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Fractal sets and their applications in Medicine

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A chi non si ferma davanti ai misteri della natura.

Introduction

"Why is geometry often described as cold and dry? One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line..."
Having always found attractive the complexity of world and the number of distinct scales of length of natural patterns that is for all practical purposes infinite, I try to write this thesis to take up the challenge of Benoit Mandelbroot[1] whose words open my introduction.
Perhaps Mandelbrot wanted to suggest to mathemathicians to "study those forms that Euclid leaves aside as being formless" because sometimes it's better to investigate "the morphology of the amorphous".
It's for these reasons that the idea to write about Fractal sets has born. Fractals are something irregular and broken, by the meaning of the Latin adjective *fractus*.
If we want to go over the etymology, it seems difficult to find a rigorous mathemathical definition of fractal set.
Mandelbrot himself thought initially to call fractal a "set whose Hausdorff dimension is slightly larger than its topologycal one". This first attempt has then been followed by other such as the simple "irregular set" or "set whose Hausdorff dimension equals to a number that is not an integer".
Although none of the cited definitions can be judged totally satisfactory, what is important is at least to draw from them the consciousness that Hausdorff measure and dimension are the necessary mathemathical instruments for describing and analysing the geometric properties of these new sets.
New because it's only with the publication of Mandelbrot *Les objets fractals* (1975) that the interest for such mysterious objects has started to increase. The first chapter of the thesis is dedicated to recall some general results of the Measure Theory and mostly to definitions and theorems about Hausdorff measure, through which a particular class of subsets of $\mathbb{R}^n$, the so-called s-sets can be defined.

[1] polish mathematician, 1924-2010, father of fractal geometry and, in particular, of Mandelbrot's fractals.

Afterwords, in the second chapter just these s-sets are investigated, with their local properties of density and tangency. The treatment is divided into two sections, one concerning the case when s is an integer and the other examining the s non integer sets, where s indicates the Hausdorff dimension.
Just sets whose Hausdorff dimension is something different from an integer number may be called Fractals. The most classical ones show also the peculiarity of being self-similar, which means that the fractals, generated iteratively, for more iterations are more complex and many substructers resembling the whole are visible. Self-similatity will be subject of the third chapter. In the same chapter, we'll specifically study three fractal sets presenting as s an irrational number and also being self-similar: Cantor set, Koch curve and Sierpinski triangle. We'll discover that the s presented as Hausdorff dimension coincides with another important mathemathical quantity: the self-similar dimension.
Both quantities can well represent the concept of fractal dimension.
From the beginning of the thesis, we'll realize that talking about fractal dimension implies doing a limit, fact which may be thought to be the reason why this dimension has always been considered suspicious. Maybe for this suspicion its physical role hasn't already been exploited at all. At least until the draft of Mandelbrot's work of 1975, when he writes laments about this.
Actually, what I want to propose for the final chapter is just the description of very recent studies in field of Medicine which recognize in images representing senile plaques or skin lesions typical fractal properties and then apply for their researches a fractal analysis.
First of all, a third quantity, the box counting, comes up by the side of Hausdorff and similar dimensions in the definition of fractal dimension, with the advantage of being very useful in practice. So, fractal dimension, here considered a morphological parameter, helps to recognize different kinds of senile plaques (diffuse and mature) formation in various animal species, not excluding human, and supports a diagnosis where the only Dermatoscopy doesn't permit to distinguish a malignant melanoma from a different skill illness.

Introduzione

“Vi sarete forse domandati perché la geometria sia così spesso considerata fredda e arida. Certamente a creare questo convincimento ha contribuito la sua inadeguatezza a descrivere le forme della natura: le nubi sferiche come palloni e montagne coniche, a punta di matita, non fan parte del panorama fisico; le linee costiere, tutte frastagliate, non sembrano certo disegnate col compasso, né si propaga, il lampo, in linea retta... ”
Attratta dalla complessità del mondo che ci circonda e dal numero praticamente infinito di scale di lunghezza che ci si presentano davanti, scrivo questa tesi con lo scopo di rispondere alla sfida di Benoit Mandelbrot[2] le cui parole aprono la mia introduzione.
Forse rivolta a qualcuno di noi matematici, è una sfida a “studiare quelle forme che la geometria euclidea tralascia come informi” lanciandosi ad investigare “la morfologia dell'amorfo”.
Ecco perché una tesi sui frattali, sugli oggetti interrotti e irregolari, come indica il nome stesso (dal latino *fractus*=rotto).
Al di là del significato etimologico, risulta difficile trovare una definizione matematica rigorosa di insieme frattale.
Mandelbrot stesso pensò inizialmente di chiamare frattale un “insieme di dimensione di Hausdorff leggermente maggiore della sua dimensione topologica”. A questo primo tentativo, ne seguirono altri come il semplice “insieme irregolare” o “insieme con misura di Hausdorff non intera ”.
Seppure nessuna delle definizioni citate può essere considerata totalmente soddisfacente, è però importante trarne la consapevolezza che la misura e la dimensione di Hausdorff sono gli strumenti matematici che permettono di descrivere ed analizzare le proprietà geometriche di questi nuovi insiemi.
Nuovi perché è solo con la pubblicazione di *Les objets fractals* di Mandelbrot (1975) che inizia a crescere l'interesse per tali oggetti misteriosi.
Il primo capitolo della tesi è dedicato, oltre a richiamare alcuni risultati generali di Teoria della Misura, alle definizioni e teoremi sulla misura di Hausdorff, grazie ai quali è possibile definire una precisa classe di sottoinsiemi di $\mathbb{R}^n$ chiamati s-sets.

[2]matematico polacco 1924-2010, padre della geometria frattale e, in particolare, dei frattali di Mandelbrot.

Successivamente, nel secondo capitolo, vengono appunto investigati gli insiemi *s-sets* con le loro proprietà di densità e tangenza, separando la trattazione per il caso s intero ed s non intero, dove s indica la dimensione di Hausdorff.
Proprio gli insiemi di dimensione di Hausdorff non intera vengono chiamati frattali. I più classici presentano anche la caratteristica di autosimilarità, ovvero i frattali, generati iterativamente, procedendo in iterati successivi, diventano via via più complicati e mostrano molte sottostrutture somiglianti all'intera di partenza. L'auto-similarità sarà argomento del capitolo 3. Nello stesso capitolo studieremo nello specifico tre frattali sia di dimensione s irrazionale e sia autosimilari: l'insieme di Cantor, le curva di Koch ed il triangolo di Sierpinski.
Scopriremo che il numero s presentato come dimensione di Hausdorff di un s-set coincide con un'altra importante quantità matematica: la dimensione di auto-similarità.
Entrambe le quantità possono rappresentare correttamente il concetto di dimensione frattale.
Fin dall'inizio della tesi ci accorgeremo che parlare della dimensione frattale implica fare un passaggio al limite, e si può pensare che per questo motivo tale dimensione sia sempre stata giudicata sospetta. Forse a causa di questo sospetto il suo ruolo fisico non è mai stato scoperto fino in fondo, almeno fino ai tempi di Mandelbrot che se ne lamenta nel suo libro precedentemente citato.
In realtà, quello che voglio proporre come capitolo finale è proprio la descrizione di recentissimi studi di natura medica che riconoscono nelle immagini raffiguranti placche senili o lesioni della pelle proprietà tipiche dei frattali e quindi applicano per le loro diagnosi un'analisi di tipo frattale.
Innanzi tutto, una terza grandezza, il box-counting, viene ad affiancare le dimensioni di Hausdorff e di autosimilarità nella definizione di dimensione frattale, con il vantaggio di risultare molto utile e maneggevole nelle applicazioni pratiche. E così, la dimensione frattale, ora considerata come parametro morfologico, da un lato interviene nell'individuare processi di formazione delle placche senili (diffuse e mature) di varie specie di mammiferi dalle quali non viene escluso l'uomo, dall'altro dà il suo contributo dove la sola Dermatoscopia non è sufficiente a riconoscere i melanomi maligni tra varie malattie della cute.

Contents

Chapter 1

Measure Theory

To build also the simplest building, before inserting the bricks, it's necessary to lay the foundations. In this chapter we furnish the ground to talk about measure of a set, from which we have to start in the intention of studying fractal sets. In the first section we collect definitions and classical results by Measure Theory, wheras in the second one we explain in detail theorems concerning Hausdorff Measure.

1.1 Recalls of general Measure Theory

Definition 1 (σ-field)**.** *Let X be a non-empty set of $\mathbb{R}^n$. A non-empty collection S of subsets of X is called $\sigma-$field if:*

(i) $\emptyset \in S$*;*

(ii) S is closed under complementation: if $E \in S \Rightarrow X - E \in S$;

(iii) S is closed under countable union: se $E_1,\ E_2,\ \cdots \in S \Rightarrow \bigcup_{j=1}^{\infty} E_j \in S$.

(i) and (ii) imply that $X \in S$.

Moreover it follows from its definition that a $\sigma-$field is closed under set difference and under countable intersection.

From the last observation we can define, given C a collection of subsets in X, the $\sigma-$**field generated by** C, denoted by $S(C)$, which is the intersection of the all $\sigma-$fields containing C.
$S(C)$ is the smallest $\sigma-$field containing C.

Definition 2 (measure)**.** *A measure μ is a function defined on S taking values in the range* $[0, \infty]$ *such that:*

(i) $\mu(\emptyset) = 0$;

(ii) Given $\{E_j\}$ *a countable sequence of disjoint sets in* S

$$\mu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j).$$

We can deduce from (ii) that, if $E, E' \subseteq S$ are $E \subseteq E'$ then $\mu(E) \leq \mu(E')$.

The following definitions of lower limit (1.1) and upper limit (1.2) of a sequence of sets E_j are necessary to enunciate the continuity of measure theorem.

$$\varliminf_{j\to\infty} E_j = \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} E_j \tag{1.1}$$

$$\varlimsup_{j\to\infty} E_j = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j \tag{1.2}$$

If $\{E_j\} \in S$, then, for the definitions of lower and upper limits and of $\sigma-$field, also $\varliminf e \varlimsup \in S$.
If $\varliminf E_j$ and $\varlimsup E_j$ are the same, we write $\lim E_j$ for the common value.

Theorem 1 (continuity of measure)**.** *[4] Let* μ *be a measure on a* $\sigma-$*field* S *of subsets of* X*. Then:*

(a) If $E_1 \subset E_2 \subset \cdots$ *is an increasing sequence of sets in* S, *then*

$$\mu(\lim_{j\to\infty} E_j) = \lim_{j\to\infty} \mu(E_j).$$

(b) Se $E_1 \supset E_2 \supset \cdots$ *is a decreasing sequence of sets in* S, *allora*

$$\mu(\lim_{j\to\infty} E_j) = \lim_{j\to\infty} \mu(E_j).$$

(c) For any sequence of sets $\{E_j\}$ *in* S,

$$\mu(\varliminf_{j\to\infty} E_j) \leq \varliminf_{j\to\infty} \mu(E_j).$$

We now introduce **outer measure**.

Definition 3 (outer measure)**.** *An outer measure* ν *on a set* X *is a function defined of all subsets of* X *taking values in* $[0, \infty]$ *such that:*

(i) $\nu(\emptyset) = 0$;

(ii) *if* $A \subset A'$ *then* $\nu(A) \leq \nu(A')$;

(iii) *for any subsets* $\{A_j\}$ *in* X,

$$\nu(\bigcup_{j=1}^{\infty} A_j) \leq \sum_{j=1}^{\infty} \nu(A_j).$$

We note that ν is defined on all subsets of X, not only on these of a $\sigma-$fiels of X and that outer measure can be interpreted as a weak definition of measure, where property (ii) is replaced by subadditivity .

Actually, there always exists a $\sigma-$field of X on which ν behaves as a true measure. Theorem 2 explains which sets form this $\sigma-$field.

Definition 4 ($\nu-$measurable set)**.** *A subset* E *of* X *is told to be* $\nu-$*measurable or measurable with respect to the outer measure* ν *if for all set* $A \subset X$

$$\nu(A) = \nu(A \cap E) + \nu(A - E).$$

Theorem 2. *[4] Let* ν *be an outer measure. The collection* M *of* $\nu-$ *measurable sets forms a* $\sigma-$*field and the restriction of* ν *to* M *is a measure.*

Definition 5 (regular outer measure)**.** *An outer measure* ν *is called regular if for every set* A *of* X *there exists a* $\nu-$*measurable set* E *containing* A *such that* $\nu(A) = \nu(E)$.

It follows from (c) of continuity measure theorem and from (ii) of the definitione of outer measure that for a regular outer measure on an increasing sequence of sets $\{A_j\}$

$$\lim_{j\to\infty} \nu(A_j) = \nu(\lim_{j\to\infty} A_j)$$

holds.

Definition 6 (metric outer measure)**.** *An outer measure* ν *is called metric if*

$$\nu(E \cup F) = \nu(E) + \nu(F)$$

with E *and* F ***positively separated*** *that means*

$$d(E, F) = inf\{d(x, y) : \quad x \in E, y \in F\} > 0.$$

Let's work in a metric space (X, d). We can think X to be the $n-$dimensional Euclidean space , $\mathbb{R}^n$, with d the usual distance function.

We recall that if we consider the collection of open sets of X, this generates a $\sigma-$field called **Borel** $\sigma-$field which includes the borel sets that are

- open sets;
- closed sets (since they are complementary of open sets);
- countable unions of closed sets ($F_\sigma - sets$);
- countable intersections of open sets ($G_\delta - sets$).

The following theorem about the measurability of Borel-sets with respect to a metric outer measure holds

Theorem 3. *[4] If ν is a metric outer measure on (X, d), then every Borel-subsets of X are $\nu-$measurable.*

Its proof is based on the following lemma.

Lemma 1. *Let ν be a metric outer measure on (X, d). Given an increasing sequence $\{A_j\}_1^\infty$ of subsets of X with*

(i) $A = \lim_{j\to\infty} \nu(A_j)$ *and such that*

(ii) $d(A_j, A - A_{j+1}) > 0 \ \forall j,$

we have

$$\nu(A) = \lim_{j\to\infty} \nu(A_j).$$

1.2 Hausdorff Measure

From the remainder of the thesis we work in Euclidean $n-$dimensional space, $\mathbb{R}^n$.

Definition 7 (diameter of a subset). *Let U a non-empty subset of $\mathbb{R}^n$. The diameter of U, that we indicate by $|U|$, is defined as*

$$|U| = \sup\{|x - y| : x, y \in U\}.$$

If $E \subset \bigcup_i U_i$ and $0 < |U_i| \leq \delta \quad \forall i$, then $\{U_i\}$ is said $\delta-$**cover** of E.

Taken E a subset of $\mathbb{R}^n$ and s a non-negative number, for $\delta > 0$ we define

$$H^s_\delta(E) = inf \sum_{i=1}^{\infty} |U_i|^s \tag{1.3}$$

where the infimum is made over all countable $\delta-$cover $\{U_i\}$ of E.

The function we have just defined is an outer measure on $\mathbb{R}^n$.

Letting δ to 0 in (1.3) we obtain:

Definition 8 (Hausdorff s-dimensional outer measure of E).

$$H^s(E) = \lim_{\delta \to 0} H^s_\delta(E) = sup_{\{\delta > 0\}} H^s_\delta(E).$$

The $lim_{\delta \to 0}$ of the definition exists but can be equal to infty, since H^s_δ increases when δ decreases.
Moreover H^s is an outer measure like H^s_δ.

Note 1. *H^s is also a metric outer measure.*

Proof. Let E and F be positively separated sets. Choosing $\delta < d(E, F)$ no set in the cover of $E \cup F$ can intersect both sets E and F, so that

$$H^s_\delta(E \cup F) = H^s_\delta(E) + H^s_\delta(F)$$

□

Definition 9 (s-dimensional Hausdorff measure). *The s-dimensional Hausdorff measure is the restriction of H^s to the $\sigma-$field of H^s-measurable sets.*

This restriction is due to Theorem 2 and Theorem 3 makes sure that it includes the Borel sets.

We note that:

1. $H^s(E)$ is a non-increasing function as s increases from 0 to ∞;
2. if $s < t$, $H^s_\delta(E) \geq \delta^{s-t} H^t_\delta(E)$ and then:
3. if $H^t(E)$ is positive, $H^s(E)$ is infinite.

We are now ready to give the second, together with the Hausdorff measure definition, of the most important definitions of the current section: the concept of Hausdorff dimension of E :

Definition 10 (Hausdorff dimension)**.** *The Hausdorff dimension of E, dimE, is the unique value to be such that*

$$H^s(E) = \infty \text{ if } 0 \leq s < dimE$$

$$H^s(E) = 0 \quad se \quad dimE < s < \infty.$$

Definition 11 (s-set)**.** *An H^s-measurable set $E \subset \mathbb{R}^n$ for which $0 < H^s(E) < \infty$ is termed an s-set.*

It directly follows from these previous definitions that the Hausdorff dimension of an $s-$set equals to s.

But it is important to realize that an $s-$set is something much more specific than a measurable set of Hausdorff dimension s and that $s-$sets aren't the only Hausdorff measurable sets whose dimension equals to s.

Many pages of this thesis will be occupied by the description of some geometric properties of $s-$sets and it will be clear how the distinction between integer s and non-integer s is linked to the possibility of defining a set E as fractal.

Intuitively, we may affirm that although the elementary geometry teaches us that an isolated point, not differently from a finite number of them, has dimension 0, that a standard curve has dimension 1, that a plane is nothing but a figure of dimension 2 and so on, figures whose dimension may be a fraction or also an irrational number can't be left out from studies.

In other words, in those places left unknown and seen only as transition, without a determined structure, the "fractal regions" can find their natural location.

Our next aim is the proof of the fact that H^s, which we have already seen to be a metric outer Hausdorff measure, shows also the regularity property. As a consequence the s-sets can be approximated by closed subsets.

Theorem 4. *[4] Let $E \subset \mathbb{R}^n$. There exists G_δ-set G containing E with $H^s(G) = H^s(E)$. In particular, H^s is a regular outer measure.*

Proof. If $H^s(E) = \infty$, $\mathbb{R}^n$ is an open set such that $H^s(E) = H^s(\mathbb{R}^n)$ the theorem is proved.

Suppose that $H^s(E) < \infty$. $\forall i = 1, 2, \cdots$ choose an open $\delta-$cover of E with $\delta = \frac{2}{i}$. This cover is $\{U_{ij}\}_j$ and be such that:

$$\sum_{j=1}^{\infty} |U_{ij}|^s < H^s_{1/i}(E) + \frac{1}{i}. \tag{1.4}$$

If we put $G = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} U_{ij}$, thus is a $G_\delta-$set containing E.

Since $\{U_{ij}\}_j$ is a $\frac{2}{i}-$cover of G, (1.3) e 3.6 imply

$$H^s_{2/i}(G) \leq H^s_{1/i}(E) + \frac{1}{i}.$$

Letting $i \to \infty$, $(\delta \to 0)$ we obtain

$$H^s(G) = H^s(E)$$

so we have found an H^s-measurable set whose outer measures equals the measure of E.
This consists in the regularity definition of an outer measure and concludes the proof.

□

Theorem 5. *[4] Any H^s-measurable set of finite measure contains an $F_\sigma-$set of equal measure.*

Proof. Let E be H^s-measurable with $H^s(E) < \infty$.

Using Theorem 4 there exist open sets $O_1, O_2, \cdots$ containing E with

$$H^s(\cap_{i=1}^{\infty} O_i - E) = H^s(\cap_{i=1}^{\infty} O_i) - H^s(E) = 0. \tag{1.5}$$

Any open subset of $\mathbb{R}^n$ is an $F_\delta-$set so suppose $O_i = \cup_{i=1}^{\infty} F_{ij}$ $\forall i$ where $\{F_{ij}\}_j$ is an increasing sequence of closed sets.

By Theorem 1 (a) applied to measure H^s

$$\lim_{j\to\infty} H^s(E\cap F_{ij}) = H^s(\lim_{j\to\infty} E\cap F_{ij}) = H^s(E\cap O_i) = H^s(E).$$

Hence, given $\epsilon > 0$, we may find j_i such that $H^s(E - F_{ij_i}) < 2^{-i}\epsilon$,
$i = 1, 2, \cdots$.

If F is the closed set $\cap_{i=1}^{\infty} F_{ij_i}$, then

$$H^s(F) \geq H^s(E\cap F) \geq H^s(E) - \sum_{i=1}^{\infty} H^s(E - F_{ij_i}) > H^s(E) - \epsilon$$

Since $F \subset \cap_{i=1}^{\infty} O_i$, then $H^s(F - E) \leq H^s(\cap_{i=1}^{\infty} O_i - E) = 0$ by (1.5).

Again for Theorem 4 $F - E$ is contained in a $G_\delta-$set G having its same measure that is $H^s(G) = 0$. So $F - G$ is an $F_\sigma-$set contained in E with

$$H^s(F - G) \geq H^s(F) - H^s(G) \geq H^s(E) - \epsilon.$$

The countable union of sets F_σ contained in E and having equal measure H^s is finally obtained taking a countable union of $F_\sigma-$sets over $\epsilon = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$.
□

The Lemma below states that any attempt to estimate Hausdorff measure of a set using a cover of sufficiently small sets gives an answer not much smaller than the actual Hausdorff measure of the given set.

Lemma 2. *Let E be H^s-measurable with $H^s(E) < \infty$. Let $\epsilon > 0$. Then there exists $\rho > 0$, dependent only on E and ϵ such that for any collection of Borel sets $\{U_i\}_{i=1}^{\infty}$ with $0 < |U_i| \leq \rho$ we have*

$$H^s(E\cap \bigcup_i U_i) < \sum_i |U_i|^s + \epsilon.$$

Proof. From the definition of H^s as the limit of H^s_δ as $\delta \to 0$ we may choose ρ such that

$$H^s(E) < \sum |W_i|^s + \frac{1}{2}\epsilon \tag{1.6}$$

for any $\rho-$cover $\{W_i\}$ of E.

Moreover, given a collection of Borel sets $\{U_i\}$ with $0 < |U_i| \leq \rho$ we may find a $\rho-$cover $\{V_i\}$ of $E - \bigcup_i U_i$ such that

$$H^s(E - \bigcup_i U_i) + \frac{1}{2}\epsilon > \sum |V_i|^s. \tag{1.7}$$

But $\{U_i\} \cup \{V_i\}$ begins a $\rho-$cover of E, so that, from (1.6)

$$H^s(E) < \sum |U_i|^s + \sum |V_i|^s + \frac{1}{2}\epsilon \tag{1.8}$$

Hence

$$H^s(E \cap \bigcup_i U_i) = H^s(E) - H^s(E - \bigcup_i U_i) \geq$$

(using (1.7) and (1.8))

$$\geq \sum |U_i|^s + \sum |V_i|^s + \frac{1}{2}\epsilon - \sum |V_i|^s + \frac{1}{2}\epsilon = \sum |U_i|^s + \epsilon$$

□

Next Lemma concerns the Hausdorff measure of sets related by a uniformly Lipschitz mapping.

Lemma 3. *Let $\psi : E \longrightarrow F$ be a surjective mapping such that*

$$|\psi(x) - \psi(y)| \leq c|x - y| \quad (x, y \in E),$$

for a constant c.

Then

$$H^s(F) \leq c^s H^s(E).$$

Proof. We have, $\forall i, \quad |\psi(U_i \cap E)| \leq c|U_i|$.

If $\{U_i\}$ is a $\delta-$cover of E, $\{\psi(U_i \cap E)\}$ is a $c\delta-$cover of F

Moreover, $\sum_i |\psi(U_i) \cap E|^s \leq c^s \sum_i |U_i|^s$ so that

$$H^s_{c\delta} \leq c^s H^s_\delta(E)$$

and, letting $\delta \longrightarrow 0$,

$$H^s(F) \leq c^s H^s(E).$$

□

The Vitali covering theorem is one of the most useful tools of geometric measure theory.
It let us select, from a sufficiently large collection of sets covering a given set E, a subcollection of disjoint sets that covers almost all of E.
Two different classes of theorems "of type Vitali" can be distinguished.
The first may be applied to any kind of cover but is valid only for few measures, like the Lebesgue measure.
The second class is extended to much more measures but with the disadvantage of giving some restrictions to the considered cover.

For the Hausdorff measure there exists a Vitali theorem which can be inserted in the second class.
Let's see how it is enunciated and proved.

Definition 12 (Vitali class)**.** *A collection of sets V is called Vitali class for E if $\forall x \in E$ and $\forall \delta > 0$ there exists $U \in V$ with $x \in U$ and $0 < |U| \leq \delta$.*

Theorem 6 (Vitali covering theorem)**.** *[4] Let E be an H^s-measurable subset of $\mathbb{R}^n$ and let V be a Vitali class of closed sets for E. Then we may select from V a finite or countable disjoint sequence $\{U_i\}$ such that*

(a) either $\sum |U_i|^s = \infty$ or $H^s(E - \bigcup_i U_i) = 0$;

(b) If $H^s(E) < \infty$, then, given $\epsilon > 0$, we may also require that

$$H^s(E) \leq \sum_i |U_i|^s + \epsilon.$$

Proof. (a) Fix $\rho > 0$ and suppose that $|U| \leq \rho \; \forall \; U \in V$.

We select the sets forming the sequence $\{U_i\}$ inductively:

let U_1 be any set of V; assume that $U_1, \cdots U_m$ have been chosen and take the set U in V whose diameter is the supremum which not intersects $U_1, \cdots U_m$. Such diameter is called d_m.

If $d_m = 0$, then $E \subset \bigcup_1^m U_i$ so the process terminates and
$H^s(E - \bigcup_1^m U_i) = 0 \Rightarrow$ (a).

Otherwise let U_{m+1} be a set in V disjoint from $\bigcup_1^m U_i$ and be such that $|U_{m+1}| \geq \frac{1}{2} d_m$.

Suppose that the process continues indefinitely and that $\sum |U_i|^s < \infty$. We want to show that the second possibility enounced in (a) is the correct proposition.

For all i let B_i be a ball with center in U_i and radius $3|U_i|$. We claim that $\forall \; k > 1$

$$E - \bigcup_1^k U_i \subset \bigcup_{k+1}^{\infty} B_i. \tag{1.9}$$

In effect, whether (1.9) is proved, with $\delta > 0$ and with k large enough to have $B_i \leq \delta$ for $i > k$, we should obtain

$$H^s_\delta(E - \bigcup_1^\infty U_i) \leq H^s_\delta(E - \bigcup_1^k) U_i \leq \sum_{k+1}^\infty |B_i|^s = 6^s \sum_{k+1}^\infty |U_i|^s$$

from which $H^s_\delta(E - \bigcup_1^\infty U_i) = 0 \ \forall \delta \ > 0$ and so$H^s(E - \bigcup_1^\infty U_i) = 0$.

We now have to show (1.9).

If $x \in E \bigcup_1^k U_i$, then there exists $U \in V$ with $x \in U$ but disjoint from $U_1, \cdots U_k$

Since $|U_i| \to 0$, then $|U| > 2|U_m|$ for some m.

By virtue of the method of selection of $\{U_i\}$, U must intersect U_i for some $k < i < m$ for which $|U| < 2|U_i|$, so $U \subset B_i$ which proves (1.9).

□

Proof. (b) To get the second part of the Theorem, suppose that ρ already chosen at the beginning of the proof (a) is the number corresponding to ϵ.

Note that E satisfies the hypothesis of Lemma 2, then if $\sum |U_i|^s = \infty$ (b) is trivial, otherwise

$$H^s(E) = H^s(E - \cup_i U_i) + H^s(E \cap \bigcup_i U_i) = \quad (a) \quad 0 + H^s(E \cap \bigcup_i U_i) <$$

$$< \sum |U_i|^s + \epsilon$$

for the Lemma 2. □

1.3 Are Hausdorff measure and Lebesgue measure comparable?

As known, $n-$dimensional Lebesgue measure extends the concept of volume in $\mathbb{R}^n$, where with volume we intend the length if we work in $\mathbb{R}^1$, the area in $\mathbb{R}^2$.

Let C be a coordinate block [1] in $\mathbb{R}^n$ of the form

$$C = [a_1, b_1) \times [a_2, b_2) \times \cdots [a_n, b_n)$$

with

$$a_i < b_i \ \forall i.$$

Definition 13 (Volume of C)**.** *Given a block C in $\mathbb{R}^n$, its volume is defined by*

$$V(C) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$$

Definition 14 ($n-$dimensional Lebesgue measure)**.** *Let E be any subset of $\mathbb{R}^n$. Its Lebesgue measure is defined as*

$$\ell^n(E) = inf \sum_i V(C_i),$$

where the infimum is made over all covers of E by a sequence $\{C_i\}$ of blocks.

Note 2. *ℓ^n is a metric outer measure on $\mathbb{R}^n$ and, if it is restricted to the Lebesgue-measurable sets (among wich the borel sets, in light of Theorem 3) becomes for Theorem 2 a true measure.*

It isn't difficult to show that in $\mathbb{R}^1$ Lebesgue measure and Hausdorff measure coincide.

Nevertheless, the introduction of Hausdorff measure is expected to give new geometric contributions in measuring a set and the first is, as already mentioned before, that we can obtain fractional or irrational dimensions for sets.

We ask, having intuited that sometimes the only length, area or volume aren't enough to describe the shape and irregularity of an object, if there exists, when $n > 1$, any link between the two measures we are interested to compare.

We anticipate that the answer will be affirmative since they differ only by a constant multiple.

[1] we call block a cartesian product of n intervals in $\mathbb{R}^n$

In order to prove the next result it's necessary to enunciate the following isodiametric inequality which says that the set of maximal volume of a given diameter is a sphere.

Theorem 7 (Isodiametric inequality)**.** *The $n-$dimensional volume of a convex closed set of diameter d is, at most,*

$$\pi^{\frac{1}{2}n}(\frac{1}{2}d)^n/(\frac{1}{2}n)!$$

that is the volume of a ball of diameter d.

Theorem 8. *[4] Let $E \subset \mathbb{R}^n$, then $\ell^n(E) = c_n H^n(E)$, with*

$$c_n = \pi^{\frac{1}{2}n}/2^n(\frac{1}{2}n)!$$

In particular $c_1 = 1$ and $c_2 = \frac{\pi}{4}$.

Proof. Suppose $H^s(E) < \infty$.

Given $\epsilon > 0$, (b) of Vitali covering theorem says that E can be covered by a sequence of closed sets $\{U_i\}$ such that

$$\sum |U_i|^n < H^n(E) + \epsilon.$$

By (7)

$$\ell^n(U_i) \leq c_n |U_i|^n,$$

so

$$\ell^n(E) \leq \sum \ell^n(U_i) < c_n H^n(E) + c_n \epsilon,$$

obtaining

$$\ell^n(E) \leq c_n H^n(E) \tag{1.10}$$

We look now for the opposite inequality.

Let $\{C_i\}$ be a collection of blocks covering E with

$$\sum_i V(C_i) < \ell^n(E) + \epsilon. \tag{1.11}$$

We may suppose that these blocks are open by expanding them slightly whilst retaining this inequality (1.11).

For all i the closed ball contained in C_i, of radius, at most, δ forms a Vitali class for C_i.

By Theorem 6, (a), there exists a collection of disjoint balls $\{B_{ij}\}_j$ in C_i of diameter at most δ with

$$H^n(C_i - \bigcup_{j=1}^{\infty} B_{ij}) = 0$$

and then with

$$H^n_\delta(C_i - \bigcup_{j=1}^{\infty} B_{ij}) = 0.$$

Since ℓ^n is a Borel measure [2]

$$\sum_{j=1}^{\infty} \ell^n(B_{ij}) = \ell^n(\bigcup_{j=1}^{\infty} B_{ij}) \leq \ell^n(C_i)$$

holds.
At the end, we have:

$$H^n_\delta(E) \leq \sum_{i=1}^{\infty} H^n_\delta(C_i) \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} H^n_\delta(B_{ij}) + \sum_{i=1}^{\infty} H^n_\delta(C_i - \bigcup_{j=1}^{\infty} B_{ij}) \leq$$

$$\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |B_{ij}|^n = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_n^{-1}\ell^n(B_{ij}) \leq$$

$$\leq c_n^{-1}\sum_{i=1}^{\infty} \ell^n(C_i) < c_n^{-1}\ell^n(E) + c_n^{-1}\epsilon \quad per \quad (1.11).$$

If we multiply first and second term of this sequence of inequalities by c_n we can write

$$c_n H^n_\delta(E) \leq \ell^n(E) + \epsilon \quad \forall\ \epsilon, \delta$$

and, if we let $\delta \to 0$,

$$c_n H^n(E) \leq \ell^n(E). \tag{1.12}$$

(1.10) and (1.12) give the thesis of the Theorem. □

[2] we call Borel measure on $\mathbb{R}^n$ any measure μ defined on the $\sigma-$field B of the Borel-sets such that $\mu(L) < \infty$ $\forall L$ limited set of $\mathbb{R}^n$.

Chapter 2

Local properties of s-sets

In this chapter we refer to particular subsets of $\mathbb{R}^n$, those we have met for the first time in Definition 11 of the first chapter, the s-sets.
We deal with the study of their local properties in terms of densities and of question about tangency existence.
After the first definitions and remarks which are valid for the all $s-$sets, we divide the treatment into two parts for integer s and non integer s as the risults concerning properties we are examining are totally distinct in the two cases.
Since the charachteristic of s, that is the Hausdorff dimension, being different from an integer is one of the salient properties of fractal sets on which this thesis is centred, the section dedicated to the $s-$sets with this charachter is faced in details.
In the section preceding it, which deals with integer s, surerly results of some analytic-geometrical interest are reported, but mainly with function of comparison and introduction to the section about non integer s. That's why in section 2.2 we'll omit the proofs of theorems.

Whenever we don't specify anything, the measure and dimension we refer to will be the Hausdorff measure and dimension.

2.1 Density and Tangency

Definition 15 (upper and lower spherical or circular densities). *Given an s-set E, in a point $x \in \mathbb{R}^n$ we define*

1. *the upper density:* $\rho^s(E,x) = \overline{\lim}_{r\to 0} \dfrac{H^s(E\cap B_r(x))}{(2r)^s}$

2. *the lower density:* $\delta^s(E,x) = \underline{\lim}_{r\to 0} \dfrac{H^s(E\cap B_r(x))}{(2r)^s}$

with $B_r(x)$ closed ball of center x, radius r and diameter $|B_r(x)| = 2r$.

Definition 16 (density of E in x). *If $\rho^s(E,x) = \delta^s(E,x)$ we say that the density of E in x esists and is indicated by $D^s(E,x)$.*

Definition 17 (regular point, regular s-set). *If $\rho^s(E,x) = \delta^s(E,x) = 1$, x is a regular point of E.*
If almost all points of $x \in E$ are regular, E is called regular itself.

Obviously a point $x \in E$ which doesn't satisfy the first part of the Definition 17 is called irregular point. A set is irregular if the density doesn't exist in almost all its points.

A second definition of upper densiy, called convex density, is useful for our next results.

Definition 18 (upper convex density). *Given an s-set E, the upper convex density in a point x of E is*

$$\rho_c^s(E,x) = \lim_{r\to 0}\{\sup \frac{H^s(E\cap U)}{|U|^s}\}$$

where the supremum is over all convex sets U with $x \in U$ and $0 < |U| \leq r$.

Note 3. *Since $B_r(x)$ is convex and, if $x \in U$ then $U \subset B_r(x)$ with $r = |U|$, we have*

$$2^{-s}\rho_c^s(E,x) \leq \rho^s(E,x) \leq \rho_c^s(E,x). \tag{2.1}$$

First step is the proof of the fact that the just defined densities, as functions of x are measurable functions.

We remind that:

- a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be measurable if $\{x : f(x) < c\}$ is a measurable set[1];
- a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called Borel- measurable if $\{x : f(x) < c\}$ is a Borel set;
- a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called upper semicontinuous if $\{x : f(x) < c\}$ is an open set $\forall\ c$.

Lemma 4. *Let E be an $s-$set. Then these propositions are true:*

(a) $H^s(E \cap B_r(x))$ is an upper semicontinuous function of x and then Borel-measurable.

[1] this definition and the following are the same if in $\{x : f(x) < c\}$ the symbol $<$ is replaced by $\leq, \geq$ e $>$.

(b) $\rho^s(E,x)$ are $\delta^s(E,x)$ are Borel-measurable functions of x.

Proof. (a) Given $r, \alpha > 0$, write $F = \{x : H^s(E \cap B_r(x)) < \alpha\}$.
For $x \in F$, if $\epsilon \to 0$, $B_{r+\epsilon}(x)$ decreases with respect to $B_r(x)$ so that for the continuity of H^s

$$H^s(E \cap B_{r+\epsilon}(x)) \to H^s(E \cap B_r(x)).$$

Then there exists ϵ such that $H^s(E \cap B_{r+\epsilon}(x)) < \alpha$ and with $|y - x| \leq \epsilon$ we have

$$B_r(y) \subset B_{r+\epsilon}(x) \text{ and } H^s(E \cap B_r(y)) < \alpha.$$

Hence F defined above is an open subset of $\mathbb{R}^n$ $\forall\ \alpha$ and so
$H^s(E \cap B_r(x))$ is an upper semicontinuous function of x. The fact that it becomes also Borel-measurable is obtained immediatly by observing that F, being open, is also a Borel set. □

Proof. (b) In ligth of the proof (a)

$$\{x : H^s(E \cap B_r(x)) < \alpha(2r)^s\}$$

is an open set, so

$$F_\rho = \{x : H^s(E \cap B_r(x)) < \alpha(2r)^s, r < \rho\}$$

is also open because it is union of such sets.

Now write the set $\{x : \delta^s(E,x) < \alpha\}$ as $\bigcap_{\rho>0} F_\rho$.

$\{x : \delta^s(E,x) < \alpha\}$ is then a $G_\delta-$set which is a borel set $\forall\ \alpha$ implying that $\delta^s(E,x)$ is Borel-measurable in x.

A similar argument establishes the measurability of $\rho^s(E,x)$. □

The two theorems which follow concern the convex density but are useful for their consequences on spheric densities thanks to the relation expressed in (2.1).

Theorem 9. *[4] If E is an s-set of $\mathbb{R}^n$ then $\rho^s_c(E,x) = 0$ for almost all x that doesn't belong to E.*

Proof. Fixed $\alpha > 0$ we show that the measurable

$$F = \{x \notin E \quad : \rho^s_c(E,x) > \alpha\}$$

has zero measure.

By the regularity oh H^s (Theorems 4 and 5) we may find, given $\delta > 0$, a closed set $E_1 \subset E$ with $H^s(E - E_1) < \delta$.

Let's write this relation that will be useful for the second part of the proof.

$$H^s(E \cap U) > \alpha |U|^s \tag{2.2}$$

For $\rho > 0$, be

$$V = \{U : \, U \text{closed and convex}, \; 0 < |U| \leq \rho, U \cap E_1 = \varnothing$$
$$\text{and for which is valid the (2.2)}\}.$$

V is a Vitali class and then we may use the Vitali covering Theorem (Theorem 6 (a)) to extract a disjoint sequence of sets $\{U_i\}$ in V such that either $\sum |U_i|^s = \infty$ or $H^s(F - \bigcup U_i) = 0$. We want to understand which of the two possibilities is correct.

From (2.2) we obtain:

$$\sum |U_i|^s < \frac{1}{\alpha} \sum H^s(E \cap U_i) = \frac{1}{\alpha} H^s(E \cap \bigcup U_i) \leq$$
$$\leq \frac{1}{\alpha} H^s(E - E_1) < \frac{\delta}{\alpha} < \infty.$$

Having excluded the firs choice, we deduce automatically that
$H^s(F - \bigcup U_i) = 0$, so

$$H^s_\rho(F) \leq H^s_\rho(F - \bigcup U_i) + H^s_\rho(F \cap \bigcup U_i) \leq H^s(F - \bigcup U_i) + \sum |U_i|^s < 0 + \frac{\delta}{\alpha}.$$

The inequality is true $\forall \;\; \delta > 0$ and $\forall \rho > 0$, than $H^s(F) = 0$ that is F is a set of Hausdorff measure zero, that concludes the proof. □

Theorem 10. *[4] If E is an s-set of $\mathbb{R}^n$ then $\rho^s_c(E, x) = 1$ for almost all x belonging to E.*

Proof. To obtain $\rho^s_c(E, x) = 1$ we first proove that $\rho^s_c(E, x) \geq 1$ and secondly that $\rho^s_c(E, x) \leq 1$.

Taken $\alpha < 1, \; \rho > 0$ we define

$$F = \{x \in E : \; H^s(E \cap U) < \alpha |U|^s \text{ for every convex set } U$$
$$\text{with } x \in U \text{ and } \; |U| \leq \rho\}. \tag{2.3}$$

F is a Borel subset of E.

For every $\epsilon > 0$ we may find a $\rho-$cover of F by convex sets $\{U_i\}$ such that

$$\sum |U_i|^s < H^s(F) + \epsilon. \tag{2.4}$$

Assuming that every U_i contains some points of F and using (2.3), it results

$$H^s(F) \leq \sum_i H^s(F \cap U_i) \leq \sum_i H^s(E \cap U_i \leq)$$

$$\leq \alpha \sum_i |U_i|^s < \alpha(H^s(F)) + \epsilon).$$

Being $\alpha < 1$ and ϵ any positive number, we deduce that $H^s(F) = 0$.

A set F of type (2.3) can be defined for every $\rho > 0$ and so

$$\rho_c^s(E, x) \geq \alpha$$

for almost all $x \in E$ and this is valid for all $\alpha < 1$.
In particular,

$$\rho_c^s(E, x) \geq 1 \text{ almost everywhere in } E. \tag{2.5}$$

The opposite inequality is obtained using the Vitali cover and the previous Theorem:

given $\alpha > 1$, be

$$F = \{x \in E \quad : \rho_c^s(E, x) > \alpha\}$$

measurable subset of E.

Moreover, we define

$$F_0 = \{x \in E \quad : \rho_c^s(E - F, x) = 0\}.$$

By Theorem 9 $H^s(F - F_0) = 0$.

By the Definition of convex density, $\rho_c^s(F, x) \geq \rho_c^s(E, x) - \rho_c^s(E - F, x) > \alpha$ if $x \in F_0$.

Then

$$V = \{U : \ U \text{ closed and convex and such that } H^s(F \cap U) > \alpha |U|^s\} \tag{2.6}$$

is a Vitali class for F_0 and we can use, this time, the proposition (b) of Theorem 6 which, given $\epsilon > 0$, finds a disjoint sequence of sets $\{U_i\}_i$ in V con $H^s(F_0) \leq \sum |U_i|^s + \epsilon$.

By (2.6)

$$H^s(F) = H^s(F_0) < \frac{1}{\alpha} \sum H^s(F \cap U_i) + \epsilon \leq \frac{1}{\alpha} H^s(F) + \epsilon.$$

Such inequality is valid for every $\epsilon > 0$, then $H^s(F) = 0$ if $\alpha > 1$ from which

$$\rho_c^s(E, x) \leq 1 \text{ almost everywhere in } E. \tag{2.7}$$

Both the inequalities (2.5) and (2.7) give the equality we need. □

By Theorem 9 using (2.1) it follows that for almost all x outside an s-set E

$$0 \leq \delta^s(E, x) \leq \rho^s(E, x) \leq 0$$

and then

$$D^s(E, x) = 0 \tag{2.8}$$

By Theorem 10 using (2.1) s it follws that for almost all x of an s-set E

$$2^{-s} \leq \rho^s(E, x) \leq 1. \tag{2.9}$$

Corollary 1. *Given F a measurable subset of an $s-$set E, then $\rho^s(F, x) = \rho^s(E, x)$ and $\delta^s(F, x) = \delta^s(E, x)$ for almost all $x \in F$.*

Proof. Since F is an $s-$set, by (2.8) both upper and lower densities are zero in almost all x outside F and in particular in $H = E - F$. As consequence

$$\rho^s(E, x) = \rho^s(F, x) + \rho^s(H, x) = \rho^s(F, x) + 0 = \rho^s(F, x).$$

$$\delta^s(E, x) = \delta^s(F, x) + \delta^s(H, x) = \delta^s(F, x) + 0 = \delta^s(F, x).$$

□

Corollary 2. *Given $E = \bigcup_j E_j$ a disjoint union of $s-$sets, with $H^s(E) < \infty$ then for all k $\rho^s(E_k, x) = \rho^s(E, x)$ and $\delta^s(E_k, x) = \delta^s(E, x)$ for almost every $x \in E_k$.*

Proof. It's an easy application of the previous Corollary with $F = E_k$. □

Corollary 3. *Given an $s-$set E,*

(a) if E is regular, any measurable subset of E of positive measure is regular;

(b) if E is irregular, any measurable subset of E of positive measure is irregular.

Proof. See Definition 17 and Corollary 1. □

Corollary 4. *The intersection of a regular set with a measurable set is regular; the intersection of an irregular set with a measurable set is irregular; the intersection of a regular set with an irregular set is a set of Hausdorff measure zero.*

Proof. The firs two statements follow immediatly from Corollary 3.
By this same Corollary, the third kind of intersection may give a set both regular and irregular, leaving as unique possibility that such intersection is a measure zero set. □

The last Corollary of this section is the decomposition theorem which enables us to treat the regular and irregular parts of an s-set indipendentely.

Corollary 5. *If E is an $s-$set,*

(a) the set of regular points of E is a regular $s-$set;

(b) the set of irregular points of E is an irregular $s-$set.

Proof. By Lemma 4 the sets of regular and irregular points are measurable, then we can apply Corollary 1 and conclude. □

The notions given till now in the current section were finalized to the following study about density of $s-$sets.

But for the question on the tangency existence we have to define a new type of density, the so called angular density.

Let:

- θ an unit vector;
- ϕ an angle;
- $S(x, \theta, \phi)$ the closed one-way infinite cone with vertex x and axis in direction θ consisting of those points y for which the segment $[x, y]$ makes an angle of, at most, ϕ with θ;
- $S_r(x, \theta, \phi) = B_r(x) \cap S(x, \theta, \phi)$ the corresponding spherical sector of radius r.

We define the upper and lower angular densities analogously to spherical densities.

Definition 19 (upper and lower angular densities). *Given an s-set E, at a point $x \in \mathbb{R}^n$ we define*

1. *the upper density:* $\rho^s(E, x, \theta, \phi) = \overline{\lim}_{r\to 0} \dfrac{H^s(E \cap S_r(x, \theta, \phi))}{(2r)^s}$

2. *the lower density:* $\delta^s(E, x, \theta, \phi) = \underline{\lim}_{r\to 0} \dfrac{H^s(E \cap S_r(x, \theta, \phi))}{(2r)^s}$

Definition 20 (s-set with a tangent). *An $s-$set $E \subset \mathbb{R}^n$ is said to have a tangent in x in direction $\pm\theta$ if $\rho^s(E, x) > 0$ and for every angle $\phi > 0$ is:*

$$\lim_{r\to 0} r^{-s} H^s(E \cap (B_r(x) - S_r(x, \theta, \phi) - S_r(x, -\theta, \phi))) = 0. \tag{2.10}$$

The line through x in direction $\pm\theta$ is of course the tangent line.

Definition 21 (s-set with a weak tangent). *An $s-$set $E \subset \mathbb{R}^n$ is said to have a weak tangent in x in direction θ if $\delta^s(E, x) > 0$ and for every angle $\phi > 0$ is:*

$$\underline{\lim}_{r\to 0} r^{-s} H^s(E \cap (B_r(x) - S_r(x, \theta, \phi) - S_r(x, -\theta, \phi))) = 0. \tag{2.11}$$

2.2 S-sets with integer s

About the $s-$sets in $\mathbb{R}^n$ when s is an integer number we present the structure in terms of density and tangency.

We have decided to start by examining the $1 - sets$ because the theory concerning them is intimately related to the rectificable curves.

Definition 22 (curve Γ)**.** *A curve Γ is the image of a continuous injection $\psi : [a, b] \to \mathbb{R}^n$, where $[a, b] \subset \mathbb{R}$ is a closed interval.*

Definition 23 (curve length)**.** *We define the length of a curve Γ as*

$$L(\Gamma) = sup \sum_{i=1}^{m} |\psi(t_i) - \psi(t_{i-1})|$$

where the supremum is made over all dissections $a = t_0 < t_1 < \cdots < t_m = b$ of $[a, b]$.

Definition 24 (rectificable curve)**.** *A curve is called rectificable if*

$$L(\Gamma) < \infty.$$

We now show a lemma which identifies the one-dimensional Hausdorff measure of a curve with its length:

Lemma 5. *If Γ is a curve, then*

$$H^1(\Gamma) = L(\Gamma).$$

As a consequence,

Corollary 6. *If Γ is a rectificable curve, then*

$$H^s(\Gamma) = \infty \text{ if } s < 1;$$

$$H^s(\Gamma) = 0 \text{ if } s > 1.$$

Lemma 6. *Let E be a compact and connetted set and let $x,\ y \in E$.*
If $|x - y| = r$, then

$$H^1(E \cap B_r(x)) \geq r.$$

Moreover

$$H^1(E) \geq |E|.$$

Lemma 5 and 6 are used to prove the following important property for the rectificable curves:

Theorem 11. *[4] Any rectificable curve is a regular $1-$set.*

It isn't hard to deduce that, if we refer to the tangency defined in Definition 21,

Theorem 12. *[4] A rectificable curve Γ has a tangent at almost all of its points.*

Theorems 11 and 12 solve the problem on density and existence of tangency limited to the rectificable curves.

We try to extend them initially to special $1-$sets, these contained in an union of rectificable curves, that we'll call $Y-$sets.

Theorem 13. *[4] An $Y-$set is a regular $1-$set and has a tangent at almost all its points.*

We have introduced the $Y - sets$, starting by the rectificable curves. In an analogous way we should transfer the properties to the compact and connetted sets E such that $H^1(E) < +\infty$.

The link is given by this remark:

Note 4. *Let E be a compact and connected set with $H^1(E) < \infty$. Then E consists on a countable union of rectificable curves, together with a set of H^1 measure zero.*

Being such E an $Y - set$, for E the Theorem 13 holds and this verifies that every compact and connected set of H^1 finite measure is regular and shows tangents in almost all its points.

We lead now with subsets of the plane.

We have defined an $Y-$set to be a subset of an union of rectificable curves; on the other hand, the sets whose intersection with every rectificable curve has measure zero are named $Z-$sets.

An irregular set is surerly a $Z-$set.

Otherwise, if it intersected a rectificable curve in some point, it would be regular at least in that part by Theorem 13. So, also its intersection with E, which is compact and connected and has $H^1(E) < \infty$ must necessarily have measure zero (see Note 4).

Next aim is to affirm that a $Z-$set is irregular, to have a precise carachterization of irregular $1-$sets.
We start by giving the following estimate for the lower density.

Theorem 14. *[4] Let E be a $Z-$set in $\mathbb{R}^2$. Then $\delta^1(E,x) \le \dfrac{3}{4}$ for almost all $x \in E$.*

The same estimate will hold for an irregular $1-$set of $\mathbb{R}^2$, having before declared that an irregular $s-$set is also a $Z-$set.

After this, what seems important to deduce from Theorem 14 is that for a $Z-$set, it will never be

$$\delta^1(E,x) = \rho^1(E,x) = D^1(E,x) = 1$$

since there will be a trivial inconsistence with the estimate we have fournished. In other words, one $Z-$set is an irregular $1-$set.

The observations we have made till now lead to enunciate one of the most important theorems of the current section:

Theorem 15 (carachterization of regular and irregular $1-$sets)**.** *[4]*

$$A\ 1-set\ is\ regular\ \Leftrightarrow\ it's\ an\ Y-set$$
$$together\ with\ a\ set\ of\ measure\ H^1\ equal\ to\ zero.$$
$$A\ 1-set\ is\ irregular\ \Leftrightarrow\ it's\ a\ Z-set;$$

The first proposition of the Theorem let us definitively answer to the question about the existence of tangents for the regular $1-$sets, by recognizing that these exist in almost all their points, thanks to Theorem 13.

Before facing the problem of tangency also for the irregular $1-$sets, we linger on the "broken" nature of irregular sets.
We remind that:

Definition 25 (totally disconnected set)**.** *A set is called totally disconnected if chosen two distinct points in it, they can't lay in the same connected component.*

Note 5. *An irregular $1-$set is totally disconnected.*

Such property will be found when we deal with $s-$sets with $0 < s < 1$ (Lemma 10) and may be interpreted as a first clue through the discovery of the irregularity intrinsic to the sets of non integer dimension. The argument will be studied in depth in the following section.

Lemma 7-9 that we are about to show are the foundamental steps necessary for the proof of the non-existence of any tangent in almost all points of an irregular 1-set.

Lemma 7. *Let θ be an unit vector of $\mathbb{R}^2$ perpendicular to a line L. Let P be a parallelogram with sides making angles ϕ to directions $\pm\theta$. Let y and z be two opposite vertices of P. Then, if d is the length of projection of P onto L,*

$$|y - z| \leq \frac{d}{\sin \phi}.$$

Lemma 8. *Let E be an irregular $1-$set of $\mathbb{R}^2$. Then, given θ and $0 < \phi < \frac{\pi}{2}$,*

$$\rho^1(E, x, \theta, \phi) + \rho^1(E, x, -\theta, \phi) \geq \frac{\sin \phi}{6}$$

for almost all $x \in E$.

Lemma 9. *Let E be an irregular $1-$set of $\mathbb{R}^2$. Then, for almost all $x \in E$,*

$$\rho^1(E, x, \theta, \phi) + \rho^1(E, x, -\theta, \phi) \geq \frac{\sin \phi}{6} > \frac{\phi}{10}$$

for all θ and for all $0 < \phi < \frac{\pi}{2}$.

Theorem 16. *[4] An irregular $1-$set of the plane hasn't any tangent (in the meaning of Definition 20) in almost all its points.*

We close the section with a Theorem that at the same time sums up the results already shown and generalizes them for the case of an $s-$set in $\mathbb{R}^n$ with s any integer number which does not necessarily coincide with 1.

Theorem 17. *[4] Let E be an $s-$set in $\mathbb{R}^n$ with integer s. Then the following propositions are equivalent:*

a) E is regular;

b) E is countably rectifiable;

b) E has a tangent at almost all of its points.

Definition 26 (countably rectifiable $s-$set [2]). *An $s-$set M is called countably rectifiable if it can be written*

$$M = \bigcup_{j=1}^{\infty} f_j(E_j) + G,$$

with $H^s(G) = 0$ and every f_j Lipschitz function $f_j : E_j \to \mathbb{R}^n$, $E_j \subset \mathbb{R}^s$.

[2]This definition can be thought as the analogous of the definition of $Y-$set when s was equal to 1.

2.3 S-sets with non integer s

At the end of the section we'll have explained that:

- every $s-$set showing a non integer s is irregular;
- the density, whose meaning has been clarified in section 2.1, fails to exist at almost all of the points of such $s-$set;
- the tangents, in the weak sense, are able to exist only for sets of points of measure zero.

We first deal with the $s-$sets with $0 < s < 1$ because obtaining the local properties for these sets seems quite simple.

Lemma 10. *An $s-$set E in $\mathbb{R}^n$ with $0 < s < 1$ is totally disconnetted.*

Proof. Let x and y be two distinct points laying in the same connected component of E.

Define a mapping $f : \mathbb{R}^n \to [0, \infty)$ by $f(z) = |z - x|$.

Since f does not increase distances, by Lemma 3,

$$H^s(f(E)) \leq H^s(E) < \infty$$

holds.

As $s < 1$, $f(E)$ is a subset of $\mathbb{R}$ of Lebesgue measure zero and has dense complement.

Choosing a number r with $r \notin E$ and $0 < r < f(y)$, we can write E as

$$E = \{z \in E : |z - x| < r\} \cup \{z \in E : |z - x| > r\}$$

and this is an open decomposition of E with x in one connected component and y in the other, contraddicting the initial assumption.

It follows that E satisfies the Definition 25 and than is totally disconnected. □

Theorem 18. *[4] If E is an $s-$set with $0 < s < 1$ the density $D^s(E, x)$ fails to exist at almost every point of E.*

Proof. Suppose the conclusion is false that is E has a measurable subset where the circular density $D^s(E,x)$ exists.

By (2.9) such density must be at least 2^{-s} and being $0 < s < 1$ we have $2^{-s} > \frac{1}{2}$.

Choosing β small enough we may find an $s-$set $F \subset E$ such that if $x \in F$ then $D^s(E,x)$ exists and $\forall\ r \leq \beta$

$$H^s(E \cap B_r(x)) > \frac{1}{2}(2r)^s. \tag{2.12}$$

holds.

By regularity of H^s we may further assume that F is closed.

Given y an accumulation point of F and η a number with $0 < \eta < 1$, we denote $A_{r,\eta}$ the anular region $B_{r(1+\eta)}(y) - B_{r(1-\eta)}(y)$.

We can then write:

$$(2r)^{-s}H^s(E \cap A_{r,\eta}) =$$

$$= (2r)^{-s}H^s(E \cap B_{r(1+\eta)}(y)) - (2r)^{-s}H^s(E \cap B_{r(1-\eta)}(y))$$

which, by Density definition, if $r \to 0$

$$\to D^s(E,y)((1+\eta)^s - (1-\eta)^s). \tag{2.13}$$

On the other hand, for arbitrary small values of r, we may find $x \in F$ with $|x-y| = r$, so

$$B_{1/2r\eta}(x) \subset A_{r,\eta}$$

and, by (2.12) replacing r with $\frac{1}{2}r\eta$ we have

$$\frac{1}{2}(r\eta)^s < H^s(E \cap B_{1/2r\eta}(x)) \leq H^s(E \cap A_{r,\eta}).$$

Considering first and last member of this inequality we can write in (2.13)

$$(2r)^{-s}\frac{1}{2}(r\eta)^s \leq D^s(E,y)((1+\eta)^s - (1-\eta)^s)$$

that is

$$2^{-(s+1)}\eta^s \leq D^s(E,y)((1+\eta)^s - (1-\eta)^s) = D^s(E,y)(2s\eta + 0(\eta^2)) \text{ for } \eta \to 0.$$

This fact, if $s < 1$ is impossible by (2.9), so we have found the contraddiction and we can conclude that a set E satisfying the hypothesis of Theorem fails to have density in almost all its points. □

Corollary 7. *Every $s-set$ with $0 < s < 1$ is irregular.*

Proof. It follows directly by the Theorem we have just proved keeping in mind the Definition of irregular $s-$set enunciated at the beginning of section 2.1. □

Theorem 19. *[4] Let θ be a unit vector and let $\phi < \frac{1}{2}\pi$. Then, if E is an $s-$set with $0 < s < 1$ we have*

$$\delta^s(E, x, \theta, \phi) = 0$$

in almost all $x \in E$.

Proof. Again in this proof we proceed by supposing to the contrary and assume that we may find β_0, $\alpha > 0$ and an $s-$set $F \subset E$ such that $\forall\ x \in F$ and $r \leq \beta_0$

$$H^s(E \cap S_r(x, \theta, \phi)) > \alpha r^s. \tag{2.14}$$

The relations (2.8) and (2.9) make then sure that we can choose an $y \in F$ for which

$$D^s(E - F, y) = 0 \text{ and } \rho^s(E, y) = c2^{-s} \text{ with } 0 < c < \infty.$$

As a consequence, given $\epsilon > 0$, there exists $\beta_1 \leq \beta_0$ such that, if $r \leq \beta_1$, then

$$H^s((E - F) \cap B_r(y)) < \epsilon r^s \tag{2.15}$$

and

$$H^s(E \cap B_r(y)) < (c + \epsilon) r^s. \tag{2.16}$$

We choose $\beta \leq \frac{1}{2}\beta_1$ such that

$$(c - \epsilon)\beta^s < H^s(E \cap B_\beta(y)). \tag{2.17}$$

Let be $x \in F \cap B_\beta(y)$ that maximizes the scalar product $x \cdot \theta$. Then

$$(F \cap B_\beta(y)) \cup (F \cap S_r(x, \theta, \phi)) \subset F \cap B_{\beta+r}(y)$$

where the union is disjoint except for the point x.

If we calculate the Hausdorff measures, we see

$$H^s(F \cap B_\beta(y)) + H^s(F \cap S_r(x, \theta, \phi)) \leq H^s(F \cap B_{\beta+r}(y))$$

and

$$H^s(F \cap B_\beta(y)) + H^s(F \cap S_r(x, \theta, \phi)) \leq$$
$$\leq H^s(E \cap B_{\beta+r}(y)) + 2H^s((E - F) \cap B_{\beta+r}(y)).$$

If $r \leq \beta$, it's also $\beta + r \leq \beta_1$ then, using (2.14)-(2.17) we have

$$(c - \epsilon)\beta^s + \alpha r^s < (c + \epsilon)(\beta + r)^s + 2\epsilon(\beta + r)^r.$$

Let's call γ the fraction r/β to obtain, for $0 < \gamma < 1$,

$$c - \epsilon + \alpha\gamma^s < (c + 3\epsilon)(1 + \gamma)^s.$$

This must hold for every $\epsilon \geq 0$, thus

$$c + \alpha\gamma^s \leq c(1 + \gamma)^s \leq c + cs\gamma$$

that contraddicts the hypothesis of α being > 0. □

The question of the existence of tangents to s-sets where $0 < s < 1$ is not of particular interest as such sets are so sparse as to make an idea of approximation by line segments raither meaningless.

Although we are of course able to find fractal sets for which $0 < s < 1$ (see the very famous Cantor set in the next chapter) we have at our disposal much more examples if we extend our study to objects for which $s > 1$.
It may be satisfactory for now to remain inside a plane where that "pathological monster, but of a concrete interest", to cite another time B. Mandelbrot, constitued by the "Koch curve", and also the "Sierpinski Triangle" live.

Next theorems are then about s-sets where $1 < s < 2$.

Let's start by asking the question of the existence of tangents at some point x belonging to an $s-$set E of the plane with non-integer s.
We'll use the answers obtained to make statements about the spherical density.

We want to prove that E does not have tangents at almost all its point neither in a weak sense, that is referring to Definition 21.

Lemma 11. *Let E an $s-$set in $\mathbb{R}^2$ with $1 < s < 2$. Thus for almost every $x \in E$*

$$\rho^s(E, x, \theta, \phi) \leq 4 \cdot 10^s \phi^{s-1}$$

for all θ and for all $\phi \leq \dfrac{\pi}{2}$.

Proof. Let's fix $\beta > 0$ and define

$$F = \{x \in E : H^s(E \cap B_r(x)) < 2^{s+1} r^s \ \forall r \leq \beta\}. \qquad (2.18)$$

We choose a point x in F, a vector θ and an angle ϕ with $0 < \phi \leq \frac{\pi}{2}$.

For $i = 1, 2, \cdots$ let

$$A_i = S_{ir\phi}(x, \theta, \phi) - S_{(i-1)r\phi}(x, \theta, \phi),$$

so that

$$S_r(x, \theta, \phi) \subset \bigcup_1^m A_i \cup \{x\}$$

for an integer $m < \frac{2}{\phi}$.

The diameter of each set A_i if r is $< \frac{\beta}{20}$ is at most $10r\phi < \beta$. Then, applying (2.18) to each A_i that contains points of F and summing, we have

$$H^s(F \cap S_r(x, \theta, \phi)) \leq 2\phi^{-1}2^{s+1}(10r\phi)^s$$

and

$$(2r)^{-s}H^s(F \cap S_r(x, \theta, \phi)) \leq 4 \cdot 10^s \phi^{s-1}$$

se $r < \frac{\beta}{20}$.

In this way, leading to limit, we would obtain

$$\rho^s(F, x, \theta, \phi) \leq 4 \cdot 10^s \phi^{s-1},$$

but

$$\rho^s(E - F) = 0$$

for almost all $x \in F$ (see Corollary 1). The proof of the lemma is till now concluded only for almost all $x \in F$, but the inequality (2.9) makes sure that almost every point of E is in F for some $\beta > 0$ and finally the lemma is proved. □

Corollary 8. *Let E be an $s-$set in $\mathbb{R}^2$ with $1 < s < 2$, then at almost all points of E no weak tangent exists.*

Proof.

$$\varliminf_{r \to 0} r^{-s}H^s(E \cap (B_r(x) - S_r(x, \theta, \phi) - S_r(x, -\theta, \phi))) \geq$$

applying both the definitions of lower spherical density and upper angular density

$$\geq \delta^s(E, x) - \rho^s(E, x, \theta, \phi) - \rho^s(E, x, -\theta, \phi) \geq$$

by Lemma 11

$$\geq \delta^s(E,x) - 4\cdot 10^s\phi^{s-1} - 4\cdot 10^s\phi^{s-1} = \delta^s(E,x) - 8\cdot 10^s\phi^{s-1}$$

$$\text{for all } \theta \text{ and } \phi < \frac{\pi}{2} \text{ and for almost every } x \in E.$$

Thus at such points, for ϕ sufficiently small, the $\underline{\lim}$ can't be $= 0$ and the Corollary is proved. □

Having recorded the non-existence of tangents in almost every point of an $s-$set E with $1 < s < 2$, it remains now to find an answer also to the second of the two questions which play their role in this chapter, that is the existence of spherical density.

We proceed by analysing results concerning the angular density and then transferring them to the circular ones.

Let's start with the following Theorem which is the analogous to Theorem 19 showed for $0 < s < 1$.

Theorem 20. *[4] Let E be an $s-$set in $\mathbb{R}^2$ with $1 < s < 2$. Then if $\phi < \frac{\pi}{2}$ the lower angular density $\delta^s(E,x,\theta,\phi)$ is zero for some θ for almost all $x \in E$.*

Proof. Fix two positive numbers α and β to define the set

$$F_0 = \{x : H^s(E\cap S_r(x,\theta,\phi)) > \alpha r^s$$
$$\forall\ r \leq \beta \text{ e } \forall\ \theta\}. \tag{2.19}$$

We want to show that $H^s(F_0) = 0$ to have the thesis.

Suppose the contrary, then by 2.9 we can find $\beta_1 \leq \beta$ and a set $F \subset F_0$ of positive measure such that if $x \in F$ and $r \leq \beta_1$ then

$$H^s(E\cap B_r(x)) < 2^{s+1}r^s. \tag{2.20}$$

Moreover we may think by the Hausdorff measure regularity F to be closed.

Let y be a point of F at which the circular density of $E - F$ is zero.
Corollary 1 makes sure that this fact is true for almost every point in F.
Hence, given $\epsilon > 0$, we may choose $\beta_2 \leq \beta_1$ such that if $r \leq \beta_2$

$$H^s((E-F)\cap B_r(y)) < \epsilon r^s. \tag{2.21}$$

We now work inside the disc $B_{\beta 2}(y)$. First we show that there are points in $B_{1/2\beta_2}(y)$ relatively remote from the set F.
To do this, suppose that for some $\gamma \leq \frac{1}{2}\beta_2$ all points of $B_{\beta 2}(y)$ are within distance γ from F.

Then if $x \in B_{\beta 2}(y)$, there exists a point z of F inside $B_\gamma(x)$.
By (2.19)

$$\alpha\gamma^s < H^s(E \cap S_\gamma(z,\theta,\phi)) \leq H^s(E \cap B_\gamma(z)) \leq H^s(E \cap B_{2\gamma}(z)). \quad (2.22)$$

holds for any θ.

If γ is $< \dfrac{1}{4}\beta_2$, so $B_{\beta_2}(y)$ contains $(\beta_2/\gamma)^2/16$ disjoints discs with centers in $B_{1/2\beta_2}(y)$ and with radius equal to 2γ.

Consequently, by (2.22) and summing over these discs, we have:

$$(\beta_2/\gamma)^2\alpha\gamma^s/16 < H^s(E \cap B_{\beta_2}(y)) < \text{ (2.20) } < 2^{s+1}\beta_2^s.$$

Looking at first and last member of the inequality, developping the calculations and considering that $1 < s < 2$ we obtain

$$\gamma > c\beta_2$$

with c dependent only on s and on α.

Thus, if γ is $\leq c\beta_2$, there will be a disc of radius γ containing $B_{\beta_2}(y)$ but not containing points of F.

Moreover we may find a disc $B_{\beta_3}(w) \subset B_{\beta_2}(y)$ with no points of F in its interior part but with its boundary containing a point ν of F_0 with

$$\beta_2 \geq \beta_3 \geq c\beta_2. \quad (2.23)$$

Let θ be the inward normal direction to $B_{\beta_3}(w)$ at ν and let β_4 be half the length of the chords of $B_{\beta_3}(w)$ through ν that make angles ϕ with θ.
Of course

$$\beta_4 = \beta_3\cos(\phi). \quad (2.24)$$

Since the sector $S_{\beta_4}(\nu,\theta,\phi)$ lies in $B_{\beta_3}(w)$, it contains no points of F other than ν.

Using all the equations (2.21), (2.23) e (2.24) we can write:

$$H^s(E \cap S_{\beta_4}(\nu,\theta,\phi)) = H^s((E-F) \cap S_{\beta_4}(\nu,\theta,\phi)) \leq$$

$$\leq H^s((E-F) \cap B_{\beta_2}(y)) < \epsilon\beta_2^s < \epsilon c_1\beta_4^s,$$

with c_1 dependent only on ϕ, α and s.

We have till now seen that it's possible to find $\nu \in F_0$, β_4 with $0 < \nu < \beta$ and a dirction θ for which

$$H^s(E \cap S_{\beta_4}(\nu,\theta,\phi)) < \epsilon c_1\beta_4^s.$$

But just this relation is in contrast to the initial definition of F_0 and we have to affirm that $H^s(F_0) = 0$. Since the proof continues to hold for all α and $\beta > 0$, this ends by remembering the definition of $\delta^s(E, x, \theta, \phi)$. $\square$

Corollary 9. *Let E be an $s-$set in $\mathbb{R}^2$ with $1 < s < 2$. Then in almost every point of E the lower angular density $\delta^s(E, x, \theta, \frac{\pi}{2})$ is zero for some θ.*

Proof. Take a sequence of angles $\{\phi_i\}$ increasing to $\frac{\pi}{2}$.

By Theorem 20 for almost all $x \in E$ we may find a sequence of directions $\{\theta_i\}$ such that $\delta^s(E, x, \theta_i, \phi_i) = 0 \; \forall \; i$.

Extracting a convergent subsequence $\{\theta_i\}$, we can assume that $\theta_i \to \theta$.

It follows that $\delta^s(E, x, \theta, \phi) = 0$ for any $\phi < \frac{\pi}{2}$.

Thus, provided x is also not one of the points excluded by result of Lemma 11, by the same Lemma we understand that $\delta^s(E, x, \theta, \frac{\pi}{2}) = 0$. $\square$

We next examine the lower angular density of regular $s-$sets, where s is not integer, in $\mathbb{R}^2$.
We'll see that it's a strategy whose aim is to affirm that actually such sets do not exist or better the only possibility for them is having measure zero.

Lemma 12. *Let E be an $s-$set in $\mathbb{R}^2$ with $1 < s < 2$. Let x be a regular point of E at which the upper convex density equals* 1, *and suppose that $\delta^s(E, x, -\theta, \frac{\pi}{2}) = 0$ for some θ. Then E has a weak tangent at x perpendicular to θ.*

Proof. By the hypothesis, $\rho_c^s(E, x) = 1$ and x is a regular point of E, so the circular density exists and is $D^s(E, x) = 1$.

Given $\eta > 0$ it's possible to find arbitrarily small values of β such that

$$H^s(E \cap B_r(x)) > 2^s r^s (1 - \eta) \text{ if } r \leq \beta, \tag{2.25}$$

$$H^s(E \cap U) < (1 + \eta)|U|^s \text{ if } x \in U \text{ and } 0 < |U| \leq 2\beta, \tag{2.26}$$

$$H^s(E \cap S_\beta(x, -\theta, \frac{\pi}{2})) < 2^s \eta \beta^s. \tag{2.27}$$

With $0 < \phi < \frac{\pi}{2}$, let L be a line through x and perpendicular to θ, wheras let M and M' be the half-lines from x forming an angle ϕ to θ.

For a fix positive integer m we construct inductively a sequence of $m+1$ semicircles S_{r_i} each of radius r_i, with center in x and diameter on L, with $\beta = r_0 > r_i > \cdots > r_m$.

For each i the semicircle S_{r_i} will have the extremes of diameter y_i and y_i' on L, and will intersect M and M' in points respectively called z_i and z_i'.

Suppose now S_{ri} has been constructed. $S_{r_{i+1}}$ is specified by taking y_{i+1} to be the point on the segment $[x, y_i]$ such that

$$d(y_{i+1}, y_i') = d(y_{i+1}, y_i) + d(y_{i+1}, z_i').$$

If we draw an arc with center y_{i+1} through z_i' to meet L in y_{i+1}', by simmetry the arc with center y_{i+1}' through z_i meets L in y_{i+1}.

We denote U_i the convex part of S_{r_i} cut off by these arcs. We have:

$$|U_i| = 2r_{i+1}. \tag{2.28}$$

We estimate the measure of the part of the set E contained between two consecutive sectors bounded by M and M'.

$$S_{r_i}(x,\theta,\phi) - S_{r_{i+1}}(x,\theta,\phi) \subset U_i - S_{r_{i+1}}(x,\theta,\frac{\pi}{2}) \subset$$

$$\subset U_i \cup S_\beta(x,-\theta,\frac{\pi}{2}) - B_{r_{i+1}}(x),$$

so

$$H^s(E \cap S_{r_i}(x,\theta,\phi) - H^s(E \cap S_{r_{i+1}}(x,\theta,\phi) \leq$$

$$\leq H^s(E \cap U_i) + H^s(E \cap S_\beta(x,-\theta,\frac{\pi}{2})) - H^s(E \cap B_{r_{i+1}}(x)) <$$

$$\text{using (2.25)-(2.27)} < (1+\eta)|U_i|^s + 2^s\eta\beta^s - 2^s r_{i+1}^s(1-\eta) \leq$$

$$\text{(2.28)} \ \leq 2^{s+1}\eta r_{i+1}^s + 2^s\eta\beta^s \leq 2^{s+2}\eta\beta^s.$$

From suc estimate, summing over all the m sectors, we have:

$$H^s(E \cap S_\beta(x,\theta,\phi) < H^s(E \cap S_{r_m}(x,\theta,\phi) + 2^{s+2}\eta\beta^s m <$$

$$\text{(2.26)} \ < 2^s(1+\eta)r_m^s + 2^{s+2}\eta\beta^s m.$$

By virtue of the construction described above, r_m/β depends only on ϕ and m, and so it tends to zero as $m \to \infty$.

Hence, given $\epsilon > 0$, we may find m indipendent of η such that, for arbitrarily small value of β and $\forall\ \eta > 0$:

$$H^s(E \cap S_\beta(x,\theta,\phi) < \beta^s(2^{s+2}\eta m + \epsilon(1+\eta)).$$

Thus, $\delta^s(E, x, \theta, \phi) = 0$ if ϕ is $< \dfrac{\pi}{2}$.

Since in the hypothesis of Theorem $\delta^s(E, x, -\theta, \dfrac{\pi}{2}) = 0$, Definition 21 makes us affirm that E has weak tangent at x in direction perpendicular to θ.

□

Corollary 10. *Let E be an $s-$set in $\mathbb{R}^2$ with $1 < s < 2$. Then E is irregular.*

Proof. At almost all points of E :

1. $\rho_c^s(E, x) = 1$ (Theorem 10);
2. $\delta^s(E, x, \theta, \dfrac{\pi}{2}) = 0$ for some θ (Corollary 9);
3. E which satisfies the first two statements has a weak tangent at almost all its regular points (Lemma 12).

The proposition 3. is inconsistent with Corollario 8 which declared the impossibility for E of having tangents unless at a subset of measure zero.

Consequently, the regular points must be only a set of measure zero, that is E is irregular in light of Definition 18. □

In order to conclude the chapter we show an important Theorem which links the results of sections 2.2 e 2.3.

Theorem 21. *[4] An $s-$set in $\mathbb{R}^n$ is necessarily irregular unless s is an integer.*

Chapter 3

The self-similarity in fractal sets

We have seen how the concepts of fractal sets and irregularity are closely linked together.
Our purpose is now to show that the degree of irregularity that we meet when dealing with fractal sets is luckily in some way "the same on different scales ".
If we have, for example, to study the features of a coastline, which can be thought as a clear manifestation of the existence of fractal sets in nature, we start examining various maps with increasing scales, so that a map shows more details than the previous one.
We probabiliy notice that all the maps, ignoring the smallest details, have the same global aspect.
In other words, the chaotic aspect of coastlines is only apparent and hides some order instead.
While an artist could express the idea painting a firework, a mathematician theorize the concept we have just tried to introduce in the property of self-similarity that is the subject of this chapter.

3.1 Self-similar sets

A set is self-similar when it's built up of pieces geometrically similar to the entire set but on a smaller scale.
Since many of the classical fractal sets are self-similar, we dedicate the current section to formalize this property through rigorous definitions and theroems and try to appreciate their consequences in calculating Hausdorff dimensions of s-sets.

Definition 27 (contraction). *A mapping*

$$\psi : \mathbb{R}^n \to \mathbb{R}^n$$

is called a contraction if

$$|\psi(x) - \psi(y)| \leq c|x - y|$$

$\forall\ x, y \in \mathbb{R}^n$, *where c is a constant* < 1.

Definition 28 (ratio of contraction). *We call ratio of contraction the infimum value of c for which the inequality holds* $\forall\ x,\ y$.

Definition 29 (set E invariant for a set of contractions). *A set $E \subset \mathbb{R}^n$ is called invariant for a set of contractions*

$$\psi_1, \cdots \psi_m$$

if we it can be written

$$E = \bigcup_1^m \psi_j(E).$$

If $\{\psi_j\}_1^m$ is a set of contractions, let ψ denote the transformation of subsets of $\mathbb{R}^n$ defined by

$$\psi(F) = \bigcup_1^m \psi_j(F),$$

while the iterates of ψ are indicated by

$$\begin{cases} \psi^0(F) = F \\ \psi^{k+1}(F) = \psi(\psi^k(F)), \text{ for } k \geq 0. \end{cases}$$

Moreover, for any sequence $\{\psi_{j_1} \cdots \psi_{j_k}\}$, with $1 \leq j_i \leq m$, and any set E we'll write

$$E_{j_1 \cdots j_k} = \psi_{j_1} \circ \psi_{j_k}(E)$$

and, given a measure μ, the notation

$$\mu_{j_1 \cdots j_k}(E)$$

means

$$\mu((\psi_{j_1 \circ \psi_{j_k}})^{-1}(E)).$$

The first results we show are valid for contractions.

We'll then define similitude as a special case of contraction and demonstrate some important specific theorems about self-similarity.

Theorem 22. *[4] Given a set of contraction $\{\psi_j\}_1^m$ on $\mathbb{R}^n$ with contraction rations $r_j < 1$,*

a) there exists a unique non-empty compact set E such that

$$E = \psi(E) = \bigcup_1^m \psi_j(E), \tag{3.1}$$

b) if F is any non-empty compact subset of $\mathbb{R}^n$ the iterates $\psi^k(F)$ converge to E in the Hausdorff metric as $k \to \infty$.

Before proving this Theorem, we need to show a result of considerable importance in geometric measure theory, due to Blaschke (1916), that is the fact that the family of all non-empty compact subsets of $\mathbb{R}^n$ is a complete[1] metric space with the Hausdorff metric.

Let's start giving the notion of Hausdorff metric.

If $E \subset \mathbb{R}^n$, the *$\delta-$parallel body* of E is the closed set of points within distance δ of E :

$$[E]_\delta = \{x \in \mathbb{R}^n : \; inf|x-y| \leq \delta\}.$$

The *Hausdorff metric* δ is defined on the collection of all non-empty compact subsets of $\mathbb{R}^n$ by

$$\delta(E,F) = \inf\{\delta : E \subset [F]_\delta \text{ and } F \subset [E]_\delta\}.$$

Theorem 23 (Blaschke selection theorem)**.** *[4] Let C be an infinite collection of non-empty compact sets all lying in a bounded portion B of $\mathbb{R}^n$. Then there exists a sequence $\{E_j\}$ of distinct sets of C convergent in the Hausdorff metric to a non-empty compact set E.*

Proof. We produce a Cauchy sequence of sets from C

Let $\{E_{1,i}\}_i$ be any sequence of distinct sets of C

For each $k > 1$ we define an infinite subsequence $\{E_{k,i}\}_i$ of $\{E_{k-1,i}\}_i$ in the following way:

if β_k is a finite collection of closed balls of diameter, at most, $1/k$, covering B, each $E_{k-1,i}$ intersects some specific combination of these balls.

So there must be an infinite subcollection $\{E_{k,i}\}_i$ of $\{E_{k-1,i}\}_i$ which all intersects precisely the same balls of β_k.

[1]A metric space M is called complete if every Cauchy sequence in M converges in M.

If F is the union of the balls of β_k in a particular combination, then

$$E_{k,i} \subset F \subset [E_{k,i}]_{1/k} \ \forall \ i$$

so that

$$\delta(E_{k,i}, F) \leq 1/k,$$

giving

$$\delta(E_{k,i}, E_{k,j}) \leq 2/k \ \forall \ i, j.$$

Letting $E_i = E_{i,i}$, we have

$$\delta(E_i, E_j) \leq 2/\min\{i, j\}, \tag{3.2}$$

so we have produced the Cauchy sequence $\{E_i\}_i$.

For the convergence, we call

$$E = \bigcap_{j=1}^{\infty} \overline{\bigcup_{i=j}^{\infty} E_i}.$$

Being the intersection of a decreasing sequence of non-empty compact sets, E is a non-empty compact self itself.

By (3.2)

$$\overline{\bigcup_{i=j}^{\infty}} E_i \subset [E_i]_{2/j},$$

so

$$E \subset [E_j]_{2/j} \ \forall \ j.$$

On the other hand, if $x \in E_j$, again by (3.2),

$$x \in [E_i]_{2/j} \text{ if } i \geq j,$$

so

$$x \in [\overline{\bigcup_{i=k}^{\infty} E_i}]_{2/j} \text{ if } k \geq j.$$

Let's choose $y_k \in \overline{\bigcup_{i=k}^{\infty} E_i}$ with $|x - y_k| \leq 2/j$.

By compactness, a subsequence of $\{y_k\}$ converges to some $y \in \mathbb{R}^n$ with $|x - y| \leq 2/j$.

But $y \in E$, so $x \in [E]_{2/j}$.

We can conclude that $E_j \subset [E]_{2/j}$ and hence that

$$\delta(E, E_j) \leq 2/j.$$

Thus E_j converges to E in the Hausdorff metric. □

We are now able to prove the Theorem 22.

Proof. Let C be the class of non-empty compact subsets of $\mathbb{R}^n$.

By the Blaschke selection theorem, (C, δ) is a complete metric space.

If $F_1, F_2 \in C$, then from the definition of δ

$$\delta(\psi(F_1), \psi(F_2)) = \delta(\bigcup_1^m \psi_j(F_1), \bigcup_1^m \psi_j(F_2)) \leq$$
$$\leq \max_j \delta(\psi_j(F_1), \psi_j(F_2)) \leq$$
$$\leq (\max_j r_j)\delta(F_1, F_2).$$

Since from the hypothesis $\max_j r_j < 1$, ψ is a contraction mapping on C.

It follows from the contraction mapping theorem for complete metric spaces that there exists a unique $E \in C$ with

$$\psi(E) = E \leftrightarrow \quad \text{(a)}.$$

Moreover,

$$\delta(\psi^k(F), E) \to 0 \text{ as } k \to \infty \text{ for any } F \in C \leftrightarrow \quad \text{(b)}.$$

□

Definition 30 (open set condition for contractions)**.** *We say that the open set condition holds for the contractions $\{\psi_j\}_1^m$ if there exists a bounded open set V such that*

$$\psi(V) = \bigcup_{j=1}^m \psi_j(V) \subset V \tag{3.3}$$

where the union is disjoint.

Note 6. *We can say that:*

1. *The sets $\{V_{j_1,\cdots j_k}\}$, where k is arbitrary form a net, that is any pair of sets from the collection are either disjoint or else have one included in the other;*

2. E is included in $\bar{V}$.

Proof. 1. If we apply the transformation $\psi_{j_1,\cdots j_k}$, where k is arbitrary, to the sets $\{V_{j_1,\cdots j_k}\}$ we obtain

$$\bigcup_{j=1}^{m} V_{j_1,\cdots j_k} \subset V_{j_1,\cdots j_k}$$

with a disjoint union.

2. Since $\{\psi^k(\bar{V})\}_k$ is a decreasing sequence of compact sets convergent to E in the Hausdorff metric (see Theorem 22), it's impossible that

$$E = \bigcap_{k=0}^{\infty} \psi^k(\bar{V})$$

has points outside $\bar{V}$.

Taking images under $\psi_{j_1,\cdots j_k}$ we have $E_{j_1,\cdots j_k} \subset \bar{V}_{j_1,\cdots j_k}$ and, of course, by continuity of contractions, $\psi_j(\bar{V}) \subset \overline{\psi_j(V)}$. □

Definition 31 (similitude)**.** *A similitude is a contraction which transforms every subset of $\mathbb{R}^n$ to a geometrically similar set.*

If ϕ is a similitude

$$|\phi(x) - \phi(y)| = r|x - y|$$

$\forall\ x, y \in \mathbb{R}^n$ where $r < 1$ is the ratio or the scale factor of the similitude.

Thus a similitude is a composition of a dilatation, a rotation, a traslation and perhaps a reflection.

Definition 32 (self-similar set)**.** *If a set $E \subset \mathbb{R}^n$ is invariant (see Definition 29) for a set of similitudes $\{\phi_j\}_1^m$ and for some s we have*

$$H^s(E) > 0 \text{ but } H^s(\phi_i(E) \cap \phi_j(E)) = 0 \text{ for } i \neq j,$$

E is self-similar.

We can repeat the definitions of invariant set E and of the open condition for a set of similitudes instead of a set of contractions, having:

Definition 33 (open set condition for similitudes)**.** *We say that the open set condition holds for the similitudes $\{\phi_j\}_1^m$ if there exists a bounded open set V such that*

$$\phi(V) = \bigcup_{j=1}^{m} \phi_j(V) \subset V \tag{3.4}$$

where the union is disjoint.

The aim is now the prove of the fact that if the open set condition holds, then the invariant for similitudes set E is self-similar and its Hausdorff dimension (see Chapter 1, Definition 10) and similarity dimension coincide.

Definition 34 (similarity dimension)**.** *The similarity dimension is the unique positive number s for which*

$$\sum_{1}^{m} r_j^s = 1 \tag{3.5}$$

where $\{r_j\}_1^m$ are the ratios of the set of similitude $\{\phi_j\}_1^m$.

Both dimensions can be interpreted as the "fractal dimension" of an $s-$set, so, if our treatment is consistent, it's necessary they are equal. The advantage is that the similarity dimension is often easier calculable than the other one.

If, in particular, $r_1 = \cdots = r_m = r$, (3.5) becomes

$$mr^s = 1,$$

$$s = \log_r(\frac{1}{m})$$

and

$$s = \frac{\log m}{\log \frac{1}{r}}. \tag{3.6}$$

We'll test this in the following section when we'll calculate the fractal dimension of Cantor set.

Lemma 13. *Let $\{V_i\}$ be a collection of disjoint open subsets of $\mathbb{R}^n$ such that each V_i contains a ball of radius $c_1\rho$ and is contained in a ball of radius $c_2\rho$. Then any ball B of radius ρ intersects, at most, $(1+2c_2)^n c_1^{-n}$ of the sets $\bar{V}_i$.*

Proof. If $\bar{V}_i$ meets B, then $\bar{V}_i$ is contained in a ball concentric with B and of radius $(1+2c_2)\rho$.

If q of the $\{V_i\}$ meet B, we sum the volumes of the corresponding interior balls, having

$$q(c_1\rho)^n \leq (1+2c_2)^n\rho^n$$

and

$$q \leq (1+2c_2)^n c_1^{-n}$$

that is the bound we wanted for q.

□

Theorem 24. *[4] Suppose the open set condition holds for the similitudes $\{\phi_j\}_1^m$ with ratios $\{r_j\}_1^m$. Then the associated compact invariant set E is an $s-$set, where s is determined by*

$$\sum_1^m r_j^s = 1;$$

in particular $0 < H^s(E) < \infty$.

Proof. First, we look for the upper bound.

Iterating (3.1) we obtain

$$E = \bigcup_{j_1,\cdots j_k} E_{j_1,\cdots j_k}$$

and

$$\sum_{j_1\cdots j_k} |E_{j_1\cdots j_k}| = \sum_{j_1\cdots j_k} |E|^s(r_{j_1}\cdots r_{j_k})^s = |E|^s.$$

As $|E_{j_1\cdots j_k}| \leq (\max_j r_j)^k|E| \longrightarrow 0$ as $k \to \infty$, we conclude that

$$H^s(E) \leq |E|^s < \infty. \tag{3.7}$$

For the lower bound, we use the Lemma 13.

Suppose the open set V which satisfies the equation (3.4) of the open set condition contains a ball of radius c_1 and is contained in a ball of radius c_2.

Take any $\rho > 0$. For each infinite sequence $\{j_1, j_2, \cdots\}$ with $1 \leq j_1 < m$, curtail the sequence at the least value of $k \geq 1$ for which

$$(\min_j r_j)\rho \leq r_{j_1}\cdots r_{j_k} \leq \rho \tag{3.8}$$

and let S denote the set of finite sequences obtained in this way.

It follows from the net property of the open sets that $\{V_{j_1\cdots j_k} : j_1, \cdots j_k \in S\}$ is a disjoint collection.

Each such $V_{j_1\cdots j_k}$ contains a ball of radius $c_1 r_{j_1\cdots j_k}$ and hence, by (3.8), one of radius $c_1(\min_j r_j)\rho$.

Similary, it's contained in a ball of radius $c_2 r_{j_1\cdots j_k}$ and so in a ball of radius

$c_2\rho$.

By Lemma 13 any ball B of radius ρ intersects, at most

$$q = (1+2c_2)^n c_1^{-n} (\min_j r_j)^{-n}$$

sets of the collection $\{\bar{V}_{j_1\cdots j_k} : j_1, \cdots j_k \in S\}$.

It can be proved that there exists a Borel measure μ with support contained in E such that $\mu(\mathbb{R}^n) = 1$ and for any measurable set F

$$\mu(F) = \sum_{j=1}^{m} r_j^s \mu(\phi_j^{-1}(F)). \tag{3.9}$$

Consequentely, $\mu_{j_1\cdots j_k}$ has support contained in $E_{j_1\cdots j_k}$ and

$$\mu_{j_1\cdots j_k} = \sum_j r_j^s \mu_{j_1\cdots j_k j}. \tag{3.10}$$

Also $\mu_{j_1\cdots j_k}(\mathbb{R}^n) = 1$ and support$(\mu_{j_1\cdots j_k}) \subset E_{j_1\cdots j_k} \subset \bar{V}$ for any $\{j_1, \cdots j_k\}$.

Iterating (3.10) as appropriate we see that

$$\mu = \sum_{j_1\cdots j_k \in S} (r_{j_1\cdots j_k})^s \mu_{j_1\cdots j_k},$$

so that

$$\mu(B) \le \sum (r_{j_1\cdots j_k})^s \mu_{j_1\cdots j_k}(\mathbb{R}^n)$$

where the sum is over those sequences $\{j_1, \cdots j_k\}$ in S for which $\bar{V}_{j_1\cdots j_k}$ intersects B.

Thus, using (3.8),

$$\mu(B) \le q\rho^s = q2^{-s}|B|^s$$

for any ball with $|B| < |V|$.

But, given a cover $\{U_i\}$ of E, we may cover E by balls $\{B_i\}$ with $|B_i| \le 2|U_i|$, so

$$1 = \mu(E) \le \sum \mu(B_i) \le q2^{-s} \sum |B_i|^s \le q \sum |U_i|^s.$$

We may now choose $\{U_i\}$ to make $\sum |U_i|^s$ arbitrarily close to $H^s(E)$, so

$$H^s(E) \ge q^{-1} > 0. \tag{3.11}$$

(3.7) and (3.11) are the bounds required for $H^s(E)$.

□

Corollary 11. *If the open set condition holds, then* $H^s(\phi_i(E) \cap \phi_j(E)) = 0 \ \ (i \neq j)$*, so, in particular,* E *is self-similar.*

Proof. Since ϕ_j are similitudes,

$$\sum_{j=1}^{m} H^s(\phi_j(E)) = \sum_{j=1}^{m} r_j^s H^s(E) = 1 \cdot H^s(E) = H^s(E).$$

By the previous Theorem, $0 < H^s(E) < \infty$, so only if $H^s(\phi_i(E) \cap \phi_j(E)) = 0, \ (i \neq j)$, we can have

$$\sum_{j=1}^{m} H^s(\phi_j(E)) = H^s(\bigcup_{1}^{m} \psi_j(E)) = \ (3.1) \ H^s(E).$$

$H^s(\phi_i(E) \cap \phi_j(E)) = 0, \ (i \neq j)$, $H^s(E) > 0$ and the invariance of E give the self-similarity. □

3.2 Cantor dust

Let's work on a real line: we meet an s-set, obviously with $0 < s < 1$, which has been sometimes mentioned through the thesis because it has non-integral Hausdorff dimension and is also self-similar: the Cantor set.
Being a set of points of the line, the study of its geometry can be easy and an advantage is that the methods may be extended to talk about sets of higher dimension.

A possible disadvantage is on the contrary due to the very few points that form this set, so that an intuitive idea of it is hard to imprint in mind.
This explains the choice of calling the section "Cantor dust" instead of the simpler "Cantor set": we'll see by the construction we now describe that Cantor set gradually "pulverizes".

Let

$$E_0 = [0, 1],$$

$$E_1 = [0, 1/3] \cup [2/3, 1],$$

$$E_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1], ...$$

So E_{j+1} is obtained by removing the open middle third of each interval in E_j; see figure 3.1.

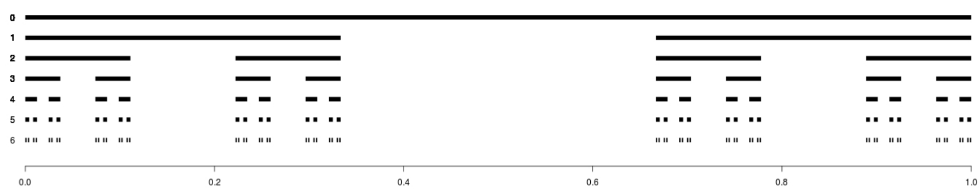


Figure 3.1: the first six steps in the construction of Cantor set

Then E_j consists on 2^j intervals, each of length 3^{-j}.

The sequence obtained is a subset of the line for which

$$E_0 \supset E_1 \supset E_2 \supset \cdots \supset E_j \supset \cdots$$

and the collection of closed intervals that occur in E_j form a net, that is, any two such intervals are either disjoint or else one is contained in the other.

Cantor's set is the closed and dense set

$$E = \bigcap_{j=0}^{\infty} E_j.$$

Although the "pulverization" is evident, we can be persuaded that some points are left and Cantor set E isn't empty.

If $[a, b]$ is a closed interval in one of the E_j, thus the extremes a, b are in all the other sets E_k, with $k > j$, and also in their intersection E.

We refer at the moment to the Lebesgue measure and try to calculate the length of E.

Theorem 25. *The Lebesgue measure of Cantor set is equal to zero.*

Proof. Every E_j consists of 2^j closed intervals whose length is 3^{-j}.

The total length of an entire E_j is $(2/3)^j$ and the Lebesgue measure of E is

$$\lim_{j\to\infty}\left(\frac{2}{3}\right)^j = 0.$$

□

This result isn't useful to give information of the geometry of Cantor dust, so we decide to involve the Hausdorff measure.

We'll procede in the following way: we explicitily calculate, or better estimate, Hausdorff dimension and measure of E using only definitions and theorems of Hausdorff measure illustrated in the first chapter.

We then recognize the fractal set to be self-similar and show that the simple computation of the similary dimension gives the same result, in agreement with the Theorem 24 of the previous section.

Theorem 26. *[7] Let E be the Cantor set described above. Then:*

a) Hausdorff dimension of Cantor set is

$$s = \frac{\log 2}{\log 3} = 0.6309...;$$

b) Hausdorff measure is $H^s(E) = 1$.

Proof. To estimate *a)* and *b)*, we show separately the upper and the lower bounds.

We shall start with the upper bound because it's as usual much simpler to find than the other one.

This is due to the definition: a judiciously chosen covering will give an upper estimate, but a lower estimate requires finding an infimum over arbitrary coverings.

Since E may be covered by the 2^j intervals of length 3^{-j} that form E_j, we see that

$$H^s_{3^{-j}}(E) \leq 2^j 3^{-sj}.$$

In order for this upper bound to be useful, it should stay bounded as $j \to \infty$.

The smallest value of s for which this happens is given by

$$2 \cdot 3^{-s} = 1$$

that is

$$3^s = 1;$$

$$\log_3(3)^s = \log_3(2);$$

$$s = \frac{\log 2}{\log 3}.$$

For this choice of s we have:

$$H^s(E) = \lim_{j \to \infty} H^s_{3^{-j}}(E) \leq 1$$

and

$$dim\ E \leq \frac{\log 2}{\log 3}. \tag{3.12}$$

To prove the opposite inequality we shall show that

$$H^s(E) \geq 1/4 \tag{3.13}$$

which will directly give

$$dim\ E \geq \frac{\log 2}{\log 3}. \tag{3.14}$$

We reduce the problem to other easier to derive, showing that they are equivalent to (3.13).

- to have (3.13) it suffices to show that

$$\sum_j |I_j|^s \geq 1/4 \tag{3.15}$$

  whenever open intervals $I_1, I_2, \cdots$ cover E;

- we may assume that there were only $I_1, \cdots I_n$ because E is compact and finitely many I_j's cover E;

- since E has no interior points we can, making I_j slightly larger if necessary, assume that the end-points of each I_j are outside E, then there is $\delta > 0$ such that the distance from all these end-points to E is at least δ. Choosing k so large that
$$\delta > 3^{-k} = |I_{k,i}|,$$
it follows that every interval $I_{k,j}$ is contained in some I_j;

- we have now to show that for any open interval I and fixed ℓ,
$$\sum_{I_{\ell,i} \subset I} |I_{\ell,i}|^s \leq 4|I|^s. \tag{3.16}$$
This gives (3.15), since
$$4\sum_j |I_j|^s \geq \sum_j \sum_{I_{k,i} \subset I_j} |I_{k,i}|^s \geq \sum_{i=1}^{2^k} |I_{k,i}|^s = 1.$$

Let then verify (3.16).

Suppose there are some intervals $I_{\ell,i}$ inside I and let n be the smallest integer for which I contains some $I_{n,i}$. Then $n \leq \ell$.

Let $I_{n,j_1}, \cdots I_{n,j_p}$ be all the $n-$th generation intervals meeting I.

Then $p \leq 4$, since otherwise I would contain some $I_{n-1,i}$.

Thus
$$4|I|^s \geq \sum_{m=1}^{p} |I_{n,j_m}|^s = \sum_{m=1}^{p} \sum_{I_{\ell,i} \subset I_{n,j_m}} |I_{\ell,i}|^s \geq \sum_{I_{\ell,i} \subset I} |I_{\ell,i}|^s$$
and (3.16) is proved, from which (3.15) and (3.13) $\Rightarrow$ for (3.12) and (3.14)
$$dim\ E = \frac{\log 2}{\log 3}.$$
Actually it is not hard to show that (3.15) can be improved to
$$\sum_j |I_j|^s \geq 1.$$
We proceed in the way illustrated above until, after a finite number of steps, we reach a covering of E by equal intervals of length 3^{-j}; these must include all the intervals of E_j so the inequality holds for this covering and also for the original covering of I_j, which gives the precise value
$$H^s(E) = 1.$$

□

Note that there is nothing special about the factor 1/3 used in the construction of the Cantor set.

We could take $0 < \lambda < 1/2$ and define

$$E_0 = [0, 1]$$

$$E_1 = [0, \lambda] \cup [1 - \lambda, 1]$$

$$...$$

and continue the process deleting from the middle of each interval in E_j an interval of length $(1 - 2\lambda)\lambda^{k-1}$. So every E_j consists of 2^j intervals, each of length λ^j. We call these Cantor sets $E(\lambda)$.

Repeating the prove we'll have that

$$dim\ E(\lambda) = \frac{\log 2}{\log(1/\lambda)}.$$

Note also that $dim\ E(\lambda)$ measures the sizes of the Cantor sets in a natural way: when λ increases, the sizes of deleted holes decrease and the set $E(\lambda)$ become larger, together with the increasing of $dim\ E(\lambda)$.

Moreover, when λ runs from 0 to 1/2, $dim\ E(\lambda)$ takes all the values between 0 and 1.

We look for the self-similarity in E.
E may be seen as a union of two copies of itself scaled of a factor equal to 1/3, or else as a union of 4 copies of itself scaled of a factor equal to 1/9, and also as a union of 16 copies scaled of 1/27, see again figure 3.1.

For example, we can write

$$E = \frac{1}{3}E \cup \left(\frac{2}{3} + \frac{1}{3}E\right),$$

or

$$E = \frac{1}{9}E \cup \left(\frac{2}{9} + \frac{1}{9}E\right) \cup \left(\frac{2}{3} + \frac{1}{9}E\right) + \left(\frac{8}{9} + \frac{1}{9}E\right).$$

After these observations, it seems clear that E satisfies the Definition 33 of open set condition and, by Corollary 11, it's self-similar.

The Cantor dust is in particular the unique compact set invariant under the similitudes of the real line

$$\phi_1(x) = x/3, \quad \phi_2(x) = (2 + x)/3$$

and, referring to Definition 34,

$$m = 2,\ r_1 = r_2 = \frac{1}{3}$$

so we can use (3.6) to calculate the similarity dimension

$$s = \frac{\log m}{\log 1/r} = \frac{\log 2}{\log 3} \approx 0,6309...,$$

that coincides with *dim E*, according with Theorem 24.

3.3 Koch coast and island

Let's go back to the comparison with the coastlines which opened the chapter and try to simplify the sequence of examined maps.

Assume the drawing of a piece of coast in scale 1/10000 to be an equilater triangle.

The new visible detail on a map representing one arm of the same coast in scale 3/10000 replaces the middle third of this arm by an headland whose shape is an equilater triangle, to give rise to 4 congruent segments.

Increasing the scale, at 9/10000, other 4 segments come up on each previous 4, the new smaller of a factor equal to 1/3, so that smaller headlands are born.

If we go on infinitely, we meet a limit called Von Koch curve, that we have named Von Koch coast in order to give a geographic (not only a mathemathic!) concept in minds.

That's another case in which a fractal set, subset of the plane, reflects well a feature of the nature and, if somebody can judge the model inaceptable, this must surerly not be because it's too irregular, but perhaps because its irregularity is too systematic.

More technically, we consider a segment P_0, for example the interval $[0, 1]$.

The following set P_1 is obtained by dividing P_0 into 3 parts and replacing the central part with a point.

This point consists of 2 arms forming an angle of $60°$ and each measures $1/3\ \ell(P_0)$, where $\ell(P_0)$ stands for the length of the segment P_0.

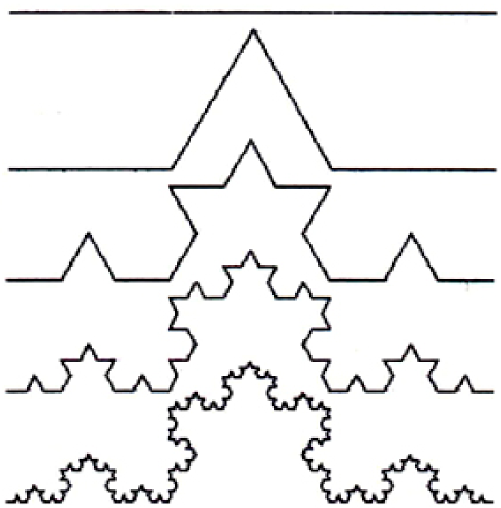
We have till now a broken line of four equal segments.

The operation described is iterated for each of these 4 segments to give P_2, another broken line of 16 segments that measure $1/9\ \ell(P_0)$.

The limit for $k \to \infty$ of the sequence $\{P_k\}$ produced is the Koch curve.

The Figure in the next page may make clear any possible imprecision left by words.

Figure 3.2: the first 5 steps in the construction of Koch curve.



For fixed k, the set P_k consists of 4^k segments whose length is $\frac{1}{3^k}$.

We examine some properties of this curve:

- Being a curve, von Koch coast's surface equals to zero;
- every set P_k consists on 4^k segments of length $1/3^k$;
- Koch curve has an infinite length. In effect, if $\ell(P_0) = \ell_0$ is the length of the initial segment,
$$\ell(P_k) = 4^k \ell_0 \frac{1}{3^k}$$
and
$$\ell(P) = \lim_{k\to\infty} \left(\frac{4}{3}\right)^k = \infty.$$
- P is self-similar, with $m = 4$ and $r_1 = \cdots r_m = \frac{1}{3}$, so we easily calculate the fractal dimension with the equation (3.6) which gives:
$$s = \frac{\log 4}{\log 3} \approx 1,2618;$$
- a curious fact is that this "piece of coast" we are about to consider, which is surerly a continuous curve, being an $s-$set laying in the plane with non integer s, hasn't tangent in any point for Corollary 8.

What happens if we assemble three copies of the Koch curve?

Let A_0 be the equilater triangle, one side of which is the unit interval.
We apply the same process defining the Koch curve, but to the three segments constituting the set A_0.

This procedure produces a star with six vertices which we may imagine as an equilater hexagon, each side of which carries an equilater triangle of side length 1/3.

The next application adds 12 smaller equilater triangles of side length 1/9. Continuing this procedure produces the contour of a set just consisting of three copies of the Koch curve.

It is known as the Koch snow flake, but the idea of it being surrounded by water and the purpuse to keep on with the geographic comparison leads me to calling it the Koch island, see Figure 3.3.

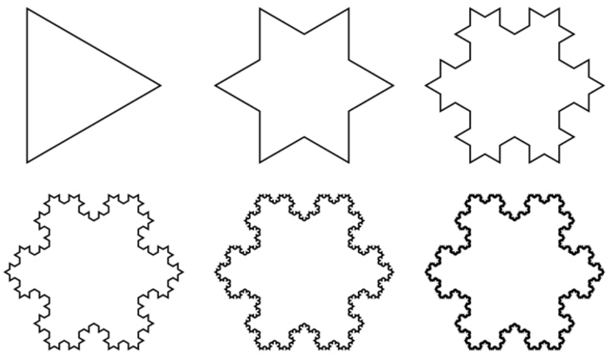


Figure 3.3: The first 6 steps in the construction of Koch island

The Koch island has infinite length because we have seen that already one third of the island's coastline has infinite length.

And what about the surface that the coast of the island surrounds? This can be naturally inscribed in a regular hexagon and so it's a finite area.

An infinite curve which contains a finit area is one of the fascinating facts met when studying fractal sets.

3.4 Sierpinski triangle

We remain inside the plane and discover a new $s-$set with $0 < s < 1$.

Take an equilater triangle T_0 with side length 1.

Connecting the mid-points of the sides by straight line segments divides T_0 into four equilater triangles of side length $\frac{1}{2}$ each.

Just as done in the construction of the Cantor set, we delete the open middle triangle to obtain the set A_1.

Continuing in this way with everyone of the 3 equilater triangles and defining T_k the compact set obtained at the $k-$th step, we achieves the Sierpinski triangle

$$T = \bigcap_{k=1}^{\infty} T_k.$$

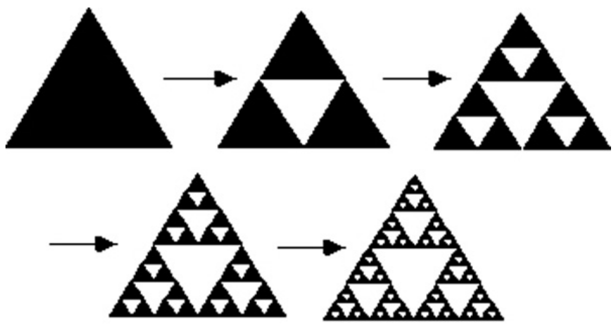


Figure 3.4: The first 5 steps in the construction of Sierpinski triangle.

In the set T_k there are 3^k triangles of side length $\left(\frac{1}{2}\right)^k$, so the calculation of area is

$$A(T_k) = \frac{\sqrt{3}}{4}\left(\frac{3}{4}\right)^k$$

so the Lebesgue measure two-dimensional of T is

$$\lim_{k\to\infty} \frac{\sqrt{3}}{4}\left(\frac{3}{4}\right)^k = 0.$$

The open triangle T_0 furnishes the set needed for the open set condition (Definition 33) and makes the Sierpinski triangle a self-similar geometric figure, whose similarity or Hausdorff dimension is

$$s = \frac{\log 3}{\log 2} \approx 1.585,$$

where $m = 3$ and $r_1 = \cdots r_m = r = \dfrac{1}{2}$.

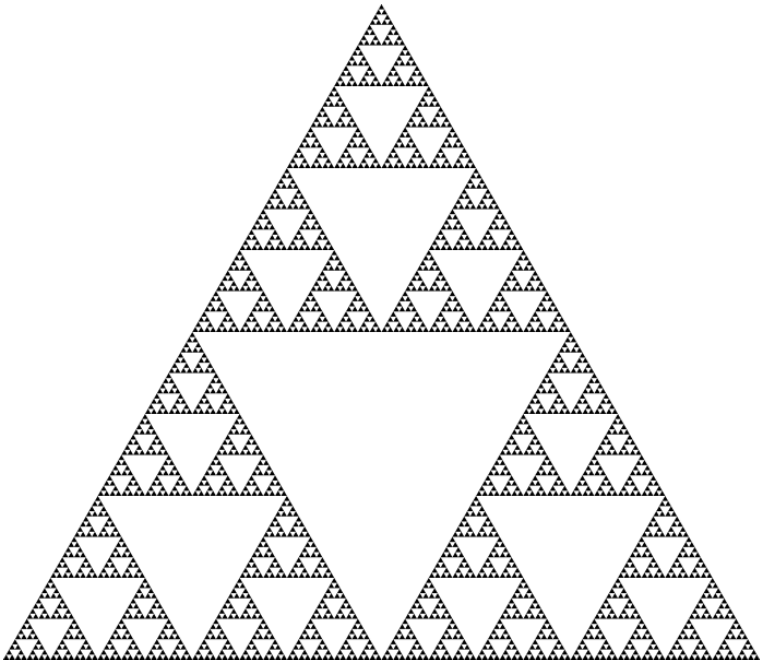


Figure 3.5: A picture of a T_k which tends to the Sierpinski triangle.

Chapter 4

Fractal analysis in Medicine

Fractal methods are commonly used in various areas of signal and image analysis.
In general they can be divided in two groups: some fractal methods take part in creations, like music or art; other ones can be used for comparison purpose by measurement methods.
In this chapter we show how this kind of measurement is applied in medicine illness recognition.

First of all, we need to explain what we mean with measurement fractal methods. These are systems based on the concept of fractal dimension.
Fractal dimension is considered to be a morphological parameter, characterizing the complexity of an object in addition to classical parameters such as size, roundness, density, ... so it is useful in comparisons and has been applied to morphological estimation in both clinical and experimental medicine.
For a practical approach, another formula to calculate the fractal dimension, to place side by side with similarity and Hausdorff dimensions, is called box-counting and is introduced in the first section.

We then present two practical approaches of using fractal methods, both in field of Medicine. The first regards the observation of senile plaques in human and other animal species; the second intends to recognize malignant melanoma among different skin lesions.

4.1 The box-counting dimension

A fractal set is well characterized by its Hausdorff dimension, which may often be hard to calculate. Think about the lower estimates we had to face to measure Cantor set (and it was only a subset of the real line!) .
When the fractal set is also self-similar, the similarity dimension has helped

us to simplify the calculations.
But it would certainly seem desirable to be able to attach this dimension not only to self-similar sets. The new approach, used in the works about Medicine we'll next show, is based on box-counting method.

Definition 35 (box-counting dimension)**.** *Let E be a non-empty bounded subset of $\mathbb{R}^n$.*
Define $N_\delta(E)$ to be the minimal number of subsets of $\mathbb{R}^n$, of diameter non exceeding $\delta > 0$, needed to cover A. The lower and upper box-counting dimension of A are defined respectively by

$$\underline{dim}_B(E) = \liminf_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}$$

and

$$\overline{dim}_B(E) = \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}.$$

If both are equal, then

$$dim_B(E) = \lim_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}.$$

For the actual computation, there exist equivalent definitions of $dim_B(E)$ where in the covering restricted class of sets are used.
In our cases (we work in $\mathbb{R}^2$) the covering consists of squares of side at most δ.
So

$$s = \frac{\log N_\delta(E)}{-\log \delta}$$

tells how the number of boxes needed to cover the considered geometric structure, which will be sometimes a plaque, sometimes a skin lesion, scales with the size of the boxes.

Surerly, the Hausdorff and self-similarity dimensions of the various fractals which we have got to know so far coincide with their box-counting dimension.

For example, if E is the Cantor set we choose

$$\delta_k = \frac{1}{3}^k, \text{ then } N_{\delta_k}(E) = 2^k,$$

and

$$dim_B(E) = \lim_{\delta_k \to 0} \frac{\log 2^k}{\log 3^k} = \lim_{k \to \infty} \frac{k \log 2}{k \log 3} = \frac{\log 2}{\log 3}.$$

In practice, the box counting method applied to an image needs the following steps:

- convertion of the images to grayscale;
- binarization of the image;
- accurate software calculation of box-dimension (the idea is that the algorithm approximates s using decreasing values of δ);
- analysis of results.

The experiments we are about to describe have in common the use of this method and diversify for the data processing and the results analysis.

4.2 A study about senile plaques in animals

In the present study,[2] the fractal dimension is applied to morphological estimation of animal and human senile plaques.

Senile plaque is one of the most characteristic histopathological changes of aged or Alzheimer's disease brain in man and is also detected in the aged brain of the mammalian species including dogs, cats, bears, monkeys and camels.
In some of these species, the plaques can be morphologically classified into two types, diffuse and mature, while some other show only diffuse type plaque.

The question asked:

With the experiment proposed we want to ask if the diffuse plaque is a precursor of the mature type in the human brain or they form separately, like it's thought to happen for species such as dogs or monkeys.
It's then necessary a comparative morphological study of senile plaques and this would also provide new concepts in the field of Alzheimer's disease.

Description of the experiment:

Seven dogs (11-17 years old), a cynomologus monkey (26 years old), an American black bear (more than 20 years old), a cat (20 years old) and a two-humped camel (more than 23 years old) and a 73-year-old woman who died from Alzheimer's disease are involved in the experiment.

Their cerebral cortices are fixed in 10% neutral buffered formalin. Four-to $6-\mu m-$thick paraffin sections were stained with periodic acidmethenamine silver (PAM).

All the cerebral tissues involved have senile plaque. Ten senile plaques are examined in each individual brain, wheras only five plaques are examined in the camel, because the total number of plaques is very low in this species.

The steps that are necessary for the box-counting (see the previous section) are applied to microscopic images of senile plaques. After the computer calculation of s when δ decreases, the values are transferred to Excell and a scatter diagram is made ($-\log \delta$ for $x-$axis and $\log N_\delta$ for $y-$axis).
The linear lines for the whole plots are made and the slope value is determined as fractal dimension (FD).

Data processing

The data processing follows these different directions:

1. first we refer to the cerebral cortices of dogs and correlate the FD values of the diffuse and mature plaques at the different ages of dogs ;
2. again in the case of the seven dogs, we look for a possible relationship between FD and size of plaques;
3. extending the observation to the other mammalian species, the study concerns the diffuse and, if present, mature plaques in a fixed mammal, but also a comparison among different species.

The comments below refer to the tables and graphics which will follow.

1. Table 1 shows FD of senile plaques in cerebral cortices from seven dogs.

   The FD value for each animal species is expressed as

   $$\text{the mean} \pm \text{ SD}$$

   where SD stands for standard deviation.

   To compare the various values, it has ben chosen the Kruskal-Wallis rank test[1] and differences are considered to be significant at p-value P< 0.5.

[1] it's a non-parametric method for testing whether samples originate from the same distribution. It is used for comparing two or more samples that are indipendent and that may have different sample sizes.

We can see by the Table 1 that the FDs of the diffuse plaques in six dogs (11-16) years old are determined to be from 1.618 to 1.690 and the mean is

$$1.656 \pm 0.046.$$

In contrast, the FD of the mature plaques in a 17−year-old dog, which was the only dog presenting mature plaques in this experiment, is

$$1.721 \pm 0.048.$$

This implies a significant difference as compared to diffuse plaque:

$$P = 0.00033.$$

2. In the graph on the right (see page 71) the size of a plaque, in terms of pixel, is represented in the $x-$axis, while in the $y-$axis the FDs of plaques of the dogs are reported.

   It's a way to discover whether the FD, that we have said to be a morphological parameter like size and other, may be put in any relationship with size.

   The result is that the fractal dimension tends to increase with size and this is true in both types of plaque: diffuse and mature.

   In order to make more evident this fact, the approximate linear lines for diffuse and mature plaques have been traced.

   These apporximate lines present different slope.

$$y = 0.0174x + 1.5918$$

   is the equation of the line approximating the values of FD of diffuse plaques, wheras

$$y = 0.0099 + 1.6622$$

   is the line for the mature plaques.

   From the comparison of the two lines, we see that mature plaques tend to have higher values of FD also when they are still small, but the FD of the diffuse plaque increases more rapidlly with size.

3. Table 2 reports the FD values of various mammalian species.

   We first notice that all the plaques of the cat and camel examined in this study are categorized as diffuse type.

   All the other species, including human, present both types of plaque.

   No significant differences are detected between the two types in the bear and monkey.

   In the dog and human on the contrary the difference is significant, with

   $$P = 0.0005$$

   in human and

   $$P = 0.0003$$

   in dog, as already seen in the previous point.

   However, the FDs of the mature plaque tend always to be greater than those of the diffuse plaques.

   Leading about diffuse plaques the lower value is absumed by feline and the difference is significant (always basing on Kruskal-Wallis rank test) if we compare with the values for

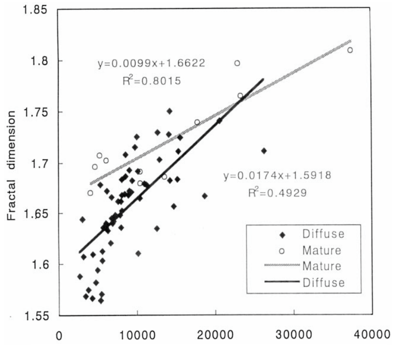
   - camel, $P = 0.0469$;
   - monkey, $P = 0.0047$
   - dog, $P = 0.0005$
   - bear, $P = 0.0002$.

   It can be interesting that P results > 0.05 in comparison between feline and human diffuse plaques.

   Moreover, the FD values of diffuse or mature plaques between other combinations of species were not significantly different.

Dog	Age	Type of plaques		Fractal dimension (mean± SD)
		Diffuse	Mature	
1)	11	✓		1.618 ± 0.040
2)	14	✓		1.622 ± 0.039
3)	15	✓		1.646 ± 0.045
4)	15	✓		1.690 ± 0.024
5)	16	✓		1.676 ± 0.037
6)	16	✓		1.686 ± 0.036
7)	17		✓	1.721 ± 0.048

Table 1: Fractal dimension of canine senile plaques



Animal	Type of plaque	Fractal dimension
Cat	Diffuse	1.468 ± 0.051
Camel	Diffuse	1.666 ± 0.025
Dog	Diffuse	1.656 ± 0.046
	Mature	1.721 ± 0.048
Bear	Diffuse	1.696 ± 0.049
	Mature	1.721 ± 0.053
Monkey	Diffuse	1.664 ± 0.040
	Mature	1.670 ± 0.032
Human	Diffuse	1.632 ± 0.041
	Mature	1.669 ± 0.030

Table 2: Fractal dimension of senile plaques in various animal species and in humans

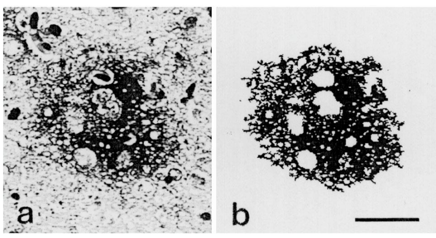


Figure a-b: gray scale microscopic image of a canine diffuse senile plaque- corresponding binary image produced on a computer

Conclusions

The significant different FDs of canine diffuse and mature plaques (Table 1 and 2) mean that these kind of plaques for dogs have different complexities. This may indicate also different origins and processes, thesis which is supported by the different slope values of the approximate lines drawn in the graphic.

Moreover, it's known that in dogs, diffuse plaques are detected as early as $7-8$ years old, and more than 40% dogs have these lesions after 15 years old.
In contrast, mature type plaques are noted only in more aged animals and their numbers are very low.

We asked at the beginning of the current section about the human senile plaque formation.
Since similary to canine plaques those of humans show significant difference in the FD value between diffuse and mature type (see Table 2), we may answer that also in human the two plaques might form in a different manner.

The maturation process of senile plaques, however, remains controversial and some authors state that diffuse plaque changes its morphology to the mature type.

Surerly it can be deduced by this study and in particular by the FDs that the original condition for plaque formation would be different among species.

Factors determining the morphology of the senile plaque of each animal species were supposed to be the chemical structure of $\beta-$amyloid protein, cerebral microenvironment during plaque formation, and metabolic speed or longevity of species.

Further analysis of senile plaque formation are required and fractal geometry would be a powerful tool for such investigations.

4.3 Support system in diagnosis of Melanoma

This second study presents a practical approach of using fractal methods in skin lesion analysis [6].
In particular, fractal analysis is used to compare different skin lesions.

Skin cancer is reaching 20% increase of diagnosed cases every year.
In dermatology it is extremely difficult to perform automatic diagnostic differentiation of malignant melanoma basing only on dermatoscopic images.
One has to bear in mind that wrong decision in the case of a malignant melanoma carries very high probability of death of the patient.

The main difficult is to diagnose a malignant lesion in early stage, because it does not exhibit melanoma's charachteristics. We want to constate if the fractal dimension may really enhance the diagnostic process for doctors and even bring tools for automatic diagnostic.

A probabilistic approach

Dermatoscopy[2] is the commonly used method for skin lesion diagnosis.
We try to briefly describe the performance of the diagnostic process.

Two indexes need to be calculeted, the **sensitivity** and **specitivity**.

The sensitivity measures the proportion of actual positives which are correctly identified as such (in our field the cases of malignant melanoma effectivly diagnosed as cancer).
Its complementary computes the cases of malignant melanoma diagnosed as other skin lesions.

The specificity measures the proportion of negative which are correctly identified as such.
Its complementary computes the cases of healthy lesions diagnosed as cancerous.

If we deal with events, we may call rispectivly A and its complementary $\neg A$ the events

$$A = \{ \text{ the lesion is a malignant melanoma}\}$$

and

$$\neg A = \{\text{the lesion is a different skin lesion}\}.$$

[2]This is a non-invasive method which allows inspection of skin lesions unobstructed by skin surface reflections.

On the other hand, we have to define B and $\neg B$ the events

$$B = \{ \text{ the lesion is a diagnosed as malignant melanoma}\}$$

and

$$\neg B = \{\text{the lesion is diagnosed as a different skin lesion}\}.$$

So that we indicate with $P(B/A)$ the probability of identifing a melanoma as cancerous if it is and with $P(\neg B/\neg A)$ the probability of excluding the cancer if effectivly the lesion isn't malignant.

$\neg B/A$ and $B/\neg A$ are the complementar events of the two we have just defined and they imply wrong diagnosis: excluding the cancer if the lesion is instead a malignant melanoma and recognizing a malignant melanoma when there isn't.

Table 3 can give order to this explanation.

In terms of probabilities of events, these definitions hold:

Definition 36 (Sensitivity)**.**

$$\textit{Sensitivity} \ = \frac{P(B/A)}{P(B/A) + P(\neg B/A)}$$

Definition 37 (Specificity)**.**

$$\textit{Specificity} = \frac{P(\neg B/\neg A)}{P(\neg B/\neg A) + P(B/\neg A)}$$

Surerly the most problem is when a doctor diagnose a lesion as not malignant when it is actually malignant.
It is better to diagnose an actually healthly lesion as malignant but both solutions don't satisfy anyone.

Table 4 sums up the values of Sensitivity and Specificity in Diagnosis taking into account various cases from the best to the worst.

In an optimistic scenario 10% of diagnosed melanoma lesions are diagnosed as healthly.

For less experiences personnel sensitivity is much lower and in the worst case this happens in about 38% of diagnosed melanoma lesions.

		Actual malignant	Actual healthy
		A	⌐A
Diagnosed as malignant	B	B/A	B/⌐A
Diagnosed as healthy	⌐B	⌐B/A	⌐B/ ⌐A

Table 3 : Events to define Sensitivity and Specificity in a probabilistic approach

	Sensitivity		**Specificity**	
	value	**complementary**	**value**	**complementary**
Experts	90%	10%	59%	41%
Dermato-logists	81%	19%	60%	40%
Trainees	85%	15%	36%	64%
General Practictio-ners	62%	38%	63%	37%

Table 4 : Sensitivity and Specificity in Diagnosis

All the mean values reported for Sensitivity and Specificity aren't promicious.
That is why doctors improve their methods, in which some mathematical factors characterising the lesion are calculated manually.

Most known and used method is called **ABCD**.
In this method four features of the dermatoscopic image are recognized and these are

- **A**symmetry;
- **B**order;
- **C**olor and
- **D**ifferential structures.

The result is called Total Dermatoscopy Score (**TDS**) and is calculated as follows:

$$\textbf{TDS} = 1.3 * \mathrm{A} + 0.1 * \mathrm{B} + 0.5 * \mathrm{C} + 0.5 * \mathrm{D}.$$

If TDS is greater than 4.75 then lesion should be handled as suspicious.

Lesion with TDS greater than 5.45 gives strong indication that this lesion is cancerous.

An alternative method is proposed by Menzies.

Scoring is based on finding positive and negative features in lesion image.
Positive features mean features that indicate that lesion is cancerous.
Menzies describes eight positive features based on melanoma's specific patterns like dots, veil or broadened network.

Menzies method describes only two negative features.
First is patterns symmetry, which means if the whole structure of lesion or color is symmetric.
Second negative feature is the color count. If lesion contains only one color then it should be recognized as benign lesion.

The few diagnosis methods used so far and their limits make the doctors improve their technology, so each parameter that makes the diagnosis accuracy greater is very helpful.
That is why we try to involve the fractal dimension.

Box-counting dimension analysis of skin lesion

We start with the observation that some typical images obtained via video dermatoscopy clearly display fractal properties: high irregularity and existence of self similar regions and structures.

Box-counting dimension is calculated as illustrated in the first section of the current chapter.

Calculated box dimension values are presented in Table 5.
Only 7 lesions in the total count of 82 analyzed are recognized as malignant.

The rest of images is recognized as healthy or as a different illness than skin cancer.
All suspicious lesions are assigned to a group following the Clark and Breslow stageing that are also indicated in Table 5.
Clark and Breslow stage values have been confirmed by lesion observation or biopsy test.

Breslow values refer to the thickness of the lesion ans is measured in mm.

Clark's levels describe depth relative to other skin structures:

- I Melanoma confined to the epiderms
- II Invasion into the papillary dermis
- III Invasion to the junction of the papillary and reticular dermis
- IV Invasion into the reticular dermis
- V Invasion into the subcutaneous fat.

Analyzing the results we can see that the Breslow and box-counting dimension are correlated.

In particular, fractal dimension tends to decrease when thickness increases.

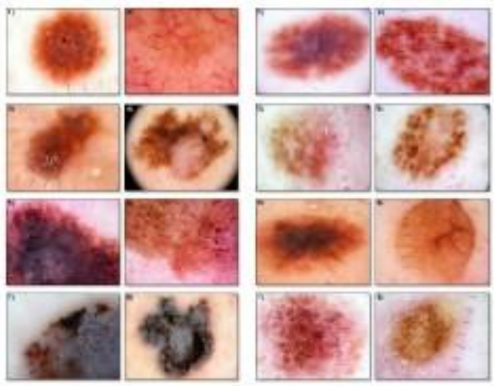
Moreover, the box-counting of the 7 lesions really identified as cancerous fournishes very simil values (from 1.20 to 1.28) and one may think to deduce a criterion based on a simil fractal dimension range to give a diagnosis about the skin lesions.

A further experiment compare fractal dimensions of various skin illness:

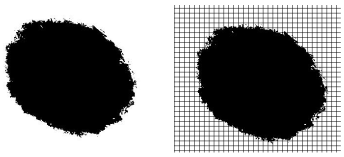
- combined,
- displastic,
- malignant,
- anginoma,
- blue,
- seborrheic and
- pigmented lesions.

The results (see graphics and comments in the next page) will show that our previous hypothesis is in someway confirmed but in other cases is too weak to be effectivly used.

Tables and graphics necessary for the data processing and to obtain our conclusions are now proposed.



Some of the skin-lesion images showing fractal properties



A skin lesion image divided into boxes.

Breslow	Clark	Fractal Dimension
0,25	II	1,28
0,25	IV	1,25
0,70	IV	1,24
0,80	IV	1,20
0,90	III	1,24
1,00	III	1,22
1,25	IV	1,21

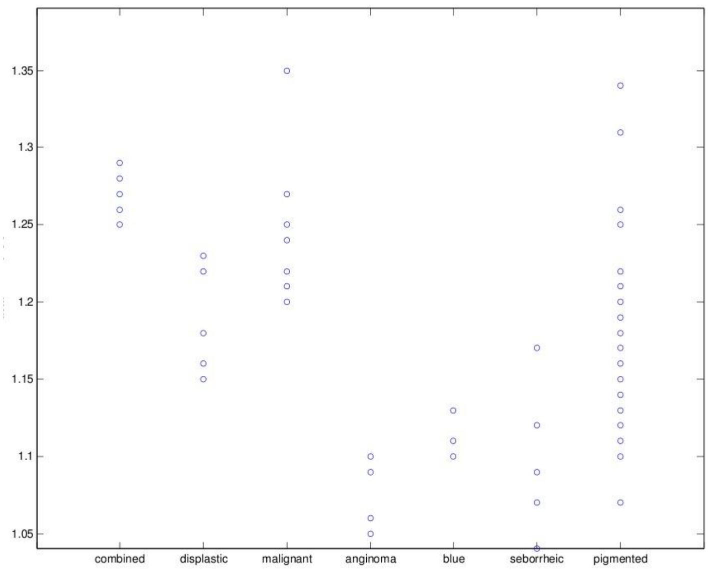
Table 5: Malignant Melanoma Fractal dimension

In this figure a comparison between melanoma and the 6 other detected skin lesion illnesses listes is done in terms of fractal dimension.

Some skin illnesses exhibit charachteristic values of box-counting dimension.

For anignoma, blue and seborrheic lesions dimension values are below 1.18.

The problem with melanoma diagnosis is that pigmented, combined and displastic lesions are covering also values of fractal dimension assigned for melanoma.



Fractal Dimension Comparison for different Skin Illnesses

Conclusions:

We were looking for a support system to improve Melanoma Diagnosis and we wanted to find it in Fractal Analysis.

The experiment shows that box-counting dimension of images of skin lesion can surerly be one of the charachteristic features in melanoma diagnosis. In effect, some of the skin lesions show distinctively different fractal dimension values.

However this charachterization should not be used as a separate parameter for skin lesion diagnosis.

More research needs to be done to check the efficiency of fractal methods together with other melanoma charachtetistics like color or assymetry.

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