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Planck Stars

Theory and Phenomenology

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Abstract

General Relativity (GR) is one of the greatest scientific achievements of the 20th century along with quantum theory.
Despite the elegance and the accordance with experimental tests, these two theories appear to be utterly incompatible at fundamental level.
Black holes provide a perfect stage to point out these difficulties. Indeed, classical GR fails to describe Nature at small radii, because nothing prevents quantum mechanics from affecting the high curvature zone, and because classical GR becomes ill-defined at $r = 0$ anyway.

Rovelli and Haggard have recently proposed a scenario where a negative quantum pressure at the Planck scales stops and reverts the gravitational collapse, leading to an effective "bounce" and explosion, thus resolving the central singularity. This scenario, called Black Hole Fireworks, has been proposed in a semiclassical framework.

The purpose of this thesis is twofold:

- Compute the bouncing time by means of a pure quantum computation based on Loop Quantum Gravity;
- Extend the known theory to a more realistic scenario, in which the rotation is taken into account by means of the Newman-Janis Algorithm.

Sommario

La Relativitá Generale costituisce, assieme alla Meccanica Quantistica, una delle piú grandi conquiste del ventesimo secolo.
Tuttavia, nonostante la loro eleganza formale e la loro compatibilitá con i risultati sperimentali, queste due teorie risultano essere fortemente incompatibili a livello fondamentale.
I buchi neri costituiscono il palcoscenico perfetto per evidenziare le difficoltá sopracitate. Infatti, come ben noto, la Relativitá Generale non risulta essere in grado di descrivere adeguatamente la Natura per piccoli valori del raggio, e risulta essere mal definita per $r = 0$.

Recentemente, Rovelli e Haggard hanno proposto un modello in cui una pressione negativa, di natura puramente quantistica, possa presentarsi una volta raggiunta la cosiddetta densitá di Planck, interrompendo cosí il collasso gravitazionale e causando un "rimbalzo" effettivo ed una conseguente esplosione, risolvendo dunque il problema della singolaritá centrale che caratterizza la teoria classica dei buchi neri.

Le finalitá della presente tesi sono le seguenti:

- Valutare il tempo di rimbalzo, ad oggi noto solo in un contesto semi-classico, per mezzo di un calcolo basato sulla Loop Quantum Gravity;
- Estendere i risultati di Rovelli e Haggard al caso rotante per mezzo dell'algoritmo Newman-Janis.

Contents

Introduction

It has been a long-standing challenge for theoretical physicists to construct a consistent theory of quantum gravity. It is well known that General Relativity itself gives some hints of its own limits, since given smooth initial data can evolve into singular field configurations.
Classically, this is not a problem at all if the singularities are hidden behind event horizons, because this would mean that these singularity are not in causal contact with the rest of the Universe. This idea, indeed, led Roger Penrose to formulate the so called "weak cosmic censorship conjecture" in 1969.
The 70s have indeed represented a turning point for black hole physics. In this period Stephen Hawking showed, under very general assumptions, that if we take into account the vacuum fluctuations in a region close to the event horizon it follows that black holes emit particles. One of the most remarkable consequences of this result is that the radiation described above is exactly thermal and contains no information about the state of the black hole. This led to the notorious problem of "information loss", since particles can fall in carrying information but what comes out is featureless thermal radiation.
It can be easily argued that this paradoxical situation would lead to non-unitary evolution of the quantum states, so that one of the basic principles of quantum mechanics would be violated.
Another great achievement obtained in the seventies is that black holes can be treated, at least formally, as thermodynamic-like systems. Indeed, they have an entropy and a temperature which are given by

$$T_H = \frac{\hbar\,\kappa}{2\pi\,k_{\mathrm{B}}}\ , \qquad S_{\mathrm{BH}} = \frac{k_{\mathrm{B}}\,A}{4\ell_{\mathrm{P}}^2}$$

where κ is the surface gravity and A is the area of the horizon.
These quantities appear to be closely related to the quantum aspects of gravitation, in the sense that they depend on both Planck's constant $\hbar$ and Newton's constant G. The Hawking temperature, as well as the Bekenstein-Hawking entropy, have been derived in many independent ways, in different settings and with different assumptions, so that they are considered robust features to be included in any complete theory of quantum gravity.
Although none of these results deal directly with the problem of the curvature singularity, which represents the emblematic example of a region characterized by pure quantum features, these quantities can still be extremely useful to determinate whether a theory of quantum gravity is, more or less, worth of trust.

Nowadays, there are different approaches to quantum gravity in which it has been possible to recover the former quantities by means of a pure quantum computation, such as Loop Quantum Gravity, String Theory, Asymptotic Safety and others.

However, despite the degree of progress of these theories, quantum gravity still remains a major unsolved challenge at the core of fundamental physics and its phenomenology is still beyond direct observations. However, recent research has brought to surface a number of different ideas for indirect tests, and the possibility of detecting effects that occur in the Planck scale regime does not appear completely out of reach. For example, the idea that quantum mechanical effects may resolve the gravitational singularity has led to the notion of Planck stars, and its associated phenomenology currently under exploration, as well as the possibility that black holes are quantum condensates resolves most of the issues encountered in the literature.

The aim of this thesis is to give a first look at which kind of predictions can be recovered by a recently proposed model for non-singular black holes. Moreover, we are also interested in the consequences deriving from the application of loop quantum gravity to the quantum core of such compact self-gravitating object.

In order to be more specific, in *Part* I: *Prologue* we present the classical description of black holes. In this context we also introduce the most peculiar paradoxes known in the (semi)classical theory of black holes, with particular regards for the paradox of information loss and for the emergence of singularities in general relativity. As discussed above, these issues lie at the very foundation of black hole physics, and represent one of the main motivation for the development of the different approaches to quantum gravity. Moreover, in the *Prologue* we introduce two different and complementary semiclassical scenarios derived by analogy with loop quantum cosmology, i.e. the Planck Star model and Black Hole Fireworks, aimed to resolve the the previously mentioned oddities of the "classical" theory.

In *Part* II: *On the Effective Metric of a Rotating Planck Star*, we present a generalization of the known results for the Planck Star model. In particular, our aim is to recover an effective description of the spacetime surrounding a rotating Planck Star in order to provide a more realistic physical description such a star-like object. To do so we use the renowned Newman-Janis Algorithm.

Finally, in *Part* III: *Black Hole Fireworks and Transition Amplitudes in Loop Quantum Gravity*, firstly we introduce some basic aspects of the Spinfoam Approach to quantum gravity. Secondly, we analyse in details the Lorentzian EPRL model also providing some generalized rules for computing transition amplitudes in loop quantum gravity. Then, we present a potential application to these techniques to the problem of the computation of the bouncing time for black hole fireworks together with a more realistic description of the boundary spin-network for this scenario. However, the implementation of these computations is left for a future study.

Acknowledgements I would like to take this opportunity to thank the various individuals to whom I am indebted, not only for their help in preparing this thesis, but also for their support and guidance throughout my studies. The particular choice of topic for this thesis proved to be very rewarding as it allowed me to explore many interrelated areas of Physics and Mathematics that are of great interest to me. Thus I would first like to extend my thanks to my supervisors, Prof. Roberto Casadio and Dr. Simone Speziale, for encouraging me to pursue this topic and for providing me with very friendly and insightful guidance when it was needed. I am also extremely grateful to Prof. Carlo Rovelli for welcoming me into his research group in Marseille. Furthermore, I would also like to thank Hal Haggard, Francesca Vidotto, Alejandro Perez and Tommaso De Lorenzo for very useful discussions.
Finally, I would like to show my gratitude to those who have not directly been part of my academic life yet, but they have been of central importance in the rest of my life. First, and foremost, my parents, Antonella and Maurizio, for their unrelenting support and for teaching me the value of things. Also, to my sisters, Giulia and Giorgia, who have had to deal with my unending rants and other oddities. I would also thank some of my dearest friends, in particular Simone, Margherita and Martina, for their help and support during some very tough moments of the last couple of years as well as during the preparation of this thesis. I also offer my thanks and apologies to my girlfriend, Claudia, for putting up with months of me working on my thesis and with my endless, uninteresting updates on the latest traumatic turn of events. You are as kind as you are beautiful.

This work is dedicated to the loving memory of the greatest man I have ever known, my grandfather Marcellino, and to my grandmother Margherita.

I have nothing to offer but blood, toil, tears, and sweat.
Winston Churchill

I don't pay attention to the world ending.
It has ended for me many times and began again in the morning.
Nayyirah Waheed

Once the game is over the king and the pawn go back into the same box.

Per quanto mi riguarda,
mi sembra di essere un ragazzo
che gioca sulla spiaggia e trova
di tanto in tanto
una pietra o una conchiglia
piú belle del solito,
mentre il grande oceano della veritá
resta sconosciuto davanti a me.
Sir Isaac Newton

Part I

Prologue

Chapter 1

Introduction to Black Holes

The aim of the first part of this thesis is to review some classical and semi-classical results known in Black Hole physics which are going to be discuss and analysed extensively later.
Most of this introductory part is based on [1; 2; 3]. We have also drawn on some ideas from the books [4; 5].

In this work we choose to use the metric signature $(+ - - -)$. Moreover, throughout most of this thesis we shall use the so called *geometrized units*, i.e. $c = \hbar = G = 1$. However, we are going to restore the values of G, $\hbar$ and c whenever necessary in order to make explicit estimates.

1.1 Spherical Symmetry

It is quite known that the only way to address the Cauchy Problem for the Einstein's Field Equations is to assume some underlying symmetries of the spacetime.
The most obvious assumption is then to consider a *spherically symmetric* gravitational field. Indeed, this assumption represents a quite good approximation for the gravitational field created by a star, at least in the far field limit.

Let us now give a proper mathematical characterization of the concept of Spherical Symmetry.
From the fundamental courses of Linear Algebra and Geometry it is well known that the set of all isometries of a metric space forms a group. Consider the line element on the 2-sphere, S^2:

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \tag{1.1}$$

The set of all the transformations that leaves this line element unchanged, i.e. the isometry group for the 2-sphere given the former metric, is called *Orthogonal Group*, $O(3)$. For sake of generality, if we consider the matrix representation of this group, we have that

$$O(n) = \left\{ A \in GL(n) \; : \; AA^T = \mathbb{I} \right\}$$

It is straightforward to conclude that $O(n)$ has has *two connected components*.
Indeed, let $A \in O(n)$. Since $\mathtt{det}A = \mathtt{det}A^T$ we have, due to the Binet's theorem, that $\mathtt{det}A = \pm 1$. So that we have one connected component that contains orthogonal matrices

whose determinant is one, known as the *Special Orthogonal Group*, $SO(n)$, and the other that contains orthogonal matrices whose determinant is minus one. Moreover, $SO(n)$ is the only subgroup of $O(n)$ connected with the identity operator.
Thus, both $SO(n)$ and $O(n)$ are Lie Group.

For our purpose, we are interested in the set of isometries that does not include reflections of the axes, indeed we are only concerned about the invariance under rotations, at least for what concerns the spherical symmetry.
Given the former discussion, we can then define a spherically symmetric spacetime as follows.

Definition 1. A spacetime is said to be **spherically symmetric** if its isometry group contains an $SO(3)$ subgroup whose orbits[1] are 2-spheres.

In such spacetime we can define the radial coordinate in an unambiguous way. Indeed, let $\mathcal{M}$ be a spherically symmetric spacetime, thus we are allowed to define the so called *area-radius function* $r : \mathcal{M} \longrightarrow \mathbb{R}$ such that $r(p) = \sqrt{A(p)/4\pi}$, $\forall p \in M$, where $A(p)$ is the area of the S^2 orbit through $p \in M$.

1.2 Static and Stationary Spacetimes

Definition 2. A spacetime $(\mathcal{M}, g)$ is said to be **stationary** if it admits a Killing vector field k which is everywhere timelike, i.e. $g(k, k) > 0$

Now, if we consider a hypersurface $\Sigma \subset \mathcal{M}$ nowhere tangent to k, we are allowed to choose the coordinate as follows:

- Let x^i, $i = 1, 2, 3$ be coordinates on Σ;
- Now assign coordinates (t, x^i) to the point parameter distance t along the integral curve of k that starts at the point with coordinates x^i on Σ

This defines a coordinate chart $(t, x^i) \equiv (t, \mathbf{x})$ at least in a neighbourhood of Σ.
In such a coordinate chart we have that $k = \partial/\partial t$. Since k is a Killing vector field, we can cast the metric of the spacetime in such a way that it shall appear as independent from the coordinate t, hence

$$ds^2 = g_{00}(\mathbf{x})\, dt^2 + 2g_{0i}(\mathbf{x})\, dtdx^i + g_{ij}(\mathbf{x})\, dx^i dx^j \tag{1.2}$$

with $g_{00}(\mathbf{x}) > 0$.

Next we need to introduce the notion of *hypersurface-orthogonality*. Let Σ be a hypersurface specified by $f(x) = 0$, where $f : \mathcal{M} \longrightarrow \mathbb{R}$ is a smooth function such that $df \neq 0$ on Σ. Then, df is normal to Σ. Indeed, if T is a vector tangent to Σ, thus $df(T) = Tf = T^\alpha\, \partial_\alpha f = 0$ because f is constant on Σ.
Any other 1-form n normal to Σ can be written as $n = gdf + fn'$ where g is a smooth function

[1] The orbit of a point under a group of diffeomorphisms is the set of points that can be reached by the starting point by acting on it with all of the diffeomorphisms.

with $g \neq 0$ on Σ and n' is a smooth 1-form.
So, if n is normal to Σ then:

$$n \wedge dn|_{\Sigma} = 0$$

Conversely,

Theorem 1 (Frobenius)**.** If $n \neq 0$ is a 1-form such that $n \wedge dn = 0$ everywhere then $\exists f, g : \mathcal{M} \to \mathbb{R}$ such that $n = g\,df$ so n is normal to surfaces of constant f i.e. n is hypersurface-orthogonal.

Given this fundamental concept, we can now define a static spacetime as follows:

Definition 3. A spacetime is **static** if it admits a hypersurface-orthogonal timelike Killing vector field.

Remark. It is straightforward to notice that a static spacetime is also stationary.

For such a spacetime, we know that k is hypersurface-orthogonal so when defining adapted coordinates we can choose Σ to be orthogonal to k. At the same time, Σ is the surface at $t = 0$, with normal dt. Thus we must have that $k_\alpha = g_{\alpha\beta}k^\beta \propto (1, \mathbf{0})$ which implies that $k_i = 0$. Hence, $k_i = g_{i\alpha}k^\alpha = g_{i0}k^0 = g_{i0}(\mathbf{x}) = 0$ from which we conclude that $g_{i0}(\mathbf{x}) = 0$.
Therefore, in adapted coordinates a static metric takes the form

$$ds^2 = g_{00}(\mathbf{x})\,dt^2 + g_{ij}(\mathbf{x})\,dx^i dx^j \tag{1.3}$$

with $g_{00}(\mathbf{x}) > 0$.

It is quite obvious to notice that *static* then means **time-independent** and **invariant under time reversal**. Both of these properties result to be fundamental in the formulation of the model of Black Hole Fireworks.

1.3 The Schwarzschild Black Hole

We are interested in determining the gravitational field of a time-independent spherical object so we assume our spacetime to be stationary and spherically symmetric. It can be shown (as a part of the Birkhoff's theorem, see below) that any such spacetime must actually be static.

The **Schwarzschild metric** (1916) is a solution to the vacuum Einstein's Field Equations, i.e. $R_{\mu\nu} = 0$, and it is given by

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{1.4}$$

where $0 < r < \infty$ is defined as above and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the round metric on S^2.

The line element (1.4) is the unique spherically symmetric solution to the vacuum Einstein's equations. This result is known as Birkhoff's theorem and it has strong implications.
More precisely,

Theorem 2 (Birkhoff)**.** Any spherically symmetric solution of the vacuum Einstein equation is isometric to the Schwarzschild solution.

Remark. It is important to notice that $k = \partial/\partial t$ is timelike for $r > 2m$ so, in this region, the Schwarzschild solution is static.

Birkhoff's theorem implies that the spacetime outside any spherical body is described by the time-independent exterior Schwarzschild solution. This is true even if the body itself is time-dependent. Moreover, the spacetime outside the star will be described by the static Schwarzschild solution even during the collapse.

Let us then study the features of the most important type of geodesics, i.e. the **radial null geodesics**. These curves are defined by the following properties:

$$ds^2 = 0\,, \qquad d\theta = d\phi = 0$$

So, the line element (1.4) reduces to

$$0 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2$$

Then, if we introduce the *Regge-Wheeler radial coordinate* r^*, defined as

$$dr^* = \frac{dr^2}{1 - \frac{2m}{r}}$$

Along a radial null geodesic we have

$$\frac{dt}{dr^*} = \pm 1$$

and then

$$t \mp r^* = const.$$

Remark. The coordinate $r^* = r^*(r)$ is often called *tortoise* coordinate, because r^* changes only logarithmically close to the horizon. This coordinate change maps the range $r \in (2m, \infty)$ of the radial coordinate onto $r^* \in \mathbb{R}$.

We can now define a new coordinate

$$v = t + r^*$$

which is manifestly constant along ingoing radial null geodesics.
If we then want to use (v, r, θ, ϕ) as coordinates, we could eliminate t from the line element (1.4) by the substitution $t = v - r^*$. Hence, the Schwarzschild metric shall be recast as follows:

$$ds^2 = \left(1 - \frac{2m}{r}\right) dv^2 - 2dvdr - r^2 d\Omega^2 \tag{1.5}$$

The latter is then the Schwarzschild metric written in terms of the so called *ingoing Eddington-Finkelstein (EF) coordinates.*
Unlike the metric components in Schwarzschild coordinates, the components of the above matrix are smooth for all $r > 0$, in particular they are smooth at $r = 2m$.

The Schwarzschild spacetime can now be extended through the surface $r = 2m$ to a new region with $r < 2m$. Moreover, it is obvious to see that the new line element is still a solution of the vacuum Einstein's equation also in this new region.

If we now call λ the affine parameter for these radial null geodesics, then one can easily deuce the geodesic equations for the Schwarzschild spacetime, i.e.

$$\frac{dt}{d\lambda} = \left(1 - \frac{2m}{r}\right)^{-1} , \qquad \frac{dr}{d\lambda} = \pm 1$$

where the upper sign corresponds to outgoing geodesics, i.e. increasing r, and the lower is for the ingoing one.
The ingoing radial null geodesics in the EF coordinates are then defined by

$$v = const. , \qquad \frac{dr}{d\lambda} = -1$$

Hence such geodesics will reach $r = 0$ in finite affine parameter. Since the metric is Ricci flat, the simplest non-trivial scalar constructed from the metric is the **Kretschmann scalar**, i.e.

$$K = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48m^2}{r^6} \tag{1.6}$$

That diverges as the radius approaches zero. Since this is a scalar, it diverges in all charts. Therefore there exists no chart for which the metric can be smoothly extended through $r = 0$. This is a clear example of a **curvature singularity**, where tidal forces become infinite and the known laws of physics break down.

Remark. Recall that for $r > 2m$, the Schwarzschild solution admits the Killing vector field $k = \partial/\partial t$. If we now change the coordinates to the ingoing EF coordinates we get that

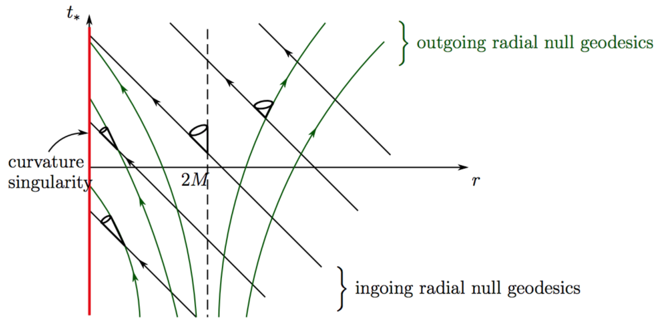
$$v = t - r^* \implies k = \frac{\partial}{\partial t} = \frac{\partial v}{\partial t} \frac{\partial}{\partial v} = \frac{\partial}{\partial v}$$

We can then use this definition in order to extend k to $r \leqslant 2m$. It is easy to see that $k^2 = g_{\mu\nu} k^\mu k^\nu = g_{vv}$, which means that k is null at $r = 2m$ and spacelike for $r < 2m$. Thus, the extended Schwarzschild solution is static only for $r > 2m$.

So far we have considered ingoing radial null geodesics ($v = const.$) in the ingoing EF coordinates. It is then interesting to study the behaviour of the outgoing radial null geodesics in the chart. The latter, for $r > 2m$, are given by $u = t - r^* = const.$.

The simpler way to understand the underling physics behind this problem is to plot the radial null geodesics on a spacetime diagram. In particular, if we define $t^* = v - r$, consequently the ingoing radial null geodesics are straight lines at 45° in the (t^*, r) plane.

This gives the so called **Finkelstein diagram** for the Schwarzschild metric in the ingoing EF coordinates.



It is clear from the plot that the outgoing radial null geodesics have increasing r if $r > 2m$. But if $r < 2m$ then r decreases for both families of null geodesics. Both reach the curvature singularity at $r = 0$ in finite affine parameter. Since nothing can travel faster than light, the same is true for radial timelike curves. Consequently, there is no signal that can be sent from a point with $r < 2m$ to a point with $r > 2m$. This is, indeed, the defying feature of a **Black Hole**. More precisely, a Black Hole is a region of an asymptotically flat spacetime from which we are not able to send a signal to infinity.

1.4 The Black Hole region

At this stage, we could try to characterize the black hole region in a more precise mathematical way. But first, we need to recall some important definitions.

Definition 4. A vector V is **causal** if it is timelike or null, i.e. $V^2 \geqslant 0$. A curve is causal if its tangent vector is everywhere causal.

Let $(\mathcal{M}, g)$ be a spacetime. At each point $p \in \mathcal{M}$, the tangent space $T_p\mathcal{M}$ is clearly isomorphic to the Minkowski spacetime. Thus, at any point of a spacetime, the metric determines two light-cones in the tangent space at that point[2]. Our aim is to regard one of these as the *future light-cone* and the other as the *past light-cone*. To do so we can, for example, pick a causal vector field and define the future light-cone to be the one in which it lies, thus

Definition 5. A spacetime is **time-orientable** if it admits a time-orientation, i.e. a causal vector field T. Then, a causal vector V is **future-directed** if it lies in the same light cone as T and past-directed otherwise.

Remark. Any other time orientation is either everywhere in the same light-cone as T or everywhere in the opposite light-cone. Hence a time-orientable spacetime admits exactly two inequivalent time-orientations.

For example, for the Schwarzschild metric in the ingoing EF coordinates $\xi = \partial/\partial r$ is globally null, indeed $\xi^2 = g_{rr} = 0\,,\ \forall p \in \mathcal{M}$, hence defines a time-orientation. Therefore we can use ξ to define our time orientation for $r > 0$.This is not the case for $k = \partial/\partial t = \partial/\partial v$ because it is

[2]It is very important to empathize that the light-cone of p is a subset of $T_p\mathcal{M}$, not $\mathcal{M}$

timelike for $r > 2m$, null for $r = 2m$ and spacelike for $r < 2m$.
All this discussion results to be important in the proof of the following fundamental property of the Schwarzschild metric in the ingoing EF coordinates:

Proposition 1. Let $x^\mu(\lambda)$ be any future-directed causal curve. If $\exists\lambda_0$ such that $r(\lambda_0) \leqslant 2m$, then $r(\lambda) \leqslant 2m\,,\ \forall\lambda \geqslant \lambda_0$.

This result implies that no future-directed causal curve connects a point with $r \leqslant 2m$ to a point with $r > 2m$. This statement makes more precise our definition of a black hole. Moreover, the boundary of the black hole region is called the **Event Horizon**. In particular, it is now straightforward to see that the surface $r = 2m$ is the event horizon for the Schwarzschild black hole.

1.4.1 White Holes

If we, instead, extend the Schwarzschild spacetime with outgoing Eddington- Finkelstein coordinates, i.e. $u = t - r^*$, the line element (1.4) becomes

$$ds^2 = \left(1 - \frac{2m}{r}\right) du^2 + 2dudr - r^2 d\Omega^2 \tag{1.7}$$

The same analysis reveals that ingoing photons emitted at $r > 2m$ or $r < 2m$ never cross $r = 2m$: they approach $r = 2m$ and hover the horizon forever. Conversely, all outgoing null geodesics escape to infinity. Looking at the light-cones, we see that everything inside $r = 2m$ is ejected. The region $r < 2m$ is then called **White Hole** and $r = 2m$ is said to be the white hole horizon. It is also important to notice that this is the exact **time reversal of a black hole**.

1.5 Extendibility & the Kruskal spacetime

In the previous sections we have shown that the Schwarzschild solution of the vacuum Einstein's equations can be analytically extended in two different ways, revealing the existence of a black hole region and a white hole region.
It is quite important then to properly empathize the concept of analytic extension in general relativity.

Definition 6. A spacetime $(\mathcal{M}, g)$ is said to be **extendible** if it is isometric to a proper subset of another spacetime $(\mathcal{M}', g')$. The latter is called an extension of $(\mathcal{M}, g)$. Otherwise, a specetime is said to be not extendible.

A canonical example is given by the relation between the Schwarzschild solution and the **Kruskal spacetime**.

Let us consider $r > 2m$. In this region we can define the **Kruskal-Szekeres coordinates** (U, V, θ, ϕ) by

$$U = -\exp\left(-u/4m\right)\,, \qquad V = \exp\left(v/4m\right) \tag{1.8}$$

thus $U < 0$ and $V > 0$.
Note that

$$UV = -\exp\left(r^*/2m\right) = \left(\frac{r}{2m} - 1\right)\exp\left(r/2m\right)\,, \qquad \frac{U}{V} = -\exp\left(t/2m\right)$$

which give us an unambiguous determination of $r = r(U, V)$ and $t = t(U, V)$.
Now, it is easy to see that the Schwarzschild spacetime can be rewritten as

$$ds^2 = \frac{32m^3 \exp(-r/2m)}{r} dUdV - r^2 d\Omega^2 \tag{1.9}$$

One can see by inspection that this new metric can be analytically extended, with non-vanishing determinant, through the surfaces $U = 0$ and $V = 0$ to the new regions $U > 0$ and $V < 0$. Moreover, from the defining equations for r and t we can see that the surface $r = 2m$ is actually two surfaces ($U = 0$ and $V = 0$) that intersect at $U = V = 0$.
Similarly, the curvature singularity at $r = 0$ corresponds to the two branches of the hyperbola given by $UV = 1$. This information, together with the causal structure of such a spacetime, can be summarized by the Carter-Penrose diagram of the Kruskal-Szekeres spacetime.

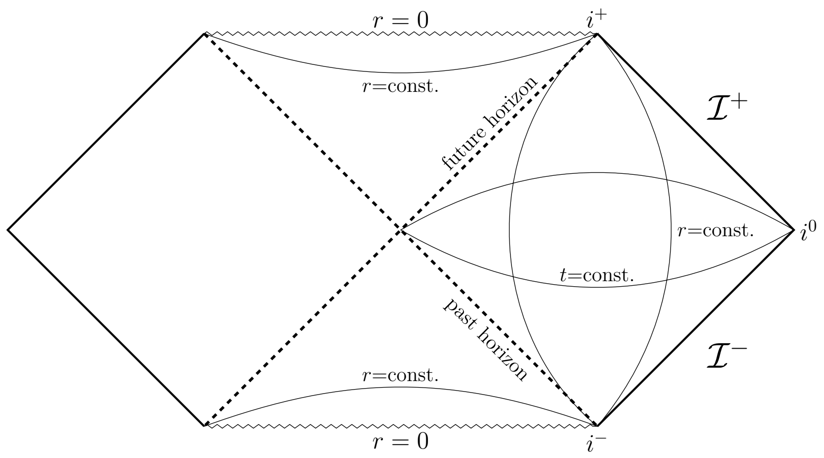


Figure 1.1: Penrose-Carter diagram of the Kruskal-Szekeres spacetime.

The Kruskal-Szekeres spacetime allows us to relate the black hole and the white hole extensions of the Schwarzschild solution. In particular, this spacetime represents the **maximal extension** of the Schwarzschild spacetime.

1.6 Some remarks about Singularities

By definition, a metric tensor is said to be singular if, in some basis, its components are not smooth or its determinant vanishes. A **coordinate singularity** can be eliminated by a change of coordinates, as it happens for the singularity at $r = 2m$ for the Schwarzschild spacetime. These are unphysical. However, if it is not possible to eliminate the bad behaviour by a change of coordinates then we have a **physical singularity**. In this chapter we have encountered such singularities while constructing the Kretschmann scalar for the Schwarzschild solution. However, it is also possible to have more general curvature singularities for which no

scalar constructed from the Riemann tensor diverges but, nevertheless, there exists no chart in which the Riemann tensor remains finite. Moreover, not all physical singularities are curvature singularities (e.g. conical singularities).
A problem in defining singularities is that they are not "places" of the spacetime with some particular features, they, indeed, do not belong to the spacetime manifold at all, due to the fact that we define spacetime as a pair $(\mathcal{M}, g)$ where g is a smooth Lorentzian metric.
A common property of singularities is that there must exist some geodesics that cannot be extended to arbitrarily large affine parameter because they "end" at the singularity. It is this property that we will use to define what we mean by "singular".

Definition 7. $p \in \mathcal{M}$ is a **future endpoint** of a future-directed causal curve $\gamma : (a, b) \to \mathcal{M}$ if, for any neighbourhood O of p, there exists $t_0 \in \mathbb{R}$ such that $\gamma(t) \in O\,,\ \forall t > t_0$. We say that γ is **future-inextendible** if it has no future endpoint. Similary for past endpoints and past inextendibility. γ is **inextendible** if it is both future and past inextendible.

Definition 8. A geodesic is complete if an affine parameter for the geodesic extends to $\pm\infty$. A spacetime is **geodesically complete** if all inextendible causal geodesics are complete.

One can easily convince himself that a spacetime that is extendible will also be geodesically incomplete. However, the Kruskal spacetime is both inextendible, being the maximal extension of the Schwarzschild spacetime, but nonetheless geodesically incomplete because one can always find a geodesic that hits $r = 0$ in finite affine parameter. **So we will regard a spacetime as singular if it is geodesically incomplete and inextendible**.

Chapter 2

Causal Structure and Predictability

Almost every physical problem can be cast as an **initial value problem**. In particular, if we are interested in knowing the state of a system at some moment in time, provided the state of the system at an earlier time and the laws of physics (encoded in a set of partial differential equations), the fact that such a problem makes sense is due to the concept of **causality**, i.e. the idea that future events can be understood as consequences of certain initial conditions coupled with the laws of physics.
In this chapter we will quickly review some of the most relevant concepts used in understanding how causality works in general relativity and how it relates with black hole physics. In this chapter we shall look at the problem of evolving matter field on a fixed background spacetime, rather the the evolution of the metric itself. Our guiding principle will be that no physical signals can travel faster then light, therefore information will only travel along null or timelike trajectories.

Definition 9. Let $(\mathcal{M}, g)$ be a time-orientable spacetime and $S \subset \mathcal{M}$. The **chronological future** of S, denoted $I^+(S)$, is the set of points of $\mathcal{M}$ which can be reached by a future-directed timelike curve starting on S. The **causal future** of S, denoted $J^+(S)$, is the union of S with the set of points of $\mathcal{M}$ which can be reached by a future-directed causal curve starting on S. The chronological past $I^-(S)$ and causal past $J^-(S)$ are defined similarly.

Definition 10. Let $S \subset \mathcal{M}$. Then S is said to be **achronal** if no two points in S are connected by a timelike curve.

Now, if we consider a *closed* achronal set $S \subset \mathcal{M}$, we can define the **future domain of dependence** of S, $D^+(S)$, as the set of all points $p \in \mathcal{M}$ such that *every* past moving inextendible[1] causal curve through p must intersect S. It is trivial to notice that, $S \subset D^+(S)$. The past domain of dependence, $D^-(S)$, is defined in a similar way. Moreover, we can also define, roughly speaking, the **future Cauchy horizon**, denoted $H^+(S)$, as the boundary of $D^+(S)$; analogously one can also define the past Cauchy horizon, $H^-(S)$. It is easy to see that $H^\pm(S)$ have to be null surfaces.

The usefulness of these definition is due to the fact that, if nothing can travel faster than light, then signals cannot propagate outside the light-cone of any $p \in \mathcal{M}$. Hence, if every curve that remains inside this light-cone must intersect S, then the informations specified on S should

[1]It basically means that the cure does not end at some *finite* point.

be sufficient to **predict** the situation at p.
The set of all points for which we can predict what happens by knowing the conditions on S is then given by $D(S) = D^+(S) \bigcup D^-(S)$, which is simply called **Domain of Dependence** of S.

Definition 11. A closed achronal surface Σ is said to be a **Cauchy surface** if $D(\Sigma) = \mathcal{M}$.

Then, given the initial data on a Cauchy surface we can predict what happens throughout all of spacetime. Nevertheless, in general $D(\Sigma) \neq \mathcal{M}$ thus solutions of hyperbolic equations will not be uniquely determined in $\mathcal{M} \backslash D(\Sigma)$ by data on Σ. Hence, given only this data, there will be infinitely many different solutions on $\mathcal{M}$ which agree within $D(\Sigma)$. Moreover, one can also define the concept of *partial Cauchy surface*, which is basically a closed, achronal and edgeless hypersurface of $\mathcal{M}$.

Definition 12. A spacetime $(\mathcal{M}, g)$ is **globally hyperbolic** if it admits a Cauchy surface.

Theorem 3. Let $(\mathcal{M}, g)$ be globally hyperbolic. Then

(i) there exists a global time function;

(ii) surfaces of constant t are Cauchy surfaces, and these all have the same topology $\mathcal{T}$;

(iii) the topology of M is $\mathbb{R} \times \mathcal{T}$.

The concept of global hyperbolicity was firstly introduced by Leray in order to consider well-posedness of the Cauchy problem for the wave equation on the manifold. In view of the initial value formulation for Einstein's equations, global hyperbolicity is seen to be a very natural condition in the context of general relativity, in the sense that given arbitrary initial data, there is a unique maximal globally hyperbolic solution of Einstein's equations.

2.1 Asymptotically Flat Spacetimes

So far we have seen that it is reasonable to define a black hole, roughly speaking, as the region of spacetime from which no information-carrying signals are allowed to escape to a distant observer. However, in order to make this definition rigorous, one must clarify what class of observers is meant and what is the geometrically invariant meaning of the term "distant".
The necessary refinement is easily achieved in the physically important case in which there is no matter and no sources of fields far from the black hole. The greater the distance from the black hole, the smaller the deviations of the spacetime geometry are from flatness. A spacetime with this property is said to be **asymptotically flat**.

2.1.1 Conformal Compactification

Consider a spacetime $(\mathcal{M}, g)$.

Definition 13. A **conformal transformation** is a map such that

$$(\mathcal{M}, g) \longrightarrow (\mathcal{M}, \bar{g}) \qquad g(x) \mapsto \bar{g}(x) = \Omega^2(x)\, g(x)$$

where Ω is a smooth positive function on $\mathcal{M}$.

The metrics g and $\bar{g}$ agree on the definitions of timelike, spacelike and null so we have that conformal transformations preserve the causal structure of the spacetime.
The idea of **conformal compactification** is to choose Ω so that points at infinity with respect to g are at finite distance with respect to the new unphysical metric $\bar{g}$. To do this we need $\Omega \to 0$ at "infinity". More precisely, we try to choose Ω so that the spacetime $(\mathcal{M}, \bar{g})$ is part of a larger unphysical spacetime $(\overline{\mathcal{M}}, \bar{g})$. $\mathcal{M}$ is then a proper subset of $\overline{\mathcal{M}}$ with $\Omega|_{\partial\mathcal{M}} = 0$ in $\overline{\mathcal{M}}$. This boundary corresponds to infinity of the physical spacetime.

2.1.2 Asymptotic Flatness

Assuming that the properties of asymptotic flat spaces in the neighborhood of infinity must be similar to those of Minkowski space, Penrose suggested the following definitions.
Firstly, let us define asymptotically simple spacetimes:

Definition 14. A spacetime $(\mathcal{M}, g)$ is said to be **asymptotically simple** if there exists another unphysical space $(\overline{\mathcal{M}}, \bar{g})$ with boundary $\partial\overline{\mathcal{M}} \equiv \mathcal{I}$ and a regular metric $\bar{g}$ such that:

i $\overline{M} \backslash \mathcal{I}$ is conformal to $\mathcal{M}$, and $\bar{g} = \Omega^2 g$ in $\mathcal{M}$;

ii $\Omega|_{\mathcal{M}} > 0$, $\Omega|_{\mathcal{I}} = 0$ and $\partial_\mu \Omega|_{\mathcal{I}} \neq 0$;

iii Each null geodesic in M begins and ends on $\mathcal{I}$.

The unphysical space $\overline{M}$, defined as above, is then called conformal Penrose space.

Let $(\mathcal{M}, g)$ be an asymptotically simple spacetime, let g be such that $\text{Ric} = 0$ in the neighborhood of $\mathcal{I}$ and assume that the natural conditions of causality and spacetime orientability are satisfied. Then, $(\mathcal{M}, g)$ has the following properties:

1. The topology of $\mathcal{M}$ is $\mathbb{R}^4$;
2. $\mathcal{I}$ is null and consists of two disconnected components, i.e. $\mathcal{I} = \mathcal{I}^+ \bigcup \mathcal{I}^-$, each diffeomorphic to $\mathbb{R} \times S^2$;
3. The generators of the surfaces $\mathcal{I}^\pm$ are the null geodesics in $\overline{\mathcal{M}}$;

The first two results tell us that the global structure of the asymptotically flat space is the same as that of Minkowski space, as we were expecting.
According to [6], in order to take into account the existence of localized regions of strong gravitational fields which do not alter the asymptotic properties of spacetime, it is sufficient to analyse the class of spaces that can be converted into asymptotically simple spaces by removing certain inner regions containing singularities of some kind and by subsequent smooth patching of the resultant holes. Such spaces are said to be **weakly asymptotically simple**.
Now, a weakly asymptotically simple spacetime is **asymptotically flat** if its metric in the neighborhood of $\mathcal{I}$ satisfies Einstein's vacuum equations.

Chapter 3

Black Holes & The Singularity Theorem

3.1 Formal definition of Black Hole

Now that we have given a proper definition of "infinity" as well as a detailed characterization of the causal structure of a Lorentzian manifold, we can then make more precise our definition of a black hole as a region of an asymptotically flat spacetime from which it is impossible to send a signal to infinity.

Definition 15 (Black Hole)**.** Let $(\mathcal{M}, g)$ be a spacetime that is asymptotically flat at null infinity. The **black hole** region is

$$\mathcal{B} = \mathcal{M} \setminus [M \cap J^-(\mathcal{I}^+)]$$

where $J^-(\mathcal{I}^+)$ is defined by means of the unphysical spacetime $(\overline{\mathcal{M}}, \bar{g})$. The **future event horizon** is then defined as $\mathcal{H}^+ = \partial\mathcal{B}$.

The definition which has been just presented can be enlightened by means of a Carter-Penrose diagram.

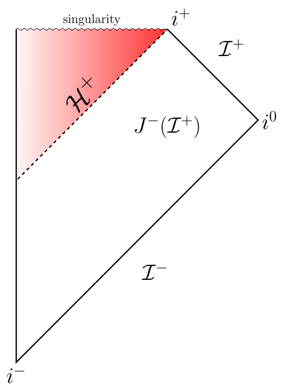


Figure 3.1: General Carter-Penrose diagram for an Eternal Black Hole. The black hole region is red-shaded while the future horizon is represented by a dashed line.

Similarly, the **white hole** region is $\mathcal{W} = \mathcal{M} \setminus [M \cap J^+(\mathcal{I}^-)]$ and the past event horizon is $\mathcal{H}^- = \partial\mathcal{W}$.

From [1], provided a time-orientable spacetime $(\mathcal{M}, g)$, we have the following fundamental theorems:

Theorem 4. Let $S \subset \mathcal{M}$. Then $\partial J^+(S)$ is an achronal 3d submanifold of $\mathcal{M}$.

Theorem 5. Let $S \subset \mathcal{M}$ be closed. Then every $p \in \partial J^+(S)$ with $pnotinS$ lies on a null geodesic γ lying entirely in $\partial J^+(S)$ and such that γ is either past-inextendible or has a past endpoint on S.

These theorems tell us that $\mathcal{H}^{\pm}$ have to be null hypersurfaces. Moreover, the time reversal of the second theorem implies that the generators of $\mathcal{H}^+$ cannot have future end- points. However, they can have past endpoints (e.g. the point in which a black hole forms in a gravitational collapse of a star). So null generators can enter $\mathcal{H}^+$ but they cannot leave it.

There is also another technical condition that will be relevant in the following.

Definition 16. An asymptotically flat spacetime $(\mathcal{M}, g)$ is said to be **strongly asymptotically predictable** if there exists an open region $\overline{V} \subset \overline{\mathcal{M}}$ such that $\{\mathcal{M} \cap J^-(\mathcal{I}^+)\}^- \subset \overline{V}$, i.e. the closure of $\mathcal{M} \cap J^-(\mathcal{I}^+)$ is contained in $\overline{V}$, and $(\overline{V}, \bar{g})$ is globally hyperbolic.

This definition implies that $(\mathcal{M} \cap \overline{V}, g)$ is a globally hyperbolic subset of $\mathcal{M}$. Roughly speaking, there is a globally hyperbolic region $\mathcal{M} \cap \overline{V}$ of spacetime consisting of the region not in $\mathcal{B}$ together with a neighbourhood of $\mathcal{H}^+$. It ensures that physics is predictable on, and outside, $\mathcal{H}^+$. A simple consequence of this definition is the result that a black hole cannot bifurcate, i.e. split into two.

3.2 The Singularity Theorem

The Schwarzschild solution of the Einstein's field equations clearly tells us that the final stage of a spherically symmetric gravitational collapse might result in the formation of a curvature singularity. It is then interesting to investigate whether such outcome is a feature of the spherical symmetry rather then a property of more general collapses. However, in 1965 Penrose formulated his notorious singularity theorem which basically states that singularities are a generic prediction of general relativity.

3.2.1 Null Hypersurfaces

Null hypersurfaces have an interesting geometry, and play an important role in general relativity. In particular, as we have seen, they represent horizons of various sorts, such as the event horizons. Let $(\mathcal{M}, g)$ be be a spacetime.

Definition 17. A null hypersurface is a hypersurface whose normal is everywhere null.

Let n be normal to a null hypersurface $\Sigma \subset \mathcal{M}$. Then any vector $X \neq 0$ tangent to the hypersurface obeys $n \cdot X \equiv g_{\mu\nu} n^\mu X^\nu$ which implies that either X is spacelike or X is parallel to n, i.e. null. In particular, note that n is also tangent to the hypersurface. Hence, the integral curves of n lie within Σ.

Proposition 2. The integral curves of a are null geodesics. These are called the **generators** of Σ.

Recalling that a *geodesic congruence* in $U \subset \mathcal{M}$ is a family of geodesics such that exactly one geodesic passes through each $p \in U$, then we can define the **null expansion scalar** θ of Σ with respect to n as a smooth function on Σ that gives a measure of the average expansion of the null generators of Σ towards the future, and it is defined as the divergence of the vector field n along Σ, i.e. $\theta = \nabla_\alpha n^\alpha$.
While θ depends on the choice of n, it does so in a simple way. Moreover, a positive rescaling of n rescales θ in the same way, i.e. $\tilde{n} = fn \;\Rightarrow\; \tilde{\theta} = f\theta$. Thus the sign of the null expansion θ does not depend on the scaling of n; therefore $\theta > 0$ implies expansion on average of the null generators, and $\theta < 0$ means contraction on average.
It is useful to understand how the null expansion varies as one moves along a null generator of Σ. Let $\lambda \to \gamma = \gamma(\lambda)$ be a null geodesic generator of Σ and assume n is scaled so that γ is affinely parameterized. Then it can be shown that the null expansion scalar $\theta = \theta(\lambda)$ along γ satisfies the propagation equation,

$$\frac{d\theta}{d\lambda} = -Ric(\gamma', \gamma') - \sigma^2 - \frac{1}{2}\theta^2 \tag{3.1}$$

where $\gamma' \equiv (dx^\mu/d\lambda)\partial_\mu = T^\mu \partial_\mu$, $Ric(\gamma', \gamma') = R_{\mu\nu}T^\mu T^\nu$ and σ, the **shear tensor**, measures the deviation from perfect isotropic expansion. Equation (3.1) is known as the **Raychaudhuri's equation** for a null geodesic congruence and, together with a timelike version, plays an important role in the proofs of the classical Hawking-Penrose singularity theorems.

Equation (3.1) shows how the curvature of space-time influences the expansion of the null generators. Here, we can see, for example, a trivial consequence of this equation.

Proposition 3. Let $(\mathcal{M}, g)$ be a spacetime which obeys the null energy condition[1] and let Σ be a smooth null hypersurface in $\mathcal{M}$. If the null generators of Σ are future geodesically complete then the null generators of σ have nonnegative expansion, $\theta \geqslant 0$.

Proof. Suppose $\theta < 0$ at $p \in \Sigma$. Let $\gamma : [0, \infty) \to \Sigma$ such that $\lambda \mapsto \gamma(\lambda)$ be the null geodesic generator of Σ passing through $p = \gamma(0)$. Let also assume γ to be affinely parametrized. Let $\theta = \theta(\lambda)$ be the null expansion of Σ along γ; hence $\theta(0) < 0$.
From the Raychaudhuri's equation and the null energy condition we have that

$$\frac{d\theta}{d\lambda} \leqslant -\frac{1}{2}\theta^2$$

which implies that $\theta(\lambda) < 0$ for all $\lambda > 0$.
Moreover, it is easy to see that the letter inequality can be recast as follows:

$$\frac{d}{d\lambda}\left(\frac{1}{\theta}\right) \geqslant \frac{1}{2}$$

thus $\theta^{-1} \to 0$, i.e. $\theta \to -\infty$ in **finite affine parameter time** ($\lambda = 2/|\theta(0)|$), which is in contradiction with the smoothness assumption for θ. □

This result is strictly connected with black hole physics, indeed it is a rudimentary form of the celebrated Hawking's area theorem.

[1]Null energy condition: $Ric(X, X) = R_{\mu\nu}X^\mu X^\nu \geqslant 0\,,\ \forall X\ :\ X^2 = 0$

3.2.2 Trapped Surfaces & Penrose Singularity Theorem

Let us begin with some definitions. Let Σ be a spacelike 2-dimensional submanifold of the spacetime $(\mathcal{M}, g)$. We are primarily interested in the case where Σ is compact (without boundary), and so we simply assume this from the outset.
Each normal space of Σ, $[T_p\Sigma]^\perp$, $p \in \Sigma$, is timelike and 2-dimensional, and hence admits two future directed null directions orthogonal to Σ. Thus, if the normal bundle is trivial, Σ admits two smooth non-vanishing future directed null normal vector fields l_+ and l_-, which are unique up to positive pointwise scaling.
We can then decompose the second fundamental form of Σ into two scalar valued **null second forms** $\chi_\pm$ related to $l_\pm$. Thus, for all $p \in \Sigma$ we have that

$$\chi_\pm : T_p\Sigma \times T_p\Sigma \to \mathbb{R}\,, \quad \chi_\pm(X,Y) := g(\nabla_X l_\pm, Y)$$

It can be proven that $\chi_\pm$ is symmetric. Thus, they can be traced with respect to the induced metric q on Σ to obtain the null mean curvatures, also known as null expansion scalars,

$$\theta_\pm = \mathtt{Tr}_q \chi_\pm = q^{ij}(\chi_\pm)_{ij} = \mathtt{div}_\Sigma l_\pm \tag{3.2}$$

Physically, θ_+ (resp., θ_-) measures the divergence of the outgoing (resp., ingoing) light rays emanating from Σ.
In regions of spacetime where the gravitational field is strong, one may have both $\theta_- < 0$ and $\theta_+ < 0$, in which case Σ is called a **trapped surface**.

Under appropriate energy and causality conditions, the occurrence of a trapped surface signals the onset of gravitational collapse. This is the implication of the Penrose singularity theorem, the first of the famous singularity theorems.

Theorem 6 (Penrose, 1965)**.** Let $(\mathcal{M}, g)$ be globally hyperbolic with a non-compact Cauchy surface Σ. Assume that the Einstein's equation and the null energy condition are satisfied and that $\mathcal{M}$ contains a trapped surface $\mathcal{T}$. Let $\theta_0 < 0$ be the maximum value of θ on $\mathcal{T}$ for both sets of null geodesics orthogonal to $\mathcal{T}$. Then at least one of these geodesics is future-inextendible and has affine length no greater than $2/|\theta_0|$.

The Einstein's equation possesses the property of **Cauchy stability**, which implies that the solution in a compact region of spacetime depend continuously on the initial data. In other words, Cauchy stability implies that if one perturbs the initial data (e.g. breaking spherical symmetry, for which we know that singularities may occur) then the resulting spacetime will also have a trapped surface, for a small enough initial perturbation. This shows that trapped surfaces occur generically in gravitational collapse.

Moreover, the Penrose's theorem can be restated equivalently as follows:

Theorem 7. A spacetime containing a trapped surface is either not globally hyperbolic or it is not geodesically complete.

The first possibility is, however, generically excluded assuming the correctness of the **strong cosmic censorship** conjecture. So, here are very good reasons to believe that gravitational collapse leads to geodesic incompleteness. Nevertheless, the singularity theorems tell us nothing about the nature of this singularity, indeed they they not forced to be curvature singularities as in the spherically symmetric case.

Chapter 4

Semiclassical aspects of Black Hole physics

It is well known that the quantum theory of fields (QFT), at lest the one used to describe the Standard Model of particle physics, is restricted inertial observers in the Minkowski spacetime. This combination is very peculiar for two reasons: first, the Minkowski spacetime has a timelike Killing vector field. Secondly, no event horizons occurs for inertial observers in this spacetime. The existence of a unique timelike Killing vector field ∂_t which has as eigenfunctions the modes $\exp(-i\omega t)$ implies that all inertial observers agree on how to split positive and negative frequency modes. This splitting selects in turn the standard **Minkowski vacuum** $|0\rangle_M$. The main feature of this vacuum state is due to the fact that no inertial observers will register particles in the vacuum state $|0\rangle_M$, due to the fact that it is invariant under Poincaré transformations.
In this chapter we shall consider the more general situation. Intuitively, we might expect that a static spacetime could be create particles if event horizon exists. Indeed, one can prove, at least theoretically, that a **thermal spectrum** of particles is created close to the horizon.

4.1 Elements of QFT in Curved Spacetime

In analogy with what we do in Minkowski space, we can perform both canonical and covariant quantisation in order to quantise classical field theories in curved spacetimes. In particular, the latter approach is useful if we are interested in quantum corrections to the stress tensor.
The expectation value of the stress-energy tensor $T_{\mu\nu}$ for the quantum field Φ in the background of a **classical** gravitational field g is given by:

$$\langle T_{\mu\nu}\rangle = \frac{1}{Z}\int \mathcal{D}\Phi\, T_{\mu\nu}\, \exp\left(iS[\Phi, g]\right)\,, \qquad Z = \exp\left(iW\right) = \int \mathcal{D}\Phi\, \exp iS[\Phi, g] \tag{4.1}$$

If we now recall the definition of the dynamical stress-energy tensor, arising from the variational principle for general relativity, i.e.

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g_{\mu\nu}} \tag{4.2}$$

then we have that

$$\langle T_{\mu\nu}\rangle = \frac{2}{\sqrt{-g}}\frac{\delta W}{\delta g^{\mu\nu}} \tag{4.3}$$

Now, having calculated $\langle T_{\mu\nu}\rangle$, one could aim at solving the Einstein equations in the semiclassical limit, i.e.

$$G_{\mu\nu} = 8\pi \langle T_{\mu\nu}\rangle \tag{4.4}$$

In this way, one discovers two effects:

- the gravitational background can produce particles;
- the gravitational background modifies the zero-point energies of the ϕ-vacuum.

It is important to notice that this approach is based on a local quantity, i.e. $\langle T_{\mu\nu}\rangle = \langle T_{\mu\nu}(x)\rangle$, thus if we can show in a specific frame that e.g. $\langle T_{\mu\nu}(x)\rangle = 0$, then any observer will agree on that. However, we will see that the expectation value for the number of particles measured in a certain vacuum state is observer dependent.

As we have already stressed, all observers in inertial frames agree on the choice of the vacuum and thus also on one and many-particle states. By contrast, in curved spacetimes no inertial system can be globally extended to cover the whole manifold, thus no unique definition of the vacuum is possible. As a consequence, the notion of particle number becomes observer dependent, thus creation of particles becomes possible as different observers may have different notions of vacuum state.
The first task in field theory is then to find a mapping between field operators defined with respect to different vacua. The relation between the two sets of field operators is provided by the so called *Bogulyubov transformation.*

The particle production can be seen as a consequence of two different cases: in the first one, the space-time is time-dependent and can perform "work" and thus create particles. The second, is the emission of a thermal spectrum of particles close to a horizon. We will consider in the next section the second case, investigating the simplest case of an accelerated observer in Minkowski space.

4.2 Uniformly accelerated observer in Special Relativity

In the following we refer to the Minkowski coordinates (t, x, y, z) as the lab frame. Let us consider an observer accelerated in positive x-direction. Her proper coordinate system (the one in the observer rest frame) is given by (τ, ξ, y, z). The world line can then be parametrized by the proper time τ and the observer has a 4-velocity vector

$$u^\alpha = \frac{dx^\alpha}{d\tau} \equiv \dot{x}(\tau)\,, \qquad u^2 \equiv u^\alpha u_\alpha = 1 \tag{4.5}$$

Hence, in the proper frame the 4-acceleration

$$a^\alpha = \ddot{x}^\alpha = \dot{u}^\alpha \tag{4.6}$$

assumes the simple form

$$a^\alpha = (0, a, 0, 0)$$

Consequently, this implies that

$$a^\alpha a_\alpha = -a^2 \tag{4.7}$$

in all frames. From now on we abandon the constant coordinates y and $z$and work in $(1+1)$-dimensional Minkowski spacetime.
The set of ordinary differential equations in (4.5) is clearly hyperbolical and has the solutions:

$$u^0(\tau) = \cosh\left(f(\tau)\right)\,, \qquad u^1(\tau) = \sinh\left(f(\tau)\right) \tag{4.8}$$

where $f(\tau)$ is a differentiable function and we assume that proper time runs into the same direction as coordinate time, $u^0 > 0$.
Deriving u^α and comparing with $a^\alpha a_\alpha = -a^2$ yields

$$f(\tau) = a\tau \tag{4.9}$$

if we choose $u^1(0) = 0$ as initial condition.
If we set $x(0) = 1/a$ and $t(0) = 0$, by integration we obtain the world line:

$$x^\mu(\tau) = (a^{-1}\sinh(a\tau),\, a^{-1}\cosh(a\tau)) \tag{4.10}$$

where we neglect the two transverse dimensions.

4.2.1 Rindler spacetime

Recall that the trajectory of an accelerated observer is given by

$$t(\tau) = \frac{1}{a}\sinh(a\tau)\,, \qquad x(\tau) = \frac{1}{a}\cosh(a\tau)$$

It describes one branch of the hyperbola $x^2 - t^2 = a^{-2}$.
To compare a quantum field in lab frame and proper (conformally flat) frame we need a coordinate transformation $t(\tau,\xi)$, $x(\tau,\xi)$. Since the accelerated frame is not inertial it cannot be a Lorentz transformation.
It can be shown that such transformation exists and it is given by:

$$t(\tau,\xi) = \frac{e^{a\xi}}{a}\sinh(a\tau)\,, \qquad x(\tau,\xi) = \frac{e^{a\xi}}{a}\cosh(a\tau) \tag{4.11}$$

Now, $(\tau,\xi) \in \mathbb{R}^2$ are then called **Rindler coordinates**, and the line element can be rewritten as

$$ds^2 = e^{2a\xi}(d\tau^2 - d\xi^2). \tag{4.12}$$

which is known as the $(1+1)$-dimensional **conformal Rindler metric**.

4.3 Massless Scalar Field in the $(1+1)$-Rindler spacetime

The action of a massless scalar field $\phi(t,x)$ is

$$S[\phi] = \frac{1}{2}\int d^2x\sqrt{-g}\, g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \tag{4.13}$$

where $d^2x\sqrt{-g}$ is the invariant measure.

Remark. The action (4.13) is conformally invariant, indeed as

$$g_{\mu\nu} \longrightarrow \overline{g}_{\mu\nu} = \Omega^2(t,x) g_{\mu\nu}$$

we have that $g^{\mu\nu}$ transforms as Ω^{-2} and while the determinant $\sqrt{-g}$ picks up a factor Ω^2.

Now, apart from a factor $\Omega^2 = e^{2a\xi}$ the Rindler spacetime is actually Minkowskian. Then actions in the lab and in the conformal Rindler coordinates then read:

$$S[\phi] = \frac{1}{2}\int dt dx \, \left[(\partial_t\phi)^2 - (\partial_x\phi)^2\right] = \frac{1}{2}\int d\tau d\xi \, \left[(\partial_\tau\phi)^2 - (\partial_\xi\phi)^2\right] \tag{4.14}$$

The corresponding equations of motion are then given by

$$\partial_t^2\phi - \partial_x^2\phi = 0\,, \qquad \partial_\tau^2\phi - \partial_\xi^2\phi = 0 \tag{4.15}$$

The general solutions are given by

$$\phi(t,x) = A(t-x) + B(t+x)\,, \qquad \phi(\tau,\xi) = C(\tau-\xi) + D(\tau+\xi) \tag{4.16}$$

where A, B, C and D are assumed to be arbitrary smooth functions.
Since the latter expressions solve the Klein-Gordon equations (4.15), one can formulate the mode expansions in both sets of coordinates. Using the dispersion relation $\omega_k = |k|$(for the 1-D spatial momentum $k^1 = k$), one obtains

$$\begin{aligned}\widehat{\phi}(t,x) &= \int_{\mathbb{R}} \frac{dk}{\sqrt{2\pi}\sqrt{2|k|}} \left\{\widehat{a}_k \exp\left(i(kx - |k|t)\right) + \widehat{a}_k^\dagger \exp\left(-i(kx-|k|t)\right)\right\} \\ \widehat{\phi}(\tau,\xi) &= \int_{\mathbb{R}} \frac{dk}{\sqrt{2\pi}\sqrt{2|k|}} \left\{\widehat{b}_k \exp\left(i(k\xi - |k|\tau)\right) + \widehat{b}_k^\dagger \exp\left(-i(k\xi-|k|\tau)\right)\right\}\end{aligned} \tag{4.17}$$

where the field ϕ has been elevated to an operator-valued distribution by means of the canonical quantization procedure. It is worth knowing that the ladder operators $\widehat{a}_k$, $\widehat{a}_k^\dagger$ and $\widehat{b}_k$, $\widehat{b}_k^\dagger$ do not agree in general. Consequently, the Rindler vacuum and the Minkowski vacuum differ, i.e. $|0\rangle_R \neq |0\rangle_M$, where

$$\widehat{a}_k\,|0\rangle_M = 0\,, \qquad \widehat{b}_k\,|0\rangle_R = 0 \qquad \forall k$$

An accelerating observer will then measure that the corresponding vacuum state $|0\rangle_R$ has the lowest possible energy; which will appear to be lower than that of the Minkowski vacuum state $|0\rangle_R$ in such reference frame. Particularly, an observer at rest in the accelerated frame will detect particles when the scalar field is in $|0\rangle_M$. Conversely, the Rindler vacuum $|0\rangle_R$ will appear excited to an observer in the lab frame. This is known as the **Unruh effect**.

4.4 Bogolyubov Transformation

It is convenient to introduce the light-cone coordinates:

$$\begin{aligned}\text{Minkowski}: &\qquad \bar{u} = t - x\,, \quad \bar{v} = t + x \\ \text{Rindler}: &\qquad u = \tau - \xi\,, \quad v = \tau + \xi\end{aligned} \tag{4.18}$$

Recalling that $t = t(\tau, \xi)$ and $x = x(\tau, \xi)$ via (4.11), then it can be proven that:

$$\bar{u} = -\frac{1}{a}\, e^{-au}\,, \qquad \bar{v} = \frac{1}{a}\, e^{-av} \tag{4.19}$$

The metric of the Minkowski spacetime can be written as follows:

$$ds^2 = dt^2 - dx^2 = d\bar{u}d\bar{v} = e^{a(v-u)}\, du dv \tag{4.20}$$

Moreover, the Klein-Gordon Equations (4.15) take the form:

$$\partial_{\bar{u}}\partial_{\bar{v}}\phi = 0\,, \qquad \partial_u\partial_v\phi = 0 \tag{4.21}$$

Then, the general solutions can be written as

$$\phi(\bar{u}, \bar{v}) = A(\bar{u}) + B(\bar{v})\,, \qquad \phi(u, v) = C(u) + D(v) \tag{4.22}$$

To obtain the light-cone mode expansion of $\phi(\bar{u}, \bar{v})$, the first mode expansion in (4.17) must be split in two integrals as follows:

$$\begin{aligned} \widehat{\phi}(t, x) &= \int_{\mathbb{R}} \frac{dk}{\sqrt{2\pi}\sqrt{2|k|}} \left\{\widehat{a}_k \exp\left(i(kx - |k|t)\right) + \widehat{a}^\dagger_k \exp\left(-i(kx - |k|t)\right)\right\} = \\ &= \left(\int_0^{+\infty} + \int_{-\infty}^0\right) \frac{dk}{\sqrt{2\pi}\sqrt{2|k|}} \left\{\widehat{a}_k \exp\left(i(kx - |k|t)\right) + \widehat{a}^\dagger_k \exp\left(-i(kx - |k|t)\right)\right\} \end{aligned} \tag{4.23}$$

Now, recalling that $\omega = |k|$ and making use of the light-cone coordinates the latter expression can be rewritten as

$$\widehat{\phi}(\bar{u}, \bar{v}) = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}\sqrt{2\omega}} \left[e^{-i\omega\bar{u}}\, \widehat{a}_\omega + h.c. + e^{-i\omega\bar{v}}\, \widehat{a}_{-\omega} + h.c.\right] \tag{4.24}$$

Then, comparing the latter with the general solutions:

$$\begin{aligned} \widehat{A}(\bar{u}) &= \int_0^\infty \frac{d\omega}{\sqrt{2\pi}\sqrt{2\omega}} \left[e^{-i\omega\bar{u}}\, \widehat{a}_\omega + h.c.\right] \\ \widehat{B}(\bar{v}) &= \int_0^\infty \frac{d\omega}{\sqrt{2\pi}\sqrt{2\omega}} \left[e^{-i\omega\bar{v}}\, \widehat{a}_{-\omega} + h.c.\right] \end{aligned} \tag{4.25}$$

Analogously,

$$\begin{aligned} \widehat{C}(u) &= \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}\sqrt{2\Omega}} \left[e^{-i\Omega u}\, \widehat{b}_\Omega + h.c.\right] \\ \widehat{D}(v) &= \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}\sqrt{2\Omega}} \left[e^{-i\Omega v}\, \widehat{b}_{-\Omega} + h.c.\right] \end{aligned} \tag{4.26}$$

Now, observe that the coordinate transformations (4.18) never mix u's and v's. One can therefore make the identifications

$$\widehat{A}(\bar{u}(u)) = \widehat{C}(u)\,, \qquad \widehat{B}(\bar{v}(v)) = \widehat{D}(v) \tag{4.27}$$

Now, if we take the Fourier Transform of both sides of the first equation we get:

$$\begin{aligned}\mathcal{F}\left[\widehat{A}(\bar{u});\,\Omega\right] &= \int_{\mathbb{R}} \frac{du}{2\pi}\, e^{i\Omega u}\, \widehat{A}(\bar{u}(u)) = \\ &= \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} \left[F(\omega,\Omega)\,\widehat{a}_\omega + F(-\omega,\Omega)\,\widehat{a}^\dagger_\omega\right]\end{aligned} \tag{4.28}$$

$$\mathcal{F}\left[\widehat{C}(u);\,\Omega\right] = \int_{\mathbb{R}} \frac{du}{2\pi}\, e^{i\Omega u}\, \widehat{C}(u) = \frac{1}{\sqrt{2|\Omega|}} \begin{cases} \widehat{b}_\Omega\,, & \Omega > 0 \\ \widehat{b}^\dagger_{|\Omega|}\,, & \Omega < 0 \end{cases} \tag{4.29}$$

where

$$F(\omega,\Omega) = \int_{\mathbb{R}} \frac{du}{2\pi} e^{i\omega u - i\omega \bar{u}(u)} \tag{4.30}$$

which has to be understood in the distributional sense.
Hence,

$$\widehat{A}(\bar{u}(u)) = \widehat{C}(u) \quad \overset{\mathcal{F}}{\Longrightarrow} \quad \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} \left[F(\omega,\Omega)\,\widehat{a}_\omega + F(-\omega,\Omega)\,\widehat{a}^\dagger_\omega\right] = \frac{1}{\sqrt{2|\Omega|}} \begin{cases} \widehat{b}_\Omega\,, & \Omega > 0 \\ \widehat{b}^\dagger_{|\Omega|}\,, & \Omega < 0 \end{cases} \tag{4.31}$$

That gives us:

$$\widehat{b}_\Omega = \int_0^\infty d\omega \left[\alpha_{\omega\Omega}\,\widehat{a}_\omega + \beta_{\omega\Omega}\,\widehat{a}^\dagger_\omega\right]\,, \qquad \Omega > 0 \tag{4.32}$$

with

$$\alpha_{\omega\Omega} = \sqrt{\frac{\Omega}{\omega}}\, F(\omega,\Omega)\,, \qquad \beta_{\omega\Omega} = \sqrt{\frac{\Omega}{\omega}}\, F(-\omega,\Omega) \tag{4.33}$$

Moreover, the relation for $\widehat{b}^\dagger_\Omega$ as well as the relations connecting $\widehat{a}_\omega$, $\widehat{a}^\dagger_\omega$ and $\widehat{b}_{-|\Omega|}$, $\widehat{b}^\dagger_{-|\Omega|}$ follow from a similar reasoning.
Such transformations are called **Bogolyubov transformation**.

Remark. The most general Bogolyubov transformation is given by

$$\widehat{b}_\Omega = \int_{\mathbb{R}} d\omega \left[\alpha_{\omega\Omega}\,\widehat{a}_\omega + \beta_{\omega\Omega}\,\widehat{a}^\dagger_\omega\right]$$

with $\alpha_{\omega\Omega}$ and $\beta_{\omega\Omega}$ arbitrary complex functions. In order to derive the corresponding normalization conditions, we use the commutation relations of the ladder operators, i.e.

$$[\widehat{a}_\omega, \widehat{a}^\dagger_{\omega'}] = \delta(\omega - \omega')\,, \qquad [\widehat{b}_\Omega, \widehat{b}^\dagger_{\Omega'}] = \delta(\Omega - \Omega')$$

from which we can deduce that

$$\int_{\mathbb{R}} d\omega\, (\alpha_{\omega\Omega}\alpha^*_{\omega\Omega'} - \beta_{\omega\Omega}\beta^*_{\omega\Omega'}) = \delta(\Omega - \Omega')$$

4.5 The Unruh Temperature

The mean number of particles the accelerated observer detects in the Minkowski vacuum is given by the Minkowski vacuum expectation value of the b-particle number operator, i.e.

$$\begin{aligned}\langle \widehat{N}_\Omega \rangle_M &= \langle 0_M | \widehat{b}^\dagger_\Omega \, \widehat{b}_\Omega | 0_M \rangle = \\ &= \langle 0_M | \int_0^\infty d\omega \left[\alpha^*_{\omega\Omega} \, \widehat{a}^\dagger_\omega + \beta^*_{\omega\Omega} \, \widehat{a}_\omega \right] \times \int_0^\infty d\omega' \left[\alpha_{\omega'\Omega} \, \widehat{a}_{\omega'} + \beta_{\omega'\Omega} \, \widehat{a}^\dagger_{\omega'} \right] | 0_M \rangle = \\ &= \int_0^\infty d\omega \, |\beta_{\omega\Omega}|^2 = \int_0^\infty d\omega \, \frac{\Omega}{\omega} \, |F(-\omega, \Omega)|^2 \end{aligned} \tag{4.34}$$

In order to proceed, we need to take a closer look at the auxiliary function $F(\omega, \Omega)$. By means of some tedious computations concerning special functions and basic complex analysis, one can prove that

$$F(\omega, \Omega) = e^{\pi\Omega/a} \, F(-\omega, \Omega) \,, \qquad \text{for } \omega, \Omega, a > 0$$

Now, considering the last remark of the previous section, in general we have that

$$\int_{\mathbb{R}} d\omega \, (\alpha_{\omega\Omega} \alpha^*_{\omega\Omega'} - \beta_{\omega\Omega} \beta^*_{\omega\Omega'}) = \delta(\Omega - \Omega')$$

Now, fixing $\Omega = \Omega'$ in the latter equation and taking advantage of properties of the auxiliary function $F(\omega, \Omega)$, one can easily get:

$$\int_0^\infty d\omega \, \frac{\Omega}{\omega} \, |F(-\omega, \Omega)|^2 = \frac{\delta(0)}{e^{2\pi\Omega/a} - 1} \tag{4.35}$$

Thus, if we “divide out” the volume factor $\delta(0)$ we thus obtain the number density

$$n_\Omega = \frac{1}{e^{2\pi\Omega/a} - 1} \,, \qquad \Omega > 0 \tag{4.36}$$

Analogously,

$$n_\Omega = \frac{1}{e^{2\pi|\Omega|/a} - 1} \,, \qquad \Omega < 0 \tag{4.37}$$

For massless 2-dimensional scalar fields $|\Omega| = E$. Thus, by analogy with the Bose-Einstein distribution:

$$n_\Omega = \frac{1}{e^{2\pi|\Omega|/a} - 1} = \frac{1}{e^{E/T} - 1} \tag{4.38}$$

we are able to deduce that the so called **Unruh Temperature** is given by

$$T = \frac{a}{2\pi}$$

Thus the Rindler horizon seems to be equipped with a thermal “atmosphere”, which temperature increases the closer an accelerated observer approaches it. In other terms, we conclude that an observer who is being accelerated by a gravitational field with strength g in relativistic units, experiences radiation with a temperature $T = g/2\pi$.

4.6 The Hawking Radiation

We can now deduce the celebrated Hawking effect by means of few simple considerations, avoiding every technical details of the formal proof.
As we have seen in the previous section, an observer moving with uniform acceleration ($a^\mu a_\mu = -a^2$) through the Minkowski vacuum observes a thermal spectrum of particles with a temperature given by:

$$T = \frac{a}{2\pi}$$

It is known in the literature (see e.g. [2]) that the region of spacetime in the vicinity of the horizon of a black hole, approximately takes the form of Rindler space. Now, the **surface gravity** κ[1], for a static Killing horizon, physically represents the acceleration, as exerted at infinity, needed to keep an object at the horizon.
Thus, from these simple considerations, we are able to deduce the famous Hawking Temperature corresponding to the thermal spectrum of a black hole:

$$T_H = \frac{\kappa}{2\pi} \tag{4.39}$$

For example, in the Schwarzschild spacetime we have that

$$\kappa = \frac{1}{4M}$$

where M is the mass of the black hole; thus,

$$T_H = \frac{1}{8\pi M} \tag{4.40}$$

Notice that the energy of the Hawking radiation must come from the black hole itself (or, more precisely, at expenses of its gravitational field). One can then estimate the rate of mass loss by using the Stefan-Boltzmann law for the rate of energy loss by a blackbody:

$$\frac{dE}{dt} = \sigma A T^4 \tag{4.41}$$

Plugging in $E = M$ with $A \propto M^2$ (from the Hawking's Area theorem) and $T_H \propto 1/M$ it gives

$$\frac{dM}{dt} \propto -\frac{1}{M^2}$$

Hence the black hole evaporates away completely in a time

$$\tau \sim M^3$$

This process of black hole evaporation leads to the **Information Paradox**. Consider gravitational collapse of matter to form a black hole which then evaporates away completely, leaving thermal radiation. It should be possible to arrange that the collapsing matter is in a definite quantum state, i.e., a pure state. However, the final state is a mixed state. Such a time evolution would then violate the unitary time evolution of quantum states, which is one of the fundamental postulates of quantum mechanics.

[1] Let k be a Killing vector field normal to the Killing horizon Σ. Then, the surface gravity κ is define as

$$k^\alpha \nabla_\alpha k^\mu = -\kappa k^\mu$$

where the equation is evaluated at the horizon.

4.7 Black Holes & Quantum Gravity

As we have seen in this chapter, some interesting new aspects appear when quantum fields play a role. They mainly concern the notions of *vacuum* and *particles*. A vacuum is only invariant with respect to Poincaré transformations, so that observers that are not related by inertial motion refer in general to different types of vacua. "Particle creation" can occur in the presence of external fields or for non-inertial observers. As we discussed, Black Holes a paradigmatic example of the second case.

Semiclassically, a black hole is supposed to emit a black-body spectrum with a characteristic temperature, known as the Hawking temperature, according to

$$T_H = \frac{\hbar\kappa}{2\pi k_B c} \tag{4.42}$$

where κ is the surface gravity of a stationary black hole, which by the **no-hair theorem** is uniquely characterized by its mass M, its angular momentum J and its electric charge Q. In the particular case of the Schwarzschild black hole, one has

$$\kappa = \frac{c^4}{4GM} \quad \Longrightarrow \quad T_H = \frac{\hbar c^3}{8\pi k_B GM} \sim 6.17\times 10^{-8}\left(\frac{M_\odot}{M}\right) K \tag{4.43}$$

This temperature is unobservationally small for solar-mass (and even bigger) black holes.

The Hawking radiation was derived in the semiclassical limit in which the gravitational field can be treated classically. According to (4.43), the black hole loses mass through its radiation and becomes hotter. After it has reached a mass of the size of the Planck mass, i.e.

$$m_P = \sqrt{\frac{\hbar c}{G}} \sim 2\times 10^{-5}\, g \sim 10^{19}\, GeV \tag{4.44}$$

the semiclassical approximation breaks down and the full theory of quantum gravity should be needed.

The consequences of the *Hawking effect*, together with the *singularity theorems*, suggest that general relativity cannot be true at the most fundamental level. As the singularity theorems and the ensuing breakdown of general relativity demonstrate, a fundamental understanding of the early Universe, concerning in particular its initial conditions near the big bang, and of the final stages of black hole evolution requires an encompassing theory. From the historical analogy of quantum mechanics, the general expectation is that this encompassing theory is a *quantum theory*.

Chapter 5

Planck Stars

Black holes are, nowadays, conventional astrophysical objects. Yet, know very little about what happen inside the classical event horizon.
Experimental observations tell us that the general theory of relativity describes well the region surrounding the horizon, at least for $r \gtrsim 3R_S$ where R_S is Schwarzschild radius. Moreover, it is also reasonable to assume that also a substantial region inside the horizon can be described in terms of the Einstein's theory of gravity, under the argument that curvature scales are still small. However, it is obvious that such a theory will fail to describe the gravitational field at very small radii because nothing will be able to prevent quantum mechanics from affecting the high curvature regions, and because classical general relativity becomes ill-defined at $r = 0$ anyway, as clearly stated by the singularity theorems under very general assumptions. Moreover, even the semiclassical description of black holes gives rise to some very fundamental problems, such as the information paradox.

A first hint about how to try to deal with this situation comes from the fact that an event horizon with (semi)classical properties is not compatible with a "point-like" source. More precisely, R. Casadio and collaborators showed (see [7]) that classical event horizons is not consistent with a spherically symmetric source (for the gravitational filed) of a radius $r \lesssim l_P = \sqrt{\hbar G/c^3} \sim 10^{-33}\, cm$. Another fundamental insight comes from Loop Quantum Cosmology. Indeed, consider the Friedmann equation suitably modified in order to incorporate a term representing the quantum gravitational effects, i.e.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho\left(1 - \frac{\rho}{\rho_P}\right) \tag{5.1}$$

where ρ is the energy density of matter and where the quantum correction term in the parenthesis is given by the ratio of ρ to the Planck scale density:

$$\rho_P = \frac{m_P}{l_P^3} = \frac{c^5}{\hbar G^2} \sim 10^{97}\, kg/m^3 \tag{5.2}$$

where $m_P = \sqrt{\hbar c/G}$ and l_P are the Planck mass and the Planck length, c is the speed of light and $\hbar$ the reduced Planck constant.
It is clear from Eq. (5.1) that Nature appears to enter the quantum gravity regime when the energy density of matter reaches some critical scale, i.e. when $\rho \sim \rho_c$. What happens in

loop quantum cosmology is that, once ρ reaches ρ_P, a collapsing spatially-compact universe bounces back into an expanding one. The bounce is due to a quantum gravitational repulsion which originates from the Heisenberg uncertainty. The most important point is that this may happen well before relevant lengths l become planckian ($l \sim l_P$). Indeed, in a matter dominated Universe the bounce happens when:

$$\rho \sim \rho_P \;\Leftrightarrow\; \frac{m}{V} \sim \frac{m_P}{l_P^3} \;\Longrightarrow\; V \sim \frac{m}{m_p}\, l_P^3 \gg l_P^3$$

So, and it deserves to be stressed again, quantum gravitational effects do not happen only over Planck volumes.

Following these leads, C. Rovelli, F. Vidotto and H. M. Haggard proposed a scenario (see [9; 10]), now known as the **Planck stars model**, by which we should have that, during a gravitational collapse, when matter reaches Planckian density, quantum gravity generates sufficient pressure to counterbalance the matter's weight, the collapse ends, and matter then bounces out. Then, quantum gravitational effects could be able to prevent the formation of a singularity and liberate the information stored into a black hole when this is still macroscopic compare with the Planck length.
This would imply the existence of a brand **new phase of the gravitational collapse** of massive objects, which could be short in proper time, but, due to the strong gravitational dilation, extremely long for an external observer.
In the following sections we will show in more details how this scenario can solve the singularity problem and the information paradox. Moreover, we will also stress how this model leads to two very different scenarios.

5.1 Singularity resolution & the quantum bounce

In this model we are assuming that the behaviour of a collapsing star is similar to the one of a collapsing spatially-compact universe in loop quantum cosmology. Hence, the energy of a collapsing star will condense into a highly compressed core with $\rho \sim \rho_P$ thus avoiding the formation of a curvature singularity due to the emergence of a quantum pressure akin to the one known in the context of loop cosmology.

Following the previous discussions, we have that the fundamental idea behind this model lies in the fact that quantum gravity is able to affect high curvature regions of the spacetime leading to an effective bounce also in the context of the gravitational collapse of massive objects. This quantum bounce then leads to two extremely different new potential phases of the life of a star:

(1) If we assume that the bouncing time is way smaller then the Hawking evaporation time of the balck hole, i.e. $\tau \ll \tau_H$, then the Hawking effect would appear to be almost irrelevant if compared with the energy blast produced by the intense quantum pressure. This, together with some other assumptions that we shall see later in further details, will lead to the Black Hole Fireworks scenario.

(2) If we assume the bouncing time to be of the order (or larger) of the Page time[1] of the balck hole, i.e. $\tau \gtrsim \tau_H/2$, then the regular core would appear to be a star-like object with a very slow dynamics (that will be introduced in the next section). This particular new phase of the gravitational collapse is called Planck Star. Moreover, by contrast with the previously discussed case, in this scenario the Hawking effect is dominant with respect to the intensity of the quantum bounce.

Now, it worth stressing that the lifetime of this new phases could be very long if measured from very far. But, at the same time, it would appear very short, $t \sim m$ (i.e. the time that light spends to hit the radius of the star), if measure from the "surface" of the quantum region. The difference is clearly due to the intense time dilation.
In other terms, in both cases the proper lifetime of the new phase is very short, indeed, from its own perspective, it is essentially a bounce. Then, due to the time dilation, an observer far away will see the bounce in a very slow motion.

5.2 Resolution of the Information Paradox

Now, if we are interested in formulating a first rudimentary effective theory of the causal structure for a Planck Star we just need to stress the fact that we can associate to this object a classical (external) event horizon and an internal trapping horizon, the latter related to the size of the Planck star.

Let us now consider the Hawking radiation and its backreaction. Due to the evaporation process, the outer horizon shrinks. Conversely, the inner horizon, i.e. the surface of the regular core of the black hole, gains energy and then it expands. Hence, there will be a certain point in time when the outer horizon and the growing internal one will meet. At this point, there is no horizon anymore, the quantum gravitational pressure is then able to tear apart the star and, consequently, all the information trapped inside can freely escape to infinity. Thus, the outer event horizon is just a temporary optical illusion. As a consequence, the Hawking pairs featuring the Hawking effect keep being maximally entangled which implies that the entropy of the radiation is purified by the information stored in the Planck star.

Clearly, the very same reasoning about the purification of the entropy of the Hawking radiation also applies to the scenario of Black Hole Fireworks.

5.3 Black Hole Fireworks

In the previous sections we have displayed the physics underlying this process, now we have to look for an effective description. For doing so, as we shall see later, we will follow two key insight: the first one, coming directly form general relativity, consist in the symmetric nature of the classical theory of gravity under time reversal. The second hint, instead, comes from the quantum theory (see [11]). The latter basically tells us that the two classical disconnected sets of solutions for a null shell collapse, i.e. the black hole and the white hole, can be connected by means of the quantum theory. In particular, in this semiclassical scenario a wave packet representing an in-falling shell undergoes a quantum bounce, tunneling into an expanding wave

[1] Half of the evaporation time.

packet.

Now, the invariance of general relativity under the inversion of the direction of time suggests that we can search for the metric of a bouncing star by gluing a collapsing region with its time reversal. In doing so, however, we shall disregard all dissipative effects, which are not time symmetric. Thus our main hypothesis is that there is a time symmetric process where a star collapses gravitationally and then bounces out. However, such a process is forbidden in the classical theory because, after the collapse, a star would not be allowed to exit its own horizon. Nevertheless, if we allow for for quantum gravitational corrections, as shown in [10], the quantum effects could be actually able to pile up, over a long time, even affecting the metric also in a small region outside the horizon. A black hole could thus be able to quantum-tunnel into a white hole without violating causality or the underlying assumptions of the semiclassical approximation.

Thus, Rovelli and Haggard were able to formulate an effective description of this scenario under the following assumptions:

(i) Spherical symmetry;

(ii) Spherical null shell collapse;

(iii) Time reversal symmetry;

(iv) Classicality at large radii and at early times;

(v) No event horizons.

Their discussion can then be summarized by the following steps:

Step one: **General considerations on the conformal description**
Because of the assumption (v), the conformal diagram of spacetime is trivial. From assumption (iii) there must be a $t = 0$ hyperplane which is the surface of reflection of the time reversal symmetry. By symmetry, the bounce must then happen at t = 0. Moreover, the metric is invariant under time reversal, so it is sufficient to describe the geometry only for the region below t = 0 (and make sure it glues well with its future).

Step two: **Conformal description of the incoming and outgoing null shells**
These are represented by the two thick lines at 45 degrees.

Step three: **Characterization of the quantum region**
There exist two significant spacetime points, Δ and $\mathcal{E}$, that lie on the boundary of the quantum region. The point Δ has $t = 0$ and is the maximal extension in space of the region where the Einstein equations are violated. Point $\mathcal{E}$, instead, is the first moment in time where this happens.

Step four: **Description of the various patch of the spacetime**
From the previous steps we get that the spacetime can be separated into three patches, respectively:

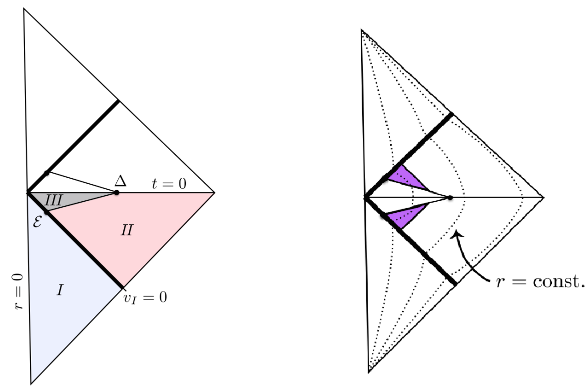
(I) The first region, inside the collapsing shell, must be flat by Bhirkoff's theorem;

(II) The second region, again by Bhirkoff's theorem, must be a portion of the metric of a mass m, namely it must be a portion of the (maximal extension of the) Schwarzschild metric;

(III) Finally, the third region is where quantum gravity becomes non-negligible. We know nothing about the metric of this region, except for the fact that it must join the rest of the spacetime.

It is important to stress, see [10] for further details, that the point Δ is a generic point in the region outside the apparent horizon, which we take on the $t = 0$ surface, so that the gluing with the future is immediate. More crucial is the position of the point $\mathcal{E}$. Remember that this is the point where the in-falling shell reaches the quantum region. Clearly this must be inside the horizon, because when the shell enters the horizon the physics is still classical.

All the discussion above can be summarized in the following conformal diagrams.



where the continuous external lines bounding the purple-shaded regions in the diagram on the right represent the spacetime positions of the apparent horizon.

We have built up the metric of a black hole tunnelling into a white hole by using the classical equations outside the quantum region. The latter is bounded by a well defined classical geometry. Thus, given the classical boundary geometry, it seems quite natural to ask whether we can perform a computation of the quantum transition amplitude between these two classical "states". Indeed, this is precisely the form of the problem that is adapted for a calculation in the Spinfoam formalism [5; 12; 13]. If this calculation can be done, we should then be able to compute from first principles the duration τ of the bounce seen from the exterior. We shall see later in this thesis an attempt to deal with this problem.

5.4 Some Phenomenology

For the case of Black Hole Fireworks, Rovelli and collaborators have also been able to give a semiclassical estimate of the bouncing time τ (see [10]), i.e.

$$\tau \sim 4km^2 \tag{5.3}$$

where $k \sim 0.05$ and m is the mass of the collapsing null shell.
A **Primordial Black Hole** of initial mass around

$$m \sim 1.2 \times 10^{23}\, kg \tag{5.4}$$

would have a semiclassical bouncing time of the order of the Hubble time, i.e. $\tau \sim t_H$, therefore it can be expected to "explode" into a white hole today.
If this happens, most of the energy of the black hole is still present at explosion time, because Hawking radiation does not have the time to consume it. The exploding object should have a total energy of the order

$$E = mc^2 \sim 10^{50}\, GeV \tag{5.5}$$

compressed into a region given by the corresponding Schwarzschild radius, i.e.

$$R_H = \frac{2Gm}{c^2} \sim 0.02\, cm \tag{5.6}$$

It has been argued in [14] that we may expect two main component of the signal from such an explosion: a lower energy signal at a wavelength of the order of the size of the exploding object and a higher energy signal which depend on the details of the liberated hole content. In this paper they also point out that a possible connection between the lower energy signal and the recently discovered Fast Radio Bursts.

Fast Radio Bursts are intense isolated astrophysical radio signals with a duration of milliseconds. The frequency of these signals is in the order of $1.3\, GHz$, namely a wavelength of $\lambda_O \sim 20\, cm$. These signals are believed to be of extragalactic origin, mostly because the observed delay of the signal arrival time with frequency agrees quite well with the dispersion due to a ionized medium, expected from a distant source. The total energy emitted in the radio by a source is estimated to be of the order $10^{41}\, GeV$. The origins and physical nature of the Fast Radio Bursts are currently unknown.
Now, if we consider a strong explosion in a small region should emit a signal with a wavelength of the order of the size of the region or somehow larger. Therefore it is reasonable to expect from the black-white hole quantum transition (also known as **Black Hole fireworks**) the production of an electromagnetic signal such that $\lambda_T \gtrsim 0.02\, cm$, as discussed before. The difference between λ_O and λ_T is around three orders of magnitude, however we must keep in mind that the model that we are using is very rudimentary. So we might hope that further theoretical refining of this scenario could lead to predictions closer to the experimental observations.

Part II

On the Effective Metric of a Rotating Planck Star

Chapter 6

Rotating Black Holes

In this chapter we summarize (in a non-rigorous way) a set of theorems and conjectures concerning black holes. Moreover, we also introduce the concept of rotating black hole and its geometrical description.

6.1 The Kerr Solution

The Schwarzschild solution, which is appropriate outside a spherical, static mass distribution, was discovered in 1916. It was not until 1963 that a solution corresponding to spinning black holes was discovered by Roy Kerr. This solution leads to the possible existence of a family of rotating, deformed black holes that are called Kerr black holes.
It is important to stress that the angular momentum is a very complicated thing in general relativity. Indeed, despite the fact that in Newtonian gravity rotation produces centrifugal effects without affecting directly the gravitational field, in the Enstein's theory of gravity the rotation of a gravitational field is itself a source for the field.

The **Kerr metric** is a vacuum solution of the Einstein's field equations and corresponds to the line element:

$$\begin{aligned} ds^2 = \left(1 - \frac{2mr}{\Sigma}\right) dt^2 + \frac{4mra\sin^2\theta}{\Sigma} d\phi dt - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 \\ - \sin^2\theta \left(r^2 + a^2 + \frac{2mra^2\sin^2}{\Sigma}\right) d\phi^2 \end{aligned} \tag{6.1}$$

where $a \in \mathbb{R}$ and $[a] = L$ in geometrized units, $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2mr + a^2$. Here, the set of coordinates (t, r, θ, ϕ) are known as **Boyer-Lindquist coordinates**.
It is easy to see that for $a = 0$ the line element (6.1) reduces to the Schwarzschild line element. Moreover, it is also asymptotically flat for $r \gg m$ and $r \gg a$.

The fact that the line element (6.1) is independent from the coordinates t and ϕ implies the existence of the Killing vectors $\xi_t = \partial_t$ and $\xi_\phi = \partial_\phi$. This tells us that we are dealing with a **stationary, axially symmetric spacetime**.

Now, for a Killing vector field $k = k^\mu\,\partial_\mu$, the Killing equation $\mathcal{L}_k\,g = 0$, that can be rewritten as $\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0$, is the statement that the symmetric part of the covariant derivative of the killing vector field vanishes, i.e. $\nabla_{(\mu}k_{\nu)} = 0$. Thus, we can define the **Komar 2-form** for the Killing vector field k as:

$$K = \frac{1}{2}\, K_{\mu\nu}\, dx^\mu \wedge dx^\nu = \nabla_\mu k_\nu \, dx^\mu \wedge dx^\nu \tag{6.2}$$

Then, it is known that for a given Killing vector field k we can associate a quantity Q_k, known as the **Komar Integral** of k, given by

$$Q_k(B) = \frac{c}{8\pi G} \int_{\partial B} \star K \tag{6.3}$$

where B is a certain spacelike region of the spacetime and $\star$ is the so called Hodge star linear map, e.g. see [2] for further details.
It can be shown that, for B large enough such that all matter is inside of it and spacetime is vacuum outside of it, $Q_k(B)$ is a **conserved** quantity and its physical interpretation will depend on the Killing vector k.

For appropriate choices of the regions B we find two conserved quantities for the Kerr spacetime:

- If $k = \xi_t$ we find that: $Q_t = m$, so the **mass of the black hole** is a conserved quantity. This should not be surprising because the conserved charge corresponding to the symmetry under time translation is nothing but the total energy;
- If $k = \xi_\phi$ we find that: $Q_\phi = ma$. It is important to notice that physically the quantity Q_ϕ can be interpreted as the **angular momentum** of the black hole, i.e. $Q_\phi = J$, thus the Kerr parameter a has a straightforward interpretation as the **angular momentum per unit of mass**, i.e. $a = J/m$.

Before entering into a more detailed discussion of the singularity and horizon structure it is important to make a remark on the geometry of the Kerr spacetime. If we consider, in Boyer-Lindquist coordinates, 2-surfaces described by $r, t = const.$, it is trivial to show that they are not described by the metric of a 2-sphere. This is going to be extremely relevant in the following.

6.1.1 Singularity and Horizon structure

It can be proven that the spacetime (6.1) has **coordinate singularities** at

$$\Delta = 0 \quad \Longleftrightarrow \quad r = r_\pm = m \pm \sqrt{m^2 - a^2} \tag{6.4}$$

And they can, indeed, be removed by a simple coordinate transformation. On the other hand, this spacetime has also a **curvature singularity** in

$$\Sigma = 0 \quad \Longleftrightarrow \quad r = 0\,, \quad \cos\theta = 0 \tag{6.5}$$

which implies that the curvature singularity is only there when $\theta = \pi/2$, i.e. when we approach $r = 0$ along the equatorial plane. To be more specific, one can show that the singularity has the topology of a ring of radius a, thus if we travel toward $r = 0$ from any other angle than

$\theta = \pi/2$, we will not encounter the singularity. Instead, we will fall through the interior of the "ring" and emerge in a new region of spacetime ($r < 0$, analytical extension of the spacetime).

Let us consider a curve in the Kerr spacetime such that $t = const.$, $\theta = \pi/2$ and $r < 0$. Under these assumptions we can rewrite the line element as:

$$ds^2 = -\left(r^2 + a^2 + \frac{2ma}{r}\right) d\chi^2 \,, \qquad d\chi = d\phi + \frac{a}{\Delta}\, dr \tag{6.6}$$

Close enough to the singularity, where $|r| \ll 1$ is small and negative, $r < 0$, the parenthesis can be negative and the curve is timelike. Since χ is a periodic coordinate with $\chi = \chi + 2\pi$, the curve is also closed. This is a so called **closed timelike curve** or, eventually, a time machine. However, this result is in contrast with the assumption of global hyperbolicity. Moreover, closed timelike curves clearly violates causality. Therefore, these regions of the spacetime are considered to be unphysical.

It is also important to notice that $r_+ \to 2m$ as $a \to 0$. So it is a coordinate singularity of the Kerr spacetime that turns into the coordinate singularity of the Schwarzschild spacetime as $a \to 0$. Following these suggestions, one can formally proof (e.g. see [1]) that r_+ corresponds to the **event horizon** of the Kerr metric. So, the property of owing an event horizon makes the Kerr metric a black hole solution of the vacuum Einstein's field equations. Thus the Kerr horizon has constant Boyer-Lindquist coordinate $r = r_+$ but it is not spherical, as we stressed before.

6.1.2 Frame Dragging and the Ergosphere

Studying the geodesic motion of test particles, it turns out that the Kerr spacetime there is a region around the outer horizon, called the **ergosphere**, in which it is impossible for any test particle to remain stationary with respect to observers at infinity. So, everything must rotate within this region. This happens because

$$g_{tt} = 1 - \frac{2mr}{\Sigma} \tag{6.7}$$

change sign and become positive in the region $r^2 - 2mr + a^2\cos^2\theta < 0$, part of which lies outside the event horizon $r = r_+$ when $a \neq 0$.
In the ergosphere, orbits of ξ_t are not timelike, so test particles cannot travel along them and remain stationary with respect to observers at infinity. In order for a curve $x^\mu = (t, r, \theta, \phi)$ to be timelike, its tangent vector $u^\mu = dx^\mu/d\tau$ must satisfy $u^2 = 1$. However, in the ergosphere every term in $u^2 = g_{\mu\nu}u^\mu u^\nu$ is positive except for $g_{t\phi}u^t u^\phi$, which means that $u^\phi = d\phi/d\tau$ must be non-zero. Any timelike worldline is therefore dragged around in the direction of rotation of the black hole. This effect is an example of **frame dragging**.

If we consider the outer horizon r_+ as a sort of black hole surface then we could conventionally consider the angular velocity of an observer which falls radially from infinity as a sort of black hole angular velocity. The angular velocity of an observer which falls radially from infinity is given by

$$\Omega = \frac{d\phi}{dt} = \frac{u^\phi}{u^t} \,, \qquad u^2 = g_{tt}(u^t)^2 + 2g_{t\phi}u^t u^\phi + g_{\phi\phi}(u^\phi)^2 = 1 \tag{6.8}$$

from which we get

$$(u^t)^2(g_{\phi\phi}\Omega^2 + 2g_{t\phi}\Omega + g_{tt}) = 1 \tag{6.9}$$

thus for u^t to be real we require that $g_{\phi\phi}\Omega^2 + 2g_{t\phi}\Omega + g_{tt} > 0$; since $g_{\phi\phi} < 0$ everywhere, the left-hand side of the inequality, as a function of Ω, gives rise to an upward pointing parabola. Hence, the allowed range of angular velocities is given by $\Omega_- < \Omega < \Omega_+$, where

$$\Omega_\pm = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{\phi\phi}} \tag{6.10}$$

Now, as $r = r_+$ we have $\Delta = 0$ thus

$$\Omega_H \equiv \Omega(r_+) = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{a}{2mr_+} = \frac{a}{r_+^2 + a^2} \tag{6.11}$$

Form the previous discussion we can learn something very important. Indeed, if we define, on the event horizon $r = r_+$, the vector field $\chi = \xi_t + \Omega_H\xi_\phi$ and if we set the condition $\chi^2(r_+) = 0$, we get that

$$\chi^2 = 0 \text{ on } r = r_+ \quad \Longrightarrow \quad \Omega_H = -\frac{g_{t\phi}}{g_{\phi\phi}} \tag{6.12}$$

and one can also prove that χ is a Killing vector field normal to the horizon.

This result is actually quite general, indeed it follows from the following

Theorem 8 (Hawking Rigidity theorem)**.** Let $(\mathcal{M}, g)$ be a stationary, asymptotically flat solution of Einstein's field equations, with matter satisfying suitable hyperbolic equations, that contains a black hole. Then the event horizon $\mathcal{H}^+$, of the black hole is a Killing horizon.

The stationary Killing field ξ_t must be tangent to $\mathcal{H}^+$. Indeed, if it is normal to $\mathcal{H}^+$ then it can be shown that the spacetime is static. Thus, if ξ_t is not normal there must exist (due to the Rigidity theorem) another Killing field χ that is normal to the horizon. It can then be further shown that there is a linear combination, ψ, of ξ_t and χ whose orbits are spacelike and closed, i.e., the spacetime is axisymmetric. Thus, a stationary black hole must be static or axisymmetric.
We can choose the normalization of χ a so that

$$\chi = \xi_t + \Omega_H\xi_\phi \tag{6.13}$$

where Ω_H is a constant that corresponds to the angular velocity of the horizon.

6.1.3 The Penrose Process

The Penrose process is a process that allows us to extract energy from a rotating black hole. Imagine sending a particle into the ergosphere. Prepare it such that that, once in the ergosphere, it decays into two particles, one of which falls into the black hole and one of which escapes the ergosphere again. Denote the energy of the initial particle by E and that of the final particles by E_1 and E_2. Conservation of four-momentum implies that $E = E_1 + E_2$. The fact that ξ_t is spacelike in the ergosphere, as well as the fact that $E = \xi_t^\mu p_\mu$ where p^μ is the four-momentum of the particle, allows us to arrange the decay such that the energy of the particle that falls into the black hole is negative: $E_1 < 0$ with respect to us. Then E_1 is negative and $E_2 > E$: the particle that re-emerges from the ergosphere has more energy than the particle we sent in.

6.2 More General Black Holes

In general, we can define a black hole as an asymptotically flat solution of Einstein's field equations in vacuum, curvature singularity concealed by the horizon. Black holes form in the gravitational collapse of stars, if they are sufficiently massive.
When a black hole forms in a gravitational collapse, since gravitational waves emission and other dissipative processes damp its violent oscillations, we can expect that, after some time, it settles down to a stationary state. Thus, **stationary black holes are considered the final outcome of gravitational collapse**.

There are some remarkable theorems on stationary black holes, derived by S. Hawking, W. Israel, B. Carter and R. Penrose which completely characterize black hole in the classical theory:

- A stationary black hole is axially symmetric;
- Any stationary, axially symmetric black hole, with no electric charge, is described by the Kerr solution.
- Any stationary, axially symmetric black hole described by the so called **Kerr-Newman solution**, which is the generalization of the Kerr solution with non-vanishing electric charge, is characterized by only three parameters: the mass M, the angular momentum $J = Ma$, and the charge Q.

All other features the star possessed before collapsing, such as a particular structure of the magnetic field, mountains, matter current, differential rotations etc, disappear in the final black hole which forms. This result has been summarized, by Penrose, with the sentence: "A black hole has no hair", and for this reason the previous **uniqueness theorems** are also called **no hair theorems**.

6.3 The Newman-Janis Algorithm

Two years after the discovery of Kerr metric, Newman and Janis presented an algorithm for converting Schwarzschild geometry into Kerr geometry. This approach basically consist in extending the Schwarzschild solution by means of a simple, analytic, complex coordinate transformation.
The reason of the success of such procedure (at least for this particular case as well as for many others) can be traced back to the behaviour of the Einstein's Field Equations.
More recently, the NJA has been invoked to explore axially symmetric inner solutions and rotating vacuum solutions in various different contexts, such as alternative theories of gravitation, non-linear electrodynamics (DBI), non-Abelian black holes, spinning loop black holes, string theory and so on.

Basically, the NJA is a technique used to **generate solutions of the Einstein's Field Equations starting from known static spherical symmetric solutions**. However, as shown in many papers, a solution obtained from the NJA may be affected by severe **pathologies** (e.g. see [17]), e.g. naked singularities, and also may not keep being a solution of the field equation from which one have started (in particular, it has been shown that this technique can actually modify the stress-energy tensor from which it started). At the same time, there are also other problems regarding a sort of **ambiguous definition of the complex coordinate transformation**.

6.3.1 The Method

As widely explained in [15] (introducing, however, a fatal mistake in the first pages of the paper that may invalid the whole discussion) and properly corrected in [18] by Caravelli and Modesto, **the NJA is a five steps procedure** with the aim of generating new solutions of the Einstein's field equations starting form known static spherically symmetric ones. However, it is still unknown whether a similar approach can be successfully applied to metrics which are not spherically symmetric.
Let us discuss those steps in more details:

1. It is given a static spherically symmetric **seed metric**, i.e.

$$\begin{aligned} ds^2 &= e^{2\Phi(r)}\,dt^2 - e^{2\lambda(r)}\,dr^2 - H(r)\,d\Omega^2 \equiv \\ &\equiv G(r)\,dt^2 - \frac{dr^2}{F(r)} - H(r)\,d\Omega^2 \end{aligned} \tag{6.14}$$

   The first step then consist in changing coordinates, in particular, we have to rewrite the line element in terms of the **advanced null coordinates**, i.e. (u, r, θ, φ), where

$$u = t - r^* \qquad \text{and} \qquad dr^* = \frac{dr}{\sqrt{FG}}$$

   The line element above then becomes,

$$ds^2 = G(r)\,du^2 + 2\sqrt{\frac{G(r)}{F(r)}}\,dudr - H(r)\,d\Omega^2 \tag{6.15}$$

   while the non zero components of the inverse metric are

$$\begin{aligned} g^{u\varphi} &= e^{-\Phi(r)-\lambda(r)}\,, \quad g^{\varphi\varphi} = -[H(r)\sin^2\theta]^{-1}\,, \\ g^{\theta\theta} &= -H(r)^{-1}\,, \quad g^{rr} = -e^{-2\lambda(r)}\,. \end{aligned}$$

2. The second step of the algorithm consist in expressing the inverse matrix element of the metric in terms of a **null tetrad**, such that

$$g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu \bar{m}^\nu - m^\nu \bar{m}^\mu \equiv 2l^{(\mu}n^{\nu)} - 2m^{(\mu}\bar{m}^{\nu)} \tag{6.16}$$

   with

$$l^2 = n^2 = m^2 = 0\,, \qquad l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1\,, \qquad l_\mu m^\mu = n_\mu m^\mu = 0 \tag{6.17}$$

where $a^2 \equiv a_\mu a^\mu$. In particular, taking into account the line element (6.15) we get:

$$\begin{aligned} l^\mu &= \delta^\mu_r \\ n^\mu &= \sqrt{\frac{F}{G}}\,\delta^\mu_u - \frac{F}{2}\,\delta^\mu_r \\ m^\mu &= \frac{1}{\sqrt{2H}}\left(\delta^\mu_\theta + \frac{i}{\sin\theta}\,\delta^\mu_\varphi\right) \end{aligned} \tag{6.18}$$

It is also useful to introduce the Newman-Penrose notation:

$$e^\mu_a \equiv (l^\mu, n^\mu, m^\mu, \bar{m}^\mu) \qquad \text{with } \ a = 1,2,3,4.$$

3. The third step is the most important and also the most ambiguous, indeed we have to replace the coordinate x^ρ with a new set of complex coordinates $\widetilde{x}^\rho \in \mathbb{C}$, i.e.

$$x^\rho \in \mathbb{R} \quad \longleftrightarrow \quad \widetilde{x}^\rho = x^\rho + i y^\rho(x^\sigma) \tag{6.19}$$

where $y^\rho = y^\rho(x^\sigma)$ are analytic real functions. It is important to stress that what we have done so far in this step is not a coordinate transformation, indeed it is just a formal substitution.
Simultaneously, we require that the null tetrad vector is replaced as follows:

$$e^\mu_a(x) \quad \longleftrightarrow \quad \widetilde{e}^\mu_a(\widetilde{x}, \bar{\widetilde{x}}) \tag{6.20}$$

becoming a function of both $\widetilde{x}^\rho$ and $\bar{\widetilde{x}}^\rho$, with the condition that the complexified null tetrad vector $\widetilde{e}^\mu_a(\widetilde{x}, \bar{\widetilde{x}})$ reduces to $e^\mu_a(x)$ when $\widetilde{x}^\rho$ is real, i.e.

$$\widetilde{e}^\mu_a(\widetilde{x}, \bar{\widetilde{x}})\Big|_{\widetilde{x}^\rho = \bar{\widetilde{x}}^\rho} = e^\mu_a(x) \tag{6.21}$$

In summary, the result of this step is to **create a new metric whose components are real functions of complex variables, taking also into account the latter constraint**, i.e.

$$g_{\mu\nu} \quad \longleftrightarrow \quad \widetilde{g}_{\mu\nu} \in \mathbb{R}\,, \qquad \text{while} \quad \widetilde{e}^\mu_a(\widetilde{x}, \bar{\widetilde{x}})\Big|_{\widetilde{x}^\rho = \bar{\widetilde{x}}^\rho} = e^\mu_a(x) \tag{6.22}$$

This step is the first part of the so called complexification procedure and it is certainly the most ambiguous part of the entire algorithm. Indeed, the choice of (6.19) and (6.20) is clearly not unique under the conditions (6.22); for example, there is no *a priori* restriction concerning the choice of the complexified null tetred vector.

4. In the forth step **a new metric is obtained by means of a complex coordinate transformation**:

$$\widetilde{x}^\rho \quad \longrightarrow \quad \widetilde{x}^\rho = x'^\rho + i\gamma^\rho(x)\,, \qquad\qquad x'^\rho,\, \gamma^\rho(x) \in \mathbb{R} \tag{6.23}$$

which leads to a transformation of the complexified null tetred vector given, as usual, by:

$$e'^\mu_a = \frac{\partial x'^\mu}{\partial \widetilde{x}^\nu}\,\widetilde{e}^\nu_a \tag{6.24}$$

So we get the new metric:

$$g'^{\mu\nu} = l'^{\mu} n'^{\nu} + l'^{\nu} n'^{\mu} - m'^{\mu} \overline{m'}^{\nu} - m'^{\nu} \overline{m'}^{\mu} \tag{6.25}$$

which is the new metric that we where looking for.

The complexification is usually chosen as:

$$\begin{aligned} r \in \mathbb{R}_0^+ \quad &\longrightarrow \quad r \in \mathbb{C}\,, \quad r = r' - ia\cos\theta\,, \quad r', a \in \mathbb{R} \\ u \in \mathbb{R} \quad &\longrightarrow \quad u \in \mathbb{C}\,, \quad u = u' + ia\cos\theta\,, \quad u', a \in \mathbb{R} \end{aligned} \tag{6.26}$$

But there is not any formal explanation which suggest to prefer this particular choice rather then some others.
By means of this specific transformation, we get the following tetrad

$$\begin{aligned} l'^{\mu} &= \delta^{\mu}_{r} \\ n'^{\mu} &= \sqrt{\frac{\widetilde{F}}{\widetilde{G}}}\,\delta^{\mu}_{u} - \frac{\widetilde{F}}{2}\,\delta^{\mu}_{r} \\ m'^{\mu} &= \frac{1}{\sqrt{2\widetilde{H}}}\left(ia\sin\theta\,(\delta^{\mu}_{u} - \delta^{\mu}_{r}) + \delta^{\mu}_{\theta} + \frac{i}{\sin\theta}\,\delta^{\mu}_{\varphi}\right) \end{aligned} \tag{6.27}$$

Thus, the non zero components of the inverse metric (6.15) become, after the transformation,

$$\begin{aligned} g^{uu} &= -\frac{a^2\sin^2\theta}{\widetilde{H}(r,\theta)}\,, \quad g^{u\varphi} = -\frac{a}{\widetilde{H}(r,\theta)}\,, \\ g^{\varphi\varphi} &= -[\widetilde{H}(r,\theta)\,\sin^2\theta]^{-1}\,, \quad g^{\theta\theta} = -\widetilde{H}(r,\theta)^{-1}\,, \\ g^{rr} &= -\frac{a^2\sin^2\theta}{\widetilde{H}(r,\theta)} - e^{-2\widetilde{\lambda}(r,\theta)}\,, \quad g^{r\varphi} = \frac{a}{\widetilde{H}(r,\theta)}\,, \\ g^{ur} &= \frac{a^2\sin^2\theta}{\widetilde{H}(r,\theta)} + e^{-\widetilde{\Phi}(r,\theta)-\widetilde{\lambda}(r,\theta)} \end{aligned} \tag{6.28}$$

where the prime have been omitted for the sake of clarity.
Thus,

$$\begin{aligned} g_{uu} &= \widetilde{G}\,, \quad g_{ur} = \sqrt{\widetilde{G}/\widetilde{F}}\,, \quad g_{\theta\theta} = -\widetilde{H}\,, \\ g_{u\varphi} &= a\sin^2\theta\left(\sqrt{\widetilde{G}/\widetilde{F}} - \widetilde{G}\right)\,, \\ g_{r\varphi} &= -a\sqrt{\widetilde{G}/\widetilde{F}}\sin^2\theta\,, \\ g_{\varphi\varphi} &= -\sin^2\theta\left[\widetilde{H} + a^2\sin^2\theta\left(2\sqrt{\widetilde{G}/\widetilde{F}} - \widetilde{G}\right)\right] \end{aligned} \tag{6.29}$$

5. The fifth and last step of the algorithm is a change of coordinates. **In some cases**, we can write the metric in the **Boyer-Lindquist form**, in which the only non-vanishing off-diagonal term is $g_{t\varphi}$. Indeed, by means of some tedious calculations (see [15] for

further details) it can be show that the former metric can be rewritten in Boyer-Lindquist coordinates as follows:

$$
{}^{(BL)}g_{\mu\nu} = \begin{pmatrix} \widetilde{G} & 0 & 0 & a\sin^2\theta\left(\sqrt{\widetilde{G}/\widetilde{F}} - \widetilde{G}\right) \\ \cdot & -\frac{\widetilde{H}}{\widetilde{H}\widetilde{F}+a^2\sin^2\theta} & 0 & 0 \\ \cdot & \cdot & -\widetilde{H} & 0 \\ \cdot & \cdot & \cdot & -\sin^2\theta\left[\widetilde{H} + a^2\sin^2\theta\left(2\sqrt{\widetilde{G}/\widetilde{F}} - \widetilde{G}\right)\right] \end{pmatrix} \tag{6.30}
$$

where $\widetilde{G} = \widetilde{G}(r,\theta)$, $\widetilde{F} = \widetilde{F}(r,\theta)$ and $\widetilde{H} = \widetilde{H}(r,\theta)$ (in particular, if we consider a spherically symmetric seed metric, we have $H(r) = r^2$ and then $\widetilde{H}(r,\theta) = \Sigma = r^2 + a^2\cos^2\theta$).

Remark. This method is quite successful when applied to (almost) any spherically symmetric solution derived in $f(R)$-gravity, it also gives the correct answer when applied to rotating dilaton-axion black hole models but, it is important to stress that although the static spherically symmetric charged dilaton black hole is also a solution to the truncated theory without axion field (i.e. Einstein-Maxwell-dilaton gravity), the result coming form the NJA is not. Moreover, whenever we apply this technique in the framework of Braneworld, it does not furnish a metric that satisfies the condition to be a valid braneworld solution, i.e. $R = 0$. It is also known that the NJA fails when applied to the Born-Infeld theory.

6.3.2 From Schwarzschild to Kerr

Consider the Schwarzschild metric:

$$
ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{dr^2}{1-\frac{2m}{r}} - r^2\,d\Omega^2 \tag{6.31}
$$

which can be written in terms of the Advanced Eddington-Finkelstein coordinates as

$$
ds^2 = \left(1 - \frac{2m}{r}\right)du^2 + 2dudr - r^2\,d\Omega^2 \tag{6.32}
$$

So we have that

$$
G(r) = F(r) = 1 - \frac{2m}{r}\,, \qquad H(r) = r^2
$$

If we apply the algorithm as prescribed above, we have to choose a complexification of the r^2 and of $1/r$ term. If we choose:

$$
r^2 \longrightarrow r\bar{r}\,, \qquad \frac{1}{r} \longrightarrow \frac{1}{2}\left(\frac{1}{r} + \frac{1}{\bar{r}}\right) = \frac{\mathtt{Re}\,r}{r\bar{r}} \tag{6.33}
$$

which, using (6.26), leads to

$$
r^2 \longrightarrow \Sigma \equiv r'^2 + a^2\cos^2\theta\,, \qquad \frac{1}{r} \longrightarrow \frac{r'}{\Sigma} \tag{6.34}
$$

From the latter we can construct the new null tetrad as:

$$\begin{aligned}
l'^{\mu} &= \delta^{\mu}_{r} \\
n'^{\mu} &= \delta^{\mu}_{u} - \frac{1}{2}\left(1 - \frac{2mr'}{\Sigma}\right)\delta^{\mu}_{r} \\
m'^{\mu} &= \frac{1}{\sqrt{2\Sigma}}\left(ia\sin\theta\,(\delta^{\mu}_{u} - \delta^{\mu}_{r}) + \delta^{\mu}_{\theta} + \frac{i}{\sin\theta}\,\delta^{\mu}_{\varphi}\right)
\end{aligned} \tag{6.35}$$

The metric defined by the latter tetrad is the Kerr metric in Kerr-Schild coordinates.

Chapter 7

Rotating Hayward & Modified-Hayward Metric

7.1 The Rotating Hayward Metric

As it is well known in the literature, the **Hayward spacetime** is one of the most famous prototypical model for non-singular black holes. In particular, we have that the line element is given by:

$$ds^2 = f(r)\,dt^2 - \frac{dr^2}{f(r)} - r^2\,d\Omega^2 \tag{7.1}$$

where

$$f(r) = 1 - \frac{2M(r)}{r}\,, \qquad M(r) = m\,\frac{r^3}{r^3 + g^3}$$

where m is the mass of the black hole and $g > 0$ is a constant measuring the deviations from the Schwarzschild solution.

The rotating Hayward spacetime has been worked out in [19], by means of the Newman-Janis algorithm, using the following complexification for the mass term:

$$\widetilde{f}(r,\theta) = 1 - \frac{2\widetilde{M}r}{\Sigma}\,, \qquad \widetilde{M}_{\gamma,\delta}\,(r,\theta) = m\,\frac{r^{3+\gamma}\Sigma^{-\gamma/2}}{r^{3+\gamma}\Sigma^{-\gamma/2} + g^3\,r^{\delta}\Sigma^{-\delta/2}} \tag{7.2}$$

where $\gamma, \delta \in \mathbb{R}$ are arbitrary parameters.
The rotating Hayward spacetime displayed in [19] is then given, in advanced coordinates, as follows

$$g_{\mu\nu} = \begin{pmatrix} \widetilde{f} & 1 & 0 & a\sin^2\theta\left(1-\widetilde{f}\right) \\ \cdot & 0 & 0 & -a\sin^2\theta \\ \cdot & \cdot & -\Sigma & 0 \\ \cdot & \cdot & \cdot & -\sin^2\theta\left[\Sigma + a^2\sin^2\theta\left(2-\widetilde{f}\right)\right] \end{pmatrix} \tag{7.3}$$

In particular, it has been pointed out in [19] that for the latter spacetime we can distinguish two classes of solutions.

1. *Complexification Type-I*: $\gamma = \delta$.
   - The spacetime is of Petrov type D;
   - The spacetime is regular everywhere for $g \neq 0$;
   - The weak energy condition, satisfied in the non-rotating case, is violated for $a \neq 0$.
2. *Complexification Type-II*: $\gamma, \delta \in \mathbb{R}\backslash\{0\}$.
   - There is no global transformation that allows us to write the new metric in Boyer-Lindquist coordinates(that can happen because we are not in the vacuum and the stress-energy tensor is not the one of a Maxwell electromagnetic field);
   - The spacetime is regular everywhere for $g \neq 0$;
   - The weak energy condition is not satisfied.

Now, let's discuss some properties of this spacetime.

Claim 4. The Rotating Hayward spacetime admits event horizons given by the roots of the following equation:

$$\widetilde{\Delta} \equiv r^2 - 2\widetilde{M}_{\gamma,\delta}\, r + a^2 = 0 \tag{7.4}$$

Proof. Basically, an event horizon for a stationary axisymmetric spacetime is a null surface with the following features:

$$\Phi(u, r, \theta, \varphi) = \Phi(r, \theta) = 0\,, \quad n_\mu \sim \partial_\mu \Phi\,, \quad g^{\mu\nu} n_\mu n_\nu = 0 \tag{7.5}$$

where n_μ is the normal to the surface.
Hence, the general condition for a surface to be an event horizon for a stationary axisymmetric spacetime is given by:

$$g^{rr}\,(\partial_r\Phi)^2 + g^{\theta\theta}\,(\partial_\theta\Phi)^2 = 0$$

Now, we can choose our coordinates (r, θ) in such a way that we can write the equation of the surface as $\Phi(r) = 0$, i.e. as a function of r alone. In this case we get

$$g^{rr}\,(\partial_r\Phi)^2 = 0$$

Thus, we see that an event horizon occurs when $g^{rr} = 0$, i.e. $\widetilde{\Delta} \equiv r^2 - 2\widetilde{M}_{\gamma,\delta}\, r + a^2 = 0$. □

Claim 5. The Rotating Hayward spacetime admits an ergosphere defined by the equation:

$$r^2 - 2\widetilde{M}_{\gamma,\delta}\, r + a^2\cos^2\theta = 0 \tag{7.6}$$

Proof. In order to evaluate the properties of the ergosphere we have to study the behavior of the norm of the timelike Killing vector $\xi_{(u)} = \xi_{(t)}$. Indeed, the ergosphere is defined as the region outside the black hole where the timelike Killing vector becomes spacelike.
Hence, the external boundary of the ergosphere is given by

$$g_{\mu\nu}\,\xi^\mu_{(u)}\,\xi^\nu_{(u)} = g_{uu} = 0 \tag{7.7}$$

Thus,

$$\widetilde{f} = 1 - \frac{2\widetilde{M}_{\gamma,\delta}\, r}{\Sigma} = 0 \quad \Longrightarrow \quad r^2 - 2\widetilde{M}_{\gamma,\delta}\, r + a^2\cos^2\theta = 0 \tag{7.8}$$

assuming $\Sigma \neq 0$. □

Claim 6. There are no Closed Timelike Curves (CTC) in the Rotating Hayward spacetime for $r \geqslant 0$.

Proof. In order to deal with the CTC's problem we have to study the norm of the Killing vector $\xi_{(\varphi)}$. Indeed, it is easy to see that

$$g_{\mu\nu}\,\xi^{\mu}_{(\varphi)}\,\xi^{\nu}_{(\varphi)} = g_{\varphi\varphi} = -\sin^2\theta\left[\Sigma + a^2\sin^2\theta\left(2-\tilde{f}\right)\right] \leqslant 0\,, \quad \forall r \geqslant 0\,,\ \theta \in [0, 2\pi) \tag{7.9}$$

which means that there are no CTC in the Rotating Hayward spacetime for $r \geqslant 0$. □

Now, in order to simplify our analysis, we may select a specific complexification for the Hayward mass. In particular, if we want to remain close to the analysis developed for the Kerr spacetime, we have to choose a complexification that allows us to write the metric (7.3) in Boyer-Lindquist coordinates. It is easy to see that this is the case when $\gamma = \delta$.
Thus, the metric (7.3) can be written in a Kerr-like form as:

$$\begin{aligned} ds^2 = &\left(1 - \frac{2M\,r}{\Sigma}\right) dt^2 + \frac{4aMr\sin^2\theta}{\Sigma}\, dt d\varphi - \frac{\Sigma}{\tilde{\Delta}}\, dr^2 \\ &- \Sigma\, d\theta^2 - \sin^2\theta\left(r^2 + a^2 + \frac{2a^2Mr\sin^2\theta}{\Sigma}\right) d\varphi^2 \end{aligned} \tag{7.10}$$

where

$$M = M(r) = m\,\frac{r^3}{r^3+g^3}\,, \qquad \tilde{\Delta} = r^2 - 2M\,r + a^2 \tag{7.11}$$

This spacetime is clearly stationary and axisymmetric, with Killing vectors $\xi_{(t)} = \partial/\partial t$ and $\xi_{(\varphi)} = \partial/\partial\varphi$. Moreover, the horizons are then given by the roots of the equation $\tilde{\Delta}(r) = 0$.

7.2 The Rotating Modified-Hayward Metric

Spacetime metrics describing "non-singular" black holes are commonly studied in the literature as effective modification to the Schwarzschild solution that mimic quantum gravity effects removing the central singularity. In [21], De Lorenzo et al. pointed out that, in order to be physically plausible, such metrics should also incorporate the 1-loop quantum corrections to the Newton potential and a non-trivial time delay between an observer at infinity and an observer in the regular center. As previously discussed, they were able to present a modification of the well-known Hayward metric that features these two properties, i.e.

$$ds^2 = G(r)F(r)\,dt^2 - \frac{dr^2}{F(r)} - r^2\,d\Omega^2 \tag{7.12}$$

with

$$F(r) = 1 - \frac{2M(r)}{r}\,,\quad M(r) = \frac{mr^3}{r^3+g^3} \qquad \text{and} \qquad G(r) = 1 - \frac{\alpha\beta m}{\alpha r^3 + \beta m} \tag{7.13}$$

where α incorporates the the **time delay** and β represents the **1-loop corrections to the Newtonian potential**.

In terms of advanced null coordinates, we have that (7.12) can be rewritten as

$$ds^2 = G(r)F(r)\,du^2 + 2\sqrt{G(r)}\,dudr - r^2\,d\Omega^2 \tag{7.14}$$

At this stage we just need to find an appropriate complexification of the functions $F(r)$ and $G(r)$. For what concerns the function $F(r)$, we could assume the one proposed in [19], i.e.

$$\widetilde{F}(r,\theta) = 1 - \frac{2\widetilde{M}r}{\Sigma}\,, \qquad \widetilde{M} = m\,\frac{r^{3+\gamma}\Sigma^{-\gamma/2}}{r^{3+\gamma}\Sigma^{-\gamma/2} + g^3\,r^{\delta}\Sigma^{-\delta/2}} \tag{7.15}$$

where $\gamma, \delta \in \mathbb{R}$ are arbitrary parameters.
However, as a first attempt we avoid complexifications of the 1-loop corrections, i.e. of the function $G(r)$.

We may choose to make the following choice for the complexification:

$$\widetilde{F}(r,\theta) = 1 - \frac{2\widetilde{M}r}{\Sigma}\,, \qquad \widetilde{G}(r,\theta) = G(r) = 1 - \frac{\alpha\beta m}{\alpha r^3 + \beta m} \tag{7.16}$$

Under these assumptions, if we follow the prescriptions stated in [18] we get:

$$\begin{aligned} g_{uu} &= G\,\widetilde{F}\,, \;\; g_{u\varphi} = a\sin^2\theta\,(\sqrt{G} - G\,\widetilde{F})\,, \\ g_{\varphi\varphi} &= -\sin^2\theta\,[\Sigma + a^2\sin^2\theta\,(2\sqrt{G} - G\,\widetilde{F})]\,, \;\; g_{\theta\theta} = -\Sigma\,, \\ g_{r\varphi} &= -a\sqrt{G}\,\sin^2\theta\,, \;\; g_{ur} = \sqrt{G} \end{aligned} \tag{7.17}$$

Now, let's discuss some properties of this spacetime.

Claim 7. The Rotating Hayward spacetime admits event horizons given by the roots of Eq. (7.4).

Claim 8. The Rotating Modified Hayward spacetime admits an ergosphere as defined in (7.6).

Proof. In order to evaluate the properties of the ergosphere we have to study the behavior of the norm of the timelike Killing vector $\xi_{(u)} = \xi_{(t)}$. Indeed, the ergosphere is defined as the region outside the black hole where the timelike Killing vector becomes spacelike.
Hence, the external boundary of the ergosphere is given by

$$g_{\mu\nu}\,\xi^{\mu}_{(u)}\,\xi^{\nu}_{(u)} = g_{uu} = 0 \quad\Longleftrightarrow\quad g_{uu} = G(r)\widetilde{F}(r,\theta) = 0 \tag{7.18}$$

Recalling that $G(r) > 0$ it is straightforward to see that the ergosphere for the Modified case is the same as before. Indeed,

$$\widetilde{F} = 1 - \frac{2\widetilde{M}_{\gamma,\delta}\,r}{\Sigma} = 0 \quad\Longrightarrow\quad r^2 - 2\widetilde{M}_{\gamma,\delta}\,r + a^2\cos^2\theta = 0 \tag{7.19}$$

assuming $\Sigma \neq 0$. □

Claim 9. There are no CTC in the Rotating Modified Hayward spacetime, for $r \geqslant 0$.

Proof. In order to deal with the CTC's problem we have to study the norm of the Killing vector $\xi_{(\varphi)}$. Indeed, it is easy to see that

$$g_{\mu\nu}\,\xi^{\mu}_{(\varphi)}\,\xi^{\nu}_{(\varphi)} = g_{\varphi\varphi} = -\sin^2\theta\,[\Sigma + a^2\sin^2\theta\,(2\sqrt{G} - G\,\widetilde{F})] \leqslant 0\,, \quad \forall r \geqslant 0\,,\ \theta \in [0, 2\pi) \tag{7.20}$$

which means that there are no CTC in the Rotating Hayward spacetime, for $r \geqslant 0$. □

Now, in order to simplify our analysis, we may select a specific complexification for the Hayward mass. In particular, if we want to remain close to the analysis developed for the Kerr spacetime, we have to choose a complexification that allows us to write the metric (7.17) in Boyer-Lindquist coordinates. It is easy to see that this is the case when $\gamma = \delta$.
Thus, the metric (7.17) can be written in a Kerr-like form as:

$$\begin{aligned} ds^2 &= G\widetilde{F}\,dt^2 + 2a\sin^2\theta\sqrt{G}\left(1 - \sqrt{G}\widetilde{F}\right)\,dtd\varphi - \frac{\Sigma}{\widetilde{\Delta}}\,dr^2 \\ &\quad - \Sigma\,d\theta^2 - \sin^2\theta\left[\Sigma + a^2\sin^2\theta\sqrt{G}\left(2 - \sqrt{G}\widetilde{F}\right)\right]\,d\varphi^2 \end{aligned} \tag{7.21}$$

where

$$M = M(r) = m\,\frac{r^3}{r^3 + g^3}\,, \qquad \widetilde{\Delta} = r^2 - 2M\,r + a^2 \tag{7.22}$$

This spacetime is clearly stationary and axisymmetric, with Killing vectors $\xi_{(t)} = \partial/\partial t$ and $\xi_{(\varphi)} = \partial/\partial\varphi$. Moreover, the horizons are then given by the roots of the equation $\widetilde{\Delta}(r) = 0$, as in the unmodified case.

It is worth noting that the equation for the horizon, for $\gamma = \delta$, can be rewritten in the following form:

$$r^5 - 2mr^4 + a^2r^3 + g^3(r^2 + a^2) = 0 \tag{7.23}$$

assuming $r \neq -g$, $m \geqslant 0$ and $g > 0$.
Using perturbation theory, we can find an approximate solution of the latter equation, under the assumption that $g \ll 1$. Hence, we find that, at the first order in $\varepsilon = g^3$, the solution is given by:

$$r^{(1)}_H = r_{0,\,H} - \frac{25g^3(r^2_{0,\,H} + a^2)}{r^2_{0,\,H}(5r_{0,\,H} - 4m - \sqrt{16m^2 - 15a^2})(5r_{0,\,H} - 4m + \sqrt{16m^2 - 15a^2})} \tag{7.24}$$

where

$$r_{0,\,H} = m + \sqrt{m^2 - a^2} \tag{7.25}$$

is the solution of the unperturbed equation, i.e. $g = 0$.
This result is useful in order to compute approximately the area of the event horizon and the angular velocity of the black hole. For further details and explicit computations refer to *Appendix* B.

7.3 Open Problems and Concluding Remarks

Starting from Eq. (7.12) we wanted to investigate an appropriate metric for a rotating Planck star by means of the NJA. What we found was the metric (7.17) which contains, however, a singularity. Indeed, the Kretschmann scalar has the following properties:

$$\begin{aligned} &\lim_{r\to 0} K = f(\alpha) \qquad \text{for } \theta \neq \frac{\pi}{2} \qquad \text{where } f(\alpha) \xrightarrow{\alpha\to 0} 0 \\ &\lim_{r\to 0} K = \infty \qquad \text{for } \theta \to \frac{\pi}{2} \end{aligned} \tag{7.26}$$

for all $\gamma, \delta \in \mathbb{R}$.
Basically, we started from a non-singular black hole and, after the application of the NJA, we obtained a singular spacetime. By contrast, if we set $G = 1$, as in the Hayward case, after the application of the NJA we recover a new metric which is still non-singular.
Unfortunately, from a physical perspective, we do not know how to explain the re-introduction of the singularity in the rotating metric by the NJA. We can just say that this is not a feature of the choice of the modification function, but it rather seem to be a property of the algorithm itself. Indeed, we proved that, independently from the choice of a function $G = G(r, \theta)$ to plug into the rotating metric (7.3), we always gain the emergence of a curvature singularity.

Now, in order to overcome this problem, we can always try to look for a suitable rotating metric following the same reasoning that we used in the static case (see [21]), i.e. we have to properly modify g_{uu}.
Indeed, let us consider:

$$\begin{aligned} ds^2 = \widetilde{f} du^2 + 2a \sin^2\theta (1 - \widetilde{f}) - 2a \sin^2\theta - \Sigma d\theta^2 \\ - \sin^2\theta \left[\Sigma + a^2 \sin^2\theta (2 - \widetilde{f}) \right] d\varphi^2 \end{aligned} \tag{7.27}$$

where $\widetilde{f}$ and Σ are defined in *Section* 7.1. Thus, if we modify g_{uu} as:

$$g_{uu} = \widetilde{f}(r, \theta) \quad \longrightarrow \quad g_{uu} = g(r) \widetilde{f}(r, \theta) \tag{7.28}$$

examining the Kretschmann scalar one can deduce that, in order to avoid the emergence of the singularity, we have to require that $g'(r = 0) = g''(r = 0) = g'''(r = 0) = 0$. Then one find that a suitable function is given by:

$$g(r) = 1 - \alpha \left[1 - \exp\left(-\frac{\beta m}{\alpha r^3} \right) \right] \tag{7.29}$$

where α represents the **time delay** and $\beta > 0$.

Part III

Black Hole Fireworks and Transition Amplitudes in Loop Quantum Gravity

Chapter 8

The Spin foam Approach to Loop Quantum Gravity

In the following section we quickly review the fundamental concepts lying at the very basis of the canonical formulation of Loop Quantum Gravity (LQG). After that we introduce the modern formulation of LQG through a covariant approach commonly known as the Spin foam formalism. In this introductory part we will mostly follow the line of resining presented in [22; 5].

Remark. In the following, in order to remain consistent with the references, we will use the metric signature $(-+++)$.

8.1 Tetrad formulation of General Relativity

Let $(\mathcal{M}, g)$ be a Lorentzian manifold.

Definition 18 (Tetrad)**.** The set of 1-form $\{e^I_\mu(x)\,,\;\; I = 0,1,2,3\}$ such that

$$g_{\mu\nu} = \eta_{IJ}\, e^I_\mu e^J_\nu\,, \qquad \eta_{IJ} \equiv \mathtt{diag}(-1,1,1,1) \tag{8.1}$$

is said to be a **tetrad**.

Clearly, a tetrad provides a local isomorphism between a general reference frame and an inertial one, characterized by the flat metric η. A local inertial frame is defined up to a Lorentz transformation, and in fact notice that the definition is invariant under

$$e^I_\mu(x) \;\longrightarrow\; \hat{e}^I_\mu(x) = \Lambda^I_J(x) e^I_\mu(x)$$

This means that the internal index I carries a representation of the Lorentz group.
From a geometrical point of view, the tetrad provides an isomorphism between the tangent bundle $T(\mathcal{M})$ and a Lorentz principal bundle $F = (\mathcal{M}, SO(1,3))$, i.e. $T(\mathcal{M}) \simeq F$. On this bundle we have a connection ω^{IJ}_μ which we can use to define covariant differentiation of the fibres,

$$D_\mu v^I(x) = \partial_\mu v^I(x) + \omega_{\mu\ J}^{\ I} v^J(x) \tag{8.2}$$

We can also define a derivative for objects equipped with both spacetime and internal lorentzian indices, such as a tetrad, i.e.

$$\mathcal{D}_\mu e^I_\nu(x) := \partial_\mu e^I_\nu(x) + \omega_{\mu\ J}^{\ I}\, e^J_\nu(x) - \Gamma^\lambda_{\nu\mu} e^I_\lambda(x) \tag{8.3}$$

As $\Gamma(g)$ is metric compatible, we can define the so called **spin-connection** as follows

Definition 19. A connection ω_μ on the Lorentz principal bundle F is called spin-connection if it is **tetrad compatible**, i.e. $\mathcal{D}_\mu e^I_\nu = 0$.

From this definition we immediately get the following

Proposition 10. Let ω_μ be a spin-connection, then

$$D_{(\mu}e^I_{\nu)} = \Gamma^\lambda_{(\nu\mu)}e^I_\lambda\,, \qquad D_{[\mu}e^I_{\nu]} = \Gamma^\lambda_{[\nu\mu]}e^I_\lambda = 0 \tag{8.4}$$

Proof. It follows trivially form the metric compatibility and from the latter definition. □

From the last proposition, one can easily deduce the following relation between the spin and Levi-Civita connections,

$$\omega^I_{\mu\,J} = e^I_\nu\,\nabla_\mu e^\nu_J \tag{8.5}$$

as well as the fact that the spin connection satisfies the **Cartan's first structure equation**, i.e.

$$d_\omega\, e^I := de^I + \omega^I_J \wedge e^J = D_\mu e^I_\nu\, dx^\mu \wedge dx^\nu = 0 \tag{8.6}$$

Given the connection, we define its **curvature** as

$$F^{IJ} := d\omega^{IJ} + \omega^I_K \wedge \omega^{KJ} \tag{8.7}$$

whose spacetime components are

$$F^{IJ}_{\mu\nu} = \partial_\mu\omega^{IJ}_\nu + \omega^I_{K\,\mu}\omega^{KJ}_\nu - \{\mu \leftrightarrow \nu\} \tag{8.8}$$

Now, given the previous relation between the spin-connection and the Levi-Civita connection, we can prove that

$$F^{IJ}_{\mu\nu}(\omega(e)) = e^{I\rho}e^{J\sigma}R_{\mu\nu\rho\sigma}(e) \tag{8.9}$$

where $R_{\mu\nu\rho\sigma}(e)$ is the Riemann tensor. This relation is known as **Cartan's second structure equation**. It shows that **general relativity is a gauge theory whose local gauge group is the Lorentz group**, and the Riemann tensor is nothing but the field-strength of the spin connection.

8.2 The Einstein-Hilbert action

The Einstein-Hilbert action can be rewritten as a functional of the tetrad in the following way (setting $16\pi G = 1$)

$$S_{EH}(e) = \frac{1}{2}\,\varepsilon_{IJKL}\int e^I \wedge e^J \wedge F^{KL}(\omega(e)) \tag{8.10}$$

Indeed,

$$\begin{aligned} S_{EH}(g(e)) &= \int d^4x\sqrt{-g}g^{\mu\nu}R_{\mu\nu} = \int d^4x\, e\, e^\mu_I e^{\nu\, I} R_{\mu\rho\nu\sigma} e^\rho_J e^{\sigma\, J} = \\ &= \int d^4x\, e\, e^\mu_I e^\rho_J F^{IJ}_{\mu\rho}(\omega(e)) = \frac{1}{4}\int d^4x\,\varepsilon_{IJKL}\,\varepsilon^{\mu\rho\alpha\beta}\, e^K_\alpha\, e^L_\beta F^{IJ}_{\mu\rho}(\omega(e)) = \\ &= \frac{1}{2}\,\varepsilon_{IJKL}\int e^I \wedge e^J \wedge F^{KL}(\omega(e)) = S_{EH}(e) \end{aligned} \tag{8.11}$$

where we have made use of the definition of tetrad, the Cartan's second structure equation and the relation between the determinants $e^2 = -g$.
A fact which plays an important role in the following is that we can lift the connection to be an independent variable, and consider the new action

$$S_{EH}(e,\omega) = \frac{1}{2}\,\varepsilon_{IJKL} \int e^I \wedge e^J \wedge F^{KL}(\omega) \tag{8.12}$$

Although it depends on extra fields, this action remarkably gives the same equations of motion as the previous one. In particular, from the variation of the action we get the following equations of motions:

$$\begin{aligned} de^I + \omega^I{}_J e^J = 0 \quad &\Longleftrightarrow \quad \text{Torsion-less} \\ F^{IJ}(\omega(e)) = 0 \quad &\Longleftrightarrow \quad \text{Einstein's Field Equations} \end{aligned} \tag{8.13}$$

If we insist on the connection being an independent variable, it is worth noting that there exists a second term that we can add to the Lagrangian that is compatible with all the symmetries:

$$\delta_{IJKL}\, e^I \wedge e^J \wedge F^{KL}(\omega) \tag{8.14}$$

where we used $\delta \equiv \delta_{I[K}\delta_{L]J}$. This term does not appear in in the ordinary second order metric since:

$$\omega_\mu = e \cdot \nabla_\mu e \quad \Longrightarrow \quad \delta_{IJKL}\, e^I \wedge e^J \wedge F^{KL}(\omega) = \varepsilon^{\mu\nu\rho\sigma}\, R_{\mu\nu\rho\sigma} = 0$$

If we then add this term, multiplied by a coupling constant $1/\gamma$, we get the so called **Holst action**:

$$S(e,\omega) = \left(\frac{1}{2}\,\varepsilon_{IJKL} + \frac{1}{\gamma}\delta_{IJKL}\right) \int e^I \wedge e^J \wedge F^{KL}(\omega) \tag{8.15}$$

Assuming non-degenerate tetrads, this action leads to the same field equations of general relativity, i.e.

$$\omega_\mu = e \cdot \nabla_\mu e\,, \qquad G_{\mu\nu}(e) = 0$$

This result is completely independent of the value of γ, which is thus a parameter irrelevant in classical vacuum general relativity. It will however turn out to play a key role in the quantum theory, where it is known as the **Immirzi parameter**.

8.3 The Hamiltonian Formulation

The ADM formalism provides a change of variables which simplifies the canonical description of Einstein-Hilbert action. Let us consider a spacetime $(\mathcal{M}, g)$ with a topology such that $\mathcal{M} \simeq \mathbb{R} \times \Sigma$ where Σ is a fixed three-dimensional manifold of arbitrary topology and spacelike signature. Under this assumption we build up two fields starting from the metric $g_{\mu\nu}$, i.e.

$$N = \sqrt{-g^{00}}\,, \qquad N_i = g_{0i} \qquad i = 1, 2, 3. \tag{8.16}$$

These two field are respectively called **Lapse** and **Shift** functions. This change of variables turns out to be extremely useful because if we write the action in terms of these variables one discovers that the Lagrangian does not de- pend on the time derivatives of Lapse and Shift (they are Lagrange multipliers), and this immediately simplifies the canonical analysis.

Moreover, N and N_i have a nice intrinsic geometrical interpretation. Indeed, if we consider the hypersurface Σ_t at $t = const.$. It is easy to see that the **Lapse** function is the proper time elapsed between a point on Σ_t and a point on $\Sigma_{t+\delta t}$, along the normal to the surface. The **Shift** function measures, instead, the shift of the spatial coordinates from one constant surface to the next, with respect to the coordinates that observers not moving on the initial surface would carry with them.
The Lapse can equivalently be defined by

$$N^2\, \mathtt{det}(q) = \mathtt{det}(g)$$

where q is the 3-metric induced on the spacelike surface Σ_t at constant time. From this definition it follows that

$$g_{00} = -N^2 + N_i N^i$$

so that the line element in the ADM variables reads

$$ds^2 = -(N^2 - N_i N^i)dt^2 + 2N_i dx^i dt + q_{ij} dx^i dx^j \tag{8.17}$$

Now, the extrinsic curvature of Σ_t is given by

$$k_{ij} = \frac{1}{2N}\left(\dot{q}_{ij} - D_{(i}N_{j)}\right) \tag{8.18}$$

where the dot indicates the derivative with respect to t and D_i is the covariant derivative of the three-metric.

Now, it is easy to see that a tetrad for (8.17) is given by

$$e_0^I = N n^I + N^a e_a^I\,, \qquad \delta_{ij} e_a^i e_b^j = g_{ab} \tag{8.19}$$

where $n^I = n^\mu e_\mu^I$ with n^μ normal to Σ_t and where the triad e_a^i is the spatial part of the tetrad. In order to simplify the discussion, it is customary to work in the *time gauge*, i.e. $e_\mu^I n^\mu = \delta_0^I$. In such a gauge, we introduce the so called **Ashtekar variables**:

- the **Densitized Triad**: $E_i^a := e\, e_i^a = \frac{1}{2}\,\varepsilon_{ijk}\,\varepsilon^{abc}\, e_b^j e_c^k$
- the **Ashtekar-Barbero connection**: $A_a^i := \gamma\omega_a^{0i} + \frac{1}{2}\,\varepsilon^i{}_{jk}\,\omega_a^{jk}$

These variables turns out to be **conjugated**. In fact, we can rewrite the Holst action in terms of the new variables as

$$S[A, E, N, N^a] = \frac{1}{\gamma}\int dt \int_\Sigma d^3x\, \left(E_i^a \dot{A}_a^i - A_0^i G_i - NH - N^a H_a\right) \tag{8.20}$$

where

$$\begin{aligned} G_i &:= D_a E_i^a \equiv \partial_a E_i^a + \varepsilon_{ijk} A_a^j E^{a\,k} \\ H_a &= \frac{1}{\gamma}\, F_{ab}^i E_i^b - \frac{1+\gamma^2}{\gamma}\, k^i{}_a G_i \\ H &= \left[F_{ab}^i - (1+\gamma^2)\varepsilon^{imn}\, k_{ma} k_{nb}\right] \frac{\varepsilon_{ikl} E^{a\,k} E^{b\,l}}{\mathtt{det}E} + \frac{1+\gamma^2}{\gamma} G^i \partial_a \frac{E_i^a}{\mathtt{det}E} \end{aligned} \tag{8.21}$$

We have three kind of constraints, the one defined by $H(A, E)$ and $H_a(A, E)$, known as the Hamiltonian and space-diffeomorphism constraints, and an extra set of constraints, defined by G_i, known in the realm of gauge theories as the Gauss constraint. It is also physically relevant to stress that the resulting algebra of the constraints is of the first class[1].
It is also easy to check that E^a_i and A^i_a transform respectively as an $SU(2)$ vector and as an $SU(2)$-connection under the gauge transformation generated by the Gauss constraints.
Summarizing, in this formulation of General Relativity the theory is described by an extended phase space of dimension $18 \cdot \infty^3$ with the fundamental Poisson bracket

$$\left\{ A^i_{\ a}(x),\, E^b_j(y) \right\} = \gamma\, \delta^b_a \delta^i_j\, \delta^3(x - y) \tag{8.22}$$

This new internal index i corresponds to the adjoint representation of $SU(2)$, and we can recover the $12 \cdot \infty^3$ dimensional phase space on the constraint surface $G_i = 0$ dividing by gauge orbits generated by G.

Now, we have to smear the algebra in order to proceed with the canonical quantization. Considering the fact that the densitised triad is a 2-form, it is natural to smear it on a surface S,

$$E_i(S) := \int_S d^2\sigma n_a E^a_i \qquad n_a = \varepsilon_{abc} \frac{\partial x^b}{\partial \sigma^1} \frac{\partial x^c}{\partial \sigma^2} \tag{8.23}$$

This quantity represents the flux of E across S.
The connection on the other hand is a 1-form, so it is natural to smear it along a 1- dimensional path. Let us consider a path γ parametrized as $\gamma\, :\, [0, 1] \to \Sigma,\ s \mapsto x^a(s)$. Then, we can associate an element of $SU(2)$ to the Ashtekar-Barbero connection defined as $A_a := A^i_{\ a}\tau_i$, where τ_i are the generator of $SU(2)$. Then we can integrate A_a along γ as

$$A^i_{\ a} \longrightarrow \int_\gamma A \tag{8.24}$$

Next, we define the **holonomy** of A along γ to be

$$h_\gamma = \mathcal{P} \exp \left(\int_\gamma A \right) \tag{8.25}$$

where $\mathcal{P}$ stands for the path-ordered product.

For a detailed discussion of the canonical quantization of the theory, described in terms of Ashtekar-Barbero variables, see [22; 13].

[1] A **first class constraint** is a dynamical quantity in a constrained Hamiltonian system whose Poisson bracket vanishes on the constraint surface with all the other constraints

8.4 Spin foam Quantization of BF Theories

It is now relevant to briefly present the general Spin foam quantization for a BF theory due to the fact, as it will appear obvious in the following discussion, that classical gravity can be regarded as a particular case of this large class of models.

First of all, let G be a compact group whose Lie algebra $\mathfrak{g}$ has an invariant inner product here denoted by $\langle,\rangle$ and $\mathcal{M}$ a d-dimensional compact, orientable manifold.
A **classical BF theory** is defined by an action

$$S[B,\omega] = \int_{\mathcal{M}} \langle B \wedge F(\omega)\rangle \tag{8.26}$$

where B is a $\mathfrak{g}$-valued $(d-2)$-form and ω is a connection on a G principal bundle over $\mathcal{M}$. One can easily notice that the theory has no local excitations or, more precisely, the gauge symmetries of the action are the local G gauge transformations

$$\delta B = [B, \alpha]\,, \qquad \delta\omega = d_\omega\, \alpha$$

where α is a $\mathfrak{g}$-valued 0-form, and the topological gauge transformation

$$\delta B = d_\omega\, \eta\,, \qquad \delta\omega = 0$$

where η is a $\mathfrak{g}$-valued 0-form. Moreover, it is worth remarking that the theory has only global or topological degrees of freedom.

The **partition function** $\mathcal{Z}$ is formally defined as

$$\mathcal{Z} := \int \mathcal{D}[B]\mathcal{D}[\omega]\, \exp\left(i \int_{\mathcal{M}} \langle B \wedge F(\omega)\rangle\right) \tag{8.27}$$

then, formally integrating over the B, we get

$$\mathcal{Z} = \int \mathcal{D}[\omega]\, \delta(F(\omega)) \tag{8.28}$$

Thus the partition function can be understood as the "volume" of the space of flat connections on $\mathcal{M}$.

Now, in order to extract some physics from this model, we want to approximate the d-dimensional manifold $\mathcal{M}$ with a suitable triangulation Δ. Thus, let us replace $\mathcal{M}$ with an arbitrary **cellular decomposition** Δ in the previous expressions. Given a cellular decomposition Δ, we can define the associated **dual 2-complex** of Δ, denoted by $\Delta^\star$, as a combinatorial object consisting of

- **vertices** $v \in \Delta^\star$, dual to d-cells in Δ;
- **edges** $e \in \Delta^\star$, dual to $(d-1)$-cells in Δ;
- **faces** $f \in \Delta^\star$, dual to $(d-2)$-cells in Δ

Hence, if we consider Δ to be a triangulation, we have that:

1. the field B is associated with Lie algebra elements B_f assigned to faces $f \in \Delta^\star$ via
$$B \longrightarrow B_f = \int_{(d-2)-cell} B$$
2. the connection ω is discretized by the assignment of group elements $g_e \in G$ to edges $e \in \Delta^\star$ as follows
$$\omega \longrightarrow g_e = \mathcal{P}\exp\left(\int_e \omega\right)$$

Thus, the discretized version of the partition function is given by

$$Z(\Delta) = \int \prod_e dg_e \prod_f dB_f \, \exp\left(iB_f U_f\right) = \int \prod_e dg_e \prod_f \delta(g_{e_1} \cdots g_{e_n}) \tag{8.29}$$

where $U_f = g_{e_1} \cdots g_{e_n}$ denotes the holonomy around faces. The last expression can then be regarded as the discrete analogue of (8.28).

Remark. The integration measure dB_f is the standard Lebesgue measure, while the integration in the group variables is done in terms of the Haar (unique and invariant) measure in G.

Now, if we recall the

Theorem 9 (Peter-Weyl)**.** A basis on the Hilbert space $L_2(G, d\mu_{Haar})$ of functions on a compact group G is given by the matrix elements of the unitary irreducible representation of the group.

one can derive a very useful formula for the Dirac delta distribution

$$\delta(g) = \sum_\rho d_\rho \, \mathtt{Tr}[\rho(g)] \tag{8.30}$$

where ρ are irreducible unitary representations of G.
Thus, inserting the resolution of the Dirac delta distribution into (8.29), we get

$$\mathcal{Z}(\Delta) = \sum_{\mathcal{C}:\{\rho\}\to\{f\}} \int \prod_e dg_e \prod_f d_{\rho_f} \, \mathtt{Tr}[\rho_f(g_{e_1} \cdots g_{e_n})] \tag{8.31}$$

If we were interested in the integration over the connection, this could be performed as follows. In a triangulation Δ, the edges $e \in \Delta^\star$ bound precisely d different faces; therefore, the g_e's appear in d different traces. Thus, the spin foam amplitude of $SO(4)$ BF theory, $Z(\Delta)$, can be recast in the following form:

$$Z_{BF}(\Delta) = \sum_{\mathcal{C}:\{\rho\}\to\{f\}} \prod_f d_{\rho_f} \prod_e P^e_{inv}(\rho_1, \ldots, \rho_d) \tag{8.32}$$

where

$$P^e_{inv}(\rho_1, \ldots, \rho_d) := \int dg_e \, \rho_1(g_e) \otimes \cdots \otimes \rho_d(g_e) \tag{8.33}$$

is a projector onto $\mathtt{Inv}[\rho_1 \otimes \cdots \otimes \rho_d]$.
In other words, the BF amplitude associated to a two complex $\Delta^\star$ is simply given by summing over the numbers obtained by the natural contraction of the network of projectors P_e according to the pattern provided defined by the two-complex $\Delta^\star$. It is also important to stress that the sum is over all possible assignments of irreducible representations of G to faces.

8.5 The Lorentzian EPRL Model

8.5.1 Elements of representation theory of $SL(2,\mathbb{C})$

It is well known in the literature that studying the representation theory of $SL(2,\mathbb{C})$ one discovers that its **unitary irreducible representations** are labelled by a positive real number $p \in \mathbb{R}^+$ and a half-integer $k \in \mathbb{N}/2$ and they act on the infinite dimensional Hilbert spaces denoted by $\mathcal{H}^{(p,k)}$, see [23]. The two **Casimirs** are $C_1 = L^2 - K^2$ and $C_2 = K \cdot L$ where L^i are the generators of an arbitrary rotation subgroup and K^i are the generators of the corresponding boosts.
Given $|p,k\rangle \in \mathcal{H}^{(p,k)}$ the Casimirs act on these vectors as follows

$$C_1\,|p,k\rangle = (k^2 - p^2 - 1)\,|p,k\rangle\,, \qquad C_2\,|p,k\rangle = pk\,|p,k\rangle$$

The definition of the EPRL model requires the introduction of an (arbitrary) subgroup $SU(2) \subset SL(2,\mathbb{C})$. This subgroup corresponds to the internal gauge group of the gravitational phase space in connection variables in the time gauge.
The link between the unitary representations of $SL(2,\mathbb{C})$ and those of $SU(2)$ is given by the decomposition

$$\mathcal{H}^{(p,k)} = \bigoplus_{j=k}^{\infty} \mathcal{H}^j \tag{8.34}$$

where $\mathcal{H}^j$ is a $d_j = 2j+1$ dimensional space that carries the spin j irreducible representation of $SU(2)$. Therefore, we can choose a basis $|p,k;j,m\rangle$ for $\mathcal{H}^{(p,k)}$, with $j = k, k+1, \ldots$ and $m = -j, \ldots, j$.

Now, we want the simplicity constraint to be fulfilled in the classical limit, i.e. we want that

$$K^i_f - \gamma L^i_f = o(\hbar) \tag{8.35}$$

where the label f makes reference to a face $f \in \Delta^\star$. Thus, in the formal limit $\hbar \to 0$ (or, more precisely, $j \to \infty$, $\hbar \to 0$ and $\hbar j = const.$) we want to recover

$$K^i - \gamma L^i = 0$$

Given this relations, in the classical limit we can rewrite the Casimirs as

$$C_1 = L^2 - K^2 = (1-\gamma^2)L^2\,, \qquad C_2 = K \cdot L = \gamma L^2 \tag{8.36}$$

Thus, from the representation theory of $SL(2,\mathbb{C})$, we get the following relations

$$\begin{cases} k^2 - p^2 - 1 = (1-\gamma^2)j(j+1) \\ pk = \gamma j(j+1) \end{cases} \tag{8.37}$$

from which we get

$$p = \gamma j\,, \qquad k = j \qquad \text{for } j \gg 1 \tag{8.38}$$

The first of these two relations is a restriction on the set of the unitary representations. The second picks out a subspace within each representation. Therefore, the states that satisfy these relations take the form

$$|p,k;j,m\rangle = |\gamma j, j; j, m\rangle \tag{8.39}$$

Notice that these states are in **one to one correspondence** with the states in the representations of $SU(2)$. We can thus introduce a map Y_γ as

$$Y_\gamma \,:\, \mathcal{H}^j \longrightarrow \mathcal{H}^{(\gamma j, j)}\,, \qquad |j;m\rangle \longmapsto |\gamma j, j; j, m\rangle \tag{8.40}$$

and all the vectors in the image of this map satisfy the simplicity constraints, in the weak sense, i.e.

$$\langle Y_\gamma \psi | K^i - \gamma L^i | Y_\gamma \phi \rangle \overset{j\to\infty}{\longrightarrow} 0 \tag{8.41}$$

Thus we assume that the states of quantum gravity are constructed from the states $|\gamma j, j; j, m\rangle$ alone.

The so called Y-map extends naturally to a map from functions over $SU(2)$ to functions over $SL(2,\mathbb{C})$. Indeed,

$$\begin{gathered} Y_\gamma \,:\, L_2[SU(2)] \longrightarrow F[SL(2,\mathbb{C})]\,, \\ \Psi[h] = \sum_{jmn} c_{jmn}\, D^{(j)}_{mn}(h) \longmapsto \Psi[g] = \sum_{jmn} c_{jmn}\, D^{(\gamma j, j)}_{jmjn}(g) \end{gathered} \tag{8.42}$$

where $D^{(j)}_{mn}(h)$ are the Wigner matrices and $D^{(p,k)}_{jmj'n}(g)$ are the matrix elements of the (p,k)-representation in the $|p,k;j,m\rangle$ basis that diagonalizes L^2 and L_z of the canonical $SU(2)$ subgroup.

And therefore we have a map from $SU(2)$ spin-networks to $SL(2,\mathbb{C})$ spin-networks. It is also well known that the Y-map is the core ingredient of the quantum gravity dynamics. Indeed, it depends on the Einstein-Hilbert action and encodes the way $SU(2)$ states transform under $SL(2,\mathbb{C})$ transformations in the theory. This, in turn, codes the dynamical evolution of the quantum states of space.

The physical states of quantum gravity are thus $SU(2)$ spin-networks, or, equivalently, their image under Y_γ. Notice that this space carries a scalar product which s well defined: the one determined by the $SU(2)$ Haar measure. The fact that the scalar product is $SU(2)$ and not $SL(2,\mathbb{C})$ invariant reflects the fact that the scalar product is associated to a boundary, and this picks up a Lorentz frame. For further informations about the spin-networks states for the 4D theory one can refer, for example, to [5].

8.5.2 The transition amplitudes of LQG

Now, following [5; 24; 25], one can rewrite the transition amplitudes of Loop Quantum Gravity in the form

$$Z_\Delta[h_l] = \int_{SL(2,\mathbb{C})} dg_{ev} \int_{SU(2)} dh_{ef} \sum_{j_f} \prod_f d_{j_f}\, \chi^{(\gamma j_f, j_f)} \left(\prod_{e\in\partial f} g_{ef}^{\epsilon_{ef}} \right) \prod_{e\in\partial f} \chi^{j_f}(h_{ef}) \tag{8.43}$$

where $\Delta^\star$ is the dual 2-complex bounded by the graph $\Gamma = \partial\Delta$ with nodes n and links l, $\chi^j(h)$ is the spin-j $SU(2)$ character and $\chi^{(p,k)}(g)$ is the $SL(2,\mathbb{C})$ character in the (p,k)-representation. Moreover, inside the $SL(2,\mathbb{C})$ character we find ϵ_{ef} which is a sign depending on the orientation of the graph Γ and g_{ef} is defined as

$$g_{ef} = \begin{cases} g_{es_e} h_{ef} g^{-1}_{et_e}\,, & \text{for the (internal) edges} \\ h_l \in SU(2)\,, & \text{for the links (i.e. boundary edges)} \end{cases} \tag{8.44}$$

Here s_e and t_e are respectively the source and target vertices of the edge e. For further details on the definition see [25].

Chapter 9

Towards computing black hole tunnelling time

9.1 Feynman rules

The classical definition of a spin foam vertex amplitude is via the evaluation of its boundary spin network. This definition applies to arbitrary vertices, and allows to immediately apply the integrability criterion conjectured by Baez and Barrett in [26] and proved by Kaminski in [27].

For every vertex whose boundary is a 3-link-connected graph, the spin foam amplitude is obtained taking:

1. a factor $D^{\gamma j,j}_{jmjn}(g)$ for all links;
2. a $SL(2,\mathbb{C})$ integral for all nodes minus a redundant one.

These rules have the advantage of being compact, but they use an explicit gauge fixing which makes appearing the $SU(2)$ subgroup that stabilizies the time gauge direction. refer to things that are not Lorentz invariant, namely the Wigner matrix for simple projected spin networks. Nonetheless, the Lorentz-invariance was proved in [25].

If this is the physical boundary, the magnetic indexes are free, or contracted with the Wigner's matrix of the boundary link holonomy. If this vertex is part of the bulk, the indexes are summed over in the next vertex, and the face spin are summed over too. In this way, one obtains the spin foam amplitude as also in [13], as a product of BF delta distributions on the faces expressed in the Plancherel decomposition and reduced by the action of the Y projector on the edges.

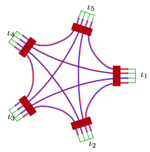
It is also important to recall that the orthogonality formula

$$\int dg D^{\rho,k}_{jmln}(g)\overline{D^{\rho',k'}_{j'm'l'n'}(g)} = \frac{\delta(\rho-\rho')}{\rho^2+k^2}\,\delta_{kk'}\delta_{jj'}\delta_{ll'}\delta_{mm'}\delta_{nn'} \tag{9.1}$$

defines a distribution in $L_2[SL(2,\mathbb{C})]$ but not in the space $\mathcal{K}$ of γ-simple projected spin networks.

The model is usually defined using a 5-valent vertex dual to a 4-simplex, and the corresponding evaluation of Feynman rules represented by the following drawing (see [13]),

$$Z = \sum_{j_f, i_e} \prod_f (2j_f + 1) \prod_v \tag{9.2}$$



The key object here is the projector on the simplicity constraints, represented by the Y map next to the integration boxes. However, the model can be defined for an arbitrary foam as discussed above.

9.2 Decomposition and boost Clebsch-Gordan

Just like in the Euclidean case, these amplitudes are given by contractions of Clebsch-Gordan (CG) coefficients. However, unlike for $SU(2)$ in the Euclidean case, the Clebsch-Gordan coefficients of the Lorentz group are not explicitly tabulated. However, only a special class of Clebsch-Gordan is needed explicitly, i.e. those concerning boosts along the z axis. This is because every group element can always be parametrized as

$$g = u\, e^{\frac{r}{2}\sigma_3}\, v^{-1} \tag{9.3}$$

where u and v are arbitrary rotations. This parametrization is clearly redundant, possessing a $U(1)$ symmetry of common rotations of u and v along the z axis. However, since this is a compact orbit, it is possible to use everywhere this parametrization and replace the $SL(2,\mathbb{C})$ integrals by

$$dg = \frac{1}{4\pi} \sinh^2 r\, dr du dv \tag{9.4}$$

where du and dv are the Haar measures for the two copies of $SU(2)$, and the normalisation comes from the explicit computations showed in [23].

The version of Eq. (9.3) in an arbitrary irreducible representation, irrep for short, is

$$D^{(\rho,k)}_{jmln}(g) = D^j_{mp}(u) d^{(\rho,k)}_{jlp}(r) D^l_{pn}(v^{-1}) \tag{9.5}$$

The matrix element of z-boosts are explicitly known via an integral form [23],

$$\begin{aligned} d^{(\rho,k)}_{jlp}(r) &= \sqrt{d_j}\sqrt{d_l}\int_0^1 dt\, d^j_{kp}(2t-1) d^l_{kp}(2t_r - 1)\left(te^{-r} + (1-t)e^{r}\right)^{i\rho - 1}, \\ t_r &= \frac{te^{-r}}{te^{-r} + (1-t)e^r} \end{aligned} \tag{9.6}$$

in terms of Wigner's $SU(2)$ matrices, $d^j_{mn}(\cos\beta)$.

For the Barrett-Crane model, only the special representations with $k = j = 0$ matter, for which we have the simple form

$$d^{(\rho,0)}_{000}(r) = \frac{\sin(\rho r)}{\rho \sinh r} \tag{9.7}$$

which allows an explicit evaluation of the Clebsch-Gordan coefficients. For the ERPL model, the situation is a bit more complicated. The relevant representations are now the so called γ-simple, for which

$$\rho = \gamma k, \qquad k = j. \tag{9.8}$$

Notice the Lorentz-breaking nature of the second condition; this is where the choice of time gauge explicitly shows up in the quantization procedure. Nonetheless, the EPRL model is perfectly covariant, see [25]. The associated d-matrix can be given a compact expression in terms of hypergeometric functions,

$$\begin{aligned} d_p^{\gamma j}(r) := d_{jjp}^{(\gamma j, j)}(r) &= d_j \frac{2j!}{(j+p)!(j-p)!} e^{-pr} \int_0^1 dt \, \frac{t^{j+p}(1-t)^{j-p}}{(te^{-r} + (1-t)e^{r})^{1+j(1-i\gamma)}} \\ &= e^{-[j(1-i\gamma)+p+1]r} {}_2F_1[j+p+1, j(1-i\gamma)+1, 2j+2, 1-e^{-2r}] \end{aligned} \tag{9.9}$$

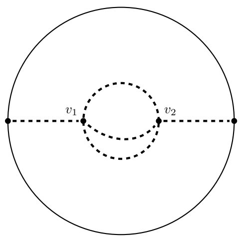
Therefore, evaluating $SL(2,\mathbb{C})$ Clebsch-Gordan amounts to evaluating integrals of these hypergeometric functions and gluing them with usual $SU(2)$ Clebsch-Gordan.
In particular, a nice feature is that one obtains a complete factorisation of the amplitude in terms of $SU(2)$ nj-symbols on the vertices glued by boost edge amplitudes:

$$Z_\sigma = \sum_{j_f, i_e} \prod_f (2j_f+1) \prod_e A_e^\gamma(j_f, i_e) \prod_v \{nj\}_v(j_f, i_e) \tag{9.10}$$

This result is, probably, known to everyone who has actually done any explicit calculation, but it does not appear easily in the literature and we think it is worth stressing.

9.3 The edge amplitude

A priori, the simplest possible boundary is given by 2 links sharing two nodes. However, the spin foam with a single vertex is not integrable. Thus one has to consider two vertices inside the two-link dipole.



Where the continuous lines corresponds to the links of the spin network, the external dots are the nodes of the spin network, the dots labelled with v_1 and v_2 are the vertices of our spin foam and the dashed lines are the edges of the spin foam.
Each vertex amplitude, in this case, corresponds to a 4-valent vertex with tetrahedral boundary graph. Therefore, we study the tetrahedral boundary graph, knowing that we could take it as a building block for the simplest contribution to the 2-link dipole, or directly for a transition amplitude with a tetrahedral boundary graph.

For this diagram, the transition amplitude is

$$A[h_{12}, \dots h_{34}] = \sum_{j_f} \prod_f (2j_f+1) \prod_e A_e^\gamma(j_f) \{6j\} \Psi_{6j}[h_{12}, \dots h_{34}] \tag{9.11}$$

and

$$A^\gamma(j_i) = \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \int d\mu(r) d^{j_1}_{p_1}(r) d^{j_2}_{p_2}(r) d^{j_3}_{p_3}(r) \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix}. \tag{9.12}$$

Then, there are a couple of interesting rewritings of this expression. First, one can swap the integral with the infinite series defining the hypergeometric function, obtaining

$$\int d\mu(r) d^{j_1}_{p_1}(r) d^{j_2}_{p_2}(r) d^{j_3}_{p_3}(r) = \sum_{n_1,n_2,n_3=0}^{\infty} \frac{(a_1)_{n_1} \cdots (b_3)_{n_3}}{(c_1)_{n_1}(b_2)_{n_2}(c_3)_{n_3}} \frac{B(\frac{1}{2}\alpha, n_1+n_2+n_3)}{n_1! n_2! n_3!} \tag{9.13}$$

where B is Euler' β function, $(a)_n$ the Pochammer symbol entering the definition of the hypergeometric series, and α the overall coefficient in the exponential of r.

The second interesting form comes from the factorisation of the p_i dependence, which allows to isolate the sums over the $SU(2)$ CG:

$$A^\gamma = \int_0^1 dt_1 dt_2 dt_3 \sum_{p_1,p_2,p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix}^2 \prod_{i=1}^{3} \left(\frac{t_i}{1-t_i}\right)^{p_i} \int d\mu(r) F(r,t_i) e^{\sum_i j_i S(r,t_i)} \tag{9.14}$$

where

$$F(r,t_i) = \prod_{i=1}^{3} \frac{1}{t_i e^{-r} + (1-t_i)e^r} \tag{9.15}$$

$$S(r,t_i) = \ln\left(\frac{t_i(1-t_i)}{t_i e^{-r} + (1-t_i)e^r}\right) + i\gamma \ln(t_i e^{-r} + (1-t_i)e^r) \tag{9.16}$$

However, in both cases it can not be evaluated explicitly, nor of course by hand nor with Mathematica.
In this form it unfortunately appears even slower to compute numerically. In this second case, we can use the fact that both $6j$ and $3m$ have known asymptotic expressions, and compute the asymptotic value of the integral at the saddle point. The intermediate step for doing this is an expression for large values of the parameters of the hypergeometric functions. Unfortunately these do not appear to be known, and the difficulty shows up immediately if one tries to compute it.

As a case study, we can take a single hypergeometric function. The relevant part is the following integral:

$$I := \int_0^1 \frac{dt}{(te^{-r} + (1-t)e^r)^{1+j(1-i\gamma)}} t^{j+p}(1-t)^{j-p} = \int_0^1 dt f(t) e^{jS(t)} \tag{9.17}$$

where

$$f(t) = \frac{1}{te^{-r} + (1-t)e^r}, \qquad S(t) = \ln\left(\frac{t(1-t)}{te^{-r} + (1-t)e^r}\right) + \frac{p}{j}\ln\left(\frac{t}{1-t}\right) + i\gamma \ln(te^{-r} + (1-t)e^r) \tag{9.18}$$

A standard saddle point approximation for all r fails because the function is not complex analytic, in particular the real part and imaginary parts of the gradient do not vanish at the same point in t. Another difficulty comes from the fact that a priori the magnetic labels can also be large, which forces us to put the second term in the action. While the first term is always negative, with maximum at fixed r given at $t = e^r/(1+e^r)$ by $r - 2\ln(1+e^r)$, the second term becomes positive at $t = 1/2$, and if p is close enough to $\pm j$, the whole real part of the action can become positive for t close enough to 1 (respectively 0), and its value grows as r goes away from zero to the right (respectively left).

A way to improve the situation is to consider projections of these functions along $SU(2)$ coherent states, as proposed in [28], and indeed key to the 4-simplex asymptotic analysis. The advantage is that one takes combinations of I with weights that are decreasing functions of p, with the result of obtaining an action with real part bounded from above by zero:

$$\langle \omega\bar{\omega}|\, Y\, |j,\zeta\rangle = \sum_{m=-j}^{j} f_{j,m}^{(\gamma j,j)}(\omega^A)\, \langle j,m|j,\zeta\rangle = \frac{2j+1}{\pi}||\omega||^{2(i\gamma j-1)} \left(\frac{\langle\omega|\zeta\rangle}{||\omega||}\right)^{2j}. \tag{9.19}$$

where the argument of the last bracket is less or equal to one.
Here,

$$|j,\zeta\rangle = \sum_m c_m(\zeta)\, |j,m\rangle, \qquad c_m(\zeta) = \frac{1}{(1+|\zeta|^2)^j}\left(\begin{array}{c} 2j \\ j+m \end{array}\right)^{1/2} \tag{9.20}$$

is a normalized Perelomov coherent state in the j irrep.
This technique is used in [28], and leads to the famous Regge-like asymptotic behaviour. That analysis shows that at the saddle point, what really matters of the group element is its z-boost. It is then interesting to combine the $SU(2)$ factorisation here discussed with the projection on the coherent states. To do so, we simply substitute the resolutions of the identity in the magnetic index p_i in (9.12) with integrals over the coherent states. In doing so, we preserve the factorised $SU(2)$ $\{nj\}$ vertex structure, and just have coherent states in the edge amplitude. This is of course slightly different from Barrett's procedure, where the coherent states are attached to the gluing of adjacent 4-simplices.
The result is now

$$A^\gamma(j_i) = \int \prod_i d\mu(\zeta_i)d\mu(\zeta_i') \sum_{p_i,q_i} \left(\begin{array}{ccc} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{array}\right) \prod_i \overline{c_{p_i}(\zeta_i)} \left(\begin{array}{ccc} j_1 & j_2 & j_3 \\ q_1 & q_2 & q_3 \end{array}\right) c_{q_i}(\zeta_i') \tag{9.21}$$
$$\times \int d\mu(r) d^{j_1}_{\zeta_1\zeta_1'}(r) d^{j_2}_{\zeta_2\zeta_2'}(r) d^{j_3}_{\zeta_3\zeta_3'}(r).$$

where the coherent state representation of the d-matrix is simply

$$D^{(\gamma j,j)}_{j\zeta\, j\zeta'}(e^{\frac{r}{2}\sigma_3}) = \sum_m \langle j,\zeta|j,m\rangle\, d^{(\gamma j,j)}_{jjm}(r)\, \langle j,m|j,\zeta'\rangle =: d^j_{\zeta\zeta'}(r). \tag{9.22}$$

This formula can be obviously generalized to a edge of arbitrary valence, where now the generalized Wigner's m-coefficients will carry a dependence on the intertwiner labels.
Although this manipulation appears as a complication at first, it is needed as explained above to perform the saddle point approximation. The interest in doing so comes also from the fact that the resulting contractions of coherent states with the CG coefficients admit very elegant and compact expressions. For instance, we have

$$\left(\begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array}\right) c_{m_1}(z_1)c_{m_2}(z_2)c_{m_3}(z_3) = \left[\frac{\prod_i (2j_i)!}{(j_1+j_2+j_3+1)!\prod_{i<j} a_{ij}!}\right]^{\frac{1}{2}} \prod_{i<j} [z_i\,|z_j\rangle^{a_{ij}} \tag{9.23}$$

where $a_{ij} = j_i + j_j - j_k$. In particular, the dependence on the spinors can be nicely exponentiated to

$$\prod_{i<j} [z_i\,|z_j\rangle^{a_{ij}} = \exp \sum_{i=1}^{3} j_i \ln o_i(z_i), \qquad o_1(z_i) := \frac{[1\,|2\rangle\,[3\,|1\rangle}{[2\,|3\rangle}, \; etc. \tag{9.24}$$

This formula, which is a special case of a more general formula proved in [29], can be easily proved using the explicit Racah formula for the $3m$-coefficients. It is also important to notice that this expression is dominated by closed configurations.

To give the explicit expression of (9.32), we recall first that (notice we are using right action so columns associated to source and rows to target nodes)

$$D^{(\gamma j,j)}_{jn\,jm}(g) = \int_{\mathbb{PC}^2} d\mu(\omega^A) \overline{f^{(\beta j,j)}_{j,n}(g\omega^A)} f^{(\gamma j,j)}_{j,m}(\omega^A), \tag{9.25}$$

Then,

$$D^{(\gamma j,j)}_{j\zeta\, j\zeta'}(g) = \frac{2j+1}{\pi} \int d\mu(\omega^A) \frac{\exp(s_w)}{||g\omega||^2 ||\omega||^2}, \qquad s[\omega, g, ; \zeta, \zeta'] = i\gamma j \ln \frac{||\omega||^2}{||g\omega||^2} + 2j\Phi, \tag{9.26}$$

where

$$\Phi := \ln \left(\frac{\langle \zeta | g\omega \rangle}{||g\omega||} \frac{\langle \omega | \zeta' \rangle}{||\omega||} \right). \tag{9.27}$$

The argument in the oscillatory part can be recognised as the boost angle of covariant twisted geometries,

$$\Xi := \ln \frac{||\omega||^2}{||\tilde{\omega}||^2}. \tag{9.28}$$

The action has real part bounded by zero, with configurations for which this maximum is attained well known from Barrett's semiclassical paper.

Notice that in the case when $g = \exp(\frac{r}{2}\sigma_3)$ is a z-boost,

$$||g\omega||^2 \equiv ||\omega||^2\, d(r,t)\,, \qquad d(r,t) := te^{-r} + (1-t)e^r \tag{9.29}$$

where

$$t = \frac{|\omega^1|^2}{||\omega||^2}, \tag{9.30}$$

and

$$\Xi = -\ln d(r,t) \tag{9.31}$$

Introducing the homogeneous coordinate $w := \omega^0/\omega^1$, we have

$$d^j_{\zeta\zeta'}(r) = \frac{2j+1}{\pi} \frac{i}{2} \int \frac{d^2 w}{(1+|w|^2)^{2+2j}} \frac{(\langle \zeta | gw \rangle \langle w | \zeta' \rangle)^{2j}}{d(r,t)^{1+j(1+i\gamma)}} \tag{9.32}$$

In any case, bringing to the exponent (9.23), we see that the 3-valent edge amplitude is characterised by an action

$$S(r, w_i, \zeta_i, \zeta_i') = \sum_{i=1}^{3} j_i \left[i\gamma \Xi_i + 2\Phi_i + \ln \overline{o_i(z_i)} + \ln o_i(z_i') \right] \tag{9.33}$$

$$= \sum_{i=1}^{3} j_i \left[\ln \frac{\langle \zeta | gw \rangle \langle w | \zeta' \rangle}{d(r,t_i)^{2(1+i\gamma)}} + \ln \overline{o_i(z_i)} + \ln o_i(z_i') \right] \tag{9.34}$$

In this form, we can compute the semiclassical limit of EPRL on an arbitrary graph. Furthermore, we bring in the action the coherent state labels using the nice factorisation property. There are many questions and potential applications of this formula, hopefully some of them actually computable.

9.4 Open Problems and Concluding Remarks

Let us now get back to the physics of the Black Hole fireworks. As we discussed in Chapter 5, this scenario can be modelled as a collapsing spherical null shell bouncing back to an outgoing spherical null shell due to quantum gravitational effects appearing when the energy density of the collapsing matter reaches a critical density.

In the previous sections we have formulated some general computational rules for the Lorentzian EPRL model. We can now rewrite our physical scenario in the time gauge in order to be able to compute the quantum transition amplitude by means of the previously discussed techniques. The time-gauge (which is the gauge in which LQG transition amplitudes are written) form of the metric of a black hole has been found by Lemaitre and It reads:

$$ds^2 = -dt^2 + \frac{2m}{r_s} dr^2 + r_s^2 d\Omega^2 \tag{9.35}$$

where $r_s = r_s(r,t)$ defined by

$$r_s = (2m)^{1/3} \left(\frac{3}{2} (r-t) \right)^{2/3} \tag{9.36}$$

The line element (9.35) shows that r_s is the Schwarzschild radial coordinate. The Lemaitre time t is related to the Schwarzschild coordinates t_s and r_s by

$$t = t_s + 2\sqrt{2mr_s} + 2m \ln \left| \frac{\sqrt{r_s/2m} - 1}{\sqrt{r_s/2m} + 1} \right| \tag{9.37}$$

The Lemaitre coordinates cover both the exterior and the interior of the black hole.
A $t = const.$ hypersurface crosses the $t_s = 0$ hypersurface at a sphere S of Schwarzschild radius $r_s(t)$, which is given by setting $t_s = 0$ in (9.37). It is convenient to parametrize the radius as

$$r_s = 2m(1 + \delta) \tag{9.38}$$

because in the following we will be interested in the regime $0 < \delta \ll 1$. In this regime (9.37) reduces to

$$t = 2m \ln \delta \tag{9.39}$$

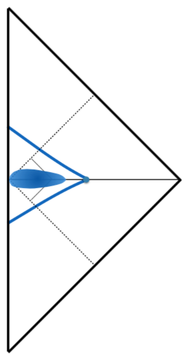
thus the Lemaitre time goes logaritmically to $-\infty$ when we approach the horizon on the $t_s = 0$ surface.
Consider then the ingoing and outgoing null shells studied in Chapter 5, to which we will refer as B_- and B_+ respectively, and the map between the extended Schwarzschild and the firework spacetime. As we discussed above, these two hypersurfaces B_- and B_+ surround the quantum region, and can be taken as initial and final surfaces for computing the tunnelling amplitude.

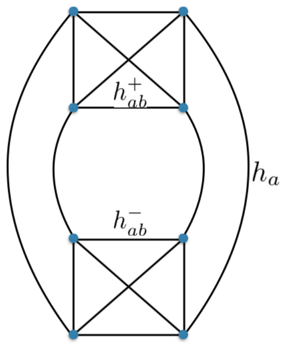
However, in order for $B = B_- \cup B_+$ to be mapped to the firework spacetime, S must be outside the radius R where the shells cross.

This radius is related to the bounce time τ by (see [10])

$$\tau = -2m \ln\left(\frac{R-2m}{2m}\right) = -2m \ln\delta = -t \tag{9.40}$$

Therefore we have the remarkable result that the Lemaitre time of the crossing point is (minus) the bounce time.
Let us choose the minimum possible radius of S, namely $R+\varepsilon$, then it can be proved that the intrinsic and extrinsic geometry of B are functions of the mass m and the bouncing time τ. We can thus compute the bouncing time $\tau(m)$ by studying when the (modulus of) the transition amplitude for a coherent boundary state $\Psi_{m,\tau}$ with the geometry of B becomes non negligible.
A more relevant description of the spin network dual to the triangulation of the $3D$ surface B is given in the following figure, where each sphere is approximated by four tetrahedrons glued together.





Hopefully, the technique discussed in the previous sections will allow us to simplify the evaluation, at least from a numerical point of view, of the relation between the bouncing time and the "black hole mass", i.e. $\tau = \tau(m)$. Such a result would provide a fundamental theoretical result which could be used to improve our theoretical predictions about the phenomenology of such objects.

Conclusions

Black holes represent some of the most interesting macroscopic objects since they are the most likely candidates to provide a bridge between classical General Relativity, and a possible quantum theory of gravity.

In the second part of this work (*Part* II *Ch.* 7) we studied the features of the Rotating Hayward spacetime as well as of the Rotating Modified-Hayward spacetime, both obtained by means of the Newman-Janis Algorithm (NJA) starting from their corresponding static spherically symmetric seed spacetimes known in the literature. In particular, in *Ch.* 7 *Sec.* 7.3 we delineated the problems rising from the application of the NJA to the modified Hayward spacetime and we also presented a way to avoid the problem of the appearance of a curvature singularity, obtaining an explicit, physically plausible, effective metric for a rotating Planck star.

In the third part of this work (*Part* III *Ch.* 9) we remarked that the Lorentzian EPRL amplitudes are given by contractions of Clebsch-Gordan coefficients of the Lorentz group. As a case study, we have considered the transition amplitude for a spinfoam graph consisting of two vertices inside a two-link dipole. For this graph we then presented a parametrization of $SL(2, \mathbb{C})$ that appears to simplify the computation in the large spin limit (semiclassical limit) due to the decomposition properties of the irrep of $SL(2, \mathbb{C})$. In order to perform the saddle point approximation needed for our analysis, however, we were forced to consider explicit projections of the $SL(2, \mathbb{C})$ irreps along $SU(2)$ coherent states. Then, recalling some properties of twisted geometries, we were able to set up the computation of the semiclassical limit of EPRL amplitudes on an arbitrary graph. In *Sec.* 9.4 we then presented a potential application for these techniques to the computation of the bouncing time for the black hole fireworks scenario. In particular, we also presented a more realistic description of the boundary spin-network for this scenario.

Appendix A

Energy Conditions

The Raychaudhuri's equation involves the Ricci tensor, which is related to the energy momentum tensor of matter via the Einstein's field equations. We will want to consider only physical matter, which implies that the energy-momentum tensor should satisfy certain conditions, such as the fact that an observer with 4-velocity u^α would measure an energy-momentum current $j^\alpha = -T^\alpha_\beta u^\beta$. One would expect physically reasonable matter not to move faster than light, so this current should be non-spacelike. This motivates:

Dominant energy condition: $J^\alpha = -T^\alpha_\beta V^\beta$ is a future-directed causal vector for all future-directed timelike vectors $V = V^\beta\,\partial_\alpha$.

For matter satisfying the dominant energy condition, one can prove that if $T_{\alpha\beta}$ is zero in some closed region S of a spacelike hypersurface Σ then $T_{\alpha\beta}$ will be zero within $D^+(S)$.

A less restrictive condition requires only that the energy density measured by all observers is positive:

Weak energy condition: $T_{\alpha\beta}V^\alpha V^\beta \geqslant 0$ for any causal vectors $V = V^\alpha\,\partial_\alpha$.

A special case of this is

Null energy condition: $T_{\alpha\beta}V^\alpha V^\beta \geqslant 0$ for any Null vectors $V = V^\alpha\,\partial_\alpha$.

Clearly, the dominant energy condition implies the weak energy condition, which implies the null energy condition. Another energy condition is

Strong energy condition: $(T_{\alpha\beta} - \frac{1}{2}\,g_{\alpha\beta}T^\lambda_\lambda)V^\alpha V^\beta \geqslant 0$ for all causal vectors $V = V^\alpha\,\partial_\alpha$.

By means of the Einstein's field equation one can find that the strong energy condition is formally equivalent to $R_{\alpha\beta}\,V^\alpha V^\beta \geqslant 0$. Physically, this condition states that *gravity is attractive.* Despite its name, the strong energy condition does not imply any of the other conditions. The strong energy condition is needed to prove some of the singularity theorems, but the dominant energy condition is the most important from a physical point of view.

Appendix B

Angular Velocity, Area of the Horizon and Perturbations

B.1 Angular Velocity

As it was established by Hawking in 1972 in the rigidity theorem, under very general conditions, if a black hole is stationary, then it must be either static or axially symmetric. This implies that the stationary spacetime of a rotating black hole is necessary axially symmetric and that it admits two Killing vector fields, $\xi_t = \xi_u$ and ξ_φ. Moreover, Hawking also proved that a linear combination of these vectors is null on the event horizon $\mathcal{H}^+$, i.e.

$$\chi = \xi_u + \Omega_H \xi_\varphi \,, \qquad \chi^2 = 0 \quad \text{on } \mathcal{H}^+ \tag{B.1}$$

where Ω_H is the angular velocity of the black hole, which vanishes if the spacetime is static. Thus the event horizon is a Killing horizon and χ is tangent to the horizon's null generators.

Thus, we can easily compute the angular velocity of the hole by means of Eq. (B.1). Indeed,

$$\chi^2 = 0 \quad \text{on } \mathcal{H}^+ \quad \Longrightarrow \quad \Omega_H^2 g_{\varphi\varphi} + 2 g_{u\varphi}\Omega_H + g_{uu} = 0 \tag{B.2}$$

Hence,

$$\Omega_H = \frac{-g_{u\varphi} \pm \sqrt{g_{u\varphi}^2 - g_{\varphi\varphi} g_{uu}}}{g_{\varphi\varphi}} \tag{B.3}$$

If we now consider the metric (7.17) we get that

$$g_{u\varphi}^2 - g_{\varphi\varphi} g_{uu} = \frac{a^2 \sin^4\theta}{\Sigma^2(r_H,\theta)} \left\{ a^4 \sin^4\theta [2 - G(r_H,\theta)] - G(r_H,\theta) \left(r_H^2 + a^2\right) \Sigma(r_H,\theta) \right\} \tag{B.4}$$

from which it follows that

$$\Omega_H = \frac{2a^3 \sin^2\theta \pm a\sqrt{a^4 \sin^4\theta[2 - G(r_H,\theta)] - G(r_H,\theta)(r_H^2 + a^2)\Sigma(r_H,\theta)}}{a^4 \sin^4\theta - (r_H^2 + a^2)\Sigma(r_H,\theta)} \tag{B.5}$$

where $r_H = \max\{r \,:\, \widetilde{\Delta}(r,\theta) = 0\}$.

B.2 Area of the Horizon

The area of the horizon for a black hole is of considerable importance because of the area theorem, which states that the horizon area of a classical black hole can never decrease in any physical process.
The horizon for the metric (7.17) is given by $r_H = \max\{r : \widetilde{\Delta}(r,\theta) =\}$, which implies that $r_H = r_H(\theta)$ in the most general case. Since the metric is stationary, the horizon is also a surface of constant t, or u. Thus, setting $dt = 0$ and $dr = {r'}_H(\theta)\, d\theta$, we get the induced metric on the horizon

$$ds^2\big|_H = -2a\sin^2\theta\, {r'}_H(\theta)\, d\theta d\varphi - \Sigma(r_H,\theta)\, d\theta^2 - \sin^2\theta\, \frac{(r_H^2+a^2)\Sigma(r_H,\theta) - a^4\sin^4\theta}{\Sigma(r_H,\theta)}\, d\varphi^2 \quad \text{(B.6)}$$

Thus, the area of the event horizon is given by:

$$\begin{aligned}\mathcal{A}_H &= \int d^2x\,\sqrt{|\mathtt{det}(g|_H)|} = \\ &= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sqrt{\left|\sin^2\theta[(r_H^2+a^2)\Sigma(r_H,\theta) - a^4\sin^4\theta] - a^2\sin^4\theta\,({r'}_H)^2(\theta)\right|} = \\ &= 2\pi \int_0^{\pi} d\theta \sqrt{\left|\sin^2\theta[(r_H^2+a^2)\Sigma(r_H,\theta) - a^4\sin^4\theta] - a^2\sin^4\theta\,({r'}_H)^2(\theta)\right|}\end{aligned} \quad \text{(B.7)}$$

B.3 Asymptotic Analysis and Perturbation Theory

Unfortunately, without an explicit expression for r_H we are not able to compute explicitly (B.5) and (B.7) for the metric (7.17).
One way to overcome this problem is to compute the solution of the equation for the horizon, i.e. $\widetilde{\Delta}(r,\theta) = 0$, up to a certain order of the parameter $g > 0$, which we might assume to be small for our purposes.

For simplicity, we will consider $\gamma = \delta$ in (7.17) that leads to

$$\widetilde{\Delta}(r,\theta) = \widetilde{\Delta}(r) = 0 \quad \Longrightarrow \quad r^2 - \frac{2mr^4}{r^3+g^3} + a^2 = 0 \quad \text{(B.8)}$$

from which it follows that

$$r^5 - 2mr^4 + a^2r^3 + g^3r^2 + a^2g^3 = 0 \quad \text{(B.9)}$$

where $m \geqslant 0$ and $g > 0$, provided that $r \neq -g$.

Now, assuming $g \ll 1$, we can rewrite Eq. (B.9) as

$$r^5 - 2mr^4 + a^2r^3 + g^3(r^2+a^2) = 0 \quad \text{(B.10)}$$

thus we could try to solve perturbatively this equation considering the term multiplied by $\varepsilon = g^3 \ll 1$ as a small perturbation. Hence, the unperturbed equation is given by

$$r^3(r^2 - 2mr + a^2) = 0 \quad \text{(B.11)}$$

which gives us

$$r_{H,0} = m \pm \sqrt{m^2 - a^2} \tag{B.12}$$

Now we recall the unperturbed equation

$$r^5 - 2mr^4 + a^2 r^3 + \varepsilon(r^2 + a^2) = 0 \tag{B.13}$$

and we propose as an Ansaz for the solution the power series

$$r_H(\varepsilon) = r_{H,0} + \varepsilon\, r_{H,1} + \varepsilon^2\, r_{H,2} + \cdots \tag{B.14}$$

that, up to the first order in ε, is given by

$$r_H(\varepsilon) = r_{H,0} + \varepsilon\, r_{H,1} + o(\varepsilon^2) \tag{B.15}$$

Now, if we plug the latter into Eq. (B.9),

$$(r_{H,0} + \varepsilon\, r_{H,1})^5 - 2m(r_{H,0} + \varepsilon\, r_{H,1})^4 + a^2(r_{H,0} + \varepsilon\, r_{H,1})^3 + \varepsilon[(r_{H,0} + \varepsilon\, r_{H,1})^2 + a^2] = 0 \tag{B.16}$$

then, using the fact that $r_{H,0}^5 - 2mr_{H,0}^4 + a^2 r_{H,0}^3 = 0$, after a simple manipulation we get that

$$r_{H,1} = -\frac{r_{H,0}^2 + a^2}{r_{H,0}^2(5r_{H,0}^2 - 8mr_0 + 3a^2)} \tag{B.17}$$

thus, the position of the event horizon up to the order one in $g^3 \ll 1$ is given by

$$r_H^{(1)} = r_{H,0} - g^3\, \frac{r_{H,0}^2 + a^2}{r_{H,0}^2(5r_{H,0}^2 - 8mr_0 + 3a^2)} \tag{B.18}$$

this gives us an explicit, although approximate, expression for the position of the event horizon for $\gamma = \delta$. However, even in this simplified case, we are not able to extract a “nice” expression for the angular velocity and the area of the horizon substituting this approximate solution into (B.5) and (B.7).

Remark. It is worth stressing that the equation of the horizon for $\gamma = \delta$ depends only on the radial coordinate. Thus, r_H does not depend on θ and the horizon surface is defined by $u = const.$ and $r = r_H = const.$. Of course, this does not implies that the horizon is 2-sphere; indeed, we can convince ourself of this fact just looking at the induced metric on $\mathcal{H}^+$.

Appendix C

$SU(2)$ and $SL(2,\mathbb{C})$ conventions

C.1 $SU(2)$ conventions

Algebra of Pauli matrices

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \tag{C.1}$$

$$\mathtt{Tr}(\sigma^i\sigma^j\sigma^k) = 2i\epsilon^{ijk}, \qquad \mathtt{Tr}(\sigma^i\sigma^j\sigma^k\sigma^l) = 2(\delta^{ij}\delta^{kl} - \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \tag{C.2}$$

Lie algebra

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \qquad J_i^{(1/2)} = \frac{\sigma_i}{2}, \qquad J_i^{(1)} = -i\epsilon_{ijk} \tag{C.3}$$

Spherical basis

$$J_\pm = J_1 \pm iJ_2, \qquad J^2 = J_3^2 + \frac{1}{2}J_+J_- + \frac{1}{2}J_-J_+ \tag{C.4}$$

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_3 \tag{C.5}$$

Conjugation

$$\epsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \epsilon\sigma\epsilon = \sigma^*, \qquad \epsilon\tau_3\epsilon = \tau_3 \tag{C.6}$$

$$\epsilon g \epsilon^T = g^* \tag{C.7}$$

$$\epsilon^{(j)}_{mn} = (-1)^{j+m}\delta_{-m,n} = (-1)^{j-n}\delta_{m,-n}, \qquad \epsilon^{-1}{}^{(j)}_{mn} = (-1)^{-j+m}\delta_{-m,n} = (-1)^{-j-n}\delta_{m,-n} \tag{C.8}$$

$$\epsilon^{(j)}_{mn}\epsilon^{-1}{}^{(j)}_{np} = \delta_{m,p}, \qquad \epsilon^{(j)}_{mn}\epsilon^{(j)}_{np} = (-)^{2j}\delta_{m,p} \tag{C.9}$$

$$|j,m] \equiv \epsilon\,|j,m\rangle = (-1)^{j+m}\,|j,-m\rangle \tag{C.10}$$

Wigner matrices

$$\begin{aligned} &D_{mn}(g^{-1}) = \overline{D_{nm}(g)}, \qquad D_{mn}(g) = (-)^{m-n}\overline{D_{-m,-n}(g)} = (\epsilon\overline{D}\epsilon^{-1})_{mn}, \\ &D_{mn}(e^{-i\omega J_3}) = \delta_{mn}e^{-im\omega} \end{aligned} \tag{C.11}$$

$$\begin{aligned} D(\alpha,\beta,\gamma) = e^{-i\alpha J_z}e^{-i\beta J_y}e^{-i\gamma J_z} \qquad & SO(3): \quad \alpha\in[0,2\pi),\ \beta\in[0,\pi),\ \gamma\in[0,2\pi) \\ & SU(2): \quad \alpha\in[0,2\pi),\ \beta\in[0,\pi),\ \gamma\in[0,4\pi) \\ & \text{(alternatively } \ \alpha\in[0,4\pi)) \end{aligned} \tag{C.12}$$

$$D(\omega,\hat{n}) = e^{-i\omega\hat{n}\cdot\vec{J}} \qquad \begin{array}{ll} SO(3): & \omega\in[0,\pi) \\ SU(2): & \omega\in[0,2\pi) \end{array} \tag{C.13}$$

Measure:

$$\int_{SU(2)} d\mu(g) = \frac{1}{16\pi^2}\int_0^{4\pi} d\gamma \int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \tag{C.14}$$

$$= \frac{1}{4\pi^2}\int_0^{2\pi} d\omega \sin^2\frac{\omega}{2} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \tag{C.15}$$

Useful trigonometric relations:

$$\cos\frac{\omega}{2} = \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}, \quad \tan\theta = \frac{\tan\frac{\beta}{2}}{\sin\frac{\alpha+\gamma}{2}}, \quad \phi = \frac{\alpha-\gamma}{2}+\frac{\pi}{2} \tag{C.16}$$

Clebsch-Gordan

$$\langle j_1 m_1 j_2 m_2 | j_1 j_2 j m\rangle = C^{jm}_{j_1 m_1 j_2 m_2}(-1)^{j_1-j_2+m}\sqrt{d_j}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} \tag{C.17}$$

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = (-1)^{-j_1+j_2+m}\frac{1}{\sqrt{d_j}}C^{j-m}_{j_1 m_1 j_2 m_2} = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j \\ -m_1 & -m_2 & -m \end{pmatrix} \tag{C.18}$$

$$= (-1)^{j_1-j_2+j_3}\sum_{JM} C^{JM}_{j_1 m_1 j_2 m_2} C^{00}_{JMjm} \tag{C.19}$$

$$D^{j_1}_{m_1 n_1}(g) D^{j_2}_{m_2 n_2}(g) = \sum_{jmn} C^{j_3 m_3}_{j_1 m_1 j_2 m_2} C^{j_3 m_3}_{j_1 m_1 j_2 m_2} D^{j}_{mn}(g) \tag{C.20}$$

$$= \sum (-1)^{2(j_1-j_2+m)} d_j \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & -n_3 \end{pmatrix} D^{j}_{mn}(g) \tag{C.21}$$

$$\sum_{n_1 n_2 n_3} C^{j_3 n_3}_{j_1 n_1 j_2 n_2} D^{j_1}_{m_1 n_1}(g) D^{j_2}_{m_2 n_2}(g)\overline{D^{j_3}_{m_3 n_3}(g)} = C^{j_3 m_3}_{j_1 m_1 j_2 m_2} \tag{C.22}$$

$$\int dg D^{j_1}_{m_1 n_1} D^{j_2}_{m_2 n_2} D^{j_3}_{m_3 n_3} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix} = \frac{(-1)^{m_3-n_3}}{d_{j_3}} C^{j_3-m_3}_{j_1 m_1 j_2 m_2} C^{j_3-n_3}_{j_1 n_1 j_2 n_2} \tag{C.23}$$

Hopf Section:

$$n(\zeta) = \frac{1}{\sqrt{1+|\zeta|^2}}\begin{pmatrix} 1 & \zeta \\ -\bar{\zeta} & 1 \end{pmatrix} \tag{C.24}$$

$$n(-1/\bar{\zeta}) = n(\zeta)\epsilon^{-1}e^{2\mathtt{arg}\zeta\tau_3} = e^{-2\mathtt{arg}\zeta\tau_3}\epsilon^{-1}n(\zeta) \tag{C.25}$$

Perelomov

$$\zeta = -\tan\frac{\Theta}{2}e^{-i\Phi}, \qquad \xi = -\frac{\Theta}{2}e^{-i\Phi}, \qquad m = (\sin\Phi, -\cos\Phi, 0) \tag{C.26}$$

$$n^{(j)}_{mn}(\zeta) = e^{\zeta J_+} e^{\ln(1+|\zeta|^2) J_0} e^{-\bar{\zeta} J_-} = e^{\xi J_+ - \bar{\xi} J_-} = e^{\frac{i}{2}\Theta m\cdot\sigma} = D^{(j)}_{mn}(\Phi, \Theta, -\Phi) = U^{(j)}_{mn}\left(\Theta, \frac{\pi}{2}, \Phi - \frac{\pi}{2}\right) \tag{C.27}$$

$$= \left[\frac{(j+m)!(j+n)!}{(j-m)!(j-n)!}\right]^{\frac{1}{2}} \sum_q \frac{(j-q)!}{(j+q)!} \frac{1}{(m-q)!(n-q)!} (1+|\zeta|^2)^q \zeta^{m-q}(-\bar{\zeta})^{n-q} \tag{C.28}$$

$$= \frac{2j!}{(j+m)!(j+n)!(j-m)!(j-n)!} \frac{\zeta^{j+m}(-\bar{\zeta})^{j+n}}{(1+|\zeta|^2)^j} \, {}_2F_1(-j-m, -j-n, -2j, \frac{1+|\zeta|^2}{|\zeta|^2}) \tag{C.29}$$

Twisted geometries and parametrization of $SU(2)$:
Let us consider the following parametrizations of $SU(2)$:

$$u = \exp\left(\frac{i}{2}\,\psi\,\vec{n}\cdot\vec{\sigma}\right) = n(\zeta)\,\exp\left(\frac{i}{2}\,\xi\,\sigma_3\right) \tag{C.30}$$

where

$$n(\zeta) = \frac{1}{\sqrt{1+|\zeta|^2}} \begin{pmatrix} 1 & \zeta \\ -\bar{\zeta} & 1 \end{pmatrix}, \qquad \zeta = \rho\, e^{i\chi}, \;\; 0 < \rho < \infty, \;\; 0 \leqslant \chi < 2\pi;$$
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{C.31}$$

in the fundamental representation, moreover $\mathbf{n} = (\cos\varphi \sin\theta, \sin\varphi\sin\theta, \cos\theta)$.

It has already been shown that these parametrizations are related as follows:

$$\begin{cases} \sin(\psi/2)\sin\theta = \rho/\sqrt{1+\rho^2} \\ \chi = \xi/2 + \pi/2 - \varphi \\ \cos\theta\,\tan(\psi/2) = \tan(\xi/2) \end{cases} \tag{C.32}$$

where we made the identifications: $\rho = |\zeta|$ and $\chi = \texttt{arg}(\zeta)$.

Claim 11. The Haar measure of $SU(2)$, in the two given parametrizations (C.30), is respectively given by:

$$\boxed{d\mu(u) = \frac{1}{4\pi^2}\sin(\psi/2)\,\sin\theta\,d\psi \wedge d\theta \wedge d\varphi = d\mu_{\mathbb{C}}(\zeta) \wedge \frac{d\xi}{4\pi}} \tag{C.33}$$

where ψ, $\varphi \in [0, 2\pi)$, $\theta \in [0,\pi)$, $\zeta \in \mathbb{C}$, $\xi \in [0, 4\pi)$ and the measure over $\mathbb{C}$ has been defined as follows

$$\boxed{d\mu_{\mathbb{C}}(\zeta) := \frac{i}{2\pi}\frac{d\zeta \wedge d\bar{\zeta}}{(1+|\zeta|^2)^2}} \tag{C.34}$$

Proof. If we consider the canonical polar parametrization of the complex plane $\mathbb{C}$, i.e. $\zeta = \rho\, e^{i\chi}$, we can easily recast $d\mu_{\mathbb{C}}(\zeta)$ as follows:

$$d\mu_{\mathbb{C}}(\zeta) = \frac{i}{2\pi}\frac{-2i\,\rho\,d\rho \wedge d\chi}{(1+\rho^2)^2} = \frac{1}{\pi}\frac{\rho}{(1+\rho^2)^2}\,d\rho \wedge d\chi \tag{C.35}$$

Thus,

$$d\mu_{\mathbb{C}}(\zeta) \wedge \frac{d\xi}{4\pi} = \frac{1}{4\pi^2}\frac{\rho}{(1+\rho^2)^2}\, d\rho \wedge d\chi \wedge d\xi \tag{C.36}$$

Now, in order to prove our statement, we just need to verify the following equivalence:

$$\sin(\psi/2)\,\sin\theta\, d\psi \wedge d\theta \wedge d\varphi = \frac{\rho}{(1+\rho^2)^2}\, d\rho \wedge d\chi \wedge d\xi \tag{C.37}$$

If we differentiate the relations shown in (C.32) we get

$$\begin{cases} d\rho = \frac{1}{2}\,(1+\rho^2)^{3/2}\,[\cos(\psi/2)\,\sin\theta\, d\psi + 2\,\sin(\psi/2)\,\cos\theta\, d\theta] \\ d\chi = \frac{d\xi}{2} - d\varphi \\ d\xi = \cos^2(\xi/2)\left[\frac{\cos\theta}{\cos^2(\psi/2)}\, d\psi - 2\,\sin\theta\tan(\psi/2)\, d\theta\right] \end{cases} \tag{C.38}$$

Now, if we substitute the letter results in the LHS of (C.37), we see that our assumption is trivially verified. □

C.2 $SL(2, \mathbb{C})$ conventions

From [23]. But denoting $\rho_{Ruhl} \equiv p = 2\rho$ in order to match with spin foam literature, we have: Fundamental representation

$$h = \mathbb{1} + \frac{i}{2}\alpha_i\sigma_i - \frac{1}{2}\eta_i\sigma_i = \mathbb{1} + \frac{i}{2}\omega^{IJ}J_{IJ}, \qquad \omega^{0i} = \eta^i, \qquad \omega^{ij} = \epsilon^{ijk}\alpha_k \tag{C.39}$$

$$J_{0i} = K_i, \qquad J_{ij} = \epsilon_{ijk}L_k \tag{C.40}$$

Irreps form a measure space

- Principal series: $(p \in \mathbb{R}, m \in \mathbb{N})$ such that $n_\pm := \frac{1}{2}(ip \pm m) \equiv i\rho \pm k \in \mathbb{N}$

  It is possible to consider only positive k since (ρ, k) and $(-\rho, -k)$ are related by complex conjugation and unitary equivalent.

- supplementary series: measure zero in the Plancherel decomposition.

Casimirs:

$$\begin{aligned} C_1 &= \frac{1}{2}J_{IJ}J^{IJ} = L^2 - K^2 = -\frac{1}{2}I_1 \mapsto \frac{1}{2}(m^2 - p^2 - 4) = k^2 - \rho^2 - 1 \\ C_2 &= \frac{1}{2}(\star J)_{IJ}J^{IJ} = -L\cdot K = -\frac{1}{4}I_2 \mapsto -\frac{1}{4}pm = -k\rho \end{aligned} \tag{C.41}$$

With these conventions, the matrix elements of a z boost $\exp(\frac{1}{2}r\sigma_3)$ read

$$d^{(\rho,k)}_{jlp}(r) = \sqrt{d_j}\sqrt{d_l}\int_0^1 \frac{dt}{d(r,t)^{1-i\rho}}\, d^{(j)}_{kp}(2t-1)\, d^{(l)}_{kp}\left(2\frac{te^{-r}}{d(r,t)} - 1\right) \tag{C.42}$$

where

$$d(r,t) = te^{-r} + (1-t)e^{r} \tag{C.43}$$

For the BC model for instance, the relevant function is

$$d^{(\rho,0)}_{000}(r) = \frac{\sin(\rho r)}{\rho \sinh r} \tag{C.44}$$

Instead, for the EPRL model we have

$$d^{\gamma j}_{p}(r) := d^{(\gamma j,j)}_{jjp}(r) = d_j \frac{2j!}{(j+p)!(j-p)!} e^{-pr} \int_0^1 \frac{dt}{d^{1+j(1-i\gamma)}} t^{j+p}(1-t)^{j-p} \tag{C.45}$$

$$= e^{-r(j(1-i\gamma)+p+1)} {}_2F_1[j+p+1, j(1-i\gamma)+1, 2j+2, 1-e^{-2r}] \tag{C.46}$$

$$= \overline{d^{\gamma j}_{-p}(r)} \tag{C.47}$$

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