ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

Scuola di Scienze
Dipartimento di Fisica e Astronomia
Corso di Laurea Magistrale in Fisica

D-Brane inflation in the non-relativistic and relativistic regimes

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Anno Accademico 2017/2018

Abstract

In this thesis we study inflation from a string theory point of view. The first part of the work is dedicated to reviewing how this mechanism can solve the main problems of the Standard Hot Big Bang (HBB) Cosmology and how it can be built in the context of General Relativity. The second part focuses on the application of the principles of string theory to inflation. In this framework an inflationary model can be viewed as an effective field theory in the super-gravity limit. In particular we concentrate on inflationary models in which the effective Lagrangian is obtained from the dimensional reduction of the Dirac-Born-Infeld (DBI) action [14], which describes the position and orientation of a $D7$-brane. This effective Lagrangian contains a non-trivial kinetic term. We study the effects of this non-trivial kinetic term on the speed of sound c_s, on the tensor-to-scalar ratio r and on the spectral index n_s in the case of monomial potentials, generalizing the work in [15] and [16]. We find that the system never enters the relativistic regime and therefore $c_s \simeq 1$. This justifies the truncation of the DBI action to second order derivatives. In the final chapter we see how the Swampland criteria [17], when applied to DBI inflation, push the system into the relativistic regime and lead to a strong decrease of c_s.

Sommario

In questa tesi studieremo l'inflazione dal punto di vista della teoria delle stringhe. La prima parte del lavoro é dedicata alla rivisitazione di come questo meccanismo possa risolvere i principali problemi della Cosmologia Standard e di come esso possa essere ottenuto nel contesto della Relativitá Generale. La seconda parte si focalizzerá sull' applicazione dei principi della teoria delle stringhe all'inflazione. In questo quadro generale un modello inflazionario puó essere visto come una teoria di campo effettiva nel limite di super-gravitá. In particolare ci concentreremo sui modelli inflazionari nei quali la Lagrangiana effettiva é ottenuta dalla riduzione dimensionale dell' azione Dirac-Born-Infeld (DBI) [14], la quale descrive la posizione e orientazione di una $D7$-brana. La Lagrangiana effettiva contiene un termine cinetico non banale. Studieremo gli effetti di questo termine cinetico non banale sulla velocitá del suono c_s, sul rapporto tensore-scalare r e sull' indice spettrale n_s nel caso dei potenziali monomiali, generalizzando i lavori in [15] e [16]. Troveremo che il sistema non entrerá mai nel regime relativistico e di conseguenza osserveremo che $c_s \simeq 1$. Questo giustifica la troncatura dell' azione DBI al secondo ordine nelle derivate. Nel capitolo finale vedremo come i criteri Swampland [17], quando applicati all' inflazione DBI, portino il sistema nel regime relativistico e conducano ad una forte diminuzione di c_s.

Contents

Introduction

The experimental analysis of the CMB spectrum, performed by COBE, Wilkinson Microwave Anistropy Probe (WMAP) and more recently by the Planck Satellite, have determined to very high precision that the CMB spectrum is characterized by an average temperature $T \sim 2.73K$ and by small inhomogeneities at the order of $\delta T/T \sim 10^{-5}$ [1]-[2]-[3]. Furthermore complementary experiments like Sloan Digital Sky Survey (SDSS) have found that the distribution of the clusters is highly homogeneous on scales larger than a few megaparsec [4]. Finally the study of supernovae Ia has shown that the Universe is undergoing accelerated expansion [5].

The principle of Standard Hot Big Bang (HBB) Cosmology based on the Cosmological Principle and General Relativity (GR) cannot explain the inhomogeneities of the CMB spectrum and the fact that they are so small [6]-[7]-[8], a fact that is related to the so called horizon, flatness and entropy problems. The mechanism of inflation arises to give a solution of these three problems.

Furthermore it allows to explain the existence of the Large Scale Structures (LSS), since it produces the seeds of the density perturbations [9].

The inflationary mechanism is based on the existence of a scalar field, called inflaton, which is coupled to gravity. The energy density of this field dominates during the inflationary epoch, producing a superluminal expansion.

Even if inflation is a successful way to solve the main problems of the Standard Cosmology, it is not clear what is its physical origin. In fact to have a complete physical explanation of this mechanism we need to use both GR and Quantum Field Theory (QFT). At present these two frameworks are inconsistent, since we do not have the tools to deal with the gravitational field in the quantum regime. Over the Planck scale $M_p \simeq 2.4 \times 10^{18}$ GeV gravity becomes non-renormalizable. Furthermore the inflationary mechanism treated as a semi-classical gravitational theory is UV sensitive.

String theory seems to be a promising way to provide a UV completion to inflation [10]-[11]. The fundamental objects of this theory are strings and branes. Strings are extended objects which are characterized by a length scale l_s. This length scale can be used to cut off the UV gravity divergences and the hope is that it will allow us to build a consistent framework of quantum gravity.

This theory has to live in ten dimensions, six of them are wrapped in an internal space, which it is too small to be detected at low energies. From the 10-dimensional theory we can extract an effective 4-dimensional theory through compactification of the internal space. In general this mechanism leads to the creation of many light gravitationally coupled scalar fields, which are called moduli. Since they can interfere with the inflationary process and with well established post inflationary physics, we need to give them a mass through the introduction of background fluxes and for non-perturbative effects. This mechanism is called moduli stabilization.

The other important objects of the theory are Dp-branes, which are $(p+1)$-hypersurfaces to which the strings can be attached. Their position, orientation and gauge-field configuration can lead to extra moduli fields. These dynamical objects are governed by the Dirac-Born-Infeld (DBI) action and the Chern-Simons (CS) action.

In general the dimensional reduction of a Dp-brane action leads to an effective field theory in which the kinetic term is non-trivial since it multiplies the potential and contains a square root which encloses all the higher derivative corrections. These structures are important because they can lead to effects on the main cosmological observables, the spectral index n_s and the tensor-to-scalar ratio r, through their effect on the speed of sound c_s.

In this thesis we study the DBI inflationary model both in the relativistic and non-relativistic regime with the aim of connecting cosmological observables to a UV theory, paying special attention to the role of the high derivative terms.

This work is organized as follows. In the chapter 1 we develop the framework of the Standard Cosmology and review the main problems which arise from it. In the chapter 2 we introduce the mechanism of inflation and the main single-field models. In the chapter 3 we give some elements of string theory. In the chapter 4 we perform the dimensional reduction of the $D7$-brane action in order to obtain the DBI inflationary model. Then we discuss the effects of the non-trivial kinetic term which arises in the 4-dimensional action. The final chapter focuses on the analysis of the Swampland criteria in the context of D-brane inflation and how they push this inflationary model into the non-Gaussianity regime.

Chapter 1

Standard Hot Big Bang Cosmology

In this first chapter we want to build the cosmology framework, known as Standard Cosmology. This framework is based on the Cosmological Principle, which states that the Universe is almost spatially homogeneous and isotropic at large distances (over than hundred of Mpc). This principle fixes some important conditions about its structure on large scales.
Since the description of the Universe involves large structures, it can be performed through GR. It is well known that this is a classical theory, based on a geometrical description of the space-time through the metric tensor $g_{\mu\nu}$, which is a set of $d(d+1)/2$ independent quantities, where d is the dimension of the space-time (for the ordinary space-time $d = 4$).
From the metric tensor we can build the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \, . \tag{1.1}$$

This line element specifies the causality relations between two arbitrary events in the space-time. We say that two events are space-like if $ds^2 < 0$. These two events cannot be connected through a light signal, therefore they are not in causal connection. Otherwise two events are time-like if $ds^2 > 0$. In this case they can be connected by a light signal, therefore they are in causal connection. Finally we say that two events are light-like if $ds^2 = 0$. GR is a gauge invariant theory since ds^2 is a scalar, that is it does not depend on the particular choice of the reference frame.
From the metric tensor we can build the Riemann tensor $R^{\mu\nu\rho\sigma}$, which is the simplest tensor that depends on the second derivative of $g_{\mu\nu}$. After an appropriate contraction we obtain the Ricci tensor

$$g_{\mu\rho} R^{\mu\nu\rho\sigma} = R^{\nu\sigma} \, . \tag{1.2}$$

The other contractions give zero or an equivalent tensor. Finally from the Ricci tensor we obtain the Ricci scalar

$$g_{\nu\sigma} R^{\nu\sigma} = \mathcal{R} \, . \tag{1.3}$$

Through the Ricci scalar we can write the action of the classical field theory associated to GR, where the classical field is $g_{\mu\nu}$, the so called Einstein-Hilbert action

$$S_{EH} = \int d^4x \sqrt{-g} \frac{\mathcal{R}}{2} \,, \tag{1.4}$$

where we have imposed (as in the remaining of the thesis) $M_p^2 \equiv \frac{1}{8\pi G} = 1$.

1.1 Homogeneous and Isotropic Metric

Starting from the principle enunciated in the introduction we want to find the structure of the Universe space-time at large distances.
Let us decompose the metric $g_{\mu\nu}$ in three parts according to

$$ds^2 = g_{00}(dx^0)^2 + 2g_{0i}dx^0dx^i + g_{ij}dx^idx^j \,. \tag{1.5}$$

It is clear that the condition of isotropy implies that the mixed term $g_{0i} = 0$, in fact if it was non zero, we can perform a particular rotation to a frame (X^0, X^i) in which the metric would have a simpler form. The metric of the Standard Cosmology must be decomposed in the following way

$$ds^2 = g_{00}(dx^0)^2 + g_{ij}dx^idx^j \,. \tag{1.6}$$

We can fix the term $g_{00} = -1$. Using this convention the line element in a resting reference system coincides with the galactic proper time.
To find the term g_{ij} we observe that in a four dimensional space, the homogeneous and isotropic manifold can be written as

$$dl^2 = a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right] \,, \tag{1.7}$$

where $a(t)$ represents a scale function, called scale factor, $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$ and K is the spatial curvature parameter. For $K = 0$ we have a flat 3-space, for $K = +1$ we have the 3-sphere and for $K = -1$ the 3-hyperboloid.
Combining (1.6) with (1.7) we find the following line element

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right] \,, \tag{1.8}$$

which is called the Friedman-Robertson-Walker (FRW) metric. The metric (1.8) represents the core structure of the Standard Cosmology, from which we find most of the next results.

1.2 Friedman Equations

After we have found the metric of our framework, we need to find the equations which rule the dynamics of the Universe. These equations must be derived from GR. Therefore from the minimal action principle applied to (1.4) we obtain the Einstein equation,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \,. \tag{1.9}$$

As we know $R_{\mu\nu}$, R are the Ricci tensor and scalar built from the FRW metric (see (1.2) and (1.3)), G is the gravitational coupling, while $T_{\mu\nu}$ is the energy-stress tensor.
It takes the following form

$$T_{\mu\nu} = (p + \rho) u_\mu u_\nu - p g_{\mu\nu} \,, \tag{1.10}$$

where p and ρ are the pressure and the density of the fluid, which are in general related by the state equation $p = p(\rho)$ depending on the value of the energy-stress tensor trace. Substituting in the Einstein equation the relations (1.10) and (1.8) we find in general ten equations. Thanks to the form of $T_{\mu\nu}$ and $g_{\mu\nu}$ we have only two independent equations which contain the scale factor $a(t)$, that is

$$\frac{\dot{a}^2}{a^2} + \frac{K}{a^2} = \frac{8\pi G}{3} \rho \,, \tag{1.11}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + 3p\right) \,. \tag{1.12}$$

Equations (1.11) and (1.12) govern the Universe in the Standard Cosmology and they are called Friedmann equations.
Introducing the Hubble parameter $H(t) = \frac{\dot{a}}{a}$ we can rewrite (1.11) as

$$H^2 + \frac{K^2}{a^2} = \frac{8\pi G}{3} \rho \,. \tag{1.13}$$

We can observe that differentiating H we find

$$\dot{H} = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = -4\pi G(\rho + p) + \frac{K}{a^2} \,, \tag{1.14}$$

which for $K = 0$ becomes

$$\dot{H} = -4\pi G(\rho + p) \,. \tag{1.15}$$

1.3 Features of the Standard Cosmology

In this section we study the main consequences of the Friedman equations.
Let us consider equation (1.13). It is clear that $H(t)$ has in general two different roots depending on the choice of the plus or minus sign, however experimental evidence fixes $H(t) \geq 0$, therefore we have

$$H(t) = \sqrt{\frac{8\pi G}{3}\rho - \frac{K}{a^2}}\,. \tag{1.16}$$

Furthermore from the definition of the Hubble parameter we infer that

$$\dot{a} \geq 0\,. \tag{1.17}$$

The scale factor is a non-decreasing function of time. If we impose the standard energy condition (SEC) $\rho + 3p > 0$, we obtain from (1.12)

$$\ddot{a} < 0\,. \tag{1.18}$$

It is clear that from (1.17) and (1.18) it must exist a time t_0, which can be viewed as the beginning in which $a(t_0) = 0$. Therefore in the Standard Cosmology the Universe begins with a singularity. We should note that in order to obtain this result we have supposed that the Friedman equations are valid in each epoch, however in the primordial epoch this can be no longer true since the Universe was so small that we cannot ignore the quantum effects. In this regime the cosmological principle cannot be applied and so the metric (1.8) is no longer valid. For this reason in a cosmology based on a complete theory of quantum gravity the singularity could not exist.
Now we want to focus on the thermodynamic evolution of the Universe. It is reasonable to think that the whole Universe can be considered as an adiabatic system, therefore from the classical thermodynamics it implies that in each epoch the entropy maintains constant, that is

$$\frac{dS}{dt}^{Universe} = 0\,. \tag{1.19}$$

Therefore the first law of thermodynamics reads for the Universe

$$d(\rho a^3) + p d(a^3) = 0\,, \tag{1.20}$$

where ρ is the total energy density of the perfect fluid. If we impose $p = \gamma\rho$ with the SEC $\gamma > -1/3$, we have from (1.20)

$$\frac{d\rho}{\rho} = -3(\gamma+1)\frac{da}{a} \quad \Longrightarrow \quad \rho = \zeta a^{-3(\gamma+1)}\,, \tag{1.21}$$

where ζ is a simple constant depending on initial conditions. Inserting (1.21) in (1.11) with $K = 0$ we find

$$a^{\frac{3\gamma+1}{2}} da = \sqrt{\frac{8\pi G\zeta}{3}} dt \quad \Longrightarrow \quad a(t) = \beta t^{\frac{2}{3(1+\gamma)}}\,, \tag{1.22}$$

where β is another constant. From (1.21) and (1.22) we observe that the scale factor and the energy density have a different behaviours depending on the value of the dominant component of the perfect fluid. For instance if the dominant component is ultra-relativistic (photons, neutrinos and others light particle) with $\gamma = 1/3$ we find

$$\rho_{UR} = \zeta a^{-4} \ , \ a_{UR}(t) = \beta t^{1/2} \, . \tag{1.23}$$

For the heavy particle, i.e. barionic matter we have $\gamma = 0$, therefore we find

$$\rho_{BM} = \zeta a^{-3} \ , \ a_{BM}(t) = t^{2/3} \, . \tag{1.24}$$

There is another component in the Universe which it must be considered according to the experimental observations, but it breaks SEC. We call it cosmological constant Λ. We prove later that this source corresponds to a state parameter $\gamma = -1$, therefore using (1.21)

$$\rho_\Lambda = \zeta \, . \tag{1.25}$$

According to the results above it is possible to show that the history of the Universe is divided in three phases: radiation epoch, matter's epoch and dark energy epoch. We can make the same considerations for open ($K = -1$) and close ($K = +1$) universes, however the integration of (1.11) becomes more difficult.
We conclude the section observing that (1.11) can be rewritten in another form

$$1 + \frac{K}{a^2 H^2} = \frac{8\pi G}{3H^2}\rho \, . \tag{1.26}$$

Introducing the two quantities

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)} \ , \ \rho_c(t) = \frac{3H^2}{8\pi G} \tag{1.27}$$

called curvature parameter and critical density, we find

$$\Omega - 1 = \frac{K}{a^2 H^2} \, . \tag{1.28}$$

It is clear that if $K = -1$ we have $\Omega < 1$, that is $\rho < \rho_c$, while if $K = +1$ we have $\Omega > 1$, that is $\rho > \rho_c$. Finally an universe is flat only for $\Omega = 1$, i.e. $\rho = \rho_c$. Curiously our Universe seems to be flat (or at least highly flat) as regard the last observations.

1.4 Friedmann Models

In this last section we want to focus on the evolution of the three kinds of universe (flat, closed and open) composed of a perfect fluid with state equation $p = \gamma\rho$ with $\gamma > -1/3$. These types of models are called Friedmann models.

1.4.1 Flat Universe

Flat universes are characterized by $K = 0$. They are important because our Universe seems to be flat. We can compute the value of the Hubble parameter using (1.22), we find

$$H(t) = \frac{2}{3(1+\gamma)t}\,. \tag{1.29}$$

We note that the Hubble parameter becomes singular when we approach to the singularity, furthermore it depends on the dominant component of the universe. If we ignore the constant factor, we can use the Hubble parameter to estimate the age T_0^{flat} of our Universe

$$T_0^{flat} \sim \frac{1}{H_0}\,. \tag{1.30}$$

1.4.2 Closed Universe

A Friedmann closed universe is characterized by $K = +1$. We can obtain the evolution of the scale factor substituting (1.21) in (1.11), as we have done in the flat case. In this case we obtain

$$\int_0^{a'} \frac{a^{\frac{3\gamma+1}{2}}}{\sqrt{1-\frac{3}{8\pi G\zeta}a^{(3\gamma+1)}}}da = \sqrt{\frac{8\pi G\zeta}{3}}t\,. \tag{1.31}$$

The integral in the left side can be rewritten using the substitution

$$a^{3\gamma+1} = (8\pi G\zeta/3)\sin^2\psi\,. \tag{1.32}$$

We obtain the following integral

$$\frac{2}{3\gamma+1}\left(\frac{8\pi G\zeta}{3}\right)^{\frac{1}{3\gamma+1}} \int_0^{\psi} d\psi' \sin^{\frac{2}{3\gamma+1}}\psi' = t\,. \tag{1.33}$$

It is possible to compute (1.33) in the case of "dust" Universe, that is when $\gamma = 0$

$$t = \frac{8\pi G\zeta}{3}\int_0^{\psi} d\psi'\, 2\sin^2\psi' = \frac{\lambda}{2}\left(\phi - \sin\phi\right)\,, \tag{1.34}$$

where $\phi = \psi/2$ and $\lambda = (8\pi G\zeta)/3$.
The scale factor follows directly from (1.32) through

$$\sin^2 2\phi = \frac{1-\cos\phi}{2}\,. \tag{1.35}$$

Therefore we find

$$a = \frac{\lambda}{2}(1-\cos\phi)\,. \tag{1.36}$$

The scale factor and the cosmological time are parameterized by a spherical angle in a closed universe. Using the definition of the curvature parameter, equation (1.28) and the fact that for the "dust" $\lambda = (8\pi G\rho_0 a_0^3)/3$, we can rewrite (1.34) and (1.36) as

$$a = \frac{a_0}{2}\frac{\Omega_0}{\Omega_0 - 1}\left(1 - \cos\phi\right) , \tag{1.37}$$

$$t = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}\left(\phi - \sin\phi\right) . \tag{1.38}$$

Since $-1 \leq \cos\phi \leq 1$ a closed universe has a maximum size when $\phi = \pi$, then it collapses depending on its curvature. The time in which it collapses is equal two times its time of maximum growth

$$t_{rec} = 2t_{max} = \frac{\pi}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} . \tag{1.39}$$

Furthermore comparing the current age of a flat Universe given by (1.30) with the current age of a closed universe given substituting (1.37) evaluated at $a = a_0$ in (1.38) we find that a closed Universe is younger than a flat Universe, that is

$$T_0^{closed} = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}\left[\cos^{-1}\left(1 - \frac{2(\Omega_0 - 1)}{\Omega_0}\right) - \sin\left(\cos^{-1}\left(1 - \frac{2(\Omega_0 - 1)}{\Omega_0}\right)\right)\right] < T_0^{flat} , \tag{1.40}$$

These conclusions are general and they can be applied to each closed universe, not only the "dust" Universe.

1.4.3 Open Universe

A Friedmann open universe is characterized by $K = -1$. It is clear that we can use the same argumentations in (1.4.2) to obtain an integral equation similar to (1.31) with a plus sign instead of a minus sign.
Using the following parameterization

$$a^{3\gamma+1} = (8\pi G\zeta/3)\sinh^2\psi , \tag{1.41}$$

we find

$$t = \frac{2}{3\gamma+1}\left(\frac{8\pi G\zeta}{3}\right)^{\frac{1}{3\gamma+1}}\int_0^{\psi} d\psi' \sinh^{\frac{2}{3\gamma+1}}\psi' . \tag{1.42}$$

Integrating for the "dust" case and following the argumentations in (1.4.2) we find

$$t = \frac{1}{2H_0}\frac{\Omega_0}{|\Omega_0 - 1|^{3/2}}\left(\sinh\phi - \phi\right) , \tag{1.43}$$

$$a = \frac{a_0}{2}\frac{\Omega_0}{|\Omega_0 - 1|}\left(\cosh\phi - 1\right) . \tag{1.44}$$

It is clear from the equation (1.44) that an open Universe does not collapse. Furthermore it is possible to compute the current age, showing that an open universe is older than a flat Universe, that is

$$T_0^{open} = \frac{1}{2H_0}\frac{\Omega_0}{|\Omega_0-1|^{3/2}}\left[\sinh\left(\cosh^{-1}\left(1+\frac{2|\Omega_0-1|}{\Omega_0}\right)\right)-\cosh^{-1}\left(1+\frac{2|\Omega_0-1|}{\Omega_0}\right)\right] > T_0^{flat}\,. \tag{1.45}$$

1.4.4 De Sitter Universe

We conclude this section considering a last type of universe, which it is not strictly a Friedmann model, but it is never the less important. Therefore let us consider a flat universe with state equation $p = -\rho$. This is called de Sitter Universe. This universe is characterized by a constant Hubble parameter, as we see substituting the state equation in (1.15). This implies that the scale factor grows exponentially

$$\dot{H} = 0 \implies H(t) = H_0 \implies a(t) = a_0 e^{H_0(t-t_0)}\,. \tag{1.46}$$

From (1.46), (1.12) and (1.25) we see that a de Sitter universe expands accelerating in each time and it does not begin with a singularity (or its singularity is located at $t \to -\infty$). We will see that the inflationary epoch is very close to be a de Sitter age.
Historically the state equation for the cosmological constant was developed to reconcile the result of the Standard Cosmology with the stationary Universe model. For this reason Einstein chose to add a new parameter in his field equation, called cosmological constant Λ. This new term can be added to the energy-stress tensor (1.10). If we want to preserve the form of the perfect fluid, we must introduce a density $\tilde{\rho}$ and a pressure $\tilde{p}$ which satisfy

$$\tilde{p} = p - \frac{\Lambda}{8\pi G} \qquad \tilde{\rho} = \rho + \frac{\Lambda}{8\pi G}\,. \tag{1.47}$$

It is clear that if we ignore the standard sources we obtain the state equation with $\gamma = -1$.

1.5 Problems of Standard Cosmology

In this section we want to discuss the main problems that arise from the principles of the Standard Cosmology.

1.5.1 Curvature Problem

Let us consider the curvature equation (1.28). Substituting in K the same relation evaluated in the current time t_0, we find

$$\Omega - 1 = \frac{a_0^2 H_0^2}{a^2 H^2}\left(\Omega_0 - 1\right)\,. \tag{1.48}$$

from the definition of the critical density and the curvature parameter in (1.27), through the relation (1.21) with $\zeta = \rho_0 a_0^{3(\gamma+1)}$, it is possible to rewrite (1.48) in the following way

$$\frac{\Omega - 1}{\Omega} = \left(\frac{a_0}{a}\right)^{-(1+3\gamma)} \frac{\Omega_0 - 1}{\Omega_0}\,. \tag{1.49}$$

Introducing the relative growth of the curvature parameter $\delta\Omega = (|\Omega - 1|)/\Omega$ we obtain

$$\delta\Omega(t) = \left(\frac{a_0}{a(t)}\right)^{-(1+3\gamma)} \delta\Omega_0\,. \tag{1.50}$$

It is clear that if $\delta\Omega_0 = 0$, then $\delta\Omega(t) = 0$ in each epoch. Furthermore if $\delta\Omega_0 = c$, we find that $\delta\Omega(t) < c$ when we preserve the SEC $1 + 3\gamma > 0$. We conclude that, a part the flat case where $\Omega_0 = 1$, the curvature of an universe is destined to grow.
Considering our Universe there are two possibilities: it is exactly flat; it is very close to be flat, $\delta\Omega_0 \sim \epsilon$, this would mean that its curvature in the past epochs was close to be zero, but not zero.
To convince ourselves as regard the second possibility, let us consider (1.49) with $\gamma = 1/3$ and the fact that after the production of the light elements (nucleosynthesis age) the scale factor is grown with a factor 10^8, we have

$$\delta\Omega_{NU} \sim (10^8)^{-2}\epsilon = \epsilon 10^{-16} \sim 0!\,. \tag{1.51}$$

The considerations above are called curvature problem or "fine-tuning" problem of our Universe.
It is important to observe that if we consider a de Sitter age, the relative growth of the curvature parameter in (1.50) becomes a decreasing function of time.

1.5.2 Entropy Problem

The entropy problem is connected with the problem stated above. In fact using the hypothesis of the adiabatic expansion and the fact that, in the early stage, our Universe was highly flat, we obtain a large value of its entropy. Let us focus on the radiation component, which dominates the early Universe. Taking the absolute value of the (1.28) and using the first of (1.23), we have

$$|\Omega - 1| = \frac{1}{a^2 H^2} \sim \frac{m_p^2}{a^2 \zeta a^{-4}}\,, \tag{1.52}$$

where $m_p^2 = 1/(8\pi G)$ is the Planck mass in natural units.
The adiabatic condition (1.20) connects the temperature of radiation with the scale factor through $T = \alpha a^{-1}$. Therefore we can find the following relation for the entropy of the early Universe

$$S_U = \frac{1}{T}(\rho + p)a^3 \sim \zeta \frac{a^{-1}}{T} = \frac{\zeta}{\alpha}\,. \tag{1.53}$$

Now substituting (1.53) in (1.52) evaluated at the Planck age, we find

$$|\Omega - 1|_{t=t_p} \sim \frac{m_p^2}{S_U T_p^2} = \frac{1}{S_U}\,. \tag{1.54}$$

Comparing with the result in (1.5.1), we find that the entropy of the Universe must be very large.

1.5.3 Horizon Problem

Let us consider a patch in the Universe located in x_0, which emits at the time t_0 a light signal radially. From (1.8) with $K = 0$ the distance traveled between x_0 and x is

$$\delta r = \int_{t_0}^{t} \frac{dt'}{a(t')}\,. \tag{1.55}$$

It is clear that this patch can communicate with only a little size of the Universe $\sim \delta r^3$ because the speed of light is finite.
Now let us imagine that t_0 coincides with the singularity $t_{singularity} = 0$, therefore we can consider the integral (1.55) as the maximum distance available to the light signal; we call it comoving particle horizon and denotes it with τ,

$$\tau = \int_{0}^{t} \frac{dt'}{a(t')}\,. \tag{1.56}$$

Substituting in (1.22) the equation (1.56) we find

$$\tau = \frac{3\gamma + 1}{2}\left[t'^{(1+3\gamma)}\right]_0^t\,. \tag{1.57}$$

If $1 + 3\gamma > 0$, then (1.57) is always finite. Since a Friedmann Universe expands at every epoch it will exist two patches that do not communicate, therefore they cannot be in equilibrium.
The problem arises when we observe that the spectrum of the CMB, the ”thermal bath” of photons produced at the decoupling age, is highly homogeneous and involves patches that cannot have communicated during the finite time between the singularity and the decoupling epoch. The high homogeneity of the CMB cannot be explained by the Standard Cosmology.

Chapter 2

Inflationary Cosmology

In this chapter we study the paradigm of inflation, showing that it can solve the problems of the HBB Cosmology reviewed in the previous chapter. First we fix the fundamental conditions, which will be used in the following of discussion. Furthermore we show that inflation can explain the existence of the large scale structures.

2.1 Conditions for Inflation

Let us consider the integral in (1.56) and rewrite it as

$$\tau = \int_0^a \frac{1}{a'H} d\log(a')\,, \tag{2.1}$$

where we have performed a change of variable and we have introduced the comoving Hubble sphere radius $R_{H,C} = 1/(aH)$.
We call (2.1) the conformal time. Using this coordinate the line element (1.8) takes the following simple form

$$ds^2 = a^2(\tau)\left(-d\tau^2 + \frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\right)\,. \tag{2.2}$$

From the equations (1.22) and (1.29) we have

$$R_{H,C}(t) \propto \frac{3\gamma+1}{2} t^{(1+3\gamma)/3(1+\gamma)}\,. \tag{2.3}$$

Comparing with the result in (1.57) we find that in the Standard Cosmology the Hubble radius always grows in time.
Since the Hubble radius grows there is in general no possibility that two arbitrary regions in the sky enter in causal contact between the singularity and the decoupling time. The

only way to avoid this problem is to suppose that there will exist a period in which the Hubble radius decreases in time, that is

$$\frac{dR_{H,C}}{dt} < 0 \,. \tag{2.4}$$

We can say that (2.4) is the main condition for inflation. Using the definition of Hubble radius in (2.4) we infer that

$$-\frac{\ddot{a}}{\dot{a}^2} < 0 \implies \ddot{a} > 0 \,, \tag{2.5}$$

therefore during the period of inflation the scale factor accelerates.
Comparing this result with equation (1.12) it is clear that $\rho + 3p < 0$, that is $1 + 3\gamma < 0$. Therefore during the period of inflation the Universe is dominated by a non-standard source. As we have discussed this source can be identified with the cosmological constant.
The period of inflation can solve each of the three problems states in section (1.5).
Firstly let us suppose that the beginning of the inflationary epoch coincides with the singularity and that it ends at the cosmological time t^*, then the comoving particle horizon is given by (1.56). However now τ has a singularity for $t = 0$; this means that between the singularity and the decoupling time there will have been enough time to allow that the different regions in the Universe enter in causal contact. This can explain the high homogeneities of the CMB spectrum. It is hard to think that the inflationary epoch starts with the singularity, but it is enough that it lasts quite to explain the high homogeneity of the spectrum.
The other two problems are evidently solved because they are connected each other and we have already observed that the flatness problem is solved when the Universe is dominated by a non-standard source, that is in the inflationary epoch. Now let us consider the condition (2.4) which can be differentiated using the product law, we find

$$-\frac{1}{H}\frac{\dot{a}}{a^2} - \frac{1}{a}\frac{\dot{H}}{H^2} < 0 \implies \epsilon_H \equiv -\frac{\dot{H}}{H^2} < 1 \,. \tag{2.6}$$

We have introduced the first inflationary parameter which depends on the model through the Hubble parameter. If the condition (2.6) is satisfied, then inflation holds.
Consequently inflation ends when

$$\epsilon_H(t^*) \sim 1 \,. \tag{2.7}$$

We can also introduce another parameter, which is useful to evaluate the variation in time of the parameter ϵ_H, which is

$$\eta_H \equiv \frac{\dot{\epsilon}_H}{\epsilon_H H} \,. \tag{2.8}$$

It is clear that in order that (2.6) holds enough to solve the cosmological problems, then the variation of ϵ_H has to be small. This can be translated to another important condition

$$|\eta_H| \equiv \frac{|\dot{\epsilon}_H|}{\epsilon_H H} \ll 1\,. \tag{2.9}$$

The conditions (2.6) and (2.9) must be satisfied by each reasonable inflationary model. It is important to observe that the condition (2.6) implies a small variation in time of the Hubble parameter, that is $\dot{H} \sim 0$. For this reason the inflationary epoch can be viewed as an almost de-Sitter epoch (in a de-Sitter epoch $\epsilon_H = 0$ at each time and so the inflationary period is eternal).
The scale factor grows almost exponentially; in fact for $H(t) \simeq H_0$, we find

$$a(t) \simeq a_0 e^{H_0(t-t_0)}\,. \tag{2.10}$$

For this reason we prefer to use the so called e-folding number N_e to estimate the duration of inflation instead of the ordinary time. It is defined by the following

$$N_e = \int_{t_0}^{t_{end}} dt H(t)\,, \tag{2.11}$$

where t_0 and t_{end} are the beginning and the end of the inflationary period.
It is possible to find that an inflationary epoch must last at least $N_e \sim 60$ to solve the horizon problem. Precise number depends on details about reheating and post inflationary evolution [13].

2.2 Single-Field Inflation Models

In this section we begin to study the field theory model, which can describe the inflationary epoch. We focus on the so called single-field models. In these models inflation is driven by a scalar field $\phi(t, \mathbf{x})$, called inflaton. It plays the role of a non-standard source, characterized by a energy-stress tensor of the type (1.10).
It is important to observe that in general the inflaton depends on the space-time coordinates, however in the discussion of the inflationary models we will suppose that the inflaton depends only on the time. This is due to the fact that inflation dynamics tends to dilute the spatial inhomogeneities.
Firstly we consider a field theory with an ordinary kinetic term. Then we will extend the discussion to the models with a non-trivial kinetic term.

2.2.1 Two-Derivative Models

Let us consider a theory with a scalar field coupled with gravity through the Einstein equation. The action of this simple model can be written as

$$S = \int d^4x\sqrt{-g}\left(\frac{\mathcal{R}}{2} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) . \tag{2.12}$$

The first part contains the gravitation through the Ricci scalar, while the second part contains the non-standard source. Choosing the time coordinate such that $g^{00} = -1$, we rewrite the action (2.12) as

$$S = \int d^4x\sqrt{-g}\left(\frac{\mathcal{R}}{2} + \frac{1}{2}\dot{\phi}^2 - V(\phi)\right) , \tag{2.13}$$

where we recognize the trivial kinetic term.
From the action (2.13) we can find the energy-stress tensor of the model and comparing with (1.10), we have

$$\rho = \left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \ , \ p = \left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right) . \tag{2.14}$$

From the Euler-Lagrange equation we find the following Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial\phi} = 0 . \tag{2.15}$$

We can substitute (2.14) in the equations (1.15) and (1.13) for $K = 0$ to obtain

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) , \tag{2.16}$$

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2 . \tag{2.17}$$

The relations (2.15), (2.16), (2.17) are the fundamental equations of the model.
Through these let us evaluate the parameter (2.6) and (2.8), finding

$$\epsilon_H = \frac{1}{2}\frac{\dot{\phi}^2}{H^2} , \tag{2.18}$$

$$\eta_H = 2\left(\frac{\ddot{\phi}}{\dot{\phi}H} - \frac{\dot{H}}{H^2}\right) \equiv 2(\epsilon_H - \delta) . \tag{2.19}$$

Now remembering the condition (2.9), we find

$$|\eta_H| = 2|\epsilon_H - \delta| \leq 2(|\epsilon_H| + |\delta|) \ll 1 , \tag{2.20}$$

which it is clearly satisfied when $|\epsilon_H| \ll 1$ and $|\delta| \ll 1$ (in this way the condition (2.6) is automatically satisfied).
Using (2.18) and the definition of δ we obtain these two remarkable conditions

$$\dot{\phi}^2 \ll V \ , \ |\ddot{\phi}| \ll |\dot{\phi}| H \, . \tag{2.21}$$

These two are called slow-roll conditions. They lead to some simplifications to (2.15) and (2.16), that is

$$H^2 \simeq \frac{1}{3} V(\phi) \, , \tag{2.22}$$

$$3H\dot{\phi} + V(\phi)_{,\phi} \simeq 0 \, . \tag{2.23}$$

Solving (2.23) respect to the velocity of the inflaton and substituting in the definition of ϵ_H we have

$$\epsilon_H \simeq \frac{1}{6} \frac{V_{,\phi}}{V H^2} = \frac{1}{2} \left(\frac{V_{,\phi}}{V} \right)^2 \equiv \epsilon_V \, . \tag{2.24}$$

Let us take the first time derivative of (2.23) and substituting it in (2.19), we find

$$\eta_H \simeq \left(\frac{V_{,\phi\phi}}{V} \right) \equiv \eta_V \, . \tag{2.25}$$

The parameters introduced in (2.24) and (2.25) are called potential slow-roll parameters (instead we refer to ϵ_H and η_H as the Hubble slow-roll parameters).
Through (2.24) and (2.25) we can say that potential slow-roll inflation occurs if and only if

$$\epsilon_V = \frac{1}{2} \left(\frac{V_{,\phi}}{V} \right)^2 \ll 1 \, , \quad |\eta_V| = \left(\frac{|V_{,\phi\phi}|}{V} \right) \ll 1 \, . \tag{2.26}$$

The conditions above are valid in the region where the potential of the model is almost flat.
It is possible to write the e-folding number (2.11) in term of ϵ_V, through

$$N_e = \int_{\phi_i}^{\phi_{end}} \frac{H}{\dot{\phi}} d\phi \simeq - \int_{\phi_i}^{\phi_{end}} \frac{3H^2}{V_{,\phi}} d\phi \simeq - \int_{\phi_i}^{\phi_{end}} \frac{V}{V_{,\phi}} d\phi \equiv \int_{\phi_i}^{\phi_{end}} \frac{1}{\sqrt{2\epsilon_V}} d\phi \, , \tag{2.27}$$

where we have used (2.22) and (2.23). The upper extreme of integration in (2.27) can be evaluated using the condition (2.7).
In the familiar case of a monomial potential $V(\phi) = V_0 \phi^n$ we have

$$\phi_{end} \sim \frac{n}{\sqrt{2}} \, . \tag{2.28}$$

The lower extreme of integration is chosen in order to have at least $N_e \sim 60$. It is important to note that in these type of models $\Delta\phi \equiv \phi_i - \phi_{end} \gg \mathcal{O}(1)$, therefore the variation of inflaton during the process is always in a trans-Planckian domain.

Now let us consider a model with a non-trivial kinetic term,

$$S=\int d^4x\sqrt{-g}\left(\frac{\mathcal{R}}{2}-\frac{1}{2}f(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi)\right)\,, \tag{2.29}$$

where $f(\phi)$ is a function which guarantees the non-canonicity. For $f(\phi)=1$ we obtain the canonical kinetic term. Through similar argumentations we have seen above, from (2.29) we obtain the following fundamental equations

$$\ddot{\phi}+\frac{1}{2}f^{-1}(\phi)f(\phi)_{,\phi}\,\dot{\phi}^2+3H\dot{\phi}+f^{-1}(\phi)V(\phi)_{,\phi}=0\,, \tag{2.30}$$

$$\rho=\left(\frac{1}{2}f(\phi)\dot{\phi}^2+V(\phi)\right)\,,\; p=\left(\frac{1}{2}f(\phi)\dot{\phi}^2-V(\phi)\right)\,. \tag{2.31}$$

Furthermore we need the Friedmann equations to evaluate the behaviour of the inflationary parameters.
Substituting (2.31) into equations (1.13) and (1.15) for $K=0$ we obtain

$$H^2=\frac{1}{3}\left(\frac{1}{2}f(\phi)\dot{\phi}^2+V(\phi)\right)\,, \tag{2.32}$$

$$\dot{H}=-\frac{1}{2}f(\phi)\dot{\phi}^2\,. \tag{2.33}$$

In this case the slow-roll conditions read

$$f(\phi)\dot{\phi}^2\ll V(\phi)\,,\quad \frac{|f(\phi)_{,\phi}|}{f(\phi)}\dot{\phi}^2\ll H|\dot{\phi}|\,,\quad |\ddot{\phi}|\ll H|\dot{\phi}|\,. \tag{2.34}$$

Therefore we have the following simplifications to the (2.30) and (2.32),

$$H^2\simeq\frac{1}{3}V(\phi)\,, \tag{2.35}$$

$$3H\dot{\phi}+f^{-1}(\phi)V(\phi)_{,\phi}\simeq 0\,. \tag{2.36}$$

We can introduce a new field Φ, which it is related to the field ϕ through

$$\frac{d\Phi}{d\phi}=\sqrt{f(\phi)}\,,\qquad \Phi=\int_0^\phi d\phi'\sqrt{f(\phi')}\,. \tag{2.37}$$

Substituting (2.37) in (2.35) and (2.36) we obtain the equivalence with the relations (2.22) and (2.23). Therefore we can introduce the same potential slow-roll parameters ϵ_V, η_V, which they satisfy to (2.26) and make the same considerations, where we have to remember that the two parameters are now expressed in term of the field Φ.
We note that if we perform this field transformation, the kinetic term in (2.29) becomes canonical as we have expected. For this reason Φ is also called canonical field.
In this paper we will use the following form of the function $f(\phi)=1+aV(\phi)$, where the parameter a can be explained only in the string framework.

2.2.2 K-Inflation Model

The two models above are similar (up to a field redefinition) because the inflation can be explained in terms of two parameters, which satisfy the potential slow-roll conditions (see (2.26)) in which $\{\epsilon_V, |\eta_V|\} \ll 1$. These conditions are strongly influenced by the shape of the potential.
However in general if we want to preserve inflation, it is enough to satisfy $\{\epsilon_H, |\eta_H|\} \ll 1$. These conditions can be satisfied without fixing constraints to the shape of the potential, but through the presence of the high derivative terms in the Lagrangian of the model. In this subsection we concentrate on these types of single-field models, which are called k-inflation models [14].
Therefore let us consider the following Lagrangian

$$p(X,\phi) = \sum_n c_n(\phi)\frac{X^n}{\Lambda^{4n-4}}\,, \tag{2.38}$$

where $X \equiv -1/2 g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is the ordinary kinetic term (up to a field redefinition), while Λ is a parameter which has the dimension of a mass. We can think that Λ is the parameter that controls the validity of the series (2.38).
It is clear that if $X^n/\Lambda^{4(n-1)} \ll 1$, the higher derivative terms can be neglected and we obtain the trivial dynamics of the two derivative models. The action of the model can be written as

$$S = \int d^4x\sqrt{-g}\left(\frac{\mathcal{R}}{2} + p(X,\phi)\right)\,. \tag{2.39}$$

The Klein-Gordon equation reads

$$\ddot{\phi}\, p_{,X} + \dot{\phi}\dot{X}\, p_{,XX} + 3H\dot{\phi}\, p_{,X} - p_{,\phi} = 0\,. \tag{2.40}$$

Maintaining the perfect fluid analogy, the pressure and the energy density are given by

$$\rho = 2Xp_{,X} - p\,, \qquad P = p(X,\phi)\,. \tag{2.41}$$

Therefore the Friedmann equations become

$$H^2 = \frac{1}{3}\left(2Xp_{,X} - p\right)\,, \tag{2.42}$$

$$\dot{H} = -Xp_{,X}\,. \tag{2.43}$$

Through these we can immediately derive the following Hubble slow-roll parameters

$$\epsilon_H = \frac{3Xp_{,X}}{2Xp_{,X} - p}\,, \tag{2.44}$$

$$\eta_H = \frac{1}{Xp_{,X}} \left[2 \left(\frac{\dot{H}}{H} \right)^2 + \frac{\dot{X}}{H} (p_{,X} + Xp_{,XX}) + \dot{\phi} \frac{X}{H} p_{,X\phi} \right] . \tag{2.45}$$

To derive these relations we have supposed that X and ϕ are two independent variables. It is important to derive the speed of sound of scalar perturbations and its variation through inflation. We have from the classical hydrodynamics

$$c_s^2 = \frac{\delta P}{\delta \rho} = \frac{p_{,X}}{2Xp_{,XX} + p_{,X}} . \tag{2.46}$$

The variation parameter can be written in analogy to η_H

$$s = \frac{\dot{c}_s}{c_s H} = \frac{2 \left[\frac{\dot{X}}{H} \left(X(p_{,XX})^2 - Xp_{,XXX}\, p_{,X} - p_{,XX}\, p_{,X} \right) + X \frac{\dot{\phi}}{H} \left(p_{,XX}\, p_{,X\phi} - p_{,XX\phi}\, p_{,X} \right) \right]}{p_{,X} \left(2Xp_{,XX} + p_{,X} \right)} . \tag{2.47}$$

If we consider $p(X, \phi) = X - V(\phi)$ we obtain the same results of the two derivative case. A special form of $p(X, \phi)$ is given by

$$p(X, \phi) = -\Lambda^4 \sqrt{1 - \frac{X}{\Lambda^4}} + \Lambda^4 . \tag{2.48}$$

As we have discussed for the form of $f(\phi)$, (2.48) arises when we apply the string theory to inflation.

2.3 Perturbations From Inflation

Inflation can also explain the existence of the cosmological structures such as galaxy clusters, star clusters and so on which are called large scale structures (LSS). These structures were formed by the perturbations to the matter density thanks to the action of gravity.

In order to explain how inflation can produce the seeds of the structures we observe today, we have to consider the fact that the inflaton in a complete quantum theory is characterized by quantum fluctuations. Therefore we can write

$$\phi(t, \mathbf{x}) = \phi^0(t) + \delta\phi(t, \mathbf{x}) , \tag{2.49}$$

$$\delta\phi(t, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \left(\delta\phi_{\mathbf{k}}(t) e^{-i\mathbf{kx}} + \bar{\delta}\phi_{\mathbf{k}}(t) e^{i\mathbf{kx}} \right) . \tag{2.50}$$

It is clear that the fluctuations of the inflaton are related to the fluctuations of the energy-stress tensor $\delta T_{\mu\nu}$. These fluctuations are related to the fluctuations of the metric $\delta g_{\mu\nu}$ through the Einstein equations (1.9). The fluctuations to the metric influence the

behaviour of the inflaton through the Euler-Lagrange equations. Therefore we conclude that the metric and the inflaton fluctuations are related.
We can write

$$\delta g_{\mu\nu}(t,\mathbf{x}) \Leftrightarrow \delta\phi(t,\mathbf{x})\,. \tag{2.51}$$

The perturbed metric can be written in terms of the scalar and the tensor perturbations as

$$g_{\mu\nu}(t,\mathbf{x}) = \begin{pmatrix} -(1+2\Phi) & 0 \\ 0 & a^2(t)((1-2\Psi)\delta_{ij}+h_{ij}) \end{pmatrix}\,, \tag{2.52}$$

where Φ and Ψ are the scalar perturbations, while h_{ij} are the tensor perturbations. These are associated to the density fluctuations and the gravitational waves, therefore we are able to detect them. The vector perturbations are diluted during the inflationary dynamics, for this reason we put them to zero.
When we study the perturbations of a quantity $\delta f(x^\mu)$ we need to know its value respect to the unperturbed background $f^0(x^\mu)$ and to the perturbed space-time $f(x^\mu)$. It is clear that these values depend on the frame that we choose for the computation (we talk about a gauge choice) and in general if we perform a frame transformation (a gauge transformation) of the type $\tilde{x}^\mu = x^\mu + \delta x^\mu$, we have $\tilde{\delta f}(\tilde{x}^\mu) \neq \delta f(x^\mu)$. In particular if we consider the quantity $f(x^\mu)$ as a scalar field, we find

$$\begin{aligned} \tilde{\delta f}(\tilde{x}^\mu) &\equiv \tilde{f}(\tilde{x}^\mu) - \tilde{f}^0(\tilde{x}^\mu) = f(x^\mu) - f^0(\tilde{x}^\mu) = \\ &= f(\tilde{x}^\mu) - \delta x^\mu \frac{\partial f}{\partial x^\mu}(\tilde{x}) - f^0(\tilde{x}^\mu) = \delta f(\tilde{x}^\mu) - \delta x^0 f'(\tilde{x}^\mu)\,, \end{aligned} \tag{2.53}$$

where we have used the property of a scalar field under a change of map and the fact that the unperturbed value f^0 is independent from the choice of the map. From (2.53) we find

$$\tilde{\delta f} = \delta f - \delta x^0 f'\,. \tag{2.54}$$

Since the inflaton is a scalar field we find

$$\tilde{\delta\phi} = \delta\phi - \phi'\delta x^0\,. \tag{2.55}$$

Furthermore from the gauge invariance of the line element ds^2 we find

$$\tilde{\Psi} = \Psi + \mathcal{H}\delta\tau\,, \tag{2.56}$$

where we have introduced the comoving Hubble parameter. It is clear from the transformation laws (2.55) and (2.56) that the quantity

$$\mathcal{R} = \Psi + \mathcal{H}\frac{\delta\phi}{\phi'} = \Psi + H\frac{\delta\phi}{\dot{\phi}} \tag{2.57}$$

is gauge invariant.
This quantity is called comoving curvature perturbation, which it is relevant because it

is connected to the gravitational potential on the hypersurface where $\delta\phi = 0$.
Now recalling the fact that the fluctuations to the inflaton and the metric are connected, one can find an equation which contains the evolution of the Fourier modes associated to the comoving curvature perturbation $\mathcal{R}_k$, the so called Mukhanov-Sasaki equation

$$v_k''(\tau) + \left(k^2 - \frac{z''(\tau)}{z(\tau)}\right) v_k(\tau) = 0\,, \tag{2.58}$$

where

$$v \equiv z\mathcal{R}\,, \qquad z(\tau) \equiv a(\tau)^2 \frac{\dot{\phi}^2}{H^2}\,. \tag{2.59}$$

This differential equation is valid only in the two derivative case. The equation for the perturbations in the k-inflation models can be found in [14]. In this section we limit ourselves to discussing only the equation (2.58), which contains all the features of our interest.
Let us consider an arbitrary mode k. If this mode satisfy $k \gg aH$, i.e. for a wavelength much smaller than the horizon, we have $k^2 \gg \frac{|z''|}{|z|}$ and so the equation (2.58) reduces to

$$v_k''(\tau) + k^2 v_k(\tau) = 0\,, \tag{2.60}$$

which it has oscillatory solutions $v_k \propto e^{\pm ik\tau}$.
However if the mode satisfy $k \ll aH$, i.e. for a wavelength much larger than the horizon, we have $k^2 \ll \frac{|z''|}{|z|}$, therefore (2.58) reduces to

$$\frac{v_k''}{v_k} = \frac{z''}{z} \approx \frac{2}{\tau^2}\,, \tag{2.61}$$

where we have used (1.56) for a de-Sitter epoch, which implies $a = (H\tau)^{-1}$. This equation has a growing solution $v_k(\tau) \propto z \propto \tau^{-1}$, which it implies that the comoving curvature mode k remains constant outside the horizon, that is

$$\mathcal{R}_k = z^{-1} v_k \propto const\,. \tag{2.62}$$

We can perform a similar considerations for the tensor perturbations, obtaining two similar equations (one for each polarization h^+,h^-), where $z = a$.

2.4 Cosmological Observables

In this section we want to introduce some quantities that are related to observational tests of inflationary models.

When we have determined the modes $\mathcal{R}_k$, h^+, h^- we can introduce the power spectra $\mathcal{P}_{\mathcal{R}}(k)$ and $\mathcal{P}_t(k)$, which are given by

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2}|\mathcal{R}_k|^2 = \frac{1}{8\pi^2}\frac{H^2}{\epsilon_H c_s}_{|_{c_s k = aH}}, \tag{2.63}$$

$$\mathcal{P}_t(k) = \frac{k^3}{2\pi^2}\left(|h_k^+|^2 + |h_k^-|^2\right) = \frac{2}{\pi^2}H^2_{|_{c_s k = aH}}, \tag{2.64}$$

where the scalar and tensor perturbations are evaluated at the exit of the horizon. As we can observe the presence of the non-trivial dynamics influenced only the amplitude of the scalar perturbations.
Through (2.63) and (2.64) it possible to define the spectral indices, which parameterize the scale dependence of the power spectra, that is

$$n_s - 1 = \frac{d\log\mathcal{P}_{\mathcal{R}}(k)}{d\log k} = -2\epsilon_H - \eta_H - s, \tag{2.65}$$

$$n_t = \frac{d\log\mathcal{P}_t(k)}{d\log k}. \tag{2.66}$$

We can introduce the tensor-to-scalar ratio

$$r = \frac{\mathcal{P}_t(k)}{\mathcal{P}_{\mathcal{R}}(k)} = 16\epsilon_H c_s. \tag{2.67}$$

The analysis of CMB spectrum gives the experimental constrains to the quantities introduced above. In fact after the end of inflation the comoving horizon begins to grow and the frozen fluctuations outside the horizon can re-enter. Perturbations at the horizon crossing are imprinted in the CMB spectrum and so they can be measured by observations. The horizon crossing is located almost $50-60$ e-folding before the end of inflation. Another important measurable quantity is the amplitude of the scalar perturbations A_s, through which we can express the power spectrum of the scalar perturbations $\mathcal{P}_{\mathcal{R}}(k)$ in the following way

$$\mathcal{P}_{\mathcal{R}}(k) = A_s\left(\frac{k}{k_*}\right)^{n_s-1}. \tag{2.68}$$

Comparing (2.63) with (2.68) we have

$$A_s = \frac{H^2}{8\pi^2\epsilon_H c_s}, \tag{2.69}$$

evaluated at the horizon crossing.
In this thesis we will use the following constrains [2]

$$r < 0.07, \qquad n_s = (0.9667 \pm 0.0080), \qquad c_s \geq 0.024 \tag{2.70}$$

with a 95% confidence level and

$$A_s = (2.207 \pm 0.076) \times 10^{-9} \tag{2.71}$$

with a 65% confidence level.

Chapter 3

Elements of String Theory

In this chapter we want to recall some important concepts useful for inflation regarding the string theory.

3.1 Framework

In the classical theories the fundamental object is the point-like particle. However it is well known that this primitive model carries to a lot of inconsistencies when we apply the theory at very large energies. These inconsistencies can be avoided in a theory in which the fundamental object is not point-like, but extended. In the string theory we have two fundamental extended objects: string and branes.
Firstly we focus on the string. Its size is given by the "string mass" or equivalently the "string length"

$$M_s^2 = \frac{1}{\alpha'} \ , \ l_s^2 = \alpha' \ , \tag{3.1}$$

where α' is the Regge slope.
The string theory can be viewed as a Quantum Field Theory, where the fundamental object has the size in (3.1). In this framework the point-like particles correspond to the different excitation modes of the string.
In a particle theory, one particle describes a world-line during its motion; in a string theory we talk about a world-sheet, specified by $x^M(\tau, \sigma)$, where $M = 0, 1, ...d - 1$ and d is the dimension of the target space-time. The first coordinate of the world-sheet τ is identified as time-like, while the other σ is identified as space-like.
A string can be viewed as a particular case of a more general object, called p-brane (a generalization of the term "membrane") which are described by a p space-like coordinates $(\sigma_1, ..., \sigma_p)$ and one time-like coordinate $\sigma_0 \equiv \tau$. A p-brane embeds a world-volume, given by $x^M(\tau, \sigma_1, ...\sigma_p)$. Therefore we can view a string as a 1-brane.
Strings can be open or closed. The space-like variables often vary in a closed interval

$0 \leq \sigma_i \leq \pi$ with $i = 1, .., p$. Otherwise it is clear that the time-like coordinate $\tau \in \mathbb{R}$ both for the open and closed strings.
The analogy with the relativistic theory of a point-like particle is evident if we recall the action of a bosonic string (or 1-brane) in a d dimensional target space-time, the so called Polyakov action

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{ab} \partial_a X^M \partial_b X^N \eta_{MN} \,, \tag{3.2}$$

where $a, b = 0, 1$ are the coordinates on the string world-sheet, η_{MN} with $M, N = 0, ..., d-1$ is the metric tensor and h^{ab} is the metric world-sheet. The string tension is given by

$$\frac{T_2}{2} \equiv \frac{1}{4\pi\alpha'} \,. \tag{3.3}$$

The theory (3.2) can be interpreted as a $(1+1)$-dimensional field theory with d bosonic fields.
In general we need to fix some boundary conditions to the string if we want to study the dynamics through the Euler-Lagrange equations. The boundary conditions are different for the open and closed string. For example if we consider a bosonic closed string we can choose

$$X^M(\tau, \sigma) = X^M(\tau, \sigma + \pi) \,. \tag{3.4}$$

In general for a bosonic open string we can choose between two possibilities. We can choose that no momentum flows outside the end of the string, that is

$$X'_M \equiv \frac{\partial X_M}{\partial \sigma} = 0 \quad at \quad \sigma = 0, \pi \,. \tag{3.5}$$

They are called Neumann boundary conditions.
Otherwise we can choose that the position of the two string ends are fixed for $M = 1, ..., d-(p+1)$, that is

$$X^M|_{\sigma=0} = X^M_0 \quad and \quad X^M|_{\sigma=\pi} = X^M_\pi \,. \tag{3.6}$$

They are called Dirichlet boundary conditions. For the other $p+1$ coordinates Neumann boundary conditions can be imposed.
Near the bosonic string action we can considered the fermionic string action, which takes the form

$$S_F = \frac{i}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{ab} \bar{\psi}^M \rho_b \partial_a \psi^N \eta_{MN} \,, \tag{3.7}$$

where ρ^a are the Clifford matrices, which satisfy

$$\{\rho^a, \rho^b\} = -2\eta^{ab} \,. \tag{3.8}$$

We can impose some boundary conditions for the two polarizations $\psi^M_+(\tau, \sigma)$ and $\psi^M_-(\tau, \sigma)$. We can choose periodic conditions such as $\psi^M_\pm(\tau, \sigma + \pi) = \psi^M_\pm(\tau, \sigma)$, from which we say

that periodic fermions are in the Ramond sector. Otherwise we can impose anti-periodic conditions $\psi_{\pm}^{M}(\tau,\sigma+\pi) = -\psi_{\pm}^{M}(\tau,\sigma)$, from which we say that anti-periodic fermions are in the Neveu-Schwarz sector.
The choice can be different between the left-moving fermions and the right-moving fermions. Therefore we have four possibilities: NS-NS,NS-R,R-NS or R-R. In a super-string theory the bosonic fields arise from the NS-NS and R-R sectors, while the fermionic fields arise from NS-R and R-NS sectors.
As regard the application of the string theory to inflation we concentrate on the Type II-B super-gravity theories. The inflationary model can be viewed as an effective field theory for energies below the string mass scale M_s. We can write the effective action of the Type II-B theory as

$$S_{IIB} = S_{NS} + S_R + S_{CS}\,. \tag{3.9}$$

The first two terms correspond to the bosonic sector and are given by

$$S_{NS} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_3|^2\right)\,, \tag{3.10}$$

$$S_R = -\frac{1}{4\kappa^2}\int d^{10}x\sqrt{-G}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right)\,, \tag{3.11}$$

in which ϕ is a bosonic field, called dilaton, G is the determinant of the metric tensor G_{MN}, $H_3 = dB_2$ and $F_1 = dC_0$ are tensor fields, R is the Ricci scalar and $2\kappa^2 = (2\pi)^7(\alpha')^4$. The other two fields are defined by the following redefinitions

$$\tilde{F}_3 = F_3 - C_0 H_3\,, \tag{3.12}$$

$$\tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 - \frac{1}{2}B_2 \wedge F_3\,, \tag{3.13}$$

where $F_3 = dC_2$ and $F_5 = dC_4$. The last part in (3.9) is the Chern-Simons term, given by

$$S_{CS} = -\frac{1}{\kappa^2}\int C_4 \wedge H_3 \wedge F_3\,. \tag{3.14}$$

In the end the self-duality of the tensor field $\tilde{F}_5 = \star\tilde{F}_5$ has to be imposed, where $\star$ is the usual Hodge operator

$$\star_d A_{\mu_1,\ldots,\mu_p} = \frac{1}{p!}\epsilon^{\nu_{p+1},\ldots,\nu_d}_{\mu_1,\ldots,\mu_p}A_{\nu_{p+1},\ldots\nu_d}\,. \tag{3.15}$$

We can write the action (3.9) in the Einstein frame, performing a Weyl transformation $G_{MN} \to G_{MN}e^{-i\phi}$, defining the axio-dilaton τ and a self-dual primitive flux G_3

$$\tau = C_0 + ie^{-\phi}\,, \tag{3.16}$$

$$G_3 = F_3 - \tau H_3\,. \tag{3.17}$$

Therefore the action (3.9) becomes

$$S_{IIB} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G_E} \left(R_E - \frac{|\partial\tau|^2}{2(\Im|\tau|)^2} - \frac{|G_3|^2}{2\Im[\tau]} - \frac{|\tilde{F}_5|^2}{4} \right) - \frac{i}{8\kappa^2} \int \frac{C_4 \wedge G_3 \wedge \tilde{G}_3}{\Im[\tau]} \,. \tag{3.18}$$

From this action we can obtain the equations of motion of the theory.

3.2 Compactification

The action discussed above has to live in 10 dimensions for consistency. Since we observe only four dimensions in the ordinary space-time $\mathcal{M}_{1,3}$, we need to hide the extra six dimensions. The idea is to consider these extra dimensions embedded in a six hermitian manifold Y_6, which it is too small to be detected. Therefore the full space, where the theory lives, has the following product structure

$$\mathcal{M} = \mathcal{M}_{1,3} \times Y_6 \,. \tag{3.19}$$

We can identify Y_6 as the Calabi-Yau manifold in order to preserve some important properties of the ordinary space-time.
We use the indexes $M, N = 0, 1, ..., 9$ to identify the general target space, while the ordinary indices $\mu, \nu = 0, 1, 2, 3$ indicate the ordinary space-time. Finally we use $u, v = 4, 5, .., 9$ to identify the coordinates of the internal space. Since the internal space is small in the sense specified above, we can integrate out its coordinates to obtain an effective four dimensional theory. This procedure is called compactification and it is the bridge between string theory and 4-D physics.
The compactification has some problems, such as the appearance of massless scalars in four dimension, which are called moduli. These scalar fields are related to the size and the shape of the internal space. Since they can couple at least gravitationally to the matter, they can generate extra forces due to the particle exchange, which is experimentally constrained. Furthermore their presence is also problematic during inflation because they can spoil the exponential expansion due to the interactions with the inflaton. For these reasons we need to introduce some mechanisms which generate potentials for these moduli in order to give them mass large enough to explain why we have not observed them yet.
This procedure is called moduli stabilization and can be done introducing some background fluxes. Since the apparition of the moduli is due to the continue deformations of the Y_6 manifold, which bring to the same effective energy, the fluxes allow to introduce an energetic cost associated to a particular deformation. In this way moduli obtain a potential and they can be stabilized.
Now let us make some considerations to the background configurations. In a vacuum

configuration we can write

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + g_{uv} dx^u dx^v \,, \tag{3.20}$$

where g_{uv} is the metric tensor of the internal space. However if some background fields are present and we want to preserve the maximal symmetry in $\mathcal{M}_{1,3}$, we obtain a warped geometry of the form

$$ds^2 = Z(Y_6)^{-1/2} g_{\mu\nu} dx^\mu dx^\nu + Z(Y_6) g_{uv} dx^u dx^v \,, \tag{3.21}$$

where Z is the warped factor, which depends only on the internal coordinates in order to preserve the Poincaré invariance. The warp factor has a non-trivial dependence on the fluxes through equations of motion, derived from the action (3.9).

3.3 D-Branes, DBI and CS actions

We have introduced the concept of p-brane in the first section. If we impose $9-p$ Dirichlet boundary conditions, we talk about a Dp-brane. This means that the coordinates x^λ with $\lambda = p+1,..,9$ are fixed. Therefore a Dp-brane can be considered as a $(p+1)$-dimensional hypersurface to which strings can be attached. We indicate indexes that are not fixed as $a, b = 0, 1, ..., p$.
The dynamics of a D-brane is related to the type of string that ends on it. In the supergravity limit the coupling of a D-brane to a massless bosonic sector is given by the so called Dirac-Infeld-Born DBI action. This action in a ten dimensional Einstein frame takes the form

$$S_{DBI} = -T_p \int d^{p+1}\zeta \sqrt{-\det\left(g_s^{1/2} G_{ab} + \mathcal{F}_{ab}\right)} \,. \tag{3.22}$$

ζ^a are the coordinates of the Dp-brane world-volume, T_p is the tension of a Dp-brane, given by the generalization of equation (3.3)

$$T_p \equiv \frac{1}{(2\pi)^p g_s (\alpha')^{(p+1)/2}} \,. \tag{3.23}$$

g_s is the string coupling constant and $\mathcal{F}_{ab}$ is a gauge-invariant field strength given by

$$\mathcal{F}_{ab} = -B_{ab} + 2\pi\alpha' F_{ab} \,. \tag{3.24}$$

G_{ab} and B_{ab} are given by a changing of map, the pull-back to the world-volume

$$G_{ab} \equiv P[G]_{ab} = \frac{\partial X^M}{\partial \zeta^a} \frac{\partial X^N}{\partial \zeta^b} G_{MN} \,, \tag{3.25}$$

$$B_{ab} \equiv P[B]_{ab} = \frac{\partial X^M}{\partial \zeta^a} \frac{\partial X^N}{\partial \zeta^b} B_{MN} \,. \tag{3.26}$$

The action (3.22) does not consider that a Dp-brane is also coupled with the bosonic sector R-R. This coupling can be described as an electromagnetic interaction through C_{p+1}, that is

$$S_{CS} = \mu_p \int_{\Sigma_{p+1}} C_{p+1} \,, \tag{3.27}$$

in which $\mu_p = g_s T_p$ and Σ_{p+1} is the Dp-brane world-volume. The action (3.27) generalizes the ordinary electromagnetic term $e \int dx_\mu A^\mu$ of the classical electrodynamics.
In general the action (3.27) is more complicated due to the fact that it can contain different potentials C_n. Therefore we need to sum over these contributions, that is

$$S_{CS} = \mu_p \int_{\Sigma_{p+1}} \sum_n C_n \wedge e^{\mathcal{F}} \,. \tag{3.28}$$

Only the $p+1$-forms contribute to the world-volume integral.
In order to find the DBI inflationary model we need to consider only the action in (3.22).

3.4 Hierarchy of mass scales

In this last section we want to recall the regime in which we use the string theory to study inflation. Inflation can be derived from the low-energy limit of the string theory in which we must suppose that the vacuum energy of the inflation is much lower than the string mass scale M_s. In this sense the inflationary model can be viewed as an effective field theory which arises after the string compactification.
In the framework of an effective field theory we must identify the so called light and heavy degrees of freedom. After we have chose a particular mass scale Λ, we say that a light state has a mass $m < \Lambda$ and so it has to be included in the theory, while an heavy state has a mass $M > \Lambda$ and so has to be integrated out. If E is the energy scale of the phenomenon we want to study (in our case is obviously inflation) in the low energy limit we have $E \ll \Lambda < M$, where M is the mass of a heavy degree of freedom.
In this regime we can write the effective Lagrangian L_{eff} as a sum of a renormalizable Lagrangian L_r and a term which contains the heavy states integrated out, that is

$$L_{eff}[\phi] = L_r[\phi] + \sum_n c_n(g) \frac{\mathcal{O}_n[\phi]}{M^{\delta_n - 4}} \,. \tag{3.29}$$

In the equation above ϕ represent the light states, while c_n are the dimensionless coefficients depending on the couplings of the theory g and $\mathcal{O}_n$ are local operators with dimension δ_n.
Therefore in the case that the energy scale E is much lower than the heavy degrees of freedom M or the couplings g of the ultraviolet theory are very little, we have a decoupling between the low energy theory and the UV theory.

Let us consider inflation as the low-energy phenomenon. Its energy scale can be fixed as the value of the Hubble scale at the time τ_*, that is

$$E \equiv H(\tau_*)\,. \tag{3.30}$$

Now since the quantum gravity is non renormalizable we have $M_s < M_P$. From the considerations above in order to study inflation as an effective string theory, we have the following hierarchy of mass scales

$$H(\tau_*) \ll M_s < M_P\,. \tag{3.31}$$

We recall that in the first assumption we have implicitly supposed that the string coupling constant g_s is very small.
In these considerations we need to introduce another important scale, the Kaluza-Klein mass M_{KK}. This mass scale defines the reference energy level of the excitation string modes. In the super-gravity limit the gap between the massless sectors and the first excitation string modes is too high, therefore we can make the following assumption

$$H(\tau_*) \ll M_{KK} < M_s\,. \tag{3.32}$$

Finally in order to avoid undesired interactions between inflaton and moduli, we have to suppose that the mass m_{moduli} of them belongs to the heavy degrees of freedom, that is

$$m_{moduli} \gg H(\tau_*)\,. \tag{3.33}$$

Equations (3.31), (3.32) with equation (3.33) define the hierarchy of mass scales we have to consider in the treatment of inflation as an effective string theory.

Chapter 4

DBI Inflationary Model

In this chapter we apply the results of the previous chapter in order to perform the dimensional reduction of the DBI action for a $D7$-brane.

4.1 Preliminary Considerations

Before performing the dimensional reduction let us make some preliminary considerations.
The space in which we perform the computation is of the type (3.19)

$$\mathcal{M}_{10} = \mathcal{M}_{1,3} \times \mathbf{T}^4 \times \mathbf{T}^2 \,. \tag{4.1}$$

where $\mathcal{M}_{1,3}$ is the ordinary Minkowski space-time, while $\mathbf{T}^4$ and $\mathbf{T}^2$ are respectively a 4-torus and a 2-torus.
We consider an 4D space with the metric tensor $g_{\mu\nu}$ given by (1.8) and a complex space which it is a product of a 4-torus and a 2−torus. To parameterize the two internal manifolds we use the following three complex combinations of x_u with $u = 4, ...9$

$$z_1 = \frac{1}{\sqrt{2}}(x_4 + ix_5)\,, \quad z_2 = \frac{1}{\sqrt{2}}(x_6 + ix_7)\,, \quad z_3 = \frac{1}{\sqrt{2}}(x_8 + ix_9)\,. \tag{4.2}$$

Therefore the 4-torus is described by $\{z_1, \bar{z}_1, z_2, \bar{z}_2\}$ and the 2-torus by $\{z_3, \bar{z}_3\}$. We use the metric G_{MN} of the form (3.21). The warp factor Z depends only on $\{z_3, \bar{z}_3\}$ and it is sourced by a form

$$\tilde{F}_5 = (1 + \star_{10}) dZ^{-1} \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \,, \tag{4.3}$$

a τ axio-dilaton introduced in (3.16) and a self-dual primitive flux G_3 of the form

$$G_3 = S_{\bar{1}\bar{1}}\; d\bar{z}_1 \wedge dz_2 \wedge dz_3 + S_{\bar{2}\bar{2}}\; dz_1 \wedge d\bar{z}_2 \wedge dz_3 + S_{\bar{3}\bar{3}}\; dz_1 \wedge dz_2 \wedge d\bar{z}_3 + G_{\bar{1}\bar{2}\bar{3}}\; d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \,. \tag{4.4}$$

The structure in (4.1) allows us to set to zero the crossed terms of the tensors. Furthermore we suppose that $B_{\mu\nu} = 0$ because we do not observe this field. As regard the strength F_{MN} field we consider only the 4-d components $F_{\mu\nu}$. Finally we suppose $B_{ij} = 0$, where i, j are the coordinates of $2-$torus, while we write the coordinates of $4-$torus as m, n.

4.2 Dimensional reduction of the DBI action

Starting from the considerations above, consider the $D7$-brane action of the form

$$S_{DBI} = -\mu_7 g_s^{-1} \int d^8\zeta \sqrt{-\det(P[E]_{ab} + \sigma F_{ab})}\,, \tag{4.5}$$

where $P[E]_{ab}$ is the tensor

$$E_{MN} = g_s^{1/2} G_{MN} - B_{MN}\,, \tag{4.6}$$

after the pull-back of the type (3.25). This form is a particular case of the action in (3.22).
It is important to recall that the two transverse coordinates are parameterized in the following way

$$x^i(x_\mu) \equiv \sigma y^i(x_\mu) = 2\pi\alpha' y^i(x_\mu)\,. \tag{4.7}$$

Performing the pull-back, using (4.7), we find

$$P[E]_{ab} = E_{ab} + \sigma E_{aj}\partial_b y^j + \sigma E_{ib}\partial_a y^i + \sigma^2 E_{ij}\partial_a y^i \partial_b y^j\,. \tag{4.8}$$

Therefore we have from (4.6) and (4.8)

$$\begin{aligned} P[E]_{ab} + \sigma F_{ab} &= g_s^{1/2} G_{ab} - B_{ab} + \sigma(g_s^{1/2} G_{aj} - B_{aj})\partial_b y^j + \\ &+ \sigma(g_s^{1/2} G_{ib} - B_{ib})\partial_a y^i + \sigma^2(g_s^{1/2} G_{ij} - B_{ij})\partial_a y^i \partial_b y^j + \sigma F_{ab}\,. \end{aligned} \tag{4.9}$$

From the boundary conditions we have imposed to F_{ab}, B_{ab} and (3.21) we find

$$\begin{aligned} P[E]_{ab} + \sigma F_{ab} &= g_s^{1/2}(Z^{-1/2} g_{\mu\nu} + Z^{1/2} g_{mn}) - B_{mn} + \sigma^2(g_s^{1/2} Z^{1/2} g_{ij})\partial_\mu y^i \partial_\nu y^j + \sigma F_{\mu\nu} = \\ &= g_s^{1/2} Z^{-1/2}(g_{\mu\nu} + \sigma^2 Z g_{ij}\partial_\mu y^i \partial_\nu y^j + \sigma g_s^{-1/2} Z^{1/2} F_{\mu\nu}) + g_s^{1/2} Z^{1/2}(g_{mn} + g_s^{-1/2} Z^{-1/2}\mathcal{F}_{mn}) \equiv \\ &\equiv N_{\mu\nu} + M_{mn}\,, \end{aligned} \tag{4.10}$$

where

$$\mathcal{F}_{mn} \equiv \sigma F_{mn} - B_{mn}\,. \tag{4.11}$$

From (4.10) we observe that $P[E]+\sigma F$ can be considered as the following 8×8 matrix

$$P[E]+\sigma F \equiv \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} . \tag{4.12}$$

Recalling the well known result

$$det \begin{pmatrix} N & -A^T \\ A^T & M \end{pmatrix} = det(N) det\left(M + AN^{-1}A^T\right) , \tag{4.13}$$

we find simply

$$\begin{aligned} det(P[E]_{ab}+\sigma F_{ab}) = g_s^4 det\left(g_{\mu\nu} + \sigma^2 Z g_{ij}\partial_\mu y^i \partial_\nu y^j + \sigma g_s^{-1/2} Z^{1/2} F_{\mu\nu}\right) \times \\ \times\, det\left(g_{mn} + g_s^{-1/2} Z^{-1/2} \mathcal{F}_{mn}\right) . \end{aligned} \tag{4.14}$$

Observing the structure of (4.14) we can identify the transverse coordinates y^i with $i = 8, 9$ as the two bosonic fields (one of them will be identified as the inflaton field) through

$$\Phi = \frac{1}{\sqrt{2}}(y^8 + iy^9) , \quad \bar{\Phi} = \frac{1}{\sqrt{2}}(y^8 - iy^9) . \tag{4.15}$$

Since

$$\begin{aligned} g_{ij}\partial_\mu y^i \partial_\nu y^j \equiv \delta_{ij}\partial_\mu y^i \partial_\nu y^j = \partial_\mu y^8 \partial_\nu y^8 + \partial_\mu y^9 \partial_\nu y^9 = 2\partial_\mu \frac{1}{\sqrt{2}}(y^8 + iy^9)\partial_\nu \frac{1}{\sqrt{2}}(y^8 - iy^9) = \\ = 2\partial_\mu \Phi \partial_\nu \bar{\Phi} , \end{aligned} \tag{4.16}$$

substituting (4.16) in (4.14), we find

$$\begin{aligned} det(P[E]_{ab}+\sigma F_{ab}) = g_s^4 det\left(g_{\mu\nu} + 2\sigma^2 Z \partial_\mu \Phi \partial_\nu \bar{\Phi} + \sigma g_s^{-1/2} Z^{1/2} F_{\mu\nu}\right) \times \\ \times\, det\left(g_{mn} + g_s^{-1/2} Z^{-1/2} \mathcal{F}_{mn}\right) . \end{aligned} \tag{4.17}$$

To simplify the computation of (4.17) we set $F_{\mu\nu}$ to zero and we supposed that $Z = 1$, finding

$$det(P[E]_{ab}+\sigma F_{ab}) = g_s^4 det\left(g_{\mu\nu} + 2\sigma^2 \partial_\mu \Phi \partial_\nu \bar{\Phi}\right) det\left(g_{mn} + g_s^{-1/2} \mathcal{F}_{mn}\right) . \tag{4.18}$$

The first determinant depends on

$$\sigma^2 \equiv 4\pi^2 \alpha'^2 = \frac{4\pi^2}{M_s^4} , \tag{4.19}$$

through (3.1).
Since we work in the hierarchy given by (3.31), (3.32), (3.33) we can use the following first order formula for the $\mathcal{M}_{1,3}$ determinant

$$det(1_{4\times 4} + \epsilon M) = 1 + \epsilon Tr M + O(\epsilon^2) . \tag{4.20}$$

Replacing $g_{\mu\nu}$ with $\delta_{\mu\nu}$ in (4.18) we have

$$det_{\mathcal{M}_{1,3}} = \left(1 + 2\sigma^2 \partial_\mu \Phi \partial^\mu \bar{\Phi} + O(\epsilon^2)\right) . \tag{4.21}$$

The second determinant depends on g_s, which has to be small. However we cannot use (4.20) because $Tr\mathcal{F} = 0$, therefore we expand the determinant at the second order in $\epsilon \equiv g_s^{1/2}$, obtaining

$$det(g_{mn})det(1 + g_s^{-1/2}\mathcal{F}) = f(\mathcal{F}) , \tag{4.22}$$

where

$$f(\mathcal{F}) = 1 + g_s^{-1}\mathcal{F}^2 + \frac{1}{4}g_s^{-2}(\mathcal{F} \wedge \mathcal{F})^2 . \tag{4.23}$$

Substituting (4.21) and (4.22) in (4.5) we find

$$S_{DBI} = -\mu_7 g_s \int d^8\zeta \sqrt{f(\mathcal{F})}\sqrt{1 + 2\sigma^2 \partial_\mu \Phi \partial^\mu \bar{\Phi}} , \tag{4.24}$$

from which we can extract the following 4-d DBI action

$$S_{DBI} = -\mu_7 V_4 g_s \int dVol_{\mathcal{M}_{1,3}} \sqrt{f(\mathcal{F})}\sqrt{1 + 2\sigma^2 \partial_\mu \Phi \partial^\mu \bar{\Phi}} . \tag{4.25}$$

In what follows we show that $f(\mathcal{F})$ can be written in terms of the scalar potential $V(\Phi, \bar{\Phi})$.
It is possible to show that

$$f(\Phi, \bar{\Phi}) = 1 + \frac{g_s \sigma^2}{4}(\mathcal{G} + \mathcal{H}) + \frac{1}{4}\left(\frac{g_s \sigma^2}{4}\right)^2 (\mathcal{G} - \mathcal{H})^2 , \tag{4.26}$$

where $\mathcal{G}$, $\mathcal{H}$ are two fluxes. For simplicity we suppose that $\mathcal{G} = \mathcal{H}$ to find

$$V(\Phi, \bar{\Phi}) = V_4 \mu_7 g_s \left(\sqrt{f(\Phi, \bar{\Phi})} - 1\right) . \tag{4.27}$$

The potential term is clearly generated by the fluxes. Therefore the final form of the DBI action is

$$S_{DBI} = -\mu_7 V_4 g_s \int dVol_{\mathcal{M}_{1,3}} \left(1 + \frac{1}{V_4 \mu_7 g_s} V(\Phi, \bar{\Phi})\right) \sqrt{1 + 2\sigma^2 \partial_\mu \Phi \partial^\mu \bar{\Phi}} . \tag{4.28}$$

In order to have a single-field model we can decompose the complex field Φ in two real scalar fields ϕ_1 and ϕ_2, one of them belongs to the heavy degrees of freedom, since it can be ignored in an effective low energy theory. Let ϕ denote the light degree of freedom. We recognize in (4.28), up to a redefinition of parameters, the characteristic terms discussed in section (2.2.1) and (2.2.2).

Through (4.19) $4\sigma^2 = M_s^{-4}$, the fact that $a \sim (V_4\mu_7 g_s)^{-1} \sim M_s^{-4}$ [15] and adding a term of the type M_s^4 we have

$$S_{DBI} = \int dVol_{\mathcal{M}_{1,3}} \left[-M_s^4 \left(1 + \frac{V(\phi)}{M_s^4} \right) \sqrt{1 + \frac{1}{2M_s^4}\partial_\mu\phi\partial^\mu\phi} + M_s^4 \right] . \tag{4.29}$$

Defining

$$X \equiv -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi \tag{4.30}$$

we obtain

$$S_{DBI} = \int dVol_{\mathcal{M}_{1,3}} \left[-M_s^4 \left(1 + \frac{V(\phi)}{M_s^4} \right) \sqrt{1 - \frac{X}{M_s^4}} + M_s^4 \right] . \tag{4.31}$$

In the following we call $1 + \frac{V(\phi)}{M_s^4}$ the potential term, while $\sqrt{1 - \frac{X}{M_s^4}}$ the relativistic term. The addictive constant has been introduced to obtain the canonical two-derivative action (2.12) in the non-relativistic limit.
If we want to maintain the perfect fluid analogy we have to consider the following action

$$S_K = S_{EH} + S_{DBI} , \tag{4.32}$$

where we have added the canonical Einstein-Hilbert term.

4.3 DBI action and cosmological observables

In this section we study in depth the effects of the non-trivial kinetic term contained in the action (4.32). As we have anticipated the particular form of the kinetic term allows a non-trivial dynamics and it produces some remarkable effects to the main cosmological observables. The aim is to compare with the results of [15] and to discern if the relativistic regime can be achieved for monomial potentials. Those potentials were central in the early development of the inflationary paradigm but have recently became disadvantaged by observational constraints on tensor-to-scalar ratio.
Firstly we study the relativistic corrections to the speed of sound c_s, the tensor-to-scalar ratio r and the spectral index n_s for a quadratic potential. Secondly we focus to the effect of the potential term to r and n_s in conjunction with the relativistic kinetic term. Since the scale M_s is considered as a variable parameter, we prefer to call it Λ. Therefore we write (4.32) in the following form

$$S_{DBI} = S_{EH} + \int d^4x\sqrt{-g} \left[-\Lambda^4 \left(1 + \frac{V(\phi)}{\Lambda^4} \right) \sqrt{1 - \frac{X}{\Lambda^4}} + \Lambda^4 \right] . \tag{4.33}$$

In this section we consider only the monomial potentials of the form

$$V(\phi) = v_0\phi^n\,, \tag{4.34}$$

where v_0 is the amplitude of the potential, while $n \in \mathbb{R}^+$.
The first part of results is presented in figures 4.2 and 4.3. In order to obtain them we have wrote a code in Mathematica (see the Appendix) which numerically solves the Cauchy problem composed of the Klein-Gordon equation (2.40) and the two Friedmann equations (2.42), (2.43) using e-folding number N_e defined in (2.11) as the time variable. Then using the numerical solution of the Cauchy problem, we can obtain the two Hubble slow-roll parameters ϵ_H and η_H from (2.44) and (2.45), while the speed of sound c_s and s are obtained from (2.46) and (2.47). Through these we can compute the evolution of r and n_s from respectively (2.65) and (2.67).
Each cosmological observables is evaluated at 60 e-folding before the end of inflation, where we have varied v_0 and Λ in order to ensure the correct normalization for the amplitude of the scalar perturbations (2.71).
Through the inspection of the figure 4.1 we observe that $X/\Lambda^4 \ll V/\Lambda^4$, therefore inflation is characterized by the slow-roll dynamics. Furthermore we have $X/\Lambda^4 \ll 1$, from which we infer that the quadratic potential does not lead to the relativistic regime. From these considerations and the square root of the equation (2.46), we find

$$c_s = \sqrt{1 - \frac{X}{\Lambda^4}} \simeq 1 - \frac{1}{2}\frac{X}{\Lambda^4}\,. \tag{4.35}$$

Since we have $X/\Lambda^4 \ll 1$, then $c_s \simeq 1$. Therefore the relativistic term leads to a small decrease of the speed of sound as we can observe in the figure 4.2.

The considerations above allow to explain the evolution of the inflaton in figure 4.2. In fact we have a slow-roll behaviour for each graphs, since $X/\Lambda^4 \ll V/\Lambda^4$. In particular for $X/\Lambda^4 \sim 0$ we have $V/\Lambda^4 \sim 0$, therefore $\phi_{end} \sim \sqrt{2}$. For higher values of X/Λ^4 we have $V/\Lambda^4 \gg 1$, therefore the kinetic term becomes non-canonical and (2.28) is no more valid.

Now let us consider the figure 4.4. The darker region corresponds to smaller values of v_0/Λ^4, while the lighter region corresponds to larger values of v_0/Λ^4. The first line corresponds to the values of n_s and r at 60 e-folding before the end of inflation, while in the second line we have evaluated the two observables at 50 e-folding before the end of inflation.

The equivalence with the results in paper [15] can be explained from the fact that for each monomial potential we have considered, the quantities $X\Lambda^4/$ and V/Λ^4 follows the same behaviour that we observe in figure 4.1. This means that monomial potentials do not lead to the relativistic regime where $X/\Lambda^4 \sim 1$.
Through these considerations, it is clear that in order to explain the behaviour of n_s and r we can considered a inflationary model in the non-relativistic limit with a non-trivial

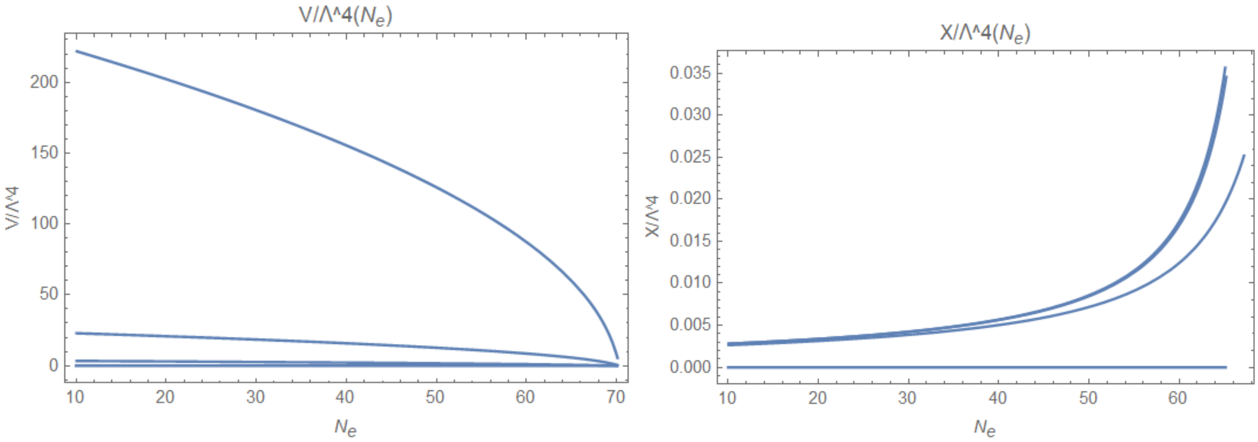

Figure 4.1: Evolution of V/Λ^4 and X/Λ^4 for $X/\Lambda^4 \sim 0$, 2.5×10^{-2}, 3.5×10^{-2}, 3.6×10^{-2} with $V(\phi)\propto\phi^2$.

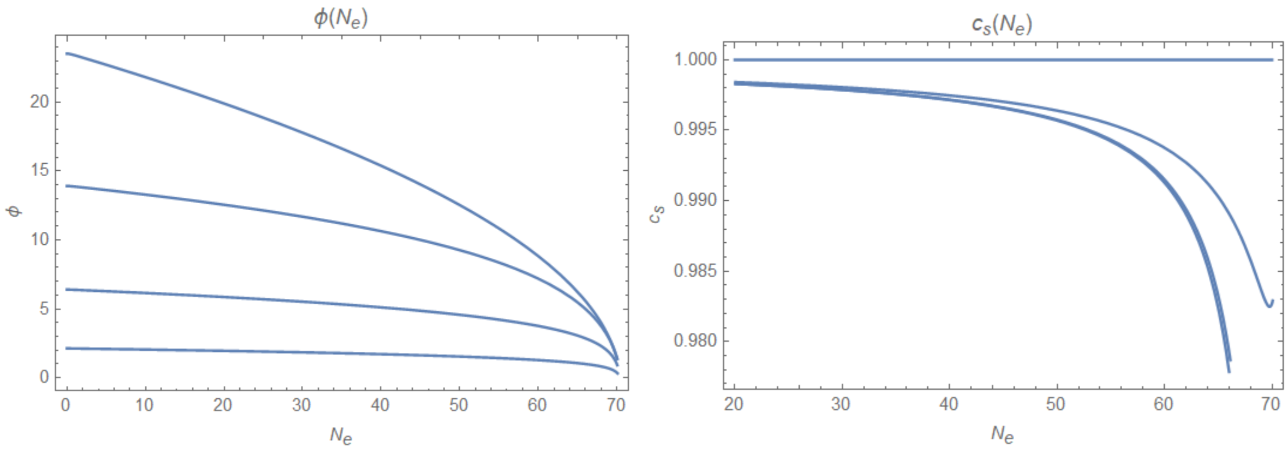

Figure 4.2: Evolution of the inflaton and the speed of sound for $X/\Lambda^4 \sim 0$, 2.5×10^{-2}, 3.5×10^{-2}, 3.6×10^{-2} with $V(\phi)\propto\phi^2$. The values of ϕ_0 have been chosen in order to have $N_e \simeq 70$.

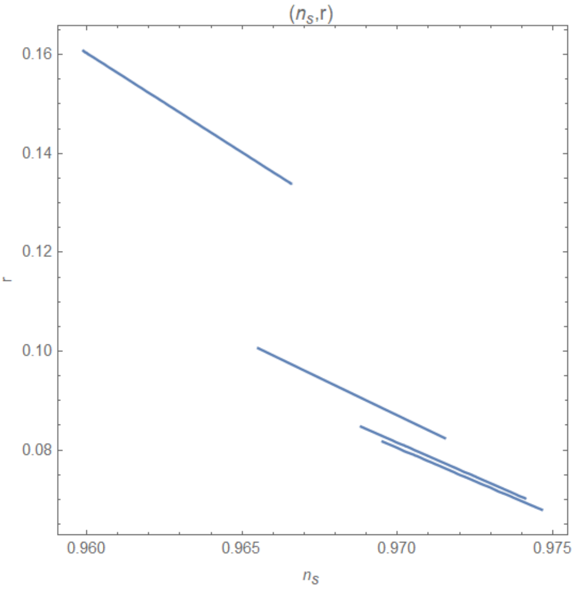

Figure 4.3: Evolution of the tensor-to-scale ratio in $\{n_s, r\}$ plane for $X/\Lambda^4 \sim 0$, 2.5×10^{-2}, 3.5×10^{-2}, 3.6×10^{-2} with $V(\phi) \propto \phi^2$.

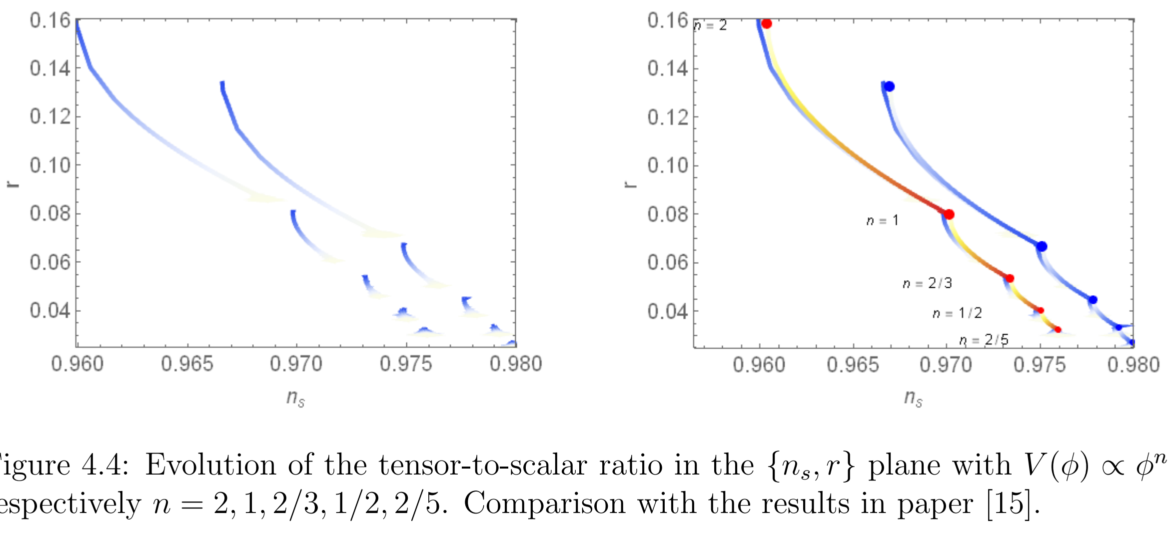

Figure 4.4: Evolution of the tensor-to-scalar ratio in the $\{n_s, r\}$ plane with $V(\phi) \propto \phi^n$, respectively $n = 2, 1, 2/3, 1/2, 2/5$. Comparison with the results in paper [15].

kinetic term. We have studied this type of model in section (2.2.1).
Therefore let us consider the action (2.29), from which we extract the following Lagrangian

$$\mathcal{L} = -\frac{1}{2} f(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi)\,. \tag{4.36}$$

In the present case we have

$$f(\phi) = 1 + \frac{V(\phi)}{\Lambda^4}\,. \tag{4.37}$$

We know that we can always recast the Lagrangian in the canonical form through the redefinition (2.37). In the present case it takes the form

$$\frac{d\psi}{d\phi} = f^{1/2}(\phi) \;\rightarrow\; \psi = g(\phi) = \int d\phi \sqrt{1 + \frac{V(\phi)}{\Lambda^4}}\,. \tag{4.38}$$

Therefore we find

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\psi\partial^\mu\psi - V(g^{-1}(\psi))\,. \tag{4.39}$$

Now the potential depends on the parameter Λ. This fact leads to a flattened potential, in fact using (4.38) we obtain

$$\frac{\partial V}{\partial \psi} = \frac{1}{f^{1/2}}\frac{\partial V}{\partial \phi} \equiv \frac{1}{\sqrt{1 + \frac{V(\phi)}{\Lambda^4}}}\frac{\partial V}{\partial \phi}\,. \tag{4.40}$$

As we have discussed in the section (2.2.1) the redefinition leads to the same slow-roll conditions of the canonical model. Therefore we can introduce the potential slow-roll parameters similar to (2.24) and (2.25)

$$\epsilon_V = \frac{1}{2}\left(\frac{V_{,\psi}}{V}\right)^2 \quad , \quad \eta_V = \frac{V_{,\psi\psi}}{V}\,. \tag{4.41}$$

We can evaluate the effect of the non-canonical term f to the slow-roll parameters above through (4.40). We find

$$\epsilon_V = \frac{1}{2f}\left(\frac{V_{,\phi}}{V}\right)^2 \quad , \quad \eta_V = \frac{1}{f}\frac{V_{,\phi\phi}}{V} - \frac{1}{f}\Lambda^{-4}V\epsilon_V\,. \tag{4.42}$$

The non-canonical term leads to a reduction of the slow-roll parameters since $f > 1$.
Now we can use the relations in (4.42) to evaluate the effect of the non-canonical term $f(\phi)$ on the scalar spectral index of the curvature perturbations n_s and the tensor-to-scalar ratio r. From (2.65) written in terms of ϵ_V and η_V with $s \simeq 0$ we find

$$n_s \simeq 1 - 6\epsilon_V + 2\eta_V = 1 - \frac{3}{f}\left(\frac{V_{,\phi}}{V}\right)^2 + \frac{2}{f}\frac{V_{,\phi\phi}}{V} - \frac{1}{f^2}\Lambda^{-4}V\left(\frac{V_{,\phi}}{V}\right)^2 =$$

$$= \frac{1}{f}\left(1 - 6\epsilon_V|_{\Lambda\sim\infty} + 2\eta_V|_{\Lambda\sim\infty}\right) + \frac{1}{f}\Lambda^{-4}V(1-2\epsilon_V)\,, \tag{4.43}$$

where we have introduced the notation $\cdot|_{\Lambda\sim\infty}$ to indicate the slow-roll parameters of the canonical ϕ model.
Then from (2.67) written in terms of ϵ_V and with $c_s \simeq 1$ we find

$$r \simeq 16\epsilon_V = \frac{8}{f}\left(\frac{V_{,\phi}}{V}\right)^2 \,. \tag{4.44}$$

Considering the monomial potentials in (4.34) with n fixed, we observe from (4.43) and (4.44) that n_s increases when $v_0/\Lambda^4 \gg 1$, similarly r decreases when $v_0/\Lambda^4 \gg 1$. For $n = 2$ this explains the results presented in figure 4.3.
Now let n be arbitrary. Through (4.38) we find

$$\psi = \frac{\phi\left[2\sqrt{1+\Lambda^{-4}v_0\phi^n} + n\,{}_2F_1(\frac{1}{2},\frac{1}{n};1+\frac{1}{n};-\Lambda^{-4}v_0\phi^n)\right]}{n+2}\,, \tag{4.45}$$

where ${}_2F_1(a,b,c,d)$ is the ordinary hyper-geometric function. Plotting (4.45) for different values of n, it is possible to see [15] that we can roughly write $\phi = \psi^{1/m(\psi)}$, where $m(\psi) > 1$. Therefore it is clear that we have a power suppression of the monomial potential when we introduce the canonically normalised field (4.38), since we have $V(\psi) \sim \psi^{n/m(\psi)}$. This power suppression increases when we increase the parameter v_0/Λ^4, therefore each graph tends to connect to each other in the lighter region as we can observe in figure 4.4.
We can prove directly this behaviour, beginning to consider the darker region for a fixed n. In this region $v_0/\Lambda^4 \ll 1$, therefore we can expand to first order (4.42), obtaining

$$\epsilon_V \simeq \frac{1}{2}(1 - v_0\Lambda^{-4}\phi^n)\frac{n^2}{\phi^2}\,, \tag{4.46}$$

$$\eta_V \simeq (1 - v_0\Lambda^{-4}\phi^n)\left[n(n-1)\phi^{-2} - \frac{n^2}{2}\Lambda^{-4}v_0\phi^{n-2}\right]. \tag{4.47}$$

Using the condition (2.7) in terms of the scalar field ϕ, we find

$$\phi_{end} = \frac{n}{\sqrt{2}}\left(1 - \left(\frac{\Lambda^{-4}v_0}{2}\frac{n}{\sqrt{2}}\right)\right). \tag{4.48}$$

Defining N_e as the number of e-foldings between the end of inflation and the epoch t_* when the fluctuations leave the horizon, we find from (2.27)

$$\phi_* = \sqrt{x} - \frac{v_0\Lambda^{-4}}{nx^2}\left(x^{\frac{n+1}{2}} + (n+1)\left(\frac{n}{\sqrt{2}}\right)^n\sqrt{x}\right), \tag{4.49}$$

where $x \equiv 2nM_e + \frac{1}{2}n^2$.
Using this expression in the expanded slow-roll parameters (4.46) and (4.47) we find

$$\epsilon_* = \frac{n^2}{2x} + \Lambda^{-4} v_0 n^2 \left(\frac{n+1}{n+2} \left(\frac{n}{\sqrt{2}} \right)^{n+2} x^{-2} - \frac{n}{2n+4} x^{\frac{1}{2}n-1} \right) , \tag{4.50}$$

$$\eta_* = \frac{n^2 - n}{x} + \Lambda^{-4} v_0 n \left(\frac{2n^2+2}{n+2} \left(\frac{n}{\sqrt{2}} \right)^{n+2} x^{-2} - \frac{3n^2}{2n+4} x^{\frac{1}{2}n-1} \right) . \tag{4.51}$$

It is clear that since $x \sim \mathcal{O}(100)$, the term $\propto x^{\frac{1}{2}n-1}$ dominates the bracket, therefore both of quantities tend to decrease when $v_0\Lambda^{-4}$ increases. For this reason the tensor-to-scalar ratio tends to decrease when $v_0\Lambda^{-4}$ increases, while the spectral index n_s tends to increase when $v_0\Lambda^{-4}$ increases. This is also true if we fix v_0/Λ^4 and we vary n.
Now if we concentrate on the lighter region, where $v_0/\Lambda^4 \gg 1$, inverting (4.45) with $f \simeq v_0/\Lambda^4$ and substituting to the monomial potential in (4.34), we find

$$V(\psi) = v_0 \left(\frac{n+2}{2\sqrt{\Lambda^{-4} v_0}} \right)^{\frac{2n}{n+2}} \psi^{\frac{2n}{n+2}} . \tag{4.52}$$

Therefore the trend of the quadratic potential $n = 2$ in the lighter region tends to connect to the trend of the linear potential $n = 1$. Similarly the trend of the linear potential tends to connect to that of the monomial potential with $n = 2/3$ and so on, proving the behaviour in figure 4.4.

We can conclude the section observing that the monomial potentials do not lead to the relativistic regime. Therefore we have a small decrease in the speed of sound c_s due to the relativistic term, while the behaviours of the spectral index and the tensor-to-scalar ratio are almost influenced by the potential term. This analysis confirm the results in [15] and extended to the higher derivative corrections the results discussed in [16].

Chapter 5

Swampland Conjecture

In this chapter we discuss two string Swampland criteria and their implication for inflation.
These criteria are thought to be useful in distinguishing effective field theories (EFTS) that admit a UV completion (i.e. can be derived from string theory) from those which do not. Two criteria have been proposed in [17], these provide an upper bound for the range traversed by the scalar field $\Delta\phi$, while a lower bound for the flatness of the potential $|V_{,\phi}|/V$ when $V > 0$. These criteria are in tension with the inflationary paradigm, which in its simplest version needs flat potentials and often requires transplanckian field excursions. In [17] it has been argued that not even inflation models based on the (intrinsically string) DBI action can lift inflation out of the Swampland as they would give rise to large non-Gaussianity. In what follows we will investigate this claim. Let us add that this topic is the subject of current research. In this chapter we will only concern ourselves with the original formulation of the criteria [17].

5.1 Swampland Criteria

CRITERION 1: the range traversed by the scalar fields $\Delta\phi$ is bounded by $\Delta \sim \mathcal{O}(1)$ in reduced mass Planck.
In particular let us consider an effective Lagrangian coupled to gravity through the ordinary Einstein-Hilbert term, that is

$$\mathcal{L} = \sqrt{-g}\left[\frac{1}{2}\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j G_{ij}(\phi) - V(\phi) + ...\right], \tag{5.1}$$

where G_{ij} is the field space metric. The swampland approach argues that there exists a finite radius $\alpha \sim \mathcal{O}(1)$ in the field space, where the effective theory in (5.1) is valid. In fact at large distance $D \gg 1$ a tower of light modes may appear

$$m \sim M_p \exp(-\alpha D), \tag{5.2}$$

invalidating the above Lagrangian as a healthy EFT. Therefore the traversed range of the field has to be bounded by $\Delta \sim \mathcal{O}(1)$, that is $\Delta\phi < \Delta \sim \mathcal{O}(1)$.

CRITERION 2: there is a lower bound on the "slope" of the potential when $V > 0$, that is

$$\frac{|\nabla_\phi V|}{V} > c \sim \mathcal{O}(1)\,. \tag{5.3}$$

This criterion is motivated by the fact that it is difficult to build a de-Sitter vacua in string theory.
It is clear that since we do not know the values of $\{c, \Delta\}$, the inflationary models are in tension with the two Swampland criteria depending on how close they are to 1. One can derive two model-independent constraints, relying only on the experimental constraints (see (2.70) and (2.71)). These turn out to require $c < 0.6$ and $c < 3.5\Delta$ [17].
In the following discussion we concentrate only on the consequences of the two criteria for the inflationary period. Let us focus on the single-field slow-roll inflation models. Criterion 1 is in tension with large field inflation models, like the chaotic monomials studied in the previous chapter. In this framework we can introduce the following potential slow-roll parameter

$$\epsilon_H \simeq \epsilon_V = \frac{1}{2}\left(\frac{|\nabla_\phi V|}{V}\right)^2\,. \tag{5.4}$$

We can estimate the relation between ϵ_V and N_e through

$$\epsilon_V \sim \frac{1}{N_e^k}\,, \tag{5.5}$$

where k is of the order of unity for the monomial potentials. From an approximation of (2.27) we have

$$\Delta\phi \sim \sqrt{2\epsilon_V}N_e \sim \sqrt{2}N_e^{1-\frac{k}{2}}\,. \tag{5.6}$$

Since we need at least $N_e = 60$ e-folding in the slow-roll inflationary models, from (5.6) we obtain that $\Delta\phi$ is greater or equal than $\mathcal{O}(1)$. The tension can grow in the case of larger values of N_e. From the fact that in the slow-roll limit (5.4) is much less than one, this is evidently in tension with the second Swampland criterion. In practice every inflationary model that relies on 2 derivative slow-roll is in tension with the second Swampland criterion. As we have anticipated above if we want to check the Swampland criteria in the relativistic regime of the DBI model, we have to consider the constraints on the non-Gaussianity, which can be recast as a lower bound of the speed of sound (see (2.70)).

5.2 DBI inflation in the relativistic regime

In this section we want to study the behaviour of the speed of sound c_s in the relativistic regime, i.e. when $X/\Lambda^4 \sim 1$. As we have observed in section (4.3) the relativistic term leads to a decrease of the speed of sound. However as we have shown we were not in the relativistic regime. This is due to the fact we have considered only the monomial potentials (4.34) which are too flat to induce large velocities.
In this section we use the following inflationary model

$$S_K = S_{EH} + \int d^4x\sqrt{-g}\, p(X,\phi)\,, \tag{5.7}$$

where $p(X,\phi)$ is given by

$$p(X,\phi) = -\Lambda^4\sqrt{1-2\frac{X}{\Lambda^4}} - V(\phi) + \Lambda^4\,. \tag{5.8}$$

This Lagrangian arises from the action of a $D3$-brane. The absence of kinetic-potential mixing term arises from the fact that we are assuming V is not generated by internal fluxes as in the previous chapter but instead by placing the brane in a warped region [18].
Firstly we want to show that the achievement of the relativistic regime $X/\Lambda^4 \sim \mathcal{O}(1)$ for the inflationary model (5.7) fixes a constraint on the steepness of the potential V. Therefore let us consider the Klein-Gordon equation (2.40) in the hypothesis $X/\Lambda^4 \ll V/\Lambda^4$ and $\ddot{\phi} \sim 0$ (we justify in the aftermath these assumptions). We find

$$3H\dot{\phi}p_{,X} - p_{,\phi} \simeq 0 \implies \dot{\phi} \simeq \frac{p_{,\phi}}{3H\, p_{,X}}\,. \tag{5.9}$$

From (5.8) we have

$$p_{,X} = \frac{1}{\sqrt{1-\frac{2X}{\Lambda^4}}}\,, \tag{5.10}$$

$$p_{\phi} = -V_{,\phi}\,. \tag{5.11}$$

Taking the absolute value of (5.9) with $\dot{\phi} = \pm\sqrt{2X}$ we can write

$$\sqrt{\frac{2X}{1-\frac{2X}{\Lambda^4}}} \simeq \frac{V_{,\phi}}{3H}\,. \tag{5.12}$$

Assuming that $H \sim \sqrt{V/3}$, then we find

$$\sqrt{\frac{2X}{1-\frac{2X}{\Lambda^4}}} \simeq \frac{V_{,\phi}}{\sqrt{V}}\,. \tag{5.13}$$

It is clear that for $X/\Lambda^4 \sim \mathcal{O}(1)$ the denominator of (5.13) tends to zero, and we find

$$\frac{V_{,\phi}}{\sqrt{V}} \gg 1\,. \tag{5.14}$$

The condition (5.14) is satisfied by the steep potentials. In this section we have chosen a family of exponential potentials of the type

$$V(\phi) = V_0 \exp(k\phi)\,, \tag{5.15}$$

where V_0 is the amplitude and k is another parameter, which it must satisfy

$$k > \mathcal{O}(1) \tag{5.16}$$

in order to verify the criterion (5.3).
The results of this section are shown in figure 5.1. In the first figure we show the behaviour of the speed of sound c_s evaluated at $N_e \simeq 60$ before the end of inflation respect to the parameter k. The values of k are chosen in order to satisfy the bound (5.16).
The second plot displays the behaviour of the same values of the speed of sound c_s respect to $\Delta\phi$, which it is defined by the following

$$\Delta\phi \equiv \phi(N_e - 60) - \phi(N_e)\,, \tag{5.17}$$

where N_e denotes the duration of inflation. Each point corresponds to each value of k in the first figure.
The values of c_s have been obtained through the variation of the parameters Λ and V_0 in order to have the correct amplitude to the power spectrum of scalar perturbations (2.71).

Through the inspection of the figure 5.2, which shows the behaviour of the quantities X/Λ^4 and V/Λ^4, we can conclude that the assumptions we have made previously hold true. Since $X/\Lambda^4 \sim \mathcal{O}(1)$, we are in the relativistic regime. This explains why $c_s \ll 1$ for a value of k. These considerations can be made for every value of k which satisfies the bound (5.16).

In order to give a quantitative explanation of the trends we observe in figure 5.1, let us consider the equation (2.11) in the differential form, written in term of the inflaton field

$$dN_e = \frac{H}{\dot{\phi}}\,d\phi\,. \tag{5.18}$$

Substituting (5.9) in (5.18) and using (2.42) we find

$$dN_e \simeq 3H^2\,\frac{p_{,X}}{p_{,\phi}}\,d\phi = 2X\,\frac{p_{,X}^2}{p_{,\phi}}d\phi - p\,\frac{p_{,X}}{p_{,\phi}}\,d\phi\,. \tag{5.19}$$

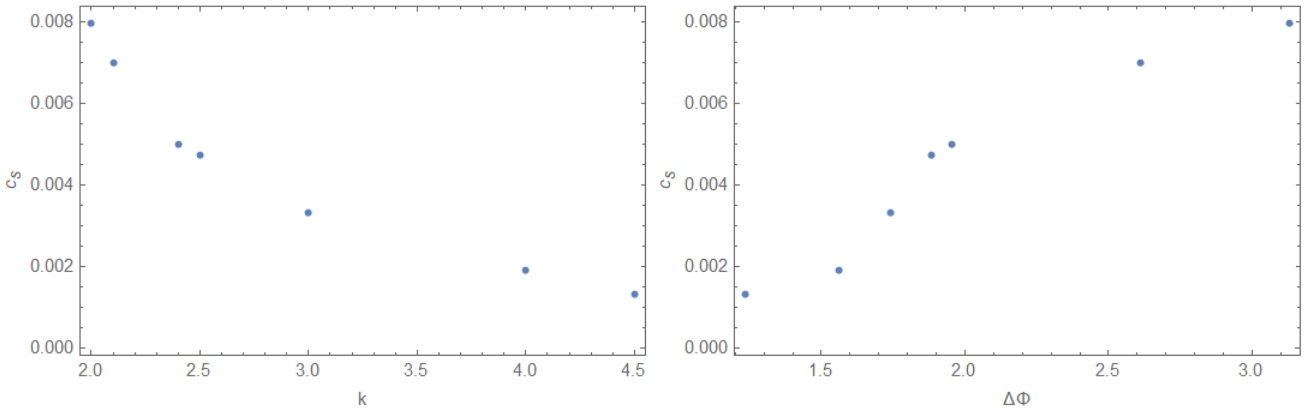

Figure 5.1: Behaviour of the speed of sound respect to k and $\Delta\phi$ at $N_e \simeq 60$ before the end of inflation.

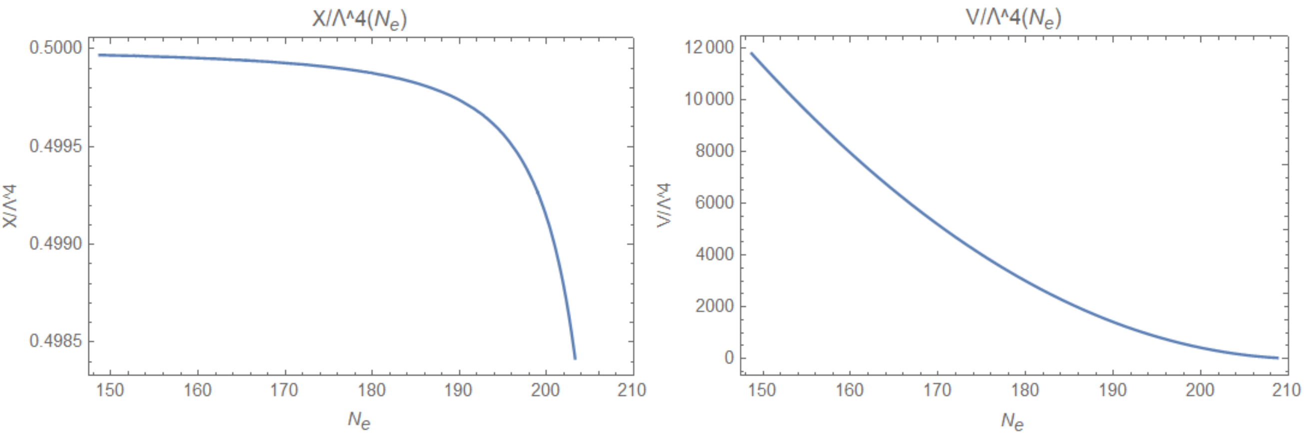

Figure 5.2: Behavior of X/Λ^4 and V/Λ^4 during the last 60 e-foldings for $k = 2$.

Through (5.10) and (5.11), the first term in (5.19) becomes

$$2X\,\frac{p_{,X}^2}{p_{,\phi}}\,d\phi = -\frac{1}{k}\frac{X}{V}\frac{1}{1-\frac{2X}{\Lambda^4}}\,d\phi \ll 1\,, \tag{5.20}$$

from the fact that $X \ll V$ and the relativistic term is almost constant. The second term is naturally decomposed in the following three parts

$$p\,\frac{p_{,X}}{p_{,\phi}}d\phi = \frac{1}{k}\frac{\Lambda^4}{V}\,d\phi + \frac{1}{k}\frac{1}{\sqrt{1-\frac{2X}{\Lambda^4}}}\,d\phi - \frac{1}{k}\frac{\Lambda^4}{V}\frac{1}{\sqrt{1-\frac{2X}{\Lambda^4}}}\,d\phi\,. \tag{5.21}$$

The first and the third term can be neglected because $V/\Lambda^4 \gg 1$, a condition required by the normalisation of the curvature perturbations. Therefore (5.19) becomes

$$dN_e \simeq -\frac{1}{k}\frac{1}{\sqrt{1-\frac{2X}{\Lambda^4}}}\,d\phi\,. \tag{5.22}$$

Now remembering the definition of ϵ_V (equation (2.24)) we have

$$-\,\frac{1}{k} = -\frac{V}{V_{,\phi}} \equiv \frac{1}{\sqrt{2\epsilon_V}}\,. \tag{5.23}$$

Furthermore taking the square root of (2.46), using (5.10) and its first derivative we find

$$c_s = \sqrt{\frac{p_{,X}}{p_{,X}+2Xp_{,XX}}} = \sqrt{1-\frac{2X}{\Lambda^4}}\,. \tag{5.24}$$

Substituting (5.23) in (5.22) and using (5.10) or (5.24) we obtain

$$dN_e \simeq \frac{1}{\sqrt{2\epsilon_V}}\,p_{,X}\,d\phi \equiv \frac{1}{\sqrt{2\epsilon_V}}\,\frac{1}{c_s}\,d\phi\,. \tag{5.25}$$

The result above coincides with the ordinary two derivative case when $p(X,\phi) \equiv X - V$ or $c_s = 1$. We can use (5.25) to give an approximate explanation of the behaviours in figure 5.1.
Let us integrate (5.22) in the following way

$$N_e \sim -\frac{1}{k}\int_{\phi_*}^{\phi_{end}} \frac{1}{\sqrt{1-\frac{2X}{\Lambda^4}}}\,d\phi\,, \tag{5.26}$$

where ϕ_* is the value of the inflaton field at 60 e-foldings before the end of inflation, while ϕ_{end} is the value of the inflaton at the end of inflation. This means that $N_e \simeq 60$.

Now since $1/\sqrt{1-\frac{2X}{\Lambda^4}}$ varies slowly in the region of integration, we can approximate the integral in this way

$$N_e \sim -\frac{1}{k}\frac{1}{\sqrt{1-\frac{2X_*}{\Lambda^4}}}\left(\phi_{end}-\phi_*\right) = \frac{\Delta\phi}{k}\frac{1}{\sqrt{1-\frac{2X_*}{\Lambda^4}}}\,, \tag{5.27}$$

where we have used (5.17) and X_* is the value of the quantity X at 60 e-foldings before the end of inflation. Through (5.24) evaluated at $*$ epoch and (5.27) we find

$$c_s^* \sim \frac{\Delta\phi}{kN_e}\,, \tag{5.28}$$

which it is in agreement with figure 5.1.
After we have computed the behaviour of the speed of sound in (5.28) we can obtain an order of magnitude estimate of the NG the amplitude through

$$f_{NL} \sim \frac{1}{c_{s,*}^2}\,. \tag{5.29}$$

Observations fix the following constraint on the amplitude of NG [6]

$$|f_{NL}| < \mathcal{O}(10)\,. \tag{5.30}$$

We have seen in the section (4.3) that monomial potentials do not lead to the relativistic regime. Therefore in this section we have found that in order to reach the relativistic regime we need to consider steep potentials. We have chosen for the analysis a family of exponential potentials, which satisfy the second Swampland criterion. These potentials lead to the regime in which $c_s \ll 1$, but they do not satisfy the constraints on the speed of sound (2.70) and on the amplitude of NG (5.30). The same applies to the cases when the first Swampland criterion is satisfied. Therefore it remains a challenge to find an inflationary model which satisfy both of the Swampland criteria conjectured without violating the constraints on the main cosmological observables and on the amplitude of NG.

Conclusions

This chapter is dedicated to a final summary of main results we have discussed in chapters (4) and (5).
We have observed throughout the thesis that the most promising idea to obtain a reasonable inflationary model is to consider it as an effective field theory of a more complicated model of string theory. This can be done only if the characteristic energy of the phenomenon is much smaller than the cut-off energy scale of the general theory. For the inflationary mechanism this is true in the super-gravity limit.
Following this idea in the chapter (4) we have derived the DBI inflationary model through the dimensional reduction of the Dirac-Born-Infeld action that describes the position and orientation of a $D7$-brane.
The DBI Lagrangian can be used in the formalism of k-inflation which we have developed in the section (2.2.2).
Applying the results of [14] we have studied the effects of the non-trivial kinetic term in the context of the monomial potentials.
As we have seen the shape of the potential has an important role in the reaching of the relativistic regime. We have observed that for the monomial potentials this regime cannot be reached, therefore we observe only a small decrease in the speed of sound, which goes like $\sim 1 - X/2\Lambda^4$.
The analysis of the non relativistic limit of the DBI model have led to a confirm of the work in [15], while the analysis to the effect to the speed of sound have extended to the higher derivative corrections the results in [16].
In the section (5.2) we have seen that the achievement of the relativistic regime can be done through the introduction of the so called steep potentials, with large $|V_{,\phi}|/V$.
This constraint in the shape of the potential agrees with the Swampland criterion on the potential enunciated in the work [17], which is related to the steepness of the potential.
We have therefore considered a family of exponential potentials dependent on a parameter k, which is taken to be large enough and in accordance with the Swampland criteria.
In the relativistic regime the speed of sound becomes $c_s \ll 1$ as expected, but it seems to be difficult to maintain its value above the constraints from non-Gaussianity while keeping $\Delta\phi < M_p$. Given the success of the inflationary paradigm in explaining observed features of our Universe and its unavoidable tension with the conjectured Swampland

Criteria, it is probably worth re-evaluating the latter rather then questioning the former.

Appendix

We include in the appendix the core fragment of the code, which we have used to obtain the numerical results discussed in the sections (4.3) and (5.2).
Let us describe briefly each line. The first line contains the Lagrangian, which we use as a pressure of the model. After that we have the algebraic equation derived from the Friedmann equation (2.42), which it gives us the initial value of the Hubble parameter H_0. The third and fourth line contain the Friedmann equation (2.43) and the Klein-Gordon equation (2.40). Each of these equation are written in term of N_e as time parameter. The fifth line contains the Cauchy problem, through which we can obtain the evolution of the inflaton ϕ and the Hubble parameter H. The last line contains the value of the duration of inflation in terms of e-folding.

```
p = -Λ4 * Sqrt[1 - 2 * X / (Λ4)] - Vinfl + Λ4;
eqH0 = NSolve[H0 == Sqrt[1 / 3 * (2 * X * D[p, X] - p)] /. {X → H0^2 * dϕ0^2 / 2, ϕ[x] → ϕ0}, H0];
eqH = {H'[x] == -(X * D[p, X]) / H[x]} /. X → H[x]^2 * ϕ'[x]^2 / 2;
eqϕ =
  {(H[x]^2 * D[p, X] + H[x]^4 * ϕ'[x]^2 * D[p, {X, 2}]) * ϕ''[x] + H[x]^3 * H'[x] * D[p, {X, 2}] * ϕ'[x]^3 +
      H[x]^2 * D[p, X, ϕ[x]] * ϕ'[x]^2 + (H[x] * H'[x] + 3 H[x]^2) * D[p, X] * ϕ'[x] - D[p, ϕ[x]] == 0} /.
   X → H[x]^2 * ϕ'[x]^2 / 2;
sol = NDSolve[{eqH, eqϕ, ϕ[0] == ϕ0, ϕ'[0] == dϕ0, H[0] == H0 /. eqH0}, {ϕ, H}, {x, 0, 10000000},
   Method → {"EventLocator", "Event" → Abs[-H'[x] / H[x]] - 1}, MaxSteps → Infinity,
   WorkingPrecision → MachinePrecision, PrecisionGoal → MachinePrecision,
   Method → {"StifnessSwitching", Method → {"ExplicitRungeKutta", Automatic}}];
Dom = ((H /. sol)[[1]]@Domain[])[[1, 2]];
```

Figure 5.3: Fragment of code written in Mathematica.

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