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Corso di Laurea in Matematica

Fibered knots and links in lens spaces

Tesi di Laurea in Topologia Algebrica

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Introduzione

Sebbene l'idea di nodo sia conosciuta sin dalla prime apparizioni dell'umanità sulla Terra (si pensi ad esempio come le popolazioni primitive usavano i nodi per realizzare le reti da pesca), un approccio matematico ai nodi, chiamato specificamente teoria dei nodi, risale a un periodo relativamente recente. Con il passare del tempo, la teoria dei nodi si è rivelata estremamente utile per moltissime applicazioni nelle più varie discipline. Un chiaro esempio ci è fornito dalla biologia molecolare, in cui è possibile studiare la struttura della molecola di DNA grazie all'utilizzo di invarianti quali il *writhing number* o il *twisting number* calcolati attraverso il modello matematico della molecola, che è approssimativamente un nastro chiuso. In aggiunta, possiamo ravvisare tracce di teoria dei nodi anche nel meccanica statistica e nella fisica quantistica. Ad esempio lo studio della teoria di Chern-Simons permette di riottenere, quasi magicamente, il polinomio di Jones associato a un nodo fissato.
Visto che negli ultimi anni la teoria dei nodi è divenuta un settore di ricerca estremamente prolifico e ormai molti matematici hanno sviluppato una notevole quantità di risultati riguardanti nodi immersi nella 3-sfera S^3, potrebbe essere particolarmente interessante estendere tali risultati a una generica varietà 3-dimensionale. Un esempio di tale generalizzazione ci è fornito da Alexander in [A], in cui si dimostra l'esistenza di una open book decomposition (si veda Definizione 3.1) per ogni 3-varietà.
Lo scopo principale di questa dissertazione è esibire un esempio di una particolare classe di nodi, detti nodi fibrati, contenuti nello spazio lenticolare $L(p, q)$. Questi particolari nodi sono tali per cui il loro complemento rispetto alla varietà ambiente risulta essere un fibrato avente per spazio base la circonferenza S^1. Al fine di determinare la loro rappresentazione in $L(p, q)$, abbiamo introdotto il concetto di open book decomposition per una data 3-varietà M, che risulta essere completamente equivalente alla nozione di nodo o link fibrato. Applicando poi le mosse di Kirby a una open book prestabilita e sfruttando la compatibilità fra le mosse di Kirby e la struttura fibrata, siamo stati in grado di ottenere una rappresentazione via mixed link diagram. Un altro risultato interessante risiede nella possibilità di tradurre i risultati ottenuti per via geometrica attraverso il linguaggio analitico delle varietà di contatto.
La tesi è strutturata come segue. Nel primo capitolo si introduce la terminologia basilare riguardante la teoria dei nodi e la topologia tridimensionale. Dopo aver riportato la definizione di link immerso in una 3-varietà, sono elencate alcune nozioni fondamentali a essa collegate, quali il diagramma di un link, il linking number, la superficie di Seifert associata a un link e via dicendo. Successivamente si introduce

la definizione di spezzamento di Heegaard relativo a una varietà tridimensionale, di cui si riportano anche due esempi nel caso della 3-sfera S^3. La sezione successiva è interamente dedicata alla chirurgia razionale, la quale ci permette di definire i diagrammi di Kirby per un link con riferimento e le mosse di Kirby associate. Nella conclusione del capitolo, l'attenzione è focalizzata sulla descrizione della spazio che gioca il ruolo di protagonista in questa tesi: lo spazio lenticolare $L(p, q)$. Dopo averne riportato alcune definizioni equivalenti, si introducono diversi metodi di rappresentazione dei link immersi in questo particolare spazio.
Nel secondo capitolo è esposta la nozione di nodo fibrato nella 3-sfera. A partire dalla definizione, si esibiscono alcuni esempi quali il nodo banale, il link di Hopf e la fibrazione di Milnor. Successivamente elenchiamo alcune proprietà rilevanti relative alla forma di Seifert di nodi fibrati. Il capitolo termina con l'introduzione di due operazioni fondamentali, il plumbing e il twisting, le quali permettono la costruzione di tutti i possibili nodi fibrati nella 3-sfera a partire dal nodo banale.
Nel terzo capitolo la nozione di link fibrato è sostituita dal concetto equivalente di open book decomposition per una generica 3-varietà. Dopo una breve digressione relativa alle differenze fra open book astratta ed effettiva, si enuncia il Lemma 3.13, il quale descrive gli effetti prodotti su una data open book da una chirurgia trasversa a ciascuna pagina. Questo lemma si rivela essenziale per i nostri fini, in quanto la sua applicazione al caso particolare di spazi lenticolari restituisce i link fibrati desiderati, come enunciato nella Proposizione 3.17 e nella Proposizione 3.8.
Riportiamo di seguito la rappresentazione via mixed link del caso generale esposto nella Proposizione 3.8.

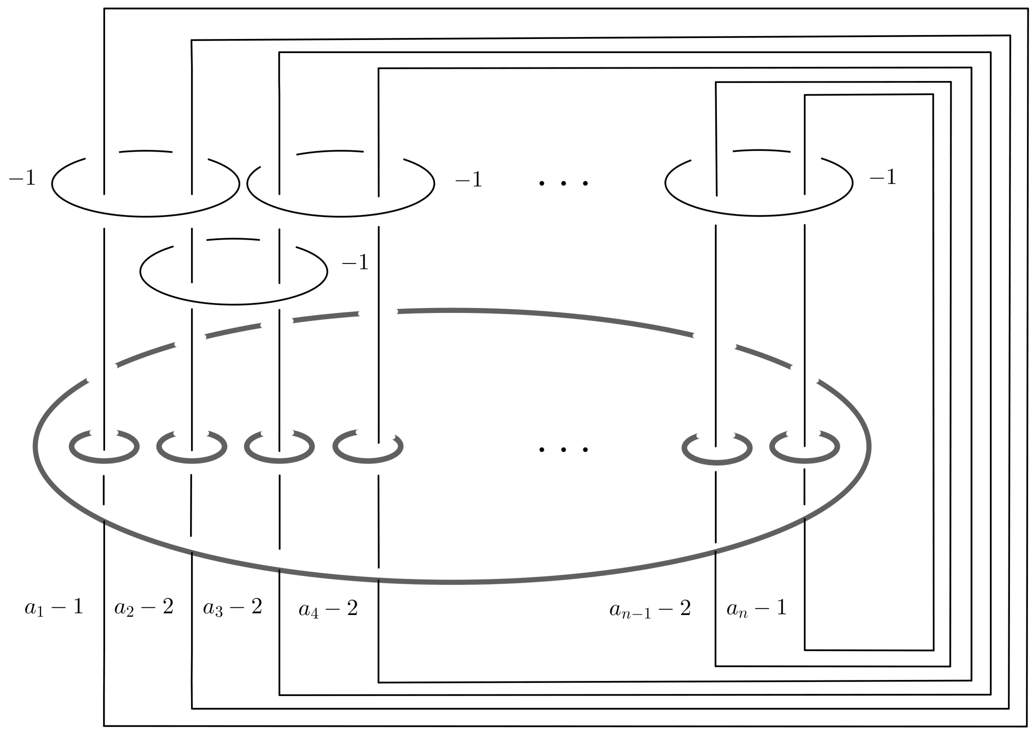

Figure 1: Esempio di link fibrato nello spazio lenticolare $L(p, q)$

Nell'ultimo capitolo, ci spostiamo verso l'ambito della geometria differenziale al

fine di introdurre il concetto di forma e di struttura di contatto per una varietà di dimensione dispari. Dopo aver riportato qualche esempio, come la struttura di contatto standard su $\mathbb{R}^3$ e sulla 3-sfera, mostriamo l'equivalenza fra l'esistenza di una struttura di contatto e l'esistenza di una open book per una varietà tridimensionale. Questo permette di tradurre tutti i risultati relativi alle open book decomposition in termini della geometria di contatto. Concludiamo il capitolo mostrando che la struttura di contatto standard sulla 3-sfera induce una struttura di contatto su ogni spazio lenticolare $L(p, q)$ per $p \neq q$ e infine supponiamo che per $p < -1$ tale struttura sia supportata dalla open book determinata nella Proposizione 3.17.
Concludiamo questa breve introduzione osservando che questa tesi lascia qualche questione aperta. Per esempio il mixed link rappresentato nella Proposizione 3.8 è abbastanza complicato e in più dipende da come scegliamo lo sviluppo in frazione continua associata al razionale p/q. Sarebbe dunque auspicabile ottenere una presentazione più semplice in cui la parte fissa del mixed link sia costituita unicamente dal nodo banale dotato del riferimento razionale p/q. Questo diagramma specifico ci permetterebbe di ottenere un punctured disk diagram in una maniera del tutto simile a quella esposta nella Sezione 3.6. Il punto fondamentale per comprendere come giungere a tale rappresentazione risiede nello studio del legame esistente fra struttura fibrata e l'applicabilità di una nuova tipologia di mosse, dette mosse di chirurgia razionale (ne è un esempio il Rolfsen twist). La nostra ipotesi è che le mosse razionali non siano compatibili con la struttura fibrata, a differenza delle mosse di Kirby, pertanto se le applicassimo al diagramma di un nodo fibrato ciò che otterremmo non sarebbe più un nodo fibrato. Una tale considerazione ci ha portati a teorizzare il bisogno di studiare strumenti matematici differenti al fine di giungere alla semplificazione desiderata.

Introduction

Even if knots have been known since the appearance of mankind (for instance we can think how primitive populations used knots to make nets), the mathematical approach to knots, called knot theory, is relatively young. Through the years, knot theory has revealed really useful for several applications to the most various disciplines. An example is given us by molecular biology, since the structure of a DNA molecule can be analyzed thanks to the use of writhing number and twisting number applied to the mathematical model of DNA presented as a closed ribbon. Moreover, knot theory arises in statistical mechanics and quantum physics. For example the study of Chern-Simons theory gives back us, in a quite magical way, the Jones polynomial associated to a given knot.
Since nowadays knot theory has become a prolific sector for scientific research and mathematicians have developed a lot of results regarding knots embedded in the 3-sphere S^3, it would be interesting trying to extend these results for a general 3-dimensional manifold. An example of this generalization is given us by Alexander in [A], where he proves the existence of an open book decomposition (see Definition 3.1) for every 3-dimensional manifold.
The main aim of this dissertation is to show an example of a particular class of knots, called fibered knots, contained in the lens space $L(p, q)$. For these particular knots, their complement with respect to the ambient manifold is a fiber bundle over the circle S^1. In order to determine their representation in $L(p, q)$, we have introduced the concept of open book decomposition of a 3-dimensional manifold M, which reveals to be completely equivalent to the notion of fibered knot, or link. By applying Kirby moves to a fixed open book of the 3-sphere S^3 and by taking advantage of the compatibility between Kirby moves and a fixed fibered structure, we have reached a representation of the desired fibered links by mixed link diagrams. Another interesting result is the possibility to translate the geometrical properties we have found by the use of the analytic language of contact manifolds.
The dissertation is structured as follows. In the first chapter we introduce the basic terminology regarding knot theory and 3-dimensional topology. First of all we report the definition of link embedded in a 3-manifold, then we briefly list other important notions related to it, such as the diagram of a link, the linking number, the Seifert surface for a link and so on. Then we introduce the concept of Heegaard splitting of a 3-dimensional manifold and we report two examples for the 3-sphere. The later section is dedicated to surgery theory, which allows us to define Kirby diagrams for a framed link and the Kirby moves. At the end of the chapter, we focus our attention on the space which plays a key role all along this dissertation: the lens space $L(p, q)$. After reporting some equivalent definitions of it, we intro-

duce several ways to represent links embedded in this particular space.
In the second chapter we expose the notion of fibered link in the 3-sphere. Starting from the definition, we exhibit several examples such as the unknot, the Hopf bands and the Milnor fibration. Then, we list some remarkable properties of fibered knots related to their Seifert form and we end the chapter by introducing two fundamental operations, plumbing and twisting, which let us to construct all the possible fibered links in the 3-sphere starting from the unknot.
In the third chapter, the notion of fibered link is substituted by the equivalent concept of open book decomposition for a general 3-manifold. After a brief digression about the differences between abstract and real open book decomposition, we state Lemma 3.13, which describes the effects produced on a given open book by performing a surgery transversal to each page. This lemma reveals to be essential for our purposes. Indeed, its application to the particular case of lens spaces returns the desired fibered links, as stated in Proposition 3.17 and Proposition 3.8. Here we report the representation by mixed link of the general case exposed in Proposition 3.8.

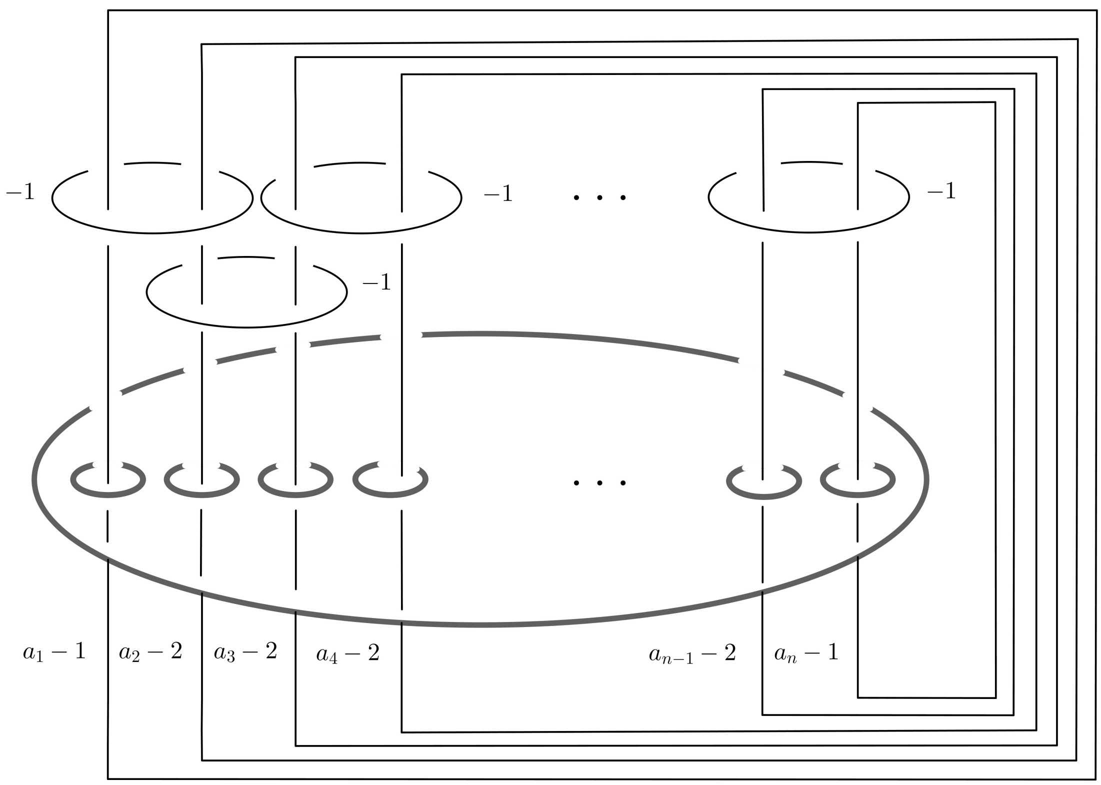

Figure 2: Example of fibered link in the lens space $L(p, q)$

In the last chapter, we move to the different setting of differential geometry in order to introduce the meaning of contact form and contact structure on an odd-dimensional manifold. After reporting some examples, e.g. the standard contact structure on $\mathbb{R}^3$ and on the 3-sphere, we show the equivalence between the existence of a contact structure and the existence of an open book for a 3-dimensional manifold. This allows us to restate all results about open book decompositions in terms of contact geometry. We conclude the chapter by showing that the standard

contact structure on the 3-sphere induces a contact structure on all the lens spaces $L(p, q)$ for $p \neq 0$ and we argue that for $p < -1$ this structure is supported by the open book decomposition found in Proposition 3.17.
We want to conclude this short introduction by observing that some open questions arise quite naturally from this dissertation. For example the mixed link reported in Proposition 3.8 is quite complicated and it depends on how we choose the continued fraction associated to p/q. It would be nice to have an easier presentation where the fixed part of the mixed link is simply the unknot with rational framing p/q. This particular diagram would give us the possibility to get a punctured disk diagram in a similar way to that one described in Section 3.6. The main point to understand how to get this presentation is to study the relation between the fibered structure and the applicability of a new kind of moves on the framed diagram, called rational surgery moves (e.g. the Rolfsen twist). Our conjecture is that the rational surgery moves are not compatible with a fixed fibered structure, so if we apply them to the diagram of a fibered knot the result will be no longer a fibered knot. This supposition leads us to conjecture that we need different mathematical tools in order to reach the desired simplification.

Chapter 1

Knots, links and 3-manifolds

In this chapter are reported the main results regarding knot theory and 3-dimensional topology. In the first section we briefly introduce the notion of link and other basic definitions related to it, such as the diagram of a link, the linking number, the Seifert surface for a link and the relation between links and braids. The second and the third sections are dedicated to Heegaard diagrams and Dehn surgery, respectively. In the last section we define lens spaces and give several presentations of links in these spaces.

1.1 Knots, links and braids

Definition 1.1. We indicate by:

- *Top* the category of topological manifolds and continuous maps,
- *Diff* the category of differentiable manifolds and differentiable maps,
- *PL* the category of piecewise linear manifolds and maps.

It is possible to show that these categories are equivalent in the 3-dimensional case (see [KS]).

We fix the following notation, which we will use all along this dissertation

$$S^n := \{(x_1, \dots x_n) \in \mathbb{R}^n | \sum_{i=1}^{n} x_i^2 = 1\},$$

$$D^n := \{(x_1, \dots x_n) \in \mathbb{R}^n | \sum_{i=1}^{n} x_i^2 \leq 1\}.$$

Definition 1.2. A *continuous curve* in a n-dimensional manifold M is a continuous function $\gamma : S^1 \to M$. We say that the curve γ is *nullhomologous* in M if its equivalence class in the first homology group $H_1(M)$ is trivial. In the same way, we say that γ is *nullhomotopic* if its equivalence class in the fundamental group $\pi_1(M, *)$ is trivial. A nullhomotopic curve is always nullhomologous in M, but the converse is not true in general.

Definition 1.3. Given a closed orientable 3-dimensional manifold M, we define a *link* L in M as a finite collection of smooth curves $\gamma_i : S^1 \to M$ whose images L_i are embedded and pairwise disjoint. Each curve is said to be a *component* of the link. A link with only one component is called a *knot*. For sake of simplicity we will refer to the link L by identifying it with the union of the images L_i of each component.

Remark 1.4. Thanks to the equivalence stated in Remark 1.4, we may alternatively define a link by setting us in *Top* or *PL*. For instance, each component of a *PL* link will be a closed simple polygonal curve. However, in *Top* it is possible to find links called *wild*, which are really hard to handle (an example is reported in Figure 1.1). Thus, the choice of smooth links is performed in order to avoid these pathological cases.

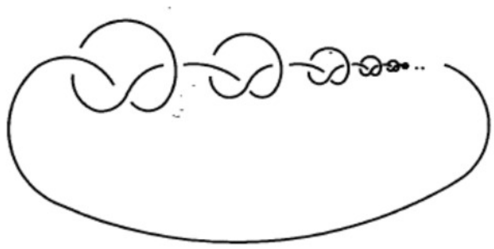

Figure 1.1: Example of wild knot

Definition 1.5. Let X be a topological space. An *ambient isotopy* between two subspaces $Y, Y' \subset X$ is a map $F : X \times [0,1] \to X$ such that each $F_t : X \to X$ given by $F_t(x) := F(t,x)$ is a homeomorphism, $F_0 = \mathrm{Id}_X$ and $F_1(Y) = Y'$.
Two different knots K_1 and K_2 in a 3-manifold M are said to be *equivalent* if there exists an ambient isotopy between them. The same definition holds for links with two or more components.

Definition 1.6. A link $L \subset M$ is *trivial* if each component bounds a disk in M in such a way that each disk is disjoint from the others.

From now on, we fix $M = S^3$ as the ambient 3-manifold to give the next definitions.

Definition 1.7. Let L be a link in $\mathbb{R}^3$ (or in $S^3 = \mathbb{R}^3 \cup \{\infty\}$) and take the orthogonal projection map $\pi : \mathbb{R}^3 \to P$, where P is a plane. The image of L under this map is called *projection* of L.
The projection map is called *regular* if it satisfies the following conditions:

1. the tangent lines to the link at all points are projected onto lines on the plane, i.e. the differential $d_m(\pi|_L)$ of the restriction $\pi|_L$ must have rank 1 for every point $m \in L$;

2. no more than two distinct points of the link are projected on one and the same point on the plane;

3. the set of double points, i.e. those on which two points project, is finite and at each crossing point the projection of the two tangents do not coincide.

We can represent a link in $\mathbb{R}^3$ using its regular projection (a theorem states that every link admits at least a regular projection, see [C]) and we can draw it as a union of smooth curves in $\mathbb{R}^2$ with gaps to indicate undercrossings and overcrossings where we have double points. The obtained drawing is called *a diagram* of the link.

Definition 1.8. Let us consider a link $L \subset S^3$ with only two components K_1 and K_2. After giving an orientation to both of them, we define the *linking number* between K_1 and K_2 as follows: we choose a diagram of L, we assign $+1$ to each right-handed crossing and -1 to each left-handed crossing (as shown in Figure 1.2) and finally we take half of the sum of the signs running over all the possible crossings.

$$\mathtt{lk}(K_1, K_2) := \frac{1}{2}\sum_c \epsilon(c), \qquad \epsilon(c) := \text{sign of the crossing } c.$$

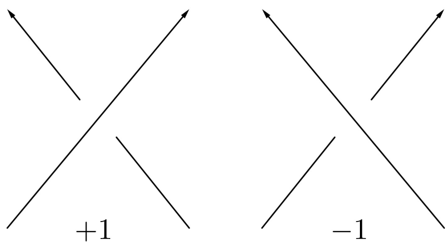

Figure 1.2: Right-handed and left-handed crossings

The given definition is independent from the chosen diagram but it depends only on the orientation of the two components K_1 and K_2. The same definition can be easily extended to a link with more components.

Remark 1.9. There exist other equivalent definitions for the linking number. For example, if we consider the first homology group $H_1(S^3 \setminus K_2) \cong \mathbb{Z}$ and we fix a generator, say $[m]$, the homology class of $[K_1] \in H_1(S^3 \setminus K_2)$ has to be written as $[K_1] = n[m]$, for a certain $n \in \mathbb{Z}$. Thus we can define $\mathtt{lk}(K_1, K_2) = n$.

Definition 1.10. An orientable surface Σ, with a given knot K as its boundary, is called *Seifert surface* for the knot.

Theorem 1.11 [Mu, Theorem 5.1.1] *Every knot admits a Seifert surface.*

Proof. Suppose that K is an oriented knot in S^3 and consider a regular diagram for it. Firstly, we draw small circles with their centers in correspondence of each crossing point in such a way that these circles contain exactly only one crossing point inside and intersect the diagram in four distinct points. After calling these points a, b, c and d respectively, as shown in Figure 1.3 (1), we remove the crossing point and connect a to d and b to c (see Figure 1.3 (2)).

In this way we change the segments ac and bd into the new segments ad and bc and, thanks to this operation called *slicing*, we remove every crossing from the

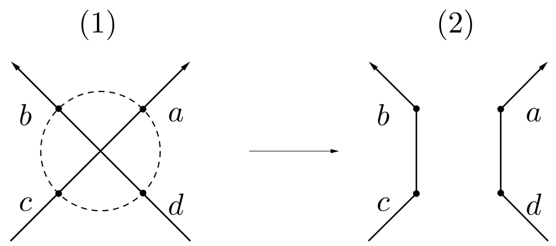

Figure 1.3: Elimination of crossings

diagram. At the end of this procedure, the initial diagram is decomposed into several closed curves called *Seifert circles*. Each circle bounds a disk and, even if the disks may be nested, they can be made disjoint by slightly pushing their interiors off the projection plane, starting with the innermost and going outward. To get a connected surface from the various disks, we need to attach to them several bands. In order to perform this attachment, we consider a crossing point which will have a certain sign, since the knot is oriented. Now we consider the square *acbd* and give it a positive or negative twist, compatible with the sign of the crossing (see Figure 1.4).

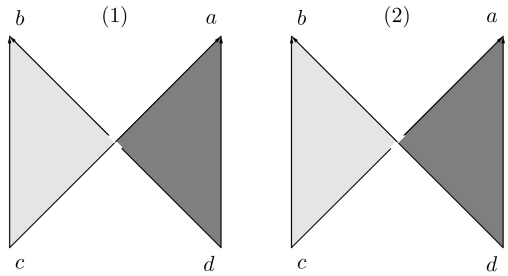

Figure 1.4: Positive and negative bands

By attaching correctly each band, we get a connected, orientable surface Σ, precisely the Seifert surface for the knot. □

Definition 1.12. The *genus* of a knot is the minimum possible genus of a Seifert surface for it.

Example 1.13. The unknot U bounds a disk, so its genus is 0.

Given a knot K, fix a Seifert surface Σ for it. Since a Seifert surface is orientable, there exists a non-vanishing normal vector field. We choose one of the two possibilities in order to distinguish the "top" side of the surface. After this choice, one picks up a simple closed oriented curve, x, on the surface and forms its *pushoff*, denoted with x^+, which is parallel to x and lies above the surface.

Definition 1.14. Chosen a basis $x_1, \ldots, x_g$ for $H_1(\Sigma)$, where g is the genus of Σ, we define the *Seifert matrix* V associated to Σ as the $2g \times 2g$ matrix with entries

$$V_{ij} = \mathtt{lk}(x_i, x_j^+).$$

Since we have several Seifert surfaces for a single knot K, the previous definition depends on the choice of the surface. How can we relate two different Seifert matrix for the same knot?

Definition 1.15. Two Seifert matrices are called *S-equivalent* if they are related by a sequence of operations of the following types:

- **type 1**: right and left multiplication by an invertible integer matrix and its transpose, i.e. V is changed into MVM^t.
- **type 2**: we add to M two rows and two columns as follows

$$\begin{pmatrix} & & & * & 0 \\ & M & & * & 0 \\ & & & * & 0 \\ * & * & * & * & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $*$ stands for a generic element.

Theorem 1.16 [Li, Theorem 3.6] *Two matrices are Seifert matrices for the same knot K if and only if they are S-equivalent.*

Definition 1.17. Let V be a Seifert matrix for a knot K, we define the *Alexander polynomial* as

$$\Delta_K(t) := \det(t^{1/2}V - t^{-1/2}V^t).$$

It easy to verify that this definition is indipendent from the choice of the Seifert matrix V, so the Alexander polynomial is a well-defined invariant of the knot K.

Definition 1.18. In $\mathbb{R}^3$ consider the points $A_i := (1/2, i/(n+1), 0)$ and $A'_i := (1/2, i/(n+1), 1)$, where $i \in \{1, 2, \ldots, n\}$. A smooth curve joining one of the points A_i to one of the points A'_j is called *ascending* if the z-coordinate increases monotonically or, equivalently, if each plane of equation $z = k, k \in [0,1]$ cuts the curve in only one point.
A *braid* in n strands is defined as a set of non-intersecting ascending smooth curves joining the points $A_1, \ldots, A_n$ to the points $A'_1, \ldots, A'_n$ in any order. Given two different braid α and β, we say that they are *equivalent* if there exists an isotopy between them fixing the endpoints.

We will use the word braid indiscriminately to mean an equivalence class of braids or a concrete representative of such a class. The set of equivalence classes of braids in n-strands has a natural structure of group. Indeed, we can define the product of two braids α and β simply by juxtaposition. More precisely, let us consider both braids contained in the cube $\mathcal{C} := \{(x,y,z) | 0 \leq x, y, z \leq 1\}$, as described in Definition 1.18. By gluing the base of the cube that contains α with the top face of the cube containing β, we get a new braid obtained by vertical juxtaposition of α and β. We indicate the product of the two braids with $\alpha\beta$ (see Figure 1.5(1)).
This operation is associative on the set of equivalence classes of braids, that is

$\alpha(\beta\gamma) = (\alpha\beta)\gamma$ for any three classes α, β, γ. The unit element is the trivial braid consisting of n parallel strand (see Figure 1.5(2)). It easy to check that the inverse braid β^{-1} is the *mirror image* β^* of β. The mirror image β^* is the image of β under the reflection with respect to base of the cube $\mathcal{C}$ (see Figure 1.5(3)).

Definition 1.19. The set of equivalence classes of braids in n-strands is a group denoted by B_n.

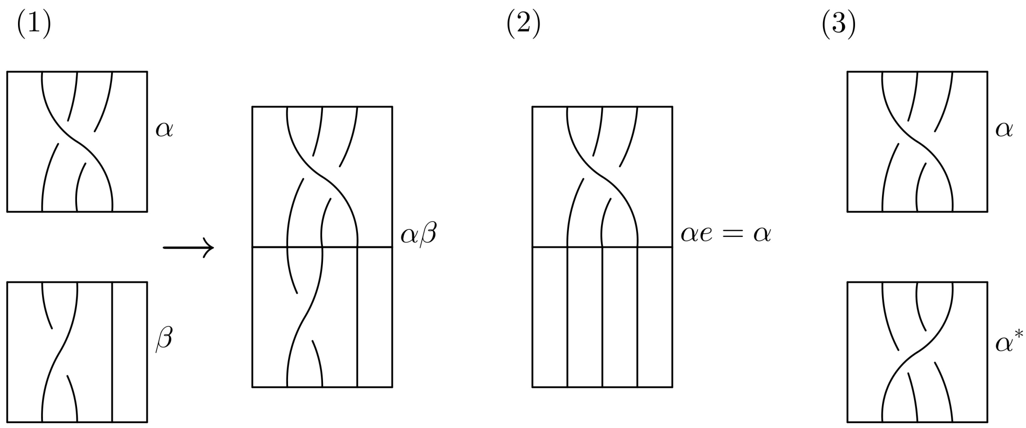

Figure 1.5: (1) Product $\alpha\beta$, (2) Unit element e,(3) Mirror image α^*

Now we want to exhibit a finite presentation of the group B_n. To this end, we have to introduce the generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$. We define the braid σ_i, with $i \in \{1, 2, \dots, n-1\}$, as the braid connecting the points A_i to A'_{i+1} and A_{i+1} to A'_i and then connecting each A_j to A'_j, for $j \neq i, i+1$, with a line segment. The points A_k and A'_k are the same of Definition 1.18. The general element σ_i is reported in Figure 1.6.
Thanks to the introduction of σ_i, we are ready to report the Artin's Theorem, which gives an explicit presentation of the braid group B_n.

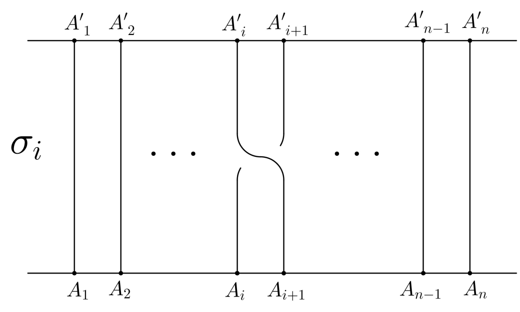

Figure 1.6: The generator σ_i

Theorem 1.20 [PS, Theorem 5.5] *The braid group B_n is isomorphic to the abstract group generated by the elements $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ which satisfy the following relations*

1. $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ *for* $i = 1, \dots, n-2$,

2. $\sigma_i\sigma_j = \sigma_j\sigma_i$ *whenever* $|i-j| \geq 2$.

Definition 1.21. Given a braid β, we can define the *closure* of β as the link $\hat{\beta}$ obtained by joining the upper points of its strands to the lower ones with arcs, respecting the order. For an example see Figure 1.7.

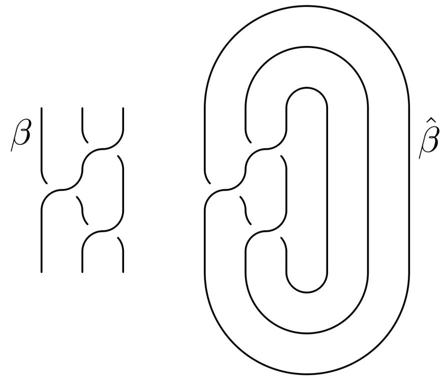

Figure 1.7: A braid and its closure

Theorem 1.22 [PS, 6.5] *Any link is the closure of some braid.*

Proof. We will give only a brief sketch of the proof, based on the so called *Alexander trick*. First of all, we consider a regular diagram for the link and we orient it. Thanks to Remark 1.4, we can suppose to represent L by a PL diagram.
We fix a point on the projection plane which does not lie on the lines determined by the segments of L (in particular it cannot lie on L). This point is called *braid axis*. The goal of the construction is to arrange for every segment of the polygon to run clockwise with respect to the chosen point. If some segment runs counter clockwise, after a suitable isotopy of the link, we can substitute it by two new segments which will both run clockwise. This procedure is illustrated in Figure1.8. By repeating the Alexander trick once for all segments running counter clockwise, we get a diagram of the link L whose segments run all clockwise. Finally, to get the desired braid, we cut along a line starting from the braid axis and transverse the segments of the diagram representing L, being careful to avoid the crossings. $\square$

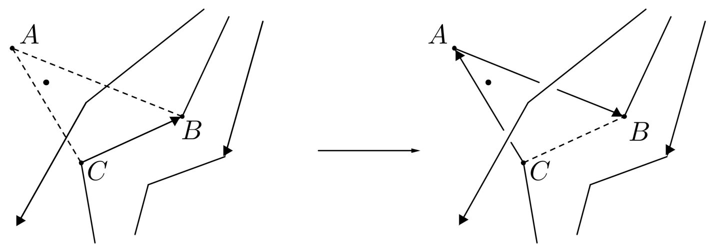

Figure 1.8: Alexander trick

1.2 Heeagaard splittings

Definition 1.23. A *handlebody* of genus g is the result of attaching g-disjoint 1-handles $D^2 \times [-1, 1]$ to the 3-ball D^3 by sewing $D^2 \times \{\pm 1\}$ to $2g$ disjoint disks on ∂D^3 in such a way that the resulting manifold is orientable with boundary. Two handlebodies of the same genus are homeomorphic and viceversa.
Let H_1 and H_2 be two handlebodies of the same genus g, and let $f : \partial H_1 \to \partial H_2$ be a homeomorphism of their boundaries. We can build a new 3-manifold by gluing the handlebodies using the map f, i.e.

$$M := H_1 \cup_f H_2.$$

The triple (H_1, H_2, f) is called *Heegaard splitting* of the manifold M. The genus of H_1 is called *genus* of the Heegaard splitting.

Definition 1.24. The *Heegaard genus* of a closed orientable 3-manifold M is the minimum over all the possible genus of its Heegaard splittings.

Example 1.25. The simplest Heegaard splitting of the 3-sphere is obtained by cutting along its equator, a 2-sphere. We obtain two copies of D^3 glued by the identity map along their boundaries.

Example 1.26. Another interesting example for the 3-sphere is the genus 1 Heegaard splitting. By regarding S^3 as the complex subspace

$$S^3 := \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 2\}$$

we can define

$$T_1 := \{(z, w) \in S^3 : |z| \leq |w|\} \quad T_2 := \{(z, w) \in S^3 : |z| \geq |w|\}.$$

It can be easily proved that both are homeomorphic to a solid torus (see Figure 1.9). Let us briefly sketch the proof. Each point of the 3-sphere can be written as $(ae^{i\alpha}, be^{i\beta})$ where $a, b \geq 0$ and $a^2 + b^2 = 2$. Since the conditions $|z| \leq |w|$ and $|z| \geq |w|$ are respectively equivalent to $|z| \leq 1$ and $|z| \geq 1$, the manifold T_1 is determined by $a \leq 1$. Hence, the assignment $(ae^{i\alpha}, be^{i\beta}) \to (a, \alpha, \beta)$ is a homeomorphism between T_1 and the solid torus $S^1 \times D^2$. An analogous reasoning is valid for T_2 considering the homeomorphism $(ae^{i\alpha}, be^{i\beta}) \to (b, \alpha, \beta)$.
Writing the points of the common boundary as $(e^{i\alpha}, e^{i\beta})$, the attaching map is given by

$$f : S^1 \times S^1 \to S^1 \times S^1 \quad f(e^{i\alpha}, e^{i\beta}) = (e^{i\beta}, e^{i\alpha}).$$

Theorem 1.27 [Sa, Theorem 1.1] *Any orientable 3-manifold has a Heegaard splitting.*

We will use this result to give a particular presentation of lens spaces $L(p, q)$ by their Heegaard splittings.
Our aim now is to give a different representation for a Heegaard splitting (H_1, H_2, f). If H_1 is a handlebody of genus g, let $D_1, \dots, D_g$ be the disks of H_1 corresponding

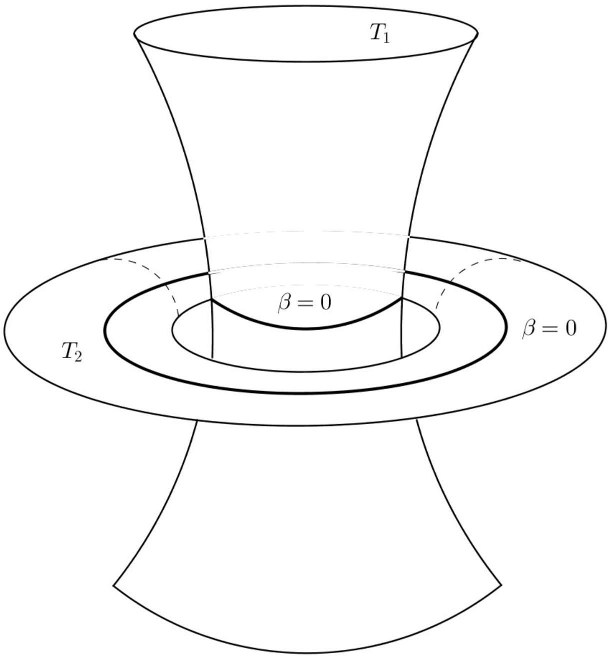

Figure 1.9: Heeagaard splitting of S^3 into solid tori

to the central disk $D_i \times \{0\}$ of each handle $D_i \times [-1, 1]$. Then each 1-handle in H_1 is a collar for D_i and $H_1 \setminus \{$*open collars of* $D_i\}$ is homeomorphic to the 3-ball. Let C_i be the boundary of D_i. Since H_2 is also a handlebody of genus g, we have in the same way a set of curves Γ_i, where each Γ_i is the boundary of the disk $\Delta_i \times \{0\}$, core disk of the handle $\Delta_i \times [-1, 1]$. Finally, we define $C_i' = f(C_i)$ as the image on ∂H_2 of C_i under the gluing map $f : \partial H_1 \to \partial H_2$ (see Figure 1.10).

Definition 1.28. The system of curves Γ_i and C_i' lying on ∂H_2 is called *system of characteristic curves* for the Heegaard splitting (H_1, H_2, f) or also *Heeagaard diagram* for the given splitting.

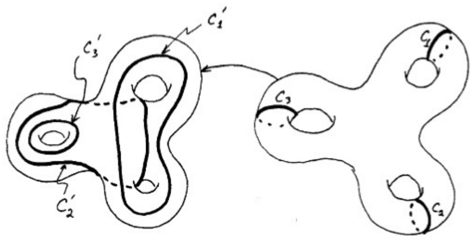

Figure 1.10: Images through the gluing map of the curve $C_1, \dots, C_g$

Theorem 1.29 [PS, Theorem 10.2] *The manifold $M = H_1 \cup_f H_2$ is completely determined by its system of characteristic curves. Moreover, if another splitting $N = H_1 \cup_h H_2$ of the same genus has characteristic curves $C_1'', \dots, C_g''$ which can be sent onto the system of curves $C_1', \dots, C_g'$ by a homeomorphism of H_2, then M and N are homeomorphic.*

1.3 Surgery and Kirby's Theorem

1.3.1 Surgery on links

Let K be a knot in a smooth closed orientable 3-dimensional manifold M. We observe that it is possible to thicken K to obtain its tubular neighborhood $\nu(K)$. By cutting along the boundary $\partial\nu(K)$, we get two different 3-manifolds: the first one is the *knot exterior*, i.e. the closure of $M - \nu(K)$, and the other one is the solid torus $\nu(K)$, which can be identified with the standard solid torus $S^1 \times D^2$. Now we can choose an arbitrary homeomorphism $h : \partial\nu(K) \to \partial(\overline{M \setminus \nu(K)})$ to sew the solid torus back. In this way, we will obtain a new manifold $\tilde{M} := \nu(K) \cup_h (\overline{M \setminus \nu(K)})$.

Definition 1.30. We say that the manifold $\tilde{M}$ is obtained from M by a *a surgery operation along* K.

The construction of the new manifold $\tilde{M}$ depends on the choice of the homeomorphism h. More precisely, $\tilde{M}$ is completely determined by the image of the meridian $\partial D^2 \times \{*\}$ of the solid torus $\nu(K)$, that is by the curve $\gamma = h(\partial D^2 \times \{*\})$. To see this, we observe that the attachment of $\nu(K)$ can be realized in two steps. First we attach $D^2 \times J$, where J is a small arc of S^1, then, to get the manifold $\tilde{M}$, we need to glue a 3-ball along its boundary S^2. But the homeomorphisms of S^2 which preserve the orientation are all isotopic to the identity.

Definition 1.31. If K is a knot embedded in S^3, the homology of the complement $V := \overline{S^3 \setminus \nu(K)}$ is given by $H_0(V) = H_1(V) \cong \mathbb{Z}$ and $H_i(V) \cong 0$, for $i \geq 2$. This allows us to define the concept of *meridian* and *parallel* relative to K.
Any generator of the group $H_1(V)$ is represented by a curve α lying on the torus ∂V. This curve is said to be a *meridian* of K. In the same way, by picking up a curve β lying on ∂V which is nullhomologous in V (see Definition 1.2) but not in ∂V, we get a curve called *longitude* (sometimes is also called *canonical longitude*). The pair $\{\alpha, \beta\}$ is a base for the first homology group $H_1(\partial V) \cong \mathbb{Z} \oplus \mathbb{Z}$.

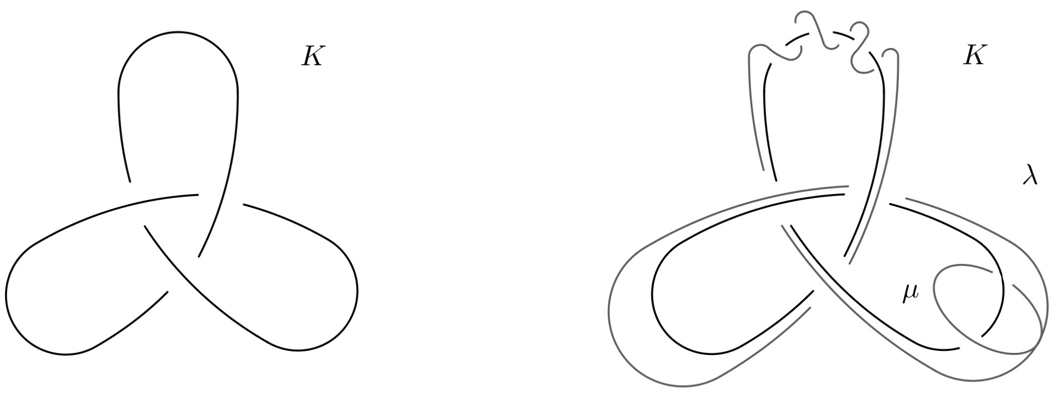

Figure 1.11: Meridian and parallel for the trefoil knot

Remark 1.32. To understand how K relates with its meridian and parallel, we analyze the associated linking numbers. It is quite clear that $\mathtt{lk}(K, \alpha) = +1$ and $\mathtt{lk}(K, \beta) = 0$.

If we now consider a curve γ on the boundary ∂V, this curve can always be written in the form $\gamma = p\alpha + q\beta$, with (p, q) pair of coprime integers, after a suitable isotopy. Moreover, the pairs (p, q) and $(-p, -q)$ refer to the same curve, since the orientation of γ is not influent. Thinking of a pair (p, q) as a reduced fraction p/q establishes a one-to-one correspondence between the isotopy classes of non trivial closed curves on ∂V except $\{\alpha\}$ and the set of rational numbers $\mathbb{Q}$. If we want to include α, we must extend $\mathbb{Q}$ with another element indicated with $1/0 = \infty$. We write $\bar{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$.

Definition 1.33. A surgery described by a pair (p, q) of coprime integers (o equivalently by a reduced fraction in $\bar{\mathbb{Q}}$) is called *rational*. The surgery is called *integral* if $q = \pm 1$.

Similarly, one can define rational and integral surgeries along a link $L \subset S^3$: the surgery coefficient along each component should be rational, respectively, integral. In general, surgery along a knot $K \subset M$ cannot be described by a rational number because there is no canonical choice of longitude if M is not a homology 3-sphere. Nevertheless, the concept of integral surgery still makes sense, it suffices to pick up a generic longitude.
A good way to represent surgeries along a knot, both rational and integral, is to draw the knot and write on the diagram the reduced fraction. We report in Figure 1.12 two examples:

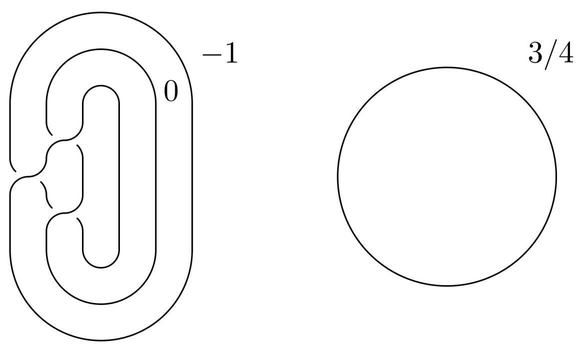

Figure 1.12: Examples of surgery diagrams

Example 1.34. If the surgery coefficient is ∞, the autohomeomorphism of the boundary torus sends the meridian to itself up to a sign. Thus, this surgery is nothing but the operation that puts the solid torus back. Therefore, the operation does not alter the 3-sphere S^3 and it is called *trivial surgery*.

Example 1.35. We notice that a 0-surgery on the unknot $U \subset S^3$ gives back $S^1 \times S^2$. The image of the meridian α through the gluing homeomorphism is the longitude β. Observing that the tubular neighborhood $\nu(U)$ and its complement are both homeomorphic to a solid torus $S^1 \times D^2$, the two solid tori are attached together along their boundaries by the identical homeomorphism. Since gluing two different copies of D^2 along S^1 returns the 2-sphere S^2, the manifold obtained by the surgery is $S^1 \times S^2$.

Theorem 1.36 [Sa, Theorem 2.1] *Every closed orientable 3-manifold can be obtained from S^3 by an integral surgery on a link $L \subset S^3$.*

1.3.2 Kirby Moves

From now on we will consider only integral surgeries on links embedded in the 3-sphere. First of all, we notice that an integral surgery along a link L is equivalent to the choice of an integer for each component. This choice is usually called a *framing* of L and L is called *framed link*.
From the previous theorem, a question could raise quite naturally: given two different framed links, how can we understand if the resulting 3-manifolds are homeomorphic?

Definition 1.37. The following operations are called *Kirby moves* and they does not change the 3-manifold represented by a framed link L

- **Move K1**:
  We add or delete an unknotted circle with framing ± 1 which does not link any other component of L (see Figure 1.13).

$L \mapsto L \sqcup$

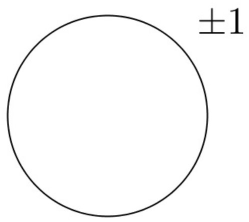

Figure 1.13: Move K1

- **Move K2**
  Let us take L_1 and L_2 two components of L with framing n_1 and n_2. Moreover, let L_2' be the curve which describes the surgery along L_2. Now, we substitute the pair $L_1 \cup L_2$ with $L_1 \cup L^{\#}$, where $L^{\#} = L_1 \#_b L_2'$ is the *sum along a band* b connecting L_1 to L_2' and disjoint from the other components, which remain unchanged after this substitution. The sum $L_1 \#_b L_2'$ is obtained as follows. We start considering a band $b = [-1, 1] \times [-\epsilon, \epsilon]$ with $\{-1\} \times [-\epsilon, \epsilon]$ lying on L_1 and $\{1\} \times [-\epsilon, \epsilon]$ lying on L_2'. Now, to get $L^{\#}$, we substitute $\{\pm 1\} \times [-\epsilon, \epsilon]$ with the sides of the band corresponding to $[-1, 1] \times \{-\epsilon, \epsilon\}$ (see Figure 1.14).

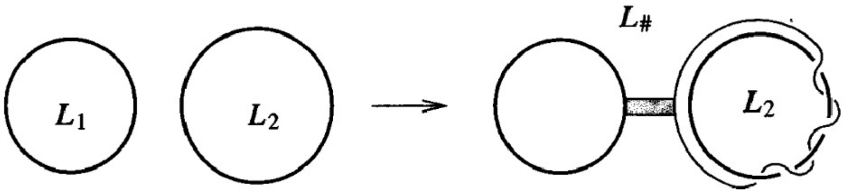

Figure 1.14: Move K2

All components but L_1 do not change their framing. The framing of the new component $L^{\#}$ is given by the formula

$$n^{\#} := n_1 + n_2 + 2\mathtt{lk}(L_1, L_2).$$

The components L_1 and L_2 have to be oriented in such a way to define a coherent orientation on $L^{\#}$. The orientation depends also on the choice of the band used in the gluing process.

Theorem 1.38 [Sa, Theorem 3.1] *The closed oriented manifolds obtained by surgery on two different framed links L and L' are homeomorphic by an orientation preserving homeomorphism if and only if the link L can be obtained from L' using a sequence of Kirby moves $K1$ and $K2$.*

Corollary 1.39 *An unknot with framing ± 1 can always be removed from a framed link L with the effect of giving to all arcs that intersect the disk bounded by the unknot a full left/right twist and changing their framings by adding ∓ 1 to each arc, assuming they represent different components of L.*

Proof. We have to repeat the application of move $K2$ to each framed component. Finally we use move $K1$ to remove the unknot framed with ± 1. $\square$

We report some applications of this corollary:

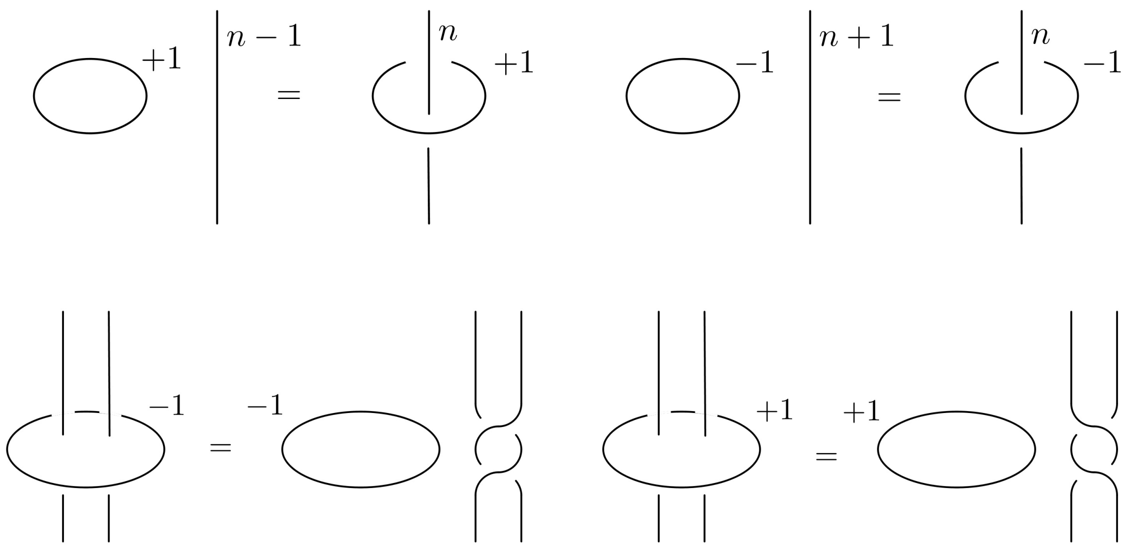

Figure 1.15: Consequences of Kirby's theorem

1.4 Lens Spaces

1.4.1 Different representations of lens spaces

In this section we will give two different presentations of lens spaces, which will play a fundamental role through this dissertation. Let us consider two different solid tori T_1 and T_2 and fix the homeomorphism $\phi_{p,q} : \partial T_1 \to \partial T_2$ which sends the meridian α to a curve isotopic to $q\alpha + p\beta$, where (p, q) is a pair of coprime integers, and α and β are the meridian and the parallel of the solid tori.

Definition 1.40. The manifold $T_1 \cup_{\phi_{p,q}} T_2$ obtained by gluing T_1 and T_2 using $\phi_{p,q}$ is the lens space $L(p, q)$.

This tells us that we have a Heegaard splitting of genus 1 for lens space, but a stronger result holds: the only 3-manifolds with Heegaard genus 1 are lens spaces. Another useful way to think of lens spaces is obtainable by applying the surgery theory: the space $L(p, q)$ is simply the result of a rational surgery along the unknot, as shown in Figure 1.16.

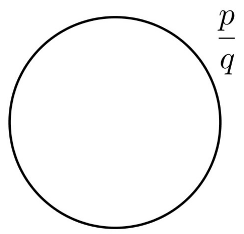

Figure 1.16: Rational p/q-surgery along the unknot

We will omit the proof of the equivalence between these two definitions, since it would not be interesting for our aims. A proof can be founded in [R].

Remark 1.41. By considering the spaces $L(p, q)$ as the result of a p/q-surgery along the unknot, it quite easy to understand that $L(p, q)$ and $L(-p, -q)$ are homeomorphic.

Actually we will not use this presentation, but we will prefer the following integral one.

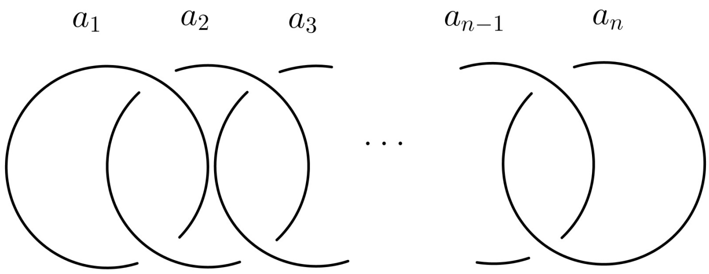

Figure 1.17: Integral surgery for $L(p, q)$

Proposition 1.42 [PS, Proposition 17.3] *Any lens space $L(p, q)$ has a surgery description given by Figure 1.17, where $p/q = [a_1, \dots, a_n]$ is a continued fraction decomposition,*

$$[a_1, \dots, a_n] = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\dots - \frac{1}{a_n}}}, \qquad a_i \in \mathbb{Z}$$

Example 1.43. It can be proved that

1. $L(1, q) = S^3$
2. $L(2, 1) = \mathbb{RP}^3$

Remark 1.44. In the case $p \neq 0$, the lens space $L(p, q)$ has another useful presentation. It can be thought as the quotient of S^3 under a properly discontinuous action of the group $\mathbb{Z}_p$. To see this, let us consider the 3-sphere as a subspace of $\mathbb{C}^2$, that is

$$S^3 := \{(z, w) \in \mathbb{C} : |z|^2 + |w|^2 = 1\}.$$

We define on it an action of the cyclic group $\mathbb{Z}_p$ as follows: let us fix a pair of coprime integer (p, q) satisfying $0 \leq q < p$ and put

$$\theta : \mathbb{Z}_p \times S^3 \to S^3, \qquad \bar{n} \cdot (z, w) := \theta(\bar{n}, (z, w)) := (e^{\frac{2\pi i n}{p}} z, e^{\frac{2\pi i n q}{p}} w).$$

This map reveals to be a free action of the group on the 3-sphere, indeed

1. $\bar{0} \cdot (z, w) = (e^0 z, e^0 w) = (z, w)$
2. $\bar{m} \cdot (\bar{n} \cdot (z, w)) = \bar{m} \cdot (e^{\frac{2\pi i n}{p}} z, e^{\frac{2\pi i n q}{p}} w) = (e^{\frac{2\pi i(n+m)}{p}} z, e^{\frac{2\pi i(n+m)q}{p}} w) = (\bar{n} + \bar{m})(z, w)$
3. $\bar{n}(z, w) = (z, w)$ implies $e^{\frac{2\pi i n}{p}} z = z$ thus $z = 0$ or $e^{\frac{2\pi i n}{p}} = 1$. In both cases we reduce to $\bar{n} = 0$.

Moreover, each element $\bar{n} \in \mathbb{Z}_p$ acts as a homeomorphism. Hence, we have an induced representation

$$\rho_\theta : \mathbb{Z}_p \to \mathrm{Homeo}(S^3).$$

Additionally, since the action is properly discontinuous, using the covering space theory we know that quotient space will be a closed 3-manifold. It can be shown that we can define the quotient space $S^3/\mathbb{Z}_p$ as the lens space $L(p, q)$ for $p \neq 0$. Moreover, from this particular presentation we easily argue that $L(p, q)$ and $L(p, q + np)$ are homeomorphic.

In order to formulate correctly the next statements, we have to underline the difference between the case $p = 0$ and the case $p \neq 0$. Indeed, we know that performing a 0-surgery along the unknot gives back us $L(0, 1)$, but thanks to Example 1.35, we recognize that $L(0, 1) \cong S^1 \times S^2$. This space is not considered a lens space, since it does not satisfy the same properties of $L(p, q)$ with $p \neq 0$.
Moreover, both Remark 1.41 and Remark 1.44 allow us to assume $0 \leq q < p$, without loss of generality. For these reasons, from now until the end of this section, we will consider only the case $0 \leq q < p$.

Proposition 1.45 *The computation of the homology groups for lens spaces returns*

1. $H_0(L(p, q)) \cong \mathbb{Z}$
2. $H_1(L(p, q)) \cong \mathbb{Z}_p$
3. $H_2(L(p, q)) \cong 0$

4. $H_3(L(p,q)) \cong \mathbb{Z}$

In particular, since they are closed, from 4. we deduce that lens spaces are all orientable.

Remark 1.46. We immediately see a difference between lens spaces and $S^1 \times S^2$. In fact, the latter has the second homology group given by $H_2(S^1 \times S^2) \cong \mathbb{Z}$, in contrast with Proposition 1.45.

Let us take two different pairs of coprime integers, say (p,q) and (p',q'), and consider the associated lens spaces. Obviously, if $p \neq p'$ they differ for the first homology group, so they can't be homeomorphic, not even of the same homotopy type. Fixing $p = p'$, is it possible to find any relation between the spaces $L(p,q)$ and $L(p,q')$?

Proposition 1.47 [Mu, Theorem 8.1.1] *$L(p,q)$ and $L(p,q')$ are:*

- *homeomorphic* $\Leftrightarrow \pm q' = q^{\pm 1} \mod p$,
- *homotopically equivalent* $\Leftrightarrow \pm qq' = m^2 \mod p$ *for a suitable* $m \in \mathbb{Z}_p$

Example 1.48. From the previous proposition, we easily understand that $L(5,2)$ and $L(5,3)$ are homeomorphic, whereas $L(5,1)$ and $L(5,2)$ are neither homeomorphic nor homotopically equivalent. On the other hand, the spaces $L(7,1)$ and $L(7,2)$ are not homeomorphic but they have the same homotopy type, since $1 \cdot 2 \cong 3^2 \mod 7$.

1.4.2 Representations of links in lens spaces

So far, we have only considered links in the 3-sphere, but for our purpose we need a way to represent links in a general 3-manifold. To do this, we will use Theorem 1.36 which allows us to represent a 3-manifold as a suitable integral surgery along a link (or a knot) in S^3.

Definition 1.49. Let M be a closed orientable 3-manifold which is obtained by an integral surgery along a framed link I in S^3. An oriented link L in M is represented as a *mixed link* in S^3 if it is given a diagram of the link $I \cup L$ in S^3. The link I is called *fixed part*, while the link L is called *moving part*.

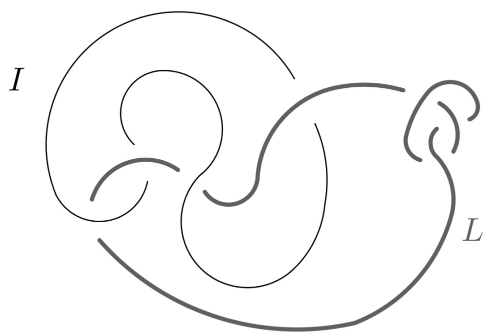

Figure 1.18: Example of mixed link

Remark 1.50. For our purpose we choose to represent the fixed part I as a closed braid by applying Theorem 1.8.

Observing that lens spaces are the result of a p/q-surgery along an unknotted circle in S^3, we can try to define an alternative representation obtained from the previous one by considering rational surgery coefficients. Therefore, a link L in a lens space is representable as a mixed link whose fixed part coincides with the unknot U and the surgery coefficient may be rational.
Recalling that $S^3 = \mathbb{R}^3 \cup \{\infty\}$, we can pick up a point p from U and suppose to identify it with ∞. By considering on $\mathbb{R}^3$ the standard coordinates system given by triple (x, y, z), we can assume that U is described by the z-axis and then we project $U \cup L$ to the xy-plane by a regular projection (see Figure 1.19).

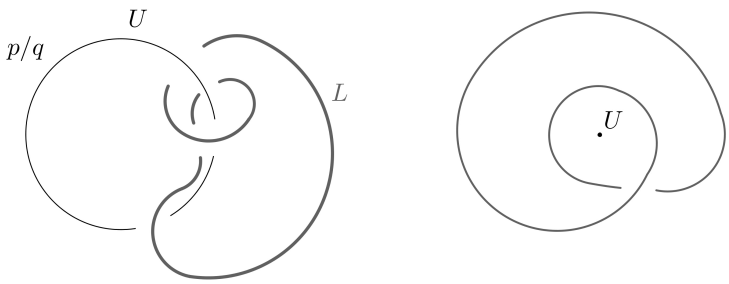

Figure 1.19: Link in $L(p, q)$: mixed link and punctured disk diagram

Definition 1.51. A *punctured disk diagram* of a link L in $L(p, q)$ is a regular projection of $L \cup U$, where U is described by a single dot on the plane.

Supposing that the diagram is all contained inside a disk, we remove a neighborhood of the dot representing U in such a way that the diagram of L lies in an annulus. Having done this, we cut along a line orthogonal to the boundary of the annulus, being careful to avoid the crossings of L. Finally, we deform the annulus to get a rectangle (Figure 1.20).

Definition 1.52. The described procedure gives back a presentation of a link in $L(p, q)$ called *band diagram*.

From a band diagram, we can easily go back to a punctured disk diagram by reversing the previous operations.

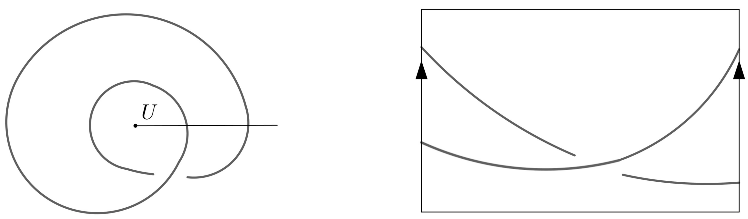

Figure 1.20: Link in $L(p, q)$: punctured disk diagram and band diagram

Remark 1.53. Even if a mixed link with rational surgery is easier to handle since we can obtain a punctured disk diagram and a band diagram from it, we choose to operate with a framed link as fixed part (see Figure 1.21). The reasons will be explained later.

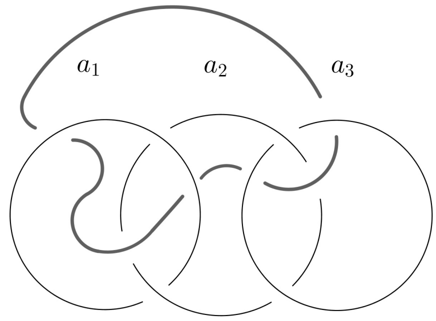

Figure 1.21: Link in $L(p, q)$: mixed link with integral surgery

Chapter 2

Fibered knots and links in S^3

In this chapter we focus on the main concept of this dissertation: the definition of fibered knot. After a short introduction where we give the definition of fibered knot in S^3, we report some examples and we illustrate some interesting results. Finally, we show the procedure that allows us to construct any fibered knot in S^3 starting from the unknot.

2.1 Definition of fibered knots and links

Definition 2.1. A map $p : E \to B$ is said to be a *locally trivial bundle* (or *fibration*) with *fiber* F if each point of B has a open neighborhood U and a homeomorphism $\varphi : p^{-1}(U) \to U \times F$ such that the following diagram commutes

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "p"] & U \times F \arrow[dl, "\pi_1"] \\ U & \end{tikzcd}$$

where π_1 is the projection onto the first factor. The map φ is called *trivializing map*. The spaces E and B are known as the *total* space and the *base* space, respectively. Each set $p^{-1}(b), b \in B$ is a fiber homeomorphic to F.
If there exists a homeomorphism ϕ between E and the product $B \times F$ commuting with the bundle projection p, we will say that the bundle is *trivial*.

Definition 2.2. A knot K in S^3 is a *fibered knot* if the two following conditions hold:

- the complement of the knot is the total space of a locally trivial bundle over the base space S^1, i.e. there exists a map $p : S^3 \setminus K \to S^1$ which is a locally trivial bundle.

- there exists a trivializing homeomorphism $\theta : \nu(K) \to S^1 \times D^2$ such that the following diagram commutes

$$\begin{tikzcd} \nu(K) \setminus K \arrow[r, "\theta"] \arrow[d, "p"] & S^1 \times (D^2 \setminus \{0\}) \arrow[dl, "\pi"] \\ S^1 & \end{tikzcd}$$

where $\pi(x, y) := \frac{y}{|y|}$.

Example 2.3. Let S^3 be the unit sphere in $\mathbb{C}^2$, and $(z_1, z_2) = (\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2})$ be two coordinate systems. If we consider the unknot $U = \{z_1 = 0\} = \{\rho_1 = 0\}$, the complement fibers:

$$\pi_U : S^3 \setminus U \to S^1, \ (z_1, z_2) \mapsto \frac{z_1}{|z_1|}.$$

In polar coordinates this fibration is just $\pi_U(\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2}) = \theta_1$.
If we represent S^3 as spanned by the rotation of the 2-sphere $\mathbb{R}^2 \cup \{\infty\}$ around the circle $l \cup \{\infty\}$, the point P generates the unknot U and each arc connecting P and P' generates a open 2-dimensional disk (see Figure 2.1). These disks exhaust the knot complement, are disjoint and parametrized by the circle $l \cup \{\infty\}$. Therefore, $S^3 \setminus U \cong S^1 \times \mathrm{int}(D^2)$.

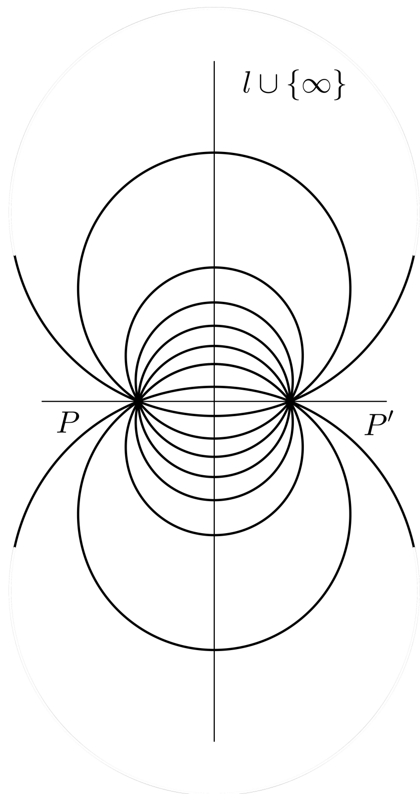

Figure 2.1: Fibering the unknot

Example 2.4. Using the previous notation, let $H^+ = \{(z_1, z_2) \in S^3 : z_1 z_2 = 0\}$ and $H^- = \{(z_1, z_2) \in S^3 : z_1 \bar{z}_2 = 0\}$. We will call H^+ *positive Hopf band* and H^- *negative Hopf band*, respectively. These bands are represented in Figure 2.2.

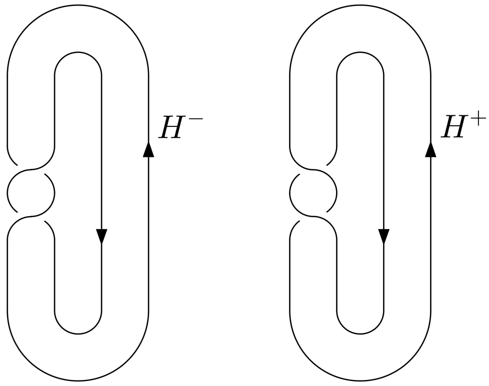

Figure 2.2: Diagrams of H^+ and H^-

The following maps give the desired fibrations

$$\pi^+ : S^3 \setminus H^+ \to S^1, \ (z_1, z_2) \mapsto \frac{z_1 z_2}{|z_1 z_2|}$$
$$\pi^- : S^3 \setminus H^- \to S^1, \ (z_1, z_2) \mapsto \frac{z_1 \bar{z}_2}{|z_1 \bar{z}_2|}.$$

In polar coordinates we have $\pi^{\pm}(\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2}) = \theta_1 \pm \theta_2$.

Example 2.5. Let us take $p : \mathbb{C}^2 \to \mathbb{C}$ a polynomial map that vanishes in $(0,0)$ and has no critical points inside S^3, except the origin. Then $B = p^{-1}(0) \cap S^3$ gives a fibered knot in S^3 with fibration

$$\pi_p : S^3 \setminus B \to S^1, \ (z_1, z_2) \mapsto \frac{p(z_1, z_2)}{|p(z_1, z_2)|}.$$

Every locally trivial bundle $p : E \to B$ has the homotopy lifting property to any CW-complex X. That is, for every two maps $f : X \to E$ and $G : X \times [0,1] \to B$ for which $pf = Gi$ (where $i : X \to X \times [0,1]$ is the inclusion $x \mapsto (x, 0)$), there exists a continuous map $\tilde{G} : X \times [0,1] \to E$ making the diagram below commutative:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "G"] \arrow[ur, "\tilde{G}"] & B \end{tikzcd}$$

Let $p : E \to S^1$ a locally trivial bundle and let $\gamma : [0, 2\pi] \to S^1$ be a path in S^1. By homotopy lifting property of p applied to the space $X = p^{-1}(\gamma(0)) = F_{\gamma(0)}$ there exists a map

$$\tilde{G} : F_{\gamma(0)} \times [0, 2\pi] \to E$$

making the diagram above commute. We choose the map $\tilde{G}$ so that the maps

$$h_t : F_{\gamma(0)} \to F_{\gamma(t)}$$

(where $F_{\gamma(t)} = p^{-1}(\gamma(t))$) given by the formula $h_t(x) = \tilde{G}(x,t)$ with $t \in [0, 2\pi]$, are homeomorphisms. It possible to check that each map h_t only changes within its homotopy class if one chooses a different path homotopic to γ relative to its endpoints, or a different map $\tilde{G}$. Thus we have a well-defined set of isomorphisms

$$(h_t)_* : H_1(F_{\gamma(0)}) \to H_1(F_{\gamma(t)}).$$

If we think of S^1 as the unit circle in the complex plane and we consider the path defined by $\gamma(t) = e^{it}, t \in [0, 2\pi]$, then the bundle E with fiber F can be described as

$$E = \frac{F \times [0, 2\pi]}{(x, 0) \sim (h(x), 2\pi)}$$

where $h = h_{2\pi}$ and $\sim$ indicates that we are identifying the corresponding points. This follows from the fact that S^1 is homeomorphic to $[0, 2\pi]$ with endpoints identified and the interval $[0, 2\pi]$ is contractible, so every bundle with fiber F over $[0, 2\pi]$ is trivial.

Definition 2.6. The homeomorphism $h = h_{2\pi} : F \to F$ is called *monodromy* and the induced map $h_* : H_1(F) \to H_1(F)$ is called *monodromy transformation.*

Definition 2.7. The quotient space

$$T_h := \frac{F \times [0, 2\pi]}{(x, 0) \sim (h(x), 2\pi)}$$

is called *mapping torus* relative to the map h. This notion allows us to say that a bundle E over S^1 can be thought of as the mapping torus of the monodromy map h.

Example 2.8. If we define the complex annulus $A := \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$, the *positive Dehn twist along the core circle* $\gamma := \{|z| = 3/2\}$ is given by

$$D_\gamma : A \to A, \quad D_\gamma(\rho e^{i\theta}) = \rho e^{i(\theta + 2\pi(\rho - 1))}.$$

Since this map is actually a homeomorphism, we can invert it and its inverse is called *negative Dehn twist along the core circle* γ. The image of a positive Dehn twist is reported in Figure 2.3.
In the case of the Hopf bands, the monodromy map consists in a Dehn twist along the core circle of the annulus, positive or negative respectively. On the other hand, the monodromy of the unknot U is simply the identity map on the disk D^2.

Proposition 2.9 [Sa, Lemma 8.1] *Let K be a fibered knot with fiber F. If we fix a basis in $H_1(F)$, let M be the matrix representing the monodromy transformation $h_* : H_1(F) \to H_1(F)$ respect to this basis. Finally let $S : H_1(F) \otimes H_1(F) \to \mathbb{Z}$ be the Seifert matrix of K. Then $M = S^{-1}S^t$.*

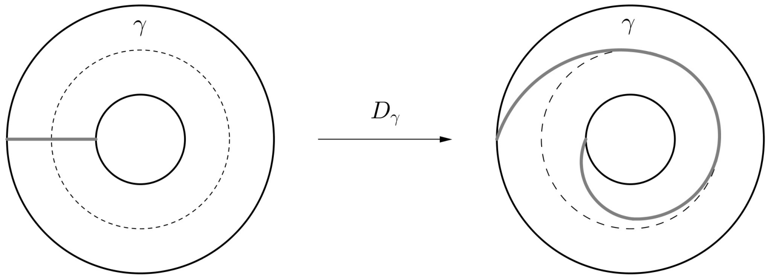

Figure 2.3: Dehn twist about the core circle

Definition 2.10. Let $p(t)$ be a Laurent polynomial in the variable t with integer coefficients, that is $p \in \mathbb{Z}[t^{-1}, t]$. We say the p is *symmetric* if it holds

$$p(t) = p(t^{-1}).$$

Given a polynomial p which is not symmetric, we can define a new polynomial $p_\sigma(t) := \pm t^\sigma p(t)$. If $p_\sigma(t)$ satisfies $p_\sigma(t) = p_\sigma(t^{-1})$ for the choice of a suitable exponent σ, we call p_σ the *symmetrized polynomial* of p.

Remark 2.11. Thanks to Definition 1.17 we immediately see that the Alexander polynomial is symmetric, that is $\Delta_K(t) = \Delta_K(t^{-1})$.

Corollary 2.12 *The Alexander polynomial of a fibered knot K equals the symmetrized characteristic polynomial of its monodromy transformation h_*.*

Proof. We have

$$\begin{aligned}\Delta_K(t) &= \det S \det(t^{1/2} I - t^{-1/2} S^{-1} S^t) = \\ &= \det S \det[(t^{-1/2} I)(tI - S^{-1} S^t)] = \\ &= \det S \det[(t^{-1/2} I)(tI - h_*)].\end{aligned}$$

Therefore, after a suitable multiplication by $\pm t^\sigma$, the characteristic polynomial of h_* is equal to the Alexander polynomial of K. Since the Alexander polynomial is symmetric by Remark 2.11, we get the desired result. □

Corollary 2.13 *If K is a fibered knot with fiber F, we have that genus(K) = genus(F), i.e. the closure $\bar{F}$ is a Seifert surface of K of minimal genus.*

Proof. On one hand, we have certainly that genus$(K) \leq$ genus(F) since $\partial F = K$. On the other hand, the degree of the Alexander polynomial of K equals the highest degree of t in the symmetrized characteristic polynomial of h_*, which in turn equals the genus of F. Since the genus of K coincides with the degree of its Alexander polynomial, we have concluded. □

Corollary 2.14 *If a knot K is fibered then its Alexander polynomial is monic.*

Proof. The top degree coefficient of $\Delta_K(t)$ equals $\pm\det h_*$. Since h_* is invertible over the integers, $\det h_* = \pm 1$. $\square$

Given a fibered knot K, we will give a procedure to determine a Heegaard splitting of S^3. From now on we think of S^1 as the interval $[0, 2\pi]$ with ends identified. Let M_1 be the closure of the preimage $p^{-1}([0,\pi]) \cong F \times [0,\pi]$. In the same way, let M_2 be the closure of $p^{-1}([\pi, 2\pi]) \cong F \times [\pi, 2\pi]$.
Both M_1 and M_2 are handlebodies of genus equal to $2\cdot\text{genus}(F)$, and their common bounday is $F' = \partial M_1 = \partial M_2 = \bar{F} \cup \bar{F}$, the union of two copies of the closure $\bar{F}$ along the knot K. The attaching map $\phi : \partial M_1 \to \partial M_2$ is an extension of the closure of the map given by

$$\begin{aligned}(x,\pi) &\mapsto (x,\pi), x \in F\\ (x,0) &\mapsto (h(x), 2\pi), x \in F,\end{aligned}$$

where h is the monodromy homeomorphism.

Example 2.15. If we consider the unknot U, the previous method gives back the Heegaard splitting of S^3 of genus 0.

2.2 Construction of fibered knots

In the previous section we have shown some examples of fibered knots in S^3, in particular the unknot. In this section we will describe two moves, plumbing and twisting, which enable us to construct all the possible fibered knots and links in S^3 starting from the unknot. From now on we will indicate a fibered link L as a pair (F, L), where F is the fiber relative to L.
Let us now introduce two constructions for fibered links:

(A) **Plumbing**

Suppose (F, L) is a fibered pair in S^3 and let α be an arc included in F with endpoints on ∂F. We can find a 3-ball $D_\alpha \subset S^3$ such that

1. $D_\alpha \cap F \subset \partial D_\alpha$,
2. there is an embedding $\varphi_\alpha : [-1,1]\times[-1,1] \to F$ with $\varphi_\alpha([-1,1]\times\{0\}) = \alpha$ and $\text{Im}(\varphi_\alpha) = D_\alpha \cap F$.

In another copy of S^3 we take a Hopf pair (F_0, H), either positive or negative, with an arc $\beta \subset F_0$ connecting the two boundary components. Again we find a 3-ball D_β with the same properties as above and its corresponding map φ_β.

Identify ∂D_α with ∂D_β by an orientation reversing map f satisfying $f(\varphi_\alpha(x,y)) = \varphi_\beta(y,x)$. Then form

$$(S^3 - \text{int}(D_\alpha)) \cup_f (S^3 - \text{int}(D_\beta))$$

and

$$F \cup_{f|\text{Im}(\varphi_\alpha)} F_0.$$

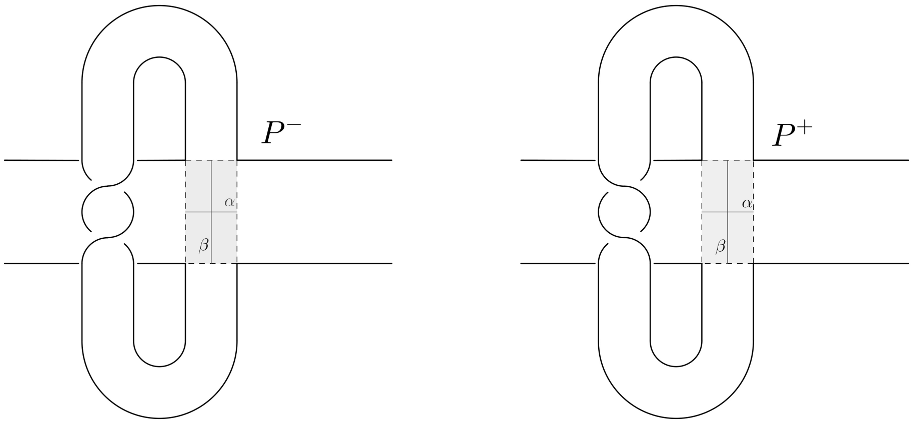

Figure 2.4: Plumbing

The result is a new surface $F' \subset S^3$ with boundary L'. In [St] it is proved that the pair (F', L') is fibered. The monodromy of this new pair may be chosen as $h \circ D_\gamma^\epsilon$, where h is the monodromy for L on F and the identity on $F' - F$, γ is the core circle of F_0 and $\epsilon = \pm 1$.

Remark 2.16. If L has only one component, L' will have two. However, we can plumb another Hopf pair of the same sign if we desire to obtain a knot L''. This operation will be called (A_1).

Remark 2.17. The described operation is local, so it can be easily extended to a general 3-manifold M. The same does not hold for the following twisting operation.

(B) **Twisting**
The twisting operation is essentially a ± 1 surgery on a simple closed non-trivial curve lying on the fiber.
Consider a fibered pair (F, L) and suppose γ is an embedded circle which is unknotted in S^3. Let δ_1, δ_2 be ± 1 and suppose $\mathtt{lk}(\gamma, \gamma^+) + \delta_1 = \delta_2$ where γ^+ is the pushoff of γ respect to a chosen orientation.
We perform a surgery along γ with coefficient δ_2. The resulting manifold is again S^3 since γ is unknotted. The fibered pair is changed into a new pair (F', L') with monodromy $h' = h \circ D_\gamma^{\delta_1}$.

We are ready to report this result of Harer.

Theorem 2.18 [H1, Theorem 1] *Let (F, L) a fibered pair in S^3. Then there exist two fibered pair (F_1, L_1) and (F_2, L_2) such that*

1. *(F_2, L_2) is constructed from the unknot by using operations of type (A),*
2. *(F_1, L_1) is obtained from (F, L) using (A),*
3. *(F_1, L_1) may be changed into (F_2, L_2) using (B).*

Even if we omit the proof of this theorem (for details see [H1]), we want to underline that a crucial point in the proof is the compatibily of the operation (A) and (B) with Kirby moves.
An equivalent formulation of this theorem states that it is possible to construct any fibered knot or link in S^3 starting from the unknot and applying a sequence of operations (A_1) and (B).

Chapter 3

Fibered knots and links in $L(p, q)$

At the beginning of this dissertation we have defined links in a general 3-manifold M as a smooth embedding of disjoint copies of S^1, the components of the link. From now on we fix the case $M = L(p, q)$, with (p, q) pair of coprime integer, in order to determine some fibered links in this particular space. Since the plumbing operation will be still valid for a general 3-manifold, if we find a fibered link L, we will be able to construct a fibered knot from it by the (A_1) operation previously described in Section 2.2.

3.1 Open Book Decompositions

In Chapter 2 we have introduced the notion of fibered link in S^3. By substituting S^3 with a 3-manifold M in Definition 2.2, we can easily generalize the concept of fibered link to every closed orientable 3-manifold.
Our aim now is to go further and give the definition of open book decomposition, which reveals to be completely equivalent to the definition of fibered link.

Definition 3.1. An *open book decomposition* of a closed oriented 3-manifold M is a pair (L, π) where

1. L is an oriented link in M called *binding* of the open book,

2. $\pi : M \setminus L \to S^1$ is a locally trivial bundle such that $\pi^{-1}(\theta)$ is the interior of a surface Σ_θ and $\partial\Sigma_\theta = L$ for all $\theta \in S^1$. The surface $\Sigma = \Sigma_\theta$ is called the *page* of the open book.

Remark 3.2. The binding L of an open book decomposition is a fibered link for M. Viceversa, given a fibered link in M, we have automatically an open book decomposition of M. So, the two concepts are equivalent. Thanks to this equivalence, we think of $M \setminus L$ as the mapping torus

$$T_h = \frac{\Sigma \times [0, 2\pi]}{(x, 0) \sim (h(x), 2\pi)},$$

where $h : \Sigma \to \Sigma$ is the monodromy map of Definition 2.6.

Definition 3.3. An *abstract open book decomposition* is a pair (Σ, h) where

1. Σ is an oriented compact surface with boundary,

2. $\phi : \Sigma \to \Sigma$ is a diffeomorphism such that ϕ is the identity in a neighborhood of $\partial\Sigma$.

First of all we observe that given an abstract open book (Σ, h) we get a 3-manifold M_h as follows:

$$M_h := T_h \cup_\phi (\bigsqcup_{|\partial\Sigma|} S^1 \times D^2)$$

where $|\partial\Sigma|$ denotes the number of boundary components of Σ and T_h is the mapping torus of h. Finally, $\cup_\phi$ means that the diffeomorphism ϕ is used to identify the boundaries of the two manifolds. For each boundary component γ of Σ the map $\phi : \partial(S^1 \times D^2) \to \gamma \times S^1$ is defined to be the unique diffeomorphism that takes $S^1 \times \{p\}$ to γ where $p \in \partial D^2$ and $\{q\} \times \partial D^2$ to $(\{q'\} \times [0, 2\pi]/\sim) \cong S^1$, where $q \in S^1$ and $q' \in \partial\Sigma$. We denote by B_h the cores of the solid tori $S^1 \times D^2$ in the definition of M_h.

Definition 3.4. Two abstract open book decomposition (Σ_1, ϕ_1) and (Σ_2, ϕ_2) are said to be *equivalent* if there is a diffeomorphism $F : \Sigma_1 \to \Sigma_2$ such that $F \circ \phi_2 = \phi_1 \circ F$.

To indicate an open book decomposition of a manifold M we will refer to the abstract open book decomposition (Σ, h), where Σ is the page and h is the monodromy, instead of using (L, π). The relation between the two pairs is given by the following lemma.

Lemma 3.5 [E, Lemma 2.4] *We have the following basic facts about abstract and non-abstract open book decompositions:*

1. *An open book decomposition (L, π) of M gives an abstract open book (Σ_π, h_π) such that (M_{h_π}, B_{h_π}) is diffeomorphic to (M, B).*

2. *An abstract open book determines M_h and an open book (B_h, π_h) up to diffeomorphism.*

3. *Equivalent open books give diffeomorphic 3-manifolds.*

Remark 3.6. The basic difference between abstract and not-abstract open book decompositions is that when discussing not-abstract decompositions we can refer to the binding and to pages up to isotopy in M, whereas when discussing abstract decompositions we discuss them up to diffeomorphism.

Remark 3.7. Following the same procedure described in Chapter 2, we can make a Heegaard splitting for a 3-manifold starting from a given open book decomposition. If $\pi : M \setminus L \to S^1$ is the locally trivial bundle associated to the decomposition, we define M_1 as the closure of $p^{-1}([0, \pi])$ and M_2 as the closure of $p^{-1}([\pi, 2\pi])$, respectively. The attaching map $\phi : \partial M_1 \to \partial M_2$ is defined by extending the map

$$(x, \pi) \mapsto (x, \pi), x \in \Sigma$$
$$(x, 0) \mapsto (h(x), 2\pi), x \in \Sigma.$$

Definition 3.8. Given two abstract open book decompositions (Σ_i, h_i) with $i = 0, 1$, let c_i be an arc properly embedded in Σ_i and R_i a rectangular neighborhood of c_i, i.e. $R_i = c_i \times [-1, 1]$. The *Murasugi sum* of (Σ_0, h_0) and (Σ_1, h_1) is the abstract open book $(\Sigma_0, h_0) * (\Sigma_1, h_1)$ with page

$$\Sigma_0 * \Sigma_1 := \Sigma_0 \cup_{R_0 = R_1} \Sigma_1,$$

where R_0 and R_1 are identified in such a way that $c_i \times \{-1, 1\} = (\partial c_{i+1}) \times [-1, 1]$, and the monodromy is $h_0 \circ h_1$.

Proposition 3.9 [E, Theorem 2.17] *The following diffeomorphism holds*

$$M_{(\Sigma_0,h_0)} \# M_{(\Sigma_1,h_1)} \cong M_{(\Sigma_0,h_0)*(\Sigma_1,h_1)}$$

where # indicates the connected sum.

Definition 3.10. A *positive* (respectively *negative*) *stabilization* of an abstract open book decomposition (Σ, h) is the open book decomposition with

1. page $S\Sigma := \Sigma \cup B$, where $B = [-1, 1] \times [0, 1]$ is a 1-handle whose sides corresponding to $\{\pm 1\} \times [0, 1]$ are attached to the boundary of Σ , and
2. monodromy $Sh := h \circ D_c^{\pm}$, where D_c is a positive (respectively negative) Dehn twist along a curve c which lies in $S\Sigma$ and intersects $\{0\} \times [0, 1]$ in B exactly once.

We denote this stabilization $S_{\pm}(\Sigma, h) = (S\Sigma, Sh)$, where $\pm$ refers to the positivity or negativity of the stabilization.

Proposition 3.11 [E, Corollary 2.21] *If we consider the open book decomposition associated to the Example 2.4 and we indicate it with $(H^{\pm}, D_{\gamma}^{\pm})$, we have*

$$S_{\pm}(\Sigma, h) = (\Sigma, h) * (H^{\pm}, D_{\gamma}^{\pm}).$$

In particular, by Proposition 3.9, it results $M_{S_{\pm}(\Sigma,h)} = M_{(\Sigma,h)}$.

Remark 3.12. Thanks to the correspondence exposed in Remark 3.2 and by the application of Proposition 3.11, we can easily understand that a positive (respectively negative) stabilization of an open book decomposition corresponds to a positive (respectively negative) plumbing operation of the associated binding.

Lemma 3.13 [O, Lemma 2.3.11] *Let (Σ, h) be an open book decomposition for a 3-manifold M.*

1) If K is an unknotted circle in M intersecting each page Σ transversely once, then the result of a 0-surgery along K is a new 3-manifold M_0 with an open book decomposition having page $\Sigma' = \Sigma - \{open\ disk\}$ (the disk is simply a neighborhood of the point of intersection of Σ with K) and monodromy $h' = h|_{\Sigma'}$.

2) If K is an unknotted circle sitting on the page Σ, then a ± 1-surgery along K gives back a new 3-manifold M_1 with open book decomposition having page Σ with monodromy $h' = h \circ D_K^{\mp}$, where D_K is the Dehn twist along K.

Proof. We will give a sketch of the proof only for 1).
Let us consider the standard open book decomposition of S^3 given by (D^2, Id) that has binding $\partial D^2 = U$ and call U' the 0-framed knot. Since U' has to be transverse to each page in order to satisfy the hypothesis, we can think of it as a line orthogonal to the disk D^2 to which we add a point at infinity (see Figure 3.1(1)).

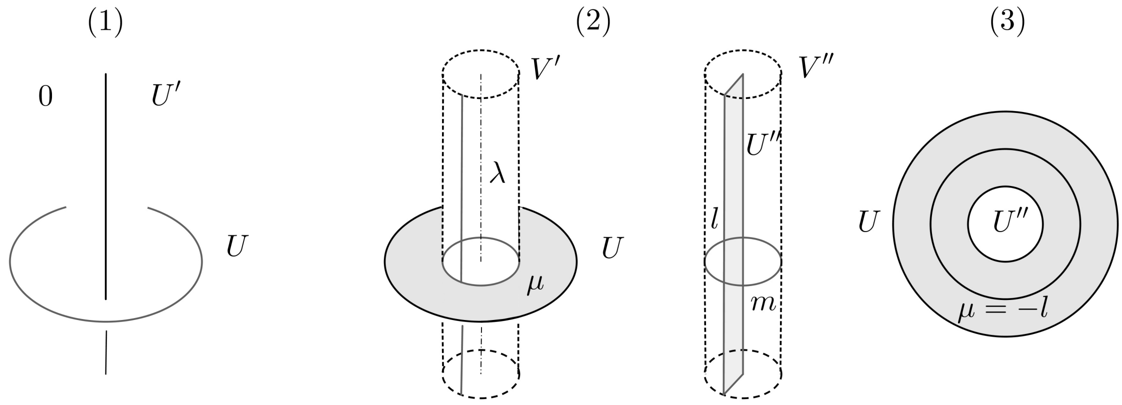

Figure 3.1: 0-surgery on D^2 in S^3

To perform a 0-surgery along U' we take a thickened neighborhood V' of U' on which lie the meridian μ and the longitude λ relative to U', we remove V' and we glue it back by sending $\mu \mapsto \lambda$. This procedure is completely equivalent to take a new solid torus V'' with meridian m and longitude l and gluing it in S^3 instead of V' using a homeomorphism of the boundaries which sends $m \mapsto \lambda$ and $l \mapsto -\mu$. The open book decomposition of the resulting manifold, $S^1 \times S^2$, will have the annulus bounded by U and U'' as page (see Figure 3.1(3)) and the identity map as monodromy. Thus 1) is proved in the particular case of a 0-surgery transverse to (D^2, Id) in S^3.
Since the previous procedure can be easily extended to the more general case with page Σ for a 3-manifold M, we have concluded. $\square$

Remark 3.14. We will try to visualize how the open book decomposition (D^2, Id) of S^3 is modified by the 0-surgery along U'. Let us recall the model presented in Example 2.3 thanks to which we think of S^3 as generated by the rotation of a 2-sphere around the circle $l \cup \{\infty\}$. Suppose to identify U' with $l \cup \{\infty\}$.
By removing a neighborhood V' of U', we puncture each page of the decomposition (see Figure 3.2(2)). The surface obtained from the page is an annulus with two boundary components, the one generated by the point P and the other one generated by the point R for example, as we can see in Figure 3.2.
By performing the 0-surgery, we attach another annulus along the boundary of each puncture. One of these annuli is generated by the arc RQ, represented in Figure 3.2(3).

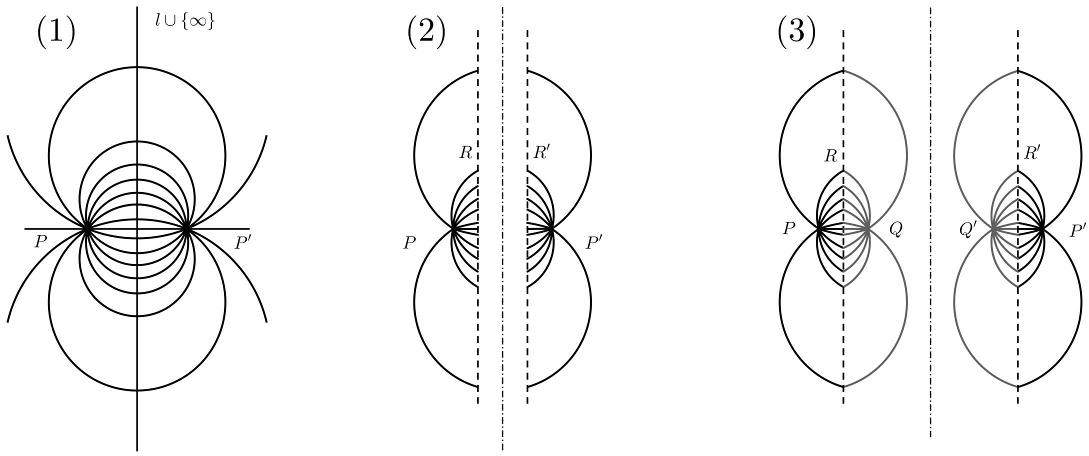

Figure 3.2: Construction of an open book decomposition for $S^1 \times S^2$

From the previous reasoning we deduce that $S^1 \times S^2$ is represented by a rotating 2-sphere. This sphere is thought of as a disk whose boundary points are identified by a reflection along a fixed axis. The identification is represented in Figure 3.3(1). In the same way the page of the decomposition is spanned by the rotation of the segment PQ around $l \cup \{\infty\}$ (see Figure 3.2(3)).
Finally, we report an alternative representation of the decomposition of $S^1 \times S^2$ given by a mixed link diagram in Figure 3.3(2).

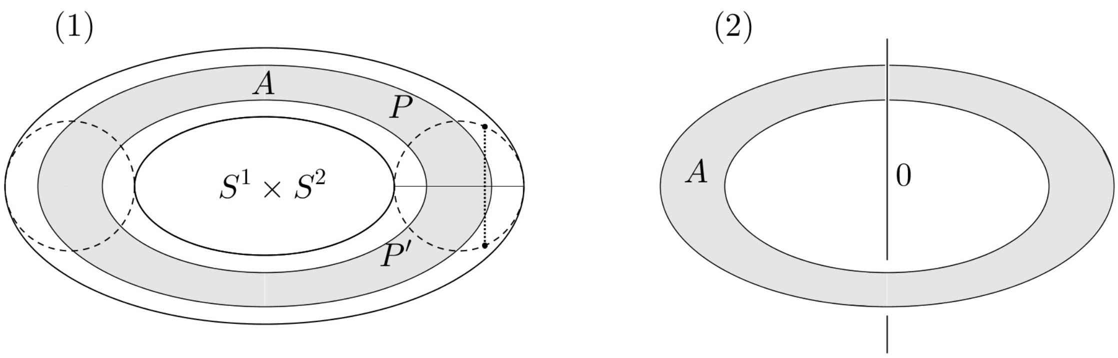

Figure 3.3: Representations of the open book decomposition for two different model of $S^1 \times S^2$

In particular part 1) of Lemma 3.13 tells us even more: it states not only that the page of the open book decomposition of M will be punctured in correspondence of each intersection point with the surgery link, but it suggests also how the page of the new manifold M_0 lies respect to the surgery link, i.e. it gives back a representation of the binding L (boundary of the annulus A in Figure 3.3(2)) as the moving part of a mixed link whose fixed part is the 0-framed knot U'.

Theorem 3.15 [OS, Theorem 9.9.1.3] *Every closed orientable 3-manifold M admits an open book decomposition.*

Proof. We can assume that M is obtained by a surgery along a link $L_M \subset S^3$ with ± 1-framing for each component (this is an easy application of Kirby moves to Theorem 1.36). Moreover, using Theorem 1.22, it is possible to represent L_M as the closure of a certain braid β.
Consider now an unknotted circle U that links to each component of L_M exactly once. We will construct an open book decomposition for M using the open book decomposition (D^2, Id) of S^3 with binding U. First of all, we remove each linking between the components of L_M, thanks to the repeated application of Kirby moves, by introducing unknots framed with ± 1 which sits on the disk bounded by U. Now, we keep using Kirby moves until each component of L_M is 0-framed and intersects transversely the disk exactly once. Finally, we apply Lemma 3.13 to obtain the desired open book for M. $\square$

Example 3.16. Let us consider the projective space $L(2,1)$ given by a surgery along the Hopf link, as illustrated in Figure 3.4(1).
We start by representing the Hopf link as a closed braid (see Figure 3.4(2)).

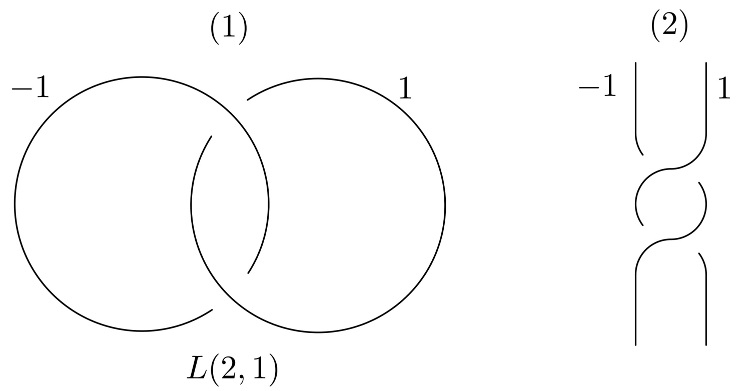

Figure 3.4: Surgery along the Hopf link

Then we remove each linking between the components of the braid by the application of Kirby moves. This application continues until the framing of both components becomes 0 (see Figure 3.5(1)).
Now we use the unknot U and the standard decomposition of S^3 to get a decomposition for $\mathbb{RP}^3$. By Lemma 3.13, after performing 0-framed surgeries, the disk is punctured twice and each component of the Hopf link becomes a binding components δ_1, δ_2. By Lemma 3.13 again, each ± 1-surgery modifies the initial monodromy with a positive or negative Dehn twist. So, the monodromy h is given by $h := D_\alpha \circ D_{\delta_1}^{-2}$.

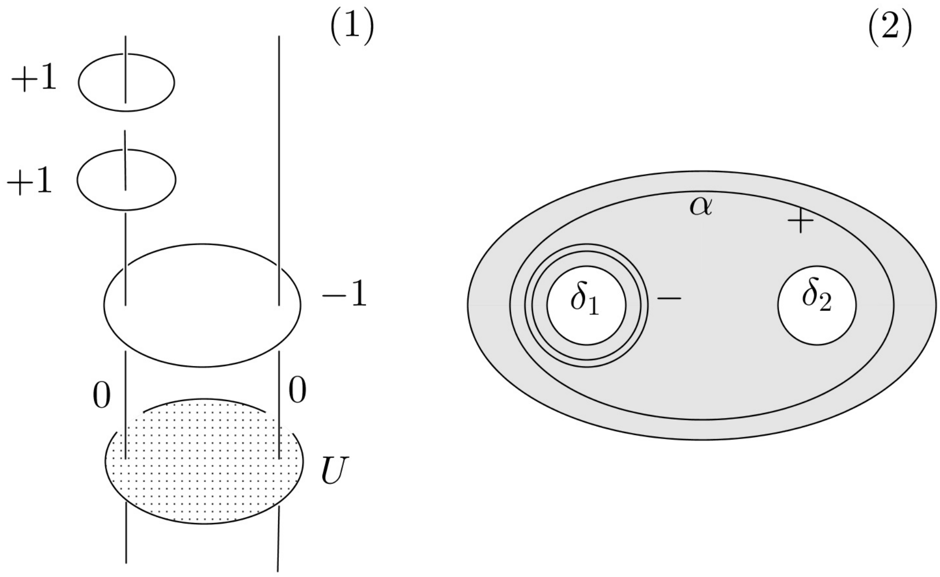

Figure 3.5: Open book decomposition for $\mathbb{RP}^3$

3.2 Example of fibered link in $L(p,q)$

Now, thanks to the notion of open book decomposition and with the help of Lemma 3.13, we are ready to show the key example of this dissertation.

Proposition 3.17 *Given $p \in \mathbb{Z}$, there exists a fibered link $L \subset L(p,1)$ represented by the mixed link diagram represented in Figure 3.6.*

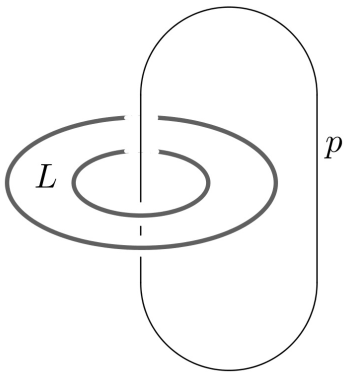

Figure 3.6: Fibered link in $L(p,1)$

Proof. Let us consider the standard open book (D^2, Id) in S^3 with binding $\partial D^2 = U$. As before, the surgery knot links U exactly once and intersect each page transversely. By adding ± 1-framed components with the help of Kirby moves, we reduce the framing of the surgery knot to be 0. Thanks to Lemma 3.13 and Remark 3.14, we know precisely how to get a mixed link representation from a 0-surgery transverse to the each page. Finally, we can remove from the representation the ± 1-components, getting back a p-framed knot. □

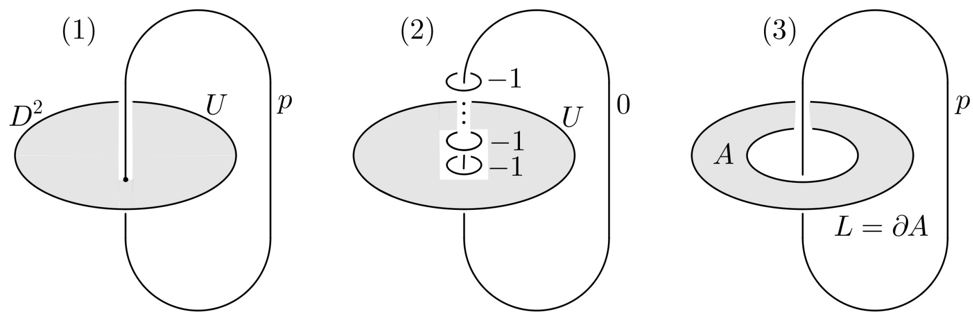

Figure 3.7: Construction of the fibered link L

Remark 3.18. In the previous proposition, p can assume the particular value $p = 0$, so the statement still holds for the space $S^1 \times S^2$.

To extend our previous results to the more general case of the lens space $L(p, q)$ with $q > 1$, we have to recall the presentation of lens spaces $L(p, q)$ given by integral surgery on the framed link of Figure 1.17. Then, by applying Theorem 1.22, we modify the chosen link in order to present it as the closure of a suitable framed braid.

Proposition 3.19 *If p/q has a continued fraction expansion given by $[a_1, a_2, \ldots, a_n]$ and $L(p, q)$ is presented by the framed link L in Figure 1.17, then there exists a fibered link in $L(p, q)$ given by Figure 3.8.*

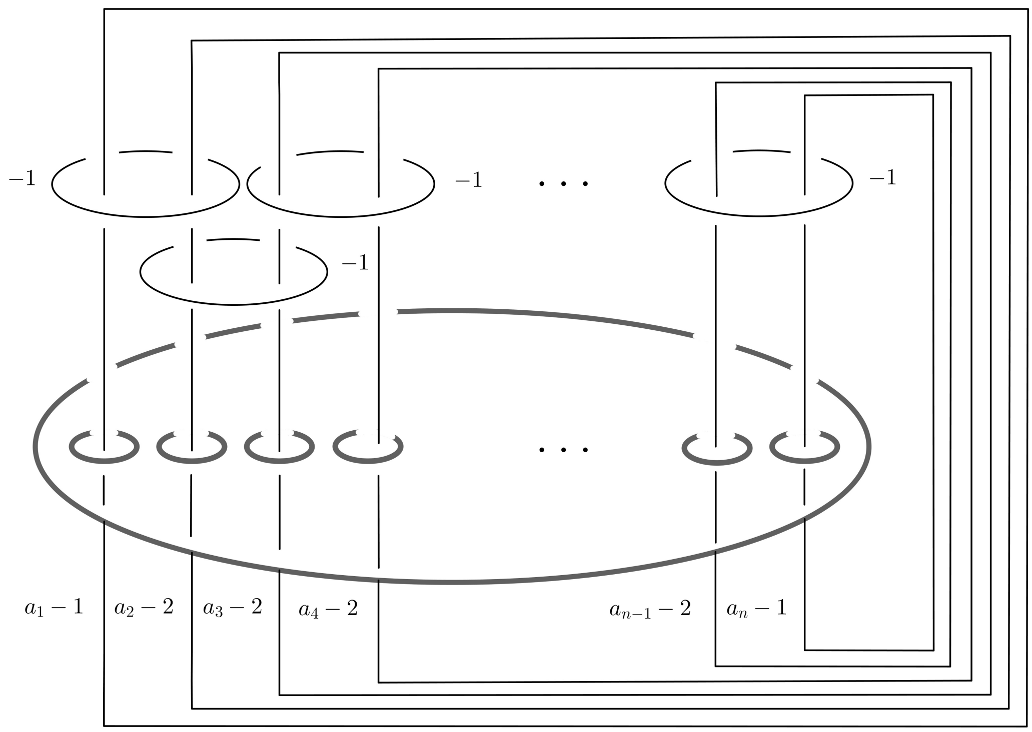

Figure 3.8: Fibered link in $L(p, q)$

Proof. The proof is a clear extension of the case $q = 1$. Let us consider the standard open book decomposition (D^2, Id) of S^3 with binding U. We require that U links to each component of the framed link exactly once and every component has to intersect the pages transversely.
Firstly, we remove each linking of L by introducing ± 1 framed circles. By applying Lemma 3.13 and Remark 3.14, we get a mixed link from each 0-framed transverse surgery. Finally, we remove every ± 1-component, to get the desired presentation. □

Example 3.20. We will give an example for the case $p/q = [a_1, a_2, a_3]$. Let us suppose to have a lens space $L(p, q)$ represented by a framed link with three components, as shown in Figure 3.9(1). The first step of the algorithm is to represent the framed link as a closed braid using Theorem 1.22 (see Figure 3.9(2)).

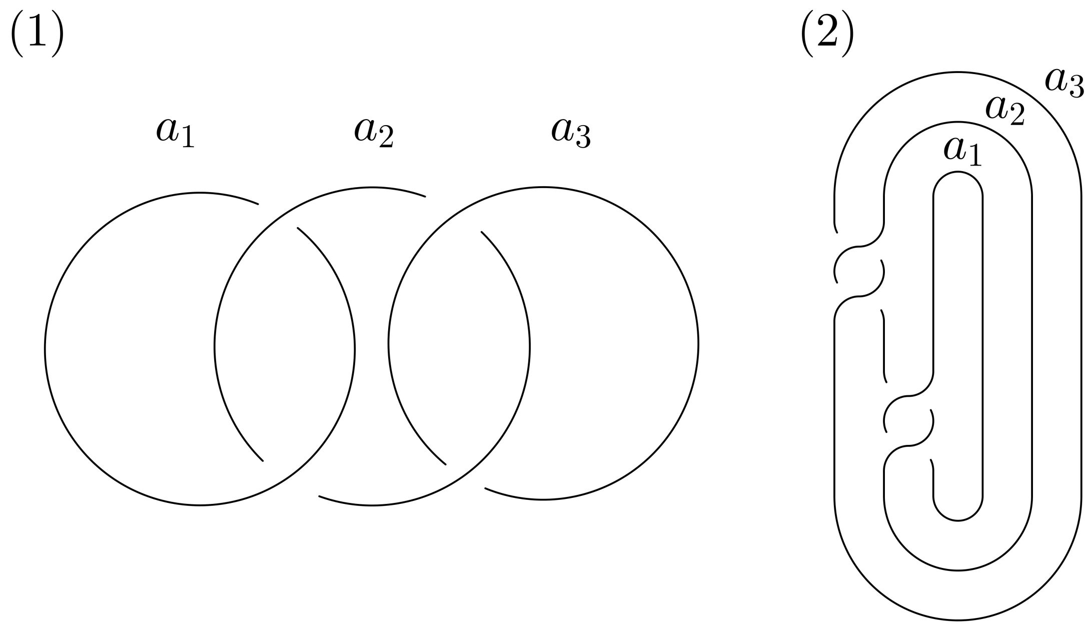

Figure 3.9: Mixed link representation of $L(p, q)$ for $p/q = [a_1, a_2, a_3]$

Then, we consider the standard open book decomposition (D^2, Id) of S^3 with D^2 in the position required in Proposition 3.8 with respect to the framed link. In order to remove each linking between the components in this particular case, we have to introduce -1-framed circles. To change $a_1 - 1, a_2 - 1$ and $a_3 - 1$ into 0, we keep introducing -1-framed circles. Their number for each braid component will be equal to $a_i - 1$ (see Figure 3.10(1)). At this point, we apply Lemma 3.13 to the page D^2. We get a page punctured three times. Finally, we remove each circle previously added in order to get back the framed link shown in Figure 3.10(2).

3.3 Construction of fibered knots in lens spaces

Our aim now is to construct a knot from the link we get in Proposition 3.17 and in Proposition 3.19. We will use the moves introduced by Harer and exposed

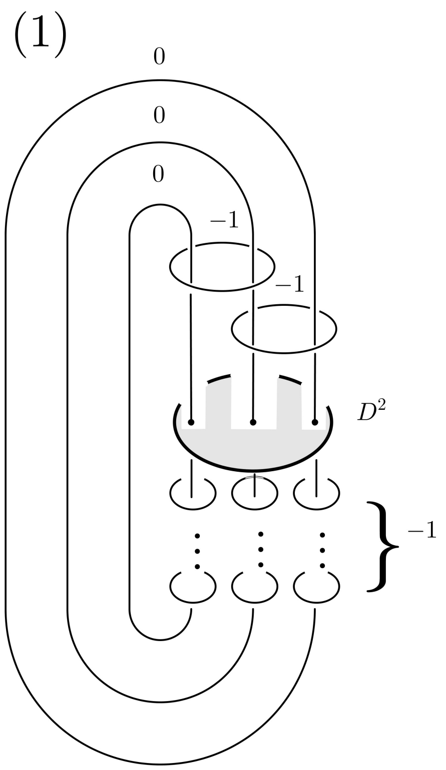

Figure 3.10: Construction of fibered link for $p/q = [a_1, a_2, a_3]$

in Section 2.2, but with some slight modifications in order to extend them to all 3-manifolds. The plumbing, as we have already said in Remark 2.17, extends easily by restricting to a suitable coordinate chart. We call this construction (A'). The same does not hold for the twisting procedure, which has to be substituted by the introduction of a new operation that we call (B'). Let (F, L) be a fibered pair in a 3-manifold M and denote by F_1, F_2 two parallel copies of F. Let $\gamma_i \subset F_i$ be embedded circles and put $\epsilon_i = \pm 1$. We require:

1. There is an oriented annulus A embedded in M with $\partial A = \gamma_1 \cup \gamma_2$.

2. Putting γ_i^+ for the oriented circle obtained by pushing off γ_i along F_i with one extra ϵ_i full twist, γ_1^+ and γ_2^+ , must intersect A algebraically one point each with opposite signs.

Surgery on γ_1 and γ_2 with framings determined by γ_1^+ and γ_2^+ will return us M and construct another fibered pair (F', L'). This is the operation (B').

Theorem 3.21 [H1, Theorem 2] *Let (F, L) and (F', L') be fibered pairs in M. Then there are pairs (F_1, L_1) and (F_2, L_2) such that*

1. *(F_2, L_2) is obtained from (F, L) using operation (A'),*

2. *(F_1, L_1) is constructed form (F', L') using (A'),*

3. *(F_1, L_1) may be changed into (F_2, L_2) using (B').*

In the particular case described in Proposition 3.17 we can apply (A') to L to get the desired fibered knot in $L(p,1)$. We represent it by the punctured disk diagram in Figure 3.11.

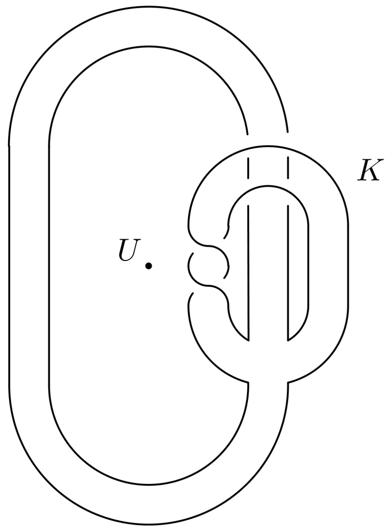

Figure 3.11: Punctured disk diagram for the fibered knot K

An analogous procedure involves the more general case of $L(p,q)$. If we consider the example reported in Proposition 3.8, we need to repeat the plumbing operation several times, once for each integer a_i that appears in the continued fraction decomposition.

Definition 3.22. If K is a knot in M completely contained in a 3-ball, we say that the knot is *local*.

Remark 3.23. Let K be a fibered knot in an orientable closed 3-dimensional manifold M. If K is local then the closure of $M - \nu(K)$ is homeomorphic to $M\#E$, where E is a knot exterior in S^3 and # means the connected sum. According to [Mo, Remark 2] the connected sum is trivial, then M has to be homeomorphic to S^3. Thus we see that any fibered knot in M except S^3 is not local. This holds in particular for $L(p,q)$ spaces, as we can see in Proposition 3.17 and Proposition 3.8. Moreover, a theorem in [H2] claims that any fibered knot in an orientable closed 3-dimensional manifold with abelian fundamental group is nullhomotopic (see Definition 1.2). Hence any fibered knot in a lens space is nullhomotopic.

Remark 3.24. From Remark 3.23 it follows easily that every fibered knot in a lens space has to be nullhomologous. We want to verify this statement by computing explicitly the homology class of the particular knot reported in Figure 3.11. Let us call the considered knot K.

Thanks to the given representation, we can suppose that K is contained in one of the two solid tori of the Heegaard splitting of $L(p,1)$. Let us denote this solid torus by T. Since T is homotopically equivalent to S^1, it follows that $H_1(T) \cong \mathbb{Z}$. If we indicate with $[T]$ the fundamental class that generates the first homology group, we must have

$$[K] = w(K)[T], \quad w(K) \in \mathbb{Z}$$

where $[K]$ is the homology class of the knot K in $H_1(T)$. We call the integer $w(K)$ the *winding number* of K with respect to the solid torus T. It is easy to see that the winding number determines completely the homology class $[K]$, by definition. It is possible to define equivalently $w(K)$ using the punctured disk diagram of K. Let D be a punctured disk diagram for K and let us identify the projection plane with $\mathbb{R}^2$. After a suitable translation, we can suppose to identify the point U which appears in Definition 1.51 with the origin. Since D avoids the origin, the diagram lies entirely in $\mathbb{R}^2 \setminus 0$, which is again homotopically equivalent to S^1. Thus, by denoting $[\mathbb{R}^2_0]$ the fundamental class which generates $H_1(\mathbb{R}^2 \setminus 0)$, it holds

$$[D] = w(K)[\mathbb{R}^2_0],$$

where $[D]$ is the homoloy class of the diagram D in $H_1(\mathbb{R}^2 \setminus 0)$. Thanks to this approach, we can compute $w(K)$ by counting the number of times that D wraps around the point U. In our particular case, this number reveals to be equal to zero (see Figure 3.12), i.e. $w(K) = 0$, then the homology class $[K]$ in $H_1(T)$ is trivial. By an easy application of the Mayer-Vietoris exact sequence to T and its complement in $L(p, 1)$, which is still a solid torus, we understand that the homology class of K in $H_1(L(p, 1)) \cong \mathbb{Z}_p$ is equal to $w(K) \mod p$. Since $w(K)$ is zero, the homology class of K in $L(p, 1)$ is trivial, as desired.

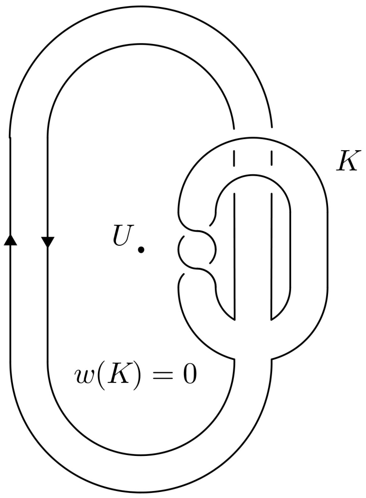

Figure 3.12: Computation of $w(K)$ from the punctured disk diagram

3.4 Lift in the 3-sphere of fibered links contained in $L(p, 1)$

Since a link contained in the 3-sphere may be easier to study rather than a link in $L(p, q)$, we might ask quite naturally how to get a *lift* in the 3-sphere of the link reported in Proposition 3.17. If $\omega_{p,q} : S^3 \to L(p, q)$ is the covering map with respect to the action of $\mathbb{Z}_p$ described in Remark 1.44 and if L is a link in $L(p, q)$, we say that a link L' contained in the 3-sphere is a lift of L if it coincides with the preimage $L' = \omega_{p,q}^{-1}(L)$. Given a band diagram of a link in $L(p, q)$, the following proposition gives back an useful method to construct its lift in S^3.

Proposition 3.25 [M, Proposition 6.4] *We define the Garnside braid Δ_n in n strands as*

$$\Delta_n := (\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1)(\sigma_{n-2}\ldots\sigma_1)\ldots\sigma_1,$$

where σ_i are the Artin's generators of the group B_n (see Theorem 1.20). Let L be a link in $L(p, q)$ in the lens space $L(p, q)$, with $0 \leq q < p$, and let B_L be a band diagram for L with n boundary points. Then a diagram for the lift L' in the 3-sphere S^3 can be found by juxtaposing p copies of B_L and closing them with the braid Δ_n^{2q} (see Figure 3.13).

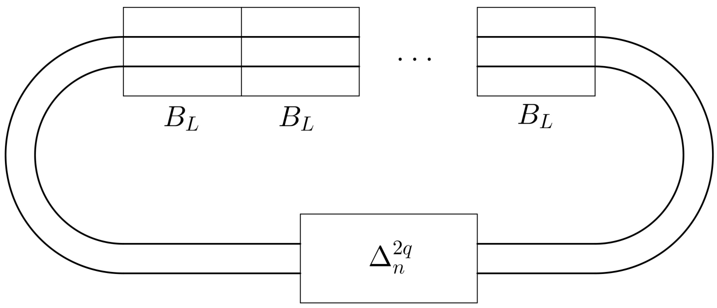

Figure 3.13: Diagram of the lift starting from the band diagram

In order to make more clear the content of Proposition 3.25, we will exhibit an example of the lift of the link in Proposition 3.17 for both $p > 1$ and $p < 1$. From now on we will indicate that link with L.
Let us consider $p > 1$, for example. We start constructing a band diagram of L by following the procedure described in Section 1.4.2 (see Figure 3.15(1)). Since we have only two components, the Garnside braid is simply the generator σ_1 of the braid group. If we now apply Proposition 3.25, we immediately see that the lift of L in S^3 reduces to the closure of the braid

$$\Lambda^+ := \Delta_2^{2q} = \sigma_1^2,$$

which is precisely the negative Hopf band H^-, according to the chosen orientation (see Figure 3.15(2)).
The case $p < 1$, results to be more delicate than the previous one. Indeed, we cannot merely apply Theorem 3.13, since this time the condition $0 \leq q < p$ is not satisfied. Since it holds $L(p, 1) = L(-|p|, 1)$ when p is negative, from now on we will consider $L(-p, 1)$ with $p > 1$. By recalling that $L(-p, 1) \cong L(p, -1)$ is homeomorphic to $L(p, p - 1)$, we can try to get a representation of the link in Figure 3.17 where the fixed part is framed by $p/p - 1$ instead of $-p$. By following Section 16.4 of [PS], it is clear that in order to change the framing from $-p$ to $p/p - 1$ we must add two extra twists to the link (see Figure 3.14).

Now, starting from the mixed link with framing $p/p - 1$, we construct firstly the punctured disk diagram and then the associated band diagram by following procedure described in Section 1.4.2. The band diagram results to be simply the

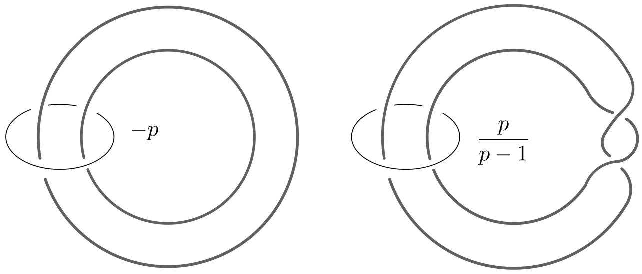

Figure 3.14: Changing framing of the fixed part: from $-p$ to $p/p-1$

braid $\Delta_2^{-2} = \sigma_1^{-2}$ (see Figure 3.16(1)). If we now apply Proposition 3.13, we see that the lift of L in S^3 is the closure of the braid

$$\Lambda^- := (\Delta_2^{-2})^p \Delta_2^{2(p-1)} = (\sigma_1^{-2})^p \sigma_1^{2(p-1)} = \sigma_1^{-2},$$

which becomes the positive Hopf band H^+ with the orientation previously fixed (see Figure 3.16(2)).

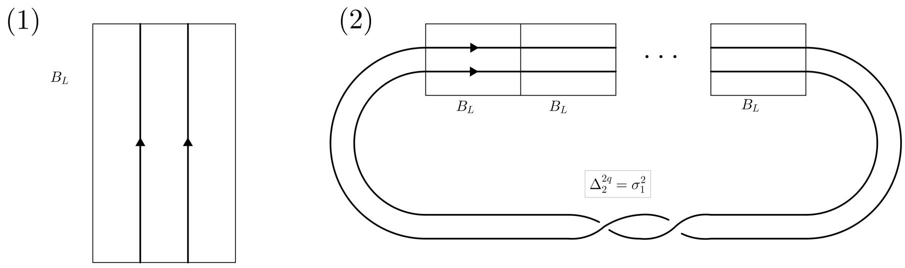

Figure 3.15: (1) Band diagram for L, (2) Lift for $p > 1$

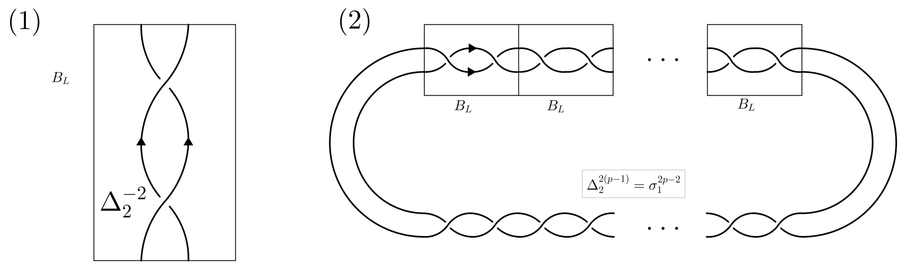

Figure 3.16: (1) Band diagram for L, (2) Lift for $p < 1$

Chapter 4

Contact structures and open book decompositions

In this chapter we introduce the notion of contact 1-form and contact structure on a $(2n+1)$-dimensional manifold M. After reporting some examples in the 3-dimensional case, we enounce the Darboux's theorem, which gives a local characterization for a contact 1-form. Then, we relate these concepts to the definition of open book decomposition by explaining when a contact form is compatible with a given open book decomposition.
Finally, starting from the equivalence stated in Giroux's theorem, we try to find a contact form for the lens space $L(p, p-1)$ that can be compatible and an open book decomposition compatible with it and equivalent to that one presented in Proposition 3.17. In particular, we recall that the space $L(p, p-1)$ is homeomorphic to $L(p, -1)$, or equivalently to $L(-p, 1)$, thanks to Remark 1.41 and Remark 1.44.

4.1 Contact 3-manifolds

Before starting with the basic definitions about contact geometry, we need to introduce the notation relative to some spaces which will be really useful for our purposes.

Definition 4.1. If we indicate by T_mM the tangent space at a point m of a n-dimensional manifold M, we define the *tangent bundle* TM as

$$TM := \bigcup_{m \in M} T_mM = \{(m, v) | m \in M, v \in T_mM\}$$

and the projection map $p : TM \to M$ given by $p(m, v) := m$. We call *cotangent bundle* T^*M the dual bundle of the tangent bundle, that is $T^*M := (TM)^*$. This space can be seen as

$$T^*M := \bigcup_{m \in M} (T_mM)^* = \{(m, \phi) | m \in M, \phi \in (T_mM)^*\}$$

and, as before, we have a projection map $\pi : T^*M \to M$ with $\pi(m, \phi) := m$.

The sets of smooth sections relative to these bundles, i.e.

$$\mathcal{X}(M) := \Gamma^\infty(TM) = \{\sigma : M \to TM | \sigma \text{ is smooth}, p \circ \sigma = id_M\}$$

$$\mathcal{A}^1(M) := \Gamma^\infty(T^*M) = \{\sigma : M \to T^*M | \sigma \text{ is smooth}, \pi \circ \sigma = id_M\}$$

are called set of *vector field* and set of *1-forms*, respectively. For more details about these fundamental notions see [Le] or [W].

Definition 4.2. Suppose that M is a manifold of dimension $2n+1$. A 1-form $\alpha \in \mathcal{A}^1(M)$ is said to be a *contact form* if $\alpha \wedge (d\alpha)^n > 0$ is nowhere zero. The $2n$-distribution $\xi \subset TM$ is a *contact structure* if locally it can be defined by a contact 1-form as $\xi = \ker \alpha$.

Remark 4.3. According to a classical result of Frobenius, the distribution $\xi = \ker \alpha$ is integrable if and only if $\alpha \wedge d\alpha = 0$. Since an integrable distribution has to be involutive, that is closed under Lie bracket, ξ will be integrable if and only if $\alpha([X_1, X_2]) = 0$ whenever $X_i \in \ker \alpha, i = 1, 2$. Now, if we suppose ξ integrable and we apply the Cartan's formula to compute $d\alpha$, we obtain

$$d\alpha(X_1, X_2) = \mathcal{L}_{X_2}(\alpha(X_1)) - \mathcal{L}_{X_1}(\alpha(X_2)) - \alpha([X_1, X_2])$$

and, thanks to the hypothesis of integrability, we get $d\alpha(X_1, X_2) = 0$ for all $X_1, X_2 \in \ker \alpha$. In particular, $d\alpha$ has to vanish on $\xi = \ker \alpha$. Therefore, the contact condition may be interpreted as a constraint which forces ξ to be *"maximally non-integrable"*.

Example 4.4. If on $\mathbb{R}^{2n+1}$ we consider the coordinate system $(x_1, y_1, \dots, x_n, y_n, z)$, the standard contact form is given by $\alpha_{st} = dz + \sum_{i=1}^n x_i dy_i$ and the associated contact structure is $\xi_{st} = \ker \alpha_{st}$.

From now on, we specialize on contact manifolds of dimension 3. We report two examples which will be fundamental for our purpose.

Example 4.5. Adopting the coordinates (ρ, θ, z) on $\mathbb{R}^3$, the form $\alpha = dz + \rho^2 d\theta$ reveals to be a contact 1-form. In fact, after the change into the classical (x, y, z)-coordinate system, we get $\alpha = dz + xdy - ydx$ and so

$$\alpha \wedge d\alpha = 2dx \wedge dy \wedge dz = 2\rho d\rho \wedge d\theta \wedge dz$$

which is clearly different from 0 all over the points where the coordinate change is defined. It is easy to verify that the contact plane of the associated distribution are generated by $\{\frac{\partial}{\partial \rho}, \rho^2 \frac{\partial}{\partial z} - \frac{\partial}{\partial \theta}\}$. The planes relative to the points of the z-axis are horizontal, i.e. parallel to the xy-plane, and as we move out along any ray perpendicular to the z-axis the planes twist in a clockwise way (see Figure 4.1). The twisting angle is an increasing function of r which converges monotonically to $\pi/2$ as $r \to \infty$. Moreover, we observe that the distribution is invariant under translation along the z-axis and under rotation on the xy-plane.

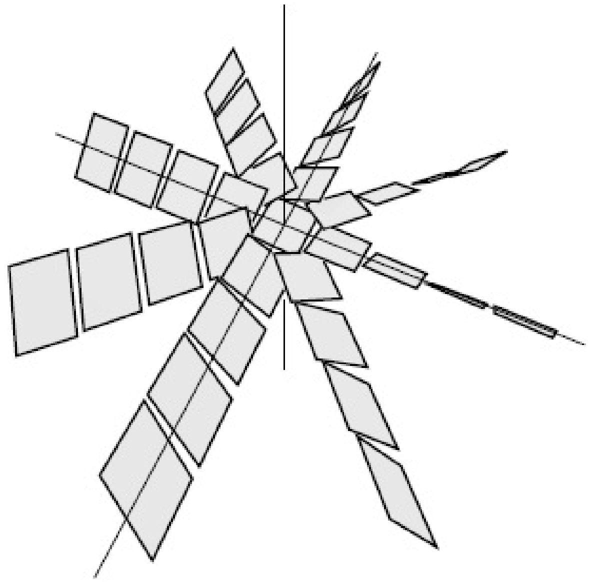

Figure 4.1: Contact planes for the $\alpha = dz + \rho^2 d\theta$

Example 4.6. If we indicate with $p = (x_1, y_1, x_2, y_2)$ a point in $\mathbb{R}^4$ and we consider the map $F : \mathbb{R}^4 \to \mathbb{R}$ defined by $F(x_1, y_1, x_2, y_2) := x_1^2 + y_1^2 + x_2^2 + y_2^2$, it is clear that $S^3 = F^{-1}(1)$ and $T_pS^3 = \ker dF_p = \ker(2x_1dx_1 + 2y_1dy_1 + 2x_2dx_2 + 2y_2dy_2)$. By identifying $\mathbb{R}^4$ with $\mathbb{C}^2$, we can define a complex structure on each tangent space of $\mathbb{R}^4$. More precisely, by putting

$$J_p : T_p\mathbb{R}^4 \to T_p\mathbb{R}^4, \quad \text{with } p \in \mathbb{R}^4$$

$$J_p(\frac{\partial}{\partial x_i}) := \frac{\partial}{\partial y_i}, \quad J_p(\frac{\partial}{\partial y_i}) := -\frac{\partial}{\partial x_i} \quad \text{for } i = 1, 2$$

we immediately see $J_p^2 = -\mathrm{Id}$. Let ξ the distribution defined by

$$\xi_p = T_pS^3 \cap J_p(T_pS^3), \quad \text{with } p \in S^3.$$

We claim that ξ is a contact structure on S^3. We want to find a contact 1-form α such that $\xi = \ker\alpha$. If we consider the 1-form $df \circ J$ and we evaluate this form on the basis $\{\partial/\partial x_i, \partial/\partial y_i\}$, we can recognize that

$$-df_p \circ J_p = 2x_1dy_1 - 2y_1dx_1 + 2x_2dy_2 - 2y_2dx_2.$$

Moreover, thanks to the equality $J_p^2 = -\mathrm{Id}$, it is possible to verify that $J_p(T_pS^3) = \ker(-df_p \circ J_p)$. By letting $\alpha := -1/2(df \circ J)|_{S^3}$, it is clear that $\xi = \ker\alpha$. In order to verify that $\alpha \wedge d\alpha > 0$, we can pick up a point $p = (x_1, y_1, x_2, y_2)$ with $x_1 \neq 0, y_1 \neq 0, y_2 \neq 0$ and choose a basis for the tangent space T_pS^3 given by

$$\{\frac{\partial}{\partial x_1} - \frac{x_1}{y_1}\frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2} - \frac{x_2}{y_2}\frac{\partial}{\partial y_2}, \frac{\partial}{\partial x_1} - \frac{x_1}{y_2}\frac{\partial}{\partial y_2}\}.$$

On this particular basis, it easy to see that $\alpha \wedge d\alpha > 0$. Hence we conclude that $\alpha = x_1dy_1 - y_1dx_1 + x_2dy_2 - y_2dx_2|_{S^3}$ is a contact form on the 3-sphere. We define $\xi_{st} := \xi = \ker\alpha$ as the *standard contact structure* on S^3. This structure can be read in polar coordinates as $\xi_{st} = \ker(\rho_1^2 d\theta_1 + \rho_2^2 \theta_2)$.

Remark 4.7. The previous example can be extended to a more general case. Let us consider an almost complex manifold (M, J) and a function $\varphi : X \to \mathbb{R}$ such that the symmetric 2-form $g_\varphi(v, w) := -dJ^* d\varphi(v, Jw)$ is a Riemannian metric (here J denotes the multiplication by i on TM and J^* is the induced map on T^*M). Then the 1-form α_φ given by

$$\alpha_\varphi(v) := -dJ^* d\varphi(\nabla_{g_\varphi}\varphi, v)$$

defines a contact form on the set $\varphi^{-1}(a)$ where a is a regular value of φ.

Definition 4.8. Two contact 3-manifolds (M, ξ) and (N, ξ') are called *contactomorphic* if there exists a diffeomorphism $F : M \to N$ such that $F_*(\xi) = (\xi')$, that is $dF_p(\xi_m) = \xi'_{F(m)}$ for all $m \in M$. If $\xi = \ker\alpha$ and $\xi' = \ker\alpha'$, the previous condition is equivalent to the existence of a nowhere vanishing function $f : M \to \mathbb{R}$ such that $F^*\alpha = f \cdot \alpha$. Two contact structures ζ and ζ' on the same manifold M are said to be *isotopic* if there is a contactomorphism $h : (M, \zeta) \to (M, \zeta')$ which is isotopic to the identity.

Example 4.9. The forms $\alpha_1 = dz + xdy$ and $\alpha_2 = dz + xdy - ydx$ define two contactomorphic structures over $\mathbb{R}^3$. An example of contactomorphism between α_1 and α_2 is given by $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ with $\varphi(x, y, z) := (x, \frac{y}{2}, z + \frac{xy}{2})$.

Definition 4.10. Let ξ be a contact structure over M given as $\xi := \ker\alpha$ for a certain contact 1-form $\alpha \in \mathcal{A}^1(M)$. The vector field R_α on M satisfying

1. $d\alpha(R_\alpha, \cdot) = 0$,
2. $\alpha(R_\alpha) = 1$

is called *Reeb vector field* of α. Equivalently, for every point $p \in M$, the Reeb vector field points in the direction where the skew-symmetric 2-form $d\alpha_p$ degenerates in T_pM and it is uniquely determined by the normalization condition $\alpha(R_\alpha) = 1$.

Theorem 4.11 [OS, Theorem 4.4.1.12] *Let M be a given 3-manifold with $N \subset M$ a connected compact subset. Consider two different contact structures ξ_0, ξ_1 on M such that $\xi_0|_N = \xi_1|_N$ as oriented 2-plane fields. Then, there exists a neighborhood U of N and a contactomorphism $\varphi : (U, \xi_0|_U) \to (U, \xi_1|_U)$ which is isotopic to id_U relative to N.*

We choose to omit the proof of the previous statement, since it won't be useful for our aim. Thanks to Theorem 4.11 we can understand the local behaviour of a contact structure ξ on a 3-dimensional manifold M, which is completely characterized by Darboux's theorem.

Theorem 4.12 [OS, Theorem 4.4.1.13] *Given a contact 3-manifold (M, ξ), for every $m \in M$ there is a neighborhood $U \subset M$ such that $(U, \xi|_U)$ is contactomorphic to $(V, \xi_{st}|_V)$ for some open set $V \subset \mathbb{R}^3$.*

Proof. It is an easy application of Theorem 4.11 to the case where N is single point. □

In conclusion, every contact structure ξ on a 3-manifold M is locally contactomorphic to the standard contact structure on $\mathbb{R}^3$. Thus, given a point $m \in M$, we can find a suitable chart (U, φ) such that $m \in U$ and the local expression of ξ in the coordinate system $\varphi(p) = (x, y, z)$, where $p \in U$, is given by $\xi_{st} = dz + xdy$.

4.2 Compatible contact structures

After the brief introduction about contact geometry, we are ready to relate the notion of contact 1-form with the definition of open book decomposition for a 3-dimensional manifold M described in Chapter 3. To this end, we present a general open book decomposition as a the pair (L, π), where L is the binding and $\pi : S^3 \setminus L \to S^1$ is the projection map (see Definition 3.1).

Definition 4.13. A contact structure ξ on M is *supported* by an open book decomposition (L, π) of M if ξ can be isotoped through contact structures so that there is a contact 1-form α for ξ such that

1. $d\alpha$ is a positive volume form on each page Σ of the open book and

2. $\alpha > 0$ when restricted to the binding L.

Equivalently, we will say that the contact structure ξ and the open book (L, π) are *compatible.*

Proposition 4.14 [E, Lemma 3.5] *Given a contact 3-manifold (M, ξ), the contact structure $\xi = \ker \alpha$ is compatible with an open book decomposition (L, π) if and only if the associated Reeb vector field R_α is positively tangent to L and transverse to the pages of π.*

Proof. Assume that R_α satisfies the hypothesis. Since R_α is positively tangent to the binding we have $\alpha > 0$ on oriented tangent vectors to L. Moreover, since R_α is positively transverse to the pages of the open book, we have $d\alpha = \iota_{R_\alpha}(\alpha \wedge d\alpha) > 0$ on the pages.
Viceversa, suppose that α is compatible with the open book (L, π). Since $d\alpha$ is a positive volume form on each page, it is clear that R_α is positively transverse to the pages. Thus it remains to check that R_α is positively tangent to L. To this end consider the coordinate system $(\varphi, (\rho, \theta))$ on a suitable neighborhood of a component of the binding L in such a way that the pages of the open book are described by the equation $\theta =$ *constant*. After changing (ρ, θ) to Cartesian coordinates (x, y), we can write

$$R_\alpha = f\frac{\partial}{\partial\varphi} + g\frac{\partial}{\partial x} + h\frac{\partial}{\partial y}$$

where f, g, h are smooth functions. We need to see that g and h are zero when $(x, y) = (0, 0)$. For instance, if $g > 0$ at some point $(c, (0, 0))$ then it will be positive in some neighborhood of this point, by continuity. In particular it will be positive at $(c, (0, \pm\varepsilon))$, for ε sufficiently small. But at $(c, (0, \varepsilon))$ the $\partial/\partial x$ component of R_α must be negative, not positive, in order to be positively transverse to the pages. Thus g and h are indeed zero along the binding. $\square$

Theorem 4.15 [OS, Theorem 9.9.2.5] *Every open book decomposition of a closed and oriented 3-manifold M admits a compatible contact structure.*

Proof. First of all, thanks to Lemma 3.5, we recall that an open book decomposition (L, π) gives back an abstract open book decomposition (Σ, h). From this observation, we know that M is diffeomorphic to

$$T_h \bigsqcup (\partial\Sigma \times D^2)$$

where T_h is the mapping torus defined in Definition 2.7. For sake of simplicity, we assume that the boundary $\partial\Sigma$ has only one component. In order to construct a contact structure on M, we start finding a contact 1-form on $\Sigma \times [0,1]$, which descends to the quotient T_h. Then, we will extend this form over the solid torus $S^1 \times D^2 \cong \partial\Sigma \times D^2$. Let (t, θ) be coordinates for a collar neighborhood C of $\partial\Sigma$ such that $t \in (1/2, 1]$ and $\partial\Sigma = \{t = 1\}$. We claim that the set $\mathscr{S}$ of 1-forms η satisfying

1. $d\eta$ is a volume form on Σ and
2. $\eta = td\theta$ near ∂F

is nonempty and convex. To prove this, choose a volume form ω on Σ with

$$\int_\Sigma \omega = 1 \quad \text{and} \quad \omega|_C = dt \wedge d\theta.$$

Let η_1 be any 1-form on Σ which equals $td\theta$ near $\partial\Sigma$. Then by Stokes' Theorem we obtain

$$\int_\Sigma (\omega - d\eta_1) = \int_\Sigma \omega - \int_\Sigma d\eta_1 = 1 - \int_{\partial\Sigma} \eta_1 = 1 - \int_{\partial\Sigma} d\theta = 0.$$

Hence the closed 2-form $\omega - d\eta_1$ represents a trivial class in cohomology and vanishes near $\partial\Sigma$. By De Rham's theorem there is a 1-form γ on Σ with

$$d\gamma = \omega - d\eta_1$$

and vanishes near $\partial\Sigma$. Define $\eta_2 = \eta_1 + \gamma$. Then $d\gamma_2 = \omega$ is a volume form on Σ and $\eta_2 = td\theta$ near $\partial\Sigma$, showing that $\mathscr{S} \neq \varnothing$. Let φ_1 and φ_2 be two 1-forms in $\mathscr{S}$. Then

$$d(\tau\varphi_1 + (1-\tau)\varphi_2) = \tau d\varphi_1 + (1-\tau)\varphi_2 > 0$$

on Σ and

$$\tau\varphi_1 + (1-\tau)\varphi_2 = td\theta$$

near $\partial\Sigma$, which allows the convexity of the set $\mathscr{S}$.
Let η be any 1-form in $\mathscr{S}$. The pullback of η along h, that is $h^*\eta$, also belongs to the set $\mathscr{S}$: $dh^*\eta = h^*d\eta$ is a volume form on Σ and $h^*\eta = \eta = td\theta$ near $\partial\Sigma$. By convexity, the 1-form

$$\tilde{\eta}_{(x,\tau)} = \tau\eta_x + (1-\tau)(h^*\eta)_x$$

is in $\mathscr{S}$ for each τ and descends to the quotient T_h where x is in the fiber and τ is in the base circle. Thus $d\tilde{\eta}$ induces a volume form when restricted to a page of our open book decomposition. Notice that when we glue the two ends of $\Sigma \times [0,1]$, the forms η and $h^*\eta$ match up on that fiber. Moreover, since h, and hence h^*, is the identity near $\partial\Sigma$, we have $\tilde{\eta}_{(x,\tau)} = td\theta$ for all $(x,\tau) = ((t,\theta),\tau)$ near $\partial T_h = \partial\Sigma \times S^1$. Let $d\tau$ be a volume form on S^1. We claim that

$$\alpha_1 = \tilde{\eta} + K\pi^* d\tau$$

is a contact form on T_h for a sufficient large real $K > 0$, where π denotes the projection of T_h onto the circle S^1, seen as quotient of $[0,1]$. To prove this claim, we pick up a point $(x,\tau) \in T_h$ and choose an oriented basis $\{u,v,w\}$ of $T_{(x,\tau)}(T_h)$ such that $d\tilde{\eta}_{(x,\tau)}(u,v) > 0$ and $d\pi_{(x,\tau)}(u) = d\pi_{(x,\tau)}(v) = 0$. This means that the vectors u and v are tangent to the fiber and w is transverse to the fibration. Thus we get

$$(\alpha_1 \wedge d\alpha_1)_{(x,\tau)}(u,v,w) = (\tilde{\eta} \wedge d\tilde{\eta})_{(x,\tau)}(u,v,w) + K_{(x,\tau)}(d\tau(d\pi_{(x,\tau)}(w))d\tilde{\eta}_{(x,\tau)}(u,v))$$

Hence we conclude that

$$(\alpha_1 \wedge d\alpha_1)_{(x,\tau)}(u,v,w) > 0$$

for $K_{(x,\tau)}$ sufficiently large since $d\tau(d\pi_{(x,\tau)}(w))d\tilde{\eta}_{(x,\tau)}(u,v)$ is positive by the choice of the oriented basis (u,v,w). By compactness of T_h, there exists a sufficiently large $K > 0$ such that $\alpha_1 \wedge d\alpha_1 > 0$ on T_h.
Now, we have to extend the 1-form near the binding. Let $D(r)$ be the 2-dimensional disk of radius r. Consider a tubular neighborhood $\nu(L)$ of the binding and suppose to have a diffeomorphism between $\nu(L) \cong \partial\Sigma \times D(3/2)$. If we use on $D(3/2)$ polar coordinates(r,ϕ), we can suppose to identify (θ,r,ϕ) with $(\theta, 2-t, \tau)$ for $1 \leq r \leq 3/2$. Then the 1-form α_1 becomes

$$\alpha_1 = (2-r)d\theta + Kd\phi$$

on $\partial\Sigma \times (D(3/2) \setminus D(1))$, since $\tilde{\eta} = td\theta$ near the boundary and $\pi^* d\tau$ is identified with $d\phi$. Note that the form $(2-r)d\theta + Kd\phi$ is a positive contact form away from $r = 0$ i.e. away from L, so it can't be extend across $r = 0$. Consider the new 1-form

$$\alpha_2 = (2-r^2)d\theta + r^2(d\phi)$$

which is a contact form near $r = 0$, since $\alpha_2 \wedge d\alpha_2 = 4rd\theta \wedge dr \wedge d\phi$ (this is simply the contact form on the solid torus obtained by restriction of the standard contact structure on the 3-sphere). To conclude, we wish to connect α_1 to α_2 by a family of contact 1-forms, that is, we need to find two functions

$$f_1, f_2 : [0, 3/2] \to \mathbb{R}$$

so that the 1-form $\alpha := f_1(r)d\theta + f_2(r)d\phi$ is a contact form on $\partial\Sigma \times D(3/2)$ such that

1. α equals α_2 near $r = 0$ and

2. α equals α_1 for $1 \leq r \leq 3/2$.

The necessary and sufficient condition for α to be a positive contact form is that $\alpha \wedge d\alpha > 0$, which equivalent to

$$f_1 f_2' - f_1' f_2 > 0$$

as shown by this simple calculation

$$(f_1 d\theta + f_2 d\phi) \wedge (f_1' dr \wedge d\theta + f_2' dr \wedge d\phi) = (f_1 f_2' - f_1' f_2)(d\theta \wedge dr \wedge d\phi).$$

To guarantee this condition, we choose two smooth functions f_1, f_2 as follows

$$f_1(r) := \begin{cases} 2 - r^2 & \text{if } 0 \leq r \leq 1/2 \\ 2 - r & \text{if } 1 \leq r \leq 3/2 \end{cases}$$

$$f_2(r) := \begin{cases} r^2 & \text{if } 0 \leq r \leq 1/2 \\ K & \text{if } 1 \leq r \leq 3/2 \end{cases}$$

and we suppose additionally that $f_1'(r) < 0$ for $1/2 \leq r \leq 1$ and $f_2'(r) > 0$ for $1/2 \leq r \leq 1$. It is clear that we can find such smooth functions. For example, we can consider two piecewise linear functions satisfying the previous conditions and then we regularize them by convolution with mollifiers.
Finally, it is easy to verify that the 1-form $\alpha = f_1(r)d\theta + f_2(r)d\phi$ is compatible with the given open book decomposition and so, we have concluded. □

Proposition 4.16 [OS, Proposition 9.9.2.7] *Any two contact structures compatible with a given open book decomposition are isotopic.*

Proof. Suppose ξ_0 and ξ_1 are two contact structures compatible with a given open book decomposition of a closed oriented 3-manifold M. Then, there are two contact forms α_0 and α_1 such that $\xi_i = \ker \alpha_i$, where $d\alpha_i$ is a positive volume form on the pages and α_i is transverse to the binding L, for $i = 0, 1$. Choose a coordinate system (θ, r, ϕ) near L in which the binding is given by $L = \{r = 0\}$ and the pages are given by $\{\phi = \text{constant}\}$. Let $\alpha = f(r)d\theta$, where $f(r)$ is a nondecreasing function function which equals to 0 for small r and which is equal 1 for $r \geq r_0$. Extend α to M as $\pi^* d\tau$, where $\pi : M \setminus L \to S^1$ is the fibration and $d\tau$ is a volume form on S^1. In this way, we get a global 1-form on M which vanishes near the binding. Then the 1-forms

$$\alpha_{i,t} = \alpha_i + t\alpha, \quad t \geq 0$$

are all contact forms. By an easy calculation, we verify that $\alpha_{i,t}$ is a contact 1-form away from the binding

$$\alpha_{i,t} \wedge d\alpha_{i,t} = \alpha_i \wedge d\alpha_i + t\pi^* d\tau \wedge d\alpha_i > 0$$

since α_i is a contact form and $d\alpha_i$ is a volume form on the pages. Moreover, for t large enough, the forms $\alpha_{t,s} = (1-s)\alpha_{0,t} + s\alpha_{1,t}$ with $0 \leq s \leq 1$ are also contact. Again, when we consider $\alpha_{s,t} \wedge d\alpha_{s,t}$ away from the binding, the only terms which are not necessarily positive are $s(1-s)\alpha_1 \wedge d\alpha_2$ and $s(1-s)\alpha_2 \wedge d\alpha_1$. On the other hand the remaining terms are positive and some of them are multiplied by the parameter t. This shows that, for t large enough, $\alpha_{s,t}$ is contact for all $0 \leq s \leq 1$ and hence $\alpha_{0,t}$ and $\alpha_{1,t}$ are isotopic, which in turn implies that α_0 and α_1 are isotopic. $\square$

4.3 Equivalence between the geometric view and the analytic approach

Given a 3-dimensional manifold M, we define the sets

$$\begin{aligned} \mathcal{F} &:= \{\text{fibered link in } M \text{ up to positive plumbing}\} \\ \mathcal{O} &:= \{\text{open book decomposition of } M \text{ up to positive stabilization}\} \\ \mathcal{C} &:= \{\text{oriented contact structures on } M \text{ up to isotopy}\}. \end{aligned}$$

First of all, we notice that Remark 3.2 and Remark 3.12 allow us to define a bijective map

$$\Pi : \mathcal{O} \to \mathcal{F}, \quad \Pi(\mathfrak{ob}) := B_{\mathfrak{ob}}$$

where $B_{\mathfrak{ob}}$ is the binding of the open book decomposition $\mathfrak{ob}$. In the same way, Proposition 4.16 says that we have a well defined map

$$\Psi : \mathcal{O} \to \mathcal{C}, \quad \Psi(\mathfrak{ob}) := \xi_{\mathfrak{ob}}$$

where $\mathfrak{ob}$ is an open book decomposition of M and $\xi_{\mathfrak{ob}}$ is the associated contact structure constructed in Theorem 4.15. Giroux's Theorem states that the map Ψ is invertible. More precisely, we have the following

Theorem 4.17 [OS, Theorem 9.9.2.11] *Two isotopic contact structures are supported by two open book decomposition which have a common positive stabilization. Equivalently it holds:*

A) For a given open book decomposition of M there exists a compatible contact structure ξ on M. Moreover, contact structures compatible with the same open book decomposition are isotopic.

B) For a given contact structure ξ on M there is a compatible open book decomposition of M. Moreover two open book decompositions compatible with a fixed contact structure admit a common positive stabilization.

To sum up, thanks to Theorem 4.17, we get the following succession of bijections

$$\mathcal{C} \leftrightarrow \mathcal{O} \leftrightarrow \mathcal{F}$$

realized by the maps Π, Ψ and their inverses. In particular, if we are interested in studying the open book decompositions of a 3-dimensional manifold, and so the fibered links, we can equivalently study the contact structures on the same manifold and try to classify the associated open book decompositions by the use of contact properties.

4.4 Example of contact structures and compatible open book decompositions

Example 4.18. Let (U, π_U) be the open book decomposition of S^3 presented in Example 2.3, where U is the unknot and

$$\pi_U : S^3 \setminus U \to S^1, \quad \pi_U(\rho_1, \theta_1, \rho_2, \theta_2) := \theta_1.$$

This open book supports the standard contact structure $\xi_{st} = \ker(\rho_1^2 d\theta_1 + \rho_2^2 \theta_2)$. To see this, notice that, for a fixed value $\omega = \theta_1$, the page $\pi_U^{-1}(\omega)$ is parametrized by

$$i_0 : D^2 \to S^3, \quad i_0(r, \theta) = (\sqrt{1 - r^2}, \omega, r, \theta).$$

Thus $di_0^*(\rho_1^2 d\theta_1 + \rho_2^2 \theta_2) = 2rdr \wedge d\theta$, which is a positive volume form on the disk. Moreover, the positively oriented tangent to U is $\partial/\partial\theta_2$ and $\alpha(\partial/\partial\theta_2) > 0$.

Example 4.19. We now consider the open book decomposition reported in Example 2.4. We want to show that ξ_{st} is compatible with the open book decomposition H^+, but it is not supported by H^-. First of all, we remember that $H^+ := \{z_1 z_2 = 0\}$, $H^- = \{z_1 \bar{z}_2 = 0\}$ and the projection maps are given by

$$\pi^+ : S^3 \setminus H^+ \to S^1, \quad \pi_+(\rho_1, \theta_1, \rho_2, \theta_2) := \theta_1 + \theta_2,$$

$$\pi^- : S^3 \setminus H^- \to S^1, \quad \pi_-(\rho_1, \theta_1, \rho_2, \theta_2) := \theta_1 - \theta_2.$$

As before, for a fixed value $\omega = \theta_1 + \theta_2$, the page $(\pi^+)^{-1}(\omega)$ is parametrized by

$$i_+ : [0,1] \times S^1 \to S^3, \quad i_+(t, \theta) := (\sqrt{1 - t^2}, \omega - \theta, t, \theta)$$

whereas the page $(\pi^-)^{-1}(\omega)$ is parametrized by

$$i_- : [0,1] \times S^1 \to S^3, \quad i_-(t, \theta) := (\sqrt{1 - t^2}, \omega + \theta, t, \theta).$$

By considering

$$\alpha^- := i_-^*(\rho_1, \theta_1, \rho_2, \theta) = (1 - t^2)d(\omega + \theta) + t^2 d\theta = d\theta$$

we immediately understand that ξ_{st} can't be supported by (H^-, π^-) since $d\alpha^- = d^2\theta = 0$ is not a volume form on $[0,1] \times S^1$. On the other hand, it results that

$$\alpha^+ := i_+^*(\rho_1, \theta_1, \rho_2, \theta) = (1 - t^2)d(\omega - \theta) + t^2 d\theta = (2t^2 - 1)d\theta$$

and $d\alpha^+ = d[(2t^2 - 1)d\theta] = 4tdt \wedge d\theta$ is a positive volume form on the annulus. Now, if we consider the component of H^+ given by $\{z_1 = 0\}$, this has $\partial/\partial\theta_2$ as tangent vector and $\alpha(\partial/\partial\theta_2) > 0$. In the same way, the component given by $\{z_2 = 0\}$ has tangent vector $\partial/\partial\theta_1$ and $\alpha(\partial/\partial\theta_1) > 0$, so we have proved that ξ_{st} is supported by (H^+, π^+).

Remark 4.20. We should not be surprised by this result, since (H^+, π^+) is simply the positive stabilization of the open book decomposition (U, π_U), which is compatible with ξ_{st}. Hence, (H^+, π^+) has to be compatible with ξ_{st} because of Theorem 4.17, whereas (H^-, π^-) can't be compatible with ξ_{st} being the negative stabilization of (U, π_U), and Theorem 4.17 does not hold any longer for negative stabilization.

Example 4.21. Our aim is to determine a suitable contact structure on the real projective space $\mathbb{RP}^3 \cong L(2,1)$ induced by the standard contact structure of S^3 and an open book decomposition compatible with it. In order to find this pair, we represent $L(2,1)$ as the quotient of the 3-sphere S^3 under the properly discontinuous action of the group $\mathbb{Z}_2$ described in Remark 1.44.
We start observing that the standard contact structure ξ_{st} on S^3 induces a contact structure on $\mathbb{RP}^3$. Indeed, if we choose the coordinates $(\rho_1, \theta_1, \rho_2, \theta_2)$ on S^3 and if we set

$$\alpha = \rho_1^2 d\theta_1 + \rho_2^2 d\theta_2,$$

we have already seen that $\xi_{st} = \ker \alpha$. This contact 1-form remains unchanged under the action of $\mathbb{Z}_2$. In fact, if we denote with $\zeta = \bar{1}$ the generator of $\mathbb{Z}_2$, we have

$$\zeta \cdot \alpha = \rho_1^2 d(\theta_1 + \pi) + \rho_2^2 d(\theta + \pi) = \rho_1^2 d\theta_1 + \rho_2^2 d\theta_2 = \alpha$$

where $\zeta \cdot \alpha$ indicates the 1-form obtained by pulling back α along the diffeomorphism induced by the action of ζ. Thus, we get a well defined contact 1-form on the projective space $\mathbb{RP}^3$. We define *standard contact structure* on $\mathbb{RP}^3$ the obtained contact structure and we indicate it with $\alpha_{2.1}$.

The next step of our analysis is to show that the open book decomposition (H^+, π^+) is stable under the $\mathbb{Z}_2$ action and this leds us to an open book decomposition of $\mathbb{RP}^3$. We start considering the component of H^+ given by $\{z_1 = 0\}$ which is parametrized by the point $(0, e^{i\theta_2}) \in S^3$ with $\theta_2 \in [0, 2\pi]$. If we keep indicating with $\zeta = \bar{1}$ the generator of $\mathbb{Z}_2$, we have

$$\zeta \cdot (0, e^{i\theta_2}) = (0, e^{i(\theta_2 + \pi)}),$$

then $\zeta \cdot (0, e^{i\theta_2})$ is still an element of $\{z_1 = 0\}$. The same result holds for the component $\{z_2 = 0\}$. Moreover, if we take back the fibration π^+ given by

$$\pi^+ : S^3 \setminus H^+ \to S^1, \qquad \pi^+(\rho_1, \theta_1, \rho_2, \theta_2) := \theta_1 + \theta_2,$$

we immediately observe that the elements of $\mathbb{Z}_2$ preserves this map. Indeed, it results

$$\pi^+(\zeta \cdot (\rho_1, \theta_1, \rho_2, \theta_2)) = \pi^+((\rho_1, \theta_1 + \pi, \rho_2, \theta_2 + \pi)) = \theta_1 + \theta_2 + 2\pi$$

which is exactly equal to $\pi^+(\rho_1, \theta_1, \rho_2, \theta_2)$, thanks to the angular periodicity. Hence we have a well defined map

$$\pi^+_{2,1} : \mathbb{RP}^3 \setminus H^+_{2,1} \to S^1, \qquad \pi^+[\rho_1, \theta_1, \rho_2, \theta_2] := \theta_1 + \theta_2$$

where $[\rho_1, \theta_1, \rho_2, \theta_2]$ stands for the equivalence class of the point $(\rho_1, \theta_1, \rho_2, \theta_2)$ in $\mathbb{RP}^3$ and $H^+_{2,1}$ indicates the image of H^+ in the projective space. With an abuse of notation we could write

$$H^+_{2,1} := H^+/\mathbb{Z}_2.$$

We now consider the annulus embedded in S^3 and parametrized by the equation

$$i_+ : [0,1] \times S^1 \to S^3, \qquad i_+(t, \theta) := (\sqrt{1 - t^2}, -\theta, t, \theta)$$

and we set $A^+ := \mathrm{Im} i_+$. Again, by choosing the generator $\zeta \in \mathbb{Z}_2$ and a point $i_+(t,\theta) \in A^+$, we have

$$\zeta \cdot (\sqrt{1-t^2}, -\theta, t, \theta) = (\sqrt{1-t^2}, -\theta+\pi, t, \theta+\pi) = (\sqrt{1-t^2}, -\theta+\pi, t, \theta-\pi)$$

and so $\zeta \cdot i_+(t,\theta) = i_+(t, \theta-\pi)$ is still a point on A^+. By using the same notation previously introduced, we set

$$A^+_{2,1} := A^+/\mathbb{Z}_2.$$

To sum what we have shown so far, it results that $(H^+_{2,1}, \pi^+_{2,1})$ is an open book decomposition of the space $\mathbb{RP}^3 \cong L(2,1)$ whose page is given by $A^+_{2,1}$. Hence, the compatibility of (H^+, π^+) with the standard contact form α on S^3 implies the compatibility of $(H^+_{2,1}, \pi^+_{2,1})$ with the standard contact structure $\alpha_{2,1}$ on $\mathbb{RP}^3$.

Example 4.22. To generalize the previous example, we start observing that the standard contact structure $\xi_{st} = \ker \alpha$ induces a contact structure on every lens space $L(p,q)$ with $p \neq 0$. Indeed, if we represent $L(p,q)$ as the quotient of S^3 under the action $\mathbb{Z}_p$ and if we still denote with ζ the generator of $\mathbb{Z}_p$, this time seen as the group of the p complex roots of unity, we have

$$\zeta^n \cdot \alpha = \rho_1^2 d(\theta_1 + \frac{2\pi n}{p}) + \rho_2^2 d(\theta + \frac{2\pi nq}{p}) = \rho_1^2 d\theta_1 + \rho_2^2 d\theta_2 = \alpha, \quad n \in \mathbb{Z}$$

which proves that we have a well defined contact 1-form on $L(p,q)$ induced by α.

Definition 4.23. The contact structure induced on $L(p,q)$ by the standard contact 1-form of S^3 under the action of $\mathbb{Z}_p$ is called *standard contact structure* on the lens space $L(p,q)$. We indicate the 1-form associated to this structure with $\alpha_{p,q}$.

Thanks to the end of Remark 1.44, we can think of $L(p,p-1)$ as the space $L(p,-1)$. In this way, the action of the generator ζ is given by

$$\zeta \cdot (\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2}) = (\rho_1 e^{i(\theta_1 + \frac{2\pi}{p})}, \rho_2 e^{i(\theta_2 - \frac{2\pi}{p})})$$

for every point $(\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2}) \in S^3$. By considering again the open book decomposition (H^+, π^+) of S^3, we immediately see that the components of H^+ are stable under the $\mathbb{Z}_p$ action. In fact, choosing for instance the component $\{z_1 = 0\}$, it results

$$\zeta \cdot (0, e^{i\theta}) = (0, e^{i(\theta_1 + \frac{2\pi}{p})}),$$

so $\zeta \cdot e^{i\theta_1}$ is still a point of $\{z_1 = 0\}$. The same is true for the other component $\{z_2 = 0\}$, but we have to change the sign in the action of ζ. In the same way, the map π^+ is stable under the action of ζ. Indeed, we have

$$\pi^+(\zeta \cdot (\rho_1, \theta_1, \rho_2, \theta_2)) = \pi^+(\rho_1, \theta_1 + \frac{2\pi}{p}, \rho_2, \theta_2 - \frac{2\pi}{p}) = \theta_1 + \frac{2\pi}{p} + \theta_2 - \frac{2\pi}{p}$$

and thanks to cancellation the result coincides with $\pi^+(\rho_1, \theta_1, \rho_2, \theta_2)$. Hence, we have a well defined map

$$\pi^+_{p,p-1} : L(p,p-1) \setminus H^+_{p,p-1} \to S^1, \qquad \pi^+_{p,p-1}[\rho_1, \theta_1, \rho_2, \theta_2] := \theta_1 + \theta_2$$

where $[\rho_1, \theta_1, \rho_2, \theta_2]$ stands for the equivalence class of the point $(\rho_1, \theta_1, \rho_2, \theta_2)$ in $L(p, p-1)$ and, with the abuse of notation previously introduced, $H^+_{p,p-1} := H^+/\mathbb{Z}_p$. Moreover, the embedded annulus A^+ introduced in the previous example, is $\mathbb{Z}_p$ stable, since it holds

$$\zeta \cdot i_+(t,\theta) = \zeta \cdot (\sqrt{1-t^2}, -\theta, t, \theta) = (\sqrt{1-t^2}, -\theta + \frac{2\pi}{p}, t, \theta - \frac{2\pi}{p})$$

which is exactly a point of the form $i_+(t, \theta - \frac{2\pi}{p}) \in A^+$. Then, by indicating with $A^+_{p,p-1} := A^+/\mathbb{Z}_p$, we have shown that the pair $(H^+_{p,p-1}, \pi^+_{p,p-1})$ is an open book decomposition for $L(p, p-1)$ with page $A^+_{p,p-1}$. Finally, the compatibility of (H^+, π^+) with the standard contact structure α of S^3 induces the compatibility between $(H^+, \pi^+_{p,p-1})$ and $\alpha_{p,p-1}$ on $L(p, p-1)$.

Remark 4.24. Both in Example 4.21 and in Example 4.22 we see that the standard contact structure on $L(p, p-1)$, where $p \geq 2$, is supported by an open book decomposition induced by (H^+, π^+) thanks the invariance under the action of $\mathbb{Z}_p$. In all the possible cases, the page of the induced open book is homeomorphic to an annulus. More precisely, the page $A^+_{p,p-1}$ is covered p-times by the annulus A^+ when we restrict the covering map $\omega_{p,p-1} : S^3 \to L(p, p-1)$ to A^+. In this way, we understand that the genus of the binding associated to the page $A^+_{p,p-1}$ is the same for every $p \geq 2$ and coincides with the genus of the fibered knot reported in Proposition 3.17. This allows us to suppose that the standard contact structure on $L(p, p-1)$ is compatible with the open book decomposition described in Proposition 3.17, but this statement needs to be verified.

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