry. Therefore, in analogy with ordinary gauge theories, we can try to make it local introducing a coordinate dependence on the supersymmetry parameters. That is, if we consider the transformation of a superfield under supersymmetry in (2.9), then to make it local we must take $\epsilon \to \epsilon(x)$.

Since the supersymmetric algebra involves the generator of space-time translations P_μ as shown in (2.1), the local version of supersymmetry contains local space-time translations which are equivalent to coordinate reparametrisations. As a consequence, the resulting theory contains General Relativity. Thus, local $\mathcal{N}$=1 supersymmetry turns out to correspond to a supersymmetric version of gravity, known as $\mathcal{N}$=1 *supergravity*. The gauge field of supergravity is the *gravitino* ψ^{μ}_{α}, a fermionic particle with spin s=3/2 that is the superpartner of the graviton $h_{\mu\nu}$. Hence, the gravitino and the graviton make up the supermultiplet of supergravity. Since the gravitino is a gauge field, it has an associated conserved supercurrent $\mathcal{J}^{\mu}_{\alpha}$. In complete analogy with A_μ associated with a local symmetry, the gravitino has a gauge freedom under the gauge transformation

$$\delta_{\text{gauge}}\psi^{\mu}_{\alpha} = \partial^{\mu}\eta_{\alpha}(x), \tag{2.32}$$

where $\eta_\alpha(x)$ is the spinorial gauge parameter.

The dynamic of the gravitino is described by the gauge invariant Rarita-Schwinger action

$$\mathcal{S}_{RS}\left[\psi\right] = \frac{1}{2}\int \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu}\gamma_5\gamma_{\nu}\partial_{\rho}\psi_{\sigma}. \tag{2.33}$$

Now, let us consider a linearised gravitational excitation

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu} \quad \text{where } \kappa^2 = \frac{8\pi}{M_p^2}. \tag{2.34}$$

The graviton $h_{\mu\nu}$ is described by the linearised Einstein-Hilbert action

$$\mathcal{S}_{EH}\left[h\right] = -\frac{1}{2}\int \mathrm{d}^4x\, h^{\mu\nu}\left(R^L_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^L\right), \tag{2.35}$$

where

$$\begin{aligned} R^L_{\mu\nu} &= \frac{1}{2}\left(\partial_{\mu}\partial_{\lambda}h^{\lambda}_{\nu} + \partial_{\nu}\partial_{\lambda}h^{\lambda}_{\mu} - \partial_{\mu}\partial_{\nu}h^{\lambda}_{\lambda} - \partial^2_{\lambda}h_{\mu\nu}\right), \\ R^L &= \eta^{\mu\nu}R^L_{\mu\nu} \end{aligned} \tag{2.36}$$

are respectively the linearised Ricci tensor and Ricci scalar. The graviton $h_{\mu\nu}$ enjoys the spin two gauge invariance under

$$\delta_{\text{gauge}}h_{\mu\nu} = \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu}. \tag{2.37}$$

Then, performing $\mathcal{S}_{RS} + \mathcal{S}_{EH}$ we find a global supersymmetric action for the supergravity supermultiplets that is invariant under the following global supersymmetric transformation,

$$\begin{aligned} \delta\psi_{\mu} &= \frac{1}{2}\left[\gamma^{\rho},\gamma^{\sigma}\right]\epsilon\,\partial_{\rho}h_{\mu\sigma}, \\ \delta h_{\mu\nu} &= -\frac{i}{2}\bar{\epsilon}\left(\gamma_{\mu}\psi_{\nu} + \gamma_{\nu}\psi_{\mu}\right), \end{aligned} \tag{2.38}$$

ϵ being the transformation parameter. Now, we promote ϵ to a space-time function, *i.e.* $\epsilon \to \epsilon(x)$. As a consequence, the total action is no longer invariant under supersymmetric transformation and it is modified as

$$\delta\left(\mathcal{S}_{RS} + \mathcal{S}_{EH}\right) = \int \mathrm{d}^4x\, \mathcal{J}^\mu \partial_\mu \epsilon, \tag{2.39}$$

where the supercurrent reads

$$\mathcal{J}^\mu = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\rho \gamma_5 \gamma_\nu \left[\gamma^\lambda, \gamma^\tau\right] \partial_\lambda h_{\tau\sigma}. \tag{2.40}$$

Thus, if we want the overall action to be invariant under local transformations, we must require the gauge field to transform as

$$\delta\psi_\mu = \frac{1}{2}\left[\gamma^\rho, \gamma^\sigma\right] \epsilon\, \partial_\rho h_{\mu\sigma} + \frac{2}{\kappa}\partial_\mu \epsilon. \tag{2.41}$$

Hence, the variation term in (2.39) can be compensated by the term

$$\mathcal{S}_{int}\left[\psi, h\right] = -\frac{\kappa}{2}\int \mathrm{d}^4x\, \mathcal{J}^\mu \psi_\mu, \tag{2.42}$$

which is the interaction term between the gravitino and the graviton. $\mathcal{S}_{int}$ can be absorbed into $\mathcal{S}_{RS}$ if we replace ∂_ρ with an appropriately defined covariant derivative $\mathcal{D}_\rho$. As a result, the invariance of the total action under local transformations is restored.

It is very useful to recast the formulation of supergravity in terms of superfields. As usual, the superspace is parametrised by the coordinates $z^M = \left(x^\mu, \theta, \bar{\theta}\right)$. The supergravity multiplet is included into a superfield of components $(e^\mu{}_a, \psi^\mu{}_a, M, b_a)$, where $e^\mu{}_a$ is the vierbein describing the metric $g_{\mu\nu} = e^a{}_\mu e_{a\nu}$ (remember that $\sqrt{-g} = e = \det e^\mu{}_a$), ψ is the gravitino, M a complex scalar auxiliary field and b_a a real vector auxiliary field. The supergravity superfield is $E = \det E^\mu{}_a$, where $E^\mu{}_a$ is the superspace generalisation of the vierbein. Then, the supergravity action ca be written in a compact way as

$$\begin{aligned} S_{\text{SUGRA}} &= -3\int \mathrm{d}^4z\, E \\ &= -\frac{1}{2}\int \mathrm{d}^4x\, e\left[R - \frac{1}{3}\bar{M}M + \frac{1}{3}b^a b_a + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left(\bar{\psi}_M \bar{\sigma}_\nu \mathcal{D}_\rho \psi_\sigma - \psi_\mu \sigma_\nu \mathcal{D}_\rho \bar{\psi}_\sigma\right)\right]. \end{aligned} \tag{2.43}$$

Notice that the auxiliary fields do not propagate, thus if we integrate them out we recover the $\mathcal{S}_{RS}+\mathcal{S}_{EH}$ action. They are only used to complete the supergravity multiplet providing an off-shell invariant action.

The next step is to couple supergravity with matter. Let us consider the total Lagrangian

$$\mathcal{L}_{\text{TOT}} = \mathcal{L}_{\text{SUGRA}} + \mathcal{L}_{\text{SUSY}}, \tag{2.44}$$

where $\mathcal{L}_{\text{SUGRA}}$ comes from the action (2.43) while the expression for $\mathcal{L}_{\text{SUSY}}$ can be found in (2.29). We want to derive the scalar potential of supergravity. For this reason, we focus on the chiral part of the total action, which reads

$$\mathcal{S} = -\frac{3}{\kappa^2}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\, E\, e^{-\frac{\kappa^2}{2}K} + \left(\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}W + h.c.\right), \tag{2.45}$$

where we restored the M_p dependence and $\mathcal{E}$ is defined by $2\mathcal{E}\mathcal{R} = E$, $\mathcal{R} = \mathcal{R}\left(R, \psi, M, b_a\right)$ being the curvature superfield. Notice that the first term of this action, when expanded in powers of κ^2 as

$$e^{-\frac{\kappa^2}{2}K} = 1 - \frac{\kappa^2}{3}K + \mathcal{O}\left(\kappa^4\right), \tag{2.46}$$

includes the pure supergravity action plus the standard kinetic term for matter fields. The fact that K appears explicitly in the pure supergravity part of the action implies that the coefficient of the Einstein term M_p depends on the chiral matter field. Hence, we switch to the Einstein frame, where M_p is constant, through a rescaling of the metric. In turn, this requires a rescaling of the fermionic fields and other complications which make the treatment quite difficult. To avoid these issues, an extra superfield φ, known as the *Weyl compensator* field, is usually introduced. It is not a physical field since it does not propagate, and it is introduced in such a way that it makes the action invariant under scale and conformal transformations. At the end of the day, the action (2.45) is modified as

$$\mathcal{S} = -3\int \mathrm{d}^4x\,\mathrm{d}^4\theta\, E\varphi\bar{\varphi}\, e^{-\frac{K}{3}} + \left(\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\varphi^3 W + h.c.\right). \tag{2.47}$$

This action is invariant under rescaling of the metric. After computing the components of the action, in order to restore a canonical Einstein term and thus obtain the standard Einstein action, φ has to be fixed to $\varphi\bar{\varphi}e^{-K/3} = M_p^2$. The fixing of the value for φ breaks explicitly the (artificial) scale-invariance and leaves only the physical fields properly normalised with standard kinetic terms.

2.2.1 Derivation of the F-term scalar potential

Now, we restrict ourselves to a flat space-time, *i.e.* $\mathcal{E} = E = 1$ and the covariant derivatives reduce to the global covariant derivatives. Consequently the action reads

$$\mathcal{S} = -3\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\varphi\bar{\varphi}\, e^{-\frac{K}{3}} + \left(\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\varphi^3 W + h.c.\right). \tag{2.48}$$

We can derive the equations of motion for the auxiliary fields F in the following way.

Recalling that

$$\begin{aligned}
&\int \mathrm{d}^2\theta \; \to -\frac{1}{4} D_a D^a, \\
&\int \mathrm{d}^4\theta \equiv \int \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \; \to \frac{1}{16} \bar{D}^2 D^2,
\end{aligned} \tag{2.49}$$

the integration over half of the superspace gives

$$\mathcal{L} = -3 \int \mathrm{d}^2\bar{\theta} \left(\bar{\varphi}\, e^{\frac{K}{3}} F^\varphi - \frac{1}{3} \varphi \bar{\varphi}\, e^{-\frac{K}{3}} K_i F^i \right) + 3\varphi^2 F^\varphi W + \varphi^3 F^i W_i + \int \mathrm{d}^2\bar{\theta}\, \bar{\varphi}^3 \bar{W}, \tag{2.50}$$

where $F^\varphi = -\frac{1}{4} D^2 \varphi$ and $F^i = -\frac{1}{4} D^2 \Phi$. Then, the integration over the rest of the superspace gives

$$\begin{aligned}
\mathcal{L} = -e^{\frac{K}{3}} &\left(3F^{\bar{\varphi}} F^\varphi - \bar{\varphi} K_{\bar{i}} F^{\bar{i}} F^\varphi - \varphi K_i F^i F^{\bar{\varphi}} - \varphi \bar{\varphi} K_{i\bar{j}} F^i F^{\bar{j}} + \frac{1}{3} \bar{\varphi} \varphi K_i F^i K_{\bar{j}} F^{\bar{j}} \right) \\
&+ 3\varphi^2 F^\varphi W + \varphi^3 F^i W_i + 3\bar{\varphi}^2 F^{\bar{\varphi}} \bar{W} + \bar{\varphi}^3 F^{\bar{i}} W_{\bar{i}}.
\end{aligned} \tag{2.51}$$

The equations of motion can be derived following $\frac{\delta \mathcal{S}}{\delta F^{\bar{\varphi}}} = 0$ and $\frac{\delta \mathcal{S}}{\delta F^i} = 0$ and they read

$$\begin{aligned}
&F^\varphi = e^{\frac{K}{3}} \bar{\varphi}^2 \bar{W} + \frac{1}{3} \varphi K_i F^i, \\
&\varphi^3 \mathcal{D}_i W + e^{-\frac{K}{3}} \varphi \bar{\varphi} K_{i\bar{j}} F^{\bar{j}} = 0,
\end{aligned} \tag{2.52}$$

where $\mathcal{D}_i W \equiv \partial_i W + M_p^{-2} (\partial_i K) W$. Hence, we solve these equations for the F terms and we plug the solutions back into (2.51). Noticing that

$$\begin{aligned}
&\int \mathrm{d}^4\theta \, K \to K_{i\bar{j}} F^i F^{\bar{j}}, \\
&\int \mathrm{d}^2\theta \, W \to F^i W_i, \\
&\int \mathrm{d}^2\bar{\theta} \bar{W} \to F^{\bar{i}} \bar{W}_{\bar{i}},
\end{aligned} \tag{2.53}$$

we come to the expression for the scalar potential, that is

$$-V = K_{i\bar{j}} F^i F^{\bar{j}} + F^i W_i + F^{\bar{i}} \bar{W}_{\bar{i}}. \tag{2.54}$$

Then, substituting the expressions for the F terms as derived in (2.52) into the potential, we can recast the potential as

$$V = \varphi^2 \bar{\varphi}^2 e^{\frac{K}{3}} \left(K^{i\bar{j}} \mathcal{D}_i W \mathcal{D}_{\bar{j}} \bar{W} + 3 \left| W \right|^2 \right). \tag{2.55}$$

It is clear that this potential is not in expressed in the canonical form, thus we must switch to the Einstein frame and introduce introduce the Weyl compensator as explained before. To determine the value of this compensator, we must require the term of the action

$$-\frac{3}{\kappa^2}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\, E\, e^{-\frac{\kappa^2}{2}K} \tag{2.56}$$

to include the Einstein-Hilbert action at leading order in a expansion in powers of κ^2 along with a single power $\varphi\bar{\varphi}$. Then, to get the canonical form the scalar component of the Weyl compensator must be $\varphi = \bar{\varphi} = e^{K/6}$. Finally, we get the standard F-term scalar potential of supergravity

$$V_F = e^K\left(K^{i\bar{j}}\,\mathcal{D}_i W \mathcal{D}_{\bar{j}}\overline{W} - 3\,|W|^2\right). \tag{2.57}$$

Notice that this potential is not positive definite. In turn, this allows for theories with supersymmetric vacua of negative vacuum energy, also called anti-de-Sitter (AdS) vacua.

2.2.2 Supersymmetry breaking in supergravity

Due to the fact that the action (2.48) enjoys the so-called *Kähler invariance* under

$$\begin{aligned} &K \to K + h(\Phi) + h*(\Phi) \\ &W \to e^{-h(\Phi)}W, \end{aligned} \tag{2.58}$$

the scalar potential (2.57) can be recast as

$$V_F = e^G\left(G^{i\bar{j}}\,G_i G_{\bar{j}} - 3\right), \tag{2.59}$$

where $G \equiv K + \ln|W|^2$. The fact that the action depends only on the invariant combination G implies that K and W in supergravity are not totally independent as they were in global supersymmetry. However, it is more convenient to work with K and W rather than G.

In supergravity, the D terms are given by $D^a = \mathrm{tr}\left(\partial_i K T^a \varphi_i\right)$, T^a being the gauge group generators. Making use of the Kähler invariance, we find and interesting relation between the F terms and the D terms, that is

$$D^a = G_i T^a \varphi_i = e^{-\frac{G}{2}} F_i T^a \varphi_i, \tag{2.60}$$

where the second equality comes from the fact that F_i can be written in terms of G_i. Thus, since $F \propto DW$, as long as W is non-vanishing F and D are proportional. Consequently, supergravity cannot be broken only by F terms or D terms independently. The supersymmetry in supergravity models is broken by the non-vanishing vev of a new auxiliary field F_g that is added by the supergravity multiplet. The non-zero vev for F_g triggers the so-called *super-Higgs mechanism*, namely the gravitino becomes massive by eating the goldstino, which provides the two extra degrees of freedom to form a massive spin-3/2 particle. Thus, having a massive gravitino and a massless graviton give the evidence of the beaking of supersymmetry. Notice that the super Higgs mechanism should not be confused with the supersymmetric extension of the standard Higgs mechanism in which a massless vector superfield eats a chiral superfield in order to receive a mass by incorporating the degrees of freedom of the chiral superfield.

The gravitino mass scales as

$$m_{3/2} \simeq \sqrt{8\pi}\frac{F}{M_p}. \tag{2.61}$$

We had to take into account the super-Higgs mechanism because in supergravity a constant FI term is not allowed.

It is noteworthy that supergravity is a non-renormalizable theory, hence it should be considered as an effective field theory having a UV cut-off below the M_p scale.

2.3 String compactifications

String theory is at present the most promising candidate for providing a UV completion of the SM, because it includes also a consistent theory of quantum gravity which is finite order by order in perturbation theory. The basic assumption of string theory is that, at the fundamental level, matter does not consist of point-particles but rather of tiny loops of strings. From these grounds, all the known physics and the theories beyond the SM such as supersymmetry can be derived. However, strings carry many other ingredients such as extra spatial dimensions as well as new particles. Due to the fact that string theory requires the existence of additional dimensions, in order to obtain the SM as a 4D effective field theory one must *compactify* the extra dimensions. This is a crucial issue if we want to be able to test string theory experimentally. For this reason, our purpose is now to compute the 4D effective action from string compactification.

We start by assuming a 10D space-time of the form $\mathbb{R}^{1,3} \times X_6$, X_6 being a 6D compact manifold. Usually one requires the string compactification to yield a supersymmetric low energy effective theory mainly because supersymmetry simplifies the calculations and provides a natural solution to the Higgs hierarchy problem. Giving that we are interested in a compact manifold that preserves the minimal amount of supersymmetry, *i.e.* $\mathcal{N} = 1$, we shall focus on the case of manifolds with $SU(3)$ as holonomy group. It turns out that such manifolds are Ricci-flat Kähler manifolds, hence corresponding to Calabi-Yau threefolds.

2.3.1 Kaluza-Klein dimensional reduction

To derive the 4D effective action of a string compactification from a 10D action one has to perform the so-called Kaluza-Klein dimensional reduction. First, let us consider the case of a free massless scalar field $\phi(x^M)$, $M = 0, \ldots, 4$ living in a 5-dimensional space. Let us label with Greek indexes the Minkowskian coordinates and set the extra dimension $x^4 = y$ defining a circle of radius r with periodicity $y \equiv y + 2\pi r$. Therefore, our space-time has the topology of $\mathcal{M}_4 \times S^1$. The 5-dimensional action for $\phi(x^M)$ is given by

$$\mathcal{S}^{(5)} = \int \mathrm{d}^5 x \, \left(\partial^\mu \phi \, \partial_\mu \phi(x^{\mu,y}) + \partial^4 \phi \, \partial_4 \phi(x^{\mu,y})\right), \tag{2.62}$$

and the field equation reads

$$\left(\partial^\mu \partial_\mu + \partial^4 \partial_4\right) \phi(x^\mu, y) = 0. \tag{2.63}$$

Due to the quantisation of the modes, the periodicity along the direction y implies a discrete Fourier expansion, which takes the form

$$\phi(x^\mu, y) = \sum_{n=-\infty}^{+\infty} \phi_n(x^\mu) e^{i\frac{ny}{r}}, \tag{2.64}$$

where n is the wave number of the fifth dimension and $\phi_n(x^\mu)$ are an infinite number of 4D scalar fields. Therefore, plugging the Fourier expansion into (2.63), the equation of motion becomes

$$\sum_{n=-\infty}^{+\infty} \left(\partial^\mu \partial_\mu - \frac{n^2}{r^2}\right) \phi_n(x^\mu) e^{i\frac{ny}{r}} \quad \Rightarrow \quad \partial^\mu \partial_\mu \phi_n(x^\mu) - \frac{n^2}{r^2} \phi_n(x^\mu) = 0, \tag{2.65}$$

which is an infinite number of Klein-Gordon equations for massive 4D fields, since each Fourier mode ϕ_n is a 4D particle with mass

$$m_n = \frac{n}{r}. \tag{2.66}$$

Due to the fact that the general solution of the field equation is a superposition of all Fourier modes, the 4-dimensional description contains an infinite Kaluza-Klein tower of massive 4- fields depending only on the Minkowskian coordinates and whose masses grow for increasing n. The massive states are called Kaluza-Klein or momentum states, since they descend from the momentum in the extra-dimension. The equation (2.66) brings to two limits. When $r \to \infty$, that is in the decompactification limit, all the Kaluza-Klein masses vanish. Otherwise, when $r \to 0$, the Kaluza-Klein masses become infinite except for the $n = 0$ mode.

Let us recover the 4-dimensional action from (2.62) by substituting the mode expansion of ϕ in the original 5-dimensional action:

$$\begin{aligned}\mathcal{S}^{(5)} &= \int \mathrm{d}^4x \int \mathrm{d}y \sum_{n=-\infty}^{+\infty} \left(\partial^\mu \phi_n(x^\mu) \partial_\mu \phi_n^*(x^\mu) - \frac{n^2}{r^2} |\phi_n|^2 \right) \\ &= 2\pi r \int \mathrm{d}^4x \, (\partial^\mu \phi_0(x^\mu) \partial_\mu \phi_0^*(x^\mu) + \partial^\mu \phi_1(x^\mu) \partial_\mu \phi_1^*(x^\mu) + \dots) \\ &= \mathcal{S}^{(4)} + \dots .\end{aligned} \tag{2.67}$$

Therefore, the 5D action reduces to a 4D action for a massless scalar field plus an infinite number of massive scalar field. Consequently, if we are interested in energies smaller than the $1/r$ scale, we shall focus only on the zero-mode action, hence $\phi(x^M) = \phi(x^\mu)$. This means that we are truncating the mass spectrum by integrating out all massive fields. In this case we speak of *dimensional reduction*. Otherwise, if we keep all massive modes, we are doing a full *compactification*, which means that the extra dimension is compact and its existence is taken into account as long as all the Fourier modes are included.

We consider now an abelian vector field $A_M(x^M)$ in 5D. It can be split into a 4-dimensional vector A_μ and a 4-dimensional scalar that we will call $A_4 \equiv \rho$. They have the following Fourier expansions

$$A_\mu = \sum_{n=-\infty}^{+\infty} A_\mu^n \, e^{i\frac{ny}{r}} \quad \text{and} \quad \rho = \sum_{n=-\infty}^{+\infty} \rho_n \, e^{i\frac{ny}{r}} . \tag{2.68}$$

The 5D action is

$$\mathcal{S}^{(5)} = \int \mathrm{d}^5x \frac{1}{g_{(5)}^2} F^{MN} F_{MN}, \tag{2.69}$$

where the field strength is given by $F_{MN} = \partial_M A_N - \partial_N A_M$, hence implying $\partial^M \partial_M A_N - \partial^M \partial_N A_M = 0$ from Maxwell's equations , as in the classic 4-dimensional electromagnetic case. If we choose the unitary gauge, *e.g.* $\partial^M A_M = 0$ with $A_0 = 0$, then we are left with only the physical degrees of freedom. In turn, Maxwell's equations become $\partial^M \partial_M A_N = 0$ and subsequently the discussion is equivalent to the scalar field case seen previously, in such a way that we have an infinite tower of massive states in 4D for each massless state in 5D. Once again, in order to recover the 4D effective action, we plug this result into the 5D action (2.69), we perform the integration over the fifth dimension and we find

$$\mathcal{S}^{(4)} = \int \mathrm{d}^4x \left(\frac{2\pi r}{g_{(5)}^2} \, F_{(0)}^{\mu\nu} F_{\mu\nu}^{(0)} + \frac{2\pi r}{g_{(5)}^2} \, \partial^\mu \rho_0 \partial_\mu \rho_0 + \dots \right) . \tag{2.70}$$

In this way, we end up with an effective 4D theory made of a massless gauge field, a massless scalar ρ_0 and an infinite tower of massive vector and scalar fields.

It is interesting to underline the relation between the 4D and 5D gauge couplings, which read

$$\frac{1}{g_{(4)}^2} = \frac{2\pi r}{g_{(5)}^2}. \tag{2.71}$$

This can be straightforwardly generalised to a D-dimensional space-time as

$$\frac{1}{g_{(4)}^2} = \frac{\mathcal{V}_{(N)}}{g_{(D)}^2}, \tag{2.72}$$

where $\mathcal{V}_{(N)} = \mathcal{V}_{(D-4)}$ is the volume of the N-dimensional compact space.

Now, let us extend these results by including them into a 5-dimensional theory of gravitation. The 5D graviton G_{MN} is composed of the usual 4D graviton $G_{\mu\nu}$ plus four 4D vectors $G_{\mu 4}$ and one 4D scalar $G_{4,4}$. The 5-dimensional Einstein-Hilbert action reads

$$\mathcal{S}_{EH}^{(5)} = \int \mathrm{d}^5 x\, \sqrt{-G}\, R_{(5)}, \tag{2.73}$$

where $R_{(5)}$ is the 5-dimensional Ricci scalar. To perform the Kaluza-Klein reduction, we can consider again the 5D space-time as made of a 4D Minkowski space-time times a 1D circle, that is $\mathcal{M}_4 \times S^1$, with a background metric of the type $\mathrm{d}s^2 = W(y)\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu - \mathrm{d}y^2$. $W(y)$ is a warped factor which we set to a constant for simplicity, and y is restricted to the interval $[0, 2\pi]$ as usual. We can write any 5D metric as

$$G_{MN} = \phi^{-1/3} \begin{pmatrix} g_{\mu\nu} - \kappa^2 \phi A_\mu A_\nu & -\kappa\phi A_\mu \\ -\kappa\phi A_\nu & \phi \end{pmatrix}. \tag{2.74}$$

Then we can make a Fourier expansion, which reads

$$G_{MN} = \phi_{(0)}^{-1/3} \begin{pmatrix} g_{\mu\nu}^{(0)} - \kappa^2 \phi_{(0)} A_\mu^{(0)} A_\nu^{(0)} & -\kappa\phi_{(0)} A_\mu^{(0)} \\ -\kappa\phi_{(0)} A_\nu^{(0)} & \phi_{(0)} \end{pmatrix} + \infty \text{ tower of massive modes,} \tag{2.75}$$

where the zero-mode expansion is the ansatz formulated by Kaluza and Klein. If we plug the zero-mode part into (2.73), we come to a 4D effective action which unifies gravitation, electro-magnetism and scalar fields and reads

$$\mathcal{S}^{(4)} = \int \mathrm{d}^4 x\, \sqrt{-g} \left(M_p^2 R_{(4)} - \frac{1}{4}\phi_{(0)} F_{\mu\nu}^{(0)} F_{(0)}^{\mu\nu} + \frac{1}{6} \frac{\partial^\mu \phi_{(0)} \partial_\mu \phi_{(0)}}{\phi_{(0)}^2} + \dots \right). \tag{2.76}$$

Notice that M_p is a derived quantity, since $M_p^2 = M_*^3 2\pi r$. We know experimentally the value of M_p, hence we can adjust M_* and r to give the right values. This action enjoys many symmetries, two of which lead to very intriguing results and explain why the interest on extra-dimensions has grown exponentially after this discover, especially in the last decades. The first symmetry we are going to analyse is the generic transformation of the fifth dimension coordinate y, which can transform into any function following from the fact that (2.76) has no dependence on y. For this reason, we take $y \to y' = F(x^\mu, y)$. From (2.75), the 5D metric restricted to the zero modes reads

$$\mathrm{d}s^2 = \phi_{(0)}^{-1/3} \left[g_{\mu\nu}^{(0)}\, \mathrm{d}x^\mu \mathrm{d}x^\nu - \phi_{(0)} \left(\mathrm{d}y - \kappa\, A_\mu^{(0)}\, \mathrm{d}x^\mu \right)^2 \right]. \tag{2.77}$$

Hence, in order to leave $\mathrm{d}s^2$ invariant, it is necessary that $F(x^{\mu,y}) = y + f(x^\mu)$, which implies that

$$\mathrm{d}y' = \mathrm{d}y + \frac{\partial f}{\partial x^\mu}\, \mathrm{d}x^\mu \quad \text{and} \quad A_\mu'^{(0)} = A_\mu^{(0)} + \frac{1}{\kappa} \frac{\partial f}{\partial x^\mu}. \tag{2.78}$$

Therefore, we derived the standard gauge transformation for a massless vector field from a general coordinate transformation in five dimensions.

The other interesting symmetry is the overall rescaling

$$\left. \begin{array}{ccc} y & \to & \lambda y \\ A_\mu^{(0)} & \to & \lambda A_\mu^{(0)} \\ \phi_{(0)} & \to & \frac{\phi_{(0)}}{\lambda^2} \end{array} \right\} \Rightarrow \ \mathrm{d}s^2 \to \lambda^{2/3}.\mathrm{d}s^2 \tag{2.79}$$

Notice that $\phi_{(0)}$, usually called *breathing mode*, is a massless field, therefore its vev $\langle \phi_{(0)} \rangle$ is arbitrary. Then, due to the fact that the fifth dimension depends only on $\phi_{(0)}$ in (2.77), the size of the fifth dimension is arbitrary. This is a major problem of theories with extra-dimensions. Indeed, it seems that all the values of the radius (or volume in general) of the extra dimensions are equally good and the theory does not provide a way to choose one over the others. This issue is usually referred to as moduli problem of extra dimensional theories. String theories share this problem. However, in string theory quantum corrections to the effective action allow to fix the value of the volume and the shape of the extra dimension obtaining a discrete but very large set of solutions. This is the so-called *landscape* of string solutions, where each one describes a different universe and ours is only one among a huge number.

We can now examine the dimensional reduction on the background of a 10D Einstein-Hilbert action. Let us consider a 10D geometry of the form

$$G_{MN}\, \mathrm{d}X^M \mathrm{d}X^N = e^{-6\phi(x)} g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + e^{2\phi(x)} \tilde{g}_{mn} \mathrm{d}y^m \mathrm{d}y^n, \tag{2.80}$$

where $\tilde{g}_{mn}$ is a reference metric with fixed volume,

$$\int_{X_6} \mathrm{d}^6 y\, \sqrt{\tilde{g}} \equiv \mathcal{V}, \tag{2.81}$$

and $e^{\phi(x)}$ is the breathing mode and it represents the variations in size of the internal space X_6 as a function of the 4D coordinates x^μ, in such a way that we are performing a local conformal rescaling of the metric. The choice of the factor $e^{-6\phi(x)}$ is *ad hoc* in order to have in 4D the gravitational action in the Einstein frame.

Now, consider the 10D Einstein-Hilbert action

$$\mathcal{S}_{EH}^{(10)} = \frac{1}{2\kappa^2} \int_{\mathbb{R}^{1,3}\times X_6} \mathrm{d}^{10}X \sqrt{-G}\, e^{-2\Phi} R_{(10)}, \tag{2.82}$$

where κ^2 is the coupling constant corresponding to the Newton constant in 10D and can be compared to the string tension as $2\kappa^2 = (2\pi)^7(\alpha')^4$, α' being proportional to the string scale, *i.e.* $(\alpha')^{-1/2} \equiv M_S$. Besides, $R_{(10)}$ is the Ricci scalar constructed from G_{MN} and $e^{-2\Phi}$ is the dilaton-dependent prefactor that goes with it. Notice that $e^{-2\Phi}$ tells how strong gravitation is in string theory, since the vev of the dilaton-dependent prefactor $e^{\langle\Phi\rangle}$ determines the string coupling constant g_s. The string coupling is then given by $e^{\Phi_0} \equiv g_s$, therefore g_s is not a free parameter of string theory. Moreover, it is the vev of a field, which means that it has to be determined dynamically.

If we go below the Kaluza-Klein scale M_{KK}, we can split the 10D integral by expressing $R_{(10)}$ in terms of $R_{(4)}$ and $\tilde{R}_{(6)}$. For this purpose, we notice that if two D-dimensional metrics g_{MN} and $\bar{g}_{MN}$ are related by the conformal rescaling $\bar{g}_{MN} = e^{2\omega(x)}g_{MN}$, then the corresponding Ricci scalars are related by

$$e^{2\omega(x)}\bar{R} = R - 2(D-1)\nabla^2\omega(x) - (D-2)(D-1)g^{MN}\nabla_M\omega(x)\nabla_N\omega(x), \tag{2.83}$$

where the Laplacians constructed from g_{MN} and $\bar{g}_{MN}$ satisfy

$$e^{2\omega(x)}\bar{\nabla}^2 = \nabla^2 + (D-2)g^{MN}\nabla_M\omega(x)\nabla_N. \tag{2.84}$$

Using these results in (2.82), we find

$$\mathcal{S}_{EH}^{(10)} = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\int_{X_6}\mathrm{d}^6y\sqrt{\tilde{g}}\, e^{-2\Phi}\left(R_{(4)} + e^{-8\phi(x)}\tilde{R}_{(6)} + 12\partial_\mu\phi(x)\partial_\nu\phi(x)\right). \tag{2.85}$$

If the string coupling $g_s \equiv e^{\langle\Phi\rangle}$ is constant over X_6, then recalling the relation (2.81), we can write the 4D Einstein-Hilbert action as

$$\mathcal{S}_{EH}^{(4)} = \frac{\mathcal{V}}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\, R_{(4)} \equiv \frac{M_p^2}{2}\int \mathrm{d}^4x\sqrt{-g}\, R_{(4)}, \tag{2.86}$$

where the equivalence comes from the requirement that the action $\mathcal{S}_{EH}^{(4)}$ found with the dimensional reduction must reproduce the usual $\mathcal{S}_{EH}$. Remarkably, this requirement implies that $M_S \sim M_p/\sqrt{\mathcal{V}}$.

In the relation (2.85), we recognize the kinetic term for a 4D scalar field $\phi(x)$. This field is a *modulus* corresponding to a space-time dependent deformation of the 10D solution. Then, the Ricci scalar $\tilde{R}_{(6)}$ yields a potential term for $\phi(x)$. If X_6 has a positive internal curvature ($\tilde{R}_{(6)} > 0$), $\tilde{R}_{(6)}$ gives a negative potential term in 4D, driving the compactification toward a small volume, whereas if the internal curvature is negative, it contributes to the potential with a positive term, thus leading to a decompactification instability. Notice that, if X_6 is a Calabi-Yau manifold as in our case, then $\tilde{R}_{(6)} \equiv 0$. This ensures that $\phi(x)$ has a vanishing potential in the 4D theory, as we will see in the next section.

2.3.2 Moduli from Calabi-Yau compactifications

To sum up, the modulus $\phi(x)$ found in the previous section can be viewed as 4D scalar fields parametrising space-time dependent variations in the compactification volume X_6. Now our purpose is to study the moduli arising from a Calabi-Yau compactification of type IIB string theory[1].

Let us start considering a 10D geometry

$$G_{MN}\, \mathrm{d}X^M \mathrm{d}X^N = \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn} \mathrm{d}y^m \mathrm{d}y^n, \tag{2.87}$$

where the six extra dimensions are parametrised by a Ricci-flat metric g_{mn} on a Calabi-Yau threefold X_6. Below M_S, we have a 10D type IIB supergravity, which has $\mathcal{N} = 2$ supersymmetries. Then, we take the low energy limit below M_{KK} and we compactify six out of ten dimensions performing a Kaluza-Klein dimensional reduction, ending up in a 4D supergravity with $\mathcal{N} = 2$ supersymmetries. In order to obtain the 4D spectrum one has to expand all the 10D fields and then keep only the zero modes of the Kaluza-Klein tower associated to each of the 10D states. The reduction of the 10D metric yelds a 4D metric $g_{\mu\nu}$, $h_{1,1}$ Kähler moduli and $h_{1,2}$ complex structure moduli[2], which are scalar fields corresponding to deformations of the metric of X_6.

Now, we want to end up with an effective field theory with $\mathcal{N} = 1$ supersymmetry due to the fact that it is the only supersymmetry able to yield realistic models, as seen previously. Hence, $\mathcal{N} = 2$ supersymmetry can be broken to $\mathcal{N} = 1$ by introducing an appropriate orientifold projection of the form

$$\mathcal{O} = (-1)^{F_L} \Omega_p \sigma, \tag{2.88}$$

where Ω_p denotes the world-sheet parity which reverses the orientation of the string world-sheet, F_L is the number of left-moving fermions and σ is an isometric and holomorphic involution of X_6 which leaves invariant the Minkowski space-time. The orientifold

[1]Five 10D superstring theories exist, and they have completely different spectra, number of supersymmetries and gauge symmetries at the perturbative level. However, they are related by dualities and their low-energy limit is a 10D supergravity theory. Type IIB string theory stands out for having a chiral spectrum and world-sheet parity as a symmetry of the theory.

[2]$h_{r,s}$ are the so-called Hodge numbers which count the number of independent (r,s) harmonic forms which can be defined on a Calabi-Yau. In turn, they allow us to compute the number of the moduli present in a given manifold. For a Calabi-Yau manifold, the only non-trivial Hodge numbers are $h_{1,1} = h_{2,2}$ and $h_{1,2} = h_{2,1}$.

projection basically removes several degrees of freedom, *i.e.* moduli and their corresponding axions, from the low energy spectrum. Therefore, after the orientifold projection, in order to determine the proper Kähler coordinates on the final moduli space, one must reassemble in $\mathcal{N} = 1$ bosonic components of chiral supermultiplets the survived scalars, that is the scalar fields which are invariant under $\mathcal{O}$. The final moduli spectrum is made up of

- axio-dilatons $S = e^{-\Phi} - iC_0$, where the vev of their real part, a dilaton, determines the value of the string coupling while the imaginary part is an axion-like field,
- complex structure moduli U^a, which parametrise the shape of the Calabi-Yau manifold,
- Kähler moduli T_i, which parametrise the size of the Calabi-Yau manifold.

The number of Kähler moduli is determined by the number of independent 4-cycles D_i present in the Calabi-Yau manifold, which in turn is given by $h_{1,1}$. The Kähler moduli can be defined as

$$T_i = \frac{1}{2} k_{ijk} t^j t^k + i b_i, \tag{2.89}$$

where k_{ijk} are the triple intersection numbers of X_6 and t^i comes from the Kähler form J. Therefore, to delve further into the relations of these moduli, we must introduce the Kähler (1,1)-form $J = t^i(x)\hat{D}_i$ ($i = 1, \ldots, h_{1,1}$), where the t^i are 2-cycle volumes and $\hat{D}_i$ are harmonic (1,1)-forms which are the dual to the 4-cycle D_i. The compactification volume $\mathcal{V}$ can be written in the terms of the t^i as

$$\mathcal{V} = \frac{1}{6} \int_{X_6} J \wedge J \wedge J = \frac{1}{6} k_{ijk} t^i t^j t^k. \tag{2.90}$$

Using the fact that τ_i, the 4-cycle volumes of the divisor D_i, are related to the 2-cycle volumes t^i by

$$\tau_i = \frac{\partial \mathcal{V}}{\partial t^i} = \frac{1}{2} c_{ijk} t^j t^k, \tag{2.91}$$

the equation (2.89) reads

$$T_i = \tau_i + i b_i. \tag{2.92}$$

Then, T_i is the complexification of the 4-cycle volumes τ_i by b_i, where the latter are defined as the integral of the 4-form C_4 over the corresponding 4-cycle D_i, namely

$$b_i = \int_{D_i} C_4. \tag{2.93}$$

We usually refer to the b_i as axions. Hence, in string compactifications axions arise from the integration of p-forms over the p-cycles of the compact manifold.

2.3.3 Supergravity effective theory

In the previous section, we demonstrated that Calabi-Yau compactifications come with many moduli representing generically deformations of the metric. They arise from the plethora of topologically distinct cycles, which are typical features of Calabi-Yau manifolds. Therefore, understanding the dynamics of moduli is crucial for the building of a reasonable effective field theory.

We can include the moduli in the low energy supergravity theory by expanding the Kähler potential in α' and string loop expansions. At leading order, it reads

$$K_0 = -2\ln(\mathcal{V}) - \ln\left(S+\bar{S}\right) - \ln\left(-i\int_{X_6}\Omega\wedge\bar{\Omega}\right), \tag{2.94}$$

where S is the axio-dilaton, Ω is a holomorphic 3-form depending on the complex structure moduli U and $\mathcal{V}$ is the Calabi-Yau volume which depends on the Kähler moduli T as follows from (2.90). Notice that K_0 is block-diagonal, and hence each term of (2.94) is independent from the others.

In order to have a scalar potential of the form (2.57), we must now examine the superpotential in the context of string compactification. In string theory the superpotential can be generated by turning on the background fluxes. The definition of the flux in string theory is the generalisation of the classic electromagnetic flux, namely it is an integral of a p-form field strength over a p-cycle. In the particular case of type IIB Calabi-Yau compactification, one can turn on two types of internal fluxes which are given by the integration of the 3-form field strengths F_3 and H_3 over 3-cycles of the Calabi-Yau X_6. Usually, it is convenient to combine F_3 and H_3 into another 3-form G_3 which reads $G_3 \equiv F_3 - iSH_3$. If the fluxes coming from F_3 and H_3 are non-vanishing, G_3 induces a scalar potential $V(U,S)$ which depends on the axio-dilaton, as it is clear from the definition of G_3, but also on the complex structure moduli, since they correspond to the volume of the 3-cycles over which the integrations of the fluxes are performed. As a consequence, both S and U-moduli can can be stabilised by turning on background 3-form fluxes. We remark that the Kähler moduli T do not appear in this scalar potential.

The flux-generated tree-level potential $V(U,S)$ can be derived from a superpotential which takes the famous Gukov-Vafa-Witten form

$$W_0 = \int_{X_6} G_3\wedge\Omega. \tag{2.95}$$

Finally, we can write the scalar potential associated to K_0 and W_0 as

$$V_F = e^{K_0}\left(K_0^{I\bar{J}}\mathcal{D}_I W_0 \mathcal{D}_{\bar{J}}\bar{W}_0 - 3\left|W_0\right|^2\right), \tag{2.96}$$

where I, J runs over all the moduli T, S, U^a. Remember that supersymmetry is preserved if all the F terms vanish, *i.e.* $\mathcal{D}_I W_0 \equiv \partial W_0 + W_0\,\partial_I K = 0$.

In order to analyse the potential (2.96), we can write down the form that the scalar potential acquires once background fluxes are turned on,

$$V_F = e^K \left(K^{S\bar{S}}\, \mathcal{D}_S W \mathcal{D}_{\bar{S}} \bar{W} + K^{U\bar{U}}\, \mathcal{D}_U W \mathcal{D}_{\bar{U}} \bar{W} + K^{i\bar{j}}\, \mathcal{D}_i W \mathcal{D}_{\bar{j}} \bar{W} - 3\,|W|^2 \right), \qquad (2.97)$$

where

$$\begin{aligned} \mathcal{D}_i W &= \frac{\partial W}{\partial T_i} + W \frac{\partial K}{\partial T_i} \equiv W_i + W K_i, \\ \mathcal{D}_{\bar{j}} \bar{W} &= \frac{\partial \bar{W}}{\partial \bar{T}_{\bar{j}}} + \bar{W} \frac{\partial K}{\partial \bar{T}_{\bar{j}}} \equiv \bar{W}_{\bar{j}} + \bar{W} K_{\bar{j}}. \end{aligned}$$

Due to the fact that the superpotential in (2.95) does not depend on T, we can recast the equation (2.97) as

$$V_F = e^K \left(K^{S\bar{S}}\, \mathcal{D}_S W \mathcal{D}_{\bar{S}} \bar{W} + K^{U\bar{U}}\, \mathcal{D}_U W \mathcal{D}_{\bar{U}} \bar{W} + \left(K^{i\bar{j}}\, K_i K_{\bar{j}} - 3 \right) |W|^2 \right). \qquad (2.98)$$

Then, we notice that $\mathcal{V}$ is a homogeneous function of degree 3/2 in the 4-cycle volumes τ_i, as follows from (2.90) and (2.91). This ensures the validity of the identity $K_0\left(\lambda \tau_i\right) \equiv K_0\left(\tau_i\right) - 3 \ln \lambda$ for every λ and every τ_i. It follows that K_0 satisfies the *no-scale* identity

$$\left(\frac{\partial^2 K_0}{\partial T_i \partial \bar{T}_{\bar{j}}} \right)^{-1} \frac{\partial K_0}{\partial T_i} \frac{\partial K_0}{\partial \bar{T}_{\bar{j}}} = 3, \qquad (2.99)$$

which implies the vanishing of the last term in (2.98) and

$$V_F = e^K \left(K^{S\bar{S}}\, \mathcal{D}_S W \mathcal{D}_{\bar{S}} \bar{W} + K^{U\bar{U}}\, \mathcal{D}_U W \mathcal{D}_{\bar{U}} \bar{W} \right) \geq 0. \qquad (2.100)$$

As the scalar potential above is positive semi-definite, it is possible to fix the axio-dilaton and the complex structure moduli at tree level by requiring the preservation of supersymmetry, namely $\mathcal{D}_S W = \mathcal{D}_U W = 0$. However, following from the fact that they are stabilised at tree level, their quantum fluctuations will only lead to subleading corrections to their vevs, so we are allowed to integrate them out and neglect them from now on.

The flatness of the scalar potential for the Kähler moduli implies that one has to keep all possible quantum corrections in order to fix them. This issue goes under the name of *moduli stabilisation* and we shall devote a large part of the following chapter to review it.

Chapter 3

Moduli Stabilisation at Large Volume

With the expression *moduli stabilisation* we refer to a mechanism which becomes essential when one has to deal with the compactification of extra dimensions. It consists briefly in finding a vacua where all the moduli have positive mass-squared. As seen in the previous Chapter, string compactifications are characterised by the widespread presence of moduli fields. From a geometrical point of view, we saw that these moduli parametrise the size and the shape of extra-dimensions, since they correspond to either deformations of the metric (Kähler moduli T_i) or deformations of the complex structure of extra-dimensions (complex structure moduli U_α). As demonstrated previously, complex structure moduli (as well as the axio-dilation) can be directly stabilised: after turning on background fluxes, a scalar potential is generated, which already gives them masses at its semi-classical tree-level form.

The problem of moduli stabilisation deals in particular with Kähler moduli. When the scalar potential is expressed at tree-level, Kähler moduli remain massless, so they would represent massless uncharged scalar particles. Since they develop an effective gravitational coupling to all ordinary particles, in turn they would give rise to long-range fifth forces, which have not been observed. For this reason, it is of primary importance to build a potential for these particles, in order to give them a mass and so avoid fifth-forces. Notice that it is important to stabilise Kähler moduli also to understand better the effective field theory resulting from string theory compactifications, since both the effective Yukawa and gauge coupling depend on their vevs.

In the next sections, we will first develop the appropriate potential to stabilise the Kähler moduli, then we will see how it behaves in a particular regime called Large Volume Scenario (LVS). We will justify this choice. Finally we will apply the theory and do the calculations for two cases, *single hole Swiss Cheese* and *Fibred* Calabi-Yaus. The dissertation is mainly based on [33, 32, 34].

3.1 Theory of moduli stabilisation

The only way to develop a potential for the Kähler moduli is to add perturbative and non-perturbative corrections to both the superpotential W_0 and the Kähler potential K_0, which are the building blocks of the supergravity potential (2.96). Let us now illustrate how this can be done.

We start with the corrections to the superpotential. We have seen that, when we turn on background fluxes, a tree-level superpotential is generated. This takes the Gukov-Vafa-Witten form given in (2.95). The Kähler moduli T_i do not appear in W_0 and so they remain precisely massless at leading semi-classical order. For this reason, in order to give them a mass and so stabilise them, one has to add non-perturbative corrections to the tree-level superpotential W_0. Perturbative terms are not allowed for W_0 since it acquires no contributions from both α' and string loop corrections to any order in perturbation theory. Non-perturbative corrections can be generated by effects on the branes wrapping the 4-cycles. These may be Euclidean D3-brane instantons or gaugino condensation in the supersymmetric gauge theories located on D7-branes. Therefore, taking into account both of these effects, the superpotential reads

$$W = W_0 + W_{np} = W_0 + \sum_i A_i\, e^{-a_i T_i}, \tag{3.1}$$

where the sum is over the 4-cycles and W_0 is independent of T_i as said before. The coefficients A_i are prefactors which depend on the complex structure moduli U_α and D-branes positions. The constants a_i are given by $a_i = \frac{2\pi}{N_i}$, N_i being the dimension of the gauge group living on the brane wrapping a certain T_i. Hence, different T_i can have different values of a_i. The new expression for W in (3.1) generates a potential for the Kähler moduli of the form

$$\begin{aligned}\delta V_{np} = e^{K_0} K_0^{i\bar{j}} \Big[& a_i A_i a_j \bar{A}_j e^{-\left(a_i T_i + a_j \overline{T}_j\right)} \\ & - \Big(a_i A_i e^{-a_i T_i} \overline{W} \partial_{\bar{j}} K_0 + a_j \bar{A}_j e^{-a_j \overline{T}_j} W \partial_i K_0 \Big) \Big] .\end{aligned} \tag{3.2}$$

However, this is not sufficient. The procedure used in the previous Chapter to express $\mathcal{V}$ as a homogeneous function of degree $\frac{3}{2}$ in τ_i ensures that the tree-level Kähler potential K_0 satisfies the no-scale identity $K_0^{i\bar{j}}\, \partial_i K_0\, \partial_{\bar{j}} K_0 = 3$. In turn, this guarantees that, if the potential (2.96) is expressed in terms of K_0, it is still completely flat for the Kähler moduli. Unlike the superpotential, the tree-level Kähler potential receives α' corrections order by order in a perturbative expansion

$$K = K_0 + K_p = K_0 + \delta K_{\alpha'}, \tag{3.3}$$

where the term $\delta K_{\alpha'}$ breaks the no-scale structure and lifts the potential. In principle K could also acquire non-perturbative corrections, but they would be subleading compared to the perturbative ones. Thus, we will neglect them. The α' corrections are basically curvature corrections coming from 10-dimensional supergravity and they correspond to higher derivative terms. Keeping only the leading α' contribution, we obtain a Kähler potential of the form

$$K = -2\ln\left(\mathcal{V} + \frac{\xi}{2\,g_s^{3/2}}\right) = -2\ln\mathcal{V} - \frac{\xi}{g_s^{3/2}\,\mathcal{V}}, \tag{3.4}$$

where ξ is a constant which controls the strength of α' corrections and is given by $\xi = -\frac{\chi(X_6)\zeta(3)}{2(2\pi)^3}$, ζ being the Riemann function and $\chi(X_6) = 2\,(h_{1,1} - h_{2,1})$ being the Euler number of the Calabi-Yau manifold X_6. Such a Kähler potential gives the following contribution to the scalar potential:

$$\delta V_{\alpha'} = e^K \frac{3\xi}{4\,g_s^{3/2}\,\mathcal{V}}|W_0|^2. \tag{3.5}$$

The contributions to K_0 coming from string loop corrections δK_{g_s} are subleading with respect to the ones corresponding to α'. However, they can become important when one has to stabilise non blow-up moduli, as we will see later on in this chapter. δK_{g_s} corrections can be separated into two types: those coming from the exchange of closed strings with Kaluza-Klein momentum between branes and those associated with the exchange of closed strings with non vanishing winding. The systematic derivation of string loop corrections can be found in [35]. Here we just write down the contribution they produce to the total potential, that is

$$\delta V_{g_s} = g_s^2 \sum_i c_{g_s}^i \left(\partial_{\tau_i}^2 K\right) \frac{W_0^2}{\mathcal{V}^2}, \tag{3.6}$$

where the coefficient $c_{g_s}^i$ depends on the complex structure moduli and could be different for every i. It is immediate to see how g_s effects are subdominant with respect to α' effects at the level of the scalar potential. If we consider just one Kähler modulus, $\mathcal{V} \sim \tau^{3/2}$ and so $\partial_{\tau_i}^2 K \sim \mathcal{V}^{-4/3}$ at tree level, which in turn implies that

$$\frac{\delta V_{\alpha'}}{\delta V_{g_s}} \sim \frac{\mathcal{V}^{1/3}}{g_s^{7/2}} \gg 1 \qquad \text{for } \mathcal{V} \gg 1 \text{ and } g_s \sim 0.1\,, \tag{3.7}$$

since $e^K \sim \mathcal{V}^{-2}$ at leading order. We took the above value for $\mathcal{V}$ because it will be the regime we will work with in this thesis. We will justify this choice in the next section. However, the expression (3.6), once used for the case we will study in the last section of this chapter, will be useful to stabilise an important modulus in the Fibre inflation case, as we will see.

As a result, the low-energy 4D supergravity potential for the Kähler moduli receives new terms which uplift the moduli directions and generate their masses. If we consider only the contributions (3.2) and (3.5) (since they will be the dominant ones in the regime taken into account), we can write the total scalar potential for Kähler moduli as

$$\begin{aligned} V &= \delta V_{np} + \delta V_{(\alpha')} = \\ &= e^K \Big[K^{i\bar{j}} \Big[a_i A_i a_j \bar{A}_j e^{-\left(a_i T_i + a_j \overline{T}_j\right)} \\ &\qquad - \left(a_i A_i e^{-a_i T_i} \overline{W} \partial_{\bar{j}} K + a_j \bar{A}_j e^{-a_j \overline{T}_j} W \partial_i K \right) \Big] + \frac{3\hat{\xi}}{4\mathcal{V}} |W_0|^2 \Big] . \end{aligned} \tag{3.8}$$

3.2 Large Volume Scenario

In literature there are two leading ideas to perform Kähler moduli stabilisation in type IIB string compactifications. The first one was proposed in [36] and it is the so-called *KKLT scenario*, which takes the name after the authors' initials, Kachru, Kallosh, Linde and Trivedi. The second one came out two years later in the paper [37] by Balasubramanian, Berglund, Conlon and Quevedo and goes under the name of *Large Volume Scenario* (LVS). Both these models succeeded thanks to the fact that they both address to regions of the parameter space where vacua result from the competition among correction terms which are (or can become) known. However, the two ideas are based on the competition of different terms and take off from distinct assumptions. We can briefly summarise the differences as follows:

- KKLT mechanism involves a competition between the tree-level superpotential W_0 (3.37), made small by fine-tuning fluxes, and the non-perturbative corrections;
- LVS works with some cycles (especially the ones controlling the overall volume) that must be exponentially larger than others, so that the competition is between α' corrections (3.5) and non-perturbative corrections.

One of the main differences between the two models lies in the value of W_0. In KKLT W_0 must be made small by hand, typically less than $\mathcal{O}(10^{-4})$, to switch α' corrections off and keep only the non-perturbative ones. Indeed, α' corrections are the most difficult ones to deal with, because they are basically unknown. Anyway, the authors of [37] start from this point, saying that in the flux landscape it is more common for W_0 to take values of order $\mathcal{O}(1)$ rather than $\mathcal{O}(10^{-4})$. Therefore, for naturalness reasons, they keep α' corrections to K but they stabilise the overall volume $\mathcal{V}$ at such large values that leading α' corrections can be controlled and become balanced by non-perturbative terms. Further orders of α' expansion are subleading and can be disregarded.

Since LVS achieves moduli stabilisation in a natural way and since the existence of an exponentially large volume makes the effective field theory treatment robust, we will work in this regime from now on in this thesis.

The main conditions required for the existence of a minimum at large volume are related to the topology of the underlying Calabi-Yau manifold X_6, because the potential (3.8) involves topological parameters. These conditions can be expressed as follows:

- $\chi(X_6)$ must be < 0, or in other words $h_{1,2} > h_{1,1} > 1$. This in turn ensures that ξ is positive and so that, as $\mathcal{V} \to \infty$, V goes to zero from below;

- there must be present at least one blow-up mode, a del Pezzo 4-cycle, resolving a point-like singularity.

The necessity of the presence of a blow-up mode comes from the fact that it brings about a single non-perturbative effect whose interplay with the leading order α' corrections naturally yields an exponentially Large volume $\mathcal{V} \sim e^{a_{blow-up}\tau_{blow-up}}$ (see next sections for concrete examples). Then the Large volume $\mathcal{V}$ that defines the scenario naturally arises as an exponentially large function of the small parameter that controls the non-perturbative effects. Moreover, we require the blow-up mode to be a del Pezzo 4-cycle mainly for one reason: a del Pezzo divisor has a local nature, so a representation can always be found where this 4-cycle enters the Calabi-Yau volume in a diagonal way. One needs this to avoid problems arising from chiral intersections with the branes supporting the visible sector (which is chiral, as well known). Otherwise, in order to prevent the breaking of the visible sector gauge group, we would be led to have $A = 0$, and in turn to $W = 0$ [34]. Therefore, LVS provides a way to construct Calabi-Yau examples including magnetised D-branes without loosing chirality and global consistency.

Another important point for LVS is that the presence of an exponentially Large volume $\mathcal{V}$ leads to a possible explanation of the hierarchies present in nature. The so-called *hierarchical problem* is a major theoretical and phenomenological issue that has no solution in the SM. Actually, it concerns two problems, which have not been solved yet. One is the absence of symmetries able to prevent the mass of the Higgs boson from getting large loop corrections, and hence from becoming way more massive than what it is at tree-level and what has been measured at LHC. In turn, this would make the electro-weak scale higher than the Planck scale after renormalization. Here comes the second problem, that is, why is the Plank scale 10^{16} times larger thank the electro-weak scale? What is the origin of such a big discrepancy?

Nevertheless, making use of LVS, one is able to express all the main scales of string compactification as different functions of the inverse of the overall volume $\mathcal{V}$. Therefore, once $\mathcal{V}$ is fixed, the hierarchy among scales arises naturally. We represented this achievement schematically in Figure 3.1.

This topic leads us to the final advantage of LVS. Since the overall volume is nearly the only free parameter, this model has good chances to make contact with experiments

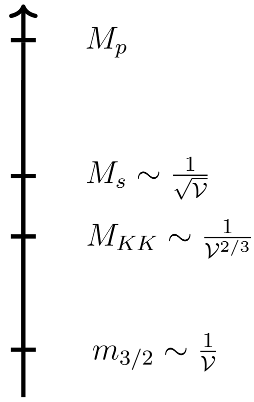

Figure 3.1: In this figure, the hierarchy among the most important physical scales is displayed. Since in LVS the mass scales can be all expressed as functions of one variable only, the hierarchy arises naturally. This can be regarded as a possible way to solve the hierarchy problem.

in two ways, via either particle phenomenology or cosmology. One can derive the dependence on $\mathcal{V}$ of the whole mass spectrum of the model, then get the scale at which the particles would show up. It is also interesting to try to determine whether these particles can be good DM candidates or not, since some of them become very light once stabilised. We will discuss this idea in Chapter 4. Alternatively, in this regime it is also possible to derive a natural model of inflation, that is Fibre inflation, as we will see in section 3.4. Making use of Fibre inflation, one can make interesting predictions on gravity waves and other cosmological parameters [38].

For all these reasons, LVS represents a good setup to work on and we will make use of it in the rest of this thesis. In the following sections, two well-known applications of LVS to moduli stabilisation are presented.

3.3 Swiss cheese Calabi-Yaus

The original example with explicit calculation of LVS in [37] makes use of the so called *single hole Swiss Cheese* geometry. This name comes from the fact that we are dealing with $h_{1,1} = 2$ Kähler moduli, where the volume of the smaller one, τ_s, controls the size of the hole and the volume of the bigger one, τ_b controls the size of the cheese. In other words, the first modulus is the blow-up mode, whereas the second modulus stands for the size of the overall volume. Thus, τ_s is a rigid cycle which is fixed small by non-perturbative corrections, whereas τ_b is stabilised large due to α' effects. We are going to show through calculations what the previous sentences mean.

The overall Calabi-Yau volume of the single hole Swiss cheese geometry is given by

$$\mathcal{V} = \frac{1}{9\sqrt{2}}\left(\tau_b^{3/2} - \tau_s^{3/2}\right). \tag{3.9}$$

In the case of this compactification, the necessary conditions given by the Large volume claim described in [39] and discussed in the previous section are readily met.

In terms of these moduli the Kähler potential (3.4) and the superpotential (3.1) read

$$K = -2\ln\left[\left(\tau_b^{3/2} - \tau_s^{3/2}\right) + \frac{\hat{\xi}}{2}\right], \tag{3.10}$$

$$W = W_0 + A_s\, e^{-a_s T_s}, \tag{3.11}$$

where $\hat{\xi} = \xi/g_s^{3/2}$, ξ being a constant including α' corrections and g_s being the string coupling. Since $\tau_b \gg \tau_s$, non-perturbative effects involving T_b can be neglected at first. Notice that, for practical convenience in our computation, we absorbed the overall factor $9\sqrt{2}$ into the quantities W_0 and A_s. Obviously, this does not affect the physics of this model in any way.

The kinetic terms for the moduli fields depend on the second derivatives of the Kähler potential, which can be written as a matrix, the so called Kähler metric. Since we work at Large volume, in computing the metric we can disregard α' corrections and other terms that are suppressed in this regime. Recalling that $\frac{\partial}{\partial T_i} = \frac{1}{2}\frac{\partial}{\partial \tau_i}$, the Kähler matrix reads

$$K_{i\bar{j}} = \begin{pmatrix} K_{b\bar{b}} & K_{b\bar{s}} \\ K_{s\bar{b}} & K_{s\bar{s}} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \frac{3}{\tau_b^2} & -\frac{9\sqrt{\tau_s}}{2\tau_b^{5/2}} \\ -\frac{9\sqrt{\tau_s}}{2\tau_b^{5/2}} & \frac{3}{2\sqrt{\tau_s}\tau_b^{3/2}} \end{pmatrix}, \tag{3.12}$$

while its inverse results in

$$K_{i\bar{j}}^{-1} = 4\begin{pmatrix} \frac{\tau_b^2}{3} & \tau_s\tau_b \\ \tau_s\tau_b & \frac{8\tau_b^{3/2}\sqrt{\tau_s}}{3} \end{pmatrix}. \tag{3.13}$$

We kept only the contributions at leading order in τ_b. As seen previously, the Large volume limit of the supergravity scalar potential (2.57) takes the form (3.8). Substituting our quantities, we can see that the total potential is given by

$$V = \frac{8a_s^2A_s^2\sqrt{\tau_s}}{3\tau_b^{3/2}}\,e^{-2a_s\tau_s} + \frac{4W_0a_sA_s\tau_s}{\tau_b^3}\,e^{-a_s\tau_s}\cos\left(a_sb_s\right) + \frac{\nu W_0^2}{\tau_b^{9/2}}, \tag{3.14}$$

where $\nu = \frac{27\sqrt{2}\hat{\xi}}{4}$. For our present purpose it is better to rewrite (3.14) extremising the axionic field as follows:

$$V = \frac{8a_s^2A_s^2\sqrt{\tau_s}}{3\tau_b^{3/2}}\,e^{-2a_s\tau_s} - \frac{4W_0a_sA_s\tau_s}{\tau_b^3}\,e^{-a_s\tau_s} + \frac{\nu W_0^2}{\tau_b^{9/2}}. \tag{3.15}$$

This potential generates the masses for the two moduli. An estimation of these masses can be performed in the following way:

$$\frac{m_{\tau_b}^2}{M_p^2} \sim K_{b\bar{b}}^{-1}\,\frac{\partial^2 V}{\partial \tau_b^2} \quad \Rightarrow \quad m_{\tau_b} \sim \frac{M_p}{\mathcal{V}^{3/2}}, \tag{3.16}$$

$$\frac{m_{\tau_s}^2}{M_p^2} \sim K_{s\bar{s}}^{-1}\,\frac{\partial^2 V}{\partial \tau_s^2} \quad \Rightarrow \quad m_{\tau_s} \sim \frac{M_p \ln \mathcal{V}}{\mathcal{V}}. \tag{3.17}$$

This is basically what we referred to as moduli stabilisation some sections above. We can immediately see that a hierarchy for the two moduli is present. In turn, this will give rise to a hierarchy between the masses for the two observable particles that come from the canonical normalisation of the moduli fields. To see it explicitly, we now review the canonical normalisation as performed in [40]. Before proceeding, it's worth stressing that the two masses mainly depend on the dimension of the volume, which follows from our use of the Large volume regime.

To perform the canonical normalisation, we first expand the moduli fields around their vevs as

$$\tau_b = \langle \tau_b \rangle + \delta\tau_b, \tag{3.18}$$

$$\tau_s = \langle \tau_s \rangle + \delta\tau_s. \tag{3.19}$$

Then we obtain the Lagrangian

$$\mathcal{L} = K_{i\bar{j}}\,\partial_\mu\,(\delta\tau_i)\,\partial^\mu\,(\delta\tau_j) - \langle V_0 \rangle - \frac{1}{2}V_{i\bar{j}}\,\delta\tau_i\delta\tau_j + \mathcal{O}(\delta\tau^3), \tag{3.20}$$

where $i, j = b, s$. If we write $\delta\tau_b$ and $\delta\tau_s$ in terms of the canonically normalised fields Φ and χ as

$$\delta\tau_i = (\vec{v}_\Phi)_i\,\frac{\Phi}{\sqrt{2}} + (\vec{v}_\chi)_i\,\frac{\chi}{\sqrt{2}}, \tag{3.21}$$

then the conditions for the Lagrangian (3.20) to take the canonical form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi + \frac{1}{2}\partial_\mu\chi\,\partial^\mu\chi - \langle V_0 \rangle - \frac{1}{2}m_\Phi^2\Phi^2 - \frac{1}{2}m_\chi^2\chi^2 \tag{3.22}$$

read:

$$K_{i\bar{j}}\,(\vec{v}_\alpha)_i\,(\vec{v}_\beta)_j = \delta^{\alpha\beta}, \qquad \frac{1}{2}V_{i\bar{j}}\,(\vec{v}_\alpha)_i\,(\vec{v}_\beta)_j = m_\alpha^2\,\delta^{\alpha\beta}. \tag{3.23}$$

These relations are satisfied when $\vec{v}_\Phi$, $\vec{v}_\chi$ (normalised according to the first of (3.23)) and m_Φ^2, m_χ^2 are, respectively, the eigenvectors and the eigenvalues of the mass-squared matrix $(M^2)_{i\bar{j}} = \frac{1}{2}K_{i\bar{k}}{}^{-1}\,V_{\bar{k}j}$. To compute the mass matrix, it is necessary to evaluate the second derivatives of the potential at the minimum. It can be shown that

$$\begin{cases} \frac{\partial V}{\partial \tau_b} = 0 \\ \frac{\partial V}{\partial \tau_s} = 0 \end{cases} \quad \Leftrightarrow \quad \begin{aligned} e^{-a_s\langle\tau_s\rangle} &= \tfrac{3W_0\sqrt{\langle\tau_s\rangle}}{4a_sA_s\langle\tau_b\rangle^{3/2}}\left(1 - \tfrac{3}{4a_s\langle\tau_s\rangle} - \tfrac{3}{(4a_s\langle\tau_s\rangle)^2} + \ldots\right), \\ \langle\tau_s\rangle^{3/2} &= \tfrac{2\nu}{3}\left(1 + \tfrac{1}{2a_s\langle\tau_s\rangle} + \tfrac{9}{(4a_s\langle\tau_s\rangle)^2} + ..\right). \end{aligned} \tag{3.24}$$

Notice also that τ_b, the modulus which controls the overall size of the Calabi-Yau, is stabilised exponentially Large:

$$\langle \tau_b \rangle^{3/2} \sim e^{a_s \langle \tau_s \rangle} \sim \langle \mathcal{V} \rangle, \tag{3.25}$$

where the second relation comes from the decompactification limit [37], *i.e.* $\mathcal{V} \to \infty$ and $a_s \tau_s \sim \ln \mathcal{V}$. Using the relations (3.24), one can show that, to the second order in an expansion in $\varepsilon = \frac{1}{a_s \tau_s}$, the mass-squared matrix is given by

$$\begin{aligned} M^2 &= \frac{1}{2} K_{i\bar{k}}^{\ -1} V_{\bar{k}j} \\ &= \frac{2 a_s \langle \tau_s \rangle W_0^2 \nu}{3 \langle \tau_b \rangle^{9/2}} \begin{pmatrix} -9(1-7\varepsilon) & 6 a_s \langle \tau_b \rangle (1 - 5\varepsilon + 16\varepsilon^2) \\ -\frac{6\sqrt{\langle \tau_b \rangle}}{\sqrt{\langle \tau_s \rangle}}(5\varepsilon + 4\varepsilon^2) & \frac{4 a_s \langle \tau_b \rangle^{3/2}}{\sqrt{\langle \tau_s \rangle}}(1 - 3\varepsilon + 6\varepsilon^2) \end{pmatrix}. \end{aligned} \tag{3.26}$$

The matrix (3.26) has one large and one small eigenvalue, m_Φ^2 and m_χ^2 respectively. For this reason, we can see that, at leading order in ε, the mass spectrum for the canonically normalised fields is given by

$$\frac{m_\Phi^2}{M_p^2} \simeq \mathrm{tr}\Big((M^2)_{i\bar{j}} \Big) \simeq \left(\frac{\ln \mathcal{V}}{\mathcal{V}} \right)^2 \tag{3.27}$$

$$\frac{m_\chi^2}{M_p^2} \simeq \frac{\det\Big((M^2)_{i\bar{j}} \Big)}{\mathrm{tr}\Big((M^2)_{i\bar{j}} \Big)} \simeq \frac{1}{\mathcal{V}^3 \ln \mathcal{V}}. \tag{3.28}$$

It is now clear that a large hierarchy of masses between the two observable particle is present, with Φ being heavier than the gravitino mass and χ lighter (recall that $m_{3/2} \sim M_p^2 \mathcal{V}^{-1}$). Finally, after computing the eigenvectors of (3.26) and replacing the results in (3.21), we come to a new expression for the original fields in terms of Φ and χ at leading order in ε

$$\delta \tau_b = \left(\sqrt{6}\, \langle \tau_b \rangle^{1/4} \langle \tau_s \rangle^{3/2} (1 - 5\varepsilon) \right) \frac{\Phi}{\sqrt{2}} + \left(\frac{2}{\sqrt{3}} \langle \tau_b \rangle \right) \frac{\chi}{\sqrt{2}} \sim \mathcal{V}^{1/6}\, \Phi + \mathcal{V}^{2/3}\, \chi, \tag{3.29}$$

$$\delta \tau_s = \left(\frac{2\sqrt{6}}{3} \langle \tau_b \rangle^{3/4} \langle \tau_s \rangle^{1/4} \right) \frac{\Phi}{\sqrt{2}} + \left(\frac{\sqrt{3}}{a_s} (1 - 2\varepsilon) \right) \frac{\chi}{\sqrt{2}} \sim \mathcal{V}^{1/2}\, \Phi + \mathcal{V}^0\, \chi. \tag{3.30}$$

The relations (3.29) and (3.30) show that, whilst $\delta\tau_b$ is mostly χ and $\delta\tau_s$ is mostly Φ, a mixing of the original fields is present, as expected. This is an important phenomenological feature because, as shown in [37], even if the large modulus τ_b has no coupling to photons, the light field χ, although mostly aligned with τ_b, does have a measurable coupling to photons due to its small component in the τ_s direction.

Now, we want to consider also the axionic partners of the moduli. Previously, for the sake of simplicity, we disregarded these components, since they are either way less massive than their real partners or suppressed, but we think it could be interesting to analyse their behaviour also in this first example of calculation. The importance of the axionic components will be eventually better understood in the following Chapter.

We have already said that axions constitute the imaginary part of the Kähler coordinates $T_b = \tau_b + ib_b$ and $T_s = \tau_s + ib_s$. If we want to analyse not only the behaviour of b_s, which could eventually have been studied previously since it was already stabilised by (3.14), but also the behaviour of b_b, we have to add a further non-perturbative term in (3.11):

$$W = W_0 + A_s\, e^{-a_s T_s} + A_b\, e^{-a_b T_b}. \tag{3.31}$$

We did not consider perturbative terms to stabilise the axions because of their shift symmetry, which forbids the presence of axion-depending terms in a perturbative expansion. However, the shift symmetry holds only at the perturbative level because, instead, it is broken by non-perturbative effects. Therefore we can only have non-perturbative axion-depending terms. Clearly, the new term is suppressed at Large volume because of (3.25), so it is correct to disregard it if one is interested only in moduli physics.

Since also the axionic partners of τ_b and τ_s receive masses after moduli stabilisation, let us compute which kind of mass do they receive. We start with rewriting the potential (3.14) adding the term for b_b:

$$\begin{aligned} V \; = \; & \frac{8a_s^2 A_s^2 \sqrt{\tau_s}}{3\tau_b^{3/2}}\, e^{-2a_s\tau_s} + \frac{4W_0 a_s A_s \tau_s}{\tau_b^3}\, e^{-a_s\tau_s} \cos\left(a_s b_s\right) \\ & + \frac{4W_0 a_b A_b \tau_b}{\tau_b^3}\, e^{-a_b\tau_b} \cos\left(a_b b_b\right) + \frac{\nu W_0^2}{\tau_b^{9/2}}. \end{aligned} \tag{3.32}$$

It is immediate to infer that $m_{b_s}^2 \simeq m_{\tau_s}^2$, therefore m_{b_s} is a rather heavy particle. As we will also see later on in this chapter, the fact that an axion whose modulus is fixed by non-perturbative terms, acquire a mass of the same order of magnitude of its modulus is a common feature of the blow-up mode. Also, this is what usually happens in the moduli stabilisation through KKLT mechanism.

A different result for b_b is found, that is

$$\frac{m_{b_b}^2}{M_p^2} \sim K_{b\bar{b}}^{-1} \frac{\partial^2 V}{\partial b_b^2} \simeq \frac{16 A_b W_0}{3\mathcal{V}^2} \left(a_b\tau_b\right)^3 e^{-a_b\tau_b} \sim e^{-\mathcal{V}}. \tag{3.33}$$

Clearly this particle is a really light one. For this reason, this axion can be regarded as a perfect candidate for fuzzy DM. We will return on this topic in the last Chapter.

3.4 Fibred Calabi-Yaus

Fibre inflation is probably the simplest and most robust family of inflationary models coming from moduli stabilisation in a Large volume regime within the type IIB string compactifications. It was first developed in [38] and it has been a promising breeding ground for stringy inflation [41, 42, 43, 44] ever since. We proceed by approaching this topic commenting on the main features and results of this model when they become manifest from calculations.

The first assumption the authors of [38] make is to have a Calabi-Yau compactification which leaves a modulus unstabilised by α' and non-perturbative leading order corrections. This requirement basically comes from the fact that they are looking for an inflaton candidate, that is, a particle that must have an almost flat potential and must be systematically light. A particle could have such features in our LVS setup if its potential is lifted by subdominant corrections. Therefore it will be stabilised by string-loop corrections, which are subleading with respect to the non-perturbative and α' ones, as shown in section 3.1.

The geometrical setting we need to construct such a model is a special kind of a Calabi-Yau manifold, *i.e.* a K3 fibration over a $\mathbb{P}^1$ base. The simplest volume of this kind can be written with two 2-cycles and must be linear in one of them, *i.e.* $\mathcal{V} \sim t_1 t_2^2$. Then, we can recast it in terms of 4-cycles by recalling that $\partial_{t_i}\mathcal{V} = \tau_i$, which leads to $\mathcal{V} = \frac{1}{2}\sqrt{\tau_1}\tau_2$. However, this volume does not satisfy one of the conditions for LVS discussed in the previous section. Therefore, it is necessary to add to the internal space a further 4-cycle modulus playing the role of the blow-up mode, whose volume we denote again as τ_s. Finally, we come to an overall Calabi-Yau volume of the form

$$\mathcal{V} = \alpha\left(\sqrt{\tau_f}\,\tau_b - \gamma\tau_s^{3/2}\right), \tag{3.34}$$

where α and γ are model depending constants (they depend on the triple intersection numbers of the manifold).

Let us briefly make a survey of the six ingredients of this model:

- $T_f = \tau_f + ib_f$: τ_f is the modulus which represents the fibration D_f over the base, and in [38] it is the inflation candidate, since it is the modulus whose potential remains flat at leading order. b_f is its corresponding axion which comes from (2.93), where the integration is over the divisor D_f, and can be stabilised thanks to small non-perturbative corrections to W_0;

- $T_b = \tau_b + ib_b$: the modulus τ_b parametrises the base (D_b) of the fibration, and mainly controls the size of the overall volume, so it can be stabilised at leading order via α' corrections. Its axionic partner b_b comes again from the integral (2.93), where the integration is over D_b, and is stabilised by non-perturbative corrections to W_0;

- $T_s = \tau_s + ib_s$: the modulus τ_s controls the size of the blow-up mode D_s and it is necessary to perform the moduli stabilisation at large volume. Both τ_s and its axionic partner b_s from (2.93) are stabilised by non-perturbative effects.

Notice that all the axionic partners are stabilised by higher-order non-perturbative effects, which in turn give rise to what the authors of [34] call *LVS Axiverse*.

Since we look for stabilisation with Large and positive $\mathcal{V}$, we work in the regime

$$\mathcal{V} \simeq \alpha \sqrt{\tau_f}\, \tau_b \gg \alpha\, \gamma\, \tau_s^{3/2} \gg 1, \tag{3.35}$$

that is the limit in which the overall volume remains Large while the blow-up cycle remains comparatively small.

We start by considering the Kähler potential and the superpotential in the most general form they may acquire in our model, *i.e.*

$$K = K_0 + \delta K_{\alpha'} + \delta K_{g_s} \tag{3.36}$$

and

$$W = W_0 + \sum_i A_i e^{-a_i T_i}, \tag{3.37}$$

where i runs over all the three 4-cycles. Therefore the F-term scalar potential originating from (3.36) and (3.37) receives several contributions which have different scaling with $\mathcal{V}$. To make our discussion simpler, we can write it as follows,

$$V = \underbrace{V(\mathcal{V}, \tau_s, b_s)}_{\mathcal{O}(\mathcal{V}^{-3})} + \underbrace{V(\tau_f)}_{\mathcal{O}\left(\mathcal{V}^{-10/3}\right)} + \underbrace{V(b_f, b_b)}_{\mathcal{O}\left(\mathcal{V}^{-4/3} e^{-\mathcal{V}^{3/2}}\right)}, \tag{3.38}$$

where we arranged it as an expansion in inverse powers of $\mathcal{V} \gg 1$. It is immediate to see that certain terms are stabilised at leading order while others are fixed by subleading corrections. Therefore, we primarily consider only the $\mathcal{O}(\mathcal{V}^{-3})$ term.

First of all, at leading order the contributions to the potential coming from (3.36) and (3.37) can be approximated as

$$K \simeq K_0 + \delta K_{\alpha'} \simeq -2 \ln \left(\mathcal{V} + \frac{\hat{\xi}}{2} \right) \tag{3.39}$$

and

$$W \simeq W_0 + A_s e^{-a_s T_s}. \tag{3.40}$$

In the Kähler potential we kept only α' corrections because of the discussion related to (3.7), which holds when working only with τ_b and τ_f. In the superpotential we are left with just one non-perturbative term because due to (3.35), the terms depending on τ_b

and τ_f are widely suppressed at Large $\mathcal{V}$ compared to the term in τ_s. The Kähler metric and its inverse coming from (3.39) take the form

$$K_{i\bar{j}} = \frac{1}{4}\begin{pmatrix} \frac{1}{\tau_f^2} & \gamma\frac{\tau_s^{3/2}}{\tau_f^{3/2}\tau_b^2} & -\frac{3\gamma}{2}\frac{\sqrt{\tau_s}}{\tau_f^{3/2}\tau_b} \\ \gamma\frac{\tau_s^{3/2}}{\tau_f^{3/2}\tau_b^2} & \frac{2}{\tau_b^2} & -3\gamma\frac{\sqrt{\tau_s}}{\sqrt{\tau_f}\tau_b^2} \\ -\frac{3\gamma}{2}\frac{\sqrt{\tau_s}}{\tau_f^{3/2}\tau_b} & -3\gamma\frac{\sqrt{\tau_s}}{\sqrt{\tau_f}\tau_b^2} & \frac{3\gamma\alpha}{2\mathcal{V}\sqrt{\tau_s}} \end{pmatrix} \tag{3.41}$$

and

$$K^{i\bar{j}} = 4\begin{pmatrix} \tau_f^2 & \gamma\tau_s^{3/2}\sqrt{\tau_f} & \tau_f\tau_s \\ \gamma\tau_s^{3/2}\sqrt{\tau_f} & \frac{\tau_b^2}{2} & \tau_b\tau_s \\ \tau_f\tau_s & \tau_b\tau_s & \frac{2}{3\gamma\alpha}\mathcal{V}\sqrt{\tau_s} \end{pmatrix}, \tag{3.42}$$

where we dropped all the terms that are suppressed compared to those shown by a factor $\sqrt{\tau_s/\tau_b}$. Then, we can write the scalar potential $V = V(\mathcal{V}, \tau_s, b_s)$ corresponding to $\mathcal{O}(\mathcal{V}^{-3})$ as

$$V = \frac{8a_s^2A_s^2\sqrt{\tau_s}}{3\alpha\gamma\mathcal{V}}\, e^{-2a_s\tau_s} + \frac{4W_0a_sA_s\tau_s}{\mathcal{V}^2}\, e^{-a_s\tau_s}\cos(a_sb_s) + \frac{3\hat{\xi}W_0^2}{4\mathcal{V}^3}. \tag{3.43}$$

Notice that, as expected, this potential leaves the directions of τ_f, b_b and b_s flat. Since we are considering $\mathcal{V}$ as a single modulus (parametrised mainly by τ_b), it is present also another combination of τ_f and τ_b, say χ, which can be regarded as parametrised mainly by τ_f. χ is independent of $\mathcal{V}$, and describes a direction along which (3.43) is completely flat.

The values of the fields that minimize the potential (3.43) can be found by solving

$$\begin{cases} \frac{\partial V}{\partial \mathcal{V}} = 0 \\ \frac{\partial V}{\partial \tau_s} = 0 \\ \frac{\partial V}{\partial b_s} = 0 \end{cases} \tag{3.44}$$

and keeping only the leading order in $a_s\tau_s \gg 1$, the vevs of the fields are given by

$$\langle\mathcal{V}\rangle \simeq \frac{3W_0\alpha\gamma}{4a_sA_s}\sqrt{\langle\tau_s\rangle}\, e^{a_s\langle\tau_s\rangle}, \qquad \langle\tau_s\rangle \simeq \left(\frac{\hat{\xi}}{2\alpha\gamma}\right)^{2/3}, \qquad \langle b_s\rangle = \frac{k_s\pi}{a_s} \quad k_s \in \mathbb{Z}. \tag{3.45}$$

Then, we can proceed with the derivation of the mass spectrum. Obviously, the directions which have not been lifted by (3.43) do not own a mass for now. Instead, for the fields stabilised by (3.43) an estimation of the masses can be performed as

$$\frac{m_{\mathcal{V}}^2}{M_p^2} \sim K_{b\bar{b}}^{-1}\frac{\partial^2 V}{\partial \mathcal{V}^2} \quad\Rightarrow\quad m_{\mathcal{V}} \sim \frac{M_p}{\mathcal{V}^{3/2}}, \tag{3.46}$$

$$\frac{m_{\tau_s}^2}{M_p^2} \simeq \frac{m_{b_s}^2}{M_p^2} \sim K_{s\bar{s}}^{-1}\frac{\partial^2 V}{\partial \tau_s^2} \quad\Rightarrow\quad m_{\tau_s} \simeq m_{b_s} \sim \frac{M_p \ln\mathcal{V}}{\mathcal{V}}. \tag{3.47}$$

Again we find that the the blow-up mode and its axionic partner acquire the same mass, therefore the axion b_s is quite heavy.

Let us now include the term of order $\mathcal{O}\left(\mathcal{V}^{-10/3}\right)$ into the scalar potential, giving a contribution of the form

$$\delta V_{g_s} = \left(\frac{\mathcal{A}}{\tau_f^2} - \frac{\mathcal{B}}{\mathcal{V}\sqrt{\tau_f}} + \frac{\mathcal{C}\,\tau_f}{\mathcal{V}^2}\right)\frac{W_0^2}{\mathcal{V}^2}. \tag{3.48}$$

This potential comes from string loop corrections and the string coupling g_s is involved in the parameters $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. $\mathcal{A}$ and $\mathcal{C}$ are defined positive, whereas the sign of $\mathcal{B}$ is undetermined [39]. Clearly, the potential (3.48) lifts the τ_f direction, so we are now able to stabilise it. Minimizing (3.48) with respect to τ_f gives

$$\tau_f^{-\frac{3}{2}} = \frac{\mathcal{B}}{8\mathcal{A}\mathcal{V}}\left[1 + (\text{sign}\mathcal{B})\sqrt{1 + \frac{32\mathcal{A}\mathcal{C}}{\mathcal{B}^2}}\right]. \tag{3.49}$$

If we take $\mathcal{B}^2 \gg 32\mathcal{A}\mathcal{C}$ [38], we find two different values of $\langle\tau_f\rangle$ according to the sign of $\mathcal{B}$:

$$\begin{aligned} \langle\tau_f\rangle &\simeq \left(-\frac{\mathcal{B}\mathcal{V}}{2\mathcal{C}}\right)^{2/3} \qquad \text{if} \quad \mathcal{B} < 0, \\ \langle\tau_f\rangle &\simeq \left(\frac{4\mathcal{A}\mathcal{V}}{\mathcal{B}}\right)^{2/3} \qquad \text{if} \quad \mathcal{B} > 0. \end{aligned} \tag{3.50}$$

Following [38], we choose the second case[1]. Now we can evaluate the second derivative of (3.48) in the vev and find the mass for τ_f, which has the form

$$\frac{m_{\tau_f}^2}{M_p^2} \sim K_{f\bar{f}}^{-1}\frac{\partial^2 V}{\partial\tau_f^2} \quad \Rightarrow \quad m_{\tau_f} \sim \frac{M_p}{\mathcal{V}^{5/3}}. \tag{3.51}$$

Therefore, τ_f turns out to be less massive than the other moduli, $m_{\tau_f} \ll m_{\mathcal{V}} < m_{\tau_s}$.

The next step consists in performing the canonical normalisation, by first neglecting string loop corrections and then including them. Our discussion will follow [45]. First of all, after fixing the blow-up mode at its vev and using the relation $a_s\langle\tau_s\rangle \simeq \ln\langle\mathcal{V}\rangle$, we can recast the potential (3.43) as

$$V \simeq \mathcal{J}\left[\frac{-\left(\ln\mathcal{V}\right)^{3/2} + \tilde{\xi}}{\mathcal{V}^3}\right], \tag{3.52}$$

[1] The authors of [38] choose the second case since the first leads to inflationary predictions which are in contrast with Planck data.

where we used $\mathcal{J} = \frac{2\alpha\gamma}{a_s^{3/2}}$ and $\tilde{\xi} = \hat{\xi}/\mathcal{J}$ for simplicity. Here the volume $\mathcal{V}$ still denotes $\mathcal{V} \simeq \sqrt{\tau_f}\,\tau_b$. The 2-dimensional inverse Kähler matrix takes the form

$$K^{i\bar{j}} = 4\begin{pmatrix} \langle\tau_f\rangle^2 & \frac{\sqrt{\langle\tau_f\rangle}}{2}(\ln\mathcal{V})^{3/2} \\ \frac{\sqrt{\langle\tau_f\rangle}}{2}(\ln\mathcal{V})^{3/2} & \frac{\langle\tau_f\rangle^2}{2} \end{pmatrix}. \tag{3.53}$$

The presence of a flat direction can be inferred by the fact that $\partial_f V = (\tau_b/2\tau_f)\,\partial_b V$. The only direction fixed by the potential is the one corresponding to the overall volume $\langle\mathcal{V}\rangle \simeq \exp\left(\tilde{\xi}^{3/2}\right)$ (in agreement with (3.45)). To derive the squared mass matrix, we must calculate the Hessian of the potential and evaluate it at the vevs:

$$\begin{aligned} V_{ff}\big|_{\text{VEV}} &= \frac{\langle\tau_b\rangle}{2\langle\tau_f\rangle}V_{fb}, \\ V_{bb}\big|_{\text{VEV}} &= \frac{2\langle\tau_f\rangle}{\langle\tau_b\rangle}V_{fb}, \end{aligned} \tag{3.54}$$

which give

$$V_{ij}\big|_{\text{VEV}} = \mathcal{J}\,\frac{9\sqrt{\ln\mathcal{V}}}{4\mathcal{V}^3}\begin{pmatrix} \frac{\langle\tau_b\rangle}{2\langle\tau_f\rangle} & 1 \\ 1 & \frac{\langle\tau_f\rangle}{2\langle\tau_b\rangle} \end{pmatrix}. \tag{3.55}$$

The subsequent mass-squared matrix reads

$$M_{ij}^2 = K^{ik}V_{kj}\big|_{\text{VEV}} = \mathcal{J}\,\frac{9\sqrt{\ln\mathcal{V}}}{2\mathcal{V}^3}\begin{pmatrix} 1+\varepsilon & 2\frac{\langle\tau_f\rangle}{\langle\tau_b\rangle}+\varepsilon \\ \frac{\langle\tau_b\rangle}{\langle\tau_f\rangle}+\varepsilon & 1+\varepsilon \end{pmatrix}, \tag{3.56}$$

with $\varepsilon \sim \mathcal{O}\left(\frac{\ln\mathcal{V}^{3/2}}{\mathcal{V}}\right) \ll 1$. It is immediate to see that the determinant of the mass-squared matrix vanishes. Thus, one of the two masses, say m_{Φ_f} is zero, as we expected from the presence of a flat direction. The other mass takes the value

$$\frac{m_{\Phi_b}^2}{M_p^2} \simeq \mathrm{tr}\left(M_{ij}^2\right) \simeq \frac{\sqrt{\ln\mathcal{V}}}{\mathcal{V}^3}. \tag{3.57}$$

The eigenvectors corresponding to these eigenvalues can be found by imposing (3.23). We can rewrite the original fields $\mathcal{V}$ and τ_f around the minimum in terms of the canonically normalised fields Φ_f and Φ_b as

$$\begin{aligned} \delta\tau_f &= -\frac{2}{\sqrt{3}}\langle\tau_f\rangle\,\delta\Phi_f + \sqrt{\frac{2}{3}}\,\langle\tau_f\rangle\,\delta\Phi_b, \\ \delta\tau_b &= \frac{\langle\tau_b\rangle}{\sqrt{3}}\,\delta\Phi_f + \sqrt{\frac{2}{3}}\,\langle\tau_b\rangle\,\delta\Phi_b. \end{aligned} \tag{3.58}$$

To make the physical interpretation of the two canonically normalised fields clearer, we notice that

$$\frac{\delta\mathcal{V}}{\mathcal{V}} = \frac{1}{2}\frac{\delta\tau_f}{\langle\tau_f\rangle} + \frac{\delta\tau_b}{\langle\tau_b\rangle} = \sqrt{\frac{3}{2}}\,\delta\Phi_b \quad \longrightarrow \quad \delta\Phi_b = \sqrt{\frac{2}{3}}\,\frac{\delta\mathcal{V}}{\mathcal{V}}, \tag{3.59}$$

which means that Φ_b plays the role of the overall volume. In turn, this is the reason why its mass is of the same order as (3.46). Moreover, if we recast the first equation in (3.58) as

$$\delta\Phi_b = \sqrt{\frac{3}{2}}\left(\frac{\delta\tau_f}{\langle\tau_f\rangle} + \sqrt{2}\,\delta\Phi_f\right), \tag{3.60}$$

and making use of (3.59), we find

$$\delta\Phi_f = \frac{1}{\sqrt{3}}\frac{\delta\mathcal{V}}{\mathcal{V}} - \frac{\sqrt{3}}{2}\frac{\delta\tau_f}{\langle\tau_f\rangle}, \tag{3.61}$$

that is, Φ_f cannot be expressed as a function of only the volume, as a term in τ_f is present as well. Therefore it turns out to be massless at this level of approximation, as expected.

Now we want to perform the same canonical normalisation but including string loop corrections. If we cast the potential in the previous form, we obtain

$$V \simeq \mathcal{J}\left[\frac{-(\ln\mathcal{V})^{3/2} + \tilde{\xi}}{\mathcal{V}^3} + \frac{y}{\mathcal{V}\sqrt{\tau_f}}\right], \tag{3.62}$$

where $y \simeq 1/\mathcal{J} \simeq (g_s \ln\mathcal{V})^{3/2}$. The differences with the previous case lie basically in the value of $\tilde{\xi}$. Since the introduction of g_s corrections in the potential induces not only a dependence on g_s in τ_f, but also a subleading dependence in $\mathcal{V}$, the volume minimum acquire a tiny τ_f-dependent shift. In [38] the loop-corrected value of $\tilde{\xi}$ is derived, and reads

$$\tilde{\xi}_{g_s} = \tilde{\xi} + \frac{3\,y}{\sqrt{\tau_f}}. \tag{3.63}$$

Evaluating the new mass-squared matrix in (3.63), we find two eigenvalues and this time both of them are non vanishing. In turn, the mass of Φ_f and Φ_b become

$$\begin{aligned} \frac{m^2_{\Phi_b}}{M_p^2} &\simeq \frac{\sqrt{\ln\mathcal{V}}}{\mathcal{V}^3}, \\ \frac{m^2_{\Phi_f}}{M_p^2} &\simeq \frac{1}{\mathcal{V}^3\sqrt{\tau_f}}. \end{aligned} \tag{3.64}$$

Notice that, due to $\langle\tau_f\rangle \simeq \mathcal{V}^{2/3}$, Φ_f acquires a mass of the same order as (3.51). Finally, the leading order string-loop-corrected canonical normalisation around the minimum looks like (3.58), but one can show that a subleading mixing is induced between Φ_f and any blow-up mode in the model [45].

Before proceeding with the next order in the $\mathcal{V}$ expansion of the scalar potential, let us summarise the mass spectrum of Fibre inflation as computed by now. All of the 3 moduli of the theory were stabilised and they acquired the following masses squared:

$$\begin{aligned} m_{\mathcal{V}}^2 &\sim \frac{M_p^2}{\mathcal{V}^3}, \\ m_{\tau_s}^2 &\sim \frac{M_p^2 \left(\ln \mathcal{V}\right)^2}{\mathcal{V}^2}, \\ m_{\tau_f}^2 &\sim \frac{M_p^2}{\mathcal{V}^{10/3}}. \end{aligned} \tag{3.65}$$

Instead, the only axion stabilised at this level is b_s, that is the one corresponding to the blow-up mode. After stabilisation this axion acquired a mass of the same order as τ_s, namely

$$m_{b_s}^2 \sim \frac{M_p^2 \left(\ln \mathcal{V}\right)^2}{\mathcal{V}^2}. \tag{3.66}$$

As already mentioned, this means that this axion is a rather heavy particle. Nevertheless, we still have two axions unstabilised, and since they will be stabilised at a higher perturbative level of the total potential with respect to b_s, they are expected to be lighter. Therefore, to see which kind of mass the remaining axions receive, we will proceed with the next and last perturbative order of the scalar potential, $\mathcal{O}\left(\mathcal{V}^{-4/3} e^{-\mathcal{V}^{3/2}}\right)$.

The two missing axions b_b and b_f enter the potential as additional non-perturbative corrections to W_0, that is

$$W \simeq W_0 + A_s e^{-a_s T_s} + A_b e^{-a_b T_b} + A_f e^{-a_f T_f}. \tag{3.67}$$

Making use of (3.67) and (3.39), we can write the total scalar potential. For simplicity, we will recast it as $V_{\text{tot}} = V_{\text{np1}} + V_{\text{np2}} + V_{\alpha'}$. The last term is given by (3.5) as usual. The other two terms take the form

$$\begin{aligned} V_{\text{np1}} =&\ e^K \left[K^{i\bar{j}} \left[a_i A_i a_j \bar{A}_j e^{-\left(a_i T_i + a_j \overline{T}_j\right)} \right]\right] \\ =&\ e^K \Bigg[\sum_{j=f,b,s} K^{jj} a_j^2 A_j^2 e^{-2a_j \tau_j} \\ &+ 2\, K^{fb} a_f A_f a_b A_b e^{-\left(a_f \tau_f + a_b \tau_b\right)} \cos(a_b b_b - a_f b_f) \\ &+ 2\, K^{fs} a_f A_f a_s A_s e^{-\left(a_f \tau_f + a_s \tau_s\right)} \cos(a_s b_s - a_f b_f) \\ &+ 2\, K^{bs} a_b A_b a_s A_s e^{-\left(a_b \tau_b + a_s \tau_s\right)} \cos(a_s b_s - a_b b_b) \Bigg] \\ \equiv&\ V_{\text{np1}}^{\text{mod}} + V_{\text{np1}}^{\text{ax}} \end{aligned} \tag{3.68}$$

and

$$
\begin{aligned}
V_{\text{np2}} = & -e^{K}\left[a_i A_i e^{-a_i T_i}\overline{W}\partial_{\bar{j}}K + a_j \bar{A}_j e^{-a_j \overline{T}_j} W \partial_i K\right] \\
= & \ 4\,e^{K}\sum_{j=f,b,s} a_j A_j W_0 e^{-a_j\tau_j}\cos(a_j b_j)\,.
\end{aligned} \tag{3.69}
$$

It is straightforward to see that not all the members of V_{tot} depend on an axionic term. Except for b_b and b_f, all the other fields are stabilised. Therefore, we can consider them as fixed in their vevs and restrict our discussion to $V_{\text{ax}} = V_{\text{np1}}^{\text{ax}} + V_{\text{np2}}$.

By minimizing V_{ax}, we find that the system

$$
\begin{cases} \frac{\partial V_{\text{ax}}}{\partial b_f} = 0 \\ \frac{\partial V_{\text{ax}}}{\partial b_b} = 0 \end{cases} \tag{3.70}
$$

has a solution if and only if the fields take the following values at the vevs:

$$
\langle b_f\rangle = \frac{k_f\pi}{a_f} \quad \text{and} \quad \langle b_b\rangle = \frac{k_b\pi}{a_b}, \qquad k_{f,b}\in\mathbb{N}. \tag{3.71}
$$

Then, we can estimate the masses for the fields b_b and b_f in the following way:

$$
\begin{aligned}
\frac{m_{b_f}^2}{M_p^2} &\simeq \frac{16 W_0 A_f a_f^3 \tau_f^3}{\mathcal{V}^2} e^{-a_f\tau_f} \sim e^{-\mathcal{V}^{2/3}}, \\
\frac{m_{b_b}^2}{M_p^2} &\simeq \frac{8 W_0 A_b a_b^3 \tau_b^3}{\mathcal{V}^2} e^{-a_b\tau_b} \sim \begin{cases} \mathcal{V}e^{-\mathcal{V}} & \text{if} \quad \tau_b\sim\tau_f \\ e^{-\mathcal{V}^{2/3}} & \text{if} \quad \tau_b\gg\tau_f \end{cases}.
\end{aligned} \tag{3.72}
$$

It turns out that the axions b_b and b_f are exponentially lighter than both b_s and the moduli. This represents a major feature of these fields and will be developed in the next Chapter. Notice that $m_{b_b}^2$ has two behaviours depending on the hierarchy between the modulus of the fibration and the modulus of the basis of the Calabi-Yau volume. When $\tau_b\sim\tau_f$, we say that the volume is *anisotropic*, whereas when $\tau_b\gg\tau_f$ the volume is *isotropic*.

We are now left with the canonical normalisation of the axionic fields. Since no mixing term between b_b and b_f is present in the Lagrangian, the procedure turns out to be rather simple. The kinetic terms of the Lagrangian for the axionic fields read

$$
\mathcal{L}_{\text{kin}}\left(b_f, b_b\right) = \frac{1}{4}\frac{\partial^2 K}{\partial\tau_f^2}\partial_\mu b_f \partial^\mu b_f + \frac{1}{4}\frac{\partial^2 K}{\partial\tau_b^2}\partial_\mu b_b \partial^\mu b_b, \tag{3.73}
$$

and the conditions for (3.74) to take the canonical form

$$
\mathcal{L}_{\text{kin}}\left(\phi_f, \phi_b\right) = \frac{1}{2}\partial_\mu\phi_f\partial^\mu\phi_f + \frac{1}{2}\partial_\mu\phi_b\partial^\mu\phi_b \tag{3.74}
$$

are

$$
\begin{aligned}
b_f &= \left(\frac{1}{2}\frac{\partial^2 K}{\partial \tau_f^2}\right)^{1/2} \phi_f = \sqrt{2}\,\tau_f \phi_f, \\
b_b &= \left(\frac{1}{2}\frac{\partial^2 K}{\partial \tau_b^2}\right)^{1/2} \phi_b = \phi_b \tau_b.
\end{aligned}
\tag{3.75}
$$

The fields b_b and b_f are dimensionless, while the canonically normalised fields ϕ_f and ϕ_b are given in M_p units.

3.4.1 A study on the axion decay constant

For later purposes it is worth spending a few words on the axion decay constant f. Let us discuss this topic considering only one axion. As well known, the axion enjoys a continuous shift symmetry with the form $b \to b + c$, where $c \in \mathbb{R}$, for instance $c = 2\pi$. Now, if we add a non-perturbative term to the superpotential W, this affects the shift symmetry of the axion. Consequently, the shift symmetry changes according to

$$
a\,b \to a\,b + 2\pi \quad \text{where} \quad a = \frac{2\pi}{N}, \;\; N \in \mathbb{N}, \tag{3.76}
$$

which in turn gives $b \to b + N$. Therefore, the shift symmetry is broken if we include non-perturbative corrections, since it varies from being continuous to being discrete.

When we canonically normalise the axionic field, we want the canonical normalised field ϕ to enjoy a shift symmetry as well. This symmetry is given by the transformation $\phi \to \phi + 2\pi f$, where the decay constant is defined as $f = \phi/\theta$. Here θ is the periodicity angle and f is given in M_p units. Therefore, the relation between the fields b and ϕ reads

$$
b = \frac{N\phi}{2\pi f} = \frac{\phi}{a\,f}, \tag{3.77}
$$

which leads to

$$
f = \frac{\phi}{a\,b}. \tag{3.78}
$$

Making use of the relations between the non-normalised and the canonically normalised fields (3.75), we can recast (3.78) as

$$
f = \left(\frac{1}{2}\frac{\partial^2 K}{\partial \tau^2}\right)^{-1/2} \frac{1}{a}. \tag{3.79}
$$

We recall that also in (3.79), f is given in M_p units. As we will see in the next Chapter, the relation (3.79) results particularly useful for the evaluation of the decay constants of the various models.

Finally, we mention that the relation (3.79) can be found in a more elegant way also via the Kaluza-Klein dimensional reduction of a 1-form.

Chapter 4

Fuzzy Dark Matter from the Large Volume Scenario

In this chapter we present the main result of this thesis, that is a survey of the possibilities to realise fuzzy dark matter (FDM) from the different models arising in the Large Volume Scenario (LVS). We will perform our study first in the well-known cases of single hole Swiss cheese (section 3.3) and Fibred Calabi-Yau (section 3.4), then we will try to generalise the results for every possible Calabi-Yau volume in the LVS.

We have already introduced FDM in section 1.3.2 and we came to the conclusion that it can represent a very promising DM candidate. To recap, with FDM we refer to models involving ULAs as CDM. In the present chapter we will consider specifically the ULAs arising from moduli stabilisation in type IIB string compactification performed in the LVS, namely the axions we found in Chapter 3. This plethora of axions is also referred to as LVS axiverse [34].

As already highlighted, the importance of FDM stands also on observational grounds. We have mentioned that in [29] the authors demonstrated that FDM with an axion mass in the range $m \sim 10^{-22} - 10^{-21}$ eV can provide a cosmic DM abundance of $\Omega_{DM} \sim 1$ in a natural way. In the same paper, they describe also very interesting astrophysical signatures and behaviours in support of FDM.

Very recently, following this work, many astrophysicists have tried to motivate their observational data on DM haloes with the presence of one, or more, ULAs. In the paper [46], the authors suppose the presence of an axion with mass $m_{18} \sim 10^{-18}$ eV, along with the axion of $m_{22} \sim 10^{-22}$ eV, as inferred from the study of the central dark compact mass of several globular clusters.

In another work [47], the authors find that the inner density profile of the nuclear star cluster in study can be fitted by a central DM presence corresponding to an axion of mass $m_{20} \sim 10^{-20}$ eV, along with the m_{22} axion (and the m_{18} axion). Moreover, they argue that the m_{22} axion is the dominant one, while the m_{20} axion would account only for a small fraction of the total DM observed. Consequently, they hypothesize a *nested* structure for the Milky Way. Here the m_{22} axionic DM makes up a structure with scale ~ 1 kpc, while the m_{20} axionic DM constitutes a smaller structure of ~ 1 pc embedded in the bigger m_{22} one. Such a concentric structure is also consistent with the observations of other galaxies, as the authors argue.

Now, our intent is to prove theoretically the hypothesis coming from the cited papers. We will examine the possibility for the axions coming from the LVS models to reproduce the astrophysical predictions. Then, we will try to figure out whether a FDM made of multiple ULAs is viable or not. The setup is the following:

- we consider the m_{22} axion as the major (almost the sole) DM constituent. Consequently, the m_{18} axion will be taken into account as a possible minor component;
- the background is the moduli stabilisation from type IIB string compactification in the LVS. Therefore, the Calabi-Yau volumes taken into account must obey the constraints given in section 3.2;
- we have to make a distinction between two cases arising from two different setups of the Calabi-Yau volume. We saw in the previous Chapter that the overall Calabi-Yau volume $\mathcal{V}$ is parametrised by a linear combination of moduli. Hence, it may happen that one modulus is much bigger than the others, leading to the so-called *isotropic case*. In the isotropic case we can approximate $\mathcal{V}$ as made up of only the leading modulus. Otherwise, if all the moduli result roughly of the same size, then we are in the so-called *anisotropic case*. As a consequence, we must consider each modulus constituting the overall volume $\mathcal{V}$ as having the same importance;
- the expected cosmic mass fraction in ALPs CDM is given by [34]

$$\frac{\Omega_{b_i} h^2}{0.112} \simeq 1.4 \left(\frac{m_i}{\text{eV}}\right)^{1/2} \left(\frac{f_i}{10^{11}\ \text{GeV}}\right)^2 \left(\frac{\theta_i}{\pi}\right)^2 , \tag{4.1}$$

  where i runs over all the axions present in the model under analysis, f is the decay constant as found at the end of section 3.4 expressed in GeV, θ is the initial misalignment angle and it is a free parameter, since ALPs are very likely to be produced before inflation (see section 1.3.2).

The following step is to consider the models reviewed in the previous Chapter, that is single hole Swiss cheese and Fibred Calabi-Yaus, together with an extension of the Fibred case. Thanks to the relation in (4.1), we will also be able to study how the axions coming from these models behave as DM. Finally, we will generalise our results for the most general form that a Calabi-Yau volume can have in the LVS.

4.1 Swiss cheese Calabi-Yaus

Let us recall the explicit expression for the Calabi-Yau volume in the single hole Swiss cheese model, that is

$$\mathcal{V} \simeq \tau_b^{3/2} - \tau_s^{3/2}. \tag{4.2}$$

In this case we have only the isotropic case, since the volume is parametrised only by τ_b, *i.e.* $\mathcal{V} \sim \tau_b^{3/2}$. We disregard the axionic partner of τ_s, since it is too heavy to behave as FDM. An estimation of the mass of the other axion present, that is b_b, was performed in (3.33), where it was found that

$$m_b^2 \simeq \frac{16 A_b W_0}{3\mathcal{V}^2} \left(a_b \tau_b\right)^3 e^{-a_b \tau_b} M_p^2. \tag{4.3}$$

The decay constant can be derived following (3.79), which gives in this case

$$f_b = \left(\frac{1}{2}\frac{\partial^2 K}{\partial \tau_b^2}\right)^{-1/2} \frac{1}{a_b} = \sqrt{\frac{3}{2}} \frac{M_p}{a_b \tau_b}. \tag{4.4}$$

We consider θ_b to acquire a natural value at first, that is $\bar{\theta}_b = \frac{\pi}{\sqrt{3}}$. We will see later whether the tuning of this parameter is worthy or not.

Now we have all the ingredients to perform the calculation for (4.1). If we impose $m_b \equiv m_{22}$ and we require that the axion b_b represents the total DM abundance, that is

$$\frac{\Omega_{b_b} h^2}{0.112} \simeq 1,$$

substituting in (4.1) the values for f_b and θ_b, we find that

$$a_b \tau_b = 64.42. \tag{4.5}$$

Let us now recall the relation (3.1) seen in the previous Chapter, that is

$$W = W_0 + W_{np} = W_0 + \sum_i A_i \, e^{-a_i T_i}. \tag{4.6}$$

It is clear that the tree-level superpotential W_0 must have a value that is, if not natural, at least much larger than the value of the perturbative corrections. If we substitute the result (4.5) into (4.6), we find that the perturbative correction scales as

$$W_{np} \sim A_b \, e^{-a_b \tau_b} \sim 10^{-28}. \tag{4.7}$$

As a consequence, we must have a value of W_0 that is at least 3 orders of magnitude bigger than 10^{-28} for the sake of consistency. At this point, if we replace the value (4.5) into (4.3), we come to the following relation

$$\frac{W_0}{\mathcal{V}^2} \simeq 10^{-77}. \tag{4.8}$$

Since $\mathcal{V}^2 \sim \tau_b^3 \sim 2.67 \times 10^5 \, a_b^{-3}$, the resulting value for W_0 is extremely unnatural, also in the highest case of $a_b = 1$. Surely, trying to tune θ_b very low in (4.1) will not help us enough. For this reason, it seems very likely that this case has to be disregarded.

As an additional proof, one can follow the other way around and calculate how much abundance of DM the axion b_b can account for. First of all, we consider again $m_b \equiv m_{22}$. If we take natural values, *i.e.* $\mathcal{O}(1)$, for W_0 and A_b, we find that $a_b \tau_b \simeq 229$. Therefore, plugging this value into (4.4) and then into (4.1), the DM abundance for b_b is

$$\frac{\Omega_{b_b} h^2}{0.112} \simeq 0.08.$$

This means that the axion b_b with mass $m_b = m_{22}$ can account only for the 8% of the total DM abundance we observe today.

4.2 Fibred Calabi-Yaus

As in the previous section, let us start by recalling the form assumed by the Calabi-Yau volume in the fibre inflation case, that is

$$\mathcal{V} \simeq \sqrt{\tau_f}\, \tau_b - \tau_s^{3/2}. \tag{4.9}$$

Both the isotropic and anisotropic cases may exist, therefore we must consider both when doing the calculation. The isotropic case is given by $\mathcal{V} \sim \tau_b^{3/2}$, whereas the anisotropic case reads $\mathcal{V} \sim \sqrt{\tau_f}\, \tau_b$.

In the case at study, three axions are present. As seen in section 3.4, the axionic partner of τ_s is very massive, so we can again neglect it. Thus, the axions we have to deal with are b_f and b_b. The explicit expressions for their masses was found in (3.72) and they read

$$m_f^2 \simeq \frac{16 W_0 A_f a_f^3 \tau_f^3}{\mathcal{V}^2} e^{-a_f \tau_f} M_p^2, \tag{4.10}$$

$$m_b^2 \simeq \frac{8 W_0 A_b a_b^3 \tau_b^3}{\mathcal{V}^2} e^{-a_b \tau_b} M_p^2. \tag{4.11}$$

We consider the axion b_b as the leading contribution to the total DM abundance, that is $m_b \equiv m_{22}$, and b_f as representing the minor DM fraction with $m_{18} \equiv m_f$. Specifically, we will take into account the following relation between the two DM abundances,

$$\frac{\Omega_{22}}{\Omega_{18}} \simeq 10^2. \tag{4.12}$$

The decay constants can be found by making use of the relation (3.79) and they read

$$f_b = \frac{M_p}{a_b \tau_b}, \tag{4.13}$$

$$f_f = \frac{M_p}{\sqrt{2}\, a_f \tau_f}. \tag{4.14}$$

Let us start the handling with the b_b axion. We select a natural value for θ_b and impose $m_b \equiv m_{22}$. Then, plugging these values and the value of f_b from (4.13) into (4.1), by requiring that b_b represents the total DM abundance, we find that

$$a_b \tau_b = 52.6. \tag{4.15}$$

From (4.15), we can derive that non perturbative corrections scale as

$$W_{np} \sim A_b \, e^{-a_b \tau_b} \sim 10^{-23}, \tag{4.16}$$

hence the tree-level potential W_0 must be at least 3 orders of magnitude bigger than 10^{-23}. Moreover, reversing the relation (4.11) and substituting the value (4.15), we find the ratio

$$\frac{W_0}{\mathcal{V}^2} \sim 10^{-82}. \tag{4.17}$$

Now, we must discriminate between the isotropic and anisotropic cases. In the isotropic case we find basically the same theoretical results as in the Swiss cheese case. Let us consider a natural value for A_b. The isotropic volume is given by $\mathcal{V}^2 \sim \tau_b^3 \sim 1.455 \times 10^5 \, a_b^{-3}$. When plugging this value into (4.17), an unnatural value of W_0 is found. Also, a fine tuning of the parameter θ_b would not be helping. Therefore, we can conclude that in general the isotropic case is not able to provide a viable FDM candidate that can account for the total DM abundance. From now on, we will neglect the isotropic case of the Calabi-Yau volume.

In the anisotropic case, the volume is given by $\mathcal{V}^2 \sim \tau_f \tau_b^2$, where τ_b is fixed to the value (4.15) and the values of τ_f and θ_f are free for now. The relation between W_0 and $\mathcal{V}^2$ from (4.17) still holds. At first instance, let us consider $m_f \equiv m_{18}$, so that to reproduce the astrophysical data from the cited papers. Imposing the relation (4.12), we find that

$$\frac{a_f \tau_f}{\theta_f} \simeq 2052. \tag{4.18}$$

If we solve $\mathcal{V}^2 \sim \tau_f \tau_b^2$ for the values (4.15) and (4.18), we fall again in the same problem as the isotropic volume, that is $\mathcal{V}$ is not big enough to lead to a natural value for W_0. For this reason, we have to look at the problem from another perspective.

For the consistency of the model, we require $W_0 \geq 10^{-20}$. Due to the relation (4.17), this requirement leads to have a volume that is at least $\mathcal{V} \sim 10^{30}$, or more. Then, since the value of τ_b is fixed, τ_f must provide the orders of magnitude missing to make up such a big volume. Consequently, from (4.10), we can conclude that the axion b_f is massless. In turn, a value of τ_f that is extremely larger than the value of τ_b leads to a geometrical setup for the volume which has a small base and a really long fibration. To sum up, if we require that the axion with $m \sim 10^{-22}$ eV is responsible of nearly the total abundance of DM observed, we must have a massless axion in the model. That is, the Fibre inflation model can provide only one axion as a FDM candidate.

An interesting feature coming from the previous calculations is that, if we require that $\mathcal{V} \sim 10^{30}$, the string scale reads

$$M_S \sim \frac{M_p}{\sqrt{\mathcal{V}}} \sim 2\,\mathrm{TeW}, \tag{4.19}$$

and so string physics could be soon tested in collider experiments.

Moreover, we investigated an additional intriguing aspect. We performed again the above calculations for a hypothetical mass of $m \simeq d \times 10^{-22}$ eV, where d is a free parameter. This value leads to a change of (4.15) to $a_b\tau_b = d^{1/4} \times 52.6$. If we suppose that $d = 10$, so that we have an axion with $m \sim 10^{-21}$ eV following [29], we come to the relations

$$a_b\tau_b = 93.5 \quad \Longrightarrow \quad \frac{W_0}{\mathcal{V}^2} \sim 10^{-63}. \tag{4.20}$$

This result is very interesting because it leads to two possible ways to face it. One can choose the most natural and common value of W_0, that is $W_0 \sim \mathcal{O}(1)$, and hence have the string scale around the TeV scale. Otherwise, one can lift the value of the string scale over the TeV for a few orders of magnitude while keeping W_0 in a natural regime as well. Therefore, we can conclude that an axion with mass $m \sim 10^{-21}$ eV follows in a quite natural way from these assumptions.

4.3 Extended Fibred case

Before generalising the results to a generic Calabi-Yau volume, we perform the calculations also on a Calabi-Yau volume that can be regarded as an extension of the Fibre inflation one and it reads

$$\mathcal{V} \simeq \sqrt{\tau_1\tau_2\tau_3} - \tau_s^{3/2}, \tag{4.21}$$

where the overall volume is now parametrised by the linear combination of the three moduli τ_1, τ_2 and τ_3. The Large volume limit leads to the regime $\mathcal{V} \simeq \sqrt{\tau_1\tau_2\tau_3} \gg \tau_s^{3/2}$. Let us consider only the anisotropic case, which is reasonable since, as seen previously, the isotropic case is not able to provide a good fitting with observable data.

The Kähler matrix derived from the above expression of the Calabi-Yau volume reads

$$K^{i\bar{j}} = 4\begin{pmatrix} \tau_1^2 & \frac{\sqrt{\tau_1\tau_2}}{\sqrt{\tau_3}}\tau_s^{3/2} & \frac{\sqrt{\tau_1\tau_3}}{\sqrt{\tau_2}}\tau_s^{3/2} & \tau_1\tau_s \\ \frac{\sqrt{\tau_1\tau_2}}{\sqrt{\tau_3}}\tau_s^{3/2} & \tau_2^2 & \frac{\sqrt{\tau_2\tau_3}}{\sqrt{\tau_1}}\tau_s^{3/2} & \tau_2\tau_s \\ \frac{\sqrt{\tau_1\tau_3}}{\sqrt{\tau_2}}\tau_s^{3/2} & \frac{\sqrt{\tau_2\tau_3}}{\sqrt{\tau_1}}\tau_s^{3/2} & \tau_3^2 & \tau_3\tau_s \\ \tau_1\tau_s & \tau_2\tau_s & \tau_3\tau_s & \frac{2}{3}\left(\mathcal{V}\sqrt{\tau_s} - \tau_s^2\right) \end{pmatrix}. \tag{4.22}$$

The masses of the thee axions b_1, b_2 and b_3 can be calculated following

$$\frac{m_{b_i}^2}{M_p^2} \sim K^{i\bar{i}}\frac{\partial^2 V}{\partial\tau_i^2}, \tag{4.23}$$

which read

$$m_{b_i}^2 \simeq \frac{16W_0A_i\left(a_i\tau_i\right)^3}{\mathcal{V}^2}e^{-a_i\tau_i}M_p^2, \qquad i=1,2,3. \tag{4.24}$$

We will not discuss the axionic partner of τ_s because it results in being too heavy to be regarded as a viable FDM constituent. The decay constants come from the relation (3.79) and take the form

$$f_i = \frac{M_p}{\sqrt{2}\,a_i\tau_i}, \qquad i=1,2,3. \tag{4.25}$$

Previously, we demonstrated that if we want the axion with mass $m_{22} \sim 10^{-22}$ eV to account for nearly the total observed DM, then we must have one massless axion. In this case, we choose b_3 as the massless axion of the model. Thus, we select b_2 as the axion with mass $m_2 \equiv m_{22}$ and b_1 as the axion with $m_1 \equiv m_{18}$. Now, let us approach the discussion imposing a hierarchy between the decay constants f_1 and f_2.

The fact that $m_{22} \equiv m_2 < m_1 \equiv m_{18}$ leads to the relation between the two moduli $a_2\tau_2 > a_1\tau_1$, which follows from the equation (4.24). In turn, it would give rise naively to a hierarchy between the decay constants with the form $f_1 > f_2$. If this relation were acceptable, by imposing the ratio between the DM abundance of the m_{22} axion and the DM abundance of the m_{18} axion to be nearly 10^2 as in (4.12), it would result in

$$\left(\frac{m_{22}}{m_{18}}\right)^{1/2}\left(\frac{f_2}{f_1}\right)^2\left(\frac{\theta_2}{\theta_1}\right)^2 = 10^2. \tag{4.26}$$

Then, given the constraints above and choosing a natural value for the initial misalignment angle θ_2, *i.e.* $\bar{\theta}_2 = \pi/\sqrt{3}$, in order to have the required value for the ratio, θ_1 must be tuned small, *i.e.* $\theta_1 \ll \bar{\theta}_2 = \pi/\sqrt{3}$. Nevertheless, this is not a big issue and the result can be explained in a very natural way by the inflationary selection already discussed in section 1.3.2.

However, the astrophysical observations supporting that the m_{22} axion constitutes nearly the total DM abundance does not go along with the relation $f_1 > f_2$. As a matter of fact, this relation means that the m_{18} axion is cosmologically produced before the one with m_{22}. Hence, how could it be possible that the observed presence of the m_{18} axion is so small compared to the one of the m_{22} axion?

To have a sensible cosmological selection we must require that $f_2 > f_1$. We intend now to show how this hierarchy can be achieved. So far, we considered only the F-term scalar potential, because it is the potential involved in the problem of Kähler moduli moduli stabilisation. We took for granted that the D terms were stabilised due to the fact that they scale as $\mathcal{V}^{-2}$ whereas the F terms scale as $\mathcal{V}^{-3}$, thus the D terms are stabilised before the F terms. However, the D-term scalar potential is useful because it induces a relation between the moduli.

The presence of non-vanishing gauge fluxes on the divisors generates a D-term scalar potential of the form

$$V_D = \frac{1}{Re(f)} \left(\xi_{FI} + \tilde{q}_{ij} \varphi^i \frac{\partial K}{\partial \varphi^i} \right)^2 , \tag{4.27}$$

where j runs over the number of the divisors supporting non-vanishing gauge fluxes, f is the gauge kinetic function, φ^i are matter fields with charge $\tilde{q}_{ij}$ under the $U(1)$ gauge symmetry. The FI term reads

$$\xi_{FI} = \frac{1}{4\pi \mathcal{V}} \sum_{i}^{h_{1,1}} q_{ij} t_i , \tag{4.28}$$

where q_{ij} is the $U(1)$-charge induced on the Kähler moduli by the presence of non-vanishing gauge fluxes. Then, if $\langle \varphi \rangle = 0$ for each charged matter field, D-term stabilisation leads to

$$V_D = 0 \quad \text{iff} \qquad \xi_{FI} = 0, \tag{4.29}$$

and so we find

$$\xi_{FI} = 0 \qquad \Longrightarrow \qquad \sum_{i}^{h_{1,1}} q_{ij} t_i = 0. \tag{4.30}$$

Recalling that the 2-cycles are related to the Kähler moduli following (2.91), it means that we have a stabilised linear combination between the moduli. Thus, D-term stabilisation introduces a proportionality between the moduli such as $\tau_2 = r\,\tau_1$, where $r \in \mathbb{R}^+$. Notice that, after D-term stabilisation, we recover the usual form of the Fibre inflation volume. Now, if we set $a_2 = m\, a_1$, where $m \in \mathbb{Z}$, we can express one decay constant in function of the other as

$$f_2 = \frac{f_1}{mr}, \tag{4.31}$$

leading to the relation between the moduli

$$mra_1\tau_1 = a_2\tau_2. \tag{4.32}$$

Thus, our need to have $f_2 > f_1$ translates into having $0 < mr < 1$. If we compare the relations for the masses of the axions b_1 and b_2 we can see that

$$\frac{m_{22}^2}{m_{18}^2} \equiv 10^{-8} = \frac{A_2}{A_1} \frac{(a_1\tau_1 mr)^3}{(a_1\tau_1)^3} e^{-a_1\tau_1(mr-1)}. \tag{4.33}$$

In turn, this means that the correct hierarchy between the masses is attained once we consider the prefactors so that $A_2 < 10^{-8} A_1$. This tuning of the prefactors allows for a reasonable cosmological selection.

The requirement $\frac{\Omega_{22}}{\Omega_{18}} \simeq 10^2$ is readily met using a natural value both for θ_1 and θ_2, *i.e.* $\theta_1 \simeq \theta_2 \simeq \pi/\sqrt{3}$,

$$\frac{\Omega_{22}}{\Omega_{18}} \simeq 10^2 \simeq \left(\frac{m_{22}}{m_{18}}\right)^{1/2} \left(\frac{f_2}{f_1}\right)^2 \simeq 10^{-2}(mr)^{-2}, \tag{4.34}$$

and it leads to $mr \simeq 10^{-2}$. Now we are able to perform the usual procedure of calculation. First, we solve

$$\frac{\Omega_{22}h^2}{0.112} \equiv 1 \simeq 1.4 \left(\frac{m_{22}}{\text{eV}}\right)^{1/2} \left(\frac{f_2}{10^{11}\ \text{GeV}}\right)^2 \left(\frac{\theta_2}{\pi}\right)^2, \tag{4.35}$$

which leads to $a_1\tau_1 = 3719.5$. To satisfy the required hierarchy between the prefactors, let us choose the values $A_1 = 1$ so that $A_2 \simeq 10^{-9}$. Plugging the results for mr and $a_1\tau_1$, as well as $A_2 \simeq 10^{-9}$ into the relation (4.24) for the m_{22} axion, we come to the usual ratio between the tree-level superpotential W_0 and the overall volume $\mathcal{V}$, which in this case is given by

$$\frac{W_0}{\mathcal{V}} \sim 10^{-79}. \tag{4.36}$$

The leading non-perturbative correction to W_0 follows from $W_{np2} = A_2\, e^{a_2\tau_2} \sim 10^{-26}$. For consistency, we must consider $W_0 \sim 10^{-22}$ at least, and consequently we find that the overall volume scales as $\mathcal{V} \sim 10^{28}$. Remarkably, this value for the volume leats to a string scale M_S which reads

$$M_S \sim \frac{M_p}{\sqrt{\mathcal{V}}} \simeq 24\,\text{TeV}. \tag{4.37}$$

The above value for the string scale is quite interesting, since it is higher than the energy scale at tested at the LHC, but it could allow for a future collider detection of string theory.

4.4 General case

The purpose of this section is to generalise the results accomplished in the previous sections to the most general case that should be able to provide viable FDM candidates from LVS moduli stabilisation. The general Calabi-Yau volume required takes the form

$$\mathcal{V} \simeq f_{\frac{3}{2}}\big(\tau_1, \ldots, \tau_{h_{1,1}-1}\big) - \tau_{\text{blow-up}}^{3/2}, \tag{4.38}$$

where the first term indicates a generic homogeneous function of $h_{1,1}-1$ moduli of degree $3/2$. This function must depend on at least two moduli due to the fact that, to realise FDM made almost entirely of an axion with $m \sim 10^{-22}$ eV, a massless axion must be present. In addition, the presence of at least one blow-up modulus is also necessary to ensure the viability of the Large volume limit.

The compactification must be anisotropic, that is the overall volume in the Large limit must exhibit a hierarchy among the moduli and none of them can be neglected. Of course, the massless modulus is the leading contribution to $\mathcal{V}$, which is the feature that allows for the presence of an exponentially Large volume and consequently a consistent tree-level superpotential that overcomes the non-perturbative corrections of at least three orders of magnitude. However, all the other moduli contributing to $\mathcal{V}$ have to be taken into account. One of these moduli shall represent the m_{22} axion which makes up almost the total DM abundance we observe today. It is noteworthy to recall from section 4.2 that if the m_{22} axion is the only massive axion present in the model, then with the Fibre inflation model we can reproduce it well enough. Moreover, if the massive axion has a mass which scales not as 10^{-22} eV but as 10^{-21} eV, Fibre inflation provides it in a very elegant and natural way. This is a rather interesting result.

Besides the modulus giving the massless axion and the modulus responsible for the m_{22} axion, the other moduli included in $\mathcal{V}$ can provide axions with masses 10^{-18} eV, 10^{-20} eV and so on. The derivation of their essential parameters can be done by making use of the equation (4.1) where the observed critical density must be plugged in Ω_{b_i}. Otherwise, using again the relation (4.1) but on the way round, if one knows the parameters f and m for a certain axion, assuming a reasonable value for θ misalignment, then it is possible to make a prediction on its cosmological abundance.

If we want the axiverse to be generated following a cosmological selection (as it is reasonable), we found that in this setup the presence of a hierarchy among the prefactors A is necessary. In particular, we expect the axions which contribute more relevantly to DM to have smaller prefactors with respect to the axions contributing less. Namely, given the axions b_1 and b_2 with critical densities Ω_1 and Ω_2 respectively, if $\Omega_1 < \Omega_2$ then $A_1 > A_2$. Since the physics and the mathematics beyond these prefactors is not well understood yet, we are not able to provide a physical interpretation of this hierarchy for now.

The last interesting feature is that this exponentially Large volume (we found $\mathcal{V} \sim 10^{30}$ in all the studied models) is able to provide a TeV string scale, namely equal or just slightly higher than the LHC scale nowadays. Consequently, we achieved a phenomenological result that can be tested, sooner or later.

Chapter 5

Conclusions and Outlook

In this thesis we tried to provide a theoretical explanation of recent astrophysical observations of the DM composition. These data seem to give the evidence for the DM to be made of one or some ULAs, where the lighter one makes up almost all the DM abundance, whereas others ULAs contribute less. Therefore, we reviewed the viability of these ULAs into the moduli stabilisation paradigm performed in the LVS. Before showing our result, let us recall what we did in this thesis.

In Chapter 1 we first introduced the DM problem from a historical point of view. Then, we reviewed many of the candidates proposed throughout the years, explaining briefly why they cannot be taken into account as leading contributions to the total DM abundance as observed by Planck and WMAP. Therefore, the last section is dedicated to the only two proposals still holding, that is WIMPs and ALPs coming respectively from supersymmetry and string theory, two of the most important theories which try to provide a UV completion to the SM. We highlighted their benefits and drawbacks and we showed that ALPs can handle the problem affecting WIMPs. We closed the Chapter with a review of ALPs, introducing the ULAs which lately have been proposed as DM constituent, hence making up the so-called FDM.

An overview of supersymmetry and string theory is provided in Chapter 2. We started with an introduction to supersymmetry and its basic features, that is superspaces, superfields, supermultiplets and supersymmetry breaking, delving into these concepts with the examples of chiral and vector superfield. Then, we analysed supergravity, the local extension of supersymmetry, and we derived its scalar potential, which is of major importance in the moduli stabilisation problem. Finally, we presented a brief review of string compactification, focusing on the features of the 4D effective field theory. In particular, we studied the derivation of the moduli from type IIB string compactification as well as their geometrical features.

Then, in Chapter 3 we faced the problem of moduli stabilisation, which consists in giving a positive mass-squared to the Kähler moduli by adding perturbative and non-perturbative corrections to the Kähler potential and the superpotential, respectively. In doing this, we worked within the regime known as LVS, which consists in taking the Large limit of the Calabi-Yau volume, thus stabilising the Kähler moduli thanks to a competition in the scalar potential between perturbative and non-perturbative corrections. Then, the LVS procedure is presented with explicit calculations in the well known cases of single-hole Swiss cheese and Fibre inflation. Furthermore, we studied also the axionic partners of the Kähler moduli present in these models.

Chapter 4 contains the original work of this thesis, consisting in the theoretical derivation and explanation of the axionic DM as observed by some groups of astrophysicists. Recently, a number of papers drew attention to certain data coming from DM haloes in galactic clusters, which can be explained with FDM made up of ULAs with masses in the range $10^{-18} - 10^{-22}$ eV. Moreover, the authors hypothesize also the presence of a hierarchy among such axions, namely the less massive one seems to be the major constituent of the total DM abundance observed. We set off from these values and we tried to justify them within the LVS. We found that, by making certain assumptions, these axions may exist in the LVS Axiverse. The assumptions needed are basically two:

- there must be present at least one massless axion in the model, so as to ensure consistency between the tree-level superpotential and its non-perturbative corrections by having an exponentially (extremely) Large volume,

- if the ULAs taken into account are more than two, a hierarchy among the prefactors of the massive ones must exist if one requires a reasonable cosmological selection for their decay constants.

Another noteworthy result is that in the Fibred Calabi-Yau models with one massless axion, one massive axion with $m \sim 10^{-21}$ eV can be explained in an elegant and natural way. Consequently, it would be interesting to fit the observed data with this axionic mass range, in order to see whether it can provide the total DM abundance by itself or not.

Our results also open up to two other stimulating questions. Now we intend to describe briefly these two topics, hoping to address in greater detail these issues in the future.

The first topic is the interplay between FDM and the so-called Dark Radiation (DR). DR can be seen as an additional relativistic matter component of the universe energy density. So far we considered only ULAs produced via the misalignment mechanism. Although being very light, such ULAs are not relativistic because they are produced dynamically, meaning that they are provided with extremely little momenta. This is the reason why they can be viable DM constituents. Nevertheless, there is another means of production for the ULAs, that is the inflaton decay or the decay of the lightest modulus which triggers the reheating after inflation. Since the inflaton and moduli in general are way more massive than ULAs, the latter are produced with a huge momentum. As a consequence, these ULAs are ultra-relativistic and, due to their exponentially light masses, they never slow down. Therefore, if DM is possible, why not DR?

As a matter of fact, there is no reason to assume *a priori* that the present-day radiation content of the universe must be made of only photons and neutrinos. DR can be seen as an excess of the effective number of neutrino species ΔN_{eff}, since $\rho_{DR} \sim \rho_\gamma \, \Delta N_{eff}$ [48]. The existence of ΔN_{eff} comes from the fact that in the SM the effective number of neutrino species is $N_{eff} = 3$ at the Big Bang nucleosynthesis time and $N_{eff,SM} = 3.046$ at the CMB time, due to the reheating after electron-positron pair-annihilation, whereas present cosmological observations do not provide a clear indication for the value of N_eff. Instead, it seems that on the average, N_eff could be larger than the SM one. Notice that $\Delta N_{eff} = N_{eff} - N_{eff,SM} > 0$ would indicate the presence of additional DR decoupled from the SM, which is relative at both the Big Bang nucleosynthesis and the CMB temperature [49]. Notice that the contribution coming from DR is lower than the ordinary radiation. Hence, an important challenge in studying DR is to control its branching ratio to prevent an over-production of ULAs and consequently $\Delta N_{eff} \gg 1$, which is in contrast with Big Bang nucleosynthesis predictions. This is where our results come in. One can use the interplay between DM and DR to constrain the ULAs making up DR by producing the majority of ULAs from the misalignment mechanism before inflation, in such a way that few moduli are left to decay after inflation. Therefore, a detailed study of the interplay between DM and DR seems in place.

The second possible outlook concerns Dark Energy (DE). As we saw in Chapter 4, one of the requirement for LVS models to reproduce FDM is the presence of at least one massless axions. Massless axions would then contribute to the vacuum energy of the universe and in turn they can be regarded as possible DE constituents as quintessence fields. As well known, an important feature of axions is their shift symmetry. In the case of axionic quintessence, the shift symmetry can protect such a light axionic mass and prevent it from getting large quantum corrections at perturbative level, in such a way to avoid one of the main challenges of constructing quintessence models. Moreover, if the quintessence field is a pseudo-scalar like an axion, it can avoid fifth-force constraints. However, a massless axion is supposed to have a trans-Planckian decay constant, as one can readily infer from the dissertation in Chapter 4. Recently in the work [50], the authors showed that the decay constant f depends on the ratio between the scale of the non-perturbative effect that gives mass to the axion and the cosmological constant, where the latter is no longer constant in quintessence models. Therefore, f can be either trans-Planckian or sub-Planckian depending on the values of these two scales. In this way, a ULA (or an almost massless axion as in our case) could both have a trans-Planckian decay constant and provide a late-time acceleration as well. We conclude that our results are able to provide a theoretical derivation of an extremely light axionic field from observational constraints. It would be very interesting to test the massless ULAs coming from our calculation in the DE scenario as described in [50] and we hope to address this topic soon.

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I want to close this thesis with some of the most meaningful words to me:

> *It's hard to stay mad when there's so much beauty in the world. Sometimes I feel like I'm seeing it all at once, and it's too much. My heart fills up like a balloon that's about to burst. And then I remember to relax, and stop trying to hold on to it, and then it flows through me like rain, and I can't feel anything but gratitude for every single moment of my stupid little life.*

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