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Black Hole Evaporation and Stress Tensor Correlations

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Abstract

La Relativitá Generale e la Meccanica Quantistica sono state le due piú grandi rivoluzioni scientifiche del ventesimo secolo.
Entrambe le teorie sono estremamente eleganti e verificate sperimentalmente in numerose situazioni. Apparentemente però, esse sono tra loro incompatibili.
Alcuni indizi per comprendere queste difficoltà possono essere scoperti studiando i buchi neri.
Essi infatti sono sistemi in cui sia la gravità, sia la meccanica quantistica sono ugualmente importanti.
L'argomento principale di questa tesi magistrale è lo studio degli effetti quantistici nella fisica dei buchi neri, in particolare l'analisi della radiazione Hawking.
Dopo una breve introduzione alla Relatività Generale, è studiata in dettaglio la metrica di Schwarzschild. Particolare attenzione viene data ai sistemi di coordinate utilizzati ed alla dimostrazione delle leggi della meccanica dei buchi neri.
Successivamente è introdotta la teoria dei campi in spaziotempo curvo, con particolare enfasi sulle trasformazioni di Bogolubov e sull'espansione di Schwinger-De Witt. Quest'ultima in particolare sarà fondamentale nel processo di rinormalizzazione del tensore energia impulso.
Viene quindi introdotto un modello di collasso gravitazionale bidimensionale.
Dimostrata l'emissione di un flusso termico di particelle a grandi tempi da parte del buco nero, vengono analizzati in dettaglio gli stati quantistici utilizzati, le correlazioni e le implicazioni fisiche di questo effetto (termodinamica dei buchi neri, paradosso dell'informazione).
Successivamente viene introdotto il tensore energia impulso rinormalizzato e viene calcolata un'espressione esplicita di quest'ultimo per i vari stati quantistici del buco nero.
Infine vengono studiate le correlazioni di questi oggetti. Queste sono molto interessanti anche dal punto di vista sperimentale: le correlazioni tra punti interni ed esterni all'orizzonte degli eventi mostrano dei picchi, i quali potrebbero presto essere misurabili nei modelli analoghi di buco nero, quali i BEC in configurazione supersonica.

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Abstract

General Relativity is one of the greatest scientific achievementes of the 20th century along with quantum theory.
These two theories are extremely beautiful and they are well verified by experiments, but they are apparently incompatible.
Hints towards understanding these problems can be derived studying Black Holes, some the most puzzling solutions of General Relativity.
The main topic of this Master Thesis is the study of Black Holes, in particular the Physics of Hawking Radiation.
After a short review of General Relativity, I study in detail the Schwarzschild solution with particular emphasis on the coordinates systems used and the mathematical proof of the classical laws of Black Hole "Thermodynamics".
Then I introduce the theory of Quantum Fields in Curved Spacetime, from Bogolubov transformations to the Schwinger-De Witt expansion, useful for the renormalization of the stress energy tensor.
After that I introduce a 2D model of gravitational collapse to study the Hawking radiation phenomenon.
Particular emphasis is given to the analysis of the quantum states, from correlations to the physical implication of this quantum effect (e.g. Information Paradox, Black Hole Thermodynamics).
Then I introduce the renormalized stress energy tensor.
Using the Schwinger-De Witt expansion I renormalize this object and I compute it analitically in the various quantum states of interest.
Moreover, I study the correlations between these objects. They are interesting because they are linked to the Hawking radiation experimental search in acoustic Black Hole models. In particular I find that there is a characteristic peak in correlations between points inside and outside the Black Hole region, which correpsonds to entangled excitations inside and outside the Black Hole.
These peaks hopefully will be measurable soon in supersonic BEC.

In this Thesis I use $c = G_N = 1$, the sign convenction (-,+,+,+) and the Reimann tensor

$$R_{\mu\nu\rho}{}^{\sigma} = +\frac{\partial \Gamma^{\sigma}_{\mu\rho}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\sigma}_{\nu\rho}}{\partial x^{\mu}} + \Gamma^{\eta}_{\mu\rho}\Gamma^{\sigma}_{\kappa\nu} - \Gamma^{\eta}_{\nu\rho}\Gamma^{\sigma}_{\mu\eta}$$

Contents

Chapter 1

General Relativity

The General Theory of Relativity formulated by Albert Einstein in 1915 is a very beautiful theory which describe gravity as a property of spacetime.
Einstein' s special relativity rejected the ether concept of a privileged inertial frame of reference, but still depends on the concept of inertial frames.
General Relativity goes beyond this concept, too: it is possible to describe the physics of a system using an arbitrary reference frame.
In this chapter, we are now going to give a short overview of General Relativity.
Firstly, we discuss the first principles of General Relativity with particular emphasis on their importance in the construction of the theory, then we study some mathematical aspects useful in the next chapters.

1.1 The Equivalence Principle

The Equivalence Principle is one of the corner stones of Einstein's theory.
It is based on the equality of the inertial mass and the gravitational mass experimentally proved for the first time by Galileo. This simple statement is very profound and it has far reaching conseguences. Einstein himself had argued that the Principle of Equivalence is his major contribute to Physics.
Let us consider the famous freely falling elevator thought experiment. If we are in an homogeneous gravitational field and we want to describe a system composed by a certain number of particle we can write:

$$m_{in}\frac{d^2\mathbf{x}}{dt^2} = \Sigma_n \mathbf{F}_n(\mathbf{x}-\mathbf{x}') + m_g\mathbf{g}$$

where $\mathbf{g}$ is the gravitational acceleration, $\mathbf{F}_n$ the non gravitational force which acts on the n-particle, m_{in} the inertial mass and m_g the gravitational mass. If we perform this change of coordinates

$$\mathbf{x} \Rightarrow \mathbf{x}^{'} = \mathbf{x} - \frac{1}{2}\mathbf{g}t^2$$

$$t^{'} = t$$

we find:

$$m_{in}\frac{d^2\mathbf{x}^{'}}{dt^2} = \sum_n \mathbf{F}_n(\mathbf{x}-\mathbf{x}^{'})$$

since $m_{in} = m_{grav}$.
Clearly, the observer in the system of reference S' (the freely falling elevator) does not measure any gravitational field.
Therefore, we understand from these equations that the gravitational force is equal to an inertial force. In particular when g is constant we can eliminate the gravitational force through a change of coordinates.
If we are in a generic gravitational field obviously we cannot simply eliminate the effects of gravity globally throught a change of coordinates but for every point we can consider a neighborhood in which g can be considered constant. Therefore we can always find *locally* a class of inertial system of references in which the Laws of Physics are those of Special Relativity.
It is important to underline that these system of reference are *local*, not global.
From the geometrical point of view the Principle of Equivalence is the analogue of the well known differential geometry theorem which states that it is always possible to approximate locally a curved manifold with a plane.
Now we are going to find the equations of motion for a generic point particle in a gravitational field. The special relativistic equation which is true in the freely falling elevator's reference frame is:

$$\frac{dp^\alpha}{ds} = 0 \Rightarrow \frac{d^2\chi^\alpha}{ds^2} = 0$$

with:

$$ds^2 = \eta_{\alpha\beta} d\chi^\alpha d\chi^\beta$$

where $\eta_{\alpha\beta} = diag(-1, +1, +1, +1)$ and $p^\alpha = m d\chi^\alpha / ds$.
These equations describe the motion of a free particle in an inertial reference system.
If we perform a general change of coordinates to a non intertial system ($\chi \Rightarrow x(\chi)$) we find:

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

where:

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial \chi^\alpha} \frac{\partial^2 \chi^\alpha}{\partial x^\mu \partial x^\nu}$$

This is the Geodesics Equation that describe the motion of a test particle subjected to a generic gravitational field.
The line element in the second reference system is:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

where:

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \chi^\alpha}{\partial x^\mu} \frac{\partial \chi^\beta}{\partial x^\nu}$$

The metric tensor describes how to measure temporal and spatial intervals.
Thus, the presence of gravity modifies the simple Minkowkian intuitive notion of distance between events.
It is important to underline that the geometry has not to be simply flat: in the General Theory of Relativity the geometry of spacetime is determined by the Einstein's Equations.
It is possible to write:

$$\Gamma^\sigma_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} \right)$$

Recalling that $\Gamma^{\sigma}_{\lambda\mu}$ is the gravitational force we can interpret the metric as the gravitational potential. Since it is a symmetric tensor in four dimension it has got 10 independent components.
The same reasoning can be applied to massless particles.
Thus, massless particles do not follow simply straight line in a graviational field.
We can define the causal structure as the set of events which can be connected by null or timelike curves.
Since nothing can travel faster than light, the null trajectories determine the causal structure of spacetime.
At every point the light cone is equal to that of special relativity locally because of the Equivalence Principle, but globally there can be very different causal configurations. Thus, in principle we can have regions in which the gravitational field is so strong that creates interesting and non obvious structures (eg. Black Holes).
Note in particular that in the first system of reference the particle moves in a straight way, in the second it is subjected to acceleration. Thus we can interpret $\Gamma^{\lambda}_{\mu\nu}$ as the gravitational force. Now we can give another mathematical statement of the Equivalence Principle:
It is always possible to find locally an inertial system in which:

$$g_{\mu\nu}(x) = \eta_{\mu\nu}$$

$$\Gamma^{\sigma}_{\lambda\mu}(x) = 0$$

The equation of the geodesics can be derived also from the minimization of the proper time between events:

$$s = \frac{1}{2}\int_a^b \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau$$

A simple calculation shows that the solution to this problem is the geodesics equation which generalizes to curved spacetime the notion of straight lines.

1.2 The Principle of General Covariance

The principle of General Covariance (see ref. [1]) also known as diffeomorphism covariance, is another fundamental principle of the General Theory of Relativity.
The essential idea is that coordinates do not exist a priori in nature, but are only artifices used in describing it, and hence they should not play any role in the formulation of fundamental physical laws.
A physical law expressed in a generally covariant fashion takes the same mathematical form in all coordinate systems and is usually expressed in terms of tensor fields.
The Principle of General Covariance says:

1. the form of physical laws under arbitrary differentiable coordinate transformations doesn't change.

2. an equation which holds in presence of gravitation agrees with the law of special relativity when the metric tensor equals the Minkoskian metric tensor and $\Gamma^{\lambda}_{\mu\nu} = 0$

Let us suppose that we are in a general gravitational field and consider any equation of motion that verify the above conditions.

Since these equations have to be true in every coordinate system we can consider a class of locally intertial systems in which the effects of gravity are absent. In this coordinate frame the equations of motion are those of Special Relativity.
But the first condition implies that the equations which holds in these systems are true in every other coordinate system.
Hence if we know the Special Relativisic Equation of motion we can write the equation in presence of gravity with the substitutions:

$$\partial_\mu \Rightarrow \nabla_\mu$$

and

$$\eta_{\mu\nu} \Rightarrow g_{\mu\nu}$$

where ∇_μ is the covariant derivative that acts eg. on vectors:

$$V^{\mu}{}_{;\lambda} = \frac{\partial V^\mu}{\partial x^\lambda} + \Gamma^\mu_{\lambda\kappa} V^\kappa$$

It is instructive to make a comparison between the Lorentz invariance Principle and the General Covariance Principle.
Any equation can be made Lorentz invariant, eg. the Newtonian second law. But the equation in the transformed system would contain the velocity of the second coordinate frame. Lorentz invariance is the requirement that these quantities cannot appear in the equations of Physics.
Let us consider now the Principle of General Covariance. Every equation can be written in a general covariant manner and two new terms enter in the discussion: the metric tensor and the affine connection. But we do not require that these quantities drop out at the end. Any Physical Principle, such as General Covariance, whose content is a limitation on the possible interactions of a particular field is called a *dynamical symmetry* (note the similarity with gauge invariance).

1.3 Curvature

We have seen in the previous sections that the metric $g_{\mu\nu}$ contains information about the gravitational field.
We know from the Principle of General Covariance that two metric $g'_{\mu\nu}$ and $g_{\mu\nu}$ linked by a differentiable coordinate transformation describe the same physical field.
It is so of fundamental importance to find an object that describes the curvature of our spacetime and which tell us if a metric describe a gravitational field: the Reimann tensor. Indeed, if every component $R_{\mu\nu\lambda\sigma} = 0$ a manifold is flat and exists a coordinate trasformation which maps globally $g_{\mu\nu}$ in the Minkoskian metric $\eta_{\mu\nu}$.
It can be demonstrated that the Riemann tensor is the only quantity which is nonlinear in the first derivatives, linear in the second derivative and is a tensor under general coordinate transformations. In a particular coordinate frame we can write (see ref. [2]):

$$R^{\lambda}{}_{\mu\nu\kappa} = -\frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\kappa} + \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\nu} - \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta} + \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta}$$

We can always put to 0 with a choice of a locally intertial frame the first derivative of the metric contained in Γ. Not such an operation is possible for the second derivatives in $\partial\Gamma$.

From the Riemann tensor we can define the *Ricci* tensor which appear in the Einstein's equations in this manner:

$$R^{\nu}{}_{\mu\nu\lambda} = R_{\mu\lambda}$$

and the *Ricci* scalar:

$$R_{\mu}{}^{\mu} = R$$

With these tensors we can build a symmetric tensor that is covariantly conserved: the *Einstein* tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

$$\nabla^{\mu} G_{\mu\nu} = 0$$

which will appear in the Einstein's equations which describe the dynamics of the gravitational field.

1.4 The stress-energy tensor

The stress energy tensor (sometimes stress energy momentum tensor or energy momentum tensor) is a tensor quantity that describes the density and flux of energy and momentum in spacetime, generalizing the stress tensor of Newtonian physics. It is an attribute of matter, radiation, and non-gravitational force fields.
Moreover the stress energy tensor is the source of the gravitational field in the Einstein's field equations of General Relativity, just as mass density is the source of such a field in Newtonian gravity. Thus it is of fundamental importance to understand correctly its properties.
In special relativity it obeys the conservation equation:

$$\partial_{\alpha} T^{\alpha\beta} = 0$$

From the Principle of general covariance we know that in presence of gravity we would have:

$$\nabla_{\alpha} T^{\alpha\beta} = 0$$

which contains also the information about the energy exchanged between the different fields and the gravitational field.
We stated previously that the stress energy tensor is an attribute of matter, radiation, and non-gravitational force fields. Infact we cannot define an energy momentum tensor for the gravitational field since it is a local object and we know from the Equivalence Principle that we could find a class of *locally* inertial systems in which gravity is absent.
We are now ready to interpret what an observer with 4-velocity v^{μ} would measure:

1. $T_{\mu\nu} v^{\mu} v^{\nu}$ is the energy density that is non negative: $T_{\mu\nu} v^{\mu} v^{\mu} \geq 0$

2. $T_{\mu\nu} v^{\mu} n^{\nu}$ is interpreted as the momentum density of matter.

3. $T_{\mu\nu} n^{\mu} n^{\nu}$ is interpreted as the stress in a particular direction.

where n_{μ} is a unit vector normal to the surface of interest and it verifies $u^{\mu} n_{\mu} = 0$. It is fundamental to understand that a physical observer measure only the full stress energy tensor, not only one component.

1.5 The Einstein's Equations

From the Newtonian theory of gravity we know that matter density creates the gravitational field.
This theory is not correct because it predicts an instantaneous action at distance which is forbidden by the Laws of Special Relativity.
We know that the matter density ρ is the T_{00} component of the stress energy tensor and so it is reasonable to hypotize that the stress-energy tensor is the relativistic source of the gravitational field.
We need another tensorial quantity $G_{\mu\nu}$ that describes the dynamics of the gravitational field.
Since not only the matter density but also the energy density ecc. creates a gravitational field we expect that the equations which describe the dynamics of the gravitational field will be non linear. This because the gravitational field transports these quantities and so "*gravity gravitates*". From the discussion above we know that the only quantity that in non linear in the first derivative and linear in the second derivative of the metric (the gravitational potential) is the Riemann's tensor.
Moreover we know that the conservation of the stress energy tensor is given by $\nabla_\mu T^{\mu\nu} = 0$
But the Riemann tensor is not conserved covariantly.
The Bianchi Identity teaches that the only quantity that is covariantly conserved is:

$$\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)_{;\mu} = 0$$

This is called the Einstein's tensor.
Now we can write the famous Einstein's Equation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

The content of these equations can be reasumed in this statement:

"*matter tells space how to curve and space tells matter how to move*".

These equations can be more formally derived from the generally covariant Einstein-Hilbert action:

$$S = \frac{1}{8\pi G_N}\int d^n x\sqrt{-g}\,(R + L_{matter})$$

where R is the Ricci scalar.
If we consider the Einstein's theory with cosmological constant we can add the term $\Lambda g_{\mu\nu}$ which is permitted since it is covariantly conserved ($\nabla_\mu g_{\mu\nu} = 0$).

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

It is also interesting to note that the the stress energy tensor's conservation law contains a great deal of information about the behaviour of matter.
It can be proved that for a perfect fluid $\nabla_\mu T^{\mu\nu} = 0$ implies the geodesics equation.
The resolution of the Einstein's field equations is a very difficult problem: it is a system of second order non linear equations.
Moreover one has to solve simultaneously for the metric and $T_{\mu\nu}$.
It is usually possible to find exact solutions only if the symmetries are strong, for example

spherical symmetry in vacuum.
In the next chapter we study the well known Schwarzsdchild Solution of the Einstein's Equations describing a spherically symmetric graviational field.

1.6 Causal Structure

As we have already noticed, gravity affects also the motion of massless particles.
Since the causal structure is determined by the light cones, the causal structure in presence of gravity can be very different from the intuitive Minkowkian structure.
The possible emergence of horizons will turn out to be a very important new feature of gravitational fields. Under normal circumstances gravity is so weak that no horizon will be seen, but some physical systems, like a star which undergoes gravitational collapse, may produce horizons.
If this happens there will be regions in space-time from which no signals can be observed.
Another important concept that will be very important in our future discussion is related to the possibility to define in a unique manner the future evolution of a system.
Consider a surface S, we call $future$ domain of dependence, denoted by $D^+(S)$:

$$D^+(S) = [p \in M : every\ past\ causal\ curve\ pass\ through\ p\ interesects\ S]$$

The *past* domain of dependence $D^-(S)$ is defined by interchanging past with future.
The $full$ domain of depence is denoted by

$$D(S) = D^+(S) \bigcup D^-(S)$$

If it verifies

$$D(S) = M$$

where M is the entire spacetime manifold, S is a $Cauchy\ Surface$.
$D(S)$ represents the complete set of events for which all conditions shoud be determined by the knowledge of conditions on S.
A spacetime which possesses a Cauchy surface S is said to be *globally hyperbolic*.

1.7 Killing Vectors

In this section we want to define a way of describing symmetries in a covariant language, which does not depend on any particular choice of the coordinate system.
Consider now a general metric $g_{\mu\nu}(x)$. It is said to be *form-invariant* under a general coordinate transformation $x \Rightarrow x^{'}$ if:

$$g^{'}_{\mu\nu}(x) = g_{\mu\nu}(x)$$

for every point x.
It is obvious from the tensor trasformation rule that we can write:

$$g_{\mu\nu}(x) = \frac{\partial x^{'\rho}}{\partial x^\mu}\frac{\partial x^{'\sigma}}{\partial x^\nu} g^{'}_{\rho\sigma}(x')$$

If the metric $g_{\mu\nu}$ is form invariant it is possible to write:

$$g_{\mu\nu}(x) = \frac{\partial x^{'\rho}}{\partial x^\mu}\frac{\partial x^{'\sigma}}{\partial x^\nu} g_{\rho\sigma}(x') \tag{1.1}$$

Any transformation which verifies the above equation is called an *isometry*.
Now if we take the infinitesimal trasformation

$$x'^{\mu} = x^{\mu} + \epsilon \xi^{\mu}$$

it is simple to expand Eq 1.1 to the first order in ϵ (see ref [1]):

$$\nabla_{\nu} \xi_{\mu} + \nabla_{\mu} \xi_{\nu} = 0$$

A vector which verifies the above equation is called a Killing vector.
It is easy to demonstrate that the quantity (the Killing Energy):

$$E_K = -u^{\mu} \xi_{\mu}$$

is conserved along a geodesic.
Indeed

$$\frac{d}{d\lambda} \left(-u^{\mu} \xi_{\mu} \right) = -\frac{D}{D\lambda} \left(u^{\mu} \xi_{\mu} \right) = -\left(\frac{Du^{\mu}}{D\lambda} \right) \xi_{\mu} - u^{\mu} \left(\frac{D\xi_{\mu}}{D\lambda} \right) = 0$$

since $\frac{Du^{\mu}}{D\lambda}$ is the geodesics equation and

$$\frac{D\xi_{\mu}}{D\lambda} = \xi_{\mu;\beta} u^{\beta} \Rightarrow u^{\mu} \xi_{\mu;\beta} u^{\beta} = 0$$

because from the Killing equation we know that $\xi_{\mu;\beta}$ is antisymmetric.
E_K will be of fundamental importance in the next chapters.

Chapter 2

The Schwarzschild Solution

In this Chapter we introduce the famous General Relativistic Schwarzschild Solution. After an analysis of its symmetries and singularities we introduce several coordinate systems and we study the global properties of this solution.
Finally we discuss the classical laws of Black Hole Mechanics.

2.1 Schwarzschild Coordinates and Basic Features

We are going to study one of the most important exact solution of the Einstein's Equations.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{2.1}$$

We are interested in a vacuum solution so every component of the stress energy tensor vanishes.
It is simple to restate Einstein's Equations as:

$$R_{\mu\nu} = 0$$

since the Ricci scalar R is 0 because of the trace of Eq. 2.1 when $T_{\mu\nu} = 0$.
Let us take the most general manifest spherically symmetric and time independent line element:

$$ds^2 = -e^{\mu(r)}dt^2 + e^{\nu(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where $\mu(r)$ and $\nu(r)$ are the functions which we want to fix.
Using the vacuum Einstein's Equations we find the famous Schwarzschild solution:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

The only parameter present in the Schwarzschild solution is M which represents the mass of the source of the gravitational field measured at infinity.
In General Relativity it is very important to understand correctly the meaning of the coordinates which are used since, as we are going to see, they are only labels for the events and they do not usually have the clear intuitive meaning which they possess in flat spacetime.
Taking

$$ds^2|_{r,t=const} = +r_0^2(d\theta^2 + \sin^2\theta d\phi^2)$$

we understand that θ and ϕ are angular coordinates of a S_2 sphere (symmetric surfaces). The r coordinate is not a common a radial coordinate: it is related to the area of the S_2 spheres

$$r = \sqrt{\frac{A}{4\pi}}$$

and it approach the intuitive notion of a radial coordinate only when $r \to \infty$.
The time coordinate t is related to the clock of a static observer at $r \to \infty$ since a static observer at $r = r_0$ measure with his clock:

$$d\tau^2 = \left(1 - \frac{2M}{r_0}\right) dt^2$$

and so $d\tau < dt$. It means that a static observer in $r = r_0$ sees the clock of an asympotic observer running faster than his.
It can be shown that the Schwarzschild solution is the only spherically symmetric vacuum solution of the Einstein's Equations (this result is called Birkohff Theorem) and thus spherical oscillations of the source do not produce gravitational radiation.
The Schwarzschild solution is asymptotically flat as the metric has the form $g_{\mu\nu} = \eta_{\mu\nu} + O(1/r^2)$ for large r and so will be possible to define several quantities of interest as we are going to see.
This spacetime does not depend on t (and it is invariant under time inversion) and thus $\xi^\mu = \partial/\partial t$ is a timelike Killing vector. Therefore we have the conserved quantity along a geodesic:

$$E = -g_{\mu\nu}\xi^\mu u^\nu = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$$

where $\xi^\mu = (1, 0, 0, 0)$ in Schwarzschild coodinates.
Note that at infinity it reduces to the usual special relativistic formula for the total energy per unit mass as measured by a static observer. From the rotational invariance we know that also $\chi^\mu = \partial/\partial\phi$ (in Schwarzschild coordinates $\chi^\mu = (0, 0, 0, 1)$) is a Killing Vector with the associated conserved quantity (choosing $\theta = \pi/2$):

$$L = g_{\mu\nu}\chi^\mu u^\nu = r^2\frac{\partial\phi}{d\tau}$$

where L is interpreted as the angular momentum/unit mass.
Note that the Schwarzschild line element is *not* well defined in $r = 0$ and $r = 2M$. It is simple but tedious to verify that the curvature invariant is (see ref. [3])

$$R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} = \frac{48M^2}{r^6}$$

From this expression we understand that $r = 0$ is a true singularity of spacetime in which curvature blows up, while $r = 2m$ is only a coordinate singularity which can be eliminated by a coordinate tranformation.
Let us consider the causal structure of our spacetime. We want to find the radial null geodesics in the Schwarzschild coordinates (t, r, θ, ϕ)

$$0 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2$$

and so

$$\frac{dr}{dt} = \pm\left(1 - \frac{2M}{r}\right)$$

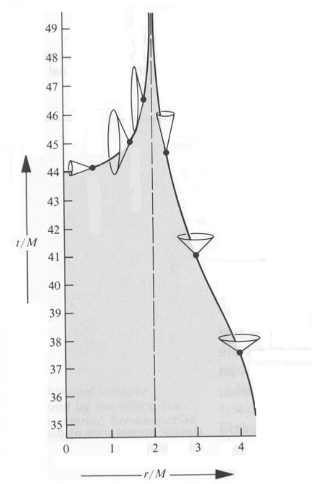

Figure 2.1: Light Cones: The Schwarzschild Spacetime

where the $\pm$ is related to the outgoing and ingoing geodesics.

From Figure 1 we can see that the light cones assume the Minkowkian form as $r \to \infty$ as expected since but have a pathological behaviour at $r = 2M$ since that past directed and future directed light rays coincide.
Inside this surface we have that the exterior spacelike coordinate r and the exterior timelike coordinate t are exchanged.
Thus it is impossible to remain static at $r = const$ and the only physical motion is along decreasing r. Every physical motion ends at the singularity $r = 0$.
Consider now a parametric line in r (we are considering only displacement in r). Outside it is a spacelike geodesic while in the interior region it is timelike.
But the tangent vector has to be paralleled propagated along a geodesic and thus it cannot change its character.
We understand from these results that the strange singular behaviour of the light cones at $r = 2M$ is an artifact of a bad choice of coordinates and it has to be improved by a coordinate transformation in order to understand the characteristics of our spacetime in this point.

2.2 Gravitational Redschift

Consider now two static observer at radius r_1 and r_2. Suppose that the observer 1 sends a signal, for example a photon with frequency ν to the second observer.
The energy measured by an observer with four velocity u^μ is

$$E = -p^\mu u_\mu$$

where p^μ is the four momentum of the photon. Since every observer follows timelike trajectories we have:

$$u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu = g_{00} u^0 u^0 = -1$$

because $u^\mu = (u^0, 0, 0, 0)$. From the above equation we find

$$u^0 = \frac{1}{\sqrt{1 - \frac{2M}{r}}}$$

and so it is easy to find

$$\frac{\nu_1}{\nu_2} = \frac{\sqrt{1 - \frac{2M}{r_2}}}{\sqrt{1 - \frac{2M}{r_1}}}$$

in particular if we take the observer 1 at infinity

$$\nu_1 = \sqrt{1 - \frac{2M}{r_2}}\nu_2 < \nu_2$$

So a photon send by the observer 2 arrives at infinity redshifted. Note in particular that the Schwarzschild radius $r = 2M$ is an infinite redschift surface.

2.3 The Eddington-Filkenstein Coordinates

Normally one would regard the Schwarzschild Solution for $r > r_0$ (with $r_0 > 2M$) as being the solution outside some spherical object of radius r_0, which is described internally by some other solution of the Einstein's equations.
But we know that sufficiently massive bodies will undergo complete gravitational collapse, therefore the region $r \leq 2M$ is physically relevant.
As we have already stated $r = 2M$ is only a coordinate singularity where no curvature invariants diverge. It would be useful to find new coordinates which are well defined there.
It is easy to find the reason because the Schwarzschild coordinates fail to cover $r = 2M$: they are associated to static observers but no one can remain static there because the scalar $a^\mu a_\mu$ (a^μ is the four accelleration) diverges when $r \to 2M$.
Consider now a *radial null* geodesic. It is defined by:

$$0 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})}$$

and thus

$$\frac{dr}{(1 - \frac{2M}{r})} = \pm dt$$

where the $\pm$ is related to the outgoing and ingoing geodesics.
We call

$$\frac{dr}{(1 - \frac{2M}{r})} = dr^* \Rightarrow r^* = \int \frac{dr}{(1 - \frac{2M}{r})} = r + 2M \ln\left(\frac{r}{2M} - 1\right)$$

the Regge-Wheeler tortoise coordinate.
It is now useful to introduce two radial null coordinates which are constant along the outgoing (u) and ingoing null geodesics (v):

$$u = t - r^*$$

$$v = t + r^*$$

Note that they reduce to the usual Minkowkian null coordinates when $r \to \infty$.
Using the coordinates (v, r, θ, ϕ) the metric takes the *advanced* Eddington-Filkenstein form:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

which is obviously non singular at $r = 2M$. Thus using different coordinates we have extended the Schwarzschild metric so that it is no longer singular at $r = 2M$.
If we introduce for semplicity

$$t' = v - r$$

we obtain

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt'^2 + \frac{4M}{r} dt' dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

and we can understand the causal structure of this representation of the Schwarzschild Solution with Figure 2.2.
In the region $r \neq 2M$ we have the same causal structure already founded in the (t, r, θ, ϕ)

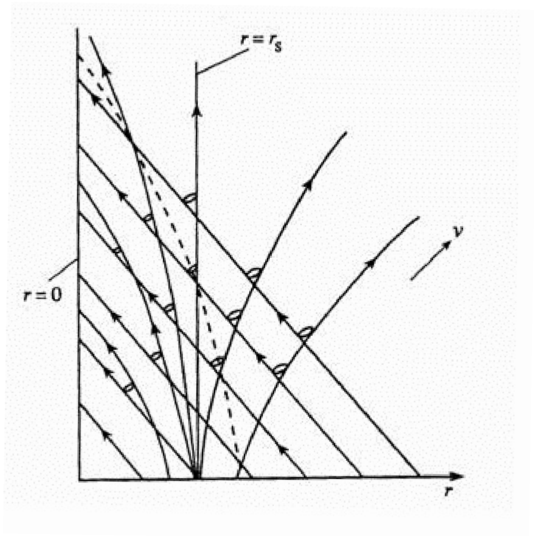

Figure 2.2: Causal Structure: Eddington Filkenstein Advanced Coordinates

diagram.
But now $r = 2M$ is locally depicted as every other point: it has no local strange behaviour. Only from a global point of view we have that it has got some interesting properties.
Note that with this change of coordinates we have lost the time reverse symmetry. The most obvious asymmetry is that of the surface $r = 2M$ which acts as a one-way membrane: null or timelike future directed geodesics cross this surface only from the outside $(r > 2M)$ to the inside $(r < 2M)$.
Let us note that the surface $r = const$ has the signature

1. $r > 2M \Rightarrow (-, +, +)$ and so it is timelike and an observer can remain at $r = const$.
2. $r < 2M \Rightarrow (+, +, +)$ and so it is spacelike and an observer cannot remain at $r = const$.

3. $r = 2M \Rightarrow (0, +, +)$ which a null like surface, only a massless particle can remain at $r = 2m$.

From these results we understand that $r = 2M$ forms the *event horizon*: every particle which passes this surface can never return to the exterior region and can only go towards the singularity.
In particular it is possible to demonstrate with a simple calculation that a geodesic entering the black hole arrives at the singularity $r = 0$ in a finite proper time.
We can also use the coordinate u instead of v. We obtain the *retarded* Eddington Filkenstein metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

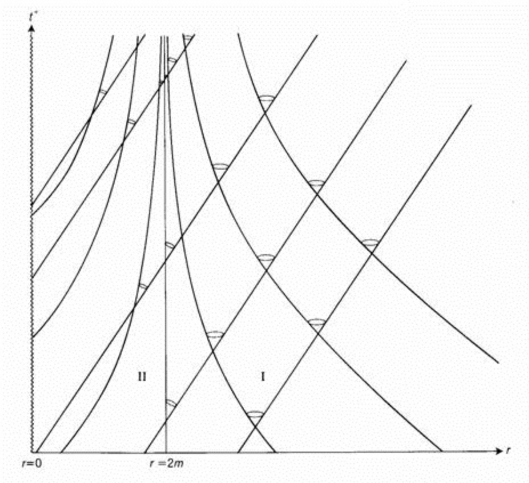

Figure 2.3: Causal Structure: Retarded Eddington Filkenstein coordinates

But this coordinate transformation seems to reverse the direction of time with respect to the retarded Eddington Filkenstein form.
Infact the $r = 2M$ surface is again a null surface which acts as a one-way membrane but it let only past directed timelike or null curves to cross itself from the outside to the inside as we can see from the Figure 2.3.
In order to understand the strange relation between the advanced and the retarded Eddington-Filkenstein metric we have to introduce the Kruskal Coordinates.

2.4 The Kruskal Coordinates

We want now to obtain the maximally extended Schwarzschild solution.
Consider $(M, g_{\mu\nu})$ in the coordinates (u, v, θ, ϕ):

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dvdu + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

In this form the two space (θ, ϕ const) is in null conformally flat coordinates, since $ds^2 = -dudv$ is flat.

The most general tranformation which leaves this space conformally flat is $V = V(v)$ and $U = U(u)$. The resulting metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)\frac{dv}{dV}\frac{du}{dU}dVdU + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

and taking

$$X = \frac{V-U}{2} \quad T = \frac{V+U}{2}$$

we find

$$ds^2 = F^2(t', x')(-dT^2 + dX^2) + r^2(T, X)(d\theta^2 + \sin^2\theta\phi^2)$$

The Kruskal choice (see ref. [3]) $V = e^{v/4M}$, $U = e^{-u/4m}$ determines the form of the metric:

$$ds^2 = \frac{32m^3}{r}e^{-r/2m}(-dT^2 + dX^2) + r^2(T, X)(d\theta^2 + \sin^2\theta\phi^2)$$

The coordinate T is always timelike and X spacelike and r is determined by

$$T^2 - X^2 = -(r - 2M)e^{r/2M}$$

The Kruskal extension is the unique analytic and locally inextendible extension of the Schwarzschild solution. A spacetime diagram is depicted in Fig. 2.4.
If we consider the light cone for $r > 2M$ we find that the outgoing light rays escape to infinity while the ingoing ones go towards the singularity.
Inside $r = 2M$ every null or timelike geodesic fall into the singularity and so $r < 2M$ is a region of no escape: the $Black\ Hole$. In the next section we will give a more formal and precise notion of such an object.
Each point inside the region II represent a 2-sphere that is a closed trapped surface.
Infact consider a 2-sphere p and other two 2-spheres formed by photons emitted (outogoing q and ingoing s) at one istant from p. If all the spheres are outside the event horizon we have that the area of q is greater than p which is greater than s.
But if p is inside the event horizon the areas of both q and s are less than the area of p and so $r < 2M$ forms a closed trapped surface.

Note that in the Kruskal coordinates the light cones take the usual minkowkian form $dT^2 - dX^2 = 0$. While the region I and II are the region of the manifold covered by the advanced Eddington-Filkenstein coordinates, the region $I^{'}$ and $II^{'}$ are related to the retarded Eddington Filkenstein coordinates: it is clear that nothing can enter in the region II' from the asympotically flat region $I^{'}$. Thus region $II^{'}$ descrive the $White\ Hole\ region$.
It is very important to note that only a part of the region I and II is important physically in a gravitational collapse. Let us consider a spherically symmetric star. Its exterior gravitational field is described by the Schwarzschild solution. If the spherical star undergo gravitational collapse, then its surface has to follow a timelike trajectory in the Schwarzschild spacetime and so only a part of the regions I and II are physically relevant for our discussion as clearly depicted in the Penrose Diagram in Figure 2.5.

It is important to recall the relations between the Kruskal and the Eddington-Filkenstein coordinates in regions I and II:

$$U = -e^{-u/4m} \quad r > 2M$$

$$U = +e^{-u/4m} \quad r < 2M$$

$$V = e^{v/4m} \quad \forall r$$

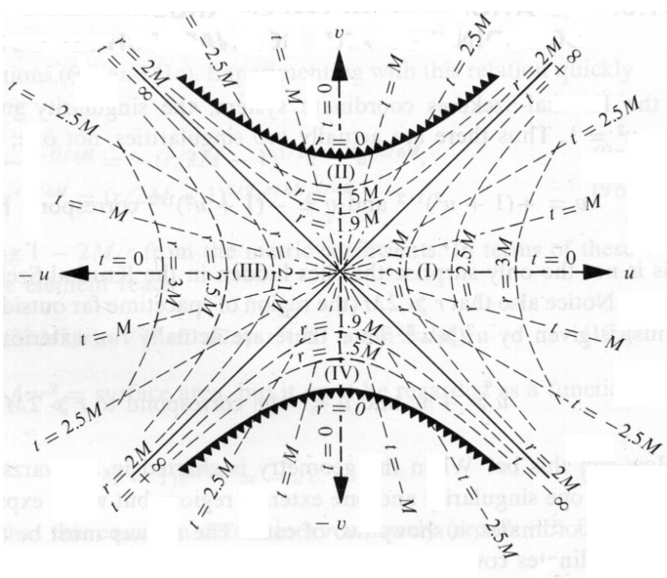

Figure 2.4: Kruskal Diagram

2.5 The Redshift Factor

It is useful to introduce an object which naturally appear when we compare the locally inertial coordinates at the future horizon with the Eddington Filkenstein coordinates.
It is possible to find (see ref [4]) that at the future horizon $H^+ = (U = 0, V = V_0)$ the locally inertial coordinates are defined by:

$$\xi^+_{H^+} = b^+_V[(V - V_0) + O((V - V_0)^3)]$$

$$\xi^-_{H^+} = \frac{1}{eb^+_V}[U + \frac{V_0}{16M^2 e}U^2 + O(U^3)]$$

where b^+_V reflects the possibility of performing arbitrary Lorentz transformations.
The comparison between the pair of *inertial* coordinates $\xi^-_{H^+}$ and u is given by

$$\frac{d\xi^-_{H^+}}{du} = \frac{1}{eb^+_V}e^{-u/4M} + O((e^{-u/4M})^2)$$

which is related to the $redshift\ factor$ for outgoing radiation. Note that it is exponentially decreasing.
Therefore a light ray emitted near the horizon arrives at infinity at *late* time $u \to \infty$ with an highly redshifted frequency $w' \propto we^{-u/4M}$.

2.6 Black Hole

We have defined in the previous section a Black Hole as a region of spacetime where gravity is so strong that any particle or light ray entering that region can never escape from it.

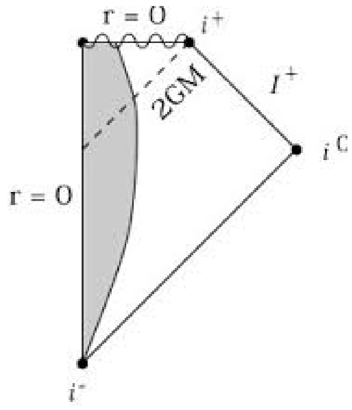

Figure 2.5: Penrose Diagram, Gravitational Collapse

It is an intuitive notion but the essence of a Black Hole is not properly captured defining a black hole in a spacetime $(M, g_{\mu\nu})$ as a subsect A such that we have $J^+(p) \subset A$ (for the definition of $J^+(p)$ see Appendix B). With this definition the causal future of any set in any spacetime would be called Black Hole.
For asympotically flat spacetimes, the impossibility of escaping to future null infinity Π^+ is an appropriate characterization of a Black Hole.
From the Penrose diagram of the Schwarzschild Spacetime it is easy to understand that the causal past of future null infinity $J^-(\Pi^+)$ (see Appendix B) does not contain the entire spacetime: the region II is not contained in $J^-(\Pi^+)$.
Let $(M, g_{\mu\nu})$ be an asympotically flat spacetime with an associated Penrose diagram $(M', g^{'}_{\mu\nu})$.
We say that $(M, g_{\mu\nu})$ is *strongly asymptotically predictable* if the unphysical spacetime $(M', g^{'}_{\mu\nu})$ there is a region $V' \subset M'$ with $\overline{M \cap J^-(\Pi^+)} \subset V^{'}$ such that $(V^{'}, g^{'}_{\mu\nu})$ is globally hyperbolic.

A strongly asymptotically predictable spacetime contains a Black Hole if M is not contained in $J^-(\Pi^+)$.
The *black hole region* B of such spacetime is defined to be

$$B = \left[M - J^-(\Pi^+)\right]$$

and the boundary of B in M

$$H = \dot{J}^-(\Pi^+) \cap M$$

is called the *event horizon*.

2.7 The Killing Energy

In the first chapter we argued that

$$E_K = -u^\mu \xi_\mu$$

where ξ_μ is a Killing vector, is conserved along the geodesic of our spacetime.
We want now to investigate the sign of this quantity. The norm of the Killing vector is

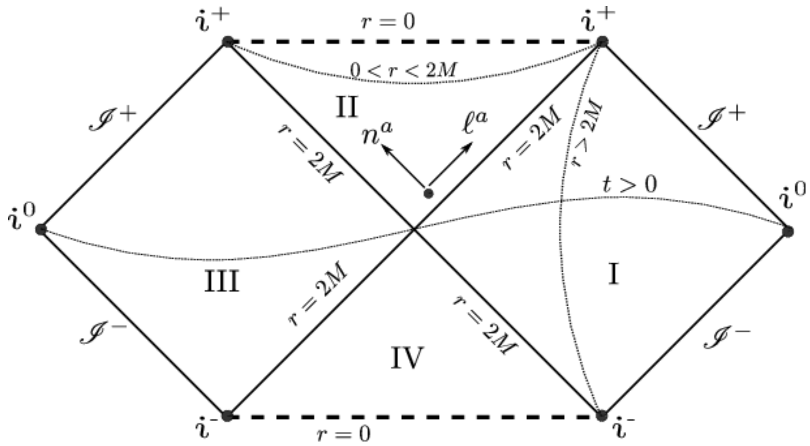

Figure 2.6: Killing Energy: n_α is a positive frequency mode, while l_α has $E_K < 0$

$$\xi^\mu \xi_\mu = g_{\mu\nu}\xi^\mu \xi^\nu = g_{00} = -\left(1 - \frac{2M}{r}\right)$$

which is

1. $r > 2M$ timelike (this means that $r = const$ is a physical motion)
2. $r = 2M$ null like (only a massless particle can remain static at the event horizon)
3. $r < 2M$ spacelike (the only possible physical motion is decreasing r)

Moreover

$$u^\mu = g^{\mu\nu} u_\mu = \frac{E}{(1 - \frac{2M}{r})} = \frac{dt}{d\lambda}$$

where λ is the parameter of our null geodesic.
Consider now the region outside the event hotizon. We have

$$\frac{dt}{d\lambda} > 0 \;\; and \;\; -\left(1 - \frac{2M}{r}\right) > 0 \Rightarrow E_K > 0$$

If we consider the region II (the Black Hole region) we find:

1. for the u outgoing geodesic
$$\frac{dt}{d\lambda} > 0 \;\; and \;\; \left(1 - \frac{2M}{r}\right) < 0 \Rightarrow E_K < 0$$
2. for the v ingoing geodesic
$$\frac{dt}{d\lambda} < 0 \;\; \left(1 - \frac{2M}{r}\right) < 0 \Rightarrow E_K > 0$$

where $u = t - r^*$ and $v = t + r^*$as we have already defined.

We recall that u are the outogoing modes. Therefore it is possible to have states with negative energy inside the Black Hole region.
But these states have to be created in the interior region since (as we have already discussed) the Killing Energy E_K is conserved along the geodesics.
This observations will be very important in the discussion on the Hawking Effect.

2.8 The Laws of Black Holes "Thermodynamics"

In this section we are going to study the classical laws of Black Hole "Thermodynamics". This laws are rigorous theorem of General Relativity and despite of their name classically are not related to thermodynamics.
The phenomenon of Black Hole evaporation which we are going to study in the next chapters will demonstrate that this analogy between thermodynamics and Black Holes is a real profound and beautiful physical result.

2.8.1 The Zertoh Law

Firstly we have to define on the horizon a quantity called κ which will be interpreted as the Black Hole surface gravity.
Consider the Killing vector $\partial/\partial t$. We have on the horizon:

$$\xi^\mu \xi_\mu = 0 \tag{2.2}$$

so in particular ξ^μ is constant on the horizon.
Thus $\nabla^\nu(\xi^\mu \xi_\mu)$ is normal to the horizon. Therefore it exists a function κ such that

$$\nabla^\nu(\xi^\mu \xi_\mu) = -2\kappa \xi^\nu \tag{2.3}$$

For the Schwarzschild Black Hole we have

$$\kappa = \frac{1}{4M} \tag{2.4}$$

We can rewrite the above Eq. 2.3 using the Killing Equation

$$\xi^\mu \nabla_\nu \xi_\mu = -\xi^\mu \nabla_\mu \xi_\nu = -k\xi_\nu \tag{2.5}$$

This is the geodesic equation in a non affine parametrization.
So in the above Eq. 2.5 we have found that κ measures the failure of the Killing parameter v (i.e. $\xi^\mu = (\partial/\partial v)$) to agree with the affine parameter λ along the null generator of the horizon.
We define *on* the horizon

$$k^\mu = e^{-kv}\xi^\mu \tag{2.6}$$

and it verifies (using the previous equations)

$$k^\mu \nabla_\mu k^\nu = e^{-2kv}\left[\xi^\mu \nabla_\mu \xi^\nu - \xi^\mu \xi^\nu \nabla_\mu(\kappa v)\right] = 0 \tag{2.7}$$

so k^μ is the affinely parametrized tangent to the null geodesic generator of the horizon.
Thus we have:

$$\frac{d\lambda}{dv} \propto e^{\kappa v} \Rightarrow \lambda \propto e^{\kappa v} \tag{2.8}$$

Since ξ^μ is and hypersurface orthogonal to the horizon we have (Froubenious Theorem):

$$\xi_{[\mu} \nabla_\nu \xi_{\lambda]} = 0 \tag{2.9}$$

where $w_{[ab]} = 1/2![w_{ab} - w_{ba}]$ and so on. Using the Killing Equation $\nabla_\mu \xi_\nu = -\nabla_\nu \xi_\mu$ we find

$$\xi_\mu \nabla_\nu \xi_\lambda = -2\xi_{[\nu} \nabla_{\lambda]} \xi_\mu \tag{2.10}$$

which is valid *on* the *horizon*. Contracting with $\nabla^\nu \xi^\lambda$

$$\xi_\mu(\nabla^\nu \xi^\lambda)(\nabla_\nu \xi_\lambda) = -2(\xi_\nu \nabla^\nu \xi^\lambda)(\nabla_\nu \xi_\mu) = -2\kappa^2 \xi_\mu \tag{2.11}$$

and finally we obtain a simple formula for κ

$$\kappa^2 = -\frac{1}{2}(\nabla^\mu \xi^\nu)(\nabla_\mu \xi_\nu)|_H \tag{2.12}$$

We can interpret the physical meaning of the surface gravity as the force which an observer at infinity must exert on a unit mass particle to mantain it stationary at the event horizon.

Having defined correctly the surface gravity κ we want now to demonstrate that κ is constant all over the horizon.
Let us recall that

$$\xi^\mu \nabla_\mu \xi_\nu = \kappa \xi_\nu \tag{2.13}$$

If we multiply

$$\xi_\mu \xi_{[\beta} \nabla_{\alpha]} \kappa + \kappa \xi_{[\beta} \nabla_{\alpha]} \xi_\mu = \xi_{[\beta} \nabla_{\alpha]} (\xi^\nu \nabla_\nu \xi_\mu) = \tag{2.14}$$

$$= (\xi_{[\beta} \nabla_{\alpha]} \xi^\nu)(\nabla_\nu \xi_\mu) + \xi^\nu \xi_{[\beta} \nabla_{\alpha]} \nabla_\nu \xi_\mu \tag{2.15}$$

$$= (\xi_{[\beta} \nabla_{\alpha]} \xi^\nu)(\nabla_\nu \xi_\mu) - \xi^\nu R_{\nu\mu[\alpha}^{\ \ \ \ \lambda} \xi_{\beta]} \xi_\lambda \tag{2.16}$$

where we have used the identity

$$\nabla_\mu \nabla_\nu \xi_\alpha = -R_{\nu\alpha\mu}{}^\beta \xi_\beta \tag{2.17}$$

Now, using Eq 2.10 and Eq 2.5 we can write

$$(\xi_{[\beta} \nabla_{\alpha]} \xi^\nu)(\nabla_\nu \xi_\mu) = -\frac{1}{2}(\xi^\nu \nabla_\beta \xi_\alpha)(\nabla_\nu \xi_\mu) = -\frac{1}{2}\kappa \xi_\mu \nabla_\beta \xi_\alpha = \tag{2.18}$$

$$= \kappa \xi_{[\beta} \nabla_{\alpha]} \xi_\mu \tag{2.19}$$

which is equal to the second term of the above Eq 2.14. Therefore using this result we can write:

$$\xi_\mu \xi_{[\beta} \nabla_{\alpha]} \kappa = \xi^\nu R_{\mu\nu[\alpha}^{\ \ \ \ \lambda} \xi_{\beta]} \xi_\lambda \tag{2.20}$$

Moreover if we multiply $\xi_\alpha \nabla_\mu \xi_\nu = -2\xi_{[\mu} \nabla_{\nu]} \xi_\alpha$ by $\xi_{[\beta} \nabla_{\lambda]}$ we find

$$(\xi_{[\beta} \nabla_{\lambda]} \xi_\alpha) \nabla_\mu \xi_\nu + \xi_\alpha \xi_{[\beta} \nabla_{\lambda]} \nabla_\mu \xi_\nu = \tag{2.21}$$

$$-2(\xi_{[\beta} \nabla_{\lambda]} \xi_{[\mu}) \nabla_{\nu]} \xi_\alpha - 2(\xi_{[\beta} \nabla_{\lambda]} \nabla_{[\nu} \xi_\alpha) \xi_{\mu]} \tag{2.22}$$

using repeatedly $\xi_\alpha \nabla_\mu \xi_\nu = -2\xi_{[\mu} \nabla_{\nu]} \xi_\alpha$ the first term of Eq. 2.21 cancels with the first term in Eq. 2.22 and, using $\nabla_\mu \nabla_\nu \xi_\alpha = -R^\beta_{\nu\alpha\mu} \xi_\beta$ we reduce the above equation in the form

$$-\xi_\alpha R_{\mu\nu[\lambda}{}^\sigma \xi_{\beta]} \xi_\sigma = 2\xi_{[\mu} R_{\nu]\alpha[\lambda}{}^\sigma \xi_{\beta]} \xi_\sigma \tag{2.23}$$

and multiplying for $g^{\alpha\lambda}$

$$-\xi_{[\mu} R_{\nu]}{}^\sigma \xi_\sigma \xi_\beta = \xi_{[\mu} R_{\nu]\alpha\beta}{}^\sigma \xi^\alpha \xi_\sigma \tag{2.24}$$

recalling Eq. 2.20:

$$\xi_\mu \xi_{[\beta} \nabla_{\alpha]} \kappa = \xi^\nu R_{\mu\nu[\alpha}^{\ \ \ \ \lambda} \xi_{\beta]} \xi_\lambda \tag{2.25}$$

we can find finally:

$$\xi_{[\beta} \nabla_{\alpha]} \kappa = -\xi_{[\beta} R_{\alpha]}^{\ \ \ \sigma} \xi_\sigma \tag{2.26}$$

Now we have to use the Einstein Equation plus the *dominant energy condition* (see appendix B).
The dominant energy condition says that the current $T_{\mu\nu}\xi^\nu$ must be null like or timelike for every physically relevant system.
Recalling $k^\mu = e^{-\kappa v} \xi^\mu$ we have

$$k_{[\mu} \nabla_{\nu]} k_\alpha = -e^{-2\kappa v} \left[\frac{1}{2} \nabla_\mu \xi_\nu + \xi_{[\mu} \nabla_{\nu]} (\kappa v) \right] \xi_\alpha \tag{2.27}$$

contracting the above equation with two vectors m^ν and n^α tangent to the horizon (so $\xi^\mu m_\mu = \xi^\mu n_\nu = 0$) we obtain

$$m^\nu n^\mu \nabla_\nu k_\mu = 0 \tag{2.28}$$

and, in the notation of appendix B $\widehat{\nabla_\mu k_\nu} = 0$.
Thus from the Raychaudri's Equation (see Appendix B) the expansion θ, the twist $w_{\mu\nu}$ and the shear $\sigma_{\mu\nu}$ of the null geodesic generators of the horizon vanish. From Eq. B.26 of appendix B we find:

$$R_{\mu\nu} k^\mu k^\nu = 0 \tag{2.29}$$

Using the Einstein's Equations together with the dominant energy condition implies

$$R_{\mu\nu} \xi^\mu \xi^\nu = 8\pi G_N \left[T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right] \xi^\mu \xi^\nu \Rightarrow R_{\mu\nu} \xi^\mu \xi^\nu = 8\pi G_N T_{\mu\nu} \xi^\mu \xi^\nu \tag{2.30}$$

since $\xi^\mu \xi_\mu = 0$ on the horizon, and then we finally find

$$T^\mu_{\ \nu} \xi^\nu \xi_\mu = 0 \tag{2.31}$$

Because of this relation we have that $T^\mu_\nu \xi^\nu$ points in the ξ^μ direction and it implies

$$\xi_{[\alpha} T_{\mu]\nu} \xi^\nu = 0 \tag{2.32}$$

Finally we find the *zeroth law of Black Hole Thermodynamics*:

$$\xi_{[\beta} \nabla_{\alpha]} \kappa = 0 \tag{2.33}$$

which states that the surface gravity κ is *constant* on the horizon.
Note the similarity with the zeroth law of thermodynamics which says that the temperature is constant throughtout a body in thermal equilibrium.

2.8.2 The First Law

Let be Σ an asymptotically flat spacelike hypersurface which intersect the horizon H on a 2-sphere which forms the boundary of Σ. It is possible to find (see ref.[5]) a simple formla for the mass of the Black Hole in a stationary, axisimmetric spacetime.

Consider a static observer. Since it is in a static spacetime the notion of stay in a place is well defined and it means to follow an orbit of the Killing vector field ξ^μ.

$$u^\mu = \frac{\xi^\mu}{V} \tag{2.34}$$

where

$$V = (-\xi^\mu \xi_\mu)^{1/2} \tag{2.35}$$

is the redschift factor. The 4-accelleration is

$$a^\mu = \frac{Du^\mu}{ds} = (\xi^\nu/V)\nabla_\nu(\xi^\mu/V) = \frac{1}{V^2}\xi^\nu\nabla_\nu\xi^\mu \tag{2.36}$$

This is the force applied on a unit mass particle by a local observer.
It is possible to prove (see ref. [5]) that an asympotic observer must exert a force which differs for a factor of V with respect to the local force.

$$F = \int_S N^\nu(\xi^\mu/V)\nabla_\mu\xi_\nu dA \tag{2.37}$$

where N^ν is the normal "outward pointing" normal to S, can be interpreted as the total outward force that must be applied to a unit surface mass density distribuited on a 2-sphere lying in the hypersurface orthogonal to ξ^μ.
Using the Killing equation $\nabla_\mu\xi_\nu = \nabla_{[\mu}\xi_{\nu]}$

$$F = \frac{1}{2}\int_S N^{\mu\nu}\nabla_\mu\xi_\nu dA = -\frac{1}{2}\int_S \epsilon_{\mu\nu\alpha\beta}\nabla^\alpha\xi^\beta \tag{2.38}$$

where $N^{\mu\nu} = 2V^{-1}\xi^{[\mu}N^{\nu]}$ is the normal bivector to the surface S and $\epsilon_{\mu\nu\alpha\beta}$ is the volume element associated with the metric. The integrand is viewed as a 2-form to be integrated on the submanifold S.
If we recall the Newtonian equation for the mass

$$M = \frac{1}{4\pi}\int_S (\vec{\nabla}\phi \cdot \vec{N}dA) \tag{2.39}$$

we find that $F = 4\pi M$. Since these 2 expression do not depend on the surfaces S and they represent the same physical quantity we can identitify the same physical quantity.

$$M = -\frac{1}{8\pi}\int_S \epsilon_{\mu\nu\alpha\beta}\nabla^\alpha\xi^\beta \tag{2.40}$$

This is the $Komar's\ equation$ for the gravitational field source's mass in a static gravitational field.
Moreover

$$M = -\frac{1}{8\pi}\int_S \alpha = -\frac{1}{8\pi}\int_\Sigma d\alpha = \tag{2.41}$$

$$= -\frac{3}{8\pi}\int_\Sigma \nabla_{[\lambda}\epsilon_{\mu\nu]\alpha\beta}\nabla^\alpha\xi^\beta = -\frac{1}{4\pi}\int_\Sigma R^\beta_\sigma\xi^\sigma\epsilon_{\beta\lambda\mu\nu} \tag{2.42}$$

$$= \frac{1}{4\pi}\int_\Sigma R_{\mu\nu}n^\mu\xi^\nu dV = 2\int_\Sigma \left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)n^\mu\xi^\nu dV \tag{2.43}$$

where we have used the identity $\nabla_{[l}(\epsilon_{mn]cd}\nabla^c\xi^d) = \frac{2}{3}R^e_f\xi^f\epsilon_{elmn}$:

$$\epsilon^{\omega\gamma\mu\nu}\nabla_\gamma[\epsilon_{\mu\nu\lambda\sigma}\nabla^\lambda\xi^\sigma] = \epsilon^{\omega\gamma\mu\nu}\epsilon_{\mu\nu\lambda\sigma}\nabla_\gamma\nabla^\lambda\xi^\sigma = \tag{2.44}$$

$$= 4\nabla_\gamma\nabla^\gamma\xi^\omega = -4R^\omega_\gamma\xi^\gamma \tag{2.45}$$

contracting with $\epsilon_{\omega lmn}$. Moreover recalling that there is the boundary Π the final formula for the mass in a static asympotically flat spacetime is

$$M = 2\int_\Sigma \left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) n^\mu\xi^\nu dV - \frac{1}{8\pi}\int_\Pi \epsilon_{\mu\nu\alpha\beta}\nabla^\alpha\xi^\beta \tag{2.46}$$

where ξ^μ is the Killing vector, n^μ is a unit vector perpendicular to the surface Σ and $\epsilon_{\mu\nu\alpha\beta}$ is the volume element.
The first integral can be regarded as the contribution to the total mass of the matter outside the event horizon while the second integral can be regarded as the mass of the Black Hole. Since we are interested in the vacuum state solution $T_{\mu\nu} = 0$ the only interesting element is the boundary integral.
We may evaluate it:

$$\int_\Pi \epsilon_{\mu\nu\alpha\beta}\nabla^\alpha\xi^\beta \tag{2.47}$$

We may express the volume element $\epsilon_{\mu\nu}$ on H as

$$\epsilon_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}N^\alpha\xi^\beta \tag{2.48}$$

where N^α is the ingoing future directed null normal to Π, which verifies $N^\mu\xi_\mu = -1$. This normalization means that if ξ^μ is tangent to a radial null outgoing geodesic then n^μ is tangent to the ingoing geodesic.
Thus:

$$\epsilon^{\mu\nu}\epsilon_{\mu\nu\alpha\beta}\nabla^\alpha\xi^\beta = N_\lambda\xi_\sigma\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\alpha\beta}\nabla^\alpha\xi^\beta = -4N_\alpha\xi_\beta\nabla^\alpha\xi^\beta = -4\kappa \tag{2.49}$$

and so

$$\int_\Pi \epsilon_{\mu\nu\alpha\beta}\nabla^\alpha\xi^\beta = \frac{1}{2}\int_\Pi (\epsilon^{\lambda\sigma}\epsilon_{\lambda\sigma\mu\nu}\nabla^\alpha\xi^\beta)\epsilon_{\mu\nu} = -2\kappa A \tag{2.50}$$

where we have used

$$\int_\Pi \epsilon_{\mu\nu} = A \tag{2.51}$$

which is the area of the event horizon.
We are interested in a law similar to the first law of thermodynamics, so it is useful to find a differential formula for M.

$$\delta M = \frac{1}{4\pi}(A\delta\kappa + \kappa\delta A) \tag{2.52}$$

It can be demonstrated (ref.[7]) that for a Schwarzschild Black Hole holds the relation:

$$8\pi\delta M = -2A\delta\kappa \Rightarrow \delta\kappa = -\frac{4\pi}{\delta MA} \tag{2.53}$$

Now we can subsistute the expression for $\delta\kappa$ in the Eq. 2.52 and we finally find

$$\delta M = \frac{1}{8\pi}\kappa\delta A \tag{2.54}$$

which is the first law of Black Hole Thermodynamics for a Schwarzschild Black Hole. Note the similarity with

$$dE = TdS \tag{2.55}$$

which is the first law of thermodynamics: M represents the same quantity since it is the total energy of the system.
Classically even if the mathematical analogy is manifest, it seems only a curious result since nothing can be emitted by a Black Hole and so it has formally temperature $T = 0$.
When we will include in the next chapters quantum effects in the discussion the situation will change.

2.8.3 The Second Law

Let $(M, g_{\mu\nu})$ be a strongly asymptotically predictable spacetime satisfying $R_{\mu\nu}k^{\mu}k^{\nu} \geq 0$ for all null k_{μ}. Let Σ_1 and Σ_2 be spacelike Cauchy surfaces for the globally hyperbolic region $V^{'}$ with $\Sigma_2 \subset I^{+}_{\Sigma_1}$ and let $\Pi_1 = H \cap \Sigma_1$ and $\Pi_2 = H \cap \Sigma_2$ where H denotes the event horizon (the boundary of the Black Hole region of $(M, g_{\mu\nu})$).
Then the area of Π_2 is greater than or equal to the area of Π_1.

Firstly we establish that the expansion θ of the null geodesics generators of H is non-negative everywhere $\theta \geq 0$.
Suppose $\theta < 0$ at $p \in H$. Let Σ be a spacelike Cauchy surface for $V^{'}$ passing throught p and consider the two-surface $\Pi = H \cap \Sigma$. Since $\theta < 0$ at p we can deform Π outward in a neightborhood of p to obtain a surface $\Pi^{'}$ on Σ which enters $J^{-}(\Pi^{+})$ and has $\theta < 0$ everywhere in $J^{-}(\Pi^{+})$. However let $K \subset \Sigma$ be a closed region lying between Π and $\Pi^{'}$ and let $q \in \Pi^{+}$ with $q \in \dot{J}^{+}(K)$.
According to theorem 2, appendix B, the null geodesic generator of $\dot{J}^{+}(K)$ on which q lies must meet $\Pi^{'}$ orthogonally.
But this is not possible since $\theta < 0$ on $\Pi^{'}$ and thus this generator will have a conjugate point before reaching q (see Theorem 1, appendix B). Thus we must have $\theta \geq 0$ everywhere on H.
So each $p \in \Pi_1$ lies on a future inextendible null geodesic γ contained in H. Since Σ_2 is a Cauchy surface γ must intersect it at the point $q \in \Pi_2$.
In this manner we obtain a map from Π_1 to Π_2. Since $\theta \geq 0$ the area of the portion of Π_2 is greater than or at least equal to the area of Π_1.
Moreover since the map need not be onto we have that new black holes may formed between Σ_1 and Σ_2 the area of Π_2 may be even larger.

This law of Black Hole Thermodynamics is very similar to the second law of thermodynamics

$$\delta S \geq 0 \tag{2.56}$$

where S is the entropy.
This equation means that the entropy has to increase for every irreversible process.

2.8.4 The Third Law

The third law of Black Hole Thermodynamics states that it is impossible to achive $\kappa = 0$ by a physical process.

The surface gravity of a Kerr-Newmann Black Hole is

$$\kappa = \frac{(M^2 - a^2 - e^2)^{1/2}}{2M[M + (M^2 - a^2 - e^2)^{1/2}] - e^2} \tag{2.57}$$

where e is the electric charge and a the angular momentum/unit mass. It is simple to see that this quantity vanishes only for $M^2 = e^2 + a^2$ which is the extremal case.
Explicit calculations show that the closer one gets to and extremal Black Hole, the harder it is to get a further step, situation similar to the third law of thermodynamics.

In the next chapters we will se that quantum mechanics will make this mathematical analogy also a physical reality.

Chapter 3

Quantum Field Theory in Curved Spacetime

After a short review of the Quantum Field Theory of a scalar field in the usual flat Minkowskian spacetime, we generalize the formalism to curved spacetime and we analize the main differences and the new characteristics.

3.1 Scalar field in flat spacetime

We start our discussion from the scalar action

$$S = \int d^4x \left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \right) \tag{3.1}$$

The Euler-Lagrangian Equation of Motion is:

$$\left(\partial_\mu\partial^\mu - m^2\right)\phi = 0 \tag{3.2}$$

We can expand the classical Klein Gordon field in the momentum representation

$$\phi(x) = \int \frac{d^4p}{(2\pi)^{3/2}} e^{ip\cdot x}\phi(p) \tag{3.3}$$

where p is the four momentum and $\phi^*(\mathbf{p})$=$\phi(-\mathbf{p})$ as ϕ is real field. Using the Klein Gordon equation

$$(p^2 + m^2)\phi(p) = 0 \Rightarrow \phi(p) = \delta(p^2 + m^2)f(k) \tag{3.4}$$

and thus we find the most general solution

$$\phi(x) = \int \frac{d^3\mathbf{k}}{[2\omega_\mathbf{k}(2\pi)^3]^{1/2}}[f_\mathbf{k}u_\mathbf{k}(x) + f^*_\mathbf{k}u^*_\mathbf{k}(x)] \tag{3.5}$$

where f are arbitrary complex functions, regular on the hyperbolic manifold $k^2 = -m^2$ which fulfills the reality condition $f(k) = f(-k)$ and

$$u_\mathbf{k}(x) = (\sqrt{2\omega_\mathbf{k}(2\pi)^3})^{-1}\exp\left(-i\omega_\mathbf{k}x^0 + i\mathbf{k}\mathbf{x}\right) \tag{3.6}$$

with $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$. These are eigenfunctions of the Minkowskian Killing vector $\partial/\partial t$ with eigenvalue $-i\omega$ with $\omega > 0$.
The surfaces $t = const$ are Cauchy surfaces for the Minkowski spacetime and so, using the scalar product we find the relations:

$$(u_{\mathbf{h}}, u_{\mathbf{k}}) = \int d^3\mathbf{x} u^*_{\mathbf{h}} i \overleftrightarrow{\partial_0} u_{\mathbf{k}} = \delta(\mathbf{h} - \mathbf{k}) \tag{3.7}$$

$$(u^*_{\mathbf{h}}, u^*_{\mathbf{k}}) = \int d^3\mathbf{x} u_{\mathbf{h}} i \overleftrightarrow{\partial_0} u^*_{\mathbf{k}} = -\delta(\mathbf{h} - \mathbf{k}) \tag{3.8}$$

Moreover

$$(u_{\mathbf{h}}, u^*_{\mathbf{k}}) = \int d^3\mathbf{x} u_{\mathbf{h}} i \overleftrightarrow{\partial_0} u_{\mathbf{k}} = 0 = (u^*_{\mathbf{h}}, u_{\mathbf{k}}) \tag{3.9}$$

From these expression we find that these normalized plane waves form a complete set of orthonormal modes with positive ($u_{\mathbf{k}}$) and negative ($u^*_{\mathbf{k}}$) norm.
The coniugate momentum is:

$$\pi = -\frac{\delta L}{\delta \partial_0 \phi} = \partial_0 \phi \tag{3.10}$$

We are now ready to quantize the system using the canonical commutation relation:

$$[\phi(t, \mathbf{x}), \pi(t, \mathbf{x}')] = i\hbar \delta^3(\mathbf{x} - \mathbf{x}') \tag{3.11}$$

We can now expand the quantum field

$$\phi(t, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{[2\omega_{\mathbf{k}}(2\pi)^3]^{1/2}} [a_{\mathbf{k}} u_{\mathbf{k}}(x) + a^\dagger_{\mathbf{k}} u^*_{\mathbf{k}}(x)] \tag{3.12}$$

where the equal time commutation relations for ϕ and π are equivalent to

$$[a_{\mathbf{h}}, a^\dagger_{\mathbf{p}}] = \delta(\mathbf{k} - \mathbf{k}') \tag{3.13}$$

$$[a_{\mathbf{h}}, a_{\mathbf{p}}] = 0 \tag{3.14}$$

$$[a^\dagger_{\mathbf{h}}, a^\dagger_{\mathbf{p}}] = 0 \tag{3.15}$$

The operator $a_{\mathbf{k}}$ is called destruction operator since it annihilates a quantum with momentum $\mathbf{k}$ while $a^\dagger_{\mathbf{k}}$ is called creation operator since it creates a quantum with momentum $\mathbf{k}$.
In the Heisenberg picture the quantum states span a Hilbert space. A convenient basis in this Hilbert space is the Fock representation.
We can now define the *vacuum* of the theory as the state annihilated by the destruction operator $a_{\mathbf{k}}$

$$a_{\mathbf{k}}|0>= 0 \quad \forall \mathbf{k} \tag{3.16}$$

The one particle state can be generated with the use of the creation operator $a^\dagger_{\mathbf{k}}$

$$a^\dagger_{\mathbf{k}}|0>= |1_{\mathbf{k}}> \tag{3.17}$$

and so one for the multiparticle states.
In the Minkowskian Theory, different inertial observer's states are linked by unitary transformation U which preserve the particle number. In particular the vacuum does not change under Poincarrè transformations

$$U|0>= |0> \tag{3.18}$$

where if we call $L_{\mu\nu}$ the angular momentum operator, $\omega_{\mu\nu}$ the matrix which contains the various parameter of the Lorentz transformation, P^μ the four momentum operator and a^μ the 4 vector of the translation:

$$U = \exp\left(ia^\mu P_\mu + \frac{i}{2}w^{\mu\nu}L_{\mu\nu}\right) \tag{3.19}$$

Moreover a Poincarré transformation links positive/negative modes to modes with the same sign of the energy.
The particle number operator is defined as:

$$N_{\mathbf{k}} = a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \tag{3.20}$$

and, once applied to a state, it gives the number of particles present in the state, which is equal for every inertial observer.
Note that the Fock space states are eigenvectors of the number operator.
Another very important object is the Feyman propagator (see ref. [6]). It is defined as

$$G_F(x, x') =< 0|T(\phi(x)\phi(x'))|0 > \tag{3.21}$$

and verifies

$$(\Box_x - m^2)G_F(x - x') = -i\hbar\delta(x - x') \tag{3.22}$$

where it depends on the difference $x - x'$ because of translation invariance.
It is easy to find

$$G_F(x - x') = i\int \frac{d^4k}{(2\pi)^4}\frac{e^{+ik\cdot(x-x')}}{k^2 + m^2 + i\epsilon} \tag{3.23}$$

3.1.1 The vacuum energy

It is easy to find, using the Noëther theorem that the stress energy tensor is

$$T_{\mu\nu} = -\frac{\delta L}{\delta\partial_{\mu\phi}}\partial_\nu\phi + g_{\mu\nu}L = \tag{3.24}$$

$$= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}L \tag{3.25}$$

and thus the Hamiltonian

$$H = T_{00} = \int d^3x\frac{1}{2}\left(\pi^2(x) + (\nabla\phi)^2 + m^2\phi^2\right) \tag{3.26}$$

Substituting the normal mode expansion we find

$$H = \sum_{\omega_{\mathbf{k}}}\frac{1}{2}\hbar\omega[a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + a_{\mathbf{k}}a^\dagger_{\mathbf{k}}] \tag{3.27}$$

It is simple, using the commutation rules to find that

$$< 0|H|0 >= \sum_{\omega_{\mathbf{k}}}\hbar\omega_{\mathbf{k}}\left(\frac{1}{2}\delta(0)\right) \tag{3.28}$$

which is clearly divergent.
This divergence, in the Minkowskian theory can be eliminated by the normal ordering prescription ($: a_{\mathbf{k}} a^{\dagger}_{\mathbf{k}} := a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}}$) and thus we have:

$$H = \sum_{\omega_{\mathbf{k}}} \hbar\omega [a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}}] \tag{3.29}$$

which is clearly finite when it acts on a multiparticle quantum state.
This is the first example of renormalization.
This reasoning can only apply in the Minkowskian theory because in a non gravitational theory we measure only the energy differences. We will see in the next chapters that the situation will change in presence of gravity.

3.2 Scalar field in curved spacetime

Let us start with the generally covariant scalar field action in curved spacetime (see ref [7]):

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[-g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) - (m^2 + \xi R(x))\phi^2\right] \tag{3.30}$$

where the non minimal coupling between the scalar field and the gravitational field represented by $\xi R\phi^2$ is the only possible local scalar coupling with the correct dimensions.
The equation of motion is

$$(\Box - m^2 - \xi R(x))\phi = 0 \tag{3.31}$$

where

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}g^{\mu\nu}\partial_\nu\phi] \tag{3.32}$$

In particular if we take

$$\xi = \frac{1}{4}[(n-2)/(n-1)] \tag{3.33}$$

where n is the number of spacetime dimesions, we have that the theory with $m = 0$ is conformally invariant.
This means that, under a conformal transformation

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu} \tag{3.34}$$

we have

$$\Box\phi \to \Box\bar{\phi} = 0 \tag{3.35}$$

We generalize the Minkowskian scalar product to

$$(\phi_1, \phi_2) = -i\int_\Sigma d\Sigma n^\mu \sqrt{-g_\Sigma}\phi_1 \overleftrightarrow{\partial_\mu} \phi_2^* \tag{3.36}$$

where n^μ is a unit vector normal to the spacelike Cauchy Σ surface in the globally hyperbolic spacetime and g_Σ is the determinant of the induced metric on the Caucly surface.
There exists a complete set of mode solutions which verify

$$(u_{\mathbf{k}}, u_{\mathbf{k}'}) = \delta_{\mathbf{k},\mathbf{k}'} \quad (u^*_{\mathbf{k}}, u^*_{\mathbf{k}'}) = -\delta_{\mathbf{k},\mathbf{k}'} \quad (u_{\mathbf{k}}, u^*_{\mathbf{k}'}) = 0 \tag{3.37}$$

In this basis we can expand the quantum field as in the Minkowskian theory

$$\phi(t, \mathbf{x}) = \sum_{\mathbf{k}} [a_{\mathbf{k}} u_{\mathbf{k}}(x) + a^\dagger_{\mathbf{k}} u^*_{\mathbf{k}}(x)] \tag{3.38}$$

where

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'} \tag{3.39}$$

while the others are 0. Moreover

$$a_{\mathbf{k}}|0>= 0 \quad \forall \mathbf{k} \tag{3.40}$$

is the vacuum state related to this quantization scheme. From this we can build the usual Fock space throught the action of the creation operator $a^\dagger_{\mathbf{k}}$.
While in flat spacetime there is a natural set of modes associated with the Poincarrè group, in curved spacetime the situation is not so simple.
In fact in curved spacetime the Poincarré group is no longer a symmetry group of the spacetime and thus in general there will not be Killing vectors which to define positive frequency modes.
Even if in certain spacetimes there could be "natural" coordinates associated with Killing vectors, these do not enjoy the same role as their Minkowski counterparts.
Moreover we know from the General Relativistic Principle of Covariance that the coordinate system is physically irrilevant. Therefore we can now expand in another complete orthonormal set of modes $v_{\mathbf{p}}(x)$ our scalar field

$$\phi(t, \mathbf{x}) = \sum_{\mathbf{p}} [b_{\mathbf{p}} v_{\mathbf{p}}(x) + b^\dagger_{\mathbf{p}} v^*_{\mathbf{p}}(x)] \tag{3.41}$$

where the creation operator $b^\dagger$ and the annihilation operator b verifies the same commutation relation of $a^\dagger$ and a.
We can now define a new vacuum

$$b_{\mathbf{p}}|0>= 0 \quad \forall \mathbf{p} \tag{3.42}$$

and a new Fock space.
We want now to understand the relations between these two different quantization schemes.

3.3 Bogolubov Transformations

Since the two sets considered in the obove section are complete we can expand the modes $v_{\mathbf{p}}$ in the first basis

$$v_{\mathbf{p}} = \sum_{\mathbf{k}} (\alpha_{\mathbf{k}\mathbf{p}} u_{\mathbf{k}} + \beta_{\mathbf{k}\mathbf{p}} u^*_{\mathbf{k}}) \tag{3.43}$$

and controversely

$$u_{\mathbf{k}} = \sum_{\mathbf{p}} (\alpha^*_{\mathbf{k}\mathbf{p}} v_{\mathbf{p}} - \beta_{\mathbf{k}\mathbf{p}} v^*_{\mathbf{p}}) \tag{3.44}$$

These relations are called *Bogolubov Transformations*. The matrices $\alpha_{\mathbf{k}\mathbf{p}}$ and $\beta_{\mathbf{k}\mathbf{p}}$ are called *Bogolubov coefficients* and they can be evaluated using the scalar product

$$\alpha_{\mathbf{k}\mathbf{p}} = (v_{\mathbf{k}}, u_{\mathbf{p}}) \tag{3.45}$$

$$\beta_{\mathbf{kp}} = -(v_{\mathbf{k}}, u^*_{\mathbf{p}}) \tag{3.46}$$

It is easy to find (substituting the Bogolubov transformed modes in the scalar field expansion)

$$a_{\mathbf{k}} = \sum_{\mathbf{p}} (\alpha_{\mathbf{pk}} b_{\mathbf{p}} + \beta^*_{\mathbf{pk}} b^\dagger_{\mathbf{p}}) \tag{3.47}$$

and

$$b^\dagger_{\mathbf{p}} = \sum_{\mathbf{k}} (\alpha^*_{\mathbf{pk}} a_{\mathbf{k}} - \beta^*_{\mathbf{pk}} a^\dagger_{\mathbf{k}}) \tag{3.48}$$

The Bogolubov coefficients verifies also:

$$\sum_{\mathbf{k}} (\alpha_{\mathbf{pk}} \alpha^*_{\mathbf{qk}} - \beta_{\mathbf{pk}} \beta^*_{\mathbf{qk}}) = \delta_{\mathbf{pq}} \tag{3.49}$$

$$\sum_{\mathbf{k}} (\alpha_{\mathbf{pk}} \beta_{\mathbf{qk}} - \beta_{\mathbf{pk}} \alpha_{\mathbf{qk}}) = 0 \tag{3.50}$$

expression derivable from the orthonormality relations.
Obviously, as long as $\beta_{\mathbf{kp}} \neq 0$ the two Fock spaces based on the choice of modes $u_{\mathbf{k}}$ and $v_{\mathbf{p}}$ are different.
For example, if we consider the action of the annihilation operator $a^\dagger_{\mathbf{k}}$ on the vacuum of the second quantization scheme we find

$$a_{\mathbf{k}}|0'> = \sum_{\mathbf{p}} \beta^*_{\mathbf{pk}} |1_{\mathbf{p}}> \neq |0'> \tag{3.51}$$

and the expectation value of the number operator $a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$ is

$$< 0'|N_{\mathbf{p}}|0'> = \sum_{\mathbf{k}} |\beta_{\mathbf{kp}}|^2 \tag{3.52}$$

and thus the vacuum state of the $v_{\mathbf{p}}$ modes contain $\sum_{\mathbf{k}} |\beta_{\mathbf{kp}}|^2$ particles in the $u_{\mathbf{p}}$ mode.
Note that if $u_{\mathbf{k}}$ are positive frequency modes with respect to some general Killing vector field ξ^μ

$$\xi^\mu \partial_\mu u_{\mathbf{k}} = -i\omega u_{\mathbf{k}} \tag{3.53}$$

and the $v_{\mathbf{p}}$ are a linear combination of only positive frequency modes $u_{\mathbf{k}}$ ($\beta_{\mathbf{kp}} = 0$), thus $b_{\mathbf{k}}|0> = a_{\mathbf{k}}|0> = 0$ which means that the vacuum state is shared by the two set of modes). But if $\beta_{\mathbf{kp}} \neq 0$ the $v_{\mathbf{p}}$ will contain positive (u) and *negative* (u^*) frequency contributions from the $u_{\mathbf{k}}$ modes.
It is interesting to write the vacuum state $|0>$ in the $|0'>$ basis. Using the Bogolubov transformation we have

$$a_{\mathbf{k}}|0> = 0 \tag{3.54}$$

and thus

$$\sum_{\mathbf{p}} (\alpha_{\mathbf{pk}} b_{\mathbf{p}} + \beta_{\mathbf{pk}} b^\dagger_{\mathbf{p}})|0> = 0 \tag{3.55}$$

Multiplying for $\alpha^{-1}_{\mathbf{kp}}$ we find

$$\left(b_{\mathbf{q}} + \sum_{\mathbf{kp}} \beta_{\mathbf{kp}} \alpha^{-1}_{\mathbf{kp}} b^{\dagger}_{\mathbf{p}}\right) |0>= 0 \tag{3.56}$$

and calling $-V_{\mathbf{pk}} = \beta_{\mathbf{kp}} \alpha^{-1}_{\mathbf{kp}}$ we finally find

$$\left(b_{\mathbf{q}} - \sum_{\mathbf{kp}} V_{\mathbf{pk}} b^{\dagger}_{\mathbf{p}}\right) |0>= 0 \tag{3.57}$$

The solution of this equation is

$$|0>= \exp\left(\frac{1}{2\hbar} \sum_{\mathbf{kp}} V_{\mathbf{kp}} b^{\dagger}_{\mathbf{k}} b^{\dagger}_{\mathbf{p}}\right) |0'> \tag{3.58}$$

This is a very important result: the vacuum state $|0>$ appear to be a collection of an *even* number of particles in the second quantization scheme.
More precisely if we would condider a charged scalar field the second quantization scheme measures a collection of particle-antiparticle states (in the case of a neutral field obviously particle and antiparticle coincides).

It is important to note that the particle concept is *global*: the particle modes are defined on the whole spacetime and so a particular oberver's specification of the field mode decomposition, and hence the particle number operator, will depend in general on the observer's past history.
This is the motivation for the introduction of local objects in our future discussion.
Moreover will be important also the concept of $Green\ function$.
As in the flat spacetime case the definition for a scalar field ϕ is

$$iG_F(x, x') =< 0|T(\phi(x)\phi(x'))|0> \tag{3.59}$$

but now is important the choice of the quantum state.

3.4 The Schwinger-De Witt Expansion

In this section we study the Schwinger-De Witt expansion of Green Functions.
This would be of fundamental importance in the calculation of the mean value quantum stress energy tensor in Chapter 5.
We know that in the regularization process of ultraviolet divergences, only the high energy behaviour of the field is important. Since high frequency probes only short distances one is led to study short distance approximations.
Let us introduce normal Reimann coordinates with respect to an origin placed at the point x.
Suppose there exists a neighborhood of this point in which there is an unique geodesic joining any point of the neighborhood of x. This is called a normal neighborhood of x.
The Reimann coordinates (see ref. [8]) y^μ at x are given by

$$y^\mu = \lambda \xi^\mu \tag{3.60}$$

where λ is the value at x of an affine parameter of the geodesic joining x to x' and ξ^μ is the tangent vector.
We choose the parameter to be $\lambda = 0$ at the origin $x = 0$.
The tangent vector in x is

$$\xi^\mu = \left(\frac{dx^\mu}{d\lambda}\right)|_x \tag{3.61}$$

Along any geodesic throught x, the tangent vector is constant or independent of λ. Therefore the geodesic equation becomes

$$\frac{d^2 y^\mu}{d\lambda^2} = 0 \tag{3.62}$$

which implies that in normal coordinates we have

$$\Gamma^\mu_{\alpha\beta}(y)\frac{dy^\alpha}{d\lambda}\frac{dy^\beta}{d\lambda} = \Gamma^\mu_{\alpha\beta}(y)\xi^\alpha(y)\xi^\beta(y) = 0 \tag{3.63}$$

Multiplying by λ^2 we find

$$\Gamma^\mu_{\alpha\beta}(y)y^\beta y^\alpha = 0 \tag{3.64}$$

We have at the point x itself $\Gamma^\mu_{\alpha\beta}(x)\xi^\beta\xi^\alpha = 0$ for every ξ^μ pointing along any geodesic throught x and, in these coordinates

$$\Gamma^\mu_{\alpha\beta}(x) = 0 \tag{3.65}$$

and we can write at the point x

$$g_{\mu\nu}(x) = \eta_{\mu\nu} \tag{3.66}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. With these coordinates we can expand the metric near the point x (see ref.[7])

$$g_{\mu\nu}(x') = \eta_{\mu\nu} + \frac{1}{3}R_{\mu\alpha\nu\beta}y^\mu y^\alpha y^\beta + \cdots \tag{3.67}$$

where all the coefficients are evaluated at $y = 0$.
Let us define

$$Ç(x, x') = \sqrt{-g}G_F(x, x') \tag{3.68}$$

and the Fourier transform

$$Ç_F(x, x') = \int \frac{d^n k}{(2\pi)^n} e^{-ik\cdot y} Ç_F(k) \tag{3.69}$$

where $k \cdot y = \eta^{\alpha\beta}k_\alpha y_\beta$. In this manner we are are working in a localized momentum space. Expanding in normal coordinates and converting in the k space we can find the solution at each adiabatic order. Thus, it can be demonstrate that (see ref.[7] and ref. [8])

$$Ç_F(k) \approx (k^2 + m^2)^{-1} + \left(\frac{1}{6} - \xi\right) R(k^2 + m^2)^{-2} + \cdots \tag{3.70}$$

where $\partial_\alpha = \partial/\partial k^\alpha$.
Substituting the above expression in the Fourier expansion we find

$$Ç_F(x, x') \approx \int \frac{d^n k}{(2\pi)^n} e^{-ik\cdot y}\left[a_0(x, x') + a_1(x, x')\left(-\frac{\partial}{\partial m^2}\right) + a_2(x, x')\left(\frac{\partial}{\partial m^2}\right)\right](k^2 - m^2)^{-1} \tag{3.71}$$

where $\approx$ indicate an asymptotic expansion and (see ref. [7])

$$a_0(x,x') = 1 \tag{3.72}$$

$$a_1(x,x') = \left(\frac{1}{6}-\xi\right)R - \frac{1}{2}\left(\frac{1}{6}-\xi\right)R_{;\alpha}y^{\alpha} - \frac{1}{3}a_{\alpha\beta}y^{\alpha}y^{\beta} \tag{3.73}$$

with the R and its derivative are evaluated at x'.
Using the representation

$$(k^2+m^2-i\epsilon)^{-1} = -i\int dse^{-is(k^2+m^2-i\epsilon)} \tag{3.74}$$

Interchanging the integrations and performing explicitly the integration in dk we find

$$\c{G}_F(x,x') = -i(4\pi)^{-n/2}\int_0^{\infty} ids(is)^{-n/2}\exp\left[-im^2s+(\sigma/2is)\right]F(x,x';is) \tag{3.75}$$

where

$$\sigma(x,x') = -\frac{1}{2}y_{\alpha}y^{\alpha} \tag{3.76}$$

which is an half of the proper distance between x and x' and

$$F(x,x';is) \approx a_0(x,x') + a_1(x,x')(is) + a_2(x,x')(is)^2 + \cdots \tag{3.77}$$

Using $\c{G}_F(x,x') = \sqrt{-g}G_F(x,x')$ we find a representation for the Green Function called the *Schwinger-De Witt expansion*:

$$G_F^{DS}(x,x') = -i\Delta^{\frac{1}{2}}(x,x')(4\pi)^{-n/2}\int_0^{\infty} ids(is)^{-n/2}\exp\left[-im^2s+(\sigma/2is)\right]F(x,x';is) \tag{3.78}$$

where Δ is the Van Vleck determinant (necessary for the General Covariance of the expression):

$$\Delta(x,x') = -\det[\partial_{\mu}\partial_{\nu}\sigma(x,x')][g(x)g(x')]^{-\frac{1}{2}} \tag{3.79}$$

In normal coordinates this reduces to the simple form

$$\Delta(x,x') = (\sqrt{-g(x)})^{-1} \tag{3.80}$$

because

$$\sigma(x,x') = \frac{1}{2}\eta_{\mu\nu}(y^{\mu}-y'^{\mu})(y^{\nu}-y'^{\nu}) \tag{3.81}$$

and thus

$$\partial_{\mu}\partial'_{\nu}\sigma = -\eta_{\mu\nu} \tag{3.82}$$

and taking normal coordinates with the origin in x' we find the previous result.
In the treatment of DeWitt, the extension of the asymptotic expansion of F to all adiabatic order is written as

$$F(x,x';is) = \sum_j a_j(x,x')(is)^j \tag{3.83}$$

with $a_0(x,x') = 1$, the other a_j derived by recursion relation.
The integral can be performed to give the adiabatic expansion of the Feynman propagator in coordinate space:

$$G_F^{DS} \approx \frac{-i\pi\Delta^{\frac{1}{2}}(x,x')}{(4\pi i)^{n/2}} \sum_{j=0}^{\infty} a_j(x,x') \left(-\frac{\partial}{\partial m^2}\right) \tag{3.84}$$

$$\left[\left(\frac{2m^2}{-\sigma}\right)^{(n-2)/4} H^{(2)}_{(n-2)/2}((2m^2\sigma)^{1/2})\right] \tag{3.85}$$

where H are the Haenkels functions of the second order and in which a small imaginary part should be substracted from σ (eg. Feymann prescription).
Note that this expansion does not determine a particular vacuum state since we did not use any global boundary condition. This means that the high energy behaviour is the *same* for almost all choice of vacuum state, a fact of considerable importance as we will see in the next chapters.

Chapter 4

Hawking Radiation

In this Chapter we apply to a 2D model of gravitational collapse the previously discussed Quantum Field Theory in Curved Spacetime.
We prove that Black Holes emitt quantum mechanically a thermal spectrum of particles.
Then we analyze the physical aspects of this process, from correlations to the information paradox.

4.1 Gravitational Collapse

Let us consider a massless scalar field in the Schwarzschild background.
The state $|0_{in}>$ corresponds to the absence of particles at $t = -\infty$.
Since we are working in the Heinsenberg picture the physical state will be always described by this quantum state.
Our first task is to compute

$$< O_{in}|N^{out}_{\mathbf{p}}|0_{in} >= \sum_{\mathbf{k}} |\beta_{\mathbf{kp}}|^2 \tag{4.1}$$

where $N^{out}_{\mathbf{p}} = a^{\dagger out}_{\mathbf{p}} a^{out}_{\mathbf{p}}$ at late times.
We start from the $Vaidya$ class of spacetimes

$$ds^2 = -\left(1 - \frac{2M(v)}{r}\right) dv^2 - 2dvdr + r^2(d\theta^2 + \sin^2\theta + d\phi^2) \tag{4.2}$$

With the stress energy tensor defined as

$$T_{vv} = \frac{L(v)}{4\pi r^2} \tag{4.3}$$

where

$$\frac{dM(v)}{dv} = L(v) \tag{4.4}$$

We can interpret it as an flux of ingoing radiation. In order to simplify our discussion we discard some of gravitational collapse's realistic feature and we can take

$$L(v) = \delta(v - v_0) \tag{4.5}$$

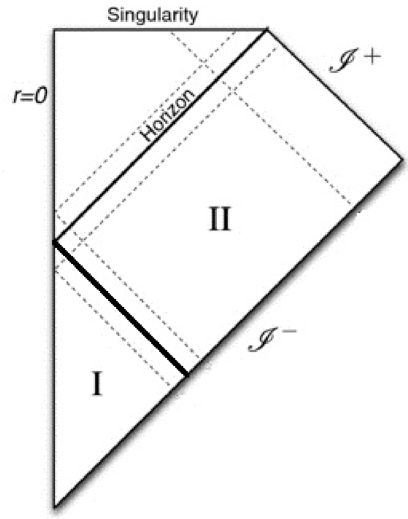

Figure 4.1: Gravitational Collapse

which describes an ingoing wave located at $v = v_0$. This gives $M(v) = M\theta(v - v_0)$. In this model the spacetime geometry can be divided in two regions:

1. $v < v_0$ A Minkowski vacuum region

$$ds^2 = -du_{in}dv_{in} + r^2(u_{in}, v_{in})(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.6}$$

where $u = t - r$ and $v = t + r$ as always. The Klein Gordon equation reads

$$\partial_\mu \partial^\mu \phi = 0 \tag{4.7}$$

whose solution is

$$f(x^\mu) = \sum_{l,m} \frac{f_l(t_{in}, r)}{r} Y_{lm}(\theta, \phi) \tag{4.8}$$

where Y_{lm} are the spherical harmonics.
For each angular momentum the Klein-Gordon equation for $f(x^\mu)$ is converted to a two dimensional wave equation for $f_l(t, r)$ with a non vanishing potential:

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2}\right) f_l(t_{in}, r) = 0 \tag{4.9}$$

2. $v > v_0$ The Schwarzschild Black Hole region

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du_{out}dv_{out} + r^2(u_{out}, v_{out})(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.10}$$

where as already defined in Chapter II $u_{out} = t_{out} - r^*$ and $v_{out} = t_{out} + r^*$.
The Klein Gordon equation reads

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \phi) = 0 \tag{4.11}$$

and for every l we have

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^{*2}}-V_l(r)\right)f_l(t_{out},r)=0 \quad (4.12)$$

where

$$V_l(r)=\left(1-\frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right] \quad (4.13)$$

We note that the potential vanishes at *both* $r=2M$ and $r=+\infty$.
Since the important physics happens *near* the horizon and the s-wave is less affected by the potential ($V_0<V_l\ \forall l$) than the $l\neq 0$ waves we can take $V_l(r)=0$.
This assumption simplifies our discussion without changing the key ideas and results.
Having discarted the potential term V_l, the outgoing and the ingoing modes are *disaccoppiate*. In particular we have no backscattering of the modes due to the curvature of spacetime and the ingoing v modes directly fall into the Black Hole.
With these approximations we have for $v<v_0$

$$\left(-\frac{\partial^2}{\partial t_{in}^2}+\frac{\partial^2}{\partial r^2}\right)f(t,r)=0\Rightarrow\partial_{u_{in}}\partial_{v_{in}}f=0 \quad (4.14)$$

with the regularity condition $\phi(t,r=0)=0$.
In the Schwarzschild region $v>v_0$ we find

$$\left(-\frac{\partial^2}{\partial t_{out}^2}+\frac{\partial^2}{\partial r^{*2}}\right)f(t,r)=0\Rightarrow\partial_{u_{out}}\partial_{v_{out}}f=0 \quad (4.15)$$

Motivated by the fact that the past null infinities Π^- coincide in the two different spacetime regions we have $v_{in}=v_{out}$ and, if we consider $v=v_0$ the matching condition reads

$$ds^2|_{v_0}=+r_{in}^2(d\theta^2+\sin^2\theta d\phi^2)=+r_{out}^2(d\theta^2+\sin^2\theta d\phi^2) \quad (4.16)$$

which means $r_{in}=r_{out}$.
The solutions the Klein Gordon equations in the different background are:

$$\frac{1}{4\pi\sqrt{\omega}}\frac{\exp\{-i\omega u_{in}\}}{r} \qquad \frac{1}{4\pi\sqrt{\omega}}\frac{\exp\{-i\omega v\}}{r} \quad (4.17)$$

and for the Schwarzschild region

$$\frac{1}{4\pi\sqrt{\omega}}\frac{\exp\{-i\omega u_{out}\}}{r} \qquad \frac{1}{4\pi\sqrt{\omega}}\frac{\exp\{-i\omega v\}}{r} \quad (4.18)$$

In particular we are interested in the two dimensional model depending only on the (t,r) variables these modes become

$$\frac{1}{\sqrt{4\pi\omega}}\exp\{-i\omega u_{in}\} \qquad \frac{1}{\sqrt{4\pi\omega}}\exp\{-i\omega v\} \quad (4.19)$$

and for the Schwarzschild region

$$\frac{1}{\sqrt{4\pi\omega}}\exp\{\pm i\omega u_{out}\} \qquad \frac{1}{\sqrt{4\pi\omega}}\exp\{-i\omega v\} \quad (4.20)$$

The normalization coefficients will be justified throught the introduction of a scalar product (next sections).

4.2 The fundamental relation

At $v = v_0$

$$r(v_0, u_{out}) = r(v_0, u_{in}) \tag{4.21}$$

Moreover

$$r^* = r + 2M \ln\left(\frac{r}{2M} - 1\right) \tag{4.22}$$

From this relation, substituting the expression in the null radial coordinates for these quantities we find

$$\frac{v_0 - u_{out}}{2} = r(v_0, u_{in}) + 2M \ln\left|\frac{r(v_0, u)}{2M} - 1\right| \tag{4.23}$$

and so

$$\frac{v_0 - u_{out}}{2} = \frac{v_0 - u_{in}}{2} + 2M \ln\left|\frac{v_0 - u_{in}}{4M} - 1\right| \tag{4.24}$$

calling $v_H = \frac{v_0}{4M} - 1$ we find

$$u_{out} = u_{in} - 4M \ln\left|\frac{v_H}{4M} - \frac{u_{in}}{4M}\right| \tag{4.25}$$

We are interested in the asymptotic behaviour of this expression:

1. $r \to \infty$

   $$u_{out} \sim u_{in} \tag{4.26}$$

   At large r there is no difference between u_{in} modes and u modes.
   A mode with positive frequency with respect to the Killing vector $\xi^\mu_{t_{in}}\partial_\mu = \partial_{t_{in}}$ is described in the *out* basis as a positive frequency mode with respect to the Killing vector $\xi^\mu_S\partial_\mu = \partial_t$ where t is the Schwarzschild time.
   Therefore the Bogolubov coefficient β is equal to 0 which means that there is no particle production for large values of r.

2. $r \to 2M$
   The argument of the logarithm vanishes

   $$\frac{v_H}{4M} = \frac{u_{in}}{4M} \tag{4.27}$$

   Taking the free parameter $v_0 = 4M$ we find as $r \to 2M$

   $$u_{out} \sim -4M \ln\left|-\frac{u_{in}}{4M}\right| \tag{4.28}$$

   and so

   $$\frac{u_{in}}{4M} = \exp\{-u_{out}/4M\} \tag{4.29}$$

   This is the definition of the Kruskal outgoing coordinate. It is important to underline that this is an *model independent* prediction.
   From this equation we find that a positive mode with respect to the Minkowskian Killing vector $\xi^\mu_{t_{in}}\partial_\mu = \partial_{t_{in}}$ becomes a mode with positive frequency with respect to the Kruskal time T_K and not the Schwarzschild time t_{out}.
   Thus we expect that the Bougoliubov coefficient $\beta \neq 0$.

Moreover the modes near the Event Horizon will have high frequencies and, considering the geometric optic approximation, they are those arriving to the asymptotic infinity at late times.
Computing the Bogolubov coefficients between the Kruskal modes and the Eddington Filkenstein modes we will find the particle spectrum for late time radiation.

4.3 Quantization in the Schwarzschild Metric and vacuum states

Let us consider the curved spacetime metric in out 2D model:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \tag{4.30}$$

As we can see from the Kruskal diagram (Figure 3, Chapter II), the surface $t = const$ is a Cauchy surface for the outer region of spacetime. Recalling the definition of scalar product

$$(\phi_1, \phi_2) = -i \int_\Sigma d\Sigma n^\mu \sqrt{-g_\Sigma} \phi_1 \overleftrightarrow{\partial_\mu} \phi_2^* \tag{4.31}$$

it easy to find

$$n_\mu = \alpha(x)(1, 0) \Rightarrow n^\mu \partial_\mu = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \partial_t \tag{4.32}$$

where $\alpha(x)$ is a normalization constant, determined by $g^{\mu\nu} n_\mu n_\nu = -1$ and

$$\sqrt{g}|_{t=const} = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \tag{4.33}$$

Therefore we find

$$\int_{2M}^\infty \frac{dr}{\left(1 - \frac{2M}{r}\right)} = \int_{-\infty}^\infty dr^* \tag{4.34}$$

and thus finally

$$(u_1, u_2) = -i \int_{-\infty}^\infty dr^* [u_1(t, r^*) \overleftrightarrow{\partial_t} u_2^*(t, r^*)] \tag{4.35}$$

It can be easyly verified that the basis with positive norm is

$$u_\omega^{R\ out} = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega u} \qquad v = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega v} \tag{4.36}$$

These modes verify

$$\xi_t^\mu \partial_\mu u = \partial_t u = -i\omega u \tag{4.37}$$

and so this basis is composed by positive norm and *positive energy* modes.
A similar analysis can be carried out for the Black Hole region, with Cauchy surface $r = const$.
The scalar product is now

$$(u_1, u_2) = -i \int_0^{2M} dr(u_1(t, r^*) \overleftrightarrow{\partial_{r*}} u_2^*(t, r^*)) (4.38)$$

and the positive norm modes are

$$u_\omega^{L\ out} = \frac{1}{\sqrt{4\pi\omega}} e^{+i\omega u} \qquad v = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega v} \tag{4.39}$$

but now

$$\xi^{t\mu}\partial_\mu u_\omega^{out} = \partial_t u_\omega^{out} = +i\omega u_\omega^{out} \tag{4.40}$$

is a *negative frequency* mode.
This means that an outgoing exicitation of the field created inside the Black Hole region *decreases* the energy of the system.
The massless scalar field can be expanded is this basis as

$$\phi(r,t) = \int d\omega [a_\omega^L u_\omega^L + a_\omega^R u_\omega^R + a_\omega^I u_\omega^I + h.c.] \tag{4.41}$$

where u^R is the outgoing modes in the asymptotically flat spacetime region, u^L the outgoing modes in the Black Hole region and u^I the ingoing modes.

4.4 Vacuum States

4.4.1 Boulware Vacuum

The vacuum state associated with these modes is the *Boulware vacuum*

$$a_\omega^i |B>= 0 \quad i = L, R, I \tag{4.42}$$

This construction coincides asymptotically with the Minkowski quantization and corresponds to the absence of particles as measured by an obsever at infinity.
We will see in the next chapter that it is not the real physical Black Hole's quantum state: it describes the *vacuum polarization* outside a spherical star.
In fact, as we will compute in the next chapter, the mean value of the stress energy tensor diverges at the horizon and thus we would have a large gravitational backreaction which would modify the causal structure of our spacetime.

4.4.2 Unruh Vacuum

We are now interested in another quantization scheme motivated by the fact that near the horizon $u_{in} \to U_K$. We consider ingoing modes described by

$$u_\omega^I = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega v} \tag{4.43}$$

and outgoing modes

$$u_\omega^K = \frac{1}{\sqrt{4\pi\omega_K}} e^{-i\omega U_K} \tag{4.44}$$

which are positive frequency and positive norm modes with respect to the scalar product defined here.
These modes are defined on the *maximally extended* spacetime.

A Cauchy surface for the outgoing modes for *both* the Black Hole region and the asymptotically flat region is the past Event Horizon H^- defined by $V = 0$.
Now the scalar product reads

$$(u_1, u_2) = -i \int_{\Re} dU_K (u_1^K \overrightarrow{\partial_{U_K}} u_2^{*K}) \tag{4.45}$$

We can expand the field in this basis

$$\phi(r,t) = \int d\omega [a^K_{\omega_K} u_K + a^I_\omega u_I + h.c.] \tag{4.46}$$

As we are going to see the Hilbert space of the two quantization schemes (Boulware and Unruh) are not unitarily equivalent.
We can define the *Unruh vacuum* as

$$a^I_\omega |U >= a^K_{\omega_K} |U >= 0 \tag{4.47}$$

This state describes well the evaporation process for late times $t \gg 1/4M$.

4.4.3 The Hartle Hawking vacuum

It is natural to consider also the vacuum state with respect to the Kruskal modes (U, V)

$$\frac{1}{\sqrt{4\pi\omega_K}} e^{-i\omega_K U_K} \qquad \frac{1}{\sqrt{4\pi\omega_K}} e^{-i\omega_K V_K} \tag{4.48}$$

which are defined in the *maximally extended* spacetime.
In this basis the field's expansion reads

$$\phi(r,t) = \int d\omega [a^{out}_{\omega_K} u^{out}_K + a^{ing}_{\omega_K} u^{ing}_K + h.c.] \tag{4.49}$$

The vacuum associated with this construction is called the Hartle-Hawking vacuum, defined by

$$a^I_{\omega_K} |U >= a^{out}_{\omega_K} |U >= 0 \tag{4.50}$$

where $a^{out}_{\omega_K}$ destroys an outgoing U particle and $a^I_{\omega_K}$ destroys an ingoing V particle.
This state describe a Black Hole in a box in thermal equilibrium with the environment.

4.5 Bogolubov Transformations

Consider a star which collapses and forms a Black Hole. A photon which starts from the past null infinity with positive frequency with respect to the Killing vector $\partial_{t_{in}}$, because of the changing of the background metric during the gravitational collapse, will arrive to the future null infinity with a modified frequency, which is a mixture of positive and negative frequency modes at I^-.
From this intuitive physical example we expect that the phenomenon of particle production will be present in this physical system.
We are interested in late time radiation, when the Black Hole has settled down. As we have already proved $u_{in} \to U_K$ near the horizon (which means late times for an asymptotic

observer, see Eddington Filkenstein diagram).
If we consider the maximally extended spacetime we can expand in the complete basis $u^{L/R}, v$ the mode u_K as

$$u_K(w_K, x) = \int_0^\infty d\omega[\alpha^L_{\omega_K\omega} u^L(\omega, x) + \beta^L_{\omega_K\omega} u^{*L}(\omega, x)] \tag{4.51}$$

$$+ \int_0^\infty d\omega[\alpha^R_{\omega_K\omega} u^R(\omega, x) + \beta^R_{\omega_K\omega} u^{*R}(\omega, x)] \tag{4.52}$$

where L and R are referred to the Black Hole region and the asymptotically flat region respectively.
The scalar product does not depend on the choice of the Cauchy surface. Therefore we can choose H^- as Cauchy surface for the outgoing modes.
We are searching for the number of particles in the *out* state and thus we have to compute the coefficient $\beta_{\omega_K\omega}$

$$< in|N^{out}_\omega|in >= \int_0^{+\infty} d\omega^{'} |\beta_{\omega\omega'}|^2 \tag{4.53}$$

However this is not a physically measurable result: there is no uncertainty in the frequency and this implies an absolute uncertain in time because of the Heinsenberg uncertain relation.
As we already state, we want to evaluate the particle production at late times when, in a realistic situation, the Black Hole has settled down to a stationary configuration.
To obtain this result we have to superpose plane waves

$$u^{out}_{jn} = \frac{1}{\sqrt{\epsilon}} \int_{j\epsilon}^{(j+1)\epsilon} d\omega e^{2\pi i\omega n/\epsilon} u^{out}_\omega \tag{4.54}$$

where $j \geq 0$ and n are integers.
These represent wave packets peacked about $u_{out} = 2\pi n/\epsilon$ with width $2\pi/\epsilon$ as easily seen from the above expression.
If we take ϵ to be small the modes are centered around $\omega = \omega_j = j\epsilon$.
Therefore $< in|N^{out}_\omega|in >$ gives the counts of a particle detector sensitive to frequencies within ϵ of ω_j which is turned on for a time interval $2\pi/\epsilon$ at time $u_{out} = 2\pi n/\epsilon$.
Recalling that in the region $r > 2M$ $u = -\frac{1}{\kappa}\ln(-U_K)$ we find

$$\alpha^R_{\omega_K jn} = (u_K(\omega_K, x), u^R_{jn}(\omega, x)) = \tag{4.55}$$

$$-i \int_{-\infty}^0 dU_K \left(\frac{e^{-i\omega_K U_K}}{\sqrt{4\pi\omega_K}}\right) \overleftrightarrow{\partial}_{U_K} \left(\frac{1}{\sqrt{\epsilon}} \int_{j\epsilon}^{(j+1)\epsilon} d\omega \frac{1}{\sqrt{4\pi\omega}} e^{2\pi in\omega/\epsilon} e^{\frac{+i\omega}{\kappa}\ln(-U_K)}\right)^* \tag{4.56}$$

$$= i \int_{-\infty}^0 dU_K \left(\frac{e^{-i\omega_K U_K}}{4\pi\sqrt{\omega_K}}\right) \left(\frac{1}{\sqrt{\epsilon}} \int_{j\epsilon}^{(j+1)\epsilon} d\omega \frac{1}{\sqrt{\omega}} e^{-i\omega(\frac{2\pi n}{\epsilon} + \frac{1}{\kappa}\ln(-U_K))}\right) \tag{4.57}$$

$$\left(-\frac{i\omega}{U_K} + i\omega_K\right) \tag{4.58}$$

calling

$$L = \frac{2\pi n}{\epsilon} + \frac{1}{\kappa}\ln(-U_K) \tag{4.59}$$

the integral over the frequencies can be computed since ω varies in a small interval

$$\alpha_{\omega_K,jn} = i\int_{-\infty}^{0} dU_K \left(\frac{e^{-i\omega_K U_K}}{4\pi\sqrt{\omega_K}}\right)\left(\frac{1}{\sqrt{\epsilon}}\frac{1}{\sqrt{\omega_j}L}e^{-i\omega_j L_j}\sin\frac{\epsilon L_j}{2}\right)\left(-\frac{i\omega_j}{U_K}+i\omega_K\right) \quad (4.60)$$

where $\omega_j = j\epsilon \approx (j+1/2)\epsilon$.
If the branch cut of the ln appearing in L lies on the negative real axis the integrand is analytic in the entire complex plane with expection to the negative real axis.
This makes us able to perform a Wick rotation (see ref. [4]).
If we Wick rotate $U_K \to iU_K$ we find

$$\alpha_{\omega_K,jn} = -\int_{-\infty}^{0} dU_K \left(\frac{e^{\omega_K U_K}}{4\pi\sqrt{\omega_K}}\right)\left(\frac{1}{\sqrt{\epsilon}}\frac{1}{\sqrt{\omega_j}L}e^{-i\omega_j L_j}\sin\frac{\epsilon L_j}{2}\right)\left(\frac{\omega}{U_K}-i\omega_K\right) \quad (4.61)$$

Since $i = e^{i\frac{\pi}{2}}$

$$L = \frac{2\pi n}{\epsilon} + \frac{1}{\kappa}\ln(-iU_K) = \frac{2\pi n}{\epsilon} + \frac{1}{\kappa}\ln(-U_K) + \left(i\frac{\pi}{2}\right) \quad (4.62)$$

The last term, when multiplied by $-i\omega/\kappa$ gives

$$\exp\left(\frac{\pi\omega}{2\kappa}\right) \quad (4.63)$$

In the same manner we can be calculate the coefficient $\beta_{\omega_K,jn}$ in which appear an exponential

$$\exp\left(-\frac{\pi\omega}{2\kappa}\right) \quad (4.64)$$

More explictly the Wick rotated β coefficient is

$$\beta_{\omega_K\omega} = -(u_K, u^*_\omega) = -\int_{-\infty}^{0} dU_K \left(\frac{e^{\omega_K U_K}}{\sqrt{4\pi\omega_K\omega\epsilon}}\right)\frac{\sin\frac{\epsilon L_j}{2}}{L}e^{+i\omega_j\left(\frac{2\pi n}{\epsilon}\right)-\frac{\pi\omega_j}{2\kappa}+i\omega_j\ln(-U_K)} \quad (4.65)$$

$$\left(\frac{\omega}{U_K+i\omega_K}\right) \quad (4.66)$$

Finally we find the fundamental relation

$$|\alpha^R_{\omega_K\omega}|^2 = e^{8\pi M\omega}|\beta^R_{\omega_K\omega}|^2 \quad (4.67)$$

Using the relation

$$\int d\omega_K(|\alpha^R_{\omega_K\omega}|^2 - |\beta^R_{\omega_K\omega}|^2) = 1 \quad (4.68)$$

which is given in Chapter 3 and substituting Eq. 4.67 we find

$$\int d\omega_K\left(e^{8\pi M\omega}-1\right)|\beta^R_{\omega_K\omega}|^2 = 1 (4.69)$$

and so

$$N^R_\omega = \sum_{\omega_K}|\beta^R_{\omega_K\omega}|^2 = \frac{1}{e^{8\pi M\omega}-1} \quad (4.70)$$

which is a thermal distribution of Bose Einstein particles. This results describe what a particle detector measures physically at great time t.

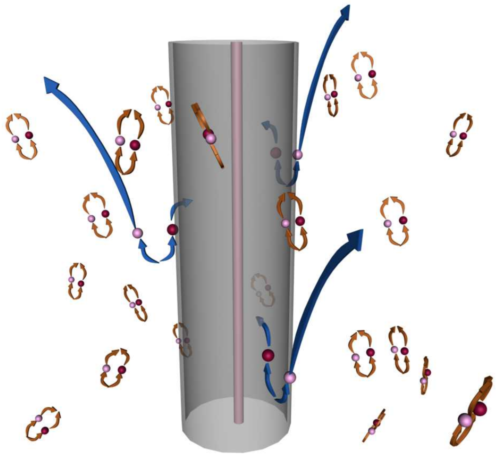

Figure 4.2: Hawking Radiation: particle pairs creation near the Black Hole horizon

This distribution corresponds to a thermal distribution of particles at the *Hawking temperature*

$$T_H = \frac{\hbar c^2}{8\pi k_B G_N M} \tag{4.71}$$

In this beautiful expression: it appears to be a great synthesis of all the physical knowledge. The Hawking temperature of a Black Hole is approximately $T_H \sim 10^{-7}\frac{M_{sun}}{M}K$ (see ref. [9]) .

Moreover it can be verified with the same calculation for the Black Hole region that

$$\sum_{\omega_K} |\beta^L_{\omega_K \omega}| = \sum_{\omega_K} |\beta^R_{\omega_K \omega}| \tag{4.72}$$

Therefore, for every outgoing particle created in the asymptotically flat region there is an outgoing particle created in the Black Hole region.
The inside outgoing particle has clearly $E_K < 0$ (see Chapter 2) and it decreases the energy of the Black Hole as required by the energy conservation principle.

4.6 Thermal Radiation

We want to investigate the Hawking radiation's physical features. If it is a thermal radiation as it seems from the previous results, the probabilities for the emission of the various particles must agree with those of thermal radiation.
We compute now

$$< in|N^{out}_{jn} N^{out}_{jn}|in >=< in|a^{out\dagger}_{jn} a^{out}_{jn} a^{out\dagger}_{jn} a^{out}_{jn}|in > \tag{4.73}$$

Using the Bougoliubov transformations we find

$$< in|N^{out}_{jn} N^{out}_{jn}|in >= \int_0^\infty d\omega' |\beta_{jn,\omega'}|^2 + 2\left(\int_0^{+\infty} d\omega' |\beta_{jn,\omega'}|^2\right)^2 \tag{4.74}$$

$$+\left|\int_0^{+\infty} d\omega' \alpha_{jn,\omega'}\beta_{jn,\omega'}\right|^2 \tag{4.75}$$

where

$$\alpha_{jn,\omega'} = \frac{1}{\sqrt{\epsilon}}\int_{j\epsilon}^{(j+1)\epsilon} d\omega e^{2\pi i\omega n/\epsilon}\alpha_{\omega\omega'} (4.76)$$

In this manner

$$\int_0^{+\infty} d\omega' \alpha_{jn,\omega'}\beta_{jn,\omega'} = \frac{1}{\epsilon}\int_{j\epsilon}^{(j+1)\epsilon} d\omega_1 e^{2\pi i\omega_1 n/\epsilon}\int_{j\epsilon}^{(j+1)\epsilon} d\omega_1 e^{2\pi i\omega_2 n/\epsilon}\int_0^{+\infty} d\omega' \alpha_{\omega_1\omega'}\beta_{\omega_2\omega'} \tag{4.77}$$

It is now necessary to compute, taking I^- as Cauchy surface

$$\int_0^{+\infty} d\omega' \alpha_{\omega_1\omega'}\beta_{\omega_2\omega'} \propto \delta(\omega_1+\omega_2) \tag{4.78}$$

Since both ω_1 and ω_2 are positive the integral is equal to 0:

$$\int_0^{+\infty} d\omega' \alpha_{jn,\omega'}\beta_{jn,\omega'} = 0 (4.79)$$

Therefore we can write

$$< in|N_{jn}^{out}N_{jn}^{out}|in >= \frac{e^{-8\pi M\omega_j}(1+e^{-8\pi M\omega_j})}{(1-e^{-8\pi M\omega_j})^2} \tag{4.80}$$

which is a thermal distribution for the expectation value of $< in|N_{jn}^{out}N_{jn}^{out}|in >$. The thermal probability is given by

$$P(N_{jn}) = (1-e^{-8\pi M\omega_j})e^{-8\pi NM\omega_j} \tag{4.81}$$

Therefore the probability to emit N particles in the mode (jn) is

$$< in|N_{jn}^{out}N_{jn}^{out}|in >= \sum_{N=0}^{+\infty} N^2 P(N_{jn}) \tag{4.82}$$

We can also compute

$$< in|N_{jn}^{out}N_{kn}^{out}|in >= \left(\int_0^{+\infty} d\omega' |\beta_{jn,\omega'}|^2\right)\left(\int_0^{+\infty} d\omega' |\beta_{kn,\omega'}|^2\right) \tag{4.83}$$

$$+\left|\int_0^{+\infty} d\omega' \beta_{jn,\omega'}\beta^*_{kn,\omega'}\right|^2 + \left|\int_0^{+\infty} d\omega' \alpha_{jn,\omega'}\beta_{kn,\omega'}\right| \tag{4.84}$$

where $j \neq k$. This results is obtained throught the use of Bougoliubov transformations.
It can be demonstated that the last two integrals are 0 (see ref.[4]) and thus the only contribution left is

$$< in|N_{jn}^{out}N_{kn}^{out}|in >= \frac{1}{e^{8\pi M\omega_j}-1}\frac{1}{e^{8\pi M\omega_k}-1} \tag{4.85}$$

which corresponds to the product of the two expectation values $< in|N_{jn}^{out}|in >$ and $< in|N_{kn}^{out}|in >$.

This corresponds to the complete *absence* of correlations between *different* modes as typically happens in thermal radiation.
The late time radiation at I^+ is described exactly by a *thermal density matrix* with the temperature given by T_H:

$$\rho_{thermal} = \prod_{jn} \sum_{N=0}^{+\infty} P(N_{jn}) |N_{jn}^{out} >< N_{jn}^{out}| \tag{4.86}$$

$$= \prod_{jn} \left(1 - e^{\frac{-\hbar\omega_j}{k_B T_H}}\right) \sum_{N=0}^{+\infty} e^{-\frac{N\hbar\omega_j}{k_B T_H}} |N_{jn}^{out} >< N_{jn}^{out}| \tag{4.87}$$

where $|N_{jn}^{out} >$ is the N particles state at I^+. As we will see in the next sections every results of a measurement at I^+ is computed using a thermal density matrix. The fact that the quantum state at I^+ is not a pure state seems to be surprising. But we have to include in our analisis the correlations between the modes across the horizon that should restore the purity of the $|in>$ quantum state.

4.7 Correlations

In this section we consider both the outgoing modes and the incoming modes at H^+ because we want to understand correctly the correlations between the different modes to get a full picture of the Hawking effect.
In the Unruh basis

$$\phi = \int_0^{+\infty} d\omega [a_{\omega_K}^K u_{\omega_K} + a_\omega^{in} u_\omega^{in} + h.c.] \tag{4.88}$$

while in the *out* Boulware basis

$$\phi = \int_0^{+\infty} d\omega (a_\omega^{Rout} u_\omega^{Rout} + + a_\omega^{Lout} u_{\omega\omega}^{L*\,int} u_\omega^{int} + a_\omega^I u_\omega^{ingoing} + h.c.) \tag{4.89}$$

where a_ω^{int} and $a_\omega^{\dagger int}$ are destruction and creation operators of incoming particles at H^+.
Since we are looking for correlations *between* emitted quanta at I^+ and quanta at H^+ we need to know the form of u_ω^{int}. Recalling the previous discussion on the Boulware quantization scheme

$$u_\omega^{int} = \frac{1}{\sqrt{4\pi\omega}} e^{+i\omega u} \tag{4.90}$$

Tracing back to I^- the emitted quanta at I^+ (see Fig. 4.1) it is easy to understand, using the Penrose Diagram technique, that the u_ω^{out} have support only in the portion of I^- $v < v_H$ and thus the emitted quanta at I^+ *cannot* see the correlations of the $|in>$ state between those points

$$< in|\phi(v < v_H)\phi(v > v_H)|in > \neq 0 \tag{4.91}$$

Since in the Black Hole region $U_K = 1/\kappa \exp\{-\kappa u\}$ the modes ca be rewrite as

$$u_\omega^{int} = -\frac{1}{\sqrt{4\pi\omega}} e^{-\frac{i\omega}{\kappa}\ln \kappa u} \tag{4.92}$$

In this manner we can define two new Bougoliubov coefficients $\gamma_{\omega\omega'}$ and $\eta_{\omega\omega'}$ relating the *int* modes to the u^K modes (recall $u_{in} \to u_K$ at late times as already proved)

$$u_\omega^{int} = \int_0^{+\infty} d\omega' (\gamma_{\omega\omega'} u_{\omega'}^K + \eta_{\omega\omega'} u_{\omega'}^{*K}) \tag{4.93}$$

$$u_\omega^{out} = \int_0^{+\infty} d\omega' (\alpha_{\omega\omega'} u_{\omega'}^K + \beta_{\omega\omega'} u_{\omega'}^{*K}) \tag{4.94}$$

where the coefficients are given by

$$\eta_{\omega\omega'} = -(u_\omega^{int}, u_{\omega'}^{*K}) = -i \int_0^{+\infty} dU_K u_\omega^{int} \overleftrightarrow{\partial_{U_K}} u_{\omega'}^K \tag{4.95}$$

$$\gamma_{\omega\omega'} = (u_\omega^{int}, u_{\omega'}^K) = -i \int_0^{+\infty} dU_K u_\omega^{int} \overleftrightarrow{\partial_{U_K}} u_{\omega'}^{*K} \tag{4.96}$$

Let us calculate $\eta_{\omega\omega'}$

$$\eta_{\omega\omega'} = -(u_\omega^{int}, u^{K*}) = +i \int_0^{+\infty} dU_K \left(\frac{e^{-\frac{i\omega}{\kappa}\ln(U_K)}}{4\pi\sqrt{\omega_K}} \overleftrightarrow{\partial}_{U_K} \left(\frac{e^{-i\omega_K U_K}}{\sqrt{\omega_K}} \right) \right) \tag{4.97}$$

$$= +i \int_0^{+\infty} dU_K \left(\frac{e^{-\frac{i\omega}{\kappa}\ln(U_K)}}{4\pi\sqrt{\omega_K}} \right) \left(\frac{e^{-i\omega_K U_K}}{\sqrt{\omega_K}} \right) \left(-i\omega_K + \frac{i\omega}{\kappa U_K} \right) \tag{4.98}$$

and Wick rotating $U \to -iU_K$

$$\int_0^{+\infty} dU_K \left(\frac{e^{-\frac{i\omega}{\kappa}\ln(-iU_K)}}{4\pi\sqrt{\omega_K}} \right) \left(\frac{e^{-\omega_K U_K}}{\sqrt{\omega_K}} \right) \left(-i\omega_K - \frac{\omega}{\kappa U_K} \right) \tag{4.99}$$

As we have done in the previous section we can write

$$\int_0^{+\infty} dU_K \left(\frac{e^{-\frac{\pi\omega}{2\kappa}} e^{-\frac{i\omega}{\kappa}\ln(U_K)}}{4\pi\sqrt{\omega_K}} \left(\frac{e^{-\omega_K U_K}}{\sqrt{\omega_K}} \right) \right) \left(-i\omega_K - \frac{\omega}{\kappa U_K} \right) \tag{4.100}$$

The same calculation can be done for $\alpha_{\omega\omega'} = (u_\omega^{out}, u_{\omega_K}^K)$ and we obtain

$$\alpha_{\omega\omega'} = -\int_0^{+\infty} dU_K \left(\frac{e^{+\frac{\pi\omega}{2\kappa}} e^{+\frac{i\omega}{\kappa}\ln(U_K)}}{4\pi\sqrt{\omega_K}} \left(\frac{e^{-\omega_K U_K}}{\sqrt{\omega_K}} \right) \right) \left(i\omega_K - \frac{\omega}{\kappa U_K} \right) \tag{4.101}$$

Thus we can write

$$\eta_{\omega\omega'} = -e^{-4\pi M\omega} \alpha^*_{\omega\omega'} \tag{4.102}$$

and, the same holds for

$$\beta_{\omega\omega'} = -e^{-4\pi M\omega} \gamma^*_{\omega\omega'} \tag{4.103}$$

For the operators

$$a_\omega^{out} = \int_0^{+\infty} d\omega' (\alpha^*_{\omega\omega'} a_{\omega'}^K - \beta^*_{\omega\omega'} a_{\omega'}^{\dagger K}) \tag{4.104}$$

$$a_\omega^{int} = \int_0^{+\infty} d\omega' (\gamma^*_{\omega\omega'} a_{\omega'}^K - \eta^*_{\omega\omega'} a_{\omega'}^{\dagger K}) \tag{4.105}$$

Thus, recalling that the Unruh vacuum for late time coincide with $|in>$

$$a^{out}_{\omega}|in>= -\int_0^{+\infty} d\omega^{'}\beta^{*}_{\omega\omega^{'}}a^{\dagger K}_{\omega^{'}}|in> \tag{4.106}$$

$$a^{int}_{\omega}|in>= \int_0^{+\infty} d\omega^{'}\eta^{*}_{\omega\omega^{'}}a^{\dagger K}_{\omega^{'}}|in> \tag{4.107}$$

and

$$a^{\dagger out}_{\omega}|in>= \int_0^{+\infty} d\omega^{'}\alpha_{\omega\omega^{'}}a^{\dagger K}_{\omega^{'}}|in> \tag{4.108}$$

$$a^{\dagger int}_{\omega}|in>= \int_0^{+\infty} d\omega^{'}\gamma_{\omega\omega^{'}}a^{\dagger K}_{\omega^{'}}|in> \tag{4.109}$$

Using Eq 4.103 we finally find

$$a^{out}_{\omega}|in>= -\int_0^{+\infty} d\omega^{'}\beta^{*}_{\omega\omega^{'}}a^{\dagger K}_{\omega^{'}}|in>= \int_0^{+\infty} d\omega^{'}e^{-4\pi M\omega}\gamma_{\omega\omega^{'}}a^{\dagger K}_{\omega^{'}}|in>= \tag{4.110}$$

$$= e^{-4\pi M\omega}a^{\dagger int}_{\omega}|in> \tag{4.111}$$

Therefore

$$(a^{out}_{\omega} - e^{-4\pi M\omega}a^{\dagger int}_{\omega})|in>= 0 \tag{4.112}$$

$$(a^{int}_{\omega} - e^{-4\pi M\omega}a^{\dagger out}_{\omega})|in>= 0 \tag{4.113}$$

These equations allows to write the *in* state in the *out* Fock space. We can write since from Eq. 4.112 and Eq. 4.113 $V_{ij} = e^{-4\pi M\omega}$

$$|in>=< out|in> \exp\left(\sum_{\omega}\hbar^{-1}e^{-4\pi M\omega}a^{\dagger int}_{\omega}a^{\dagger out}_{\omega}\right)|out> \tag{4.114}$$

$$=< out|in> \prod_{\omega}\sum_{N} e^{-4\pi NM\omega}\frac{1}{N!\hbar^{N}}(a^{\dagger int}_{\omega})^{N}(a^{\dagger out}_{\omega})^{N}|out> \tag{4.115}$$

$$=< out|in> \prod_{\omega}\sum_{N} e^{-4\pi NM\omega}|N^{out}_{\omega}> \otimes |N^{int}_{\omega}> \tag{4.116}$$

where $|N^{out}_{\omega}>$ and $|N^{int}_{\omega}>$ are the N-particles state with frequency ω at I^{+} and H^{+} rispectively. Requiring that $< in|in>= 1$ we find

$$< out|in>= \prod_{\omega}\sqrt{1 - e^{-8\pi M\omega}} \tag{4.117}$$

and so finally

$$|in>= \prod_{\omega}\sqrt{1 - e^{-8\pi M\omega}}\sum_{N}^{+\infty} e^{-4\pi NM\omega}|N^{out}_{\omega}> \otimes |N^{int}_{\omega}> \tag{4.118}$$

This expression is given for continuum normalization states. If we consider wave packets

$$|in>=\prod_{jn}\sqrt{1-e^{-8\pi M\omega_j}}\sum_{N}^{+\infty}e^{-4\pi NM\omega_j}|N_{jn}^{out}>\otimes|N_{jn}^{int}> \quad (4.119)$$

From this equation we understand that we have an independent emission in frequency of entangled quantum states representing outgoing and ingoing radiation. There are *no* correlations between particles emitted in different modes.
Moreover late times for the asymptotic observer ($n\to\infty$) means early time for horizon states $|N_{jn}^{int}>$.
Therefore the correlations ensuring the purity of the *in* vacuum state take place between the late time outgoing quanta and the early time incoming quanta entering the horizon.
The probabilities of occupation of each state are independent and are of the form

$$P(N_{jn})=|<in|(|N_{jn}^{out}>\otimes|N_{jn}^{int}>)|^2 \quad (4.120)$$

$$=(1-e^{-8\pi M\omega_j})e^{-8\pi NM\omega_j} \quad (4.121)$$

which agrees with the thermal results.

4.8 Thermal Density Matrix

Let us consider the mean value of the operator O at future null infinity I^+

$$<in|O|in>=\prod_{jnj'n'}\sqrt{1-e^{-8\pi NM\omega_j}}\sqrt{1-e^{-8\pi N'M\omega_j}}\sum_{NN'}e^{-4\pi N'M\omega_{j'}}e^{-4\pi NM\omega_{j'}} \quad (4.122)$$

$$<N_{j'n'}^{'out}|O|N_{jn}^{'out}><N_{j'n'}^{'int}|N_{jn}^{out}> \quad (4.123)$$

and since $<N_{j'n'}^{'int}|N_{jn}^{'out}>=\delta_{N'N}\delta_{j'j}\delta_{n'n}$ we find

$$<in|O|in>=\prod_{jn}(1-e^{-8\pi NM\omega_j})\sum_{N=0}^{+\infty}e^{-8\pi NM\omega_j}<N_{j'n'}^{out}|O|N_{jn}^{out}> \quad (4.124)$$

$$=Tr\{\rho_{thermal}O\} \quad (4.125)$$

Therefore the $|in>$ vacuum state is described as a thermal state for every physical measurement at I^+.
Tracing the degrees of freedom going down the Black Hole the pure quantum state $|in>$ is described by a thermal density matrix:

$$\rho_{thermal}=Tr_{int}|in><in|=\prod_{jn}(1-e^{-8\pi M\omega_j})\sum_{N=0}^{+\infty}e^{-8\pi MN\omega_j}|N_{jn}^{out}><N_{jn}^{out}| \quad (4.126)$$

4.8.1 Motivating the Exactly Black Body Radiation

There are *two* very important elements which carry us to this result:

1. *divergent redshift* property of the Black Hole Horizon
$$|\alpha_{jn,\omega'}| = e^{4\pi M\omega_j}|\beta_{jn,\omega'}| \tag{4.127}$$
which derive from
$$u_{out}(u_{in}) \approx -4M\ln(-u_{in}/4M) \tag{4.128}$$

2. The exponential appearing in the expression of the $|in>$ vacuum state in the *out* basis
$$|in> = <out|in> \exp\left(\frac{1}{2\hbar}\sum_{ij} V_{ij} a_i^{\dagger out} a_j^{\dagger out}\right)|out> \tag{4.129}$$
with the factor $V_{ij} = e^{-4\pi M\omega_j}$ which is responsible of the black body factor $\exp(-N\hbar\omega_j/k_B T_H)$ which carry to the thermal probability distribution at the temperature T_H.

4.9 The Vacuum States Physical Interpretation

We want now to understand the structure of the vacuum states previously defined

1. *Bolware Vacuum*
As we have put $V_l = 0$ the ingoing and the outgoing modes are disaccoppiate:
$$|B> = |B>_{in} \otimes |B>_{out} \tag{4.130}$$
If a photon is sent from past null infinity to the future null infinity in the spacetime of a static star it would not modify its frequency, thus it arrive at I^+ without any contribution from the negative norm modes at I^-. This means $\beta = 0$.
Therefore, as previously stated, it describe the vacuum outside a static spherical star.

2. *Unruh Vacuum*
The Unruh vacuum is defined in the maximally extended spacetime
$$|U> = |U>_{in} \otimes |U>_{out} \tag{4.131}$$
with $|U>_{out} = |B>_{in}$. This condion means that there are no particles at I^- as in the Boulware case.
As we have proved in the above sections the Unruh state represent a thermal flux of particles at I^+. Therefore it describes the gravitational collapse state at late times.

3. *Hartle-Hawking state*
The Hartle-Hawking state is defined on the maximally extended spacetime, too.
$$|HH> = |HH>_{out} \otimes |HH>_{in} \tag{4.132}$$
with the propriety $|HH>_{out} = |U>_{out}$.
This condition means that $|HH>_{out}$ describe a thermal flux of particles at I^+. Moreover $|HH>_{in}$ describes a thermal flux of particles entering the Black Hole. Thus, as previously stated this vacuum state describes a Black Hole in a box in equilibrium with its environment.

4.10 Black Hole Thermodynamics

In Chapter 2 we proved mathematically the classical Laws of Black Holes Thermodynamics. They are only mathematically related to the ordinary law of Thermodynamics because classically a Black Hole only absorbs particles and so its formal temperature is equal to 0. Moreover with the introduction of quantum effects we have found that a Black Hole emits particles just a Black Body with temperature T_H

$$T_H = \frac{\hbar\kappa}{ck_B 2\pi} \tag{4.133}$$

and this implies the beautiful exact relation between entropy and the area of a Black Hole

$$S_{BH} = \frac{k_B c^3 A}{G_N \hbar 4} \tag{4.134}$$

which is called the *Beckenstein-Hawking entropy*.
While we have a clear physical understanding of the Hawking Temperature we have not a clear and direct derivation of the entropy of a Black Hole based on first principles.
This derivation has to be done counting the quantum degrees of freedom of the Black Hole.
There are a good number of derivations (eg. String Theory, Loop Quantum Gravity...) but none of them is complete.
There are also physical reasons in favor of the interpretation of S_{BH} as the ordinary entropy of the Black Hole.
Beckenstein proposed a generalized second law of thermodynamics: in presence of a Black Hole

$$S^{'} = S + S_{BH} \tag{4.135}$$

cannot decreases in a physical process, where S is the entropy of the matter fields.
This relation has never been proved but there are no known counterexample.
Note that

$$\frac{S_{BH}}{k_B} = \frac{A}{4l_P^2} \approx 10^{+77} \tag{4.136}$$

where $l_P^2 \approx 10^{-66}\ cm^2$ is the squared Planck lenght. S_{BH} is much bigger than the entropy of the star which produces a Black Hole.

4.11 The Transplanckian Problem

We proved in chapter II that a modes with asymptotic frequency ω' would start near the horizon with a frequency ω_{LI} as measured by a locally inertial observer related to the asymptotic measurement as

$$\omega \approx \omega_{LI} e^{-u/4M} \tag{4.137}$$

Our derivation is thus mathematically and physically not coerent: our theory is a good approximation to quantum gravity only for energy $E \ll M_P$, but this derivation involves modes which have arbitrarily large *transplanckian* frequencies with respect to freely falling observers.
However the Hawking evaporation process can be derived using different methods and in different physical situations.

A particular mention to the Bose Einstein Condensates which, under certain physical conditions (supersonic flow) can simulate a Black Hole (see Chapter 6). The equation which the sonic perturbations waves obey are the same of the gravitational setting. Moreover in these systems we have a clear understanding of the microscopic quantum theory.
It has been proved that Hawking Radiation exists theoretically also in these systems.
These models are very important also from the experimental point of view: throught the measurement of the correlations between the modes in the outer region and inside the Sonic Black Hole region physicists are trying to experimentally prove the existence of the Hawking Radiation.

4.12 Black Hole Evaporation

The derivation of the Black Hole evaporation process given in the above sections has got a strong limitation: it is based on the fixed background approximation. This is not consistent with the energy conservation law. In fact the energy radiated by the Black Hole must be balanced by a decreasing of the mass and a correction to the background metric.
The lifetime of a Schwarzschild Black Hole can be estimated

$$\frac{dM}{dt} = -\beta \frac{m_P^3}{t_P} \frac{1}{M^2} \tag{4.138}$$

where $\beta \approx 10^{-5}$ is a dimensionless constant. This leads to

$$M(t) = \left(M_0^3 - 3\beta \frac{m_P^3}{t_P} t \right) \tag{4.139}$$

where M_0 is the initial mass. The Black Hole undergoes a complete evaporation after a time interval

$$\Delta t = \frac{t_P}{3\beta} \left(\frac{M_0}{m_P} \right) \tag{4.140}$$

where M_0 is the initial Black Hole mass and m_P the Planck mass. The semiclassical approximation cannot be used where $m \simeq m_P$ is the Planck mass and so the final state of the Black Hole is actually not known.
Note that the Black Hole Evaporation process causes a great violation of the Barion Number: the energy radiated by the Black Hole carries 0 baryon number since $k_B T_H$ is much less than the mass of any barion until the final stage of the evaporation .

4.13 The Information Paradox

There is a very important paradox related to the phaenomenon of Black Hole evaporation: the $information\ paradox$ (see ref. [4] and ref. [10]).
Classically the existence of the event horizon does not permit to an external observer to know the full detail of the star which forms a Black Hole since the only parameter accessible to him are the energy E, the angular momentum J and the charge Q.
However this information is not lost, it is only inside the Black Hole.
If we consider quantum effects the situation is worst. A state $|in>$ with Cauchy surface

Σ_{in} in Minkowski spacetime is mapped by an *unitary* operator to a final state $|out>$ with Cauchy surface Σ_{out}. If we write

$$|in>=\sum_k c_k^{in}|\psi_k> \tag{4.141}$$

where $|\psi_k>$ are the basis vectors, we can determine exactly

$$|out>=\sum_j c_j^{out}|\psi_j> \tag{4.142}$$

from the coefficients of the $|in>$ state.
This discussion is changed if the causal structure is not simple as in the Minkowskian theory. Consider for example the Schwarzschild Black Hole metric. The final Cauchy surface is splitted:

$$\Sigma_{out}=\Sigma_{int}\cup\Sigma_{ext} \tag{4.143}$$

where Σ_{int} is placed inside the Black Hole. We can repeat the above discussion and the initial state $|in>$ is unitarily mapped in the final state $|out>$ but, considering measures performed by external observers a new phenomenon emerges.
From the previous analysis we know that the $|in>$ state can be regarded as a flux of pairs of entangled particles, one falling into the Black Hole and the other emitted to future null infinity.
The outgoing flux at late times approach a thermal radiation flux.
If we consider only the exterior region we *cannot* describe this state as a pure state of the form $|out>=\sum_j c_j^{out}|\psi_j>$ because we cannot know the value of c_j^{out}. We can only know the probabilities for finding a state $|\psi_k>$. Thus the only thing that we can compute is the probability distribution $P(N)$ and we cannot define a pure state in the exterior region.
It is the causal structure of the Black Hole the ultimate responsible of this *breackdown of quantum predictability*.
Recalling that

$$\rho_{thermal}=Tr_{int}|in><in|=\prod_{jn}(1-e^{-8\pi M\omega_j})\sum_{N=0}^{+\infty}e^{-8\pi MN\omega_j}|N_{jn}^{out}><N_{jn}^{out}| \tag{4.144}$$

we understand that for an external observer the correlations between the interior and the exterior region are lost because of the tracing over the internal degrees of freedom (althought these correlations exists).
The initial state does not determine an unique final state as expressed by the expansion of the density matrix.
Therefore we can only assign probabilities $P(\xi_i)$ to each possible final state $|\xi_i>$ as given by the density matrix ρ

$$\rho=\sum_i P(\xi_i)|\xi_i><\xi_i| \tag{4.145}$$

Finally the correlations are necessarily are *lost* in the singularity when the Black Hole ceases to exist.
The above problem is called *information paradox* because on the basis of the principles of General Relativity and Quantum Mechanics we arrive to a conclusion which contradicts one of the fundamental rules of quantum mechanics: the unitary evolution of quantum systems. Nowadays there is no full satisfactory solution to this paradox.

Chapter 5

The renormalized stress energy tensor

In the previous chapters we pointed out that the particle concept in curved spacetime is not usually related to the intuitive notion of subatomic particle.
In fact, particles are globally definited objects while the instruments used to perform measurements are local.
It is so very interesting to study objects like the stress energy tensor which are defined locally (i.e. at a point), in order to understand the physical features of an Evaporating Black Hole. Moreover the stress-energy tensor is also the source of gravitation. We will find that this object suffers divergences which have to be renormalized in order to find definite physical predictions.
After that we will compute the renormalized stress energy tensor in the quantum states of interest for describing Black Holes.

5.1 The stress energy tensor

As we have computed in Chapter 3, the vacuum energy in Minkowski spacetime is divergent.
This is not a real problem in a non gravitational theory since in these we measure only energy differences.
When the theory involves also gravitation the situation is worst. In a gravitational theory energy is source of gravitation and it will contain information about the spacetime curvature. Moreover, we cannot simply substract the Minkowski zero point energy from the stress energy tensor in our curved spacetime since this difference is infinite, too.
Therefore we need more involved procedures to find physical sensible results.
In the semiclassical theory studied in this thesis, the gravitational force is described by General Relativity while the matter fields obey to the rules of Quantum Field Theory.
It would be very interesting to couple the Einstein tensor to the *expectation value* of the stress energy tensor in the quantum state obtaining the semiclassical field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N < T_{\mu\nu} > \tag{5.1}$$

where Λ is the cosmological constant.
This theory is a good approximation to quantum gravity at scales $\ll L_P$.
The classical Einstein action n-dimesions is

$$S_{E.H.} = \int d^n x \sqrt{-g} \left(\frac{1}{16\pi G_N} (R - 2\Lambda) + S_{matter} \right) \tag{5.2}$$

We have

$$-\frac{2}{\sqrt{-g}} \frac{\delta S_g}{\delta g^{\mu\nu}} = (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}) \tag{5.3}$$

$$-\frac{2}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}} = T_{\mu\nu} \tag{5.4}$$

Since we are in search of the expectation value of $T_{\mu\nu}$ we postulate the existence of an object W such that

$$-\frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g^{\mu\nu}} =< T_{\mu\nu} > \tag{5.5}$$

We define in the Minkowskian theory the *generating functional*

$$Z[J] = \int D\phi \exp\{+iS_{matter}[\phi] + i \int J(x)\phi(x) d^n x\} \tag{5.6}$$

which is interpreted physically as the vacuum persistence amplitude $< 0, out|0, in >$. If the classical source J is equal to 0

$$Z[0] =< 0, out|0, in >=< 0|0 >= 1 \tag{5.7}$$

Path integral quatization works in curved spacetime, too, but in this case we can have $< 0, out|0, in > \neq 0$ also if $J = 0$.
We have clearly

$$\delta Z[0] = i \int D\phi \delta S_{matter} e^{iS_{matter}[\phi]} \tag{5.8}$$

$$= i < 0, out|\delta S_{matter}|0, in > \tag{5.9}$$

Therefore

$$-\frac{2}{\sqrt{-g}} \frac{\delta Z[0]}{\delta g^{\mu\nu}} = i < 0, out|T_{\mu\nu}|0, in > \tag{5.10}$$

If we define W as

$$Z[0] = e^{+iW} \Rightarrow W = -i \ln < 0, out|0, in > \tag{5.11}$$

we finally find

$$-\frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g^{\mu\nu}} = \frac{< 0, out|T_{\mu\nu}|0, in >}{< 0, out|0, in >} \tag{5.12}$$

Since

$$Z[0] = \int D\phi \exp\{i \int \{\frac{1}{2}\phi(\partial_\mu \partial^\mu - m^2)\phi\}\} = N(\det(K))^{-\frac{1}{2}} \tag{5.13}$$

where K is the kinetic element and N is a normalization constant.
We know that $K^{-1}_{xy} = -G_F(x, z)$

$$Z[0] \propto \det(G_F)^{\frac{1}{2}} \tag{5.14}$$

where the proportionality constant is metric independent and so we can ignore it

$$W = -i \ln Z[0] = -\frac{1}{2} tr[\ln{(-G_F)}] \tag{5.15}$$

G_F, the Feynman propagator, can be interpreted as an operator acting on the vectors $|x>$ normalized

$$< x|x^{'} >= \delta^n(x - x^{'}) \frac{1}{\sqrt{-g}} \tag{5.16}$$

and thus

$$G_F(x, x') =< x|G_F|x^{'} > \tag{5.17}$$

The trace in this space is

$$tr(G_F) = \int d^n x \sqrt{-g} < x|G_F|x > \tag{5.18}$$

We can use the proper time representation

$$G_F = -K^{-1} = -i \int_0^{+\infty} e^{-iKs} ds \tag{5.19}$$

Using the Schwinger-De Witt expansion (see Chapter 3) we find

$$< x|e^{-iKs}|x^{'} >= +i(4\pi)^{-n/2} \Delta^{1/2}(x, x^{'}) e^{-im^2 s + \sigma/2is} F(x, x^{'}, is)(is)^{-n/2} \tag{5.20}$$

If K has got a small imaginary part

$$\int_\Lambda^{+\infty} e^{-iKs}(is)^{-1} i ds = \gamma + \ln(-i\Lambda K) + O(x) \tag{5.21}$$

where γ is the Euler constant.
In this manner we find

$$\ln(-G_F) = -\ln(K) = \int_0^{+\infty} e^{-iKs}(is)^{-1} i ds \tag{5.22}$$

that is a correct results up to the addiction of a metric independent and infinite constant that we can ignore.
Thus we finally obtain

$$< x| \ln(-G_F)|x^{'} >= -\int_{m^2}^{+\infty} G_F(x, x^{'}) dm^2 \tag{5.23}$$

where the integral with respect to m^2 brings the power $(is)^{-1}$.
Finally we can write

$$W = \frac{i}{2} \int d^n x \sqrt{-g}^{\frac{1}{2}} \lim_{x^{'} \to x} \int_{m^2}^{+\infty} dm^2 G_F^{DS}(x, x^{'}) \tag{5.24}$$

If we exchange the two integrals we obtain

$$W = \frac{i}{2} \int_{m^2}^{+\infty} dm^2 \int d^n x \sqrt{-g}^{\frac{1}{2}} G_F^{DS}(x, x) \tag{5.25}$$

Note that the second integral is a one-loop Feynman diagram. This is the motivation for the W's name *one-loop* $effective$ *action*.
From the above quantity we can find the $effective$ *Lagrangian density* L_{eff}

$$W = \int \sqrt{-g} L_{eff} d^n x \tag{5.26}$$

where

$$L_{eff} = \frac{i}{2} \lim_{x' \to x} \int_{m^2}^{+\infty} dm^2 G_F^{DS}(x, x^{'}) (5.27)$$

and

$$G_F^{DS}(x, x') = -i \Delta^{\frac{1}{2}}(x, x')(4\pi)^{-n/2} \int_0^\infty ids(is)^{-n/2} \exp\left[-im^2 s + (\sigma/2is)\right] F(x, x'; is) \tag{5.28}$$

where

$$F(x, x'; is) \approx a_0(x, x') + a_1(x, x')(is) + a_2(x, x')(is)^2 + \cdots \tag{5.29}$$

Inspection of the above quantities shows that L_{eff} diverges at lower end of the s integral because $\sigma/2s$ goes to 0 when $x^{'} \to x$.
The divergent term of the Lagrangian is

$$L_{div} = - \lim_{x' \to x} \frac{\Delta(x, x^{'})^{\frac{1}{2}}}{8\pi} \int_0^{+\infty} \frac{ds}{s^2} e^{-i(m^2 s + \frac{\sigma}{2s})} [a_0 + a_1(is) + a_2(is)^2] \tag{5.30}$$

Using dimensional regularization, in n dimension this is equal to

$$= \frac{1}{2}(4\pi)^{-n/2} \left(\frac{m}{\mu}\right)^{n-4} \sum_{j=0}^{+\infty} a_j(x) m^{4-2j} \Gamma(j - \frac{n}{2}) \tag{5.31}$$

where μ is an arbitrary mass scale μ, necessary to mantain the correct dimensionality of this object away from $d = 4$.
Obviously these divergences are the same which afflict $< T_{\mu\nu} >$.
The terms a_0, a_1 and a_2 are purely geometrical and local in the limit $x \to x^{'}$. As we already proved in Chapter 3, these coefficients are built from $R_{\mu\nu\alpha\beta}$ and its contraction.
Since the divergent terms are purely geometrical and local they can be viewed as a contribution to the *gravitational* Lagrangian.
These divergences appear because of the ultraviolet behaviour of the field modes. High energy means short distances and so these modes probe only the local geometry: they are insensitive to the global features of the spacetime. Moreover they are *independent* also on the quantum state of interest.
The divergences, once renormalized, are not present in the effective action L_{eff}, which includes the contributions from the long wavelenght part which also probes the global structure of spacetime.

5.2 Wald's axioms

The renormalization method used there to find $< T_{\mu\nu} >$ has to verify the Wald Axioms (see ref. [7])

1. Preserve general covariance $\nabla_\mu < T^{\mu\nu} >= 0$
2. Satisfy causality requirements
3. standard results for off-diagonal elements
4. Agree with the standard procedure of normal ordering in flat spacetime

The first condition is necessary for being the right hand side of the semiclassical Einstein Equations.
The causality requirement means that the metric changes of spacetime outside the past light cone cannot affect $< T_{\mu\nu} >$.
Consider two renormalized stress energy tensors satisfying the conditions 1-3. We want to prove that

$$U_{\mu\nu} = T_{\mu\nu} - \bar{T}_{\mu\nu} \tag{5.32}$$

is a local conserved tensor.
Moreover if $|\Pi_\pm >= 1/\sqrt{2}(|\psi > \pm|\phi >)$

$$< \Pi_+|U_{\mu\nu}|\Pi_- >= 0 \tag{5.33}$$

because $< T_{\mu\nu} >$ must gives standard results for off-diagonal elements and thus $< \pi|T_{\mu\nu}|\psi >=< \pi|\bar{T}_{\mu\nu}|\psi >$. In this way

$$< \phi|U_{\mu\nu}|\phi > - < \psi|U_{\mu\nu}|\psi >= 0 \ \ \forall\psi, \phi \tag{5.34}$$

and thus the diagonal elements expectation values are equal.
This condition implies

$$U_{\mu\nu} = u_{\mu\nu} 1 \tag{5.35}$$

where 1 is the identity operator and $u_{\mu\nu}$ a c-number tensor field.
Therefore $u_{\mu\nu}$ must be a local tensor. If we take

$$< in|U_{\mu\nu}(p)|in >= u_{\mu\nu} \tag{5.36}$$

and

$$< out|U_{\mu\nu}(p)|out >= u_{\mu\nu} \tag{5.37}$$

The condition 2 requires that it has to depend only on the geometry in the causal past of p, while $u_{\mu\nu}(p)$ in the last equation is similarly restricted by the geometry in the causal future. Hence these two objects have to depend only on the intersection of past and future null cones: $u_{\mu\nu}(p)$ is a local tensor at p.
The convarince conservation reads

$$u^{\mu\nu}_{;\nu} = 0 \tag{5.38}$$

We have proved that the renormalized stress energy tensor $< T_{\mu\nu} >$ is unique to within a local conserved tensor.
This tensor is a function of the local geometry and it more properly belongs to the left hand of the gravitational field equations.
A physical measurement can resolve the ambiguity related to its coefficients, which is found to be zero.

5.3 Conformal Anomalies

Let us consider a classical action invariant under conformal transformation

$$g_{\mu\nu}(x) \to \bar{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}(x) \tag{5.39}$$

The definition of functional differentiation gives

$$S[\bar{g}_{\mu\nu}] - S[g_{\mu\nu}] = \int \frac{\delta S[\bar{g}_{\mu\nu}]}{\delta \bar{g}^{\rho\sigma}} \delta \bar{g}^{\rho\sigma} d^n x \tag{5.40}$$

Using the identity

$$\delta \bar{g}^{\mu\nu} = -2\bar{g}^{\mu\nu}\Omega^{-1}(x)\delta\Omega(x) \tag{5.41}$$

we find

$$S[\bar{g}_{\mu\nu}] - S[g_{\mu\nu}] = -\int \sqrt{-g}T^{\rho}_{\rho}[\bar{g}_{\mu\nu}]\Omega^{-1}(x)\delta\Omega(x)d^n x \tag{5.42}$$

and thus we finally obtain

$$T^{\rho}_{\rho} = -\frac{\Omega(x)}{\sqrt{-g}} \frac{\delta S[\bar{g}_{\mu\nu}]}{\delta \Omega(x)}|_{\Omega=1} \tag{5.43}$$

It is clear that if the classical action is invariant under conformal transformations then the classical stress energy tensor is traceless.
When we consider the quantum theory this reasoning no longer applies as we are going to see below.
We study the model described by this action

$$S = \frac{1}{8\pi G_N} \int d^2x\sqrt{-g}R - \int d^2x\sqrt{-g}\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{5.44}$$

The variation of the gravitational lagrangian in 2D gives a null value. Thus in $2D$ the Einstein Hilbert action produces a non dynamical theory.
The model which we are studying is conformally invariant. Indeed, the scalar field's mass is equal to 0. Mass introduces a fixed lenghts in the theory and conformal transformations are rescaling of lenghts, therefore $m = 0$ is a necessary condition for a conformally invariant model.
We can put $m \to 0$ in the expansion in the $j = 0$ term of the expansion since it is positive in the two dimensional case. These term therefore vanishes.
The divergence is

$$\frac{1}{2}\frac{1}{(4\pi)^{n-2}}a_1(x)\Gamma(1-\frac{n}{2}) \tag{5.45}$$

The divergent part of the effective action is

$$W_{div} = \frac{1}{2}\frac{1}{(4\pi)^{n-2}}\Gamma(1-\frac{n}{2})\int d^n x\sqrt{-g}a_1(x) \tag{5.46}$$

$$= \frac{1}{2}\frac{1}{(4\pi)^{n-2}}\Gamma(1-\frac{n}{2})\int d^n x\sqrt{-g}\left(-\frac{R}{6}\right) \tag{5.47}$$

since $a_1(x) = \frac{R}{6}$ in the conformally invariant case since the non minimal coupling $\xi = 0$. Using dimensional regularization (μ is an arbitrary mass parameter necessary to mantain the correct dimensions)

$$< T^{\mu}_{\mu} >_{div} = -\frac{2}{\sqrt{-g}}g^{\mu\nu}\frac{\delta W_{div}}{\delta g^{\mu\nu}} = \tag{5.48}$$

$$= -\frac{2}{\sqrt{-g}} g^{\mu\nu} \left(\frac{1}{12} \left(\frac{m}{\mu} \right)^{n-2} \frac{1}{(4\pi)^{n-2}} \Gamma(1 - \frac{n}{2}) \frac{\delta}{\delta g^{\mu\nu}} \int d^n x \sqrt{-g} R \right) = \tag{5.49}$$

$$= -\frac{2}{\sqrt{-g}} g^{\mu\nu} \left(-\frac{1}{12} \left(\frac{m}{\mu} \right)^{n-2} \frac{1}{(4\pi)^{n-2}} \left(\frac{2}{(n-2)} \right) (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) \right) = \tag{5.50}$$

$$= -\frac{2}{\sqrt{-g}} \left(-\frac{1}{12} \left(\frac{m}{\mu} \right)^{n-2} \frac{1}{(4\pi)^{n-2}} \left(\frac{2}{(n-2)} \right) g^{\mu\nu} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) \right) = \tag{5.51}$$

$$= -\frac{2}{\sqrt{-g}} \left(-\frac{1}{12} \left(\frac{m}{\mu} \right)^{n-2} \frac{1}{(4\pi)^{n-2}} \left(\frac{2}{(n-2)} \right) (R - \frac{1}{2} R n) \right) = \tag{5.52}$$

$$= -\frac{1}{24\pi} \left(\frac{m}{\mu} \right)^{n-2} R \tag{5.53}$$

If we return to the original dimension of our model $D = 2$ we finally find

$$< T^\mu_\mu >_{div} = -\frac{R}{24\pi} \tag{5.54}$$

Since W is conformally invariant in the massless case, and

$$L_{ren} = L_{eff} - L_{div} \tag{5.55}$$

the expectation value of the *total* stress tensor is zero

$$< T^\mu_\mu > |_{m=0,\xi=0} = -\frac{\Omega(x)}{\sqrt{-g}} \frac{\delta W[\bar{g}_{\mu\nu}]}{\delta \Omega(x)} |_{m=0,\xi=0} \tag{5.56}$$

Therefore since $< T^\mu_\mu >_{div}$ has acquired a trace, also $< T^\mu_\mu >_{ren}$ is finite and non zero

$$< T^\mu_\mu >_{ren} = - < T^\mu_\mu >_{div} = \frac{R}{24\pi} \tag{5.57}$$

which is called the *trace anomaly*. An anomaly is a classical symmetry which does not survive in the quantized theory. Thus the physical quantities, obtained using the dimensional regularization with $D \neq 2$ mantain a non null value even if we put again $D = 2$ at the end of calculation.
This is due to the non-conformality of W_{div} away from $D = 2$ which leaves a finite contribution to the trace.
Note that the trace anomaly is local, quantum state independent and it depends only on the geometry at x.

5.4 Computation of the renormalized stress energy tensor

A two dimensional curved spacetime is conformally flat since we can always write its line element as

$$ds^2 = -C(t, x)(dt^2 - dx^2) \tag{5.58}$$

In this case turn out that from the trace anomaly we can compute the entire stress energy tensor.
Recalling that

$$W[\bar{g}_{\mu\nu}] = W[g_{\mu\nu}] - \int \sqrt{-\bar{g}} T^{\rho}_{\rho}[\bar{g}_{\mu\nu}] \Omega^{-1}(x) \delta\Omega(x) d^n x \tag{5.59}$$

which is obtained from Eq 5.42 with the substitution $S \to W_{ren}$. Using the identity

$$\bar{g}^{\nu\sigma} \frac{\delta}{\delta \bar{g}^{\mu\sigma}} = g^{\nu\sigma} \frac{\delta}{\delta g^{\mu\sigma}} \tag{5.60}$$

we find

$$< T^{\nu}_{\mu}[\bar{g}_{\rho\sigma}] >_{ren} = \left(\frac{g}{\bar{g}}\right)^{1/2} < T^{\nu}_{\mu}[g_{\rho\sigma}] >_{div} \tag{5.61}$$

$$- \frac{2}{\sqrt{-\bar{g}(x)}} \bar{g}^{\nu\sigma} \frac{\delta}{\delta \bar{g}(x)^{\mu\sigma}} \int \sqrt{-\bar{g}(x^{'})} < T^{\rho}_{\rho}[\bar{g}_{\rho\sigma}(x^{'})] >_{ren} \Omega^{-1}(x^{'}) \delta\Omega(x^{'}) d^n x^{'} \tag{5.62}$$

The trace is totally anomalous, local and state independent.
Using dimensional regularization, since

$$< T^{\rho}_{\rho}[\bar{g}_{\kappa\lambda}] >_{ren} = - < T^{\rho}_{\rho}[\bar{g}_{\kappa\lambda}] >_{div} \tag{5.63}$$

$$= \frac{\Omega}{\sqrt{-\bar{g}(x)}} \frac{\delta W_{div}[\bar{g}_{\kappa\lambda}]}{\delta\Omega(x)} \tag{5.64}$$

Therefore we can write (using $\delta\Omega(x^{'})/\delta\Omega(x) = \delta(x - x^{'})$)

$$< T^{\nu}_{\mu}[\bar{g}_{\kappa\lambda}] >_{ren} = \left(\frac{g}{\bar{g}}\right)^{1/2} < T^{\nu}_{\mu}[g_{\kappa\lambda}] >_{ren} \tag{5.65}$$

$$- \frac{2}{\sqrt{-\bar{g}(x)}} \bar{g}^{\nu\sigma} \frac{\delta}{\delta \bar{g}^{\mu\sigma}(x)} W_{div}[\bar{g}_{\kappa\lambda}] \tag{5.66}$$

$$+ \frac{2}{\sqrt{-\bar{g}(x)}} g^{\nu\sigma} \frac{\delta}{\delta g^{\mu\sigma}(x)} W_{div}[g_{\kappa\lambda}] \tag{5.67}$$

Recalling that

$$W_{div} = - \left[\frac{1}{24\pi(n-2)}\right] \int \sqrt{-g(x^{'})} R(x^{'}) d^n x^{'} \tag{5.68}$$

It is easy now to find

$$< T^{\rho}_{\rho}[\bar{g}_{\kappa\lambda}] >_{ren} = \left(\frac{g}{\bar{g}}\right)^{1/2} < T^{\rho}_{\rho}[g_{\kappa\lambda}] >_{ren} \tag{5.69}$$

$$+ \left[\frac{1}{12\pi(n-2)}\right] [(\bar{R}^{\nu}_{\mu} - \frac{1}{2}\bar{R}\delta^{\nu}_{\mu}) - (R^{\nu}_{\mu} - \frac{1}{2}R\delta^{\nu}_{\mu})] \tag{5.70}$$

and using the identities

$$\bar{R}^{\nu}_{\mu} = \Omega^{-2} R^{\nu}_{\mu} - (n-2)\Omega^{-1}(\Omega^{-1})_{;\mu\rho} g^{\rho\nu} + (n-2)^{-1}\Omega^{-n}(\Omega^{n-2})_{;\rho\sigma} g^{\rho\sigma} \delta^{\nu}_{\mu} \tag{5.71}$$

$$\bar{R} = \Omega^{-2}R - 2(n-1)\Omega^{-3}(\Omega)_{;\mu\nu}g^{\mu\nu} + (n-1)(n-4)\Omega^{-4}\Omega_{;\mu}(\Omega)_{;\nu}g^{\mu\nu} \tag{5.72}$$

we can easyly find

$$< T_\mu^\nu[\bar{g}_{\kappa\lambda}] >_{ren} = \left(\frac{g}{\bar{g}}\right)^{1/2} < T_\mu^\nu[g_{\kappa\lambda}] >_{ren} \tag{5.73}$$

$$+\frac{1}{12\pi}[(\Omega^{-3}\Omega_{;\rho\mu} - 2\Omega^{-4}\Omega_{;\rho}\Omega_{;\mu})g^{\rho\nu}] \tag{5.74}$$

$$+\delta_\mu^\nu g^{\rho\sigma}(\frac{3}{2}\Omega^{-4}\Omega_{;\rho}\Omega_{;\sigma} - \Omega^{-3}\Omega_{;\rho\sigma})] \tag{5.75}$$

Since all the two dimensional spacetime are conformally flat we can evaluate the renormalized stress energy tensor in curved spacetime in a simple manner using the known results of Minkowskian theory.
Writing in null coordinates

$$ds^2 = -C(u,v)dudv \tag{5.76}$$

and using the identity

$$R = \nabla_\mu \nabla^\mu \ln C(u,v) = \frac{4}{C}\partial_u \partial_v \ln C \tag{5.77}$$

$$< T_\mu^\nu[\bar{g}_{\kappa\lambda}] >_{ren} = \left(\frac{g}{\bar{g}}\right)^{1/2} < T_\mu^\nu[\eta_{\kappa\lambda}] >_{ren} \tag{5.78}$$

$$+\theta_\mu^\nu - \frac{1}{48\pi}R\delta_\mu^\nu \tag{5.79}$$

where

$$\theta_{uu} = -\frac{1}{12\pi}\sqrt{C}\partial_u^2 \frac{1}{\sqrt{C}} \tag{5.80}$$

$$\theta_{vv} = -\frac{1}{12\pi}\sqrt{C}\partial_v^2 \frac{1}{\sqrt{C}} \tag{5.81}$$

$$\theta_{uv} = \theta_{vu} = 0 \tag{5.82}$$

Thus, in the next section, using the results founded in flat spacetime with the usual methods we can find the stress energy tensors related to the various quantum states of interest in our model.
The flat spacetime is the usual Minkowski vacuum if and only if the curved spacetime is conformal to all, or only a part of the Minkowski spacetime. In the first case only θ_μ^ν would give contributions to the expectation value of the stress energy tensor.

5.5 Boulware state

As already analized the Boulware state is defined by the modes

$$\frac{1}{4\pi\sqrt{\omega}}e^{\pm i\omega u} \qquad \frac{1}{4\pi\sqrt{\omega}}e^{-i\omega v} \tag{5.83}$$

where $\pm$ is referred to the Black Hole region/asymptotically flat region's modes.
Related to these modes there are the creation and the destruction operator which define the Boulware Vacuum:

$$a^i_\omega|B>=0 \quad i=L,R,I \tag{5.84}$$

By definition of normal ordering

$$<B|:T_{uu}(u):|B>=<B|:T_{vv}(v):|B>=0 \tag{5.85}$$

and thus, applying Eq. 5.77 we easily find the expression for the generally covariant stress energy tensor mean value

$$<B|T_{uu}|B>=<B|T_{vv}(v)|B>=\frac{\hbar}{24\pi}\left[-\frac{M}{r^3}+\frac{3}{2}\frac{M^2}{r^4}\right] \tag{5.86}$$

$$<B|T_{uv}|B>=-\frac{\hbar}{24\pi}\left(1-\frac{2M}{r}\right)\frac{M}{r^3} \tag{5.87}$$

These expressions are clearly time independent.
If $r\to\infty$ these expressions reduce to the usual Minkowski vacuum as expected.
Let us calculate the behaviour of these mean values at $r=2M$:

$$<B|T_{uu}|B>=<B|:T_{vv}(v):|B>\approx-\frac{\hbar}{768\pi M^2} \tag{5.88}$$

$$<B|T_{uv}|B>\approx 0 \tag{5.89}$$

These expressions are clealy finite but the coordinates used are ill defined at the event horizon.
We expect that the measurement of a freely falling observer must be finite.
Let us consider a freely falling observer in our 2D model.

$$u^\mu=\left(\frac{E}{1-\frac{2M}{r}},-\sqrt{E^2-\left(\frac{2M}{r}\right)}\right) \tag{5.90}$$

This observer would measure an energy density

$$E=T_{\mu\nu}u^\mu u^\nu\propto\frac{1}{\left(1-\frac{2M}{r}\right)^2}\to-\infty \tag{5.91}$$

The vacuum polarization diverges at the event horizon.
Motivated by the fact that $<T_{\mu\nu}>$ has to be finite with respect to the locally intertial sistem such as the Kruskal frame (U,V) and that the Eddington Filkenstein-Kruskal change of coordinates is singular at the horizon

$$\frac{dv}{dV}\sim\frac{1}{r-2M} \tag{5.92}$$

$$\frac{du}{dU}\sim\frac{1}{r-2M} \tag{5.93}$$

we find the regularity condition which a well behaved stress energy tensor has to verify in order to describe a regular physical state

$$<T_{VV}>\propto|<T_{vv}>|<\infty \tag{5.94}$$

$$< T_{UV} > \propto \frac{1}{r - 2M} | < T_{uv} > | < \infty \tag{5.95}$$

$$< T_{UU} > \propto \frac{1}{(r - 2M)^2} | < T_{uu} > | < \infty \tag{5.96}$$

The regularity condition for the past event horizon are similar with $u \leftrightarrow v$.
Clearly the regularity conditions at the horizon are not fullfilled for the $|B>$ state.
Therefore we can interpret it as the *vacuum polarization* outside a static spherical star whose radius is bigger than $2M$.
In this way the physical region of the Schwarzschild spacetime does not contain any causal horizon.

5.6 The Hartle Hawking State

The Hartlee Hawking vacuum state is clearly regular at the horizon since it is associated to the modes

$$\frac{1}{\sqrt{4\pi\omega}} e^{-i\omega V} \qquad \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega U} \tag{5.97}$$

where U, V are the Kruskal coordinates.
This state is defined in the maximally extended spacetime. As we analized in the previous chapter the restriction of this state to the external region implies a tracing on the interior degrees of freedom.
This produce a *mixed* state which can be described by a thermal density matrix

$$|H>_{r>2M} \Rightarrow \rho = \overrightarrow{\rho} \otimes \overleftarrow{\rho} \tag{5.98}$$

where

$$\overrightarrow{\rho} = \prod_{\omega} \left(1 - e^{-8\pi M\omega}\right) \sum_{N} e^{-8\pi N M\omega} |N_\omega><N_\omega| \tag{5.99}$$

as previously proved and $|N_\omega>$ is the Fock state with N outgoing particles with frequency ω constructed from $|B>$

$$|N_\omega> = \hbar^{-N/2}(N)^{-1/2}(a^\dagger_\omega)^N|B> \tag{5.100}$$

In the same manner we can define the ingoing density matrix $\overleftarrow{\rho}$.
Let us compute the normal ordered stress energy tensor components.
Recalling that $T_{uu} = \partial_u\phi\partial_u\phi$ we find

$$< H| : T_{uu} : |H> = \sum_{\omega\omega'} < H| \frac{-\sqrt{\omega\omega'}}{4\pi} e^{-i(\omega' + \omega)u} a_\omega a_{\omega'} |H> + \tag{5.101}$$

$$+ < H| \sum_{\omega\omega'} \frac{-\sqrt{\omega\omega'}}{4\pi} e^{-i(\omega - \omega')u} a^\dagger_{\omega'} a_\omega |H> + h.c. \tag{5.102}$$

Using the Bougoliubov transformation on the creation and desctruction operators defined in Chapter 3 we can easily find that the only non vanishing elements ($\sum_i \alpha_{ij}\beta_{ik} = 0$) are

those deriving from the $a^{\dagger}_{\omega'} a_{\omega}$ and its hermitian conjugate.
We find

$$\sum_{\omega\omega'} \frac{\sqrt{\omega\omega'}}{4\pi} \sum_i e^{-i(\omega-\omega')u} \beta^*_{\omega' i} \beta \omega i \tag{5.103}$$

Using the definition of scalar product and taking I^- as Cauchy surface it is easy to find that

$$\sum_i \beta_{\omega i} \beta^*_{\omega' i} = \frac{1}{e^{8\pi M\omega} - 1} \delta(\omega - \omega') \tag{5.104}$$

Therefore we finally find

$$< H| : T_{uu} : |H >= \frac{1}{2\pi} \sum_{\omega} \left(\frac{\omega e^{-8\pi M\omega}}{1 - e^{-8\pi M\omega}} \right) = \frac{\hbar}{768\pi M^2} \tag{5.105}$$

The same reasoning can be applied to $< H| : T_{vv} : |H >$ and we find

$$< H| : T_{vv} : |H >= \frac{\hbar}{768\pi M^2} \tag{5.106}$$

Adding the vacuum polarization terms we find

$$< H|T_{uu}|H >=< H|T_{vv}|H >= \frac{\hbar}{768\pi M^2} \left(1 - \frac{2M}{r}\right) \left[1 + \frac{4M}{r} + \frac{12M^2}{r}\right] \tag{5.107}$$

$$< H|T_{uv}|H >= -\frac{\hbar}{24\pi} \left(1 - \frac{2M}{r}\right) \frac{M}{r^3} \tag{5.108}$$

The normal ordered quantities on the horizon cancel with the vacuum polarization terms.
Asymptotically we have

$$< H|T_{uu}|H >=< H|T_{vv}|H >\approx \frac{\hbar}{768\pi M^2} \tag{5.109}$$

$$< H|T_{uu}|H >\approx 0 \tag{5.110}$$

Therefore $|H >$ is a *thermal state* and at I^+ it describe radiation at the temperature T_H

$$T_H = \frac{\hbar}{8\pi k_B M} \tag{5.111}$$

In fact since the mean value of the stress energy tensor at infinity can be found using the thermal density matrix

$$Tr[T_{uu} \vec{\rho}] = Tr[: T_{uu} : \vec{\rho}] = \int_0^{+\infty} d\omega \sum_{N=0}^{+\infty} <\vec{N_\omega} \mid : T_{uu} : \vec{\rho} \mid \vec{N_\omega}> \tag{5.112}$$

since when $r \to +\infty$ the vacuum polarization terms go to 0.
In this manner

$$<\vec{N_\omega} \mid \vec{\rho} \mid \vec{N'_{\omega'}}>= (1 - e^{-8\pi M\omega}) e^{-8N\pi\omega} \delta_{NN'} \delta(\omega - \omega') \tag{5.113}$$

and thus

$$Tr[: T_{uu} : \vec{\rho}] = \int_0^{+\infty} d\omega \sum_{N=0}^{+\infty} <\vec{N_\omega} \mid : T_{uu} : \mid N_\omega > (1 - e^{-\vec{8\pi M\omega}}) e^{-8N\pi M\omega} \quad (5.114)$$

It is easy to verify using the canonical commutation relations that

$$<\vec{N_\omega} \mid : T_{uu} : \mid \vec{N_\omega}> = \frac{\hbar N\omega}{2\pi} \quad (5.115)$$

and using the identity

$$\sum_{N=0}^{+\infty} N e^{-8N\pi M\omega} = \frac{e^{-8\pi M\omega}}{(1 - e^{-8\pi M\omega})^2} \quad (5.116)$$

we finally find the expected result

$$Tr[: T_{uu} : \vec{\rho}] = \frac{\hbar}{2\pi} \int_0^{+\infty} d\omega \frac{\omega e^{-8\pi M\omega}}{(1 - e^{-8\pi M\omega})} = \frac{\hbar}{768\pi M^2} \quad (5.117)$$

Moreover, the scalar Green Function reads (see next Chapter for an explicit calculation of G^+ which differs by a 2 factor)

$$D_K = -\frac{1}{2\pi} \ln\left[\Delta U_K \Delta V_K\right] \quad (5.118)$$

Transforming in Schwarzschild coordinates we find

$$D_K(x^{''}, x^{'}) = -\frac{1}{2\pi} \ln[\cosh \kappa(t^{''} - t^{'}) - \cosh \kappa(r^{''} - r^{'})] + \cdots \quad (5.119)$$

This expression is invariant under the transformation

$$t^{''} \to t^{''} + \frac{2\pi i n}{\kappa} \quad (5.120)$$

where n is an integer number and κ is the surface gravity.
Thus, D_K is periodic in imaginary time with period $2\pi/\kappa$. This is a feature of thermal Green Function with temperature $\kappa/2\pi k_B$, the Hawking temperature.
With this analysis we prove that the Hartle-Hawking vacuum is a *thermal state* at temperature T_H.
This state describes a Black Hole in a box in thermal equilibrium with his own radiation. We have chosen these coordinate system and not another with the same asymptotic behaviour of Kruskal's because these are the only coordinates which make the mean values of the stress energy tensors time independent.

5.7 The Unruh state

The Unruh state was introduced in the previous chapter to reproduce late time radiation since, at late time a mode with positive frequency with respect to the inertial time t_{in} becomes a modes with positive frequency with respect to the Kruskal time T_K near the

horizon.
It is constructed from the modes

$$\frac{1}{\sqrt{4\pi\omega}}e^{-i\omega v} \qquad \frac{1}{\sqrt{4\pi\omega_K}}e^{-i\omega U} \tag{5.121}$$

Like the Hartle Hawking state it is defined in the maximally extended spacetime. The restriction of the Unruh vacuum to the asymptotically flat region is described by a mixed state and again

$$|U>|_{r>2M} \Leftrightarrow \overrightarrow{\rho} \tag{5.122}$$

where again

$$\overrightarrow{\rho} = \prod_{\omega} \left(1 - e^{-8\pi M\omega}\right) \sum_{N} e^{-8\pi NM\omega} |N_\omega> < N_\omega| \tag{5.123}$$

The time orderd components are, as easily verified

$$< U| : T_{uu}(u) : |U> = \frac{\hbar}{768\pi M^2} \tag{5.124}$$

$$< U| : T_{vv}(v) : |U> = 0 \tag{5.125}$$

Adding the vacuum polarization terms

$$< U|T_{uu}(u)|U> = \frac{\hbar}{768\pi M^2}\left(1 - \frac{2M}{r}\right)^2 \left[1 + \frac{4M}{r} + \frac{12M^2}{r}\right] \tag{5.126}$$

$$< U|T_{vv}|U> = \frac{\hbar}{24\pi}\left[-\frac{M}{r^3} + \frac{3M^2}{2r^4}\right] \tag{5.127}$$

which is equal to $< B|T_{vv}|B >$. Finally

$$< U|T_{uv}|U> = -\frac{\hbar}{24\pi}\left(1 - \frac{2M}{r}\right)\frac{M}{r^3} \tag{5.128}$$

which is equal to $< B|T_{uv}|B >$.
Again the regularity conditions are fully satisfied by these expressions on the future event horizon while we have a divergence if we consider the past horizon. Since this state is physical only at late times, this divergence is not physically relevant.
Therefore we interpret the Unruh state as the state which describe the quantum state of matter in the gravitational collapse in the late time and near horizon limit (since $u_{in} \to u_K$ if $r \to 2M$ and thus $|in > \approx |U >$).
If $r \to +\infty$

$$< U|T_{uu}|U> = \frac{\hbar}{768\pi M^2} = +\frac{\hbar\kappa^2}{48\pi} \tag{5.129}$$

which represents a constant thermal flux with temperature

$$T_H = \frac{\hbar}{8\pi k_B M} \tag{5.130}$$

Moreover, at the event horizon

$$< U|T_{vv}|U> = -\frac{\hbar}{768\pi M^2} = -\frac{\hbar\kappa^2}{48\pi} \tag{5.131}$$

This is a *negative* and equal to minus the Hawking flux at infinity.
It represent a null flux of negative energy entering the Black Hole. Thus the Black Hole loses mass not emitting particles but throught this negative energy flux.
As the Black Hole loses mass, it becomes hotter (because of its negative specifit heat) and it will radiate particles with greater mass until the possible disappearing.
Note that when the Black Hole's area shrinks there is no violation of the second law of Black Hole mechanics since the expectation value of the stress energy tensor is negative and so it does not satisfy the energy condition necessary (see chapter II).
Moreover this does not imply a violation of the second law of Thermodynamics since

$$\Delta S_{Tot} = \Delta S_{BH} + \Delta S_{HR} \geq 0 \tag{5.132}$$

Therefore, even if the area of the Black Hole shrinks, and so $\Delta S_{BH} < 0$, the total entropy increases because of the contribution of the Hawking Radiation $\Delta S_{HR} > 0$.

5.8 Infalling observer

It is interesting to consider an infalling observer. For an asymptotic observer, if we ignore backreaction effects, the Black Hole emits an infinite number of particles during the infinite time the observer uses to reach the horizon.
Intuitively it appear that the infalling observer would meet an infinite amount of radiation which has to destroy it.
This is resolved considering the differences between the particle number and the energy density.
When the observer approaches the Black Hole, the well definite concept of particle breaks down.
Morever, the observer would be inside the characteristic wavelenght of a particle.
Therefore we do not need to worry about the number of particle measured by the infalling observer.
The energy density however has a local significance.
The Hawking radiation flux is divergent at the horizon but there is also a contribution from the gravitational vacuum polarization which is negatively divergent, too.
An infalling observer cannot distinguish operationally between these two contributions. The difference between these two infinities gives a $finite$ result at the Event Horizon.

Chapter 6

Stress Energy Tensor Correlations

6.1 Introduction

In this chapter we compute an explicit expression of the Stress Energy Tensor 2 point function, which describes the fluctuations of quantum fields in curved spacetime.
There are various motivations for studying this quantity. On the one hand it contains important information regarding the quantum fields, on the other hand the fluctuations described by the stress energy 2 point function are related to the search on an experiamental proof in the analogue models of Black Holes.

6.2 Black Hole Analogue Models

In this section we prove that, under certain physical conditions, the behaviour of a condensed matter system can simulate a Black Hole (see ref. [12]).
Let us consider a simple fluid model, irrotational such that $\mathbf{v} = \nabla\psi$ (where $\mathbf{v}$ is the velocity and ψ a scalar function) and homentropic ($P = P(\rho)$ where P is the pressure and ρ is the density).
The action reads

$$S = -\int d^4x \left[\rho\frac{d\psi}{dt} + \frac{1}{2}\rho(\nabla\psi)^2 + u(\rho)\right] \tag{6.1}$$

where u is the internal energy.
The variations of the action give the two fundamental laws of fluid dynamics which are the continuity equation

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{6.2}$$

and the Bernulli Equation

$$\partial_t\psi + \frac{1}{2}\mathbf{v}^2 + +\mu(\rho) = 0 \tag{6.3}$$

where $\mu = \frac{du}{d\rho}$.
Once applied the gradient operator to the above equation, we find the Euler's equation

$$\frac{d\mathbf{v}}{dt} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{\rho}\nabla P = 0 \tag{6.4}$$

where $P = \int \rho d\mu$.
In order to study the fluctuations we expand the equation of motion at the first order around the stationary configurations

$$\rho \to \rho_0 + \rho_1 \tag{6.5}$$

$$\mathbf{v} \to \mathbf{v}_0 + \mathbf{v}_1 \tag{6.6}$$

$$\psi \to \psi_0 + \psi_1 \tag{6.7}$$

This describe a mean flow configuration, which is the unperturbed solutions $(\rho_0, \mathbf{v}_0, \psi_0)$, with perturbations $(\rho_1, \mathbf{v}_1, \psi_1)$.
Expanding the action to the linear order, discarting the higher orders contributions we find

$$S = S_0 + S_2 = S_0 - \int d^4x \left[\frac{1}{2}\rho_0(\nabla\psi_1)^2 + \frac{\rho_0}{2c^2}(\partial_t\psi_1 + \mathbf{v}\cdot\nabla\psi_1)^2 \right] \tag{6.8}$$

where c is the speed of sound and is defined as $c = d\mu/d\rho|_{\rho_0}$.
The above action gives the equation of motion for the perturbation of the velocity potential

$$-\partial_t \left[\frac{\rho_0}{2c^2}(\partial\psi_1 + \mathbf{v}_0\cdot\nabla\psi_1) \right] + \nabla\cdot\left\{ \mathbf{v}_0 \left[-\frac{\rho_0}{c^2}(\partial_t\psi_1 + \mathbf{v}_0\cdot\nabla\psi_1) + \rho_0\nabla\psi_1 \right] \right\} = 0 \tag{6.9}$$

Rewritting the obove equation with a four dimensional notation we find

$$\partial_\mu(f^{\mu\nu}\partial_\nu\psi_1) = 0 \tag{6.10}$$

$$f^{\mu\nu} = -\frac{\rho}{c^2}\begin{pmatrix} +1 & +v_0^j \\ v_0^i & c^2 - (\delta_{ij} - v_0^i v_0^j) \end{pmatrix}$$

Setting

$$f^{\mu\nu} = \sqrt{-g}g^{\mu\nu} \tag{6.11}$$

we finally obtain the action

$$S_2 = -\frac{1}{2}\int d^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\psi_1\partial_\nu\psi_2 \tag{6.12}$$

and the related equation of motion

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi) = 0 \tag{6.13}$$

This is clearly the Klein Gordon equation for a massless field in an effective spacetime metric $g_{\mu\nu}$. The acoustic line element can be rewritten as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\frac{\rho_0}{c}[(c^2 - v_0)^2dt^2 + 2v_0^i dt dx^i - \delta_{ij}(dx^i dx^j)] \tag{6.14}$$

where we have used the acoustic metric

$$g_{\mu\nu} = \frac{\rho}{c^2}\begin{pmatrix} -(c^2 - v_0^2) & -v_0^j \\ -v_0^i & +\mathbf{1} \end{pmatrix}$$

which resembles the Schwarzschild solution expressed in the Painleve-Gullstrand coordinates multiplied by a conformal factor which does not modify the causal structure and hence, in

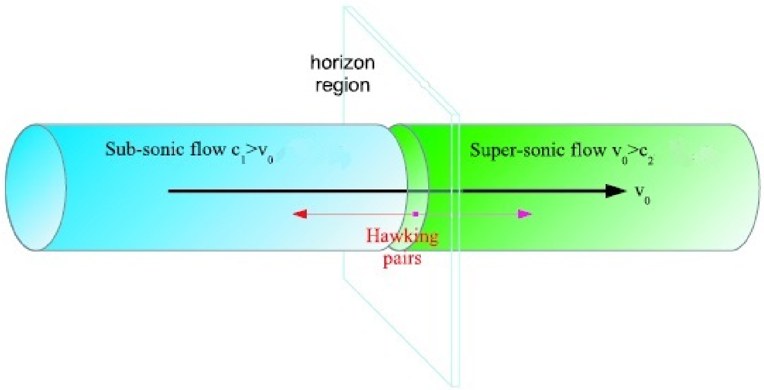

Figure 6.1: Acoustic Black Hole

this case, it does not affect the propagation of sound waves.
In particular, note that the spatial sections obtained through the condition $dt = 0$ are flat.
It is quite remarkable that even though the underlying fluid dynamics is *Newtonian*, so nonrelativistic, and takes place in flat space-plus-time, the fluctuations (sound waves) are governed by a curved (3+1)-dimensional Lorentzian (pseudo-Riemannian) spacetime geometry.
For general relativists this observation describes a very simple and concrete physical model for certain classes of Lorentzian spacetimes, including Black Holes.
Let us analyze the behaviour of the soundwaves in this simple model.
On one hand if $c^2 > v_0^2$ the perturbations behaves normally and $g_{tt} > 0$. On the other hand, if $c^2 < v_0^2$ the fluid exihibits a supersonic behaviour and $g_{tt} < 0$.
The similarity with the Black Hole behaviour is manifest: the perturbation in the region in which $v_0^2 < c^2$ can be propagated in both direction, upstream and downstram, while in the $v_0^2 > c^2$ region they are forced to follow a downstream direction, similarly to a particle in the Black Hole Region.
The surface in which $v_0^2 = c^2$ is called Acoustic Event Horizon.

These model exhibits an emission of thermal spectrum of phonons with the same characteristics of the gravitational counterpart (see ref. [11]). In particular they have a thermal spectrum at the acusting Hawking temperature T_H proportional to the acoustic surface gravity

$$\kappa = \frac{1}{2}\frac{d(c^2 - v^2)}{dx}|_{Horizon} \tag{6.15}$$

evaluated on the horizon.

Indeed, there is a production of couples of phonons, one going downstram and the other going upstream, a situation very similar to that in Capter 4, section Correlations.
Moreover, in these models it is possible to study in detail problems related to the quantum field theoretical derivation of the Hawking effect, as the Transplankian problem.
Indeed, the fluid approximation breacks down at lenghts comparable with the atomic spacing.
But in the analogue models we clearly understand the underlying microscopical quantum

theory while we do not know the microscopic quantum gravity theory. If we do not use the fluid approximation, it is possible to find that the emission of phonons is still present.
Therefore, thorugh this studies physicists have understood that Hawking Radiation is a very robust result and modifications to the dispersion relations do not change the main result as can be verified.

6.3 Point separation

The product of two tempered distribution $\phi(x)^2$ at the same spacetime point is not well defined. Therefore, to obtain a well defined and physically meaningfull expression of the stress energy tensor expectation value we have to use some regularization process in order to identify the divergences.
Point separation regularization is well suited for the present analysis.
Since $\phi(x)\phi(x^{'})$ is a well defined quantity if $x \neq x^{'}$ we can consider the expecatation value of this expression and after that we can take the limit $x \to x^{'}$ to find the divergences present in this expression.
It is interesting to note that in the calculation of the main object of this chapter, the stress energy 2 point function, the divergences cancel and therefore there is no need to renormalize.

6.3.1 The Stress Energy Tensor

The matter action for our model is

$$S_m = -\int d^2x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x)\right) \tag{6.16}$$

The stress energy tensor can be derived using

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \tag{6.17}$$

Using the identity

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{6.18}$$

it is easy to compute

$$-\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = \frac{2}{\sqrt{-g(x)}}\int d^2y\frac{1}{4}\sqrt{-g(y)}g_{\alpha\beta}\frac{\delta g^{\alpha\beta}(y)}{g^{\mu\nu}(x)}g^{\rho\sigma}\partial_\rho\phi(y)\partial_\sigma\phi(y) \tag{6.19}$$

$$-\frac{2}{\sqrt{-g(x)}}\int d^2y\frac{1}{4}\sqrt{-g(y)}\left(+\frac{1}{2}g_{\mu\nu}\frac{\delta g^{\rho\sigma}(y)}{\delta g^{\mu\nu}(x)}\partial_\rho\phi(y)\partial_\sigma\phi(y)\right) \tag{6.20}$$

Using $\frac{\partial g^{\alpha\beta}(y)}{\partial g^{\mu\nu}(x)} = \delta^\alpha_\rho\delta^\beta_\sigma\delta^{(2)}(x-y)$ we finally find

$$\frac{2}{\sqrt{-g(x)}}\int d^2y\frac{1}{4}\sqrt{-g(y)}g_{\mu\nu}\delta^{(2)}(x-y)g^{\rho\sigma}(y)\partial_\rho\phi(y)\partial_\sigma\phi(y) \tag{6.21}$$

$$-\frac{2}{\sqrt{-g(x)}}\int d^2y\frac{1}{4}\sqrt{-g(y)}\frac{1}{2}g_{\mu\nu}\delta^{(2)}(x-y)\partial_\rho\phi(y)\partial_\sigma\phi(y) \tag{6.22}$$

Therefore the final expression reads

$$T_{\mu\nu} = \partial_\mu\phi(x)\partial_\nu\phi(x) - \frac{1}{2}g_{\mu\nu}\partial^\rho\phi(x)\partial_\rho\phi(x) \tag{6.23}$$

6.4 The Stress Energy 2 points function

In this section we derive a formula which make us able to find a well defined quantity for the stress energy 2 point function defined

$$< \tilde{T}_{\mu\nu}(x)\tilde{T}_{\rho\sigma}(y) >=< T_{\mu\nu}(x)- < T_{\mu\nu}(x) >, T_{\rho\sigma}(y)- < T_{\rho\sigma}(y) >> \quad (6.24)$$

The mean value of the operator $< \tilde{T}_{\mu\nu}(x) >$ defined at one point, as we have already seen in the previous chapter, is a divergent quantity. Instead, we will show that the stress energy tensor 2 point function is well defined for every $y \neq x$.
To regularize the stress energy two point function we take the point separated expression (see ref. [11])

$$(x, y) \rightarrow (x, x^{'}, y, y^{'}) \quad (6.25)$$

After the computation of the main expression we take the limit $x \rightarrow x^{'}$ and $y \rightarrow y^{'}$.
This will allow us to express our quantity of interest as a differential operator acting on a four point Green function. This can be reexpressed as a product of two point functions through the Wick Theorem.
From the viewpoint of the stress energy tensor, the separation of points is an artificial construct. This implies that, in the transition between classical field and the quantum one, *neither* point should be favoured.
Therefore we choice to symmetrize the product of the operators.

$$\phi_{cl}(x)\phi_{cl}(y) \Rightarrow \frac{1}{2}\{\hat{\phi}(x), \hat{\phi}(y)\} = \frac{1}{2}(\hat{\phi}(x)\hat{\phi}(y) + \hat{\phi}(y)\hat{\phi}(x)) \quad (6.26)$$

Therefore, the point separated stress energy tensor operator can be defined as

$$T_{\mu\nu} = \frac{1}{2}\Sigma_{\mu\nu}\{\hat{\phi}(x), \hat{\phi}(y)\} \quad (6.27)$$

where $\Sigma_{\mu\nu}$ is the differential operator acting on the fields

$$\Sigma_{\mu\nu} = \partial^x_\mu \partial^y_\nu - \frac{1}{2} g_{\mu\nu} \partial^{x\rho} \partial^y_\rho \quad (6.28)$$

This is a well defined and finite quantity.
The expectation value of the point-separated stress energy operator can be computed using

$$G^{Schwinger}(x, y) =< \{\hat{\phi}(x), \hat{\phi}(y)\} > \quad (6.29)$$

and thus

$$< T_{\mu\nu} >= \frac{1}{2}\Sigma_{\mu\nu} G^{Schwinger}(x, y) \quad (6.30)$$

Let us return to our main problem: the evaluation of the stress energy tensor 2 point function.

$$< \tilde{T}_{\mu\nu}(x)\tilde{T}_{\rho\sigma}(y) >= \lim_{x^{'} \rightarrow x} \lim_{y^{'} \rightarrow y} \Sigma_{\mu\nu}\Sigma_{\rho\sigma} G(x, x^{'}, y, y^{'}) \quad (6.31)$$

where

$$G(x, x^{'}, y, y^{'}) = \frac{1}{4}[< \{\{\hat{\phi}(x), \hat{\phi}(x^{'})\}, \{\hat{\phi}(y), \hat{\phi}(y^{'})\}\} > \quad (6.32)$$

$$-2 < \{\hat{\phi}(x), \phi(x^{'})\}\{\hat{\phi}(y), \hat{\phi}(y^{'})\} >] \quad (6.33)$$

Applying the Wick Theorem for free fields we can to rexpress as a sum of product of Wightman functions $G_{xy} =< \phi(x)\phi(y) >$

$$< \hat{\phi}(x)\hat{\phi}(x^{'})\hat{\phi}(y)\hat{\phi}(y^{'}) >= G_{xy'}G_{yx'} + G_{xx'}G_{yy'} + G_{xy}G_{y'x'} \tag{6.34}$$

Using these results we can compute the 4 point function $G(x, x^{'}, y, y^{'})$ and we find

$$G(x, x^{'}, y, y^{'}) = G_{xy'}G_{x'y} + G_{xy}G_{x'y'} + G_{yx'}G_{y'x} + G_{xy}G_{y'x'} \tag{6.35}$$

Now, the limit $(x^{'}, y^{'}) \to (x, y)$ is well defined and we finally find

$$< \tilde{T}_{\mu\nu}(x)\tilde{T}_{\rho\sigma}(y) >= \lim_{x'\to x}\lim_{y'\to y} \Sigma_{\mu\nu}\Sigma_{\rho\sigma}[G_{xy'}G_{x'y}+G_{xy}G_{x'y'}+G_{yx'}G_{y'x}+G_{yx}G_{y'x'}] \tag{6.36}$$

Therefore the divergence in the left part of the stress energy tensor two point function cancels with the second part's divergence in Eq. 6.23 and we obtain a well defined object.

6.5 Calculation of Wightman functions

In this section we compute the Wightman function, defined by

$$G^{+}(x, x^{'}) =< 0|\hat{\phi}(x)\hat{\phi}(x^{'})|0 > \tag{6.37}$$

As proved in the previous section, it is necessary for the computation of the stress energy 2 point function.

6.5.1 Unruh State

We use the field mode expansion to find the explicit expression for the Wightman function in the coordinate space for the Unruh state:

$$\hat{\phi}(x) = \sum_{\omega} \frac{1}{\sqrt{4\pi\omega}}[a^{K}_{\omega}e^{-i\omega U_K} + a^{\dagger K}_{\omega}e^{+i\omega U_K} + a_{\omega}e^{-i\omega v} + a^{\dagger}_{\omega}e^{+i\omega v}] \tag{6.38}$$

where U_K is the outgoing Kruskal coordinate and v is the ingoing Eddington-Filkenstein coordinate.
Therefore we compute

$$< U|\sum_{\omega} \frac{1}{\sqrt{4\pi\omega}}[a^{K}_{\omega}e^{-i\omega U_K} + a_{\omega}e^{-i\omega v}]\sum_{\omega'}[\frac{1}{\sqrt{4\pi\omega'}}a^{\dagger K}_{\omega'}e^{+i\omega' U_K} + a^{\dagger}_{\omega'}e^{+i\omega v'}]= \tag{6.39}$$

$$= \int_{0}^{+\infty} \frac{d\omega}{4\pi\omega}\left[e^{-i\omega(U_K-U^{'}_K)} + e^{-i\omega(v-v^{'})}\right] \tag{6.40}$$

where we have used the canonical commutation relations between the creation and the destruction operators.
This is clearly an infrared divergent quantity. Using an infrared regulator λ in order to control the divergence we find

$$\int_{\lambda}^{+\infty} \frac{d\omega}{4\pi\omega}\left[e^{-i\omega(U_K-U^{'}_K)} + e^{-i\omega(v-v^{'})}\right] \tag{6.41}$$

$$= -\frac{\hbar}{4\pi}(2\gamma) + \ln[\lambda^2|(\Delta U_K \Delta v)|] \tag{6.42}$$

where γ is the Euler constant and $\Delta U_K = U_K - U_K^{'}$ while $\Delta v = v - v^{'}$.
We can rewrite the previous equation as

$$G_U^+(x, x^{'}) = C - \frac{\hbar}{4\pi}\ln[|(\Delta U_K \Delta v)| \tag{6.43}$$

where C is a divergent quantity. This infrared divergence is a feature of massless scalar models in $2D$. It is not a real problem for our purpuse since we are going to apply derivative operators to this expression and thus we can be discard C.

6.5.2 Hartle-Hawking State

In the same way, in the Hartle-Hawking state, we can expand the state in the basis

$$\hat{\phi}(x) = \sum_\omega \frac{1}{\sqrt{4\pi\omega}}[\overrightarrow{a}_\omega^K e^{-i\omega U_K} + \overrightarrow{a}_\omega^{K\dagger} e^{+i\omega U_K} + \overrightarrow{a}_\omega^K e^{-i\omega V_K} + \overrightarrow{a}_\omega^{K\dagger} e^{+i\omega V_K}] \tag{6.44}$$

where U_K, V_K are the Kruskal coordinates.
Following the Unruh State's computation we find

$$G_{HH}^+(x, x^{'}) = C - \frac{\hbar}{4\pi}\ln[|(\Delta U_K \Delta V_K)| \tag{6.45}$$

where $\Delta U_K = U_K - U_K^{'}$ and $\Delta V_K = V_K - V_K^{'}$. Again, since we are going to apply derivative operators to this expression the infrared divergent constant C is not a problem.

6.5.3 Boulware State

Expanding in the Boulware basis the field $\phi(x)$ we have found in Chapter IV

$$\phi(r, t) = \int \frac{d\omega}{\sqrt{4\pi\omega}}[a_\omega^L e^{+i\omega u} + a_\omega^R e^{-i\omega u} + a_\omega^I e^{-i\omega v} + h.c.] \tag{6.46}$$

where a_ω^L and a_ω^R are the destruction operator for modes inside and outside the Event Horizon.
In particular, using the same reasoning of the above discussion it is possible to find

$$G_B^+(x, x^{'}) = C - \frac{\hbar}{4\pi}\ln[|(\Delta u \Delta u \Delta v)| \tag{6.47}$$

6.6 External Correlations - Boulware state

Obviously, since the Boulware state is physical only outside a spherical star and a freely falling observer measures a divergent energy density at the Event Horizon, the computation of the correlations between points in the Black Hole region and points outside it previously described is not of physical intertest.
Indeed, the divergent energy density at the horizon would largely modify the causal structure

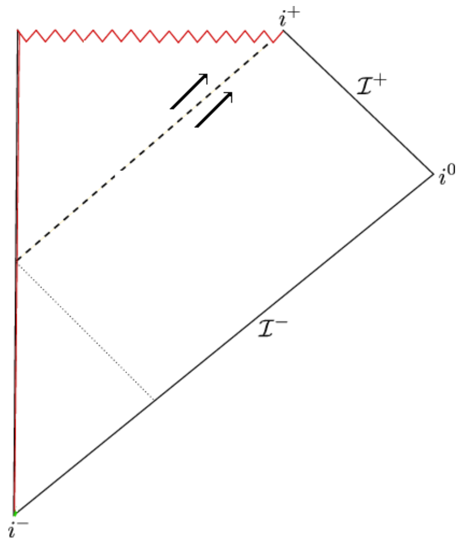

Figure 6.2: In this Penrose Diagram is depicted the peak in the correlations between the ougoing modes

of our spacetime because of the strong beackreaction.
Recalling that

$$G_B^+(x, x^{'}) = C - \frac{\hbar}{4\pi} \ln[|(\Delta u \Delta u \Delta v)| \tag{6.48}$$

and applying Eq. 6.35 it is easy to find the explicit expression for the stress tensor 2 point function

$$< B|T_{uu}(x)T_{uu}(x^{'})|B >= \left(\frac{\hbar}{2\pi}\right)^2 \frac{1}{(u-u^{'})^4} \tag{6.49}$$

Therefore, correlations decrease with distance as $1/x^4$ as expected.
The same behaviour for the vv component

$$< B|T_{vv}(x)T_{vv}(x^{'})|B >= \left(\frac{\hbar}{2\pi}\right)^2 \frac{1}{(v-v^{'})^4} \tag{6.50}$$

6.7 Correlations - Unruh State

It is now simple to find the expression for the stress energy 2 point function in the Unruh state applying Eq. 6.35.
In particular, we are interested in the correlations between points inside and outside the Black Hole region.
Indeed, this correlations will be very different from the usual decreasing law as we are going to prove.
This analysis shows us a peak related to the outgoing particles inside and outside the Black Hole.
The measurement of these correlations is very important for an experimental proof of the Hawking Radiation in condensed matter systems since the exact spectrum of the phonons emitted by the acoustic Black Hole is difficult to measure because of the great noise present.

In the gravitational setting this measurement is clearly impossible and moreover it not physical since it does not exist any observer able to perform at the same time measurements inside and outside the Black Hole region.
We want to compute

$$< U|\tilde{T}_{\mu\nu}(x)\tilde{T}_{\rho\sigma}(y)|U > \tag{6.51}$$

The differential operator related to the stress energy tensor found above is

$$\Sigma_{\mu\nu} = \partial_{\mu}^{(x)}\partial_{\nu}^{(x)} - \frac{1}{2}g_{\mu\nu}(\partial^{(x)\rho}\partial_{\rho}^{(x)}) \tag{6.52}$$

In particular the point separated component uu is

$$\Sigma_{uu} = \partial_{u}^{(x)}\partial_{u}^{(x^{'})} \tag{6.53}$$

because $g_{uu} = 0$.
In this manner

$$< T_{uu}(x)T_{uu}(y) >= \lim_{x'\to x}\lim_{y'\to y} \partial_u^x \partial_{u'}^{x'} \partial_u^y \partial_u^{y'} [G_{xy'}G_{x'y} + G_{xy}G_{x'y'} + G_{yx'}G_{y'x} + G_{xy}G_{y'x'}] \tag{6.54}$$

Recalling the previously found Green Function in the Unruh state

$$G_U^+(x,x^{'}) = -\frac{\hbar}{4\pi}\ln[|(\Delta U_K \Delta v)| = -\frac{\hbar}{4\pi}\ln[|(e^{-\kappa u} + e^{-\kappa u^{'}})(v - v^{'})| \tag{6.55}$$

with $u = t - r^*$ placed outside the horizon and $u^{'}$ inside.
We have discarted the infrared divergent constant and we used the definition of the Kruskal coordinate

$$U_K = -e^{-\kappa u} \qquad r > 2M \tag{6.56}$$

$$U_K^{'} = e^{-\kappa u} \qquad r < 2M \tag{6.57}$$

It is easy to verify that, considering x inside the Black Hole region and $x^{'}$ in the Asymptotically flat region

$$G_U^+(x^{'},x) = -\frac{\hbar}{4\pi}\ln[| - (e^{-\kappa u^{'}} + e^{-\kappa u})(v^{'} - v)| = \tag{6.58}$$

$$= -\frac{\hbar}{4\pi}\ln[|(e^{-\kappa u^{'}} + e^{-\kappa u})(v - v^{'})| = G_U^+(x,x^{'}) \tag{6.59}$$

Therefore, using

$$\frac{\partial}{\partial u} = \frac{\partial U_K}{\partial u}\frac{\partial}{\partial U_K} = -\kappa U_K\frac{\partial}{\partial U_K} \tag{6.60}$$

and applying Eq. 6.46 we find

$$< U|T_{uu}(x)T_{uu}(x^{'})|U >= \left(\frac{\hbar\kappa^2}{2\pi}\right)^2 \frac{U_K^2 U_K^{'2}}{(U_K - U_K^{'})^4} \tag{6.61}$$

Transforming to the Eddington Filkenstein coordinates we find the beautiful result

$$< U|T_{uu}(x)T_{uu}(x^{'})|U >= \left(\frac{\hbar\kappa^2}{8\pi^2}\right)^2 \frac{1}{\cosh^4(\frac{\kappa(u-u')}{2})} \tag{6.62}$$

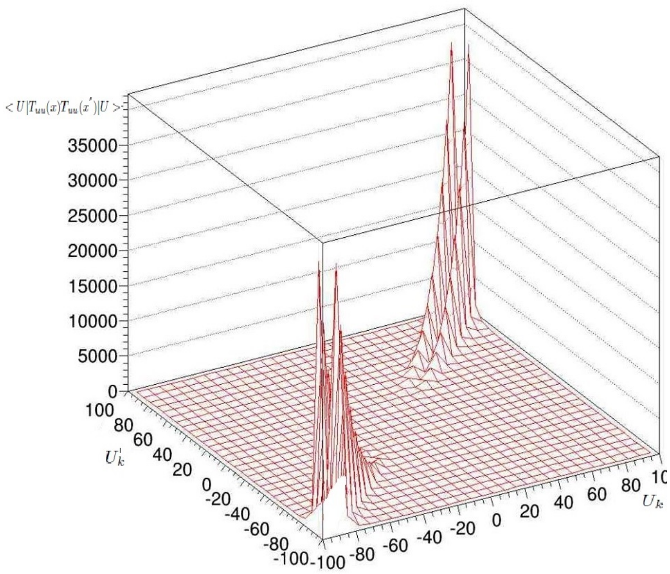

Figure 6.3: Correlations, component $< T_{uu}(x)T_{uu}(x^{'}) >$, peak in the correlations between interior and exterior points
The z axis has to be multiplied by $(\hbar k^2/8\pi)^2$

Firstly we note the presence of the ultraviolet divergence in $U^{'} = U$ where inside the Event Horizon $U^{'} = -U = e^{-\kappa u}$.
From Eq. 6.56 we find a strong peak as we can see from Figure 6.3.
This peak in the correlations between the stress energy tensors describes two massless particles, one going to infinity, the other falling towards the singularity as expected.
Indeed, this is related to the representation of the $|U>$ state in the Boulware basis, and thus to the analysis in $Chapter\ IV$, $section: Correlations$.
If we consider points outside the Event Horizon, since

$$U_K = -e^{-\kappa u} \tag{6.63}$$

and so, using the result in Eq. 6.60 we find

$$< U|T_{uu}(x)T_{uu}(x^{'})|U >= \left(\frac{\hbar\kappa^2}{8\pi^2}\right)^2 \frac{1}{\sinh^4(\frac{\kappa(u-u^{'})}{2})} \tag{6.64}$$

which does not show any peak.
Thus the peak in the stress energy correlations is a characteristic feature of the entangled outgoing particles inside and outside the Black Hole (see Eq. 4.114).
Indeed, these correlations are searched experimentally in the Black Hole analogue models.
It is interesting to study also the correlation between the ingoing components of the stress

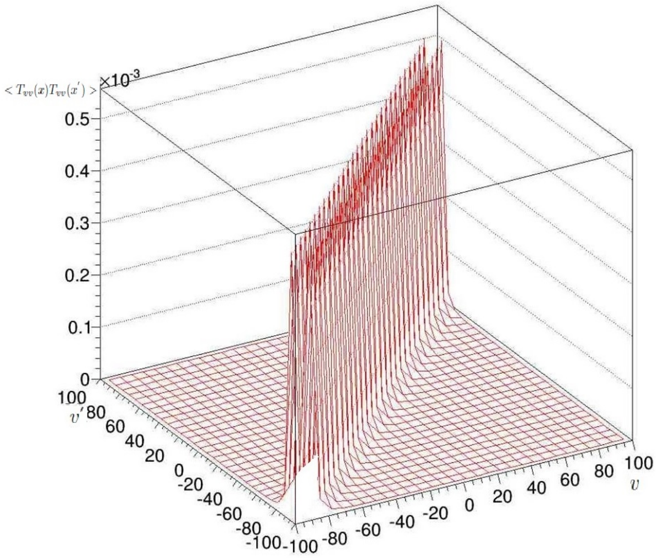

Figure 6.4: Correlations, component $< T_{vv}(x)T_{vv}(x^{'}) >$
The z axis has to be multiplied by $(\hbar/2\pi^2)$

energy tensors which is, using the same above reasoning

$$< U|T_{vv}(x)T_{vv}(x^{'})|U >= \left(\frac{\hbar^2}{4\pi^2}\right)\frac{1}{(v-v')^4} \tag{6.65}$$

Note again the presence of the ultraviolet divergence $v = v^{'}$. The correlations falls as $1/x^4$ as it can be seen from Figure 6.4.

6.8 Correlations - Hartle Hawking State

In this section we compute the 2 point stress energy function in the Hartle-Hawking vacuum.
We recall the previously computed Wightman Function in the Hartle-Hawking state

$$G^{+}_{HH}(x,x^{'}) = -\frac{\hbar}{4\pi}\ln[|(\Delta U_K \Delta V_K)| \tag{6.66}$$

it is easy to verify that

$$G^{+}_{HH}(x,x^{'}) = G^{+}_{HH}(x^{'},x) \tag{6.67}$$

where x is placed inside the horizon and $x^{'}$ outside the horizon.
Recalling the previously found green function in the Unruh state we can rewrite it in the Eddington Filkenstein coordinates

$$G^{+}_{HH}(x,x^{'}) = -\frac{\hbar}{4\pi}\ln[|(e^{-\kappa u} + e^{-\kappa u^{'}})(e^{\kappa v} - e^{\kappa v^{'}})| \tag{6.68}$$

where we have discarted the infrared divergent constant and we used the definition of the Kruskal coordinate

$$U_K = -e^{-\kappa u} \qquad r > 2M \tag{6.69}$$

$$U_K^{'} = e^{-\kappa u} \qquad r < 2M \tag{6.70}$$

$$V_K = +e^{\kappa v} \qquad \forall r \tag{6.71}$$

The general equation for the 2 point stress energy correlations is

$$< \tilde{T}_{\mu\nu}(x)\tilde{T}_{\rho\sigma}(y) >= \lim_{x^{'}\to x}\lim_{y^{'}\to y} \Sigma_{\mu\nu}\Sigma_{\rho\sigma}G(x,x^{'},y,y^{'}) \tag{6.72}$$

Again the uu component of the point separated differential operator $\Sigma_{\mu\nu}$ is

$$\Sigma_{uu} = \partial_u^x \partial_u^{x^{'}} \tag{6.73}$$

Thus

$$< T_{uu}(x)T_{uu}(x^{'}) >= \lim_{x^{'}\to x}\lim_{y^{'}\to y} \partial_u^x \partial_{u^{'}}^{x^{'}} \partial_u^y \partial_u^{y^{'}} [G_{xy^{'}}G_{x^{'}y} + G_{xy}G_{x^{'}y^{'}} + G_{yx^{'}}G_{y^{'}x} + G_{xy}G_{y^{'}x^{'}}] \tag{6.74}$$

This gives

$$< HH|\tilde{T}_{uu}(x)\tilde{T}_{uu}(x^{'})|HH >= \left(\frac{\hbar\kappa^2}{2\pi}\right)^2 \frac{U_K^2 {U^{2}}^{'}_{K}}{(U_K - U_K^{'})^4} \tag{6.75}$$

Again in the Eddington Filkenstein coordinates

$$< HH|\tilde{T}_{uu}(x)\tilde{T}_{uu}(x^{'})|HH >= \left(\frac{\hbar\kappa^2}{8\pi^2}\right)^2 \frac{1}{\cosh^4(\frac{\kappa(u-u^{'})}{2})} \tag{6.76}$$

Therefore this result is equal to the the previously computed Unruh's case (see Figure 6.3). The peak describes the outgoing entangled exctitations inside and outside the Black Hole region.

As in the Unruh's case, this is related to the representation of the $|U>$ state in the Boulware basis, and thus to the analysis in *Chapter IV*, *section* : *Correlations*.

If we consider points outside the Event Horizon, since

$$U_K = -e^{-\kappa u} \tag{6.77}$$

and so, using the result in Eq. 6.68 we find

$$< HH|T_{uu}(x)T_{uu}(x^{'})|HH >= \left(\frac{\hbar\kappa^2}{8\pi^2}\right)^2 \frac{1}{\sinh^4(\frac{\kappa(u-u')}{2})} \tag{6.78}$$

which does not describe any peak.

6.9 Analysis of the correlations

In this Chapter we computed the stress energy tensor correlations between points inside and outside the Black Hole region.
This analysis shows a rich and elegant correlations structure.
The main result is the peak present in the expression for $< T_{uu}(x)T_{uu}(x^{'}) >$ in the Unruh State and in the Hartle-Hawking state. These maxima in the two expressions correspond to the outgoing entangled particles.
Moreover, we find the presence of an ultraviolet divergence placed at $U = U^{'}$. This is related to the coincidence point limit. In addiction, it would be really interesting to find a way to renormalize $< T^2_{uu}(x) >$ in order to study the behaviour of the quantum fields at the same point.

Conclusion

In this thesis we have analyzed the Physics of Black Holes and Hawking Radiation.
We have found that Black Holes obey classical laws mathematically very similar to those of Thermodynamics. Adding quantum effects, through the use of the semiclassical theory of Quantum Field in Curved Spacetime, we have found that Black Holes emitt particles with a thermal spectrum. In particular there are no correlations between particles with different frequency.
Moreover, we have found that a measurement performed by an external observer can be described by a thermal density matrix because of the tracing over the interior degrees of freedom.
This is due to the causal structure of the Black Hole spacetime and has a very important implication: the Information Paradox.
This states that an initial pure state evolves in a mixed state, in presence of an evaporating Black Hole. This obviously is in contrast with the Quantum Mechanics principle of unitary evolution.
Then, in order to understand better the meaning of the quantum states of interest, we study the mean value of the stress energy tensor.
Finally we have studied the correlations between these objects, with particular emphasis on the correlations between points inside and outside the horizon. Indeed, they are related to the Acoustic Models of Black Holes and they exhibit a peak that, hopefully soon, could be measured and this would be the proof of the existence of Hawking radiation.
Moreover, it would be interesting to study the coincidence limit of the stress energy correlations. To study this object, it would be necessary to renormalize $< T^2(x) >$, because of the presence of the ultraviolet divergence $\lim_{x \to x'} < T(x)T(x^{'}) >$.

Appendix A

Penrose Diagrams

In order to understand correctly the meaning of a General Relativistic solution, it is of fundamental importance to find a technique which permits us to understand intuitively the causal structure of a spacetime geometry: the Penrose Diagram.
The main idea is to codificate the entire spacetime structure in an unphysical spacetime $g'_{\mu\nu}$ which conserves the causal properties of the true physical solution.
This can be done throught the use of a *conformal* transformation

$$ds^2 \rightarrow ds^{'2} = \Omega^2(x^\mu)ds^2 \tag{A.1}$$

Note that Ω depends on the spacetime point and it is in general not vanishing and positive. This mathematical transformation changes the distance between points but the character (timelike, spacelike or null) is not modified.
The geodesic equation

$$\frac{dx^\mu}{d\lambda}\nabla_\mu\frac{dx^\nu}{d\lambda} = 0 \tag{A.2}$$

becomes

$$\frac{dx^\mu}{d\lambda}\nabla'_\mu\frac{dx^\nu}{d\lambda} = 2\frac{dx^\nu}{d\lambda}\frac{dx^\alpha}{d\lambda}\nabla_\alpha \ln\Omega - \left(g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\right)g^{\nu\mu}\nabla_\mu \ln\Omega \tag{A.3}$$

where ∇' is the covariant derivative with respect to the unphysical metric g'.
In the null case this is the geodesic equation in a non affine parametrization λ.
It can be verified that the new affine parameter is

$$\frac{d\lambda'}{d\lambda} = c\Omega^2 \tag{A.4}$$

where $c = const$.
In particular we can represent points at infinity within a finite diagram using

$$\Omega^2 \rightarrow 0 \tag{A.5}$$

asymptotically. In this manner an infinite distance in the physical metric is codified in a finite distance in the unphysical metric $g'_{\mu\nu}$.

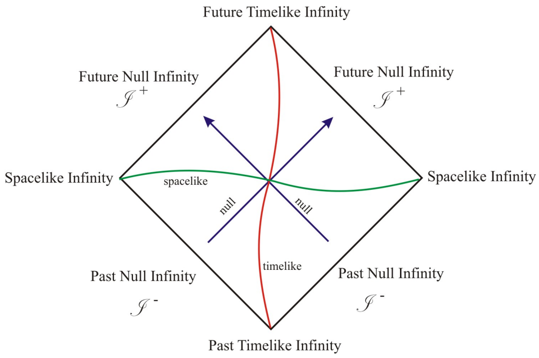

Figure A.1: Penrose Diagram: Minkowski Spacetime

A.1 The Minkowskian Penrose diagram

Let us consider the Minkowski metric

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{A.6}$$

which, using the null coordinates $u = t - r$ and $v = t + r$ can be rewritten as

$$ds^2 = -dudv + r^2(u,v)(d\theta^2 + \sin^2\theta d\phi^2) \tag{A.7}$$

With the substitution

$$u = \tan\bar{u} \qquad v = \tan\bar{v} \tag{A.8}$$

and thus

$$ds^2 = -\frac{1}{(2\cos\bar{u}\cos\bar{v})^2}(d\bar{u}d\bar{v} + r^2(d\theta^2 + \sin^2\theta d\phi^2)) \tag{A.9}$$

and taking $\Omega = (2\cos\bar{u}\cos\bar{v})^2$

$$d\bar{s}^2 = -d\bar{u}d\bar{v} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{A.10}$$

Let us define $t' = (\bar{v}+\bar{u})/2$ and $r' = (\bar{v}-\bar{u})/2$. We can define five different types of infinity:

1. *timelike future* i^+
   $t \to \infty,\ r = const \Rightarrow u, v \to \infty \Rightarrow \bar{u}, \bar{v} = \pi/2 \Rightarrow t^{'} = \frac{\pi}{2}, r^{'} = 0$

2. *timelike past* i^-
   $t \to -\infty,\ r = const \Rightarrow u, v \to -\infty \Rightarrow \bar{u}, \bar{v} = -\pi/2 \Rightarrow t^{'} = -\frac{\pi}{2}, r^{'} = 0$

3. *spacelike infinity* i^0
   $t = const,\ r \to \infty \Rightarrow u \to -\infty, , v \to +\infty \Rightarrow \bar{u} = -\pi/2, \bar{v} = +\pi/2 \Rightarrow t^{'} = 0, r^{'} = \pi/2$

4. *null future infinity* I^+
   $t \to \infty,\ r \to \infty$ with $t - r = const \Rightarrow u = const, , v \to +\infty \Rightarrow \bar{u} = const, \bar{v} = +\pi/2 \Rightarrow t^{'} = -r^{'} + \pi/2$

5. *null past infinity* I^-
   $t \to -\infty,\ r \to \infty$ with $t + r = const \Rightarrow v = const, , u \to -\infty \Rightarrow \bar{v} = const, \bar{u} = -\pi/2 \Rightarrow t^{'} = +r^{'} - \pi/2 = 0$

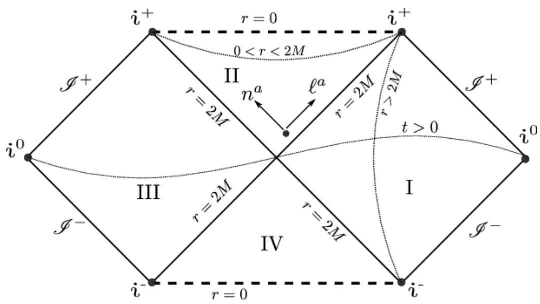

Figure A.2: Penrose Diagram: Schwarzschild Diagram

The Penrose Diagram of the Minkowski spacetime is depicted in Figure A.1.
Note that an uniformely accellerated observer cannot see a part of the Minkowski spacetime and, in this case, there is the presence of an event horizon, the *Rindler* horizon.

A.2 The Penrose Diagram for the Schwarzschild spacetime

We are now ready to find the Pensore diagram for the Schwarzschild spacetime. Recalling the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{A.11}$$

Using the previously defined Kruskal Coordinates (see Chapter II) U and V we can rewrite

$$ds^2 = -\frac{32m^2 e^{-r/4M}}{r} dU dV + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{A.12}$$

Again

$$\bar{U} = 4M \tan U \qquad \bar{V} = 4M \tan V \tag{A.13}$$

This leads to

$$d\bar{s}^2 = -\frac{32M^3 e^{-r/2M}}{r(\cos\bar{U}\cos\bar{V})^2} d\bar{U} d\bar{V} + r^2(\bar{U}\bar{V})(d\theta^2 + \sin^2\theta d\phi^2) \tag{A.14}$$

and, if we choose

$$\Omega^2 = \frac{r e^{r/2M}(\cos\bar{U}\cos\bar{V})^2}{8M^3} \tag{A.15}$$

and we find the unphysical line element

$$d\bar{s} = -4 d\bar{U} d\bar{V} + \frac{r^3 e^{r/2M}(\cos\bar{U}\cos\bar{V})^2}{8M^3}(d\theta^2 + \sin^2\theta d\phi^2) \tag{A.16}$$

We note that, from the relation between the Kruscal coordinates and the Schwarzschild coordinates

$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = -\tan\bar{U}\tan\bar{V} \tag{A.17}$$

and so the point $r = 0$ is made of two disconnected vertial horizontal curves

$$\tan \bar{U} \tan \bar{V} = 1 \Longrightarrow \bar{U} + \bar{V} = \pm \frac{\pi}{2} \tag{A.18}$$

Defining $T' = (\bar{U} + \bar{V})/2$ we have $r = 0 \Rightarrow T' = \frac{\pi}{4}$.
Morover the condition $r > 0$ becomes

$$-\frac{\pi}{2} \leq \bar{U} + \bar{V} \leq +\frac{\pi}{2} \tag{A.19}$$

The Schwarzschild Penrose diagram is depicted in Figure A.2.
Note that the horizons are formed by null geodesics and are located at $\bar{U} = 0$ (H^+ the future horizon) and $\bar{V} = 0$ (H^- the past horizon).
Moreover the singularity $r = 0$ is clearly spacelike, and so every observer falling in the region II must encounter it in a finite proper time.
The surfaces divede the diagram in 4 region:

1. I Asympotically flat spacetime
   Asympotically the structure is equal to Minkowski's.

2. II Black Hole Region
   The region of no escape. It contains the Black Hole singularity $r = 0$.

3. III White Hole region It contains the White Hole singularity $r = 0$

4. IV Asympotically flat spacetime
   It is causally disconnected from the region I.

Appendix B

Global Methods

B.1 Future and Past

We define λ to be a future directed timelike curve if at each $p \in \lambda$ the tangent vector is a future directed timelike vector.
The same definition applies to a future directed *causal* curve, but now the tangent vector can be also null like.
The *chronological* future of $p \in M$ is defined

$$I^+(p) = [p \in M \ : \ \exists \ a \ future \ directed \ timelike \ curve : \lambda(0) = p, \ \lambda(1) = q] \tag{B.1}$$

where q is the final point of the curve.
If we consider a subset $S \subset M$ we define $I^+(S)$ as

$$I^+(S) = \cup_{p \in S} I^+(p) \tag{B.2}$$

The *causal* future of p is defined as the chronological future with the substitution of the words timelike curve with causal curve. In particular, for a subset $S \subset M$ we have

$$J^+(S) = \cup_{p \in S} J^+(p) \tag{B.3}$$

The definition for the past are obvious.

We want now to define the notion of extendibility of a continuous curve.
Consider a future directed causal curve $\lambda(t)$. We say that $p \in M$ is a $future\ endpoint$ of λ if for every neighborhood O of p there exists a t_0 such that $\lambda(t) \in O \ \forall t > t_0$.
A curve is called $future\ inextendible$ if it has no future endpoint.

B.2 Timelike and null like congruences

Let M be a manifold and $O \subset M$ be open subset. A *congruence* is a family of curves such that throught each $p \in O$ passes precisely one curve of this family.
Consider now a smooth congruence of timelike geodesics. Consider now the tangents normalized to unit lenghts $\xi^\mu \xi_\mu = -1$. Then we define

$$B_{\mu\nu} = \nabla_\mu \xi_\nu \tag{B.4}$$

which is purely spatial

$$B_{\mu\nu}\xi^{\mu} = B_{\mu\nu}\xi^{\nu} = 0 \tag{B.5}$$

Let us define

$$h_{\mu\nu} = g_{\mu\nu} + \xi_{\mu}\xi_{\nu} \tag{B.6}$$

which is a spatial metric since $\xi^{\mu}h_{\mu\nu} = \xi^{\nu}h_{\mu\nu} = 0$. Thus $h^{\mu}_{\nu} = g^{\mu\lambda}h_{\nu\lambda}$ is the projection operator onto the subspace of the tangent vectors perpendicular to ξ^{μ}. We define the expansion θ, the shear $\sigma_{\mu\nu}$ and the twist $\omega_{\mu\nu}$:

$$\theta = B^{\mu\nu}h_{\mu\nu} \tag{B.7}$$

$$\sigma_{\mu\nu} = B_{(\mu\nu)} - \frac{1}{3}\theta h_{\mu\nu} \tag{B.8}$$

$$\omega_{\mu\nu} = B_{[\mu\nu]} \tag{B.9}$$

and so

$$B_{\mu\nu} = \frac{1}{3}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} \tag{B.10}$$

θ measures the average expansion of 2 nearby geodesics, while $\omega_{\mu\nu}$ theri rotation and $\sigma_{\mu\nu}$ their shear.
We can find

$$\xi^{\lambda}\nabla_{\lambda}B_{\mu\nu} = \xi^{\lambda}\nabla_{\lambda}\nabla_{\nu}\xi_{\mu} = \xi^{\lambda}\nabla_{\nu}\nabla_{\lambda}\xi_{\mu} + R_{\lambda\nu\mu}{}^{\sigma}\xi^{\lambda}\xi_{\sigma} = \tag{B.11}$$

$$= \nabla_{\nu}(\xi^{\lambda}\nabla_{\lambda}\xi_{\mu}) - (\nabla_{\nu}\xi^{\lambda})(\nabla_{\lambda}\xi_{\mu}) + R^{\sigma}_{\lambda\nu\mu}\xi^{\lambda}\xi_{\sigma} = \tag{B.12}$$

$$= -B^{\lambda}_{\nu}B_{\mu\lambda} + R^{\sigma}_{\lambda\nu\mu}\xi^{\lambda}\xi_{\sigma} \tag{B.13}$$

and taking the trace we find the *Raychaudhuri's equation*:

$$\xi^{\lambda}\nabla_{\lambda}\theta = \frac{d\theta}{d\tau} = -\frac{1}{3}\theta^{2} - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}\xi^{\mu}\xi^{\nu} \tag{B.14}$$

Consider now the last term of the equation, we find

$$R_{\mu\nu}\xi^{\mu}\xi^{\nu} = 8\pi\left[T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right]\xi^{\mu}\xi^{\nu} = 8\pi\left[T_{\mu\nu}\xi^{\mu}\xi^{\nu} + \frac{1}{2}T\right] \tag{B.15}$$

Note that $T_{\mu\nu}\xi^{\mu}\xi^{\nu}$ is the energy density measured by an observer with 4-velocity ξ^{μ}.
It is belived that, for any physically realistic classical matter we have

$$T_{\mu\nu}\xi^{\mu}\xi^{\nu} \geq 0 \tag{B.16}$$

for all timelike ξ^{μ}, which means that the energy density is non negative.
This is called *weak energy condition*.
This condition means that the matter always has a converging effect on congruences of null geodesics.
This property is very important in the demonstration of the Singularity Theorems that prove the necessary presence of singularities in varius physical settings like cosmology and gravitational collapse.

Moreover it is also physically reasonable that the right hand of Eq. B.15 does not become negative. This means

$$T_{\mu\nu}\xi^{\mu}\xi^{\nu} \geq -\frac{1}{2}T \tag{B.17}$$

for all unit timelike vectors ξ^{μ}. This is known as the *strong energy condition*.
There is also another energy condition: the *dominant energy condition*. Consider an observer with 4-velocity ξ^{μ}, then $T^{\mu}_{\nu}\xi^{\nu}$ is a future directed timelike or null vector.
$T^{\mu}_{\nu}\xi^{\nu}$ physically represent the energy momentum vector of matter as seen by him and this condition can be interpreted as saying that the speed of the energy flow must be always less than the speed of light.

B.3 Null congruences

Consider a congruence of null geodesics with tangent k^{μ}.
For every deviation vector η^{μ} we have

$$\xi^{\mu}\nabla_{\mu}(\xi_{\nu}\eta^{\nu}) = 0 \tag{B.18}$$

and thus $k^{\nu}\eta_{\nu}$ does not vary along the geodesic. This implies that we can decompose η_{ν} is the sum of a vector not orthogonal to k_{ν} which is parallelly propagated along the geodesic plus a vector perpendicular to k_{ν}. In our discussion only the perpendicular vector is useful and thus we can consider $\eta^{\mu}k_{\mu} = 0$. Note that if a vector $\eta^{'}_{\nu}$ verifies $\eta_{\nu} - \eta^{'}_{\nu} = const k_{\nu}$ it represents the same physical displacement.
Since this restriction is independent from the first, we have that the interesting space is 2 dimensional and it is composed by the class of equivalence of vectors satisfing $\eta^{'}_{\nu} - \eta_{\nu} = ck_{\nu}$.
More generally the space of the tensors $T^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m}$ which are equal to zero when they are contracted with k_{μ} or k^{μ} forms a subspace called $\hat{V}_p$.
The tensor field

$$B_{\mu\nu} = \nabla_{\nu}k_{\mu} \tag{B.19}$$

also satisfies the above propertiy, and thus gives rise to $\widehat{B_{\mu\nu}}$. We can decompose $\widehat{B_{\mu\nu}}$

$$\widehat{B}_{\mu\nu} = \frac{1}{2}\theta\widehat{h}_{\mu\nu} + \widehat{\sigma}_{\mu\nu} + \widehat{\omega}_{\mu\nu} \tag{B.20}$$

$$\theta = \widehat{h}^{\mu\nu}\widehat{B}_{\mu\nu} \tag{B.21}$$

$$\widehat{\sigma}_{\mu\nu} = \widehat{B}_{(\mu\nu)} - \frac{1}{2}\theta\widehat{h}_{\mu\nu} \tag{B.22}$$

$$\widehat{\omega}_{\mu\nu} = \widehat{B}_{[\mu\nu]} \tag{B.23}$$

and thus θ, $\widehat{\sigma_{\mu\nu}}$ and $\widehat{\omega_{\mu\nu}}$ have the interpretation as the expansion, shear and twist of the congruence.
The change of the numerical factor $\frac{1}{2}$ with respect to the previous equation is due to the two dimensionality of the vector space.
With the same derivation of the previous section we find

$$k^{\lambda}\nabla_{\lambda}B_{\mu\nu} + B^{\lambda}_{\nu}B_{\mu\lambda} = R_{\lambda\nu\mu}{}^{\sigma}k_{\sigma}k^{\lambda} \tag{B.24}$$

and thus, since $k^\mu B_{\mu\nu} = 0$ we find

$$k^\lambda \nabla_\lambda \widehat{B}_{\mu\nu} + \widehat{B}_\nu^\lambda \widehat{B}_{\mu\lambda} = \widehat{R_{\lambda\sigma\mu\nu} k^\lambda} k^\sigma \tag{B.25}$$

and taking the trace we find

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \widehat{\sigma}_{\mu\nu}\widehat{\sigma}^{\mu\nu} + \widehat{\omega}^{\mu\nu}\widehat{\omega}_{\mu\nu} - R_{\lambda\sigma} k^\lambda k^\sigma \tag{B.26}$$

If we take $\widehat{\nabla_\mu k_\mu} = 0$ and thus $\widehat{B_{\mu\nu}} = 0$ we find

$$R_{\mu\nu} k^\mu k^\nu = 0 \tag{B.27}$$

which is the result used in Chapter 2. Note that the above equation is very similar to the Raychaudhuri's equation. Using the Einstein's equations we find

$$R_{\mu\nu} k^\mu k^\nu = 8\pi G_N T_{\mu\nu} k^\mu k^\nu \tag{B.28}$$

B.4 Conjugate Points

Consider now a manifold M and let be γ a geodesic with tangent v^μ. A solution η^μ of the geodesic deviation equation

$$v^\mu \nabla_\mu (v^\nu \nabla_\nu \eta^\lambda) = -R^\lambda_{\mu\nu\sigma} \eta^\nu v^\mu v^\sigma \tag{B.29}$$

is called a Jacoby field.
A pair of points $p, q \in \gamma$ are called *conjugate* points if exists a Jacoby field which is not 0 everywhere but vanish at both p and q.
Intuitively p and q are conjugate if two nearby geodesics intersects γ at p and q.
We will enunciate two theorems, for the demonstration see eg. Wald.

Theorem 1.
Let $(M, g_{\mu\nu})$ a spacetime which verifies $R_{\mu\nu} k^\mu k^\nu \geq 0$ for every null k^μ.
Let μ be a null geodesic and $p \in \mu$. Suppose the convergence θ of the null geodesics emanating from p attains negative value θ_0 at $r \in \mu$. Then whitin affine lenght $\lambda \leq 2/|\theta_0|$ from r, there exists a point q conjugate to p along μ assuming that μ extends that far.

Theorem 2.
Let $(M, g_{\mu\nu})$ a globally hyperbolic spacetime and let K be a compact two dimensional spacelike submanifold of M.
Then every $p \in \dot{I}^+(K)$ lies on a future directed null geodesics starting from K which is orthogonal to K and has no point conjugate to K between K and p.

Bibliography

[1] Steven Weinberg , *Gravitation and Cosmology.* Joh WileySons (1971)

[2] Robert M. Wald, *General Relativity.* University of Chicago Press (1984)

[3] Hawking, Ellis, *The Large Structure of Spacetime.* Cambridge University Press (1973)

[4] Fabbri, Navarro-Salas , *Modelling Black Hole Evaporation.* Imperial College Press (2005)

[5] Bardeen, Carter, Hawking, *The four Laws of Black Hole Mechanics* (1973)

[6] Peskin, Schroeder *An introduction to Quantum Field Theory.* Addison Weysley Advanced Book Program (1995)

[7] Birrel, Davies *Quantum Field Theory in Curved Spacetime* Cambridge Monographs on Mathematical Physics (1982)

[8] Parker, Toms *Quantum Field Theory in Curved Spacetime, Quantized Fields and Gravity* Cambridge Monographs on Mathematical Physics (2009)

[9] Hawking, *Black Hole explosions?* *Nature* 248, 30-31 (01 March 1974) doi:10.1038/248030a0

[10] Hawking, *Breakdown of Quantum Predictability in Gravitational Collapse* Phys. Rev D 14,2460 (15 November 1976)

[11] Phillips, Hu *Noise Kernel in Stochastic Gravity and Stress Energy Bi-Tensor of Quantum Fields in Curved Spacetime* Phys. Rev. D, Oct 5, 2000

[12] Balbinot, Fabbri, Fagnocchi, Parentani *Hawking Radiation from Acoustic Black Holes, Short Distance and Backreaction Effects* Rivista del Nuovo Cimento Vol.28, N.3, DOI 10.1393/ncr/i2006-10001-9 (2006)