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Stochastic fundamental solutions for a class of degenerate SPDEs

Tesi di Laurea in Equazioni Differenziali Stocastiche

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Alla mia famiglia

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Introduction and motivations

This thesis focuses on a broad, possibly degenerate class of *stochastic partial differential equations* (SPDEs). Such equations, combining elements of PDEs and Itô's stochastic calculus, are well suitable to describe systems with distributed parameters in the presence of random perturbations. In probability theory, interest in equations of this type arose by the problem of filtering diffusion processes.

The filtering problem

Suppose that $T \geq 0$ is given and let us fix a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$. Let W_t be a standard $\mathbb{R}^{d+d'}$-valued Wiener process and consider a diffusion process Z_t, which is a solution of a system of Itô equations

$$dZ_t = b(t, Z_t)dt + \sigma(t, Z_t)dW_t, \quad t \in [0, T]$$

Suppose that one part of the components of the diffusion Z_t is observable, call it Y_t, and the other is not, call it X_t. Assume (without loss of generality) that X_t consists of the first d coordinates of Z_t and d' of the remaining d coordinates.

Let $\mathcal{F}_t^Y$ be the completion of $\sigma(Y_s,\ s \leq t)$, which defines the filtration of the observations on Y and let $T_1, T_2 \in [0, T]$. Then, for any bounded and measurable real valued function f on $\mathbb{R}^d$, it is well known that $E\left[f(X_{T_2})|\mathcal{F}_{T_1}^Y\right]$ is the best, in the mean square, $\mathcal{F}_{T_1}^Y$-measurable estimate for $f(X_{T_2})$, that is, intuitively, the best estimate for $f(X_{T_2})$ given the information extracted from Y up until the time T_1.

The problem of calculating $E\left[f(X_{T_2})|\mathcal{F}_{T_1}^Y\right]$ is called the problem of *filtering* if $T_1 = T_2$, the problem of *interpolation* if $T_2 < T_1$ and that of *extrapolation* if $T_2 > T_1$.

Under natural assumptions one can find that

$$E\left[f(X_t) \mid \mathcal{F}_t^Y\right] = \int_{\mathbb{R}^d} f(x)p_t(x)dx$$

namely, there exists the conditional density of X_t given $\mathcal{F}_t^Y$ and it turns out that p_t satisfies a SPDE of the form

$$dp_t(x) = L_t p_t(x)dt + M_t p_t(x)dW_t, \tag{0.0.1}$$

where L_t is a second order elliptic operator and M_t is a first order operator. The coefficients of L_t and M_t depend on t, x and Y_t: therefore *they are random and typically not smooth w.r.t. t.*

A very particular case is when $Y \equiv 0$: in this case $M_t \equiv 0$ and (0.0.1) reduces to the classical *backward Kolmogorov equation* for the *deterministic* transition density p_t of X_t.
In the general case of (0.0.1), p_t is the *stochastic* fundamental solution of (0.0.1) and is the conditional transition density of the process X_t.

It was recently discovered that one can derive filtering equations by means of a 'direct' PDE approach (see [24], or [14] where distribution valued processes are considered). To give an elementary idea of this approach, consider the Itô's stochastic equation

$$dx_t = b(\sigma_t)dt + \sigma(x_t)dW_t,$$

say in one dimension with nonrandom coefficients satisfying appropriate conditions. Let $x_t(x)$ be a solution of this equation starting at x. Take a smooth and bounded function $\varphi(x)$ and define $v(t,x) = E[\varphi(x_t(x))]$. Then under appropriate conditions v satisfies Kolmogorov's equation

$$\begin{cases} \partial_t u = \frac{1}{2}\sigma^2 u_{xx} + b u_x, & t \geq 0,\ x \in \mathbb{R}, \\ u(0,x) = \varphi(x). \end{cases} \tag{0.0.2}$$

One can derive this fact in two ways. Historically, the first one is based on proving probabilistically that v is smooth enough and then using the Markov property (see, for instance, [9]). This way Kolmogorov's equation is derived even if the diffusion can degenerate. However, if the process is nondegenerate, one can use a 'direct' approach consisting of taking the solution u of (0.0.2), the existence of which is garanteed by the theory of PDEs, and applying Itô's formula to $u(t-T, x_t(x))$. Then one gets

$$u(0, x_T(x)) = u(T, x_0(x)) + \int_0^T \sigma(x_s(x)) u_x(x_s(x)) dW_s. \tag{0.0.3}$$

By taking the expectations on (0.0.3), and noting that $u(0, x_T) = \varphi(x_T)$ and $x_0(x) = x$, it is easily proved that $u = v$ and hence v satisfies (0.0.2).

The general case can be treated following the same approach. Take the filtering equation, take its solutions, whose existence needs to be previoulsy investigated, and then apply Itô's formula to appropriately chooses functions.

This way of arguing strongly motivates the study of equations of the type (0.0.1).

The aim of this thesis is to prove existence, regularity and estimates of a solution p_t to (0.0.1) when L_t is a Kolmogorov type operator satisfying the *weak Hörmander condition*.

Our approach is based on the parametrix method for which the natural functional setting is that of stochastic Hölder spaces. We recall that [4] considers classical solutions in Hölder spaces to uniformly parabolic SPDEs with coefficients that are functions of t only, independent of x. For other old references on SPDEs in Hilbert spaces (i.e. with infinite dimensional noise), see [4]: different notions of weak solution (integrating in (t,x) against a test function), strong solution (integral in t and pointwise a.e. in x) and classical solution (integral in t and pointwise in x).

The parametrix method has been recently revisited in [5] and [1] under the perspective of probabilistic and financial applications.

The following example well describes the basic ideas behind the arguments brought up in the thesis.

Constant coefficient SPDEs

Let $\hat{u} = \hat{u}_t(x)$ be a solution of the heat equation

$$d\hat{u}_t(x) = \frac{a^2}{2}\partial^2_{xx}\hat{u}_t(x)dt, \qquad t > \tau,\ x \in \mathbb{R}. \tag{0.0.4}$$

By the Îto formula we have that

$$u_t(x) := \hat{u}_t\left(x + \alpha\left(W_t - W_\tau\right)\right), \qquad t > \tau,$$

solves the one-dimensional SPDE

$$du_t(x) = \frac{\sigma^2}{2}\partial_{xx}u_t(x)dt + \alpha\partial_x u_t(x)dW_t, \qquad \sigma^2 := a^2 + \alpha^2. \tag{0.0.5}$$

The other way round, starting from a solution $u_t(x)$ of the SPDE (0.0.5), the Itô-Wentzell change of variable $X_t(x) = x - \alpha W_t$ transforms $u_t(x)$ into a solution of (0.0.4).

Now, let Γ denote the Gaussian fundamental solution of (0.0.4)

$$\Gamma\left(t, x; \tau, \xi\right) = \frac{1}{\sqrt{2\pi a^2(t-\tau)}}\exp\left(-\frac{(x-\xi)^2}{2a^2(t-\tau)}\right), \qquad t > \tau,\ x, \xi \in \mathbb{R}. \tag{0.0.6}$$

Then

$$\begin{aligned} p\left(t, x; \tau, \xi\right) &:= \Gamma\left(t, x + \alpha(W_t - W_\tau); \tau, \xi\right) \\ &= \frac{1}{\sqrt{2\pi a^2(t-\tau)}}\exp\left(-\frac{(x + \alpha(W_t - W_\tau) - \xi)^2}{2a^2(t-\tau)}\right), \qquad t > \tau \geq 0,\ x, \xi \in \mathbb{R}, \end{aligned} \tag{0.0.7}$$

is the stochastic fundamental solution of (0.0.5): more precisely, for any $\varphi \in C_b(\mathbb{R})$, we have that

$$u_t(x) := \int_{\mathbb{R}} p\left(t, x; \tau, \xi\right)\varphi(\xi)d\xi,$$

solves the stochastic Cauchy problem

$$\begin{cases} du_t(x) = \frac{\sigma^2}{2}\partial_{xx}u_t(x)dt + \alpha\partial_x u_t(x)dW_t, & t > \tau,\ x \in \mathbb{R}, \\ u_\tau(x) = \varphi(x), & x \in \mathbb{R}. \end{cases}$$

Indeed, by the stochastic Fubini's theorem we have

$$du_t(x) = \int_{\mathbb{R}} dp\left(t, x; \tau, \xi\right)\varphi(\xi)d\xi$$

(by the Îto formula)

$$= \int_{\mathbb{R}}\left(\frac{\sigma^2}{2}\partial_{xx}p\left(t, x; \tau, \xi\right)\varphi(\xi)dt + \alpha\partial_x p\left(t, x; \tau, \xi\right)\varphi(\xi)dW_t\right)d\xi$$

(again by the standard and stochastic Fubini's theorems)

$$= \frac{\sigma^2}{2}\partial_{xx}u_t(x)dt + \alpha\partial_x u_t(x)dW_t.$$

Moreover, we have

$$|u_t(x) - \varphi(x_0)| \leq \int_{\mathbb{R}} p\,(t, x; \tau, \xi)\,|\varphi(\xi) - \varphi(x_0)|\,d\xi$$

(by the change of variable $\eta = \frac{\xi - x - \alpha(W_t - W_\tau)}{\sqrt{2\pi a^2(t-\tau)}}$)

$$= \int_{\mathbb{R}} e^{-\eta^2}\left|\varphi\left(\eta\sqrt{2\pi a^2(t-\tau)} + x + \alpha(W_t - W_\tau)\right) - \varphi(x_0)\right| d\xi$$

which converges to zero as $(t, x) \to (\tau, x_0)$ by the dominated convergence theorem, because the integrand converges pointwisely and is dominated by the integrable function $2\|\varphi\|_\infty e^{-\eta^2}$. This means that $u_t(x)$ is a continuous function up to $t = \tau$.

Remark 0.0.1. *The stochastic fundamental solution* p *in* (0.0.7) *has distinctive properties compared to the Gaussian deterministic fundamental solution* (0.0.6)*. In particular, the asymptotic behaviour near the pole of* p *is affected by the presence of the Brownian motion: this was studied in [21] in the more general framework of Riemannian manifolds.*

In this dissertation we are only going to consider the forward problem. This is due to some adaptability problems that arise when trying to solve stochastic backward equations. To better exemplify what we have stated here we briefly consider a possible backward problem based on the framework of the previous example.

Backward or forward?

As in the case of *ordinary* differential equations (ODEs and SDEs), moving forward or backward in time makes the difference. The forward SPDE (with constant coefficients) (0.0.5) has fundamental solution

$$p\,(t, x; \tau, \xi) = \Gamma\,(t, x + \alpha(W_t - W_\tau); \tau, \xi)\,, \qquad t > \tau.$$

Notice that $t \mapsto p\,(t, x; \tau, \xi)$ is an adapted process. Notice also the damping effect of the stochastic component on the diffusion coefficient: σ^2 in the SPDE (0.0.5) corresponds to $\sigma^2 - \alpha^2$ in the related PDE; this causes some concern about the ellipticity condition and forces to impose assumptions like

$$\sigma^2 - \alpha^2 > 0.$$

Analogously, the *backward SPDE*

$$du_t(x) = -\frac{\sigma^2}{2}\partial_{xx}u_t(x)dt + \alpha\partial_x u_t(x)dW_t, \qquad t < T, \tag{0.0.8}$$

can be converted in the *backward PDE*

$$du_t(x) = -\frac{\sigma^2 + \alpha^2}{2} \partial_{xx} u_t(x) dt, \qquad t < T.$$

By analogy, the fundamental solution of (0.0.8) should be

$$p_B(t, x; T, y) = \frac{1}{\sqrt{2\pi(\sigma^2 + \alpha^2)(T - t)}} \exp\left(-\frac{(x + \alpha(W_t - W_T) - y)^2}{2(\sigma^2 + \alpha^2)(T - t)}\right), \qquad t < T.$$

However, $t \mapsto p_B(t, x; T, y)$ is NOT an adapted process. Thus it seems that an ad-hoc notion of solution and, more generally, a theory for backward SPDEs (analogous to that of BSDEs) has to be developed: in this regard see [18] where only the case of x-independent coefficients has been considered. Moreover, in this case, the stochastic component has a reinforcing effect on the diffusion coefficient: σ^2 in the SPDE (0.0.5) corresponds to $\sigma^2 + \alpha^2$ in the related PDE and no additional ellipticity conditions have to be imposed as soon as $\sigma > 0$.

We will consider the forward case as in the stream of literature initiated by Kunita, Chow, Krylov, Rozovskii among others (cf. [7], [2], [19], [20], [13], [15], [3]).

Chapter 1

General setting and main results

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with an increasing filtration $(\mathcal{F}_t)_{t\geq 0}$ of complete with respect to $(\mathcal{F}, P)$ σ-fields $\mathcal{F}_t \subset \mathcal{F}$. Let $d_1 \in \mathbb{N}$ and $W^1, \ldots, W^{d_1}$ be independent one-dimensional Wiener processes with respect to $(\mathcal{F}_t)$.

Notations: $d \in \mathbb{N}$, $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ and $D = (\partial_{x_1}, \ldots, \partial_{x_d})$. Moreover $m_0 \leq d$ is fixed (cf. Assumptions 1.1.2 and 1.1.1).

We consider the differential operator

$$L_t u_t(x) = \frac{1}{2} \sum_{i,j=1}^{m_0} a_t^{ij}(x) D_{ij} u_t(x) + \sum_{i=1}^{m_0} a_t^i(x) D_i u_t(x) + c_t(x) u_t(x)$$

with coefficients:

$$a_t^{ij}(x) = a_t^{ij}(x, \omega), \ a_t^i(x) = a_t^i(x, \omega), \ c_t(x) = c_t(x, \omega), \qquad x \in \mathbb{R}^d, \ \omega \in \Omega.$$

Let $\sigma_t = \sigma_t(x, \omega)$ be a random vector field

$$\sigma_t = \left(\sigma_t^1, \ldots, \sigma_t^d\right).$$

Notice that we will often omit the dependence on ω: so, for instance, we write $\sigma_t(x)$ rather than $\sigma_t(x, \omega)$. We define the differential operator L_{σ_t} acting as

$$L_{\sigma_t} u_t(x) := \sum_{i=1}^{d} \sigma_t^i(x) D_i u_t(x).$$

We consider the vector field

$$\sigma_t^0(x, \omega) = Bx + b_t(\omega),$$

with coefficients linearly dependent on x, and set

$$Y_t = \partial_t - L_{\sigma_t^0}.$$

We say that

$$K_t := L_t - Y_t \tag{1.0.1}$$

is a Kolmogorov-type operator.

Next we consider the vector fields $L_{\sigma_t^1}, \dots, L_{\sigma_t^{d_1}}$ with coefficients $\sigma_t^{ki} = \sigma_t^{ki}(\omega)$, *independent of* x, and such that $\sigma_t^{ki} \equiv 0$ for $i = m_0 + 1, \dots, d$. Let $f_t = f_t(x, \omega)$ be a bounded and continuous function in (t, x). We are interested in the following Kolmogorov-type SPDE

$$d_Y u_t = (L_t u_t + f_t)dt + L_{\sigma_t^k} u_t dW_t^k, \tag{1.0.2}$$

where and below the summation convention over repeated indices is enforced regardless of whether they stand at the same level or at different ones.

The actual meaning of (1.0.2) needs to be specified. Given an open subset $D \subseteq \mathbb{R}^d$, denote by $\mathcal{B}(D)$ the Borel σ-field of D and denote by $\mathcal{S}_T$ the predictable σ-field in $[0, T] \times \Omega$.

Definition 1.0.1. *A real valued function* u *on* $[0, T] \times \Omega \times D$*,* $\mathcal{S}_T \otimes \mathcal{B}(D)$*-measurable, is a solution to the equation 1.0.2 if* u*,* $\partial_{x_i} u$*,* $\partial_{x_i x_j} u$ *are continuous in (t,x), for* $i, j = 1, \cdots, m_0$ *for almost any* ω *and it holds that*

$$u_t(\gamma_t(x)) = u_0(x) + \int_0^t (L_s u_s + f_s)(\gamma_s(x))ds + \int_0^t \left(L_{\sigma_s^k} u_s\right)(\gamma_s(x))dW_s^k, \tag{1.0.3}$$

where $t \mapsto \gamma_t(x)$ *denotes the integral curve, starting from* x *at time* 0*, of* $-L_{\sigma_t^0}$*: more precisely,* $\gamma_t(x) \equiv \gamma_{0,t}(x)$ *where*

$$\gamma_{\tau,t}(x) := e^{-(t-\tau)B}\left(x - \int_\tau^t e^{(s-\tau)B} b_s ds\right) \tag{1.0.4}$$

is absolutely continuous as a function of the variable t *and solves*

$$\begin{cases} \frac{d}{dt}\gamma_{\tau,t}(x) = -B\gamma_{\tau,t}(x) - b_t & a.e. \\ \gamma_{\tau,\tau}(x) = x. \end{cases}$$

Example 1.0.2 (**Langevin**)**.** *Let* $d = 2$*,* $m_0 = d_1 = 1$ *and*

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then

$$Y = \partial_t - x\partial_y, \qquad (x, y) \in \mathbb{R}^2,$$

and we have the following SPDE

$$d_Y u_t(x, y) = \frac{a_t(x, y)}{2}\partial_{xx} u_t(x, y)dt + \sigma_t \partial_x u_t(x, y)dW_t. \tag{1.0.5}$$

If (1.0.5) *has a smooth (in the spatial variables) solution* u*, then* (1.0.5) *can be rewritten in the more familiar Ito sense*

$$du_t(x, y) = \left(\frac{a_t(x, y)}{2}\partial_{xx} u_t(x, y) + x\partial_y u_t(x, y)\right)dt + \sigma_t \partial_x u_t(x, y)dW_t.$$

In the deterministic case $\sigma_t \equiv 0$ *and* $a_t(x, y) \equiv 1$*,* (1.0.5) *reduces to the following degenerate Kolmogorov PDE, known as Langeving equation:*

$$\partial_t u = \frac{1}{2}\partial_{xx} u + x\partial_y u.$$

The main goal is to construct and estimate the *stochastic fundamental solution* of the Kolmogorov-type SPDE (1.0.2). The dissertation follows two steps:

i) we consider the case when $L_{\sigma_t^k} \equiv 0$, that is we have a deterministic PDE with coefficients that are measurable functions of t, Hölder continuous with respect to x: the treatise is developed by an extension of the parametrix method. The arguments and results are reported with details in Chapter 3;

ii) as in [12], we use the Ito-Wentzell formula to reduce the SPDE to a PDE to which the results of Step i) apply. This is discussed in Chapter 2.

1.1 Assumptions and results

We assume the following structural hypothesis on L_t to hold:

Assumption 1.1.1. *The matrix $B := (b_{ij})_{1\le i,j\le d}$ has constant real entries and takes the block-form*

$$B = \begin{pmatrix} * & * & \cdots & * & * \\ B_1 & * & \cdots & * & * \\ 0 & B_2 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_\nu & * \end{pmatrix} \tag{1.1.1}$$

where each B_i is a $(m_i \times m_{i-1})$-matrix of rank m_i with

$$m_0 \ge m_1 \ge \cdots \ge m_\nu \ge 1, \qquad \sum_{i=0}^{\nu} m_i = d,$$

and the blocks denoted by "$*$" are arbitrary.

Assumption 1.1.2. *The coefficients $a_t^{ij} = a_t^{ji}, a_t^i, c_t, b_t$, for $1 \le i,j \le m_0$, are bounded and measurable in t functions such that*

$$\mu^{-1}|\xi|^2 < \sum_{i,j=1}^{m_0} \left(a_t^{ij}(x) - \alpha_t^{ij}\right)\xi_i\xi_j < \mu|\xi|^2, \qquad \xi \in \mathbb{R}^{m_0},\ t \in [0,T],\ x \in \mathbb{R}^d,$$

for some positive constant μ, with α_t^{ij} as in (2.0.7).

Assumption 1.1.3. *The second order coefficients take the form:*

$$a_t^{ij}(x) = A_t^{ij}(q^{ij}(t,x))$$

where $A_t^{ij} = A_t^{ij}(q,\omega)$ are measurable in t and Hölder continuous in q in the Euclidean sense[1]*, while $q^{ij} = q^{ij}(t,x,\omega) \in C_B^\alpha(\mathbb{R}^{d+1})$ (see Definition 3.1.5). We also set $A_t^{ij} \equiv 0$ for i, $j \ge m_0+1$.*

[1]The main example is that of linear functions

$$A_t(q,\omega) = a_t(\omega)q + b_t(\omega), \qquad q \in \mathbb{R}.$$

Notation 1.1.4. *We will denote with A^0_t the (positive definite) restriction $(a^{ij}_t)_{i,j=1,\cdots,m_0}$ of the second order coefficients matrix.*

Now we can state the main result of the thesis.

Theorem 1.1.5. *(i) Under Assumptions 1.1.1, 1.1.2 and 1.1.3 there exists a fundamental solution $p_t = p_t(x, \omega; \zeta)$ to* (1.0.2).

(ii) Let p^ε_t be the fundamental solution to the constant coefficient Kolmogorov-type SPDE

$$d_Y u_t = L^\varepsilon_t u_t dt + L_{\sigma^k_t} u_t dW^k_t,$$

with

$$L^\varepsilon_t u_t(x) := \frac{1}{2}(\mu + \varepsilon) \sum_{j=1}^{m_0} D^2_j u_t(x)$$

Then the following estimates hold: for every positive ε and T, there exists a positive constant C, only dependent on ε, T, μ and B such that

$$p_t(x; \zeta) \le C p^\varepsilon_t(x; \zeta),$$
$$\partial_{x_i} p_t(x; \zeta) \le \frac{C}{\sqrt{t-\tau}} p^\varepsilon_t(x; \zeta),$$
$$\partial_{x_i x_j} p_t(x; \zeta) \le \frac{C}{t-\tau} p^\varepsilon_t(x; \zeta).$$

for any $i, j = 1, \cdots, m_0$ and $z = (t, x), \zeta = (\tau, \xi) \in \mathbb{R}^{d+1}$ with $0 < t - \tau < T$.

Chapter 2

Random mappings and Itô-Wentzell formula

We begin this Chapter with a technical result concerning stochastic integration depending on a parameter.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with an increasing filtration $(\mathcal{F}_t)_{t\geq 0}$ of complete with respect to $(\mathcal{F}, P)$ σ-fields $\mathcal{F}_t \subset \mathcal{F}$, satisfying the usual hypotesis. Hereafter we will use the classic notations:

$\mathbb{L}^2$: the family of real measurable processes $\psi = \{\psi(t,\omega)\}_{t\geq 0}$ on Ω adapted to $\mathcal{F}_t$ such that, for every $T > 0$,

$$\|\psi\|^2_{L^2(\Omega\times[0,T])} = E\left[\int_0^T \psi_s^2 ds\right] < \infty.$$

$\mathbb{L}^2_{loc}$: the family of real measurable processes $\psi = \{\psi(t,\omega)\}_{t\geq 0}$ on Ω adapted to $\mathcal{F}_t$ such that, for every $T > 0$,

$$\|\psi\|^2_{L^2([0,T])} = \int_0^T \psi_s^2 ds < \infty \quad a.s.$$

$\mathcal{M}^2_c$: the complete metric space of continuous square integrable martingale $M = \{M(t,\omega)\}_{t\in[0,T]}$ such that $M_0 = 0$ a.s. equipped with the seminorm

$$[|M|]_T := \left(E\left[\sup_{0\leq t\leq T} |M_t|^2\right]\right)^{\frac{1}{2}}$$

or equivalently $\|M_T\|_{L(\Omega)}$.

We recall that the stochastic integral of a process in $\mathbb{L}^2$ with respect to a Brownian motion is well defined as an element of $\mathcal{M}^2_c$.

This is a slight variation to the stochastic Fubini's theorem in [8].

Lemma 2.0.1 (A Fubini's type theorem for stochastic integrals). *Let W be a one dimensional Wiener process with respect to $(\mathcal{F}_t)$ and let $\{\varphi(t,x,\omega)\}$, $t \in [0,\infty)$, $x \in \mathbb{R}^d$ a family of real random variables such that:*

(i) $((t,\omega),x) \in ([0,\infty)\times\Omega)\times\mathbb{R}^d \to \varphi(t,x,\omega)$ *is* $\mathcal{S}\otimes\mathcal{B}(\mathbb{R}^d)$*-measurable;*

(ii) There exists a non-negative Borel-measurable function $f(x)$ *such that* [1]

$$|\varphi(t,x,\omega)| \le f(x)\;\forall\, x,t,w \quad \text{and} \quad \int_{\mathbb{R}^d} f(x)dx < \infty.$$

By (i) and (ii), $I_\varphi(t,x) := \int_0^t \varphi_s(x)dW_s \in \mathcal{M}_2^c$ *is well defined. We assume further that*

(iii) $(x,w) \to \int_0^t \varphi_s(x)dW_s$ *is* $\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{F}$*-measurable for each* $t \ge 0$*. Then*

$$t \to \int_{\mathbb{R}^d} \varphi_s(x)dx \in \mathbb{L}_2$$

and we have

$$\int_0^t \left(\int_{\mathbb{R}^d} \varphi_s(x)dx\right) dW_s = \int_{\mathbb{R}^d}\left(\int_0^t \varphi_s(x)dW_s\right)dx \tag{2.0.1}$$

Proof. It is clear that $\int_{\mathbb{R}^d}\varphi(s,x,\omega)dx$ is predictable and bounded. Hence it is obvious that

$$E\left[\int_0^t \left(\int_{\mathbb{R}^d}\varphi(s,x,\omega)dx\right)^2 dt\right] < \infty.$$

Thus the left hand side of (2.0.1) is well defined as an element in $\mathcal{M}_2^c$. On the other hand, the map $x \to \int_0^t \varphi_s(x)dW_s$ is $\mathcal{B}(\mathbb{R}^d)$-measurable by assumption (iii) and for every $T > 0$

$$\begin{aligned} &E\left[\int_{\mathbb{R}^d}\max_{0\le t\le T}\left|\int_0^t \varphi_s(x)dW_s\right| dx\right] \\ &\quad \le \int_{\mathbb{R}^d}\left(E\left[\max_{0\le t\le T}\left|\int_0^t \varphi_s(x)dW_s\right|^2\right]\right)^{1/2} dx \\ &\quad \le 2\int_{\mathbb{R}^d}\left(E\left[\left|\int_0^T \varphi_s(x)dW_s\right|^2\right]\right)^{1/2} dx \end{aligned}$$

(by Doob's inequality)

$$= 2\int_{\mathbb{R}^d}\left(E\left[\int_0^T |\varphi_s(x)|^2 ds\right]\right)^{1/2} dx$$

(by Itô's isometry)

$$= 2\sqrt{T}\int_{\mathbb{R}^d} f(x)dx < \infty$$

[1] The result can still be proved under the weaker assumption (see e.g. Lemma 2.6 in [10] or [22])

$$\int_X \left(\int_0^T |\varphi(x,t)|^2 dt\right)^{1/2} d\mu_X < \infty$$

Hence

$$\int_{\mathbb{R}^d} \max_{0\le t\le T} \left| \int_0^t \varphi_s(x) dW_s \right| dx < \infty \quad \text{a.s.}$$

and this implies that

$$t \to \int_{\mathbb{R}^d} \int_0^t \varphi_s(x) dW_s dx$$

is continuous a.s. Thus the right-hand side of (2.0.1) is well defined and defines an $(\mathcal{F}_t)$-adapted process. It is also square-integrable because

$$\begin{aligned} E & \left[\left(\int_{\mathbb{R}^d} (\int_0^t \varphi_s(x) dW_s) dx \right)^2 \right] \\ & = \int_{\mathbb{R}^d} dx_1 \int_{\mathbb{R}^d} dx_2 E \left[\int_0^t \varphi_s(x_1) dW_s \int_0^t \varphi_s(x_2) dW_s \right] \\ & = \int_{\mathbb{R}^d} dx_1 \int_{\mathbb{R}^d} dx_2 E \left[\int_0^t \varphi_s(x_1) \varphi_s(x_2) ds \right] \\ & \le t \left(\int_{\mathbb{R}^d} f(x) dx \right)^2 < \infty \end{aligned}$$

It is an $(\mathcal{F}_t)$ martingale because if $t > s > 0$ and $A \in \mathcal{F}_s$ then

$$\begin{aligned} E & \left[I_A \int_{\mathbb{R}^d} \int_s^t \varphi_s(x) dW_s dx \right] \\ & \int_{\mathbb{R}^d} E \left[I_A \int_s^t \varphi_s(x) dW_s \right] dx = 0. \end{aligned}$$

Similarly, if $N \in \mathcal{M}_2$, then

$$\begin{aligned} E & \left[I_A \left(\int_{\mathbb{R}^d} \int_s^t \varphi_u(x) dW_u dx \right) (N_t - N_s) \right] \\ & = \int_{\mathbb{R}^d} E \left[I_A \int_s^t \varphi_u(x) dW_u (N_t - N_s) \right] dx \\ & = \int_{\mathbb{R}^d} E \left[I_A \int_0^t \varphi_u(x) d < W, N >_u \right] dx \\ & = E \left[I_A \int_s^t \left(\int_{\mathbb{R}^d} \varphi_u(x) dx \right) d < W, N >_u \right] \end{aligned}$$

Thus $t \to \int_{\mathbb{R}^d} \int_0^t \varphi_s(x) dW_s dx = L_t$ is an element in $\mathcal{M}_2^c$ such that, for every $N \in \mathcal{M}_2$,

$$< N, L >_t = \int_0^t \left(\int_{\mathbb{R}^d} \varphi_s(x) dx \right) d < W, N >_s$$

Then necessarily $L_t = \int_0^t (\int_{\mathbb{R}^d} \varphi_s(x) dx) dW_s$. This completes the proof. □

Now let $d_1 \in \mathbb{N}$ and $W^1, \dots, W^{d_1}$ be independent one-dimensional Wiener processes with respect to $(\mathcal{F}_t)$. We introduce the mapping

$$X_t(x) = x - \int_0^t \sigma_s^k dW_s^k$$

where σ_t^k, for $k = 1, \ldots, d_1$, are as in Section 1, and define the operation "hat" which transforms any function $u_t(x)$ into

$$\hat{u}_t(x) = u_t(X_t(x)).$$

We have

$$D_i \hat{u}_t(x) = \widehat{D_i u_t}(x), \qquad i = 1, \ldots, d, \tag{2.0.2}$$

and

$$\widehat{L_{\sigma_t^0} u_t}(x) = L_{\hat{\sigma}_t^0} \hat{u}_t(x), \tag{2.0.3}$$

where

$$\hat{\sigma}_t^0(x) = Bx + \hat{b}_t, \qquad \hat{b}_t := b_t - \int_0^t B\sigma_s^k dW_s^k.$$

Now we are in the position to state the version of the Itô-Wentzell formula we need. The Itô-Wentzell formula, going back to A. Wentzell [23], allowes to construct the differential of a composition of two random processes, while the classical Itô formula and its generalizations only allows to determine the differential of a deterministic function of a random process. This can be used to make random change of coordinates for stochastic equations in such a way that the stochastic terms in such equations would disappear.

We point out the very recent and relevant contributions by Krylov in [12] and [11] where the Hörmander's theorem for SPDEs is proved under the *strong* Hörmander condition. This is done by using a generalized Itô-Wentzell formula for distribution valued precesses [10], and studying the reduced analytical equation with coefficients measurable in time.

Our version of the Itô-Wentzell formula concerns more familiar real valued processes. Nevertheless, it has to deal with the more uncommon version of the 'differential' that involves all the field Y, i.e it somehow includes some spatial derivatives to non-deterministic quantities.

Theorem 2.0.2 (Itô-Wentzell formula). *Let f, u, g^k, for $k = 1, \ldots, m_0$ be some real valued functions on $\Omega \times [0,T] \times \mathbb{R}^d$ such that*

(i) $u, f, g^k,\ k = 1, \cdots m_0$ are $\mathcal{S} \otimes B(\mathbb{R}^d)$-measurable;

(ii) For any ω the functions u, $D_j u$ and $D_{ij} u$, for $i, j = 1, \cdots m_0$ are continuous functions of (t,x). For almost any (ω, t) the functions f_t, g_t^k, u_t, $D_j g_t^k$, for $k = 1, \cdots d'$, $j = 1, \cdots m_0$ are continuous functions of x;

(iii) For $k = 1, \cdots d'$, $x \in \mathbb{R}^d$, $g^k(x), f(x) \in \mathbb{L}^2_{loc}$.

Assume that

$$d_Y u_t(x) = f_t(x)dt + g_t^k(x) dW_t^k. \tag{2.0.4}$$

(in the sense of (1.0.3)*). Then we have*

$$d_{\hat{Y}} \hat{u}_t(x) = \left(\hat{f}_t(x) + \frac{1}{2} \alpha_t^{ij} D_{ij} \hat{u}_t(x) - L_{\sigma_t^k} \hat{g}_t^k(x) \right) dt + \left(\hat{g}_t^k(x) - L_{\sigma_t^k} \hat{u}_t(x) \right) dW_t^k \tag{2.0.5}$$

where

$$\hat{Y}_t = \partial_t - L_{\hat{\sigma}_t^0} \tag{2.0.6}$$

and

$$\alpha_t^{ij} = \sum_{k=1}^{d_1} \sigma_t^{ki}\sigma_t^{kj}. \tag{2.0.7}$$

Proof. We consider the case in which u_t is differentiable with respect to x_k for $k = 1, \cdots, d$. Under this assumption we may rewrite (2.0.4) in the more familiar Itô sense

$$du_t(x) = \Big(f_t(x) + L_{\sigma_t^0} u_t(x)\Big)\, dt + \Big(g_t^k(x) + L_{\sigma_t^0} u_t\Big)\, dW_t^k.$$

Also note that by (2.0.3) it suffices to prove the statement for $L_{\sigma_t^0} u_t \equiv 0$.

Take $\Phi \in C_0^\infty$, which is non negative, radially symmetric, with unit integral and support in B_γ. Then, for all $x, y \in \mathbb{R}^d$, un application of the standard Itô formula shows that

$$u_t(x)\varphi(X_t(y) - x) = u_0(x)\Phi(y - x) + \int_0^t F_s(x) d_s + \sum_{k=1}^{d'} \int_0^t G_s^k(x) dW_s^k$$

where

$$\begin{aligned} F_s(x) =& \Phi(X_s(y) - x) f_s(x) + \frac{1}{2} u_s(x) \alpha_s^{ij} \left[D_{ij}\Phi\right](X_s(y) - x) + \sum_{k=1}^{d'} g_s^k(x) \left[L_{\sigma_s^k}\Phi\right](X_s(y) - x), \\ G_s^k(x) =& \Phi(X_s(y) - x) g_s(x) + u_s(x) \left[L_{\sigma_s^k}\Phi\right](X_s(y) - x). \end{aligned}$$

with α_t^{ij} as defined in (2.0.7), for all $t \in [0, T]$. Now we integrate on $\mathbb{R}^d$ with respect to x. Note that

$$\int_{\mathbb{R}^d} \left(\int_0^t |G_s^k(x)|^2 ds\right)^{\frac{1}{2}} dx < \infty$$

Indeed, we have

$$|\Phi(X_s(x) - x) g_s^k(x)| \leq I_{B_\gamma + M} \|\Phi\|_\infty |g_s^k(x)|$$

where $M = \{X_s(x), s \in [0, t]\}$ is a compact set and I_K denotes the identity function on the set K. Then, by assumption (iii) and the continuity of g_s^k with respect to the spatial variables we have

$$\begin{aligned} &\int_{\mathbb{R}^d} \left(\int_0^t |\Phi(X_s(x) - x) g_s^k(x)|^2 ds\right)^{\frac{1}{2}} dx \\ &\qquad \leq c \int_{B_\gamma + M} \left(\int_0^t |g_s^k(x)|^2 ds\right)^{\frac{1}{2}} \leq c \int_{B_\gamma + M} \sqrt{H(x)} dx < \infty \end{aligned}$$

The same argument still works for $L_\sigma^k u_s$, since $D_{ij}u$ are continuous functions of (t, x) and σ^k are bounded and measurable functions independent of x.

Then, we may apply the stochastic Fubini's Theorem 2.0.1 as well as the standard one, and get

$$\int_{\mathbb{R}^d} u_t(x)\varphi(X_t(y)-x)dx = \int_{\mathbb{R}^d} u_0(x)\Phi(y-x)dx + \int_0^t \int_{\mathbb{R}^d} F_s(x)dxds + \sum_{k=1}^{d'} \int_0^t \int_{\mathbb{R}^d} G_s^k(x)dxdW_s^k$$

We fix $t \in [0,T]$ and use this formula with $\Phi_\varepsilon := \varepsilon^{-d}\Phi(\frac{x}{\varepsilon})$, $\varepsilon > 0$, in place of Φ and integrate by parts the integrals of F and G with respect to x. Then, using the notation $h^{(\varepsilon)} = h * \Phi_\varepsilon$ we find

$$\begin{aligned} u_t^{(\varepsilon)}(X_t(y)) = u_0^{(\varepsilon)}(y) + \sum_{k=1}^{d'} \int_0^t \left(g_s^{k\,(\varepsilon)}(X_s(y)) - \left[L_{\sigma_s^k} u_s^{(\varepsilon)}\right](X_s(y))\right) dW_s^k \\ + \int_0^t \left(f_s^{(\varepsilon)}(X_s(y)) + \frac{1}{2}\alpha_s^{ij}\left[D_{ij}u_s^{(\varepsilon)}\right](X_s(y)) - \sum_{k=1}^{d'} \left[L_{\sigma_s^k} g_s^{k\,(\varepsilon)}\right](X_s(y))\right) ds \end{aligned} \tag{2.0.8}$$

Now we let $\varepsilon \to 0$. By the continuity assumptions we have $u_t^{(\varepsilon)}(X_t(y)) \to u_t(X_t(y))$ for every $\omega \in \Omega$. Analogously, for almost any $\omega \in \Omega$, $s \in [0,t]$ we have $f_s^{(\varepsilon)}(x) \to f_s(x)$, $D_{ij}u_s^{(\varepsilon)}(x) \to D_{ij}u_s(x)$ and $L_{\sigma_s^k}(g_s^k)^{(\varepsilon)}(x) \to L_{\sigma_s^k} g_s^k(x)$ uniformly in compact sets in $\mathbb{R}^d$. Thus, given the coefficients σ^k, α^{ij} are bounded functions of t and by (2.0.2) we may infer that the Lebesgue integral in (2.0.8) converges (a.s) to the one in (2.0.5).

On the other hand, again by the continuity assumptions, $g_s^{k\,(\varepsilon)}$ and $L_{\sigma_s^k} u_s^{(\varepsilon)}$ converge in $\mathbb{L}^2_{loc}$ to g_s^k and $L_{\sigma_s^k} u_s$ respectively. This implies that the stochastic integral in (2.0.8) converges to the one in (2.0.5) in probability. □

Corollary 2.0.3. *Assume that u satisfies* (1.0.2). *Then*

$$d_{\hat{Y}}\hat{u}_t(x) = \hat{L}_t\hat{u}_t(x)dt \tag{2.0.9}$$

with $\hat{Y}_t$ *defined in* (2.0.6) *and*

$$\hat{L}_t\hat{u}_t(x) = \frac{1}{2}\sum_{i,j=1}^{m_0}\left(\hat{a}_t^{ij}(x) - \alpha_t^{ij}\right) D_{ij}\hat{u}_t(x) + \sum_{i=1}^{m_0} \hat{a}_t^i(x) D_i\hat{u}_t(x) + \hat{c}_t(x)\hat{u}_t(x) + \hat{f}_t(x).$$

Proof. By Definition 1.0.1 we may apply the Itô-Wentzell formula (2.0.5) with $f_t = L_t u_t$ and $g_t^k = L_{\sigma_t^k} u_t$ in (2.0.4). Since the coefficients σ_t^k are independent of x we get

$$d_{\hat{Y}}\hat{u}_t(x) = \left(\widehat{L_t u_t}(x) - \frac{1}{2}\alpha_t^{ij} D_{ij}\hat{u}_t(x)\right) dt$$

from which (2.0.9) easily follows. □

Relying on the Itô-Wentzell formula and the results for the deterministic case with time-measurable coefficients we can now easily prove the main statement 1.1.5.

Proof of theorem 1.1.5. By the results in Chapter 3 (see Theorem 3.0.1) there exists a fundamental solution Γ to the Kolmogorov equation 2.0.9. Thus by Corollary 2.0.3 it suffices to make the reverse random change of variables $(t, x) \to (t, X_t^{-1}(x))$ to get the stochastic fundamental solution to the (1.0.2) for $t \geq 0$.

Also note that the estimates 1.1.5, 1.1.5 and 1.1.5 are a direct transposition of the analogous ones ((3.0.6), 3.0.7, 3.0.1) found in the deterministic case.

Chapter 3

The deterministic problem

In this chapter we construct the fundamental solution for the Kolmogorov equation

$$d_Y u_t = L_t u_t dt \tag{3.0.1}$$

under Assumptions 1.1.1, 1.1.2 and 1.1.3. This is done by adapting the classical parametrix method, which goes back to Levi [17]. It consists on the approximation of the fundamental solution of a differential equation through an iterative process.

We remark that the only results available in the literature based on the parametrix method, are proved under the assumption of Hölder regularity of the coefficients in the time-variable t. Thus these results may be of independent interest also in the determistic case.

Here we state the main result of the chapter.

Theorem 3.0.1. *Assume that L_t in* (3.0.1) *verifies hypotheses 1.1.1, 1.1.2 and 1.1.3. Then there exists a fundamental solution Γ with the following properties:*

(i) $\Gamma(\cdot;\zeta) \in L^1_{\text{loc}}(\mathbb{R}^{d+1}) \cap C(\mathbb{R}^{d+1} \setminus \{\zeta\})$ $\forall \zeta \in \mathbb{R}^{d+1}$;

(ii) $\Gamma(\cdot;\zeta)$ *is a solution to* (3.0.1) *in* $\mathbb{R}^{d+1} \setminus \{\zeta\}$ $\forall \zeta \in \mathbb{R}^{d+1}$ *(in the sense of Definition 1.0.1)*

(iii) Let $g \in C(\mathbb{R}^d)$ *such that*

$$|g(x)| \le C_0 e^{C_0|x|^2}, \quad \forall x \in \mathbb{R}^d, \tag{3.0.2}$$

for some positive constant C_0, then there exists

$$\lim_{\substack{(t,x)\to(\tau,y)\\ t>\tau}} \int_{\mathbb{R}^d} \Gamma(t,x;\tau,\xi) g(\xi) d\xi = g(y), \quad \forall y \in \mathbb{R}^d,\ \tau \in \mathbb{R}$$

(iv) Let $g \in C(\mathbb{R}^d)$ *verifying* (3.0.2) *and let f be a continuous function in the strip* $S_{T_0,T_1} =]T_0, T_1[\times\mathbb{R}^d$, *such that*

$$|f(t,x)| \le C_1 e^{C_1|x|^2} \quad \forall x \in S_{T_0,T_1} \tag{3.0.3}$$

and for any compact set $M \subset \mathbb{R}^d$ there exists a positive constant C such that

$$|f(t,x) - f(t,y)| \le C|x-y|_B^\beta, \quad \forall x, y \in M,\ t \in]T_0, T_1[,$$

for some $\beta \in]0,1[$; then there exists $T \in]T_0, T_1]$ such that the function

$$u(t,x) = \int_{\mathbb{R}^d} \Gamma(t,x;\tau,\xi) g(\xi) d\xi + \int_{T_0}^t \int_{\mathbb{R}^d} \Gamma(t,x;\tau,\xi) f(\tau,\xi) d\xi d\tau \tag{3.0.4}$$

is a solution to the Cauchy problem

$$\begin{cases} d_Y u_t = (L_t u_t + f_t)dt & \text{in } S_{T_0,T} \\ u(\cdot, T_0) = g & \text{in } \mathbb{R}^d \end{cases} \tag{3.0.5}$$

(v) Let Γ^ε be the fundamental solution to the constant coefficients Kolmogorov equation

$$d_Y u_t = L_t^\varepsilon u_t dt$$

with

$$L_t^\varepsilon u_t(x) := \frac{1}{2}(\mu + \varepsilon) \sum_{j=1}^{m_0} D_j^2 u_t(x)$$

where $\varepsilon > 0$ and μ is as in 1.1.2: then for every positive ε and T, there exists a constant C, only dependent on μ, B, ε and T such that

$$\Gamma(z;\zeta) \le C\Gamma^\varepsilon(z;\zeta), \tag{3.0.6}$$

$$\partial_{x_i}\Gamma(z;\zeta) \le \frac{C}{\sqrt{t-\tau}}\Gamma^\varepsilon(z;\zeta), \tag{3.0.7}$$

$$\partial_{x_i x_j}\Gamma(z;\zeta) \le \frac{C}{t-\tau}\Gamma^\varepsilon(z;\zeta).$$

for any $i, j = 1, \cdots, m_0$ and $z, \zeta \in \mathbb{R}^{d+1}$ with $0 < t - \tau < T$.

The chapter is organized as follows: in the next Section we give the fundamental solution for the case where the coefficients are only time dependent and give some Gaussian estimates. In Section 3.2 we define the candidate solution for the general equation through an adaptation of the parametrix method. In Section 3.3 we provide some potential estimates whose complete proofs will be given in Section A. Finally, we will be able to prove the main Theorems 3.0.1 and 1.1.5 in the Appendix 3.4.

3.1 Estimates of the fundamental solution of Kolmogorov PDEs with time dependent coefficients

We start by introducing some general notation. For any symmetric and positive definite matrix $\mathcal{C} = (\mathcal{C}^{ij})_{1 \le i,j \le d}$, we denote by

$$\Gamma^{\text{heat}}(\mathcal{C}, x) = \frac{1}{\sqrt{(2\pi)^d \det \mathcal{C}}} \exp\left(-\frac{1}{2}\langle \mathcal{C}^{-1}x, x\rangle\right), \qquad x \in \mathbb{R}^d, \tag{3.1.1}$$

the fundamental solution of the d-dimensional heat equation: Γ^{heat} is a smooth function and satisfies

$$\partial_t \Gamma^{\text{heat}}(t\,\mathcal{C}, x) = \frac{1}{2} \sum_{i,j=1}^{d} \mathcal{C}^{ij} D_{ij} \Gamma^{\text{heat}}(t\,\mathcal{C}, x), \qquad t > 0,\ x \in \mathbb{R}^d.$$

Next we consider the Kolmogorov operator with *coefficients dependent only on the time variable t (not on the space variable x)*:

$$\bar{K} := \frac{1}{2} \sum_{i,j=1}^{m_0} \bar{A}_t^{ij} D_{ij} - Y. \tag{3.1.2}$$

In case of constant coefficients, Assumptions 1.1.2 and 1.1.1 are equivalent to the hypoellipticity of $\bar{K}$: in fact, they are equivalent to the Hörmander's condition, which in our setting reads:

$$\text{rank Lie}\left(\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y\right)(t, x) = d + 1, \qquad \text{for all } (t, x) \in \mathbb{R}^{d+1},$$

where $\text{Lie}\left(\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y\right)$ denotes the Lie algebra generated by the vector fields $\partial_{x_1}, \dots, \partial_{x_{m_0}}$ and Y (see Proposition 2.1 in [16]). In general we have the explicit expression of the fundamental solution.

Lemma 3.1.1. *Under Assumptions 1.1.2 and 1.1.1, the fundamental solution of $\bar{K}$ in* (3.1.2) *is*

$$\bar{\Gamma}(t, x; \tau, \xi) = e^{-(t-\tau)\text{Tr}B} \Gamma^{\text{heat}}(\mathcal{C}_{\tau,t}, x - \gamma_{\tau,t}(\xi)), \qquad t > \tau,\ x, \xi \in \mathbb{R}^d, \tag{3.1.3}$$

with Γ^{heat} as in (3.1.1) *and*

$$\gamma_{\tau,t}(\xi) = e^{-(t-\tau)B} \left(\xi - \int_\tau^t e^{(s-\tau)B} b_s ds \right), \qquad \textit{(as in (1.0.4))}$$

$$\mathcal{C}_{\tau,t} = \int_\tau^t e^{(t-s)B} \bar{A}_s e^{(t-s)B^*} ds.$$

Proof. Assumption 1.1.1 implies that $\mathcal{C}_{\tau,t}$ is positive definite for $t > \tau$. Indeed, $\mathcal{C}_{\tau,t}$ is positive semi-definite and non-decreasing in $t - \tau \geq 0$ because $\bar{A}_s \geq 0$. By contradiction, suppose there exist $t > \tau$ and $\xi \in \mathbb{R}^d \setminus \{0\}$ such that $\langle \mathcal{C}_{\tau,t} \xi, \xi \rangle = 0$: then we have

$$\langle \bar{A}_s e^{(t-s)B^*} \xi, e^{(t-s)B^*} \xi \rangle = 0 \qquad \text{for a.e. } s \in [\tau, t].$$

This implies that $\bar{A}_s e^{(t-s)B^*} \xi = 0$ for a.e. $s \in [\tau, t]$, that is

$$\sum_{k=0}^{\infty} \frac{(t-s)^k}{k!} \bar{A}_s (B^*)^k \xi = 0,$$

and we deduce that

$$\bar{A}_s (B^*)^k \xi = 0 \qquad k \geq 0, \text{ for a.e. } s \in [\tau, t]. \tag{3.1.4}$$

Identity (3.1.4) with $k = 0$ implies $\xi_1, \dots, \xi_{m_0} = 0$. On the other hand, by Assumption 1.1.1, for $1 \leq k \leq \nu$ we have

$$\bar{A}_s (B^*)^k = \begin{pmatrix} * & C_{k,s} & \mathbf{0}_{m_0 \times (\dots)} \\ \mathbf{0}_{(d-m_0) \times (\dots)} & \mathbf{0}_{(d-m_0) \times m_k} & \mathbf{0}_{(d-m_0) \times (\dots)} \end{pmatrix} \tag{3.1.5}$$

where $C_{k,s} = A_s^0 B_1^* \cdots B_k^*$ is a $m_0 \times m_k$ matrix of rank m_k, $\mathbf{0}_{i\times j}$ denotes a $i \times j$ null matrix and $*$ denotes a generic block. Then (3.1.5) and (3.1.4) for $k = 1$ imply that $\xi_{m_0+1}, \dots, \xi_{m_0+m_1} = 0$. Repeating the argument for $k = 2, \dots, \nu$ we get $\xi = 0$.

Thus $\bar{\Gamma}(t, x; \tau, \xi)$ in (3.1.3) is well defined. Moreover, since Γ^{heat} is a smooth function and $\bar{A}_t$, b_t are bounded and measurable in t by assumption, then $\bar{\Gamma}(t, x; \tau, \xi)$ is absolutely continuous in t, smooth in x and a direct computation shows that

$$\bar{K}\bar{\Gamma}(t, x; \tau, \xi) = 0, \qquad x, \xi \in \mathbb{R}^d, \text{ a.e. } t > \tau.$$

The previous differential equation has to be interpreted in the integral sense

$$\bar{\Gamma}(t, \gamma_{s,t}(x); \tau, \xi) = \bar{\Gamma}(s, x; \tau, \xi) + \frac{1}{2} \sum_{i,j=1}^{m_0} \int_s^t \bar{A}_\varsigma^{ij} \cdot \left(\partial_{ij}\bar{\Gamma}\right)(\varsigma, \gamma_\varsigma(x); \tau, \xi) d\varsigma, \qquad x, \xi \in \mathbb{R}^d, \ t > s > \tau,$$

where $\partial_{ij}\bar{\Gamma}(\varsigma, y; \tau, \xi) \equiv \partial_{y_i y_j}\bar{\Gamma}(\varsigma, y; \tau, \xi)$; equivalently, we can write it with the differential notation as in (1.0.2):

$$d_Y \bar{\Gamma}(t, x; \tau, \xi) = \frac{1}{2} \sum_{i,j=1}^{m_0} \bar{A}_t^{ij} \cdot \left(\partial_{ij}\bar{\Gamma}\right)(t, x; \tau, \xi) dt.$$

Moreover, let us set

$$u(t, x) := \int_{\mathbb{R}^d} \bar{\Gamma}(t, x; \tau, \xi) \varphi(\xi) d\xi, \qquad x \in \mathbb{R}^d, \ t > \tau.$$

An application of the dominated convergence theorem shows that

$$\lim_{\substack{(t,x)\to(\tau,\xi) \\ t>\tau}} u(t, x) = \varphi(\xi), \qquad \xi \in \mathbb{R}^d,$$

for any bounded and continuous function φ. Thus $u(t, x)$ solves the Cauchy problem

$$\begin{cases} \bar{K}u(t, x) = 0, & x \in \mathbb{R}^d, \ \text{ a.e. } t > \tau, \\ u(\tau, x) = \varphi(x) & x \in \mathbb{R}^d, \end{cases}$$

that is, $\bar{\Gamma}$ is the fundamental solution of $\bar{K}$. □

Remark 3.1.2. *Lemma 3.1.15 states that $(t, x) \mapsto \bar{\Gamma}(t, x; \tau, \xi)$ is the fundamental solution of $\bar{K}$; on the other hand, it is well known that $(\tau, \xi) \mapsto \bar{\Gamma}(t, x; \tau, \xi)$ is the fundamental solution of the adjoint operator*

$$\bar{K}^* u(\tau, \xi) = \frac{1}{2} \sum_{i,j=1}^{m_0} \bar{A}_\tau^{ij} D_{ij} u(\tau, \xi) + Y u(\tau, \xi) + (\mathrm{Tr} B)\, u(\tau, \xi).$$

In general, adjoint operators are more natural from a probabilistic perspective because they are linked with the theory of stochastic differential equations (SDEs). Precisely

$$\frac{1}{2} \sum_{i,j=1}^{m_0} \bar{A}_t^{ij} D_{ij} u + Y$$

is the infinitesimal generator of the d-dimensional linear SDE

$$d\bar{X}_t = -\left(B\bar{X}_t + b_t\right)dt + \sigma_t dW_t,$$

where W is a standard m_0-dimensional Brownian motion, σ is a $(d \times m_0)$-matrix such that $\sigma_t^{ij} \equiv 0$ for $i = m_0 + 1, \dots, d$ and $\bar{A}_t := \sigma_t \sigma_t^*$ satisfies Assumption 1.1.2.

3.1.1 Geometric framework

When the coefficients are constant and $b_t \equiv 0$, operator $\bar{K}$ has remarkable invariance properties that are crucial in the analysis of existence and regularity issues: these properties were first studied in [16]. In our more general setting, these properties do not hold anymore but there is still a Lie group structure that provides the natural geometric and functional framework for the study of $\bar{K}$.

Lemma 3.1.3. *For any $(\tau, \xi) \in \mathbb{R}^{d+1}$, we denote by $\ell^Y_{(\tau,\xi)}$ the left-translation in $\mathbb{R}^{d+1}$ defined as*

$$\ell^Y_{(\tau,\xi)}(t,x) := (\tau,\xi) \circ_Y (t,x) := \left(t+\tau, x + e^{-tB}\left(\xi + \int_0^t e^{sB} b_s ds\right)\right).$$

Then we have

$$\bar{\Gamma}(t,x;\tau,\xi) = e^{-(t-\tau)\mathrm{Tr}B}\Gamma^{\mathrm{heat}}\left(\mathcal{C}_{\tau,t}, \pi_d\left((\tau,\xi)^{-1} \circ_Y (t,x)\right)\right), \qquad t > \tau,\ x,\xi \in \mathbb{R}^d. \tag{3.1.6}$$

Proof. It suffices to check that

$$(\tau,\xi)^{-1} = \left(-\tau, -e^{\tau B}\left(\xi - \int_\tau^0 e^{(s-\tau)B} b_s ds\right)\right)$$

and

$$y - \gamma_{\tau,t}(x) = \pi_d\left((\tau,\xi)^{-1} \circ_Y (t,x)\right),$$

where we denoted by $\pi_d(t,x) := x$ the projection on $\mathbb{R}^d$. □

Next we introduce a family of dilations in $\mathbb{R}^{d+1}$ that are natural for the study of $\bar{K}$. Let $z = (t,x) \in \mathbb{R}^{d+1}$, define

$$\mathcal{D}(r)z := \left(r^2 t, \mathcal{D}_0(r)x\right), \qquad r > 0,$$

where

$$\mathcal{D}_0(r) := \mathrm{diag}(rI_{m_0}, r^3 I_{m_1}, \dots, r^{2\nu+1} I_{m_\nu}), \qquad r > 0,$$

and I_{m_i} denotes the $(m_i \times m_i)$-identity matrix. The natural number

$$Q := m_0 + 3m_1 + \cdots + (2\nu+1)m_\nu$$

is usually called the *homogeneous dimension* of $\mathbb{R}^d$ with respect to $(\mathcal{D}_0(r))_{r>0}$, since that the Jacobian $J\mathcal{D}_0(r)$ equals r^Q.

Remark 3.1.4. *Let us consider the Kolmogorov operator*

$$\bar{K}_0 := \frac{1}{2}\sum_{i=1}^{m_0} \partial_{x_i x_i} - Y_0, \qquad Y_0 = \partial_t - \langle Bx, D\rangle,$$

with $\bar{A}_t^{ij} \equiv 1$ *and* $b_t \equiv 0$. *It is proved in [16], Proposition 2.2, that* $\bar{K}_0$ *is* $\mathcal{D}(r)$*-homogeneous of degree two if and only if all the* $*$*-blocks of* B *in* (1.1.1) *are null: in that case, we have*

$$\bar{K}_0 \left(u(\mathcal{D}(r)z)\right) = r^2 \left(\bar{K}_0 u\right)(\mathcal{D}(r)z), \qquad r > 0.$$

A $\mathcal{D}(r)$-homogeneous norm is defined as follows:

$$\|(t,x)\|_B = |t|^{1/2} + |x|_B, \qquad |x|_B = \sum_{j=1}^{d} |x_j|^{1/q_j},$$

where $(q_j)_{1\le j\le d}$ are the integers such that

$$\mathcal{D}_0(r) = \operatorname{diag}\left(r^{q_1}, \cdots, r^{q_d}\right).$$

Based on the previous definitions of intrinsic translations and dilations, the following functional spaces turn out to provide the natural framework for the study of Kolmogorov operators.

Definition 3.1.5. *Let* $\alpha \in]0,1[$ *and* O *be a domain of* $\mathbb{R}^{d+1}$. *We denote by* $C_Y^\alpha(O)$ *the Hölder space of functions on* O *such that*

$$|u(t,x) - u(s,y)| \le C\|(s,y)^{-1} \circ_Y (t,x)\|_B^\alpha, \qquad (t,x),(s,y) \in O,$$

for some positive constant C.

By Assumption 1.1.3, $A_t^{ij} \in C^\gamma(C_B^\alpha(\mathbb{R}^{d+1}), \mathbb{R})$. Thus, given $t \ge 0$, $z, \zeta \in \mathbb{R}^{d+1}$ we have

$$|A_t^{ij}(q^{ij}(z)) - A_t^{ij}(q^{ij}(z))| \le C\|\zeta^{-1} \circ_Y z\|_B^{\gamma\alpha}$$

Hereafter, the exponent product will be more conveniently noted as α.

Example 3.1.6. *The function* $f(t,x) = |x|$ *is Lipschitz continuous in* $\mathbb{R}^3$. *Now, consider the Langevin operator in Example 1.0.2, then, for* (t,x), $(\tau,\xi) \in \mathbb{R}^3$

$$\begin{aligned}\left\|(\tau,\xi)^{-1} \circ_Y (t,x)\right\|_B &= \left\|\left(t-\tau, x - e^{-(t-\tau)B}\xi\right)\right\|_B = \\ &= |t-\tau|^{\frac{1}{2}} + |x_1 - \xi_1| + |x_2 + (t-\tau)\xi_1 - \xi_2|^{\frac{1}{3}}\end{aligned}$$

so that, for fixed $z = (t,x) = 0$ *and, for example* $\zeta = (\tau,\xi)$ *along the line*

$$\begin{cases} \xi_1 = 1 \\ \xi_2 = -\tau \end{cases}$$

we have

$$\frac{|f(z) - f(\zeta)|}{\|\zeta^{-1} \circ_Y z\|_B^\alpha} = \frac{|\xi|}{\left(|\tau|^{\frac{1}{2}} + 1\right)^\alpha} = \frac{\sqrt{1+\tau^2}}{\left(|\tau|^{\frac{1}{2}} + 1\right)^\alpha} \longrightarrow \infty$$

when τ *tends to infinity along the line, for every* $\alpha \in]0,2[$.

Example 3.1.7. *As in the previous example, in* $\mathbb{R}^3$

$$\left\|(\tau,\xi)^{-1}\circ_Y (t,x)\right\|_B = |t-\tau|^{\frac{1}{2}} + |x_1-\xi_1| + |x_2+(t-\tau)\xi_1-\xi_2|^{\frac{1}{3}}.$$

Let

$$z = \left(t, -\frac{x_2-\xi_2}{t-\tau}, x_2\right), \qquad \zeta = \left(\tau, -\frac{x_2-\xi_2}{t-\tau}, \xi_2\right) \tag{3.1.7}$$

with $x_2, \xi_2 \in \mathbb{R}$ *and* $t \neq \tau$. *We notice that*

$$\zeta^{-1}\circ_Y z = (t-\tau, 0, 0).$$

and therefore

$$\left\|(\tau,\xi)^{-1}\circ_Y (t,x)\right\|_B = |t-\tau|^{\frac{1}{2}}$$

for any $x_2, \xi_2 \in \mathbb{R}$ *and* $t \neq \tau$*: in other terms, points that are very distant in the Euclidean sense, can be very close in the intrinsic sense. It follows that, if a function* $f(t,x_1,x_2) = f(x_2)$ *depends only on* x_2 *and belongs to* C^α_Y*, then it must be constant: indeed, for* z, ζ *as in* (3.1.7)*, we have*

$$|f(x_2)-f(\xi_2)| = |f(z)-f(\zeta)| \le C|t-\tau|^{\frac{1}{2}}$$

for any $x_2, \xi_2 \in \mathbb{R}$ *and* $t \neq \tau$.

3.1.2 Gaussian estimates

Given the Kolmogorov operator $\bar{K}$ in (1.0.1), for any fixed $w \in \mathbb{R}^{d+1}$ we denote by $\Gamma_w(t,x;\tau,\xi)$ the fundamental solution of the Kolmogorov operator with time-dependent coefficients $\bar{A}^{ij}_t(w) = A^{ij}_t(q(w))$.

$$\begin{aligned} K_w &:= \frac{1}{2}\sum_{i,j=1}^{m_0} \bar{A}^{ij}_t(w)\partial_{x_i x_j} - Y \\ &= \frac{1}{2}\sum_{i,j=1}^{m_0} \bar{A}^{ij}_t(w)\partial_{x_i x_j} + \langle Bx + b_t, D\rangle - \partial_t. \end{aligned}$$

The explicit expression of $\Gamma_w(t,x;\tau,\xi)$ is given in (3.1.3).

Notation 3.1.8. *Given* B *in the form* (1.1.1)*, we denote by* $\hat{B}$ *the matrix obtained by substituting the* $*$*-blocks with null blocks. We also set*

$$I := \begin{pmatrix} I_{m_0} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \hat{Y} = \partial_t - \langle \hat{B}x, D\rangle,$$

and, for $w \in \mathbb{R}^{d+1}$ *and* $0 \le \tau < t \le T$,

$$\begin{aligned} \mathcal{C}_{\tau,t,w} &= \int_\tau^t e^{(t-s)B}\bar{A}_s(w)e^{(t-s)B^*}ds, & \mathcal{C}_t &= \int_0^t e^{(t-s)B} I e^{(t-s)B^*}ds \\ \hat{\mathcal{C}}_{\tau,t,w} &= \int_\tau^t e^{(t-s)\hat{B}}\bar{A}_s(w)e^{(t-s)\hat{B}^*}ds, & \hat{\mathcal{C}}_t &= \int_0^t e^{(t-s)\hat{B}} I e^{(t-s)\hat{B}^*}ds. \end{aligned}$$

Remark 3.1.9. *Assumption 1.1.2 yields the following comparison between the quadratic forms associated to* $\mathcal{C}_{\tau,t,w}$ *and* $\mathcal{C}_{t-\tau}$

$$\mu^{-1}\mathcal{C}_{t-\tau} \le \mathcal{C}_{\tau,t,w} \le \mu\,\mathcal{C}_{t-\tau}$$

for $w \in \mathbb{R}^{d+1}$ *and* $0 \le t-\tau \le T$*. Since* $\mathcal{C}_{t-\tau}$ *is symmetric and positive definite for* $t > \tau$*, analogous estimates hold for* $\mathcal{C}^{-1}_{\tau,t,w}$*,* $\hat{\mathcal{C}}_{\tau,t,w}$ *and* $\hat{\mathcal{C}}^{-1}_{\tau,t,w}$ *in terms of* $\mathcal{C}^{-1}_{t-\tau}$*,* $\hat{\mathcal{C}}_{\tau,t}$ *and* $\hat{\mathcal{C}}^{-1}_{t-\tau}$ *respectively.*

Proposition 3.1.10. *For every* $z, \zeta, w \in \mathbb{R}^{d+1}$*, with* $0 < t-\tau \le T$ *it holds that*

$$\frac{1}{\mu^d}\Gamma^-(t,x;\tau,\xi) \le \Gamma_w(t,x;\tau,\xi) \le \mu^d\Gamma^+(t,x;\tau,\xi),$$

where μ *is the constant in Assumption 1.1.2 and* Γ^-, Γ^+ *are the fundamental solutions of the Kolmogorov operators*

$$\bar{K}^- = \frac{1}{2\mu}\sum_{i=1}^{m_0}\partial_{x_ix_i} - Y, \qquad \bar{K}^+ = \frac{\mu}{2}\sum_{i=1}^{m_0}\partial_{x_ix_i} - Y,$$

respectively.

Proof. By Remark 0.0.1, we have

$$\det\mathcal{C}_{\tau,t,w} \ge \mu^{-d}\det\mathcal{C}_{t-\tau}, \qquad \exp\left(-\frac{1}{2}\langle\mathcal{C}^{-1}_{\tau,t,w}\eta,\eta\rangle\right) \le \exp\left(-\frac{1}{2}\langle\mathcal{C}^{-1}_{t-\tau}\eta,\eta\rangle\right)$$

for any $t > \tau$ and $\eta \in \mathbb{R}^d$. Given (t,x), $(\tau,\xi) \in \mathbb{R}^{d+1}$, for convenience, we set $\eta = \pi_d((\tau,\xi)^{-1} \circ_Y (t,x))$ and $c_d = (2\pi)^{-d/2}$. Then we have:

$$\begin{aligned}\Gamma_w(t,x;\tau,\xi) &= \frac{c_d e^{-(t-\tau)\mathrm{Tr}B}}{\sqrt{\det\mathcal{C}_{\tau,t,w}}}\exp\left(-\frac{1}{2}\langle\mathcal{C}^{-1}_{\tau,t,w}\eta,\eta\rangle\right)\\ &\le \mu^{d/2}\frac{c_d e^{-(t-\tau)\mathrm{Tr}B}}{\sqrt{\det\mathcal{C}_{t-\tau}}}\exp\left(-\frac{1}{2\mu}\langle\mathcal{C}^{-1}_{t-\tau}\eta,\eta\rangle\right) = \mu^d\Gamma^+(t,x;\tau,\xi).\end{aligned}$$

The other inequality is analogous. □

Lemma 3.1.11. *We have*

$$\hat{\mathcal{C}}_t = \mathcal{D}_0(\sqrt{t})\hat{\mathcal{C}}_1\mathcal{D}_0(\sqrt{t}), \qquad \hat{\mathcal{C}}_t^{-1} = \mathcal{D}_0\left(\frac{1}{\sqrt{t}}\right)\hat{\mathcal{C}}_1^{-1}\mathcal{D}_0\left(\frac{1}{\sqrt{t}}\right)$$

Proof. See Proposition 2.3 in [16].

The next lemma is proved following the argumets in [16].

Lemma 3.1.12. *There exists a positive constant* C*, only dependent on the general constants (*μ*,* B*,* $\|\bar{A}\|_\infty$*) and* T*, such that*

$$(1 - C(t-\tau))\hat{\mathcal{C}}_{\tau,t,w} \le \mathcal{C}_{\tau,t,w} \le (1 + C(t-\tau))\hat{\mathcal{C}}_{\tau,t,w} \tag{3.1.8}$$

for any $w \in \mathbb{R}^{d+1}$ *and* $0 < t-\tau \le T$*.*

Proof. Let $t > 0$ and $s \in]0, t[$.

$$e^{(t-s)B} = \sum_{k=0}^{\nu} \frac{(t-s)^k B^k}{k!} + O((t-s)^{\nu+1}) \text{ as } (t-s) \to 0.$$

Then we have

$$\begin{aligned} e^{(t-s)B} A_s(w) e^{(t-s)B^*} &= \sum_{k_1,k_2=0}^{\infty} \frac{(t-s)^{k_1+k_2}}{k_1!k_2!} B^{k_1} A_s(w)(B^*)^{k_2} \\ &= \sum_{n=0}^{\infty} (t-s)^n \sum_{k=0}^{n} \frac{1}{k!(n-k)!} B^k A_s(w)(B^*)^{n-k} \\ &= \sum_{n=0}^{2\nu} \frac{(t-s)^n}{n!} F_n + O((t-s)^{2\nu+1}) \text{ as } (t-s) \to 0. \end{aligned}$$

where

$$F_n = \sum_{k=0}^{n} \binom{n}{k} B^k A_s(w)(B^*)^{n-k}$$

Let us study the block decomposition of F_n. We have

$$B^j A_s(y) = \begin{pmatrix} *(1)A_s^0(w) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ *(1)A_s^0(w) & 0 & \cdots & 0 \\ C_{j0} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $C_{j0} = B_j \cdots B_1 A_s^0(w)$ lies in the $(j+1)$-th row and has maximum rank.
The $*(1)$-blocks are obtained by multiplying j blocks of B whose at least one is a $*$-block. Note that when $K_t = K_t^0$ all the $*(1)$-blocks are null.

$$B^j A_s(w)(B^*)^i = \begin{pmatrix} *(2) & \cdots & *(2) & *(2) & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ *(2) & \cdots & *(2) & *(2) & 0 & \cdots & 0 \\ *(2) & \cdots & *(2) & C_{ji} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad (3.1.9)$$

where $C_{ji} = B_j \cdots B_1 A_s^0(w) B_1^* \cdots B_i^*$ lies in the (j,i) block of the partition, and each $*(2)$-block is obtained this way:

$$*(2) = (\sum_{l=1}^{j} *(1)_{k_l}) A_s^0(w) (\sum_{h=1}^{i} *(1)_{k_h})$$

As the $*(1)$ ones they are null when $K_t = K_t^0$. Note also that the dependence on s and y lies only in the factor $A_s^0(w)$.
By (3.1.9) we can deduce that the block partition $(F_n)_{ji},\ i,j = 0;\cdots m_0$ of the matrix F_n has the following properties:

i) if $j+i>n$ then the block (j,i) is null;

ii) if $j+i=n$ then $(F_n)_{ji} = B_j\cdots B_1 A_s^0(w) B_1^*\cdots B_i^*$ has maximum rank;

iii) if $j+i<n$ then $(F_n)_{ji}$ is a sum of $*(2)$ blocks (then is null for the operator K_t^0).

Hence, for every $0\le i,j\le \nu$,

$$\left(e^{(t-s)B}A_s(w)e^{(t-s)B^*}\right)_{ji} = \binom{k}{j} B_j\cdots B_i A_s^0(w) B_1\cdots B_i \frac{(t-s)^k}{k!}\cdot \\ \cdot\left(1+*_1^{(3)}(t-s)+\cdots+*_{2\nu-k}^{(3)}(t-s)^{2\nu-k}\right)+O((t-s)^{2\nu-k}) \qquad (3.1.10)$$

where $k=i+j$ and the $*_l^{(3)}$ blocks are sum of $*^{(2)}$ blocks and they have the same properties. Therefore, when $K_t = K_t^0$, (3.1.10) holds with every $*^{(3)}$ block equal to zero and without the remainder $O((t-s)^{2\nu+1})$, being $\hat{B}$ nilpotent. Thus we may eventually infer that

$$\frac{\langle(\mathcal{C}_{\tau,t,w}-\hat{\mathcal{C}}_{\tau,t,w})x,\ x\rangle}{\langle\hat{\mathcal{C}}_{\tau,t,w}x,\ x\rangle} = O(1)(t-\tau) \text{ for } (t-\tau)\to 0$$

which is equivalent to the statement we wanted to prove. □

Remark 3.1.13. *As a consequence of Lemma 3.1.8 and Proposition 3.1.10, there is $\delta>0$ only dependent on the general constants such that*

$$\frac{1}{2\mu}\hat{\mathcal{C}}_{t-\tau} \le \frac{1}{2}\hat{\mathcal{C}}_{\tau,t,w} \le \mathcal{C}_{\tau,t,w} \le 2\hat{\mathcal{C}}_{\tau,t,w} \le 2\mu\,\hat{\mathcal{C}}_{t-\tau}$$

for any $w\in\mathbb{R}^{d+1}$ and $0<t-\tau<\delta$. Analogous estimates also hold for $\mathcal{C}_{\tau,t,w}^{-1}$.

Remark 3.1.14. *For $0<t-\tau\le T$ there exist two positive constants C_1, C_2 only dependent on the general constants and T such that*

$$C_1(t-\tau)^Q \le \det\mathcal{C}_{\tau,t,\zeta} \le C_2(t-\tau)^Q \qquad (3.1.11)$$

Indeed, for $s=t-\tau<\delta$ as in Remark 3.1.13 we have

$$(2\mu)^{-d}s^Q\det\hat{\mathcal{C}}_1 \le \det\hat{\mathcal{C}}_{\tau,t,\zeta} \le (2\mu)^d s^Q \det\hat{\mathcal{C}}_1$$

On the other hand, for $\delta\le s\le T$ we can write

$$0<\frac{\mu^{-d}}{T^Q}\left(\min_{s\in[\delta,T]}\det\mathcal{C}_s\right) \le \frac{\mu^{-d}\det\mathcal{C}_s}{T^Q} \le \frac{\det\mathcal{C}_{\tau,t,\zeta}}{s^Q} \le \frac{\mu^{-d}\det\mathcal{C}_s}{\delta^Q} \le \frac{\mu^d}{\delta^Q}\left(\max_{s\in[\delta,T]}\det\mathcal{C}_s\right) < +\infty$$

Then (3.1.11) directly follows.

Lemma 3.1.15. *For every $T > 0$, there exists a positive constant C, only dependent on μ, B and T, such that*

$$\left|\left(\mathcal{C}_{\tau,t,w}^{-1}y\right)_i\right| \leq \frac{C}{\sqrt{t-\tau}}\left|\mathcal{D}_0\left(\frac{1}{\sqrt{t-\tau}}\right)y\right|, \tag{3.1.12}$$

$$\left|\left(\mathcal{C}_{\tau,t,w}^{-1}\right)_{ij}\right| \leq \frac{C}{t-\tau} \tag{3.1.13}$$

for every $i, j = 1, \cdots, m_0,\ t \in]\tau, T],\ y \in \mathbb{R}^d$.

Proof. Let δ as in Remark 3.1.13 and let $t \in]\tau, \delta]$. Recall that $\left(\mathcal{D}_0(\lambda)y\right)_i = \lambda y_i$ for $i = 1, \cdots m_0$. Then we have

$$\begin{aligned}
\left|\left(\mathcal{C}_{\tau,t,w}^{-1}y\right)_i\right| &\leq \left|\left(\left(\mathcal{C}_{\tau,t,w}^{-1} - \hat{\mathcal{C}}_{\tau,t,w}^{-1}\right)y\right)_i\right| + \left|\left(\hat{\mathcal{C}}_{\tau,t,w}^{-1}y\right)_i\right| = \\
&= \frac{1}{\sqrt{t-\tau}}\left|\left(\mathcal{D}_0\left(\sqrt{t-\tau}\right)\left(\mathcal{C}_{\tau,t,w}^{-1} - \hat{\mathcal{C}}_{\tau,t,w}^{-1}\right)\mathcal{D}_0\left(\sqrt{t-\tau}\right)\mathcal{D}_0\left(\frac{1}{\sqrt{t-\tau}}\right)y\right)_i\right| + \\
&+ \frac{1}{\sqrt{t-\tau}}\left|\left(\mathcal{D}_0\left(\sqrt{t-\tau}\right)\hat{\mathcal{C}}_{\tau,t,w}^{-1}\mathcal{D}_0\left(\sqrt{t-\tau}\right)\mathcal{D}_0\left(\frac{1}{\sqrt{t-\tau}}\right)y\right)_i\right| = \\
&= \mathrm{I}_1 + \mathrm{I}_2
\end{aligned}$$

We note that

$$\begin{aligned}
&\left\|\mathcal{D}_0\left(\sqrt{t-\tau}\right)\hat{\mathcal{C}}_{\tau,t,w}^{-1}\mathcal{D}_0\left(\sqrt{t-\tau}\right)\right\| \\
&\quad = \sup_{|\xi|=1}\left|\langle\hat{\mathcal{C}}_{\tau,t,w}^{-1}\mathcal{D}_0\left(\sqrt{t-\tau}\right)\xi, \mathcal{D}_0\left(\sqrt{t-\tau}\right)\xi\rangle\right| \\
&\quad \leq \mu\sup_{|\xi|=1}\left|\langle\hat{\mathcal{C}}_{t-\tau}^{-1}\mathcal{D}_0\left(\sqrt{t-\tau}\right)\xi, \mathcal{D}_0\left(\sqrt{t-\tau}\right)\xi\rangle\right| \\
&\quad = \mu\sup_{|\xi|=1}\left|\langle\hat{\mathcal{C}}_1^{-1}\xi, \xi\rangle\right| = \mu\left\|\hat{\mathcal{C}}_1^{-1}\right\|
\end{aligned}$$

by Lemma 3.1.11. Therefore:

$$\begin{aligned}
\mathrm{I}_2 &\leq \frac{1}{\sqrt{t-\tau}}\left\|\mathcal{D}_0\left(\sqrt{t-\tau}\right)\hat{\mathcal{C}}_{\tau,t,w}^{-1}\mathcal{D}_0\left(\sqrt{t-\tau}\right)\right\|\left|\mathcal{D}_0\left(\frac{1}{\sqrt{t-\tau}}\right)y\right| \\
&\leq \frac{\mu}{\sqrt{t-\tau}}\left\|\hat{\mathcal{C}}_1^{-1}\right\|\left|\mathcal{D}_0\left(\frac{1}{\sqrt{t-\tau}}\right)y\right|
\end{aligned}$$

On the other hand we have

$$\left\|\mathcal{D}_0\left(\sqrt{t-\tau}\right)\left(\mathcal{C}_{\tau,t,w}^{-1} - \hat{\mathcal{C}}_{\tau,t,w}^{-1}\right)\mathcal{D}_0\left(\sqrt{t-\tau}\right)\right\| \leq \left\|\mathcal{D}_0\left(\sqrt{t-\tau}\right)\hat{\mathcal{C}}_{\tau,t,w}^{-1}\mathcal{D}_0\left(\sqrt{t-\tau}\right)\right\|$$

by Remark 3.1.13 since $t \in]\tau, \delta]$. Then we also get:

$$\mathrm{I}_1 \leq \frac{\mu}{\sqrt{t-\tau}}\left\|\hat{\mathcal{C}}_1^{-1}\right\|\left|\mathcal{D}_0\left(\frac{1}{\sqrt{t-\tau}}\right)y\right|$$

Assume now that $t \in [\delta, T]$:

$$\begin{aligned}\left|\left(\mathcal{C}_{\tau,t,w}^{-1} z\right)_i\right| &\leq \frac{1}{\sqrt{t-\tau}} \left|\left(\mathcal{D}_0\left(\sqrt{t-\tau}\right)\mathcal{C}_{\tau,t,w}^{-1}\mathcal{D}_0\left(\sqrt{t-\tau}\right)\mathcal{D}_0\left(\frac{1}{\sqrt{t-\tau}}\right) y\right)_i\right| \\ &\leq \frac{\mu}{\sqrt{t-\tau}} \sup_{\delta\leq t\leq T} \left\|\mathcal{D}_0\left(\sqrt{t-\tau}\right)\mathcal{C}_{t-\tau}^{-1}\mathcal{D}_0\left(\sqrt{t-\tau}\right)\right\| \left|\mathcal{D}_0\left(\frac{1}{\sqrt{t-\tau}}\right) y\right|\end{aligned}$$

This completes the proof for (3.1.12). (3.1.13) follows as a consequence. Indeed

$$\left|\left(\mathcal{C}_{\tau,t,w}^{-1}\right)_{ij}\right| = \left|\left(\mathcal{C}_{\tau,t,w}^{-1} e_j\right)_i\right| \leq \frac{C}{\sqrt{t-\tau}} \left|\mathcal{D}_0\left(\frac{1}{\sqrt{t-\tau}}\right) e_j\right| = \frac{C}{t-\tau}$$

□

Proposition 3.1.16. *Given $\varepsilon > 0$ and a polynomial function p, there exists a constant C, only dependent on ε, μ, B and p such that, if we set $(s,\omega) = (\tau,\xi)^{-1} \circ_Y (t,x)$ and $\eta = \mathcal{D}_0(\frac{1}{\sqrt{s}})(\omega)$ then we have*

$$|p(|\eta|)|\Gamma_w(t,x;\tau,\xi) \leq C\Gamma^\varepsilon(t,x;\tau,\xi) \tag{3.1.14}$$

for any $z,\zeta,w \in \mathbb{R}^{d+1}$, where Γ^ε denotes the fundamental solution of the Kolmogorov operator:

$$\bar{K}^\varepsilon = \frac{\mu+\varepsilon}{2}\sum_{i=1}^{m_0}\partial_{x_i x_i} - Y.$$

Proof. By Lemma 3.1.12 we may consider $t_0 > 0$ such that (3.1.8) holds and

$$(1 - C_0 t_0)^2 \geq \frac{\mu + \frac{\varepsilon}{2}}{\mu + \varepsilon}$$

were C_0 is the constant in (3.1.8). We first prove (3.1.14) for $s \in [0, t_0]$.
By Remark 3.1.9 we have

$$\begin{aligned}|p(|\eta|)|\Gamma_w(t,x;\tau,\xi) &\leq |p(|\eta|)|\frac{c_d \mu^{\frac{d}{2}} e^{-s\mathrm{Tr}B}}{\sqrt{\det \mathcal{C}_s}} \exp\left(-\frac{1}{2\mu}\langle \mathcal{C}_s^{-1}\omega,\omega\rangle\right) \\ &\leq |p(|\eta|)|\frac{c_d \mu^{\frac{d}{2}} e^{-s\mathrm{Tr}B}}{\sqrt{\det \mathcal{C}_s}} \exp\left(-\frac{(1-C_0t_0)}{2\mu}\langle \hat{\mathcal{C}}_1^{-1}\eta,\eta\rangle\right) \leq\end{aligned}$$

(applying Lemmas 3.1.12 and 3.1.11)

$$\begin{aligned}&\leq \frac{Ce^{-s\mathrm{Tr}B}}{\sqrt{\det \mathcal{C}_s}} \exp\left(-\frac{(1-C_0t_0)}{2\left(\mu+\frac{\varepsilon}{2}\right)}\langle \hat{\mathcal{C}}_1^{-1}\eta,\eta\rangle\right) \\ &\leq \frac{Ce^{-s\mathrm{Tr}B}}{\sqrt{\det \mathcal{C}_{t-\tau}}} \exp\left(-\frac{(1-C_0t_0)^2}{2\left(\mu+\frac{\varepsilon}{2}\right)}\langle \mathcal{C}_s^{-1}\omega,\omega\rangle\right) \leq\end{aligned}$$

(applying Lemmas 3.1.12 and 3.1.11 again)

$$\leq \frac{Ce^{-s\mathrm{Tr}B}}{\sqrt{\det \mathcal{C}_s}} \exp\left(-\frac{1}{2\left(\mu+\varepsilon\right)}\langle \mathcal{C}_s^{-1}\omega,\omega\rangle\right) \leq$$

(noting that $(\mu+\varepsilon)\mathcal{C}_{t-\tau}$ is the covariance matrix for the operator $\bar{K}^{\varepsilon}$)

$$\leq C_1\Gamma^{\varepsilon}(s,\omega;0,0) = C_1\Gamma^{\varepsilon}(t,x;\tau,\xi).$$

Now consider the case $s \geq t_0$. This yields $|\eta| \leq c|\omega|$, then by Proposition 3.1.10:

$$|p(|\eta|)|\,\Gamma_w(t,x;\tau,\xi) \leq C\,|p(|\omega|)|\,\Gamma^{+}(t,x;\tau,\xi)$$

where C is only dependent of μ, B and ε. Then

$$\begin{aligned} |p(|\eta|)|\,\Gamma_w(t,x;\tau,\xi) &\leq C\,|p(|\omega|)|\,\frac{c_d\mu^{\frac{d}{2}}}{\sqrt{\det\hat{\mathcal{C}}_s}}\exp\left(\frac{1}{2\mu}\langle\hat{\mathcal{C}}_s^{-1}\omega,\omega\rangle\right)\\ &\leq C_1\frac{c_d\mu^{\frac{d}{2}}}{\sqrt{\det\hat{\mathcal{C}}_s}}\exp\left(\frac{1}{2(\mu+\varepsilon)}\langle\hat{\mathcal{C}}_s^{-1}\omega,\omega\rangle\right)\end{aligned}$$

with C_1 also dependent of p

$$= C_2\Gamma^{\varepsilon}(s,\omega;0,0) = C_2\Gamma^{\varepsilon}(t,x;\tau,\xi).$$

□

Proposition 3.1.17. *For every $\varepsilon > 0$ and $T > 0$, there exists a positive constant C, only dependent on μ, B, ε and T, such that*

$$\begin{aligned} |\partial_{x_i}\Gamma_\theta(t,x;\tau,\xi)| &\leq \frac{C}{\sqrt{t-\tau}}\Gamma^{\varepsilon}(t,x;\tau,\xi),\\ \left|\partial_{x_ix_j}\Gamma_\theta(t,x;\tau,\xi)\right| &\leq \frac{C}{t-\tau}\Gamma^{\varepsilon}(t,x;\tau,\xi),\end{aligned}$$

for every x, $\xi, \in \mathbb{R}^d$, $0 < t-\tau < T$, $\theta \in \mathbb{R}^{d+1}$, and for every i, $j = 1,\cdots,m_0$.

Proof. Let $(s,\omega) = (\tau,\xi)^{-1}\circ_Y (t,x)$, then by (3.1.6) we have

$$\begin{aligned}\partial_{x_i}\Gamma_\theta(t,x;\tau,\xi) &= \partial_{x_i}\Gamma^{\text{heat}}\left(\mathcal{C}_{\tau,t,\theta},\omega\right)e^{-s\text{Tr}B}\\ &= -\frac{1}{2}\left(\mathcal{C}^{-1}_{\tau,t,\theta}\omega\right)_i\Gamma^{\text{heat}}\left(\mathcal{C}_{\tau,t,\theta},\omega\right)e^{-s\text{Tr}B}\end{aligned}$$

Finally, by Propositions 3.1.16 and 3.1.17:

$$\begin{aligned}|\partial_{x_i}\Gamma_\theta(t,x;\tau,\xi)| &= \frac{1}{2}\left|\left(\mathcal{C}^{-1}_{\tau,t,\theta}\omega\right)_i\right|\Gamma_\theta(t,x;\tau,\xi)\\ &\leq \frac{C}{\sqrt{s}}\left|\left(\mathcal{D}\left(\frac{1}{\sqrt{s}}\right)\omega\right)\right|\Gamma_\theta(t,x;\tau,\xi)\\ &\leq \frac{c}{\sqrt{t-\tau}}\Gamma^{\varepsilon}(t,x;\tau,\xi)\end{aligned}$$

The other estimate uses (3.1.13) and is analogous. □

3.2 Introduction to the parametrix

In this section we construct a fundamental solution Γ for the operator $\bar{K}$ in (1.0.1) using the so-called *parametrix method*, under Assumptions 1.1.1, 1.1.2 and 1.1.3. The parametrix method goes back to Levi [17] for elliptic equations and was first used by Dressel [6] to construct the fundamental solution of uniformly parabolic equations. The idea is that we start with a parametrix (principal part or leading term of the approximation)

$$Z(z;\zeta) := \Gamma_\zeta(z;\zeta)$$

where for convenience we set $z = (t, x)$ and $\zeta = (\tau, \xi)$. According to Levi's method, we look for the fundamental solution Γ in the form

$$\Gamma(z;\zeta) = Z(z;\zeta) + \int_{S_{\tau,t}} Z(z;w)\varphi(w;\zeta)dw, \quad S_{\tau,t} = \mathbb{R}^d \times]\tau, t[.$$

Then we put $\bar{K}\Gamma(z;\zeta) = 0$ and we are left with an integral equation to determine $\varphi(z;\zeta)$. By the method of successive approximations, we find

$$\varphi(z;\zeta) = \sum_{k=1}^{+\infty} (\bar{K}Z)_k(z;\zeta), \tag{3.2.1}$$

with

$$(\bar{K}Z)_1(z;\zeta) = \bar{K}Z(z;\zeta),$$
$$(\bar{K}Z)_{k+1}(z;\zeta) = \int_{S_{\tau,t}} \bar{K}Z(z;w)(\bar{K}Z)_k(w;\zeta)dw, \quad k \in \mathbb{N}.$$

N.B. As already seen for the time-dependent coefficients case (see Lemma 3.1.1), the equalities written above are not to be intended pointwise (indeed, generally, Z is not differentiable in t, nor in x_j, for $j = m_0 + 1, \cdots, m_0$). However we can still write $\bar{K}Z$ meaning:
$\bar{K}Z = \bar{L}Z - YZ$ where YZ indicates the Lie derivative

$$YZ(t,x) := \frac{d}{ds} Z(\gamma(s))_{|s=0}$$

where γ is the integral curve of Y starting from (t, x), wich is defined a.e.

The rest of the Section is devoted the the proof of the following.

Proposition 3.2.1. *There exists $k_0 \in \mathbb{N}$ such that,*

(i) The function $(KZ)_k(\cdot;\zeta)$ is $L^\infty(S_{\tau,T})$ for $k \geq k_0$, $T > \tau$.

(ii) The series

$$\sum_{k=k_0}^{\infty} (KZ)_k(\cdot;\zeta)$$

converges in $L^\infty(S_{\tau,T})$.

(iii) The function φ, as defined in (3.2.1) *solves the integral equation*

$$\varphi(z,\zeta) = (KZ)(z;\zeta) + \int_\tau^t \int_{\mathbb{R}^d} (KZ)(z;s,y)\varphi(s,y;\zeta)dyds \tag{3.2.2}$$

a.e. in $S_{\tau,T}$.

Lemma 3.2.2. *Let $\varepsilon > 0$ and $T > 0$. It holds that*

$$\left|(\bar{K}Z)_k(t,x;\tau,\xi)\right| \le \frac{M_k}{(t-\tau)^{1-\alpha k/2}}\Gamma^\varepsilon(t,x;\tau,\xi) \tag{3.2.3}$$

a.e. in $S_{\tau,\tau+T}$ for every $k \in \mathbb{N}$, $\zeta \in \mathbb{R}^{d+1}$ where

$$M_k = C^k \frac{\Gamma_E^k\left(\frac{\alpha}{2}\right)}{\Gamma_E\left(\frac{\alpha k}{2}\right)},$$

Γ_E is the Euler Gamma function and C is a positive constant only dependent on ε, T, μ, B and the L^∞-norm of the coefficients.

Proof. As usual, we set $z = (t,x)$ and $\zeta = (\tau,\xi)$. The estimate is proved by an inductive argument. For $k = 1$ and $z \neq \zeta$, we have

$$\left|\bar{K}Z(z;\zeta)\right| = \left|(\bar{K} - K_\zeta)Z(z;\zeta)\right| \le I_1 + I_2 + I_3$$

where

$$I_1 = \frac{1}{2}\sum_{i,j=1}^{m_0}\left|\bar{A}_t^{ij}(z) - \bar{A}_t^{ij}(\zeta)\right|\left|\partial_{x_i x_j} Z(z;\zeta)\right|, \quad I_2 = \sum_{i=1}^{m_0}\left|a_t^i(x)\right|\left|\partial_{x_i} Z(z;\zeta)\right|, \quad I_3 = \left|c_t(x)Z(z;\zeta)\right|.$$

We study I_1 first: by Assumption 1.1.3 we have

$$|\bar{A}_t^{ij}(z) - \bar{A}_t^{ij}(\zeta)| \le C\|(\tau,\xi)\circ_Y (t,x)\|^\alpha = C(t-\tau)^{\frac{\alpha}{2}}\|(1,\eta)\|^\alpha$$

where $\eta = \mathcal{D}_0\left(\frac{1}{\sqrt{t-\tau}}\right)(x - \gamma_{\tau,t}(\xi))$.
Hence, by Proposition 3.1.17 we infer

$$I_1 \le c_1\|(1,\eta)\|^\alpha \frac{\Gamma^{\varepsilon/2}(z;\zeta)}{(t-\tau)^{1-\alpha/2}} \le C_1\frac{\Gamma^\varepsilon(z;\zeta)}{(t-\tau)^{1-\alpha/2}}$$

Since the coefficients are bounded function and by Proposition 3.1.17 we also have

$$I_2 \le m_0\|\bar{A}\|_\infty c_2\frac{\Gamma^\varepsilon(z;\zeta)}{\sqrt{t-\tau}} \le C_2 T^{1-(\alpha+1)/2}\frac{\Gamma^\varepsilon(z;\zeta)}{(t-\tau)^{1-\alpha/2}}$$
$$I_3 \le \|c\|_\infty c_3\Gamma^\varepsilon(z;\zeta) \le C_3 T^{1-\alpha/2}\frac{\Gamma^\varepsilon(z;\zeta)}{(t-\tau)^{1-\alpha/2}}$$

This concludes the proof for $k = 1$.

We now assume that (3.2.3) holds for k and prove it for $k+1$. We have

$$\begin{aligned}\left|(\bar{K}Z)_{k+1}(z;\zeta)\right| &= \left|\int_\tau^t \int_{\mathbb{R}^d} \bar{K}Z(t,x;s,y)(\bar{K}Z)_k(s,y;\tau,\xi)dyds\right| \\ &\leq \int_\tau^t \frac{M_1}{(t-s)^{1-\alpha/2}} \frac{M_k}{(s-\tau)^{1-\alpha k/2}} \int_{\mathbb{R}^d} \Gamma^\varepsilon(t,x;s,y)\Gamma^\varepsilon(s,y;\tau,\xi)dyds\end{aligned}$$

(by the inductive hypothesis)

$$\leq \Gamma^\varepsilon(x,t;\tau,\xi) \int_\tau^t \frac{M_1}{(t-s)^{1-\alpha/2}} \frac{M_k}{(s-\tau)^{1-\alpha k/2}} ds$$

(by the reproduction property for Γ^ε)

$$= \frac{M_1\, M_k}{(t-\tau)^{1-(k+1)\alpha/2}} \Gamma^\varepsilon(z;\zeta) \int_0^1 \frac{dr}{(1-r)^{1-\alpha/2} r^{1-\alpha k/2}}$$

by the change of variable $s = \tau + r(t-\tau)$.
Note that the integral above is the Beta function $B(\frac{\alpha k}{2}, \frac{\alpha}{2})$ which is related to the Euler Gamma by the equality: $\Gamma_E(x+y)B(x,y) = \Gamma_E(x)\Gamma_E(y)$. Then

$$\int_0^1 \frac{dr}{(1-r)^{1-\alpha/2} r^{1-\alpha k/2}} = \frac{\Gamma_E(\frac{k\alpha}{2})\Gamma_E(\frac{\alpha}{2})}{\Gamma_E(\frac{(k+1)\alpha}{2})}$$

and this concludes the proof. □

Proof of Proposition 3.2.1. (i) By Lemma 3.2.2 and Remark 3.1.14 we have

$$\begin{aligned}\left|(\bar{K}Z)_k(z;\zeta)\right| &\leq C \frac{M_k}{(t-\tau)^{1-\frac{\alpha k}{2}} \sqrt{\det \mathcal{C}_{t-\tau}}} \exp\left(-\frac{1}{2(\mu+\varepsilon)} \langle \mathcal{C}_{t-\tau}\omega, \omega\rangle\right) \\ &\leq C' M_k (t-\tau)^{\frac{\alpha k}{2}-1-\frac{Q}{2}}\end{aligned}$$

for a.e. $t \in]\tau, T[$, $x, \xi \in \mathbb{R}^d$, for some constant C'. Then it suffices $k_0 \geq \frac{Q+2}{\alpha}$.

(ii) By the previous result, noting that the power series

$$\sum_{k\geq 1} M_{k_0+k} s^k$$

has radius of convergence equal to infinity.

(iii) By construction, for a.e. $t \in]\tau, T[$, $x, \xi \in \mathbb{R}^d$ we have

$$\begin{aligned}&\int_\tau^t \int_{\mathbb{R}^d} (\bar{K}Z)(z;s,y)\varphi(s,y;\zeta)dyds \\ &\quad = \sum_{k\geq 1} \int_\tau^t \int_{\mathbb{R}^d} (\bar{K}Z)(z;s,y)(\bar{K}Z)_k(z;s,y)dyds \\ &\quad = \sum_{k\geq 1} (\bar{K}Z)_{k+1}(z;\zeta) = \varphi(z,\zeta) - (\bar{K}Z)(z;\zeta)\end{aligned}$$

□

A straightforward consequence of the Lemma 3.2.2 is the following

Corollary 3.2.3. *For every* $\zeta \in \mathbb{R}^{d+1}$ *and* $T > \tau$ *there exists a positive constant* C *such that*

$$|\varphi(z;\zeta)| \leq C \frac{\Gamma^{\varepsilon}(z;\zeta)}{(t-\tau)^{1-\frac{\alpha}{2}}} \qquad a.e. \text{ in } S_{\tau,T} \tag{3.2.4}$$

Corollary 3.2.4. *Let us denote*

$$J(z;\zeta) := \int_{S_{\tau,t}} Z(z;w)\varphi(w;\zeta)dw$$

the approssimation term in Levi's method. For every $\varepsilon > 0$ *and* $T > \tau$ *there exists a positive constant* C *such that*

$$|J(z;\zeta)| \leq C(t-\tau)^{\frac{\alpha}{2}} \Gamma^{\varepsilon}(z;\zeta) \tag{3.2.5}$$

in $S_{\tau,T}$ *and the fundamental solution* Γ *verifies:*

$$\Gamma(z;\zeta) \leq C\Gamma^{\varepsilon}(z;\zeta) \tag{3.2.6}$$

for any $\zeta \in \mathbb{R}^{d+1}$ *in* $S_{\tau,T}$.

Proof. By (3.2.3) and by the reproduction property of Γ^{ε} we have

$$\begin{aligned} |J(z;\zeta)| &\leq C \int_{S_{\tau,t}} \Gamma^{\varepsilon}(t,x;s,y) \frac{\Gamma^{\varepsilon}(s,y;\tau,\xi)}{(s-\tau)^{1-\frac{\alpha}{2}}} dyds \\ &= C\Gamma^{\varepsilon}(z;\zeta) \int_{\tau}^{t} \frac{ds}{(s-\tau)^{1-\frac{\alpha}{2}}} \end{aligned}$$

from wich (3.2.5) follows. Together with the estimate of Z in Proposition 3.1.10 this implies (3.2.6). □

3.3 Potential estimates

We consider the potential

$$V_f(z) = \int_{S_{T_0,t}} Z(z,\zeta)f(\zeta)d\zeta, \quad S_{T_0,t} =]T_0, t[\times \mathbb{R}^d, \tag{3.3.1}$$

where $f \in C(S_{T_0,T_1})$ satisfies the growth condition:

$$|f(t,x)| \leq Ce^{C|x|^2}, \quad \forall (t,x) \in S_{T_0,T_1}$$

and Z denotes the parametrix of (1.0.1).

In this section we are going to show some regularity properties of V_f, briefly discussing the main arguments and ideas used to prove them. Complete proofs for the following results are postponed at the end of the chapter as they can be skipped at a first reading.

We first note that the integral (3.3.1) is well posed, i.e. is convergent in the strip $S_{T_0,T}$ for some $T \in]T_0, T_1]$. Indeed, by the growth estimate for f and Proposition 3.1.10 we have

$$
\begin{aligned}
|V_f(t,x)| &\leq C_1 \int_{T_0}^{t} \int_{\mathbb{R}^d} \Gamma^+(t,x,\tau,\xi) e^{C|\xi|^2} d\xi d\tau \\
&\leq C_2 \int_{T_0}^{t} \int_{\mathbb{R}^d} \frac{1}{\sqrt{\det \mathcal{C}_{\tau,t}}} \exp\left(-\frac{1}{2\mu} \langle \mathcal{C}_{\tau,t}^{-1} \omega, \omega \rangle + C|\xi|^2 \right) d\xi d\tau
\end{aligned}
$$

(with the usual notation $(s,\omega) = \zeta^{-1} \circ_Y z = (t-\tau, x - \gamma_{\tau,t}(\xi))$)

$$
\leq C_3 \int_{T_0}^{t} \int_{\mathbb{R}^d} \exp\left(-\frac{|\eta|^2}{2\mu} + C \left| e^{(t-\tau)B} \left(x - \mathcal{C}_{\tau,t}^{\frac{1}{2}} \eta - \int_{\tau}^{t} e^{(s-\tau)B} b_s ds \right) \right|^2 \right) d\eta d\tau
$$

(by the change of variable $\eta = \mathcal{C}_{\tau,t}^{-\frac{1}{2}} \omega$)

$$
\leq C_4 (t - T_0) e^{C_4 |x|^2}
$$

for some positive constant C_4, assuming $t \in]T_0, T]$ with $T - T_0$ suitably small.

The first result we need is the following.

Proposition 3.3.1. *There exists $\partial_{x_i} V_f \in C(S_{T_0,T})$ for $i = 1, \ldots, m_0$ and it holds that*

$$
\partial_{x_i} V_f(t,x) = \int_{T_0}^{t} \int_{\mathbb{R}^d} \partial_{x_i} Z(t,x,\tau,\xi) f(\tau,\xi) d\xi d\tau \tag{3.3.2}
$$

By Proposition 3.1.17 we can prove the absolutely convergence of the integral in (3.3.2) using the above arguments. We can then prove (3.3.2) for

$$
V_{f,\delta}(t,x) = \int_{T_0}^{t-\delta} \int_{\mathbb{R}^d} Z(t,x,\tau,\xi) f(\tau,\xi) d\xi d\tau, \quad 0 < \delta < t - T_0, \tag{3.3.3}
$$

using Lebesque's dominated convergence and let $\delta \to 0$.

The next result concerns existence and continuity of the second order derivatives.

Proposition 3.3.2. *Let f a continuous function in the strip S_{T_0,T_1} verifying the growth condition 3.3 and the regularity condition*

$$
|f(t,x) - f(t,y)| \leq C \, |x - y|_B^{\beta} \tag{3.3.4}
$$

for all $x, y \in M$, $t \in]T_0, T_1[$ for any compact subset M of $\mathbb{R}^d$. Then there exists $\partial_{x_i x_j} V_f \in C(S_{T_0,T})$ for $i, j = 1, \ldots, m_0$ and it holds that

$$
\partial_{x_i x_j} V_f(t,x) = \int_{T_0}^{t} \int_{\mathbb{R}^d} \partial_{x_i x_j} Z(t,x,\tau,\xi) f(\tau,\xi) d\xi d\tau \tag{3.3.5}
$$

To prove the existence of the integral in (3.3.4) we rely on the regularity properties of f and Z, so the main idea is to split the integral in the spatial variable as follows:

$$\begin{aligned}&\int_{\mathbb{R}^d} \partial_{x_i x_j} Z(t,x,\tau,\xi) f(\tau,\xi) d\xi \\ &= \int_{\mathbb{R}^d} \partial_{x_i x_j} Z(t,x,\tau,\xi) \left(f(\tau,\xi) - f(\tau,y) \right) d\xi \\ &+ f(\tau,y) \int_{\mathbb{R}^d} \partial_{x_i x_j} \left(Z(t,x,\tau,\xi) - \Gamma_w(t,x,\tau,\xi) \right)_{|w=y} d\xi \\ &+ f(\tau,y) \int_{\mathbb{R}^d} \partial_{x_i x_j} \Gamma_y(t,x,\tau,\xi)_{|w=y} d\xi \\ &= I_1 + I_2 + I_3\end{aligned}$$

where τ is fixed in $]T_0, t[$, choosing $y = \gamma_{t,\tau}(\xi)$. This way, by Proposition 3.1.16 we could make up the singularity in $\tau = t$ in I_1 by

$$|f(t,x) - f(t,y)| \le C\, |x-y|_B^\beta \le C_1 (t-\tau)^{\beta/2} |\eta|_B^{\beta/2}$$

with the usual notation for η. Similarly, we can handle the singularity in I_2 by the regularity properties of Z. Thus a more in depth study of these properties is required (see Lemma A.0.2).

On the other hand, if we limit ourselves to integrate on a ball B_R centered in the origin, we can reduce I_3 to

$$-f(\tau,y) \sum_{k=1}^d \int_{\partial B_R} \partial_{x_i} \Gamma_w(t,x;\tau,\xi)_{|w=y} \left(e^{-(t-\tau)B} \right)_{kj} \nu_k d\sigma(\xi)$$

where ν is the outher normal to ∂B_R, by an integration by parts. The integral above can be treated as in Proposition 3.3.1. The convergence of the remaining integral on $\mathbb{R}^d \setminus B_R$ for a suitable $R > 0$ relies on the asymptotic behaviour of the Gaussian for $|x| \to \infty$.

Then, the actual proof of (3.3.4) will proceed similarly to the previous Proposition.

Now we state the last Proposition of this section.

Proposition 3.3.3. *Under the hypotheses of Proposition 3.3.2 there exists the derivative YV_f for a.e. $t \in]T_0, T[$, $x \in \mathbb{R}^d$ and it holds that*

$$YV(z) = \int_{S_{T_0,t}} YZ(z;\zeta) f(\zeta) d\zeta + f(z)$$

As for Proposition 3.3.1, we first consider the integral function (3.3.3), and by definition (see 3.2) we write the incremental ratio

$$\begin{aligned}\frac{V_\delta(\widetilde{\gamma}(s);\tau,\xi) - V_\delta(t,x;\tau,\xi)}{s} &= \int_\tau^{t-\delta} \int_{\mathbb{R}^d} \frac{Z(\widetilde{\gamma}(s);\tau,\xi) - Z(t,x;\tau,\xi)}{s} f(\tau,\xi) d\xi d\tau \\ &+ \frac{1}{s} \int_{t-\delta}^{t+s-\delta} \int_{\mathbb{R}^d} Z(\widetilde{\gamma}(s);\tau,\xi) f(\tau,\xi) d\xi d\tau = I_1 + I_2\end{aligned}$$

where $\widetilde{\gamma}$ is the integral curve of Y, starting from (t,x). Taking the limit for $s \to 0$ it turns out that the following equality holds.

$$YV_{\delta,f} = \int_{\tau}^{t-\delta}\int_{\mathbb{R}^d} YZ(t,x;\tau,\xi)f(\tau,\xi)d\xi d\tau + \int_{\mathbb{R}^d} Z(t,x;t-\delta,\xi)f(t-\delta,\xi)d\xi$$

From there we get to the thesis by taking $\delta \to 0$, using the fact that $Z(z;\zeta) = \Gamma_\zeta(z;\zeta)$ with Γ_ζ being the fundamental solution of the operator K_ζ.

3.4 Proof of the main Theorem

One more preliminary result is needed.

Lemma 3.4.1. *For every $\varepsilon > 0$ and $T > 0$ there exists a positive constant C such that*

$$|\varphi(t,x;\tau,\xi) - \varphi(t,y;\tau,\xi)| \leq C\frac{|x-y|_B^{\frac{\alpha}{2}}}{(t-\tau)^{1-\frac{\alpha}{4}}}\left(\Gamma^\varepsilon(t,x;\tau,\xi) + \Gamma^\varepsilon(t,y;\tau,\xi)\right) \qquad (3.4.1)$$

for any $(\tau,\xi) \in \mathbb{R}^{d+1}$, $x,y \in \mathbb{R}^d$, a.e $t \in]\tau,\tau+T]$.

Proof. Set $w = (t,y)$. By (3.2.4) we have

$$\left|\bar{K}Z(z;\zeta) - \bar{K}Z(w;\zeta)\right| \leq \frac{C}{(t-\tau)^{1-\frac{\alpha}{2}}}\left(\Gamma^\varepsilon(z;\zeta) + \Gamma^\varepsilon(w;\zeta)\right) \qquad \text{(a.e.)}$$

Thus, for $|x-y|_B \geq \sqrt{t-\tau}$ we get

$$\left|\bar{K}Z(z;\zeta) - \bar{K}Z(w;\zeta)\right| \leq C\frac{|x-y|_B^{\frac{\alpha}{2}}}{(t-\tau)^{1-\frac{\alpha}{4}}}\left(\Gamma^\varepsilon(z;\zeta) + \Gamma^\varepsilon(w;\zeta)\right) \qquad \text{(a.e.)}, \qquad (3.4.2)$$

In the case $|x-y|_B < \sqrt{t-\tau}$ we first prove the following preliminary estimates:

$$\begin{aligned}
&|Z(z;\zeta) - Z(w;\zeta)| \leq \frac{C}{\sqrt{t-\tau}}\Gamma^{\varepsilon/2}(z;\zeta),\\
&|\partial_{x_i}Z(z;\zeta) - \partial_{x_i}Z(w;\zeta)| \leq C\frac{|x-y|_B}{t-\tau}\Gamma^{\varepsilon/2}(z;\zeta),\\
&\left|\partial_{x_ix_j}Z(z;\zeta) - \partial_{x_ix_j}Z(w;\zeta)\right| \frac{|x-y|_B}{(t-\tau)^{\frac{3}{2}}}\Gamma^{\varepsilon/2}(z;\zeta).
\end{aligned} \qquad (3.4.3)$$

Consider the third estimate in 3.4.3. By the mean value theorem, we have

$$|\partial_{x_ix_j}Z(z,\zeta) - \partial_{x_ix_j}Z(w,\zeta)| \leq \max_{\rho\in[0,1]}\sum_{i,j=1}^{d}\left|\partial_{x_hx_ix_j}Z\left(t,x+\rho(x-y);\tau,\xi\right)(x-y)_h\right|$$

Denoting $(s,\omega) = (\tau,\xi)^{-1}\circ_Y (t,x)$ and $\mathcal{C} = \mathcal{C}_{\tau,t,\zeta}$,

$$\begin{aligned}
\partial_{x_hx_ix_j}Z(z;\zeta) &= Z(z;\zeta)\big(\mathcal{C}_{ih}^{-1}(\mathcal{C}^{-1}\omega)_j - (\mathcal{C}^{-1}\omega)_i\mathcal{C}_{hj}^{-1} - \mathcal{C}_{ij}^{-1}(\mathcal{C}^{-1}\omega)_h\\
&\quad + (\mathcal{C}^{-1}\omega)_h(\mathcal{C}^{-1}\omega)_i(\mathcal{C}^{-1}\omega)_j\big)\\
&= Z(z;\zeta)\left(a_h(\omega) + b_h(\omega) + c_h(\omega) + d_h(\omega)\right).
\end{aligned}$$

Put $\nu = x - y$, $\widetilde{\omega} = \omega + \rho\nu$. Then, by Lemma 3.1.15 we get

$$\left|\sum_{i,j=1}^{d} a_h(\widetilde{\omega})\nu_h\right| = \left|\sum_{i,j=1}^{d} \mathcal{C}_{ih}^{-1}\nu_h(\mathcal{C}^{-1}\widetilde{\omega})_j\right| = \left|(\mathcal{C}^{-1}\nu)_i\right| \left|(\mathcal{C}^{-1}\widetilde{\omega})_j\right|$$
$$\leq \left|\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)\nu\right| \left|\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)\widetilde{\omega}\right|$$

Since $\nu \leq \sqrt{s}$, we have $|\mathcal{D}_0(1/\sqrt{s})\nu| \leq C|\mathcal{D}_0(1/\sqrt{s})\nu|_B = C|\nu|_B/\sqrt{s}$, therefore

$$\left|\sum_{i,j=1}^{d} \nu_h a_h(\widetilde{\omega})\right| \leq C\frac{|\nu|_B|\widetilde{\eta}|}{s^{3/2}},$$

where $\widetilde{\eta} = \mathcal{D}_0(1/\sqrt{s})\widetilde{\omega}$. The same estimate holds for b_h and c_h. By the same arguments we also have

$$\left|\sum_{i,j=1}^{d} \nu_h d_h(\widetilde{\omega})\right| \leq C\frac{|\nu|_B|\widetilde{\eta}|^3}{s^{3/2}},$$

Collecting all the terms and using Proposition 3.1.17 we obtain

$$|\partial_{x_ix_j}Z(z,\zeta) - \partial_{x_ix_j}Z(w,\zeta)| \leq C\frac{|\nu|_B(|\widetilde{\eta}| + |\widetilde{\eta}|^3)}{s^{3/2}}Z(t, x + \bar{\rho}\nu; \tau, \xi)$$
$$\leq C\frac{|x-y|_B}{s^{3/2}}\Gamma^{\varepsilon/3}(t, x + \bar{\rho}\nu; \tau, \xi) \leq C\frac{|x-y|_B}{s^{3/2}}\Gamma^{\varepsilon/2}(t, x; \tau, \xi) \qquad (3.4.4)$$

This concludes the proof of the third inequality in (3.4.3) for $|x - y|_B \leq \sqrt{t - \tau}$. The first two can be proved similarly. Next we are going to deduce from (3.4.4) a similar estimate to (3.4.2).

$$\begin{aligned}
&|\bar{K}Z(z;\zeta) - \bar{K}Z(w;\zeta)| \\
&= \Big| \sum_{i,j=1}^{m_0} a_t^{ij}(x)\partial_{x_ix_j}Z(z;\zeta) + \sum_{i=1}^{m_0} a_t^i(x)\partial_{x_i}Z(z;\zeta) \\
&\quad - \sum_{i,j=1}^{m_0} a_t^{ij}(y)\partial_{x_ix_j}Z(w;\zeta) + \sum_{i=1}^{m_0} a_t^i(y)\partial_{x_i}Z(w;\zeta) \\
&\quad + YZ(z;\zeta) - YZ(w;\zeta) + c_t(x)Z(w;\zeta) - c_t(y)Z(z;\zeta) \\
&\quad + \bar{K}_\zeta Z(z;\zeta) - \bar{K}_\zeta Z(w;\zeta)\Big| \\
&\leq \sum_{i,j=1}^{m_0} \left|a_t^{ij}(x) - a_t^{ij}(y)\right| \left|\partial_{x_ix_j}Z(z;\zeta)\right| \\
&\quad + \sum_{i,j=1}^{m_0} \left|a_t^{ij}(x) - \bar{A}_t^{ij}(\zeta)\right| \left|\partial_{x_ix_j}Z(z,\zeta) - \partial_{x_ix_j}Z(w,\zeta)\right| \\
&\quad + \sum_{i=1}^{m_0} \left|a_t^i(x) - a_t^i(w)\right| |\partial_{x_i}Z(w;\zeta)| \\
&\quad + \sum_{i=1}^{m_0} \left|a_t^i(w)\right| |\partial_{x_i}Z(z,\zeta) - \partial_{x_i}Z(w,\zeta)| \\
&\quad + |c_t(x) - c_t(y)||Z(z;\zeta)| + |c_t(x)||Z(z;\zeta) - Z(w;\zeta)|
\end{aligned}$$

Note that we have $w^{-1} \circ_Y x = (0, x - y)$. Then, by Proposition 3.1.17, by (3.4.3) and the regularity properties of the coefficients we get

$$
\begin{aligned}
&|\bar{K}Z(z;\zeta) - \bar{K}Z(w;\zeta)| \\
&\quad \le C\Big(\frac{|x-y|_B^\alpha}{t-\tau}\Gamma^{\varepsilon/2}(w;\zeta) + \|\zeta \circ_Y z\|_B^\alpha \frac{|x-y|_B}{(t-\tau)^{3/2}}\Gamma^{\varepsilon/2}(z;\zeta) \\
&\quad + \frac{|x-y|_B^\alpha}{\sqrt{t-\tau}}\Gamma^{\varepsilon/2}(w;\zeta) + \frac{|x-y|_B}{(t-\tau)^{3/2}}\Gamma^{\varepsilon/2}(z;\zeta) \\
&\quad + |x-y|_B^\alpha \Gamma^{\varepsilon/2}(w;\zeta) + \frac{|x-y|_B}{\sqrt{t-\tau}}\Gamma^{\varepsilon/2}(z;\zeta)\Big)
\end{aligned}
$$

Since

$$
\|\zeta^{-1} \circ_Y z\|_B^\alpha \le (t-\tau)^{\frac{\alpha}{2}} \left(1 + \left| \mathcal{D}_0\left(\frac{1}{\sqrt{t-\tau}}\right)(x - \gamma_{\tau,t}(\xi))\right|_B\right)^\alpha
$$

we may adapt Proposition 3.1.16 to deduce

$$
|\bar{K}Z(z;\zeta) - \bar{K}Z(w;\zeta)| \le C\left(\frac{|x-y|_B}{(t-\tau)^{3/2}} + \frac{|x-y|_B^\alpha}{t-\tau}\right)(\Gamma^\varepsilon(z;\zeta) + \Gamma^\varepsilon(w;\zeta)) \tag{3.4.5}
$$

On the other hand, if $|x-y|_B \le \sqrt{t-\tau}$, it holds that

$$
\begin{aligned}
&\frac{|x-y|_B}{(t-\tau)^{(3-\alpha)/2}} + \frac{|x-y|_B^\alpha}{t-\tau} \le \qquad (3.4.6)\\
&\quad \le \frac{|x-y|_B}{(t-\tau)^{(3-\alpha)/2}}\left(\frac{|x-y|_B}{\sqrt{t-\tau}}\right)^{-1+\frac{\alpha}{2}} + \frac{|x-y|_B^\alpha}{t-\tau}\left(\frac{|x-y|_B}{\sqrt{t-\tau}}\right)^{-\frac{\alpha}{2}} \\
&\quad = 2\frac{|x-y|_B^{\alpha/2}}{(t-\tau)^{1-\alpha/2}}
\end{aligned}
$$

Combining (3.4.2), (3.4.5) and (3.4.6), we finally get

$$
\left|\bar{K}Z(z;\zeta) - \bar{K}Z(w;\zeta)\right| \le C\frac{|x-y|_B^{\frac{\alpha}{2}}}{(t-\tau)^{1-\frac{\alpha}{4}}}(\Gamma^\varepsilon(z;\zeta) + \Gamma^\varepsilon(w;\zeta)) \quad \text{(a.e)} \tag{3.4.7}
$$

Let M_1 be the constant in (3.2.3) such that

$$
\left|(\bar{K}Z)(t,x;\tau,\xi)\right| \le \frac{M_1}{(t-\tau)^{1-\alpha/2}}\Gamma^\varepsilon(t,x;\tau,\xi) \qquad \text{(a.e)}
$$

Then, by (3.4.7) we can prove by induction (similarly to Lemma 3.2.2) that

$$
\left|(\bar{K}Z)_k(z;\zeta) - (\bar{K}Z)_k(w;\zeta)\right| \le \widetilde{M}_k \frac{|x-y|_B^{\frac{\alpha}{2}}}{(t-\tau)^{1-\frac{\alpha}{4}}}(\Gamma^\varepsilon(z;\zeta) + \Gamma^\varepsilon(w;\zeta)) M_1^k (t-\tau)^k \quad \text{'a.e'}
$$

where

$$
\widetilde{M}_k = C_0 \Gamma_E^k\left(\frac{\alpha}{2}\right)\frac{\Gamma_E^k\left(\frac{\alpha}{4}\right)}{\Gamma_E^k\left(\frac{\alpha}{2}\left(k+\frac{1}{2}\right)\right)}
$$

for some positive constant C_0. Therefore, by Remark 3.1.14 we get

$$\left|(\bar{K}Z)_k(z;\zeta) - (\bar{K}Z)_k(w;\zeta)\right| \leq C\widetilde{M}_k \frac{|x-y|_B^{\frac{\alpha}{2}}}{(t-\tau)^{1-\frac{\alpha}{4}-k-\frac{Q}{2}}} \quad (a.e)$$

arguing as for Proposition 3.2.1, items (i).
Since $1 - \frac{\alpha}{4} - k - \frac{Q}{2} < 0$ for $k \geq 1$, the thesis follows since the power series $\sum_{k\geq 1} \widetilde{M}_k s^k$ has radius of convergence equal to infinity. □

Proof of Theorem 3.0.1. Let Γ be the function defined in Section 3.2:

$$\Gamma(z;\zeta) = Z(z;\zeta) + \int_{S_{\tau,t}} Z(z;w)\varphi(w;\zeta)dw \tag{3.4.8}$$

(i) By Corollary 3.2.4 and Proposition 3.2.1 we may infer that $\Gamma(\cdot;\zeta) \in L^1_{\text{loc}}(\mathbb{R}^{d+1}) \cap C(\mathbb{R}^{d+1} \setminus \{\zeta\})$ $\forall \zeta \in \mathbb{R}^{d+1}$;

(ii) Thanks to estimate (3.2.4) and Lemma 3.4.1 we can apply Propositions 3.3.1, 3.3.2 to conclude that the following derivatives exist and are continuous functions for $z \neq \zeta$:

$$\partial_{x_i}\Gamma(z;\zeta) = \partial_{x_i}Z(z;\zeta) + \int_{S_{\tau,t}} \partial_{x_i}Z(z;w)\varphi(w,\zeta)dw,$$
$$\partial_{x_ix_j}\Gamma(z;\zeta) = \partial_{x_ix_j}Z(z;\zeta) + \int_{S_{\tau,t}} \partial_{x_ix_j}Z(z;w)\varphi(w,\zeta)dw$$

for every $i,j = 1,\dots,m_0$. Morover, by Proposition 3.3.3 the following derivative is well defined a.e.

$$Y\Gamma(z;\zeta) = \partial_{x_i}Z(z;\zeta) + \int_{S_{\tau,t}} YZ(z;w)\varphi(w,\zeta)dw + \varphi(z;\zeta)$$

Then we can directly obtain

$$\bar{K}\Gamma(z;\zeta) = \partial_{x_i}Z(z;\zeta) + \int_{S_{\tau,t}} \bar{K}Z(z;w)\varphi(w,\zeta)dw - \varphi(z;\zeta) = 0$$

a.e. for $z \neq \zeta$, since φ satisfies the integral equation (3.2.2). This implies

$$d_Y\Gamma(z,\zeta) = L\Gamma(z,\zeta)dt$$

in the integral sense (1.0.3).

(iii) We write

$$\Gamma(z;\zeta) = Z(z;\zeta) + J(z;\zeta)$$

and evaluate the two limits separately: we have

$$\int_{\mathbb{R}^d} Z(t,x;\tau,\xi)g(\xi)d\xi$$
$$= \int_{\mathbb{R}^d} \left[\Gamma_{(\tau,\xi)}(t,x;\tau,\xi) - \Gamma_{(\tau,x)}(t,x;\tau,\xi)\right] g(\xi)d\xi + \int_{\mathbb{R}^d} \Gamma_{(\tau,x)}(t,x;\tau,\xi)g(\xi)d\xi$$

Since $\Gamma_{(\tau,x)}$ is the fundamental solution of $L_{(\tau,x)}$

$$\lim_{\substack{(t,x)\to(\tau,y)\\ t>\tau}} \int_{\mathbb{R}^d} \Gamma_{(\tau,x)}(t,x;\tau,\xi)g(\xi)d\xi = g(y), \quad \forall y \in \mathbb{R}^d,\ \tau \in \mathbb{R}$$

Now let $\gamma : [0,1] \to \mathbb{R}^{d+1}$ a continuous path in $\mathbb{R}^{d+1}$ such that $\gamma(0) = (\tau, y)$. We need to prove that

$$\lim_{\sigma\to 0} \int_{\mathbb{R}^d} \left[\Gamma_{(\tau,\xi)}(\gamma(\sigma);\tau,\xi) - \Gamma_{(\tau,x(\sigma))}(\gamma(\sigma);\tau,\xi)\right] g(\xi)d\xi = \lim_{\sigma\to 0} I = 0 \tag{3.4.9}$$

By Lemma A.0.2 we have

$$|I| \le C \int_{\mathbb{R}^d} |\xi - x(\sigma)|_B^{\alpha} \Gamma^{\varepsilon}(\gamma(\sigma);\tau,\xi) e^{c|\xi|^2} d\xi$$

Arguing as for (3.3.1) we can see that the last integral is well defined, thus we get (3.4.9) since Γ^ε is the fundamental solution of $\bar{K}^\varepsilon$. On the other hand, by (3.2.5) we know that

$$|J(z;\zeta)| \le C(t-\tau)^{\frac{\alpha}{2}} \Gamma^{\varepsilon}(z;\zeta) \qquad \text{'a.e' in } S_{\tau,T}$$

therefore we can similarly prove that

$$\lim_{\substack{(t,x)\to(\tau,y)\\ t>\tau}} \int_{\mathbb{R}^d} J(t,x;\tau,\xi)g(\xi)d\xi = 0, \quad \forall y \in \mathbb{R}^d,\ \tau \in \mathbb{R}$$

(iv) By the results in section 3.2, the function $u(t,x)$ in (3.0.4) is well defined in $S_{T_0,T}$ for $T - T_0$ suitably small. We set

$$V(z) = \int_{S_{T_0,T}} \Gamma(z;\zeta) f(\zeta) d\zeta,$$

and prove that

$$d_Y V = (LV + f)dt \text{ in } S_{T_0,T}$$

By (3.4.8) we can write $V = V_f + V_{\hat{f}}$ where V_f is the potential in (3.3.1) and

$$\hat{f}(z) = \int_{S_{T_0,t}} \varphi(z;\zeta) f(\zeta) d\zeta$$

We aim to apply Propositions 3.3.1, 3.3.2 and 3.3.2 to the potential $V_{\hat{f}}$. To do that we need to check that $\hat{f}$ verifies the growth estimate (3.0.3) and the regularity condition (3.0.3). By (3.2.4) we have

$$|\hat{f}(z)| \le \int_{S_{T_0,T}} \frac{\Gamma^{\varepsilon}(z;\zeta)}{(t-\tau)^{1-\alpha/2}} |f(\zeta)| d\zeta \le C(t-T_0)^{\frac{\alpha}{2}} e^{C|x|^2}$$

proceeding as in the proof of Proposition 3.3.2. On the other hand, by Lemma 3.4.1

$$\begin{aligned}
&|\hat{f}(t,x) - \hat{f}(t,y)| \\
&\le \int_{T_0}^{t} \int_{\mathbb{R}^d} |\varphi(t,x;\tau,\xi) - \varphi(t,y;\tau,\xi)||f(\tau,\xi)| d\xi d\tau \\
&\le C|x-y|_B^{\frac{\alpha}{2}} \int_{T_0}^{t} \frac{1}{(t-\tau)^{1-\alpha/4}} \int_{\mathbb{R}^d} \left(\Gamma^{\varepsilon}(t,x;\tau,\xi) + \Gamma^{\varepsilon}(t,y;\tau,\xi)\right) |f(\tau,\xi)| d\xi d\tau \\
&\le C(t-\tau)^{\frac{\alpha}{4}} |x-y|_B^{\frac{\alpha}{2}} e^{C(|x|^2+|y|^2)}.
\end{aligned}$$

Therefore we can apply Propositions 3.3.1, 3.3.2 and 3.3.2 and we get

$$\begin{aligned}
\bar{K}V(z) &= \bar{K}V_f(z) + \bar{K}V_{\hat{f}}(z) \\
&= -f(z) - \hat{f}(z) + \int_{S_{T_0,t}} \bar{K}Z(z;\zeta)\left(f(\zeta) + \hat{f}(\zeta)\right) d\zeta \\
&= -f(z) + \int_{S_{T_0,t}} f(\zeta)\left(-\varphi(z;\zeta) + \bar{K}Z(z;\zeta) + \int_{S_{\tau,t}} \bar{K}Z(z; w\varphi(w))dw\right) d\zeta \\
&= -f(z)
\end{aligned}$$

by (3.2.2). Morover, by Corollary 3.2.4

$$|V(z)| \le C \int_{S_{T_0,t}} \Gamma^{\varepsilon}(z;\zeta)|f(\zeta)|d\zeta \le C(t-T_0)e^{C|x|^2}$$

arguing as in the proof of Proposition 3.3.2. Therefore, by item (iii), $u(t,x) \in C([T_0,T[\times\mathbb{R}^d)$ and $u(T_0,\cdot) = g(\cdot)$. This provides (3.0.5).

(iv) Estimate (3.0.6) has been already given in Corollary 3.2.4. Now consider (3.0.7). By Proposition 3.1.17 and the estimate (3.2.4) we have

$$\begin{aligned}
\left|\partial_{x_j}\Gamma(z;\zeta)\right| &\le C\frac{\Gamma^{\varepsilon}(z;\zeta)}{\sqrt{t-\tau}} + C\int_{\tau}^{t}\int_{\mathbb{R}^d} \frac{\Gamma^{\varepsilon}(t,x;\sigma,y)}{(t-s)^{1/2}}\frac{\Gamma^{\varepsilon}(s,y;\tau,\xi)}{(s-\tau)^{1-\alpha/2}}d\xi ds \\
&\le C\frac{\Gamma^{\varepsilon}(z;\zeta)}{\sqrt{t-\tau}} + C\int_{\tau}^{t} \frac{1}{(t-s)^{1/2}}\frac{1}{(s-\tau)^{1-\alpha/2}}d\xi ds \le C\frac{\Gamma^{\varepsilon}(z;\zeta)}{\sqrt{t-\tau}}
\end{aligned}$$

for any $j = 1,\cdots,m_0$ and $z,\zeta \in \mathbb{R}^{d+1}$ with $0 < t-\tau < T$. Finally, by Propositions 3.1.17, 3.3.2 we have

$$\begin{aligned}
\left|\partial_{x_i x_j}\Gamma(z;\zeta)\right| &\le C\frac{\Gamma^{\varepsilon}(z;\zeta)}{t-\tau} + \left|\int_{S_{\tau,t}} \partial_{x_i x_j}Z(z;w)\varphi(w;\zeta)dw\right| \\
&\le C\frac{\Gamma^{\varepsilon}(z;\zeta)}{t-\tau} + C\int_{\tau}^{t} \frac{1}{(t-s)^{\alpha/4}}\frac{1}{(s-\tau)^{\alpha/4}}ds \le C\frac{\Gamma^{\varepsilon}(z;\zeta)}{t-\tau}
\end{aligned}$$

This is done by repeating the argument in the proof of Proposition 3.3.2 to manage the singularity in the integral, using estimates (3.2.4) and (3.4.1). □

Appendix A

Proofs of Propositions 3.3.1, 3.3.2 and 3.3.3

Here we give the complete proofs of the Propositions stated in Section 3.3. The notation is intended to be the same.

Proof of Proposition 3.3.1. We first note that the integral in (3.3.2) is absolutely convergent. Indeed by Proposition 3.1.17 we have

$$\begin{aligned}\int_{T_0}^{t}\int_{\mathbb{R}^d} \left|\partial_{x_i} Z(t,x,\tau,\xi) f(\tau,\xi)\right| d\xi d\tau \\ \le C_1 \int_{T_0}^{t} \frac{1}{\sqrt{t-\tau}} \int_{\mathbb{R}^d} \Gamma^{+}(t,x,\tau,\xi) e^{C|\xi|^2} d\xi d\tau \le C_2 \frac{1}{\sqrt{t-T_0}} e^{C_3|x|^2}. \qquad \text{(A.0.1)}\end{aligned}$$

arguing as for the well posedness of (3.3.1) in section 3.3. Next we prove (3.3.2). Let

$$V_{f,\delta}(t,x) = \int_{T_0}^{t-\delta}\int_{\mathbb{R}^d} Z(t,x,\tau,\xi) f(\tau,\xi) d\xi d\tau, \quad 0<\delta<t-T_0,$$

By Lebesgue's theorem we have

$$\lim_{\delta\to 0^+} V_{f,\delta}(t,x) = V_f(t,x), \qquad \text{(A.0.2)}$$

$$\partial_{x_i} V_{f,\delta}(t,x) = \int_{T_0}^{t-\delta} \partial_{x_i} Z(t,x,\tau,\xi) f(\tau,\xi) d\xi d\tau, \quad i=1,\cdots m_0. \qquad \text{(A.0.3)}$$

Morover, by (A.0.1) and (A.0.3) we have

$$\begin{aligned}\partial_{x_i} V_{f,\delta}(t,x) - \int_{T_0}^{t} \partial_{x_i} Z(t,x,\tau,\xi) f(\tau,\xi) d\xi d\tau \\ = \int_{t-\delta}^{t} \partial_{x_i} Z(t,x,\tau,\xi) f(\tau,\xi) d\xi d\tau \le C\sqrt{\delta} e^{C|x|^2}\end{aligned}$$

so that

$$\lim_{\delta\to 0^+}\partial_{x_i}V_{f,\delta}(t,x)=\int_{T_0}^{t}\partial_{x_i}Z(t,x,\tau,\xi)f(\tau,\xi)d\xi d\tau,\quad i=1,\cdots m_0.$$

uniformly in $]T_0,T]\times B_R$. Together with (A.0.2) this proves (3.3.2) □

Lemma A.0.1. *There exists a positive constant C such that, for every $z\in S_{[T_0,T_1]}$*

$$\|z^{-1}\|_B\le C\|z\|_B$$

Proof. Let $z=(t,x)$, $t>0$. We have

$$\|z^{-1}\|_B\le|t|^{\frac{1}{2}}+\left|e^{tB}x\right|_B+\left|\int_t^0 e^{sB}b_s ds\right|_B$$

Since t takes values in a bounded interval, the thesis directly follows by noting

$$\left|e^{tB}x\right|_B\le\sup_{|\xi|_B=1}\left|e^{tB}\xi\right|_B|x|_B\le c_1|x|_B$$
$$\left|\int_t^0 e^{sB}b_s ds\right|_B\le 2\int_0^{\sqrt{t}}\left|e^{\tau^2 B}b_{\tau^2}\tau\right|_B d\tau\le c_2\sqrt{t}$$

The proof for the case $t<0$ is analogous. □

Lemma A.0.2. *For every positive ε and T there exists a positive constant C such that*

$$\begin{aligned}|\Gamma_\zeta(z,\zeta)-\Gamma_\theta(z,\zeta)|&\le C\|\theta^{-1}\circ_Y\zeta\|^\alpha\Gamma^\varepsilon(z,\zeta),\\ |\partial_{x_i}\Gamma_\zeta(z,\zeta)-\partial_{x_i}\Gamma_\theta(z,\zeta)|&\le C\frac{\|\theta^{-1}\circ_Y\zeta\|^\alpha}{\sqrt{t-\tau}}\Gamma^\varepsilon(z,\zeta),\\ \left|\partial_{x_ix_j}\Gamma_\zeta(z,\zeta)-\partial_{x_ix_j}\Gamma_\theta(z,\zeta)\right|&\le C\frac{\|\theta^{-1}\circ_Y\zeta\|^\alpha}{t-\tau}\Gamma^\varepsilon(z,\zeta),\end{aligned}$$

for $i,j=1,\cdots,m_0$ and $z,\zeta,\theta\in\mathbb{R}^{d+1}$ with $0<t-\tau\le T$.

Proof. We first note that

$$\begin{aligned}\partial_{x_j}\Gamma_\theta(z,\zeta)&=-(\mathcal{C}^{-1}_{\tau,t,\theta}\omega)_j\Gamma_\theta(z,\zeta)\\ \partial_{x_i,x_j}\Gamma_\theta(z,\zeta)&=\left((\mathcal{C}^{-1}_{\tau,t,\theta}\omega)_i(\mathcal{C}^{-1}_{\tau,t,\theta}\omega)_j-(\mathcal{C}^{-1}_{\tau,t,\theta})_{ij}\right)\Gamma_\theta(z,\zeta)\end{aligned}$$

where $(s,\omega) = (\tau,\xi) \circ_Y (t,x)$ as usual. Then the thesis follows from the following estimates

$$\left|\frac{1}{\sqrt{\det\mathcal{C}_{\tau,t,\zeta}}} - \frac{1}{\sqrt{\det\mathcal{C}_{\tau,t,\theta}}}\right| \leq C\frac{\|\theta^{-1}\circ_Y\zeta\|^\alpha}{\sqrt{\det\mathcal{C}_{\tau,t,\zeta}}}, \tag{A.0.4}$$

$$\left|\exp\left(-\frac{1}{2}\langle\mathcal{C}_{\tau,t,\zeta}^{-1}\omega,\omega\rangle\right) - \exp\left(-\frac{1}{2}\langle\mathcal{C}_{\tau,t,\theta}^{-1}\omega,\omega\rangle\right)\right| \tag{A.0.5}$$
$$\leq C\|y^{-1}\circ_Y\zeta\|^\alpha \exp\left(-\frac{1}{2(\mu+\varepsilon)}\langle\mathcal{C}_s^{-1}\omega,\omega\rangle\right)$$

$$\left|(\mathcal{C}_{\tau,t,\zeta}^{-1}\omega)_j - (\mathcal{C}_{\tau,t,\theta}^{-1}\omega)_j\right| \leq \frac{C}{\sqrt{s}}\|\theta^{-1}\circ_Y\zeta\|^\alpha|\eta| \tag{A.0.6}$$

$$\left|(\mathcal{C}_{\tau,t,\zeta}^{-1})_{ij} - (\mathcal{C}_{\tau,t,\theta}^{-1})_{ij}\right| \leq \frac{C}{s}\|\theta^{-1}\circ_Y\zeta\|^\alpha \tag{A.0.7}$$

$$\left|(\mathcal{C}_{\tau,t,\zeta}^{-1}\omega)_i(\mathcal{C}_{\tau,t,\zeta}^{-1}\omega)_j - (\mathcal{C}_{\tau,t,\theta}^{-1}\omega)_i(\mathcal{C}_{\tau,t,\theta}^{-1}\omega)_j\right| \leq \frac{C}{s}\|\theta^{-1}\circ_Y\zeta\|^\alpha|\eta|^2, \tag{A.0.8}$$

Consider (A.0.4): by Remark 3.1.14 we can write

$$\begin{aligned}
\left|\frac{1}{\sqrt{\det\mathcal{C}_{\tau,t,\zeta}}} - \frac{1}{\sqrt{\det\mathcal{C}_{\tau,t,\theta}}}\right| &\leq \frac{C}{\sqrt{\det\mathcal{C}_{t,\tau,\zeta}}}\left|\frac{\det\mathcal{C}_{\tau,t,\zeta} - \det\mathcal{C}_{\tau,t,\theta}}{s^Q}\right| \\
= \frac{C}{\sqrt{\det\mathcal{C}_{t,\tau,\zeta}}}&\left|\det\left(\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)\mathcal{C}_{\tau,t,\zeta}\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)\right) - \det\left(\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)\mathcal{C}_{\tau,t,\theta}\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)\right)\right| \\
\leq \frac{C}{\sqrt{\det\mathcal{C}_{t,\tau,\zeta}}}&\left\|\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)\left(\mathcal{C}_{\tau,t,\zeta} - \mathcal{C}_{\tau,t,\theta}\right)\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)\right\|
\end{aligned}$$

Note that

$$\sup_{|y|=1}\left|\langle\left(\mathcal{C}_{\tau,t,\zeta} - \mathcal{C}_{\tau,t,\theta}\right)\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)y, \mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)y\rangle\right|$$
$$\leq C\|\theta^{-1}\circ_Y\zeta\|^\alpha\left\|\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)\mathcal{C}_s\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)\right\| \leq C'\|\theta^{-1}\circ_Y\zeta\|^\alpha \tag{A.0.9}$$

Indeed, if we take again $s < \delta$ as in Remark 3.1.13

$$\left\|\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)\mathcal{C}_s\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)\right\| \leq 2\left\|\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)\hat{\mathcal{C}}_s\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)\right\| \leq 2\|\hat{\mathcal{C}}_1\|$$

On the other hand, for $\delta \leq s \leq T$ we can take the Maximum of the norm, that is limited. This conludes the proof for (A.0.4). Now consider (A.0.5). We have

$$\begin{aligned}
&\left|\exp\left(-\frac{1}{2}\langle\mathcal{C}_{\tau,t,\zeta}^{-1}\omega,\omega\rangle\right) - \exp\left(-\frac{1}{2}\langle\mathcal{C}_{\tau,t,\theta}^{-1}\omega,\omega\rangle\right)\right| \\
&\quad\leq \left|\langle(\mathcal{C}_{\tau,t,\zeta}^{-1} - \mathcal{C}_{\tau,t,\theta}^{-1})\omega,\omega\rangle\right|\exp\left(-\frac{1}{2\mu}\langle\mathcal{C}_s^{-1}\omega,\omega\rangle\right) \\
&\quad\leq \left\|\mathcal{D}_0(\sqrt{s})(\mathcal{C}_{\tau,t,\zeta}^{-1} - \mathcal{C}_{\tau,t,\theta}^{-1})\mathcal{D}_0(\sqrt{s})\right\||\eta|^2\exp\left(-\frac{1}{2\mu}\langle\mathcal{C}_s^{-1}\omega,\omega\rangle\right) \\
&\quad\leq C\left\|\mathcal{D}_0(\sqrt{s})(\mathcal{C}_{\tau,t,\zeta}^{-1} - \mathcal{C}_{\tau,t,\theta}^{-1})\mathcal{D}_0(\sqrt{s})\right\|\exp\left(-\frac{1}{2(\mu+\varepsilon)}\langle\mathcal{C}_s^{-1}\omega,\omega\rangle\right)
\end{aligned}$$

by Proposition 3.1.16. Moreover, by (A.0.9)

$$\begin{aligned}
&\left\|\mathcal{D}_0(\sqrt{s})(\mathcal{C}^{-1}_{\tau,t,\zeta}-\mathcal{C}^{-1}_{\tau,t,\theta})\mathcal{D}_0(\sqrt{s})\right\| \\
&\quad\le \left\|\mathcal{D}_0(\sqrt{s})\mathcal{C}^{-1}_{\tau,t,\zeta}\mathcal{D}_0(\sqrt{s})\right\| \left\|\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)(\mathcal{C}_{\tau,t,\zeta}-\mathcal{C}_{\tau,t,\theta})\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)\right\| \left\|\mathcal{D}_0(\sqrt{s})\mathcal{C}^{-1}_{\tau,t,\theta}\mathcal{D}_0(\sqrt{s})\right\| \\
&\quad\le C\|\zeta^{-1}\circ_Y\theta\|^{\alpha} \qquad \text{(A.0.10)}
\end{aligned}$$

Next we consider (A.0.6). By (A.0.10) we have

$$\begin{aligned}
&\left|(\mathcal{C}^{-1}_{\tau,t,\zeta}\omega)_j-(\mathcal{C}^{-1}_{\tau,t,\theta}\omega)_j\right| = \\
&\quad= \left|\langle\left(\mathcal{C}^{-1}_{\tau,t,\zeta}-\mathcal{C}^{-1}_{\tau,t,\theta}\right)\omega,e_j\rangle\right| = \left|\langle\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)\left(\mathcal{C}^{-1}_{\tau,t,\zeta}-\mathcal{C}^{-1}_{\tau,t,\theta}\right)\mathcal{D}_0\left(\frac{1}{\sqrt{s}}\right)\eta,\frac{e_j}{\sqrt{s}}\rangle\right| \\
&\quad\le \left\|\mathcal{D}_0(\sqrt{s})\left(\mathcal{C}^{-1}_{\tau,t,\zeta}-\mathcal{C}^{-1}_{\tau,t,\theta}\right)\mathcal{D}_0(\sqrt{s})\right\|\frac{|\eta|}{\sqrt{s}} \le \|\zeta^{-1}\circ_Y\theta\|^{\alpha}\frac{|\eta|}{\sqrt{s}}
\end{aligned}$$

By substituting w with e_i we also get (A.0.7).
Finally, consider (A.0.8): we have

$$\begin{aligned}
&\left|(\mathcal{C}^{-1}_{\tau,t,\zeta}\omega)_i(\mathcal{C}^{-1}_{\tau,t,\zeta}\omega)_j-(\mathcal{C}^{-1}_{\tau,t,\theta}\omega)_i(\mathcal{C}^{-1}_{\tau,t,\theta}\omega)_j\right| = \\
&\quad\le \left|\langle\left(\mathcal{C}^{-1}_{\tau,t,\zeta}-\mathcal{C}^{-1}_{\tau,t,\theta}\right)\omega,e_i\rangle\langle\mathcal{C}^{-1}_{\tau,t,\theta}\omega,e_j\rangle\right| + \left|\langle\mathcal{C}^{-1}_{\tau,t,\theta}\omega,e_j\rangle\langle\left(\mathcal{C}^{-1}_{\tau,t,\zeta}-\mathcal{C}^{-1}_{\tau,t,\theta}\right)\omega,e_i\rangle\right| \\
&\quad\le \frac{C}{s}\|\theta^{-1}\circ_Y\zeta\|^{\alpha}|\eta|^2
\end{aligned}$$

with the analogous arguments. □

Proof of Proposition 3.3.2. We first show that the integral in (3.3.5) exists. Fixed $R>0$ consider $x\in\mathbb{R}^d$ such that $|x|<R$. For a suitable $R_1>R$ to be determined later we set

$$\begin{aligned}
&\int_{T_0}^{t}\int_{\mathbb{R}^d}\partial_{x_ix_j}Z(t,x,\tau,\xi)f(\tau,\xi)d\xi d\tau \\
&\qquad= \int_{T_0}^{t}\int_{B_{R_1}}\partial_{x_ix_j}Z(t,x,\tau,\xi)f(\tau,\xi)d\xi d\tau + \int_{T_0}^{t}\int_{\mathbb{R}^d\setminus B_{R_1}}\partial_{x_ix_j}Z(t,x,\tau,\xi)f(\tau,\xi)d\xi d\tau \\
&\qquad= K_1+K_2.
\end{aligned}$$

where B_{R_1} denotes the ball in $\mathbb{R}^d$ centered at the origin with radius $R_1>0$.
Consider K_1. For fixed $\tau\in]T_0,t[$ and $y\in\mathbb{R}^d$, denoting $\theta=(\tau,y)$ we have

$$\begin{aligned}
&\int_{B_{R_1}}\partial_{x_ix_j}Z(t,x,\tau,\xi)f(\tau,\xi)d\xi \qquad \text{(A.0.11)}\\
&\quad= \int_{B_{R_1}}\partial_{x_ix_j}Z(t,x,\tau,\xi)\left(f(\tau,\xi)-f(\tau,y)\right)d\xi \\
&\quad+ f(\tau,y)\int_{B_{R_1}}\partial_{x_ix_j}\left(Z(t,x,\tau,\xi)-\Gamma_w(t,x,\tau,\xi)\right)_{|w=\theta}d\xi \\
&\quad+ f(\tau,y)\int_{B_{R_1}}\partial_{x_ix_j}\Gamma_y(t,x,\tau,\xi)_{|w=\theta}d\xi \\
&\quad= I_1+I_2+I_3
\end{aligned}$$

First consider I_1. Let $y = \gamma_{t,\tau}(x)$: by Proposition 3.1.17 and the hypothesis on f we get

$$|I_1| \le C \int_{\mathbb{R}^d} \frac{\Gamma^{\frac{\varepsilon}{2}}(t,x,\tau,\xi)}{t-\tau} \left|\xi - \gamma_{t,\tau}(x)\right|_B^\beta d\xi$$

By Lemma A.0.1 we have

$$\left|\xi - \gamma_{t,\tau}(x)\right|_B \le C \left|x - \gamma_{\tau,t}(\xi)\right|_B = C\sqrt{t-\tau}\,|\eta|_B$$

where $\eta = \mathcal{D}_0\left((t-\tau)^{-\frac{1}{2}}\right)(x - \gamma_{\tau,t}(\xi))$ with the usual notation. Therefore, by Proposition 3.1.16 we deduce

$$|I_1| \le C \int_{\mathbb{R}^d} \frac{\Gamma^{\varepsilon}(t,x,\tau,\xi)}{(t-\tau)^{1-\frac{\beta}{2}}} d\xi \le \frac{C_1}{(t-\tau)^{1-\frac{\beta}{2}}}$$

Consider now I_2: by Lemma A.0.2 and the growth estimate for f we have

$$\begin{aligned} |I_2| &\le C_1 \int_{B_{R_1}} \frac{\Gamma^{\varepsilon}(t,x,\tau,\xi)}{t-\tau} \left|\xi - \gamma_{t,\tau}(x)\right|_B^\beta d\xi \\ &\le C_2 e^{C_2|x|^2} \int_{\mathbb{R}^d} \frac{\Gamma^{\varepsilon}(t,x,\tau,\xi)}{(t-\tau)^{1-\frac{\alpha}{2}}} d\xi \le \frac{C}{(t-\tau)^{1-\frac{\alpha}{2}}} \end{aligned}$$

repeating the argument above.

Next consider I_3. For $w \in \mathbb{R}^d$ we have

$$\begin{aligned} \partial_{x_j}\Gamma_w(t,x,\tau,\xi) &= -\left(\mathcal{C}_{\tau,t,w}^{-1}(x - \gamma_{\tau,t}(\xi))\right)_j \Gamma_w(t,x,\tau,\xi), \\ \partial_{\xi_j}\Gamma_w(t,x,\tau,\xi) &= \sum_{k=1}^d \left(\mathcal{C}_{\tau,t,w}^{-1}(x - \gamma_{\tau,t}(\xi))\right)_k \left(e^{-(t-\tau)B}\right)_{kj} \\ &= \sum_{k=1}^d -\partial_{x_k}\Gamma_w(t,x,\tau,\xi)\left(e^{-(t-\tau)B}\right)_{kj} \end{aligned}$$

Thus it holds that

$$\nabla_x \Gamma_w(t,x,\tau,\xi) = -\nabla_\xi \Gamma_w(t,x,\tau,\xi)\left(e^{(t-\tau)B}\right)$$

Therefore we have

$$\begin{aligned} &\int_{B_{R_1}} \partial_{x_i x_j}\Gamma_w(t,x,\tau,\xi)_{|w=\theta} d\xi \\ &\quad = -\sum_{K=1}^d \int_{B_{R_1}} \partial_{x_i \xi_k}\Gamma_w(t,x,\tau,\xi)_{|w=\theta}\left(e^{-(t-\tau)B}\right)_{kj} d\xi \\ &\quad = -\sum_{K=1}^d \int_{\partial B_{R_1}} \partial_{x_i}\Gamma_w(t,x,\tau,\xi)_{|w=\theta}\left(e^{-(t-\tau)B}\right)_{kj} \nu_k d\sigma(\xi) \end{aligned}$$

by the integration by parts formula and denoting by ν the outer normal to ∂B_{R_1}. Thus, by Proposition 3.1.17 we can conclude that

$$|I_3| \le \frac{C}{\sqrt{t-\tau}}$$

Now we consider K_2. With fixed x, there exists a positive constant C such that

$$|x - \gamma_{\tau,t}(\xi)| \geq R_2 > 0$$

for $|\xi| \geq R_1$ with R_1 suitably large. Then

$$\begin{aligned}|K_2| \leq & C \int_{T_0}^{t} \int_{\mathbb{R}^d \setminus B_{R_1}} \frac{\Gamma^{\varepsilon}(t,x,\tau,\xi)}{t-\tau} e^{C_1|\xi|^2} d\xi d\tau \\ & \leq Ce^{|x|^2} \int_{T_0}^{t} \int_{|\omega| \geq R_1} \frac{1}{(t-\tau)^{\frac{Q}{2}+1}} \exp\left(-\frac{\langle \mathcal{C}_{t-\tau}^{-1}\omega, \omega \rangle}{2\mu} + C_2|\omega|^2 \right) d\omega d\tau\end{aligned}$$

by the change of variable $\omega = x - \gamma_{\tau,t}(\xi)$ and (3.1.11). The last integral surely converges provided that $T - T_0$ is suitably small by Lemma 3.1.12.

This proved the existence of the integral. Next we prove (3.3.5). Set

$$V_f(z) = V_f^{(1)}(z) + V_f^{(2)}(z)$$

where

$$\begin{aligned} V_f^{(1)}(t,x) &= \int_{T_0}^{t} \int_{B_{R_1}} Z(t,x,\tau,\xi) f(\tau,\xi) d\xi d\tau, \\ V_f^{(2)}(t,x) &= \int_{T_0}^{t} \int_{\mathbb{R}^d \setminus B_{R_1}} Z(t,x,\tau,\xi) f(\tau,\xi) d\xi d\tau \end{aligned}$$

We aim to prove that

$$\partial_{x_i x_j} V_f^{(1)}(t,x) = \int_{T_0}^{t} \int_{B_{R_1}} \partial_{x_i x_j} Z(t,x,\tau,\xi) f(\tau,\xi) d\xi d\tau, \tag{A.0.12}$$

$$\partial_{x_i x_j} V_f^{(2)}(t,x) = \int_{T_0}^{t} \int_{\mathbb{R}^d \setminus B_{R_1}} \partial_{x_i x_j} Z(t,x,\tau,\xi) f(\tau,\xi) d\xi d\tau \tag{A.0.13}$$

Equation (A.0.13) follows from Lebesgue's theorem. To prove (A.0.12) we set

$$V_{f,\delta}^{(1)}(t,x) = \int_{T_0}^{t-\delta} \int_{B_{R_1}} Z(t,x,\tau,\xi) f(\tau,\xi) d\xi d\tau, \quad 0 < \delta < t - T_0.$$

By the dominated convergence theorem and Proposition 3.3.1

$$\begin{aligned} \lim_{\delta \to 0^+} \partial_{x_j} V_{f,\delta}^{(1)}(t,x) &= \lim_{\delta \to 0^+} \int_{T_0}^{t-\delta} \int_{B_{R_1}} \partial_{x_j} Z(t,x,\tau,\xi) f(\tau,\xi) d\xi d\tau \\ &= \int_{T_0}^{t} \int_{B_{R_1}} \partial_{x_j} Z(t,x,\tau,\xi) f(\tau,\xi) d\xi d\tau = \partial_{x_j} V_f^{(1)}(t,x). \end{aligned}$$

Hence, to show (A.0.12) it suffices to prove that

$$\lim_{\delta \to 0^+} \partial_{x_i x_j} V_{f,\delta}^{(1)}(t,x) = \int_{T_0}^{t} \int_{B_{R_1}} \partial_{x_i x_j} Z(t,x,\tau,\xi) f(\tau,\xi) d\xi d\tau,$$

uniformly on $]T_0, T] \times B_{R_1}$. We have

$$\partial_{x_i x_j} V^{(1)}_{f,\delta}(t,x) - \int_{T_0}^{t} \int_{B_{R_1}} \partial_{x_i x_j} Z(t,x,\tau,\xi) f(\tau,\xi) d\xi d\tau$$
$$= \int_{t-\delta}^{t} \left(J_1(\tau) + J_2(\tau) + J_3(\tau)\right) d\tau$$

where

$$J_1(\tau) = \int_{B_{R_1}} \partial_{x_i x_j} Z(t,x,\tau,\xi) \left(f(\tau,\xi) - f(\tau,y)\right) d\xi$$
$$J_2(\tau) = f(\tau,y) \int_{B_{R_1}} \partial_{x_i x_j} \left(Z(t,x,\tau,\xi) - \Gamma_w(t,x,\tau,\xi)_{|w=y}\right) d\xi$$
$$J_2(\tau) = f(\tau,y) \int_{B_{R_1}} \partial_{x_i x_j} \Gamma_w(t,x,\tau,)_{|w=y} d\xi$$

Proceding as for the estimates of I_1, I_2, I_3 in (A.0.11), by choosing $y = \gamma_{t,\tau}(x)$ we get

$$\int_{t-\delta}^{t} |J_1(\tau) + J_2(\tau) + J_3(\tau)| \, d\tau$$
$$\leq C \int_{t-\delta}^{t} \left(\frac{1}{(t-\tau)^{1-\frac{\beta}{2}}} + \frac{1}{(t-\tau)^{1-\frac{\alpha}{2}}} + \frac{1}{\sqrt{t-\tau}} \right) d\tau$$
$$\leq C \left(\delta^{\frac{\beta}{2}} + \delta^{\frac{\alpha}{2}} + \delta^{\frac{1}{2}} \right)$$

□

Proof of Proposition 3.3.3. As in the proof of Proposition 3.3.2 we split the domain of the integral in $]T_0, t[\times B_{R_1}$ and $]T_0, t[\times(\mathbb{R}^d \setminus B_{R_1})$. We consider the former. Set

$$V_\delta(t,x) = \int_{\tau}^{t-\delta} \int_{B_{R_1}} Z(t,x;\tau,\xi) f(\tau,\xi) d\xi d\tau$$

and consider the integral path of Y starting from z:

$$\widetilde{\gamma} : \mathbb{R} \to \mathbb{R}^{d+1}, \quad \widetilde{\gamma}(s) = (t(s), x(s)) = (t+s, \gamma_{t,t+s}(x))$$

Clearly $\frac{d}{ds}\widetilde{\gamma}(s) = (1, -Bx(s) - b_s) = Y(\gamma(s))$ a.e and $\widetilde{\gamma}(0) = z$.
Let $|s| < \frac{\delta}{2}$, then we have

$$\frac{V_\delta(\widetilde{\gamma}(s);\tau,\xi) - V_\delta(t,x;\tau,\xi)}{s} = \int_{\tau}^{t-\delta} \int_{B_{R_1}} \frac{Z(\widetilde{\gamma}(s);\tau,\xi) - Z(t,x;\tau,\xi)}{s} f(\tau,\xi) d\xi d\tau$$
$$+ \frac{1}{s} \int_{t-\delta}^{t+s-\delta} \int_{B_{R_1}} Z(\widetilde{\gamma}(s);\tau,\xi) f(\tau,\xi) d\xi d\tau = I_1 + I_2$$

We should put s to 0. Formally

$$\lim_{s\to 0} \frac{Z(\widetilde{\gamma}(s);\tau,\xi) - Z(t,x;\tau,\xi)}{s} = \frac{d}{ds} Z(\widetilde{\gamma}(s);\tau,\xi)_{|s=0} = YZ(t,x;\tau,\xi) \quad \text{'t-a.e'}$$

Morover, since Z is the parametrix and by Proposition 3.1.17 we have

$$\frac{1}{s}\left(Z(\widetilde{\gamma}(s);\tau,\xi)-Z(t,x;\tau,\xi)\right)$$
$$=\frac{1}{s}\int_t^{t+s}\sum_{i,j=1}^{m_0}\bar{a}_\sigma^{ij}\partial_{x_ix_j}Z(\widetilde{\gamma}(s);\tau,\xi)d\sigma\leq\frac{C}{s}\int_t^{t+s}\frac{\Gamma^\varepsilon(\widetilde{\gamma}(s);\tau,\xi)}{t+\sigma-\tau}d\sigma$$

Then for $|s|<\frac{\delta}{2}$ and $T_0<\tau<t-\delta$ the integrand is a bounded function of $\xi\in]T_0,t-\delta[\times\mathbb{R}^d$. Thus we have

$$\lim_{s\to 0}\int_\tau^{t-\delta}\int_{B_{R_1}}\frac{Z(\widetilde{\gamma}(s);\tau,\xi)-Z(t,x;\tau,\xi)}{s}f(\tau,\xi)d\xi d\tau=\int_\tau^{t-\delta}\int_{B_{R_1}}YZ(t,x;\tau,\xi)f(\tau,\xi)d\xi d\tau$$

On the other hand

$$\int_{B_{R_1}}Z(t,x;t-\delta,\xi)f(t-\delta,\xi)d\xi-\frac{1}{s}\int_{t-\delta}^{t+s-\delta}\int_{B_R}Z(\widetilde{\gamma}(s);\tau,\xi)f(\tau,\xi)d\xi d\tau$$
$$=\int_0^1\int_{B_{R_1}}\left(Z(t,x;t-\delta,\xi)-Z(\widetilde{\gamma}(s);\varrho s+t-\delta,\xi)\right)f(\xi,t-\delta)d\xi d\varrho$$
$$+\int_0^1\int_{B_{R_1}}Z(\widetilde{\gamma}(s);\varrho s+t-\delta,\xi)\left(f(t-\delta,\xi)-f(\xi,\varrho s+t-\delta)\right)d\xi d\varrho$$

(setting $\varrho=\frac{t+\delta-\tau}{s}$)

$$=I(z,s)+J(z,s)$$

An application of the dominated convergence theorem shows that

$$\lim_{s\to 0}I(z,s)=0,\quad \lim_{s\to 0}J(z,s)=0$$

So far we proved the following:

$$YV_{\delta,f}=\int_\tau^{t-\delta}\int_{B_{R_1}}YZ(t,x;\tau,\xi)f(\tau,\xi)d\xi d\tau+\int_{B_{R_1}}Z(t,x;t-\delta,\xi)f(t-\delta,\xi)d\xi$$

Now, since f is a continuous and bounded function on $B_R\times]T_0,T_1[$ we have

$$\lim_{\delta\to 0^+}\int_{B_{R_1}}Z(t,x;t-\delta,\xi)f(t-\delta,\xi)d\xi=f(t,x)$$

and morover

$$\left|\int_{t-\delta}^t\int_{B_R}YZ(t,x,\tau,\xi)f(\tau,\xi)\right|$$
$$\leq\sum_{ij=1}^{m_0}\int_{t-\delta}^t\int_{B_R}\left|\bar{A}_t^{ij}(\tau,\xi)\partial x_ix_jZ(t,x;\tau,\xi)f(\tau,\xi)\right|d\xi d\tau$$
$$\leq C\|\bar{A}\|_\infty(\delta^{\beta/2}+\delta^{\alpha/2}+\delta^{1/2})$$

proceeding as for Proposition 3.3.2, so that, for a.e $t \in]T_0, T_1[$

$$\lim_{\delta\to 0^+} YV_{\delta,f} = \int_{\tau}^{t}\int_{B_{R_1}} YZ(t,x;\tau,\xi)f(\tau,\xi)d\xi d\tau + f(t,x).$$

□

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