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SCUOLA DI SCIENZE MATEMATICHE, FISICHE E NATURALI
Corso di Laurea in Matematica

A mathematical introduction to Kerr black holes

Tesi di Laurea in Matematica

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Introduzione

La *relatività generale* è una teoria fisica formulata da Albert Einstein. Questa teoria è stata pubblicata nel 1916. Essa cambia profondamente il concetto di *interazione gravitazionale* rispetto a come era stato formulato da Newton. Se prima era descritta come azione a distanza tra corpi massivi, ora viene descritta tramite la curvatura dello spaziotempo (in particolare, ogni corpo massivo curva lo spaziotempo). La relatività generale è riuscita a conciliare il principio di relatività galileiana con le equazioni di Maxwell. Il principio di relatività galileiana asserisce che le leggi della fisica sono le stesse per tutti i sistemi inerziali che equivale matematicamente a dire che tutte le leggi della fisica sono simmetriche rispetto alle trasformazioni galileiane. Il punto cruciale delle equazioni di Maxwell risiede nell'implicazione che la luce viaggi costantemente a velocità c, indipendentemente dal sistema di riferimento. La teoria della relatività generale è fondata su un concetto di *spaziotempo*, descritto come uno spazio pseudo-riemanniano a quattro dimensioni (la quarta dimensione è il tempo). Le equazioni di campo di Einstein, per ogni punto dello spaziotempo, legano la curvatura al tensore energia-momento T, che descrive la quantità di materia ed energia nel punto. Di queste equazioni è stata dimostrata l'unicità, sotto l'ipotesi di co-varianza generale. Posti a zero la costante cosmologica e il tensore energia-momento, queste equazioni hanno delle soluzioni utilizzate per lo studio dei buchi neri. Un buco nero è descritto come una regione dello spaziotempo con un campo gravitazionale talmente forte da creare un orizzonte degli eventi. Un orizzonte degli eventi è una regione chiusa dello spaziotempo dalla quale niente può fuoriuscire (ovvero, qualsiasi particella si trovi dentro a questa regione avrebbe bisogno di una velocità superiore a quella della luce per poterne uscire). In particolare, delle quattro soluzioni per i buchi neri, noi ci occuperemo della soluzione di Kerr.

La soluzione di Kerr descrive un buco nero scarico e rotante. Poiché i buchi neri possono essere descritti interamente da tre parametri (massa, carica elettrica e momento angolare) e considerando che finora non sono mai stati osservati buchi neri con una carica elettrica, probabilmente la soluzione di Kerr è la soluzione più generale corrispondente al caso reale. In particolare, andremo a studiare delle particolari regioni dello spaziotempo (posto ovviamente il tempo costante) chiamate *ergosfere* e *orizzonti degli eventi*. Per analizzare queste regioni, dovremo studiare le traiettorie delle particelle massive e dei fotoni (che, in particolare, risultano essere geodetiche). Tramite lo studio delle geodetiche, vedremo che ci saranno regioni dove tutte le geodetiche vengono curvate e seguono lo stesso senso di rotazione del buco nero (che chiameremo *ergosfere*), e regioni dove invece le geodetiche possono solo entrare ma non uscire (orizzonte degli eventi esterno) o possono solo uscire, ma non entrare (orizzonte degli eventi interno). Nel primo capitolo sono introdotti gli strumenti matematici necessari per studiare i buchi neri di Kerr, che consistono in tensori, geodetiche e vettori di Killing. Nel secondo capitolo viene enunciata la metrica di Kerr e studiata superficialmente, ovvero studieremo l'effetto che la rotazione del corpo ha sullo spaziotempo, la curvatura dello spaziotempo e enunceremo uno dei più importanti "teoremi" riguardanti i buchi neri. Nel terzo capitolo vengono studiati approfonditamente i due orizzonti degli eventi, le due ergosfere, l'esistenza di un cosiddetto orizzonte di Killing e il moto dei fotoni lungo il piano equatoriale.

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Chapter 1

Definitions

1.1 Tensors

Definition 1.1. *Tensor*
A (m, n) tensor is a multilinear application

$$T : \overbrace{V^* \times ... \times V^*}^{\text{m}} \times \overbrace{V \times ... \times V}^{\text{n}} \longrightarrow \mathbb{R}$$

which is linear in each of its arguments, where V denote a vector space and V^* is the corresponding dual space of covectors.
Let $\{e_j\}$ be a basis for V. From that, we define $\{e^i\}$ a basis for V^* as follows:

$$e^i(e_j) = \begin{cases} 1 & for \quad i = j \\ 0 & for \quad i \neq j \end{cases}$$

If we apply a tensor T to this base, we find a (n+m)-dimensional array of components.

$$T^{i_1 ... i_n}_{j_1 ... j_m} \equiv T(e^{i_1}, ..., e^{i_n}, e_{j_1}, ...e_{j_m})$$

A different choice of basis will yield different components.
Let us now explore the change of basis for a general (m,n) tensor
Let $A = \{e_j\}, B = \{\epsilon_j\}$ be two different coordinate basis.
There must exist two matrices $C, D = C^{-1} : \begin{cases} A = CB \\ B = DA \end{cases}$

Now, the transformation law for a tensor is:

$$U^{j_1 \dots j_m}_{i_1 \dots i_n} = D^{j_1}_{l_1} \dots D^{j_m}_{l_n} C^{t_1}_{i_1} \dots C^{t_m}_{i_n} T^{l_1 \dots l_n}_{t_1 \dots t_m}$$

$$T^{j_1 \dots j_m}_{i_1 \dots i_n} = C^{j_1}_{l_1} \dots C^{j_m}_{l_n} D^{t_1}_{i_1} \dots D^{t_m}_{i_n} U^{l_1 \dots l_n}_{t_1 \dots t_m}$$

Definition 1.2. *Metric Tensor*
A metric tensor g is a (0,2) tensor, symmetric

$$g(v, w) = g(w, v) = g_{ij} v^i w^j = v \cdot w$$

and non-degenerate

$$[g(v, w) = 0 \ \forall w \in T_P \Leftrightarrow v = 0] \Longleftrightarrow det[g_{ij}] \neq 0$$

everywhere.
For every point in a local chart, the metric tensor can be written as a symmetric matrix with nonzero determinant.

Definition 1.3. *Christoffel symbols of the first kind* : $[jk, i] := \frac{1}{2} \left[\partial_j g_{ki} + \partial_k g_{ij} - \partial_i g_{jk}\right]$

Definition 1.4. *Christoffel symbols of the second kind* : $\Gamma^l_{jk} = \Gamma^l_{kj} := g^{li}[jk, i]$
We can rewrite Chrystoffel symbols of the second kind in a more explicit way:
$\Gamma^i_{jk} := \frac{1}{2} g^{il}(g_{jl,k} + g_{lk,j} - g_{jk,l})$
where the comma denotes a partial derivative.
It is worth nothing that Christoffel symbols of the second kind can also be defined in another way:
Given $\{e_i\}$ a local coordinate basis, Γ^l_{jk} are the unique coefficients such that the equation

$$\nabla_i e_j = \Gamma^k_{ij} e_k$$

holds.

1.1.1 Riemann tensor and its contractions

Definition 1.5. *Riemann Tensor*
Let M be a differentiable manifold, provided of a *connection* ∇. Let X,Y,Z be vector fields on M. We then define Riemann Tensor the following (1,3) tensor:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

In the index notation, we can write the Riemann tensor as R^i_{jkl}
Rising and lowering indexes, we can obtain:

- $R_{abcd} = g_{ak} R^{kbcd}$
- $R^{abcd} = g^{bj} g^{ck} g^{dl} R^{ajkl}$

Properties

- is block symmetric $R_{ijkl} = R_{klij}$
- is antisymmetric in respect to the swap of the first and the second couple of indexes $R_{ijkl} = -R_{jikl} = -R_{ijlk}$

The general form of the Riemann tensor is given by:

$$R_{ijkl} = \frac{1}{2} \left(g_{il;jk} - g_{ik;jl} + g_{jk;il} - g_{jl;ik} \right)$$

where ; denotes the covariant derivative.

Definition 1.6. *Ricci Tensor*
Let M be a differentiable manifold , with a *connection* ∇. We then define the Ricci tensor as a contraction of Riemann tensor

$$R_{ij} = R^k_{ikj}$$

Properties

- it is symmetric $R_{ij} = R_{ji}$

The Ricci tensor can be written in terms of the Riemann curvature tensor and the Christoffel symbols:

$$R_{ij} = R^t{}_{itj} = \partial_t \Gamma^t_{ij} - \partial_j \Gamma^t_{ti} + \Gamma^t_{t\lambda}\Gamma^\lambda_{ji} - \Gamma^t_{j\lambda}\Gamma^\lambda_{ti}$$

Definition 1.7. *Scalar Curvature*
We define the scalar curvature R as Ricci tensor trace. To be able to define the trace, we must raise an index of the Ricci tensor, to have a (1,1) tensor. This can be done through the metric g.

$$R = tr_g R_{ij} = g^{ij} R_{ij} = R^j{}_j$$

The scalar curvature is an invariant, under a change of coordinates.
Given a coordinate system and a metric tensor, we can write the scalar curvature as:

$$R = g^{ij}(\Gamma^t_{ij,t} - \Gamma^t_{it,j} + \Gamma^\lambda_{ij}\Gamma^t_{t\lambda} - \Gamma^\lambda_{it}\Gamma^t_{j\lambda})$$

Definition 1.8. *Einstein tensor*
The Einstein tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

It expresses the curvature of a Riemannian manifold. The most explicit form of the Einstein tensor is:

$$G_{ab} = \left(\delta^\nu_a \delta^\mu_b - \frac{1}{2} g_{ab} g^{\nu\mu}\right)\left(\Gamma^i_{\nu\mu,i} - \Gamma^i_{\nu i,\mu} + \Gamma^i_{ij}\Gamma^j_{\nu\mu} - \Gamma^i_{\mu j}\Gamma^j_{i\nu}\right)$$

where δ^i_j is the Kronecker tensor.
Properties:

- Bianchi identity:

  $$\nabla_\mu G^{\mu\nu} = 0$$

  tells us that the Einstein tensor is covariantly conserved, and reduces the number of independent parameters on which the Einstein tensor depends.

Definition 1.9. *Kretschmann Scalar*
The Kretschmann scalar is defined as: $K = R_{abcd}R^{abcd}$

$$R_{abcd}R^{abcd} = g_{at}R^{tbcd}g^{bj}g^{ck}g^{dl}R^{ajkl}$$

This scalar is a quadratic invariant (it does not depend on the reference frame). This implies that if Kretschmann scalar diverges for a specific frame, it will diverge for all frames. If this happens, there must exist a singularity.
We can't use the scalar curvature to find singularities, because of the vacuum field equations: if the Ricci tensor is zero, the scalar curvature (defined as the contraction of Ricci tensor) will be zero too.

Definition 1.10. *Einstein Field Equations*:
The Einstein field equations are a set of 10 equations. They describe the gravity as a consequence of a curved space-time (curved by energy and matter).
In particular, on the left side we have the Einstein tensor, that expresses the curvature of a Riemannian manifold (the space-time manifold). On the right side we have $T_{\mu\nu}$, that is the Stress-Energy tensor (this tensor describes the quantity of matter and energy in a point). The result, will be the curvature of the manifold in a point as a result of the matter and energy in that point.
Set the cosmological constant $\Lambda = 0$ and $c = 1$, the equations can be written as:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}$$

There is an important case: if we have a vacuum space (that means no energy and no matter in a small neighbourhood of the point considered), we must have by definition $T_{\mu\nu} = 0$. If this occurs, we find the so called *vacuum field equations.* If we look at the Einstein Field Equations, we can see that an obvious solution is

$$R_{\mu\nu} = 0$$

In particular, this is the only solution. In the vacuum case, we can immediately note that the scalar curvature R must be zero too, because it is defined as a contraction of the Ricci tensor.

1.2 Geodesics

Definition 1.11. *Geodesic*
Let M be a metric space, I an interval of $\mathbf{R}$. A curve $\gamma : I \to M$ is a *geodesic* if, given J a neighbourhood of $x \in I$

$$d(\gamma(t), \gamma(s)) = |t - s| \ \ \forall \ t, s \in J$$

Given a Riemannian manifold M with a metric tensor g, we can define the length of a curve $\gamma : [a, b] \to M$:

$$L(\gamma) = \int_a^b \sqrt{g(\dot\gamma(t))g(\dot\gamma(t))}dt$$

We can define a geodesic also using *covariant derivatives*: a differentiable curve γ with $\dot\gamma$ differentiable is a geodesic if

$$\nabla_{\dot\gamma}\dot\gamma = 0$$

Geometrically, is equivalent to say that the tangent vector is parallel transported along γ.

Now, geodesics are a sort of generalisation of the notion of *straight line*, if we consider instead of $\mathbb{R}^n$ a general curved space (it is immediate to see that the definition of geodesic equals the definition of a straight line, if we consider $M = \mathbb{R}^n$).

1.2.1 Geodesics derivation from The Principle of Equivalence

In a free falling coordinate system, we suppose that all particles have null acceleration (in the neighbourhood of a given point).
Using the local coordinate system (T, X^1, X^2, X^3) and the general coordinate system (t, x^1, x^2, x^3) we have:

$$\frac{d^2X^\mu}{dT^2} = 0$$

From the chain rule, we find that

$$\frac{dX^\mu}{dT} = \frac{\partial X^\mu}{\partial x^\nu}\frac{dx^\nu}{dT}$$

then we differentiate again in respect to T:

$$0 = \frac{d^2X^\mu}{dT^2} = \frac{\partial X^\mu}{\partial x^\nu}\frac{d^2x^\nu}{dT^2} + \frac{\partial^2 X^\mu}{\partial x^\nu \partial x^\alpha}\frac{dx^\nu}{dT}\frac{dx^\alpha}{dT} \quad \Rightarrow \quad \frac{\partial X^\mu}{\partial x^\nu}\frac{d^2x^\nu}{dT^2} = -\frac{\partial^2 X^\mu}{\partial x^\nu \partial x^\alpha}\frac{dx^\nu}{dT}\frac{dx^\alpha}{dT}$$

Multiplying for $\frac{\partial x^\beta}{\partial X^\mu}$ we find:

$$\frac{d^2 x^\beta}{dT^2} = -\left[\frac{\partial^2 X^\mu}{\partial x^\nu \partial x^\alpha}\frac{\partial x^\beta}{\partial X^\mu}\right]\frac{dx^\nu}{dT}\frac{dx^\alpha}{dT}$$

Using the chain rule, we can swap T with t, we will have:

$$\frac{d^2 x^\beta}{dt^2} = -\left[\frac{\partial^2 X^\mu}{\partial x^\nu \partial x^\alpha}\frac{\partial x^\beta}{\partial X^\mu}\right]\frac{dx^\nu}{dt}\frac{dx^\alpha}{dt} + \left[\frac{\partial^2 X^\mu}{\partial x^\nu \partial x^\alpha}\frac{\partial x^0}{\partial X^\mu}\right]\frac{dx^\nu}{dt}\frac{dx^\beta}{dt}\frac{dx^\alpha}{dt}$$

Finally, we can rewrite the equation as:

$$\frac{d^2 x^\beta}{dt^2} = -\Gamma^\beta_{\nu\alpha}\frac{dx^\nu}{dt}\frac{dx^\alpha}{dt} + \Gamma^0_{\nu\alpha}\frac{dx^\nu}{dt}\frac{dx^\beta}{dt}\frac{dx^\alpha}{dt}$$

1.2.2 Length from a metric tensor

Length of a curve γ in a manifold M where is defined a metric tensor g is given by

$$L(\gamma_1, \gamma_2) = \int_{\gamma_1}^{\gamma_2} \sqrt{|g(\vec{v}, \vec{v})|}d\lambda = \int_{\gamma_1}^{\gamma_2} \sqrt{|g_{ij}v^i v^j|}d\lambda$$

Using Euler-Lagrange equations, we can find a particle motion (in particular, we once again arrive to the geodesic equations)

$$\frac{\partial L(x,v)}{\partial x^k} - \frac{d}{ds}\left(\frac{\partial L(x,v)}{\partial v^k}\right) = 0 \quad with \quad L(x,v) = \sqrt{|g_{ij}(x)v^i v^j|}$$

1.3 Killing vector field

Definition 1.12. *Invariance*

Let T be an (m,n) tensor on a manifold M, V a vector field. V is an *invariance* of T if satisfies:

$$\mathcal{L}_V T = 0$$

where $\mathcal{L}$ is the Lie derivative.

Definition 1.13. *Killing vector field*
A Killing vector field is a vector field on a riemannian (or pseudo-riemannian) metric, that preserves the metric.
If we have a metric g on the manifold M, a vector field V is a Killing vector field if:

$$\mathcal{L}_V g = 0$$

This is called *Killing equation*

The Killing equation can be expressed showing the components:

$$(\mathcal{L}_V g)_{ij} = V_{i;j} + V_{j;i} = 0$$

Theorem 1.3.1. *In a n-dimensional manifold M, the number of existing Killing vector fields is* $\leqslant \frac{n(n+1)}{2}$

Chapter 2

Kerr Metric

2.1 Kerr metric properties

We first write the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G_n T_{\mu\nu} \tag{2.1}$$

where:
$R_{\mu\nu}$ is the Ricci tensor
R is the scalar curvature
$g_{\mu\nu}$ is the metric tensor
$T_{\mu\nu}$ is the stress-energy tensor
G_n is a constant that has been found as integration constant. If we use the dimensional analysis, its dimension must be $\frac{[L]}{[M]}$. In particular, to be able to recover the newtonian physics, this must be the gravitational constant.
There are four known solutions of the Einstein equations that describe regions called *black holes*, because the curvature is so high that even the electromagnetic waves are trapped within. The four solutions are:

- Schwarzschild (uncharged, non rotating)
- Reissner-Nosdtröm (charged, non rotating)
- Kerr (uncharged, rotating)

- Kerr-Newman (charged, rotating)

We will study now Kerr black holes.
A Kerr black hole has an axial symmetry, and it is described by two parameters, M and J. M is the total mass of the black hole, J its total angular momentum.
Due to J dimension M^2, it is common to define $a := \frac{J}{M}$. In this way the 2 new parameters M and a have the same dimension.
The parameter a will then be the angular momentum per unit mass.
We now write the Kerr metric in Boyer–Lindquist coordinates :

$$\boldsymbol{ds^2} = \frac{2Mr - \rho^2}{\rho^2}\boldsymbol{dt^2} \;-\; \frac{4aMr\sin^2\theta}{\rho^2}\boldsymbol{dtd\phi} \\ +\; \frac{(r^2+a^2)^2 \;-\; a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\boldsymbol{d\phi^2} \;+\; \frac{\rho^2}{\Delta}\boldsymbol{dr^2} \;+\; \rho^2\boldsymbol{d\theta^2} \tag{2.2}$$

where

$$\Delta := r^2 - 2Mr + a^2$$

$$\rho^2 := r^2 + a^2\cos^2\theta$$

We remind that r, θ, ϕ are the spherical coordinates.
Observation: we can write an orthonormal 1-form basis for this metric:

$$e^1 = \frac{\rho}{\sqrt{\Delta}}\boldsymbol{dr}$$
$$e^2 = \rho\boldsymbol{d\theta}$$
$$e^3 = \frac{\sin\theta}{\rho}[(r^2+a^2)\boldsymbol{d\phi} - a\boldsymbol{dt}]$$
$$e^4 = \frac{\sqrt{\Delta}}{\rho}(\boldsymbol{dt} - a\sin^2\theta\boldsymbol{d\phi})$$

demonstration is very easy (only calculations), but very long.

Theorem 2.1.1 (Papapetrou)**.** If there is a real non- singular axis of rotation then the coordinates can be chosen so that there is only one off-diagonal component of the metric. We call a metric like this a *quasi-diagonalizable* metric.

In particular, all cross terms between $\{dr, d\theta\}$ can be eliminated by the following transformations:

$$\begin{cases} dt' = dt + Adr + Bd\theta \\ d\phi' = d\phi + Cdr + Dd\theta \end{cases}$$

A similar thing will happen for crossed terms beween $\{dt, d\phi\}$:

$$\begin{cases} dr' = dr + Adt + Bd\phi \\ d\theta' = d\theta + Cdt + Dd\phi \end{cases}$$

what Papapetrou proved is that if the axis is regular, then dt' and $d\phi'$ (or dr' and $d\theta'$) are perfect differentials.

One can immediately note that:

- for $a \longrightarrow 0$ the metric equals the Schwarzschild's one:

$$\boldsymbol{ds^2} = -\left(1 - \frac{2M}{r}\right)\boldsymbol{dt^2} \ + \ \frac{1}{1-\frac{2M}{r}}\boldsymbol{dr^2} \ + \ r^2\sin^2\theta\boldsymbol{d\phi^2} \ + \ r^2\boldsymbol{d\theta^2}$$

- the metric is independent from t and ϕ, so there must exist two commuting Killing vectors, denoted by ∂_ϕ and ∂_t.
- it is axisymmetric: the metric does not depend on ϕ
- it is not static: it is not invariant under time-reversal transformation $t \rightarrow -t$
- It is invariant for simultaneous inversion of t and ϕ

$$\begin{cases} t \rightarrow -t \\ \phi \rightarrow -\phi \end{cases}$$

  This could be expected: the time reversal of a rotating object will make the object rotate in the opposite direction

- In the limit $r \to \infty$, the Kerr metric reduces to Minkowski metric. This means that the Kerr spacetime is asymptotically flat.

- For $M \to 0$ the metric is:

$$ds^2 = -dt^2 + \frac{r^2 + a^2 \cos^2\theta}{r^2 + a^2} dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2 + r^2 + a^2 \sin^2\theta d\phi^2$$

This metric is the metric of flat space, using spheroidal coordinates:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

where

$$\begin{aligned} x &= \sqrt{r^2 + a^2} \sin\theta\cos\phi \\ y &= \sqrt{r^2 + a^2} \sin\theta \sin\phi \\ z &= r\cos\theta \end{aligned}$$

Demonstration:

$$\begin{aligned} dx &= \frac{r}{\sqrt{r^2+a^2}} \sin\theta\cos\phi dr + \sqrt{r^2+a^2}\cos\theta\cos\phi d\theta - \sqrt{r^2+a^2}\sin\theta\sin\phi d\phi \\ dy &= \frac{r}{\sqrt{r^2+a^2}} \sin\theta\sin\phi dr + \sqrt{r^2+a^2}\cos\theta\sin\phi d\theta + \sqrt{r^2+a^2}\sin\theta\cos\phi d\phi \\ dz &= \cos\theta dr - r\sin\theta d\theta \end{aligned}$$

then, we simply put this in $dx^2 + dy^2 + dz^2$

Fixing time and radius, for $a \neq 0$, the metric is not the S^2 metric (differently from Schwarzschild's case)

$$\boldsymbol{ds^2} = \frac{(r^2+a^2)^2 \; - \; a^2\Delta\sin^2\theta}{\rho^2} \sin^2\theta \boldsymbol{d\phi^2} + \; \rho^2 \boldsymbol{d\theta^2}$$

This metric describes an ellipsoid.

2.1.1 No-hair theorem

The no-hair theorem postulates that the black holes solutions of the Einstein Field Equations (see Chapter 1) can be completely described using only 3 parameters. More, those parameters are observable by an external observer. Those parameters are:

- Mass
- Angular momentum
- Charge

The term "hair" stands for the matter forming the black hole and the matter falling into it. All the informations about the matter inside the event horizon become completely and permanently inaccessible. For example, all the informations about number and nature of particles that formed the star that collapsed. There is still no rigorous mathematical proof of this theorem. This is why it's often called *ho-hair conjecture*. Although the Schwarzschild metric this conjecture has been partially solved by Hawking, Carter and Robinson, the full proof of this theorem (even for the Schwarzschild case) is still far from complete. Regarding Kerr black holes, the theorem asserts that the Kerr metric is the only black hole solution of the Einstein Field Equations with the following characteristics:

- Rotating and uncharged
- The event horizon is regular
- Out from the event horizon, the space-time is stationary, has an axial symmetry and is asymptotically flat

2.2 Dragging of inertial frames

The most important characteristic of Kerr metric is the off-diagonal term

$$g_{t\phi} = g_{\phi t} = -a\frac{2Mr\sin^2\theta}{\rho^2}$$

that is $\frac{1}{2}$ the coefficient in $dtd\phi$ in the metric, because the element line is symmetric and contains

$$g_{t\phi}dtd\phi + g_{\phi t}d\phi dt = 2g_{t\phi}dtd\phi$$

In fact, we can diagonalise the metric for a chart, but we can't diagonalise it globally, for all of the manifold.
We'll see that this term leads to an intrinsic definition of "angular momentum".
We remind that Kerr metric is independent of ϕ and t. P_ϕ and P_t will then be conserved.
Rising indexes, we have:

- $P^\phi = g^{\phi\alpha}P_\alpha = g^{\phi\phi}P_\phi + g^{\phi t}P_t \qquad P^\phi = m\frac{d\phi}{d\tau}$
- $P^t = g^{t\alpha}P_\alpha = g^{tt}P_t + g^{t\phi}P_\phi \qquad P^t = m\frac{dt}{d\tau}$

Consider now a zero angular-momentum particle (i.e. $P_\phi = 0$).
We find a non null "angular velocity" [1] for the particle:

$$\omega(r,\theta) := \frac{d\phi}{dt} = \frac{P^\phi}{P^t} = \frac{g^{\phi t}}{g^{tt}} =^{2} \frac{2aMr\sin^2\theta}{(r^2+a^2)^2 \ - \ a^2(r^2-2Mr+a^2)\sin^2\theta}$$

2.2.1 Properties

The equation above tells us that $\omega \sim r^{-3}$, then the effect weakens with distance. Regarding θ, it's clear that is symmetric (as we expected, since the source is axially symmetric) and obviously independent from ϕ. This effect is called "dragging of inertial frames".
The *dragging of inertial frames* is a general result: each metric with $g_{\phi t} \neq 0$ will have $\omega(r,\theta) \neq 0$.

[1] The so called "angular velocity" of the particle is not properly an angular velocity in the present case. If we have an observer placed far from the center of coordinates who drops a particle, he will see the particle falling in straight line (see Chapter 3)

[2] result demonstrated in Chapter 3

- ω has the same sign of J.

$$\omega = \frac{2aMr\sin^2\theta}{(r^2+a^2)^2 \; - \; a^2(r^2-2Mr+a^2)\sin^2\theta}$$

$(r^2+a^2)^2 \; - \; a^2(r^2-2Mr+a^2)\sin^2\theta =$
$= r^4+2r^2a^2+a^4-a^2r^2\sin^2\theta+2a^2Mr\sin^2\theta-a^4\sin^2\theta =$
$= a^4\cos^2\theta+r^4+r^2a^2(2-\sin^2\theta)+2a^2Mr\sin^2\theta > 0$
(In this result we used $r > 0$)

We now rewrite the metric in a different form:

$$ds^2 = \rho^2(r,\theta)\left[\frac{d^2r}{\Delta(r)}+d^2\theta\right]+(r^2+a^2)\sin^2\theta\cdot d^2\phi-d^2t+\left[\frac{2mr}{\rho^2(r,\theta)}\right](dt-a\sin^2\theta\cdot d\phi)^2$$

Where $\rho^2(r,\theta) = r^2+a^2\cos^2\theta > 0$
Let us express the above metric in a different coordinate chart:

$$\begin{cases} y = \cos\theta \\ |y| < 1 \\ \rho^2(r,y) := \rho^2(r,\theta) \end{cases}$$

$$\boldsymbol{ds^2} = (r^2+a^2y^2)\left(\frac{\boldsymbol{d^2r}}{\Delta(r)}+\frac{\boldsymbol{d^2y}}{1-y^2}\right)+(r^2+a^2)(1-y^2)\boldsymbol{d\phi^2}-\boldsymbol{d^2t} + \frac{2mr}{r^2+a^2y^2}(\boldsymbol{dt}-a(1-y^2)\boldsymbol{d\phi})^2 \quad (2.3)$$

With some algebra (that we will avoid, since it is very long and not interesting) it can be found:

$$g := det[g_{ij}] = -\left(r^2+a^2y^2\right)^2(1-y^2)$$

2.2.2 Riemann Tensor

The nonzero components of the Riemann Curvature Tensor are:

$$R_{1414} = -2R_{2424} = -2R_{3434} = 2R_{1212} = 2R_{1313} = -R_{2323} = -\frac{2mr(r^2 - 3a^2y^2)}{(r^2 + a^2y^2)^3}$$

$$R_{2341} = R_{1342} = -R_{1243} = \frac{2may(3r^2 - 3a^2y^2)}{(r^2 + a^2y^2)^3}$$

Knowing the connection, the components above can be calculated from:

$$R^i_{jmn}(x) = \partial_m \Gamma^i_{nj} - \partial_n \Gamma^i_{mj} + \Gamma^i_{mh}\Gamma^h_{nj} + \Gamma^i_{nh}\Gamma^h_{mj}$$

The Kretschmann scalar for the metric is

$$K = R^{abcd}R_{abcd} = \frac{48m^2(r^2 - a^2y^2)\left[(r^2 - a^2y^2)^2 - 16r^2a^2y^2\right]}{(r^2 + a^2y^2)^6}$$

K in undefined for $r^2 + a^2y^2 = 0$

Solutions are

$$\begin{cases} r = 0 & y \neq 0 & a = 0 \\ r = 0 & y = 0 \ \ (\Rightarrow \theta = \frac{\pi}{2}) & a \neq 0 \end{cases}$$

These points must be singularities (see Chapter 1, Kretschmann scalar)

Chapter 3

Ergoshpere and Horizons

3.1 Ergosphere

We will now study the motion of photons emitted far from the singularity, in the equatorial plane $\theta = 0$ in two directions, $+\phi, -\phi$, both tangent to a circle of constant r. We know that photons in free-fall follow geodesics.
Because they're massless, we have $p_\mu p^\mu = 0$.
The nonzero components will be only dt and $d\phi$ and, since $ds^2 = 0$, we have:

$$0 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2$$

The solution to this equation is:

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}$$

If $g_{tt} = 0$, the solutions are

$$\frac{dt}{d\phi} = \begin{cases} 0 \\ -2\frac{g_{t\phi}}{g_{\phi\phi}} \end{cases}$$

The nonzero solution describes the angular velocity of the photons dropped in direction $+\phi$.
The null solution is very important. It describes the region where photons have null angular velocity, that happens only when they change direction.

After this region, the photon is forced to move in the same direction of the rotation of the black hole, because it has the same sign of the parameter a.
Since photons travel with speed c, every other particle (included massive ones, that cannot reach c) will be forced to behaviour like the photons.
This region is called the *static limit*
The surface $g_{tt} = 0$ lies outside the horizon (we'll see that after).

The equation

$$g_{tt} = 0$$

describes the so called *ergosurface*

Since $g_{tt} = -\frac{\Delta - a^2 \sin^2\theta}{\rho^2}$, the previous equation turns into:

$$\frac{\Delta - a^2 \sin^2\theta}{\rho^2} = 0$$

We now only have to solve this simple equation:
$r^2 - 2Mr + a^2 - a^2 \sin^2\theta$
The solution is:

$$r = \mathcal{R}_{\pm}(\theta) := m \pm \sqrt{m^2 - a^2 \cos^2\theta}$$

We can now rewrite the metric in function of those solutions:

$g_{11} = \frac{\rho^2}{(r-r_-)(r-r_+)}$
$g_{22} = \rho^2 > 0$
$g_{33} = \left[r^2 + a^2 + \frac{2ma^2 r \sin^2\theta}{\rho^2}\right] \sin^2\theta > 0$
$g_{34} = -\frac{2mar \sin^2\theta}{\rho}^2$
$g_{44} = -\frac{[r-\mathcal{R}_+(\theta)][r-\mathcal{R}_-(\theta)]}{\rho^2}$

where $r_{\pm} = m \pm \sqrt{m^2 - a^2}$ are the solutions of $\Delta(r, \theta) = 0$

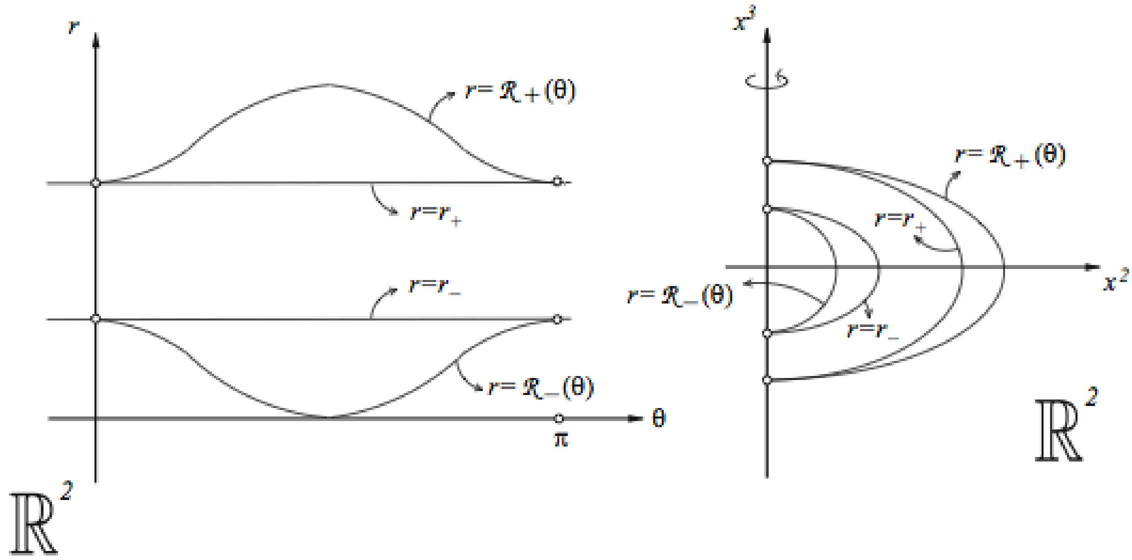


Figure 3.1: These graphics represent the behaviour of the horizons in the submanifold

$$\begin{cases} \phi = \frac{2}{\pi} \\ t = const. \end{cases}$$

Now we can find the eigenvalues of g_{ij}, through $det[g_{ij} - \lambda\delta_{ij}] = 0$. The eigenvalues will be then:

$$\lambda_1 = \frac{\rho^2}{(r - r_+)(r - r_-)}$$

$$\lambda_2 = \rho^2 > 0$$

$$\lambda_3 = \frac{1}{2}\left(g_{33} + g_{44} + \sqrt{(g_{33} - g_{44})^2 + 4(g_{34})^2}\right)$$

$$\lambda_4 = \frac{1}{2}\left(g_{33} + g_{44} - \sqrt{(g_{33} - g_{44})^2 + 4(g_{34})^2}\right)$$

We notice that the signature of the metric is +2. However, some of the metric tensor components and some of the eigenvalues change sign, but the signature always remains +2.

The hypersurface $r = \mathcal{R}_+(\theta)$ is called *stationary limit surface*, or the *outer ergosurface*. The domain $r_+ < r < \mathcal{R}_+(\theta)$ is called *ergosphere*.

Consider now the Killing vector field $\partial_t = \frac{\partial}{\partial x^4}$. We derive from the metric above that

$$\mathbf{g}\left[\frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^4}\right] = \mathbf{g}_{44} = -\frac{[r - \mathcal{R}_+(\theta)][r - \mathcal{R}_-(\theta)]}{\rho^2}$$

The Killing vector $\frac{\partial}{\partial x^4}$ becomes space-like for $\mathcal{R}_-(\theta) < r < \mathcal{R}_+(\theta)$.
The curvature singularity can be very easily calculated starting from the line element of the Kerr metric, and lies in

$$\begin{cases} r = 0_+ \\ \theta = \frac{\pi}{2} \\ t = x^4 = const. \end{cases}$$

Now, if we put this conditions in the metric, we find:

$$(dl)^2 = a^2(d\phi)^2$$

This metric is a ring of radius a, that lies in the equatorial plane of $\mathbb{R}^3$.
It is called *ring singularity*
This is, by construction, a similar thing of the singularity in Schwarzschild metric. Since a single point cannot have rotation or angular momentum, the result is a ring. We could expect that, since the ring is the most simple geometrical object that can support an angular momentum. This ring is zero in thickness, but has a radius of a. Note that the radius will be then proportional at the angular momentum J, another result that could be expected.
In Schwarzschild black holes, a geodesic entering in the event horizon can't avoid the singularity, which lies in the future world line of everything within the horizon.
In a Kerr black hole, thing are qualitatively different. We'll see that a particle falling into the outer event horizon won't be able to cross the inner event horizon.

3.2 Kerr Horizons

The horizon is a particular region of space-time. If we pick an observer placed at $r >> 0$, he will be able to communicate with all of the space-time, except for a region. The surface that delimits this region is called *event horizon*.
In particular, in this region the 4-acceleration must diverge, and this condition is the one we can start from to find the horizon.
Let us consider a free falling static observer. Due to the spin he must have, he will have:

$$u^\mu = (u^t, 0, 0, u^\phi)$$

$$\omega = \frac{d\phi}{dt} = \frac{p^\phi}{p^t}$$

This implies $p^\phi = \omega p^t \Rightarrow u^\phi = \omega u^t$.
The first equation becomes:

$$u^\mu = (u^t, 0, 0, \omega u^t)$$

Then, we apply the normalisation identity for the 4-velocity

$$u^\alpha u_\alpha = -1$$

finding

$$u_t u^t + \omega u_\phi u^t = -1$$

We now can rise indexes

$$g_{\mu t} u^\mu u^t + \omega g_{\mu\phi} u^\mu u^t = -1$$

Now, only t and ϕ will be non zero for the angular velocity, then we can rewrite it in a more explicit way:

$$g_{tt} u^t u^t + g_{t\phi} u^\phi u^t + \omega g_{t\phi} u^t u^t + \omega g_{\phi\phi} u^\phi u^t = -1$$

Using $u^\phi = \omega u^t$:

$$g_{tt} u^t u^t + g_{t\phi} \omega u^t u^t + \omega g_{t\phi} u^t u^t + \omega g_{\phi\phi} \omega u^t u^t = -1$$

$$(g_{tt} + 2\omega g_{t\phi} + \omega^2 g_{\phi\phi}) u^t u^t = -1$$

Finally,

$$u^t = \frac{1}{\sqrt{-(g_{tt} + 2\omega g_{t\phi} + \omega^2 g_{\phi\phi})}}$$

now, we need to calculate the 4-acceleration

$$\overline{\alpha} = \nabla_{\overline{u}}\overline{u}$$

where ∇ is the covariant derivative. More specifically:

$$\alpha^k \overline{e_k} = u^\nu (u^k_{,\nu} + \Gamma^k_{\mu\nu} u^\nu)\overline{e_k}$$

$$\alpha^k = u^t u^t (\Gamma^k_{tt} + 2\omega\Gamma^k_{t\phi} + \omega^2 \Gamma^k_{\phi\phi})$$

Now we can choose a coordinate frame of reference, we have:

$$\Gamma^l_{jk} := \frac{1}{2} g^{il}(g_{jl,k} + g_{lk,j} - g_{jk,l})$$

With some algebra and considering only the non-null components of the 4-velocity, we can use the equation above into the 4-acceleration, finding:

$$\alpha^k = \frac{u^t u^t}{2} g^{kl}(g_{tt} + 2\omega g_{t\phi} + \omega^2 g_{\phi\phi})_{,l}$$

now we can use $u^t = \frac{1}{\sqrt{-(g_{tt}+2\omega g_{t\phi}+\omega^2 g_{\phi\phi})}}$

$$\alpha^k = -\frac{g^{kl}u^t u^t \left(\frac{1}{u^t u^t}\right)_{,l}}{2}$$

one can find that

$$g^{kl}u^t u^t \left(\frac{1}{u^t u^t}\right)_{,l} = g^{rr}u^t u^t \left(\frac{1}{u^t u^t}\right)_{,r} + g^{\theta\theta}u^t u^t \left(\frac{1}{u^t u^t}\right)_{,\theta}$$

because the other components are null.

Finally, we have

$$\alpha = \sqrt{\alpha^k \alpha_k} = \frac{u^t u^t}{2}\sqrt{g^{rr}\left[\left(\frac{1}{u^t u^t}\right)_{,r}\right]^2 + g^{\theta\theta}\left[\left(\frac{1}{u^t u^t}\right)_{,\theta}\right]^2}$$

Now, we have to find where acceleration diverges.
To do so, we need to calculate the derivatives. For the sake of simplicity, we rewrite the acceleration as:

$$\alpha = \sqrt{\alpha^k \alpha_k} = \frac{u^t u^t}{2\rho} \sqrt{\Delta \left[\left(\frac{1}{u^t u^t} \right)_{,r} \right]^2 + \left[\left(\frac{1}{u^t u^t} \right)_{,\theta} \right]^2}$$

$$\left[\left(\frac{1}{u^t u^t} \right)_{,\theta} \right]^2 = \frac{4a^4 \sin^2 cos^2\theta[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta - \Delta\rho^2]^2}{\rho^4[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta]^2}$$

$$\left[\left(\frac{1}{u^t u^t} \right)_{,r} \right]^2 = \frac{4[\rho^2(r-M)(r^2+a^2)^2 - \Delta[r\Delta a^2 \sin^2\theta + 2r\rho^2(r^2+a^2-r(r^2+a^2)2)]]}{\Delta^2\rho^4[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta]^2}$$

The 4-acceleration diverges for

$$\Delta = 0$$

We now have the radius of the horizon.
Remember that $\Delta = r^2 + 2Mr + a^2$, the solution is

$$r_{horizon} = M \pm \sqrt{M^2 - a^2}$$

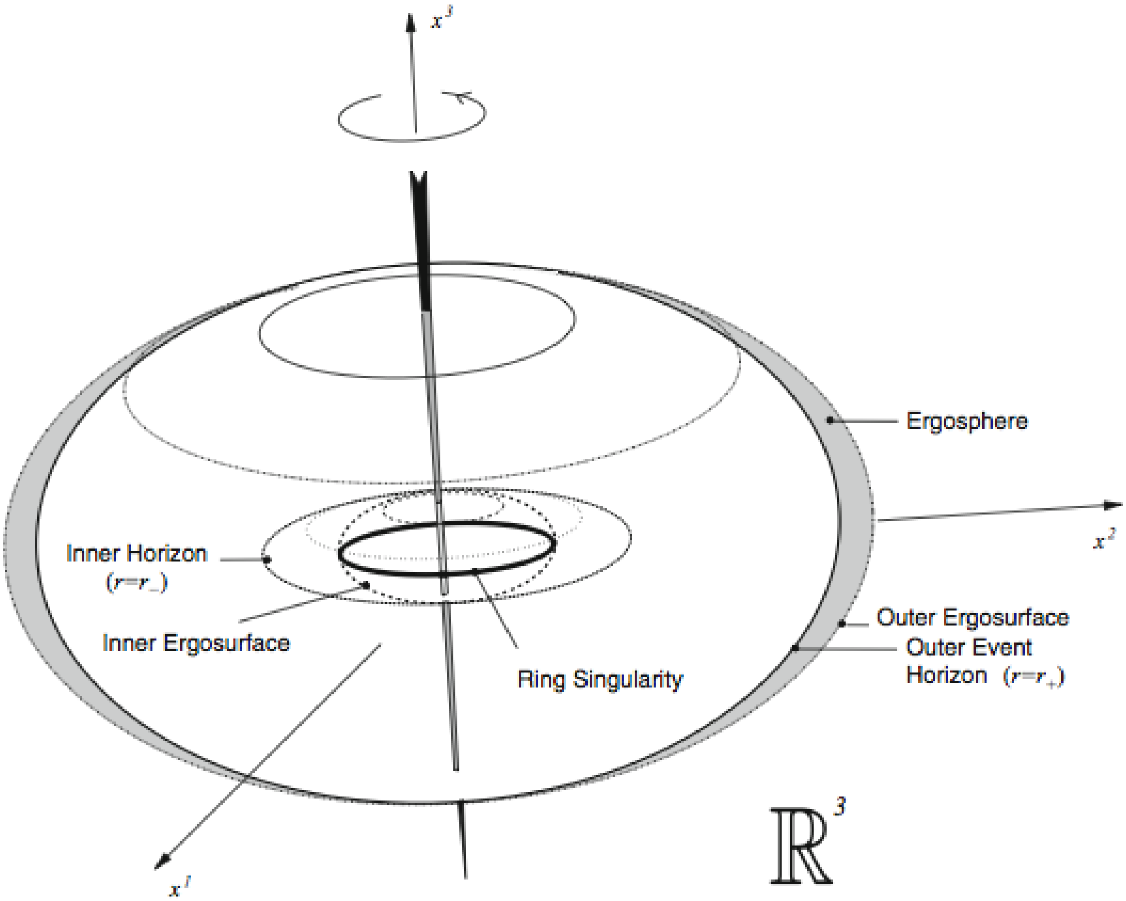
We can find that there are 2 regions where the 4-acceleration diverges, then there must exist two different horizons. The horizon

$$r_- = M - \sqrt{M^2 - a^2}$$

is the *internal horizon* (Cauchy horizon). The other:

$$r_+ = M + \sqrt{M^2 - a^2}$$

is called *external horizon.*
In the graphic below we can see the ergosurface, the ergosphere and the two horizons.



Of course, when $\theta = \pm\frac{\pi}{2}$ the ergosphere and the horizon coincide (it is the rotation axis).

Now, we can write the equations governing the outer event horizon

$$(x^1, x^2, x^3) = (r_+ \sin\theta\cos\phi, r_+ \sin\theta\sin\phi, r_+ \cos\theta)$$

the inner event horizon

$$(x^1, x^2, x^3) = (r_+ - \sin\theta\cos\phi, r_- \sin\theta\sin\phi, r_- \cos\theta)$$

the outer ergosurface

$$(x^1, x^2, x^3) = (\mathcal{R}_+ \sin\theta\cos\phi, \mathcal{R}_+ \sin\theta\sin\phi, \mathcal{R}_+ \cos\theta)$$

and the inner ergosurface

$$(x^1, x^2, x^3) = (\mathcal{R}_- \sin\theta\cos\phi, \mathcal{R}_- \sin\theta\sin\phi, \mathcal{R}_- \cos\theta)$$

The domain for those surfaces is:

$$D = \left\{(\theta, \phi) \in \mathbb{R}^2 : 0 < \theta < \pi, -\pi < \phi < \pi\right\}$$

It is clear from the graphic that we can take 5 charts to describe the Kerr manifold.

- The first is for $r > \mathcal{R}_+$, the space time here is asymptotically flat and has an axial symmetry. The signature of the metric is - + + +

- The second is the ergosphere: $r_+ < r < \mathcal{R}_+$ where all particles (massive and massless) are forced to spin in the same direction of the spinning of the black hole. Signature still is - + + +.

- The third, is $r_- < r < r_+$. Here the metric signature changes, becoming + - - -. In the horizon r_+, a particle can enter inside that region but it can't go out. A similar thing happens in r_-, that is a very important region. Here, a particle can't go inside, but can escape from inside this region. This has great consequences. Particles that went through r_+ now are trapped between the two horizons. Because of the gravity, particles will tend to accumulate at r slightly greater than r_-. This effect is known as *mass inflation*. In particular, if we consider an astronomical object that collapses, the matter will be distributed between two places: some matter will form the singularity (inside $\mathcal{R}_-$) and the rest will accumulate at r_-. This implies that we cannot consider the space as a vacuum space-time outside the singularity, and it extremely complicates the calculations (the Ricci tensor won't be 0, because there will be no vacuum solution). Actually, there is no solution that fully includes the mass inflation.

- The fourth is when $\mathcal{R}_- < r < r_-$. From here, metric changes again, returning - + + +. Again, particles are forced to go in the direction of the black hole as soon as they reach $\mathcal{R}_-$

- The fifth is when $r < \mathcal{R}_-$, where lies the singularity.

The full proof that r_+ is really an horizon consists in verifying that no geodesic can escape from inside r_+.

3.2.1 Area

The horizon is a surface of constant r and t. Then the horizon will have an intrinsic metric whose line element comes from the line element, with $dt = dr = 0$:

$$ds^2 = \frac{(r^2 + a^2)^2 - a^2\Delta}{\rho^2} \sin^2\theta d\phi^2 + \rho^2 d\theta^2$$

The proper area of this surface is given by:

$$A = \int \sqrt{det[g_{ij}]}$$

$$A(r) = \int_0^{2\pi} d\phi \int_0^{\pi} \sqrt{(r^2 + a^2)^2 - a^2\Delta} \sin\theta d\theta$$

Since the square root does not depend on ϕ or θ, and

$$A = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta = 4\pi$$

is the area of a unit two-sphere, we can immediately find:

$$Ar = 4\pi\sqrt{(r^2 + a^2)^2 - a^2\Delta}$$

Now, remember that the horizon is the region where $\Delta = 0$

$$A(r_+) = 4\pi(r_+^2 + a^2) = 8\pi(M^2 + \sqrt{M^4 - M^2a^2})$$

3.2.2 Apparent singularities

An apparent singularity is a singularity that only appears because of a reference issue. In that case, there may be an apparent divergence , removable by changing coordinates. To be sure our singularity is a real singularity we must find some invariant. One we could use is the Kretschmann scalar, defined as a total contraction of the Riemann tensor: $K = R^{abcd}R_{abcd}$. K is does not depend on the reference frame chosen. The curvature scalar is also another invariant, but the problem is that it may be null. If occurs so, it is useless for our purpose.
There are some coordinates that remove apparent singularities.
One of them is the Eddington Finkelstein coordinate frame (χ, r, ν, θ):

$$d\nu_{\pm} = dt \pm \frac{r^2 + a^2}{\Delta} dr$$

$$d\chi_{\pm} = d\phi \pm \frac{a}{\Delta} dr$$

The metric will be then

$$\boldsymbol{ds^2} = \rho^2 d\theta^2 \mp 2a \sin^2\theta \boldsymbol{dr d\chi_{\pm}} \pm 2\boldsymbol{dr d\nu_{\pm}} + \frac{((r^2 + a^2)^2 - \Delta a^2 \sin^2\theta)\sin^2\theta}{\rho^2} \boldsymbol{d\chi^2_{\pm}} - 4m \frac{mr \sin^2\theta \boldsymbol{d\chi^2_{\pm} d\nu_{\pm}}}{\rho^2} - \left(1 - \frac{2mr}{\rho^2}\right) \boldsymbol{d\nu^2_{\pm}} \quad (3.1)$$

In this reference frame, we find that near $r = r_+$ the metric is very smooth. The singularity must be an apparent singularity. We will see that photons can go through it, so it will be proved that $r = r_+$ is not a real singularity, but only a metric problem.

3.2.3 Killing horizon

A Killing horizon is a null hypersurface Σ, defined by the vanishing of the norm of a Killing vector field ξ.
In case of flat space time, Killing vectors are null. Therefore, the hypersurface defined from them is null too (i.e. we have no killing horizons).

$$\xi_{\pm} = k + \left(\frac{a}{r_{\pm}^2 + a^2}\right) m$$

is the vector field, derived from the Eddington Finkelstein coordinate frame (is a linear combination of k and m, that are both Killing vector fields).
We define

$$k = \sqrt{g_{\nu\nu}} \partial_{\nu}$$
$$m = \sqrt{g_{\xi\xi} \partial_{\xi}}$$

Our surface Σ_+ will be the surface $r = r_+$, and Σ_- will be the surface $r = r_-$.
For all $f \neq 0$, we have

$$l_{\pm} = f g^{\nu\mu} \partial_{\mu} = -\frac{r_{\pm}^2 + a^2}{r \pm a^2 \cos^2\theta} f \left(\partial_{\nu} + \frac{a}{r_{\pm}^2 + a^2} \partial_{\chi}\right)$$

It is a vector field normal to $\Sigma_{\pm}$ because

$$\left(\partial_{\nu} + \frac{a}{r_{\pm}^2 + a^2} \partial_{\chi}\right) = \xi$$

Then, $\Sigma_{\pm}$ is the Killing horizon of the Killing vector field ξ.
Now, we can calculate the angular velocity

$$\omega(r) = \frac{2aMr}{(r^2 + a^2)^2 - a^2 \Delta \sin^2\theta}$$

$$\omega(r_{\pm}) = \frac{2aMr_{\pm}}{(r_{\pm}^2 + a^2)^2} = \frac{2aM(M \pm \sqrt{M^2 - a^2})}{(2M^2 \pm 2M\sqrt{M^2 - a^2})^2} =$$

$$= \frac{a}{2M(M \pm \sqrt{M^2 - a^2}} = \frac{a}{r_{\pm}^2 + a^2}$$

Finally, we know the angular velocity of the particle

$$\omega(r_{\pm}) = \frac{a}{r_{\pm}^2 + a^2}$$

3.3 Equatorial photon motion in the Kerr metric

We now study the motion of a photon in the equatorial plane. We shall see that differs from the Schwarzschild's case. To do so, we need to find the inverse of the metric:

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2$$

To make calculations easier, we write here the Kerr metric using a matrix (thing that we can do, see Chapter 1, metric tensor definition):

$$g_{\mu\nu} = \begin{pmatrix} -\frac{\Delta - a^2\sin^2\theta}{\rho^2} & 0 & 0 & -\frac{2aMr\sin^2\theta}{\rho^2} \\ 0 & -\frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ -\frac{2aMr\sin^2\theta}{\rho^2} & 0 & 0 & \sin^2\theta\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2} \end{pmatrix}$$

Since the only off-diagonal term is the $dtd\phi$ one, we can immediately calculate:

$$g^{rr} = \frac{1}{g_{rr}} = \frac{\Delta}{\rho^2}$$

$$g^{\theta\theta} = \frac{1}{g_{\theta\theta}} = \frac{1}{\rho^2}$$

Now, it only remains the matrix

$$\begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{t\phi} & g_{\phi\phi} \end{pmatrix}$$

The inverse is simply:

$$\frac{1}{D}\begin{pmatrix} g_{\phi\phi} & -g_{t\phi} \\ -g_{t\phi} & g_{tt} \end{pmatrix}$$

where

$$D = g_{tt}g_{\phi\phi} - g_{t\phi}^2$$

is the determinant.
Then, we have an angular velocity

$$\omega = \frac{g^{\phi t}}{g^{tt}} = \frac{\frac{-g_{t\phi}}{D}}{\frac{g_{\phi\phi}}{D}} = -\frac{g_{t\phi}}{g_{\phi\phi}}$$

Using the Kerr metric, we find:

$$D = -\Delta \sin^2\theta$$

$$g^{tt} = -\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2\Delta}$$
$$g^{t\phi} = -a\frac{2Mr}{\rho^2\Delta}$$
$$g^{\phi\phi} = \frac{\Delta - a^2\sin^2\theta}{\rho^2\Delta\sin^2\theta}$$

Then, the frame dragging will be

$$\omega = \frac{2Mra}{(r^2+a^2)^2 - a^2\sin^2\theta}$$

Once again, the denominator is always positive, then ω has the same sign of a (and the same sign of J) and it falls off to the infinity as r^{-3}.

A photon with motion in the equatorial plane has $d\theta = 0$, but photons with trajectories not included in the equatorial plane could have different orbits (because of the axially but not spherically simmetry, unlike the Schwarzschild case) A photon with $p^\theta = 0$ with trajectory in the equatorial plane at time=0 will have always $p^\theta = 0$,because the metric is reflection symmetry respect to $\theta = \frac{\pi}{2}$.

$$E = -p_t$$

$$L = p_\phi$$

are constants for the motion.
Then, the motion will be determined by the equation $p \cdot p = 0$

$$\left(\frac{dr}{d\lambda}\right)^2 = g^{rr}(-g^{tt})\left[E^2 - 2\omega EL + \frac{g^{\phi\phi}}{g^{tt}}L^2\right]$$

For $\theta = \frac{\pi}{2}$ we have:

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{(r^2+a^2)^2 - a^2\Delta}{r^4}\left[E^2 - \frac{4MRa}{(r^2+a^2)^2 - a^2\Delta}EL - \frac{r^2-2Mr}{(r^2+a^2)^2 - a^2\Delta}L^2\right]$$

In Schwarzschild we established a potential V^2 and we found the solution as $(\frac{dr}{d\lambda})^2 = E^2 - V^2$. This time, we cannot do so, because of the present of the EL term off the diagonal.
We can instead search for V_+ *and* V_-,

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{(r^2+a^2)^2 - a^2\Delta}{r^4}(E - V_+)(E - V_-)$$

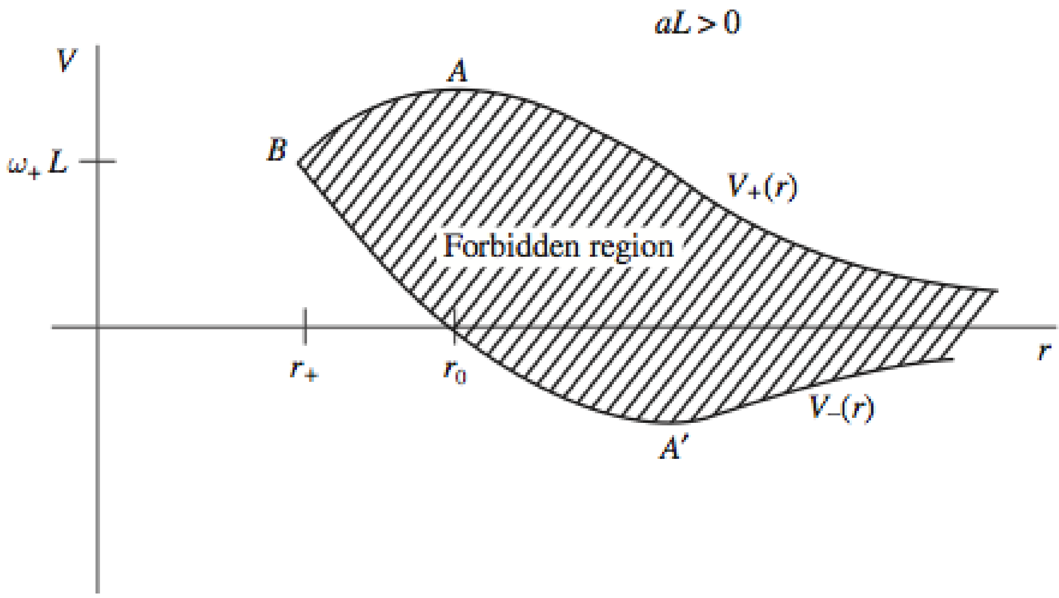
Using equations above, we find that

$$V_\pm(r) = \left[\omega \pm \sqrt{\omega^2 - \frac{g^{\phi\phi}}{g^{tt}}}\right] L$$

This equation has complex solutions when $\Delta < 0$. In that region does not exist solutions $\frac{dr}{d\lambda} = 0$, so a photon can't turn back, whatever energy he got. Then, $\Delta = 0$ marks the horizon in the equatorial plane. More generally, the equation

$$\Delta = 0$$

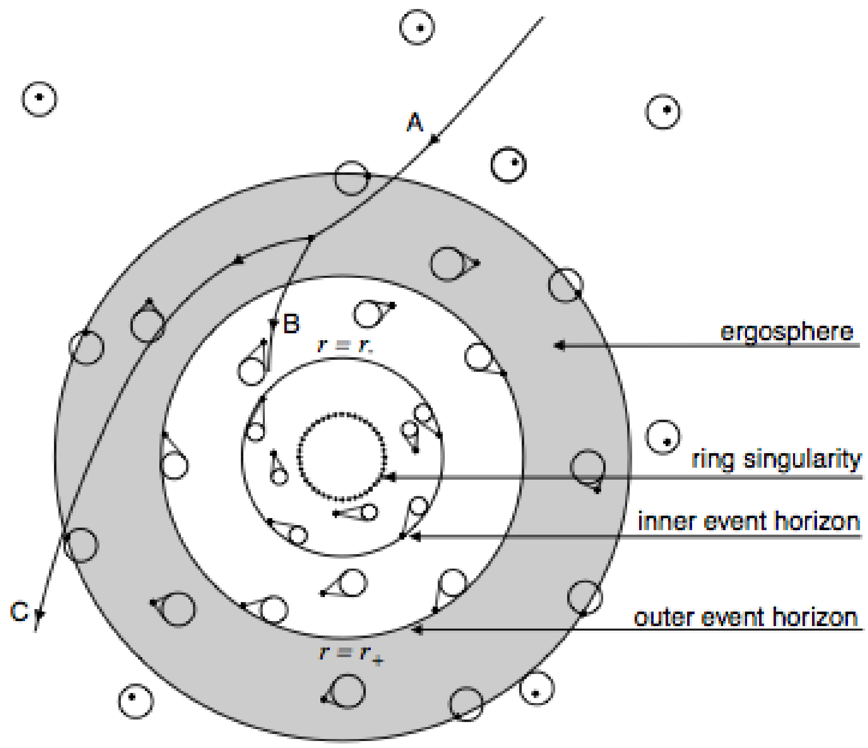
is the general equation that describes the horizon for every trajectory.
In this graphic is plotted $V \pm (r)$, for $aL > 0$, that means the photon has the angular



momentum in the same direction of the black hole. In this image we set $r > r_+$, to focus what happens outside the horizon. For $r >> r_+$, the curve goes asymptotically to zero with a force of $\frac{1}{r}$. Here, the metric is almost the Schwarzschild metric, because the rotation effect weakens with distance. For r near r_+, the rotation is strong, and introduces new effects. V_- through the ergosphere (here named r_0) changes sign, and exactly in the ergosphere is zero. In the horizon, $V_- = V_+$. It is clear that a photon cannot go in the forbidden region $V_- < E < V_+$ because of the equation:

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{(r^2+a^2)^2 - a^2\Delta}{r^4}(E - V_+)(E - V_-)$$

There is a curious fact we will not demonstrate. A particle A entering in the ergosphere starting from infinity, can split itself into 2 particles B and C such that C arrives at the infinity with more energy than A had at the beginning. The following picture shows also local light wavefronts.



Now, the angular velocity of a particle gains a strange meaning. If we place an observer far from the singularity and we let a particle fall from the observer, the observer will see the particle falling in a straight line, because both massive and massless particles follows geodesics, that are curved. There is no chance that the observer could see such angular velocity. The angular velocity and the angular momentum in the Kerr manifold are something intrinsic, they cannot be studied by direct observations. Another interesting thing, is that the event horizon does not "feel" the angular momentum as one can imagine. It does depend on the parameter a (that is $a = \frac{J}{M}$), but is a sphere. The axial symmetry that featured Kerr black holes falls, regarding the event horizon. This effect is not present in the ergosurface, which has an ellipsoidal shape, as might be expected.

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