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INFLATION AND PRIMORDIAL BLACK HOLES IN MATTER DOMINATION

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Abstract

The nature of dark matter is still today one of the biggest mysteries in physics. Most of the efforts made in the last decades focused on explaining it in terms of non-baryonic particles but, to date, any attempt to detect them has failed. In this thesis we will study a different interpretation of dark matter in terms of Primordial Black Holes (PBHs), i.e. black holes that are believed to form when large density perturbations, produced during the inflationary epoch, reenter the horizon and collapse because of their own gravitational force. Black holes produced in this way are referred to as “Primordial” since their production mechanism has a primordial origin during inflation. These large density fluctuations are produced if the inflationary potential V possesses enough tuning freedom to feature a slow-roll plateau followed by a near inflection point that greatly enhances the power spectrum of scalar perturbations. We examine PBH formation during a radiation dominated (RD) epoch and during a matter dominated (MD) epoch driven by a gravitationally coupled scalar field φ (modulus) which decays before Big Bang Nucleosynthesis in order to preserve its successful predictions. We will require in both cases that the produced PBHs constitute 100% of dark matter today. In the MD case, we find that the mass m_φ of the scalar field affects the enhancement required in the curvature power spectrum: for large modulus masses, this enhancement turns out to be smaller than the one in the RD case, meaning that in the case of horizon reentry during moduli domination the potential requires less tuning to produce the same amount of PBHs. Therefore, we focus on PBH production during a MD epoch. We then introduce a model of string inflation called “Fibre Inflation”, that works particularly well for our purposes since it naturally leads to a post-inflationary epoch of MD driven by an axion-like modulus. In this model the inflationary potential has enough tuning freedom to induce a period of ultra slow-roll that enhances the density perturbations at the required PBH scales: we employ this potential to obtain a numerical estimate of the scalar power spectrum and the other inflationary observables.

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Chapter 1

Inflation and PBH Dark Matter

1.1 Dark Matter: a brief introduction

Dark Matter (DM) is defined as hypothetical matter that results invisible (i.e. "dark") in the entire electromagnetic spectrum: it does not interact with electromagnetic waves, regardless of their energy, making its gravitational effects on the surroundings the only way to detect it indirectly.

The hypothesis of DM has a long history that we will briefly summarize (see Ref. [1] for details); it has its roots in two main astronomical anomalies, initially observed decades ago:

1. *Flat galaxies rotation curves.* A galaxy rotation curve is a plot of the galaxy's stars (and gas) orbital velocity versus their radial distance from the center of the galaxy. The firsts observations on rotation curves were made in the '60s, after the development and improvement of radio astronomy, but it was in the early '70s that the studies became enough to reveal a strange feature: rotation curves tend to be flat. To understand why this feature was, and still is, quite surprising, we need to imagine our galaxy as a non-solid disk made of stars and gas, where most of the mass is concentrated in the center: the rotational curve of such a disk is expected to decrease with the square root of the radial distance, similarly to what happens with the planets of our Solar system[1]. If, instead, velocity is constant with the radius, as flat rotation curves imply, the mass should increase linearly with the distance from the center, which is in contrast with our observations of galaxies' baryonic matter[2]. Today, the most accepted explanation of this observational discrepancy

[1] And this is why this behaviour is often referred to as "Keplerian".

[2] In nuclear physics, "baryonic matter" refers to matter made out of baryons, such as protons, neutrons and all other objects that are formed by a combination of these two particles. However, cosmologists often broaden the definition to indicate all of the matter that is made out of atoms, hence including electrons (which are leptons, not baryons!); we will use the term "baryonic matter" in this enlarged way.

is DM: the matter in galaxies actually increases with the radius, but we cannot notice it since most of it is dark. However in the '70s, when the first flat rotation curves where found, what these implied was quite uncertain and not seen as very urgent; yes, some astronomers like Morton Roberts and Arnold Rots advanced in their works the hypothesis of invisible matter present at large distances from the center of galaxies, but no one suggested that this was a sensational nor crucial discovery.

2. *High velocities dispersion in galaxy clusters.* The astronomer Fritz Zwicky was the first one to find out this anomaly: in the early '30s, studying the Coma cluster, he observed that the velocity dispersion of the galaxies in it is so high, compared to the total baryonic mass, that it should not be stable. Since a bigger mass would stabilize the system, Zwicky claimed that the cluster's stability was due to additional invisible matter in it. When years later, in the late '50s, the astronomical community acquired a bigger interest in galaxy clusters, mass discrepancies analogous to the one in Coma were found in many other clusters. As interest grew, ideas on how to solve the problem started to flourish, and DM became just one among them: conferences were organized to discuss various hypothesis about the nature of the discrepancy between clusters' masses and their velocities dispersions; even if Zwicky's explanation in terms of additional dark mass was taken into account, it did not look more appealing nor less problematic than any of the alternatives proposed.

Hence, we can state that these two phenomenological anomalies we just discussed are what made the scientific community think about DM for the first time. However, none of these observations at this point was employed to support evidence for the presence of extra matter or to falsify any other hypothesis: the experimental and theoretical constraints available at that time where still too few to allow a single interpretation of these anomalies.

Things changed years later, when discovery and observations of Quasars suggested that our Universe is not a fixed and unchanging stage for the events to take place, but it evolves over time: General Relativity (GR) ceased being a purely mathematical theory and met with astronomy, leading, in the late '60s, to modern cosmology. Scientists started to look at Friedmann's equations with great interest; these equations tell us that our Universe can be open, flat or closed, depending on how gravitational pull and cosmological expansion are balanced. At that time there was a priori theoretical belief, later supported by observations and experiments, that our Universe should be flat; since Einstein's cosmological constant was believed to be $\Lambda = 0$, this implied that the energy density of our Universe should be $\rho_{\text{crit}} \sim 10^{-26}\,\text{kg/m}^3$. Suddenly, the total mass of the Universe became one of the most important parameters: measurements on the average masses of galaxies, made by considering luminous matter only, produced for our Universe an energy density $\rho \sim 10^{-28}\,\mathrm{kg/m^3}$, a value two orders of magnitude inferior to the one needed for flatness.

A search for extra matter begun and, in 1974, a very important paper was published by the physicist James Peeble and the astronomers Jeremiah Ostriker and Amos Yahil. In it, the authors stated that the reason the measured density of the Universe was so inferior to the critical one was that the masses of galaxies were strongly underestimated; in fact, supposing that galaxies have a component of dark matter and that this latter is well more than 10 times the amount of baryonic matter, one can obtain $\Omega = \rho/\rho_{\mathrm{crit}} = 1$. In support of their hypothesis, the authors brought galaxies rotation curves and clusters' velocities dispersion: the two anomalies went from being two apparently different problems with uncertain solutions, to being interpreted as the evidence for the existence of dark matter. Many papers similar to this latter were published by different research groups in the following months, and the existence of DM quickly became accepted by most of the scientific community.

For our current understanding of the Universe, DM should constitute roughly 85% of all matter there is, and about 25% of the total energy. Nevertheless, we should be careful: although the existence of DM is today generally accepted, we should refer to it as a hypothesis since, to date, its nature is still not clear. Most of the efforts made in the last decades focused on explaining it in terms of new elementary particles, such as weak interacting massive particles (WIMPs), axions, gravitationally interacting massive particles (GIMPs) and so on. However, to date, any attempt to detect these theoretical non-baryonic particles has failed. Also, we should at least mention that a minority of astrophysicists and theoretical physicists are not convinced by the DM hypothesis: they support theories of modified gravity on cosmological scales, such as MOND (Modified Newtonian dynamics) or TeVeS (Tensor–vector–scalar gravity). These theories successfully explain the flatness of galaxies rotation curves and a variety of galactic phenomena that are difficult to interpret in terms of DM; however, unlike this latter, MOND and TeVeS do not explain anomalies observed in galaxy clusters, and cosmological models built around them are very problematic. Moreover, none of these theories has been experimentally tested yet.

The purpose of this dissertation is to study an alternative explanation to DM, that does not rely on elementary particles nor modifications to general relativity: Primordial Black Holes (PBHs). However, before proceeding to explain this alternative solution, we should take a detour to review the basics of inflation.

1.2 Inflation

1.2.1 Problems of the Hot Big Bang Model

In the '70s, the Hot Big Bang (BB) Model was already considered the best theory of the origin and evolution of the Universe by the majority of the scientific community. Nonetheless, the model presented problems[3] of various nature, three of them being:

1. *The Cosmological Horizon problem.* Our Universe satisfies the Cosmological Principle: it is homogeneous and isotropic on scales where galaxies appear point-like, which are the scales relevant in Cosmology. This can be noticed in many ways, mainly by observations of the Cosmic Microwave Background (CMB), which tell us that our universe is in thermal equilibrium at $T \sim 2.7\,\mathrm{K}$ everywhere. However, in a BB model this should not be the case, since the singularity at the beginning of time automatically implies that the region causally connected to any observer, i.e. the particle horizon, is finite. To better understand this, we are going to build an example that reproduces the situation. Let us pick a direction: since information does not propagate instantaneously, the most distant objects that we are able to observe in that direction are those whose photons emitted at decoupling time just arrived in our position. Let us call (A) the distant region where these objects are located, and assume that the CMB observations tell us that $T_{(\mathrm{A})} \sim 2.7\,\mathrm{K}$. Now, let us look in the opposite direction: once again, the most distant objects in that direction are the ones whose photons emitted at decoupling time just reached us; let us call (B) the region where these objects are located and assume that, even in (B), $T_{(\mathrm{B})} \sim 2.7\,\mathrm{K}$. To reach thermal equilibrium, different regions of a system must be able to interact, so that inhomogeneities and anisotropies can be dissipated. But interactions require the exchange of signals, like light signals, that cannot travel instantaneously from a region to the other, and hence two regions can be at thermal equilibrium only if they have had enough time to exchange information. However, this last statement does not seem to apply to the two regions of the Universe we considered, and this is the so-called "Horizon problem": having the same average temperature, which is also the same of the rest of the universe, we can affirm that the two distant regions (A) and (B) are in thermal equilibrium; but we can also state that it is impossible that (A) and (B) have been able to exchange signals, since they are so far apart that the most ancient light signals emitted from these regions just arrived to us, and we are only half-way between the two.

2. *The Cosmological Flatness problem.* The Friedmann's equations tell us that there are three possible local geometries for our Universe, depending on the value of the

[3] Two of these problems, the evolution of the Universe before the Planck time and the so-called "coincidence problem", are still unsolved up to date.

so-called density parameter $\Omega = \rho/\rho_{\text{crit}}$: spherical geometry $\Omega > 1$, hyperbolic geometry $0 \leq \Omega < 1$ and flat geometry $\Omega = 1$. A spherical Universe is overdense since $\rho > \rho_{\text{crit}}$ and, because of this, gravitational attraction wins against cosmological expansion: sooner or later the Universe will cease its expansion and begin to contract until a Big Crunch occurs; because of this, the more time passes the bigger Ω gets and, consequently, the more spherical our universe becomes. Conversely, in a hyperbolic Universe we find $\rho < \rho_{\text{crit}}$, and cosmological expansion wins against gravitational pull: our Universe expands forever, and the more time passes the faster this expansion gets; this implies that $\Omega \to 0$ and, consequently, our hyperbolic Universe becomes more and more hyperbolic each instant. At last, in a flat Universe, we find $\rho = \rho_{\text{crit}}$, that implies a perfect balance between gravity and cosmic expansion; differently from the spherical and hyperbolic cases, that happen for infinite values of Ω, the flat universe only occurs for the specific value $\Omega = 1$. The value of the density parameter today, Ω_0, can be inferred in many ways, for example by observations of the very little anisotropies (i.e. variations with direction) in the CMB; these studies tell us that $|\Omega_0^{-1} - 1| \approx 0.01$, hence our Universe today looks practically flat. From the Friedmann's equations, one could show that the Universe can be this flat today only if, at Planck time, $|\Omega_{\text{p}}^{-1} - 1| \approx 10^{-62}$. This last result is exactly what constitutes the "Flatness Problem": the only way for the Universe to be this flat today is that it was born with a curvature parameter that differed at most of 10^{-62} from 1 at Planck time; but, since Ω_{p} can potentially assume values in the range $[0, +\infty[$, the idea that the Universe is born with an Ω_{p} so close to 1 appears like an enormous fine-tuning.

3. *The Magnetic Monopole problem.* The Grand Unification Theory (GUT) is a model of Quantum Field Theory (QFT) which essentially states that once, when the energy in our Universe was extremely high, the strong, weak and electromagnetic interactions were merged into a single force. This force had its own coupling constant and possessed a larger gauge symmetry compared to the ones we observe in today's fundamental interactions, consequently it had more force carriers. Then, at $T \sim 10^{15}$ GeV, a phase transition occurred: this large symmetry spontaneously broke, and the unified force split into the strong interaction and electroweak (EW) interaction. The GUT theory looks very elegant and we would like to include it in the early history of our Universe. Unfortunately, this seems impossible to realize because the theory implies, during the early times in which it is supposed to hold, the production of very exotic particles such as magnetic monopoles. This is the "Magnetic Monopole problem": these monopoles are heavy and stable and it can be estimated, if the GUT era happened, that today their numerical density should approximately be equal to the numerical density of photons, which of course is ridiculous considering that we are literally surrounded by photons but we have never observed such monopoles in the history of physics.

The three problems we just listed are quite serious; the only one that seems to have an immediate solution is the third since, to eliminate it, it is sufficient to say that the GUT theory does not apply to reality. However, in this way we are renouncing to an elegant and logical interpretation of the early universe while the problems of the horizon and the flatness still remain unsolved.

1.2.2 An elegant solution: Inflation

These three problems have all been solved in the early '80s by admitting the existence of an additional epoch, between the GUT and EW symmetry breaking, during which the universe went through an accelerated expansion that allowed it to increase at least $e^{60} \approx 10^{26}$ times along each direction. How can this additional epoch, called inflation, solve all of these different problems at once?

Let us suppose that we have an observer located in the point O. We define the particle horizon[4] as the set of points capable of sending light signals that could have been received by the observer up to some generic time t[5]; this set of point is located inside a sphere centered upon O, whose proper radius can be derived from the Friedmann-Robertson-Walker metric (FRW) as

$$R_{\mathrm{H}}(t) = a(t) \int_0^t \frac{\mathrm{d}t'}{a(t')} \tag{1.2.1}$$

where we have used the units $\hbar = c = 1$, that we will employ for the rest of the dissertation. We also define the Hubble radius R_{c} as the proper distance from O of an object moving with the cosmological expansion at the velocity of light with respect to O

$$R_{\mathrm{c}}(t) = \frac{1}{H(t)} \tag{1.2.2}$$

where $H(t) = \dot{a}/a$ is the Hubble parameter; the content of the spherical surface with radius R_{c} is called Hubble sphere, or proper cosmological horizon. The two radii above are not the same, essentially because the particle horizon is a quantity that takes all the past history of the observer into account, while the Hubble radius is defined instantaneously. For example, a point can be outside an observer's Hubble sphere, but inside his particle horizon; also, once a point is inside the particle horizon it stays within this horizon forever, while the Hubble sphere can shrink and expand so that it crosses the same point multiple times. At early times $t \sim 0$ however, one could demonstrate that $R_H(t \sim 0) = R_{\mathrm{c}}(t \sim 0)$ and, since from this moment onward we are going to consider early times, we will use the mathematical expression (1.2.2) and refer to the cosmological

[4] Also referred to as the observer horizon.

[5] The set of points in question can be said to have the possibility to be causally connected with the observer in O at time t.

horizon. An observer in O however does not measure $R_{\rm c}$ but its comoving version[6]

$$r_{\rm c} = \frac{R_{\rm c}}{a} = \frac{1}{aH} \tag{1.2.3}$$

which is the radius of the comoving cosmological horizon; its derivative results

$$\dot{r}_{\rm c} = -\frac{\ddot{a}}{\dot{a}^2} \tag{1.2.4}$$

which means that $r_{\rm c}(t)$ increases in the early times of a BB model, since the condition for the existence of the singularity is that $\ddot{a} < 0$.

Consider an observer Alice, situated in the origin O, and a comoving scale l_0 that enters her cosmological horizon at a time t_1, so that $r_{\rm c}(t_1) = l_0$: from this time onward, processes can causally connect this region, hence it can reach thermal equilibrium, homogeneity and isotropy with the smaller scales that were already causally connected. Now suppose that, for a time interval $[t_{\rm in}, t_{\rm end}]$ with $t_1 < t_{\rm in} < t_{\rm end}$, the Universe goes through an accelerated expansion $\ddot{a} > 0$: equation (1.2.4) tells us that the cosmological horizon shrinks in this epoch and, if it lasts enough, at some time t_2 such that $t_{\rm in} < t_2 < t_{\rm end}$ Alice will measure $r_{\rm c}(t_2) = l_0$. In other words the scale l_0 is now escaping the cosmological horizon[7]. After $t_{\rm end}$ we suppose that the Universe resumes its usual decelerated expansion: Alice's comoving cosmological horizon starts to grow once again and, at some time $t_3 > t_{\rm end}$, the scale l_0 will enter her horizon a second time, this time being already in thermal equilibrium with the rest of the universe and, hence, homogeneous and isotropic! In fact, for times $t_2 < t < t_3$ the scale l_0 was not causally connected with the others, hence no observed physical process could change its proprieties. Everything is clear to Alice, because she has been in the origin O since before t_1; but a second observer, say Bob, that appears in O after $t_{\rm end}$, is surprised: at t_3 he believes that the scale l_0 is entering the horizon for the first time, but the fact that this scale is already at thermal equilibrium with the others makes him claim that the BB model has a cosmological horizon problem. Today, we are in the same situation as Bob in the example, where l_0 are the CMB scales: by admitting that in the early universe there has been an epoch of accelerated expansion called inflation, we can affirm that the CMB scales already entered the comoving cosmological horizon in the past, and this is why we observe them homogeneous and isotropic as the rest of the Universe.

Of course, inflation solves the horizon problem only as long as today's scales are included in the ones that exited the horizon, i.e. $r_{\rm c}(t_0) \leq r_{\rm c}(t_{\rm in})$. We also do not want the inflation solution to be a fine-tuning, hence the condition we should really impose is $r_{\rm c}(t_0) \ll r_{\rm c}(t_{\rm in})$. With some algebra, this last condition can be translated into a condition for the scale factor

$$a(t_{\rm end}) = a(t_{\rm in})\, e^N, \quad N \gg 60 \tag{1.2.5}$$

[6] From now on, we will often omit time dependencies.

[7] We stress out that the cosmological horizon, differently from the particle one, can be abandoned.

where N is the so-called "number of e-foldings".

The problem of flatness is solved by requiring that Ω today is closer to 1 than it was at the beginning of inflation; this translates, once again, in a condition for the number of e-foldings

$$\frac{\Omega_{\text{in}}^{-1} - 1}{\Omega_0^{-1} - 1} \geq 1 \Leftrightarrow N \geq 60 \tag{1.2.6}$$

that is just a weaker version of the condition (1.2.5); hence, the flatness problem is automatically solved by imposing $N \gg 60$, necessary for the horizon problem to be solved.

Finally, the problem of magnetic monopoles can be solved by simply imposing that the inflation era begins after the GUT transition, during which the unified force gets split. Now new monopoles cannot be produced anymore, and the pre-existing ones get smeared during inflation on a volume that becomes at least e^{180} times bigger than the initial one: their numerical density quickly goes to zero, and this is why we have not observed a single magnetic monopole here on Earth and, probably, will never be able to.

1.2.3 Inflation from scalar fields

At this point we have understood why inflation is needed in our BB model, but we have not explained the physics behind it yet: how can we achieve this early epoch of accelerated expansion? Many different versions of the inflationary mechanism have been proposed during the years. The first one, dating back to 1981, is Guth's "False Vacuum Inflation": in this model the Universe undergoes a first-order phase transition[8] triggered by the cooling due to cosmic expansion. The Universe hence passes from a metastable false vacuum, whose non-zero energy drives inflation, to a true vacuum at lower energy, and this transition ends inflation: the energy released in the process expands the Universe. However this model was quickly discarded because of bubble nucleation, a process that occurs in every first-order phase transition: when the temperature reaches a critical value, the system begins to stochastically form and dissolve little regions (bubbles) where the transition has already occurred. This process continues until an unusually large region forms: it is so big that its probability to shrink is low, and over time it absorbs all the other little regions, growing until its dimensions are comparable to the system, which marks the end of the transition. If the thermodynamic system is the Universe, bubble nucleation cannot occur because of cosmic expansion: the bubbles are taken away from each other, making their nucleation almost impossible, which produces a highly chaotic Universe that does not match our current observations. Even if Guth's model was discarded, inflation captured a lot of attention and many other inflationary models were proposed in the following years. A class of models that is affirmed today is

[8]First-order phase transitions are characterized by a metastable state and the release of a finite amount of energy all at once.

the so-called "Slow-roll Inflation", and it was introduced by Linde in 1983; we are now going to explain this mechanism of inflation.

In slow-roll inflation (also called "chaotic inflation") our Universe is endowed with a scalar field ϕ, called inflaton, whose non-zero energy $V(\phi)$ permeates the Universe. Immediately after the Planck time, ϕ is just a component among others, contributing to the total energy density of the Universe with its effective energy density ρ_ϕ; but if we admit that, some time after the GUT transition, ϕ becomes the dominant component and also $\rho_\phi \sim V(\phi) \sim \text{const}$, the result is that the Universe expands exponentially and for enough time to explain the homogeneity and isotropy of the CMB scales. After this epoch, the field dissipates the energy left by decaying into Standard Model particles, reheating the Universe, and the usual decelerated expansion takes place again.

The action of the slow-roll inflation class of models describes a scalar field minimally coupled to gravity

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{16\pi G}\mathcal{R} + \mathcal{L}_\Phi\right] \tag{1.2.7}$$

where $\mathcal{R}$ is the 4-dimensional Ricci scalar derived by the metric $g_{\mu\nu}$ with signature $(-,+,+,+)$, and

$$\mathcal{L}_\Phi = \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + V(\Phi) \tag{1.2.8}$$

is the inflaton $\Phi(t,\mathbf{x})$ Lagrangian density. It is convenient to use a perturbative approach to the problem, writing

$$\Phi(t,\mathbf{x}) = \phi(t) + \delta\phi(t,\mathbf{x}) \tag{1.2.9}$$

$$g_{\mu\nu}(t,\mathbf{x}) = \eta_{\mu\nu} + \delta g_{\mu\nu}(t,\mathbf{x}) \tag{1.2.10}$$

where $\phi(t)$ is the background/homogeneous part of the field, $\delta\phi(t,\mathbf{x})$ represents its fluctuations $\delta\phi \ll \phi$, $\eta_{\mu\nu}$ is the Minkowski metric and $\delta g_{\mu\nu}(t,\mathbf{x})$ its local perturbations. We will now study the background field, whose time evolution can be derived by the Euler-Lagrange equations of motion for a classical relativistic wave field[9]

$$\partial_\alpha\frac{\delta(\mathcal{L}_0 a^3)}{\delta\partial_\alpha\phi} - \frac{\delta(\mathcal{L}_0 a^3)}{\delta\phi} = 0 \tag{1.2.11}$$

where we decomposed the Lagrangian density (1.2.8) as

$$\mathcal{L}_\Phi = \mathcal{L}_0(\phi,\eta_{\mu\nu}) + \mathcal{L}_1(\delta\phi,\delta g_{\mu\nu}) + \mathcal{L}_2(\delta\phi,\delta g_{\mu\nu}) + \dots \tag{1.2.12}$$

The solution to (1.2.11), which describes the dynamic of the inflaton, is the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \tag{1.2.13}$$

[9] $\mathcal{L}_0 a^3$ represents the Lagrangian of the system.

where the dot indicates derivative over t, while the prime derivative over ϕ. But ϕ is not the only dynamical component of our study, since the geometry of spacetime on cosmological scales also evolves: the equations describing the behaviour of our Universe are the Friedmann's equations[10]

$$\dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3p) \tag{1.2.14}$$

$$H^2 + \frac{\mathrm{k}}{a^2} = \frac{8\pi G}{3}\rho \tag{1.2.15}$$

where $\mathrm{k} = 0, \pm 1$ is the parameter that discriminates between the three possible kinds of universe, while ρ and p are the energy density and pressure of the content of the Universe.

We have said that inflation is an era during which the comoving Hubble radius decreases; for (1.2.3), inflation implies

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} < 0 \tag{1.2.16}$$

or, analogously, $\ddot{a} > 0$. We now define the parameter

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = -\frac{1}{H}\frac{\mathrm{d}H}{\mathrm{d}N} \tag{1.2.17}$$

where, in the last step we employed, $H\mathrm{d}t = a^{-1}\mathrm{d}a = \mathrm{d}\ln a = \mathrm{d}N$; this parameter tells us the fractional change of the Hubble parameter per e-folding. By evaluating the left term of (1.2.16) we can see that inflation can occur if and only if $\epsilon < 1$, i.e. if the Hubble parameter changes slowly during inflation. To sum up:

$$\text{Inflation} \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} < 0 \quad \Leftrightarrow \quad \ddot{a} > 0 \quad \Leftrightarrow \quad \epsilon < 1$$

To solve the horizon problem, it is necessary that inflation occurs, but it is not sufficient: as we saw, inflation must last at least for 60 e-foldings. Inflation lasts for long if, once started, ϵ stays below 1 and does not change sensibly. To make sure this happens, we introduce another new parameter

$$\eta \equiv \frac{\dot{\epsilon}}{H\epsilon} = \frac{1}{\epsilon}\frac{\mathrm{d}\epsilon}{\mathrm{d}N} \tag{1.2.18}$$

where in the last step we employed the same relation we used above. η represents the fractional change of ϵ per e-folding: for $|\eta| < 1$ we find that ϵ does not change significantly over time and inflation persists.

[10]These equations are just Einstein's equations with the FRW metric.

The energy-momentum tensor of any scalar field can be characterised by an effective energy density and an effective pressure; for the homogeneous part of the inflaton

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{1.2.19}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) = \mathcal{L}_0 \tag{1.2.20}$$

We have introduced ϕ in our model of the Universe because we want it to govern the energy density of the Universe so that it can produce inflation; this is achieved by simply substituting these last two expressions into (1.2.14) and (1.2.15)

$$\dot{H} + H^2 = -\frac{4\pi G}{3}(2\dot{\phi}^2 - 2V) \tag{1.2.21}$$

$$H^2 = \frac{4\pi G}{3}(\dot{\phi}^2 + 2V) \tag{1.2.22}$$

where, in the second equation, we also put k = 0 since, as we explained before, during inflation the space-time gets strongly flattened. Substituting (1.2.22) into (1.2.21) gives

$$\dot{H} = -4\pi G\dot{\phi}^2 \tag{1.2.23}$$

that leads to a new expression for ϵ

$$\epsilon = 4\pi G\frac{\dot{\phi}^2}{H^2} \tag{1.2.24}$$

Now, using (1.2.22)

$$\epsilon = \frac{3\dot{\phi}^2}{\dot{\phi}^2 + 2V} \tag{1.2.25}$$

and we find that $\epsilon < 1 \Leftrightarrow \dot{\phi}^2 < V(\phi)$: this means that inflation occurs only when the kinetic energy of the background field is subleading to its potential. We now define a third parameter

$$\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} = -\frac{1}{\dot{\phi}}\frac{\mathrm{d}\dot{\phi}}{\mathrm{d}N} \tag{1.2.26}$$

where we used the usual relation for the most right term; this last parameter measures the fractional change of $\dot{\phi}$ per e-folding. The parameters ϵ, η and δ are called Hubble slow-roll parameters. If we now evaluate $\dot{\epsilon}$ from (1.2.24) and substitute into (1.2.18), making use of the parameter δ we find

$$\eta = 2(\epsilon - \delta) \tag{1.2.27}$$

This last equation tells us another condition for inflation in terms of the inflaton field, even though it is not immediate to notice it. We understood that, during inflation, we must have $\epsilon < 1$, but it is even better to require that $\epsilon \ll 1$. For this reason, the only way we could have $|\eta| \ll 1$ (necessary for inflation to last long enough) is if $|\delta| \ll 1$ too; for (1.2.26), this last condition implies that $|\ddot{\phi}| \ll H|\dot{\phi}|$. To sum up, the two conditions necessary to obtain inflation and solve the horizon problem, are

$$\epsilon \ll 1 \Rightarrow \dot{\phi}^2 \ll V(\phi), \quad |\delta| \ll 1 \Rightarrow |\ddot{\phi}| \ll H|\dot{\phi}| \tag{1.2.28}$$

Up to this point, we have studied the conditions to obtain inflation, but we still have not solved the equations of motion (1.2.13), (1.2.21) and (1.2.22); to do so, we will now use the conditions (1.2.28) to simplify the problem. This passage is called the slow-roll approximation, since equation (1.2.13) is analogous to the classical equation that describes the motion of a marble under the force $-V'$ with friction $3H\dot{\phi}$: the condition $|\ddot{\phi}| \ll H|\dot{\phi}|$ simplifies (1.2.13) into

$$3H\dot{\phi} \approx -V' \tag{1.2.29}$$

that is the equivalent of requiring that the rolling of the marble occurs at approximately constant speed even in the presence of the force, due to the high friction that is present. Using $\dot{\phi}^2 \ll V(\phi)$, (1.2.22) becomes

$$H^2 \approx \frac{8\pi G}{3} V \tag{1.2.30}$$

Thanks to these last two approximated equations, we can find out which are the constraints that our inflationary potential V must satisfy in order for ϕ to obey (1.2.28) and produce inflation. From (1.2.24), first using (1.2.29) and then (1.2.30), we find

$$\epsilon \approx \frac{1}{4\pi G}\left(\frac{V'}{V}\right)^2 \equiv \epsilon_V \tag{1.2.31}$$

Furthermore, taking the time derivative of (1.2.29) and using the chain rule on V' gives

$$3\dot{H}\dot{\phi} + 3H\ddot{\phi} \approx -V''\dot{\phi} \tag{1.2.32}$$

which can be used along with (1.2.30) to get

$$\delta + \epsilon \approx \frac{1}{8\pi G}\frac{V''}{V} \equiv \eta_V \tag{1.2.33}$$

The parameters defined in (1.2.31) and (1.2.33) are called slow-roll parameters, and their relation to the Hubble slow-roll parameters previously defined[11] implies that our model produces inflation if its potential $V(\phi)$ is such that

$$\epsilon_V \equiv \frac{1}{4\pi G}\left(\frac{V'}{V}\right)^2 \ll 1, \quad \eta_V \equiv \frac{1}{8\pi G}\frac{V''}{V} \ll 1 \tag{1.2.34}$$

[11] We stress out that $\epsilon \approx \epsilon_V$ and $\eta \approx \eta_V$ is valid only in the slow-roll regime.

for a certain range of values of ϕ that we call inflationary epoch. The conditions (1.2.34) are the ones used to test models of inflation.

The enhancement of the scale factor during inflation is measured by the number of e-foldings

$$N \equiv \int_{a_{\rm in}}^{a_{\rm end}} \mathrm{d}\ln a = \int_{t_{\rm in}}^{t_{\rm end}} H(t)\,\mathrm{d}t \tag{1.2.35}$$

where $\epsilon(t_{\rm in}) = \epsilon(t_{\rm end}) = 1$, i.e. these two times mark the interval of inflation. Now, to use the inflaton ϕ as clock, we use (1.2.29) and (1.2.30) to write

$$H\mathrm{d}t = \frac{H}{\dot\phi}\mathrm{d}\phi \approx -\frac{3H^2}{V'}\mathrm{d}\phi \approx -8\pi G\frac{V}{V'}\mathrm{d}\phi \approx -\sqrt{\frac{16\pi G}{\epsilon_V}}\mathrm{d}\phi$$

hence

$$N = -\sqrt{16\pi G}\int_{\phi_{\rm in}}^{\phi_{\rm end}} \frac{\mathrm{d}\phi}{\sqrt{\epsilon_V}} \gg 60 \tag{1.2.36}$$

where $\phi_{\rm in}$ and $\phi_{\rm end}$ are the inflaton values corresponding to the times $t_{\rm in}$ and $t_{\rm end}$. From the last equation we can see that, the flatter is the potential, the greater is N and, therefore, the longer is the inflationary epoch.

1.2.4 Perturbations during inflation

Up until now we studied the dynamics of the homogeneous part of the inflaton field, that is the main part of the inflationary mechanism and explains the high grade of homogeneity and isotropy of the CMB; nonetheless, the CMB presents small temperature fluctuations $\delta T/T \sim 10^{-5}$. These can be traced back to density perturbations $\delta = \delta\rho/\rho$ of the same magnitude, that were present in our universe when the scales corresponding to the CMB, $k_{\rm CMB}$ exited the comoving horizon during inflation. These density perturbations, in turn, were seeded by inflaton perturbations $\delta\phi(t, \mathbf{x})$, that caused the Universe to inflate by different amounts in different regions. Therefore, it is time to study these perturbations that, as we will discuss in the next Section, play a crucial role in the new explanation of dark matter we are proposing.

A priori, we have five different scalar modes to deal with, four of which are metric perturbations (δg_{00}, δg_{0i}, δg_{ii}, δg_{ij}) and one is the scalar field perturbation $\delta\phi$. However, not all of these modes are physical degrees of freedom: two modes are removed because of the gauge invariance of (1.2.7) under time and space translations, while other two are removed because of Bianchi identities. Hence, only one scalar mode of the initial five is physical; to derive the action for this mode, we make use of the comoving gauge, defined by the vanishing value of the momentum density $\delta T_{0i} = 0$. Only in the slow-roll approximation, this gauge fixing can be written as

$$\delta\phi = 0 \tag{1.2.37}$$

which means that the fluctuations of the field have been moved in the metric, that can be written as

$$\delta g_{ij} = a^2(1 - 2\zeta)\delta_{ij} + a^2 h_{ij} \tag{1.2.38}$$

where the physical degrees of freedom are h_{ij}, a traceless ($h^i_i = 0$), transverse ($\nabla_i h^{ij} = 0$) tensor, and the scalar ζ. ζ represents the curvature perturbations, that translate to density perturbations δ when the inflaton decays during the reheating epoch; δ and ζ are proportional, as we will see in (2.1.3). h_{ij} represents the primordial Gravitational Waves (GW) produced by inflation perturbations.

The non-dynamical perturbations δg_{00} and δg_{0i} can also be expressed in terms of ζ: by doing so and substituting into (1.2.7) we arrive to the quadratic action for ζ[12]

$$S = \frac{1}{2}\int \mathrm{d}t\mathrm{d}\mathbf{x}\, a^3 \frac{\dot\phi^2}{H^2}\left[\dot\zeta^2 - \frac{1}{a^2}(\partial_i\zeta)^2\right] + \ldots \tag{1.2.39}$$

where we put $8\pi G = 1$ and the ellipses denote that there are higher order corrections that we are ignoring. To express this last action in canonical form, we define the Mukhanov-Sasaki variable

$$v \equiv z\zeta \tag{1.2.40}$$

where

$$z^2 \equiv a^2\frac{\dot\phi^2}{H^2} = 2\epsilon a^2 \tag{1.2.41}$$

where in the last step we employed (1.2.24). Switching to comoving Hubble time, we get

$$S = \frac{1}{2}\int \mathrm{d}\tau\mathrm{d}\mathbf{x}\left[(v')^2 - (\partial_i v)^2 + \frac{z''}{z}v^2\right] \tag{1.2.42}$$

where the prime, from this moment onward, stands for derivative over τ. This latter is the normalized action for the scalar field $v(\tau, \mathbf{x})$, which is massive since we can recognize a quadratic mass term $-\frac{1}{2}m^2_{\text{eff}}v^2$, where the mass is:

$$m^2_{\text{eff}}(\tau) \equiv -\frac{z''}{z} = -\frac{H}{a\dot\phi}\frac{\partial^2}{\partial\tau^2}\frac{a\dot\phi}{H} \tag{1.2.43}$$

Since the background ϕ rolls towards the minimum of the potential over time, $\dot\phi$ also changes over time and so does the effective mass m_{eff}. Aside for this particular mass, the action (1.2.42) is analogous to the one for the usual massive free scalar field. The Euler-Lagrange equations (1.2.11) give

$$v'' - \nabla^2 v - \frac{z''}{z}v = 0 \tag{1.2.44}$$

[12]A quadratic action similar to (1.2.39) can be derived for h_{ij}, meaning that at this order ζ and h_{ij} are independent variables with decoupled equations of motion; since we are not interested in h_{ij} for the purposes of this dissertation, we will ignore it from now on.

Now, defining the Fourier modes[13]

$$v(\tau,\mathbf{x}) \equiv \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{3/2}}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, v_{\mathbf{k}}(\tau) \tag{1.2.45}$$

we obtain the so-called Mukhanov-Sasaki equation

$$v''_{\mathbf{k}} + \underbrace{\left(k^2 - \frac{z''}{z}\right)}_{\omega_k^2(\tau)} v_{\mathbf{k}} = 0 \tag{1.2.46}$$

The fact that $z(\tau)$ depends on the background, $\phi(t)$ $a(t)$, makes it difficult to generally solve the Mukhanov-Sasaki equation. This is the reason why, up to the end of this Section, we will discuss the case of a de Sitter space $a = (H\tau)^{-1}$, a peculiar background that allows to find approximated analytical solutions. If an analytical approach is too difficult one can always solve the equation numerically, which is what we are going to do in Chapter 3 in order to compute the power spectrum of curvature perturbations. For a de Sitter background, the effective frequency $\omega_k^2(\tau)$ reduces to

$$\omega_k^2(\tau) = k^2 - \frac{2}{\tau^2} \tag{1.2.47}$$

Before proceeding to the solution of the equation, let us consider the two limits to (1.2.46). If $k^2 \gg |z''/z|^2 \sim \tau^{-2} = (aH)^2$, this means that we are on sub-horizon scales $\lambda \ll r_c$, and the equation reduces to

$$v''_{\mathbf{k}} + k^2 v_{\mathbf{k}} = 0 \tag{1.2.48}$$

whose solutions are oscillatory, $v_{\mathbf{k}} \sim e^{\pm ik\tau}$. The opposite to this is the limit of super-horizon scales, $k^2 \ll |z''/z|^2$, that leads to

$$v''_{\mathbf{k}} - \frac{z''}{z} v_{\mathbf{k}} = 0 \tag{1.2.49}$$

whose solutions are $v \sim \tau^{-1}$ and $v \sim \tau^2$. Only the growing solution is of interest in the study of perturbations; τ runs from large negative values to zero, hence this solution is the former. Translating this last result to the curvature perturbations $\zeta_{\mathbf{k}} = z^{-1} v_{\mathbf{k}}$ highlights a very important propriety

$$\lim_{k \ll aH} \dot{\zeta}_{\mathbf{k}} = 0 \tag{1.2.50}$$

[13] We should stress that this decomposition is three-dimensional, i.e. the time variable does not change transferring to the Fourier space.

i.e. curvature perturbations freeze on super-horizon scales[14]. This means that, to study the CMB predictions, one can completely ignore the physics from the moment the scale k_{CMB} exited the horizon and today, because essentially nothing happened on that scale during this enormous energy gap.

The most general solution of (1.2.46) is written as a mode expansion

$$v_{\mathbf{k}} = a^-_{\mathbf{k}} v_k(\tau) + a^+_{-\mathbf{k}} v^*_k(\tau) \tag{1.2.51}$$

where $v_k(\tau)$ and $v^*_k(\tau)$ are complex conjugates and linearly independent solutions of (1.2.46); the subscript k indicates that these two functions, called mode functions, are the same for every mode with $|\mathbf{k}| = k$, and their normalization is chosen so that their Wronskian gives $-i$

$$W[v_k, v^*_k] = v'_k v^*_k - v_k v^{*\prime}_k \equiv -i \tag{1.2.52}$$

Instead, the $a^\pm_{\mathbf{k}}$ are two time-independent integration constants such that

$$a^-_{\mathbf{k}} = \frac{W[v^*_k, v_{\mathbf{k}}]}{W[v^*_k, v_k]}, \quad a^+_{\mathbf{k}} = (a^-_{\mathbf{k}})^* \tag{1.2.53}$$

where the relation between $a^+_{\mathbf{k}}$ and $a^-_{\mathbf{k}}$ comes from the reality of $v(\tau, \mathbf{x})$, that can now be written as

$$\begin{aligned} v(\tau, \mathbf{x}) &= \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{3/2}} [a^-_{\mathbf{k}} v_k(\tau) + a^+_{-\mathbf{k}} v^*_k(\tau)]\, e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{3/2}} a^-_{\mathbf{k}} v_k(\tau)\, e^{i\mathbf{k}\cdot\mathbf{x}} + \text{c.c.} \end{aligned} \tag{1.2.54}$$

We now proceed to the canonical quantization of the theory: the scalar field v and its conjugate momentum $\pi \equiv v'$ are promoted from real valued distributions defined on our spacetime to operatorial valued distributions $\hat{v}$, $\hat{\pi}$ on the same spacetime that satisfy the standard equal-time commutation relations

$$[\hat{v}(\tau, \mathbf{x}), \hat{\pi}(\tau, \mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}), \tag{1.2.55}$$

$$[\hat{v}(\tau, \mathbf{x}), \hat{v}(\tau, \mathbf{y})] = [\hat{\pi}(\tau, \mathbf{x}), \hat{\pi}(\tau, \mathbf{y})] = 0 \tag{1.2.56}$$

This means that the integration constants (1.2.53) also become operators $\hat{a}^\pm_k$ and the field operator $\hat{v}(\tau, \mathbf{x})$ is expanded as

$$\hat{v}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{3/2}} \left[\hat{a}^-_{\mathbf{k}} v_k(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}^+_{\mathbf{k}} v^*_k(\tau) e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \tag{1.2.57}$$

[14] We will see in Chapter 3 that, in regime of ultra slow-roll, this is not true anymore and perturbations can grow even on super-horizon scales

Substituting this last expression into (1.2.55) and (1.2.56) we get the following commutation relations

$$[\hat{a}^-_{\mathbf{k}}, \hat{a}^+_{\mathbf{p}}] = \delta(\mathbf{k}-\mathbf{p}), \quad [\hat{a}^-_{\mathbf{k}}, \hat{a}^-_{\mathbf{p}}] = [\hat{a}^+_{\mathbf{k}}, \hat{a}^+_{\mathbf{p}}] = 0 \tag{1.2.58}$$

which imply that $\hat{a}^{\pm}_k$ are creation and annihilation operators. These operators act on the Hilbert space of one particle states $\mathcal{H}_1$

$$\mathcal{H}_1 = \{|\mathbf{k}\rangle = \hat{a}^+_{\mathbf{k}}|0\rangle,\, \mathbf{k} \in \mathbb{R}^3\} \tag{1.2.59}$$

where the vacuum is defined as

$$\hat{a}^-_{\mathbf{k}}|0\rangle = 0 \tag{1.2.60}$$

It can be shown from (1.2.58) and (1.2.60) that these states are normalized and satisfy the completeness relation

$$\langle \mathbf{k}|\mathbf{p}\rangle = \delta(\mathbf{k}-\mathbf{p}), \quad \int \mathrm{d}\mathbf{k}|\mathbf{k}\rangle\langle\mathbf{k}| = \mathbb{I}_1 \tag{1.2.61}$$

Many particle states, belonging to $\mathcal{H}_n = \mathcal{H}_1^{\otimes n}$, are obtained by repeated application of creation operators[15], e.g.

$$|m\mathbf{k}_1, n\mathbf{k}_2, \ldots\rangle = \frac{1}{\sqrt{m!n!\ldots}}(\hat{a}^+_{\mathbf{k}1})^m(\hat{a}^+_{\mathbf{k}2})^n \ldots |0\rangle \tag{1.2.62}$$

The space on which the operator distributions $\hat{v}(\tau,\mathbf{x})$ and $\hat{\pi}(\tau,\mathbf{x})$ act is the Fock space

$$\mathcal{F} = \mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_n \oplus \ldots \tag{1.2.63}$$

that contains all the possible single and many particle states.

Our construction, however, is not unique: consider the following functions

$$u_k(\tau) = \alpha_k v_k(\tau) + \beta_k v_k^*(\tau) \tag{1.2.64}$$

where α_k and β_k are complex constants. Being a linear combination of solutions, the $u_k(\tau)$ are solutions of (1.2.46); imposing that $|\alpha_k|^2 - |\beta_k|^2 = 1$, i.e. $W[u_k, u_k^*] \equiv -i$, these functions become a new set of mode functions, different from the $v_k(\tau)$ previously defined. Hence, we can expand the quantized scalar field in terms of these new mode functions

$$\hat{v}(\tau,\mathbf{x}) = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{3/2}}\left[\hat{b}^-_{\mathbf{k}} u_k(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{b}^+_{\mathbf{k}} u_k^*(\tau) e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \tag{1.2.65}$$

where the $\hat{b}^{\pm}_{\mathbf{k}}$ are a new set of creation annihilation operators satisfying (1.2.58). By comparing (1.2.65) with (1.2.57), one can find the following transformations between the two different sets, called Bogolyubov transformations

$$\begin{aligned}\hat{a}^-_{\mathbf{k}} &= \alpha_k^* \hat{b}^-_{\mathbf{k}} + \beta_k \hat{b}^+_{-\mathbf{k}} \\ \hat{a}^+_{\mathbf{k}} &= \alpha_k \hat{b}^+_{\mathbf{k}} + \beta_k^* \hat{b}^-_{-\mathbf{k}}\end{aligned} \tag{1.2.66}$$

[15] This is propriety is known as cyclic propriety of the vacuum state.

Of course, both set of operators can be use to build a base of $\mathcal{H}_1$, defining two different vacua

$$\hat{a}^-_{\mathbf{k}}|0\rangle_a = 0, \quad \hat{b}^-_{\mathbf{k}}|0\rangle_b = 0 \tag{1.2.67}$$

b-states are different from a-states, as can be inferred by evaluating the b-void expectation value of the number operator of the a-particles

$$\begin{aligned} {}_b\langle 0|\hat{N}^{(a)}_{\mathbf{k}}|0\rangle_b &= {}_b\langle 0|\hat{a}^+_{\mathbf{k}}\hat{a}^-_{\mathbf{k}}|0\rangle_b \\ &= b_b\langle 0|(\alpha_k\hat{b}^+_{\mathbf{k}} + \beta^*_k\hat{b}^-_{-\mathbf{k}})(\alpha^*_k\hat{b}^-_{\mathbf{k}} + \beta_k\hat{b}^+_{-\mathbf{k}})|0\rangle_b \\ &= |\beta_k|^2\,\delta(0) \end{aligned} \tag{1.2.68}$$

where the divergent factor $\delta(0)$ is due to the fact that we are considering an infinite volume; however, the mean density of a-particles in the b-vacuum is finite and not zero. In other words, our definition of vacuum is ambiguous: $|0\rangle_{a/b}$ are not physical vacua since they describe excited states of the other base, and thus we require some additional physical input to define the true vacuum. This additional input, for a de Sitter spacetime, is given by the Bunch-Davies (BD) initial conditions for the mode functions

$$\lim_{\tau\to-\infty} v_k(\tau) = \frac{1}{\sqrt{2k}}\,e^{-ik\tau} \tag{1.2.69}$$

Since these are exactly the mode functions that define the physical vacuum in a Minkowski spacetime, condition (1.2.69) is equivalent to require that the evolution of the perturbations $\hat{v}$ in de Sitter spacetime is the same as in flat spacetime for scales deep inside the horizon, i.e. when $k \gg aH$ (sub-horizon). This defines a preferable set of modes and a unique physical vacuum, the Bunch-Davies vacuum.

We will now apply the formalism we just reviewed to de Sitter spacetime: using the expression for effective frequencies in de Sitter (1.2.47), we can solve the Mukhanov-Sasaski equation (1.2.46) analytically, obtaining

$$v_k(\tau) = \alpha\frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) + \beta\frac{e^{ik\tau}}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right) \tag{1.2.70}$$

The BD conditions for the vacuum uniqueness impose $v_k \sim e^{-ik\tau}$, so that $\alpha = 1$ and $\beta = 0$

$$v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) \tag{1.2.71}$$

At super-horizon scales the dynamics is

$$\lim_{k\tau\to 0} v_k(\tau) = \frac{e^{-ik\tau}}{i\sqrt{2}}\cdot\frac{1}{k^{3/2}\tau} \tag{1.2.72}$$

We now introduce an observable quantity, the power spectrum $P_v(k)$, by evaluating the two-point correlation function

$$\begin{aligned}\langle \hat{v}_{\mathbf{k}} \hat{v}_{\mathbf{p}} \rangle &= \langle 0 | \hat{v}_{\mathbf{k}} \hat{v}_{\mathbf{p}} | 0 \rangle \\ &= \langle 0 | (\hat{a}^-_{\mathbf{k}} v_k + \hat{a}^+_{-\mathbf{k}} v^*_k)(\hat{a}^-_{\mathbf{p}} v_p + \hat{a}^+_{-\mathbf{p}} v^*_p) | 0 \rangle \\ &= v_k v^*_p \langle 0 | \hat{a}^-_{\mathbf{k}} \hat{a}^+_{-\mathbf{p}} | 0 \rangle \\ &= v_k v^*_p \langle 0 | [\hat{a}^-_{\mathbf{k}}, \hat{a}^+_{-\mathbf{p}}] | 0 \rangle \\ &= |v_k|^2 \delta(\mathbf{k} + \mathbf{p}) \\ &\equiv P_v \delta(\mathbf{k} + \mathbf{p}) \end{aligned} \tag{1.2.73}$$

On super-horizon scales, (1.2.72) tells us that

$$P_v = \frac{1}{2k^3} \frac{1}{\tau^2} = \frac{1}{2k^3} (aH)^2 \tag{1.2.74}$$

This power spectrum is referred to the canonically-normalized field; the one for curvature perturbations can simply be obtained for this latter by dividing for z^2

$$P_\zeta = \frac{1}{z^2} P_v \tag{1.2.75}$$

Given any point in time and any momentum scale k, which can be sub/super-horizon depending on the Hubble radius $(aH)^{-1}$ at that time, the power spectrum P_ζ tells us the variance (or variability) of the density on that scale, i.e. it tells us the amplitude of the fluctuations on that scale. We should mention that, strictly speaking, the curvature fluctuations $\zeta = z^{-1} v$ are ill-defined in the de Sitter spacetime, since $z^2 \sim \epsilon$ vanishes in this limit and, therefore, V is flat and inflation has no end. Hence de Sitter spacetime is just an idealization: our spacetime deviates from it during inflation because ϵ assumes little but finite values. Because of this little deviation, we refer to our Universe as quasi-de Sitter.

In a quasi-de Sitter spacetime ζ is well defined and the curvature power spectrum (1.2.75) results

$$P_\zeta = \frac{1}{z^2} P_v = \frac{1}{4k^3} \frac{H^2}{\epsilon} \tag{1.2.76}$$

where in the second step we made use of (1.2.41). As we have said, at a certain point in time, one can compute the power spectrum for any scale k. However, only one scale at each time is interesting to us, and it is the one that crosses the horizon at that particular time. For example, we are interested in a particular scale k_{CMB}: when $k_{\text{CMB}} > aH$, i.e. k_{CMB} is sub-horizon, the curvature perturbations on this scale can still evolve and, therefore, they are not the same we measure today through temperature observations. When $k_{\text{CMB}} = aH$ however, the scale crosses the horizon and the curvature perturbations freeze out: from this moment until horizon reentry, that for the CMB scales in particular is today, perturbations on that scale cannot evolve anymore. This implies that the fluctuations we observe today through the measurement of temperature fluctuations, are the same that were present in an epoch far away in the past, when the energy of the universe was 19 orders of magnitude bigger than today.

For the curvature perturbations at horizon crossing, the power spectrum becomes just a function of k

$$P_\zeta(k) = \frac{1}{4k^3}\frac{H^2}{\epsilon}\bigg|_{k=aH} \tag{1.2.77}$$

This last expression has the dimensions $[\text{length}]^{-3}$; its dimensionless version is defined as

$$\Delta_s^2(k) \equiv \frac{k^3}{2\pi^2}P_\zeta(k) = \frac{1}{8\pi^2}\frac{H^2}{\epsilon}\bigg|_{k=aH} \tag{1.2.78}$$

A little dependence on time, via H and ϵ, is present in a quasi-de Sitter Universe, and causes the power spectrum to deviate from the scale-invariant form $\Delta_s^2 \sim k^0$. To quantify this deviation with respect to the perfect de Sitter limit, one introduces the scalar spectral index

$$n_s - 1 \equiv \frac{\mathrm{d}\ln\Delta_s^2}{\mathrm{d}\ln k} \tag{1.2.79}$$

and, with some algebra, it is possible to show that

$$n_s - 1 = -2\epsilon - \eta \tag{1.2.80}$$

1.3 Primordial Black Holes as Dark Matter

As we have said, today the existence of DM is commonly accepted; nevertheless, its nature is still one of the biggest problem remained open in physics, and neither new non-baryonic particles nor theories of modified gravity have been proven to be the answer to this problem. However, there is a third possible way to explain DM that we have not discussed yet, and it appears is quite simple compared to the others: Black Holes.

A Black Hole (BH) is a region where spacetime got extremely bent by the theoretically infinite energy density of its singularity, situated at its center; the gravitational pull resulting from this deformation of spacetime is so extreme that anything that gets closer to the singularity than a certain radial distance, called Schwarzschild radius, will fall into it without possibility of escape. This implies that not even the fastest signal in the Universe, a light signal, can escape a BH once its Event Horizon, i.e. the surface with radius equal to the Schwarzschild radius, has been crossed. Hence BHs look "dark", as their name suggests, and this makes them valid candidates for DM[16].

[16]Actually, in principle BHs can be detected because they emit radiation at very low temperature. However this effect will be take into account in Fig. 1.1 and we will notice that it does not exclude all possible masses of BHs.

To date, the gravitational collapse of stars is the only mechanism to produce BHs that we have been able to observe. Nonetheless it is believed that there is another way to form BHs, tied to the evolution of the early Universe: it was first theorized by Hawking in 1971 [2], and we are now going to explain it.

Looking at our Universe on cosmological scales, where galaxies appear like particles of a cosmological fluid, we can associate to it an average energy density $\bar{\rho}$. We can now consider a particular scale of length from our position, $k = \lambda^{-1}$, and evaluate the medium density of the Universe on this scale, ρ_k: the quantity $\delta\rho_k = \rho_k - \bar{\rho}$ tells us the deviation, from the total average, for this region. Today, our measurements on the scales of the Cosmic Microwave Background (CMB) establish that $\delta\rho_{\mathrm{CMB}} \sim 10^{-5}\bar{\rho}$; however we will show in the following chapters that, during inflation, it is theoretically possible to have an inflationary potential such that, on momentum scales k_{PBH} bigger than the current ones k_{CMB}, fluctuations of $\delta\rho \sim 0.1\bar{\rho}$ or even larger can be produced. In this way, when after inflation the scales k_{PBH} reenter the horizon, these large fluctuations could collapse due to their strong internal gravitational force, producing the so called Primordial Black Holes (PBHs). There are two important differences between PBHs and astrophysical ones to keep in mind:

- Astrophysical BHs have a mass bound $M > 3M_\odot$, that comes from the fact that only very massive stars can collapse into BHs. PBHs do not have such a bound: we could theoretically produce stellar/super-massive PBHs as well as very light ones. However, below we will see that there is a series of observational constraints that limits the amount of PBHs that can be produced for each mass.

- Astrophysical BHs surely cannot have been produced at before the first stars were born, i.e. before the Universe was 200 million years old. Instead PBHs can be produced in post-inflationary epochs before the Big Bang Nucleosynthesis and, in principle, even after; in this latter case however they would destroy the BBN model and, therefore, we assume that all the PBHs are produced before BBN occurs.

PBHs and their proprieties have been theoretically studied since Hawking's paper back in the '70s. However, the idea that they could be the main constituents of DM is quite recent: it descends from LIGO detections of Gravitational Waves (GWs) emitted by binary BH mergers [3–7], that triggered a renewed interest in them. Initially, this renewed interest in PBHs was limited to the solar mass range because the merging BH masses, that span from 7 to a few tens of $M_\odot$, turned out to be larger than expected; this led to the publication of several articles aimed at explaining the nature of these binary systems, some of which (see Ref. [8–10]) point out that the inferred mass range can be explained by the cohalescence of PBHs, without violating the bound that the PBH abundance today has to be equal to or less than the total DM abundance. After that, this same bound has gained the attention of the astrophysical and theoretical community, that started wondering if these BHs originated from large primordial fluctuations produced during inflation could constitute a significant amount of the present DM content of the Universe.

The mass of a PBH depends only on the scale k at which the perturbations that seed its formation cross the horizon: the relation has been found in [11] as (for PBH formation in an epoch of radiation dominance)

$$M_{\mathrm{k}} \simeq 5\gamma \cdot 10^{18} \left(\frac{k}{7 \cdot 10^{13}\,\mathrm{Mpc}^{-1}} \right)^{-2} \mathrm{g} \tag{1.3.1}$$

where γ is a numerical factor between 0 and 1, whose value depends on the details of the gravitational collapse, and measures the efficiency of the process (for the moment, we will not assign a value to it). Since inflation involved all the range of scales k that are currently observable, there is a vast interval of masses M_{k} that are a priori possible for PBHs contributing to the amount of DM. However, the fractional abundance of PBHs over the total DM content at present time, $f_{\mathrm{PBH}}(M_{\mathrm{k}}) = \Omega_{\mathrm{PBH,0}}(M_{\mathrm{k}})/\Omega_{\mathrm{DM,0}}$, has an upper limit due to constraints of various nature on this wide range of masses M_{k}. For example, due to Hawking's radiation, PBHs lighter than $10^{-18} M_{\odot}$ cannot be part of current DM, because they are completely evaporated by now (see Ref. [12]); hence, $f_{\mathrm{PBH}}(M_{\mathrm{k}} \leq 10^{-18} M_{\odot}) = 0$.

Figure 1.1 illustrates all the constraints on f_{PBH}, showing its upper limit for different values of M_{k}. The bounds have been derived under the assumption that f_{PBH} is monochromatic, an appropriate approximation for narrow mass distributions like the one that we will consider, and all of them have been explained in great detail in [13]; also, they all come with a certain degree of uncertainty, due for example to various assumptions made on cosmological parameters in order to estimate them.

In Chapter 2, we will consider a particular PBH mass M_{f} in the interval $10^{-16} M_{\odot} \leq M_{\mathrm{f}} \leq 10^{-11} M_{\odot}$, since this latter results to be the only window where constraints of Fig. 1.1 allow PBHs to constitute a significant fraction of the total DM abundance. Imposing $f_{\mathrm{PBH}}(M_{\mathrm{f}}) = 1$, i.e. imposing that PBHs with mass M_{k} constitute 100% of the DM today, we will find, via the Power Spectrum $P_{\mathrm{k}} \sim (\delta\rho/\rho)^2$, the enhancement of the fluctuations necessary for that mass M_{f}. This procedure will be done in two different cases: in the first case, the scale corresponding to the enhanced fluctuations reenters the horizon in an epoch of Radiation Domination (RD), while in the second case the same scale reenters in an epoch of Matter Domination (MD). This last epoch is assumed to occur because of a temporary domination of heavy scalar particles, like the moduli originated from String Theory. As we already mentioned, since the BBN model has been tested experimentally and works really well, this new matter dominated epoch that we are going to consider has to end before the BBN time.

After all this, in Chapter 3, we will examine a particular single-field model of string inflation, called "Fibre Inflation" (see Ref. [23]), and tune its potential in order to reproduce the features of the Power Spectrum previously estimated in the MD Section of

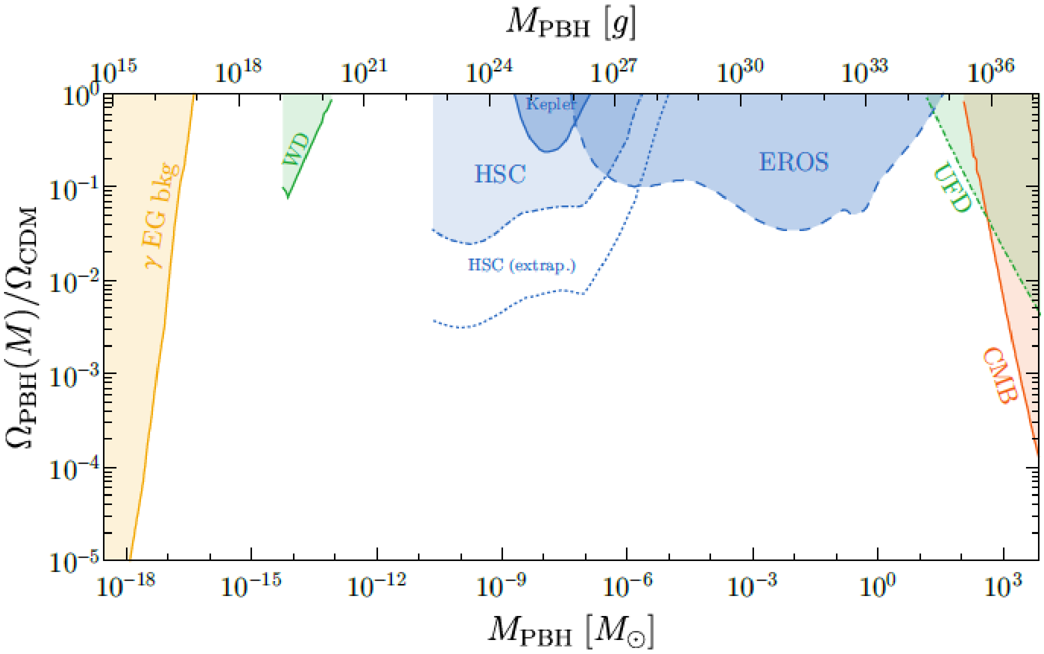

Figure 1.1: Experimental constraints on the fraction of PBHs with mass $M_{\rm PBH}$ over the total DM. In yellow, the observations of extra-galactic γ-ray background [14]; in green, dynamical constraints from White Dwarfs and Ultra-Faint Dwarf galaxies [15]; in blue, micro- and milli-lensing observations from EROS [16], Kepler [17] and Subaru HSC [18]; in orange, constraints from the CMB observations [19–21]. This figure was adapted from the one in [22] to be in agreement with the most recent observations.

Chapter 2. As we have already mentioned, the inflationary potential V needs to be tuned so that the fluctuations turn out to be enhanced on a particular scale k and produce PBHs at horizon reentry. If we assume that V is approximately constant during inflation, which is the case of standard slow-roll, we can write $P_{\rm k} \sim H^2/\epsilon_V$[17], where $\epsilon_V \sim (V'/V)^2$ is the slow-roll parameter that measures the flatness of the potential[18]. Then, one might think that $P_{\rm k}$ can be enhanced simply by taking $\epsilon_V \to 0$, i.e. imposing V to be flat on the momentum scales k where we want the amplification to take place. Unfortunately things are more complicated than this: Ref. [24] showed that a single-field inflationary potential that produces a PBH population capable of accounting for a significant portion of the total DM today, must feature a near-inflection point after the slow-roll plateau. The authors of [25] pointed out that, close to this near-inflection point, the slow-roll approximation ceases to be valid and our system enters an ultra slow-roll regime, where the inflaton decelerates. In this regime the approximated expression of $P_{\rm k}$ that we mentioned

[17]See (1.2.76).
[18]See (1.2.31).

above is not valid anymore, and that calls for a more careful analysis of our model, where the observables will be computed from solutions to the Mukhanov-Sasaki equation, with a numerical approach.

Chapter 2

PBH Production

PBHs are produced by the gravitational collapse of very large density perturbations which reenter the horizon. We will examine the reentry of these perturbations in two post-inflationary epochs: Radiation[1] Domination (RD), and Matter[2] Domination (MD) driven by a gravitationally coupled scalar field. Notice that this last epoch is assumed to be post-inflationary and to end before Big Bang Nucleosynthesis (BBN) since this model does not have theoretical problems and is in agreement with observations. Before we start, let us introduce some important quantities in our study of PBH production:

- $f_{\rm PBH}(M_{\rm f})$ is the energy density of PBHs with mass $M_{\rm f}$ over the total Dark Matter (DM) energy density at present time. One of our objectives is to determine an expression for this quantity in terms of the power spectrum $P_\zeta(k)$, which is our main observable quantity. This is crucial because imposing $f_{\rm PBH} = 1$, i.e. that PBHs constitute all of the DM today, we can find out how big the amplitude of the density perturbations $(\delta\rho/\rho)^2 \sim P_\zeta(k)$ has to be at horizon reentry in order to produce enough PBHs with mass $M_{\rm f}$ to explain all of the DM today.

- $\beta(M_{\rm f})$ is the energy density of PBHs with mass $M_{\rm f}$ over the total energy density at the time of formation.

- γ is a numerical factor between 0 and 1 which represents the fraction of the horizon mass that collapses due to gravitational force, forming a black hole; its value depends on the details of the gravitational collapse.

- $\Delta N_{\rm CMB}^{\rm PBH}$ is the distance, in e-foldings, between the following two events that occurred during inflation: the horizon exit of the scale $k_{\rm CMB}$, corresponding to the

[1]In the context of this cosmological dissertation, the term "radiation" refers to a fluid of ultra-relativistic particles, whose kinetic energy is such that $p = \frac{1}{3}\rho$.

[2]Analogously, the term "matter" refers to a fluid of non-relativistic particles, whose kinetic energy is completely negligible compared to the energy due to mass. Therefore, its equation of state is $p = 0$.

perturbations we observe today in the CMB, and the exit of the scale $k_{\rm PBH}$, that corresponds to the large density perturbations that collapsed into PBHs at horizon re-entry.

2.1 PBH formation in Radiation Domination

Let us begin our study of PBH production by searching for an expression for $\beta_{\rm RD}(M_{\rm f})$: since it represents the fraction of total energy density in PBHs with mass $M_{\rm f}$ evaluated at the time of PBH formation[3], we could simply write it as

$$\beta_{\rm RD}(M_{\rm f}) \equiv \frac{1}{\gamma}\frac{\rho_{\rm PBH}(M_{\rm f})}{\rho_{\rm tot}}\bigg|_{\rm f} \tag{2.1.1}$$

However $\beta_{\rm RD}(M_{\rm f})$ can also be interpreted as the probability that large enough density fluctuations collapse into PBHs. Assuming spherical collapse and that the density perturbations δ reentering the horizon follow a Gaussian distribution, in RD we get

$$\begin{aligned}\beta_{\rm RD}(M_{\rm f}) &= \int_{\delta_{\rm c}}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma(M_{\rm f})} e^{-\frac{\delta^2}{2\sigma(M_{\rm f})^2}}\,{\rm d}\delta \\ &\sim \frac{1}{\sqrt{2\pi}}\frac{\sigma(M_{\rm f})}{\delta_{\rm c}}\, e^{-\frac{\delta_{\rm c}^2}{2\sigma(M_{\rm f})^2}}\end{aligned} \tag{2.1.2}$$

where $\delta_{\rm c} \approx 0.41$ represents the minimum fluctuation needed for a fraction γ of the horizon mass $M_{\rm H}$ to undergo gravitational collapse and produce PBHs; the last step in the expression above is a good approximation if $\sigma(M_{\rm f}) \ll \delta_{\rm c}$, and we will show below that this holds in our case. The width of this Gaussian distribution is set by the power spectrum of density perturbations, i.e. $\sigma(M_{\rm f})^2 \sim \langle\delta\delta\rangle \sim P_\delta(k_{\rm f})$, where in the comoving gauge (1.2.37) δ can be expressed in terms of the curvature perturbations ζ as

$$\delta = 2\,\frac{1+w}{3w+5}\,\zeta \xrightarrow[w=1/3]{\rm RD} \delta = \frac{4}{9}\,\zeta \tag{2.1.3}$$

Being that $\delta \sim \zeta$, density and curvature perturbations are practically the same quantity. For this reason, even if from this moment onward our calculations will be done in terms of ζ, we will often refer to perturbations as density fluctuations.

Now, before studying $f_{\rm PBH}$, we are going to take a detour and review some cosmology. We begin by reminding that, for the horizon mass

$$M_{\rm H} = \rho_{\rm tot} V = \frac{4\pi}{3}\frac{\rho_{\rm tot}}{H^3} = \frac{4\pi}{3}\frac{3H^2M_{\rm p}^2}{H^3} = 4\pi\frac{M_{\rm p}^2}{H} \tag{2.1.4}$$

[3]We will always make the assumption that the formation of PBHs occurs exactly at horizon reentry.

where $M_{\rm p} = (8\pi G)^{-1/2}$ is the reduced Planck mass, in units $\hbar = c = 1$, and we used the Friedmann equation (1.2.15) for a flat universe, i.e. $\rho_{\rm tot} = 3H^2 M_{\rm p}^2$. When a sufficiently large density perturbation reenters the horizon, it collapses under its own gravity and forms a black hole, whose mass at formation $M_{\rm f}$ is just a fraction γ (depending on the detail of the gravitational collapse) of the horizon mass; hence

$$M_{\rm f} = \gamma M_{\rm H,f} = 4\pi\gamma \frac{M_{\rm p}^2}{H_{\rm f}} \tag{2.1.5}$$

The energy density for a gas of ultra-relativistic particles (i.e. radiation) at temperature T is (see Ref. [26])

$$\rho_{\rm rad} = \frac{\pi^2}{30}\, g_*(T)\, T^4 \tag{2.1.6}$$

where $g_*(T)$ represents the total relativistic degrees of freedom of the fluid. Since our universe is very flat we have $\rho_{\rm crit} \equiv 3H^2 M_{\rm p}^2 \simeq \rho_{\rm tot}$ and therefore

$$\Omega_i \equiv \frac{\rho_i}{\rho_{\rm crit}} \simeq \frac{\rho_i}{\rho_{\rm tot}} \tag{2.1.7}$$

for any component i of the Universe. Then, using $\rho_{\rm rad} = \Omega_{\rm rad}\, \rho_{\rm tot} = 3\, \Omega_{\rm rad}\, H^2 M_{\rm p}^2$ in (2.1.6) we find that the temperature of the cosmological fluid behaves as

$$T = \left(\frac{90\, \Omega_{\rm rad}}{\pi^2 g_*(T)}\right)^{1/4} \sqrt{H M_{\rm p}} \tag{2.1.8}$$

The entropy density s of a system of relativistic particles in thermal equilibrium is

$$s = \frac{\rho_{\rm rad} + p}{T} = \frac{4}{3}\frac{\rho_{\rm rad}}{T} \tag{2.1.9}$$

where we used the equation of state for radiation $p_{\rm rad} = \frac{1}{3}\rho_{\rm rad}$. The conservation of comoving entropy $sa^3 = \text{const}$, which holds assuming that there is no entropy production, can be combined with (2.1.9), (2.1.6) and (2.1.8), so that

$$g_*(T)\, a^3\, T^3 = \text{const} \quad \Leftrightarrow \quad g_*(T)^{1/4}\, a^3\, \Omega_{\rm rad}^{3/4}\, H^{3/2} = \text{const} \tag{2.1.10}$$

Using this last expression, the Hubble parameter at formation $H_{\rm f}$ can be related to the one at present time H_0 as

$$H_{\rm f} = H_0 \left(\frac{g_{*0}}{g_{*\rm f}}\right)^{1/6} \left(\frac{a_0}{a_{\rm f}}\right)^2 \left(\frac{\Omega_{\rm rad,0}}{\Omega_{\rm rad,f}}\right)^{1/2} \simeq H_0 \frac{\Omega_{\rm rad,0}^{1/2}}{a_{\rm f}^2} \left(\frac{g_{*0}}{g_{*\rm f}}\right)^{1/6} \tag{2.1.11}$$

In the last step we have used the fact that we are assuming PBH formation occurs during RD, hence $\Omega_{\rm rad,f} \simeq 1$; we have also set $a_0 = 1$. We can now write the PBH mass at formation $M_{\rm f}$ using (2.1.5), (2.1.11), the horizon crossing relation $k = aH$ and then (2.1.10); we get

$$\begin{aligned} M_{\rm f} &\simeq 4\pi \frac{M_{\rm p}^2}{H_0}\,\gamma\,\Omega_{\rm rad,0}^{-1/2}\left(\frac{g_{*\rm f}}{g_{*0}}\right)^{1/6}\left(\frac{a_{\rm f}k_{\rm f}}{a_0 k_0}\right)^2\left(\frac{k_0}{k_{\rm f}}\right)^2 \\ &= \gamma\, M_{\rm H,0}\,\Omega_{\rm rad,0}^{-1/2}\left(\frac{g_{*\rm f}}{g_{*0}}\right)^{1/6}\left(\frac{a_{\rm f}^2 H_{\rm f}}{a_0^2 H_0}\right)^2\left(\frac{k_0}{k_{\rm f}}\right)^2 \\ &\simeq \gamma\, M_{\rm H,0}\,\Omega_{\rm rad,0}^{1/2}\left(\frac{g_{*0}}{g_{*\rm f}}\right)^{1/6}\left(\frac{k_0}{k_{\rm f}}\right)^2 \end{aligned} \tag{2.1.12}$$

We should now clarify that this relation is valid only for the fraction of the horizon mass that collapses in PBHs, $M_{\rm f}$; a more general version of (2.1.12), valid for the horizon mass $M_{{\rm H},*}$ at any momentum scale k during RD, can be obtained from this latter simply dividing by γ.

Now let us go back to search an expression for the fraction of the total DM energy density in PBHs with mass $M_{\rm f}$ today, $f_{\rm PBH}(M_{\rm f})$. It is defined as

$$f_{\rm PBH}(M_{\rm f}) \equiv \frac{\rho_{\rm PBH,0}(M_{\rm f})}{\rho_{\rm DM,0}} = \frac{\Omega_{\rm PBH,0}(M_{\rm f})}{\Omega_{\rm DM,0}} = \frac{\Omega_{\rm rad,0}}{\Omega_{\rm DM,0}}\frac{\rho_{\rm PBH,0}(M_{\rm f})}{\rho_{\rm rad,0}} \tag{2.1.13}$$

To express $f_{\rm PBH}(M_{\rm f})$ in terms of our observable quantity $P_\zeta(k) \equiv P_k$, we will write it in terms of $\beta_{\rm RD}(M_{\rm f})$. We recall that the energy density redshifts as

$$\rho \sim \begin{cases} a^{-4} & \text{for radiation} \\ a^{-3} & \text{for matter} \end{cases} \tag{2.1.14}$$

Using the fact that PBHs redshift as matter, we can determine their abundance today

$$\begin{aligned} \rho_{\rm PBH,0}(M_{\rm f}) &= \rho_{\rm PBH,f}(M_{\rm f})\left(\frac{a_{\rm f}}{a_0}\right)^3 \\ &= \gamma\,\beta_{\rm RD}(M_{\rm f})\,\rho_{\rm tot,f}\left(\frac{a_{\rm f}}{a_0}\right)^3 \\ &\simeq \gamma\,\beta_{\rm RD}(M_{\rm f})\,\rho_{\rm rad,f}\left(\frac{a_{\rm f}}{a_0}\right)^3 \end{aligned} \tag{2.1.15}$$

where we also used (2.1.1) and the fact that we are assuming reentry in RD, hence $\rho_{\rm tot,f} \simeq \rho_{\rm rad,f}$. Now, using this last result with (2.1.6) and (2.1.10), the relation (2.1.13) becomes

$$\begin{aligned} f_{\rm PBH}(M_{\rm f}) &\simeq \gamma\,\beta_{\rm RD}(M_{\rm f})\frac{\Omega_{\rm rad,0}}{\Omega_{\rm DM,0}}\frac{\rho_{\rm rad,f}}{\rho_{\rm rad,0}}\left(\frac{a_{\rm f}}{a_0}\right)^3 \\ &= \gamma\,\beta_{\rm RD}(M_{\rm f})\frac{\Omega_{\rm rad,0}}{\Omega_{\rm DM,0}}\left(\frac{g_{*\rm f}a_{\rm f}^3T_{\rm f}^3}{g_{*0}a_0^3T_0^3}\right)\frac{T_{\rm f}}{T_0} = \gamma\,\beta_{\rm RD}(M_{\rm f})\frac{\Omega_{\rm rad,0}}{\Omega_{\rm DM,0}}\frac{T_{\rm f}}{T_0} \end{aligned} \tag{2.1.16}$$

We can express (2.1.16) in terms of the mass M_f: using in order (2.1.10), $k = aH$, (2.1.4), (2.1.5) and (2.1.12), we arrive at

$$\begin{aligned}
\frac{T_\mathrm{f}}{T_0} &= \left(\frac{g_{*\mathrm{f}}}{g_{*0}}\right)^{-1/3} \frac{a_0}{a_\mathrm{f}} \\
&= \left(\frac{g_{*0}}{g_{*f}}\right)^{1/3} \frac{H_\mathrm{f} k_\mathrm{f}^{-1}}{H_0 k_0^{-1}} \\
&= \left(\frac{g_{*0}}{g_{*f}}\right)^{1/3} \frac{\gamma\, M_{\mathrm{H},0}}{M_\mathrm{f}} \frac{k_0}{k_f} \\
&\simeq \left(\frac{g_{*0}}{g_{*f}}\right)^{1/3} \frac{\gamma\, M_{\mathrm{H},0}}{M_\mathrm{f}} \frac{\gamma^{-1/2}}{\Omega_{\mathrm{rad},0}^{1/4}} \left(\frac{M_\mathrm{f}}{M_{\mathrm{H},0}}\right)^{1/2} \left(\frac{g_{*0}}{g_{*\mathrm{f}}}\right)^{-1/12} \\
&= \left(\frac{g_{*0}}{g_{*\mathrm{f}}\, \Omega_{\mathrm{rad},0}}\right)^{1/4} \left(\frac{\gamma\, M_{\mathrm{H},0}}{M_\mathrm{f}}\right)^{1/2}
\end{aligned} \tag{2.1.17}$$

that, substituted into (2.1.16), leads us to the expression of f_PBH in terms of the PBH mass M_f in the case of formation during RD:

$$f_\mathrm{PBH}(M_\mathrm{f}) \simeq \gamma^{3/2}\, \beta_\mathrm{RD}(M_\mathrm{f})\, \frac{\Omega_{\mathrm{rad},0}^{3/4}}{\Omega_{\mathrm{DM},0}} \left(\frac{g_{*0}}{g_{*\mathrm{f}}}\right)^{1/4} \left(\frac{M_{\mathrm{H},0}}{M_\mathrm{f}}\right)^{1/2} \tag{2.1.18}$$

We can now quickly estimate the amplification required in P_k for PBHs with mass M_f to constitute 100% of DM today. We assume that only the Standard Model (SM) degrees of freedom are present, so that $g_{*0} = 3.38$ and $g_{*\mathrm{f}} = 106.75$ (see Ref. [26], page 55); today we have $\Omega_{\mathrm{DM},0} \approx 0.23$, $\Omega_{\mathrm{rad},0} \approx 8.24 \cdot 10^{-5}$ and $H_0 \approx 6.363 \cdot 10^{-61} M_\mathrm{p}$[4], so that using (2.1.4) we find $M_{\mathrm{H},0} \approx 4.35 \cdot 10^{22} M_\odot$; setting $\gamma = 1$, $f_\mathrm{PBH}(M_\mathrm{f}) = 1$ and a PBH mass of $M_\mathrm{f} = 10^{-15} M_\odot$, which according to Fig. 1.1 is a value in the mass window that allows 100% of DM as PBHs, eq. (2.1.18) gives $\beta_\mathrm{RD}(M_\mathrm{f}) \approx 9.6 \cdot 10^{-17}$. This, substituted in the approximated version of (2.1.2), leads to $\sigma \approx 0.049$ and, therefore, $\zeta \approx 0.11$; this value is $\ll \zeta_\mathrm{c}$ justifying our use of the approximated expression of $\beta_\mathrm{RD}(M_\mathrm{f})$. Finally,

$$P_k \simeq \zeta^2 \approx 1.26 \cdot 10^{-2} \tag{2.1.19}$$

Today, i.e. at CMB scales k_CMB, we measure $P_k \approx 2 \cdot 10^{-9}$. This means that the only way PBHs of mass $10^{-15} M_\odot$ produced during a RD era could explain the 100% of the DM today is that, during inflation, the power spectrum underwent a 7 orders of magnitude enhancement. It is easy to understand that a very specific inflation potential is required to realize a power spectrum with such an enhancement, especially considering that the peak value is not the only constraint on the power spectrum.

[4] This value for H_0 was obtained using $H_0 = 72.5\,(\mathrm{km/s})/\mathrm{Mpc}$ and the conversion table [27].

During inflation the fluctuations on the CMB scales, $k_{\rm CMB}$, leave the horizon before the enhanced fluctuations, that are produced when the scale $k_{\rm PBH} > k_{\rm CMB}$ leaves the horizon; but what is the distance in e-foldings $\Delta N_{\rm CMB}^{\rm PBH}$ between these two horizon crossings? This is not a free parameter, and can be determined as follows.

First of all, we recall that the scale factor at a time t can be casted in terms of the number of e-foldings at that same time, $N(t)$, as

$$a(t) = a_0\, e^{N(t)} \tag{2.1.20}$$

In our case of interest, this leads to the definition of $\Delta N_{\rm CMB}^{\rm PBH}$

$$a_{\rm PBH} = a_{\rm CMB}\, e^{\Delta N_{\rm CMB}^{\rm PBH}} \quad \Rightarrow \quad \Delta N_{\rm CMB}^{\rm PBH} \equiv \ln\left(\frac{a_{\rm PBH}}{a_{\rm CMB}}\right) \tag{2.1.21}$$

If we now multiply and divide by $H_{\rm end}$, where the subscript indicates the end of inflation, in the slow-roll approximation $H \simeq$ const we find

$$\Delta N_{\rm CMB}^{\rm PBH} = \ln\left(\frac{a_{\rm PBH}\, H_{\rm end}}{a_{\rm CMB}\, H_{\rm end}}\right) = \ln\left(\frac{a_{\rm PBH}\, H_{\rm PBH}}{a_{\rm CMB}\, H_{\rm CMB}}\right) = \ln\left(\frac{k_{\rm PBH}}{k_{\rm CMB}}\right)$$

The mode $k_{\rm PBH}$, which left the horizon when inflation was almost over is the one that, reentering the horizon during RD, triggers the PBHs formation $\Rightarrow k_{\rm PBH} = k_{\rm f} = a_{\rm f} H_{\rm f}$. Then

$$\Delta N_{\rm CMB}^{\rm PBH} = \ln\left(\frac{k_{\rm f}}{k_{\rm CMB}}\right)$$

We can now write $\Delta N_{\rm CMB}^{\rm PBH}$ as a function of $M_{\rm f}$ by employing (2.1.12)

$$\Delta N_{\rm CMB}^{\rm PBH} = \ln\left(\frac{k_0}{k_{\rm CMB}}\right) + \frac{1}{4}\ln\Omega_{\rm rad,0} + \frac{1}{2}\ln\gamma - \frac{1}{12}\ln\left(\frac{g_{*\rm f}}{g_{*0}}\right) - \frac{1}{2}\ln\left(\frac{M_{\rm f}}{M_{\rm H,0}}\right) \tag{2.1.22}$$

Using $k_0 = H_0 \approx 2.4\cdot 10^{-4}{\rm Mpc}^{-1}$, $k_{\rm CMB} \approx 0.05$, $\Omega_{\rm rad,0} \approx 8.24\cdot 10^{-5}$ and $M_{\rm H,0} \approx 4.35\cdot 10^{22} M_\odot$, the expression above simplifies to

$$\Delta N_{\rm CMB}^{\rm PBH} \simeq 18.4 + \frac{1}{2}\ln\gamma - \frac{1}{12}\ln\left(\frac{g_{*\rm f}}{g_{*0}}\right) - \frac{1}{2}\ln\left(\frac{M_{\rm f}}{M_\odot}\right) \tag{2.1.23}$$

that, for the values of γ, g_{*0}, $g_{*\rm f}$ and $M_{\rm f}$ listed above (2.1.19), gives

$$\Delta N_{\rm CMB}^{\rm PBH} \approx 35.3 \tag{2.1.24}$$

i.e., $k_{\rm PBH}$ leaves the horizon approximately 35 e-foldings after the CMB modes.

The power spectrum constraints that we found, (2.1.19) and (2.1.24), are quite strong. As we briefly explained in Section 1.3, in order to obtain such a high peak in $P_{\rm k}$ the inflationary potential must exhibit a plateau followed by a near-inflection point, which leads the inflaton into an ultra slow-roll regime. In this regime the inflation decelerates, enhancing the power spectrum of scalar perturbations and triggering efficient black hole production with a peaked mass distribution. Such a particular shape of the inflationary potential requires a very high tuning freedom, which is the big downside behind the apparently simple idea of dark matter as PBHs. Some interesting examples of articles that examined PBH production in RD with great detail, with the objective of explaining all of the DM today, are [28] and [29]. The latter in particular was written by my thesis supervisors, Michele Cicoli and Francisco Gil Pedro, and examines PBH production by a single-field inflationary string model named Fibre Inflation [23], which we will briefly present in Chapter 3.

The question we should ask ourselves at this point is: is it possible to lower the amount of fine-tuning required in the inflationary potential and make PBH production more "natural"? What if, for example, we suppose PBHs to be produced during a Matter Dominated (MD) era before BBN?

2.2 Matter Domination due to Heavy Scalars

We already mentioned that, aside from RD, we would like to study PBH production in an epoch dominated by matter. We have also mentioned how, in order to preserve the well working BBN model, this MD epoch has to occur before Nucleosynthesis. These two aspects seem apparently incompatible, because we are used to think to matter as the combination of baryonic matter and dark matter: baryonic matter cannot decouple from radiation before BBN, and if we want to explain all of the dark matter in terms of PBHs it is nonsensical to assume that the MD epoch in which we are trying to study the PBH production is driven by dark matter. However we recall that, in cosmology, the term "matter" has a broader meaning than this latter: it refers to anything that has an equation of state $p = 0$, so that its energy density redshifts as $\rho \sim a^{-3}$. Gravitationally coupled scalar fields (moduli) have this equation of state and, admitting their existence, we will now see the process by which these fields temporarily dominate the energy density of the Universe after inflation.

Let us consider a generic massive scalar field (modulus) φ; its potential can be approximately expressed as

$$V(\varphi) \sim \frac{1}{2}m_\varphi^2(\varphi-\varphi_0)^2 - H^2(\varphi-\varphi_0)^2 + \frac{1}{M_{\rm p}^{2n}}(\varphi-\varphi_0)^{4+2n} \tag{2.2.1}$$

where φ_0 is the true vacuum-expectation-value of the field (i.e. its value today), m_φ is its mass, H is the Hubble parameter and n takes the number of dimensions into account. During inflation $H \gg m_\varphi$ and, therefore, the last two terms dominate. Hence, the minimum of the potential is situated at

$$\langle\varphi\rangle_{\rm inf} \sim \varphi_0 + M_{\rm p}\left(\frac{H_{\rm inf}}{M_{\rm p}}\right)^{1/(1+n)}, \qquad H \gg m_\varphi \tag{2.2.2}$$

where $H_{\rm inf} \simeq$ const is the value of the Hubble parameter during inflation. This last result holds assuming that the induced squared-mass parameter of φ during inflation is negative.

After the end of inflation H ceases to be constant and starts to decrease; this causes the first term of (2.2.1) to dominate over the others, and the minimum of the potential can now be found at

$$\langle\varphi\rangle_0 \sim \varphi_0, \qquad H \ll m_\varphi \tag{2.2.3}$$

Therefore, because of inflation, a shift of the minimum occurred between the inflationary epoch and today, whose entity is

$$|\Delta\varphi| \equiv |\langle\varphi\rangle_0 - \langle\varphi\rangle_{\rm inf}| \sim M_{\rm p}\left(\frac{H_{\rm inf}}{M_{\rm p}}\right)^{1/(1+n)} \lesssim M_{\rm p} \tag{2.2.4}$$

Now, the evolution of the moduli after inflation is described by the following equation

$$\ddot\varphi + (3H + \Gamma_\varphi)\dot\varphi + \frac{\partial V}{\partial\varphi} = 0 \tag{2.2.5}$$

where, in the friction term, the modulus decay rate appears

$$\Gamma_\varphi = D_\varphi \frac{m_\varphi^3}{M_{\rm p}^2} \tag{2.2.6}$$

This last expression reflects the fact that the modulus is gravitationally coupled ($\Gamma_\varphi \sim M_{\rm p}^{-2} \sim G$); D_φ is a constant depending on the model, whose value is close to unity.

During inflation, the minimum is (2.2.2), and the modulus is positioned on it. Then, as we already mentioned, after the end of inflation H decreases as t^{-1} and this causes the minimum to start moving towards (2.2.3). However, the modulus does not immediately follow the minimum: immediately after inflation H is still important, and the friction term in (2.2.5) is not negligible at all. The decay ratio is Planck suppressed, and the equation becomes

$$\ddot\varphi + \mathcal{O}(1)\frac{1}{t}\dot\varphi + \frac{\partial V}{\partial\varphi} = 0 \tag{2.2.7}$$

where we employed the fact that, in the epochs after inflation[5], $H \sim t^{-1}$. Because of this, in the first moments after inflation φ moves so slow towards the new minimum that

[5]During the reheating epoch we have matter domination driven by oscillation of the inflaton field, hence $H \sim \frac{2}{3t}$. After the inflaton decay, the thermalization induces a RD era, hence $H \sim \frac{1}{2t}$.

it can be almost considered stationary. This goes on until the time $t_{\rm osc}^{-1} \sim H \sim m_\varphi$: now all the terms in the potential have the same weight, and H is so small that the friction term in (2.2.7) has not the same impact as before and can be ignored. Therefore the field starts moving towards the new minimum, and the initial displacement causes it to freely oscillate around φ_0. This is described by

$$\ddot{\varphi} + \frac{\partial V}{\partial \varphi} = 0 \tag{2.2.8}$$

These oscillations make the field evolve as pressure-less matter, with an energy density $\rho_\varphi = \frac{1}{2} m_\varphi^2 f_\varphi^2 \sim a^{-3}$, where f_φ is the amplitude of the oscillations. The time $t_{\rm osc}$ occurs not long after the inflaton reheating, when the universe is dominated by radiation; since this last component redshifts faster than matter ($\rho_{\rm rad} \sim a^{-4}$), if enough energy is initially stored in the scalar condensate it will quickly grow to dominate the total energy density, leading to a MD era. This MD era driven by moduli ceases at the time when $H \simeq \Gamma_\varphi$, i.e. when the age of the Universe becomes compatible with the lifetime of the modulus. At this point the modulus decades into Standard Model particles, a new reheating occurs and the radiation returns the dominant component of the Universe. As we will notice in Chapter 3 while studying Fibre Inflation, moduli fields are a natural consequence of string inflationary models.

2.3 PBH formation in Matter Domination

Now that we explained how an epoch of Matter Domination (MD) can take place before BBN, thanks to scalar fields lighter than the inflaton (moduli) that temporarily dominate the energy density of the Universe, we are going to assume that the mode corresponding to the peak in the power spectrum $P_{\rm k}$ reenters the horizon in a MD era. We will show below that the expressions of $f_{\rm PBH}(M_{\rm f})$ and $\Delta N_{\rm CMB}^{\rm PBH}$ are different from their RD counterparts, the main difference being the appearance of the field mass m_φ, an additional parameter beside the PBH mass $M_{\rm f}$. We will find out that this additional parameter can help reducing the amount of enhancement required to explain all of the DM today in terms of PBHs.

We begin by studying $\beta_{\rm MD}(M_{\rm f})$, whose definition is the same as in RD

$$\beta_{\rm MD}(M_{\rm f}) \equiv \frac{1}{\gamma} \frac{\rho_{\rm PBH}(M_{\rm f})}{\rho_{\rm tot}}\bigg|_{\rm f} \tag{2.3.1}$$

However, in MD an analogous of (2.1.2) does not exist: a detailed expression of the fluctuations threshold in terms of w[6] has been found in [30] and shows that, for $w = 0$,

[6] w is the cosmological parameter used to express the cosmological fluid equation in the compact form $p = w\rho$, $c = 1$; for matter $w = 0$, for radiation $w = \frac{1}{3}$.

$\zeta_c \to 0$, i.e. any perturbation can collapse into PBHs, no matter how little it is; this result does not make any sense, which means that (2.1.2) is not a correct expression for $\beta_{\rm MD}$. The correct relation that links $\beta_{\rm MD}$ with the fluctuations has been obtained in [31]:

$$\beta_{\rm MD}(M_{\rm f}) = 0.2055\, \sigma(M_{\rm f})^{13/2} \tag{2.3.2}$$

where, again, $\sigma(M_{\rm f})$ represents the density fluctuations; this expression was found in a semi-analytical way by studying in detail the gravitational collapse of PBHs and its non-spherical effects, inhomogeneity and anisotropy, that play an important role in MD. Since we want to work in terms of the curvature perturbations ζ we make use of (2.1.3) with $w = 0$, which tells us that $\sigma(M_{\rm f})^2 \sim P_\delta \sim (\frac{2}{5})^2 P_\zeta$, and finally get

$$\beta_{\rm MD}(M_f) \simeq 5.3236 \cdot 10^{-4}\, P_\zeta^{13/4} \tag{2.3.3}$$

Before analyzing the $f_{\rm PBH}$ expression, let us explain with some detail the non-standard cosmological history that we are assuming to occur

1. *Inflation* ($t < t_{\rm end}$): Our Universe is dominated by a scalar field, the inflaton ϕ, which causes it to expand faster than the speed of light while its energy density $\rho_{\rm tot} = 3H^2 M_{\rm p}^2 \simeq$ const. This epoch, needed to solve various problem in the hot Big Bang model, ceases at $t_{\rm end}$.

2. *Reheating* ($t_{\rm end} < t < t_{\rm rh}$): This is an epoch of matter domination $\rho_{\rm tot} \simeq \rho_{\rm m} \sim a^{-3}$ that occurs after the end of inflation, driven by the oscillations of the inflaton field around the minimum of its potential.

3. *First radiation dominated era* ($t_{\rm rh} < t < t_{\rm dom}$): The standard hot Big Bang theory begins when the inflaton decays into Standard Model particles. We assume that an instantaneous thermalisation of these particles occurs at time $t_{\rm rh}$, so that they form a thermal bath which immediately starts to dominate the energy density of the Universe. Like in any radiation epoch, $\rho_{\rm tot} \simeq \rho_{\rm rad} \sim a^{-4} g_*^{-1/3}$.

4. *Moduli dominated era* ($t_{\rm dom} < t < t_{\rm dec}$): At some point $t_{\rm dom}$ the energy density of a scalar field φ, called modulus field, starts to dominate over radiation. Our Universe experience a matter dominated era because of the oscillations of this field around its new minimum. We assume that PBHs are produced at some time $t_{\rm f}$ during this epoch. At $t_{\rm dec}$ the oscillations of the modulus cease: an instantaneous decay of the modulus field occurs, ending the matter dominated epoch.

5. *Second radiation dominated era* ($t_{\rm dec} < t < t_{\rm eq}$): The modulus decays in Standard Model particles and triggers a second thermalisation of the Universe: the radiation regains energy density, becoming the dominant component of the Universe once again. However this cannot last forever, since matter redshifts slower than radiation: at the time $t_{\rm eq}$ the baryonic matter produced by BBN has the same weight

of radiation on the energy balance of the Universe and, after that, the ordinary matter dominated era occurs.

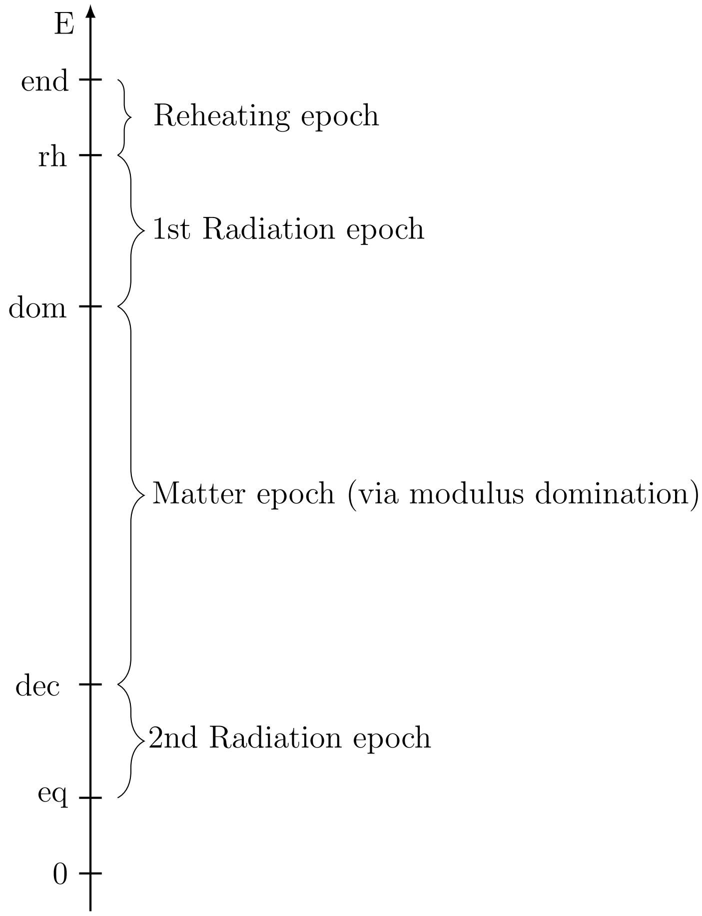

Figure 2.1: Scheme of the epochs between the end of inflation ("end") and today ("0"), on the energy axis; time's arrow is reversed respect to this axis. Formation of PBHs, i.e. the moment when the mode corresponding to the peak in $P_{\rm k}$ reenters the horizon, occurs somewhere between $t_{\rm dom}$ and $t_{\rm dec}$.

Now, our starting expression for $f_{\rm PBH}$ is always (2.1.13), that we report here for convenience

$$f_{\rm PBH}(M_{\rm f}) = \frac{\Omega_{\rm rad,0}}{\Omega_{\rm DM,0}} \frac{\rho_{\rm PBH,0}(M_{\rm f})}{\rho_{\rm rad,0}} \tag{2.3.4}$$

Since PBHs redshift exactly as the background during the modulus-dominated era, using (2.1.1) we have

$$\gamma\, \beta_{\rm MD}(M_{\rm f}) = \frac{\rho_{\rm PBH,f}(M_{\rm f})}{\rho_{\rm tot,f}} = \frac{\rho_{\rm PBH,dec}(M_{\rm f})}{\rho_{\rm tot,dec}} \simeq \frac{\rho_{\rm PBH,dec}(M_{\rm f})}{\rho_{\rm rad,dec}} \tag{2.3.5}$$

which implies

$$\rho_{\rm PBH,dec}(M_{\rm f}) \simeq \gamma\, \beta_{\rm MD}(M_{\rm f}) \rho_{\rm rad,dec} \tag{2.3.6}$$

Then, following the exact same steps of (2.1.15)-(2.1.16), we arrive at expression

$$f_{\rm PBH}(M_{\rm f}) \simeq \gamma\, \beta_{\rm MD}(M_{\rm f})\, \frac{\Omega_{\rm rad,0}}{\Omega_{\rm DM,0}} \frac{T_{\rm dec}}{T_0} \tag{2.3.7}$$

which is very similar to the expression obtained in RD, with the decay temperature of the moduli $T_{\rm dec}$ instead of the temperature at formation $T_{\rm f}$. This little difference reflects the fact that, when produced in a moduli dominated era, the PBHs do not start to redshift with respect to the background since the formation time, but only after the MD epoch is over.

Before studying (2.3.7), let us take a detour to express $f_{\rm PBH}$ in terms of the mass at formation $M_{\rm f}$. Since $\rho_{\rm tot} = 3\, H^2 M_{\rm p}^2$ is valid in every scenario, (2.1.14) determines that

$$H \sim \begin{cases} a^{-2} & \text{in RD} \\ a^{-3/2} & \text{in MD} \end{cases} \tag{2.3.8}$$

(2.1.5) is also background-independent, hence

$$M_{\rm f} = 4\pi\gamma \frac{M_{\rm p}^2}{H_{\rm f}} = 4\pi\gamma \frac{M_{\rm p}^2}{H_{\rm dec}} \frac{H_{\rm dec}}{H_{\rm f}} = \gamma\, M_{\rm H,dec} \frac{H_{\rm dec}}{H_{\rm f}} = \gamma\, M_{\rm H,dec} \left(\frac{a_{\rm f}}{a_{\rm dec}}\right)^{3/2}$$

and rewriting the scale factor ratio in terms of the wave-number

$$\frac{k_{\rm dec}}{k_{\rm f}} = \frac{a_{\rm dec} H_{\rm dec}}{a_{\rm f} H_{\rm f}} = \frac{a_{\rm dec}}{a_{\rm f}} \left(\frac{a_{\rm dec}}{a_{\rm f}}\right)^{-3/2} = \left(\frac{a_{\rm f}}{a_{\rm dec}}\right)^{1/2}$$

we obtain

$$M_{\rm f} = \gamma\, M_{\rm H,dec} \left(\frac{k_{\rm dec}}{k_{\rm f}}\right)^3 \tag{2.3.9}$$

$M_{\rm H,dec}$ is not an observable quantity, so we want to remove it from our formula by expressing it in terms of $M_{\rm H,0}$. This can be done by using (2.1.12), which holds[7] because $t_{\rm dec}$ can be considered a moment of radiation domination. The result is:

$$\begin{aligned} M_{\rm f} &\simeq \gamma\, M_{\rm H,0}\, \Omega_{\rm rad,0}^{1/2} \left(\frac{g_{*0}}{g_{*\rm dec}}\right)^{1/6} \left(\frac{k_0}{k_{\rm dec}}\right)^2 \left(\frac{k_{\rm dec}}{k_{\rm f}}\right)^3 \\ &= \gamma\, M_{\rm H,0}\, \Omega_{\rm rad,0}^{1/2} \left(\frac{g_{*0}}{g_{*\rm dec}}\right)^{1/6} \left(\frac{k_0}{k_{\rm dec}}\right)^2 \left(\frac{k_{\rm dec}}{k_{\rm dom}}\right)^3 \left(\frac{k_{\rm dom}}{k_{\rm f}}\right)^3 \end{aligned} \tag{2.3.10}$$

[7] As long as we subtract the γ factor, as explained under the aforementioned expression.

where we also multiplied and divided by $k_{\rm dom}^3$ in the last step. We now define $N_{\rm MD}$ the duration in e-foldings of the moduli dominated era, and $N_{\rm f}$ the distance in e-foldings between $t_{\rm dom}$ and the moment $t_{\rm f}$ when the peak in $P_{\rm k}$ crosses the horizon. Relation (2.3.8) implies

$$k \sim \begin{cases} a^{-1} & \text{in RD} \\ a^{-1/2} & \text{in MD} \end{cases} \tag{2.3.11}$$

and, therefore, we can use (2.1.20) as follows:

$$a_{\rm dec} = a_{\rm dom} e^{N_{\rm MD}} \Rightarrow \frac{k_{\rm dec}}{k_{\rm dom}} = e^{-\frac{1}{2}N_{\rm MD}} \tag{2.3.12}$$

$$a_{\rm f} = a_{\rm dom} e^{N_{\rm f}} \Rightarrow \frac{k_{\rm dom}}{k_{\rm f}} = e^{\frac{1}{2}N_{\rm f}} \tag{2.3.13}$$

so that (2.3.10) becomes

$$M_{\rm f} = \gamma\, M_{\rm H,0}\, \Omega_{\rm rad,0}^{1/2} \left(\frac{g_{*\rm dec}}{g_{*0}}\right)^{-1/6} \left(\frac{k_0}{k_{\rm dec}}\right)^2 e^{-\frac{3}{2}(N_{\rm MD}-N_{\rm f})} \tag{2.3.14}$$

Now let us take a look at the temperature ratio in (2.3.7): the first steps are analogous[8] to the ones done in (2.1.17), because for $t_{\rm dec} < t < t_0$ we are in a RD era. This leads to

$$\begin{aligned} \frac{T_{\rm dec}}{T_0} &= \left(\frac{g_{*\rm dec}}{g_{*0}}\right)^{-1/3} \frac{a_0}{a_{\rm dec}} \\ &= \left(\frac{g_{*0}}{g_{*\rm dec}}\right)^{1/3} \frac{H_{\rm dec} k_{\rm dec}^{-1}}{H_0 k_0^{-1}} \\ &= \left(\frac{g_{*0}}{g_{*\rm dec}}\right)^{1/3} \frac{M_{\rm H,0}}{M_{\rm H,dec}} \frac{k_0}{k_{\rm dec}} \end{aligned}$$

that, expressing the momentum ratio using (2.3.10) and $M_{\rm H,dec}$ using (2.3.9), gives

$$\begin{aligned} \frac{T_{\rm dec}}{T_0} &\simeq \left(\frac{g_{*0}}{g_{*\rm dec}}\right)^{1/3} \frac{M_{\rm H,0}}{M_{\rm H,dec}} \gamma^{-1/2} \left(\frac{M_{\rm f}}{M_{\rm H,0}}\right)^{1/2} \Omega_{\rm rad,0}^{-1/4} \times \\ &\quad \times \left(\frac{g_{*0}}{g_{*\rm dec}}\right)^{-1/12} e^{\frac{3}{4}(N_{\rm MD}-N_{\rm f})} \\ &= \left(\frac{g_{*0}}{g_{*\rm dec}\,\Omega_{\rm rad,0}}\right)^{1/4} \left(\frac{\gamma\, M_{\rm H,0}}{M_{\rm f}}\right)^{1/2} e^{-\frac{3}{4}(N_{\rm MD}-N_{\rm f})} \end{aligned} \tag{2.3.15}$$

Finally, substituting into (2.3.7), we get

$$f_{\rm PBH}(M_{\rm f}) \simeq \gamma^{3/2} \beta_{\rm MD}(M_{\rm f}) \frac{\Omega_{\rm rad,0}^{3/4}}{\Omega_{\rm DM,0}} \left(\frac{g_{*0}}{g_{*\rm dec}}\right)^{1/4} \left(\frac{M_{\rm H,0}}{M_{\rm f}}\right)^{1/2} e^{-\frac{3}{4}(N_{\rm MD}-N_{\rm f})} \tag{2.3.16}$$

[8] With (2.1.12) divided by γ, because it is used for the horizon mass $M_{\rm H,dec}$.

Examining the exponential term above we can see that the greater is the difference $N_{\rm MD} - N_{\rm f}$, i.e. the earlier PBHs form during the moduli-dominated era, the smaller is this factor and, hence, the greater is the peak required in $P_{\rm k}$ to obtain $f_{\rm PBH}(M_{\rm f}) = 1$. The reason a term like this, that acts as a "dilution" factor[9] for PBHs, is present in MD, is that both the PBHs and the background behave as matter: PBH energy density does not increase relatively to the background energy density as it happens when radiation dominates. How can we lower this "dilution" effect so that the peak in the power spectrum and, consequentially, the amount of fine tuning in the potential can be reduced? Looking at the exponent expression, it seems that the best way to do so is to shorten the modulus dominated epoch; this can be done by increasing the field decay rate, as we will now show.

Given that a generic modulus φ couples democratically to all degrees of freedom with gravitational strength, at its decay it generally produces both visible ("vis") and hidden ("hid") sector particles, with decay rates respectively:

$$\Gamma_{\varphi\to\rm vis} = c_{\rm vis}\Gamma_0 \qquad \text{and} \qquad \Gamma_{\varphi\to\rm hid} = c_{\rm hid}\Gamma_0 \tag{2.3.17}$$

where

$$\Gamma_0 = \frac{1}{48\pi}\frac{m_\varphi^3}{M_{\rm p}^2} \tag{2.3.18}$$

The total decay rate of φ is simply

$$\Gamma_\varphi^{\rm tot} = (c_{\rm vis} + c_{\rm hid})\Gamma_0 \equiv c_{\rm tot}\Gamma_0 \tag{2.3.19}$$

(we refer to $\Gamma_\varphi^{\rm tot}$ as "total decay width"); therefore, to reduce "dilution" as much as possible and lower the amount of enhancement required, we should choose the heaviest modulus mass m_φ allowed by our inflationary model.

The equation (2.3.16) was useful to explain this latter aspect of PBH production in MD driven by moduli, but it is useless in order to evaluate the peak in the power spectrum, since we obviously do not know the values of $N_{\rm MD}$ and $N_{\rm f}$; thus, we go back to equation (2.3.7).

To make use of (2.3.7) we need to determine the decay temperature of the modulus $T_{\rm dec}$. At decay, the modulus energy density gets transferred into visible degrees of freedom, which thermalise, and hidden sector degrees of freedom, which we assume to be relativistic particles with only gravitational couplings (like ultra-light axions which generally emerge in 4D string models). Hence, visible sector particles reach thermal

[9] The reason we are writing the word dilution between commas is that the term is used improperly: it should indicate decrease of energy density by entropy production but this is not what is happening here. However, one could demonstrate that the exponential factor in (2.3.16) can be written proportional to $(s_{\rm gen}/s_{\rm ini})_{t=\Gamma^{-1}}$, which is exactly the definition of dilution factor (see Ref. [32]); this is why we are using the word dilution anyways.

equilibrium at an energy:

$$\rho_{\rm vis} = \frac{c_{\rm vis}}{c_{\rm tot}}\rho_{\rm tot} = 3\,\frac{c_{\rm vis}}{c_{\rm tot}} H_{\rm dec}^2 M_{\rm p}^2 = \frac{\pi^2}{30}\,g_{*\rm dec}T_{\rm dec}^4 \tag{2.3.20}$$

Thus

$$T_{\rm dec} = \left(\frac{90}{\pi^2 g_{*\rm dec}}\frac{c_{\rm vis}}{c_{\rm tot}}\right)^{1/4}\sqrt{H_{\rm dec}M_{\rm p}} \tag{2.3.21}$$

The decay of φ occurs when the age of the Universe is approximately equal to the modulus lifetime. Since the these two times are inversely proportional to H and $\Gamma_\varphi^{\rm tot}$ respectively, we find (see Ref. [33]) that the decay of the modulus occurs when $3H \simeq 4\Gamma_\varphi^{\rm tot}/3$. Thus, we can express $T_{\rm dec}$ from the total decay width:

$$T_{\rm dec} = \left(\frac{40\,c_{\rm vis}\,c_{\rm tot}}{\pi^2 g_{*\rm dec}}\right)^{1/4}\sqrt{\Gamma_0 M_{\rm p}} \tag{2.3.22}$$

We have not talked about the couplings $c_{\rm vis}$ and $c_{\rm hid}$ yet. The hidden sector degrees of freedom give an extra contribution $\Delta N_{\rm eff}$ to the effective number of neutrinos $N_{\rm eff}$ as [33]:

$$\Delta N_{\rm eff} = \frac{43}{7}\frac{c_{\rm hid}}{c_{\rm vis}}\left(\frac{g_{*\nu}}{g_{*\rm dec}}\right)^{1/3} \tag{2.3.23}$$

where $g_{*\nu} = 10.75$ is the number of relativistic degrees of freedom at neutrino decoupling. Present observational bounds fix a stringent upper bound on (2.3.23) of order $\Delta N_{\rm eff} \lesssim 0.6$. This, in turn, imposes an upper bound on the ratio of the two couplings:

$$\frac{c_{\rm hid}}{c_{\rm vis}} = \frac{7}{42}\left(\frac{g_{*\rm dec}}{g_{*\nu}}\right)^{1/3}\Delta N_{\rm eff} \lesssim 0.2 \qquad \text{for} \qquad g_{*\rm dec} \lesssim 106.75 \tag{2.3.24}$$

In what follows we shall set $c_{\rm vis} = 40$ and $c_{\rm hid} = 1$, so that $\Delta N_{\rm eff} \approx 0.07$ and observations are not contradicted. Now, using (2.3.18) on (2.3.22), we determine the expression for the decay temperature in terms of m_φ:

$$T_{\rm dec} \simeq 0.115\left(\frac{c_{\rm vis}^2}{g_{*\rm dec}}\right)^{1/4} m_\varphi\sqrt{\frac{m_\varphi}{M_{\rm p}}} \tag{2.3.25}$$

The modulus mass has both an upper and a lower limit to its value: the upper limit $m_\varphi \lesssim m_\phi$ is due to the fact that the modulus φ has to be lighter than the inflaton ϕ (in the Fibre Inflation model we will describe in the next Section, $m_\phi \simeq 5\cdot 10^{13}$ GeV); the second bound is imposed by the fact that the beginning of BBN is located at $T_{\rm BBN} \simeq 3\,\text{MeV}$ and, if we do not want to spoil the model, we must have that $T_{\rm dec} \gtrsim T_{\rm BBN}$, that for our values of the couplings implies[10] $m_\varphi \gtrsim 50\,\text{TeV}$.

[10] For $T_{\rm dec} \simeq 3\text{MeV}$, Ref. [26] tells us that $g_{*\rm dec} = 10.75$.

If we now combine (2.3.7) and (2.3.25), we get:

$$\beta_{\rm MD}(M_{\rm f}) \simeq 1.28\cdot 10^{-7}\gamma^{-1} g_{*\rm dec}^{1/4}\left(\frac{50\,\rm TeV}{m_\varphi}\right)^{3/2} f_{\rm PBH}(M_{\rm f}) \tag{2.3.26}$$

where we used $\Omega_{\rm DM,0}=0.23$, $\Omega_{\rm rad,0}=8\cdot 10^{-5}$, $T_0 = 9.645\cdot 10^{-32}\,M_{\rm p}$ and $c_{\rm vis}=40$. Notice how this last expression is almost independent on the PBH mass $M_{\rm f}$.

If we focus on the case where the whole DM content of the Universe today is given by PBHs with a definite mass $M_{\rm f}$, i.e. $f_{\rm PBH}=1$, and we set $\gamma=1$ and $g_{*\rm dec}=106.75$ (which is correct for $T_{\rm dec}\gtrsim 30\,\rm GeV \Leftrightarrow m_\varphi \gtrsim 3.5\cdot 10^7\,GeV$), the expression (2.3.26) combined with (2.3.3) gives the required enhancement of the power spectrum in terms of the modulus mass:

$$P_\zeta \simeq 16.3\cdot\left(\frac{1\,\rm GeV}{m_\varphi}\right)^{6/13} \qquad \text{for} \qquad m_\varphi \gtrsim 3.5\cdot 10^7\rm GeV \tag{2.3.27}$$

We can easily see that, the greater is the modulus mass, the smaller is the peak required, which is in agreement with our previous discussion on the duration of the modulus epoch. A plot of this last expression is shown in Fig. 2.2; for $m_\varphi = 3.5\cdot 10^7\,\rm GeV$ we obtain $P_\zeta \approx 5\cdot 10^{-3}$, while $m_\varphi = 1\cdot 10^{13}\,\rm GeV$ gives $P_\zeta\approx 2\cdot 10^{-5}$. Even if the enhancement required can be up to 1000 times smaller than the one in radiation dominance (2.1.19), the near-inflection point after the slow roll plateau in the potential is still necessary.

As in RD, we shall determine the second constraint on the power spectrum, the distance in e-foldings between the moments of CMB modes $k_{\rm CMB}$ exit and peak exit $k_{\rm PBH}$, $\Delta N_{\rm CMB}^{\rm PBH}$. The first steps are exactly the same we did in RD, thus we start directly by

$$\Delta N_{\rm CMB}^{\rm PBH} = \ln\left(\frac{k_{\rm f}}{k_{\rm CMB}}\right) = \ln\left(\frac{k_{\rm f}}{k_{\rm dec}}\right) + \ln\left(\frac{k_{\rm dec}}{k_0}\right) + \ln\left(\frac{k_0}{k_{\rm CMB}}\right) \tag{2.3.28}$$

The first term can be expanded using (2.3.10), so that

$$\begin{aligned}\Delta N_{\rm CMB}^{\rm PBH} = \ln\left(\frac{k_0}{k_{\rm CMB}}\right) + \frac{1}{3}\ln\left(\frac{k_{\rm dec}}{k_0}\right) + \frac{1}{3}\ln\gamma + \frac{1}{6}\ln\Omega_{\rm rad,0} + \\ -\frac{1}{3}\ln\left(\frac{M_{\rm f}}{M_{\rm H,0}}\right) - \frac{1}{18}\ln\left(\frac{g_{*\rm dec}}{g_{*0}}\right)\end{aligned} \tag{2.3.29}$$

The ratio $k_{\rm dec}/k_0$ in the above equation can also be expanded using (2.1.11) and (2.1.10)

$$\frac{k_{\rm dec}}{k_0} = \frac{a_{\rm dec}H_{\rm dec}}{a_0 H_0} = \left(\frac{g_{*\rm dec}}{g_{*0}}\right)^{1/6}\left(\frac{\Omega_{\rm rad,0}}{\Omega_{\rm rad,dec}}\right)^{1/2}\frac{T_{\rm dec}}{T_0} \tag{2.3.30}$$

where $\Omega_{\rm rad,dec}\simeq 1$; substituting into (2.3.29) we obtain

$$\Delta N_{\rm CMB}^{\rm PBH} \simeq \ln\left(\frac{k_0}{k_{\rm CMB}}\right) + \frac{1}{3}\ln\left(\frac{T_{\rm dec}}{T_0}\right) + \frac{1}{3}\ln\Omega_{\rm rad,0} + \frac{1}{3}\ln\gamma - \frac{1}{3}\ln\left(\frac{M_{\rm f}}{M_{\rm H,0}}\right) \tag{2.3.31}$$

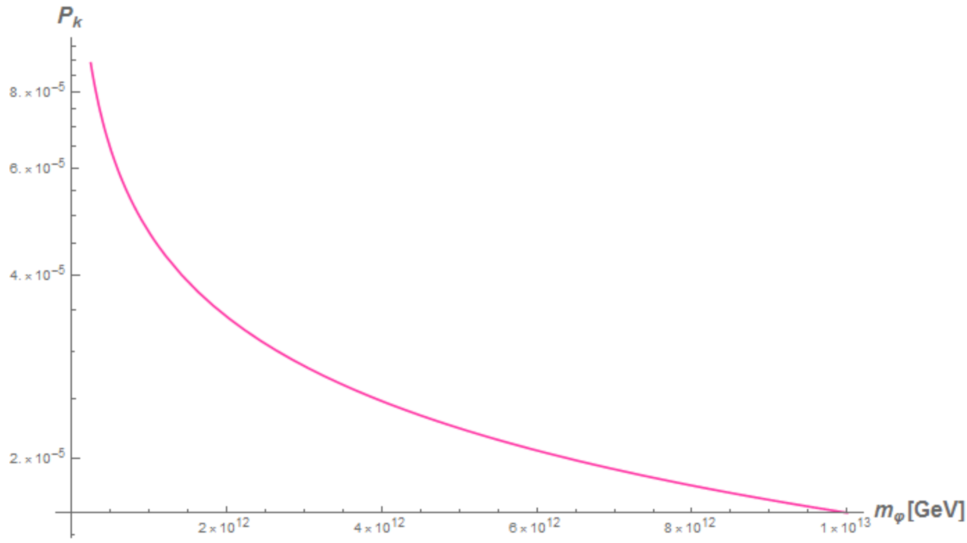

Figure 2.2: Plot of the power spectrum as a function of the modulus mass in the mass range $3.5 \cdot 10^7$ GeV $\leq m_\varphi \leq 1 \cdot 10^{13}$GeV to explain all of the dark matter today in terms of PBHs with mass M_f. The enhancement required in the power spectrum greatly reduces by increasing the modulus mass m_φ (i.e. by shortening the duration of the modulus epoch).

At this point, we make use of (2.3.25) to express T_dec in terms of the modulus mass. Summing all the constant factors we get

$$\Delta N_\mathrm{CMB}^\mathrm{PBH} \simeq 16.8 + \frac{1}{3}\ln\gamma - \frac{1}{12}\ln g_{*\mathrm{dec}} + \frac{1}{2}\ln\left(\frac{m_\varphi}{50\,\mathrm{TeV}}\right) - \frac{1}{3}\ln\left(\frac{M_\mathrm{f}}{M_\odot}\right) \tag{2.3.32}$$

where $k_0 = 2.4 \cdot 10^{-4}\,\mathrm{Mpc}^{-1}$, $k_\mathrm{CMB} = 0.05\,\mathrm{Mpc}^{-1}$, $\Omega_{\mathrm{rad},0} = 8.24 \cdot 10^{-5}$, $M_{\mathrm{H},0} = 4.35 \cdot 10^{22}\,M_\odot$, $T_0 = 9.645 \cdot 10^{-32} M_\mathrm{p}$. Equation (2.3.32) shows that, as anticipated, the distance in e-foldings is a function of both the PBH mass M_f and the modulus mass m_φ.

If we now set $\gamma = 1$, $g_{*\mathrm{dec}} = 106.75$ (i.e. $m_\varphi \gtrsim 3.5 \cdot 10^7$ GeV) and $M_\mathrm{f} = 10^{-15}\,M_\odot$, a value for which it is possible to have 100% of DM as PBHs, we find

$$\Delta N_\mathrm{CMB}^\mathrm{PBH} \simeq 27.9 + \frac{1}{2}\ln\left(\frac{m_\varphi}{50\,\mathrm{TeV}}\right) \qquad \text{for} \qquad m_\varphi \gtrsim 3.5 \cdot 10^7\mathrm{GeV} \tag{2.3.33}$$

i.e. the greater is the modulus mass, the greater is the distance in e-foldings between the peak and the CMB modes exit. Fig. 2.3 shows a plot of (2.3.33); for $m_\varphi = 3.5 \cdot 10^7$ GeV we obtain $\Delta N_\mathrm{CMB}^\mathrm{PBH} \approx 31.2$, while $m_\varphi = 1 \cdot 10^{13}$ GeV gives $\Delta N_\mathrm{CMB}^\mathrm{PBH} \approx 37.5$.

The relations (2.3.27) and (2.3.33) represent the two constraints that the power spectrum of curvature perturbations P_ζ must satisfy so that all of the dark matter today can be constituted of PBHs with mass M_f produced during a matter dominated era driven by a modulus with mass m_φ. We are now going to examine a particular class of string inflationary models called "Fibre Inflation" and, subsequently, we will derive the entire power spectrum and all the other inflationary observables from its inflationary potential.

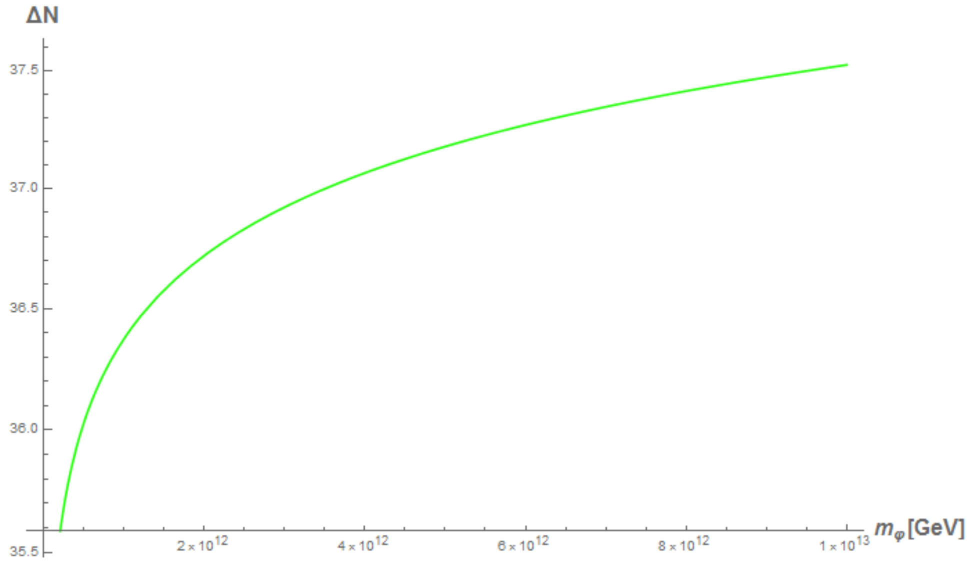

Figure 2.3: Plot of $\Delta N^{\rm PBH}_{\rm CMB} = \Delta N^{\rm PBH}_{\rm CMB}(M_{\rm f}, m_\varphi)$ in the mass range $3.5 \cdot 10^7\,\text{GeV} \leq m_\varphi \leq 1 \cdot 10^{13}\text{GeV}$ and for a PBH mass $M_{\rm f} = 10^{-15} M_\odot$. The distance in e-foldings increases with the mass of the modulus m_φ.

Chapter 3

PBHs from Fibre Inflation

In this chapter we shall study PBH formation within the framework of a particularly promising string inflationary model called *Fibre Inflation*. Before discussing the details, we present a very brief summary of the features of the 4-dimensional supergravity effective action.

3.1 Supergravity effective action

It is probably fair to claim that type IIB string compactifications are the best tools to make string theory in contact with observations. In fact, type IIB models manage both to stabilise the moduli and to build Standard Model-like constructions via systems of intersecting branes. The moduli in the 4-dimensional effective supergravity theory which are relevant for our discussion are the so-called Kähler moduli. These fields descend from the 10-dimensional metric and parametrize deformations of the extra dimensions in size. They are complex scalar fields defined as:

$$T_i = \tau_i + i\, b_i, \quad i = 1, ..., h^{1,1}(X) \tag{3.1.1}$$

where $h^{1,1}(X)$ is a Hodge number of the Calabi-Yau threefold X which is used as our compactification manifold[1]. The fields b_i are axions which enjoy perturbative shift symmetries, while the τ_i are saxions, the supersymmetric counterparts of the axions, which control the size of internal 4-cycles.

Now we consider a particular Calabi-Yau manifold which is suitable to derive inflation, and define its dimensionless volume $\mathcal{V}$ as:

$$\mathcal{V} = \sqrt{\tau_1}\tau_2 - \tau_3^{3/2} \tag{3.1.2}$$

[1]The reason why X is a Calabi-Yau manifold is to obtain an effective theory with $N = 1$ supersymmetry (after orientifolding) which can be chiral, and so compatible with observations.

and, introducing the string length

$$l_{\rm s} = 2\pi\sqrt{\alpha'} \tag{3.1.3}$$

we can define the Calabi-Yau 6-dimensional volume $\mathbb{V}$ as

$$\mathbb{V} = \mathcal{V}\, l_{\rm s}^6 = \mathcal{V}\,(2\pi)^6\,\alpha'^3 \tag{3.1.4}$$

Our supersymmetric low-energy 4D theory is described by a Kähler potential K and a superpotential W; the starting Kähler potential is

$$\begin{aligned} K &= -2\ln\left(\mathcal{V} + \frac{\xi}{2}\right) \\ &\simeq -2\ln\mathcal{V} - \frac{\xi}{\mathcal{V}} + \dots \\ &= -2\ln\mathcal{V} - \frac{\xi}{\mathbb{V}}(2\pi)^6\,\alpha'^3 + \dots \\ &= K_{\rm tree} + K_{\alpha'} + \dots \end{aligned} \tag{3.1.5}$$

where in the second step we assumed $\frac{\xi}{\mathcal{V}} \ll 1$, where ξ is an $\mathcal{O}(1)$ constant determined by the topological properties of the underlying Calabi-Yau manifold. $K_{\rm tree}$ is the classical part of K, while $K_{\alpha'}$ represents the leading order quantum correction, related to the length of the string. In the limit $\alpha' \to 0$ the length of string goes to zero, i.e. strings become point-like, and $K_{\alpha'}$ also vanishes. Therefore $K_{\alpha'}$ takes into account the fact that, even if we are in a low-energy 4-dimensional theory where all fundamental particles appear point-like, they are strings at the fundamental level and this produces small corrections to the behaviour predicted by the classical theory.

The starting superpotential is instead:

$$W = W_0 + A_3\, e^{-a_3 T_3} = W_{\rm tree} + W_{\rm np} \tag{3.1.6}$$

where W_0 is a constant tree-level contribution while $W_{\rm np}$ represents non-perturbative corrections with both A_3 and a_3 $\mathcal{O}(1)$ constants. The superpotential receives no contributions at any finite order in α' and $g_{\rm s}$, i.e. it cannot receive perturbative corrections. There can be higher non-perturbative corrections to W that, for the moment, we ignore.

From K and W we can compute the potential of our supergravity theory V_F[2] as follows:

$$V_F = e^K \left(K^{i\bar{j}} D_i W\, D_{\bar{j}}\overline{W} - 3|W|^2\right) \tag{3.1.7}$$

where the covariant derivarive is

$$D_i W \equiv \partial_i W + W\partial_i K \tag{3.1.8}$$

[2]Actually, this is the F-term scalar potential in supergravity.

and ∂_i indicates the derivative with respect to the moduli (3.1.1), while $K^{i\bar{j}}$ is the inverse of the Kähler metric, defined as

$$K_{\bar{\imath}j} = \partial_{\bar{\imath}}\,\partial_j K \tag{3.1.9}$$

Calculating the potential, one finds out that it is a function $V_F = V_F(\tau_3, b_3, \mathcal{V})$. However, since τ_3 is way smaller[3] than τ_1 and τ_2, the volume can be approximated as

$$\mathcal{V} \simeq \sqrt{\tau_1}\tau_2 \quad \text{for} \quad \tau_3 \ll \tau_1, \tau_2 \tag{3.1.10}$$

This means that, at this level of approximation, only τ_3, b_3 and the combination $\mathcal{V} \simeq \sqrt{\tau_1}\tau_2$ have mass[4]. Hence, since there is no potential for the other fields, we find $m_{b_1} = m_{b_2} = m_\chi = 0$, where χ is the precise combination of τ_1 and τ_2 that is orthogonal to $\mathcal{V}$.

Massless particles are generally quite problematic, since they mediate forces that have never been observed[5], which modify the gravitational interaction at long range. Axions however are peculiar particles, whose interactions depend on spin. Since gravity has always been tested on macroscopic systems (e.g. stars, planets etc.), that have no spin because they are constituted of many particles with random spins that cancel each others, we can admit the existence of axions without contradicting our observations on gravity. This is not true for the saxions τ_i, whose interactions are spin-independent. Therefore, the only way to cancel fifth forces mediated by the massless saxion χ is to add an additional correction $K_{g_s} \ll K_{\alpha'}$ to the Kähler potential

$$K \simeq K_{\text{tree}} + K_{\alpha'} + K_{g_s}(\tau_1, \tau_2) \tag{3.1.11}$$

called "string loop correction", that has the effect of generating a subleading potential $V_{\text{sub}} = V(\chi)$[6]; this causes χ to acquire a mass $m_\chi \neq 0$, $m_\chi \ll m_{\mathcal{V}} < m_{b_3} \sim m_{\tau_3}$.

3.2 Inflationary potential

As we have seen in Chapter 1, to produce inflation one needs an approximately flat potential, similar to a cosmological constant. In string theory we have many fields in our potential: to obtain single-field inflation we need a potential that, considering only first order corrections, is flat along one particular direction (i.e. a field which is at leading orde massless). This is the case of χ: this field does not acquire a mass from the $K_{\alpha'}$ corrections, hence the potential looks completely flat along its direction at this level of approximation; but adding the second order correction K_{g_s}, the potential gets slightly lifted along the χ direction, that now is not flat anymore but only much less steep

[3]For this reason, in the literature it is also referred to as "tau-small" τ_s.

[4]The actual physical fields are different from τ_3, b_3 and $\mathcal{V}$ since they have to be canonically normalized.

[5]Also known as "Fifth forces".

[6]To be precise, V_{sub} is also a function of $\mathcal{V}$; however, $\mathcal{V}$ has already been fixed at leading order, so it behaves like a constant in V_{sub}.

than the previously lifted directions. Therefore, χ constitutes a good inflaton, the only problem with it being that it is not canonically normalized. Hence, the actual inflaton is the canonical version of χ, called ϕ. When in the previous chapters we said that we need to tune the inflationary potential so that all of the dark matter today can be explained in terms of PBHs, we were not referring to the whole string model potential $V = V_{\rm lead} + V_{\rm sub}$ with all fields which evolve dynamically, but just to the part of it along ϕ when all the other fields are fixed at their minimum, i.e. to $V = V_0 + V_{\rm sub}$. Thus, from this moment onward we will refer to this potential as $V_{\rm inf}$.

Computing the inflationary potential via (3.1.7), one finds that (see Ref. [29], [23])

$$V_{\rm inf}(\chi) = \frac{W_0^2}{\mathcal{V}^3}\left[\frac{C_{\rm up}}{\mathcal{V}^{1/3}} - \frac{C_{\rm W}}{\sqrt{\chi}} + \frac{A_{\rm W}}{\sqrt{\chi} - B_{\rm W}} + \frac{\chi}{\mathcal{V}}\left(D_{\rm W} - \frac{G_{\rm W}}{1 + R_{\rm W}\frac{\chi^{3/2}}{\mathcal{V}}}\right)\right] \tag{3.2.1}$$

where we considered both α' and $g_{\rm s}$ corrections to the Kähler potential. $C_{\rm up}$, $C_{\rm W}$, $A_{\rm W}$, $B_{\rm W}$, $D_{\rm W}$, $G_{\rm W}$ and $R_{\rm W}$ are the coefficients that measure the weight of the various contributions to the inflationary potential, and they depend on microscopic parameters of string theory. Let us take a closer look at these contributions: the first term is just a constant that takes into account that we are in the minimum of $V_F(\tau_3, b_3, \mathcal{V})$; the second and the third term dominate at small field values, while the fourth term, being proportional to χ, dominates at large field values and is responsible for destroying the slow-roll plateau for large values of χ; the fifth term, that is proportional to $-\chi$ for small field values and to $-\chi^{-1/2}$ for large field values, is the one that is responsible for the near inflection point and, thus, for the production of amplified scalar perturbations that will collapse forming PBHs at horizon reentry. The potential above can be expressed in term of the inflaton ϕ, the canonically normalized counterpart of χ, as

$$\chi = e^{\frac{2}{\sqrt{3}}\phi} = \langle\chi\rangle e^{\frac{2}{\sqrt{3}}\hat{\phi}} \tag{3.2.2}$$

where in the second step we have expanded ϕ around its minimum as $\phi = \frac{\sqrt{3}}{2}\ln\langle\chi\rangle + \hat{\phi}$. Substituting this result in the expression above of the inflationary potential, one finds that

$$V_{\rm inf}(\hat{\phi}) = V_0\left[C_1 - e^{-\frac{1}{\sqrt{3}}\hat{\phi}}\left(1 - \frac{C_6}{1 - C_7 e^{-\frac{1}{\sqrt{3}}\hat{\phi}}}\right) + C_8 e^{\frac{2}{\sqrt{3}}\hat{\phi}}\left(1 - \frac{C_9}{1 + C_{10}e^{\sqrt{3}\hat{\phi}}}\right)\right] \tag{3.2.3}$$

This potential has enough tuning freedom to induce a slow-roll plateau (necessary for inflation to occur) and a near-inflection point such that no more than 5-6 e-foldings of inflation occur on it; this last requirement is important, because otherwise the CMB scales would also be enhanced, which is not in agreement with observations. To produce all of these features and obtain a potential like the one shown in Fig. 3.1, the coefficients should be such that (C_1 has to be adjusted to obtain a nearly Minkowski vacuum after the end of inflation)

$$C_6 = \frac{A_{\rm W}}{C_{\rm W}} \sim \mathcal{O}(1), \quad C_7 = \frac{B_{\rm W}}{\varepsilon^{1/3}\mathcal{V}^{1/3}} \sim \mathcal{O}(1), \quad C_8 = \varepsilon \frac{D_{\rm W}}{C_{\rm W}} \ll 1,$$
$$C_9 = \frac{G_{\rm W}}{D_{\rm W}} \sim \mathcal{O}(1), \quad C_{10} = \varepsilon R_{\rm W} \ll 1\,. \tag{3.2.4}$$

where we have also defined ε by parameterising the inflaton minimum as $\langle\chi\rangle \equiv \varepsilon\mathcal{V}$.

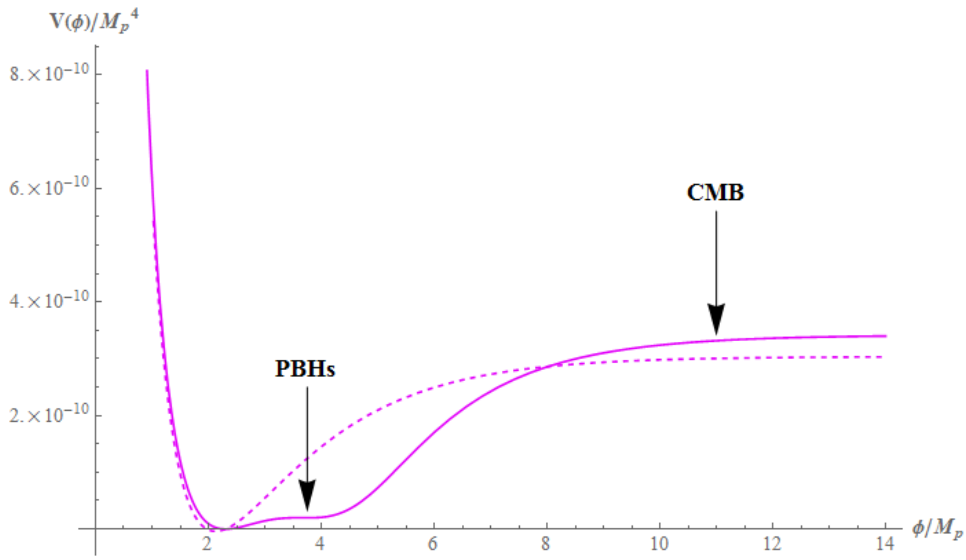

Figure 3.1: Solid line: Inflationary potential featuring a slow-roll plateau followed by a near-inflection point which enhances the scalar power spectrum and triggers PBH formation at horizon reentry; it has been obtained from the parameter set $\mathcal{P}_2$ of Tab. 3.1. The arrows indicate where the inflaton is located when the modes corresponding to CMB and PBHs exit the horizon. Dashed line: potential that has been obtained from the solid one by setting $G_{\rm W} = 0$, i.e. by canceling the term responsible for the near-inflection point; for this reason, it cannot lead to PBH production.

3.3 Reheating and Dark Radiation

After the end of inflation the inflaton oscillates around the minimum of its potential $V_{\rm sub}(\phi)$ and, once the minimum has been reached, ϕ decays into particles that get thermalised, forming an ultra-relativistic gas that begins to dominate the energy density of the Universe. In a non-stringy model, these particles are just the Standard Model particles we are familiar with; however, in our string inflationary scenario, ϕ could also decay into the massless axions b_1 and b_2. Being that $m_{b_1} = m_{b_2} = 0$ and that they do not interact with standard radiation, hence contributing to the effective number of neutrinos $N_{\rm eff}$, these particles are often referred to as "dark radiation". The quantity ΔN_{eff} measures the deviation of N_{eff} from 3, and it is observationally bounded to be $\Delta N_{\text{eff}} \ll 0.6$. How many of these massless axions are produced, i.e. how much they influence N_{eff}, depends on the branching ratios associated to the inflaton decay width into these hidden sector particles $\Gamma_{\phi\to\text{hid}} = \Gamma_{\phi\to b_1,b_1} + \Gamma_{\phi\to b_2,b_2}$. This has been obtained in a recent article (see Ref. [33]): depending on the presence of absence of gauge fluxes on D7-brane wrapping the inflaton 4-cycle, this ratio goes to zero and the dark radiation does not give an extra contribution to N_{eff}, while in other situations too many of these particles are produced and the deviation ΔN_{eff} from the observed value is too big.

3.4 Solution: Axions and Matter Domination

To solve the problem we just mentioned, one can give mass to the axions by considering two additional corrections to the superpotential:

$$W = W_0 + A_3\, e^{-a_3 T_3} + A_1\, e^{-a_1 T_1} + A_2\, e^{-a_2 T_2} \tag{3.4.1}$$

For the sake of simplicity, we choose $m_{b_1} = m_{b_2}$, so that we can identify $b_1 \sim b_2 \sim \varphi$ and $0 < m_\varphi < m_\phi \ll m_{\mathcal{V}} < m_{b_3} \sim m_{\tau_3}$. Now that the axion φ is massive and that there are no more hidden sector particles left massless, φ will decay only into Standard Model particles and $\Delta N_{\text{eff}} = 0$. However, the fact that $m_\varphi \neq 0$ has another consequence: this field is gravitationally coupled and, via the process explained in Section 2.2, it will dominate over the energy density of the Universe leading to a MD epoch before BBN[7]. Therefore, φ is our modulus field. The value of its mass is controlled by the parameters A_1, A_2, a_1 and a_2 and can be chosen to take different values. The only requirements are that φ decays before BBN, which translates into $m_\varphi \gtrsim 50$ TeV, and that this axionic field is lighter than the inflaton whose mass in Fibre Inflation is of order $m_\phi \simeq 5 \cdot 10^{13}\,\text{GeV} > m_\varphi$.

3.5 PBH formation

It is now time to compute our inflationary observables, starting with the power spectrum of scalar perturbations P_k. We cannot make use of the slow-roll approximation (1.2.77) because it does not hold during the e-foldings of ultra slow-roll (see Ref. [25]), causing the enhancement to be underestimated. We are therefore forced to make use of the

[7]As we have already mentioned, the choice $m_{b_1} = m_{b_2}$ has been made for simplicity, but it is not the only way to solve the problem of effective neutrinos. In fact one could lift only one of the axions, say b_1, leaving the other massless. This reproduces the effect we just explained, but with an important difference: at t_{dec} the lifted axion could decay into the massless one. At this point, one should evaluate the branching ratio $\Gamma_{b_1\to b_2 b_2}$ and make sure that $\Delta N_{\text{eff}} \ll 0.6$.

definition of $P_v(k)$ from the two-point correlation function (1.2.73). However, to make an appropriate comparison with observations, we will consider the dimensionless analogous of $P_v(k)$ for the curvature perturbation ζ_k, that we will indicate as P_k and can be obtained combining (1.2.73), (1.2.75) and (1.2.78)[8]

$$P_k \equiv \frac{k^3}{2\pi^2}|\zeta_k|^2 = \frac{k^3}{2\pi^2}\left|\frac{v_k}{z}\right|^2 \tag{3.5.1}$$

We recall that the $v_k = v_k(\tau)$ are the mode functions of the normalized curvature perturbations, that are obtained by solving the Mukhanov-Sasaki equation (1.2.46) with the Bunch-Davies initial conditions (1.2.69). We report the equation here for convenience

$$v_k''(\tau) + \left(k^2 - \frac{z''}{z}\right)v_k(\tau) = 0 \qquad \text{with} \quad \lim_{k\tau\to-\infty} v_k(\tau) = \frac{1}{\sqrt{2k}}e^{-ik\tau} \tag{3.5.2}$$

where τ represents the comoving conformal time $\tau = (aH)^{-1}$, $z(\tau)^2 = 2\epsilon a^2$ and ϵ is one of the Hubble slow-roll parameters

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \qquad \eta \equiv \frac{\dot{\epsilon}}{H\epsilon}, \qquad \kappa \equiv \frac{\dot{\eta}}{H\eta} \tag{3.5.3}$$

Since the effective mass z''/z turns out to be dependent on τ through the Hubble parameters

$$\frac{z''}{z} = (aH)^2\left[2 - \epsilon + \frac{3}{2}\eta - \frac{1}{2}\epsilon\eta + \frac{1}{4}\eta^2 + \frac{1}{2}\eta\kappa\right] \tag{3.5.4}$$

the solution of (3.5.2) turns out to be rather complicate, involving the Henkel functions[9] of first kind

$$v_k(\tau) = \frac{\sqrt{-\pi\eta}}{2}H_\nu^{(1)}(-k\tau) \tag{3.5.5}$$

where the index ν is determined from (3.5.4) once a given background is chosen. Now that we have the mode functions v_k we can find the expression for the dimensionless scalar power spectrum (3.5.1) in the superhorizon limit $k\tau \to 0$, which is the only regime of interest for us

$$P_k \simeq \frac{H^2}{8\pi^2\epsilon}\frac{2^{2\nu-1}|\Gamma(\nu)|^2}{\pi}\left(\frac{k}{aH}\right)^{3-2\nu} \tag{3.5.6}$$

On CMB scales, this expression is constrained by experimental observations:

$$P_{k=k_{\text{CMB}}} \simeq 2\cdot 10^{-9} \tag{3.5.7}$$

[8]When looking at these three equations in Chapter 1, we need to be careful: since here we are not always in slow-roll, one shall use the expressions that are valid in every scenario and not just in the slow-roll approximation.

[9]The Henkel functions of first kind are defined as $H_\nu^{(1)} = J_\nu + iY_\nu$, where J_ν is a Bessel function of the first kind and Y_ν a Bessel function of the second kind

Moreover, since we want PBHs with mass M_{f} to constitute all of the DM today, we have two more constraints on P_k: the enhancement required for perturbations to collapse in so many PBHs to explain all of the DM, and how long before the end of inflation the enhanced perturbations must be produced. These last two requirements depend on the dominant component of the Universe at the moment of horizon reentry and have been obtained in Chapter 2 for both a radiation and a modulus dominated epoch.

We will now focus on the matter case, since the case of radiation has already been analyzed by my supervisors in [29] and clearly requires a greater amount of fine-tuning. Being that the greater is the mass of the modulus the smaller is the enhancement required, we assume the larger value admissible $m_\varphi = 1 \cdot 10^{13}\,\mathrm{GeV}$. For the PBH mass, we choose the usual value $M_{\mathrm{f}} = 10^{-15}\,M_\odot$ based on the experimental bounds shown in Fig. 1.1. The constraints on the spectrum for these two values of the masses are

$$P_{k=k_{\mathrm{PBH}}} \simeq 2 \cdot 10^{-5}, \qquad \Delta N_{\mathrm{CMB}}^{\mathrm{PBH}} \simeq 37.5 \tag{3.5.8}$$

To obtain a spectrum that respects (3.5.7)-(3.5.8) we first tuned the potential (3.2.1) with any of the three sets of parameters listed in Tab. 3.1. Then we solved the equation (3.5.2) numerically, which for the parameter set $\mathcal{P}_2$ gave us the solid line plot shown in Fig. 3.2; the dashed plot in this same figure shows the slow-roll estimate of this spectrum, that has been obtained via (1.2.78). As expected, the result in the slow-roll approximation differs significantly from the numerical one only for the enhanced modes produced during the ultra slow-roll regime; in particular, the slow-roll spectrum seriously underestimates the enhancement necessary for perturbations to collapse into PBHs with mass M_{f} at horizon reentry.

Tab. 3.2 shows the computational values of the scalar spectral index n_{s}, which for all the sets results a bit too red compared to the Planck reference value. Another observable shown in this table is the tensor to scalar ratio r, defined as the ratio between the amplitude of tensor perturbations and the amplitude P_ζ of the scalar perturbations; the values inferred from our model of inflation are large enough to give us the hope that they could be measured by the next generation of cosmological observations.

The ultra slow-roll regime can be seen clearly by looking at Fig. 3.3 which shows, for the potential descending from the set $\mathcal{P}_2$, the behaviour of the three Hubble slow-roll parameters during the last e-foldings of inflation. In Sec. 1.2 we explained that, in order to obtain standard slow-roll inflation, one should have both $\epsilon < 1$ and $|\eta| < 1$; as can be seen from Fig. 3.3, this is exactly the case during the first e-foldings of inflation. However, between 10 and 13 e-foldings before the end of inflation, the behaviour of the parameters drastically changes because of the near inflection point in the potential: ϵ rapidly decreases towards zero, a behaviour that strongly enhances the power spectrum of curvature perturbations, while also causing $|\eta| \gg 1$ and a deceleration of the inflaton field. This is exactly the ultra slow-roll regime we mentioned many times before.

In Figure 3.4 we plot the evolution of the curvature perturbations on the scales corresponding to CMB and PBHs. In Sec. 1.2 we explained that, in a context of slow

	$C_{\rm W}$	$A_{\rm W}$	$B_{\rm W}$	$G_{\rm W}/\langle\mathcal{V}\rangle$	$R_{\rm W}/\langle\mathcal{V}\rangle$	$\langle\chi\rangle$	$\langle\mathcal{V}\rangle$
$\mathcal{P}_1$	1/10	2/100	1	$1.303355 \cdot 10^{-3}$	$6.58724 \cdot 10^{-3}$	3.89	107.3
$\mathcal{P}_2$	4/100	2/100	1	$3.08044 \cdot 10^{-5}$	$7.071067 \cdot 10^{-4}$	14.30	1000
$\mathcal{P}_3$	1.978/100	1.65/100	1.01	$9.25746 \cdot 10^{-8}$	$1.414 \cdot 10^{-5}$	168.03	$5 \cdot 10^4$

Table 3.1: Examples of coefficients used in the potential (3.2.1) to obtain a scalar power spectrum that satisfies the constraints (3.5.7)-(3.5.8). $G_{\rm W}$ is the parameter that influences the height of the peak, while $R_{\rm W}$ affects the distance in e-foldings between the peak and the CMB modes; many digits have been necessary in these two parameters to obtain the correct power spectrum. In all the sets, $D_{\rm W} = 0$. The last two columns show geometrical compactification data.

	$n_{\rm s}$	r	$\Delta N^{PBH}_{\rm CMB}$	$P_{k=k_{\rm PBH}}$
$\mathcal{P}_1$	0.9505	0.013	37.5	$2.1011 \cdot 10^{-5}$
$\mathcal{P}_2$	0.9494	0.013	37.5	$2.5279 \cdot 10^{-5}$
$\mathcal{P}_3$	0.9502	0.013	37.5	$2.4874 \cdot 10^{-5}$

Table 3.2: Inflationary observables for each parameter set of Tab. 3.1. The spectral index $n_{\rm s}$ has been obtained numerically, evaluating the derivative of the spectrum at CMB modes exit, while the tensor to scalar ratio r has been computed from the slow-roll approximation, valid at CMB scales.

roll, we expect perturbations to freeze-out at horizon crossing, so that their amplitude stays constant until their reentry occurs after the end of inflation. In our model however, the ultra slow-roll regime takes over at approximately 13 e-foldings before the end of inflation, causing a super-horizon growth which is determined by the ratio $\frac{k}{aH}$ evaluated at the onset of the ultra slow-roll period. For the CMB scales this quantity is so small that the super-horizon growth is not visible in Fig. 3.4, meaning that the slow-roll freeze out is a very good approximation for perturbations corresponding to these scales. This is not the case for small scales exiting the horizon immediately before the peak in the power spectrum, which undergo a significant super-horizon growth during the e-foldings of ultra slow-roll. Finally, let us note that even for scales reentering after the ultra slow-roll period the super-horizon behaviour deviates from slow-roll; this occurs because, as can be seen from Fig. 3.3, for $N_{\rm e} < 10$, we have $\epsilon < 1$ but $|\eta| \sim \mathcal{O}(1)$.

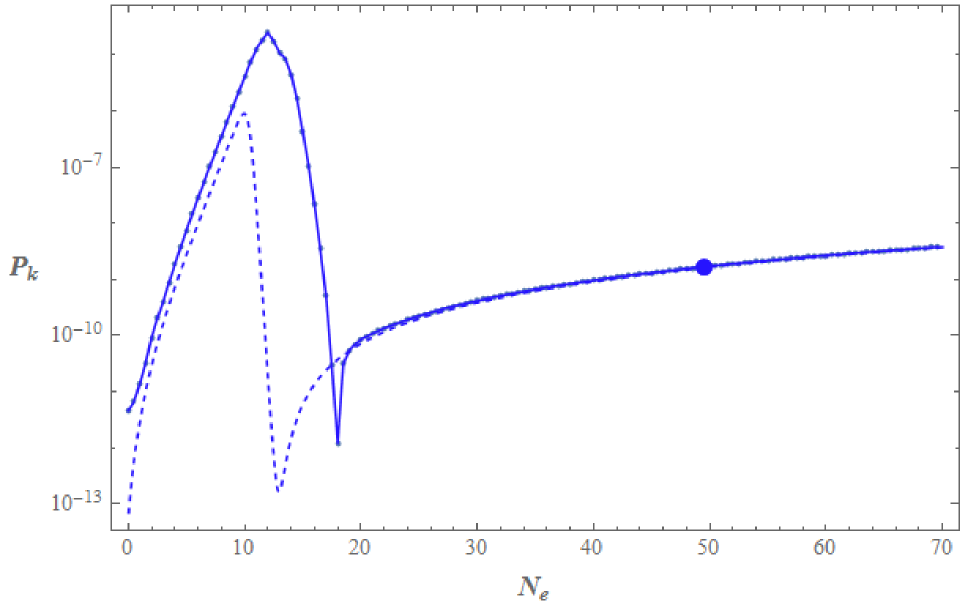

Figure 3.2: Solid line: Plot of the dimensionless power spectrum of curvature perturbations (3.5.1) as a function of the number of e-foldings before the end of inflation, obtained by solving (3.5.2) numerically. This spectrum descends from the potential in Fig. 3.1, i.e. the one obtained from the parameter set $\mathcal{P}_2$. Dashed line: slow-roll estimate (1.2.78). The dot corresponds to CMB scales. This plot correctly reproduces (3.5.7).

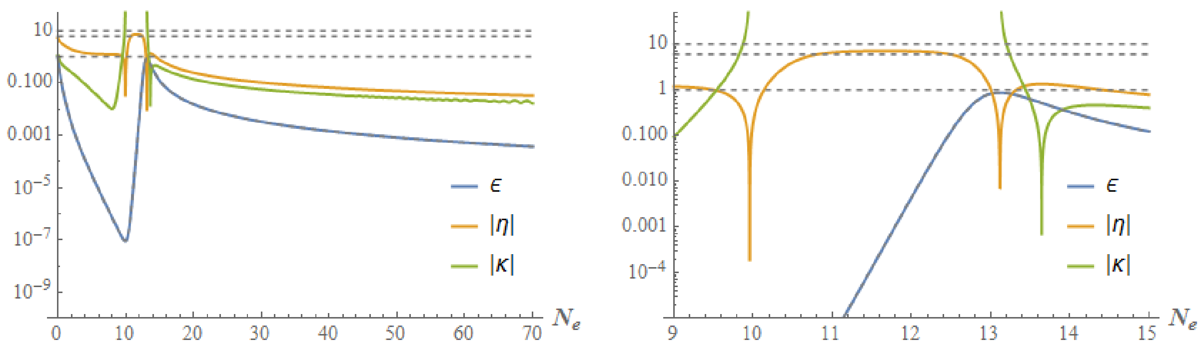

Figure 3.3: Left: Evolution of the Hubble parameters in terms of the number of e-foldings before the end of inflation for the parameter set $\mathcal{P}_2$ of Tab. 3.1. This plot clearly shows that the background evolves from slow-roll ($N_e \gtrsim 13$) to ultra slow-roll ($10 \lesssim N_e \lesssim 13$). Right: zoom on the ultra slow-roll interval. The three dashed lines correspond to the values 1, 6 and 10.

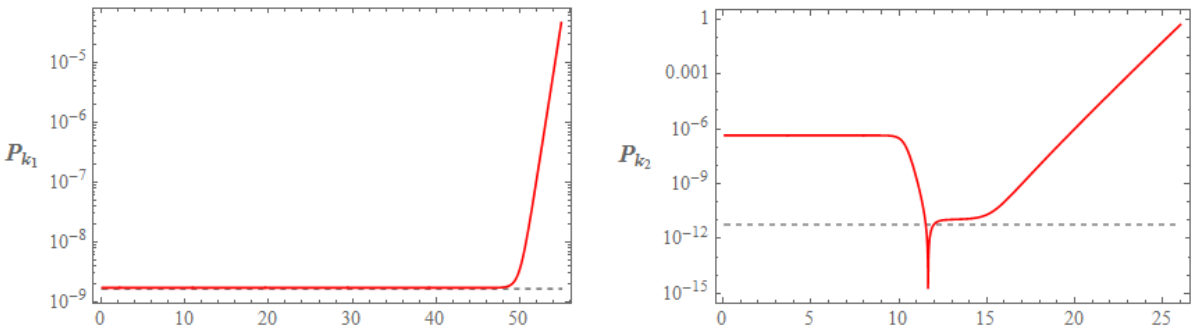

Figure 3.4: Left: evolution of the scalar perturbations corresponding to CMB scales, that exit the horizon approximately 50 e-foldings before the end of inflation. This super-horizon evolution is practically the same that follows from the slow-roll estimate (1.2.77), represented by the dashed line. Right: evolution of the perturbations corresponding to a scale that exits the horizon approximately 15 e-foldings before the end of inflation. We can see that, for $10 \lesssim N_e \lesssim 13$, the perturbations undergo a super-horizon enhancement due to the ultra slow-roll behaviour of the inflaton.

Chapter 4

Conclusions

In this thesis, we have discussed an alternative hypothesis on the nature of dark matter that has recently gained a lot of attention. It relies on black holes that form when large density perturbations, produced during the inflationary epoch, reenter the horizon and collapse under their own gravitational force. These black holes are referred to as "primordial" because this formation process has its origin in large density perturbations produced during the inflationary epoch. These PBHs could in principle have any mass, but experimental constraints of various nature tell us that only PBHs that at the moment of formation have a mass $10^{-16}\,M_{\odot} \leq M_{\rm f} \leq 10^{-11}\,M_{\odot}$ could explain a significant fraction (or even all) of the dark matter content of our Universe today. The post-inflationary epoch of horizon reentry of PBH scales can be a "radiation" dominated epoch, where the products of the inflaton decay thermalised and came to dominate the energy density, or an epoch of "matter" domination, in which a light and gravitationally coupled scalar field temporarily dominates the Universe via oscillations around its minimum.

In Sec. 2.1 we started studying PBH formation in the case of radiation domination. Our results tell us that PBHs with a mass of $M_{\rm f} = 10^{-15}\,M_{\odot}$ constitute 100% of the DM today if the power spectrum of curvature perturbations is of order $P_{\zeta} \sim 10^{-2}$ on the scales that, at reentry, trigger PBH production. This is an enhancement of 7 orders of magnitude with respect to the value measured on the CMB scales $P_{\zeta} = 2 \cdot 10^{-9}$, and only an inflationary potential with an high tuning freedom could produce such a spectrum.

The extreme tuning required in radiation domination made us wonder if a better result could be obtained by producing PBHs during matter domination. We first reviewed the mechanism that allows gravitationally coupled scalar fields to dominate in a post-inflationary epoch (Sec. 2.2), and then we repeated our calculations for a modulus driven matter dominated era (Sec. 2.3). In this latter case, we found that the required enhancement in the perturbations depends on the mass of the modulus that drives the matter dominated epoch: the greater is this mass, the smaller is the amplitude of the fluctuations needed. Therefore we hypothesized the largest possible modulus mass $m_{\varphi} = 1 \cdot 10^{13}\,\text{GeV}$ and, for the same PBH mass studied in the radiation scenario, we found that a value of

$P_\zeta \sim 10^{-5}$ is necessary for perturbations reentering in a modulus dominated epoch to form enough PBHs to explain all of the DM today. Thus, an enhancement with respect to the perturbations on the CMB scales is still needed; however, the height of the peak has been significantly reduced compared to the radiation case, and this implies that the tuning of the potential required in matter domination is smaller and PBH formation is more "natural".

After showing that in matter domination the constraints are less stringent compared to the radiation case, we started studying a model of single field string inflation called Fibre Inflation. The reason we turned to string theory is that epochs of matter domination driven by scalar fields lighter than the inflaton (moduli) are a natural consequence of string inflation. Moreover, the Fibre Inflation potential has enough tuning freedom to feature a slow-roll plateau followed by a near inflection point; this last feature is indispensable to produce the enhancement required in the fluctuation and trigger PBH production at horizon reentry. In Sec. 3.1-3.4 we briefly reviewed the main characteristics of Fibre Inflation (how inflation and modulus domination descend from the model, how the inflationary potential is obtained and its key features). Then, we tuned the potential to obtain, through a numerical approach, a power spectrum compatible with the requests that PBHs have been produced during a modulus dominated era and constitute all of the DM there is today (Sec. 3.5). The reason we employed a numerical approach is that the slow-roll estimate of the spectrum does not hold, because of the near inflection point which drives our inflaton into an ultra slow-roll regime.

The idea of DM in terms of PBHs is quite simple compared to other alternatives based on new non-baryonic particles or theories of modified gravity; however, it is not easy to implement. In the case of PBH production during radiation domination, the tuning freedom required in the inflationary potential is so high that it excludes many inflationary models a priori. The case of PBH production during moduli dominated epochs requires less fine tuning. Given that these periods are a natural consequence of 4-dimensional models based on string theory, a detection of PBHs could be considered as an interesting hint in favour of string theory because of the higher naturalness of their production in a modulus dominated epoch compared to a radiation epoch. Moreover, we have seen how string compactifications feature the generic presence of very light axion-like fields. Thus we conclude that another very strong indirect evidence of string theory would be the discovery of axion-like particles, for example via axion-photon conversion in very intense magnetic fields.

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