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Forward Implied Volatility Expansions in LSV models

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Ai miei genitori
Alla mia pulce Fabiana

Table of Contents

CHAPTER 3

Introduction

In this work we address the problem of finding formulas for efficient and reliable analytical approximation for the calculation of forward implied volatility in LSV models, a problem which is reduced to the calculation of option prices as an expansion of the price of the same financial asset in a Black-Scholes dynamic.

The motivation for this work comes from financial mathematics, where the prices of financial derivatives is reduced to the calculation of these expectations. The speed of calculation of rates and calibration procedures is a very strong operational constraint and we provide real-time tools (or at least more competitive than Monte Carlo simulations, in the case of multi-dimensional diffusion) to meet these needs.

Our approach involves an expansion of the differential operator, whose solution represents the price in local stochastic volatility dynamics. Further calculations then allow to obtain an expansion of the implied volatility without the aid of any special function or expensive from the computational point of view, in order to obtain explicit formulas fast to calculate but also as accurate as possible.

In the first chapter we introduce briefly the dynamics of black-schole and the problem of implied volatility, explaining the importance of this value, but also the difficulty of building models that are able to effectively simulate the behavior of financial securities, and that provide implied volatility surfaces that are close to the empirical data.

The second chapter of the thesis focuses on the work of Lorig, Pagliarini and Pascucci, while in the third chapter we try to implement codes that allow to verify the consistency of this method, but also to numerically analyze what are the most convenient choices for a practical use of these formulas.

Finally, in the fourth chapter we attempt to extend this approach to the case of forward implied volatility, performing some numerical tests on the Heston model.

CHAPTER 1

B&S Model and IV problem

1 The Black&Scholes Model

In the early 1970s, Fischer Black, Myron Scholes, and Robert Merton achieved a major breakthrough in the pricing of stock options ([BS73] and [Mer73]). This involved the development of what has become known as the Black&Scholes (or Black-Scholes-Merton) model. This model has had a huge influence on the way that traders price and hedge options. It has also been pivotal to the growth and success of financial engineering in the last 30 years.

1.1 Lognormal Property of Stock Prices

The model of stock price behaviour used by Black, Scholes and Merton is a model which assumes that percentage changes in the stock price S in a short period of time are normally distributed; given:

μ: Expected return on stock per year

σ: Volatility of the stock price per year

The mean of the return in a period of time Δt is $\mu\,\Delta t$, and the standard deviation of the return is $\sigma\sqrt{\Delta t}$, so that

$$\frac{\Delta S}{S} \sim \mathcal{N}_{\mu\,\Delta t,\,\sigma^2\,\Delta t} \tag{1.1}$$

where ΔS is the change in the stock price S in time Δt, and $\mathcal{N}_{m,v}$ denotes a normal distribution with mean m and variance v. Formula (1.1) implies that:

$$\ln(S_T) - \ln(S_{t_0}) \sim \mathcal{N}_{\left(\mu+\frac{\sigma^2}{2}\right)(T-t_0),\,\sigma^2(T-t_0)} \tag{1.2}$$

and

$$\ln(S_T) \sim \mathcal{N}_{\ln(S_{t_0})+\left(\mu+\frac{\sigma^2}{2}\right)(T-t_0),\,\sigma^2(T-t_0)} \tag{1.3}$$

where S_T is the stock price at the future time T, while S_{t_0} is the present stock price (at time t_0). Formula (1.3) shows tha $\ln(S_T)$ is normally distribuited, so that S_T has a lognormal distribution. The mean of $\ln(S_T)$ is $\ln(S_{t_0}) + \left(\mu - \sigma^2 \big/ 2\right)(T - t_0)$ and the standard deviation is $\sigma^2(T - t_0)$; thus from (1.3) and the properties of the lognormal distribution we have this equation for the expected value of S_T:

$$\mathbb{E}[\![S_T]\!] = S_{t_0}\, e^{\mu\,(T-t_0)} \tag{1.4}$$

1.2 Black-Scholes-Merton differential equations

The Black-Scholes-Merton differential equation is an equation that must be satisfied by the price of any derivative dependent on a non-dividend-paying stock. They involve setting up a riskless portfolio consisting of a position in the derivative and a position in the stock. In absence of arbitrage opportunities, the return from the portfolio must be the risk-free interest rate, r. This leads to the Black-Scholes-Merton differential equation.

In the Black&Scholes (BS) model, the market consists of a locally non-risky asset, the bond B, and a risky asset, the stock S. The bond price verifies the equation

$$dB_t = r\,B_t\,dt \tag{1.5}$$

where r is the short-term (or locally risk-free) interest rate, assumed to be a constant. Thus the bond B has a deterministic behaviour; from now on let's consider $B_{t_0} = 1$, then

$$B_t = e^{r\,(t-t_0)} \tag{1.6}$$

The price of the stock S is a geometric Brownian process, verifying the equations

$$dS_t = \mu\,S_t\,dt + \sigma\,S_t\,dW_t \tag{1.7}$$

where the drift $\mu \in \mathbb{R}$ is the average rate of return, while $\sigma \in \mathbb{R}_{>0}$ is the volatility. $(W_t)_{t\in[t_0,T]}$ is a real Brownian motion on the probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$. Finally we recall that the solution of the previous SDE has an explicit expression:

$$S_t = S_{t_0}\, e^{\sigma\, W_{(t-t_0)} + \left(\mu - \frac{\sigma^2}{2}\right)(t-t_0)} \tag{1.8}$$

Suppose now that f is the price of a call option or other derivative contingent on S. The value of f must be some function of S and t. Hence, form Ito's formula:

$$df = \left(\frac{\partial f}{\partial S_t}\,\mu\,S_t + \frac{\partial f}{\partial t} + \frac{1}{2}\,\frac{\partial^2 f}{\partial S_t^2}\,\sigma^2\,S_t^2\right)dt + \frac{\partial f}{\partial S_t}\,\sigma\,S_t\,dW_t \tag{1.9}$$

It follows that a portfolio of the stock and the derivative can be constructed so that Wiener process is eliminated; the holder of this portfolio is short one derivative and long an amount $\frac{\partial f}{\partial S}$ of shares. Define Π as the value of the portfolio. By definition:

$$\Pi = -f + \frac{\partial f}{\partial S} S \tag{1.10}$$

and

$$d\Pi = \left(-\frac{\partial f}{\partial S_t} + \frac{1}{2} \frac{\partial^2 f}{\partial S_t^2} \sigma^2 S_t^2 \right) dt \tag{1.11}$$

Because this equation does not involve the Wiener process, the portfolio must be riskless, so it must instantaneously earn the same rate of return as other short-term risk-free securities. If it earned more than this return, arbitrageurs could make a riskless profit by borrowing money to buy the portofolio; if it earned less, they could make a riskless profit by shorting the portfolio and buying risk-free bonds. It follows that:

$$d\Pi = r\,\Pi\, dt \tag{1.12}$$

Substituting (1.10) and (1.11) in (1.12) we obtain the Black-Scholes-Merton differential equation:

$$\frac{\partial f}{\partial t} + r S_t \frac{\partial f}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2} = r f \tag{1.13}$$

This equation has many solutions, corresponding to all the different derivatives that can be defined with S as the underlying variable. The particular derivative that is obtained when the equation is solved depends on the boundary conditions used. These specify the values of the derivative at the boundaries of possible values of S and t. In the case of European call and put options, these boundary conditions are:

$$\text{(call)} \quad f(S_T) = (S_T - K)^+ \tag{1.14}$$

$$\text{(put)} \quad f(S_T) = (K - S_T)^+ \tag{1.15}$$

where K is the strike price, and T is the maturity.

One point that should be emphasized about the port folio used in the derivation of (1.13) is that it is not permanently riskless: it's riskless only for an infinitesimally short period of time. To keep the portfolio riskless, it is therefore necessary fo frequently change the relative proportions of the derivative and the stock in the portfolio composition.

In the last part of this work we consider forward start European option, which basically can be considered as forwards on normal plain vanilla options. More precisely, a Call option of this kind is an option which begins at some specified future date $t_i > t_0$, the forward date, and with an expiration further in the future $t_i + T$ with $T > 0$, the premium being paid in advance at the initial date t_0. We can distinguish two types of payoffs:

$$f(S_{t_i+T}, S_{t_i}) = \left(\frac{S_{t_i+T}}{S_{t_i}} - K\right)^+ \tag{1.16}$$

(type A) for a given strike $K > 0$; it's essentially an option on the return of the asset in the time interval $[t_i, t_i + T]$.

$$f(S_{t_i+T}, S_{t_i}) = (S_{t_i+T} - K\, S_{t_i})^+ \tag{1.17}$$

(type B) with $K > 0$, which can be view as an option with a stochastic strike determined only at the forward date t_i; this looks like a spread option with the same underlying but considered at different dates.

1.3 Risk-Neutral Valuation

The risk-neutral valuation is with no doubt the most important tool for the analysis of derivatives. The key point is that the Black-Scholes-Merton differential equations doesn't involve any variables that are affected by risk; the variables that do appear are the current stock price, time, stock price volatility and the risk-free rate of interest.

Because the Black-Schole-Merton differential equation is independent of risk preferences, they cannot affect its solution, so the very simple assumption that all investors are risk neutral can be made. In a world where investors are risk neutral, the expected return on all investment assets is the risk-free rate of interest, r. The reason is that risk-neutral investors do not require a premium to induce them to take risks. Thus, it is also true that the present value of any cash flow in a risk-neutral world can be obtained by discounting its expected value at the risk-free rate.

It is important to appreciate that risk-neutral valuation is merely an artificial device for obtaining solutions to the Black-Scholes differential equation. The obtained solutions are valid in all worlds, not just those where investors are risk neutral, because changes in the expected growth rate in the stock price, and in the discount rate off set each other exactly.

Assuming that interest rates are constant and equal to r, and considering a European call option with strike K that matures at time T, we know that the value of the contract at maturity is

$$(S_T - K)^+ \tag{1.18}$$

From the risk-neutral valuation argument, the value of the forward contract at time t_0 is its expected value at time T in a risk-neutral world discounted at the risk-free rate of interest. Denoting the value of the forward contract with the function F, this means that:

$$F(S_T) = \mathbb{E}[\![(S_T - K)^+]\!]$$

and

$$F(S_{t_0}) = e^{-r(T-t_0)}\,\mathbb{E}[\![(S_T - K)^+]\!] \tag{1.19}$$

where $\mathbb{E}$ denotes as usual the expected value in a risk-neutral world.

1.4 Black&Scholes Pricing Formulas

The Black-Scholes formulas for the prices at time t of an European call option on a non-dividend-paying stock and of the relative put option are:

$$c_t = S_t\,\Phi(d_1) - K\,e^{-r(T-t)}\,\Phi(d_2) \tag{1.20}$$

$$p_t = K\,e^{-r(T-t)}\,\Phi(-d_2) - S_t\,\Phi(-d_1) \tag{1.21}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy \tag{1.22}$$

is the cumulative probability distribution function for a stardardized normal distribution, and

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{1.23}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{1.24}$$

As you will see, it's important to explain that we usually work only on call options, because we can easily obtain the price of the relative put option, thanks to the put-call parity formula:

$$c_t = p_t + S_t - \mathrm{K}e^{-r(T-t)} \tag{1.25}$$

which can be derived from (1.20) and (1.21), or making a simple reasoning on a market free from arbitrage opportunities that consists of a bond B and a stock S that is the underlying asset of a call option c and of a put option p, both of European type with maturity T and strike K. If we now consider two portfolios of this kind:

$$X_t = c_t + \frac{K}{B_T}B_t \quad \text{and} \quad Y_t = p_t + S_t$$

we can notice that they both have the same final value, so from the no-arbitrage principle, they must have the same value at every time t; from here follows the put-call parity formula.

One way of deriving the Black&Scholes formulas is by solving the differential equation (1.13) subject to a boundary condition, for example (1.14); another approach is to use risk-neutral valuation. Given an European call option, as we have seen before, the expected value of the option at time t in a risk-neutral world is

$$c_t = F(S_t) = e^{-r(T-t)}\, \mathbb{E}[\![(S_T - K)^+]\!] \tag{1.26}$$

which can computed analitically even in a computer algebra program such as Wolfram's *Mathematica*:

- first of all let's consider the underlying asset dynamic $S_t = S_{t_0}\, e^{X_t}$ where

$$dX_t = \mu\, dt + \sigma\, dW_t \quad \text{and} \quad X_{t_0} = 0 \tag{1.27}$$

so the density function of the process X_t is the normal distribution $\mathcal{N}_{\mu t, \sigma^2 t}$;

- secondly, in order to guarantee the absence of arbitrage opportunities, we must assure the martingale condition on the discounted process $\tilde{X}_t = -r\,t + X_t$, which means that the expected value of e^{X_t} is equal to the value of the risk-free bond with the same initial value, i.e. $B_t = S_{t_0}\, e^{r\,t}$: so we must put $\mu = r - \frac{\sigma^2}{2}$; thus $X_t \sim \mathcal{N}_{\left(r-\frac{\sigma^2}{2}\right)t, \sigma^2 t}$;

- finally we can compute the call price with this code in which we limit the interval of integration in order to have $(S_{t_0}\, e^{x_t} - K)^+ = (S_{t_0}\, e^{x_t} - K)$, and where compare the function $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt$:

Input Code:

$$e^{-r\,(T-t_0)} \int_{\mathrm{Log}\left[\frac{S_{t_0}}{K}\right]}^{\infty} (e^x\, S_{t_0} - K)\; \mathrm{PDF}\left[\mathrm{NormalDistribution}\left[\left(r - \frac{\sigma^2}{2}\right)(T - t_0),\, \sigma\sqrt{T - t_0}\right],\, x\right] dx$$

Output:

$$\frac{1}{2}\left(S_{t_0}\left(\mathrm{Erf}\left[\frac{(2\,r+\sigma^2)\,(T-t_0)-2\,\mathrm{Log}\left[\frac{K}{S_{t_0}}\right]}{2\sqrt{2}\,\sigma\sqrt{T-t_0}}\right]+1\right) - K\,e^{-r\,(T-t_0)}\left(1-\mathrm{Erf}\left[\frac{2\,\mathrm{Log}\left[\frac{K}{S_{t_0}}\right]+(\sigma^2-2\,r)\,(T-t_0)}{2\sqrt{2}\,\sigma\sqrt{T-t_0}}\right]\right)\right)$$

and it can be easily proved that this formula is equivalent to (1.20).

1.5 Consistence of Black&Scholes Formulas

When the initial stock price S_{t_0} becomes very large, a call option is almost certain to be exercised. It then becomes very similar to a forward contract with delivery price K, so we expect the call price to be

$$S_{t_0} - K\, e^{-r(T-t_0)} \tag{1.28}$$

This is, in fact, the call price given by equation (1.20) because, when S_{t_0} becomes very large, both d_1 and d_2 become very large, and $\Phi(d_1)$ and $\Phi(d_2)$ go close to 1.

When the stock price becomes very large, the price of an European put option p approaches zero. This is consistent with equation (1.21), because $\Phi(-d_1)$ and $\Phi(-d_2)$ are both close to zero in this case.

Consider next what happens when the volatility σ approaches zero. Because the stock is virtually riskless, its price will grow at rate r to $S_{t_0}\, e^{r(T-t_0)}$ at time T, so discounting the payoff from a call option

$$e^{-r(T-t_0)}\left(S_{t_0}\, e^{r(T-t_0)} - K\right)^+ = \left(S_{t_0} - K\, e^{-r(T-t_0)}\right)^+ \tag{1.29}$$

and again this is consistent with equation (1.20).

2 Implied Volatility

At a first glance, the works of Black and Scholes ([BS73]) and Merton ([Mer73]) on how to price an option seem now irrelevant: the market in doing it for us! Such is however not the case. First, there exist more complex options, often called exotic options, a simple example that we shall consider later is a barrier option. These are not actively traded and therefore need to be priced. More importantly the work of Black and Scholes and Merton is still of utmost importance because of the paradigm they proposed to price options.

Their fundamental contribution was to see that the risk contained in an option (the uncertainty about stock prices in the future) could be exactly synthesized in a self financing portfolio. In other words, they provide a 'trick' to change risks in the future into portfolio strategies of stocks and bonds. These portfolios have simply to be rebalanced continuously and in a precise way but without injecting cash. This means that the seller of an option will exactly meet his or her obligations at maturity by simply holding at each time a certain quantities of stocks and bonds. This is true in every state of the world, whatever happens to the stock! The fair price of the option ought then to be the initial cost of such a replicating strategy.

As we shall see in the next paragraphs, the hypotheses under which the Black-&Scholes formula was established are wrong. However, the idea of dynamically hedging the risk is still the main methodology to price options.

The first problem when we price an option in the Black&Scholes model is the choice of the parameter σ that is not directly observable. The first idea could be to use a value of σ obtained from an estimate on the historical data on the underlying asset, i.e. the so-called historical volatility. Actually, the most widespread and simple approach is that of using directly, where it is available, the implied volatility of the market: we see, however, that this approach is not free from problems.

The Black&Scholes formula in [BS73] and [Mer73] is obtained under the assumption that the stock can be modeled as a geometric Brownian motion with constant volatility. Since option quote prices are available and the only unknown parameter is the volatility, if the Black&Scholes assumption were true, we could find the volatility of the stock by simply inverting Black&Scholes formula.

In the Black&Scholes model the price of a European Call option is a function of the form

$$C_{\mathrm{BS}} = C_{\mathrm{BS}}(\sigma, S_{t_0}, K, T, r)$$

where σ is the volatility, S_{t_0} is the current price of the underlying asset, K is the strike, T is the maturity and r is the short-term rate. Actually the price can also be expressed in the form

$$C_{\mathrm{BS}} = S_{\varphi}\left(\sigma, \frac{S_{t_0}}{K}, T, r\right)$$

where ϕ is a function whose expression can be easily deduced from the Black&Scholes formula. The number $m = \frac{S_{t_0}}{K}$ is usually called "moneyness" of the option: if $S > K$, we say that that the Call option is "in the money", since we are in a situation of potential profit; if
$S < K$, the Call option is "out of the money" and has null intrinsic value; finally, if $S = K$, we say that the option is "at the money".

Of all the parameters determining the Black-Scholes price, the volatility σ, as we have already said, is the only one that is not directly observable. We recall that

$$\sigma \mapsto C_{\mathrm{BS}}(\sigma, S_{t_0}, K, T, r)$$

is a strictly increasing function and therefore invertible: having fixed all the other parameters, a Black&Scholes price of the option corresponds to every value of σ; conversely, a unique value of the volatility σ^* is associated to every value C^* on the interval $]\,0, S[$ (the interval to which the price must belong by arbitrage arguments). We set

$$\sigma^* = \mathrm{IV}(C^*, S_{t_0}, K, T, r)$$

where σ^* is the unique value of the volatility parameter such that

$$C^* = C_{\mathrm{BS}}(\sigma^*, S_{t_0}, K, T, r)$$

The function

$$C^* \mapsto \mathrm{IV}(C^*, S_{t_0}, K, T, r)$$

is called implied volatility function.

2.1 The Implied Volatility Problem

The concept of implied volatility is so important and widespread that, in financial markets, the plain vanilla options are commonly quoted in terms of implied volatility, rather than explicitly by giving their price. As a matter of fact, using the implied volatility is convenient for various reasons. First of all, since the put and call prices are increasing functions of the volatility, the quotation in terms of the implied volatility immediately gives the idea of the “cost” of the option. Analogously, using the implied volatility makes it easy to compare the prices of options on the same asset, but with different strikes and maturities.

The main problem with the implied volatility is that, for options with different maturities and different strikes but written on the same stock, one should find the same implied volatility, i.e., the volatility of the stock which is unique. Such is not the case. At a given maturity, options with different strikes trade at different implied volatilities. When plotted against strikes, implied volatilities exhibit a smile or a skew effect. So, the implied volatility surface relative to the prices obtained by the Black&Scholes model considering different strikes and maturities should be flat and coincide with the graph of the function that is constant and equal to σ; on the contrary, for an empirical implied volatility surface, inferred from quoted prices in real markets, the result is generally quite different: it is well known that the market prices of European options on the same underlying asset have implied volatilities that vary with strike and maturity.

Typically every section, with T fixed, of the implied volatility surface takes a particular form that is usually called "smile" or "skew". Generally we can say that market quotation tends to give more value (greater implied volatility) to the extreme cases "in" or "out of the money". This reflects that some situations in the market are perceived as more risky, in particular the case of extreme falls or rises of the quotations of the underlying asset.

Also the dependence on T, the time to maturity, is significant in the analysis of the implied volatility: this is called the term-structure of the implied volatility. Typically when we get close to maturity ($T \to 0^+$), we see that the smile or the skew become more marked.

Other characteristic features make definitely different the implied volatility surface of the market from the constant Black&Scholes volatility: for example, in Figure 1.1 we show the dependence of the implied volatility of options on the S&P500 index, with respect to the so-called “deviation from trend” of the underlying asset, defined as the difference between the current price and a weighted mean of historical prices. Intuitively this parameter indicates if there have been sudden large movements of the quotation of the underlying asset.

Finally we note that the implied volatility depends also on time in absolute terms: indeed, it is well known that the shape of the implied volatility surface on the S&P500 index has significantly changed from the beginning of the eighties until today. The market crash of 19 October 1987 may be taken as the date marking the end of flat volatility surfaces. This also reflects the fact that, though based on the same mathematical and probabilistic tools, the modeling of financial and, for instance, physical phenomena are essentially different: indeed, the financial dynamics strictly depends on the behaviour and beliefs of investors and therefore, differently from the general laws in physics, may vary drastically over time.

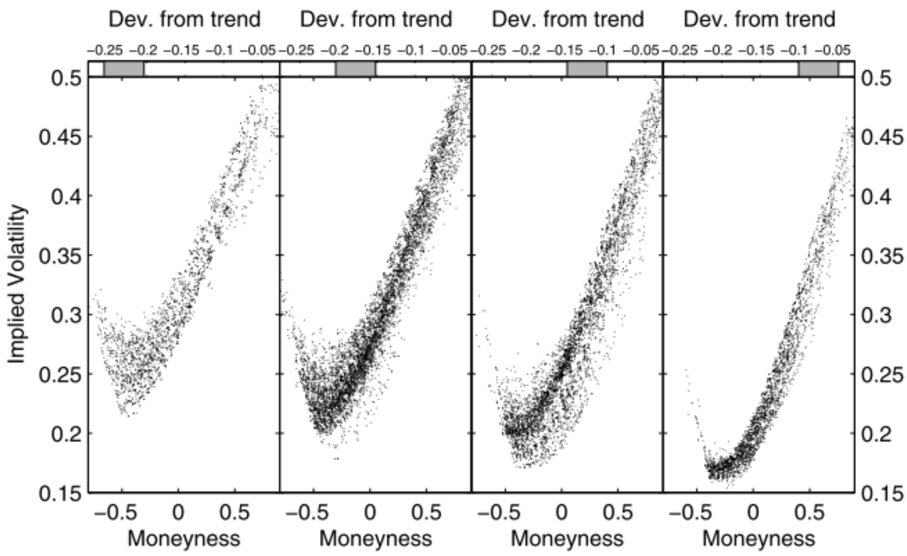


Fig. 1.1. Effect of the deviation from trend on the implied volatility. The volatility smiles for options on the S&P500 index are grouped for different values of the deviation, as indicated on top of each box

The Black&Scholes model and its pricing formula are wrong: the analysis of the implied volatility surface makes it evident that the Black&Scholes model is not realistic. As often quoted, the implied volatility is the wrong number to put in the wrong formula to obtain the right price: more precisely, we could say that nowadays Black&Scholes is the language of the market (since prices are quoted in terms of implied volatility), but usually it is not the model really used by investors to price and hedge derivatives. It comes at first as a surprise to see this apparently irrelevant number being constantly used by traders. Why should it deserve so much attention?

A first answer is pointed out in [Lee05]. "[...] *it is helpful to regard the Black-&Scholes implied volatility as a language in which to express an option price. Use of this language does not entail any belief that volatility is actually constant. A relevant analogy is the quotation of a discount bond price by giving its yield to maturity, which is the interest rate such that the observed bond price is recovered by the usual constant interest rate bond pricing formula. In no way does the use or study of bond yields entail a belief that interest rates are actually constant. As yield to maturity is just an alternative way of expressing a bond price, so is implied volatility just an alternative way of expressing an option price.*

The language of implied volatility is, moreover, a useful alternative to raw prices. It gives a metric by which option prices can be compared across different strikes, maturities, underlyings, and observation times; and by which market prices can be compared to assessments of fair value. It is a standard in industry, to the extent that traders quote option prices in "vol" points, and exchanges update implied volatility indices in real time."

Indeed the use of the Black&Scholes model poses some not merely theoretical problem: for instance, let us suppose that, despite all the evidence against the

Black-Scholes model, we wish to use it anyway. Then we have seen that we have to face the problem of the choice of the volatility parameter for the model. If we use the historical volatility, we might get quotations that are "out of the market", especially when compared with those obtained from the market-volatility surface in the extreme "in" and "out of money" regions. On the other hand, if we want to use the implied volatility, we have to face the problem of choosing one value among all the values given by the market, since the volatility surface is not "flat". Evidently, if our goal is to price and hedge a plain vanilla option, with strike, say, K and maturity, say, T, the most natural idea is to use the implied volatility corresponding to $(K,\ T)$. But the problem does not seem to be easily solvable if we are interested in the pricing and hedging of an exotic derivative: for example, if the derivative does not have a unique maturity (e.g. a Bermudan option) or if a fixed strike does not appear in the payoff (e.g., an Asian option with floating strike).

These problems make it necessary to introduce more sophisticated models than the Black&Scholes one, that can be calibrated in such a way that it is possible to price plain vanilla options in accordance with the implied volatility surface of the market. In this way such models can give prices to exotic derivatives that are consistent with the market Call and Put prices. This result is not particularly difficult and can be obtained by various models with non-constant volatility. A second goal that poses many more delicate questions and is still a research topic consists in finding a model that gives the "best" solution to the hedging problem and that is stable with respect to perturbations of the value of the parameters involved ([SST04] and [Con06]).

Instead of a constant volatility, one can posit a volatility that is a function of the spot process itself. This way, the spot process is a Markov process solution of a stochastic differential equation, and we obtain the so-called local volatility (LV) models ([Bre06], [Cre03], [DK94], [DKC96], [DKK96], [Dup93], [Dup94], [Eng06], [Gat06], [Reb04], [Wil06]). A more general approach consists in a volatility being a stochastic process on its own. In such a case the spot process alone is no longer Markov and such models are often called stochastic volatility (SV) models ([DK98], [HKLW02], [Hes93], [HW87]). Finally, people have further proposed to introduce jumps in the spot process (Lévy models: [AA00], [Mer76]). These idea can of course be combined with any of the other ones.

Stochastic volatility processes were introduced by Hull and White [HW87]. In it the volatility itself is a process that satisfies a stochastic differential equation. The most famous stochastic volatility models are the Heston model [Hes93] and the SABR model [HKLW02]. How these models cause the volatility skew in the market is discussed in ([DK98], [HKLW02]).

Jump-diffusion models were first introduced by Merton [Mer76]. These models incorporate discontinuous jumps in the underlying asset price. This resembles reality were events can have sudden impacts on asset prices. How this explains the volatility skew is described in [AA00].

The local volatility model assumes the volatility is a deterministic function of the asset price and time. It came into existence when Dupire ([Dup93], [Dup94]) showed that, in the presence of volatility skews, consistent models can be built if the asset price process is assumed to have the following dynamics under the risk neutral probability measure $\mathbb{Q}$

$$\frac{dS_t}{S_t} = (r_t - q_t)\, dt + \sigma(t, S_t)\, dW_t$$

where the volatility is now a deterministic function of time and the asset price and r_t and q_t denote the continuously compounded short rate and dividend respectively. In this case the diffusion process is usually referred to as local volatility.

These models attempt to explain the various empirical deviations from the Black&Scholes model by introducing additional degrees of freedom in the model such as a local volatility function, a stochastic diffusion coefficient, jump intensities, jump amplitudes etc. However, these additional parameters describe the infinitesimal stochastic evolution of the underlying asset while the market usually quotes options directly in terms of their market-implied volatilities which are global quantities.

In order to see whether the model reproduces empirical observations, one has to relate these two representations: the infinitesimal description via a stochastic differential equation on one hand, and the market description via implied volatilities on the other hand.

However, in the majority of these models it is impossible to compute directly the shape of the implied volatility surface in terms of the model parameters. Although it is possible to compute the implied volatility surface numerically, these numerical studies show that simple jump processes and one factor stochastic volatility models do not reproduce correctly the profiles of empirically observed implied volatility surfaces and smiles ([BCC97], [DS99], [Tom01]).

This problem is also reflected in the difficulty in 'calibrating' model parameters simultaneously to a set of liquid option prices on a given date: if the number of input option prices exceeds the number of parameters (which should be the case for a parsimonious model) a conflict arises between different calibration constraints since the implied volatility pattern predicted by the model does not correspond to the empirically observed one. This problem, already present at the static level, becomes more acute if one examines the consistency of model dynamics with those observed in the options market. While a model with a large number of parameters, such as a non-parametric local volatility function, may calibrate well the strike profile and term structure of options on a given day, the same model parameters might give a poor fit at the next date, creating the need for constant recalibration of the model. Examples of such inconsistencies over time have been documented for the implied-tree approach by Dumas et al ([DFW98]). This time instability of model parameters leads to large variations in sensitivities and hedge parameters, which is problematic for risk management applications.

The inability of models based on the underlying asset to describe dynamic behaviour of option prices or their implied volatilities is not, however, simply due to the mis-specification of the underlying stochastic process. There is a deeper reason: since the creation of organized option markets in 1973, these markets have become increasingly autonomous and option prices are driven, in addition to movements in the underlying asset, also by internal supply and demand in the options market.

This fact is also supported by recent empirical evidence of violations of qualitative dynamical relations between options and their underlying ([BCC00]). This observation can be accounted for by introducing sources of randomness which are specific to the options market and which are not present in the underlying asset dynamics.

CHAPTER 2

Lorig, Pagliarani and Pascucci's Work

When it comes to the options market, particularly the Foreign Exchange (FX) options market, the valuation of barrier options and other exotic options is different from that of vanilla options as prices of these exotic options are not only dependent on the dynamics of the vanilla market quotes.

Neither local volatility (LV) nor stochastic volatility (SV) models are able to fit empirically observed implied volatility levels over the full range of strikes an maturities. This has led to the development of local- stochastic volatility (LSV) models, which combine the features of LV and SV models by describing the instantaneous volatility of an underlying S by a function $f(S_t, Z_t)$ where Z is some auxiliary, possibly multidimensional, stochastic process (see, for instance, [Lip02], [AN04], [Ewa05], [HeLa09] and [Cla10]).

Local volatility and stochastic volatility models are actually calibrated to only market vanilla options, and hence do not have the flexibility to capture the dynamics of exotic options. Therefore, hybrid use of both models was proposed. For example, a linear average of prices from both local volatility and stochastic volatility models was employed by practitioners as to price exotic options (see [TF10]), whereas there is no theocratical point of this approach.

The general dynamics of the hybrid stochastic-local volatility model are

$$\begin{aligned} dS_t &= \mu_1(S_t, t)\,dt + L(S_t, t)\,\sigma_1(S_t, Z_t, t)\,dW_t^1, \\ dZ_t &= \mu_2(Z_t, t)\,dt + \sigma_2(Z_t, t)\,dW_t^2, \\ dW_t^1 \cdot dW_t^2 &= \rho\,dt \end{aligned} \tag{2.1}$$

with the correlation $-1 < \rho < 1$. $L(S_t, t)$ is called leverage function, which is to be determined by market information in order to control the weights of local volatility and stochastic volatility. If the price process of the new LSV model could mimic that of the local volatility model, then the two models should generate the same pricing results for European options. In order to achieve this goal, we need to match the diffusion terms of the two models. We refer to [Gyo86], [Kle02], [Dup04] and [Tac11] for more theoretical backgrounds.

Compared to their LV and SV counterparts, LSV models produce implied volatility surfaces that more closely match those observed in the market. However, LSV models rarely allow for exact formulas for option prices. Thus, LSV models present two challenges. First, given an LSV model, can one find accurate closed-form approximations for option prices? Second, given approximate option prices, can one find accurate closed-form approximations for implied volatilities?

In the class of the stochastic-local volatility model, there are different viewpoints of modeling and calibration approaches. Jex, Henderson and Wang ([JHW99]) used a trinomial tree method. Lipton ([Lip02]) proposed a so-called universal volatility model, which combines local volatility and stochastic volatility. Ren, Madan and Qian ([RMQ07]) used a log-normal model for spot process and volatility process with zero correlation. Tataru & Fisher ([TF10]) also suggested a term-structure model with log-normal process for volatility, which they believe describes the realistic market dynamics. Moreover, both Ren et al. ([RMQ07]) and Tataru and Fisher ([TF10]) constructed the volatility process as a stochastic multiplier (around 1) imposed on local volatility. Henry-Labordère ([HeLa09]) and van der Stoep, Grzelak and Oosterlee ([vdSGO13]) proposed Monte Carlo based approaches for calibration. Choi, Fouque and Kim ([CFK12]) used a stochastic volatility following a mean-reverting Ornstein-Uhlenbeck process. Deelstra & Rayée ([DR12]) suggested a LSV model with stochastic interest rate to price long-dated FX options.

In the area of pricing, there have been a number or recent developments. An exhaustive review of LSV pricing approximations would be prohibitive. However, we also mention the following: Pagliarani and Pascucci ([PP13]) added jumps to a LSV model, but also a general local volatility to the Heston model obtaining a Fourier-like representation for approximate option prices. Rather than non-paramet ric construction of the leverage function, Murex ([Mur11]) and Wystup ([Wys11]) constructed a parametric approach with a forward process. Lorig ([Lor12]) added multi scale stochastic volatility to general scalar diffusions, and thus obtains analytically tractable eigenfunction approximations for options prices. Finally, Lorig, Pagliarani and Pascucci ([LPP13]) proposed an asymptotic expansion for the transition probability of general parametric hybrid volatility models.

Typically, unobservable LSV (or SV or LV) model parameters are obtained by calibrating these models to implied volatilities that are observed on the market. To do this, one must find model-induced implied volatilities over a range of strikes and maturities. Computing model-induced implied volatilities from option prices by inverting the Black Scholes formula numerically is a computationally intensive task, and therefore, not suitable for the purposes of calibration. For this reason closed-form approximations for model-induced implied volatilities are needed. A number of different approach have been taken for computing approximate implied volatilities in LV, SV and LSV models.

Concerning LV models, perhaps the earliest and most well-known implied volatility result is due to Hagan and Woodward ([HW99]), who used singular perturbation methods to obtain an implied volatility expansion for general LV models. For certain models (e.g., CEV) they obtained closed-form approximations. More recently, Lorig ([Lor13]) used regular perturbation methods to obtain an implied volatility expansion when a LV model can be written as a regular perturbation around Black-Scholes. Jacquier and Lorig ([JL13]) extended and refined the results of Lorig ([Lor13]) to find closed-form approximations of implied volatility for local Lévy-type models with jumps. Gatheral, Hsu, Laurence, Ouyang, and Wang ([GHLO12]) examined the small-time asymptotics of implied volatility for LV models using heat kernel methods.

There is no shortage of implied volatility results for SV models either. Fouque, Lorig, and Sircar ([FLS12]) (see also [FPSS11]) derived an asymptotic expansion for general multiscale stochastic volatility models using combined singular and regular perturbation theory. Forde and Jacquier ([FJ11]) used the Freidlin-Wentzell theory of large deviations for SDEs to obtain the small-time behavior of implied volatility for general stochastic volatility models with zero correlation. Their work added mathematical rigor to previous work by Lewis ([Lew07]). Forde and Jacquier ([FJ09]) used large deviation techniques to obtain the small-time behavior of implied volatility in the Heston model (with correlation). They further refined these results in [FJL12].

Concerning LSV models, perhaps the most well-known implied volatility result is due to Hagan, et al. ([HKLW02]), who used WKB approximation methods to obtain implied volatility asymptotics in a LSV model with a CEV-like factor of local volatility and a GBM-like factor of non-local volatility (i.e., the SABR model). More recently, Henry-Labordère ([HeLa05]) used a heat kernel expansion on a Riemann manifold to derive first order asymptotics for implied volatility for any LSV model. As an example, he introduced the λ-SABR model, which is a LSV model with a mean reverting non-local factor of volatility, and obtained closed form asymptotic formulas for implied volatility in this setting. See also [HeLa09a]. There are also some model-free results concerning the extreme-strike behavior of implied volatility. Most notably, we mention the work of Lee ([Lee04]) and Gao and Lee ([GL11]).

In this chapter we talk about local stochastic volatility (LSV) models, and in particular, we're going to obtain a price and an implied volatility expansion for these models; in doing so, we define the problem and the method of resolution as presented in the works of Lorig, Pagliarani and Pascucci ([LPP13] and [LPP13a]), following also other works of the same authors ([LPP13b] and [PP12]) and proposing some method's improvements.

In order to guarantee consistence in our work, we use from the beginning the notation presented in [LPP13b] which is more general and flexible, but whereas in all of these papers the risk-free interest rate is set to zero, here we consider a deterministic interest rates.

In the next chapter we consider the particular case covered in [LPP13a] and we implement the obtained implied volatility expansion in some numerical tests, optimizing the code and going further in the computation than in ([LPP13]) in order to prove whether the approximation converges and at which order we could and should stop.

3 Local Stochastic Volatility (LSV) models

As in [LPP13], we consider general LSV models. For these models, they derive a family of closed-form asymptotic expansions for transition densities, option prices and implied volatilities. The presented method most closely follows that of [LPP13c] who derive a family of density and option price expansions for scalar Lévy-type processes. Lorig, Pagliarani, and Pascucci in [LPP13c] use a very general technique, the so-called Adjoint Expansion method, first introduced by Pagliarani and Pascucci in [PP12] and Pagliarani, Pascucci, and Riga in [PPR13] (see also [CFP10] for previous related results).

The major contributions of the method considerd here are as follows:

- Whereas Pagliarani and Pascucci ([PP12]) expand the coeffcients of a scalar diffusion as a Taylor series about an arbitrary point, i.e. $f(x) = \sum_n a_n(x - \overline{x})$, in order to achieve their approximation result, here is expanded the diffusion coefficients of a multi-dimensional diffusion in an arbitrary basis, i.e. $f(x, y) = \sum_n \sum_h c_{n,h} B_{n,h}(x, y)$. Thus, the results of [PP12] is not only extended from one to multiple dimensions, but are also considered more general expansions.
- An explicit formula for the nth term in transition density and option-price expansions is provided. The terms in the density expansion appear as Hermite polynomials multiplied by Gaussian kernels and thus, can be computed extremely quickly. In [LPP13c] the nth term of the transition density is given as a Fourier transform, which is computationally more intensive. In [PPR13], no general formula for the nth term appears.
- A closed-form approximations for implied volatility in a general local-stochastic volatility setting is provided. In the next chapter we show (through a series of numerical experiments) that this implied volatility approximation performs favorably when compared to other well-known implied volatility approximations (e.g., Hagan and Woodward ([HW99]) for CEV or Forde, Jacquier, and Lee ([FJL12]) for Heston).
- Many of the above-mentioned implied volatility approximations rely on some special structure for the underlying diffusion (e.g., fast-or slow-varying volatility, or some particular Riemannian geometry which allows for closed-form computation of geodesics). When these structures are absent, the associated implied volatility expansions will not work. By contrast, the implied volatility approximation presented by Lorig, Pagliarani and Pascucci in [LPP13] works for any LSV model (actually, by the Adjoint Expansion method, jumps can be added as well). Thus, in addition to being highly accurate, this approach is quite general and includes several models of great interest for the financial industry. For instance, to the best of our

knowledge, here is given the first approximation formula for implied volatilities in the 3/2 stochastic volatility model. Of late, the 3/2 model has attracted much interest due to its ability match market prices for both European-style options as well as variance and volatility derivatives ([BB12]).

- Here, Lorig, Pagliarani and Pascucci provide a general result showing how to pass in a model-free way from a price expansion to an implied volatility expansion.

3.1 General local-stochastic volatility models

For simplicity, we assume a frictionless market, no arbitrage, no dividends and a risk-free interest rate r. We take, as given, an equivalent martingale measure $\mathbb{Q}$, chosen by the market on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{Q})$. The filtration $\{\mathcal{F}_t, t \geq 0\}$ represents the history of the market. All stochastic processes defined below live on this probability space and all expectations are taken with respect to $\mathbb{Q}$. The risk-neutral dynamics of our market are described by the following d-dimensional Markov diffusion

$$\begin{cases} S_t = \mathbb{I}_{\{\zeta > t\}}\, e^{X_t}, \\ dX_t = \mu(t, X_t)\, dt + \Sigma(t, X_t)\, dW_t, \quad X_0 = x_0 \in \mathbb{R}^d. \end{cases} \tag{3.1}$$

Here, $W \in \mathbb{R}^m$ is a standard m-dimensional Brownian motion, the function $\mu : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and the function $\Sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$. The components of X could represent a number of things, for example interest rates, asset prices, economic indicators, or functions of these quantities. We also introduce a stopping time $\zeta \geq 0$, which is given by

$$\zeta = \inf \left\{ t \geq 0 : \int_0^t \gamma(s, X_s)\, ds \geq \epsilon \right\},$$

with ϵ exponentially distributed and independent of X. The stopping time ζ could represent the default time of an asset, the arrival of an economic shock, etc.

We assume that SDE (3.1) has a unique strong solution and that functions Σ and μ are smooth. Suffcient conditions for the existence of a unique strong solution can be found, for example, in [IW89]. We also assume that the coeffcients are such that $\mathbb{E}\left[\!\left[S_t^{(i)} \right]\!\right] < \infty$ for all $t \in [0, \infty)$ and for all $i = 1, \ ..., d$.

Equation (3.1) includes virtually all one-factor stochastic volatility models, all local stochastic volatility models, all one-factor local-stochastic volatility models, but also every local-stochastic volatility models.

4 Transition density and option pricing PDE

Denote by V the no-arbitrage price of European derivative expiring at time T with payoff

$$H(X_T)\,\mathbb{1}_{\{\zeta>T\}} + G(X_T)\,\mathbb{1}_{\{\zeta\leq T\}} = (H(X_T) - G(X_T))\,\mathbb{1}_{\{\zeta>T\}} + G(X_T).$$

It is well known (see, for instance, [JYC09]) that for $t < T$

$$V_t = e^{-\int_t^T r(s)\,ds}\left(\mathbb{E}[\![G(X_T) \mid X_t]\!] + \mathbb{1}_{\{\zeta>t\}}\,\mathbb{E}\left[\!\!\left[e^{-\int_t^T \gamma(s,\,X_s)\,ds}(H(X_T) - G(X_T) \mid X_t\right]\!\!\right]\right).$$

Then, to value a European-style option, one must compute functions of the form

$$u(t, x) := \mathbb{E}\left[\!\!\left[e^{-\int_t^T (r(s)+\gamma(s,\,X_s))\,ds} h(X_T)\,\middle|\, X_t = x\right]\!\!\right] \tag{4.1}$$

The function u defined in (4.1) is $C^{1,2}([0, T), \mathbb{R}^d)$, then u satisfies the Kolmogorov Backward equation

$$\begin{cases} (\partial_t + \mathcal{A})\,u = 0, & x \in \mathbb{R}^d,\ t \in [0, T) \\ u(T, x) = h(x), & x \in \mathbb{R}^d, \end{cases} \tag{4.2}$$

where the operator $\mathcal{A}$ is given explicitly by

$$\mathcal{A} = \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} C^{i,j}(t, x)\,\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{d}\mu_i(t, x)\,\partial_{x_i} - (r(s) + \gamma(t, x)). \tag{4.3}$$

We shall assume henceforth that the operator $\mathcal{A}$ is parabolic, which is typically the case in financial applications.

The matrix C is defined in the following lemma.

Lemma (1)

Let us set

$$C = \Sigma\,\Sigma^T$$

then we have

$$\langle X^i, X^j\rangle_t = \int_0^t C^{i,j}(s, X_s)\,ds \qquad t \geq 0$$

or in differential notation,

$$d\langle X\rangle_t = C(t, X_t)\,dt\,.$$

The computation of $\langle X^i,\ X^j\rangle_t$ can be handled by applying the following "rules":

$$d\langle X^i, X^j\rangle_t = dX_t^i\,dX_t^j\,,$$

and computing the product on the right-hand side using the following formal rules:

$$dt\,dt = dt\,dW^i = dW^i\,dt = 0, \qquad dW^i\,dW^j = \delta_{i\,j}\,dt\,.$$

If one have a LSV model where the sources of uncertainty are correlated, the method here presented is still valid, and it’s sufficient to make some extra calculations: first of all we recall the definition of the correlated Brownian motion.

Definition (2)

Let us consider a $(d \times m)$-dimensional matix γ, whose components $\gamma^{i,j}$ are progressively measurable processes and whose rows γ^i are such that

$$\|\gamma^i(t, x)\| = 1 \qquad t \geq 0,\ x \in \mathbb{R}^d,\ a.s.$$

The process B_t given by

$$dB_t = \gamma(t, X_t)\,dW_t$$

is called correlated Brownian motion.

Every component of B is a real Brownian motion and by Lemma (1) we have

$$\langle B^i, B^j\rangle_t = \int_0^t \rho^{i,j}(s, X_s)\,ds$$

where $\rho = \gamma\,\gamma^T$ is called correlation matrix of B.

In what follows we omit the notation (t, x) in every function, but it’s considered to be implied; let's consider two matrices, a $(d \times d)$-dimensional diagonal matrix $\mathcal{D}$ and a $(d \times m)$-dimensional matrix $\mathcal{H}$, such that

$$\mathcal{D}\,\mathcal{H} = \Sigma$$

so we define their element in this way

$$\mathcal{D}^{i\,j} = \|\Sigma^i\|\,\delta_{i\,j}\,, \qquad \mathcal{H}^{i\,j} = \frac{\Sigma^{i\,j}}{\|\Sigma^i\|},$$

$$(\mathcal{D}\,\mathcal{H})^{i\,j} = \sum_{k=1}^{d} \mathcal{D}^{i\,k}\,\mathcal{H}^{k\,j} = \sum_{k=1}^{d} \|\Sigma^i\|\,\delta_{i\,k}\,\frac{\Sigma^{k\,j}}{\|\Sigma^k\|} = \|\Sigma^i\|\,\delta_{i\,i}\,\frac{\Sigma^{i\,j}}{\|\Sigma^i\|} = \Sigma^{i\,j}\,.$$

Now we can construct our correlated Brownian motion as in Definition (2):

$$dB_t = \mathcal{H}\,dW_t$$

so we say that B is a correlated Brownian motion with correlations matrix

$$\mathcal{R} = \mathcal{H}\,\mathcal{H}^T, \qquad \mathcal{R}^{i,j}(t, X_t)\,dt = d\langle B^i, B^j\rangle_t$$

so SDE (3.1) becomes

$$d X_t = \mu(t, X_t)\, d t + \mathcal{D}(t, X_t)\, d B_t\,.$$

In this case, in addition to the vector μ, the observer knows only the matrix $\mathcal{D}$ and the correlation matrix $\mathcal{R}$; so the operator $\mathcal{A}$ becomes:

$$\mathcal{A} = \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\tilde{C}^{i\,j}(t, x)\,\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{d}\mu_i(t, x)\,\partial_{x_i} - (r(s) + \gamma(t, x)). \qquad (4.4)$$

where the matrix $\tilde{C}$ is such that $\tilde{C}^{i\,j} = \mathcal{D}^{i\,i}\,\mathcal{D}^{j\,j}\,\mathcal{R}^{i\,j}$. This could be shown easily in the following computation:

$$\left(\Sigma\,\Sigma^T\right)^{i\,j} = C^{i\,j} = \sum_{l=1}^{m}\Sigma^{i\,l}\left(\Sigma^T\right)^{l\,j} = \sum_{l=1}^{m}\Sigma^{i\,l}\,\Sigma^{j\,l} =$$

$$= \sum_{l=1}^{m}\sum_{h=1}^{d}\mathcal{D}^{i\,h}\,\mathcal{H}^{h\,l}\sum_{k=1}^{d}\mathcal{D}^{j\,k}\,\mathcal{H}^{k\,l} = \sum_{l=1}^{m}\mathcal{D}^{i\,i}\,\mathcal{H}^{i\,l}\,\mathcal{D}^{j\,j}\,\mathcal{H}^{j\,l} =$$

$$= \mathcal{D}^{i\,i}\,\mathcal{D}^{j\,j}\sum_{l=1}^{m}\mathcal{H}^{i\,l}\,\mathcal{H}^{j\,l} = \mathcal{D}^{i\,i}\,\mathcal{D}^{j\,j}\sum_{l=1}^{m}\mathcal{H}^{i\,l}\left(\mathcal{H}^T\right)^{l\,j} = \mathcal{D}^{i\,i}\,\mathcal{D}^{j\,j}\,\mathcal{R}^{i\,j}.$$

Example (3)

If we consider $d = m = 2$ and

$$\mathcal{D}(t, X_t) := \begin{pmatrix} \sigma(t, X_t) & 0 \\ 0 & \beta(t, X_t) \end{pmatrix}$$

$$\mathcal{R}(t, X_t) := \begin{pmatrix} 1 & \rho(t, X_t) \\ \rho(t, X_t) & 1 \end{pmatrix}$$

we obtain

$$\tilde{C}(t, X_t) := \begin{pmatrix} (\sigma(t, X_t))^2 & \rho\,(t, X_t)\,\sigma\,(t, X_t)\,\beta\,(t, X_t) \\ \rho\,(t, X_t)\,\sigma(t, X_t)\,\beta(t, X_t) & (\beta(t, X_t))^2 \end{pmatrix}$$

and the operator $\mathcal{A}$ defined as in (4.4), which is exactly the case considered in [LPP13] and [LPP13a].

5 Expansion basis

In what follows, it will be convenient to express the operator $\mathcal{A}$ using multi-index notation

$$\mathcal{A} = \sum_{|\alpha|\le 2} \mathbb{a}_\alpha(t,x)\,\mathbb{D}^\alpha, \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \tag{5.1}$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d, \quad |\alpha| = \sum_{i=1}^{d} \alpha_i, \quad \mathbb{D}^\alpha \equiv \mathbb{D}_x^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_d}^{\alpha_d},$$

and where $\mathbb{a}_\alpha(.,.)$ are given by setting (5.1) equal to (4.3).

Throughout the rest of this section, for any $\alpha \in \mathbb{N}_0^d$ with $|\alpha|\le 2$, we fix once and for all a family $(\mathbb{a}_n^\alpha)_{n\in\mathbb{N}_0}$ of functions $\mathbb{a}_n^\alpha : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ such that:

1. for all multi-indices α with $|\alpha|\le 2$ we have

$$\mathbb{a}_\alpha = \sum_{n=0}^{\infty} \mathbb{a}_n^\alpha \tag{5.2}$$

pointwise or in some norm;

2. for any t the functions $\mathbb{a}_n^\alpha(t,.)$ are polynomials of degree d_n^α with $d_0^\alpha = 0$;
3. for any x, the functions $\mathbb{a}_n^\alpha(.,x)$ belong to $L^\infty([0,T])$.

In such conditions we say that $(\mathbb{a}_n^\alpha)_{n\in\mathbb{N}_0}$ is an LPP-expansion of the coefficient of $\mathcal{A}$.

5.1 Taylor Expansion

Assume the coeffcients a $\mathbb{a}_\alpha(t,.)$ to be analytic. Then, for any fixed $\bar{x} \in \mathbb{R}^d$, we define $\mathbb{a}_n^\alpha$ as the n-th order term of the spatial Taylor expansion of $\mathbb{a}_\alpha$ around $\bar{x}$, i.e.

$$\mathbb{a}_n^\alpha(.,x) = \sum_{|\beta|=n} \frac{\mathbb{D}^\beta \mathbb{a}_\alpha(.\,,\bar{x})}{\beta!}(x-\bar{x})^\beta, \quad n \ge 0, \ |\alpha| \le 2,$$

where as usual $\beta\,! = \beta_1! \ldots \beta_d!$ and $x^\beta = x_1^{\beta_1} \ldots x_d^{\beta_d}$. The expansion proposed in [LPP13] and [LPP13a] is the particular case where d = 2.

5.2 Enhanced Taylor Expansion

In the previous example, the n-order term of the LPP-expansion of the coeff-cients coincides with the n-order term of the Taylor expansion. More generally, we may want to define the n-order term of the LPP-expansion as a higher (or lower!) order Taylor expansion. Indeed this can be motivated by the fact that much of the precision of the approximation is achieved through the convolution process. So now we may assume that $\mathbb{a}_0^\alpha(t,x) = \mathbb{a}_\alpha(t,\bar{x})$ and $\mathbb{a}_n^\alpha$ are defined in such a way that, for any increasing sequence $(N_i)_{i\ge 1}$ with $N_i \in \mathbb{N}$ we have

$$\mathbb{a}_n^{\alpha}(., x) = \sum_{|\beta|=N_{n-1}+1}^{N_n} \frac{\mathbb{D}^{\beta}\mathbb{a}_{\alpha}(.\,, \bar{x})}{\beta!}(x-\bar{x})^{\beta}, \quad n \geq 0, \ |\alpha| \leq 2,$$

Note that for any n the $\sum_{k=0}^{n} \mathbb{a}_k^{\alpha}(t, x)$, is equal to the N_n-th order Taylor expansion, where in general N_n may be different from n.

5.3 Hermite Expansion

Hermite expansions are useful when the volatility function is not smooth: a remarkable example is given by the Dupire's local volatility formula for models with jumps (see [FGY13]). In some cases (e.g. the VG model) the density has singularities and therefore it is natural to approximate it in some L^p norm rather than in the pointwise sense. For the Hermite expansion centered at $\bar{x}$, one sets

$$\mathbb{a}_n^{\alpha}(., x) = \sum_{|\beta|=n} \langle H_{\beta}(. - \bar{x}), \mathbb{a}_{\alpha}(t, .)\rangle_{\Gamma}\, H_{\beta}(x - \bar{x}), \quad n \geq 1, \ |\alpha| \leq 2,$$

where the inner product $\langle .\,, .\rangle_{\Gamma}$ is an integral over $\mathbb{R}^d$ with a Gaussian weighting centered at $\bar{x}$ and the functions $H_{\beta}(x) = H_{\beta_1}(x_1) \ldots H_{\beta_d}(x_d)$ with H_n being the n-th one-dimensional Hermite polynomial (properly normalized so that $\langle H_{\alpha}, H_{\beta}\rangle_{\Gamma} = \delta_{\alpha,\beta}$).

6 Density and option price expansions

Now we can explain the main idea behind the construction of the approximation for the solution of (4.2): let us consider an LPP-expansion $(\mathbb{a}_n^{\alpha})$ of the coefficients of $\mathcal{A}$. Assuming that (5.2) holds, the operator $\mathcal{A}$ can be formally written as

$$\mathcal{A} = \sum_{n=0}^{\infty} \mathcal{A}_n, \qquad \mathcal{A}_n := \sum_{|\alpha|\leq 2} \mathbb{a}_n^{\alpha}(t, x)\, \mathbb{D}^{\alpha}. \tag{6.1}$$

In light of the above expansion for $\mathcal{A}$, we also expand the pricing function u as follows

$$u = \sum_{n=0}^{\infty} u_n \tag{6.2}$$

Inserting (6.1) and (6.2) into (4.2) we find that the functions $(u_n)_{n\geq 0}$ satisfy the sequence of backward Cauchy problems

$$\begin{cases} (\partial_t + \mathcal{A}_0)\, u_0(t, x) = 0, & x \in \mathbb{R}^d, t < T \\ u_0(T, x) = h(x), & x \in \mathbb{R}^d, \end{cases} \tag{6.3}$$

$$\begin{cases} (\partial_t + \mathcal{A}_0)\, u_n(t, x) = -\sum_{k=1}^{n} \mathcal{A}_k\, u_{n-k}(t, x), & x \in \mathbb{R}^d, t < T \\ u_0(T, x) = 0, & x \in \mathbb{R}^d, \end{cases} \tag{6.4}$$

Now notice that, since $\mathbb{a}_0^\alpha$ are assumed to depend only on t, then $(\partial_t + \mathcal{A}_0)$ is a heat operator with time-dependent coeffcients. More precisely, we have

$$\mathcal{A}_0 = \frac{1}{2} \langle \nabla_x, C(t) \nabla_x \rangle + \langle m(t), \nabla_x \rangle + \gamma(t), \tag{6.5}$$

where the $n \times n$-matrix C, the n-dimensional vector m and the scalar γ are determined by setting (6.5) equal to $\mathcal{A}_0$ in (6.1). Hereafter we assume that the matrix C(t) is positive definite, uniformly for t $\in$ [0, T].

Example (4)

If d=2 we have

$$C(t) = \begin{pmatrix} 2\,\mathbb{a}_0^{(2,0)}(t) & \mathbb{a}_0^{(1,1)}(t) \\ \mathbb{a}_0^{(1,1)}(t) & 2\,\mathbb{a}_0^{(0,2)}(t) \end{pmatrix},$$

$$m(t) = \left(\mathbb{a}_0^{(1,0)}(t),\ \ \mathbb{a}_0^{(0,1)}(t) \right),\ \gamma(t) = \mathbb{a}_0^{(0,0)}(t)$$

Proposition (5)

Then the leading term u_0 in (6.2) is given by

$$u_0(t, x) = e^{-\int_t^T (r(s)+\gamma(s))\, ds} \int_{\mathbb{R}^d} \Gamma(t, x; T, y)\, h(y)\, dy, \quad t < T, \quad x \in \mathbb{R}^d \tag{6.6}$$

where $\Gamma(t, x; T, y)$ is the d-dimensional Gaussian density function

$$\Gamma(t, x; T, y) = \frac{1}{\sqrt{(2\pi)^d\, |\boldsymbol{C}(t, T)|}}\, e^{-\frac{1}{2} \left\langle \boldsymbol{C}^{-1}(t,T)\, (y-(x+\boldsymbol{m}(t,T))),(y-(x+\boldsymbol{m}(t,T))) \right\rangle} \tag{6.7}$$

with covariance matrix $\boldsymbol{C}$ and mean vector $x + \boldsymbol{m}$ given by:

$$\boldsymbol{C}(t, T) = \int_t^T C(s)\, ds, \quad \boldsymbol{m}(t, T) = \int_t^T m(s)\, ds. \tag{6.8}$$

The next theorem provides an explicit representation for each u_n of (6.2), in terms of a finite sum of partial derivatives of u_0 with respect to the spatial variables.

Theorem (6)

For any n ≥ 1, the n-th term u_n of the approximation of u is given by

$$u_n(t, x) = \mathcal{L}_n(t, T)\, u_0(t, x), \quad t < T,\ x \in \mathbb{R}^d, \tag{6.9}$$

with

$$\mathcal{L}_n(s_0, T) := \sum_{h=1}^{n} \int_{s_0}^{T} ds_1 \int_{s_1}^{T} ds_2 \ldots \int_{s_{h-1}}^{T} ds_h \sum_{i \in I_{n,h}} \mathcal{G}_{i_1}(s_0, s_1) \ldots \mathcal{G}_{i_h}(s_0, s_h) \tag{6.10}$$

where

$$I_{n,h} = \left\{ i = (i_1, \ \ldots, i_h) \in \mathbb{N}^h \,\middle|\, i_1 + \ldots + i_h = n \right\}, \quad 1 \le h \le n \tag{6.11}$$

and $\mathcal{G}_n(t, s)$ *is the operator*

$$\mathcal{G}_n(t, s) := \sum_{|\alpha| \le 2} \mathbb{a}_n^{\alpha}(s, \mathcal{M}(t, s))\, \mathcal{D}^{\alpha}, \tag{6.12}$$

with

$$\mathcal{M}(t, s) = x + \boldsymbol{m}(t, s) + \boldsymbol{C}(t, s)\, \nabla_x. \tag{6.13}$$

The proof of Theorem (6) lies on some symmetry properties of Gaussian kernels, combined with other very general relations such as the classical Chapman-Kolmogorov equation and the standard Duhamel's principle: as such, the proof opens the way to possible extensions to other more general expansions not necessarily based on Gaussian kernels.

The complete proof is postponed into Appendix A.

7 Implied volatility expansions

European call and put prices are commonly quoted in units of implied volatility rather than in units of currency. In fact, in the financial industry, model parameters for the risk-neutral dynamics of a security are routinely obtained by calibrating to the market's implied volatility surface. Because calibration requires computing implied volatilities across a range of strikes and maturities and over a large set of model parameters, it is extremely useful to have a method of computing implied volatilities quickly.

Firstly, we show how to pass in a general and model-independent way from an expansion of option prices to an expansion of implied volatilities. Then, we show that when call option prices can be computed as a series whose terms are as given in Theorem (6), the terms in the corresponding implied volatility expansion can be computed explicitly (i.e., without special functions or integrals). As such, approximate implied volatilities can be computed even faster than approximate option prices, which require the special function N, the standard normal CDF (cumulative density function).

7.1 Implied volatility expansions from price expansions

To begin our analysis, we assume that one has a model for the log of the underlying $X = \log(S)$. We fix a time to maturity $t > 0$, an initial value $X_0 = x_0$ and a call option payoff $H(X_t) = h\left(X_t^{(1)}\right) = \left(e^{X_t^{(1)}} - e^k\right)^+$, where k is the log-strike and $X_t^{(1)}$ is the first component of X_t. The goal is to find the implied volatility for this particular call option. To ease notation, we will suppress much of the dependence on (t, x, k). However, the reader should keep in mind that the implied volatility of the option under consideration does depend on (t, x, k), even if this is not explicitly indicated.

Definition (7)

For a fixed (t,x,k), with the function $u^{\mathrm{BS}} : \mathbb{R}^+ \to \mathbb{R}^+$ we denote the Black&Scholes price.

For the same (t,x,k) fixed, the implied volatility corresponding to a call price $u \in \left(\left(e^{x^{(1)}} - e^k\right)^+, e^{x^{(1)}}\right)$, where $x^{(1)}$ is the first component of x, is defined as the unique strictly positive real solution σ of the equation

$$u^{\mathrm{BS}}(\sigma) = u. \tag{7.1}$$

Notice that $\left[u^{\mathrm{BS}}\right]^{-1}$ is an analytic function on its domain $\left(\left(e^{x^{(1)}} - e^k\right)^+, e^{x^{(1)}}\right)$. For any $u \in \left(\left(e^{x^{(1)}} - e^k\right)^+, e^{x^{(1)}}\right)$, we denote by ρ_u the radius of convergence of the Taylor series of $\left[u^{\mathrm{BS}}\right]^{-1}$ about u.

The main result of the Section is the following Theorem:

Theorem (8)

Assume that the call price u admits an expansion of the form

$$u = u^{\mathrm{BS}}(\sigma_0) + \sum_{n=1}^{\infty} u_n , \tag{7.2}$$

for some positive σ_0 and some sequence $(u_n)_{n\geq 1}$ where $u_n \in \mathbb{R}$ for all n. If

$$\left|u - u^{\mathrm{BS}}(\sigma_0)\right| < \rho_{u^{\mathrm{BS}}(\sigma_0)}, \tag{7.3}$$

then the implied volatility $\sigma := \left[u^{BS}\right]^{-1}(u)$ is given by

$$\sigma = \sigma_0 + \sum_{n=1}^{\infty} \sigma_n. \tag{7.4}$$

The sequence $(\sigma_n)_{n\geq 1}$ is defined recursively by

$$\sigma_n = U_n(\sigma_0) - \frac{1}{n!}\sum_{h=2}^{n} A_h(\sigma_0)\, B_{n,h}(\sigma_1, 2!\,\sigma_2, 3!\,\sigma_3, \,\ldots, (n-h+1)!\,\sigma_{n-h+1}) \tag{7.5}$$

where with $B_{n.h}$ we denote the (n, h)-th partial Bell polynomial and

$$U_n(\sigma_0) := \frac{u_n}{\partial_\sigma u^{\mathrm{BS}}(\sigma_0)}, \quad n \geq 1, \tag{7.6}$$

$$A_n(\sigma_0) := \frac{\partial_\sigma^n u^{\mathrm{BS}}(\sigma_0)}{\partial_\sigma u^{\mathrm{BS}}(\sigma_0)}, \quad n \geq 2. \tag{7.7}$$

Proof. We Define $u(\epsilon)$ an analytic function of ϵ by

$$u(\epsilon) := u^{\mathrm{BS}}(\sigma_0) + \sum_{n=1}^{\infty} \epsilon^n u_n, \quad \epsilon \in [0, 1]. \tag{7.8}$$

Note that $\sigma(\epsilon) := \left[u^{\mathrm{BS}}\right]^{-1}(u(\epsilon))$ is the composition of two analytic functions; it is therefore an analytic function of ϵ and admits an expansion about $\epsilon = 0$ of the form

$$\sigma(\epsilon) = \sigma_0 + \sum_{n=1}^{\infty} \epsilon^n \sigma_n, \quad \sigma_n = \frac{1}{n!}\,\partial_\epsilon^n \sigma(\epsilon)\,|_{\epsilon=0}\,. \tag{7.9}$$

which by (7.3) is convergent for any $\epsilon \in [0, 1]$. By (7.8) we also have

$$u_n = \frac{1}{n!}\,\partial_\epsilon^n u^{\mathrm{BS}}(\sigma(\epsilon))\,|_{\epsilon=0}\,. \tag{7.10}$$

We compute the n-th derivative of the composition of the two functions in (7.10) by applying the Bell polynomial version of the Faa di Bruno's formula, which can be found in [Rio46] and in [Joh02].

We have

$$u_n = \frac{1}{n!}\sum_{h=1}^{n} \partial_\sigma^h u^{\mathrm{BS}}(\sigma_0)\, B_{n,h}\big(\partial_\epsilon \sigma(\epsilon), \partial_\epsilon^2 \sigma(\epsilon), \,\ldots, \partial_\epsilon^{n-h+1} \sigma(\epsilon)\big)\,|_{\epsilon=0}\,. \tag{7.11}$$

Theorem (8) follows by inserting (7.9) into (7.11) and solving for σ_n.

In the following Proposition, we will show that the coefficients A_n in (7.7) can be computed explicitly using an iterative algorithm. In particular, each $A_n(\sigma)$ is a rational function of σ and no special functions appear in its expression.

Proposition (9)

Let us define the differential operator

$$\mathcal{J} := t\left(\partial_x^2 - \partial_x\right). \tag{7.12}$$

Then

$$A_n(\sigma) = \frac{P_n(\mathcal{J})\, u^{\mathrm{BS}}(\sigma)}{\partial_\sigma u^{\mathrm{BS}}(\sigma)}, \tag{7.13}$$

where P_n is a polynomial function of order n defined recursively by

$$P_0(\mathcal{J}) = 1,$$
$$P_1(\mathcal{J}) = \sigma \mathcal{J},$$
$$P_n(\mathcal{J}) = \sigma \mathcal{J}\, P_{n-1}(\mathcal{J}) + (n-1)\, \mathcal{J}\, P_{n-2}(\mathcal{J}), \quad n \geq 2.$$

Moreover, the coefficients $A_n(\sigma_0)$ can be expressed explicitly in terms of Hermite polynomials.

Proof. First, we recall the classical relation between the Delta, Gamma and Vega for European options in the Black-Scholes setting

$$\partial_\sigma u^{\mathrm{BS}}(\sigma) = \sigma \mathcal{J}\, u^{\mathrm{BS}}(\sigma). \tag{7.14}$$

Next, using the product rule for derivatives we compute

$$\partial_\sigma^{n+1} u^{\mathrm{BS}} = \partial_\sigma^n\left(\partial_\sigma u^{\mathrm{BS}}\right) = \partial_\sigma^n\left(\sigma \mathcal{J}\, u^{\mathrm{BS}}\right) = \sum_{h=0}^{n} \binom{n}{h} \left(\partial_\sigma^h \sigma\right)\left(\mathcal{J}\, \partial_\sigma^{n-h} u^{\mathrm{BS}}\right) = \left(\sigma \mathcal{J}\, \partial_\sigma^n + n\, \partial_\sigma^{n-1} \mathcal{J}\right) u^{\mathrm{BS}}. \tag{7.15}$$

Equation (7.13) follows from (7.7) and (7.15). Now, to show that each of the $A_n(\sigma)$ can be expressed as a sum of Hermite polynomials, we observe that

$$\frac{\partial_x^n \exp\left(-\left(\frac{x-a}{b}\right)^2\right)}{\exp\left(-\left(\frac{x-a}{b}\right)^2\right)} = \frac{(-1)^n}{b^n} H_n\left(\frac{x-a}{b}\right), \quad a \in \mathbb{R},\ b > 0, \tag{7.16}$$

where H_n is the n-th Hermite polynomial, defined as usual:

$$H_n(x) := (-1)^n \frac{\partial_x^n \exp\left(-x^2\right)}{\exp\left(-x^2\right)}.$$

Moreover using the Black&Scholes formula for call options (1.20), a direct computation shows

$$\mathcal{J}\, u^{\mathrm{BS}}(\sigma) = \frac{e^k \sqrt{t}}{\sigma \sqrt{2\pi}} \exp\left(-\left(\frac{x - k - \sigma^2 t/2}{\sigma \sqrt{2t}}\right)^2\right). \tag{7.17}$$

Thus, using (7.14) and (7.17) we obtain

$$\frac{\mathcal{J}^{n+1}\, u^{\mathrm{BS}}(\sigma)}{\partial_\sigma u^{\mathrm{BS}}(\sigma)} = \frac{\mathcal{J}^{n}\, \mathcal{J}\, u^{\mathrm{BS}}(\sigma)}{\sigma \mathcal{J}\, u^{\mathrm{BS}}(\sigma)} = \frac{\mathcal{J}^{n} \exp\left(-\left(\frac{x-k-\sigma^2 t/2}{\sigma\sqrt{2\,t}}\right)^2\right)}{\sigma \exp\left(-\left(\frac{x-k-\sigma^2 t/2}{\sigma\sqrt{2\,t}}\right)^2\right)} =$$

$$\frac{t^n}{\sigma}\sum_{h=0}^{n}\binom{n}{h}(-1)^h\,\frac{\partial_x^{2\,n-h}\exp\left(-\left(\frac{x-k-\sigma^2 t/2}{\sigma\sqrt{2\,t}}\right)^2\right)}{\exp\left(-\left(\frac{x-k-\sigma^2 t/2}{\sigma\sqrt{2\,t}}\right)^2\right)},$$

where, in the last equality, we have used the binomial expansion of $(\partial_{\mathrm{xx}}-\partial_x)^n$. Finally, using (7.16) with $a = k + \frac{\sigma^2 t}{2}$ and $b = \sigma\sqrt{2\,t}$, we obtain for $n \geq 1$

$$\frac{\mathcal{J}^{n}\, u^{\mathrm{BS}}(\sigma)}{\partial_\sigma u^{\mathrm{BS}}(\sigma)} = \sum_{h=0}^{n-1}\binom{n-1}{h}\frac{t^{\frac{h}{2}}}{\sigma\left(\sigma\sqrt{2}\right)^{2\,(n-1)-h}}\,H_{2\,(n-1)-h}\left(\frac{x-k-\sigma^2\,t/2}{\sigma\sqrt{2\,t}}\right) \tag{7.18}$$

Combining (7.13) with (7.18), we conclude that $A_n(\sigma)$ can be expressed as a sum of Hermite polynomials, In particular, computing $A_n(\sigma)$ does not involve any special functions or integration.

□

Below using (7.5) and Proposition (9), we provide explicit expressions for σ_n for $n \leq 3$. For simplicity, we remove the argument σ_0 from $U_n(\sigma_0)$. We have

$$\begin{aligned}
\dot{}_1 &= U_1,\\
\dot{}_2 &= U_2 - \tfrac{1}{2}\left(\tfrac{(k-x)^2}{t\,\sigma_0^3} - \tfrac{t\,\sigma_0}{4}\right)U_1^2,\\
\dot{}_3 &= U_3 + \tfrac{1}{48}\left(2\,t\,U_1^3 + t^2\,\sigma_0^2\,U_1^3 + 12\,t\,\sigma_0\,U_1\,U_2\right)\\
&\quad + \tfrac{1}{6\,t\,\sigma_0^4}\left(3\,U_1^3 - t\,\sigma_0^2\,U_1^3 - 6\,\sigma_0\,U_1\,U_2\right)(k-x)^2 + \tfrac{1}{3\,t^2\,\sigma_0^6}\,U_1^3(k-x)
\end{aligned}$$

where the $(U_n)_{n\geq 1}$ are as given in (7.6).

7.2 Implied volatility when option prices are given by Theorem (6)

We now consider the specific case where the sequence of (u_n) is as given in Theorem (6). We will show that, in this particular setting, the expansion (7.4) is convergent and approximate implied volatilities can be computed without any numerical integration or special functions.

First of all, we must set to zero the default parameter γ, because in order to verify the equality $u_0 = u^{\mathrm{BS}}(\sigma_0)$, which means that u_0 is a Black-Scholes price, we must cancel γ because in the B&S model there's no default parameter. Now let us make this observation:

Remark (10)

From (6.6), one can easily show that $u_0 = u^{\mathrm{BS}}\left(\sqrt{\boldsymbol{C}_{1,1}(t, T)}\right)$, where $\boldsymbol{C}(t, T)$ was already defined in (6.8). Then, our expansion for the price of a European call option (6.2) in the general local-stochastic volatility setting (3.1) becomes

$$u = u^{\mathrm{BS}}(\sigma_0) + \sum_{n=1}^{\infty} u_n, \qquad \sigma_0 = \sqrt{\boldsymbol{C}_{1,1}(t, T)} \tag{7.19}$$

From (7.19), it is clear that our option price expansion is of the form (7.2). Therefore, we can use Theorem (8) to find approximate implied volatilities.

Note that, in general, computing approximate implied volatilities using Theorem (8) requires numerical integration, as U_n appearing on the right-hand side of (7.5) contains u_n, which usually must be computed as a numerical integral. However, as the following Proposition shows, when the sequence of (u_n) are as given in Theorem (6), the sequence of (U_n) appearing in (7.5) can be computed explicitly, with no numerical integration and no special functions.

Proposition (11)

Let the sequence of (u_n) be as given in Theorem (6). Then U_n, defined in (7.6), are given by

$$U_n(\sigma_0) = \sum_{h=0}^{N^{(n)}} D_h^{(n)}\, H_h\left(\frac{x - k - \sigma_0^2\, t/2}{\sigma_0 \sqrt{2\, t}}\right)$$

where $\sigma_0 = \sqrt{\boldsymbol{C}_{1,1}(t, T)}$, the sequence of coeffcients $\left(D_h^{(n)}\right)$ are (t,x,y)-dependent constants, and each $N^{(n)}$ $(n \in \mathbb{N})$ is a finite positive integer.

Proof. From Theorem (6), one can deduce that every u_n is of the form

$$u_n = \sum_{h=0}^{N^{(n)}} C_h^{(n)}\, \partial_x^h\left(\partial_x^2 - \partial_x\right) u^{\mathrm{BS}}(\sigma_0), \qquad \sigma_0 = \sqrt{\boldsymbol{C}_{1,1}(t, T)}\,, \tag{7.20}$$

where the sequence of $\left(C_h^{(n)}\right)$ are (t, x, y)-dependent constants and $N^{(n)}$ is a finite positive integer for every n. Both the sequence of coeffcients $\left(C_h^{(n)}\right)$ and the limit of the sum $N^{(n)}$ depend on the choice of the expansion basis and can be computed explicitly using (6.9). However (and we shall emphasize the following) independent of the choice of basis function, the general form (7.20) always holds; this is due to the fact that in every basis choice, the first base is always equal to 1. Now, using (7.20) we compute

$$U_n(\sigma_0) = \sum_{h=0}^{N^{(n)}} C_h^{(n)} \frac{\partial_x^h \left(\partial_x^2 - \partial_x\right) u^{\mathrm{BS}}(\sigma_0)}{\partial_\sigma u^{\mathrm{BS}}(\sigma_0)}$$

by (7.6),

$$= \sum_{h=0}^{N^{(n)}} C_h^{(n)} \frac{\partial_x^h \mathcal{J}\, u^{\mathrm{BS}}(\sigma_0)}{t\,\sigma_0\, \mathcal{J}\, u^{\mathrm{BS}}(\sigma_0)}$$

by (7.12),

$$= \sum_{h=0}^{N^{(n)}} C_h^{(n)} \frac{\partial_x^h \exp\left(-\left(\frac{x-k-\sigma_0{}^2\, t/2}{\sigma_0\, \sqrt{2\, t}}\right)^2\right)}{t\,\sigma_0 \exp\left(-\left(\frac{x-k-\sigma_0{}^2\, t/2}{\sigma_0\, \sqrt{2\, t}}\right)^2\right)}$$

by (7.17),

$$= \sum_{h=0}^{N^{(n)}} D_h^{(n)}\, H_h\!\left(\frac{x - k - \sigma_0{}^2\, t\,/\,2}{\sigma_0\, \sqrt{2\, t}}\right),$$

by (7.16), where we have absorbed some powers of t and σ_0 into $D_h^{(n)}$.

□

To review, when the sequence of (u_n) is as given in Theorem (6), then using Theorem (8) and Propositions (9) and (11), approximate implied volatilities can be computed as a sum of Hermite polynomials in log-moneyness: $(k - x)$. We emphasize: No numerical integration or special functions are required. Approximate implied volatilities can therefore be computed even more quickly than approximate option prices (which require a normal CDF).

Remark (12)

Proposition (11) holds for any choice of the basis functions. However, for the Taylor expansion basis, condition (7.3) is satisfied for any t small enough. Therefore the expansion (7.4) is convergent for short maturities.

We define the n-th order approximation of implied volatility as

$$\sigma^{(n)} := \sum_{h=0}^{n} \sigma_h. \tag{7.21}$$

CHAPTER 3

Numerical Tests on Implied Volatility

8 A Taylor series approach

In this chapter we consider an LSV model as given in [LPP13a], so we consider a frictionless market, no arbitrage, interest rates equal to r and no dividends; here we have an asset S whose risk-neutral dynamics are given by:

$$S_t = \mathbb{1}_{\{\zeta > t\}}\, e^{X_t}$$

$$dX_t = \mu(t, X_t, Y_t)\, dt + \sigma(t, X_t, Y_t)\, dW_t, \quad X_0 = x \in \mathbb{R},$$

$$dY_t = \alpha(t, X_t, Y_t)\, dt + \beta(t, X_t, Y_t)\, dB_t, \quad Y_0 = y \in \mathbb{R},$$

$$d\langle W, B\rangle_t = \rho(X_t, Y_t)\, dt \qquad |\rho| < 1,$$

where ζ is a stopping time which represents a possible default event

$$\zeta = \inf\left\{t \geq 0 : \int_0^t \gamma(s, X_s, Y_t)\, ds \geq \epsilon\right\},$$

with ϵ exponentially distributed and independent of X. As we already said, the stopping time ζ could represent the default time of an asset, the arrival of an economic shock, etc.

As we know, the asset price S must be a martingale, and in this case the martingale condition becomes:

$$\mu(t, x, y) = r - \frac{1}{2}\sigma^2(t, x, y) + \gamma(t, x, y)$$

This model virtually includes all local volatility models, all one-factor stochastic volatility models, and all one-factor local-stochastic volatility models.

In this case, the operator $\mathcal{A}$ becomes:

$$\mathcal{A} = a(t, x, y)\left(\partial_x^2 - \partial_x\right) +$$
$$\alpha(t, x, y)\,\partial_y + b(t, x, y)\,\partial_y^2 + c(t, x, y)\,\partial_{x,y} + (\gamma(t, x, y) + r)\,(\partial_x - 1),$$

and where the functions a, b, c are defined as

$$a(t, x, y) = \frac{1}{2}\,\sigma^2(t, x, y), \quad b(t, x, y) = \frac{1}{2}\,\beta^2(t, x, y),$$
$$c(t, x, y) = \rho(t, x, y)\,\sigma(t, x, y)\,\beta(t, x, y),$$

and finally we consider the Taylor series around fixed point $(\overline{x}, \overline{y})$ as the expansion basis of our approximation.

We use results of previous chapter to compute approximate model-induces implied volatilities under four different model dynamics in which European option prices can be computed explicitly:

- CEV local volatility model
- Quadratic local volatility model
- Heston stochastic volatility model
- 3/2 stochastic volatility model

For these models we consider zero interest rates ($r = 0$) and a log-spot x equal to zero, even if they compare in the formulas presented: this because formulas proposed are for generic models and generic parameters.

8.1 CEV local volatility model

In the Constant Elasticity Variance (CEV) local volatility model of Cox ([Cox75]), the dynamics of the underlying S are given by

$$dS_t = \delta S_t^{\beta-1}\,S_t\,dW_t, \qquad S_0 = s > 0$$

The parameter β controls the relationship between volatility and price. When $\beta < 1$, volatility increases as $S \to 0^+$. This feature, referred to as the leverage effect, is commonly observed in equity markets. When $\beta < 1$, one also observes a negative at-the-money skew in the model-induced implied volatility surface. Like the leverage effect, a negative at-the-money skew is commonly observed in equity options markets. The origin is attainable when $\beta < 1$. In order to prevent the process S from taking negative values, one typically specifies zero as an absorbing boundary. Hence, the state space of S is $[0, \infty)$. In log notation $X := \operatorname{Log} S$, we have the following dynamics

$$dX_t = -\frac{1}{2}\,\delta^2\,e^{2(\beta-1)X_t}\,dt + \delta e^{(\beta-1)X_t}\,dW_t, \quad X_0 = x := \log s.$$

The generator of X is given by

$$\mathcal{A} = \frac{1}{2}\,\delta^2\,e^{2(\beta-1)x}\left(\partial_x^2 - \partial_x\right).$$

Thus we have

$$a(x, y) = \frac{1}{2}\,\delta^2\,e^{2(\beta-1)x}, \quad b(x, y) = 0, \quad c(x, y) = 0, \quad \alpha(x, y) = 0.$$

In this case we have computed our approximation till 8th order of the Taylor series considering generic value of $\bar{x}$, but the obtained formulas are too long to be written down, so we present here only the first ones:

$$\sigma_0 = e^{(-1+\beta)\,\bar{x}}\,\delta$$

$$\sigma_1 = -\frac{1}{2}\,e^{(-1+\beta)\,\bar{x}}\,(-1+\beta)\,\delta\,(-k + x - 2\,(x - \bar{x}))$$

$$\begin{aligned}\sigma_2 = -\frac{1}{96}\,e^{(-1+\beta)\,\bar{x}}\,(-1+\beta)^2\,\delta\,\Big(&-4\,r^2\,t^2 + 8\,r\,t\,(-k+x) - \\ &8\,(-k+x)^2 - 4\,e^{2\,(-1+\beta)\,\bar{x}}\,t\,\delta^2 + e^{4\,(-1+\beta)\,\bar{x}}\,t^2\,\delta^4 + 48\,(-k+x)\,(x-\bar{x}) - 48\,(x-\bar{x})^2\Big)\end{aligned}$$

$$\begin{aligned}\sigma_3 = &\frac{1}{192}\,e^{(-1+\beta)\,\bar{x}}\,(-1+\beta)^3\,\delta\,(-k + x - 2\,(x-\bar{x})) \\ &\Big(-4\,r^2\,t^2 + 8\,r\,t\,(-k+x) - 12\,e^{2\,(-1+\beta)\,\bar{x}}\,t\,\delta^2 + 5\,e^{4\,(-1+\beta)\,\bar{x}}\,t^2\,\delta^4 + 16\,(-k+x)\,(x-\bar{x}) - 16\,(x-\bar{x})^2\Big)\end{aligned}$$

$$\begin{aligned}\sigma_4 = \frac{1}{92\,160}\,e^{(-1+\beta)\,\bar{x}}\,(-1+\beta)^4\,\delta\,\Big(&-144\,r^4\,t^4 - 192\,r^3\,t^3\,(-k+x) - 128\,(-k+x)^4 + \\ &1296\,e^{4\,(-1+\beta)\,\bar{x}}\,t^2\,\delta^4 - 928\,e^{6\,(-1+\beta)\,\bar{x}}\,t^3\,\delta^6 + 67\,e^{8\,(-1+\beta)\,\bar{x}}\,t^4\,\delta^8 + 16\,r\,t\,(-k+x) \\ &\Big(-32\,(-k+x)^2 + e^{2\,(-1+\beta)\,\bar{x}}\,t\,\delta^2\,\Big(-64 + 21\,e^{2\,(-1+\beta)\,\bar{x}}\,t\,\delta^2\Big) + 240\,(-k+x)\,(x-\bar{x}) - 240\,(x-\bar{x})^2\Big) - 8\,r^2 \\ &t^2\,\Big(-104\,(-k+x)^2 + e^{2\,(-1+\beta)\,\bar{x}}\,t\,\delta^2\,\Big(-56 + 29\,e^{2\,(-1+\beta)\,\bar{x}}\,t\,\delta^2\Big) + 240\,(-k+x)\,(x-\bar{x}) - 240\,(x-\bar{x})^2\Big) + \\ &16\,(-k+x)^2\,\Big(e^{2\,(-1+\beta)\,\bar{x}}\,t\,\delta^2\,\Big(238 - 177\,e^{2\,(-1+\beta)\,\bar{x}}\,t\,\delta^2\Big) + 240\,(x-\bar{x})^2\Big) - \\ &480\,(-k+x)\,\Big(e^{2\,(-1+\beta)\,\bar{x}}\,t\,\delta^2\,\Big(36 - 25\,e^{2\,(-1+\beta)\,\bar{x}}\,t\,\delta^2\Big) + 16\,(x-\bar{x})^2\Big)\,(x-\bar{x}) + \\ &17\,280\,e^{2\,(-1+\beta)\,\bar{x}}\,t\,\delta^2\,(x-\bar{x})^2 - 12\,000\,e^{4\,(-1+\beta)\,\bar{x}}\,t^2\,\delta^4\,(x-\bar{x})^2 + 3840\,(x-\bar{x})^4\Big)\end{aligned}$$

$$\begin{aligned}\sigma_5 = &\\ &-\frac{1}{184\,320}\,e^{(-1+\beta)\,\bar{x}}\,(-1+\beta)^5\,\delta\,(-k + x - 2\,(x-\bar{x}))\,\Big(-144\,r^4\,t^4 - 192\,r^3\,t^3\,(-k+x) + 6480\,e^{4\,(-1+\beta)\,\bar{x}}\,t^2\,\delta^4 - \\ &6496\,e^{6\,(-1+\beta)\,\bar{x}}\,t^3\,\delta^6 + 603\,e^{8\,(-1+\beta)\,\bar{x}}\,t^4\,\delta^8 + 16\,r\,t\,(-k+x) \\ &\Big(8\,(-k+x)^2 + 3\,e^{2\,(-1+\beta)\,\bar{x}}\,t\,\delta^2\,\Big(-64 + 35\,e^{2\,(-1+\beta)\,\bar{x}}\,t\,\delta^2\Big) + 80\,(-k+x)\,(x-\bar{x}) - 80\,(x-\bar{x})^2\Big) - 8\,r^2 \\ &t^2\,\Big(-64\,(-k+x)^2 + e^{2\,(-1+\beta)\,\bar{x}}\,t\,\delta^2\,\Big(-168 + 145\,e^{2\,(-1+\beta)\,\bar{x}}\,t\,\delta^2\Big) + 80\,(-k+x)\,(x-\bar{x}) - 80\,(x-\bar{x})^2\Big) +\end{aligned}$$

$$32\,(-k+x)^2\left(e^{2(-1+\beta)\overline{x}}\,t\,\delta^2\left(87-130\,e^{2(-1+\beta)\overline{x}}\,t\,\delta^2\right)+16\,(x-\overline{x})^2\right)+256\,(-k+x)^3\,(x-\overline{x})-$$
$$32\,(-k+x)\left(5\,e^{2(-1+\beta)\overline{x}}\,t\,\delta^2\left(108-125\,e^{2(-1+\beta)\overline{x}}\,t\,\delta^2\right)+48\,(x-\overline{x})^2\right)(x-\overline{x})+$$
$$17280\,e^{2(-1+\beta)\overline{x}}\,t\,\delta^2\,(x-\overline{x})^2-20000\,e^{4(-1+\beta)\overline{x}}\,t^2\,\delta^4\,(x-\overline{x})^2+768\,(x-\overline{x})^4\Big)$$

Considering instead $\overline{x} = x$, which means that we are expanding around initial value, the computational cost decreases, and as a matter of fact in this case we have reached the 9th order of our approximation. As we shall see, these calculations are only useful for a numerical analysis of our method, because the degree of accuracy of the results is high enough already from the fourth or fifth order, except for very long maturities; thus, in a practical use of our method it is necessary to perform the calculations only for the first five orders of approximation.

As proposed by Bompis in ([Bom13]), for the numerical experiments we choose a spot value S_0 equal to 1 (so $x = 0$), null interest rates ($r = 0$) and we test two values of δ: firstly we set $\delta = 0.25$ and we consider either $\beta = 0.8$ (a priori close to the log-normal case) or $\beta = 0.2$ (a priori close to the normal case). Then we investigate the case of a larger volatility with $\delta = 0.4$ and $\beta = 0.5$.

First of all we consider a vector of times to maturity t:

3M	5M	Y	1.5Y	2Y	3Y	5Y	10Y
0.25	0.5	1	1.5	2	3	5	10

Next we create a table of strikes expressed in basis points (bp), in such a way that the range of prices is realistic, which means that both call-prices for very high strikes and put-prices for very low strikes should not be less than 10^{-2}. To create this table, we numerically compute call and put prices using Cox's method ([Cox75]). So every value of this table is of the form BP = 100 e^k, where k is the log-strike as in our formulas.

8.1.1 First Set: δ = 0.25, β = 0.8

For the first set of parameters ($\delta = 0.25$, $\beta = 0.8$) and referring to European call option, we have a table in which columns 1 and 2 represent a far in-the-money (FITM) situation, columns 3,4 and 5 represent an in-the-money (ITM) situation, columns 6,7 and 8 represent an at-the-money (ATM) situation, columns 9,10,11 represent an out-of-the-money (OTM) situation, and finally columns 12 and 13 represent a far out-of-the-money (FOTM) situation.

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.25	88	90	92	94	96	98	100	102	105	107	109	112	114
0.5	81	84	87	90	94	97	100	104	108	112	116	120	124
1	72	77	81	86	91	95	100	107	114	120	127	134	141
1.5	65	71	77	82	88	94	100	109	119	128	137	147	156
2	59	66	73	80	86	93	100	112	124	136	147	159	171
3	50	58	67	75	83	92	100	117	133	150	167	183	200
5	39	49	59	70	80	90	100	126	152	178	204	230	256
10	24	37	49	62	75	87	100	151	202	252	303	354	405

In what follows, for problems of layout, we will present only incomplete matrices, indicating which columns we consider; here we have respectively call and put prices (numerically computed) using the corresponding elements of the previous table:

	1	3	5	7	9	11	13
0.25	0.1297	0.09841	0.07161	0.04984	0.02969	0.01861	0.009726
0.5	0.1997	0.1509	0.1029	0.07044	0.03956	0.02050	0.009857
1	0.2905	0.2163	0.1475	0.09949	0.04944	0.02373	0.009953
1.5	0.3605	0.2611	0.1855	0.1217	0.05703	0.02554	0.01019
2	0.4200	0.3032	0.2139	0.1403	0.06224	0.02627	0.01004
3	0.5096	0.3673	0.2585	0.1715	0.07114	0.02668	0.009876
5	0.6200	0.4547	0.3173	0.2202	0.08046	0.02841	0.009996
10	0.7703	0.5731	0.4176	0.3076	0.09120	0.02939	0.01004

	1	3	5	7	9	11	13
0.25	0.009702	0.01841	0.03161	0.04984	0.07969	0.1086	0.1497
0.5	0.009720	0.02088	0.04285	0.07044	0.1196	0.1805	0.2499
1	0.01054	0.02629	0.05748	0.09949	0.1894	0.2937	0.4200
1.5	0.01049	0.03112	0.06546	0.1217	0.2470	0.3955	0.5702
2	0.01001	0.03318	0.07385	0.1403	0.3022	0.4963	0.7200
3	0.009552	0.03726	0.08854	0.1715	0.4011	0.6967	1.010
5	0.01003	0.04466	0.1173	0.2202	0.6005	1.068	1.570
10	0.01034	0.06306	0.1676	0.3076	1.111	2.059	3.060

We numerically compute the corresponding table of exact implied volatilities in %

	1	3	5	7	9	11	13
0.25	25.3216	25.2097	25.1029	25.0006	24.8789	24.7858	24.6745
0.5	25.5319	25.3511	25.1563	25.0013	24.8094	24.6320	24.4673
1	25.8331	25.5332	25.2392	25.0026	24.6763	24.4096	24.1532
1.5	26.0968	25.6632	25.3249	25.0038	24.5713	24.2247	23.9081
2	26.3482	25.8006	25.3842	25.0051	24.4708	24.0537	23.6871
3	26.7820	26.0229	25.4766	25.0075	24.3007	23.7463	23.3132
5	27.4435	26.3562	25.5748	25.0120	23.9785	23.2698	22.7327
10	28.7688	26.8523	25.7499	25.0221	23.3017	22.3472	21.6810

If we consider $\bar{x} = x$, we can compute our approximated implied volatily up to 9th order. In the following matrix, every element is a vector of elements corresponding to our implied volatility computed at n-th order for $n = 0, \ ..., 9$.

	1	3	5	7	9	11	13
0.25	25. 25.3196 25.3216 25.3216 25.3216 25.3216 25.3216 25.3216 25.3216 25.3216	25. 25.2085 25.2097 25.2097 25.2097 25.2097 25.2097 25.2097 25.2097 25.2097	25. 25.1021 25.1028 25.1029 25.1029 25.1029 25.1029 25.1029 25.1029 25.1029	25. 25. 25.0006 25.0006 25.0006 25.0006 25.0006 25.0006 25.0006 25.0006	25. 24.878 24.8789 24.8789 24.8789 24.8789 24.8789 24.8789 24.8789 24.8789	25. 24.7846 24.7858 24.7858 24.7858 24.7858 24.7858 24.7858 24.7858 24.7858	25. 24.6724 24.6745 24.6745 24.6745 24.6745 24.6745 24.6745 24.6745 24.6745

0.5	25.	25.	25.	25.	25.	25.	25.
	25.5268	25.3482	25.1547	25.	24.8076	24.6289	24.4622
	25.5318	25.3511	25.1563	25.0013	24.8094	24.6321	24.4674
	25.5319	25.3511	25.1563	25.0013	24.8094	24.632	24.4673
	25.5319	25.3511	25.1563	25.0013	24.8094	24.632	24.4673
	25.5319	25.3511	25.1563	25.0013	24.8094	24.632	24.4673
	25.5319	25.3511	25.1563	25.0013	24.8094	24.632	24.4673
	25.5319	25.3511	25.1563	25.0013	24.8094	24.632	24.4673
	25.5319	25.3511	25.1563	25.0013	24.8094	24.632	24.4673
	25.5319	25.3511	25.1563	25.0013	24.8094	24.632	24.4673
1	25.	25.	25.	25.	25.	25.	25.
	25.8213	25.5268	25.2358	25.	24.6724	24.4025	24.141
	25.8328	25.5331	25.2391	25.0026	24.6764	24.4098	24.1534
	25.8331	25.5332	25.2392	25.0026	24.6763	24.4096	24.1532
	25.8331	25.5332	25.2392	25.0026	24.6763	24.4096	24.1532
	25.8331	25.5332	25.2392	25.0026	24.6763	24.4096	24.1532
	25.8331	25.5332	25.2392	25.0026	24.6763	24.4096	24.1532
	25.8331	25.5332	25.2392	25.0026	24.6763	24.4096	24.1532
	25.8331	25.5332	25.2392	25.0026	24.6763	24.4096	24.1532
	25.8331	25.5332	25.2392	25.0026	24.6763	24.4096	24.1532
1.5	25.	25.	25.	25.	25.	25.	25.
	26.077	25.6534	25.3196	25.	24.5651	24.213	23.8883
	26.0962	25.6629	25.3248	25.0038	24.5715	24.225	23.9086
	26.0967	25.6632	25.3249	25.0038	24.5713	24.2247	23.9081
	26.0968	25.6632	25.3249	25.0038	24.5713	24.2247	23.9081
	26.0968	25.6632	25.3249	25.0038	24.5713	24.2247	23.9081
	26.0968	25.6632	25.3249	25.0038	24.5713	24.2247	23.9081
	26.0968	25.6632	25.3249	25.0038	24.5713	24.2247	23.9081
	26.0968	25.6632	25.3249	25.0038	24.5713	24.2247	23.9081
	26.0968	25.6632	25.3249	25.0038	24.5713	24.2247	23.9081
2	25.	25.	25.	25.	25.	25.	25.
	26.3191	25.7868	25.3771	25.	24.4622	24.0368	23.6588
	26.3473	25.8001	25.384	25.005	24.4711	24.0543	23.6878
	26.3481	25.8005	25.3842	25.005	24.4708	24.0537	23.687
	26.3482	25.8006	25.3842	25.0051	24.4708	24.0537	23.6871
	26.3482	25.8006	25.3842	25.0051	24.4708	24.0537	23.6871
	26.3482	25.8006	25.3842	25.0051	24.4708	24.0537	23.6871
	26.3482	25.8006	25.3842	25.0051	24.4708	24.0537	23.6871
	26.3482	25.8006	25.3842	25.0051	24.4708	24.0537	23.6871
	26.3482	25.8006	25.3842	25.0051	24.4708	24.0537	23.6871
3	25.	25.	25.	25.	25.	25.	25.
	26.7329	26.0012	25.4658	25.	24.2871	23.7179	23.2671
	26.7804	26.022	25.4762	25.0074	24.3013	23.7473	23.3146
	26.7818	26.0229	25.4766	25.0074	24.3007	23.7462	23.3131
	26.782	26.0229	25.4766	25.0075	24.3007	23.7463	23.3133
	26.782	26.0229	25.4766	25.0075	24.3007	23.7463	23.3132
	26.782	26.0229	25.4766	25.0075	24.3007	23.7463	23.3132
	26.782	26.0229	25.4766	25.0075	24.3007	23.7463	23.3132
	26.782	26.0229	25.4766	25.0075	24.3007	23.7463	23.3132
	26.782	26.0229	25.4766	25.0075	24.3007	23.7463	23.3132
5	25.	25.	25.	25.	25.	25.	25.
	27.354	26.3191	25.5579	25.	23.9532	23.2176	22.65
	27.4399	26.3543	25.574	25.012	23.9798	23.272	22.7356
	27.4431	26.3561	25.5748	25.012	23.9784	23.2696	22.7324
	27.4435	26.3562	25.5748	25.012	23.9785	23.2698	22.7328
	27.4435	26.3562	25.5748	25.012	23.9785	23.2698	22.7327
	27.4435	26.3562	25.5748	25.012	23.9785	23.2698	22.7327
	27.4435	26.3562	25.5748	25.012	23.9785	23.2698	22.7327
	27.4435	26.3562	25.5748	25.012	23.9785	23.2698	22.7327
	27.4435	26.3562	25.5748	25.012	23.9785	23.2698	22.7327

10	25.	25.	25.	25.	25.	25.	25.
	28.5678	26.7834	25.7192	25.	23.2423	22.2286	21.5032
	28.7595	26.8478	25.7481	25.022	23.3054	22.353	21.6882
	28.7677	26.8519	25.7497	25.022	23.3014	22.3466	21.6801
	28.7688	26.8523	25.7499	25.0221	23.3017	22.3473	21.6811
	28.7688	26.8523	25.7499	25.0221	23.3017	22.3472	21.6811
	28.7688	26.8523	25.7499	25.0221	23.3017	22.3472	21.681
	28.7688	26.8523	25.7499	25.0221	23.3017	22.3472	21.681
	28.7688	26.8523	25.7499	25.0221	23.3017	22.3472	21.681
	28.7688	26.8523	25.7499	25.0221	23.3017	22.3472	21.681

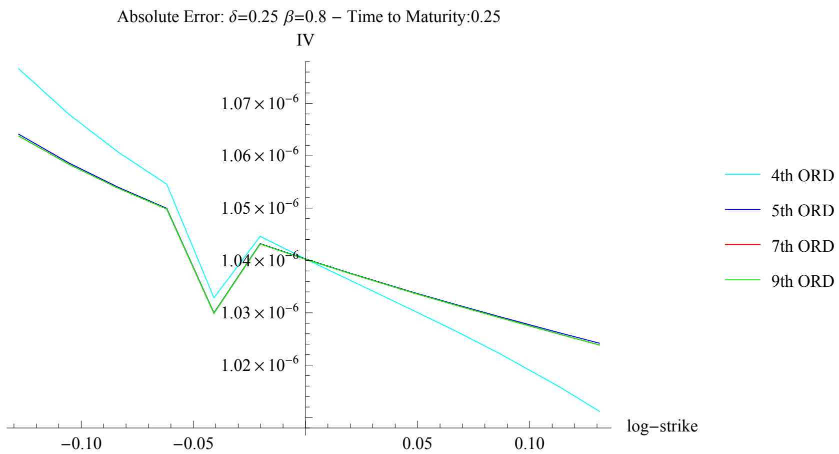
but in order to compare our approximation with the values calculated numerically, we must observe absolute errors between these values (since these absolute errors are a difference between two percentages of the price, they are automatically a percentage error relative to the price).

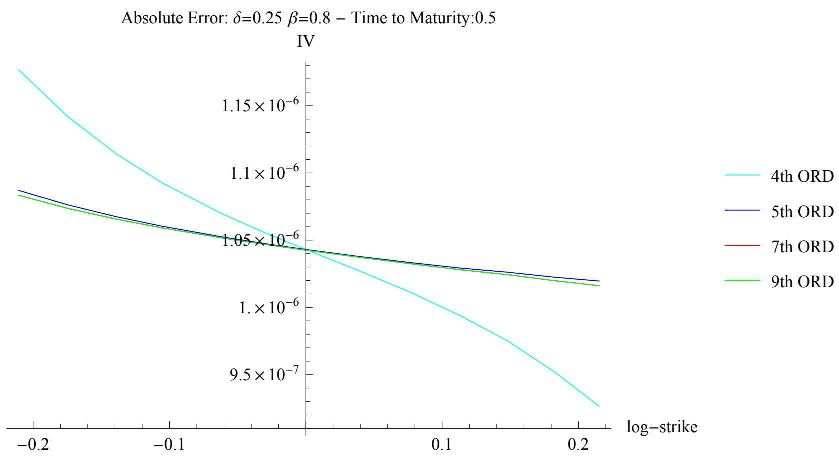
	1	3	5	7	9	11	13
0.25	3.22×10^{-1}	2.1×10^{-1}	1.03×10^{-1}	6.5×10^{-4}	1.21×10^{-1}	2.14×10^{-1}	3.26×10^{-1}
	2.04×10^{-3}	1.25×10^{-3}	7.97×10^{-4}	6.5×10^{-4}	8.39×10^{-4}	1.25×10^{-3}	2.06×10^{-3}
	2.64×10^{-5}	1.76×10^{-5}	9.13×10^{-6}	1.18×10^{-6}	8.24×10^{-6}	1.54×10^{-5}	2.39×10^{-5}
	1.61×10^{-6}	1.37×10^{-6}	1.21×10^{-6}	1.18×10^{-6}	1.23×10^{-6}	1.34×10^{-6}	1.57×10^{-6}
	1.08×10^{-6}	1.06×10^{-6}	1.03×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	1.01×10^{-6}
	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.03×10^{-6}	1.02×10^{-6}
	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.03×10^{-6}	1.02×10^{-6}
	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.03×10^{-6}	1.02×10^{-6}
	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.03×10^{-6}	1.02×10^{-6}
	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.03×10^{-6}	1.02×10^{-6}
0.5	5.32×10^{-1}	3.51×10^{-1}	1.56×10^{-1}	1.29×10^{-3}	1.91×10^{-1}	3.68×10^{-1}	5.33×10^{-1}
	5.08×10^{-3}	2.96×10^{-3}	1.64×10^{-3}	1.29×10^{-3}	1.76×10^{-3}	3.07×10^{-3}	5.07×10^{-3}
	8.51×10^{-5}	5.63×10^{-5}	2.57×10^{-5}	1.58×10^{-6}	2.78×10^{-5}	5.46×10^{-5}	7.93×10^{-5}
	3.84×10^{-6}	2.61×10^{-6}	1.8×10^{-6}	1.58×10^{-6}	1.85×10^{-6}	2.6×10^{-6}	3.68×10^{-6}
	1.18×10^{-6}	1.11×10^{-6}	1.07×10^{-6}	1.04×10^{-6}	1.01×10^{-6}	9.75×10^{-7}	9.27×10^{-7}
	1.09×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.03×10^{-6}	1.02×10^{-6}
	1.08×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.08×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.08×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.08×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
1	8.33×10^{-1}	5.33×10^{-1}	2.39×10^{-1}	2.57×10^{-3}	3.24×10^{-1}	5.9×10^{-1}	8.47×10^{-1}
	1.18×10^{-2}	6.43×10^{-3}	3.38×10^{-3}	2.57×10^{-3}	3.9×10^{-3}	7.15×10^{-3}	1.21×10^{-2}
	2.64×10^{-4}	1.68×10^{-4}	7.59×10^{-5}	3.15×10^{-6}	9.5×10^{-5}	1.74×10^{-4}	2.58×10^{-4}
	1.38×10^{-5}	7.74×10^{-6}	4.14×10^{-6}	3.15×10^{-6}	4.66×10^{-6}	8.24×10^{-6}	3.44×10^{-6}
	1.73×10^{-6}	1.38×10^{-6}	1.17×10^{-6}	1.05×10^{-6}	8.86×10^{-7}	6.93×10^{-7}	9.52×10^{-6}
	1.14×10^{-6}	1.1×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	8.89×10^{-6}
	1.11×10^{-6}	1.08×10^{-6}	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	8.92×10^{-6}
	1.11×10^{-6}	1.08×10^{-6}	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	8.92×10^{-6}
	1.11×10^{-6}	1.08×10^{-6}	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	8.92×10^{-6}
	1.11×10^{-6}	1.08×10^{-6}	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	8.92×10^{-6}

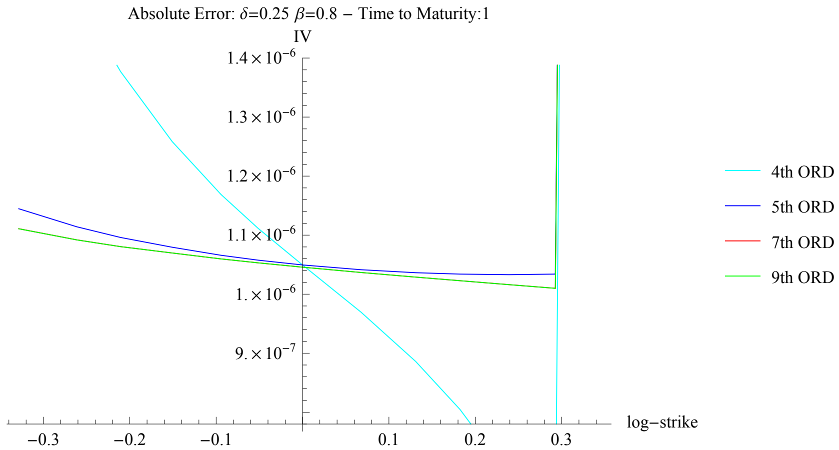
	1.1	6.63×10^{-1}	3.25×10^{-1}	3.82×10^{-3}	4.29×10^{-1}	7.75×10^{-1}	1.09
	1.98×10^{-2}	9.82×10^{-3}	5.33×10^{-3}	3.82×10^{-3}	6.15×10^{-3}	1.17×10^{-2}	1.98×10^{-2}
	5.18×10^{-4}	3.1×10^{-4}	1.52×10^{-4}	5.67×10^{-6}	1.86×10^{-4}	3.36×10^{-4}	4.71×10^{-4}
	3.25×10^{-5}	1.61×10^{-5}	8.35×10^{-6}	5.67×10^{-6}	9.54×10^{-6}	1.84×10^{-5}	3.02×10^{-5}
1.5	$3.\times10^{-6}$	1.9×10^{-6}	1.4×10^{-6}	1.06×10^{-6}	6.11×10^{-7}	6.03×10^{-8}	7.44×10^{-7}
	1.26×10^{-6}	1.14×10^{-6}	1.09×10^{-6}	1.06×10^{-6}	1.05×10^{-6}	1.07×10^{-6}	1.11×10^{-6}
	1.14×10^{-6}	1.09×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.02×10^{-6}	$1.\times10^{-6}$	9.86×10^{-7}
	1.14×10^{-6}	1.09×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.01×10^{-6}	9.96×10^{-7}
	1.13×10^{-6}	1.09×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.01×10^{-6}	9.95×10^{-7}
	1.13×10^{-6}	1.09×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.01×10^{-6}	9.95×10^{-7}
	1.35	8.01×10^{-1}	3.84×10^{-1}	5.05×10^{-3}	5.29×10^{-1}	9.46×10^{-1}	1.31
	2.91×10^{-2}	1.38×10^{-2}	7.17×10^{-3}	5.05×10^{-3}	8.6×10^{-3}	1.69×10^{-2}	2.83×10^{-2}
	8.38×10^{-4}	4.95×10^{-4}	2.31×10^{-4}	9.09×10^{-6}	3.01×10^{-4}	5.37×10^{-4}	7.4×10^{-4}
	5.61×10^{-5}	2.88×10^{-5}	7.41×10^{-6}	9.09×10^{-6}	1.77×10^{-5}	3.36×10^{-5}	5.46×10^{-5}
2	1.94×10^{-7}	2.82×10^{-6}	4.84×10^{-6}	1.08×10^{-6}	1.18×10^{-6}	1.03×10^{-6}	2.7×10^{-6}
	3.64×10^{-6}	1.23×10^{-6}	5.46×10^{-6}	1.08×10^{-6}	2.13×10^{-6}	1.15×10^{-6}	1.26×10^{-6}
	3.93×10^{-6}	1.11×10^{-6}	5.51×10^{-6}	1.05×10^{-6}	2.06×10^{-6}	9.91×10^{-7}	9.62×10^{-7}
	3.96×10^{-6}	1.1×10^{-6}	5.51×10^{-6}	1.05×10^{-6}	2.07×10^{-6}	$1.\times10^{-6}$	9.89×10^{-7}
	3.96×10^{-6}	1.1×10^{-6}	5.51×10^{-6}	1.05×10^{-6}	2.07×10^{-6}	$1.\times10^{-6}$	9.86×10^{-7}
	3.96×10^{-6}	1.1×10^{-6}	5.51×10^{-6}	1.05×10^{-6}	2.07×10^{-6}	$1.\times10^{-6}$	9.86×10^{-7}
	1.78	1.02	4.77×10^{-1}	7.46×10^{-3}	6.99×10^{-1}	1.25	1.69
	4.91×10^{-2}	2.17×10^{-2}	1.08×10^{-2}	7.46×10^{-3}	1.36×10^{-2}	2.83×10^{-2}	4.61×10^{-2}
	1.64×10^{-3}	9.3×10^{-4}	4.32×10^{-4}	1.83×10^{-5}	5.79×10^{-4}	1.03×10^{-3}	1.37×10^{-3}
	1.46×10^{-4}	6.43×10^{-5}	2.92×10^{-5}	1.83×10^{-5}	3.7×10^{-5}	7.86×10^{-5}	1.23×10^{-4}
3	1.33×10^{-5}	5.8×10^{-6}	2.84×10^{-6}	1.14×10^{-6}	1.45×10^{-6}	4.84×10^{-6}	9.17×10^{-6}
	2.28×10^{-6}	1.52×10^{-6}	1.24×10^{-6}	1.14×10^{-6}	1.22×10^{-6}	1.49×10^{-6}	1.85×10^{-6}
	1.31×10^{-6}	1.16×10^{-6}	1.09×10^{-6}	1.05×10^{-6}	9.99×10^{-7}	9.42×10^{-7}	8.82×10^{-7}
	1.21×10^{-6}	1.12×10^{-6}	1.08×10^{-6}	1.05×10^{-6}	1.02×10^{-6}	9.97×10^{-7}	9.82×10^{-7}
	1.2×10^{-6}	1.12×10^{-6}	1.08×10^{-6}	1.05×10^{-6}	1.02×10^{-6}	9.91×10^{-7}	9.71×10^{-7}
	1.19×10^{-6}	1.12×10^{-6}	1.08×10^{-6}	1.05×10^{-6}	1.02×10^{-6}	9.91×10^{-7}	9.72×10^{-7}
	2.44	1.36	5.75×10^{-1}	1.2×10^{-2}	1.02	1.73	2.27
	8.95×10^{-2}	3.72×10^{-2}	1.7×10^{-2}	1.2×10^{-2}	2.53×10^{-2}	5.22×10^{-2}	8.28×10^{-2}
	3.59×10^{-3}	1.96×10^{-3}	8.27×10^{-4}	4.43×10^{-5}	1.32×10^{-3}	2.21×10^{-3}	2.87×10^{-3}
	3.91×10^{-4}	1.65×10^{-4}	6.87×10^{-5}	4.43×10^{-5}	1.03×10^{-4}	2.12×10^{-4}	3.19×10^{-4}
5	3.98×10^{-5}	1.57×10^{-5}	6.13×10^{-6}	1.38×10^{-6}	7.61×10^{-6}	1.81×10^{-5}	3.07×10^{-5}
	5.02×10^{-6}	2.62×10^{-6}	1.64×10^{-6}	1.38×10^{-6}	1.87×10^{-6}	2.87×10^{-6}	3.96×10^{-6}
	1.66×10^{-6}	1.3×10^{-6}	1.14×10^{-6}	1.06×10^{-6}	9.14×10^{-7}	7.62×10^{-7}	6.13×10^{-7}
	1.29×10^{-6}	1.16×10^{-6}	1.09×10^{-6}	1.06×10^{-6}	1.02×10^{-6}	9.97×10^{-7}	9.87×10^{-7}
	1.24×10^{-6}	1.14×10^{-6}	1.09×10^{-6}	1.06×10^{-6}	$1.\times10^{-6}$	9.7×10^{-7}	9.45×10^{-7}
	1.24×10^{-6}	1.14×10^{-6}	1.09×10^{-6}	1.06×10^{-6}	1.01×10^{-6}	9.73×10^{-7}	9.49×10^{-7}
	3.77	1.85	7.5×10^{-1}	2.21×10^{-2}	1.7	2.65	3.32
	2.01×10^{-1}	6.89×10^{-2}	3.07×10^{-2}	2.21×10^{-2}	5.95×10^{-2}	1.19×10^{-1}	1.78×10^{-1}
	9.33×10^{-3}	4.55×10^{-3}	1.85×10^{-3}	1.28×10^{-4}	3.71×10^{-3}	5.74×10^{-3}	7.17×10^{-3}
	1.09×10^{-3}	4.3×10^{-4}	1.87×10^{-4}	1.28×10^{-4}	3.55×10^{-4}	6.67×10^{-4}	9.16×10^{-4}
10	6.57×10^{-5}	3.7×10^{-5}	1.57×10^{-5}	2.26×10^{-6}	3.01×10^{-5}	5.45×10^{-5}	7.86×10^{-5}
	1.3×10^{-5}	3.11×10^{-6}	2.68×10^{-6}	2.26×10^{-6}	3.33×10^{-6}	2.25×10^{-6}	1.97×10^{-6}
	6.07×10^{-6}	$8.\times10^{-7}$	1.16×10^{-6}	1.07×10^{-6}	1.02×10^{-6}	1.99×10^{-6}	4.03×10^{-6}
	1.32×10^{-6}	9.49×10^{-7}	1.08×10^{-6}	1.07×10^{-6}	8.86×10^{-7}	4.47×10^{-7}	3.08×10^{-7}
	4.44×10^{-7}	1.09×10^{-6}	1.09×10^{-6}	1.06×10^{-6}	1.02×10^{-6}	1.11×10^{-6}	1.32×10^{-6}
	1.03×10^{-6}	1.14×10^{-6}	1.1×10^{-6}	1.06×10^{-6}	9.68×10^{-7}	8.83×10^{-7}	7.75×10^{-7}

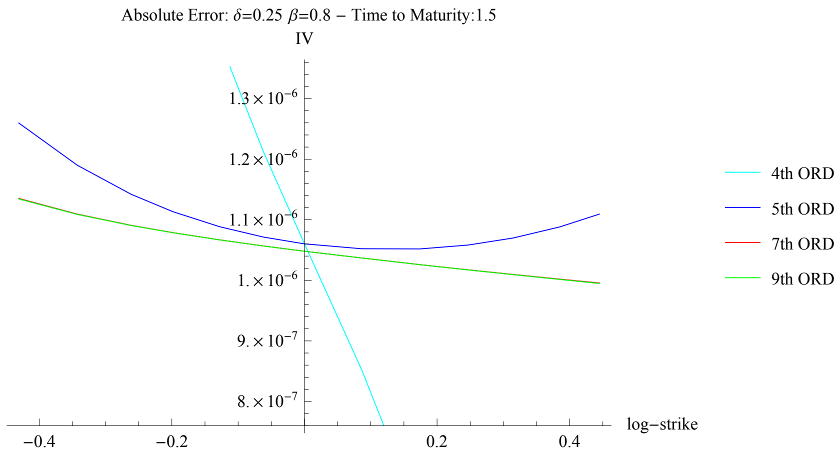
With this set of parameters, the method seems to converge, given that the error continues to decrease in going on with the computation; on the other hand, we note that we can stop at the fourth or fifth order, even for long maturities.

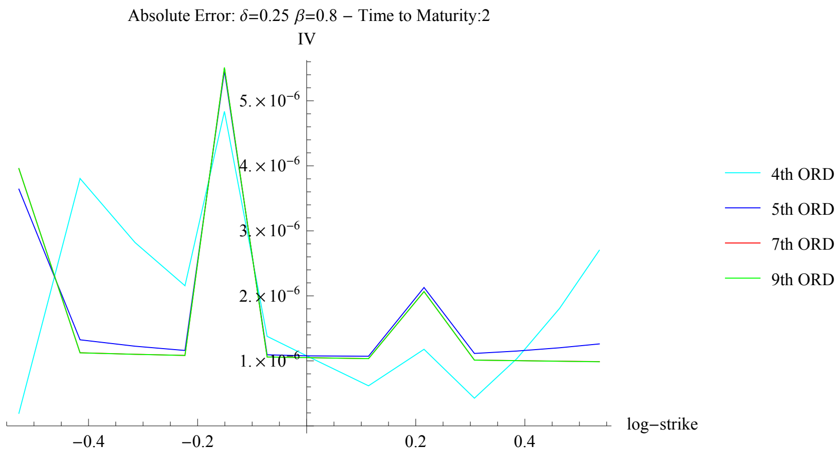
In the following graphs we plotted the absolute errors for some orders of approximation and for each maturity considered, in order to prove a kind of convergence, at least for the first orders: we see is in fact a progressive decrease of the absolute errors, but we also note that the values obtained at the fifth order generally are already quite optimal for a practical use of these formulas; however, another very important fact to notice the behaviour of the fourth order OTM for small maturities: its errors are significantly smaller than those of following orders. Even if this is not generally true for every set of parameters, this is a first clue that our method doesn't converge.

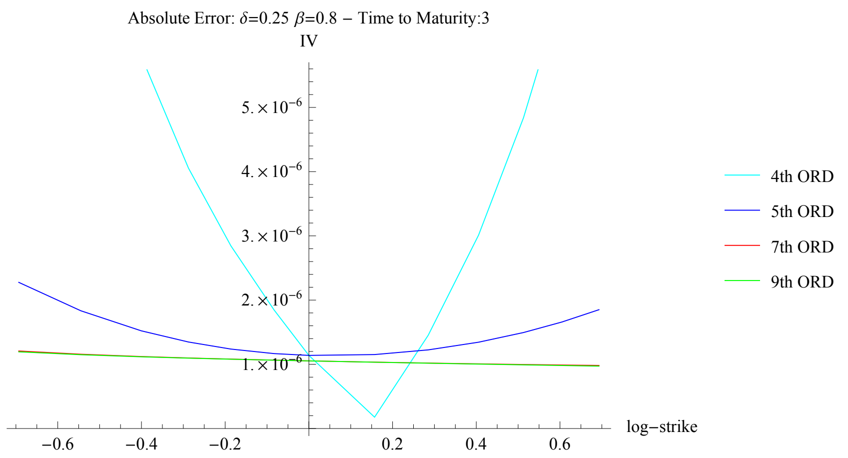


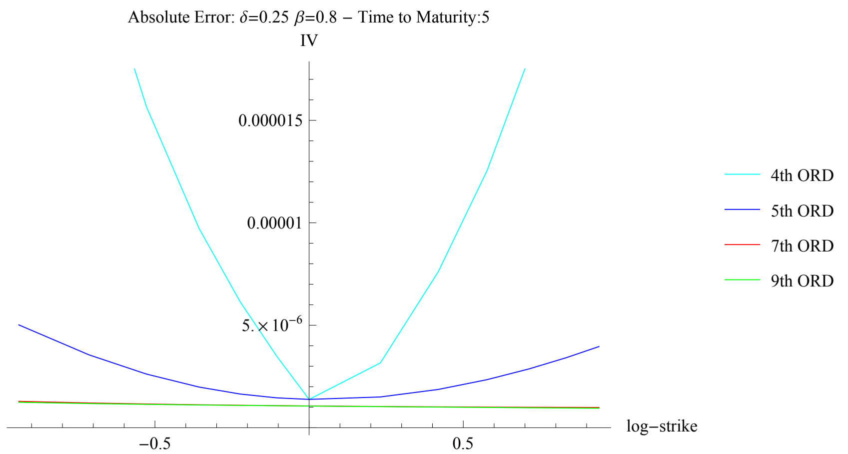


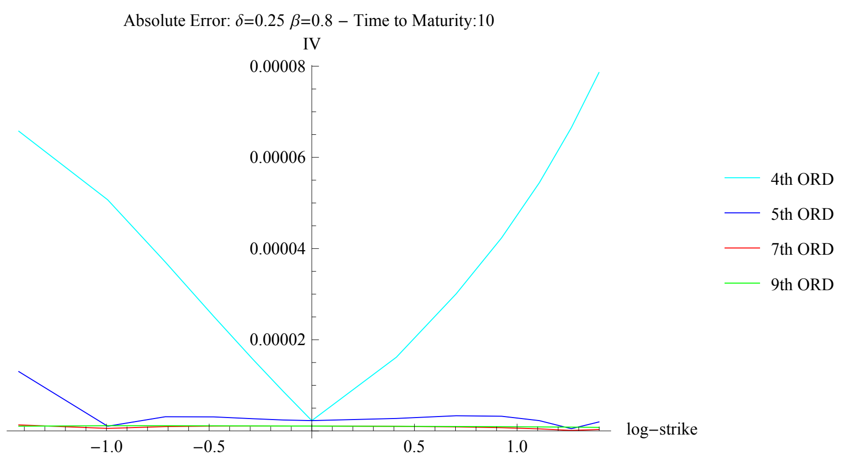

Absolute Error: δ=0.25 β=0.8 − Time to Maturity:1
IV
log−strike
4th ORD
5th ORD
7th ORD
9th ORD


Absolute Error: δ=0.25 β=0.8 − Time to Maturity:1.5
IV
log−strike
4th ORD
5th ORD
7th ORD
9th ORD


Absolute Error: δ=0.25 β=0.8 − Time to Maturity:2
IV
log−strike
4th ORD
5th ORD
7th ORD
9th ORD


Absolute Error: δ=0.25 β=0.8 − Time to Maturity:3
IV
5.×10^−6
4.×10^−6
3.×10^−6
2.×10^−6
1.×10^−6
log−strike
−0.6
−0.4
−0.2
0.2
0.4
0.6
4th ORD
5th ORD
7th ORD
9th ORD


Absolute Error: δ=0.25 β=0.8 − Time to Maturity:5
IV
0.000015
0.00001
5.×10^−6
log−strike
−0.5
0.5
4th ORD
5th ORD
7th ORD
9th ORD


Absolute Error: δ=0.25 β=0.8 − Time to Maturity:10
IV
0.00008
0.00006
0.00004
0.00002
log−strike
−1.0
−0.5
0.5
1.0
4th ORD
5th ORD
7th ORD
9th ORD

Another possible choice of $\overline{x}$ is the middle point between the log-price x and the log-strike k; in this case we obtain very smaller absolute errors since first orders:

	1	3	5	7	9	11	13
	1.47×10^{-5}	3.74×10^{-4}	5.88×10^{-4}	6.5×10^{-4}	5.42×10^{-4}	3.26×10^{-4}	8.14×10^{-5}
	1.47×10^{-5}	3.74×10^{-4}	5.88×10^{-4}	6.5×10^{-4}	5.42×10^{-4}	3.26×10^{-4}	8.14×10^{-5}
	1.16×10^{-6}	1.17×10^{-6}	1.16×10^{-6}	1.18×10^{-6}	1.16×10^{-6}	1.14×10^{-6}	1.11×10^{-6}
	1.16×10^{-6}	1.17×10^{-6}	1.16×10^{-6}	1.18×10^{-6}	1.16×10^{-6}	1.14×10^{-6}	1.11×10^{-6}
0.25	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.03×10^{-6}	1.02×10^{-6}
	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.03×10^{-6}	1.02×10^{-6}
	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.03×10^{-6}	1.02×10^{-6}
	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.03×10^{-6}	1.02×10^{-6}
	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.03×10^{-6}	1.02×10^{-6}
	5.12×10^{-4}	5.29×10^{-4}	1.16×10^{-3}	1.29×10^{-3}	1.02×10^{-3}	3.33×10^{-4}	6.74×10^{-4}
	5.12×10^{-4}	5.29×10^{-4}	1.16×10^{-3}	1.29×10^{-3}	1.02×10^{-3}	3.33×10^{-4}	6.74×10^{-4}
	1.45×10^{-6}	1.52×10^{-6}	1.58×10^{-6}	1.58×10^{-6}	1.51×10^{-6}	1.4×10^{-6}	1.3×10^{-6}
	1.45×10^{-6}	1.52×10^{-6}	1.58×10^{-6}	1.58×10^{-6}	1.51×10^{-6}	1.4×10^{-6}	1.3×10^{-6}
0.5	1.08×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.08×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.08×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.08×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.08×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.82×10^{-3}	8.42×10^{-4}	2.27×10^{-3}	2.57×10^{-3}	1.76×10^{-3}	6.6×10^{-5}	2.45×10^{-3}
	1.82×10^{-3}	8.42×10^{-4}	2.27×10^{-3}	2.57×10^{-3}	1.76×10^{-3}	6.6×10^{-5}	2.45×10^{-3}
	2.55×10^{-6}	2.87×10^{-6}	3.14×10^{-6}	3.15×10^{-6}	2.79×10^{-6}	2.32×10^{-6}	7.94×10^{-6}
	2.55×10^{-6}	2.87×10^{-6}	3.14×10^{-6}	3.15×10^{-6}	2.79×10^{-6}	2.32×10^{-6}	7.94×10^{-6}
1	1.11×10^{-6}	1.08×10^{-6}	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	8.92×10^{-6}
	1.11×10^{-6}	1.08×10^{-6}	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	8.92×10^{-6}
	1.11×10^{-6}	1.08×10^{-6}	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	8.92×10^{-6}
	1.11×10^{-6}	1.08×10^{-6}	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	8.92×10^{-6}
	1.11×10^{-6}	1.08×10^{-6}	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	8.92×10^{-6}
	3.74×10^{-3}	1.2×10^{-3}	3.28×10^{-3}	3.82×10^{-3}	2.39×10^{-3}	5.22×10^{-4}	4.53×10^{-3}
	3.74×10^{-3}	1.2×10^{-3}	3.28×10^{-3}	3.82×10^{-3}	2.39×10^{-3}	5.22×10^{-4}	4.53×10^{-3}
	4.33×10^{-6}	5.08×10^{-6}	5.66×10^{-6}	5.67×10^{-6}	4.74×10^{-6}	3.61×10^{-6}	3.03×10^{-6}
	4.33×10^{-6}	5.08×10^{-6}	5.66×10^{-6}	5.67×10^{-6}	4.74×10^{-6}	3.61×10^{-6}	3.03×10^{-6}
1.5	1.14×10^{-6}	1.1×10^{-6}	1.08×10^{-6}	1.06×10^{-6}	1.04×10^{-6}	1.02×10^{-6}	$1.\times10^{-6}$
	1.14×10^{-6}	1.1×10^{-6}	1.08×10^{-6}	1.06×10^{-6}	1.04×10^{-6}	1.02×10^{-6}	$1.\times10^{-6}$
	1.13×10^{-6}	1.09×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.01×10^{-6}	9.95×10^{-7}
	1.13×10^{-6}	1.09×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.01×10^{-6}	9.95×10^{-7}
	1.13×10^{-6}	1.09×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.01×10^{-6}	9.95×10^{-7}
	6.34×10^{-3}	1.28×10^{-3}	4.31×10^{-3}	5.05×10^{-3}	2.86×10^{-3}	1.44×10^{-3}	7.05×10^{-3}
	6.34×10^{-3}	1.28×10^{-3}	4.31×10^{-3}	5.05×10^{-3}	2.86×10^{-3}	1.44×10^{-3}	7.05×10^{-3}
	1.71×10^{-6}	8.05×10^{-6}	2.52×10^{-6}	9.09×10^{-6}	8.25×10^{-6}	5.17×10^{-6}	4.44×10^{-6}
	1.71×10^{-6}	8.05×10^{-6}	2.52×10^{-6}	9.09×10^{-6}	8.25×10^{-6}	5.17×10^{-6}	4.44×10^{-6}
2	3.93×10^{-6}	1.13×10^{-6}	5.48×10^{-6}	1.08×10^{-6}	2.09×10^{-6}	1.02×10^{-6}	9.97×10^{-7}
	3.93×10^{-6}	1.13×10^{-6}	5.48×10^{-6}	1.08×10^{-6}	2.09×10^{-6}	1.02×10^{-6}	9.97×10^{-7}
	3.96×10^{-6}	1.1×10^{-6}	5.51×10^{-6}	1.05×10^{-6}	2.07×10^{-6}	$1.\times10^{-6}$	9.86×10^{-7}
	3.96×10^{-6}	1.1×10^{-6}	5.51×10^{-6}	1.05×10^{-6}	2.07×10^{-6}	$1.\times10^{-6}$	9.86×10^{-7}
	3.96×10^{-6}	1.1×10^{-6}	5.51×10^{-6}	1.05×10^{-6}	2.07×10^{-6}	$1.\times10^{-6}$	9.86×10^{-7}

	1.23×10^{-2}	1.42×10^{-3}	6.4×10^{-3}	7.46×10^{-3}	3.57×10^{-3}	3.99×10^{-3}	1.26×10^{-2}
	1.23×10^{-2}	1.42×10^{-3}	6.4×10^{-3}	7.46×10^{-3}	3.57×10^{-3}	3.99×10^{-3}	1.26×10^{-2}
	1.41×10^{-5}	1.62×10^{-5}	1.85×10^{-5}	1.83×10^{-5}	1.35×10^{-5}	8.82×10^{-6}	8.45×10^{-6}
	1.41×10^{-5}	1.62×10^{-5}	1.85×10^{-5}	1.83×10^{-5}	1.35×10^{-5}	8.82×10^{-6}	8.45×10^{-6}
3	1.27×10^{-6}	1.21×10^{-6}	1.17×10^{-6}	1.14×10^{-6}	1.08×10^{-6}	1.03×10^{-6}	9.99×10^{-7}
	1.27×10^{-6}	1.21×10^{-6}	1.17×10^{-6}	1.14×10^{-6}	1.08×10^{-6}	1.03×10^{-6}	9.99×10^{-7}
	1.19×10^{-6}	1.12×10^{-6}	1.08×10^{-6}	1.05×10^{-6}	1.02×10^{-6}	9.92×10^{-7}	9.72×10^{-7}
	1.19×10^{-6}	1.12×10^{-6}	1.08×10^{-6}	1.05×10^{-6}	1.02×10^{-6}	9.92×10^{-7}	9.72×10^{-7}
	1.19×10^{-6}	1.12×10^{-6}	1.08×10^{-6}	1.05×10^{-6}	1.02×10^{-6}	9.91×10^{-7}	9.72×10^{-7}
	2.49×10^{-2}	1.74×10^{-3}	1.07×10^{-2}	1.2×10^{-2}	3.68×10^{-3}	9.9×10^{-3}	2.43×10^{-2}
	2.49×10^{-2}	1.74×10^{-3}	1.07×10^{-2}	1.2×10^{-2}	3.68×10^{-3}	9.9×10^{-3}	2.43×10^{-2}
	3.68×10^{-5}	3.96×10^{-5}	4.56×10^{-5}	4.43×10^{-5}	2.85×10^{-5}	1.8×10^{-5}	2.13×10^{-5}
	3.68×10^{-5}	3.96×10^{-5}	4.56×10^{-5}	4.43×10^{-5}	2.85×10^{-5}	1.8×10^{-5}	2.13×10^{-5}
5	1.52×10^{-6}	1.48×10^{-6}	1.44×10^{-6}	1.38×10^{-6}	1.22×10^{-6}	1.1×10^{-6}	1.03×10^{-6}
	1.52×10^{-6}	1.48×10^{-6}	1.44×10^{-6}	1.38×10^{-6}	1.22×10^{-6}	1.1×10^{-6}	1.03×10^{-6}
	1.24×10^{-6}	1.14×10^{-6}	1.09×10^{-6}	1.06×10^{-6}	1.01×10^{-6}	9.75×10^{-7}	9.5×10^{-7}
	1.24×10^{-6}	1.14×10^{-6}	1.09×10^{-6}	1.06×10^{-6}	1.01×10^{-6}	9.75×10^{-7}	9.5×10^{-7}
	1.24×10^{-6}	1.14×10^{-6}	1.09×10^{-6}	1.06×10^{-6}	1.01×10^{-6}	9.73×10^{-7}	9.49×10^{-7}
	6.61×10^{-2}	3.78×10^{-3}	2.03×10^{-2}	2.21×10^{-2}	9.09×10^{-4}	2.94×10^{-2}	5.57×10^{-2}
	6.61×10^{-2}	3.78×10^{-3}	2.03×10^{-2}	2.21×10^{-2}	9.09×10^{-4}	2.94×10^{-2}	5.57×10^{-2}
	1.15×10^{-4}	1.09×10^{-4}	1.32×10^{-4}	1.28×10^{-4}	6.39×10^{-5}	4.73×10^{-5}	8.55×10^{-5}
	1.15×10^{-4}	1.09×10^{-4}	1.32×10^{-4}	1.28×10^{-4}	6.39×10^{-5}	4.73×10^{-5}	8.55×10^{-5}
10	8.93×10^{-7}	2.01×10^{-6}	2.29×10^{-6}	2.26×10^{-6}	1.65×10^{-6}	1.29×10^{-6}	1.1×10^{-6}
	8.93×10^{-7}	2.01×10^{-6}	2.29×10^{-6}	2.26×10^{-6}	1.65×10^{-6}	1.29×10^{-6}	1.1×10^{-6}
	1.21×10^{-6}	1.14×10^{-6}	1.1×10^{-6}	1.07×10^{-6}	9.86×10^{-7}	9.41×10^{-7}	9.1×10^{-7}
	1.21×10^{-6}	1.14×10^{-6}	1.1×10^{-6}	1.07×10^{-6}	9.86×10^{-7}	9.41×10^{-7}	9.1×10^{-7}
	1.31×10^{-6}	1.16×10^{-6}	1.1×10^{-6}	1.06×10^{-6}	9.8×10^{-7}	9.37×10^{-7}	9.08×10^{-7}

Doing a direct comparison between the two method, we notice that, even if the method with generic $\bar{x}$ has an higher computational cost, it garantees better results since first orders:

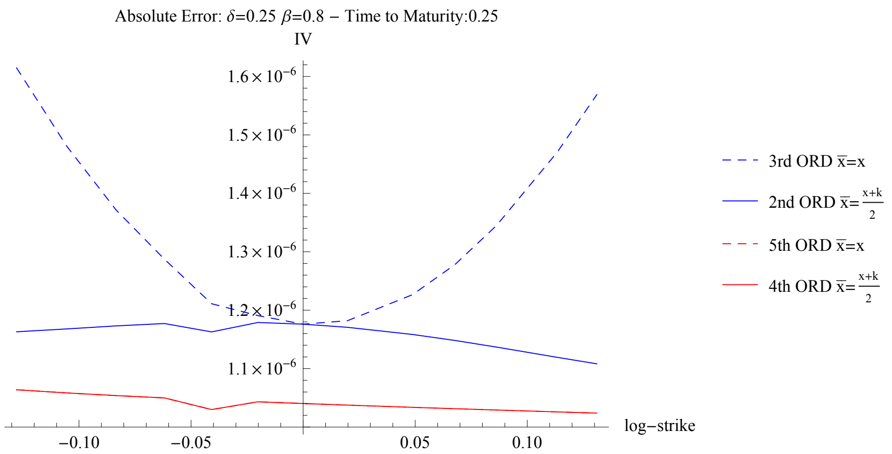
	1		7		13	
	3.22×10^{-1}	1.47×10^{-5}	6.5×10^{-4}	6.5×10^{-4}	3.26×10^{-1}	8.14×10^{-5}
	2.04×10^{-3}	1.47×10^{-5}	6.5×10^{-4}	6.5×10^{-4}	2.06×10^{-3}	8.14×10^{-5}
	2.64×10^{-5}	1.16×10^{-6}	1.18×10^{-6}	1.18×10^{-6}	2.39×10^{-5}	1.11×10^{-6}
	1.61×10^{-6}	1.16×10^{-6}	1.18×10^{-6}	1.18×10^{-6}	1.57×10^{-6}	1.11×10^{-6}
0.25	1.08×10^{-6}	1.06×10^{-6}	1.04×10^{-6}	1.04×10^{-6}	1.01×10^{-6}	1.02×10^{-6}
	1.06×10^{-6}	1.06×10^{-6}	1.04×10^{-6}	1.04×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.06×10^{-6}	1.06×10^{-6}	1.04×10^{-6}	1.04×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.06×10^{-6}	1.06×10^{-6}	1.04×10^{-6}	1.04×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.06×10^{-6}	1.06×10^{-6}	1.04×10^{-6}	1.04×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.06×10^{-6}	0	1.04×10^{-6}	0	1.02×10^{-6}	0

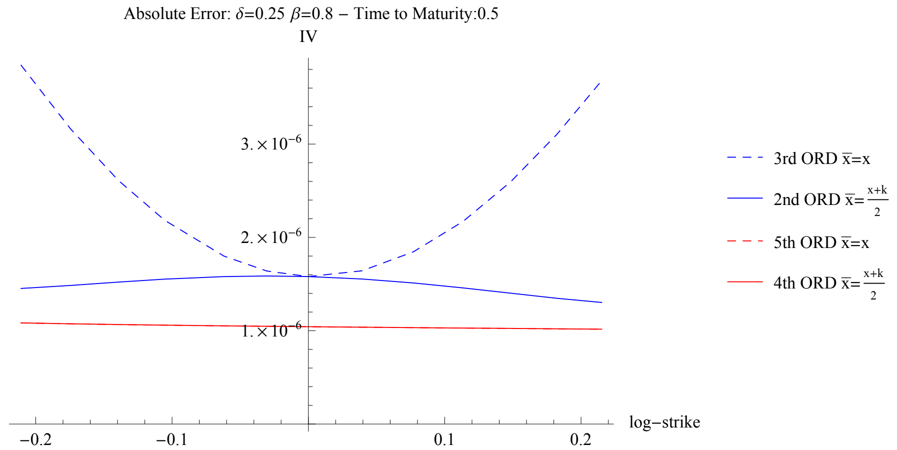
	5.32×10^{-1}	5.12×10^{-4}	1.29×10^{-3}	1.29×10^{-3}	5.33×10^{-1}	6.74×10^{-4}
	5.08×10^{-3}	5.12×10^{-4}	1.29×10^{-3}	1.29×10^{-3}	5.07×10^{-3}	6.74×10^{-4}
	8.51×10^{-5}	1.45×10^{-6}	1.58×10^{-6}	1.58×10^{-6}	7.93×10^{-5}	1.3×10^{-6}
	3.84×10^{-6}	1.45×10^{-6}	1.58×10^{-6}	1.58×10^{-6}	3.68×10^{-6}	1.3×10^{-6}
0.5	1.18×10^{-6}	1.08×10^{-6}	1.04×10^{-6}	1.04×10^{-6}	9.27×10^{-7}	1.02×10^{-6}
	1.09×10^{-6}	1.08×10^{-6}	1.04×10^{-6}	1.04×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.08×10^{-6}	1.08×10^{-6}	1.04×10^{-6}	1.04×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.08×10^{-6}	1.08×10^{-6}	1.04×10^{-6}	1.04×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.08×10^{-6}	1.08×10^{-6}	1.04×10^{-6}	1.04×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.08×10^{-6}	0	1.04×10^{-6}	0	1.02×10^{-6}	0
	8.33×10^{-1}	1.82×10^{-3}	2.57×10^{-3}	2.57×10^{-3}	8.47×10^{-1}	2.45×10^{-3}
	1.18×10^{-2}	1.82×10^{-3}	2.57×10^{-3}	2.57×10^{-3}	1.21×10^{-2}	2.45×10^{-3}
	2.64×10^{-4}	2.55×10^{-6}	3.15×10^{-6}	3.15×10^{-6}	2.58×10^{-4}	7.94×10^{-6}
	1.38×10^{-5}	2.55×10^{-6}	3.15×10^{-6}	3.15×10^{-6}	3.44×10^{-6}	7.94×10^{-6}
1	1.73×10^{-6}	1.11×10^{-6}	1.05×10^{-6}	1.05×10^{-6}	9.52×10^{-6}	8.92×10^{-6}
	1.14×10^{-6}	1.11×10^{-6}	1.05×10^{-6}	1.05×10^{-6}	8.89×10^{-6}	8.92×10^{-6}
	1.11×10^{-6}	1.11×10^{-6}	1.05×10^{-6}	1.05×10^{-6}	8.92×10^{-6}	8.92×10^{-6}
	1.11×10^{-6}	1.11×10^{-6}	1.05×10^{-6}	1.05×10^{-6}	8.92×10^{-6}	8.92×10^{-6}
	1.11×10^{-6}	1.11×10^{-6}	1.05×10^{-6}	1.05×10^{-6}	8.92×10^{-6}	8.92×10^{-6}
	1.11×10^{-6}	0	1.05×10^{-6}	0	8.92×10^{-6}	0
	1.1	3.74×10^{-3}	3.82×10^{-3}	3.82×10^{-3}	1.09	4.53×10^{-3}
	1.98×10^{-2}	3.74×10^{-3}	3.82×10^{-3}	3.82×10^{-3}	1.98×10^{-2}	4.53×10^{-3}
	5.18×10^{-4}	4.33×10^{-6}	5.67×10^{-6}	5.67×10^{-6}	4.71×10^{-4}	3.03×10^{-6}
	3.25×10^{-5}	4.33×10^{-6}	5.67×10^{-6}	5.67×10^{-6}	3.02×10^{-5}	3.03×10^{-6}
1.5	$3.\times10^{-6}$	1.14×10^{-6}	1.06×10^{-6}	1.06×10^{-6}	7.44×10^{-7}	$1.\times10^{-6}$
	1.26×10^{-6}	1.14×10^{-6}	1.06×10^{-6}	1.06×10^{-6}	1.11×10^{-6}	$1.\times10^{-6}$
	1.14×10^{-6}	1.13×10^{-6}	1.05×10^{-6}	1.05×10^{-6}	9.86×10^{-7}	9.95×10^{-7}
	1.14×10^{-6}	1.13×10^{-6}	1.05×10^{-6}	1.05×10^{-6}	9.96×10^{-7}	9.95×10^{-7}
	1.13×10^{-6}	1.13×10^{-6}	1.05×10^{-6}	1.05×10^{-6}	9.95×10^{-7}	9.95×10^{-7}
	1.13×10^{-6}	0	1.05×10^{-6}	0	9.95×10^{-7}	0
	1.35	6.34×10^{-3}	5.05×10^{-3}	5.05×10^{-3}	1.31	7.05×10^{-3}
	2.91×10^{-2}	6.34×10^{-3}	5.05×10^{-3}	5.05×10^{-3}	2.83×10^{-2}	7.05×10^{-3}
	8.38×10^{-4}	1.71×10^{-6}	9.09×10^{-6}	9.09×10^{-6}	7.4×10^{-4}	4.44×10^{-6}
	5.61×10^{-5}	1.71×10^{-6}	9.09×10^{-6}	9.09×10^{-6}	5.46×10^{-5}	4.44×10^{-6}
2	1.94×10^{-7}	3.93×10^{-6}	1.08×10^{-6}	1.08×10^{-6}	2.7×10^{-6}	9.97×10^{-7}
	3.64×10^{-6}	3.93×10^{-6}	1.08×10^{-6}	1.08×10^{-6}	1.26×10^{-6}	9.97×10^{-7}
	3.93×10^{-6}	3.96×10^{-6}	1.05×10^{-6}	1.05×10^{-6}	9.62×10^{-7}	9.86×10^{-7}
	3.96×10^{-6}	3.96×10^{-6}	1.05×10^{-6}	1.05×10^{-6}	9.89×10^{-7}	9.86×10^{-7}
	3.96×10^{-6}	3.96×10^{-6}	1.05×10^{-6}	1.05×10^{-6}	9.86×10^{-7}	9.86×10^{-7}
	3.96×10^{-6}	0	1.05×10^{-6}	0	9.86×10^{-7}	0
	1.78	1.23×10^{-2}	7.46×10^{-3}	7.46×10^{-3}	1.69	1.26×10^{-2}
	4.91×10^{-2}	1.23×10^{-2}	7.46×10^{-3}	7.46×10^{-3}	4.61×10^{-2}	1.26×10^{-2}
	1.64×10^{-3}	1.41×10^{-5}	1.83×10^{-5}	1.83×10^{-5}	1.37×10^{-3}	8.45×10^{-6}
	1.46×10^{-4}	1.41×10^{-5}	1.83×10^{-5}	1.83×10^{-5}	1.23×10^{-4}	8.45×10^{-6}
3	1.33×10^{-5}	1.27×10^{-6}	1.14×10^{-6}	1.14×10^{-6}	9.17×10^{-6}	9.99×10^{-7}
	2.28×10^{-6}	1.27×10^{-6}	1.14×10^{-6}	1.14×10^{-6}	1.85×10^{-6}	9.99×10^{-7}
	1.31×10^{-6}	1.19×10^{-6}	1.05×10^{-6}	1.05×10^{-6}	8.82×10^{-7}	9.72×10^{-7}
	1.21×10^{-6}	1.19×10^{-6}	1.05×10^{-6}	1.05×10^{-6}	9.82×10^{-7}	9.72×10^{-7}
	1.2×10^{-6}	1.19×10^{-6}	1.05×10^{-6}	1.05×10^{-6}	9.71×10^{-7}	9.72×10^{-7}
	1.19×10^{-6}	0	1.05×10^{-6}	0	9.72×10^{-7}	0

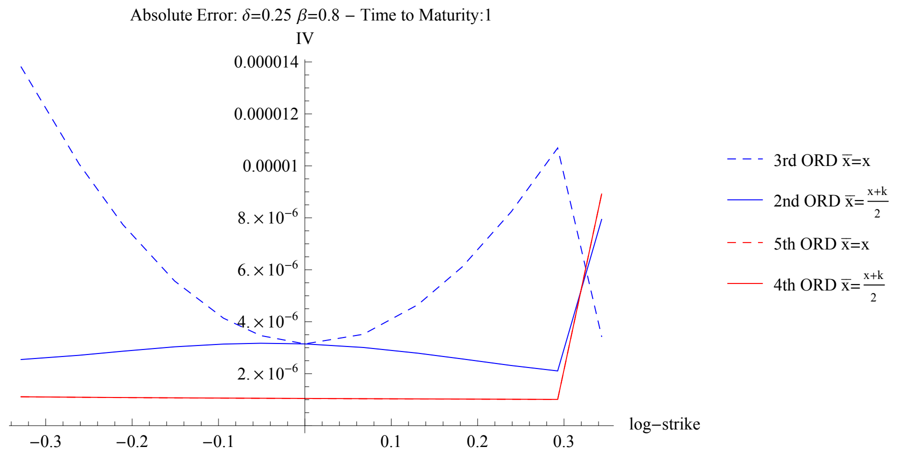
5	2.44	2.49×10^{-2}	1.2×10^{-2}	1.2×10^{-2}	2.27	2.43×10^{-2}
	8.95×10^{-2}	2.49×10^{-2}	1.2×10^{-2}	1.2×10^{-2}	8.28×10^{-2}	2.43×10^{-2}
	3.59×10^{-3}	3.68×10^{-5}	4.43×10^{-5}	4.43×10^{-5}	2.87×10^{-3}	2.13×10^{-5}
	3.91×10^{-4}	3.68×10^{-5}	4.43×10^{-5}	4.43×10^{-5}	3.19×10^{-4}	2.13×10^{-5}
	3.98×10^{-5}	1.52×10^{-6}	1.38×10^{-6}	1.38×10^{-6}	3.07×10^{-5}	1.03×10^{-6}
	5.02×10^{-6}	1.52×10^{-6}	1.38×10^{-6}	1.38×10^{-6}	3.96×10^{-6}	1.03×10^{-6}
	1.66×10^{-6}	1.24×10^{-6}	1.06×10^{-6}	1.06×10^{-6}	6.13×10^{-7}	9.5×10^{-7}
	1.29×10^{-6}	1.24×10^{-6}	1.06×10^{-6}	1.06×10^{-6}	9.87×10^{-7}	9.5×10^{-7}
	1.24×10^{-6}	1.24×10^{-6}	1.06×10^{-6}	1.06×10^{-6}	9.45×10^{-7}	9.49×10^{-7}
	1.24×10^{-6}	0	1.06×10^{-6}	0	9.49×10^{-7}	0
10	3.77	6.61×10^{-2}	2.21×10^{-2}	2.21×10^{-2}	3.32	5.57×10^{-2}
	2.01×10^{-1}	6.61×10^{-2}	2.21×10^{-2}	2.21×10^{-2}	1.78×10^{-1}	5.57×10^{-2}
	9.33×10^{-3}	1.15×10^{-4}	1.28×10^{-4}	1.28×10^{-4}	7.17×10^{-3}	8.55×10^{-5}
	1.09×10^{-3}	1.15×10^{-4}	1.28×10^{-4}	1.28×10^{-4}	9.16×10^{-4}	8.55×10^{-5}
	6.57×10^{-5}	8.93×10^{-7}	2.26×10^{-6}	2.26×10^{-6}	7.86×10^{-5}	1.1×10^{-6}
	1.3×10^{-5}	8.93×10^{-7}	2.26×10^{-6}	2.26×10^{-6}	1.97×10^{-6}	1.1×10^{-6}
	6.07×10^{-6}	1.21×10^{-6}	1.07×10^{-6}	1.07×10^{-6}	4.03×10^{-6}	9.1×10^{-7}
	1.32×10^{-6}	1.21×10^{-6}	1.07×10^{-6}	1.07×10^{-6}	3.08×10^{-7}	9.1×10^{-7}
	4.44×10^{-7}	1.31×10^{-6}	1.06×10^{-6}	1.06×10^{-6}	1.32×10^{-6}	9.08×10^{-7}
	1.03×10^{-6}	0	1.06×10^{-6}	0	7.75×10^{-7}	0

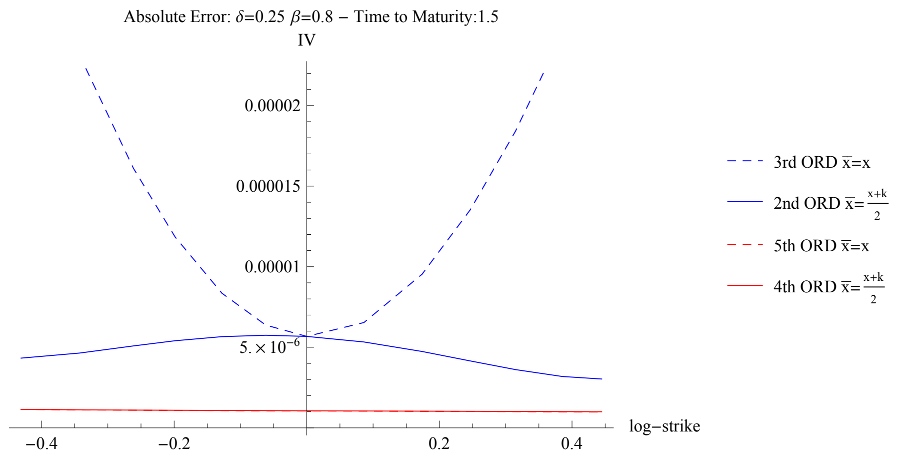
in this matrix, every element is a sub-matrix where the first column is relative to the choice of the initial point $\overline{x} = x$, and the second is relative to the choice $\overline{x} = \frac{(x+k)}{2}$ (as a matter of fact, in the second column, due to computation cost problems, we stop at 8th order instead at 9th). From these results we can see that in fact the midpoint ensures lower errors already at zeroth order, as it is quite intuitive to think; it may be noted also that for any considered value, both methods allow to reach a precision of at least 10^{-6}, which is well above of practical needs.

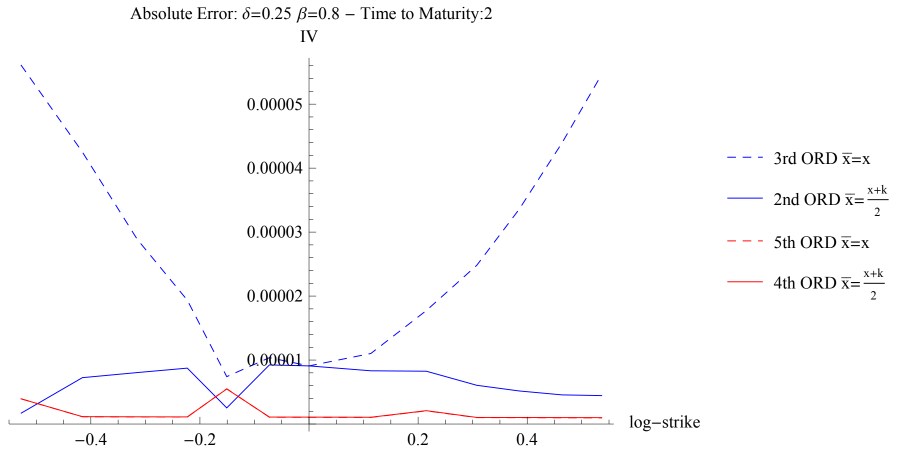
As we have said, however, we want to compare the lower orders of the two methods to see which produces better results with less effort.

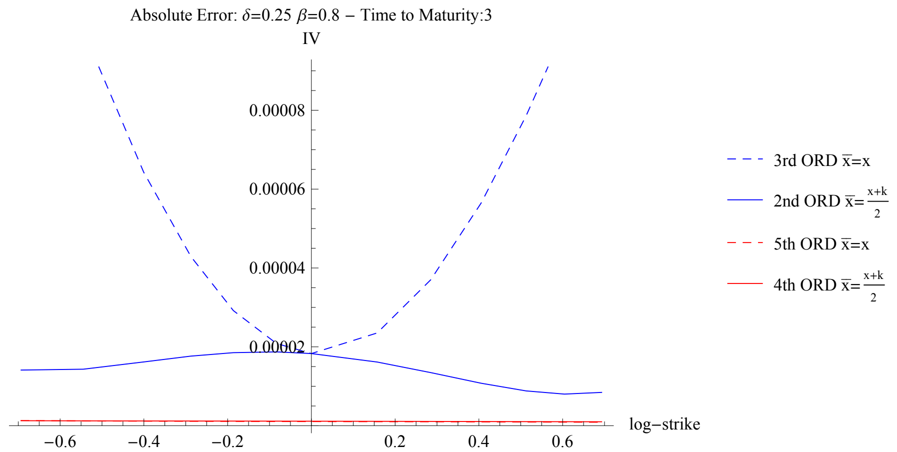


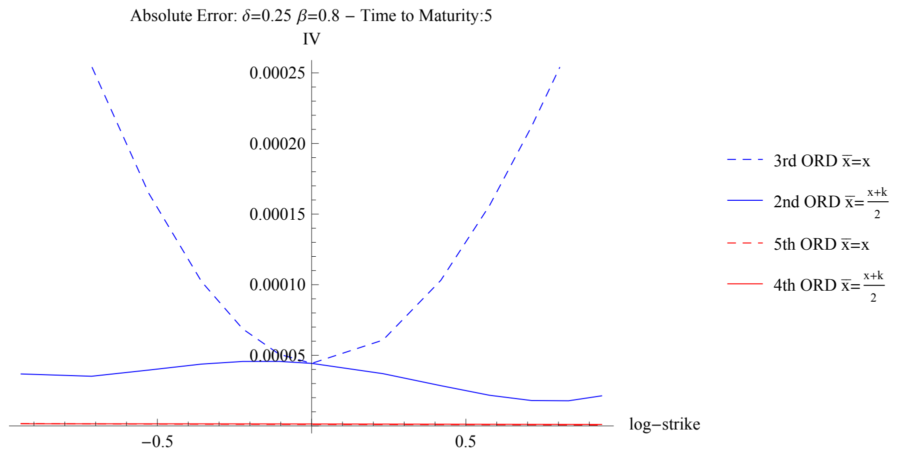

Absolute Error: δ=0.25 β=0.8 – Time to Maturity:0.5
IV
3. × 10^−6
2. × 10^−6
1. × 10^−6
log–strike
−0.2
−0.1
0.1
0.2
3rd ORD x̄=x
2nd ORD x̄=(x+k)/2
5th ORD x̄=x
4th ORD x̄=(x+k)/2

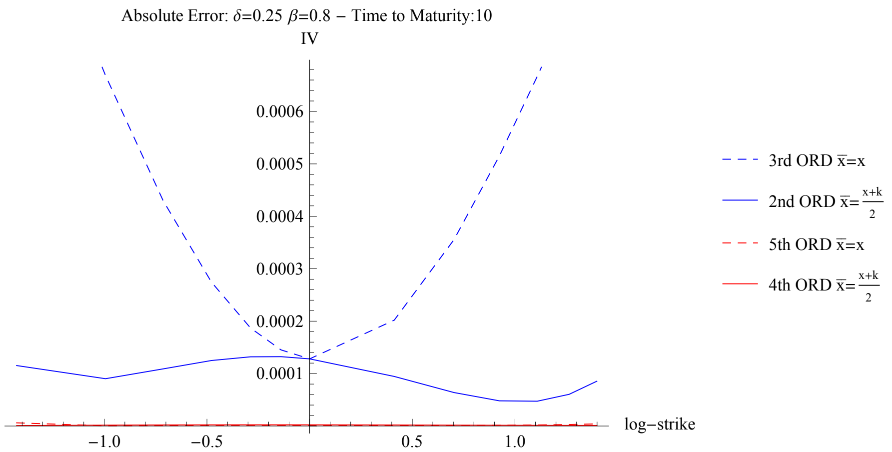

Absolute Error: δ=0.25 β=0.8 – Time to Maturity:1
IV
0.000014
0.000012
0.00001
8. × 10^−6
6. × 10^−6
4. × 10^−6
2. × 10^−6
log–strike
−0.3
−0.2
−0.1
0.1
0.2
0.3
3rd ORD x̄=x
2nd ORD x̄=(x+k)/2
5th ORD x̄=x
4th ORD x̄=(x+k)/2


Absolute Error: δ=0.25 β=0.8 – Time to Maturity:1.5
IV
0.00002
0.000015
0.00001
5. × 10^−6
log–strike
−0.4
−0.2
0.2
0.4
3rd ORD x̄=x
2nd ORD x̄=(x+k)/2
5th ORD x̄=x
4th ORD x̄=(x+k)/2


Absolute Error: δ=0.25 β=0.8 – Time to Maturity:2
IV
0.00005
0.00004
0.00003
0.00002
0.00001
log–strike
–0.4
–0.2
0.2
0.4
3rd ORD $\bar{x}$=x
2nd ORD $\bar{x}=\frac{x+k}{2}$
5th ORD $\bar{x}$=x
4th ORD $\bar{x}=\frac{x+k}{2}$


Absolute Error: δ=0.25 β=0.8 – Time to Maturity:3
IV
0.00008
0.00006
0.00004
0.00002
log–strike
–0.6
–0.4
–0.2
0.2
0.4
0.6
3rd ORD $\bar{x}$=x
2nd ORD $\bar{x}=\frac{x+k}{2}$
5th ORD $\bar{x}$=x
4th ORD $\bar{x}=\frac{x+k}{2}$


Absolute Error: δ=0.25 β=0.8 – Time to Maturity:5
IV
0.00025
0.00020
0.00015
0.00010
0.00005
log–strike
–0.5
0.5
3rd ORD $\bar{x}$=x
2nd ORD $\bar{x}=\frac{x+k}{2}$
5th ORD $\bar{x}$=x
4th ORD $\bar{x}=\frac{x+k}{2}$



First of all, in these graphs we compare the second order of the method with the midpoint and the third order with the initial point : although in this second method we consider the following order, its lower computational cost, due to the cancellation of many factors during the calculation, makes it comparable from a computational complexity point of view, to a lower order of the method with the midpoint. Nevertheless, the graphs show that the second order with the midpoint is always significantly better than the third order with the starting point.

A similar comparison can also be done between the fifth order with the starting point and the fourth order with the middle point : in this case we see that the method with the midpoint unable to compete with the major orders of the other method.

8.1.2 Second Set: $\delta = 0.25$, $\beta = 0.2$

In the second set of parameter we have ($\delta = 0.25$, $\beta = 0.2$). Proceeding analogously to the previous case we create the matrix of the strikes

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.25	87	89	91	94	96	98	100	102	104	106	109	111	113
0.5	80	83	87	90	93	97	100	104	107	111	115	118	122
1	68	73	79	84	89	95	100	106	112	118	123	129	135
1.5	58	65	72	79	86	93	100	108	116	124	131	139	147
2	51	59	67	76	84	92	100	110	119	128	138	148	157
3	38	48	59	69	79	90	100	112	125	138	150	162	175
5	20	33	47	60	73	87	100	118	135	152	170	188	205
10	7	22	38	54	69	84	100	128	156	184	211	239	267

With these strikes we can numerically compute call and put prices, ensuring that they are admissible

	1	3	5	7	9	11	13
0.25	0.1392	0.1069	0.07220	0.04986	0.03258	0.01759	0.01000
0.5	0.2106	0.1530	0.1105	0.07049	0.04140	0.02018	0.009715
1	0.3303	0.2364	0.1631	0.09964	0.05196	0.02557	0.01037
1.5	0.4296	0.3074	0.2029	0.1220	0.05995	0.02694	0.009876
2	0.5002	0.3598	0.2328	0.1408	0.06794	0.02814	0.009942
3	0.6300	0.4443	0.2937	0.1722	0.07882	0.03045	0.009869
5	0.8097	0.5712	0.3764	0.2219	0.09549	0.03405	0.01002
10	0.9409	0.6922	0.4793	0.3118	0.1215	0.03896	0.009848

	1	3	5	7	9	11	13
0.25	0.009247	0.01690	0.03220	0.04986	0.07258	0.1076	0.1400
0.5	0.01059	0.02302	0.04048	0.07049	0.1114	0.1702	0.2297
1	0.01030	0.02635	0.05311	0.09964	0.1720	0.2556	0.3604
1.5	0.009637	0.02736	0.06292	0.1220	0.2200	0.3369	0.4799
2	0.01024	0.02985	0.07283	0.1408	0.2579	0.4081	0.5799
3	0.009983	0.03431	0.08370	0.1722	0.3288	0.5304	0.7599
5	0.009721	0.04121	0.1064	0.2219	0.4455	0.7341	1.060
10	0.01086	0.07216	0.1693	0.3118	0.6815	1.149	1.680

Then we numerically obtain exact implied volatilities

	1	3	5	7	9	11	13
0.25	26.4308	25.9666	25.4214	25.0104	24.6198	24.1575	23.8067
0.5	27.3249	26.4430	25.7554	25.0208	24.3487	23.6460	23.0805
1	29.1198	27.4861	26.2312	25.0416	23.9201	23.0193	22.1477
1.5	30.9581	28.5202	26.6130	25.0623	23.5972	22.4417	21.3837
2	32.5131	29.3502	26.8859	25.0829	23.3680	21.9732	20.8075
3	36.2697	30.8728	27.5945	25.1239	22.9295	21.2395	19.8819
5	45.1832	33.7623	28.5717	25.2046	22.2614	20.1746	18.5885
10	56.7246	36.4540	29.4028	25.3780	21.0481	18.4309	16.5685

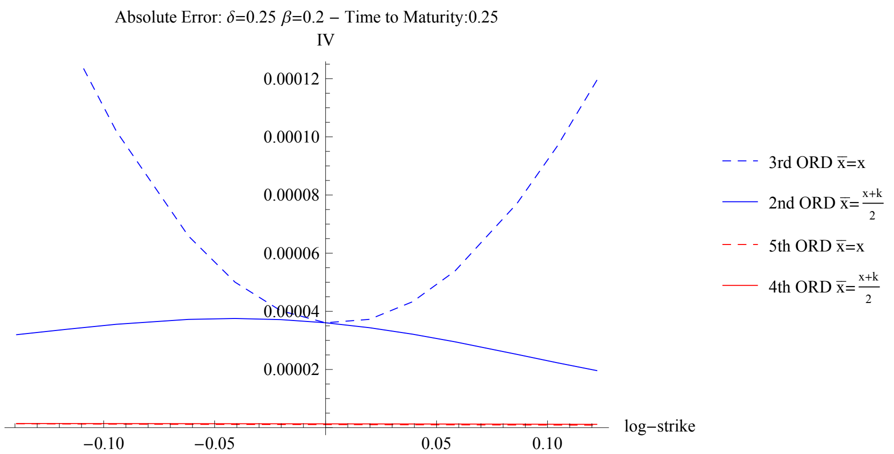
Finally we compute the absolute error between exact values of the implied volatility and our approximated ones (as before, in every sub-matrix setting we have $\overline{x} = x$ in the first column, and $\overline{x} = \frac{x+k}{2}$).

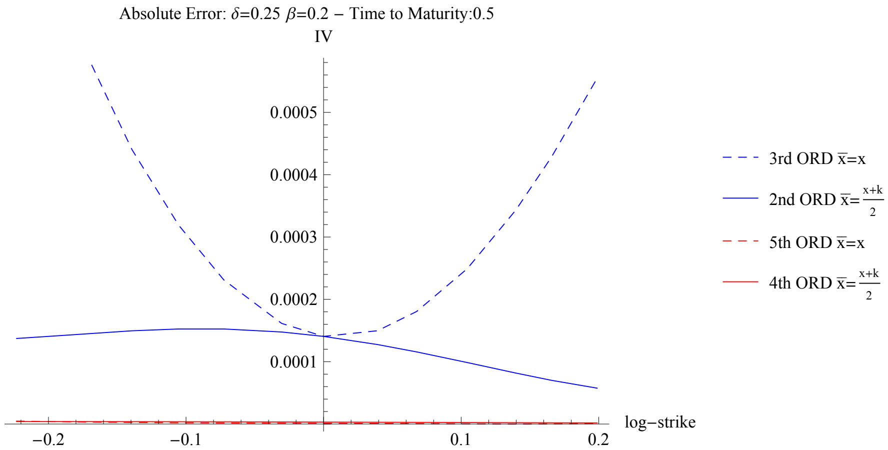
	1		7		13	
0.25	1.43	1.38×10^{-3}	1.04×10^{-2}	1.04×10^{-2}	1.19	$5.\times10^{-4}$
	3.81×10^{-2}	1.38×10^{-3}	1.04×10^{-2}	1.04×10^{-2}	2.89×10^{-2}	$5.\times10^{-4}$
	1.9×10^{-3}	3.2×10^{-5}	3.61×10^{-5}	3.61×10^{-5}	1.4×10^{-3}	1.96×10^{-5}
	1.74×10^{-4}	3.2×10^{-5}	3.61×10^{-5}	3.61×10^{-5}	1.19×10^{-4}	1.96×10^{-5}
	1.77×10^{-5}	1.4×10^{-6}	1.3×10^{-6}	1.3×10^{-6}	9.77×10^{-6}	1.13×10^{-6}
	2.96×10^{-6}	1.4×10^{-6}	1.3×10^{-6}	1.3×10^{-6}	2.14×10^{-6}	1.13×10^{-6}
	1.34×10^{-6}	1.12×10^{-6}	1.04×10^{-6}	1.04×10^{-6}	8.51×10^{-7}	9.9×10^{-7}
	1.14×10^{-6}	1.12×10^{-6}	1.04×10^{-6}	1.04×10^{-6}	1.01×10^{-6}	9.9×10^{-7}
	1.12×10^{-6}	1.11×10^{-6}	1.04×10^{-6}	1.04×10^{-6}	9.86×10^{-7}	9.89×10^{-7}
	1.11×10^{-6}	0	1.04×10^{-6}	0	9.89×10^{-7}	0

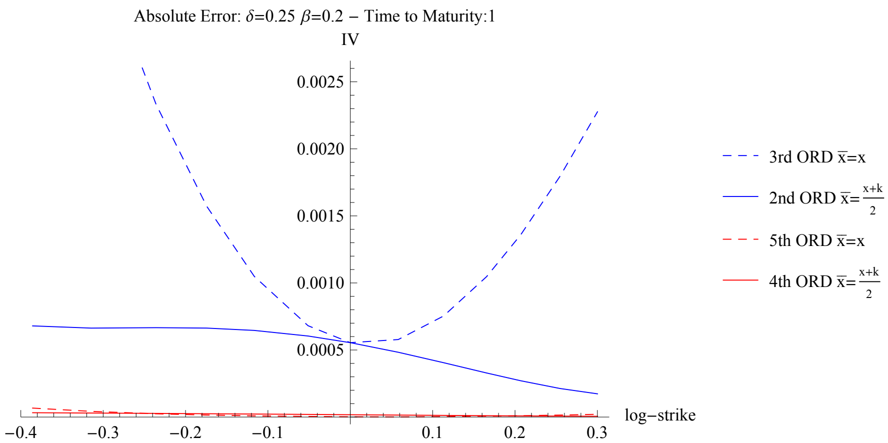
	2.32	9.18×10^{-3}	2.08×10^{-2}	2.08×10^{-2}	1.92	7.99×10^{-3}
	9.34×10^{-2}	9.18×10^{-3}	2.08×10^{-2}	2.08×10^{-2}	6.9×10^{-2}	7.99×10^{-3}
	6.37×10^{-3}	1.37×10^{-4}	1.41×10^{-4}	1.41×10^{-4}	4.35×10^{-3}	5.76×10^{-5}
	8.68×10^{-4}	1.37×10^{-4}	1.41×10^{-4}	1.41×10^{-4}	5.55×10^{-4}	5.76×10^{-5}
0.5	1.22×10^{-4}	3.94×10^{-6}	3.1×10^{-6}	3.1×10^{-6}	7.12×10^{-5}	1.79×10^{-6}
	2.06×10^{-5}	3.94×10^{-6}	3.1×10^{-6}	3.1×10^{-6}	1.19×10^{-5}	1.79×10^{-6}
	4.69×10^{-6}	1.24×10^{-6}	1.09×10^{-6}	1.09×10^{-6}	9.22×10^{-7}	9.78×10^{-7}
	1.86×10^{-6}	1.24×10^{-6}	1.09×10^{-6}	1.09×10^{-6}	1.31×10^{-6}	9.78×10^{-7}
	1.31×10^{-6}	1.16×10^{-6}	1.05×10^{-6}	1.05×10^{-6}	8.88×10^{-7}	9.6×10^{-7}
	1.19×10^{-6}	0	1.05×10^{-6}	0	9.75×10^{-7}	0
	4.12	5.02×10^{-2}	4.16×10^{-2}	4.16×10^{-2}	2.85	2.44×10^{-2}
	2.63×10^{-1}	5.02×10^{-2}	4.16×10^{-2}	4.16×10^{-2}	1.49×10^{-1}	2.44×10^{-2}
	2.38×10^{-2}	6.8×10^{-4}	5.55×10^{-4}	5.55×10^{-4}	1.23×10^{-2}	1.73×10^{-4}
	5.03×10^{-3}	6.8×10^{-4}	5.55×10^{-4}	5.55×10^{-4}	2.28×10^{-3}	1.73×10^{-4}
1	1.06×10^{-3}	3.23×10^{-5}	1.72×10^{-5}	1.72×10^{-5}	4.17×10^{-4}	5.5×10^{-6}
	2.46×10^{-4}	3.23×10^{-5}	1.72×10^{-5}	1.72×10^{-5}	8.93×10^{-5}	5.5×10^{-6}
	6.67×10^{-5}	3.43×10^{-6}	1.81×10^{-6}	1.81×10^{-6}	2.04×10^{-5}	1.11×10^{-6}
	2.02×10^{-5}	3.43×10^{-6}	1.81×10^{-6}	1.81×10^{-6}	6.54×10^{-6}	1.11×10^{-6}
	7.13×10^{-6}	1.46×10^{-6}	1.1×10^{-6}	1.1×10^{-6}	6.72×10^{-7}	9.33×10^{-7}
	3.19×10^{-6}	0	1.1×10^{-6}	0	1.41×10^{-6}	0
	5.96	1.28×10^{-1}	6.23×10^{-2}	6.23×10^{-2}	3.62	4.58×10^{-2}
	5.11×10^{-1}	1.28×10^{-1}	6.23×10^{-2}	6.23×10^{-2}	2.36×10^{-1}	4.58×10^{-2}
	5.42×10^{-2}	1.99×10^{-3}	1.24×10^{-3}	1.24×10^{-3}	2.26×10^{-2}	3.17×10^{-4}
	1.49×10^{-2}	1.99×10^{-3}	1.24×10^{-3}	1.24×10^{-3}	5.19×10^{-3}	3.17×10^{-4}
1.5	4.02×10^{-3}	1.48×10^{-4}	5.47×10^{-5}	5.47×10^{-5}	1.16×10^{-3}	1.21×10^{-5}
	1.16×10^{-3}	1.48×10^{-4}	5.47×10^{-5}	5.47×10^{-5}	2.96×10^{-4}	1.21×10^{-5}
	3.89×10^{-4}	1.87×10^{-5}	4.77×10^{-6}	4.77×10^{-6}	8.57×10^{-5}	1.55×10^{-6}
	1.42×10^{-4}	1.87×10^{-5}	4.77×10^{-6}	4.77×10^{-6}	2.85×10^{-5}	1.55×10^{-6}
	5.59×10^{-5}	4.05×10^{-6}	1.41×10^{-6}	1.41×10^{-6}	8.6×10^{-6}	9.43×10^{-7}
	2.38×10^{-5}	0	1.41×10^{-6}	0	4.38×10^{-6}	0
	7.51	2.14×10^{-1}	8.29×10^{-2}	8.29×10^{-2}	4.19	6.53×10^{-2}
	7.8×10^{-1}	2.14×10^{-1}	8.29×10^{-2}	8.29×10^{-2}	3.18×10^{-1}	6.53×10^{-2}
	9.44×10^{-2}	4.43×10^{-3}	2.18×10^{-3}	2.18×10^{-3}	3.38×10^{-2}	4.9×10^{-4}
	3.06×10^{-2}	4.43×10^{-3}	2.18×10^{-3}	2.18×10^{-3}	8.95×10^{-3}	4.9×10^{-4}
2	9.71×10^{-3}	4.58×10^{-4}	1.26×10^{-4}	1.26×10^{-4}	2.28×10^{-3}	2.19×10^{-5}
	3.21×10^{-3}	4.58×10^{-4}	1.26×10^{-4}	1.26×10^{-4}	6.57×10^{-4}	2.19×10^{-5}
	1.24×10^{-3}	7.74×10^{-5}	1.24×10^{-5}	1.24×10^{-5}	2.18×10^{-4}	2.42×10^{-6}
	5.23×10^{-4}	7.74×10^{-5}	1.24×10^{-5}	1.24×10^{-5}	7.99×10^{-5}	2.42×10^{-6}
	2.33×10^{-4}	1.74×10^{-5}	2.49×10^{-6}	2.49×10^{-6}	2.99×10^{-5}	1.02×10^{-6}
	1.1×10^{-4}	0	2.49×10^{-6}	0	1.36×10^{-5}	0
	1.13×10^{1}	5.45×10^{-1}	1.24×10^{-1}	1.24×10^{-1}	5.12	1.04×10^{-1}
	1.59	5.45×10^{-1}	1.24×10^{-1}	1.24×10^{-1}	4.78×10^{-1}	1.04×10^{-1}
	2.26×10^{-1}	1.51×10^{-2}	4.79×10^{-3}	4.79×10^{-3}	5.86×10^{-2}	8.98×10^{-4}
	9.27×10^{-2}	1.51×10^{-2}	4.79×10^{-3}	4.79×10^{-3}	1.87×10^{-2}	8.98×10^{-4}
3	3.68×10^{-2}	1.53×10^{-3}	4.03×10^{-4}	4.03×10^{-4}	5.64×10^{-3}	4.92×10^{-5}
	1.37×10^{-2}	1.53×10^{-3}	4.03×10^{-4}	4.03×10^{-4}	1.9×10^{-3}	4.92×10^{-5}
	5.75×10^{-3}	2.65×10^{-4}	5.41×10^{-5}	5.41×10^{-5}	7.43×10^{-4}	5.73×10^{-6}
	2.37×10^{-3}	2.65×10^{-4}	5.41×10^{-5}	5.41×10^{-5}	3.15×10^{-4}	5.73×10^{-6}
	7.43×10^{-4}	6.53×10^{-4}	1.06×10^{-5}	1.06×10^{-5}	1.41×10^{-4}	1.49×10^{-6}
	4.83×10^{-5}	0	1.06×10^{-5}	0	6.87×10^{-5}	0

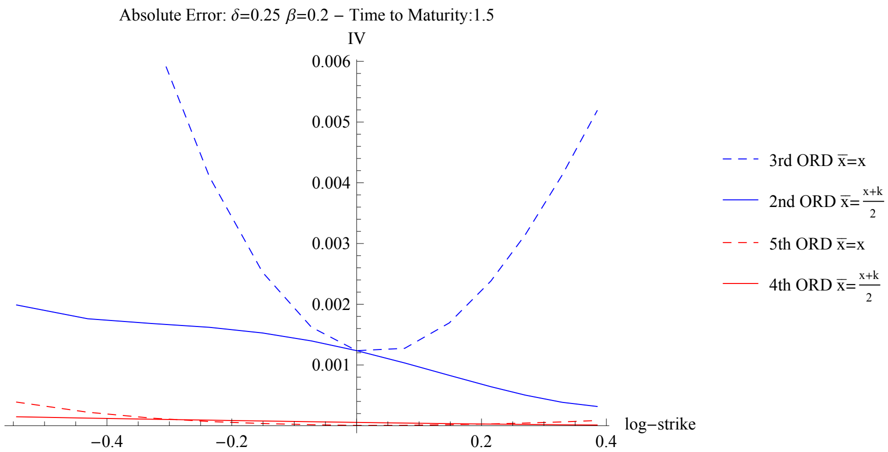
	2.02×10^{1}	2.41	2.05×10^{-1}	2.05×10^{-1}	6.41	1.72×10^{-1}
	4.09	2.41	2.05×10^{-1}	2.05×10^{-1}	7.67×10^{-1}	1.72×10^{-1}
	4.43×10^{-1}	1.51×10^{-1}	1.26×10^{-2}	1.26×10^{-2}	1.12×10^{-1}	1.89×10^{-3}
	9.31×10^{-2}	1.51×10^{-1}	1.26×10^{-2}	1.26×10^{-2}	4.39×10^{-2}	1.89×10^{-3}
5	8.54×10^{-2}	$2.\times10^{-1}$	1.57×10^{-3}	1.57×10^{-3}	1.56×10^{-2}	1.29×10^{-4}
	2.03×10^{-1}	$2.\times10^{-1}$	1.57×10^{-3}	1.57×10^{-3}	6.05×10^{-3}	1.29×10^{-4}
	2.38×10^{-1}	1.62×10^{-1}	2.59×10^{-4}	2.59×10^{-4}	2.68×10^{-3}	1.95×10^{-5}
	2.46×10^{-1}	1.62×10^{-1}	2.59×10^{-4}	2.59×10^{-4}	1.23×10^{-3}	1.95×10^{-5}
	2.45×10^{-1}	9.27×10^{-2}	2.28×10^{-5}	2.28×10^{-5}	5.62×10^{-4}	4.33×10^{-6}
	2.39×10^{-1}	0	2.28×10^{-5}	0	2.38×10^{-4}	0
	3.17×10^{1}	1.57×10^{1}	3.78×10^{-1}	3.78×10^{-1}	8.43	3.1×10^{-1}
	5.13	1.57×10^{1}	3.78×10^{-1}	3.78×10^{-1}	1.39	3.1×10^{-1}
	4.65	1.11	2.64×10^{-2}	2.64×10^{-2}	2.48×10^{-1}	4.93×10^{-3}
	5.63	1.11	2.64×10^{-2}	2.64×10^{-2}	1.15×10^{-1}	4.93×10^{-3}
10	5.95	1.02×10^{1}	5.76×10^{-3}	5.76×10^{-3}	4.06×10^{-2}	3.7×10^{-4}
	6.17	1.02×10^{1}	5.76×10^{-3}	5.76×10^{-3}	9.53×10^{-3}	3.7×10^{-4}
	5.52	3.28×10^{1}	1.06×10^{-2}	1.06×10^{-2}	3.37×10^{-3}	$7.\times10^{-5}$
	4.22	3.28×10^{1}	1.06×10^{-2}	1.06×10^{-2}	1.11×10^{-2}	$7.\times10^{-5}$
	2.45	5.21×10^{1}	1.09×10^{-2}	1.09×10^{-2}	1.66×10^{-2}	9.64×10^{-6}
	1.25×10^{-2}	0	1.09×10^{-2}	0	2.18×10^{-2}	0

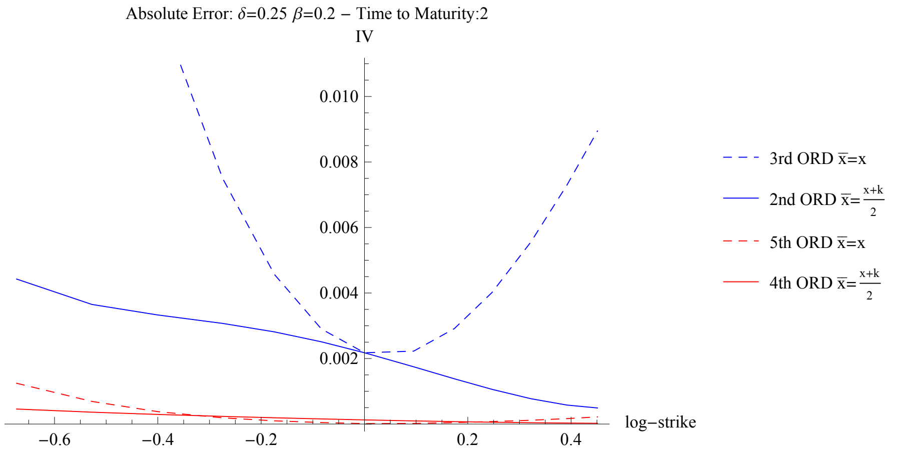
Compared to the previous set, we have a widespread worsening of the values, in particular far ITM and especially for long maturities. There still remains a guaranteed improvement by choosing the middle point rather than the starting point, but as you can see from the table, considering a 10-year maturity and moving us far ITM, also the midpoint does not lead to acceptable approximations, with an error well above 10%, and that continues to rise. This shows again the non-convergence of our method, in fact, as we shall see in some other cases, it is useless to calculate many orders of our approximation, both from a computational point of view, but also and especially because the value of the implied volatility tends to explode.

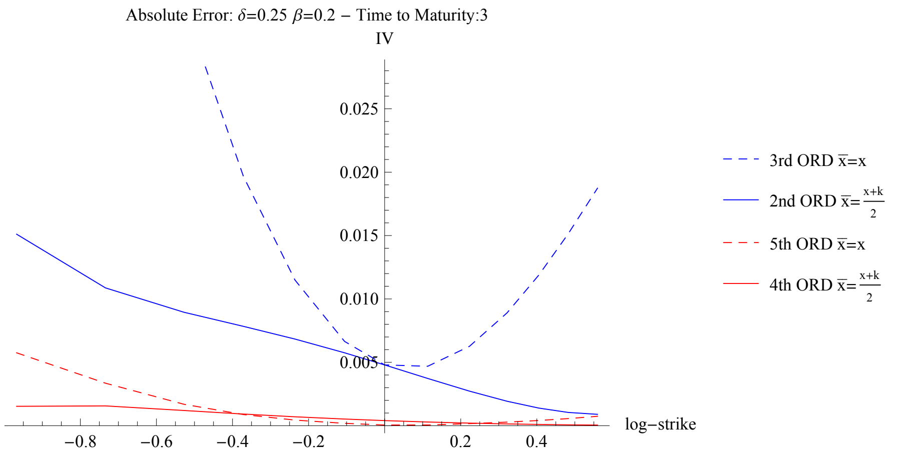


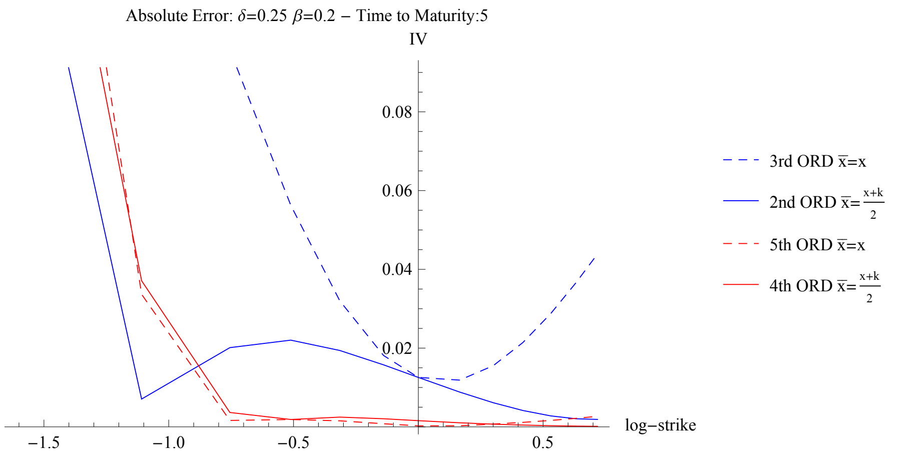

Absolute Error: δ=0.25 β=0.2 – Time to Maturity:0.5
IV
0.0005
0.0004
0.0003
0.0002
0.0001
log–strike
−0.2
−0.1
0.1
0.2
3rd ORD x̄=x
2nd ORD x̄=(x+k)/2
5th ORD x̄=x
4th ORD x̄=(x+k)/2

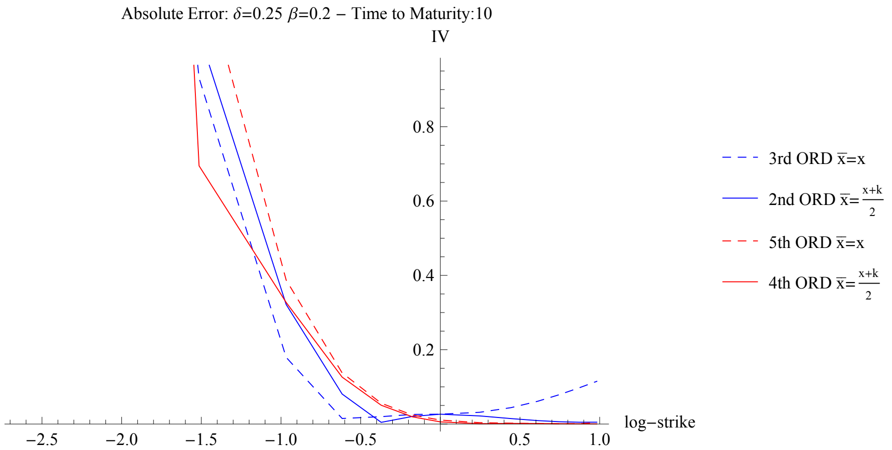

Absolute Error: δ=0.25 β=0.2 – Time to Maturity:1
IV
0.0025
0.0020
0.0015
0.0010
0.0005
log–strike
−0.4
−0.3
−0.2
−0.1
0.1
0.2
0.3
3rd ORD x̄=x
2nd ORD x̄=(x+k)/2
5th ORD x̄=x
4th ORD x̄=(x+k)/2


Absolute Error: δ=0.25 β=0.2 – Time to Maturity:1.5
IV
0.006
0.005
0.004
0.003
0.002
0.001
log–strike
−0.4
−0.2
0.2
0.4
3rd ORD x̄=x
2nd ORD x̄=(x+k)/2
5th ORD x̄=x
4th ORD x̄=(x+k)/2


Absolute Error: δ=0.25 β=0.2 − Time to Maturity:2
IV
0.010
0.008
0.006
0.004
0.002
log−strike
−0.6
−0.4
−0.2
0.2
0.4
3rd ORD x̄=x
2nd ORD x̄=(x+k)/2
5th ORD x̄=x
4th ORD x̄=(x+k)/2


Absolute Error: δ=0.25 β=0.2 − Time to Maturity:3
IV
0.025
0.020
0.015
0.010
0.005
log−strike
−0.8
−0.6
−0.4
−0.2
0.2
0.4
3rd ORD x̄=x
2nd ORD x̄=(x+k)/2
5th ORD x̄=x
4th ORD x̄=(x+k)/2


Absolute Error: δ=0.25 β=0.2 − Time to Maturity:5
IV
0.08
0.06
0.04
0.02
log−strike
−1.5
−1.0
−0.5
0.5
3rd ORD x̄=x
2nd ORD x̄=(x+k)/2
5th ORD x̄=x
4th ORD x̄=(x+k)/2



As in the previous set, a comparison between comparable models from the computational point of view shows that the choice of the middle point is once again more performant. As already guessed from the table, however, when considering long maturities our approximation explodes ITM. At first one might also think that this behavior is due to the choice of strikes too extreme, but we see that when considering a 5 or 10-years maturity, this abnormal behavior occurs as soon as we are in a ITM state.

8.1.3 Third Set: $\delta = 0.4$, $\beta = 0.5$

In the third set of parameter we have ($\delta = 0.4$, $\beta = 0.5$). As before, we create the matrix of the strikes

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.25	77	81	85	88	92	96	100	104	109	114	118	122	127
0.5	65	71	77	82	88	94	100	108	115	122	130	138	145
1	49	58	66	74	83	92	100	113	125	138	151	163	176
1.5	39	49	59	70	80	90	100	117	134	151	168	185	202
2	31	42	54	66	77	88	100	121	142	164	185	206	227
3	20	33	47	60	73	87	100	129	158	186	215	244	273
5	9	24	39	54	70	85	100	143	187	230	273	317	360
10	3	19	35	52	68	84	100	177	254	330	407	484	561

then we compute call and put prices

	1	3	5	7	9	11	13
0.25	0.2400	0.1738	0.1245	0.07969	0.04424	0.02235	0.01029
0.5	0.3599	0.2581	0.1794	0.1126	0.05667	0.02548	0.01028
1	0.5197	0.3734	0.2516	0.1588	0.07196	0.02764	0.009813
1.5	0.6202	0.4488	0.3006	0.1940	0.08191	0.03035	0.01005
2	0.7002	0.5043	0.3442	0.2234	0.09033	0.03129	0.009976
3	0.8100	0.5844	0.4094	0.2721	0.1012	0.03385	0.01007
5	0.9195	0.6829	0.4897	0.3475	0.1169	0.03577	0.009949
10	0.9788	0.7762	0.6075	0.4766	0.1399	0.03853	0.009975

	1	3	5	7	9	11	13
0.25	0.009953	0.02383	0.04449	0.07969	0.1342	0.2024	0.2803
0.5	0.009876	0.02815	0.05939	0.1126	0.2067	0.3255	0.4603
1	0.009654	0.03337	0.08155	0.1588	0.3220	0.5376	0.7698
1.5	0.01024	0.03882	0.1006	0.1940	0.4219	0.7104	1.030
2	0.01016	0.04426	0.1142	0.2234	0.5103	0.8813	1.280
3	0.01000	0.05441	0.1394	0.2721	0.6812	1.184	1.740
5	0.009500	0.07294	0.1897	0.3475	0.9869	1.766	2.610
10	0.008796	0.1262	0.2875	0.4766	1.680	3.109	4.620

Even with this set of parameters we have really extreme strikes in the first two columns, but, as we can see in the last matrix, the corresponding put-price is close enough to 10^{-2}.

We compute exact values of the implied volatility:

	1	3	5	7	9	11	13
0.25	42.6906	41.6659	40.8572	40.0166	39.1599	38.3823	37.6713
0.5	44.5073	42.7104	41.3281	40.0329	38.6483	37.4607	36.4242
1	47.6641	44.3863	41.9665	40.0648	37.8650	36.0679	34.6551
1.5	50.3351	45.6469	42.3853	40.0958	37.2218	35.1011	33.4365
2	53.1210	46.6698	42.8216	40.1256	36.6930	34.2426	32.4286
3	58.6938	48.3208	43.4544	40.1818	35.7310	32.9364	30.8803
5	68.9979	50.5961	44.0136	40.2770	34.2488	30.9319	28.6602
10	78.0078	51.4828	44.3241	40.3608	31.6487	27.7744	25.3465

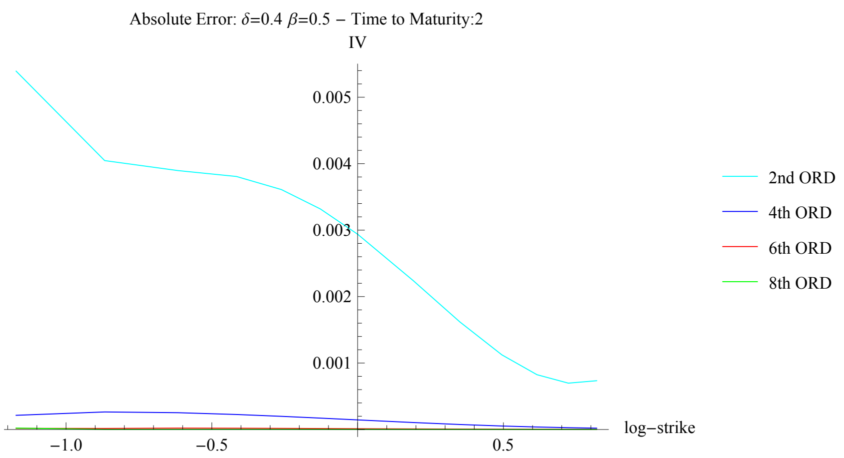
then we calculate our approximation with both choices of $\overline{x}$, and we calculate absolute errors.

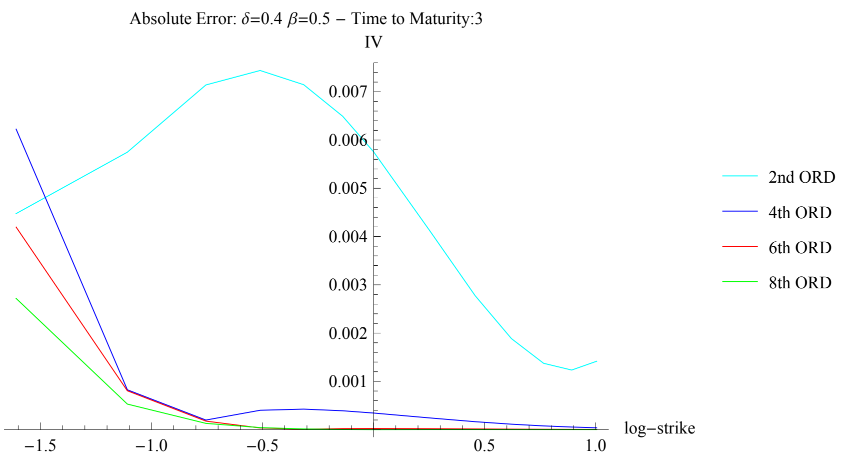
	1		7		13	
0.25	2.69	1.03×10^{-2}	1.66×10^{-2}	1.66×10^{-2}	2.33	8.59×10^{-3}
	7.7×10^{-2}	1.03×10^{-2}	1.66×10^{-2}	1.66×10^{-2}	6.14×10^{-2}	8.59×10^{-3}
	3.56×10^{-3}	4.73×10^{-5}	5.67×10^{-5}	5.67×10^{-5}	2.69×10^{-3}	2.55×10^{-5}
	3.48×10^{-4}	4.73×10^{-5}	5.67×10^{-5}	5.67×10^{-5}	2.51×10^{-4}	2.55×10^{-5}
	3.6×10^{-5}	2.23×10^{-6}	2.06×10^{-6}	2.06×10^{-6}	2.17×10^{-5}	1.74×10^{-6}
	5.64×10^{-6}	2.23×10^{-6}	2.06×10^{-6}	2.06×10^{-6}	4.08×10^{-6}	1.74×10^{-6}
	2.28×10^{-6}	1.79×10^{-6}	1.67×10^{-6}	1.67×10^{-6}	1.25×10^{-6}	1.56×10^{-6}
	1.86×10^{-6}	1.79×10^{-6}	1.67×10^{-6}	1.67×10^{-6}	1.6×10^{-6}	1.56×10^{-6}
	1.8×10^{-6}	1.79×10^{-6}	1.66×10^{-6}	1.66×10^{-6}	1.56×10^{-6}	1.56×10^{-6}
	1.79×10^{-6}	0	1.66×10^{-6}	0	1.56×10^{-6}	0
0.5	4.51	4.1×10^{-2}	3.29×10^{-2}	3.29×10^{-2}	3.58	2.75×10^{-2}
	$2.\times10^{-1}$	4.1×10^{-2}	3.29×10^{-2}	3.29×10^{-2}	1.4×10^{-1}	2.75×10^{-2}
	1.22×10^{-2}	2.05×10^{-4}	2.17×10^{-4}	2.17×10^{-4}	7.9×10^{-3}	7.64×10^{-5}
	1.8×10^{-3}	2.05×10^{-4}	2.17×10^{-4}	2.17×10^{-4}	1.08×10^{-3}	7.64×10^{-5}
	2.61×10^{-4}	5.97×10^{-6}	4.75×10^{-6}	4.75×10^{-6}	1.41×10^{-4}	2.51×10^{-6}
	4.33×10^{-5}	5.97×10^{-6}	4.75×10^{-6}	4.75×10^{-6}	2.3×10^{-5}	2.51×10^{-6}
	9.52×10^{-6}	2.01×10^{-6}	1.74×10^{-6}	1.74×10^{-6}	2.21×10^{-6}	1.53×10^{-6}
	3.4×10^{-6}	2.01×10^{-6}	1.74×10^{-6}	1.74×10^{-6}	2.21×10^{-6}	1.53×10^{-6}
	2.2×10^{-6}	1.88×10^{-6}	1.67×10^{-6}	1.67×10^{-6}	1.37×10^{-6}	1.51×10^{-6}
	1.95×10^{-6}	0	1.67×10^{-6}	0	1.54×10^{-6}	0

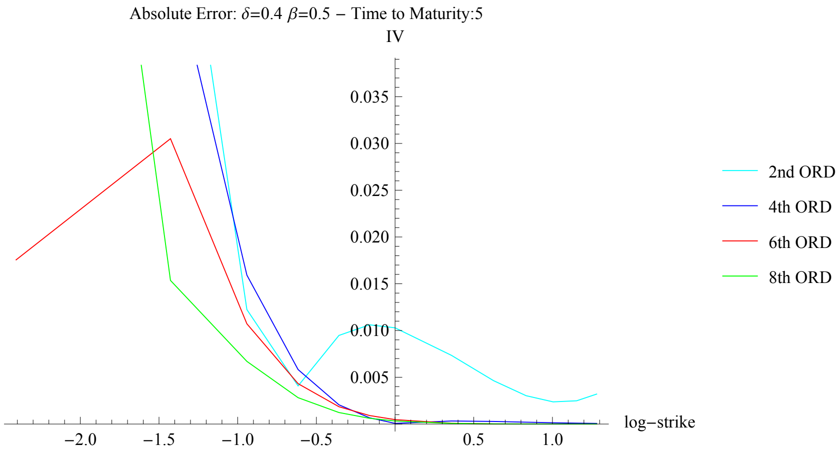
	7.66	1.45×10^{-1}	6.48×10^{-2}	6.48×10^{-2}	5.34	7.31×10^{-2}
	5.31×10^{-1}	1.45×10^{-1}	6.48×10^{-2}	6.48×10^{-2}	3.08×10^{-1}	7.31×10^{-2}
	4.25×10^{-2}	1.04×10^{-3}	8.22×10^{-4}	8.22×10^{-4}	2.21×10^{-2}	2.35×10^{-4}
	9.24×10^{-3}	1.04×10^{-3}	8.22×10^{-4}	8.22×10^{-4}	4.31×10^{-3}	2.35×10^{-4}
1	1.93×10^{-3}	4.29×10^{-5}	2.42×10^{-5}	2.42×10^{-5}	7.9×10^{-4}	6.17×10^{-6}
	4.24×10^{-4}	4.29×10^{-5}	2.42×10^{-5}	2.42×10^{-5}	1.61×10^{-4}	6.17×10^{-6}
	1.1×10^{-4}	4.74×10^{-6}	2.65×10^{-6}	2.65×10^{-6}	3.62×10^{-5}	1.62×10^{-6}
	3.19×10^{-5}	4.74×10^{-6}	2.65×10^{-6}	2.65×10^{-6}	1.1×10^{-5}	1.62×10^{-6}
	1.08×10^{-5}	2.26×10^{-6}	1.73×10^{-6}	1.73×10^{-6}	1.17×10^{-6}	1.45×10^{-6}
	4.72×10^{-6}	0	1.73×10^{-6}	0	2.2×10^{-6}	0
	1.03×10^{1}	2.82×10^{-1}	9.58×10^{-2}	9.58×10^{-2}	6.56	1.16×10^{-1}
	9.19×10^{-1}	2.82×10^{-1}	9.58×10^{-2}	9.58×10^{-2}	4.67×10^{-1}	1.16×10^{-1}
	8.62×10^{-2}	2.75×10^{-3}	1.75×10^{-3}	1.75×10^{-3}	3.85×10^{-2}	4.57×10^{-4}
	2.26×10^{-2}	2.75×10^{-3}	1.75×10^{-3}	1.75×10^{-3}	8.98×10^{-3}	4.57×10^{-4}
1.5	5.62×10^{-3}	1.46×10^{-4}	$7.\times10^{-5}$	$7.\times10^{-5}$	1.93×10^{-3}	1.24×10^{-5}
	1.37×10^{-3}	1.46×10^{-4}	$7.\times10^{-5}$	$7.\times10^{-5}$	4.45×10^{-4}	1.24×10^{-5}
	3.91×10^{-4}	1.42×10^{-5}	5.81×10^{-6}	5.81×10^{-6}	1.18×10^{-4}	1.98×10^{-6}
	1.17×10^{-4}	1.42×10^{-5}	5.81×10^{-6}	5.81×10^{-6}	3.57×10^{-5}	1.98×10^{-6}
	3.51×10^{-5}	2.98×10^{-6}	2.01×10^{-6}	2.01×10^{-6}	8.92×10^{-6}	1.43×10^{-6}
	1.06×10^{-5}	0	2.01×10^{-6}	0	4.59×10^{-6}	0
	1.31×10^{1}	4.86×10^{-1}	1.26×10^{-1}	1.26×10^{-1}	7.57	1.59×10^{-1}
	1.41	4.86×10^{-1}	1.26×10^{-1}	1.26×10^{-1}	6.26×10^{-1}	1.59×10^{-1}
	1.43×10^{-1}	5.39×10^{-3}	2.94×10^{-3}	2.94×10^{-3}	5.63×10^{-2}	7.34×10^{-4}
	4.2×10^{-2}	5.39×10^{-3}	2.94×10^{-3}	2.94×10^{-3}	1.47×10^{-2}	7.34×10^{-4}
2	1.12×10^{-2}	2.14×10^{-4}	1.45×10^{-4}	1.45×10^{-4}	3.44×10^{-3}	2.02×10^{-5}
	2.47×10^{-3}	2.14×10^{-4}	1.45×10^{-4}	1.45×10^{-4}	8.33×10^{-4}	2.02×10^{-5}
	5.26×10^{-4}	1.77×10^{-5}	1.2×10^{-5}	1.2×10^{-5}	2.31×10^{-4}	2.57×10^{-6}
	2.65×10^{-5}	1.77×10^{-5}	1.2×10^{-5}	1.2×10^{-5}	6.71×10^{-5}	2.57×10^{-6}
	9.18×10^{-5}	2.14×10^{-5}	2.61×10^{-6}	2.61×10^{-6}	1.62×10^{-5}	1.45×10^{-6}
	9.91×10^{-5}	0	2.61×10^{-6}	0	4.95×10^{-6}	0
	1.87×10^{1}	1.12	1.82×10^{-1}	1.82×10^{-1}	9.12	2.38×10^{-1}
	2.6	1.12	1.82×10^{-1}	1.82×10^{-1}	9.23×10^{-1}	2.38×10^{-1}
	2.65×10^{-1}	4.48×10^{-3}	5.76×10^{-3}	5.76×10^{-3}	9.32×10^{-2}	1.42×10^{-3}
	7.17×10^{-2}	4.48×10^{-3}	5.76×10^{-3}	5.76×10^{-3}	2.74×10^{-2}	1.42×10^{-3}
3	7.02×10^{-3}	6.23×10^{-3}	3.44×10^{-4}	3.44×10^{-4}	6.68×10^{-3}	3.81×10^{-5}
	1.25×10^{-2}	6.23×10^{-3}	3.44×10^{-4}	3.44×10^{-4}	1.41×10^{-3}	3.81×10^{-5}
	1.34×10^{-2}	4.2×10^{-3}	2.37×10^{-5}	2.37×10^{-5}	2.44×10^{-4}	4.28×10^{-6}
	1.14×10^{-2}	4.2×10^{-3}	2.37×10^{-5}	2.37×10^{-5}	4.82×10^{-5}	4.28×10^{-6}
	9.34×10^{-3}	2.72×10^{-3}	2.23×10^{-7}	2.23×10^{-7}	1.06×10^{-4}	1.58×10^{-6}
	7.54×10^{-3}	0	2.23×10^{-7}	0	9.59×10^{-5}	0
	2.9×10^{1}	4.03	2.77×10^{-1}	2.77×10^{-1}	1.13×10^{1}	3.79×10^{-1}
	4.92	4.03	2.77×10^{-1}	2.77×10^{-1}	1.47	3.79×10^{-1}
	1.8×10^{-1}	2.97×10^{-1}	1.03×10^{-2}	1.03×10^{-2}	1.64×10^{-1}	3.2×10^{-3}
	5.81×10^{-1}	2.97×10^{-1}	1.03×10^{-2}	1.03×10^{-2}	4.9×10^{-2}	3.2×10^{-3}
5	6.69×10^{-1}	1.87×10^{-1}	6.86×10^{-5}	6.86×10^{-5}	6.94×10^{-3}	6.46×10^{-5}
	6.45×10^{-1}	1.87×10^{-1}	6.86×10^{-5}	6.86×10^{-5}	3.86×10^{-3}	6.46×10^{-5}
	5.42×10^{-1}	1.76×10^{-2}	4.71×10^{-4}	4.71×10^{-4}	5.11×10^{-3}	6.2×10^{-6}
	4.34×10^{-1}	1.76×10^{-2}	4.71×10^{-4}	4.71×10^{-4}	4.68×10^{-3}	6.2×10^{-6}
	3.38×10^{-1}	1.37×10^{-1}	3.27×10^{-4}	3.27×10^{-4}	3.95×10^{-3}	1.16×10^{-6}
	2.51×10^{-1}	0	3.27×10^{-4}	0	3.28×10^{-3}	0

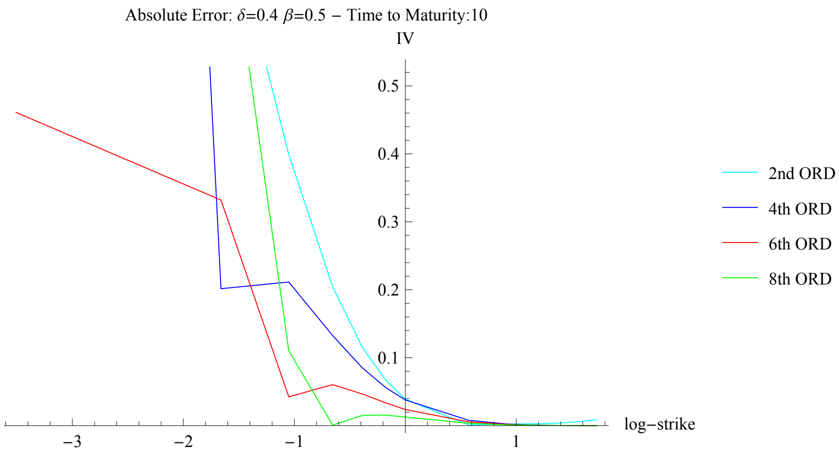
10	3.8×10^{1}	1.81×10^{1}	3.61×10^{-1}	3.61×10^{-1}	1.47×10^{1}	6.44×10^{-1}
	2.94	1.81×10^{1}	3.61×10^{-1}	3.61×10^{-1}	2.59	6.44×10^{-1}
	7.7	6.32	3.92×10^{-2}	3.92×10^{-2}	2.86×10^{-1}	8.91×10^{-3}
	8.29	6.32	3.92×10^{-2}	3.92×10^{-2}	1.03×10^{-3}	8.91×10^{-3}
	7.38	6.18	3.8×10^{-2}	3.8×10^{-2}	1.26×10^{-1}	6.96×10^{-5}
	5.42	6.18	3.8×10^{-2}	3.8×10^{-2}	1.6×10^{-1}	6.96×10^{-5}
	2.43	4.61×10^{-1}	2.39×10^{-2}	2.39×10^{-2}	1.58×10^{-1}	6.44×10^{-5}
	1.06	4.61×10^{-1}	2.39×10^{-2}	2.39×10^{-2}	1.51×10^{-1}	6.44×10^{-5}
	4.81	5.07×10^{1}	1.28×10^{-2}	1.28×10^{-2}	1.45×10^{-1}	4.1×10^{-5}
	8.79	0	1.28×10^{-2}	0	1.42×10^{-1}	0

This choice of parameters, as well as the previous set in 8.1.2, shows the limits of our approximation, especially for long maturity: from the table in fact denotes a worsening of both methods far ITM, but OTM the midpoint method continues to ensure good results already at the fourth order, and paradoxically better than those obtained ATM.

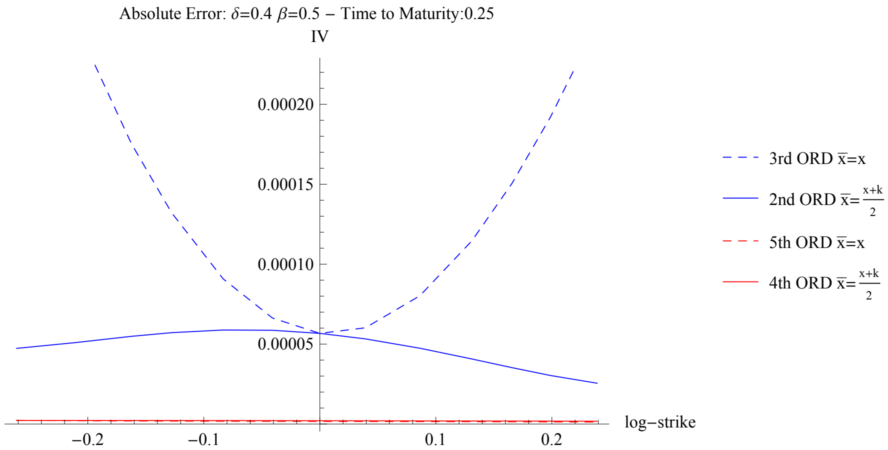




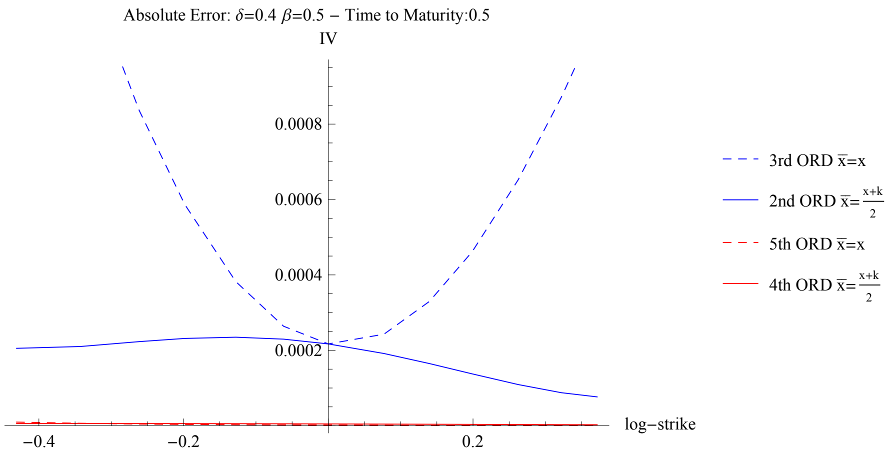




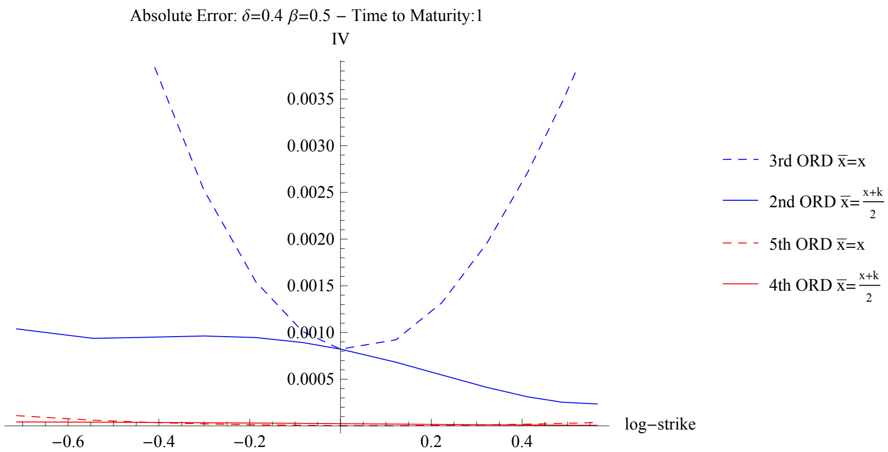
In these graphs of the midpoint method, we can see how our approximation explodes starting from a 3-year maturity, but only in the ITM region, as in the following graphs where we compare the two methods.



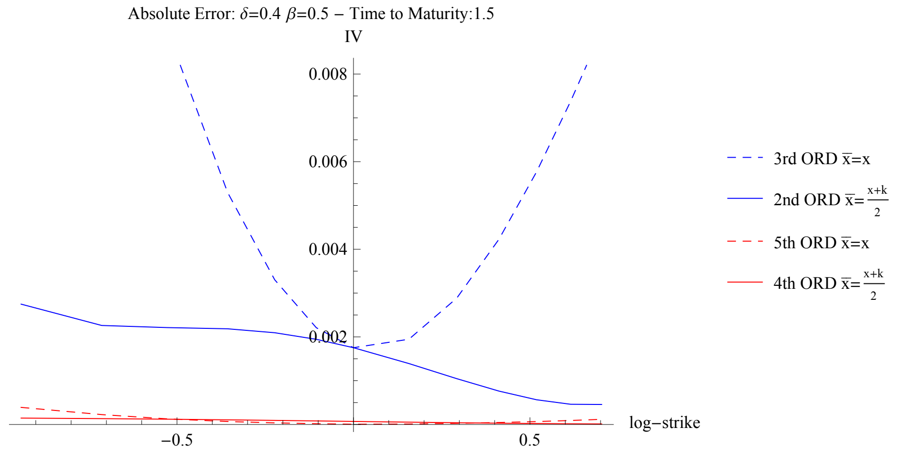
Absolute Error: δ=0.4 β=0.5 – Time to Maturity:0.25
IV
0.00020
0.00015
0.00010
0.00005
−0.2
−0.1
0.1
0.2
log−strike
3rd ORD $\bar{x}$=x
2nd ORD $\bar{x}=\frac{x+k}{2}$
5th ORD $\bar{x}$=x
4th ORD $\bar{x}=\frac{x+k}{2}$

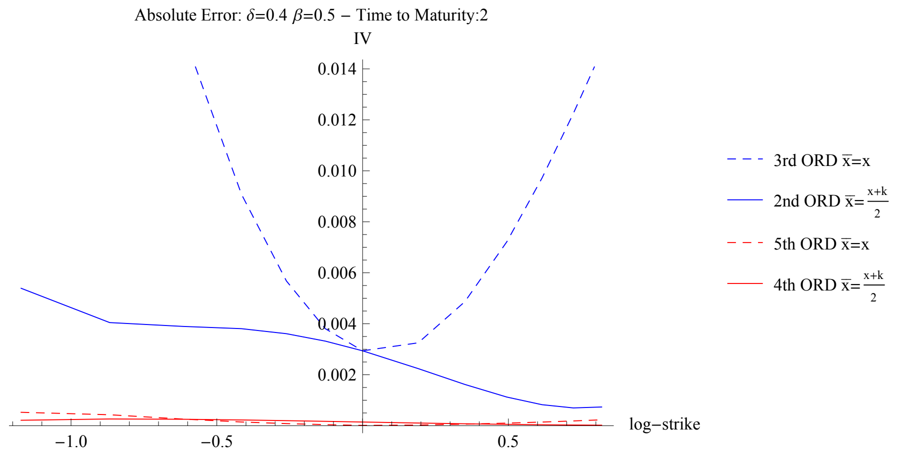


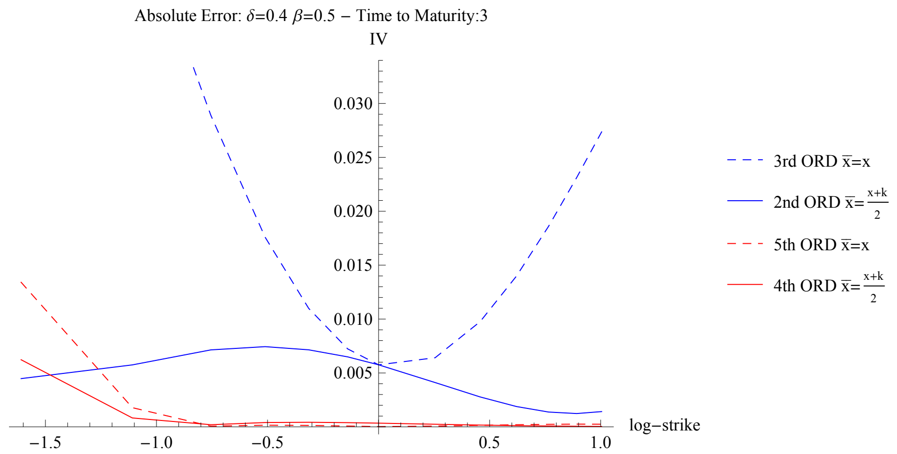
Absolute Error: δ=0.4 β=0.5 – Time to Maturity:0.5
IV
0.0008
0.0006
0.0004
0.0002
−0.4
−0.2
0.2
log−strike
3rd ORD $\bar{x}$=x
2nd ORD $\bar{x}=\frac{x+k}{2}$
5th ORD $\bar{x}$=x
4th ORD $\bar{x}=\frac{x+k}{2}$

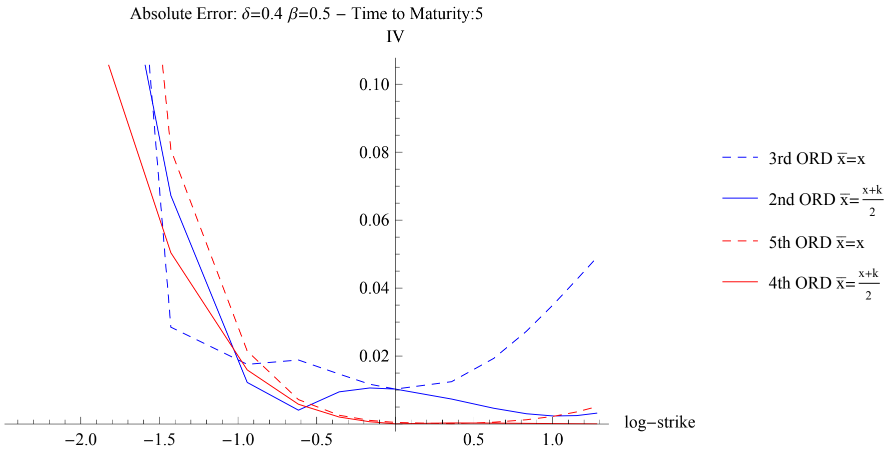


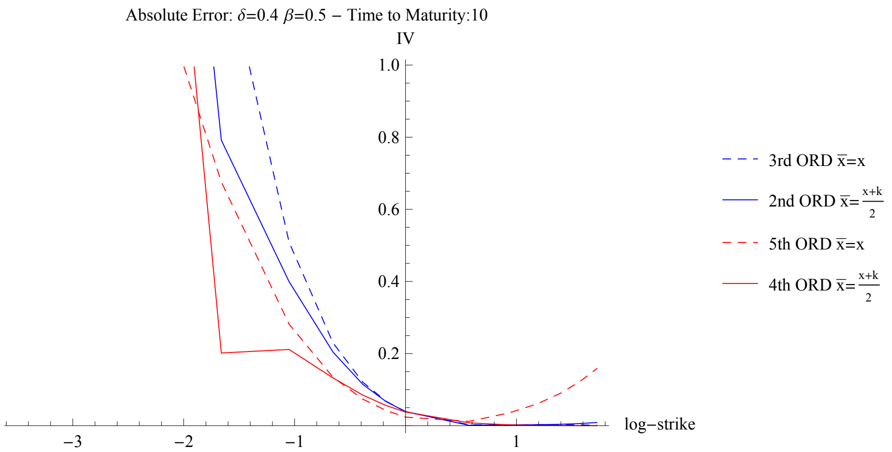
Absolute Error: δ=0.4 β=0.5 – Time to Maturity:1
IV
0.0035
0.0030
0.0025
0.0020
0.0015
0.0010
0.0005
−0.6
−0.4
−0.2
0.2
0.4
log−strike
3rd ORD $\bar{x}$=x
2nd ORD $\bar{x}=\frac{x+k}{2}$
5th ORD $\bar{x}$=x
4th ORD $\bar{x}=\frac{x+k}{2}$


Absolute Error: δ=0.4 β=0.5 – Time to Maturity:1.5
IV
0.008
0.006
0.004
0.002
log–strike
–0.5
0.5
3rd ORD $\bar{x}$=x
2nd ORD $\bar{x}=\frac{x+k}{2}$
5th ORD $\bar{x}$=x
4th ORD $\bar{x}=\frac{x+k}{2}$


Absolute Error: δ=0.4 β=0.5 – Time to Maturity:2
IV
0.014
0.012
0.010
0.008
0.006
0.004
0.002
log–strike
–1.0
–0.5
0.5
3rd ORD $\bar{x}$=x
2nd ORD $\bar{x}=\frac{x+k}{2}$
5th ORD $\bar{x}$=x
4th ORD $\bar{x}=\frac{x+k}{2}$


Absolute Error: δ=0.4 β=0.5 – Time to Maturity:3
IV
0.030
0.025
0.020
0.015
0.010
0.005
log–strike
–1.5
–1.0
–0.5
0.5
1.0
3rd ORD $\bar{x}$=x
2nd ORD $\bar{x}=\frac{x+k}{2}$
5th ORD $\bar{x}$=x
4th ORD $\bar{x}=\frac{x+k}{2}$





8.2 Quadratic local volatility model

In the Quadratic local volatility model, the dynamics of the underlying S are given by

$$d S_t = \left(\frac{\delta}{S_t} \; \frac{(e^{uu} - S_t)\left(e^{ll} - S_t\right)}{e^{uu} - e^{ll}} \right) S_t \, d W_t, \quad S_0 = s > 0, \qquad s < e^{ll} < e^{uu}.$$

Note that volatility increases as $S \to 0^+$, which is consistent with the leverage effect and which results in a negative at-the-money skew in the model-induced implied volatility surface. The left-hand root e^{ll} of the polynomial $(e^{uu} - S_t)\left(e^{ll} - S_t\right)$ is an unattainable boundary for S. The origin, however, is attainable. In order to prevent the process S from taking negative values,one typically specifies zero as an absorbing boundary.

Hence, the state space of S is $\left[0, e^{ll}\right)$. In log notation $X := \operatorname{Log} S$, we have the following dynamics

$$dX_t =$$
$$-\frac{1}{2}\left(\frac{\delta}{e^{X_t}}\,\frac{\left(e^{\mathrm{uu}}-e^{X_t}\right)\left(e^{\mathrm{ll}}-e^{X_t}\right)}{e^{\mathrm{uu}}-e^{\mathrm{ll}}}\right)^2+\frac{\delta}{e^{X_t}}\,\frac{\left(e^{\mathrm{uu}}-e^{X_t}\right)\left(e^{\mathrm{ll}}-e^{X_t}\right)}{e^{\mathrm{uu}}-e^{\mathrm{ll}}}\,dW_t$$
$$X_0 = x := \mathrm{Log}\, s.$$

The generator of X is given by

$$\mathcal{A}=\frac{1}{2}\left(\frac{\delta}{e^{x}}\,\frac{\left(e^{\mathrm{uu}}-e^{x}\right)\left(e^{\mathrm{ll}}-e^{x}\right)}{e^{\mathrm{uu}}-e^{\mathrm{ll}}}\right)^2\left(\partial_x^2-\partial_x\right).$$

Thus, we identify

$$a(t,x,y)=\frac{1}{2}\left(\frac{\delta}{e^{x}}\,\frac{\left(e^{\mathrm{uu}}-e^{x}\right)\left(e^{\mathrm{ll}}-e^{x}\right)}{e^{\mathrm{uu}}-e^{\mathrm{ll}}}\right)^2,$$
$$b(t,x,y)=0,\quad c(t,x,y)=0,\quad \alpha(t,x,y)=0$$

Applying our method to this model, we obtained the formulas of the implied volatility approximation up to the eighth order considering a generic point $\bar{x}$, and the ninth order if we set $\bar{x}$ equal to the starting point x. In fact, we have found general formulas that are valid for any LV model. Here for example we write them up to the fourth order.

$$\sigma_0 = \sqrt{2}\,\sqrt{a[0,0]}$$

$$\sigma_1 = -\frac{a[1,0]\,(-k+x-2\,(x-\bar{x}))}{2\sqrt{2}\,\sqrt{a[0,0]}}$$

$$\sigma_2 = \frac{1}{48\sqrt{2}\,a[0,0]^{3/2}}\Big(r^2 t^2 a[1,0]^2 - 2\,r\,t\,(-k+x)\,a[1,0]^2 - 6\,t\,a[0,0]\,a[1,0]^2 - t^2 a[0,0]^2 a[1,0]^2 + 16\,t\,a[0,0]^2 a[2,0] + (-k+x)^2\left(-6\,a[1,0]^2+16\,a[0,0]\,a[2,0]\right) + 12\,(-k+x)\left(a[1,0]^2-4\,a[0,0]\,a[2,0]\right)(x-\bar{x}) - 12\,a[1,0]^2\,(x-\bar{x})^2 + 48\,a[0,0]\,a[2,0]\,(x-\bar{x})^2\Big)$$

$$\sigma_3 = \frac{1}{192\sqrt{2}\,a[0,0]^{5/2}}\,(-k+x-2\,(x-\bar{x}))\Big(-6\,t\,a[0,0]\,a[1,0]^3 + t^2 a[0,0]^2 a[1,0]^3 + 32\,t\,a[0,0]^2 a[1,0]\,a[2,0] + 8\,t^2 a[0,0]^3 a[1,0]\,a[2,0] + r^2 t^2 a[1,0]\left(3\,a[1,0]^2 - 8\,a[0,0]\,a[2,0]\right) + 2\,r\,t\,(-k+x)\,a[1,0]\left(-3\,a[1,0]^2+8\,a[0,0]\,a[2,0]\right) - 96\,t\,a[0,0]^3 a[3,0] -$$

$$
\begin{aligned}
&4\,(-k+x)^2\left(3\,a[1,0]^3-10\,a[0,0]\,a[1,0]\,a[2,0]+12\,a[0,0]^2\,a[3,0]\right)+\\
&12\,(-k+x)\left(a[1,0]^3-4\,a[0,0]\,a[1,0]\,a[2,0]+8\,a[0,0]^2\,a[3,0]\right)(x-\bar{x})-\\
&12\,a[1,0]^3\,(x-\bar{x})^2+48\,a[0,0]\,a[1,0]\,a[2,0]\,(x-\bar{x})^2-96\,a[0,0]^2\,a[3,0]\,(x-\bar{x})^2\Big)
\end{aligned}
$$

Then we can obtain the formulas of each LV model replacing the functions a, b, c, α, with the specific functions of the model considered.

However, the formulas for the quadratic volatility model are significantly longer than those of the CEV, so we can write here only the formulas up to third order.

$$
\sigma_0 = e^{-\bar{x}}\sqrt{\frac{\left(-e^{\mathrm{ll}}+e^{\bar{x}}\right)^2\left(-e^{\mathrm{uu}}+e^{\bar{x}}\right)^2}{\left(e^{\mathrm{ll}}-e^{\mathrm{uu}}\right)^2}}\;\delta
$$

$$
\sigma_1 = \frac{e^{-\bar{x}}\sqrt{\frac{\left(e^{\mathrm{ll}}-e^{\bar{x}}\right)^2\left(e^{\mathrm{uu}}-e^{\bar{x}}\right)^2}{\left(e^{\mathrm{ll}}-e^{\mathrm{uu}}\right)^2}}\left(e^{\mathrm{ll}+\mathrm{uu}}-e^{2\bar{x}}\right)\delta\,\left(-k+x-2\,(x-\bar{x})\right)}{2\left(-e^{\mathrm{ll}}+e^{\bar{x}}\right)\left(-e^{\mathrm{uu}}+e^{\bar{x}}\right)}
$$

$$
\begin{aligned}
\sigma_2 = {} & \frac{1}{96\left(e^{\mathrm{ll}}-e^{\mathrm{uu}}\right)^6\sqrt{\frac{\left(e^{\mathrm{ll}}-e^{\bar{x}}\right)^2\left(e^{\mathrm{uu}}-e^{\bar{x}}\right)^2}{\left(e^{\mathrm{ll}}-e^{\mathrm{uu}}\right)^2}}}\,e^{-5\bar{x}}\,\delta\,\Big(-8\,e^{4\bar{x}}\left(e^{\mathrm{ll}}-e^{\mathrm{uu}}\right)^4\left(e^{\mathrm{ll}+\mathrm{uu}}-e^{2\bar{x}}\right)^2 r\,t\,(-k+x)+\\
&8\,e^{4\bar{x}}\left(e^{\mathrm{ll}}-e^{\mathrm{uu}}\right)^4\left(e^{\mathrm{ll}+\mathrm{uu}}+e^{2\bar{x}}-2\,e^{\mathrm{ll}+\bar{x}}\right)\left(e^{\mathrm{ll}+\mathrm{uu}}+e^{2\bar{x}}-2\,e^{\mathrm{uu}+\bar{x}}\right)(-k+x)^2+\\
&4\,e^{2\bar{x}}\left(e^{\mathrm{ll}}-e^{\mathrm{uu}}\right)^2\left(e^{\mathrm{ll}}-e^{\bar{x}}\right)^2\left(e^{\mathrm{uu}}-e^{\bar{x}}\right)^2\left(e^{\mathrm{ll}+\mathrm{uu}}+e^{2\bar{x}}-2\,e^{\mathrm{ll}+\bar{x}}\right)\left(e^{\mathrm{ll}+\mathrm{uu}}+e^{2\bar{x}}-2\,e^{\mathrm{uu}+\bar{x}}\right)t\,\delta^2-\\
&\left(e^{\mathrm{ll}+\mathrm{uu}}-e^{2\bar{x}}\right)^2 t^2\Big(e^{2(\mathrm{ll}+\mathrm{uu})}\delta^2+e^{4\bar{x}}\delta^2-2\,e^{2\,\mathrm{ll}+\mathrm{uu}+\bar{x}}\delta^2-2\,e^{\mathrm{ll}+2\,\mathrm{uu}+\bar{x}}\delta^2-2\,e^{\mathrm{ll}+3\bar{x}}\delta^2-\\
&\qquad 2\,e^{\mathrm{uu}+3\bar{x}}\delta^2+e^{2(\mathrm{ll}+\bar{x})}\left(-2\,r+\delta^2\right)+e^{2(\mathrm{uu}+\bar{x})}\left(-2\,r+\delta^2\right)+4\,e^{\mathrm{ll}+\mathrm{uu}+2\bar{x}}\left(r+\delta^2\right)\Big)\\
&\ \Big(e^{2(\mathrm{ll}+\mathrm{uu})}\delta^2+e^{4\bar{x}}\delta^2-2\,e^{2\,\mathrm{ll}+\mathrm{uu}+\bar{x}}\delta^2-2\,e^{\mathrm{ll}+2\,\mathrm{uu}+\bar{x}}\delta^2-2\,e^{\mathrm{ll}+3\bar{x}}\delta^2-2\,e^{\mathrm{uu}+3\bar{x}}\delta^2-\\
&\qquad 4\,e^{\mathrm{ll}+\mathrm{uu}+2\bar{x}}\left(r-\delta^2\right)+e^{2(\mathrm{ll}+\bar{x})}\left(2\,r+\delta^2\right)+e^{2(\mathrm{uu}+\bar{x})}\left(2\,r+\delta^2\right)\Big)-\\
&48\,e^{4\bar{x}}\left(e^{\mathrm{ll}}-e^{\mathrm{uu}}\right)^4\left(-e^{\mathrm{ll}}+e^{\bar{x}}\right)\left(-e^{\mathrm{uu}}+e^{\bar{x}}\right)\left(e^{\mathrm{ll}+\mathrm{uu}}+e^{2\bar{x}}\right)(-k+x)\,(x-\bar{x})+\\
&48\,e^{4\bar{x}}\left(e^{\mathrm{ll}}-e^{\mathrm{uu}}\right)^4\left(-e^{\mathrm{ll}}+e^{\bar{x}}\right)\left(-e^{\mathrm{uu}}+e^{\bar{x}}\right)\left(e^{\mathrm{ll}+\mathrm{uu}}+e^{2\bar{x}}\right)(x-\bar{x})^2\Big)
\end{aligned}
$$

8.2.1 First Set: $\delta = 0.02$, uu = 15, ll = 2.2

For the Quadratic model we consider as a first set of parameters: ($x = 0$, $\delta = 0.02$, uu = 15, ll = 2.2). As in CEV model we consider a vector of times to maturity

3 M	5 M	Y	1.5 Y	2 Y	3 Y	5 Y	10 Y
0.25	0.5	1	1.5	2	3	5	10

and we create a matrix of strikes expressed in bp.

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.25	94	95	96	97	98	99	100	101	102	103	104	105	106
0.5	89	91	93	94	96	98	100	102	104	106	107	109	111
1	81	84	87	90	94	97	100	103	106	109	112	115	118
1.5	75	79	83	88	92	96	100	104	108	112	116	120	124
2	69	74	79	84	90	95	100	105	110	114	119	124	129
3	60	67	73	80	87	93	100	106	113	119	125	132	138
5	43	52	62	72	81	90	100	109	117	126	135	143	152
10	10	25	40	55	70	85	100	113	126	140	153	166	179

Now we can compute numerically call and put prices using the method proposed in Lemma 3.1 of ([And11]).

	1	3	5	7	9	11	13
0.25	0.07066	0.05598	0.04304	0.03202	0.02297	0.01584	0.01048
0.5	0.1202	0.08879	0.06817	0.04528	0.02795	0.01846	0.009829
1	0.1997	0.1493	0.09868	0.06403	0.03823	0.02077	0.01017
1.5	0.2600	0.1916	0.1252	0.07842	0.04447	0.02249	0.01001
2	0.3197	0.2325	0.1497	0.09055	0.04869	0.02479	0.01012
3	0.4103	0.2956	0.1886	0.1109	0.05700	0.02680	0.009941
5	0.5801	0.4086	0.2592	0.1432	0.07258	0.02940	0.01022
10	0.9103	0.6333	0.3900	0.2024	0.09557	0.03498	0.01022

	1	3	5	7	9	11	13
0.25	0.01066	0.01598	0.02304	0.03202	0.04297	0.05584	0.07048
0.5	0.01022	0.01879	0.02817	0.04528	0.06795	0.08846	0.1198
1	0.009685	0.01933	0.03868	0.06403	0.09823	0.1408	0.1902
1.5	0.01005	0.02162	0.04518	0.07842	0.1245	0.1825	0.2500
2	0.009666	0.02253	0.04971	0.09055	0.1487	0.2148	0.3001
3	0.01035	0.02562	0.05861	0.1109	0.1870	0.2768	0.3899
5	0.01006	0.02864	0.06924	0.1432	0.2426	0.3794	0.5302
10	0.01026	0.03333	0.09000	0.2024	0.3556	0.5650	0.8002

and the corresponding values of the exact implied volatility

	1	3	5	7	9	11	13
0.25	16.6182	16.4252	16.2373	16.0543	15.8760	15.7021	15.5325
0.5	17.1300	16.7216	16.4298	16.0586	15.7061	15.4530	15.1296
1	18.0345	17.3539	16.6324	16.0671	15.5440	15.0581	14.6050
1.5	18.7964	17.8123	16.8411	16.0756	15.3856	14.7592	14.1869
2	19.6426	18.3010	17.0563	16.0841	15.2307	14.5450	13.8587
3	21.1195	19.1067	17.3972	16.1013	15.0081	14.1361	13.3079
5	24.9134	20.8504	18.1322	16.1358	14.7306	13.5097	12.5402
10	47.2059	26.1319	19.7494	16.2236	14.1491	12.5276	11.2940

Finally we compute our approximating formulas considering $\bar{x}$=x, and we calculate absolute errors.

	1	3	5	7	9	11	13
	5.68×10^{-1}	3.75×10^{-1}	1.87×10^{-1}	4.24×10^{-3}	1.74×10^{-1}	3.48×10^{-1}	5.18×10^{-1}
	9.75×10^{-3}	6.74×10^{-3}	4.93×10^{-3}	4.24×10^{-3}	4.62×10^{-3}	5.99×10^{-3}	8.31×10^{-3}
	4.81×10^{-4}	3.16×10^{-4}	1.59×10^{-4}	1.11×10^{-5}	1.3×10^{-4}	2.66×10^{-4}	3.96×10^{-4}
	3.27×10^{-5}	2.1×10^{-5}	1.39×10^{-5}	1.11×10^{-5}	1.25×10^{-5}	1.76×10^{-5}	2.6×10^{-5}
0.25	2.54×10^{-6}	1.49×10^{-6}	7.05×10^{-7}	6.13×10^{-8}	5.39×10^{-7}	1.18×10^{-6}	1.95×10^{-6}
	2.32×10^{-7}	1.36×10^{-7}	8.16×10^{-8}	6.13×10^{-8}	7.13×10^{-8}	1.09×10^{-7}	1.74×10^{-7}
	2.33×10^{-8}	1.26×10^{-8}	5.6×10^{-9}	5.34×10^{-10}	4.09×10^{-9}	9.56×10^{-9}	1.7×10^{-8}
	2.54×10^{-9}	1.4×10^{-9}	7.6×10^{-10}	5.34×10^{-10}	6.43×10^{-10}	1.07×10^{-9}	1.85×10^{-9}
	2.57×10^{-10}	1.55×10^{-10}	6.46×10^{-11}	6.31×10^{-12}	4.6×10^{-11}	1.13×10^{-10}	2.14×10^{-10}
	1.2×10^{-11}	2.01×10^{-11}	9.62×10^{-12}	6.31×10^{-12}	7.8×10^{-12}	1.46×10^{-11}	2.68×10^{-11}
	1.08	6.71×10^{-1}	3.8×10^{-1}	8.49×10^{-3}	3.44×10^{-1}	5.97×10^{-1}	9.2×10^{-1}
	2.82×10^{-2}	1.65×10^{-2}	1.13×10^{-2}	8.49×10^{-3}	9.96×10^{-3}	1.36×10^{-2}	2.14×10^{-2}
	1.89×10^{-3}	1.15×10^{-3}	6.54×10^{-4}	4.46×10^{-5}	5.09×10^{-4}	8.94×10^{-4}	1.38×10^{-3}
	1.99×10^{-4}	1.08×10^{-4}	6.68×10^{-5}	4.46×10^{-5}	5.53×10^{-5}	8.15×10^{-5}	1.36×10^{-4}
0.5	2.29×10^{-5}	1.16×10^{-5}	5.95×10^{-6}	4.9×10^{-7}	4.26×10^{-6}	8.2×10^{-6}	1.48×10^{-5}
	3.07×10^{-6}	1.48×10^{-6}	8.26×10^{-7}	4.9×10^{-7}	6.44×10^{-7}	1.04×10^{-6}	1.9×10^{-6}
	4.59×10^{-7}	2.05×10^{-7}	9.72×10^{-8}	8.53×10^{-9}	6.55×10^{-8}	1.35×10^{-7}	2.71×10^{-7}
	7.48×10^{-8}	3.19×10^{-8}	1.61×10^{-8}	8.53×10^{-9}	1.19×10^{-8}	2.08×10^{-8}	4.24×10^{-8}
	1.31×10^{-8}	5.23×10^{-9}	2.31×10^{-9}	2.06×10^{-10}	1.48×10^{-9}	3.25×10^{-9}	7.15×10^{-9}
	2.46×10^{-9}	9.37×10^{-10}	4.32×10^{-10}	2.06×10^{-10}	3.03×10^{-10}	5.77×10^{-10}	1.28×10^{-9}
	1.98	1.3	5.82×10^{-1}	1.7×10^{-2}	5.06×10^{-1}	9.92×10^{-1}	1.45
	8.26×10^{-2}	4.7×10^{-2}	2.39×10^{-2}	1.7×10^{-2}	1.98×10^{-2}	3.09×10^{-2}	4.87×10^{-2}
	7.34×10^{-3}	4.66×10^{-3}	2.06×10^{-3}	1.78×10^{-4}	1.45×10^{-3}	2.88×10^{-3}	4.17×10^{-3}
	1.22×10^{-3}	6.56×10^{-4}	2.89×10^{-4}	1.78×10^{-4}	2.18×10^{-4}	3.72×10^{-4}	6.07×10^{-4}
1	2.09×10^{-4}	1.05×10^{-4}	3.81×10^{-5}	3.92×10^{-6}	2.38×10^{-5}	5.44×10^{-5}	9.4×10^{-5}
	4.11×10^{-5}	1.95×10^{-5}	7.27×10^{-6}	3.92×10^{-6}	5.03×10^{-6}	9.58×10^{-6}	1.71×10^{-5}
	9.14×10^{-6}	4.01×10^{-6}	1.26×10^{-6}	1.36×10^{-7}	7.21×10^{-7}	1.82×10^{-6}	3.48×10^{-6}
	2.2×10^{-6}	9.07×10^{-7}	2.88×10^{-7}	1.36×10^{-7}	1.84×10^{-7}	3.93×10^{-7}	7.75×10^{-7}
	5.7×10^{-7}	2.2×10^{-7}	6.04×10^{-8}	6.56×10^{-9}	3.21×10^{-8}	8.89×10^{-8}	1.86×10^{-7}
	1.57×10^{-7}	5.73×10^{-8}	1.57×10^{-8}	6.56×10^{-9}	9.3×10^{-9}	2.2×10^{-8}	4.79×10^{-8}
	2.75	1.76	7.91×10^{-1}	2.55×10^{-2}	6.64×10^{-1}	1.29	1.86
	1.5×10^{-1}	8.06×10^{-2}	3.85×10^{-2}	2.55×10^{-2}	3.01×10^{-2}	4.87×10^{-2}	7.82×10^{-2}
	1.59×10^{-2}	9.72×10^{-3}	4.27×10^{-3}	4.02×10^{-4}	2.78×10^{-3}	5.48×10^{-3}	7.83×10^{-3}
	3.39×10^{-3}	1.73×10^{-3}	7.13×10^{-4}	4.02×10^{-4}	4.94×10^{-4}	8.71×10^{-4}	1.43×10^{-3}
1.5	7.37×10^{-4}	3.43×10^{-4}	1.21×10^{-4}	1.32×10^{-5}	6.81×10^{-5}	1.56×10^{-4}	2.71×10^{-4}
	1.8×10^{-4}	7.9×10^{-5}	2.75×10^{-5}	1.32×10^{-5}	1.7×10^{-5}	3.35×10^{-5}	6.01×10^{-5}
	5.04×10^{-5}	2.03×10^{-5}	6.07×10^{-6}	6.88×10^{-7}	3.06×10^{-6}	7.84×10^{-6}	1.5×10^{-5}
	1.52×10^{-5}	5.68×10^{-6}	1.66×10^{-6}	6.88×10^{-7}	9.3×10^{-7}	2.06×10^{-6}	4.1×10^{-6}
	4.93×10^{-6}	1.71×10^{-6}	4.41×10^{-7}	4.96×10^{-8}	2.03×10^{-7}	5.72×10^{-7}	1.21×10^{-6}
	1.7×10^{-6}	5.52×10^{-7}	1.38×10^{-7}	4.96×10^{-8}	7.01×10^{-8}	1.74×10^{-7}	3.8×10^{-7}

2	3.59	2.25	1.01	3.41×10^{-2}	8.19×10^{-1}	1.51	2.19
	2.44×10^{-1}	1.24×10^{-1}	5.53×10^{-2}	3.41×10^{-2}	4.08×10^{-2}	6.49×10^{-2}	1.07×10^{-1}
	2.9×10^{-2}	1.7×10^{-2}	7.34×10^{-3}	7.14×10^{-4}	4.49×10^{-3}	8.33×10^{-3}	1.2×10^{-2}
	7.49×10^{-3}	3.6×10^{-3}	1.39×10^{-3}	7.14×10^{-4}	8.89×10^{-4}	1.52×10^{-3}	2.54×10^{-3}
	1.96×10^{-3}	8.4×10^{-4}	2.81×10^{-4}	3.13×10^{-5}	1.45×10^{-4}	3.13×10^{-4}	5.54×10^{-4}
	5.61×10^{-4}	2.27×10^{-4}	7.3×10^{-5}	3.13×10^{-5}	4.07×10^{-5}	7.71×10^{-5}	1.41×10^{-4}
	1.86×10^{-4}	6.83×10^{-5}	1.91×10^{-5}	2.17×10^{-6}	8.64×10^{-6}	2.07×10^{-5}	4.06×10^{-5}
	6.64×10^{-5}	2.24×10^{-5}	5.99×10^{-6}	2.17×10^{-6}	2.95×10^{-6}	6.24×10^{-6}	1.27×10^{-5}
	2.55×10^{-5}	7.92×10^{-6}	1.88×10^{-6}	2.08×10^{-7}	7.58×10^{-7}	1.99×10^{-6}	4.31×10^{-6}
	1.04×10^{-5}	2.98×10^{-6}	6.72×10^{-7}	2.08×10^{-7}	2.95×10^{-7}	6.94×10^{-7}	1.56×10^{-6}
3	5.07	3.06	1.35	5.12×10^{-2}	1.04	1.91	2.74
	4.59×10^{-1}	2.16×10^{-1}	9.03×10^{-2}	5.12×10^{-2}	6.1×10^{-2}	9.99×10^{-2}	1.65×10^{-1}
	6.6×10^{-2}	3.63×10^{-2}	1.51×10^{-2}	1.61×10^{-3}	8.25×10^{-3}	1.53×10^{-2}	2.16×10^{-2}
	2.19×10^{-2}	9.71×10^{-3}	3.48×10^{-3}	1.61×10^{-3}	1.96×10^{-3}	3.42×10^{-3}	5.64×10^{-3}
	7.34×10^{-3}	2.82×10^{-3}	8.86×10^{-4}	1.05×10^{-4}	3.9×10^{-4}	8.54×10^{-4}	1.49×10^{-3}
	2.62×10^{-3}	9.45×10^{-4}	2.79×10^{-4}	1.05×10^{-4}	1.32×10^{-4}	2.54×10^{-4}	4.56×10^{-4}
	1.1×10^{-3}	3.54×10^{-4}	9.16×10^{-5}	1.09×10^{-5}	3.4×10^{-5}	8.3×10^{-5}	1.59×10^{-4}
	4.92×10^{-4}	1.44×10^{-4}	3.49×10^{-5}	1.09×10^{-5}	1.4×10^{-5}	3.03×10^{-5}	6.04×10^{-5}
	2.37×10^{-4}	6.34×10^{-5}	1.37×10^{-5}	1.57×10^{-6}	4.36×10^{-6}	1.17×10^{-5}	2.48×10^{-5}
	1.21×10^{-4}	2.96×10^{-5}	5.95×10^{-6}	1.57×10^{-6}	2.06×10^{-6}	4.93×10^{-6}	1.08×10^{-5}
5	8.86	4.8	2.08	8.58×10^{-2}	1.32	2.54	3.51
	1.25	4.86×10^{-1}	1.8×10^{-1}	8.58×10^{-2}	9.75×10^{-2}	1.68×10^{-1}	2.69×10^{-1}
	2.27×10^{-1}	1.04×10^{-1}	4.07×10^{-2}	4.47×10^{-3}	1.63×10^{-2}	3.18×10^{-2}	4.31×10^{-2}
	1.06×10^{-1}	3.8×10^{-2}	1.21×10^{-2}	4.47×10^{-3}	4.99×10^{-3}	9.07×10^{-3}	1.43×10^{-2}
	5.16×10^{-2}	1.47×10^{-2}	4.12×10^{-3}	4.86×10^{-4}	1.2×10^{-3}	2.84×10^{-3}	4.72×10^{-3}
	2.44×10^{-2}	6.54×10^{-3}	1.67×10^{-3}	4.86×10^{-4}	5.34×10^{-4}	1.06×10^{-3}	1.8×10^{-3}
	1.41×10^{-2}	3.27×10^{-3}	7.31×10^{-4}	8.35×10^{-5}	1.64×10^{-4}	4.36×10^{-4}	7.83×10^{-4}
	8.68×10^{-3}	1.77×10^{-3}	3.58×10^{-4}	8.35×10^{-5}	8.98×10^{-5}	1.99×10^{-4}	3.69×10^{-4}
	5.72×10^{-3}	1.03×10^{-3}	1.86×10^{-4}	1.99×10^{-5}	3.27×10^{-5}	9.66×10^{-5}	1.88×10^{-4}
	3.99×10^{-3}	6.34×10^{-4}	1.05×10^{-4}	1.99×10^{-5}	2.07×10^{-5}	5.08×10^{-5}	1.02×10^{-4}
10	3.12×10^{1}	1.01×10^{1}	3.7	1.74×10^{-1}	1.9	3.52	4.76
	1.04×10^{1}	1.81	4.8×10^{-1}	1.74×10^{-1}	1.85×10^{-1}	3.16×10^{-1}	4.98×10^{-1}
	3.24	5.51×10^{-1}	1.57×10^{-1}	1.79×10^{-2}	4.1×10^{-2}	7.81×10^{-2}	1.03×10^{-1}
	2.52	3.12×10^{-1}	6.71×10^{-2}	1.79×10^{-2}	1.72×10^{-2}	2.97×10^{-2}	4.53×10^{-2}
	2.43	1.85×10^{-1}	3.34×10^{-2}	3.85×10^{-3}	5.22×10^{-3}	1.2×10^{-2}	1.92×10^{-2}
	1.73	1.2×10^{-1}	1.92×10^{-2}	3.85×10^{-3}	3.26×10^{-3}	5.83×10^{-3}	9.2×10^{-3}
	1.64	8.99×10^{-2}	1.22×10^{-2}	1.3×10^{-3}	1.17×10^{-3}	3.03×10^{-3}	5.02×10^{-3}
	1.71	7.24×10^{-2}	8.49×10^{-3}	1.3×10^{-3}	9.4×10^{-4}	1.74×10^{-3}	2.87×10^{-3}
	2.	6.26×10^{-2}	6.34×10^{-3}	6.11×10^{-4}	3.58×10^{-4}	1.02×10^{-3}	1.71×10^{-3}
	2.25	5.75×10^{-2}	5.07×10^{-3}	6.11×10^{-4}	3.51×10^{-4}	6.35×10^{-4}	1.02×10^{-3}

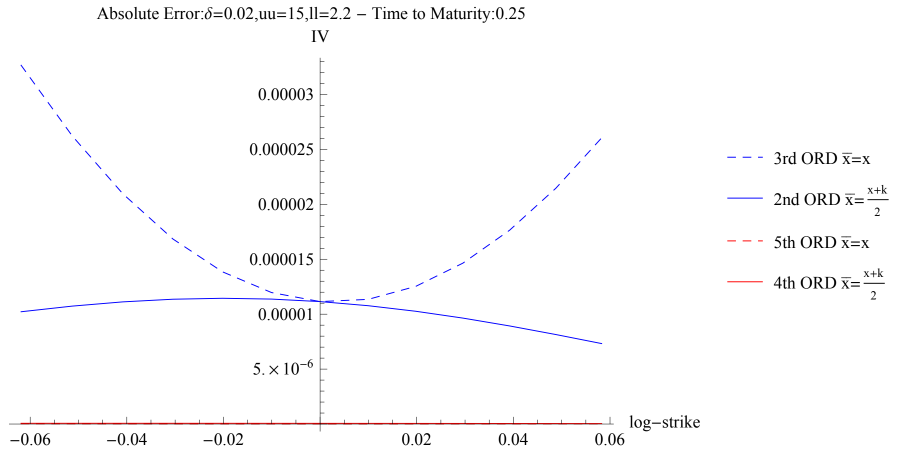
As in CEV model, we can also consider $\overline{x} = \frac{x+k}{2}$, and in the next table we compare the two methods: as before, the first column corresponds to the initial point method, while the second column corresponds to the midpoint method.

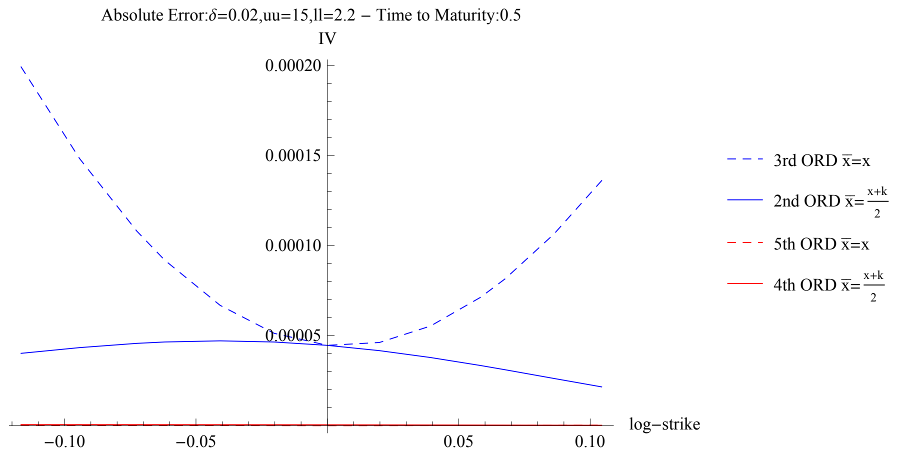
We see that with this choice of parameters we obtain excellent results with almost all strikes considered: the only exception is the 10-year maturity when we are in the ITM region, in which our approximation explodes rather than producing smaller errors.

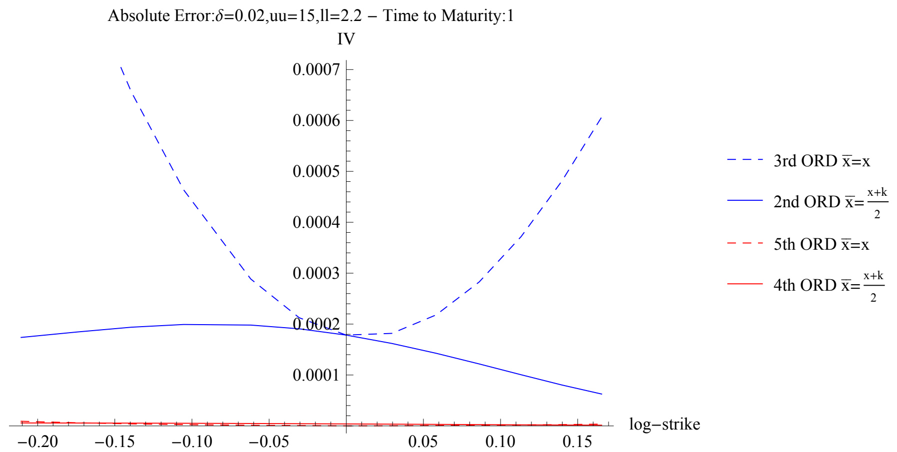
	1		7		13	
0.25	5.68×10^{-1}	1.03×10^{-3}	4.24×10^{-3}	4.24×10^{-3}	5.18×10^{-1}	7.19×10^{-4}
	9.75×10^{-3}	1.03×10^{-3}	4.24×10^{-3}	4.24×10^{-3}	8.31×10^{-3}	7.19×10^{-4}
	4.81×10^{-4}	1.02×10^{-5}	1.11×10^{-5}	1.11×10^{-5}	3.96×10^{-4}	7.32×10^{-6}
	3.27×10^{-5}	1.02×10^{-5}	1.11×10^{-5}	1.11×10^{-5}	2.6×10^{-5}	7.32×10^{-6}
	2.54×10^{-6}	6.55×10^{-8}	6.13×10^{-8}	6.13×10^{-8}	1.95×10^{-6}	4.15×10^{-8}
	2.32×10^{-7}	6.55×10^{-8}	6.13×10^{-8}	6.13×10^{-8}	1.74×10^{-7}	4.15×10^{-8}
	2.33×10^{-8}	5.84×10^{-10}	5.34×10^{-10}	5.34×10^{-10}	1.7×10^{-8}	3.55×10^{-10}
	2.54×10^{-9}	5.84×10^{-10}	5.34×10^{-10}	5.34×10^{-10}	1.85×10^{-9}	3.55×10^{-10}
	2.57×10^{-10}	4.3×10^{-11}	6.31×10^{-12}	6.3×10^{-12}	2.14×10^{-10}	4.01×10^{-12}
	1.2×10^{-11}	0	6.31×10^{-12}	0	2.68×10^{-11}	0
0.5	1.08	3.03×10^{-3}	8.49×10^{-3}	8.49×10^{-3}	9.2×10^{-1}	2.74×10^{-3}
	2.82×10^{-2}	3.03×10^{-3}	8.49×10^{-3}	8.49×10^{-3}	2.14×10^{-2}	2.74×10^{-3}
	1.89×10^{-3}	4.02×10^{-5}	4.46×10^{-5}	4.46×10^{-5}	1.38×10^{-3}	2.15×10^{-5}
	1.99×10^{-4}	4.02×10^{-5}	4.46×10^{-5}	4.46×10^{-5}	1.36×10^{-4}	2.15×10^{-5}
	2.29×10^{-5}	5.78×10^{-7}	4.9×10^{-7}	4.9×10^{-7}	1.48×10^{-5}	2.48×10^{-7}
	3.07×10^{-6}	5.78×10^{-7}	4.9×10^{-7}	4.9×10^{-7}	1.9×10^{-6}	2.48×10^{-7}
	4.59×10^{-7}	1.22×10^{-8}	8.53×10^{-9}	8.53×10^{-9}	2.71×10^{-7}	4.15×10^{-9}
	7.48×10^{-8}	1.22×10^{-8}	8.53×10^{-9}	8.53×10^{-9}	4.24×10^{-8}	4.15×10^{-9}
	1.31×10^{-8}	3.46×10^{-10}	2.06×10^{-10}	2.06×10^{-10}	7.15×10^{-9}	8.53×10^{-11}
	2.46×10^{-9}	0	2.06×10^{-10}	0	1.28×10^{-9}	0
1	1.98	2.12×10^{-2}	1.7×10^{-2}	1.7×10^{-2}	1.45	1.14×10^{-2}
	8.26×10^{-2}	2.12×10^{-2}	1.7×10^{-2}	1.7×10^{-2}	4.87×10^{-2}	1.14×10^{-2}
	7.34×10^{-3}	1.74×10^{-4}	1.78×10^{-4}	1.78×10^{-4}	4.17×10^{-3}	6.26×10^{-5}
	1.22×10^{-3}	1.74×10^{-4}	1.78×10^{-4}	1.78×10^{-4}	6.07×10^{-4}	6.26×10^{-5}
	2.09×10^{-4}	5.83×10^{-6}	3.92×10^{-6}	3.92×10^{-6}	9.4×10^{-5}	1.42×10^{-6}
	4.11×10^{-5}	5.83×10^{-6}	3.92×10^{-6}	3.92×10^{-6}	1.71×10^{-5}	1.42×10^{-6}
	9.14×10^{-6}	2.78×10^{-7}	1.36×10^{-7}	1.36×10^{-7}	3.48×10^{-6}	4.57×10^{-8}
	2.2×10^{-6}	2.78×10^{-7}	1.36×10^{-7}	1.36×10^{-7}	7.75×10^{-7}	4.57×10^{-8}
	5.7×10^{-7}	1.77×10^{-8}	6.56×10^{-9}	6.56×10^{-9}	1.86×10^{-7}	1.95×10^{-9}
	1.57×10^{-7}	0	6.56×10^{-9}	0	4.79×10^{-8}	0
1.5	2.75	4.6×10^{-2}	2.55×10^{-2}	2.55×10^{-2}	1.86	2.25×10^{-2}
	1.5×10^{-1}	4.6×10^{-2}	2.55×10^{-2}	2.55×10^{-2}	7.82×10^{-2}	2.25×10^{-2}
	1.59×10^{-2}	4.52×10^{-4}	4.02×10^{-4}	4.02×10^{-4}	7.83×10^{-3}	1.1×10^{-4}
	3.39×10^{-3}	4.52×10^{-4}	4.02×10^{-4}	4.02×10^{-4}	1.43×10^{-3}	1.1×10^{-4}
	7.37×10^{-4}	2.49×10^{-5}	1.32×10^{-5}	1.32×10^{-5}	2.71×10^{-4}	3.69×10^{-6}
	1.8×10^{-4}	2.49×10^{-5}	1.32×10^{-5}	1.32×10^{-5}	6.01×10^{-5}	3.69×10^{-6}
	5.04×10^{-5}	1.95×10^{-6}	6.88×10^{-7}	6.88×10^{-7}	1.5×10^{-5}	1.71×10^{-7}
	1.52×10^{-5}	1.95×10^{-6}	6.88×10^{-7}	6.88×10^{-7}	4.1×10^{-6}	1.71×10^{-7}
	4.93×10^{-6}	2.03×10^{-7}	4.96×10^{-8}	4.96×10^{-8}	1.21×10^{-6}	1.05×10^{-8}
	1.7×10^{-6}	0	4.96×10^{-8}	0	3.8×10^{-7}	0

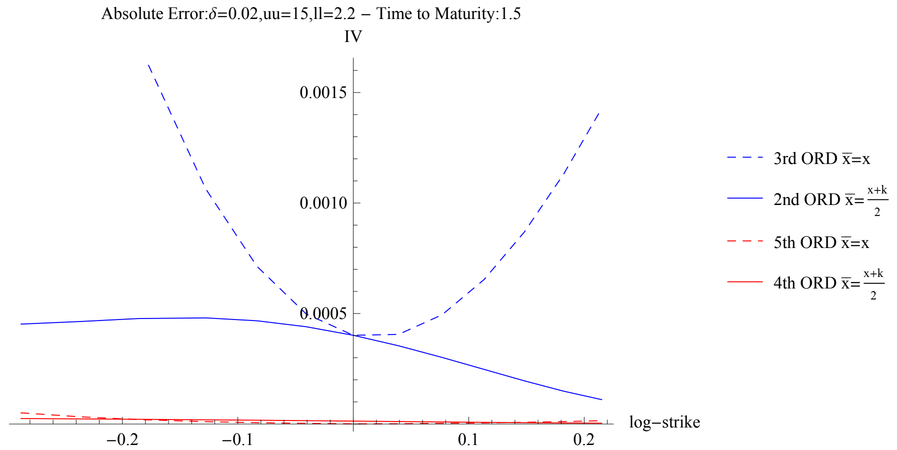
2	3.59	8.72×10^{-2}	3.41×10^{-2}	3.41×10^{-2}	2.19	3.35×10^{-2}
	2.44×10^{-1}	8.72×10^{-2}	3.41×10^{-2}	3.41×10^{-2}	1.07×10^{-1}	3.35×10^{-2}
	2.9×10^{-2}	9.51×10^{-4}	7.14×10^{-4}	7.14×10^{-4}	1.2×10^{-2}	1.65×10^{-4}
	7.49×10^{-3}	9.51×10^{-4}	7.14×10^{-4}	7.14×10^{-4}	2.54×10^{-3}	1.65×10^{-4}
	1.96×10^{-3}	7.61×10^{-5}	3.13×10^{-5}	3.13×10^{-5}	5.54×10^{-4}	7.19×10^{-6}
	5.61×10^{-4}	7.61×10^{-5}	3.13×10^{-5}	3.13×10^{-5}	1.41×10^{-4}	7.19×10^{-6}
	1.86×10^{-4}	8.77×10^{-6}	2.17×10^{-6}	2.17×10^{-6}	4.06×10^{-5}	4.3×10^{-7}
	6.64×10^{-5}	8.77×10^{-6}	2.17×10^{-6}	2.17×10^{-6}	1.27×10^{-5}	4.3×10^{-7}
	2.55×10^{-5}	1.34×10^{-6}	2.08×10^{-7}	2.08×10^{-7}	4.31×10^{-6}	3.4×10^{-8}
	1.04×10^{-5}	0	2.08×10^{-7}	0	1.56×10^{-6}	0
3	5.07	1.83×10^{-1}	5.12×10^{-2}	5.12×10^{-2}	2.74	5.74×10^{-2}
	4.59×10^{-1}	1.83×10^{-1}	5.12×10^{-2}	5.12×10^{-2}	1.65×10^{-1}	5.74×10^{-2}
	6.6×10^{-2}	3.05×10^{-3}	1.61×10^{-3}	1.61×10^{-3}	2.16×10^{-2}	2.8×10^{-4}
	2.19×10^{-2}	3.05×10^{-3}	1.61×10^{-3}	1.61×10^{-3}	5.64×10^{-3}	2.8×10^{-4}
	7.34×10^{-3}	4.15×10^{-4}	1.05×10^{-4}	1.05×10^{-4}	1.49×10^{-3}	1.73×10^{-5}
	2.62×10^{-3}	4.15×10^{-4}	1.05×10^{-4}	1.05×10^{-4}	4.56×10^{-4}	1.73×10^{-5}
	1.1×10^{-3}	8.31×10^{-5}	1.09×10^{-5}	1.09×10^{-5}	1.59×10^{-4}	1.47×10^{-6}
	4.92×10^{-4}	8.31×10^{-5}	1.09×10^{-5}	1.09×10^{-5}	6.04×10^{-5}	1.47×10^{-6}
	2.37×10^{-4}	2.2×10^{-5}	1.57×10^{-6}	1.57×10^{-6}	2.48×10^{-5}	1.63×10^{-7}
	1.21×10^{-4}	0	1.57×10^{-6}	0	1.08×10^{-5}	0
5	8.86	6.13×10^{-1}	8.58×10^{-2}	8.58×10^{-2}	3.51	$1.\times10^{-1}$
	1.25	6.13×10^{-1}	8.58×10^{-2}	8.58×10^{-2}	2.69×10^{-1}	$1.\times10^{-1}$
	2.27×10^{-1}	2.07×10^{-2}	4.47×10^{-3}	4.47×10^{-3}	4.31×10^{-2}	5.43×10^{-4}
	1.06×10^{-1}	2.07×10^{-2}	4.47×10^{-3}	4.47×10^{-3}	1.43×10^{-2}	5.43×10^{-4}
	5.16×10^{-2}	6.09×10^{-3}	4.86×10^{-4}	4.86×10^{-4}	4.72×10^{-3}	5.09×10^{-5}
	2.44×10^{-2}	6.09×10^{-3}	4.86×10^{-4}	4.86×10^{-4}	1.8×10^{-3}	5.09×10^{-5}
	1.41×10^{-2}	2.85×10^{-3}	8.35×10^{-5}	8.35×10^{-5}	7.83×10^{-4}	6.58×10^{-6}
	8.68×10^{-3}	2.85×10^{-3}	8.35×10^{-5}	8.35×10^{-5}	3.69×10^{-4}	6.58×10^{-6}
	5.72×10^{-3}	1.75×10^{-3}	1.99×10^{-5}	1.99×10^{-5}	1.88×10^{-4}	1.11×10^{-6}
	3.99×10^{-3}	0	1.99×10^{-5}	0	1.02×10^{-4}	0
10	3.12×10^{1}	7.87	1.74×10^{-1}	1.74×10^{-1}	4.76	1.97×10^{-1}
	1.04×10^{1}	7.87	1.74×10^{-1}	1.74×10^{-1}	4.98×10^{-1}	1.97×10^{-1}
	3.24	4.37	1.79×10^{-2}	1.79×10^{-2}	1.03×10^{-1}	1.23×10^{-3}
	2.52	4.37	1.79×10^{-2}	1.79×10^{-2}	4.53×10^{-2}	1.23×10^{-3}
	2.43	7.66	3.85×10^{-3}	3.85×10^{-3}	1.92×10^{-2}	1.9×10^{-4}
	1.73	7.66	3.85×10^{-3}	3.85×10^{-3}	9.2×10^{-3}	1.9×10^{-4}
	1.64	2.28×10^{1}	1.3×10^{-3}	1.3×10^{-3}	5.02×10^{-3}	4.19×10^{-5}
	1.71	2.28×10^{1}	1.3×10^{-3}	1.3×10^{-3}	2.87×10^{-3}	4.19×10^{-5}
	2.	8.03×10^{1}	6.11×10^{-4}	6.11×10^{-4}	1.71×10^{-3}	1.2×10^{-5}
	2.25	0	6.11×10^{-4}	0	1.02×10^{-3}	0

We can also observe that as in the CEV model, in this case the choice of the midpoint guarantees better results with lower orders, as we can see in the following plots; thus, for a practical usage of these formulas, the fourth or fifth order with midpoint is on average the best choice.

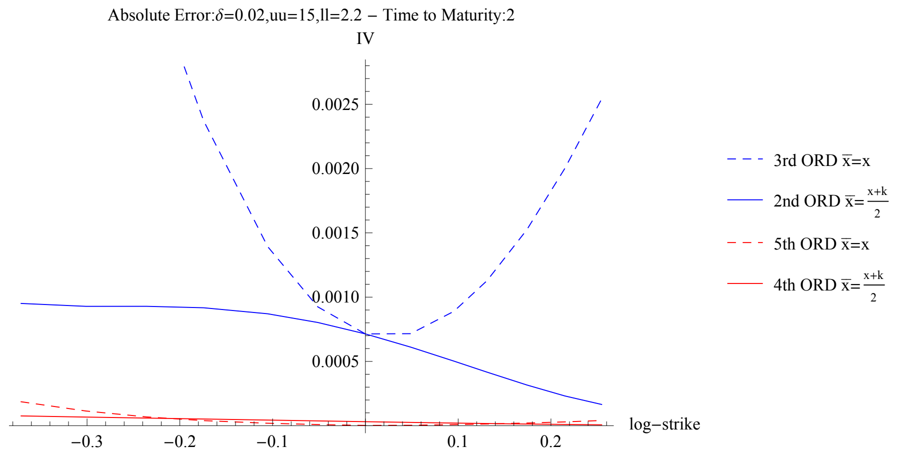

Absolute Error:δ=0.02,uu=15,ll=2.2 – Time to Maturity:0.25
IV
0.00003
0.000025
0.00002
0.000015
0.00001
5. × 10^−6
log–strike
−0.06
−0.04
−0.02
0.02
0.04
0.06
3rd ORD x̄=x
2nd ORD x̄=(x+k)/2
5th ORD x̄=x
4th ORD x̄=(x+k)/2


Absolute Error:δ=0.02,uu=15,ll=2.2 – Time to Maturity:0.5
IV
0.00020
0.00015
0.00010
0.00005
log–strike
−0.10
−0.05
0.05
0.10
3rd ORD x̄=x
2nd ORD x̄=(x+k)/2
5th ORD x̄=x
4th ORD x̄=(x+k)/2

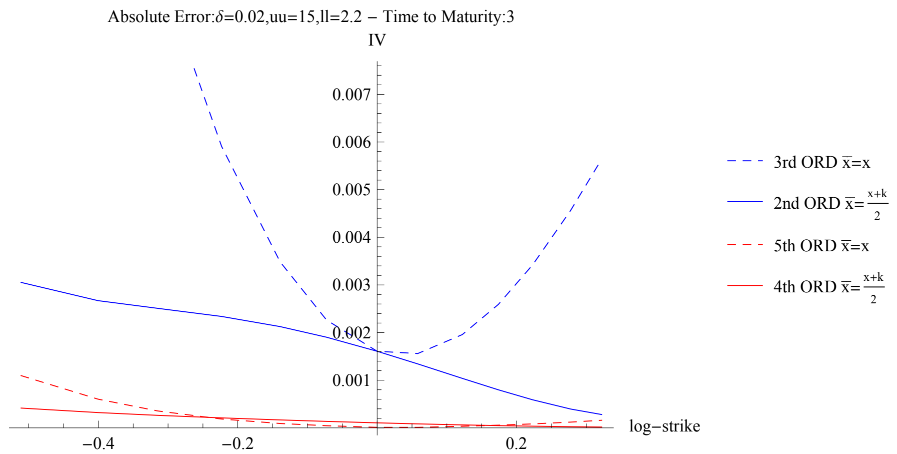

Absolute Error:δ=0.02,uu=15,ll=2.2 – Time to Maturity:1
IV
0.0007
0.0006
0.0005
0.0004
0.0003
0.0002
0.0001
log–strike
−0.20
−0.15
−0.10
−0.05
0.05
0.10
0.15
3rd ORD x̄=x
2nd ORD x̄=(x+k)/2
5th ORD x̄=x
4th ORD x̄=(x+k)/2



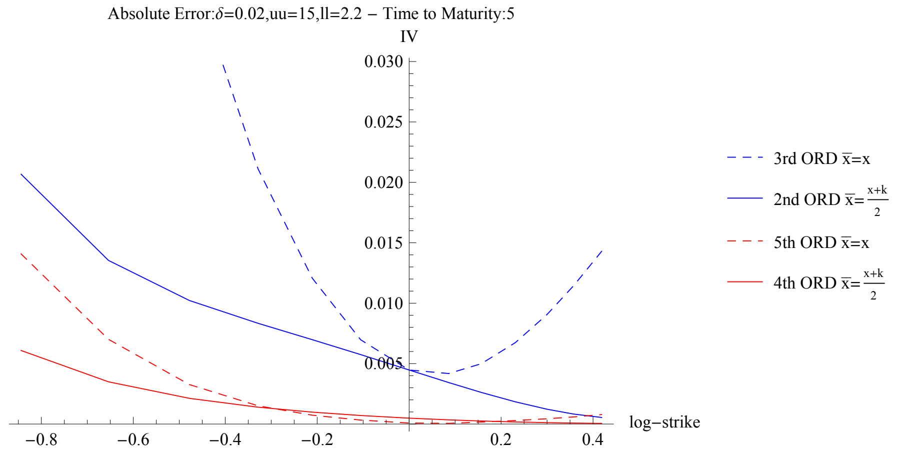
Absolute Error:δ=0.02,uu=15,ll=2.2 − Time to Maturity:1.5
IV
0.0015
0.0010
0.0005
log−strike
−0.2
−0.1
0.1
0.2
3rd ORD x̄=x
2nd ORD x̄=(x+k)/2
5th ORD x̄=x
4th ORD x̄=(x+k)/2

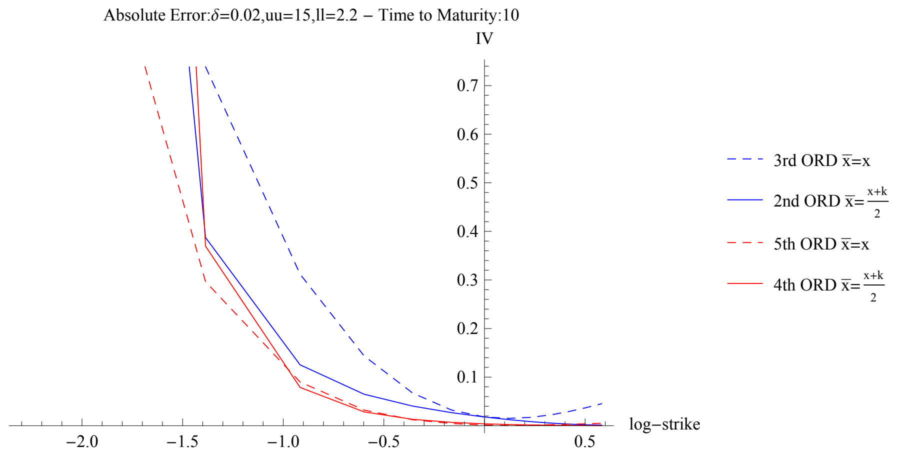


Absolute Error:δ=0.02,uu=15,ll=2.2 − Time to Maturity:2
IV
0.0025
0.0020
0.0015
0.0010
0.0005
log−strike
−0.3
−0.2
−0.1
0.1
0.2
3rd ORD x̄=x
2nd ORD x̄=(x+k)/2
5th ORD x̄=x
4th ORD x̄=(x+k)/2



Absolute Error:δ=0.02,uu=15,ll=2.2 − Time to Maturity:3
IV
0.007
0.006
0.005
0.004
0.003
0.002
0.001
log−strike
−0.4
−0.2
0.2
3rd ORD x̄=x
2nd ORD x̄=(x+k)/2
5th ORD x̄=x
4th ORD x̄=(x+k)/2





However, it is appropriate to specify that the goodness of these results is strongly linked to the choice of parameters, in fact choosing ($\delta = 0.05$, uu = 2.2, ll = 2) in next section we have obtained errors well above acceptable limits.

8.2.2 Second Set: δ = 0.05, uu = 2.2, ll = 2

In this section we consider as we said (δ=0.05,uu=2.2,ll=2). First of all we create the table of strikes.

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.25	10	25	40	55	70	85	100	115	130	145	160	175	190
0.5	10	25	40	55	70	85	100	115	130	145	160	175	190
1	10	25	40	55	70	85	100	115	130	145	160	175	190
1.5	10	25	40	55	70	85	100	115	130	145	160	175	190
2	10	25	40	55	70	85	100	115	130	145	160	175	190
3	10	25	40	55	70	85	100	115	130	145	160	175	190
5	10	25	40	55	70	85	100	115	130	145	160	175	190
10	10	25	40	55	70	85	100	115	130	145	160	175	190

We have set 10 bp and 190 bp as limitis of our range because the value of the asset S is confined between 0 and the left root ll, and not considering European call and put prices, as a matter of fact these prices never becomes smaller that 10^{-2} in this range, as we can see in the two following tables.

	1	3	5	7	9	11	13
0.25	0.9708	0.7201	0.4975	0.3126	0.1730	0.08085	0.03016
0.5	1.072	0.8379	0.6261	0.4420	0.2902	0.1735	0.09196
1	1.244	1.020	0.8124	0.6251	0.4607	0.3217	0.2099
1.5	1.386	1.164	0.9567	0.7655	0.5929	0.4414	0.3127
2	1.509	1.288	1.079	0.8835	0.7047	0.5441	0.4037
3	1.717	1.494	1.280	1.078	0.8899	0.7162	0.5593
5	2.031	1.802	1.580	1.367	1.165	0.9739	0.7961
10	2.498	2.256	2.020	1.790	1.567	1.353	1.148

	1	3	5	7	9	11	13
0.25	0.07081	0.1201	0.1975	0.3126	0.4730	0.6808	0.9302
0.5	0.1720	0.2379	0.3261	0.4420	0.5902	0.7735	0.9920
1	0.3437	0.4196	0.5124	0.6251	0.7607	0.9217	1.110
1.5	0.4860	0.5643	0.6567	0.7655	0.8929	1.041	1.213
2	0.6093	0.6877	0.7785	0.8835	1.005	1.144	1.304
3	0.8174	0.8937	0.9803	1.078	1.190	1.316	1.459
5	1.131	1.202	1.280	1.367	1.465	1.574	1.696
10	1.598	1.656	1.720	1.790	1.867	1.953	2.048

Now we numerically compute the exact implied volatility corresponding to these prices.

	1	3	5	7	9	11	13
0.25	1516.161	280.797	202.474	160.939	133.709	113.911	98.593
0.5	3541.393	735.526	212.555	165.707	136.358	115.517	99.620
1	2100.000	9480.363	242.331	177.464	142.419	119.043	101.818
1.5	2100.000	10798.438	317.469	194.134	149.845	123.075	104.231
2	2100.000	12493.8663	13514.119	221.982	159.295	127.719	106.862
3	2100.000	1431.074	2540.429	3242.972	191.738	139.540	112.764
5	2100.000	2505.786	8834.002	1835.552	3683.198	206.870	128.666
10	2100.000	6514.533	2261.573	919.668	547.501	4046.632	5644.426

Then we compute our approximation, and in the following table we have the absolute errors.

	1	3	5	7	9	11	13
0.25	9.21×10^{2} 9.21×10^{2} 1.43×10^{3} 1.43×10^{3} 1.84×10^{3} 1.84×10^{3} $2.\times10^{3}$ $2.\times10^{3}$ 1.06×10^{5}	6.77 6.77 1.39×10^{1} 1.39×10^{1} 2.09×10^{1} 2.09×10^{1} 4.56×10^{1} 4.56×10^{1} 1.3×10^{2}	6.48 6.48 3.33 3.33 2.83 2.83 3.67 3.67 6.47	4.23 4.23 1.24 1.24 7.19×10^{-1} 7.19×10^{-1} 6.46×10^{-1} 6.46×10^{-1} 7.97×10^{-1}	1.63 1.63 5.1×10^{-1} 5.1×10^{-1} 2.3×10^{-1} 2.3×10^{-1} 1.59×10^{-1} 1.59×10^{-1} 1.5×10^{-1}	9.18×10^{-1} 9.18×10^{-1} 1.98×10^{-1} 1.98×10^{-1} 7.83×10^{-2} 7.83×10^{-2} 4.51×10^{-2} 4.51×10^{-2} 3.47×10^{-2}	3.32 3.32 6.82×10^{-2} 6.82×10^{-2} 2.57×10^{-2} 2.57×10^{-2} 1.32×10^{-2} 1.32×10^{-2} 8.74×10^{-3}
0.5	2.95×10^{3} 2.95×10^{3} 4.95×10^{3} 4.95×10^{3} 1.56×10^{4} 1.56×10^{4} 1.09×10^{5} 1.09×10^{5} 3.08×10^{4}	4.62×10^{2} 4.62×10^{2} 4.88×10^{2} 4.88×10^{2} 5.71×10^{2} 5.71×10^{2} 8.74×10^{2} 8.74×10^{2} 1.92×10^{3}	1.66×10^{1} 1.66×10^{1} 1.41×10^{1} 1.41×10^{1} 2.12×10^{1} 2.12×10^{1} 4.75×10^{1} 4.75×10^{1} 1.41×10^{2}	9. 9. 5.03 5.03 5.44 5.44 8.98 8.98 2.01×10^{1}	4.28 4.28 2.14 2.14 1.82 1.82 2.36 2.36 4.14	6.88×10^{-1} 6.88×10^{-1} 9.61×10^{-1} 9.61×10^{-1} 6.9×10^{-1} 6.9×10^{-1} 7.38×10^{-1} 7.38×10^{-1} 1.06	2.29 2.29 4.22×10^{-1} 4.22×10^{-1} 2.73×10^{-1} 2.73×10^{-1} 2.53×10^{-1} 2.53×10^{-1} 3.11×10^{-1}
1	1.5×10^{3} 1.5×10^{3} 9.89×10^{3} 9.89×10^{3} 5.89×10^{5} 5.89×10^{5} 5.86×10^{7} 5.86×10^{7} 6.38×10^{9}	9.21×10^{3} 9.21×10^{3} 9.36×10^{3} 9.36×10^{3} 9.73×10^{3} 9.73×10^{3} 9.27×10^{3} 9.27×10^{3} 2.26×10^{4}	4.63×10^{1} 4.63×10^{1} 6.24×10^{1} 6.24×10^{1} 1.41×10^{2} 1.41×10^{2} 4.3×10^{2} 4.3×10^{2} 1.36×10^{3}	2.08×10^{1} 2.08×10^{1} 2.09×10^{1} 2.09×10^{1} 3.85×10^{1} 3.85×10^{1} 1.04×10^{2} 1.04×10^{2} 3.6×10^{2}	1.03×10^{1} 1.03×10^{1} 8.94 8.94 1.36×10^{1} 1.36×10^{1} 3.07×10^{1} 3.07×10^{1} 9.17×10^{1}	4.21 4.21 4.26 4.26 5.5 5.5 1.06×10^{1} 1.06×10^{1} 2.7×10^{1}	9.08×10^{-2} 9.08×10^{-2} 2.13 2.13 2.42 2.42 4.03 4.03 8.9
1.5	1.5×10^{3} 1.5×10^{3} 2.08×10^{4} 2.08×10^{4} 3.51×10^{6} 3.51×10^{6} 9.43×10^{8} 9.43×10^{8} 2.93×10^{11}	1.05×10^{4} 1.05×10^{4} 1.09×10^{4} 1.09×10^{4} 1.06×10^{4} 1.06×10^{4} 2.9×10^{3} 2.9×10^{3} 1.82×10^{5}	1.21×10^{2} 1.21×10^{2} 1.79×10^{2} 1.79×10^{2} 3.94×10^{2} 3.94×10^{2} 8.79×10^{2} 8.79×10^{2} 3.32×10^{3}	3.74×10^{1} 3.74×10^{1} 4.98×10^{1} 4.98×10^{1} 1.14×10^{2} 1.14×10^{2} 3.54×10^{2} 3.54×10^{2} 1.15×10^{3}	1.78×10^{1} 1.78×10^{1} 2.08×10^{1} 2.08×10^{1} 4.19×10^{1} 4.19×10^{1} 1.21×10^{2} 1.21×10^{2} 4.37×10^{2}	8.25 8.25 $1.\times10^{1}$ $1.\times10^{1}$ 1.77×10^{1} 1.77×10^{1} 4.54×10^{1} 4.54×10^{1} 1.51×10^{2}	2.32 2.32 5.18 5.18 8.11 8.11 1.85×10^{1} 1.85×10^{1} 5.51×10^{1}
2	1.5×10^{3} 1.5×10^{3} 3.63×10^{4} 3.63×10^{4} 1.19×10^{7} 1.19×10^{7} 6.16×10^{9} 6.16×10^{9} 3.76×10^{12}	1.22×10^{4} 1.22×10^{4} 1.29×10^{4} 1.29×10^{4} 8.18×10^{3} 8.18×10^{3} 3.68×10^{3} 3.68×10^{3} 1.14×10^{5}	1.33×10^{4} 1.33×10^{4} 1.34×10^{4} 1.34×10^{4} 1.38×10^{4} 1.38×10^{4} 1.28×10^{4} 1.28×10^{4} 2.43×10^{4}	6.53×10^{1} 6.53×10^{1} 9.79×10^{1} 9.79×10^{1} 2.34×10^{2} 2.34×10^{2} 6.57×10^{2} 6.57×10^{2} 2.18×10^{2}	2.72×10^{1} 2.72×10^{1} 3.83×10^{1} 3.83×10^{1} $9.\times10^{1}$ $9.\times10^{1}$ 2.86×10^{2} 2.86×10^{2} 9.64×10^{2}	1.29×10^{1} 1.29×10^{1} 1.83×10^{1} 1.83×10^{1} 3.92×10^{1} 3.92×10^{1} 1.18×10^{2} 1.18×10^{2} 4.35×10^{2}	4.95 4.95 9.6 9.6 1.86×10^{1} 1.86×10^{1} 5.13×10^{1} 5.13×10^{1} 1.8×10^{2}

	1.5×10^{3}	1.16×10^{3}	2.34×10^{3}	3.09×10^{3}	5.97×10^{1}	2.47×10^{1}	1.09×10^{1}
	1.5×10^{3}	1.16×10^{3}	2.34×10^{3}	3.09×10^{3}	5.97×10^{1}	2.47×10^{1}	1.09×10^{1}
	8.1×10^{4}	2.89×10^{3}	2.66×10^{3}	3.18×10^{3}	9.81×10^{1}	4.34×10^{1}	2.25×10^{1}
	8.1×10^{4}	2.89×10^{3}	2.66×10^{3}	3.18×10^{3}	9.81×10^{1}	4.34×10^{1}	2.25×10^{1}
3	6.47×10^{7}	4.22×10^{4}	2.19×10^{3}	3.45×10^{3}	2.43×10^{2}	1.13×10^{2}	5.61×10^{1}
	6.47×10^{7}	4.22×10^{4}	2.19×10^{3}	3.45×10^{3}	2.43×10^{2}	1.13×10^{2}	5.61×10^{1}
	8.06×10^{10}	1.28×10^{6}	1.15×10^{4}	2.79×10^{3}	6.24×10^{2}	3.67×10^{2}	1.87×10^{2}
	8.06×10^{10}	1.28×10^{6}	1.15×10^{4}	2.79×10^{3}	6.24×10^{2}	3.67×10^{2}	1.87×10^{2}
	1.21×10^{14}	3.39×10^{7}	1.66×10^{5}	2.77×10^{4}	1.56×10^{3}	9.47×10^{2}	6.79×10^{2}
	1.5×10^{3}	2.23×10^{3}	8.64×10^{3}	1.68×10^{3}	3.55×10^{3}	9.2×10^{1}	2.68×10^{1}
	1.5×10^{3}	2.23×10^{3}	8.64×10^{3}	1.68×10^{3}	3.55×10^{3}	9.2×10^{1}	2.68×10^{1}
	2.25×10^{5}	7.32×10^{3}	9.62×10^{3}	$2.\times10^{3}$	3.69×10^{3}	1.61×10^{2}	6.6×10^{1}
	2.25×10^{5}	7.32×10^{3}	9.62×10^{3}	$2.\times10^{3}$	3.69×10^{3}	1.61×10^{2}	6.6×10^{1}
5	5.27×10^{8}	4.77×10^{5}	8.26×10^{3}	9.79×10^{2}	3.87×10^{3}	3.57×10^{2}	1.9×10^{2}
	5.27×10^{8}	4.77×10^{5}	8.26×10^{3}	9.79×10^{2}	3.87×10^{3}	3.57×10^{2}	1.9×10^{2}
	1.92×10^{12}	6.45×10^{7}	2.91×10^{5}	1.44×10^{4}	4.23×10^{2}	1.83×10^{2}	4.84×10^{2}
	1.92×10^{12}	6.45×10^{7}	2.91×10^{5}	1.44×10^{4}	4.23×10^{2}	1.83×10^{2}	4.84×10^{2}
	8.5×10^{15}	9.68×10^{9}	3.14×10^{6}	1.26×10^{5}	7.42×10^{4}	1.76×10^{4}	2.51×10^{3}
	1.5×10^{3}	6.24×10^{3}	2.07×10^{3}	7.63×10^{2}	4.15×10^{2}	3.93×10^{3}	5.54×10^{3}
	1.5×10^{3}	6.24×10^{3}	2.07×10^{3}	7.63×10^{2}	4.15×10^{2}	3.93×10^{3}	5.54×10^{3}
	9.03×10^{5}	2.74×10^{4}	6.31×10^{3}	2.22×10^{3}	1.06×10^{3}	4.26×10^{3}	5.73×10^{3}
	9.03×10^{5}	2.74×10^{4}	6.31×10^{3}	2.22×10^{3}	1.06×10^{3}	4.26×10^{3}	5.73×10^{3}
10	8.75×10^{9}	9.45×10^{6}	4.64×10^{5}	5.59×10^{4}	9.28×10^{3}	2.25×10^{3}	5.47×10^{3}
	8.75×10^{9}	9.45×10^{6}	4.64×10^{5}	5.59×10^{4}	9.28×10^{3}	2.25×10^{3}	5.47×10^{3}
	1.32×10^{14}	6.38×10^{9}	7.34×10^{7}	2.82×10^{6}	1.21×10^{5}	1.02×10^{4}	3.66×10^{3}
	1.32×10^{14}	6.38×10^{9}	7.34×10^{7}	2.82×10^{6}	1.21×10^{5}	1.02×10^{4}	3.66×10^{3}
	2.45×10^{18}	5.11×10^{12}	1.3×10^{10}	1.39×10^{8}	5.44×10^{5}	1.91×10^{4}	1.25×10^{5}

With this choice of parameters, our approximation provides unacceptable values, even for short maturities both ITM and OTM, so it is unnecessary to plot the results.

8.3 Heston stochastic volatility model

Perhaps the most well-known stochastic volatility model is that of Heston ([Hes93]). In the Heston model, the dynamics of the underlying S are given by

$$dS_t = \sqrt{Z_t}\, S_t\, dW_t, \qquad S_0 = s > 0,$$

$$dZ_t = \kappa(\theta - Z_t)\, dt + \delta \sqrt{Z_t}\, dB_t, \quad Z_0 = z > 0,$$

$$d\langle W.B\rangle_t = \rho\, dt.$$

Although it is not required, one typically sets $\rho < 0$ in order to capture the leverage effect. In log notation $(X, Y) := (\operatorname{Log} S, \operatorname{Log} Z)$ we have the following dynamics

$$dX_t = -\frac{1}{2} e^{Y_t}\, dt + e^{\frac{1}{2} Y_t}\, dW_t, \qquad X_0 = x := \operatorname{Log} s,$$

$$d Y_t = \left(\left(\kappa\,\theta - \frac{1}{2}\,\delta^2\right) e^{-Y_t} - \kappa\right) d t + \delta e^{-\frac{1}{2}\,Y_t}\, d B_t,\quad Y_0 = y = \operatorname{Log} z,$$

$$d\langle W.B\rangle_t = \rho\, d t.$$

The generator of (X,Y) is given by

$$\mathcal{A} = \frac{1}{2}\,e^{y}\left(\partial_x^2 - \partial_x\right) + \left(\left(\kappa\,\theta - \frac{1}{2}\,\delta^2\right) e^{-y} - \kappa\right)\partial_y + \frac{1}{2}\,\delta^2\,e^{-y}\,\partial_y^2 + \rho\,\delta\,\partial_{x,y}.$$

Thus we identify

$$a(x,\,y) = \frac{1}{2}\,e^{y},\quad b(x,\,y) = \frac{1}{2}\,\delta^2\,e^{-y},$$

$$c(x,\,y) = \rho\,\delta,\quad \alpha(x,\,y) = \left(\left(\kappa\,\theta - \frac{1}{2}\,\delta^2\right) e^{-y} - \kappa\right)$$

As in the case of LV models, even with LSV models we calculated generic formulas: we have calculated our approximation up to the fifth order with a generic $\bar{x}$, and up to the sixth order if we set $\bar{x}$ equal to the initial log-price x; substituting in these formulas the specific functions of the Heston model we obtain

$$\sigma_0 = e^{\frac{\bar{y}}{2}}$$

$$\sigma_1 = \frac{1}{8}\,e^{-\frac{\bar{y}}{2}}\left(-2\,(-k+x)\,\delta\,\rho - t\left(\delta^2 - 2\,\theta\,\kappa + 2\,r\,\delta\,\rho\right) + e^{\bar{y}}\,(-2\,t\,\kappa + t\,\delta\,\rho + 4\,(y-\bar{y}))\right)$$

$$\begin{aligned}
\sigma_2 = {} & \\
& \frac{1}{384}\,e^{-\frac{3\bar{y}}{2}}\Big(16\,(-k+x)^2\,\delta^2 - 3\,t^2\,\delta^4 + 12\,t^2\,\delta^2\,\theta\,\kappa - 12\,t^2\,\theta^2\,\kappa^2 - 20\,t\,(-k+x)\,\delta^3\,\rho + 40\,t\,(-k+x)\,\delta\,\theta\,\kappa\,\rho - \\
& \quad 40\,(-k+x)^2\,\delta^2\,\rho^2 - 8\,r^2\,t^2\,\delta^2\left(-2+5\,\rho^2\right) - 4\,r\,t\,\delta\left(5\,t\left(\delta^2 - 2\,\theta\,\kappa\right)\rho + 4\,(-k+x)\,\delta\left(-2+5\,\rho^2\right)\right) + \\
& \quad e^{\bar{y}}\Big(-2\,t^2\,(-2\,\kappa+\delta\,\rho)\left(\delta^2 - 2\,\theta\,\kappa - 2\,r\,\delta\,\rho\right) + 4\,t\Big(\delta^2\left(8 + (1-k+x)\,\rho^2 + 6\,(y-\bar{y})\right) + \\
& \qquad 2\,\delta\,\rho\,(-(-k+x)\,\kappa + 6\,r\,(y-\bar{y})) - 12\,\theta\,\kappa\,(y-\bar{y})\Big) + 48\,(-k+x)\,\delta\,\rho\,(y-\bar{y})\Big) + \\
& \quad 4\,e^{2\bar{y}}\left(t^2\left(5\,\kappa^2 - 5\,\delta\,\kappa\,\rho + \delta^2\left(-1+2\,\rho^2\right)\right) + 6\,t\,(-2\,\kappa+\delta\,\rho)\,(y-\bar{y}) + 12\,(y-\bar{y})^2\right)\Big)
\end{aligned}$$

$$\begin{aligned}
\sigma_3 = {} & \\
& \frac{1}{3072}\,e^{-\frac{5\bar{y}}{2}}\Big(64\,t\,(-k+x)^2\,\delta^4 - 3\,t^3\,\delta^6 - 128\,t\,(-k+x)^2\,\delta^2\,\theta\,\kappa + 18\,t^3\,\delta^4\,\theta\,\kappa - 36\,t^3\,\delta^2\,\theta^2\,\kappa^2 + 24\,t^3\,\theta^3\,\kappa^3 + \\
& \quad 160\,(-k+x)^3\,\delta^3\,\rho - 42\,t^2\,(-k+x)\,\delta^5\,\rho + 168\,t^2\,(-k+x)\,\delta^3\,\theta\,\kappa\,\rho - 168\,t^2\,(-k+x)\,\delta\,\theta^2\,\kappa^2\,\rho - \\
& \quad 184\,t\,(-k+x)^2\,\delta^4\,\rho^2 + 368\,t\,(-k+x)^2\,\delta^2\,\theta\,\kappa\,\rho^2 - 256\,(-k+x)^3\,\delta^3\,\rho^3 - \\
& \quad 32\,r^3\,t^3\,\delta^3\,\rho\left(-5+8\,\rho^2\right) - 8\,r^2\,t^2\,\delta^2\left(12\,(-k+x)\,\delta\,\rho\left(-5+8\,\rho^2\right) + t\left(\delta^2 - 2\,\theta\,\kappa\right)\left(-8+23\,\rho^2\right)\right) - \\
& \quad 2\,r\,t\,\delta\left(21\,t^2\left(\delta^2 - 2\,\theta\,\kappa\right)^2\rho + 48\,(-k+x)^2\,\delta^2\,\rho\left(-5+8\,\rho^2\right) + 8\,t\,(-k+x)\,\delta\left(\delta^2 - 2\,\theta\,\kappa\right)\left(-8+23\,\rho^2\right)\right) + \\
& \quad e^{\bar{y}}\Big(t^2\left(\delta^2 - 2\,\theta\,\kappa\right)\left(t\,\delta^3\,\rho - 2\,t\,\delta\,\theta\,\kappa\,\rho + 4\,\theta\,\kappa\,(t\,\kappa - 18\,(y-\bar{y})) + 2\,\delta^2\left(8 - t\,\kappa + 10\,\rho^2 + 18\,(y-\bar{y})\right)\right) + \\
& \qquad 4\,t\,(-k+x)\,\delta\,\rho
\end{aligned}$$

$$
\begin{aligned}
&\left(5\,t\,\delta^3\,\rho - 10\,t\,\delta\,\theta\,\kappa\,\rho + 20\,\theta\,\kappa\,(t\,\kappa - 6\,(y - \bar{y})) + 2\,\delta^2\left(8 - 5\,t\,\kappa + 9\,\rho^2 + 30\,(y - \bar{y})\right)\right) + \\
&\quad 4\,r\,t\,\delta\left(t\,\rho\left(5\,t\,\delta^3\,\rho - 10\,t\,\delta\,\theta\,\kappa\,\rho + 20\,\theta\,\kappa\,(t\,\kappa - 6\,(y - \bar{y})) + 2\,\delta^2\left(8 - 5\,t\,\kappa + 9\,\rho^2 + 30\,(y - \bar{y})\right)\right) + \right. \\
&\qquad 4\,(-k + x)\,\delta\left(-t\,(2\,\kappa - \delta\,\rho)\left(-2 + 7\,\rho^2\right) + 12\left(-2 + 5\,\rho^2\right)(y - \bar{y})\right)\Big) + \\
&\quad 8\,r^2\,t^2\,\delta^2\left(-t\,(2\,\kappa - \delta\,\rho)\left(-2 + 7\,\rho^2\right) + 12\left(-2 + 5\,\rho^2\right)(y - \bar{y})\right) + \\
&\quad 8\,(-k + x)^2\,\delta^2\left(-t\,(2\,\kappa - \delta\,\rho)\left(-2 + 7\,\rho^2\right) + 12\left(-2 + 5\,\rho^2\right)(y - \bar{y})\right)\Big) + \\
&4\,e^{2\,\bar{y}}\left(t^3\left(2\,\theta\,\kappa^3 + 2\,\delta\,(3\,r - \theta)\,\kappa^2\,\rho - \delta^2\,\kappa\left(\kappa + 6\,r\,\rho^2\right) + \delta^3\,\rho\left(\kappa + r\left(-2 + 4\,\rho^2\right)\right)\right) + \right. \\
&\quad t^2\left(\delta\,\rho\left(3\,\delta\,\rho\,(-2\,\kappa + \delta\,\rho) + (-k + x)\left(6\,\kappa^2 - 6\,\delta\,\kappa\,\rho + \delta^2\left(-2 + 4\,\rho^2\right)\right)\right) + \right. \\
&\qquad 2\,(-2\,\kappa + \delta\,\rho)\left(\delta^2 - 2\,\theta\,\kappa - 2\,r\,\delta\,\rho\right)(y - \bar{y})\Big) - 4\,t\left(\delta^2\left(8 + (1 - k + x)\,\rho^2 + 3\,(y - \bar{y})\right) + \right. \\
&\qquad \delta\,(-2\,(-k + x)\,\kappa\,\rho + 6\,r\,\rho\,(y - \bar{y})) - 6\,\theta\,\kappa\,(y - \bar{y})\Big)(y - \bar{y}) - 24\,(-k + x)\,\delta\,\rho\,(y - \bar{y})^2\Big) + \\
&2\,e^{3\,\bar{y}}\left(t^3\left(-12\,\kappa^3 + 18\,\delta\,\kappa^2\,\rho - 4\,\delta^2\,\kappa\left(-3 + 4\,\rho^2\right) + \delta^3\,\rho\left(-6 + 5\,\rho^2\right)\right) + \right. \\
&\quad 8\,t^2\left(5\,\kappa^2 - 5\,\delta\,\kappa\,\rho + \delta^2\left(-1 + 2\,\rho^2\right)\right)(y - \bar{y}) + 24\,t\,(-2\,\kappa + \delta\,\rho)\,(y - \bar{y})^2 + 32\,(y - \bar{y})^3\Big)\Big)
\end{aligned}
$$

In ([Hes93]), Heston computed explicitly the characteristic function of X_t

$$\eta(t, x, y, \lambda) := \mathrm{Log}\,\mathbb{E}_{x,y}\left[e^{i\,\lambda\,X_t}\right] = i\,\lambda\,x + C(t, \lambda) + D(t, \lambda)\,e^y,$$

$$C(t, \lambda) = \kappa\,\frac{\theta}{\delta^2}\left((\kappa - \rho\,\delta\,i\,\lambda + d(\lambda))\,t - 2\,\mathrm{Log}\left[\frac{1 - f(\lambda)\,e^{d(\lambda)\,t}}{1 - f(\lambda)}\right]\right),$$

$$D(t, \lambda) = \frac{\kappa - \rho\,\delta\,i\,\lambda + d(\lambda)}{\delta^2}\;\frac{1 - e^{d(\lambda)\,t}}{1 - f(\lambda)\,e^{d(\lambda)\,t}},$$

$$f(\lambda) = \frac{\kappa - \rho\,\delta\,i\,\lambda + d(\lambda)}{\kappa - \rho\,\delta\,i\,\lambda - d(\lambda)},$$

$$d(\lambda) = \sqrt{\delta^2\left(\lambda^2 + i\,\lambda\right) + (\kappa - \rho\,i\,\lambda\,\delta)^2}\,.$$

Thus, the price of an European call option can be computed using standard Fourier methods

$$u(t, x, y) = \frac{1}{2\,\pi}\int_{\mathbb{R}} d\lambda_r\,e^{\eta(t,x,y,\lambda)}\,\hat{h}(\lambda),$$

$$\hat{h}(\lambda) = \frac{-e^{k - i\,k\,\lambda}}{i\,\lambda + \lambda^2}, \quad \lambda = \lambda_r + i\,\lambda_i, \quad \lambda_i < -1.$$

Note, since the call option payoff $h(x) = \left(e^x - e^k\right)^+$ is not in $L^1(\mathbb{R})$, its Fourier transform $\hat{h}(\lambda)$ must be computed in a generalized sense by fixing an imaginary component of the Fourier variable $\lambda_i < -1$. Using the previous formula of $u(t, x, y)$, the exact implied volatility σ can be computed numerically.

8.3.1 First Set by Ribeiro and Poulsen

The first set of parameters considered is as given by Ribeiro and Poulsen in ([RP13]), so we have $(\kappa = 1, \theta = 0.08,\ \delta = 0.39,\ \rho = -0.93,\ x = 0,\ y = 2\,\mathrm{Log}(0.245))$.

First of all we consider a vector of times to maturity shorter than before beacuse LSV models are more difficult to approximate at long maturities, in fact we will see that our method can’t be used for maturities longer than 1 or 2 years, and that we obtain good results only for maturities shorter than a year.

≈1M	3M	6M	1.Y	1.5Y	2Y	3Y	5Y
0.1	0.25	0.5	1	1.5	2	3	5

and we create a matrix of strikes expressed in basis points (bp).

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.1	94	95	96	97	98	99	100	101	102	102	103	104	105
0.25	86	88	91	93	95	98	100	102	103	105	107	108	110
0.5	76	80	84	88	92	96	100	103	105	108	111	113	116
1	63	69	75	82	88	94	100	104	108	112	116	120	124
1.5	55	62	70	78	85	92	100	105	111	116	121	127	132
2	48	57	65	74	83	91	100	107	113	120	127	133	140
3	38	48	59	69	79	90	100	109	119	128	137	147	156
5	38	48	59	69	79	90	100	115	129	144	159	173	188

To check the admissibility of these values, we numerically compute call and put prices using the characteristic function as defined in ([Hes93]) and than using Fourier methods.

	1	3	5	7	9	11	13
0.1	0.07082	0.05578	0.04235	0.03077	0.02120	0.01719	0.01069
0.25	0.1500	0.1086	0.07922	0.04834	0.03348	0.01827	0.01029
0.5	0.2495	0.1801	0.1185	0.06779	0.04293	0.02089	0.009156
1	0.3797	0.2735	0.1716	0.09503	0.05582	0.02745	0.01022
1.5	0.4603	0.3279	0.2110	0.1161	0.06430	0.03147	0.01036
2	0.5299	0.3794	0.2393	0.1347	0.07546	0.03232	0.01018
3	0.6298	0.4415	0.2915	0.1684	0.08778	0.03781	0.009612
5	0.6305	0.4658	0.3365	0.2261	0.1147	0.04452	0.01027

	1	3	5	7	9	11	13
0.1	0.01082	0.01578	0.02235	0.03077	0.04120	0.04719	0.06069
0.25	0.01000	0.01865	0.02922	0.04834	0.06348	0.08827	0.1103
0.5	0.009533	0.02011	0.03850	0.06779	0.09293	0.1309	0.1692
1	0.009700	0.02347	0.05160	0.09503	0.1358	0.1874	0.2502
1.5	0.01027	0.02787	0.06104	0.1161	0.1743	0.2415	0.3304
2	0.009850	0.02945	0.06934	0.1347	0.2055	0.3023	0.4102
3	0.009833	0.03146	0.08151	0.1684	0.2778	0.4078	0.5696
5	0.01055	0.05580	0.1265	0.2261	0.4047	0.6345	0.8903

Now we numerically compute the corresponding exact volatilities using Fourier prices.

	1	3	5	7	9	11	13
0.1	26.4519	25.7834	25.0984	24.3942	23.6673	23.2942	22.5258
0.25	28.6691	27.1509	25.8952	24.2514	23.2114	21.7408	20.5581
0.5	30.9509	28.7281	26.4527	24.0610	22.4702	20.4238	18.5902
1	33.1609	30.1114	26.8893	23.8763	21.8010	19.6558	17.4762
1.5	33.9733	30.4313	27.0892	23.8396	21.5608	19.4796	17.1115
2	34.7029	30.8063	26.9724	23.9888	21.7937	19.3489	16.9145
3	36.1483	29.9554	27.1799	24.5620	22.0946	19.6375	16.6877
5	28.4428	28.8170	27.5302	25.6980	23.0257	20.2249	16.9747

In our approximation formulas for Heston model, doesn't appear the value of $\overline{x}$, so the choice of this parameter is irrelevant, on other hand, we can make different choices on $\overline{y}$.

The first natural choice is to set $\overline{y} = y$, and calculating absolute errors we have

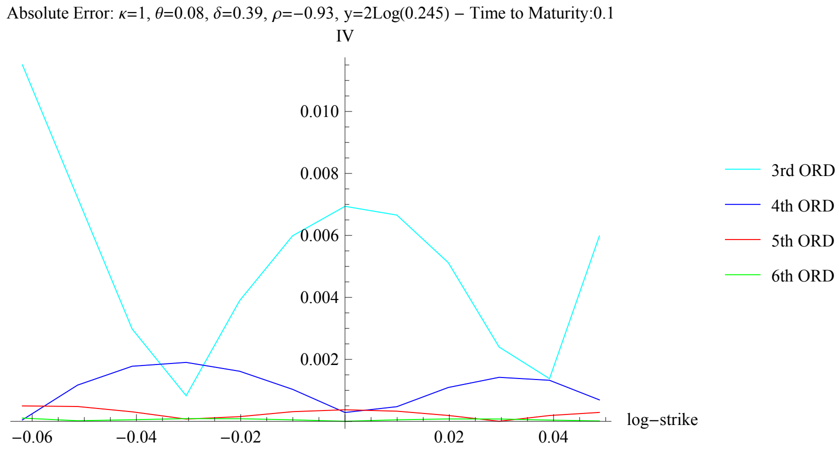
	1	3	5	7	9	11	13
	1.95	1.28	5.98×10^{-1}	1.06×10^{-1}	8.33×10^{-1}	1.21	1.97
	3.45×10^{-1}	4.56×10^{-1}	5.34×10^{-1}	5.77×10^{-1}	5.84×10^{-1}	5.71×10^{-1}	5.15×10^{-1}
	6.99×10^{-2}	6.2×10^{-2}	4.14×10^{-2}	1.31×10^{-2}	1.78×10^{-2}	3.25×10^{-2}	5.64×10^{-2}
0.1	1.15×10^{-2}	2.98×10^{-3}	3.9×10^{-3}	6.94×10^{-3}	5.12×10^{-3}	2.4×10^{-3}	5.98×10^{-3}
	4.66×10^{-5}	1.78×10^{-3}	1.62×10^{-3}	2.86×10^{-4}	1.09×10^{-3}	1.42×10^{-3}	6.93×10^{-4}
	4.98×10^{-4}	3.08×10^{-4}	1.52×10^{-4}	3.74×10^{-4}	1.9×10^{-4}	2.29×10^{-6}	2.89×10^{-4}
	1.06×10^{-4}	5.24×10^{-5}	8.48×10^{-5}	1.83×10^{-6}	7.82×10^{-5}	7.57×10^{-5}	1.26×10^{-5}
	4.17	2.65	1.4	2.49×10^{-1}	1.29	2.76	3.94
	2.95×10^{-1}	8.69×10^{-1}	1.21	1.46	1.51	1.45	1.29
	2.05×10^{-1}	2.89×10^{-1}	2.38×10^{-1}	8.14×10^{-2}	3.58×10^{-2}	1.86×10^{-1}	2.7×10^{-1}
0.25	1.29×10^{-1}	7.01×10^{-2}	4.34×10^{-3}	4.03×10^{-2}	3.26×10^{-2}	2.15×10^{-2}	9.01×10^{-2}
	5.49×10^{-2}	5.13×10^{-3}	2.05×10^{-2}	4.82×10^{-3}	8.49×10^{-3}	9.42×10^{-3}	1.57×10^{-2}
	1.59×10^{-2}	9.61×10^{-3}	2.35×10^{-3}	6.13×10^{-3}	3.81×10^{-3}	2.92×10^{-3}	6.98×10^{-4}
	5.15×10^{-4}	2.41×10^{-3}	2.38×10^{-3}	5.11×10^{-6}	2.06×10^{-3}	9.1×10^{-4}	1.02×10^{-4}
	6.45	4.23	1.95	4.39×10^{-1}	2.03	4.08	5.91
	2.9×10^{-1}	1.19	2.28	2.98	3.19	3.2	3.
	1.76×10^{-1}	8.12×10^{-1}	7.76×10^{-1}	3.2×10^{-1}	7.82×10^{-2}	5.88×10^{-1}	1.
0.5	4.83×10^{-1}	4.62×10^{-1}	9.17×10^{-2}	1.37×10^{-1}	9.81×10^{-2}	1.76×10^{-1}	6.15×10^{-1}
	5.79×10^{-1}	1.11×10^{-1}	1.23×10^{-1}	4.25×10^{-2}	4.15×10^{-2}	1.36×10^{-2}	2.53×10^{-1}
	5.66×10^{-1}	5.79×10^{-2}	4.1×10^{-2}	5.22×10^{-2}	3.16×10^{-2}	1.54×10^{-2}	7.81×10^{-2}
	4.61×10^{-1}	8.02×10^{-2}	2.51×10^{-2}	1.21×10^{-3}	2.78×10^{-2}	8.31×10^{-3}	4.31×10^{-3}
	8.66	5.61	2.39	6.24×10^{-1}	2.7	4.84	7.02
	1.61	1.8	4.49	6.21	6.98	7.48	7.77
	4.14×10^{-1}	2.27	2.48	1.25	2.4×10^{-2}	1.52	3.2
1	1.57	2.4	6.97×10^{-1}	3.71×10^{-1}	2.46×10^{-1}	7.11×10^{-1}	2.59
	4.55	1.37	7.21×10^{-1}	3.32×10^{-1}	2.28×10^{-1}	2.39×10^{-1}	9.65×10^{-1}
	9.22	1.33×10^{-1}	5.3×10^{-1}	3.95×10^{-1}	1.78×10^{-1}	4.03×10^{-1}	3.89×10^{-1}
	1.56×10^{1}	1.42	2.64×10^{-1}	1.01×10^{-1}	2.13×10^{-1}	3.04×10^{-1}	1.5
	9.47	5.93	2.59	6.6×10^{-1}	2.94	5.02	7.39
	2.4	2.98	6.82	9.59	1.12×10^{1}	1.23×10^{1}	1.31×10^{1}
	6.62×10^{-1}	4.53	4.97	2.89	3.49×10^{-1}	2.38	5.61
1.5	4.2	5.98	2.1	5.87×10^{-1}	4.07×10^{-1}	1.51	5.49
	1.51×10^{1}	4.	1.97	9.26×10^{-1}	8.88×10^{-1}	1.25	1.37
	3.84×10^{1}	1.26	2.36	1.09	1.45×10^{-1}	2.33	3.65
	8.13×10^{1}	8.07	1.02	8.21×10^{-1}	4.36×10^{-1}	2.53	1.01×10^{1}

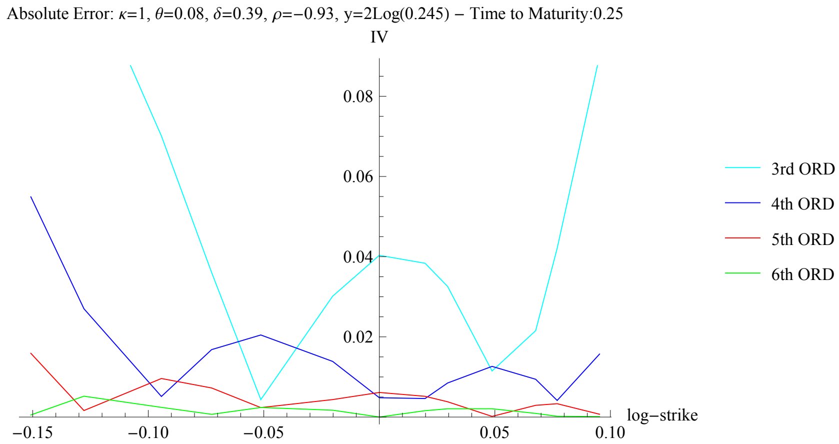
2	1.02×10^{1} 3.3 1.46 8.17 3.69×10^{1} 1.17×10^{2} 3.06×10^{2}	6.31 4.03 7.3 1.19×10^{1} 1.01×10^{1} 2.88 2.83×10^{1}	2.47 9.24 8.36 4.16 3.88 6.64 3.17	5.11×10^{-1} 1.32×10^{1} 5.21 5.27×10^{-1} 1.98 2.25 2.83	2.71 1.55×10^{1} 1.49 6.25×10^{-1} 2.07 3.94×10^{-2} 6.67×10^{-1}	5.15 1.74×10^{1} 3.12 2.8 3.52 7.83 1.1×10^{1}	7.59 1.85×10^{1} 7.61 9.03 1.02 1.19×10^{1} 3.45×10^{1}
3	1.16×10^{1} 3.66 2.88 2.47×10^{1} 1.33×10^{2} 5.41×10^{2} 1.78×10^{3}	5.46 6.43 1.65×10^{1} 2.97×10^{1} 3.06×10^{1} 2.57×10^{1} 1.66×10^{2}	2.68 1.45×10^{1} 1.73×10^{1} 1.27×10^{1} 1.02×10^{1} 2.94×10^{1} 9.55	6.2×10^{-2} 2.06×10^{1} 1.22×10^{1} 1.01 5.45 5.53 1.53×10^{1}	2.41 2.45×10^{1} 5.3 5.06×10^{-1} 8.52 5.02 1.61	4.86 2.73×10^{1} 2.02 5.6 1.36×10^{1} 3.41×10^{1} 6.18×10^{1}	7.81 2.91×10^{1} 9.84 1.87×10^{1} 1.83 5.05×10^{1} 1.83×10^{2}
5	3.94 2.3 3.53×10^{1} 1.24×10^{2} 3.63×10^{2} 5.31×10^{2} 1.17×10^{3}	4.32 1.9×10^{1} 4.58×10^{1} 8.26×10^{1} 3.43×10^{1} 3.12×10^{2} 8.14×10^{2}	3.03 2.85×10^{1} 4.36×10^{1} 4.13×10^{1} 3.81×10^{1} 1.32×10^{2} 1.43×10^{2}	1.2 3.54×10^{1} 3.64×10^{1} 1.48×10^{1} 1.62×10^{1} 8.17 1.17×10^{2}	1.47 4.21×10^{1} 2.31×10^{1} 5.62 4.26×10^{1} 4.54×10^{1} 3.41×10^{1}	4.28 4.71×10^{1} 7.85 2.15×10^{1} 6.31×10^{1} 2.11×10^{2} 5.95×10^{2}	7.53 $5.\times10^{1}$ 6.49 5.38×10^{1} 1.75×10^{1} 2.75×10^{2} 1.47×10^{3}

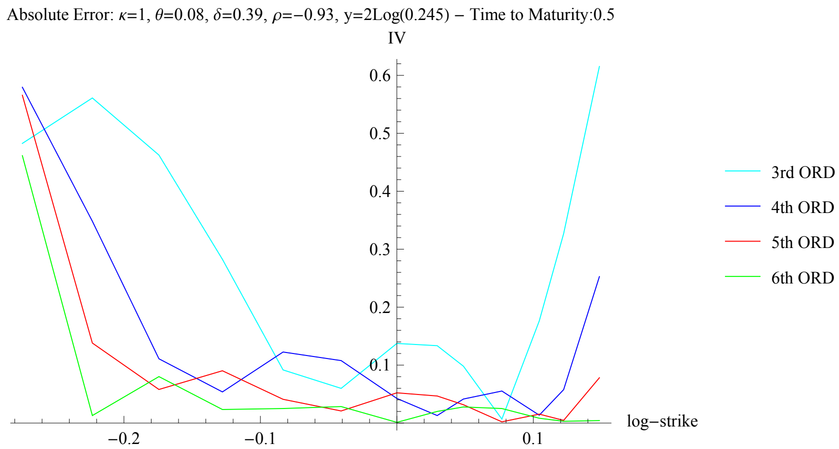
Each element of this matrix is a vector of 7 values, because with the choice of the initial point we were able to calculate our expansion up to the sixth order, then each vector represents the absolute errors committed by our approximation at order n with $n = 0, \ ..., \ 6$.

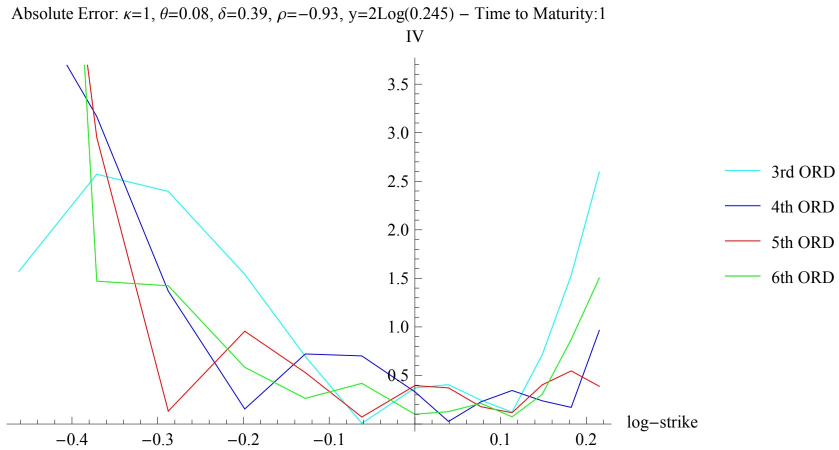
First, we see that results are worse than those obtained in LV models, although here we considered shorter maturities; we can notice that for maturities inferior to 1 year, the method seems to converge in the orders considered here both ATM and OTM, but we get errors of less than percentage point even ITM.

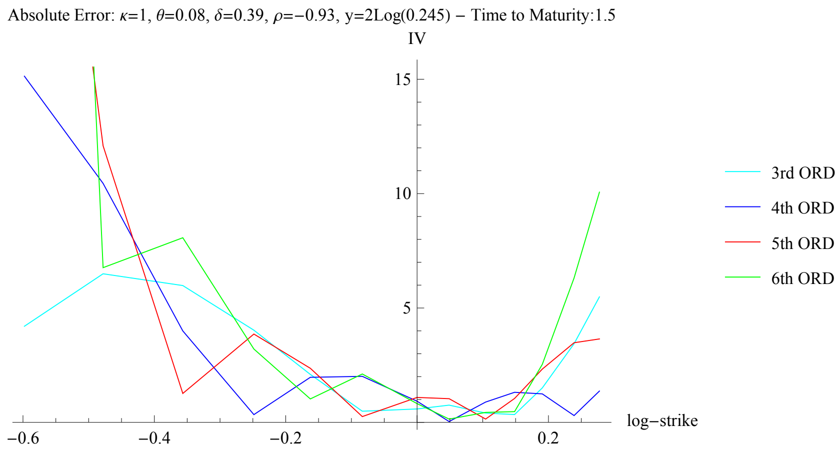
Problems come out with maturities equal or greater than 1 year: the method explodes almost always producing errors of a few percent or even tens of percentage points, so if before the choice to stop the computation to a certain order of approximation was convenient from a computational point of view, here becomes necessary for the admissibility of the results; thus we note that for maturities of 1 or 2 years we should stop at the fourth or fifth order, while for longer maturities the results seem unusable in any case.

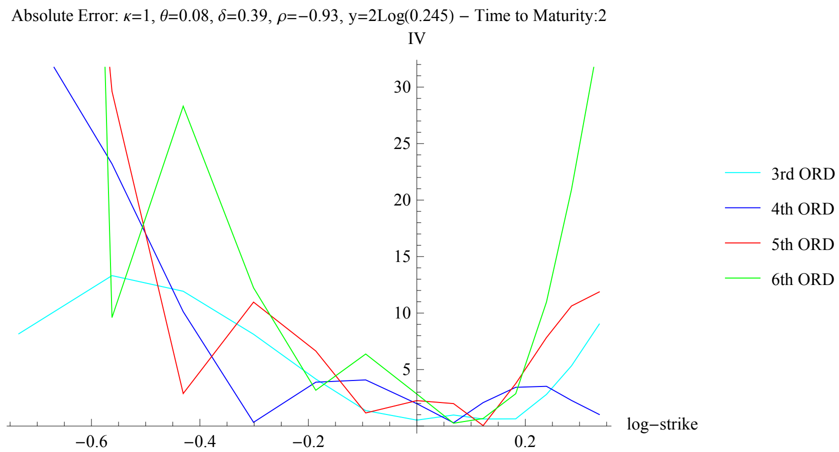

Absolute Error: κ=1, θ=0.08, δ=0.39, ρ=−0.93, y=2Log(0.245) − Time to Maturity:0.1
IV
0.010
0.008
0.006
0.004
0.002
log−strike
−0.06
−0.04
−0.02
0.02
0.04
3rd ORD
4th ORD
5th ORD
6th ORD

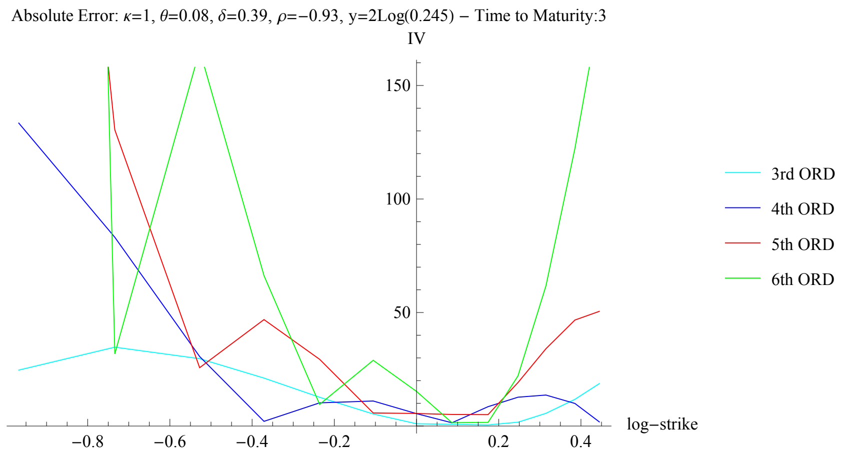

Absolute Error: κ=1, θ=0.08, δ=0.39, ρ=−0.93, y=2Log(0.245) − Time to Maturity:0.25
IV
0.08
0.06
0.04
0.02
log−strike
−0.15
−0.10
−0.05
0.05
0.10
3rd ORD
4th ORD
5th ORD
6th ORD

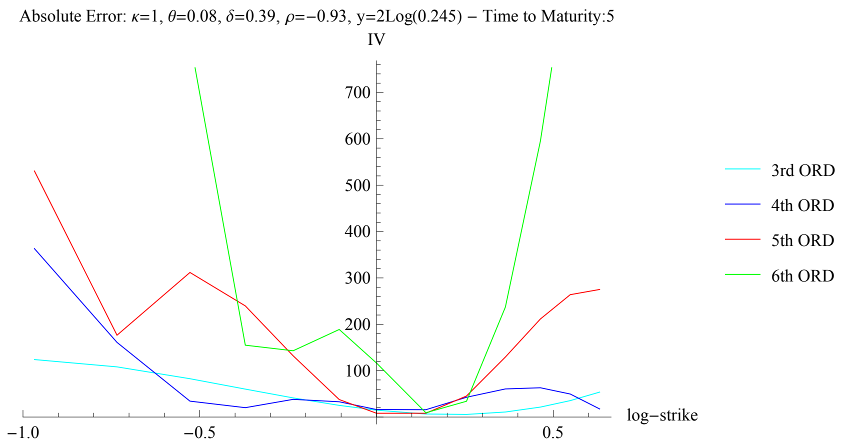

Absolute Error: κ=1, θ=0.08, δ=0.39, ρ=−0.93, y=2Log(0.245) − Time to Maturity:0.5
IV
0.6
0.5
0.4
0.3
0.2
0.1
log−strike
−0.2
−0.1
0.1
3rd ORD
4th ORD
5th ORD
6th ORD


Absolute Error: κ=1, θ=0.08, δ=0.39, ρ=−0.93, y=2Log(0.245) − Time to Maturity:1
IV
log−strike
3rd ORD
4th ORD
5th ORD
6th ORD


Absolute Error: κ=1, θ=0.08, δ=0.39, ρ=−0.93, y=2Log(0.245) − Time to Maturity:1.5
IV
log−strike
3rd ORD
4th ORD
5th ORD
6th ORD


Absolute Error: κ=1, θ=0.08, δ=0.39, ρ=−0.93, y=2Log(0.245) − Time to Maturity:2
IV
log−strike
3rd ORD
4th ORD
5th ORD
6th ORD





In these graphs we see a kind of inversion, in fact while in the first graphics errors of low orders are bigger than those of the approximation at the fifth and sixth order, in latter graphics this behavior is reversed because we see that the best approximations are those of the third and fourth order; however we must point out that they are still not usable values.

In order to try to improve the results, we can make two choices for:

1) the drif of Z_t process has limit θ as time tends to infinity, so we can set $\bar{z}$ equal to the midpoint between the initial value z and θ; then in log notation we have $\bar{y} = \mathrm{Log}\left(\frac{e^y+\theta}{2}\right)$;

2) the drift of Z_t is a solvable ODE with solution $Z(t) = e^{-t\kappa}(z-\theta+e^{t\kappa}\theta)$, so we can set $\bar{z}$ equal to the integral mean of this function, and in log notation we have: $\bar{y} = \mathrm{Log}\left(\frac{e^y-\theta+e^{-t\kappa}(-e^y+\theta)+t\,\theta\,\kappa}{t\,\kappa}\right)$.

In the following matrix, every sub-matrix has three columns, the first corresponds to the initial point method, the second to the midpoint method, the third to the integral mean method.

	1			7			13		
	1.95	7.97×10^{-3}	1.76	1.06×10^{-1}	2.07	3.02×10^{-1}	1.97	3.93	2.17
	3.45×10^{-1}	6.52×10^{-1}	3.7×10^{-1}	5.77×10^{-1}	7.15×10^{-1}	5.84×10^{-1}	5.15×10^{-1}	5.18×10^{-1}	5.07×10^{-1}
	6.99×10^{-2}	6.57×10^{-2}	6.99×10^{-2}	1.31×10^{-2}	2.38×10^{-2}	8.68×10^{-3}	5.64×10^{-2}	6.33×10^{-2}	5.78×10^{-2}
0.1	1.15×10^{-2}	3.4×10^{-3}	1.01×10^{-2}	6.94×10^{-3}	5.7×10^{-3}	6.98×10^{-3}	5.98×10^{-3}	1.44×10^{-2}	7.03×10^{-3}
	4.66×10^{-5}	3.39×10^{-3}	5.65×10^{-4}	2.86×10^{-4}	1.2×10^{-3}	1.15×10^{-4}	6.93×10^{-4}	2.39×10^{-3}	3.81×10^{-4}
	4.98×10^{-4}	5.11×10^{-5}	5.46×10^{-4}	3.74×10^{-4}	1.96×10^{-4}	3.71×10^{-4}	2.89×10^{-4}	2.8×10^{-4}	2.64×10^{-4}
	1.06×10^{-4}	0	0	1.83×10^{-6}	0	0	1.26×10^{-5}	0	0
	4.17	2.21	3.7	2.49×10^{-1}	2.21	7.14×10^{-1}	3.94	5.9	4.41
	2.95×10^{-1}	9.37×10^{-1}	4.4×10^{-1}	1.46	1.69	1.5	1.29	1.26	1.27
	2.05×10^{-1}	3.33×10^{-1}	2.42×10^{-1}	8.14×10^{-2}	8.34×10^{-3}	5.72×10^{-2}	2.7×10^{-1}	2.55×10^{-1}	2.69×10^{-1}
0.25	1.29×10^{-1}	9.71×10^{-2}	1.27×10^{-1}	4.03×10^{-2}	3.88×10^{-2}	4.09×10^{-2}	9.01×10^{-2}	1.14×10^{-1}	9.72×10^{-2}
	5.49×10^{-2}	2.03×10^{-3}	4.12×10^{-2}	4.82×10^{-3}	4.66×10^{-3}	2.32×10^{-3}	1.57×10^{-2}	3.88×10^{-2}	2.2×10^{-2}
	1.59×10^{-2}	1.47×10^{-2}	5.03×10^{-3}	6.13×10^{-3}	4.91×10^{-3}	6.02×10^{-3}	6.98×10^{-4}	1.21×10^{-2}	3.4×10^{-3}
	5.15×10^{-4}	0	0	5.11×10^{-6}	0	0	1.02×10^{-4}	0	0
	6.45	4.49	5.6	4.39×10^{-1}	2.4	1.29	5.91	7.87	6.76
	2.9×10^{-1}	8.43×10^{-1}	$2.\times10^{-1}$	2.98	3.36	3.12	3.	2.97	2.96
	1.76×10^{-1}	7.53×10^{-1}	4.56×10^{-1}	3.2×10^{-1}	1.57×10^{-1}	2.42×10^{-1}	1.	9.17×10^{-1}	9.66×10^{-1}
0.5	4.83×10^{-1}	6.22×10^{-1}	5.85×10^{-1}	1.37×10^{-1}	1.38×10^{-1}	1.4×10^{-1}	6.15×10^{-1}	6.58×10^{-1}	6.37×10^{-1}
	5.79×10^{-1}	3.19×10^{-1}	4.86×10^{-1}	4.25×10^{-2}	4.14×10^{-3}	2.45×10^{-2}	2.53×10^{-1}	3.46×10^{-1}	2.98×10^{-1}
	5.66×10^{-1}	6.12×10^{-2}	3.11×10^{-1}	5.22×10^{-2}	4.66×10^{-2}	5.07×10^{-2}	7.81×10^{-2}	1.66×10^{-1}	1.19×10^{-1}
	4.61×10^{-1}	0	0	1.21×10^{-3}	0	0	4.31×10^{-3}	0	0
	8.66	6.7	7.2	6.24×10^{-1}	2.58	2.08	7.02	8.98	8.48
	1.61	3.45×10^{-1}	1.52×10^{-1}	6.21	6.89	6.7	7.77	7.87	7.82
	4.14×10^{-1}	1.46	1.04	1.25	9.92×10^{-1}	1.05	3.2	2.94	3.01
1	1.57	2.85	2.66	3.71×10^{-1}	3.79×10^{-1}	3.82×10^{-1}	2.59	2.67	2.66
	4.55	3.61	3.98	3.32×10^{-1}	1.75×10^{-1}	2.12×10^{-1}	9.65×10^{-1}	1.33	1.25
	9.22	3.41	4.68	3.95×10^{-1}	3.69×10^{-1}	3.79×10^{-1}	3.89×10^{-1}	1.83×10^{-1}	5.57×10^{-2}
	1.56×10^{1}	0	0	1.01×10^{-1}	0	0	1.5	0	0
	9.47	7.51	7.58	6.6×10^{-1}	2.62	2.55	7.39	9.35	9.28
	2.4	2.25×10^{-1}	1.35×10^{-1}	9.59	1.06×10^{1}	1.05×10^{1}	1.31×10^{1}	1.34×10^{1}	1.34×10^{1}
	6.62×10^{-1}	2.64	2.54	2.89	2.61	2.62	5.61	5.04	5.06
1.5	4.2	7.09	7.04	5.87×10^{-1}	5.91×10^{-1}	5.93×10^{-1}	5.49	5.52	5.51
	1.51×10^{1}	1.23×10^{1}	1.24×10^{1}	9.26×10^{-1}	5.62×10^{-1}	5.74×10^{-1}	1.37	2.14	2.12
	3.84×10^{1}	1.59×10^{1}	1.65×10^{1}	1.09	1.03	1.03	3.65	1.76	1.81
	8.13×10^{1}	0	0	8.21×10^{-1}	0	0	1.01×10^{1}	0	0
	1.02×10^{1}	8.24	7.99	5.11×10^{-1}	2.47	2.73	7.59	9.55	9.8
	3.3	9.76×10^{-3}	4.27×10^{-1}	1.32×10^{1}	1.44×10^{1}	1.46×10^{1}	1.85×10^{1}	1.89×10^{1}	1.9×10^{1}
	1.46	3.71	4.27	5.21	4.98	4.97	7.61	6.59	6.46
2	8.17	1.39×10^{1}	1.43×10^{1}	5.27×10^{-1}	4.98×10^{-1}	4.84×10^{-1}	9.03	8.88	8.87
	3.69×10^{1}	3.08×10^{1}	2.96×10^{1}	1.98	1.31	1.24	1.02	2.23	2.36
	1.17×10^{2}	5.3×10^{1}	4.66×10^{1}	2.25	2.13	2.11	1.19×10^{1}	7.4	6.95
	3.06×10^{2}	0	0	2.83	0	0	3.45×10^{1}	0	0
	1.16×10^{1}	9.69	9.01	6.2×10^{-2}	1.9	2.58	7.81	9.77	1.05×10^{1}
	3.66	8.91×10^{-1}	2.42	2.06×10^{1}	2.25×10^{1}	2.32×10^{1}	2.91×10^{1}	2.98×10^{1}	3.01×10^{1}
	2.88	6.61	9.3	1.22×10^{1}	1.23×10^{1}	1.24×10^{1}	9.84	7.57	6.82
3	2.47×10^{1}	3.81×10^{1}	4.04×10^{1}	1.01	1.27	1.43	1.87×10^{1}	1.78×10^{1}	1.76×10^{1}
	1.33×10^{2}	1.11×10^{2}	$1.\times10^{2}$	5.45	3.82	3.31	1.83	1.47×10^{-1}	3.33×10^{-1}
	5.41×10^{2}	2.59×10^{2}	1.89×10^{2}	5.53	5.29	5.15	5.05×10^{1}	3.56×10^{1}	3.17×10^{1}
	1.78×10^{3}	0	0	1.53×10^{1}	0	0	1.83×10^{2}	0	0
	3.94	1.98	8.69×10^{-1}	1.2	7.62×10^{-1}	1.88	7.53	9.49	1.06×10^{1}
	2.3	8.07	1.12×10^{1}	3.54×10^{1}	3.85×10^{1}	4.03×10^{1}	$5.\times10^{1}$	5.14×10^{1}	5.23×10^{1}
	3.53×10^{1}	4.62×10^{1}	5.13×10^{1}	3.64×10^{1}	3.8×10^{1}	3.92×10^{1}	6.49	5.07×10^{-1}	2.7
5	1.24×10^{2}	1.23×10^{2}	1.2×10^{2}	1.48×10^{1}	1.67×10^{1}	1.81×10^{1}	5.38×10^{1}	$5.\times10^{1}$	4.9×10^{1}
	3.63×10^{2}	2.41×10^{2}	1.84×10^{2}	1.62×10^{1}	1.05×10^{1}	7.63	1.75×10^{1}	1.99×10^{1}	2.04×10^{1}
	5.31×10^{2}	1.22×10^{1}	1.79×10^{2}	8.17	8.04	7.63	2.75×10^{2}	2.09×10^{2}	1.81×10^{2}
	1.17×10^{3}	0	0	1.17×10^{2}	0	0	1.47×10^{3}	0	0

We see that these different choices of $\overline{y}$ don't always produce better values, thus we choose the first method (expansion around initial point) due to its minor computational cost.

8.3.2 Second Set by Pascucci

The second set of parameters is proposed by Pascucci, and we have $(\kappa = 1, \theta = 0.3,\ \delta = 0.8,\ \rho = -0.7,\ x = 0,\ y = \mathrm{Log}(\theta))$;

In this set, given $y = \mathrm{Log}(\theta)$, we see that even with the midpoint method and the method with the integral mean, $\overline{y}$ always coincides with the initial value y, so we will only consider the method with the initial point because of its computational cost.

With this set of parameters we consider this table of strikes expressed in bp.

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.1	80	83	87	90	93	97	100	103	107	110	113	117	120
0.25	65	71	77	82	88	94	100	106	112	118	125	131	137
0.5	51	59	67	76	84	92	100	110	119	129	139	148	158
1	35	46	57	68	78	89	100	116	132	148	163	179	195
1.5	27	39	51	64	76	88	100	122	144	166	188	210	232
2	23	36	49	62	74	87	100	128	157	185	213	242	270
3	17	31	45	58	72	86	100	133	167	200	233	267	300
5	11	26	41	56	70	85	100	133	167	200	233	267	300

Then we numerically compute Fourier prices for European call and put options

	1	3	5	7	9	11	13
0.1	0.2096	0.1515	0.1085	0.06804	0.03880	0.02203	0.01022
0.25	0.3597	0.2565	0.1753	0.1053	0.05634	0.02468	0.01003
0.5	0.5003	0.3618	0.2365	0.1446	0.07015	0.02768	0.01010
1	0.6597	0.4708	0.3193	0.1960	0.08186	0.03005	0.009986
1.5	0.7396	0.5366	0.3625	0.2340	0.08998	0.03013	0.009884
2	0.7804	0.5665	0.3994	0.2659	0.09485	0.03101	0.01008
3	0.8401	0.6191	0.4517	0.3194	0.1312	0.05265	0.02129
5	0.9001	0.6816	0.5265	0.4039	0.2262	0.1310	0.07759

	1	3	5	7	9	11	13
0.1	0.009619	0.02151	0.03848	0.06804	0.1088	0.1520	0.2102
0.25	0.009664	0.02652	0.05530	0.1053	0.1763	0.2747	0.3800
0.5	0.01030	0.03176	0.07653	0.1446	0.2602	0.4177	0.5901
1	0.009724	0.04079	0.09932	0.1960	0.4019	0.6601	0.9600
1.5	0.009628	0.04662	0.1225	0.2340	0.5300	0.9101	1.330
2	0.01043	0.05646	0.1394	0.2659	0.6648	1.161	1.710
3	0.01010	0.06909	0.1717	0.3194	0.8012	1.383	2.021
5	0.01013	0.09161	0.2265	0.4039	0.8962	1.461	2.078

and the corresponding exact implied volatilities.

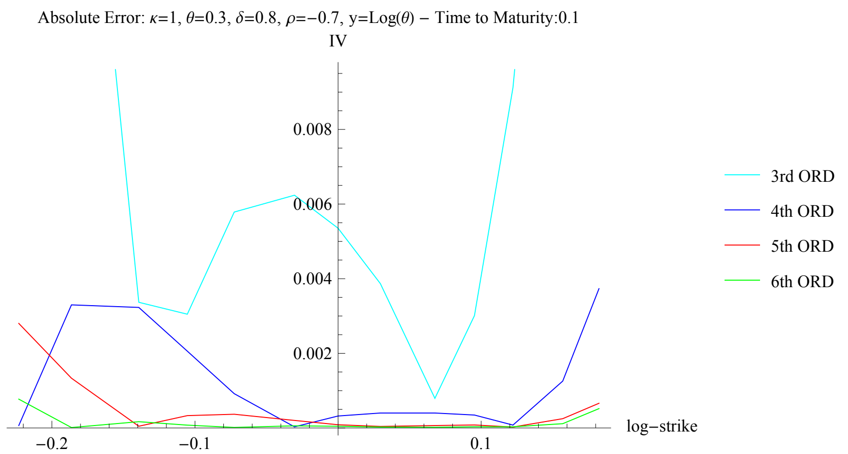
	1	3	5	7	9	11	13
0.1	59.3749	57.4122	55.7995	53.9965	52.2807	50.8866	49.3637
0.25	62.6016	59.0201	56.0034	52.9565	50.1661	47.4973	45.4576
0.5	65.1244	60.0348	55.3859	51.5276	47.5726	44.3076	42.1863
1	67.2568	59.7090	54.2581	49.6378	44.5101	41.2433	39.3597
1.5	67.3294	58.8485	52.9059	48.6054	43.0630	39.8029	38.1237
2	66.3222	57.4101	52.0719	48.0366	42.2414	39.1230	37.5144
3	64.8343	55.6842	50.9278	47.5392	42.4361	39.5896	37.8969
5	63.4838	53.5445	50.0072	47.4109	43.7258	41.5763	40.1461

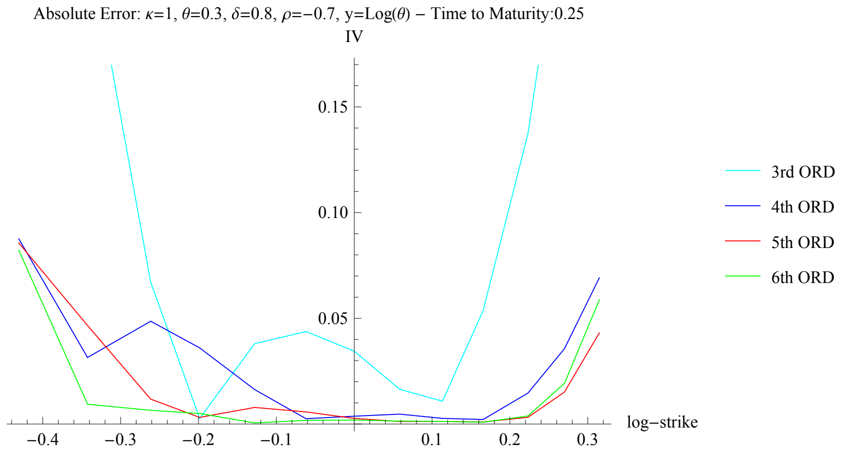
Now we compute our approximated implied volatility, as we said only for the method with the initial point ($\overline{y} = y$), then we calculate absolute errors.

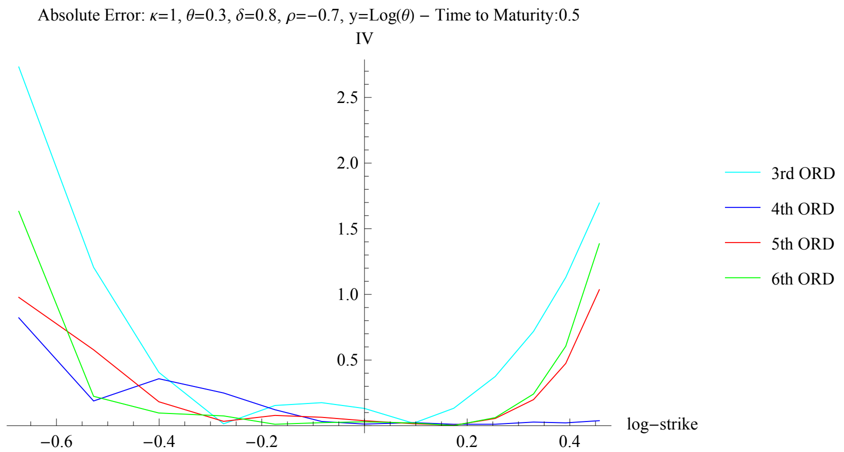
	1	3	5	7	9	11	13
0.1	4.6 7.43×10^{-1} 2.31×10^{-1} 4.56×10^{-2} 6.53×10^{-5} 2.8×10^{-3} 7.73×10^{-4}	2.64 9.24×10^{-1} 1.31×10^{-1} 3.37×10^{-3} 3.23×10^{-3} 4.67×10^{-5} 1.68×10^{-4}	1.03 1.02 6.81×10^{-2} 5.79×10^{-3} 9.19×10^{-4} 3.68×10^{-4} 1.44×10^{-5}	7.76×10^{-1} 1.07 1.04×10^{-2} 5.35×10^{-3} 3.21×10^{-4} 8.67×10^{-5} 4.53×10^{-5}	2.49 1.08 4.31×10^{-2} 7.94×10^{-4} 4.01×10^{-4} 6.51×10^{-5} 1.92×10^{-5}	3.89 1.08 $1.\times10^{-1}$ 9.12×10^{-3} 7.76×10^{-5} 2.5×10^{-5} 3.17×10^{-5}	5.41 1.1 2.01×10^{-1} $4.\times10^{-2}$ 3.73×10^{-3} 6.62×10^{-4} 5.18×10^{-4}
0.25	7.83 1.43 1.13 5.4×10^{-1} 8.76×10^{-2} 8.57×10^{-2} 8.22×10^{-2}	4.25 2.18 6.65×10^{-1} 6.71×10^{-2} 4.86×10^{-2} 1.18×10^{-2} 6.48×10^{-3}	1.23 2.57 3.43×10^{-1} 3.8×10^{-2} 1.63×10^{-2} 7.85×10^{-3} 5.27×10^{-4}	1.82 2.79 7.24×10^{-2} 3.45×10^{-2} 3.7×10^{-3} 2.66×10^{-3} 1.87×10^{-3}	4.61 2.9 1.78×10^{-1} 1.08×10^{-2} 2.68×10^{-3} 1.2×10^{-3} 1.22×10^{-3}	7.27 3.04 5.46×10^{-1} 1.37×10^{-1} 1.46×10^{-2} 3.16×10^{-3} 3.8×10^{-3}	9.31 3.34 1.11 4.19×10^{-1} 6.92×10^{-2} 4.31×10^{-2} 5.87×10^{-2}
0.5	1.04×10^{1} 2.36 3.44 2.73 8.22×10^{-1} 9.78×10^{-1} 1.63	5.26 4.25 2.16 4.06×10^{-1} 3.58×10^{-1} 1.82×10^{-1} 9.67×10^{-2}	6.14×10^{-1} 5.38 1.11 1.54×10^{-1} 1.22×10^{-1} 7.84×10^{-2} 1.1×10^{-2}	3.24 5.98 3.25×10^{-1} 1.31×10^{-1} 1.24×10^{-2} 3.79×10^{-2} 2.97×10^{-2}	7.2 6.47 5.76×10^{-1} 1.33×10^{-1} 1.07×10^{-2} 3.14×10^{-3} 1.22×10^{-3}	1.05×10^{1} 7.17 1.83 7.19×10^{-1} 2.77×10^{-2} 1.99×10^{-1} 2.42×10^{-1}	1.26×10^{1} 8.33 3.58 1.7 3.83×10^{-2} 1.04 1.39
1	1.25×10^{1} 4.09 1.04×10^{1} 1.32×10^{1} 6.37 1.17×10^{1} 3.05×10^{1}	4.94 9.01 6.64 1.75 2.56 1.69 1.75	5.14×10^{-1} 1.16×10^{1} 3.92 4.36×10^{-1} 1.02 7.15×10^{-1} 3.44×10^{-1}	5.13 1.33×10^{1} 1.56 4.56×10^{-1} 5.74×10^{-3} 3.7×10^{-1} 2.79×10^{-1}	1.03×10^{1} 1.53×10^{1} 1.64 6.61×10^{-1} 5.64×10^{-1} 7.99×10^{-1} 7.77×10^{-1}	1.35×10^{1} 1.74×10^{1} 5.08 2.3 1.68 4.27 5.44	1.54×10^{1} 2.01×10^{1} 9.15 4.31 4.6 1.37×10^{1} 2.01×10^{1}
1.5	1.26×10^{1} 6.75 1.94×10^{1} 2.96×10^{1} 1.38×10^{1} 4.97×10^{1} 1.32×10^{2}	4.08 1.45×10^{1} 1.3×10^{1} 4.19 7.92 5.91 9.38	1.87 1.88×10^{1} 8.06 6.77×10^{-1} 2.99 2.38 1.7	6.17 2.15×10^{1} 4.21 9.75×10^{-1} 1.43×10^{-3} 1.11 7.65×10^{-1}	1.17×10^{1} 2.53×10^{1} 1.94 9.32×10^{-1} 3.09 4.73 4.89	1.5×10^{1} 2.88×10^{1} 7.84 3.41 8.37 1.99×10^{1} 2.82×10^{1}	1.66×10^{1} 3.25×10^{1} 1.37×10^{1} 5.74 1.84×10^{1} 5.28×10^{1} 8.89×10^{1}

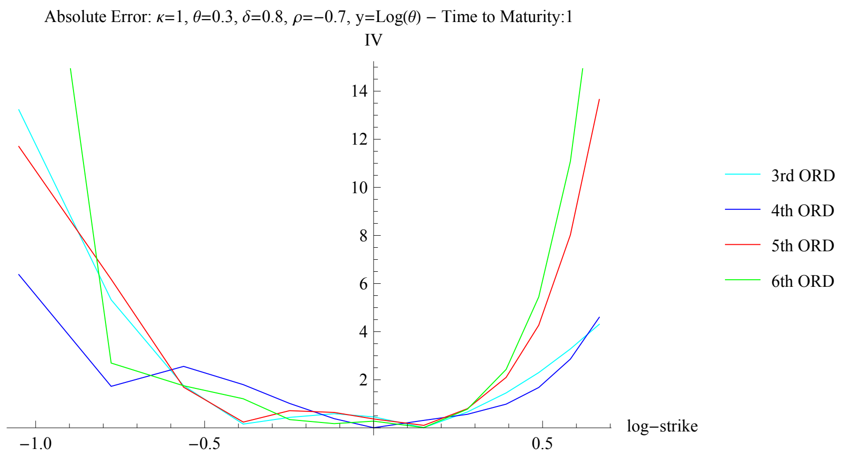
	1.15×10^{1}	2.64	2.7	6.74	1.25×10^{1}	1.56×10^{1}	1.73×10^{1}
	1.09×10^{1}	2.13×10^{1}	2.65×10^{1}	3.01×10^{1}	3.59×10^{1}	4.06×10^{1}	4.5×10^{1}
	3.01×10^{1}	2.06×10^{1}	1.41×10^{1}	8.65	9.24×10^{-1}	9.04	1.66×10^{1}
2	4.74×10^{1}	6.59	5.69×10^{-1}	1.58	7.87×10^{-1}	3.67	5.96
	1.26×10^{1}	1.66×10^{1}	6.61	1.22×10^{-1}	8.72	2.17×10^{1}	4.33×10^{1}
	1.28×10^{2}	1.04×10^{1}	5.09	2.17	1.49×10^{1}	5.47×10^{1}	1.33×10^{2}
	2.85×10^{2}	2.91×10^{1}	5.67	1.51	1.64×10^{1}	8.61×10^{1}	2.55×10^{2}
	1.01×10^{1}	9.12×10^{-1}	3.84	7.23	1.23×10^{1}	1.52×10^{1}	1.69×10^{1}
	2.01×10^{1}	3.58×10^{1}	4.31×10^{1}	4.81×10^{1}	5.61×10^{1}	6.18×10^{1}	6.65×10^{1}
	5.85×10^{1}	4.19×10^{1}	3.17×10^{1}	2.37×10^{1}	9.46	1.3	1.05×10^{1}
3	1.01×10^{2}	1.61×10^{1}	1.67	1.84	8.27×10^{-1}	1.71	3.51
	4.17	4.81×10^{1}	1.99×10^{1}	2.11	2.48×10^{1}	5.45×10^{1}	9.47×10^{1}
	4.93×10^{2}	3.18×10^{1}	1.16×10^{1}	3.35	4.92×10^{1}	1.48×10^{2}	3.11×10^{2}
	9.38×10^{2}	1.33×10^{2}	2.56×10^{1}	5.62	4.89×10^{1}	2.46×10^{2}	6.6×10^{2}
	8.71	1.23	4.77	7.36	1.1×10^{1}	1.32×10^{1}	1.46×10^{1}
	4.45×10^{1}	6.82×10^{1}	7.83×10^{1}	8.48×10^{1}	9.43×10^{1}	1.01×10^{2}	1.06×10^{2}
	1.41×10^{2}	1.09×10^{2}	9.23×10^{1}	8.01×10^{1}	6.1×10^{1}	4.72×10^{1}	3.6×10^{1}
5	2.75×10^{2}	6.05×10^{1}	2.6×10^{1}	1.37×10^{1}	6.53	5.83	6.15
	1.87×10^{2}	1.84×10^{2}	8.74×10^{1}	2.8×10^{1}	5.53×10^{1}	1.23×10^{2}	1.95×10^{2}
	2.57×10^{3}	1.89×10^{2}	3.06×10^{1}	5.23×10^{1}	2.08×10^{2}	4.38×10^{2}	7.52×10^{2}
	3.18×10^{3}	7.58×10^{2}	9.2×10^{1}	1.06×10^{2}	8.62	5.11×10^{2}	1.41×10^{3}

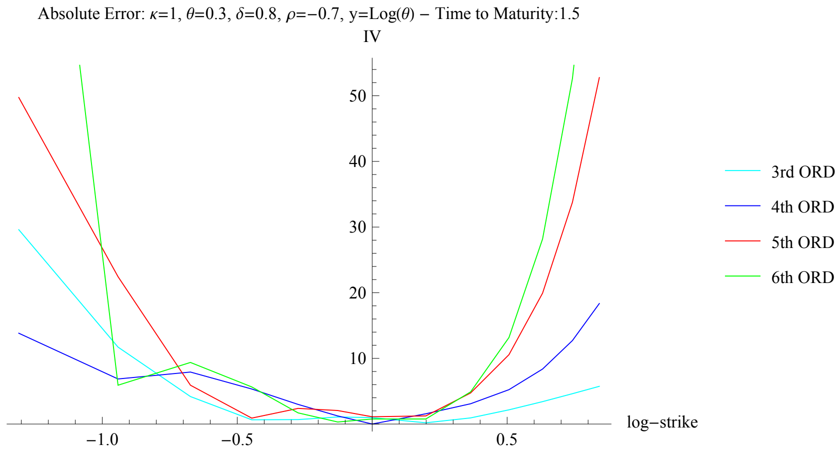
We notice that even with this set of parameters we good results only for short maturities, as we can see even in these graphs.

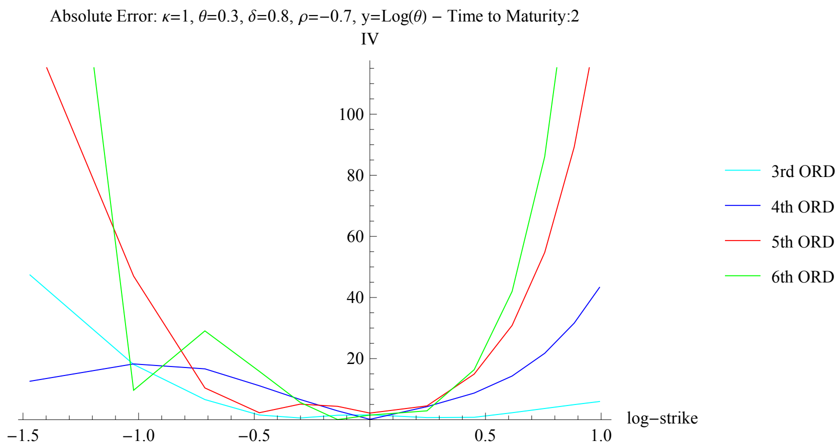


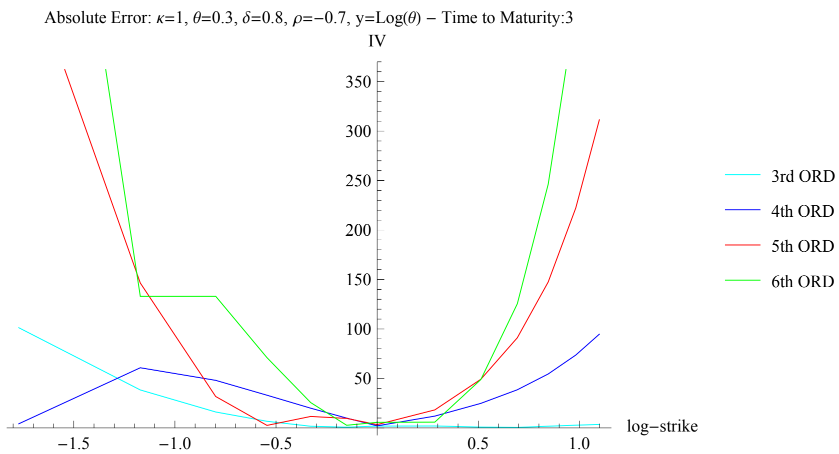

Absolute Error: κ=1, θ=0.3, δ=0.8, ρ=−0.7, y=Log(θ) − Time to Maturity:0.25
IV
0.15
0.10
0.05
log−strike
−0.4
−0.3
−0.2
−0.1
0.1
0.2
0.3
3rd ORD
4th ORD
5th ORD
6th ORD

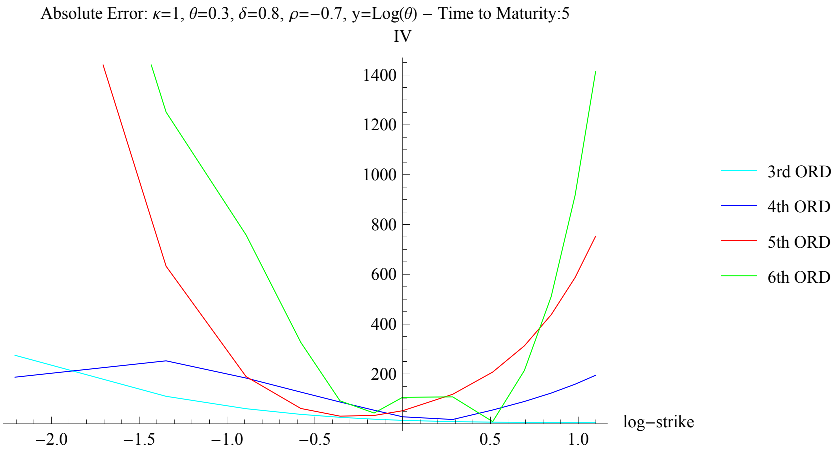

Absolute Error: κ=1, θ=0.3, δ=0.8, ρ=−0.7, y=Log(θ) − Time to Maturity:0.5
IV
2.5
2.0
1.5
1.0
0.5
log−strike
−0.6
−0.4
−0.2
0.2
0.4
3rd ORD
4th ORD
5th ORD
6th ORD


Absolute Error: κ=1, θ=0.3, δ=0.8, ρ=−0.7, y=Log(θ) − Time to Maturity:1
IV
14
12
10
8
6
4
2
log−strike
−1.0
−0.5
0.5
3rd ORD
4th ORD
5th ORD
6th ORD


Absolute Error: κ=1, θ=0.3, δ=0.8, ρ=−0.7, y=Log(θ) − Time to Maturity:1.5
IV
log−strike
3rd ORD
4th ORD
5th ORD
6th ORD


Absolute Error: κ=1, θ=0.3, δ=0.8, ρ=−0.7, y=Log(θ) − Time to Maturity:2
IV
log−strike
3rd ORD
4th ORD
5th ORD
6th ORD


Absolute Error: κ=1, θ=0.3, δ=0.8, ρ=−0.7, y=Log(θ) − Time to Maturity:3
IV
log−strike
3rd ORD
4th ORD
5th ORD
6th ORD



8.3.3 Third Set by Bakshi, Cao and Chen

The third set of parameters considered is as given by Bakshi, Cao and Chen in ([BCC97]), so we have $\left(\kappa = 1.15, \theta = \frac{0.04}{1.15}, \ \delta = 0.39, \ \rho = -0.64, \ x = 0, \ y = 2\,\mathrm{Log}(0.2)\right)$.

Now we create a matrix of strikes expressed in bp

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.1	96	97	97	98	99	99	100	101	101	102	103	103	104
0.25	90	92	93	95	97	98	100	101	102	104	105	106	107
0.5	83	86	89	92	94	97	100	102	104	106	108	110	112
1	74	78	83	87	91	96	100	103	106	109	112	115	118
1.5	67	72	78	84	89	94	100	104	108	112	116	120	124
2	62	68	75	81	87	94	100	105	110	115	120	125	130
3	55	62	70	78	85	92	100	107	114	120	127	134	141
5	39	49	59	70	80	90	100	111	121	132	143	153	164

then we numerically compute call and put prices of Fourier

	1	3	5	7	9	11	13
0.1	0.05056	0.04327	0.03031	0.02472	0.01978	0.01185	0.008843
0.25	0.1096	0.08500	0.05606	0.03801	0.02803	0.01638	0.01080
0.5	0.1796	0.1283	0.09005	0.05179	0.03228	0.01826	0.009424
1	0.2703	0.1914	0.1284	0.07006	0.04126	0.02175	0.01051
1.5	0.3401	0.2423	0.1549	0.08400	0.04679	0.02299	0.01033
2	0.3901	0.2744	0.1792	0.09600	0.05096	0.02357	0.009952
3	0.4603	0.3271	0.2099	0.1167	0.05736	0.02579	0.009916
5	0.6204	0.4361	0.2609	0.1490	0.07884	0.03482	0.009980

	1	3	5	7	9	11	13
0.1	0.01056	0.01327	0.02031	0.02472	0.02978	0.04185	0.04884
0.25	0.009552	0.01500	0.02606	0.03801	0.04803	0.06638	0.08080
0.5	0.009580	0.01832	0.03005	0.05179	0.07228	0.09826	0.1294
1	0.01030	0.02138	0.03840	0.07006	0.1013	0.1417	0.1905
1.5	0.01005	0.02228	0.04488	0.08400	0.1268	0.1830	0.2503
2	0.01008	0.02439	0.04921	0.09600	0.1510	0.2236	0.3100
3	0.01029	0.02707	0.05986	0.1167	0.1974	0.2958	0.4199
5	0.01038	0.02608	0.06092	0.1490	0.2888	0.4648	0.6500

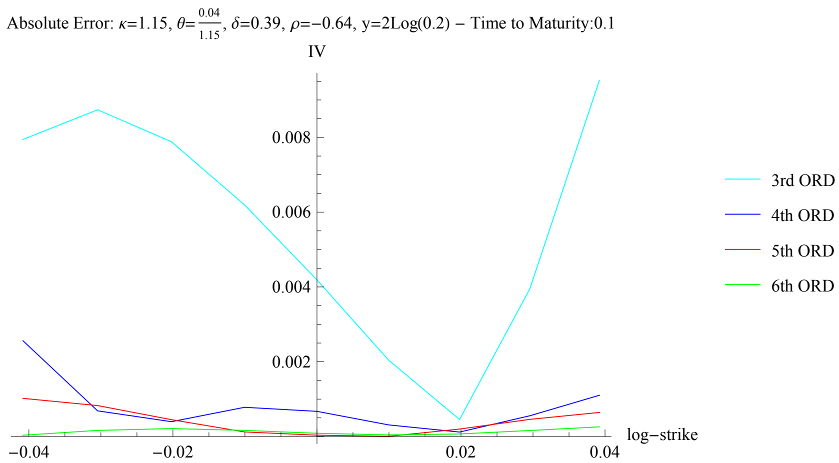
and the corresponding exact implied volatilities

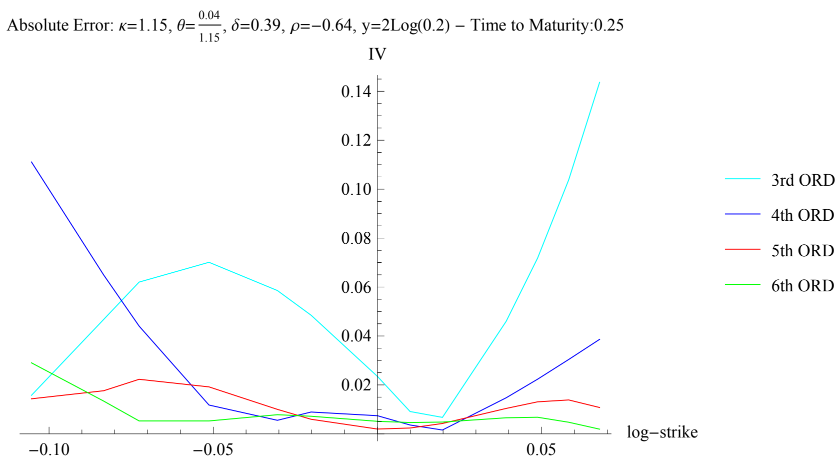
	1	3	5	7	9	11	13
0.1	20.8514	20.5375	19.9098	19.5976	19.2877	18.6812	18.3885
0.25	22.1212	21.2130	19.9853	19.0641	18.4626	17.6127	17.1079
0.5	23.2380	21.5344	20.0918	18.3722	17.3012	16.3866	15.7154
1	24.0045	21.7211	19.7221	17.5843	16.3363	15.3634	14.7517
1.5	24.3455	21.8086	19.4109	17.2243	15.8844	14.8918	14.3152
2	24.3397	21.5905	19.2765	17.0562	15.6568	14.6525	14.0915
3	24.0332	21.2940	18.9430	16.9522	15.4781	14.5175	13.9322
5	27.6529	21.7978	17.4399	16.8007	16.6548	15.9243	14.0942

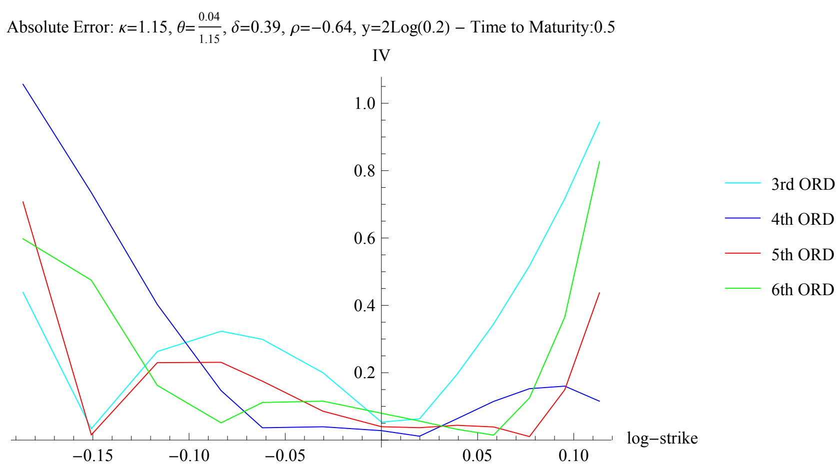
Now we compute our approximated implied volatilities setting $\overline{y} = y$, and we calculate absolute errors.

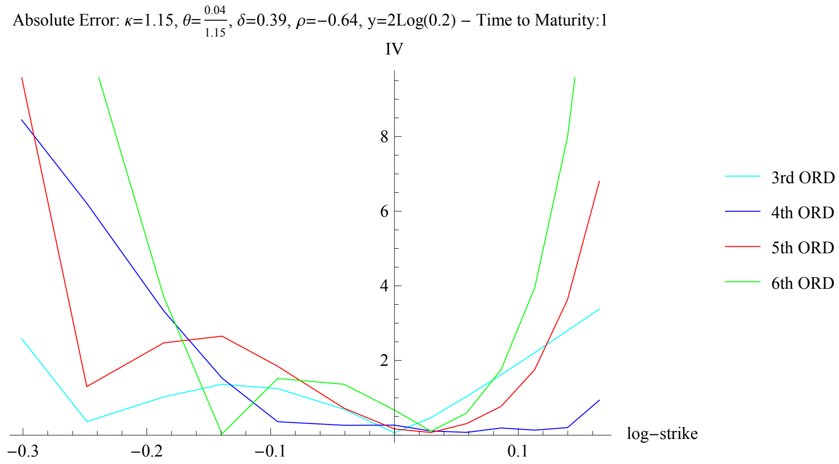
	1	3	5	7	9	11	13
	8.51×10^{-1}	5.38×10^{-1}	9.02×10^{-2}	4.02×10^{-1}	7.12×10^{-1}	1.32	1.61
	6.66×10^{-1}	6.75×10^{-1}	6.84×10^{-1}	6.86×10^{-1}	6.86×10^{-1}	6.91×10^{-1}	$7.\times10^{-1}$
	7.5×10^{-2}	5.14×10^{-2}	1.32×10^{-2}	2.96×10^{-3}	1.87×10^{-2}	5.49×10^{-2}	7.95×10^{-2}
0.1	7.94×10^{-3}	8.74×10^{-3}	6.19×10^{-3}	4.19×10^{-3}	2.04×10^{-3}	3.95×10^{-3}	9.52×10^{-3}
	2.56×10^{-3}	6.87×10^{-4}	7.82×10^{-4}	6.73×10^{-4}	3.11×10^{-4}	5.55×10^{-4}	1.1×10^{-3}
	1.02×10^{-3}	8.28×10^{-4}	1.22×10^{-4}	3.58×10^{-5}	4.36×10^{-6}	4.59×10^{-4}	6.44×10^{-4}
	3.45×10^{-5}	1.63×10^{-4}	1.64×10^{-4}	8.69×10^{-5}	4.34×10^{-5}	1.61×10^{-4}	2.59×10^{-4}
	2.12	1.21	1.47×10^{-2}	9.36×10^{-1}	1.54	2.39	2.89
	1.55	1.67	1.76	1.78	1.8	1.85	1.94
	5.89×10^{-1}	3.62×10^{-1}	1.26×10^{-1}	1.61×10^{-2}	1.08×10^{-1}	2.75×10^{-1}	4.34×10^{-1}
0.25	1.57×10^{-2}	6.2×10^{-2}	5.86×10^{-2}	2.35×10^{-2}	6.73×10^{-3}	7.19×10^{-2}	1.44×10^{-1}
	1.11×10^{-1}	4.4×10^{-2}	5.54×10^{-3}	7.42×10^{-3}	1.57×10^{-3}	2.23×10^{-2}	3.86×10^{-2}
	1.43×10^{-2}	2.23×10^{-2}	$1.\times10^{-2}$	1.97×10^{-3}	4.24×10^{-3}	1.31×10^{-2}	1.08×10^{-2}
	2.9×10^{-2}	5.28×10^{-3}	7.82×10^{-3}	5.12×10^{-3}	4.81×10^{-3}	6.78×10^{-3}	1.92×10^{-3}
	3.24	1.53	9.18×10^{-2}	1.63	2.7	3.61	4.28
	2.86	3.34	3.6	3.81	3.97	4.23	4.69
	2.21	1.25	5.98×10^{-1}	7.09×10^{-2}	5.21×10^{-1}	1.08	1.83
0.5	4.38×10^{-1}	2.63×10^{-1}	2.99×10^{-1}	5.34×10^{-2}	1.95×10^{-1}	5.17×10^{-1}	9.44×10^{-1}
	1.06	4.03×10^{-1}	3.65×10^{-2}	2.8×10^{-2}	6.32×10^{-2}	1.53×10^{-1}	1.16×10^{-1}
	7.07×10^{-1}	2.3×10^{-1}	1.75×10^{-1}	3.97×10^{-2}	4.39×10^{-2}	1.04×10^{-2}	4.37×10^{-1}
	5.97×10^{-1}	1.62×10^{-1}	1.12×10^{-1}	7.96×10^{-2}	3.2×10^{-2}	1.25×10^{-1}	8.26×10^{-1}

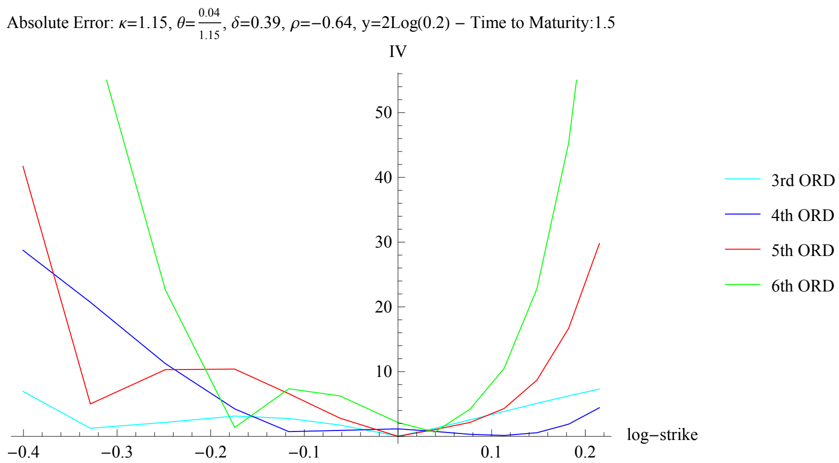
1	4. 5.49 7.17 2.59 8.45 9.59 1.31×10^{1}	1.72 6.79 4.25 1.02 3.33 2.47 3.71	2.78×10^{-1} 7.66 2.04 1.25 3.6×10^{-1} 1.85 1.52	2.42 8.46 1.61×10^{-1} 6.41×10^{-2} 2.68×10^{-1} 1.66×10^{-1} 6.73×10^{-1}	3.66 9.03 1.61 1.03 7.46×10^{-2} 3.09×10^{-1} 5.94×10^{-1}	4.64 9.78 3.2 2.21 1.35×10^{-1} 1.76 3.95	5.25 1.08×10^{1} 5.03 3.38 9.36×10^{-1} 6.8 1.48×10^{1}
1.5	4.35 8.17 1.42×10^{1} 6.95 2.87×10^{1} 4.17×10^{1} 8.56×10^{1}	1.81 1.04×10^{1} 8.79 2.17 1.12×10^{1} 1.03×10^{1} 2.26×10^{1}	5.89×10^{-1} 1.21×10^{1} 4.19 2.77 7.33×10^{-1} 6.58 7.36	2.78 1.35×10^{1} 1.43×10^{-1} 3.58×10^{-2} 1.16 1.1×10^{-2} 2.14	4.12 1.46×10^{1} 2.67 2.55 3.18×10^{-1} 2.15 4.2	5.11 1.58×10^{1} 5.54 5.1 5.63×10^{-1} 8.7 2.28×10^{1}	5.68 1.73×10^{1} 8.6 7.3 4.42 2.98×10^{1} 8.32×10^{1}
2	4.34 1.12×10^{1} 2.26×10^{1} 1.17×10^{1} 6.62×10^{1} 9.46×10^{1} 3.26×10^{2}	1.59 1.44×10^{1} 1.4×10^{1} 4.32 2.26×10^{1} 3.15×10^{1} 5.75×10^{1}	7.24×10^{-1} 1.67×10^{1} 7.35 4.86 9.96×10^{-1} 1.64×10^{1} 2.31×10^{1}	2.94 1.88×10^{1} 1.07 2.04×10^{-1} 2.96 8.56×10^{-1} 4.93	4.34 2.04×10^{1} 3.38 5.09 8.13×10^{-1} 5.79 1.35×10^{1}	5.35 2.21×10^{1} 7.75 9.67 1.36 2.39×10^{1} 7.82×10^{1}	5.91 2.4×10^{1} 1.22×10^{1} 1.34×10^{1} 1.01×10^{1} 8.28×10^{1} 2.94×10^{2}
3	4.03 1.8×10^{1} 4.3×10^{1} $2.\times10^{1}$ 2.03×10^{2} 1.94×10^{2} 1.93×10^{3}	1.29 2.28×10^{1} 2.78×10^{1} 1.07×10^{1} 5.78×10^{1} 1.45×10^{2} 1.75×10^{2}	1.06 2.65×10^{1} 1.55×10^{1} 9.21 3.99 4.8×10^{1} 1.17×10^{2}	3.05 2.96×10^{1} 5.17 2.61 9.48 5.5 1.6×10^{1}	4.52 3.22×10^{1} 3.38 1.53×10^{1} 1.46 1.28×10^{1} 5.28×10^{1}	5.48 3.46×10^{1} 1.07×10^{1} 2.59×10^{1} 3.39 7.47×10^{1} 3.95×10^{2}	6.07 3.73×10^{1} 1.82×10^{1} 3.47×10^{1} 1.93×10^{1} 3.12×10^{2} 1.68×10^{3}
5	7.65 3.27×10^{1} 1.11×10^{2} 1.07×10^{2} 1.14×10^{3} 2.03×10^{3} 2.62×10^{4}	1.8 3.97×10^{1} 7.37×10^{1} 2.44×10^{1} 2.43×10^{2} 1.05×10^{3} 1.15×10^{3}	2.56 4.49×10^{1} 4.62×10^{1} 1.6×10^{1} 5.08×10^{1} 1.78×10^{2} $9.\times10^{2}$	3.2 5.12×10^{1} 2.32×10^{1} 2.57×10^{1} 3.31×10^{1} 2.21×10^{1} 8.3×10^{1}	3.35 5.7×10^{1} 2.96 7.17×10^{1} $6.\times10^{1}$ 1.39×10^{2} 2.29×10^{2}	4.08 6.15×10^{1} 1.43×10^{1} 1.12×10^{2} 1.08×10^{2} 1.64×10^{2} 3.41×10^{3}	5.91 6.39×10^{1} 2.72×10^{1} 1.39×10^{2} 4.64×10^{1} 1.57×10^{3} 1.65×10^{4}

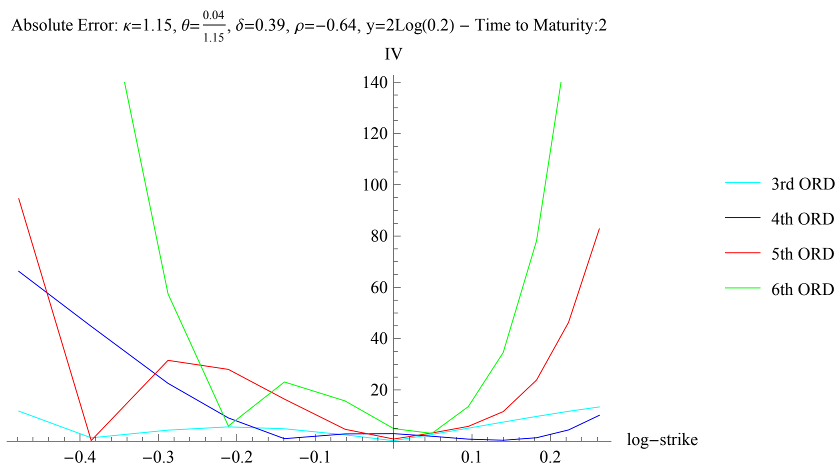

Absolute Error: κ=1.15, $\theta=\frac{0.04}{1.15}$, δ=0.39, ρ=−0.64, y=2Log(0.2) − Time to Maturity:0.1
IV
0.008
0.006
0.004
0.002
log−strike
−0.04
−0.02
0.02
0.04
3rd ORD
4th ORD
5th ORD
6th ORD

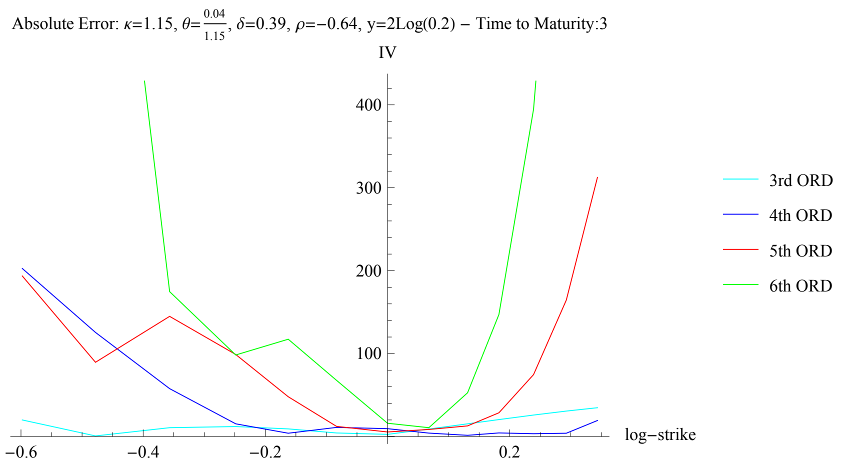

Absolute Error: κ=1.15, $\theta=\frac{0.04}{1.15}$, δ=0.39, ρ=−0.64, y=2Log(0.2) − Time to Maturity:0.25
IV
0.14
0.12
0.10
0.08
0.06
0.04
0.02
log−strike
−0.10
−0.05
0.05
3rd ORD
4th ORD
5th ORD
6th ORD

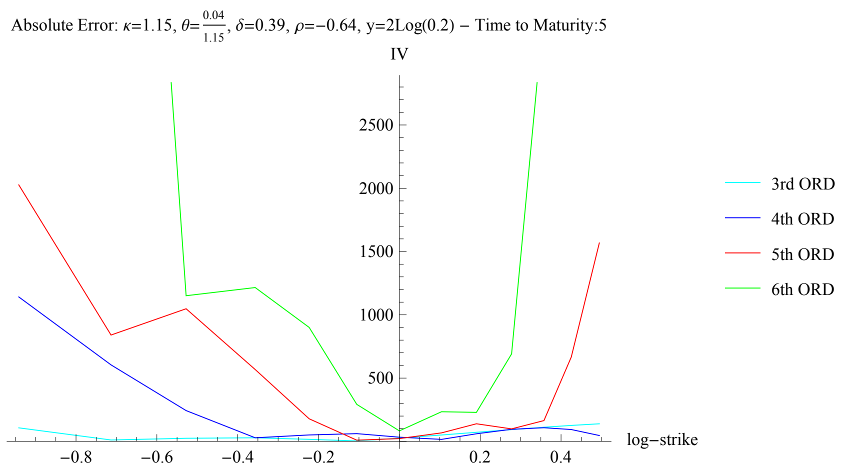

Absolute Error: κ=1.15, $\theta=\frac{0.04}{1.15}$, δ=0.39, ρ=−0.64, y=2Log(0.2) − Time to Maturity:0.5
IV
1.0
0.8
0.6
0.4
0.2
log−strike
−0.15
−0.10
−0.05
0.05
0.10
3rd ORD
4th ORD
5th ORD
6th ORD


Absolute Error: κ=1.15, $\theta=\frac{0.04}{1.15}$, δ=0.39, ρ=−0.64, y=2Log(0.2) − Time to Maturity:1
IV
8
6
4
2
−0.3
−0.2
−0.1
0.1
log−strike
3rd ORD
4th ORD
5th ORD
6th ORD


Absolute Error: κ=1.15, $\theta=\frac{0.04}{1.15}$, δ=0.39, ρ=−0.64, y=2Log(0.2) − Time to Maturity:1.5
IV
50
40
30
20
10
−0.4
−0.3
−0.2
−0.1
0.1
0.2
log−strike
3rd ORD
4th ORD
5th ORD
6th ORD


Absolute Error: κ=1.15, $\theta=\frac{0.04}{1.15}$, δ=0.39, ρ=−0.64, y=2Log(0.2) − Time to Maturity:2
IV
140
120
100
80
60
40
20
−0.4
−0.3
−0.2
−0.1
0.1
0.2
log−strike
3rd ORD
4th ORD
5th ORD
6th ORD





From the previous table and these graphs we can conclude that the behavior of our approximation is the same of the previous cases. Let us now consider the other two choices for the value of $\overline{y}$, so in the next matrix, every element is a sub-matrix where the first column corresponds to the initial point method, the second corresponds to the midpoint method, and the third corresponds to the integral mean method.

	1			7			13		
	8.51×10^{-1}	1.51	9.24×10^{-1}	4.02×10^{-1}	2.61×10^{-1}	3.3×10^{-1}	1.61	9.48×10^{-1}	1.54
	6.66×10^{-1}	6.28×10^{-1}	6.61×10^{-1}	6.86×10^{-1}	6.91×10^{-1}	6.85×10^{-1}	$7.\times10^{-1}$	7.48×10^{-1}	7.04×10^{-1}
	7.5×10^{-2}	1.02×10^{-1}	7.79×10^{-2}	2.96×10^{-3}	1.94×10^{-2}	6.33×10^{-4}	7.95×10^{-2}	6.2×10^{-2}	7.77×10^{-2}
0.1	7.94×10^{-3}	6.11×10^{-4}	7.13×10^{-3}	4.19×10^{-3}	4.84×10^{-3}	4.29×10^{-3}	9.52×10^{-3}	1.29×10^{-3}	8.67×10^{-3}
	2.56×10^{-3}	3.23×10^{-3}	2.68×10^{-3}	6.73×10^{-4}	2.66×10^{-4}	6.36×10^{-4}	1.1×10^{-3}	3.51×10^{-4}	9.19×10^{-4}
	1.02×10^{-3}	4.4×10^{-4}	9.68×10^{-4}	3.58×10^{-5}	2.84×10^{-5}	2.76×10^{-5}	6.44×10^{-4}	6.57×10^{-4}	6.29×10^{-4}
	3.45×10^{-5}	0	0	8.69×10^{-5}	0	0	2.59×10^{-4}	0	0
	2.12	2.78	2.29	9.36×10^{-1}	2.73×10^{-1}	7.64×10^{-1}	2.89	2.23	2.72
	1.55	1.44	1.52	1.78	1.78	1.78	1.94	2.01	1.95
	5.89×10^{-1}	6.73×10^{-1}	6.1×10^{-1}	1.61×10^{-2}	3.66×10^{-2}	3.05×10^{-3}	4.34×10^{-1}	4.04×10^{-1}	4.27×10^{-1}
0.25	1.57×10^{-2}	9.9×10^{-2}	3.55×10^{-2}	2.35×10^{-2}	2.88×10^{-2}	2.5×10^{-2}	1.44×10^{-1}	1.09×10^{-1}	1.35×10^{-1}
	1.11×10^{-1}	1.09×10^{-1}	1.12×10^{-1}	7.42×10^{-3}	5.64×10^{-3}	7.03×10^{-3}	3.86×10^{-2}	1.92×10^{-2}	3.37×10^{-2}
	1.43×10^{-2}	4.87×10^{-2}	2.25×10^{-2}	1.97×10^{-3}	3.06×10^{-3}	2.25×10^{-3}	1.08×10^{-2}	4.31×10^{-3}	8.92×10^{-3}
	2.9×10^{-2}	0	0	5.12×10^{-3}	0	0	1.92×10^{-3}	0	0
	3.24	3.9	3.55	1.63	9.65×10^{-1}	1.31	4.28	3.62	3.97
	2.86	2.65	2.76	3.81	3.8	3.8	4.69	4.8	4.74
	2.21	2.42	2.31	7.09×10^{-2}	2.3×10^{-2}	2.78×10^{-2}	1.83	1.81	1.82
0.5	4.38×10^{-1}	8.15×10^{-1}	6.07×10^{-1}	5.34×10^{-2}	7.55×10^{-2}	6.39×10^{-2}	9.44×10^{-1}	8.35×10^{-1}	8.95×10^{-1}
	1.06	1.04	1.06	2.8×10^{-2}	2.58×10^{-2}	2.72×10^{-2}	1.16×10^{-1}	4.22×10^{-2}	4.38×10^{-2}
	7.07×10^{-1}	1.2	9.25×10^{-1}	3.97×10^{-2}	4.74×10^{-2}	4.32×10^{-2}	4.37×10^{-1}	5.86×10^{-1}	5.06×10^{-1}
	5.97×10^{-1}	0	0	7.96×10^{-2}	0	0	8.26×10^{-1}	0	0
	4.	4.67	4.54	2.42	1.75	1.88	5.25	4.59	4.71
	5.49	5.12	5.19	8.46	8.42	8.43	1.08×10^{1}	1.09×10^{1}	1.09×10^{1}
	7.17	7.69	7.59	1.61×10^{-1}	1.92×10^{-2}	4.78×10^{-2}	5.03	5.12	5.1
1	2.59	3.96	3.68	6.41×10^{-2}	1.47×10^{-1}	1.31×10^{-1}	3.38	3.17	3.21
	8.45	8.93	8.85	2.68×10^{-1}	2.92×10^{-1}	2.88×10^{-1}	9.36×10^{-1}	1.7	1.54
	9.59	1.49×10^{1}	1.38×10^{1}	1.66×10^{-1}	2.06×10^{-1}	1.98×10^{-1}	6.8	8.4	8.06
	1.31×10^{1}	0	0	6.73×10^{-1}	0	0	1.48×10^{1}	0	0
	4.35	5.01	5.04	2.78	2.11	2.08	5.68	5.02	4.99
	8.17	7.67	7.64	1.35×10^{1}	1.35×10^{1}	1.35×10^{1}	1.73×10^{1}	1.75×10^{1}	1.75×10^{1}
	1.42×10^{1}	1.52×10^{1}	1.52×10^{1}	1.43×10^{-1}	2.85×10^{-1}	2.93×10^{-1}	8.6	8.9	8.91
1.5	6.95	9.87	$1.\times10^{1}$	3.58×10^{-2}	2.03×10^{-1}	2.11×10^{-1}	7.3	7.13	7.12
	2.87×10^{1}	3.13×10^{1}	3.15×10^{1}	1.16	1.27	1.28	4.42	6.39	6.5
	4.17×10^{1}	6.31×10^{1}	6.43×10^{1}	1.1×10^{-2}	5.61×10^{-2}	5.97×10^{-2}	2.98×10^{1}	3.67×10^{1}	3.71×10^{1}
	8.56×10^{1}	0	0	2.14	0	0	8.32×10^{1}	0	0
	4.34	5.	5.15	2.94	2.28	2.13	5.91	5.25	5.1
	1.12×10^{1}	1.06×10^{1}	1.04×10^{1}	1.88×10^{1}	1.87×10^{1}	1.87×10^{1}	2.4×10^{1}	2.42×10^{1}	2.43×10^{1}
	2.26×10^{1}	2.4×10^{1}	2.43×10^{1}	1.07	1.17	1.2	1.22×10^{1}	1.28×10^{1}	1.29×10^{1}
2	1.17×10^{1}	1.63×10^{1}	1.74×10^{1}	2.04×10^{-1}	5.67×10^{-2}	1.17×10^{-1}	1.34×10^{1}	1.35×10^{1}	1.35×10^{1}
	6.62×10^{1}	7.41×10^{1}	7.59×10^{1}	2.96	3.23	3.29	1.01×10^{1}	1.38×10^{1}	1.48×10^{1}
	9.46×10^{1}	1.45×10^{2}	1.59×10^{2}	8.56×10^{-1}	8.23×10^{-1}	8.15×10^{-1}	8.28×10^{1}	1.03×10^{2}	1.08×10^{2}
	3.26×10^{2}	0	0	4.93	0	0	2.94×10^{2}	0	0
	4.03	4.7	4.99	3.05	2.38	2.09	6.07	5.4	5.11
	1.8×10^{1}	1.72×10^{1}	1.69×10^{1}	2.96×10^{1}	2.94×10^{1}	2.94×10^{1}	3.73×10^{1}	3.75×10^{1}	3.76×10^{1}
	4.3×10^{1}	4.52×10^{1}	4.64×10^{1}	5.17	5.04	4.99	1.82×10^{1}	1.98×10^{1}	2.06×10^{1}
3	$2.\times10^{1}$	2.72×10^{1}	3.1×10^{1}	2.61	2.19	2.	3.47×10^{1}	3.65×10^{1}	3.74×10^{1}
	2.03×10^{2}	2.34×10^{2}	2.49×10^{2}	9.48	1.02×10^{1}	1.06×10^{1}	1.93×10^{1}	2.61×10^{1}	2.97×10^{1}
	1.94×10^{2}	3.27×10^{2}	4.05×10^{2}	5.5	5.91	6.14	3.12×10^{2}	3.92×10^{2}	4.36×10^{2}
	1.93×10^{3}	0	0	1.6×10^{1}	0	0	1.68×10^{3}	0	0
	7.65	8.32	8.76	3.2	2.54	2.09	5.91	5.24	4.8
	3.27×10^{1}	3.14×10^{1}	3.05×10^{1}	5.12×10^{1}	5.09×10^{1}	5.08×10^{1}	6.39×10^{1}	6.42×10^{1}	6.44×10^{1}
	1.11×10^{2}	1.16×10^{2}	1.2×10^{2}	2.32×10^{1}	2.2×10^{1}	2.12×10^{1}	2.72×10^{1}	3.18×10^{1}	3.51×10^{1}
5	1.07×10^{2}	1.3×10^{2}	1.49×10^{2}	2.57×10^{1}	2.55×10^{1}	2.55×10^{1}	1.39×10^{2}	1.51×10^{2}	1.61×10^{2}
	1.14×10^{3}	1.33×10^{3}	1.48×10^{3}	3.31×10^{1}	3.34×10^{1}	3.35×10^{1}	4.64×10^{1}	5.26×10^{1}	5.72×10^{1}
	2.03×10^{3}	3.11×10^{3}	4.09×10^{3}	2.21×10^{1}	2.27×10^{1}	2.31×10^{1}	1.57×10^{3}	$2.\times10^{3}$	2.37×10^{3}
	2.62×10^{4}	0	0	8.3×10^{1}	0	0	1.65×10^{4}	0	0

We notice that even with this third set of parameters, a different choice of the point of expansion $\overline{y}$ doesn't produce significantly better results, so we can only consider the first method due to its smaller computation cost.

8.4 Three Halves stochastic volatility model

We consider now the 3/2 stochastic volatility model. The risk-neutral dynamics of the underlying S in this setting are given by

$$dS_t = \sqrt{Z_t}\, S_t\, dW_t, \quad S_0 = s > 0$$

$$dZ_t = Z_t\big(\kappa(\theta - Z_t)\, dt + \delta\sqrt{Z_t}\, dB_t\big), \quad Z_0 = z > 0,$$

$$d\langle W, B\rangle_t = \rho\, dt.$$

As in all stochastic volatility models, one typically sets $\rho < 0$ in order to capture the leverage effect. The 3/2 model is noteworthy in that it does not fall into the affine class of Duffie, Pan, and Singleton ([DPS00]), and yet it still allows for European option prices to be computed in semi-closed form (as a Fourier integral). Notice however that the characteristic function (given in (8.1) below) involves special functions such as the Gamma and the confluent hypergeometric functions. Therefore, Fourier pricing methods are not an efficient means of computed prices. The importance of the 3/2 model in the pricing of options on realized variance is well documented by Drimus ([Dri12]). In particular, the 3/2 model allows for upward-sloping implied volatility of variance smiles while Heston's model leads to downward-sloping volatility of variance smiles, in disagreement with observed skews in variance markets.

In log notation $(X, Y) := (\log S, \log Z)$ we have the following dynamics

$$dX_t = -\frac{1}{2}\, e^{Y_t} + e^{\frac{1}{2} Y_t}\, dW_t, \quad X_0 = x := \log s,$$

$$dY_t = \left(\kappa\big(\theta - e^{Y_t}\big) - \frac{1}{2}\,\delta^2\, e^{Y_t}\right) dt + \delta e^{\frac{1}{2} Y_t}\, dB_t, \quad Y_0 = y := \log z,$$

$$d\langle W, B\rangle_t = \rho\, dt.$$

The generator of (X, Y) is given by

$$\mathcal{A} = \frac{1}{2}\, e^y\big(\partial_x^2 - \partial_x\big) + \left(\kappa(\theta - e^y) - \frac{1}{2}\,\delta^2\, e^y\right)\partial_y + \frac{1}{2}\,\delta^2\, e^y\,\partial_y^2 + \rho\,\delta\, e^y\,\partial_{x,y}\,.$$

Thus we identify

$$a(x, y) = \frac{1}{2}\, e^y, \quad b(x, y) = \frac{1}{2}\,\delta^2\, e^y,$$

$$c(x, y) = \rho\,\delta\, e^y, \quad \alpha(x, y) = \kappa(\theta - e^y) - \frac{1}{2}\,\delta^2\, e^y.$$

Our expansion formulas goes till 6th order if we set $\bar{x} = x$ and $\bar{y} = y$, and only till 5th order with generic $\overline{x}$ and $\overline{y}$. We present here only the first formulas.

$$\sigma_0 = e^{\frac{\overline{y}}{2}}$$

$$\sigma_1 = -\frac{1}{8} e^{\frac{\overline{y}}{2}} \left(e^{\overline{y}}\, t \left(\delta^2 + 2\kappa - \delta\rho\right) + 2\,(-t\,\theta\,\kappa + r\,t\,\delta\,\rho + (-k + x)\,\delta\,\rho) - 4\,(y - \overline{y})\right)$$

$$\begin{aligned}
\sigma_2 = {} & \frac{1}{384} e^{\frac{\overline{y}}{2}} \\
& \Big(e^{2\overline{y}}\, t^2 \left(13\,\delta^4 + 52\,\kappa^2 - 26\,\delta^3\rho - 52\,\delta\,\kappa\,\rho + 4\,\delta^2 \left(-1 + 13\,\kappa + 4\,\rho^2\right)\right) - 4\,\Big(-5\,t^2\theta^2\kappa^2 + 6\,t\,(-k + x)\,\delta\,\theta\,\kappa\,\rho + {} \\
& \quad 2\,r^2 t^2 \delta^2 \left(-2 + \rho^2\right) + 2\,(-k + x)^2 \delta^2 \left(-2 + \rho^2\right) + 2\,r\,t\,\delta \left(3\,t\,\theta\,\kappa\,\rho + 2\,(-k + x)\,\delta \left(-2 + \rho^2\right)\right)\Big) + {} \\
& \quad 4\,e^{\overline{y}}\, t \,\Big(-18\,t\,\theta\,\kappa^2 + 7\,(-k + r\,t + x)\,\delta^3\rho + \delta\,(14\,r\,t + 14\,(-k + x) + 9\,t\,\theta)\,\kappa\,\rho - {} \\
& \qquad \delta^2 \left(-8 + 7\,(1 - k + x)\,\rho^2 + t \left(9\,\theta\,\kappa + 7\,r\,\rho^2\right)\right)\Big) - {} \\
& \quad 24 \left(3\,e^{\overline{y}}\, t \left(\delta^2 + 2\kappa - \delta\rho\right) + 2\,(-t\,\theta\,\kappa + r\,t\,\delta\,\rho + (-k + x)\,\delta\,\rho)\right)(y - \overline{y}) + 48\,(y - \overline{y})^2\Big)
\end{aligned}$$

$$\begin{aligned}
\sigma_3 = {} & \frac{1}{3072} e^{\frac{\overline{y}}{2}} \\
& \Big(-t\,\Big(e^{3\overline{y}}\, t^2 \Big(35\,\delta^6 + 280\,\kappa^3 - 105\,\delta^5\rho - 420\,\delta\,\kappa^2\rho - 2\,\delta^3\rho \left(-16 + 210\,\kappa + 29\,\rho^2\right) + 2\,\delta^4 \Big(-16 + 105\,\kappa + {} \\
& \qquad\qquad 64\,\rho^2\Big) + 4\,\delta^2\kappa \left(-16 + 105\,\kappa + 64\,\rho^2\right)\Big) + 8\,\theta\,\kappa\,\Big(-3\,t^2\theta^2\kappa^2 + 5\,t\,(-k + x)\,\delta\,\theta\,\kappa\,\rho + {} \\
& \qquad\quad 2\,r^2 t^2 \delta^2 \left(-2 + \rho^2\right) + 2\,(-k + x)^2 \delta^2 \left(-2 + \rho^2\right) + r\,t\,\delta \left(5\,t\,\theta\,\kappa\,\rho + 4\,(-k + x)\,\delta \left(-2 + \rho^2\right)\right)\Big) - {} \\
& \quad 4\,e^{\overline{y}} \Big(2\,r^2 t^2 \delta^2 \left(\delta^2 + 2\kappa - \delta\rho\right) \left(-8 + \rho^2\right) + 2\,(-k + x)^2 \delta^2 \left(\delta^2 + 2\kappa - \delta\rho\right) \left(-8 + \rho^2\right) + {} \\
& \qquad 6\,(-k + x)\,\delta\,\rho \left(14\,t\,\theta\,\kappa^2 - 7\,t\,\delta\,\theta\,\kappa\,\rho + \delta^2 \left(-4 + 7\,t\,\theta\,\kappa + 3\,\rho^2\right)\right) - {} \\
& \qquad 3\,t\,\theta\,\kappa \left(22\,t\,\theta\,\kappa^2 - 11\,t\,\delta\,\theta\,\kappa\,\rho + \delta^2 \left(-16 + 11\,t\,\theta\,\kappa + 14\,\rho^2\right)\right) + 2\,r\,t\,\delta\,\Big(2\,(-k + x)\,\delta \\
& \qquad\quad \left(\delta^2 + 2\kappa - \delta\rho\right) \left(-8 + \rho^2\right) + 3\,\rho \left(14\,t\,\theta\,\kappa^2 - 7\,t\,\delta\,\theta\,\kappa\,\rho + \delta^2 \left(-4 + 7\,t\,\theta\,\kappa + 3\,\rho^2\right)\right)\Big)\Big) + {} \\
& \quad 2\,e^{2\overline{y}}\, t\,\Big(-260\,t\,\theta\,\kappa^3 + 45\,(-k + r\,t + x)\,\delta^5\rho + 20\,\delta\,(9\,r\,t + 9\,(-k + x) + 13\,t\,\theta)\,\kappa^2\rho + {} \\
& \qquad 2\,\delta^3\rho \left(-60 + 65\,t\,\theta\,\kappa + 45\,\rho^2 + 2\,r\,t \left(-4 + 45\,\kappa + 14\,\rho^2\right) + (-k + x) \left(-8 + 90\,\kappa + 28\,\rho^2\right)\right) - {} \\
& \qquad 5\,\delta^4 \left(t \left(13\,\theta\,\kappa + 18\,r\,\rho^2\right) + 6 \left(-4 + 3\,(1 - k + x)\,\rho^2\right)\right) - {} \\
& \qquad 20\,\delta^2\kappa \left(-12 + 9\,(1 - k + x)\,\rho^2 + t \left(9\,r\,\rho^2 + \theta \left(-1 + 13\,\kappa + 4\,\rho^2\right)\right)\right)\Big)\Big) + {} \\
& 4\,\Big(5\,e^{2\overline{y}}\, t^2 \left(13\,\delta^4 + 52\,\kappa^2 - 26\,\delta^3\rho - 52\,\delta\,\kappa\,\rho + 4\,\delta^2 \left(-1 + 13\,\kappa + 4\,\rho^2\right)\right) - {} \\
& \quad 4\,\Big(-5\,t^2\theta^2\kappa^2 + 6\,t\,(-k + x)\,\delta\,\theta\,\kappa\,\rho + 2\,r^2 t^2 \delta^2 \left(-2 + \rho^2\right) + {} \\
& \qquad 2\,(-k + x)^2 \delta^2 \left(-2 + \rho^2\right) + 2\,r\,t\,\delta \left(3\,t\,\theta\,\kappa\,\rho + 2\,(-k + x)\,\delta \left(-2 + \rho^2\right)\right)\Big) + {} \\
& \quad 12\,e^{\overline{y}}\, t\,\Big(-18\,t\,\theta\,\kappa^2 + 7\,(-k + r\,t + x)\,\delta^3\rho + \delta\,(14\,r\,t + 14\,(-k + x) + 9\,t\,\theta)\,\kappa\,\rho - {} \\
& \qquad \delta^2 \left(-8 + 7\,(1 - k + x)\,\rho^2 + t \left(9\,\theta\,\kappa + 7\,r\,\rho^2\right)\right)\Big)\Big)\,(y - \overline{y}) - {} \\
& 48 \left(9\,e^{\overline{y}}\, t \left(\delta^2 + 2\kappa - \delta\rho\right) + 2\,(-t\,\theta\,\kappa + r\,t\,\delta\,\rho + (-k + x)\,\delta\,\rho)\right)(y - \overline{y})^2 + {} \\
& 64 \\
& \;(y - \overline{y})^3\Big)
\end{aligned}$$

To the best of our knowledge, the above formula is the first explicit implied volatility expansion for the 3/2 model. The characteristic function of X_t is given, for example, in Proposition 3.2 of Baldeaux and Badran's work ([BB12]). We have

$$\mathbb{E}_{x,y}\, e^{i\lambda X_t} = e^{i\lambda x}\, \frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} \left(\frac{2}{\delta^2 z}\right)^{\alpha} M\left(\alpha, \gamma, -\frac{2}{\delta^2 z}\right),$$

$$z = \frac{e^y}{\kappa\theta}\left(e^{\kappa\theta t} - 1\right), \quad \gamma = 2\left(\alpha + 1 - \frac{p}{\delta^2}\right),$$

$$\alpha = -\left(\frac{1}{2} - \frac{p}{\delta^2}\right) + \left(\left(\frac{1}{2} - \frac{p}{\delta^2}\right)^2 + 2\,\frac{q}{\delta^2}\right)^{\frac{1}{2}},$$

$$p = -\kappa + i\,\delta\,\rho\,\lambda, \quad q = \frac{1}{2}\left(i\,\lambda + \lambda^2\right),$$

where Γ is a Gamma function and M is a confluent hypergeometric function. Thus, the price of a European call option can be computed using standard Fourier methods

$$u(t, x, y) = \frac{1}{2\pi}\int_{\mathbb{R}} d\lambda_r\, \hat{h}(\lambda)\, \mathbb{E}_{x,y}\, e^{i\lambda X_t}, \quad \lambda = \lambda_r + i\,\lambda_i, \quad \lambda_i < -1, \tag{8.2}$$

where $\hat{h}(\lambda)$ for an European call option is, as defined before

$$\hat{h}(\lambda) = \frac{-e^{k - i\,k\,\lambda}}{i\,\lambda + \lambda^2}.$$

Using $u(t, x, y)$, the exact implied volatility σ can be computed numerically.

For our numerical test we consider an initial log-spot $x = 0$, and a vector of times to maturity

≈1 M	≈2 M	3 M	6 M	9 M	1. Y	1.5 Y	2 Y
0.08	0.17	0.25	0.5	0.75	1	1.5	2

8.4.1 First Set by Baldeaux and Badran modified #1

In the first case we use parameters values as proposed by Baldeaux and Badran in ([BB12]), so we have ($\kappa = 30.84$, $\theta = 0.48^2$, $\rho = -0.55$, $y = \mathrm{Log}\left(0.19^2\right)$), but we set $\delta = 10$ instead of 70.56.

First of all we create strikes matrix expressed in bp

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.08	96	97	97	98	99	99	100	100	101	102	102	102	103
0.17	91	92	94	96	97	98	100	101	103	104	105	107	108
0.25	87	89	91	94	96	98	100	102	104	106	107	109	111
0.5	77	81	85	88	92	96	100	104	107	111	115	118	122
0.75	70	75	80	85	90	95	100	105	111	116	121	127	132
1	65	71	77	82	88	94	100	107	114	121	128	135	142
1.5	57	64	71	78	86	93	100	110	120	130	141	151	161
2	52	60	68	76	84	92	100	114	127	140	154	168	181

then we compute corresponding call and put prices

	1	3	5	7	9	11	13
0.08	0.04938	0.04201	0.02903	0.02351	0.01868	0.01454	0.01108
0.17	0.09981	0.07581	0.05468	0.03710	0.02346	0.01656	0.009179
0.25	0.1402	0.1070	0.07077	0.04724	0.02923	0.01932	0.01033
0.5	0.2403	0.1722	0.1202	0.07253	0.04242	0.02049	0.009850
0.75	0.3102	0.2243	0.1503	0.09214	0.04798	0.02387	0.009972
1	0.3603	0.2576	0.1772	0.1085	0.05463	0.02443	0.009857
1.5	0.4399	0.3199	0.2123	0.1355	0.06416	0.02601	0.01009
2	0.4904	0.3550	0.2431	0.1579	0.06874	0.02703	0.009929

	1	3	5	7	9	11	13
0.08	0.009379	0.01201	0.01903	0.02351	0.02868	0.03454	0.04108
0.17	0.009815	0.01581	0.02468	0.03710	0.05346	0.06656	0.08918
0.25	0.01017	0.01699	0.03077	0.04724	0.06923	0.08932	0.1203
0.5	0.01032	0.02221	0.04023	0.07253	0.1124	0.1705	0.2298
0.75	0.01020	0.02430	0.05034	0.09214	0.1580	0.2339	0.3300
1	0.01033	0.02763	0.05720	0.1085	0.1946	0.3044	0.4299
1.5	0.009908	0.02989	0.07231	0.1355	0.2642	0.4360	0.6201
2	0.01040	0.03499	0.08311	0.1579	0.3387	0.5670	0.8199

and the exact implied volatility by numerical integration

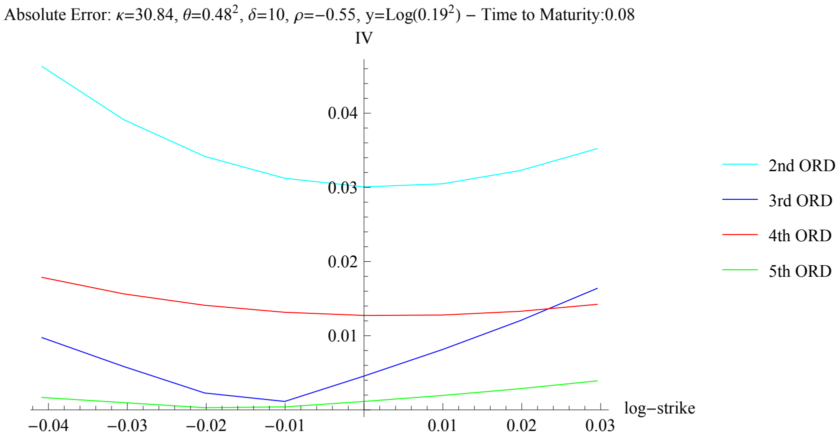
	1	3	5	7	9	11	13
0.08	22.0039	21.6903	21.1088	20.8403	20.5862	20.3464	20.1206
0.17	25.2075	24.2234	23.3414	22.5611	21.8812	21.4828	20.9660
0.25	27.2799	26.0280	24.6489	23.6973	22.8813	22.3576	21.7751
0.5	30.4996	28.6050	27.1587	25.7483	24.7255	23.7922	23.1711
0.75	31.5292	29.6597	28.0672	26.7300	25.5337	24.6767	23.9623
1	31.8317	29.9839	28.5685	27.2827	26.0781	25.1487	24.4530
1.5	31.9912	30.3392	28.9322	27.8720	26.6811	25.7462	25.0853
2	31.8333	30.2994	29.1185	28.1791	26.9741	26.1019	25.4604

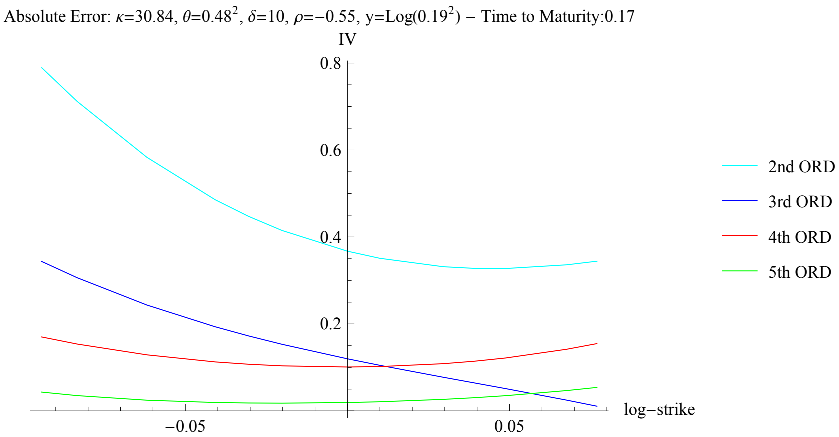
Now we compute our approximated implied volatilities using the initial point method ($\bar{x} = x$, $\bar{y} = y$), which means that we are expanding around initial values of X_t and Y_t; in the following table we show absolute errors.

	1	3	5	7	9	11	13
	3.	2.69	2.11	1.84	1.59	1.35	1.12
	3.84×10^{-1}	3.41×10^{-1}	2.93×10^{-1}	2.87×10^{-1}	2.93×10^{-1}	3.1×10^{-1}	3.39×10^{-1}
	4.63×10^{-2}	3.91×10^{-2}	3.12×10^{-2}	3.01×10^{-2}	3.05×10^{-2}	3.23×10^{-2}	3.52×10^{-2}
0.08	9.74×10^{-3}	5.85×10^{-3}	1.14×10^{-3}	4.56×10^{-3}	8.14×10^{-3}	1.2×10^{-2}	1.64×10^{-2}
	1.79×10^{-2}	1.56×10^{-2}	1.32×10^{-2}	1.27×10^{-2}	1.28×10^{-2}	1.33×10^{-2}	1.42×10^{-2}
	1.67×10^{-3}	9.8×10^{-4}	4.01×10^{-4}	1.14×10^{-3}	1.94×10^{-3}	2.86×10^{-3}	3.91×10^{-3}
	5.19×10^{-3}	4.74×10^{-3}	4.35×10^{-3}	4.4×10^{-3}	4.59×10^{-3}	4.94×10^{-3}	5.46×10^{-3}

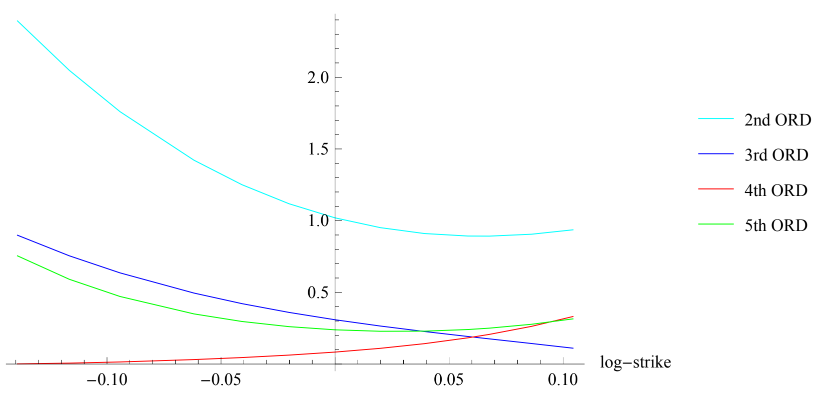
0.17	6.21	5.22	4.34	3.56	2.88	2.48	1.97
	4.43×10^{-1}	3.06×10^{-1}	2.45×10^{-1}	2.6×10^{-1}	3.52×10^{-1}	4.56×10^{-1}	6.76×10^{-1}
	7.89×10^{-1}	5.83×10^{-1}	4.47×10^{-1}	3.67×10^{-1}	3.32×10^{-1}	3.27×10^{-1}	3.44×10^{-1}
	3.44×10^{-1}	2.43×10^{-1}	1.72×10^{-1}	1.2×10^{-1}	7.73×10^{-2}	5.12×10^{-2}	1.07×10^{-2}
	1.7×10^{-1}	1.29×10^{-1}	1.07×10^{-1}	1.01×10^{-1}	1.08×10^{-1}	1.22×10^{-1}	1.55×10^{-1}
	4.31×10^{-2}	2.45×10^{-2}	1.81×10^{-2}	1.91×10^{-2}	2.64×10^{-2}	3.49×10^{-2}	5.39×10^{-2}
	1.09×10^{-2}	2.18×10^{-3}	1.35×10^{-2}	2.57×10^{-2}	4.18×10^{-2}	5.66×10^{-2}	8.8×10^{-2}
0.25	8.28	7.03	5.65	4.7	3.88	3.36	2.78
	2.13×10^{-1}	2.9×10^{-1}	2.72×10^{-1}	1.57×10^{-1}	5.15×10^{-2}	2.71×10^{-1}	6.47×10^{-1}
	2.39	1.76	1.25	1.02	9.1×10^{-1}	8.92×10^{-1}	9.36×10^{-1}
	8.98×10^{-1}	6.36×10^{-1}	4.21×10^{-1}	3.09×10^{-1}	2.27×10^{-1}	1.75×10^{-1}	1.1×10^{-1}
	9.91×10^{-4}	1.44×10^{-2}	4.51×10^{-2}	8.33×10^{-2}	1.42×10^{-1}	2.07×10^{-1}	3.31×10^{-1}
	7.54×10^{-1}	4.71×10^{-1}	2.97×10^{-1}	2.39×10^{-1}	2.29×10^{-1}	2.5×10^{-1}	3.15×10^{-1}
	4.38×10^{-1}	2.75×10^{-1}	1.4×10^{-1}	5.89×10^{-2}	2.09×10^{-2}	9.37×10^{-2}	2.27×10^{-1}
0.5	1.15×10^{1}	9.6	8.16	6.75	5.73	4.79	4.17
	5.04	4.35	3.73	2.96	2.22	1.27	3.43×10^{-1}
	1.1×10^{1}	7.51	5.56	4.31	3.86	3.9	4.31
	8.32×10^{-1}	8.86×10^{-1}	8.32×10^{-1}	7.26×10^{-1}	6.39×10^{-1}	5.84×10^{-1}	5.89×10^{-1}
	1.28×10^{1}	8.17	5.67	3.9	2.86	1.83	7.92×10^{-1}
	1.62×10^{1}	8.5	5.4	3.87	3.59	4.23	5.65
	5.45	2.86	1.95	1.52	1.56	2.25	3.67
0.75	1.25×10^{1}	1.07×10^{1}	9.07	7.73	6.53	5.68	4.96
	1.14×10^{1}	9.73	8.25	6.83	5.3	3.91	2.35
	2.15×10^{1}	1.46×10^{1}	1.05×10^{1}	8.29	7.46	7.74	8.9
	1.59×10^{1}	1.24×10^{1}	9.82	7.97	6.6	5.86	5.5
	6.4×10^{1}	4.09×10^{1}	2.76×10^{1}	2.01×10^{1}	1.55×10^{1}	1.27×10^{1}	9.68
	4.69×10^{1}	1.87×10^{1}	8.49	5.5	6.56	1.09×10^{1}	1.97×10^{1}
	1.63×10^{2}	9.29×10^{1}	5.91×10^{1}	4.22×10^{1}	3.38×10^{1}	3.28×10^{1}	3.84×10^{1}
1	1.28×10^{1}	1.1×10^{1}	9.57	8.28	7.08	6.15	5.45
	1.78×10^{1}	1.53×10^{1}	1.32×10^{1}	1.11×10^{1}	8.92	6.82	4.8
	3.17×10^{1}	2.12×10^{1}	1.55×10^{1}	1.23×10^{1}	1.12×10^{1}	1.2×10^{1}	1.41×10^{1}
	5.06×10^{1}	3.87×10^{1}	3.11×10^{1}	2.53×10^{1}	2.1×10^{1}	1.86×10^{1}	1.76×10^{1}
	1.73×10^{2}	1.08×10^{2}	7.53×10^{1}	5.55×10^{1}	4.4×10^{1}	3.78×10^{1}	3.27×10^{1}
	2.95×10^{1}	2.11×10^{1}	2.99×10^{1}	2.69×10^{1}	1.6×10^{1}	2.8	3.21×10^{1}
	9.41×10^{2}	5.31×10^{2}	3.51×10^{2}	2.5×10^{2}	1.97×10^{2}	1.84×10^{2}	2.01×10^{2}
1.5	1.3×10^{1}	1.13×10^{1}	9.93	8.87	7.68	6.75	6.09
	3.08×10^{1}	2.67×10^{1}	2.31×10^{1}	2.03×10^{1}	1.67×10^{1}	1.34×10^{1}	1.06×10^{1}
	5.19×10^{1}	3.44×10^{1}	2.43×10^{1}	1.98×10^{1}	1.83×10^{1}	2.06×10^{1}	2.51×10^{1}
	1.99×10^{2}	1.55×10^{2}	1.24×10^{2}	1.04×10^{2}	8.72×10^{1}	7.76×10^{1}	7.36×10^{1}
	6.53×10^{2}	4.08×10^{2}	2.74×10^{2}	2.1×10^{2}	1.69×10^{2}	1.55×10^{2}	1.48×10^{2}
	7.85×10^{2}	7.5×10^{2}	6.3×10^{2}	5.28×10^{2}	4.04×10^{2}	2.75×10^{2}	1.25×10^{2}
	9.02×10^{3}	5.19×10^{3}	3.31×10^{3}	2.46×10^{3}	1.93×10^{3}	1.81×10^{3}	1.91×10^{3}
2	1.28×10^{1}	1.13×10^{1}	1.01×10^{1}	9.18	7.97	7.1	6.46
	4.31×10^{1}	3.76×10^{1}	3.33×10^{1}	2.97×10^{1}	2.46×10^{1}	2.05×10^{1}	1.69×10^{1}
	6.87×10^{1}	4.43×10^{1}	3.18×10^{1}	2.6×10^{1}	2.46×10^{1}	2.9×10^{1}	3.64×10^{1}
	4.75×10^{2}	3.73×10^{2}	3.08×10^{2}	2.65×10^{2}	2.21×10^{2}	1.99×10^{2}	1.89×10^{2}
	1.56×10^{3}	9.63×10^{2}	6.65×10^{2}	5.15×10^{2}	4.16×10^{2}	3.96×10^{2}	3.97×10^{2}
	4.74×10^{3}	3.95×10^{3}	3.26×10^{3}	2.75×10^{3}	2.15×10^{3}	1.67×10^{3}	1.18×10^{3}
	$4.\times10^{4}$	2.3×10^{4}	1.54×10^{4}	1.16×10^{4}	9.09×10^{3}	8.64×10^{3}	9.18×10^{3}

As might be expected, the 3/2 model is difficult to be approximated with explicit formulas, in fact we see that even for maturities of less than one years, we have problems, especially ITM. In particular, we see that up to 3 months we get absolute errors of less than one percentage point, but there was an explosion of the approximation from 9 months, which is not a long time.

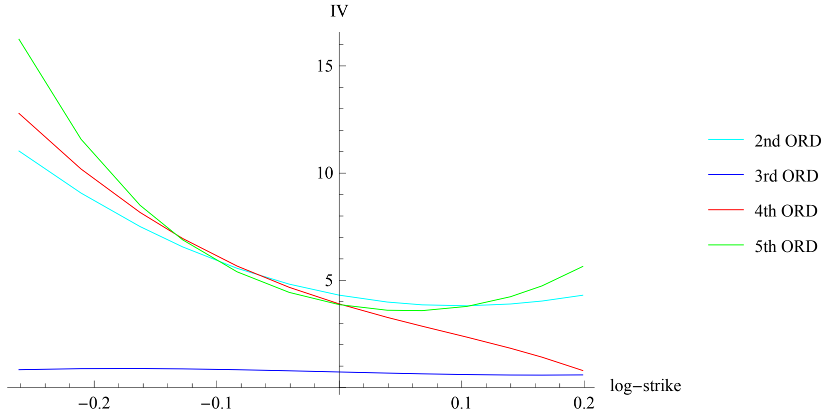




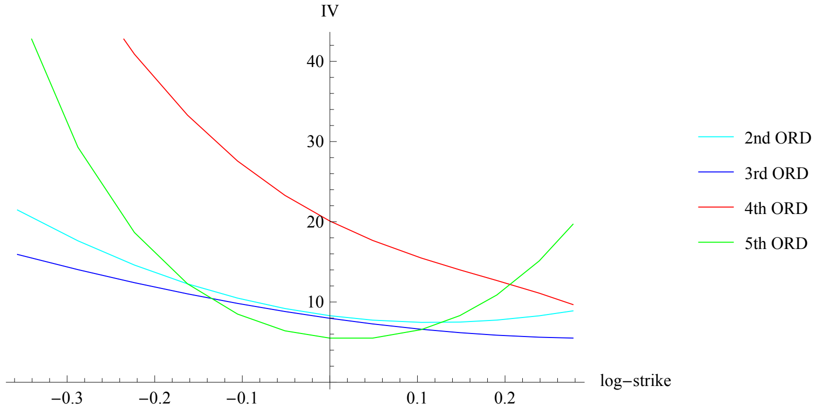
Absolute Error: κ=30.84, θ=0.48^2, δ=10, ρ=−0.55, y=Log(0.19^2) − Time to Maturity:0.25

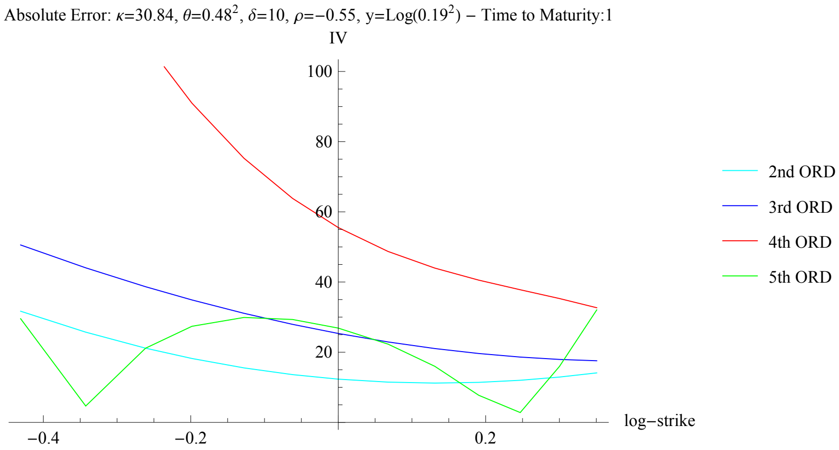


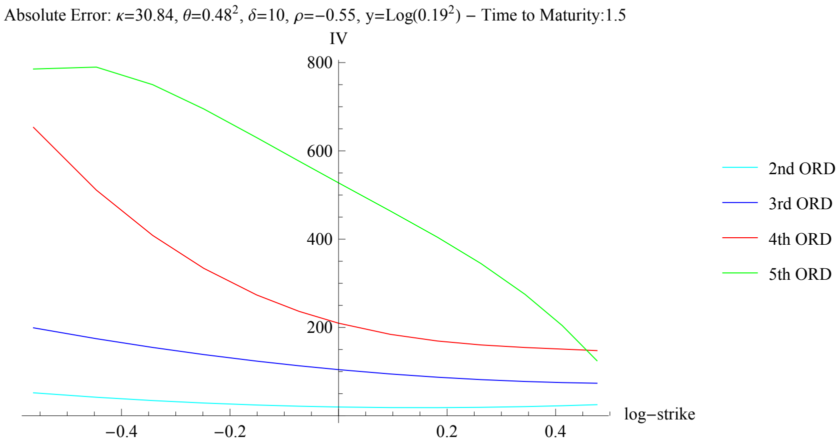
Absolute Error: κ=30.84, θ=0.48^2, δ=10, ρ=−0.55, y=Log(0.19^2) − Time to Maturity:0.5

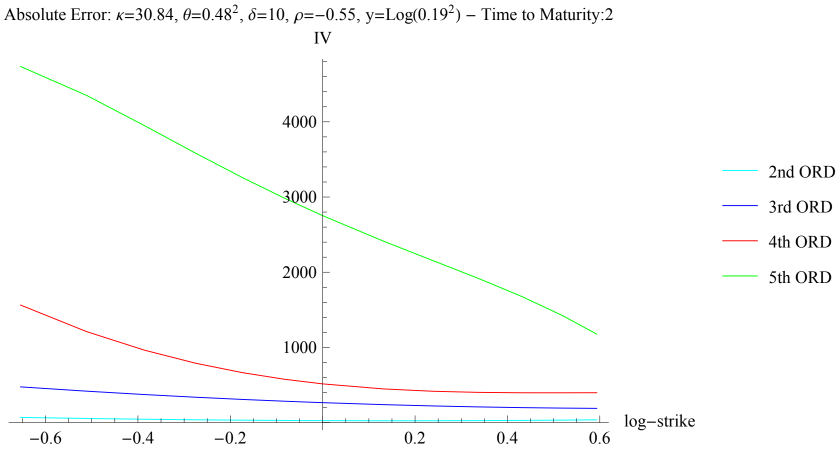


Absolute Error: κ=30.84, θ=0.48^2, δ=10, ρ=−0.55, y=Log(0.19^2) − Time to Maturity:0.75









As in Heston model, we propose two other choices of $\overline{y}$:

1) midpoint method: $\overline{y} = \mathrm{Log}\left(\frac{e^{y}+\theta}{2}\right)$

2) integral mean method: $\overline{y} = \mathrm{Log}\left(\frac{-\mathrm{Log}(\theta)+\mathrm{Log}\left(-e^{y}+e^{y+t\,\theta\,\kappa}+\theta\right)}{t\,\kappa}\right)$

In the following table we present a comparison between these three methods:

	1			7			13		
	3.	1.45×10^{1}	5.76×10^{-1}	1.84	1.57×10^{1}	5.87×10^{-1}	1.12	1.64×10^{1}	1.31
	3.84×10^{-1}	1.02×10^{1}	5.5×10^{-1}	2.87×10^{-1}	1.11×10^{1}	5.89×10^{-1}	3.39×10^{-1}	1.19×10^{1}	7.4×10^{-1}
	4.63×10^{-2}	1.16×10^{1}	1.85×10^{-1}	3.01×10^{-2}	1.27×10^{1}	2.08×10^{-1}	3.52×10^{-2}	1.35×10^{1}	2.6×10^{-1}
0.08	9.74×10^{-3}	1.66×10^{1}	8.7×10^{-2}	4.56×10^{-3}	1.82×10^{1}	1.12×10^{-1}	1.64×10^{-2}	1.96×10^{1}	1.47×10^{-1}
	1.79×10^{-2}	2.73×10^{1}	7.03×10^{-2}	1.27×10^{-2}	$3.\times10^{1}$	7.87×10^{-2}	1.42×10^{-2}	3.24×10^{1}	9.91×10^{-2}
	1.67×10^{-3}	4.94×10^{1}	4.51×10^{-2}	1.14×10^{-3}	5.45×10^{1}	5.47×10^{-2}	3.91×10^{-3}	5.94×10^{1}	7.14×10^{-2}
	5.19×10^{-3}	0	0	4.4×10^{-3}	0	0	5.46×10^{-3}	0	0
	6.21	1.13×10^{1}	7.69×10^{-1}	3.56	1.39×10^{1}	1.88	1.97	1.55×10^{1}	3.47
	4.43×10^{-1}	1.41×10^{1}	1.56	2.6×10^{-1}	1.61×10^{1}	2.08	6.76×10^{-1}	1.84×10^{1}	3.07
	7.89×10^{-1}	2.45×10^{1}	1.7	3.67×10^{-1}	2.82×10^{1}	1.81	3.44×10^{-1}	3.23×10^{1}	2.49
0.17	3.44×10^{-1}	5.31×10^{1}	1.33	1.2×10^{-1}	6.15×10^{1}	1.75	1.07×10^{-2}	7.15×10^{1}	2.56
	1.7×10^{-1}	1.32×10^{2}	1.87	1.01×10^{-1}	1.54×10^{2}	2.28	1.55×10^{-1}	1.82×10^{2}	3.34
	4.31×10^{-2}	3.66×10^{2}	2.65	1.91×10^{-2}	4.29×10^{2}	3.19	5.39×10^{-2}	5.11×10^{2}	4.75
	1.09×10^{-2}	0	0	2.57×10^{-2}	0	0	8.8×10^{-2}	0	0
	8.28	9.22	7.89×10^{-2}	4.7	1.28×10^{1}	3.5	2.78	1.47×10^{1}	5.43
	2.13×10^{-1}	1.68×10^{1}	3.07	1.57×10^{-1}	2.02×10^{1}	4.69	6.47×10^{-1}	2.35×10^{1}	6.67
	2.39	3.77×10^{1}	5.63	1.02	4.46×10^{1}	6.39	9.36×10^{-1}	5.23×10^{1}	8.64
0.25	8.98×10^{-1}	1.03×10^{2}	7.82	3.09×10^{-1}	1.23×10^{2}	9.87	1.1×10^{-1}	1.47×10^{2}	1.38×10^{1}
	9.91×10^{-4}	3.25×10^{2}	1.47×10^{1}	8.33×10^{-2}	3.91×10^{2}	1.86×10^{1}	3.31×10^{-1}	4.73×10^{2}	2.67×10^{1}
	7.54×10^{-1}	1.14×10^{3}	3.22×10^{1}	2.39×10^{-1}	1.37×10^{3}	3.96×10^{1}	3.15×10^{-1}	1.69×10^{3}	5.77×10^{1}
	4.38×10^{-1}	0	0	5.89×10^{-2}	0	0	2.27×10^{-1}	0	0
	1.15×10^{1}	6.	4.07	6.75	1.08×10^{1}	8.82	4.17	1.33×10^{1}	1.14×10^{1}
	5.04	2.31×10^{1}	1.67×10^{1}	2.96	3.15×10^{1}	2.43×10^{1}	3.43×10^{-1}	3.89×10^{1}	3.12×10^{1}
	1.1×10^{1}	8.9×10^{1}	6.15×10^{1}	4.31	1.11×10^{2}	7.8×10^{1}	4.31	1.39×10^{2}	1.01×10^{2}
0.5	8.32×10^{-1}	3.88×10^{2}	2.43×10^{2}	7.26×10^{-1}	4.85×10^{2}	3.06×10^{2}	5.89×10^{-1}	6.2×10^{2}	4.04×10^{2}
	1.28×10^{1}	1.93×10^{3}	1.08×10^{3}	3.9	2.43×10^{3}	1.38×10^{3}	7.92×10^{-1}	3.18×10^{3}	1.87×10^{3}
	1.62×10^{1}	1.06×10^{4}	5.4×10^{3}	3.87	1.34×10^{4}	6.9×10^{3}	5.65	1.8×10^{4}	9.59×10^{3}
	5.45	0	0	1.52	0	0	3.67	0	0
	1.25×10^{1}	4.97	7.38	7.73	9.77	1.22×10^{1}	4.96	1.25×10^{1}	1.49×10^{1}
	1.14×10^{1}	2.86×10^{1}	4.19×10^{1}	6.83	4.17×10^{1}	5.62×10^{1}	2.35	5.28×10^{1}	6.83×10^{1}
	2.15×10^{1}	1.54×10^{2}	2.43×10^{2}	8.29	$2.\times10^{2}$	3.05×10^{2}	8.9	2.57×10^{2}	3.77×10^{2}
0.75	1.59×10^{1}	9.14×10^{2}	1.62×10^{3}	7.97	1.16×10^{3}	2.02×10^{3}	5.5	1.54×10^{3}	2.56×10^{3}
	6.4×10^{1}	6.05×10^{3}	1.21×10^{4}	2.01×10^{1}	7.76×10^{3}	1.53×10^{4}	9.68	1.05×10^{4}	1.99×10^{4}
	4.69×10^{1}	4.42×10^{4}	$1.\times10^{5}$	5.5	5.7×10^{4}	1.27×10^{5}	1.97×10^{1}	7.96×10^{4}	1.7×10^{5}
	1.63×10^{2}	0	0	4.22×10^{1}	0	0	3.84×10^{1}	0	0
	1.28×10^{1}	4.67	9.45	8.28	9.22	1.4×10^{1}	5.45	1.21×10^{1}	1.68×10^{1}
	1.78×10^{1}	3.43×10^{1}	7.18×10^{1}	1.11×10^{1}	5.14×10^{1}	9.17×10^{1}	4.8	6.62×10^{1}	1.09×10^{2}
	3.17×10^{1}	2.35×10^{2}	5.76×10^{2}	1.23×10^{1}	3.1×10^{2}	7.08×10^{2}	1.41×10^{1}	4.06×10^{2}	8.54×10^{2}
1	5.06×10^{1}	1.74×10^{3}	5.38×10^{3}	2.53×10^{1}	2.23×10^{3}	6.63×10^{3}	1.76×10^{1}	3.01×10^{3}	8.22×10^{3}
	1.73×10^{2}	1.43×10^{4}	5.68×10^{4}	5.55×10^{1}	1.84×10^{4}	7.07×10^{4}	3.27×10^{1}	2.56×10^{4}	8.99×10^{4}
	2.95×10^{1}	1.28×10^{5}	6.54×10^{5}	2.69×10^{1}	1.67×10^{5}	8.21×10^{5}	3.21×10^{1}	2.4×10^{5}	1.07×10^{6}
	9.41×10^{2}	0	0	2.5×10^{2}	0	0	2.01×10^{2}	0	0
	1.3×10^{1}	4.51	1.16×10^{1}	8.87	8.63	1.58×10^{1}	6.09	1.14×10^{1}	1.85×10^{1}
	3.08×10^{1}	4.63×10^{1}	1.35×10^{2}	2.03×10^{1}	7.04×10^{1}	1.65×10^{2}	1.06×10^{1}	9.15×10^{1}	1.9×10^{2}
	5.19×10^{1}	4.4×10^{2}	1.69×10^{3}	1.98×10^{1}	5.93×10^{2}	2.02×10^{3}	2.51×10^{1}	7.86×10^{2}	2.37×10^{3}
1.5	1.99×10^{2}	4.49×10^{3}	2.47×10^{4}	1.04×10^{2}	5.86×10^{3}	2.98×10^{4}	7.36×10^{1}	8.03×10^{3}	3.6×10^{4}
	6.53×10^{2}	5.05×10^{4}	4.06×10^{5}	2.1×10^{2}	6.59×10^{4}	4.96×10^{5}	1.48×10^{2}	9.37×10^{4}	6.15×10^{5}
	7.85×10^{2}	6.18×10^{5}	7.25×10^{6}	5.28×10^{2}	8.12×10^{5}	8.96×10^{6}	1.25×10^{2}	1.2×10^{6}	1.14×10^{7}
	9.02×10^{3}	0	0	2.46×10^{3}	0	0	1.91×10^{3}	0	0
	1.28×10^{1}	4.67	1.29×10^{1}	9.18	8.32	1.66×10^{1}	6.46	1.1×10^{1}	1.93×10^{1}
	4.31×10^{1}	5.99×10^{1}	2.01×10^{2}	2.97×10^{1}	8.91×10^{1}	2.38×10^{2}	1.69×10^{1}	1.16×10^{2}	2.71×10^{2}
	6.87×10^{1}	7.13×10^{2}	3.41×10^{3}	2.6×10^{1}	9.6×10^{2}	3.99×10^{3}	3.64×10^{1}	1.28×10^{3}	4.62×10^{3}
2	4.75×10^{2}	9.16×10^{3}	6.78×10^{4}	2.65×10^{2}	1.2×10^{4}	$8.\times10^{4}$	1.89×10^{2}	1.65×10^{4}	9.52×10^{4}
	1.56×10^{3}	1.29×10^{5}	1.51×10^{6}	5.15×10^{2}	1.69×10^{5}	1.81×10^{6}	3.97×10^{2}	2.43×10^{5}	2.2×10^{6}
	4.74×10^{3}	1.98×10^{6}	3.62×10^{7}	2.75×10^{3}	2.61×10^{6}	4.4×10^{7}	1.18×10^{3}	3.9×10^{6}	5.51×10^{7}
	$4.\times10^{4}$	0	0	1.16×10^{4}	0	0	9.18×10^{3}	0	0

We see that the best choice still remains the initial point method, due to its minor computation cost (as a matter of fact we have been able to compute the 6th order of the approximation, while for generic expansion points we had to stop at 5th order) but also to the fact that other two methods produce greater or at least equal errors.

8.4.2 Second Set by Baldeaux and Badran modified #2

In this second case we use parameters values as proposed by Baldeaux and Badran in ([BB12]), so we have ($\kappa = 30.84$, $\theta = 0.48^2$, $\rho = -0.55$, $y = \mathrm{Log}(0.19^2)$), but this time we set $\delta = 30$ instead of 70.56.

First of all we create strikes matrix expressed in bp

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.08	97	98	98	98	99	100	100	100	101	101	101	102	102
0.17	95	96	97	98	98	99	100	101	101	102	103	103	104
0.25	94	95	96	97	98	99	100	101	102	102	103	104	105
0.5	90	92	93	95	97	98	100	101	103	104	105	107	108
0.75	88	90	92	94	96	98	100	102	103	105	107	108	110
1	86	88	91	93	95	98	100	102	104	106	108	110	112
1.5	82	85	88	91	94	97	100	103	106	108	111	114	117
2	79	82	86	90	93	96	100	104	107	110	114	118	121

then we compute corresponding call and put prices

	1	3	5	7	9	11	13
0.08	0.03906	0.03172	0.02506	0.01920	0.01421	0.01421	0.01013
0.17	0.05975	0.04448	0.03756	0.02545	0.02036	0.01222	0.009152
0.25	0.07058	0.05506	0.04117	0.02927	0.01963	0.01569	0.009536
0.5	0.1094	0.08476	0.05584	0.03800	0.02403	0.01692	0.009301
0.75	0.1302	0.09746	0.06864	0.04486	0.03078	0.01709	0.01027
1	0.1504	0.1094	0.08056	0.05079	0.03264	0.01951	0.01080
1.5	0.1898	0.1397	0.09611	0.06100	0.03526	0.02065	0.009865
2	0.2198	0.1610	0.1103	0.06975	0.04033	0.02118	0.01008

	1	3	5	7	9	11	13
0.08	0.009063	0.01172	0.01506	0.01920	0.02421	0.02421	0.03013
0.17	0.009749	0.01448	0.01756	0.02545	0.03036	0.04222	0.04915
0.25	0.01058	0.01506	0.02117	0.02927	0.03963	0.04569	0.05954
0.5	0.009426	0.01476	0.02584	0.03800	0.05403	0.06692	0.08930
0.75	0.01020	0.01746	0.02864	0.04486	0.06078	0.08709	0.1103
1	0.01039	0.01941	0.03056	0.05079	0.07264	0.09951	0.1308
1.5	0.009841	0.01968	0.03611	0.06100	0.09526	0.1307	0.1799
2	0.009817	0.02102	0.04027	0.06975	0.1103	0.1612	0.2201

and the exact implied volatility by numerical integration

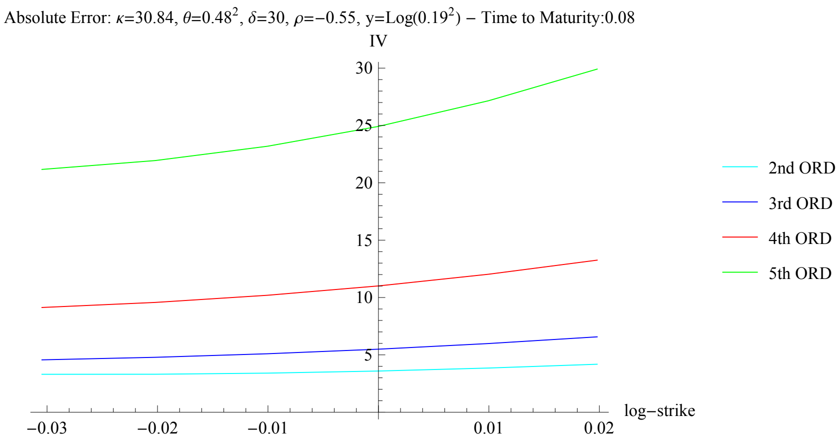
	1	3	5	7	9	11	13
0.08	18.6240	18.0494	17.5131	17.0188	16.5698	16.5698	16.1694
0.17	17.4098	16.5817	16.1932	15.4764	15.1518	14.5789	14.3335
0.25	16.5623	15.8865	15.2548	14.6763	14.1600	13.9280	13.5207
0.5	15.5677	14.8887	14.0475	13.4768	12.9689	12.6700	12.2868
0.75	14.8754	14.1968	13.5652	12.9899	12.6014	12.1481	11.8609
1	14.5150	13.8301	13.3212	12.7407	12.3265	11.9617	11.6494
1.5	14.1603	13.5527	12.9971	12.4965	12.0542	11.7324	11.4037
2	13.9120	13.3466	12.8356	12.3785	11.9756	11.6266	11.3308

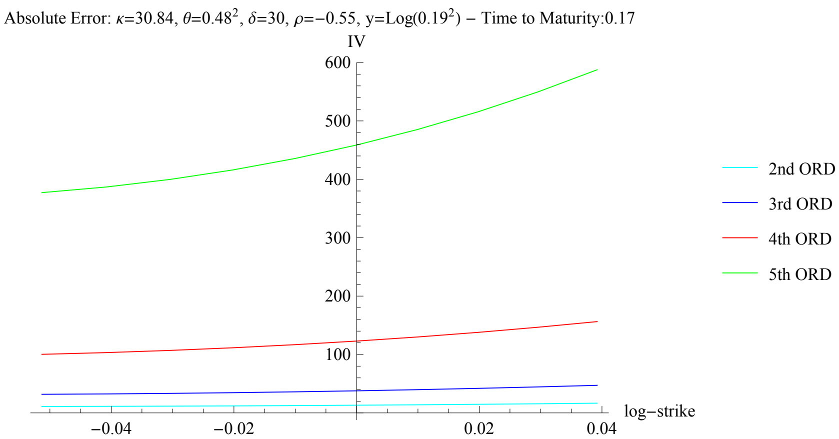
Now we compute our approximated implied volatilities with the initial point method and in the following table we show absolute errors.

	1	3	5	7	9	11	13
	3.76×10^{-1}	9.51×10^{-1}	1.49	1.98	2.43	2.43	2.83
	1.25	1.48	1.73	2.03	2.36	2.36	2.73
	3.3	3.31	3.41	3.59	3.85	3.85	4.18
0.08	4.57	4.79	5.1	5.5	5.99	5.99	6.57
	9.13	9.57	1.02×10^{1}	1.1×10^{1}	1.2×10^{1}	1.2×10^{1}	1.33×10^{1}
	2.12×10^{1}	2.19×10^{1}	2.32×10^{1}	2.49×10^{1}	2.72×10^{1}	2.72×10^{1}	2.99×10^{1}
	5.07×10^{1}	5.28×10^{1}	5.6×10^{1}	6.05×10^{1}	6.62×10^{1}	6.62×10^{1}	7.33×10^{1}
	1.59	2.42	2.81	3.52	3.85	4.42	4.67
	2.91	3.71	4.13	5.	5.45	6.42	6.93
	1.1×10^{1}	1.15×10^{1}	1.2×10^{1}	1.31×10^{1}	1.39×10^{1}	1.56×10^{1}	1.66×10^{1}
0.17	3.18×10^{1}	3.34×10^{1}	3.46×10^{1}	3.78×10^{1}	3.98×10^{1}	4.45×10^{1}	4.72×10^{1}
	$1.\times10^{2}$	1.07×10^{2}	1.11×10^{2}	1.23×10^{2}	1.3×10^{2}	1.47×10^{2}	1.56×10^{2}
	3.77×10^{2}	$4.\times10^{2}$	4.16×10^{2}	4.59×10^{2}	4.85×10^{2}	5.5×10^{2}	5.88×10^{2}
	1.53×10^{3}	1.62×10^{3}	1.69×10^{3}	1.86×10^{3}	1.97×10^{3}	2.24×10^{3}	2.41×10^{3}
	2.44	3.11	3.75	4.32	4.84	5.07	5.48
	5.24	6.22	7.2	8.21	9.24	9.77	1.09×10^{1}
	2.15×10^{1}	2.27×10^{1}	2.44×10^{1}	2.66×10^{1}	2.91×10^{1}	3.05×10^{1}	3.35×10^{1}
0.25	8.6×10^{1}	8.97×10^{1}	9.53×10^{1}	1.03×10^{2}	1.11×10^{2}	1.17×10^{2}	1.28×10^{2}
	3.65×10^{2}	3.84×10^{2}	4.11×10^{2}	4.44×10^{2}	4.85×10^{2}	5.09×10^{2}	5.61×10^{2}
	1.78×10^{3}	1.87×10^{3}	$2.\times10^{3}$	2.17×10^{3}	2.38×10^{3}	2.49×10^{3}	2.77×10^{3}
	9.46×10^{3}	9.93×10^{3}	1.06×10^{4}	1.15×10^{4}	1.27×10^{4}	1.34×10^{4}	1.49×10^{4}
	3.43	4.11	4.95	5.52	6.03	6.33	6.71
	1.34×10^{1}	1.53×10^{1}	1.77×10^{1}	1.95×10^{1}	2.13×10^{1}	2.26×10^{1}	2.44×10^{1}
	7.83×10^{1}	8.38×10^{1}	9.28×10^{1}	1.01×10^{2}	1.09×10^{2}	1.15×10^{2}	1.25×10^{2}
0.5	5.31×10^{2}	5.57×10^{2}	6.07×10^{2}	6.54×10^{2}	7.08×10^{2}	7.48×10^{2}	8.13×10^{2}
	3.91×10^{3}	4.11×10^{3}	4.48×10^{3}	4.84×10^{3}	5.26×10^{3}	5.57×10^{3}	6.09×10^{3}
	3.17×10^{4}	3.34×10^{4}	3.65×10^{4}	3.96×10^{4}	4.32×10^{4}	4.59×10^{4}	5.05×10^{4}
	2.81×10^{5}	2.95×10^{5}	3.23×10^{5}	3.51×10^{5}	3.84×10^{5}	4.1×10^{5}	4.54×10^{5}
	4.12	4.8	5.43	6.01	6.4	6.85	7.14
	2.34×10^{1}	2.62×10^{1}	2.9×10^{1}	3.16×10^{1}	3.35×10^{1}	3.6×10^{1}	3.79×10^{1}
	1.79×10^{2}	1.92×10^{2}	2.08×10^{2}	2.25×10^{2}	2.38×10^{2}	2.57×10^{2}	2.71×10^{2}
0.75	1.64×10^{3}	1.74×10^{3}	1.86×10^{3}	2.01×10^{3}	2.13×10^{3}	2.31×10^{3}	2.45×10^{3}
	1.67×10^{4}	1.77×10^{4}	1.9×10^{4}	2.05×10^{4}	2.19×10^{4}	2.38×10^{4}	2.54×10^{4}
	1.87×10^{5}	1.98×10^{5}	2.13×10^{5}	2.31×10^{5}	2.47×10^{5}	2.71×10^{5}	2.9×10^{5}
	2.26×10^{6}	2.4×10^{6}	2.58×10^{6}	2.81×10^{6}	3.01×10^{6}	3.32×10^{6}	3.58×10^{6}
	4.48	5.17	5.68	6.26	6.67	7.04	7.35
	3.38×10^{1}	3.76×10^{1}	4.04×10^{1}	4.39×10^{1}	4.65×10^{1}	4.91×10^{1}	5.16×10^{1}
	3.22×10^{2}	3.48×10^{2}	3.7×10^{2}	3.99×10^{2}	4.23×10^{2}	4.49×10^{2}	4.75×10^{2}
1	3.7×10^{3}	3.95×10^{3}	4.19×10^{3}	4.52×10^{3}	4.81×10^{3}	5.12×10^{3}	5.45×10^{3}
	4.79×10^{4}	5.11×10^{4}	5.42×10^{4}	5.86×10^{4}	6.27×10^{4}	6.7×10^{4}	7.18×10^{4}
	6.77×10^{5}	7.22×10^{5}	7.68×10^{5}	8.34×10^{5}	8.96×10^{5}	9.64×10^{5}	1.04×10^{6}
	1.03×10^{7}	1.1×10^{7}	1.17×10^{7}	1.28×10^{7}	1.38×10^{7}	1.49×10^{7}	1.61×10^{7}
	4.84	5.45	6.	6.5	6.95	7.27	7.6
	5.48×10^{1}	5.97×10^{1}	6.43×10^{1}	6.87×10^{1}	7.28×10^{1}	7.61×10^{1}	7.99×10^{1}
	7.38×10^{2}	7.9×10^{2}	8.44×10^{2}	8.99×10^{2}	9.55×10^{2}	$1.\times10^{3}$	1.06×10^{3}
1.5	1.19×10^{4}	1.27×10^{4}	1.35×10^{4}	1.45×10^{4}	1.54×10^{4}	1.63×10^{4}	1.74×10^{4}
	2.17×10^{5}	2.31×10^{5}	2.47×10^{5}	2.65×10^{5}	2.84×10^{5}	3.02×10^{5}	3.24×10^{5}
	4.3×10^{6}	4.58×10^{6}	4.91×10^{6}	5.29×10^{6}	5.72×10^{6}	6.11×10^{6}	6.61×10^{6}
	9.12×10^{7}	9.72×10^{7}	1.05×10^{8}	1.13×10^{8}	1.23×10^{8}	1.32×10^{8}	1.44×10^{8}

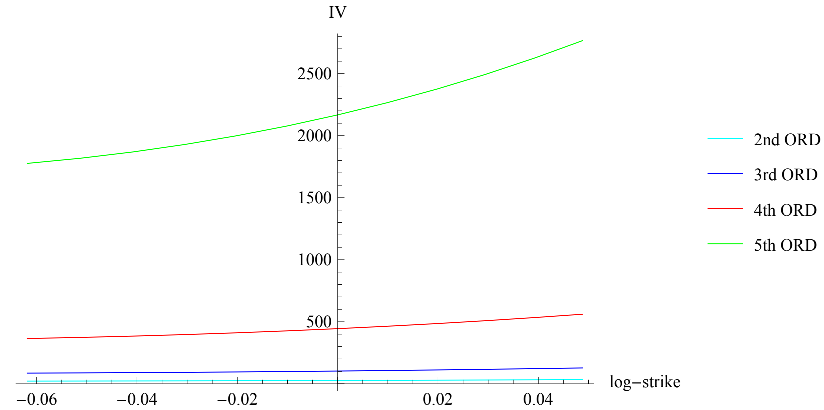
	5.09	5.65	6.16	6.62	7.02	7.37	7.67
	7.67×10^{1}	8.28×10^{1}	8.84×10^{1}	9.36×10^{1}	9.85×10^{1}	1.03×10^{2}	1.08×10^{2}
	1.34×10^{3}	1.42×10^{3}	1.51×10^{3}	1.6×10^{3}	1.69×10^{3}	1.78×10^{3}	1.86×10^{3}
2	2.77×10^{4}	2.95×10^{4}	3.13×10^{4}	3.33×10^{4}	3.53×10^{4}	3.74×10^{4}	3.95×10^{4}
	6.48×10^{5}	6.89×10^{5}	7.35×10^{5}	7.85×10^{5}	8.38×10^{5}	8.94×10^{5}	9.52×10^{5}
	1.65×10^{7}	1.75×10^{7}	1.88×10^{7}	2.02×10^{7}	2.16×10^{7}	2.32×10^{7}	2.49×10^{7}
	4.47×10^{8}	4.76×10^{8}	5.11×10^{8}	5.51×10^{8}	5.95×10^{8}	6.43×10^{8}	6.94×10^{8}

With this set of parameters we have an explosion of our approximation, as can be seen in the following plots, where the graphs of the fifth and sixth order are always over those of the lower orders.

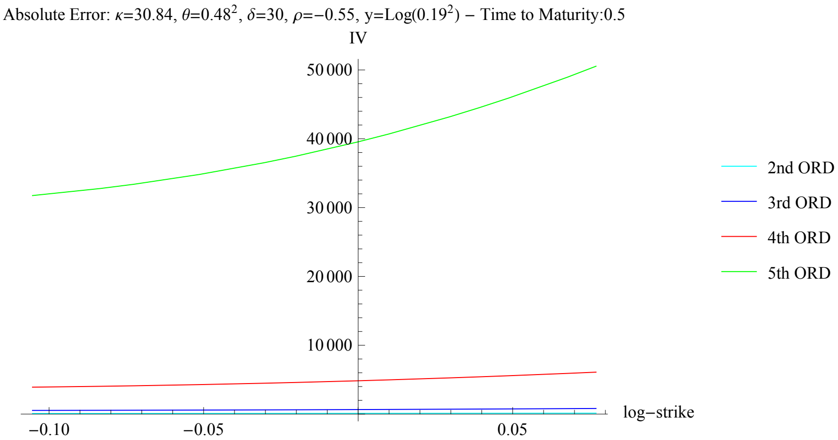




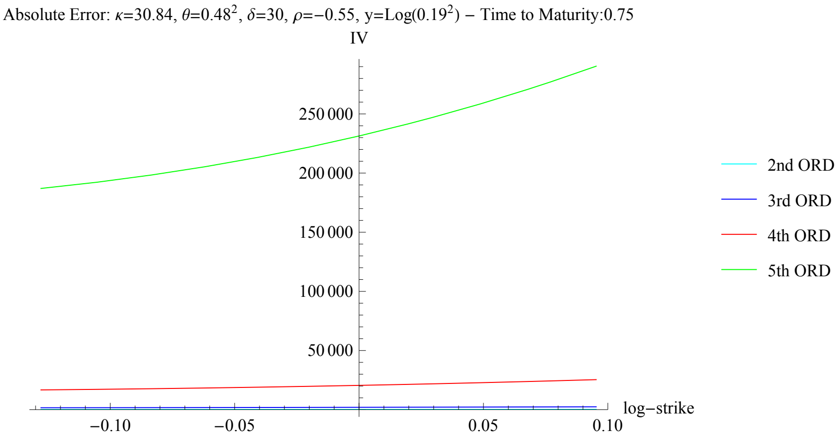
Absolute Error: κ=30.84, θ=0.48^2, δ=30, ρ=−0.55, y=Log(0.19^2) − Time to Maturity:0.25

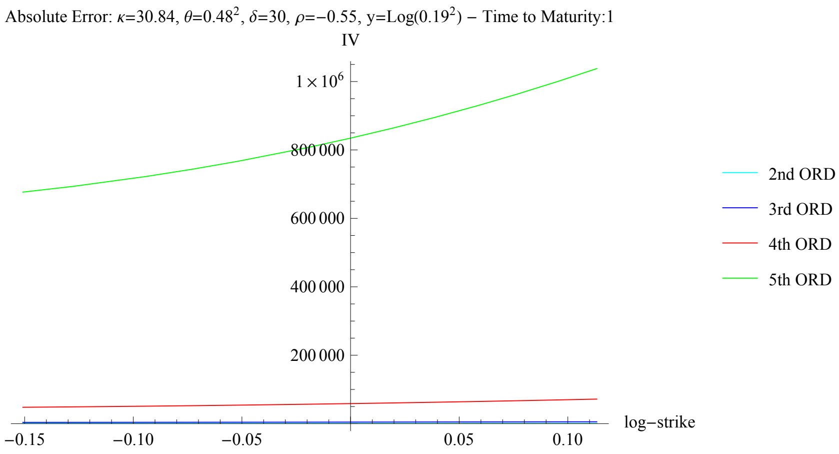


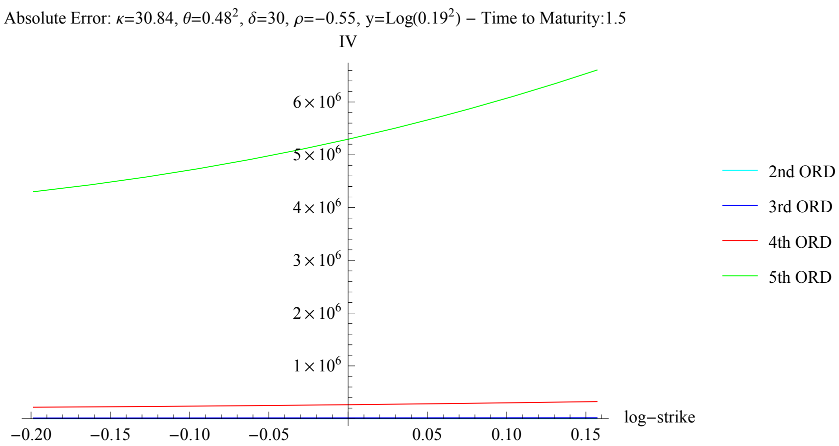
Absolute Error: κ=30.84, θ=0.48^2, δ=30, ρ=−0.55, y=Log(0.19^2) − Time to Maturity:0.5

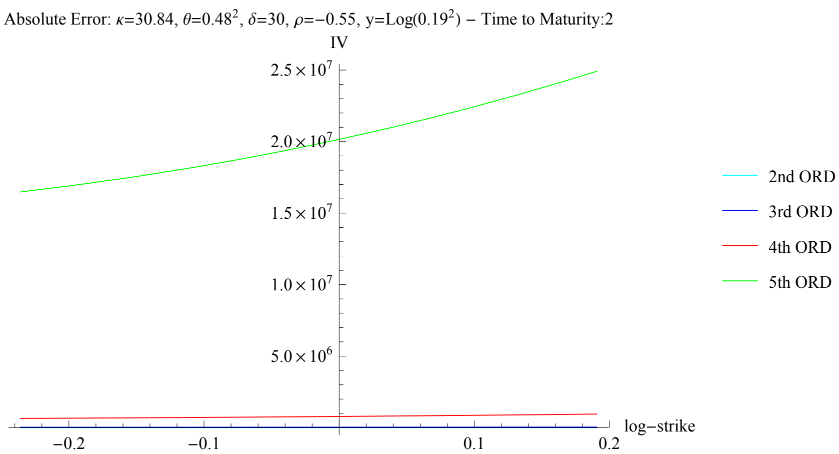


Absolute Error: κ=30.84, θ=0.48^2, δ=30, ρ=−0.55, y=Log(0.19^2) − Time to Maturity:0.75









In the next matrix we make a comparison between all three methods: initial point, midpoint and integral mean.

	1			7			13		
	3.76×10^{-1}	1.79×10^{1}	2.8	1.98	1.95×10^{1}	4.41	2.83	2.03×10^{1}	5.26
	1.25	4.38×10^{1}	3.66	2.03	4.67×10^{1}	4.75	2.73	4.89×10^{1}	5.65
	3.3	1.75×10^{2}	7.43	3.59	1.87×10^{2}	8.47	4.18	1.96×10^{2}	9.64
0.08	4.57	8.73×10^{2}	1.47×10^{1}	5.5	9.39×10^{2}	1.71×10^{1}	6.57	9.88×10^{2}	1.96×10^{1}
	9.13	5.01×10^{3}	3.52×10^{1}	1.1×10^{1}	5.42×10^{3}	4.13×10^{1}	1.33×10^{1}	5.73×10^{3}	4.76×10^{1}
	2.12×10^{1}	3.18×10^{4}	9.63×10^{1}	2.49×10^{1}	3.45×10^{4}	1.12×10^{2}	2.99×10^{1}	3.66×10^{4}	1.3×10^{2}
	5.07×10^{1}	0	0	6.05×10^{1}	0	0	7.33×10^{1}	0	0
	1.59	1.91×10^{1}	7.03	3.52	2.1×10^{1}	8.96	4.67	2.22×10^{1}	1.01×10^{1}
	2.91	8.71×10^{1}	1.69×10^{1}	5.	9.29×10^{1}	2.01×10^{1}	6.93	9.77×10^{1}	2.3×10^{1}
	1.1×10^{1}	6.25×10^{2}	6.19×10^{1}	1.31×10^{1}	6.69×10^{2}	7.13×10^{1}	1.66×10^{1}	7.06×10^{2}	8.1×10^{1}
0.17	3.18×10^{1}	5.6×10^{3}	2.74×10^{2}	3.78×10^{1}	6.03×10^{3}	3.16×10^{2}	4.72×10^{1}	6.4×10^{3}	3.6×10^{2}
	$1.\times10^{2}$	5.72×10^{4}	1.39×10^{3}	1.23×10^{2}	6.19×10^{4}	1.61×10^{3}	1.56×10^{2}	6.63×10^{4}	1.85×10^{3}
	3.77×10^{2}	6.4×10^{5}	7.85×10^{3}	4.59×10^{2}	6.97×10^{5}	9.13×10^{3}	5.88×10^{2}	7.5×10^{5}	1.06×10^{4}
	1.53×10^{3}	0	0	1.86×10^{3}	0	0	2.41×10^{3}	0	0
	2.44	1.99×10^{1}	1.06×10^{1}	4.32	2.18×10^{1}	1.25×10^{1}	5.48	2.3×10^{1}	1.37×10^{1}
	5.24	1.27×10^{2}	4.16×10^{1}	8.21	1.34×10^{2}	4.67×10^{1}	1.09×10^{1}	1.41×10^{2}	5.1×10^{1}
	2.15×10^{1}	1.26×10^{3}	2.41×10^{2}	2.66×10^{1}	1.34×10^{3}	2.69×10^{2}	3.35×10^{1}	1.41×10^{3}	2.95×10^{2}
0.25	8.6×10^{1}	1.56×10^{4}	1.73×10^{3}	1.03×10^{2}	1.67×10^{4}	1.94×10^{3}	1.28×10^{2}	1.76×10^{4}	2.15×10^{3}
	3.65×10^{2}	2.19×10^{5}	1.42×10^{4}	4.44×10^{2}	2.35×10^{5}	1.61×10^{4}	5.61×10^{2}	2.5×10^{5}	1.79×10^{4}
	1.78×10^{3}	3.34×10^{6}	1.29×10^{5}	2.17×10^{3}	3.62×10^{6}	1.46×10^{5}	2.77×10^{3}	3.88×10^{6}	1.65×10^{5}
	9.46×10^{3}	0	0	1.15×10^{4}	0	0	1.49×10^{4}	0	0
	3.43	2.09×10^{1}	1.9×10^{1}	5.52	2.3×10^{1}	2.11×10^{1}	6.71	2.42×10^{1}	2.23×10^{1}
	1.34×10^{1}	2.52×10^{2}	2.09×10^{2}	1.95×10^{1}	2.66×10^{2}	2.21×10^{2}	2.44×10^{1}	2.76×10^{2}	2.31×10^{2}
	7.83×10^{1}	4.65×10^{3}	3.45×10^{3}	1.01×10^{2}	4.92×10^{3}	3.68×10^{3}	1.25×10^{2}	5.13×10^{3}	3.86×10^{3}
0.5	5.31×10^{2}	1.06×10^{5}	7.06×10^{4}	6.54×10^{2}	1.13×10^{5}	7.59×10^{4}	8.13×10^{2}	1.19×10^{5}	8.02×10^{4}
	3.91×10^{3}	2.71×10^{6}	1.63×10^{6}	4.84×10^{3}	2.91×10^{6}	1.76×10^{6}	6.09×10^{3}	3.08×10^{6}	1.88×10^{6}
	3.17×10^{4}	7.51×10^{7}	4.07×10^{7}	3.96×10^{4}	8.13×10^{7}	4.44×10^{7}	5.05×10^{4}	8.66×10^{7}	4.76×10^{7}
	2.81×10^{5}	0	0	3.51×10^{5}	0	0	4.54×10^{5}	0	0
	4.12	2.16×10^{1}	2.4×10^{1}	6.01	2.35×10^{1}	2.59×10^{1}	7.14	2.46×10^{1}	2.7×10^{1}
	2.34×10^{1}	3.8×10^{2}	4.72×10^{2}	3.16×10^{1}	3.98×10^{2}	4.9×10^{2}	3.79×10^{1}	4.11×10^{2}	5.04×10^{2}
	1.79×10^{2}	1.02×10^{4}	1.44×10^{4}	2.25×10^{2}	1.07×10^{4}	1.5×10^{4}	2.71×10^{2}	1.11×10^{4}	1.55×10^{4}
0.75	1.64×10^{3}	3.39×10^{5}	5.43×10^{5}	2.01×10^{3}	3.58×10^{5}	5.7×10^{5}	2.45×10^{3}	3.74×10^{5}	5.91×10^{5}
	1.67×10^{4}	1.26×10^{7}	2.29×10^{7}	2.05×10^{4}	1.34×10^{7}	2.42×10^{7}	2.54×10^{4}	1.41×10^{7}	2.52×10^{7}
	1.87×10^{5}	5.04×10^{8}	1.04×10^{9}	2.31×10^{5}	5.39×10^{8}	1.1×10^{9}	2.9×10^{5}	5.7×10^{8}	1.16×10^{9}
	2.26×10^{6}	0	0	2.81×10^{6}	0	0	3.58×10^{6}	0	0
	4.48	2.2×10^{1}	2.68×10^{1}	6.26	2.38×10^{1}	2.85×10^{1}	7.35	2.49×10^{1}	2.96×10^{1}
	3.38×10^{1}	5.09×10^{2}	7.67×10^{2}	4.39×10^{1}	5.3×10^{2}	7.91×10^{2}	5.16×10^{1}	5.46×10^{2}	8.09×10^{2}
	3.22×10^{2}	1.8×10^{4}	3.48×10^{4}	3.99×10^{2}	1.88×10^{4}	3.59×10^{4}	4.75×10^{2}	1.94×10^{4}	3.68×10^{4}
1	3.7×10^{3}	7.83×10^{5}	1.94×10^{6}	4.52×10^{3}	8.22×10^{5}	2.02×10^{6}	5.45×10^{3}	8.55×10^{5}	2.08×10^{6}
	4.79×10^{4}	3.8×10^{7}	1.21×10^{8}	5.86×10^{4}	4.03×10^{7}	1.26×10^{8}	7.18×10^{4}	4.21×10^{7}	1.31×10^{8}
	6.77×10^{5}	1.99×10^{9}	8.08×10^{9}	8.34×10^{5}	2.12×10^{9}	8.49×10^{9}	1.04×10^{6}	2.22×10^{9}	8.83×10^{9}
	1.03×10^{7}	0	0	1.28×10^{7}	0	0	1.61×10^{7}	0	0
	4.84	2.23×10^{1}	2.95×10^{1}	6.5	2.4×10^{1}	3.11×10^{1}	7.6	2.51×10^{1}	3.22×10^{1}
	5.48×10^{1}	7.66×10^{2}	1.38×10^{3}	6.87×10^{1}	7.95×10^{2}	1.41×10^{3}	7.99×10^{1}	8.17×10^{2}	1.44×10^{3}
	7.38×10^{2}	$4.\times10^{4}$	1.03×10^{5}	8.99×10^{2}	4.16×10^{4}	1.06×10^{5}	1.06×10^{3}	4.29×10^{4}	1.09×10^{5}
1.5	1.19×10^{4}	2.57×10^{6}	9.58×10^{6}	1.45×10^{4}	2.69×10^{6}	9.88×10^{6}	1.74×10^{4}	2.79×10^{6}	1.01×10^{7}
	2.17×10^{5}	1.84×10^{8}	9.86×10^{8}	2.65×10^{5}	1.93×10^{8}	1.02×10^{9}	3.24×10^{5}	2.02×10^{8}	1.05×10^{9}
	4.3×10^{6}	1.41×10^{10}	1.08×10^{11}	5.29×10^{6}	1.49×10^{10}	1.13×10^{11}	6.61×10^{6}	1.56×10^{10}	1.17×10^{11}
	9.12×10^{7}	0	0	1.13×10^{8}	0	0	1.44×10^{8}	0	0
	5.09	2.26×10^{1}	3.08×10^{1}	6.62	2.41×10^{1}	3.24×10^{1}	7.67	2.52×10^{1}	3.34×10^{1}
	7.67×10^{1}	1.03×10^{3}	$2.\times10^{3}$	9.36×10^{1}	1.06×10^{3}	2.04×10^{3}	1.08×10^{2}	1.09×10^{3}	2.07×10^{3}
	1.34×10^{3}	7.09×10^{4}	2.1×10^{5}	1.6×10^{3}	7.34×10^{4}	2.14×10^{5}	1.86×10^{3}	7.55×10^{4}	2.18×10^{5}
2	2.77×10^{4}	6.02×10^{6}	2.71×10^{7}	3.33×10^{4}	6.26×10^{6}	2.78×10^{7}	3.95×10^{4}	6.48×10^{6}	2.84×10^{7}
	6.48×10^{5}	5.68×10^{8}	3.89×10^{9}	7.85×10^{5}	5.95×10^{8}	4.01×10^{9}	9.52×10^{5}	6.19×10^{8}	4.11×10^{9}
	1.65×10^{7}	5.73×10^{10}	5.96×10^{11}	2.02×10^{7}	6.04×10^{10}	6.17×10^{11}	2.49×10^{7}	6.32×10^{10}	6.35×10^{11}
	4.47×10^{8}	0	0	5.51×10^{8}	0	0	6.94×10^{8}	0	0

Even in this case we don't have any improvement changing the value of $\overline{y}$.

8.4.3 Third Set by Drimus modified

In this third case we use parameters values as proposed by Drimus in ([Dri12]), so we have ($\kappa = 22.84$, $\theta = 0.4669^2$, $\rho = -0.99$, $y = \mathrm{Log}\left(0.245^2\right)$) but we consider $\delta = 20$ instead of 8.56.

First of all we create strikes matrix expressed in bp

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.08	95	96	97	98	98	99	100	100	101	102	102	102	103
0.17	91	92	94	96	97	98	100	101	102	102	103	104	105
0.25	89	91	93	94	96	98	100	101	102	103	104	105	106
0.5	84	87	89	92	95	97	100	102	103	104	106	108	109
0.75	81	84	87	90	94	97	100	102	104	106	108	110	112
1	78	82	85	89	93	96	100	102	105	107	109	112	114
1.5	74	78	83	87	91	96	100	103	106	110	113	116	119
2	70	75	80	85	90	95	100	104	108	112	116	120	124

then we compute corresponding call and put prices

	1	3	5	7	9	11	13
0.08	0.05990	0.04419	0.03697	0.02419	0.01876	0.01405	0.01009
0.17	0.09963	0.07444	0.05162	0.03215	0.02156	0.01709	0.009895
0.25	0.1200	0.08623	0.06320	0.03693	0.02623	0.01743	0.01063
0.5	0.1698	0.1265	0.07980	0.04752	0.03194	0.01963	0.01071
0.75	0.2002	0.1481	0.09392	0.05562	0.03548	0.02023	0.009894
1	0.2299	0.1689	0.1069	0.06261	0.03817	0.02340	0.01085
1.5	0.2702	0.1923	0.1310	0.07469	0.04622	0.02277	0.01045
2	0.3098	0.2227	0.1464	0.08513	0.04893	0.02437	0.01002

	1	3	5	7	9	11	13
0.08	0.009904	0.01419	0.01697	0.02419	0.02876	0.03405	0.04009
0.17	0.009630	0.01444	0.02162	0.03215	0.04156	0.04709	0.05989
0.25	0.01001	0.01623	0.02320	0.03693	0.04623	0.05743	0.07063
0.5	0.009787	0.01646	0.02980	0.04752	0.06194	0.07963	0.1007
0.75	0.01015	0.01811	0.03392	0.05562	0.07548	0.1002	0.1299
1	0.009865	0.01886	0.03691	0.06261	0.08817	0.1134	0.1508
1.5	0.01021	0.02227	0.04098	0.07469	0.1062	0.1528	0.2005
2	0.009752	0.02267	0.04640	0.08513	0.1289	0.1844	0.2500

and the exact implied volatility by numerical integration

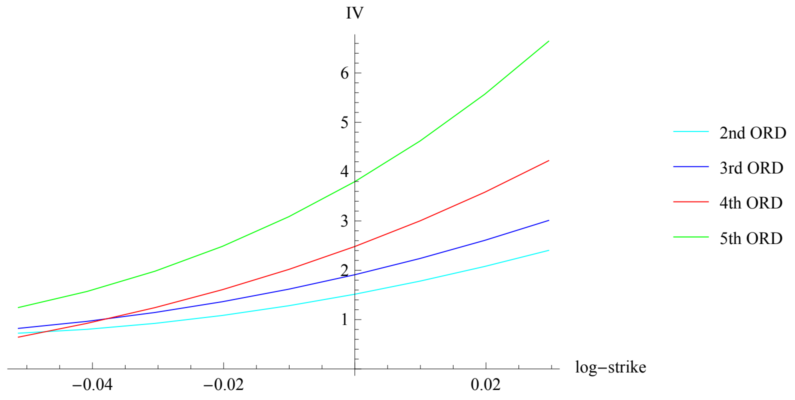
	1	3	5	7	9	11	13
0.08	25.5616	23.8769	23.0516	21.4391	20.6535	19.8828	19.1277
0.17	25.0293	23.1715	21.3420	19.5490	18.3791	17.8032	16.6717
0.25	24.0517	22.0165	20.5063	18.5211	17.5441	16.5802	15.6320
0.5	22.3008	20.5806	18.5395	16.8566	15.8557	14.8631	13.8809
0.75	21.1187	19.5120	17.6695	16.1111	15.0815	14.0594	13.0463
1	20.4941	18.9347	17.1988	15.7095	14.6588	13.8250	12.7912
1.5	19.5507	18.0247	16.7245	15.3089	14.3866	13.3277	12.4323
2	19.0818	17.6828	16.3685	15.1177	14.1521	13.2110	12.2897

Now we compute our approximated implied volatilities and in the following table we show absolute errors.

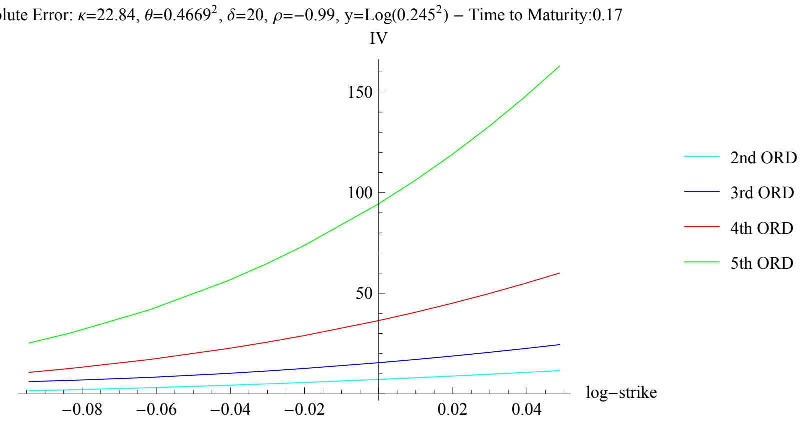
	1	3	5	7	9	11	13
	1.06	6.23×10^{-1}	1.45	3.06	3.85	4.62	5.37
	7.53×10^{-1}	8.87×10^{-2}	5.07×10^{-1}	1.34	1.77	2.19	2.62
	7.25×10^{-1}	9.23×10^{-1}	1.08	1.51	1.78	2.08	2.4
0.08	8.22×10^{-1}	1.14	1.36	1.91	2.24	2.61	3.01
	6.44×10^{-1}	1.24	1.61	2.48	3.	3.58	4.22
	1.24	1.98	2.48	3.8	4.62	5.57	6.64
	1.98	3.25	4.11	6.38	7.84	9.54	1.15×10^{1}
	5.29×10^{-1}	1.33	3.16	4.95	6.12	6.7	7.83
	1.55	5.3×10^{-1}	2.51	4.41	5.64	6.25	7.45
	1.55	3.04	4.92	7.17	8.83	9.72	1.16×10^{1}
0.17	6.13	8.17	1.13×10^{1}	1.55×10^{1}	1.88×10^{1}	2.06×10^{1}	2.45×10^{1}
	1.07×10^{1}	1.71×10^{1}	2.56×10^{1}	3.64×10^{1}	4.5×10^{1}	4.97×10^{1}	$6.\times10^{1}$
	2.53×10^{1}	4.17×10^{1}	6.45×10^{1}	9.45×10^{1}	1.19×10^{2}	1.33×10^{2}	1.63×10^{2}
	7.34×10^{1}	1.17×10^{2}	1.81×10^{2}	2.7×10^{2}	3.44×10^{2}	3.86×10^{2}	4.82×10^{2}
	4.48×10^{-1}	2.48	3.99	5.98	6.96	7.92	8.87
	8.13×10^{-1}	2.48	4.82	7.79	9.21	1.06×10^{1}	1.2×10^{1}
	3.46	7.64	1.12×10^{1}	1.64×10^{1}	1.92×10^{1}	2.21×10^{1}	2.52×10^{1}
0.25	1.76×10^{1}	2.54×10^{1}	3.33×10^{1}	4.64×10^{1}	5.39×10^{1}	6.19×10^{1}	7.06×10^{1}
	4.99×10^{1}	7.7×10^{1}	1.04×10^{2}	1.47×10^{2}	1.73×10^{2}	2.01×10^{2}	2.31×10^{2}
	1.55×10^{2}	2.53×10^{2}	3.5×10^{2}	5.13×10^{2}	6.11×10^{2}	7.19×10^{2}	8.39×10^{2}
	5.7×10^{2}	9.29×10^{2}	1.3×10^{3}	1.95×10^{3}	2.34×10^{3}	2.79×10^{3}	3.3×10^{3}
	2.2	3.92	5.96	7.64	8.64	9.64	1.06×10^{1}
	4.19	9.48	1.54×10^{1}	1.99×10^{1}	2.25×10^{1}	2.5×10^{1}	2.74×10^{1}
	2.25×10^{1}	3.79×10^{1}	5.71×10^{1}	7.35×10^{1}	8.34×10^{1}	9.35×10^{1}	1.04×10^{2}
0.5	1.36×10^{2}	1.92×10^{2}	2.74×10^{2}	3.5×10^{2}	$4.\times10^{2}$	4.51×10^{2}	5.05×10^{2}
	7.44×10^{2}	1.04×10^{3}	1.49×10^{3}	1.93×10^{3}	2.22×10^{3}	2.53×10^{3}	2.86×10^{3}
	4.19×10^{3}	6.02×10^{3}	8.8×10^{3}	1.16×10^{4}	1.35×10^{4}	1.56×10^{4}	1.79×10^{4}
	2.57×10^{4}	3.75×10^{4}	5.62×10^{4}	7.57×10^{4}	8.92×10^{4}	1.04×10^{5}	1.2×10^{5}
	3.38	4.99	6.83	8.39	9.42	1.04×10^{1}	1.15×10^{1}
	1.24×10^{1}	1.94×10^{1}	2.7×10^{1}	3.29×10^{1}	3.66×10^{1}	4.02×10^{1}	4.36×10^{1}
	7.2×10^{1}	1.05×10^{2}	1.42×10^{2}	1.75×10^{2}	1.96×10^{2}	2.17×10^{2}	2.38×10^{2}
0.75	5.17×10^{2}	7.04×10^{2}	9.45×10^{2}	1.17×10^{3}	1.32×10^{3}	1.48×10^{3}	1.64×10^{3}
	3.92×10^{3}	5.3×10^{3}	7.17×10^{3}	8.98×10^{3}	1.03×10^{4}	1.16×10^{4}	1.3×10^{4}
	3.14×10^{4}	4.31×10^{4}	5.95×10^{4}	7.58×10^{4}	8.77×10^{4}	$1.\times10^{5}$	1.14×10^{5}
	2.69×10^{5}	3.76×10^{5}	5.3×10^{5}	6.87×10^{5}	8.04×10^{5}	9.31×10^{5}	1.07×10^{6}
	4.01	5.57	7.3	8.79	9.84	1.07×10^{1}	1.17×10^{1}
	2.09×10^{1}	2.98×10^{1}	3.9×10^{1}	4.63×10^{1}	5.11×10^{1}	5.48×10^{1}	5.93×10^{1}
	1.5×10^{2}	2.06×10^{2}	2.68×10^{2}	3.2×10^{2}	3.57×10^{2}	3.85×10^{2}	4.21×10^{2}
1	1.32×10^{3}	1.75×10^{3}	2.28×10^{3}	2.76×10^{3}	3.1×10^{3}	3.39×10^{3}	3.74×10^{3}
	1.26×10^{4}	1.68×10^{4}	2.21×10^{4}	2.71×10^{4}	3.09×10^{4}	3.41×10^{4}	3.81×10^{4}
	1.3×10^{5}	1.75×10^{5}	2.34×10^{5}	2.92×10^{5}	3.37×10^{5}	3.75×10^{5}	4.24×10^{5}
	1.42×10^{6}	1.94×10^{6}	2.65×10^{6}	3.37×10^{6}	3.94×10^{6}	4.42×10^{6}	5.06×10^{6}

1.5	4.95	6.48	7.78	9.19	1.01×10^{1}	1.12×10^{1}	1.21×10^{1}
	4.11×10^{1}	5.35×10^{1}	6.34×10^{1}	7.34×10^{1}	7.96×10^{1}	8.63×10^{1}	9.16×10^{1}
	4.19×10^{2}	5.39×10^{2}	6.39×10^{2}	7.46×10^{2}	8.14×10^{2}	8.9×10^{2}	9.53×10^{2}
	5.09×10^{3}	6.54×10^{3}	7.83×10^{3}	9.28×10^{3}	1.02×10^{4}	1.13×10^{4}	1.23×10^{4}
	6.9×10^{4}	8.95×10^{4}	1.09×10^{5}	1.31×10^{5}	1.46×10^{5}	1.64×10^{5}	1.79×10^{5}
	1.01×10^{6}	1.33×10^{6}	1.64×10^{6}	2.01×10^{6}	2.27×10^{6}	2.58×10^{6}	2.86×10^{6}
	1.57×10^{7}	2.11×10^{7}	2.65×10^{7}	3.3×10^{7}	3.77×10^{7}	4.33×10^{7}	4.84×10^{7}
2	5.42	6.82	8.13	9.38	1.03×10^{1}	1.13×10^{1}	1.22×10^{1}
	6.15×10^{1}	7.63×10^{1}	8.92×10^{1}	1.01×10^{2}	1.09×10^{2}	1.17×10^{2}	1.24×10^{2}
	8.2×10^{2}	1.01×10^{3}	1.19×10^{3}	1.35×10^{3}	1.47×10^{3}	1.59×10^{3}	1.7×10^{3}
	1.29×10^{4}	1.6×10^{4}	1.91×10^{4}	2.2×10^{4}	2.43×10^{4}	2.65×10^{4}	2.86×10^{4}
	2.26×10^{5}	2.85×10^{5}	3.45×10^{5}	4.04×10^{5}	4.52×10^{5}	4.99×10^{5}	5.45×10^{5}
	4.29×10^{6}	5.49×10^{6}	6.75×10^{6}	8.05×10^{6}	9.11×10^{6}	1.02×10^{7}	1.12×10^{7}
	8.63×10^{7}	1.12×10^{8}	1.41×10^{8}	1.7×10^{8}	1.95×10^{8}	2.2×10^{8}	2.46×10^{8}

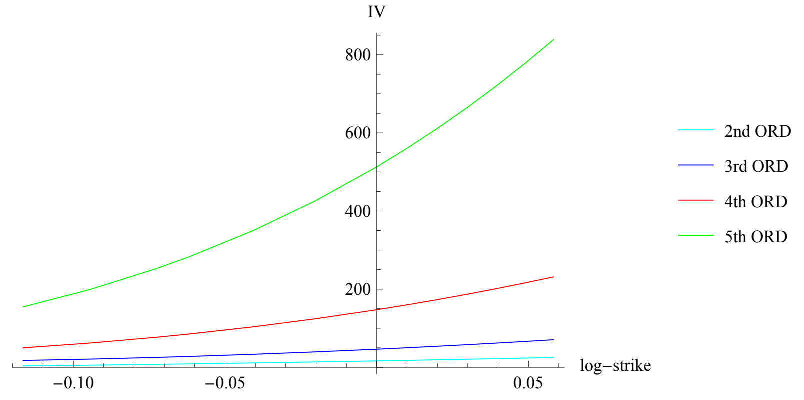
Absolute Error: κ=22.84, θ=0.4669^2, δ=20, ρ=−0.99, y=Log(0.245^2) − Time to Maturity:0.08



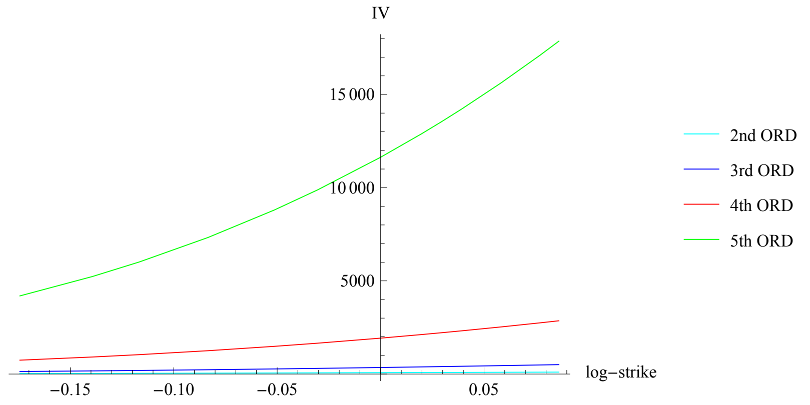
Absolute Error: κ=22.84, θ=0.4669^2, δ=20, ρ=−0.99, y=Log(0.245^2) − Time to Maturity:0.17



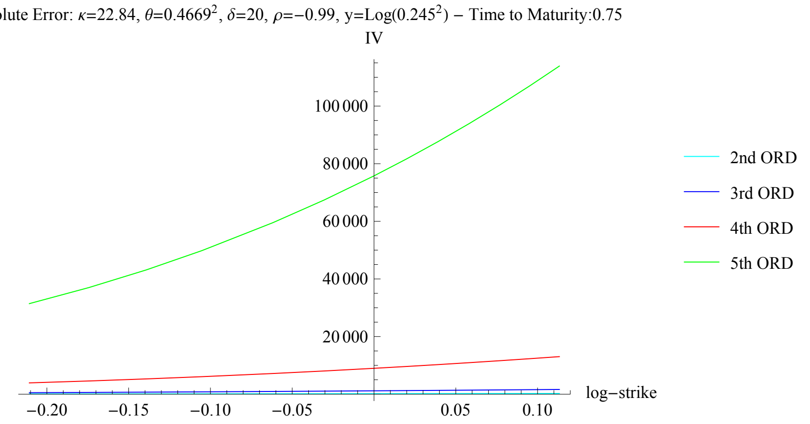
Absolute Error: κ=22.84, θ=0.4669^2, δ=20, ρ=−0.99, y=Log(0.245^2) − Time to Maturity:0.25

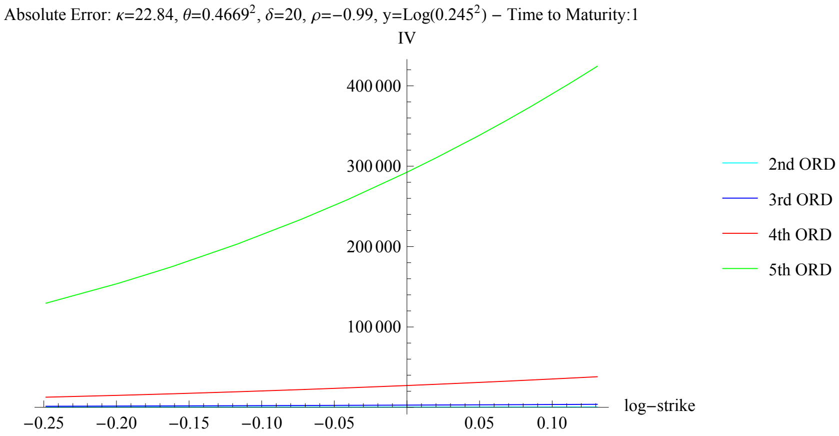


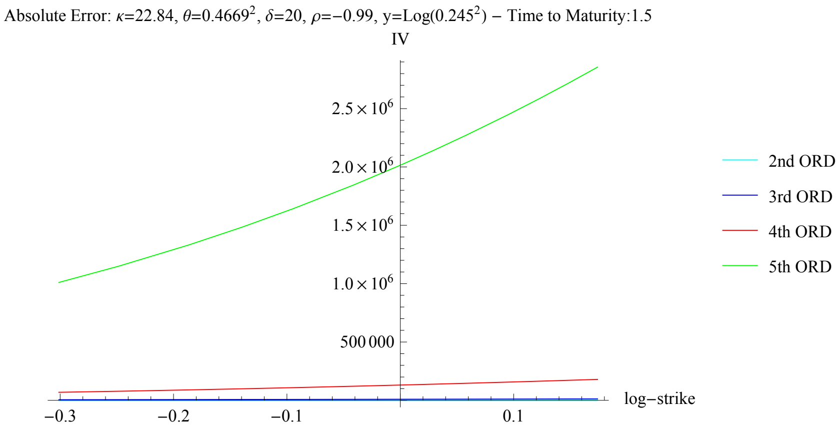
Absolute Error: κ=22.84, θ=0.4669^2, δ=20, ρ=−0.99, y=Log(0.245^2) − Time to Maturity:0.5

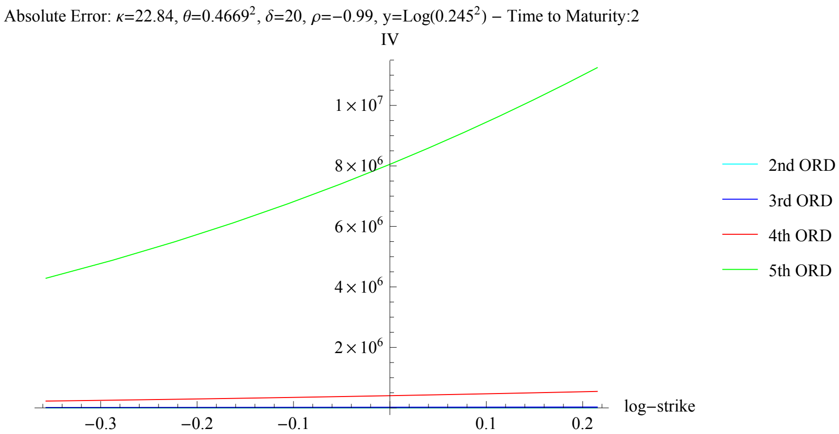


Absolute Error: κ=22.84, θ=0.4669^2, δ=20, ρ=−0.99, y=Log(0.245^2) − Time to Maturity:0.75









In the next matrix we make a comparison between all three methods: initial point, midpoint and integral mean.

	1			7			13		
	1.06	1.17×10^{1}	7.4×10^{-1}	3.06	1.58×10^{1}	4.86	5.37	1.82×10^{1}	7.17
	7.53×10^{-1}	1.49×10^{1}	2.98×10^{-1}	1.34	2.02×10^{1}	2.85	2.62	2.34×10^{1}	4.39
	7.25×10^{-1}	2.96×10^{1}	1.43	1.51	4.08×10^{1}	3.12	2.4	4.79×10^{1}	4.55
0.08	8.22×10^{-1}	7.33×10^{1}	2.01	1.91	1.04×10^{2}	4.32	3.01	1.24×10^{2}	6.34
	6.44×10^{-1}	2.08×10^{2}	2.61	2.48	3.03×10^{2}	6.64	4.22	3.7×10^{2}	1.02×10^{1}
	1.24	6.49×10^{2}	4.53	3.8	9.79×10^{2}	1.15×10^{1}	6.64	1.22×10^{3}	1.83×10^{1}
	1.98	0	0	6.38	0	0	1.15×10^{1}	0	0
	5.29×10^{-1}	1.23×10^{1}	3.33	4.95	1.77×10^{1}	8.81	7.83	2.06×10^{1}	1.17×10^{1}
	1.55	2.94×10^{1}	4.14	4.41	4.13×10^{1}	1.19×10^{1}	7.45	4.74×10^{1}	1.59×10^{1}
	1.55	1.04×10^{2}	1.11×10^{1}	7.17	1.49×10^{2}	2.49×10^{1}	1.16×10^{1}	1.74×10^{2}	3.38×10^{1}
0.17	6.13	4.55×10^{2}	3.19×10^{1}	1.55×10^{1}	6.7×10^{2}	6.72×10^{1}	2.45×10^{1}	8.01×10^{2}	9.28×10^{1}
	1.07×10^{1}	2.26×10^{3}	9.36×10^{1}	3.64×10^{1}	3.45×10^{3}	2.09×10^{2}	$6.\times10^{1}$	4.21×10^{3}	2.97×10^{2}
	2.53×10^{1}	1.23×10^{4}	3.03×10^{2}	9.45×10^{1}	1.95×10^{4}	7.16×10^{2}	1.63×10^{2}	2.43×10^{4}	1.05×10^{3}
	7.34×10^{1}	0	0	2.7×10^{2}	0	0	4.82×10^{2}	0	0
	4.48×10^{-1}	1.32×10^{1}	6.1	5.98	1.88×10^{1}	1.16×10^{1}	8.87	2.17×10^{1}	1.45×10^{1}
	8.13×10^{-1}	4.47×10^{1}	1.33×10^{1}	7.79	6.07×10^{1}	2.51×10^{1}	1.2×10^{1}	6.85×10^{1}	3.09×10^{1}
	3.46	2.18×10^{2}	4.46×10^{1}	1.64×10^{1}	3.02×10^{2}	8.12×10^{1}	2.52×10^{1}	3.47×10^{2}	1.02×10^{2}
0.25	1.76×10^{1}	1.3×10^{3}	1.82×10^{2}	4.64×10^{1}	1.86×10^{3}	3.31×10^{2}	7.06×10^{1}	2.18×10^{3}	4.27×10^{2}
	4.99×10^{1}	8.82×10^{3}	8.19×10^{2}	1.47×10^{2}	1.31×10^{4}	1.56×10^{3}	2.31×10^{2}	1.57×10^{4}	2.06×10^{3}
	1.55×10^{2}	6.53×10^{4}	4.04×10^{3}	5.13×10^{2}	1.01×10^{5}	8.05×10^{3}	8.39×10^{2}	1.23×10^{5}	1.09×10^{4}
	5.7×10^{2}	0	0	1.95×10^{3}	0	0	3.3×10^{3}	0	0
	2.2	1.5×10^{1}	1.27×10^{1}	7.64	2.04×10^{1}	1.81×10^{1}	1.06×10^{1}	2.34×10^{1}	2.11×10^{1}
	4.19	9.61×10^{1}	7.23×10^{1}	1.99×10^{1}	1.23×10^{2}	9.71×10^{1}	2.74×10^{1}	1.36×10^{2}	1.09×10^{2}
	2.25×10^{1}	8.64×10^{2}	5.71×10^{2}	7.35×10^{1}	1.12×10^{3}	7.79×10^{2}	1.04×10^{2}	1.26×10^{3}	8.9×10^{2}
0.5	1.36×10^{2}	9.42×10^{3}	5.45×10^{3}	3.5×10^{2}	1.27×10^{4}	7.72×10^{3}	5.05×10^{2}	1.45×10^{4}	$9.\times10^{3}$
	7.44×10^{2}	1.16×10^{5}	5.88×10^{4}	1.93×10^{3}	1.61×10^{5}	8.65×10^{4}	2.86×10^{3}	1.88×10^{5}	1.03×10^{5}
	4.19×10^{3}	1.54×10^{6}	6.89×10^{5}	1.16×10^{4}	2.23×10^{6}	1.05×10^{6}	1.79×10^{4}	2.64×10^{6}	1.28×10^{6}
	2.57×10^{4}	0	0	7.57×10^{4}	0	0	1.2×10^{5}	0	0
	3.38	1.62×10^{1}	1.71×10^{1}	8.39	2.12×10^{1}	2.21×10^{1}	1.15×10^{1}	2.42×10^{1}	2.52×10^{1}
	1.24×10^{1}	1.52×10^{2}	1.68×10^{2}	3.29×10^{1}	1.86×10^{2}	2.03×10^{2}	4.36×10^{1}	2.04×10^{2}	2.22×10^{2}
	7.2×10^{1}	1.99×10^{3}	2.32×10^{3}	1.75×10^{2}	2.47×10^{3}	2.85×10^{3}	2.38×10^{2}	2.74×10^{3}	3.14×10^{3}
0.75	5.17×10^{2}	3.15×10^{4}	3.89×10^{4}	1.17×10^{3}	4.04×10^{4}	4.91×10^{4}	1.64×10^{3}	4.57×10^{4}	5.51×10^{4}
	3.92×10^{3}	5.6×10^{5}	7.28×10^{5}	8.98×10^{3}	7.42×10^{5}	9.49×10^{5}	1.3×10^{4}	8.53×10^{5}	1.08×10^{6}
	3.14×10^{4}	1.07×10^{7}	1.47×10^{7}	7.58×10^{4}	1.47×10^{7}	1.98×10^{7}	1.14×10^{5}	1.72×10^{7}	2.3×10^{7}
	2.69×10^{5}	0	0	6.87×10^{5}	0	0	1.07×10^{6}	0	0
	4.01	1.68×10^{1}	1.98×10^{1}	8.79	2.16×10^{1}	2.46×10^{1}	1.17×10^{1}	2.45×10^{1}	2.75×10^{1}
	2.09×10^{1}	2.08×10^{2}	2.81×10^{2}	4.63×10^{1}	2.49×10^{2}	3.26×10^{2}	5.93×10^{1}	2.71×10^{2}	3.49×10^{2}
	1.5×10^{2}	3.58×10^{3}	5.68×10^{3}	3.2×10^{2}	4.35×10^{3}	6.67×10^{3}	4.21×10^{2}	4.77×10^{3}	7.21×10^{3}
1	1.32×10^{3}	7.43×10^{4}	1.39×10^{5}	2.76×10^{3}	9.3×10^{4}	1.68×10^{5}	3.74×10^{3}	1.04×10^{5}	1.84×10^{5}
	1.26×10^{4}	1.73×10^{6}	3.81×10^{6}	2.71×10^{4}	2.23×10^{6}	4.71×10^{6}	3.81×10^{4}	2.52×10^{6}	5.24×10^{6}
	1.3×10^{5}	4.31×10^{7}	1.12×10^{8}	2.92×10^{5}	5.74×10^{7}	1.42×10^{8}	4.24×10^{5}	6.61×10^{7}	1.6×10^{8}
	1.42×10^{6}	0	0	3.37×10^{6}	0	0	5.06×10^{6}	0	0
	4.95	1.77×10^{1}	2.29×10^{1}	9.19	2.2×10^{1}	2.72×10^{1}	1.21×10^{1}	2.49×10^{1}	$3.\times10^{1}$
	4.11×10^{1}	3.25×10^{2}	5.27×10^{2}	7.34×10^{1}	3.76×10^{2}	5.86×10^{2}	9.16×10^{1}	4.06×10^{2}	6.19×10^{2}
	4.19×10^{2}	8.26×10^{3}	1.76×10^{4}	7.46×10^{2}	9.67×10^{3}	1.97×10^{4}	9.53×10^{2}	1.05×10^{4}	2.1×10^{4}
1.5	5.09×10^{3}	2.53×10^{5}	7.13×10^{5}	9.28×10^{3}	3.04×10^{5}	8.16×10^{5}	1.23×10^{4}	3.36×10^{5}	8.79×10^{5}
	6.9×10^{4}	8.66×10^{6}	3.22×10^{7}	1.31×10^{5}	1.07×10^{7}	3.76×10^{7}	1.79×10^{5}	1.2×10^{7}	4.1×10^{7}
	1.01×10^{6}	3.18×10^{8}	1.55×10^{9}	2.01×10^{6}	4.02×10^{8}	1.85×10^{9}	2.86×10^{6}	4.58×10^{8}	2.04×10^{9}
	1.57×10^{7}	0	0	3.3×10^{7}	0	0	4.84×10^{7}	0	0
	5.42	1.82×10^{1}	2.45×10^{1}	9.38	2.22×10^{1}	2.84×10^{1}	1.22×10^{1}	2.5×10^{1}	3.13×10^{1}
	6.15×10^{1}	4.42×10^{2}	7.77×10^{2}	1.01×10^{2}	5.04×10^{2}	8.5×10^{2}	1.24×10^{2}	5.41×10^{2}	8.94×10^{2}
	8.2×10^{2}	1.48×10^{4}	3.62×10^{4}	1.35×10^{3}	1.71×10^{4}	3.99×10^{4}	1.7×10^{3}	1.85×10^{4}	4.22×10^{4}
2	1.29×10^{4}	6.03×10^{5}	2.05×10^{6}	2.2×10^{4}	7.1×10^{5}	2.3×10^{6}	2.86×10^{4}	7.79×10^{5}	2.46×10^{6}
	2.26×10^{5}	2.72×10^{7}	1.29×10^{8}	4.04×10^{5}	3.29×10^{7}	1.47×10^{8}	5.45×10^{5}	3.66×10^{7}	1.59×10^{8}
	4.29×10^{6}	1.31×10^{9}	8.68×10^{9}	8.05×10^{6}	1.63×10^{9}	1.01×10^{10}	1.12×10^{7}	1.84×10^{9}	1.1×10^{10}
	8.63×10^{7}	0	0	1.7×10^{8}	0	0	2.46×10^{8}	0	0

9 An Enhanced Taylor approach

In this section we present numerical test on the same model already discussed in the previous section, but using an enhanced Taylor approach, which means that we use expansion basis as defined in section 5.2:

$$\mathbb{a}_n^{\alpha}(.,x) = \sum_{|\beta|=N_{n-1}+1}^{N_n} \frac{D^{\beta}\mathbb{a}_{\alpha}(.\,,\bar{x})}{\beta!}(x-\bar{x})^{\beta}, \quad n \geq 0,\ |\alpha| \leq 2,$$

in particular we set the function N_n

$$N_0 = 0\,,$$
$$N_n = 2\,n\,,\quad n \geq 1.$$

thus in every order of our approximation formula we add two orders of the Taylor series, so from now on we refer to this particular choice as the 2n-Taylor approximation.

Using generic points of expansion we have computed formulas till the third order for a generic LSV model and till the fifth order for a generic LV model, which contain Taylor expansions up to sixth and tenth order respectively; for a generic LV model with expansion point equal to the initial point (so $\bar{x} = x$), we have reached the sixth order (so twelfth Taylor series order).

9.1 CEV local volatility model

As for any LV model, for CEV model we have computed our approximation with the Enhanced Taylor approach till fifth order, and we present here the first formulas

$$\sigma_0 = e^{(-1+\beta)\,\bar{x}}\,\delta$$

$$\sigma_1 = \frac{1}{6}\,e^{(-1+\beta)\,\bar{x}}\,(-1+\beta)\,\delta\,\Big(2\,(-k+x)^2\,(-1+\beta) + e^{2\,(-1+\beta)\,\bar{x}}\,t\,(-1+\beta)\,\delta^2 + (-k+x)\,(-3-6\,(-1+\beta)\,(x-\bar{x})) + 6\,(x-\bar{x}) + 6\,(-1+\beta)\,(x-\bar{x})^2\Big)$$

$$\sigma_2 = -\frac{1}{1440}\,e^{(-1+\beta)\,\bar{x}}\,(-1+\beta)^2\,\delta\,\Big(176\,(-k+x)^4\,(-1+\beta)^2 + 180\,e^{2\,(-1+\beta)\,\bar{x}}\,t\,\delta^2 + 3\,e^{4\,(-1+\beta)\,\bar{x}}\,t^2\,\delta^4 + 24\,e^{4\,(-1+\beta)\,\bar{x}}\,t^2\,\beta\,\delta^4 - 12\,e^{4\,(-1+\beta)\,\bar{x}}\,t^2\,\beta^2\,\delta^4 + 4\,e^{6\,(-1+\beta)\,\bar{x}}\,t^3\,\delta^6 - 8\,e^{6\,(-1+\beta)\,\bar{x}}\,t^3\,\beta\,\delta^6 + 4\,e^{6\,(-1+\beta)\,\bar{x}}\,t^3\,\beta^2\,\delta^6 - 360\,(-k+x)^3\,(-1+\beta)\,(1+2\,(-1+\beta)\,(x-\bar{x})) + 8\,r\,t\,(-k+x)\,\Big(15 + 16\,(-k+x)^2\,(-1+\beta)^2 + 2\,e^{2\,(-1+\beta)\,\bar{x}}\,t\,\delta^2 - 4\,e^{2\,(-1+\beta)\,\bar{x}}\,t\,\beta\,\delta^2 + 2\,e^{2\,(-1+\beta)\,\bar{x}}\,t\,\beta^2\,\delta^2 -$$

$$
\begin{array}{l}
\quad 30\,(-k+x)\,(-1+\beta)\,(1+2\,(-1+\beta)\,(x-\overline{x}))+60\,(-1+\beta)\,(x-\overline{x})+60\,(-1+\beta)^2\,(x-\overline{x})^2\bigr)-\\
4\,r^2\,t^2\,\bigl(15+16\,(-k+x)^2\,(-1+\beta)^2+4\,e^{2\,(-1+\beta)\,\overline{x}}\,t\,\delta^2-8\,e^{2\,(-1+\beta)\,\overline{x}}\,t\,\beta\,\delta^2+4\,e^{2\,(-1+\beta)\,\overline{x}}\,t\,\beta^2\,\delta^2-\\
\quad 30\,(-k+x)\,(-1+\beta)\,(1+2\,(-1+\beta)\,(x-\overline{x}))+60\,(-1+\beta)\,(x-\overline{x})+60\,(-1+\beta)^2\,(x-\overline{x})^2\bigr)+\\
8\,(-k+x)^2\,\bigl(45-2\,e^{2\,(-1+\beta)\,\overline{x}}\,t\,(-1+\beta)^2\,\delta^2+2\,e^{4\,(-1+\beta)\,\overline{x}}\,t^2\,(-1+\beta)^2\,\delta^4+\\
\quad 120\,(-1+\beta)\,(x-\overline{x})+120\,(-1+\beta)^2\,(x-\overline{x})^2\bigr)-30\,(-k+x)\,\bigl(e^{4\,(-1+\beta)\,\overline{x}}\,t^2\,(-1+\beta)\,\delta^4+\\
\quad 2\,\bigl(12+e^{4\,(-1+\beta)\,\overline{x}}\,t^2\,(-1+\beta)^2\,\delta^4\bigr)\,(x-\overline{x})+24\,(-1+\beta)\,(x-\overline{x})^2+16\,(-1+\beta)^2\,(x-\overline{x})^3\bigr)-\\
60\,e^{4\,(-1+\beta)\,\overline{x}}\,t^2\,\delta^4\,(x-\overline{x})+60\,e^{4\,(-1+\beta)\,\overline{x}}\,t^2\,\beta\,\delta^4\,(x-\overline{x})+720\,(x-\overline{x})^2+\\
60\,e^{4\,(-1+\beta)\,\overline{x}}\,t^2\,\delta^4\,(x-\overline{x})^2-120\,e^{4\,(-1+\beta)\,\overline{x}}\,t^2\,\beta\,\delta^4\,(x-\overline{x})^2+60\,e^{4\,(-1+\beta)\,\overline{x}}\,t^2\,\beta^2\,\delta^4\,(x-\overline{x})^2-\\
480\,(x-\overline{x})^3+480\,\beta\,(x-\overline{x})^3+240\,(x-\overline{x})^4-480\,\beta\,(x-\overline{x})^4+240\,\beta^2\,(x-\overline{x})^4\bigr)
\end{array}
$$

For the CEV model we make a comparison between our normal approximation and the 2n-Taylor approximation, but only considering the midpoint method, because we have seen that produce better results.

9.1.1 First Set: $\delta = 0.25$, $\beta = 0.8$

As in 8.1.1 we ($\delta = 0.25$, $\beta = 0.8$) and we use the same table of strikes expressed in pb

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.25	88	90	92	94	96	98	100	102	105	107	109	112	114
0.5	81	84	87	90	94	97	100	104	108	112	116	120	124
1	72	77	81	86	91	95	100	107	114	120	127	134	141
1.5	65	71	77	82	88	94	100	109	119	128	137	147	156
2	59	66	73	80	86	93	100	112	124	136	147	159	171
3	50	58	67	75	83	92	100	117	133	150	167	183	200
5	39	49	59	70	80	90	100	126	152	178	204	230	256
10	24	37	49	62	75	87	100	151	202	252	303	354	405

and the corresponding exact implied volatilities

	1	3	5	7	9	11	13
0.25	25.3216	25.2097	25.1029	25.0006	24.8789	24.7858	24.6745
0.5	25.5319	25.3511	25.1563	25.0013	24.8094	24.6320	24.4673
1	25.8331	25.5332	25.2392	25.0026	24.6763	24.4096	24.1532
1.5	26.0968	25.6632	25.3249	25.0038	24.5713	24.2247	23.9081
2	26.3482	25.8006	25.3842	25.0051	24.4708	24.0537	23.6871
3	26.7820	26.0229	25.4766	25.0075	24.3007	23.7463	23.3132
5	27.4435	26.3562	25.5748	25.0120	23.9785	23.2698	22.7327
10	28.7688	26.8523	25.7499	25.0221	23.3017	22.3472	21.6810

We now compute our 2n-Taylor approximation, and in the following table we show absolute errors

	1	3	5	7	9	11	13
0.25	1.47×10^{-5}	3.74×10^{-4}	5.88×10^{-4}	6.5×10^{-4}	5.42×10^{-4}	3.26×10^{-4}	8.14×10^{-5}
	4.1×10^{-3}	2.88×10^{-3}	2.19×10^{-3}	1.95×10^{-3}	2.22×10^{-3}	2.82×10^{-3}	$4.\times10^{-3}$
	1.04×10^{-6}	1.05×10^{-6}	1.07×10^{-6}	1.1×10^{-6}	1.06×10^{-6}	1.03×10^{-6}	1.01×10^{-6}
	6.57×10^{-8}	$6.\times10^{-7}$	7.59×10^{-7}	8.11×10^{-7}	7.56×10^{-7}	5.83×10^{-7}	3.27×10^{-8}
	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.03×10^{-6}	1.02×10^{-6}
	1.06×10^{-6}	1.05×10^{-6}	1.03×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.03×10^{-6}	1.02×10^{-6}

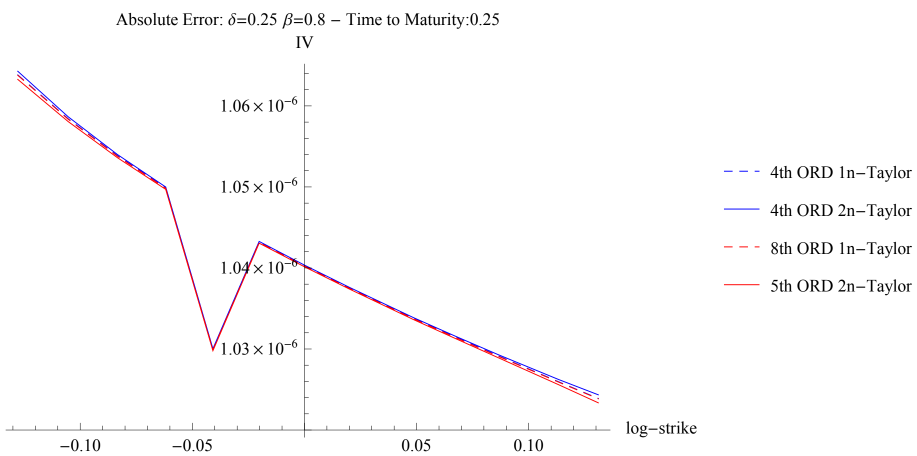
0.5	5.12×10^{-4}	5.29×10^{-4}	1.16×10^{-3}	1.29×10^{-3}	1.02×10^{-3}	3.33×10^{-4}	6.74×10^{-4}
	9.84×10^{-3}	6.54×10^{-3}	4.47×10^{-3}	3.91×10^{-3}	4.56×10^{-3}	6.46×10^{-3}	9.33×10^{-3}
	1.05×10^{-6}	1.01×10^{-6}	1.2×10^{-6}	1.26×10^{-6}	1.13×10^{-6}	9.57×10^{-7}	1.05×10^{-6}
	4.93×10^{-6}	1.3×10^{-6}	6.79×10^{-8}	1.27×10^{-7}	1.29×10^{-7}	1.4×10^{-6}	4.77×10^{-6}
	1.09×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.03×10^{-6}	1.02×10^{-6}
	1.08×10^{-6}	1.06×10^{-6}	1.05×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	1.02×10^{-6}	1.01×10^{-6}
1	1.82×10^{-3}	8.42×10^{-4}	2.27×10^{-3}	2.57×10^{-3}	1.76×10^{-3}	6.6×10^{-5}	2.45×10^{-3}
	2.26×10^{-2}	1.4×10^{-2}	9.2×10^{-3}	7.85×10^{-3}	9.66×10^{-3}	1.43×10^{-2}	2.13×10^{-2}
	1.43×10^{-6}	7.59×10^{-7}	1.58×10^{-6}	1.87×10^{-6}	1.23×10^{-6}	6.86×10^{-7}	7.85×10^{-6}
	3.13×10^{-5}	9.82×10^{-6}	3.62×10^{-6}	2.62×10^{-6}	4.14×10^{-6}	1.14×10^{-5}	4.14×10^{-5}
	1.2×10^{-6}	1.1×10^{-6}	1.07×10^{-6}	1.05×10^{-6}	1.04×10^{-6}	1.04×10^{-6}	8.83×10^{-6}
	1.02×10^{-6}	1.06×10^{-6}	1.05×10^{-6}	1.04×10^{-6}	1.02×10^{-6}	9.93×10^{-7}	9.01×10^{-6}
1.5	3.74×10^{-3}	1.2×10^{-3}	3.28×10^{-3}	3.82×10^{-3}	2.39×10^{-3}	5.22×10^{-4}	4.53×10^{-3}
	3.77×10^{-2}	2.15×10^{-2}	1.43×10^{-2}	1.18×10^{-2}	1.49×10^{-2}	2.27×10^{-2}	3.4×10^{-2}
	3.06×10^{-6}	2.6×10^{-7}	2.02×10^{-6}	2.84×10^{-6}	1.22×10^{-6}	3.56×10^{-7}	5.01×10^{-6}
	$9.\times10^{-5}$	2.44×10^{-5}	1.01×10^{-5}	7.19×10^{-6}	1.13×10^{-5}	3.14×10^{-5}	8.43×10^{-5}
	1.53×10^{-6}	1.16×10^{-6}	1.1×10^{-6}	1.07×10^{-6}	1.06×10^{-6}	1.1×10^{-6}	1.36×10^{-6}
	7.15×10^{-7}	1.02×10^{-6}	1.04×10^{-6}	1.03×10^{-6}	9.98×10^{-7}	9.15×10^{-7}	5.97×10^{-7}
2	6.34×10^{-3}	1.28×10^{-3}	4.31×10^{-3}	5.05×10^{-3}	2.86×10^{-3}	1.44×10^{-3}	7.05×10^{-3}
	5.52×10^{-2}	3.01×10^{-2}	1.94×10^{-2}	1.58×10^{-2}	2.04×10^{-2}	3.19×10^{-2}	4.75×10^{-2}
	2.24×10^{-6}	6.84×10^{-7}	$4.\times10^{-6}$	4.1×10^{-6}	2.01×10^{-6}	2.23×10^{-7}	1.11×10^{-5}
	2.02×10^{-4}	4.88×10^{-5}	2.58×10^{-5}	1.36×10^{-5}	2.11×10^{-5}	6.48×10^{-5}	1.71×10^{-4}
	2.72×10^{-6}	1.3×10^{-6}	5.43×10^{-6}	1.11×10^{-6}	2.15×10^{-6}	1.27×10^{-6}	2.04×10^{-6}
	5.3×10^{-6}	9.14×10^{-7}	5.58×10^{-6}	$1.\times10^{-6}$	$2.\times10^{-6}$	7.31×10^{-7}	1.66×10^{-7}
3	1.23×10^{-2}	1.42×10^{-3}	6.4×10^{-3}	7.46×10^{-3}	3.57×10^{-3}	3.99×10^{-3}	1.26×10^{-2}
	9.37×10^{-2}	4.77×10^{-2}	2.96×10^{-2}	2.38×10^{-2}	3.17×10^{-2}	5.16×10^{-2}	7.53×10^{-2}
	2.57×10^{-5}	3.88×10^{-6}	3.85×10^{-6}	7.31×10^{-6}	2.87×10^{-7}	1.75×10^{-6}	3.42×10^{-5}
	5.7×10^{-4}	1.23×10^{-4}	4.56×10^{-5}	3.19×10^{-5}	5.5×10^{-5}	1.8×10^{-4}	4.5×10^{-4}
	7.23×10^{-6}	1.87×10^{-6}	1.36×10^{-6}	1.26×10^{-6}	1.31×10^{-6}	2.16×10^{-6}	5.49×10^{-6}
	5.36×10^{-6}	3.92×10^{-7}	8.51×10^{-7}	8.95×10^{-7}	7.62×10^{-7}	2.46×10^{-7}	$4.\times10^{-6}$
5	2.49×10^{-2}	1.74×10^{-3}	1.07×10^{-2}	1.2×10^{-2}	3.68×10^{-3}	9.9×10^{-3}	2.43×10^{-2}
	1.75×10^{-1}	8.37×10^{-2}	4.92×10^{-2}	$4.\times10^{-2}$	5.63×10^{-2}	9.14×10^{-2}	1.31×10^{-1}
	9.57×10^{-5}	1.73×10^{-5}	6.75×10^{-6}	1.51×10^{-5}	7.46×10^{-6}	1.47×10^{-5}	1.29×10^{-4}
	1.95×10^{-3}	3.72×10^{-4}	1.26×10^{-4}	9.06×10^{-5}	1.79×10^{-4}	5.98×10^{-4}	1.43×10^{-3}
	3.9×10^{-5}	5.08×10^{-6}	2.4×10^{-6}	2.01×10^{-6}	2.54×10^{-6}	8.01×10^{-6}	2.68×10^{-5}
	4.01×10^{-5}	2.63×10^{-6}	3.08×10^{-8}	3.19×10^{-7}	3.93×10^{-7}	6.59×10^{-6}	2.77×10^{-5}
10	6.61×10^{-2}	3.78×10^{-3}	2.03×10^{-2}	2.21×10^{-2}	9.09×10^{-4}	2.94×10^{-2}	5.57×10^{-2}
	4.22×10^{-1}	1.71×10^{-1}	$1.\times10^{-1}$	8.21×10^{-2}	1.24×10^{-1}	1.96×10^{-1}	2.66×10^{-1}
	5.62×10^{-4}	1.15×10^{-4}	7.02×10^{-6}	2.5×10^{-5}	4.9×10^{-5}	1.61×10^{-4}	7.01×10^{-4}
	1.06×10^{-2}	1.44×10^{-3}	5.03×10^{-4}	3.66×10^{-4}	9.21×10^{-4}	$3.\times10^{-3}$	6.39×10^{-3}
	4.76×10^{-4}	3.33×10^{-5}	1.21×10^{-5}	8.83×10^{-6}	1.68×10^{-5}	8.11×10^{-5}	2.54×10^{-4}
	5.24×10^{-4}	2.86×10^{-5}	7.68×10^{-6}	5.02×10^{-6}	1.44×10^{-5}	8.72×10^{-5}	2.84×10^{-4}

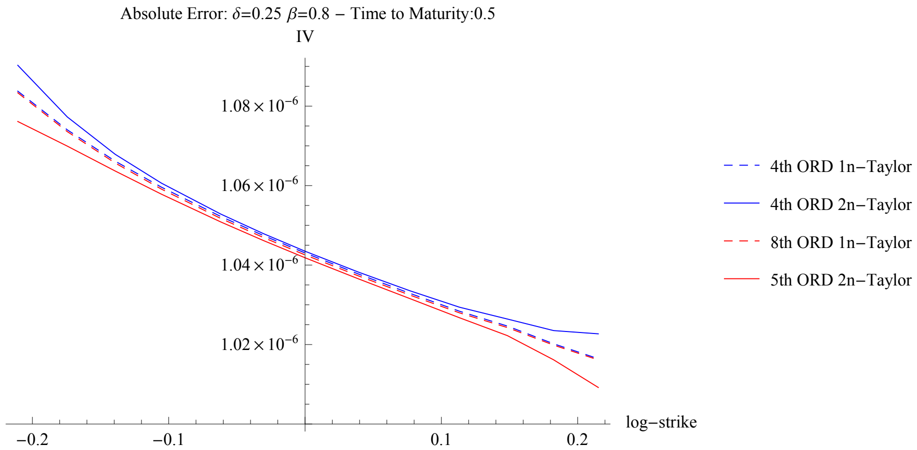
In order to make a comparison between 2n-Taylor approximation and the normal one, we create the following matrix, where every element has the first column referring to our normal approximation, while the second column corresponds to the 2n-Taylor approximation.

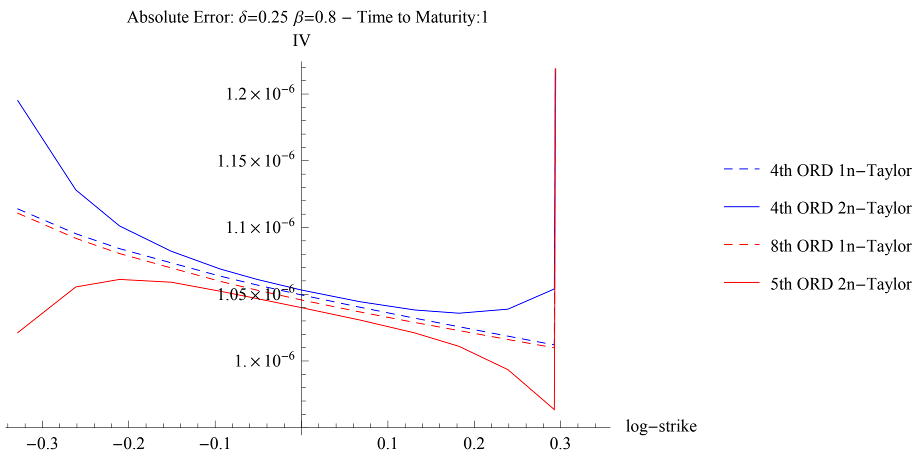
	1		7		13	
	1.47×10^{-5}	1.47×10^{-5}	6.5×10^{-4}	6.5×10^{-4}	8.14×10^{-5}	8.14×10^{-5}
	1.47×10^{-5}	4.1×10^{-3}	6.5×10^{-4}	1.95×10^{-3}	8.14×10^{-5}	$4.\times10^{-3}$
	1.16×10^{-6}	1.04×10^{-6}	1.18×10^{-6}	1.1×10^{-6}	1.11×10^{-6}	1.01×10^{-6}
	1.16×10^{-6}	6.57×10^{-8}	1.18×10^{-6}	8.11×10^{-7}	1.11×10^{-6}	3.27×10^{-8}
0.25	1.06×10^{-6}	1.06×10^{-6}	1.04×10^{-6}	1.04×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.06×10^{-6}	1.06×10^{-6}	1.04×10^{-6}	1.04×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.06×10^{-6}	0	1.04×10^{-6}	0	1.02×10^{-6}	0
	1.06×10^{-6}	0	1.04×10^{-6}	0	1.02×10^{-6}	0
	1.06×10^{-6}	0	1.04×10^{-6}	0	1.02×10^{-6}	0
	5.12×10^{-4}	5.12×10^{-4}	1.29×10^{-3}	1.29×10^{-3}	6.74×10^{-4}	6.74×10^{-4}
	5.12×10^{-4}	9.84×10^{-3}	1.29×10^{-3}	3.91×10^{-3}	6.74×10^{-4}	9.33×10^{-3}
	1.45×10^{-6}	1.05×10^{-6}	1.58×10^{-6}	1.26×10^{-6}	1.3×10^{-6}	1.05×10^{-6}
	1.45×10^{-6}	4.93×10^{-6}	1.58×10^{-6}	1.27×10^{-7}	1.3×10^{-6}	4.77×10^{-6}
0.5	1.08×10^{-6}	1.09×10^{-6}	1.04×10^{-6}	1.04×10^{-6}	1.02×10^{-6}	1.02×10^{-6}
	1.08×10^{-6}	1.08×10^{-6}	1.04×10^{-6}	1.04×10^{-6}	1.02×10^{-6}	1.01×10^{-6}
	1.08×10^{-6}	0	1.04×10^{-6}	0	1.02×10^{-6}	0
	1.08×10^{-6}	0	1.04×10^{-6}	0	1.02×10^{-6}	0
	1.08×10^{-6}	0	1.04×10^{-6}	0	1.02×10^{-6}	0
	1.82×10^{-3}	1.82×10^{-3}	2.57×10^{-3}	2.57×10^{-3}	2.45×10^{-3}	2.45×10^{-3}
	1.82×10^{-3}	2.26×10^{-2}	2.57×10^{-3}	7.85×10^{-3}	2.45×10^{-3}	2.13×10^{-2}
	2.55×10^{-6}	1.43×10^{-6}	3.15×10^{-6}	1.87×10^{-6}	7.94×10^{-6}	7.85×10^{-6}
	2.55×10^{-6}	3.13×10^{-5}	3.15×10^{-6}	2.62×10^{-6}	7.94×10^{-6}	4.14×10^{-5}
1	1.11×10^{-6}	1.2×10^{-6}	1.05×10^{-6}	1.05×10^{-6}	8.92×10^{-6}	8.83×10^{-6}
	1.11×10^{-6}	1.02×10^{-6}	1.05×10^{-6}	1.04×10^{-6}	8.92×10^{-6}	9.01×10^{-6}
	1.11×10^{-6}	0	1.05×10^{-6}	0	8.92×10^{-6}	0
	1.11×10^{-6}	0	1.05×10^{-6}	0	8.92×10^{-6}	0
	1.11×10^{-6}	0	1.05×10^{-6}	0	8.92×10^{-6}	0
	3.74×10^{-3}	3.74×10^{-3}	3.82×10^{-3}	3.82×10^{-3}	4.53×10^{-3}	4.53×10^{-3}
	3.74×10^{-3}	3.77×10^{-2}	3.82×10^{-3}	1.18×10^{-2}	4.53×10^{-3}	3.4×10^{-2}
	4.33×10^{-6}	3.06×10^{-6}	5.67×10^{-6}	2.84×10^{-6}	3.03×10^{-6}	5.01×10^{-6}
	4.33×10^{-6}	$9.\times10^{-5}$	5.67×10^{-6}	7.19×10^{-6}	3.03×10^{-6}	8.43×10^{-5}
1.5	1.14×10^{-6}	1.53×10^{-6}	1.06×10^{-6}	1.07×10^{-6}	$1.\times10^{-6}$	1.36×10^{-6}
	1.14×10^{-6}	7.15×10^{-7}	1.06×10^{-6}	1.03×10^{-6}	$1.\times10^{-6}$	5.97×10^{-7}
	1.13×10^{-6}	0	1.05×10^{-6}	0	9.95×10^{-7}	0
	1.13×10^{-6}	0	1.05×10^{-6}	0	9.95×10^{-7}	0
	1.13×10^{-6}	0	1.05×10^{-6}	0	9.95×10^{-7}	0
	6.34×10^{-3}	6.34×10^{-3}	5.05×10^{-3}	5.05×10^{-3}	7.05×10^{-3}	7.05×10^{-3}
	6.34×10^{-3}	5.52×10^{-2}	5.05×10^{-3}	1.58×10^{-2}	7.05×10^{-3}	4.75×10^{-2}
	1.71×10^{-6}	2.24×10^{-6}	9.09×10^{-6}	4.1×10^{-6}	4.44×10^{-6}	1.11×10^{-5}
	1.71×10^{-6}	2.02×10^{-4}	9.09×10^{-6}	1.36×10^{-5}	4.44×10^{-6}	1.71×10^{-4}
2	3.93×10^{-6}	2.72×10^{-6}	1.08×10^{-6}	1.11×10^{-6}	9.97×10^{-7}	2.04×10^{-6}
	3.93×10^{-6}	5.3×10^{-6}	1.08×10^{-6}	$1.\times10^{-6}$	9.97×10^{-7}	1.66×10^{-7}
	3.96×10^{-6}	0	1.05×10^{-6}	0	9.86×10^{-7}	0
	3.96×10^{-6}	0	1.05×10^{-6}	0	9.86×10^{-7}	0
	3.96×10^{-6}	0	1.05×10^{-6}	0	9.86×10^{-7}	0

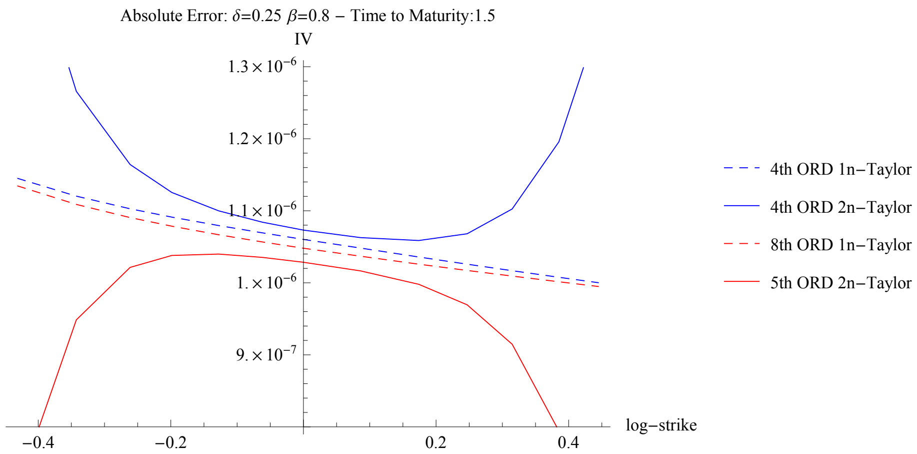
	1.23×10^{-2}	1.23×10^{-2}	7.46×10^{-3}	7.46×10^{-3}	1.26×10^{-2}	1.26×10^{-2}
	1.23×10^{-2}	9.37×10^{-2}	7.46×10^{-3}	2.38×10^{-2}	1.26×10^{-2}	7.53×10^{-2}
	1.41×10^{-5}	2.57×10^{-5}	1.83×10^{-5}	7.31×10^{-6}	8.45×10^{-6}	3.42×10^{-5}
	1.41×10^{-5}	5.7×10^{-4}	1.83×10^{-5}	3.19×10^{-5}	8.45×10^{-6}	4.5×10^{-4}
3	1.27×10^{-6}	7.23×10^{-6}	1.14×10^{-6}	1.26×10^{-6}	9.99×10^{-7}	5.49×10^{-6}
	1.27×10^{-6}	5.36×10^{-6}	1.14×10^{-6}	8.95×10^{-7}	9.99×10^{-7}	$4.\times10^{-6}$
	1.19×10^{-6}	0	1.05×10^{-6}	0	9.72×10^{-7}	0
	1.19×10^{-6}	0	1.05×10^{-6}	0	9.72×10^{-7}	0
	1.19×10^{-6}	0	1.05×10^{-6}	0	9.72×10^{-7}	0
	2.49×10^{-2}	2.49×10^{-2}	1.2×10^{-2}	1.2×10^{-2}	2.43×10^{-2}	2.43×10^{-2}
	2.49×10^{-2}	1.75×10^{-1}	1.2×10^{-2}	$4.\times10^{-2}$	2.43×10^{-2}	1.31×10^{-1}
	3.68×10^{-5}	9.57×10^{-5}	4.43×10^{-5}	1.51×10^{-5}	2.13×10^{-5}	1.29×10^{-4}
	3.68×10^{-5}	1.95×10^{-3}	4.43×10^{-5}	9.06×10^{-5}	2.13×10^{-5}	1.43×10^{-3}
5	1.52×10^{-6}	3.9×10^{-5}	1.38×10^{-6}	2.01×10^{-6}	1.03×10^{-6}	2.68×10^{-5}
	1.52×10^{-6}	4.01×10^{-5}	1.38×10^{-6}	3.19×10^{-7}	1.03×10^{-6}	2.77×10^{-5}
	1.24×10^{-6}	0	1.06×10^{-6}	0	9.5×10^{-7}	0
	1.24×10^{-6}	0	1.06×10^{-6}	0	9.5×10^{-7}	0
	1.24×10^{-6}	0	1.06×10^{-6}	0	9.49×10^{-7}	0
	6.61×10^{-2}	6.61×10^{-2}	2.21×10^{-2}	2.21×10^{-2}	5.57×10^{-2}	5.57×10^{-2}
	6.61×10^{-2}	4.22×10^{-1}	2.21×10^{-2}	8.21×10^{-2}	5.57×10^{-2}	2.66×10^{-1}
	1.15×10^{-4}	5.62×10^{-4}	1.28×10^{-4}	2.5×10^{-5}	8.55×10^{-5}	7.01×10^{-4}
	1.15×10^{-4}	1.06×10^{-2}	1.28×10^{-4}	3.66×10^{-4}	8.55×10^{-5}	6.39×10^{-3}
10	8.93×10^{-7}	4.76×10^{-4}	2.26×10^{-6}	8.83×10^{-6}	1.1×10^{-6}	2.54×10^{-4}
	8.93×10^{-7}	5.24×10^{-4}	2.26×10^{-6}	5.02×10^{-6}	1.1×10^{-6}	2.84×10^{-4}
	1.21×10^{-6}	0	1.07×10^{-6}	0	9.1×10^{-7}	0
	1.21×10^{-6}	0	1.07×10^{-6}	0	9.1×10^{-7}	0
	1.31×10^{-6}	0	1.06×10^{-6}	0	9.08×10^{-7}	0

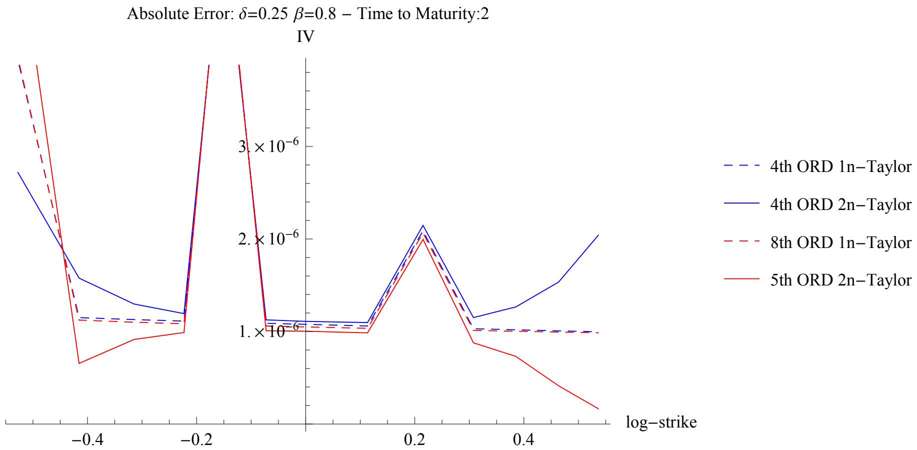
In the following plots we compare the 2n-Taylor approximation at the fourth order with our normal approximation at the eighth order because they are equivalent from a computational point of view due to the fact that they contain the same Taylor expansions; we also insert the fifth order of the 2n-Taylor approx. which is our maximum order in the enhanced Taylor approach, and the fourth order of our normal approximation.

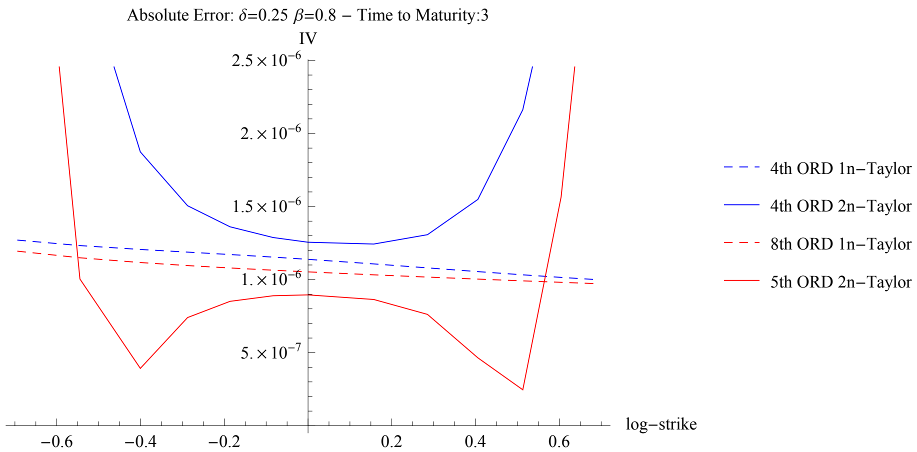


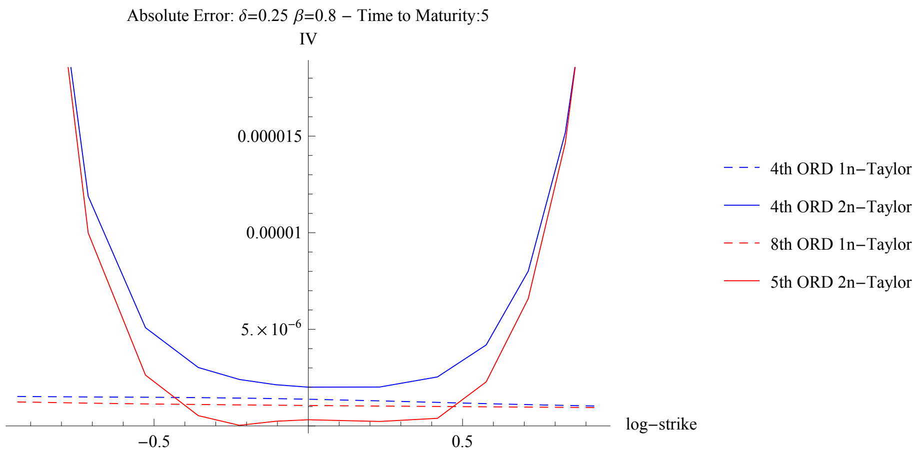

Absolute Error: δ=0.25 β=0.8 – Time to Maturity:0.5
IV
1.08 × 10^−6
1.06 × 10^−6
1.04 × 10^−6
1.02 × 10^−6
−0.2
−0.1
0.1
0.2
log−strike
4th ORD 1n−Taylor
4th ORD 2n−Taylor
8th ORD 1n−Taylor
5th ORD 2n−Taylor

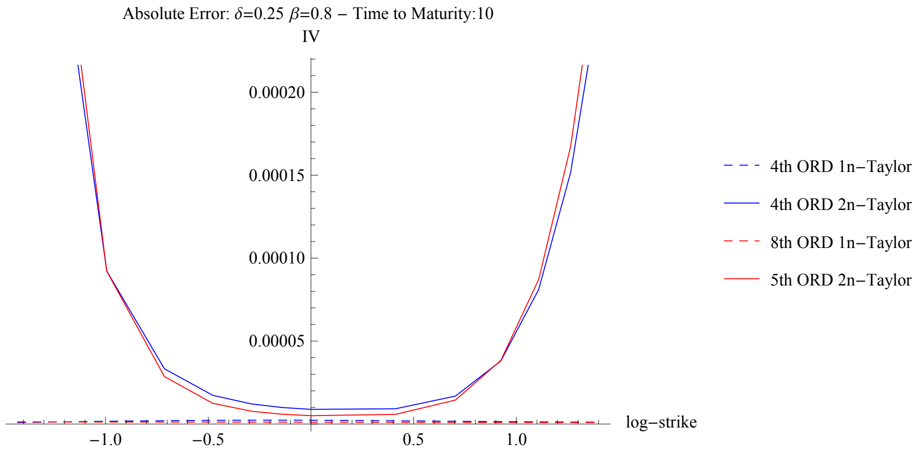

Absolute Error: δ=0.25 β=0.8 – Time to Maturity:1
IV
1.2 × 10^−6
1.15 × 10^−6
1.1 × 10^−6
1.05 × 10^−6
1. × 10^−6
−0.3
−0.2
−0.1
0.1
0.2
0.3
log−strike
4th ORD 1n−Taylor
4th ORD 2n−Taylor
8th ORD 1n−Taylor
5th ORD 2n−Taylor


Absolute Error: δ=0.25 β=0.8 – Time to Maturity:1.5
IV
1.3 × 10^−6
1.2 × 10^−6
1.1 × 10^−6
1. × 10^−6
9. × 10^−7
−0.4
−0.2
0.2
0.4
log−strike
4th ORD 1n−Taylor
4th ORD 2n−Taylor
8th ORD 1n−Taylor
5th ORD 2n−Taylor


Absolute Error: δ=0.25 β=0.8 – Time to Maturity:2
IV
3. × 10⁻⁶
2. × 10⁻⁶
1. × 10⁻⁶
log–strike
–0.4
–0.2
0.2
0.4
4th ORD 1n–Taylor
4th ORD 2n–Taylor
8th ORD 1n–Taylor
5th ORD 2n–Taylor


Absolute Error: δ=0.25 β=0.8 – Time to Maturity:3
IV
2.5 × 10⁻⁶
2. × 10⁻⁶
1.5 × 10⁻⁶
1. × 10⁻⁶
5. × 10⁻⁷
log–strike
–0.6
–0.4
–0.2
0.2
0.4
0.6
4th ORD 1n–Taylor
4th ORD 2n–Taylor
8th ORD 1n–Taylor
5th ORD 2n–Taylor


Absolute Error: δ=0.25 β=0.8 – Time to Maturity:5
IV
0.000015
0.00001
5. × 10⁻⁶
log–strike
–0.5
0.5
4th ORD 1n–Taylor
4th ORD 2n–Taylor
8th ORD 1n–Taylor
5th ORD 2n–Taylor



We see that the approximation 2n-Taylor works well for short-term maturities, but for longer maturities can not compete with our normal approximation: in particular the fourth order of the 2n-Taylor can not compete with the eighth order the 1n-Taylor (which contains the same factors for the Taylor series), but neither with the fourth order of the 1n-Taylor.

9.1.2 Second Set: $\delta = 0.25$, $\beta = 0.2$

As in 8.1.2 we have ($\delta = 0.25$, $\beta = 0.2$), we consider the following table of strikes

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.25	87	89	91	94	96	98	100	102	104	106	109	111	113
0.5	80	83	87	90	93	97	100	104	107	111	115	118	122
1	68	73	79	84	89	95	100	106	112	118	123	129	135
1.5	58	65	72	79	86	93	100	108	116	124	131	139	147
2	51	59	67	76	84	92	100	110	119	128	138	148	157
3	38	48	59	69	79	90	100	112	125	138	150	162	175
5	20	33	47	60	73	87	100	118	135	152	170	188	205
10	7	22	38	54	69	84	100	128	156	184	211	239	267

and the corresponding matrix of exact implied volatilities

	1	3	5	7	9	11	13
0.25	26.4308	25.9666	25.4214	25.0104	24.6198	24.1575	23.8067
0.5	27.3249	26.4430	25.7554	25.0208	24.3487	23.6460	23.0805
1	29.1198	27.4861	26.2312	25.0416	23.9201	23.0193	22.1477
1.5	30.9581	28.5202	26.6130	25.0623	23.5972	22.4417	21.3837
2	32.5131	29.3502	26.8859	25.0829	23.3680	21.9732	20.8075
3	36.2697	30.8728	27.5945	25.1239	22.9295	21.2395	19.8819
5	45.1832	33.7623	28.5717	25.2046	22.2614	20.1746	18.5885
10	56.7246	36.4540	29.4028	25.3780	21.0481	18.4309	16.5685

Now we compute our 2n-Taylor approximation and absolute errors from exact values

	1	3	5	7	9	11	13
0.25	1.38×10^{-3}	5.49×10^{-3}	9.8×10^{-3}	1.04×10^{-2}	8.92×10^{-3}	4.6×10^{-3}	$5.\times10^{-4}$
	7.8×10^{-2}	5.35×10^{-2}	3.62×10^{-2}	3.13×10^{-2}	3.28×10^{-2}	4.25×10^{-2}	5.54×10^{-2}
	$5.\times10^{-6}$	7.36×10^{-7}	1.22×10^{-5}	1.53×10^{-5}	1.05×10^{-5}	2.72×10^{-8}	2.91×10^{-6}
	3.5×10^{-4}	1.52×10^{-4}	7.24×10^{-5}	5.76×10^{-5}	6.15×10^{-5}	1.03×10^{-4}	1.95×10^{-4}
	4.28×10^{-6}	2.28×10^{-6}	1.66×10^{-6}	1.52×10^{-6}	1.5×10^{-6}	1.7×10^{-6}	2.39×10^{-6}
	2.14×10^{-6}	4.55×10^{-9}	5.82×10^{-7}	6.69×10^{-7}	6.43×10^{-7}	3.72×10^{-7}	4.37×10^{-7}
0.5	9.18×10^{-3}	1.09×10^{-2}	1.91×10^{-2}	2.08×10^{-2}	1.62×10^{-2}	5.26×10^{-3}	7.99×10^{-3}
	1.91×10^{-1}	1.15×10^{-1}	7.91×10^{-2}	6.25×10^{-2}	6.66×10^{-2}	8.98×10^{-2}	1.22×10^{-1}
	1.01×10^{-5}	9.45×10^{-6}	3.91×10^{-5}	5.81×10^{-5}	3.03×10^{-5}	1.11×10^{-5}	4.45×10^{-6}
	2.11×10^{-3}	6.98×10^{-4}	3.39×10^{-4}	2.34×10^{-4}	2.54×10^{-4}	4.87×10^{-4}	1.05×10^{-3}
	4.52×10^{-5}	1.24×10^{-5}	6.67×10^{-6}	4.89×10^{-6}	4.77×10^{-6}	7.33×10^{-6}	1.75×10^{-5}
	4.55×10^{-5}	9.31×10^{-6}	3.51×10^{-6}	1.98×10^{-6}	2.05×10^{-6}	5.09×10^{-6}	1.67×10^{-5}
1	5.02×10^{-2}	1.42×10^{-2}	3.83×10^{-2}	4.16×10^{-2}	2.81×10^{-2}	6.04×10^{-3}	2.44×10^{-2}
	5.46×10^{-1}	2.88×10^{-1}	1.72×10^{-1}	1.25×10^{-1}	1.34×10^{-1}	1.77×10^{-1}	2.47×10^{-1}
	3.27×10^{-4}	1.11×10^{-4}	1.39×10^{-4}	2.28×10^{-4}	7.74×10^{-5}	5.07×10^{-5}	1.1×10^{-4}
	1.72×10^{-2}	4.35×10^{-3}	1.58×10^{-3}	9.38×10^{-4}	1.04×10^{-3}	1.97×10^{-3}	4.66×10^{-3}
	9.57×10^{-4}	1.56×10^{-4}	5.47×10^{-5}	3.22×10^{-5}	$3.\times10^{-5}$	5.07×10^{-5}	1.56×10^{-4}
	1.07×10^{-3}	1.52×10^{-4}	4.42×10^{-5}	2.38×10^{-5}	2.35×10^{-5}	4.83×10^{-5}	1.7×10^{-4}
1.5	1.28×10^{-1}	9.51×10^{-3}	5.83×10^{-2}	6.23×10^{-2}	3.82×10^{-2}	1.22×10^{-3}	4.58×10^{-2}
	1.1	5.25×10^{-1}	2.73×10^{-1}	1.88×10^{-1}	1.99×10^{-1}	2.67×10^{-1}	3.73×10^{-1}
	2.4×10^{-3}	3.6×10^{-4}	3.03×10^{-4}	5.09×10^{-4}	1.27×10^{-4}	1.09×10^{-4}	4.62×10^{-4}
	6.79×10^{-2}	1.43×10^{-2}	3.91×10^{-3}	2.11×10^{-3}	2.31×10^{-3}	4.74×10^{-3}	1.14×10^{-2}
	7.18×10^{-3}	8.52×10^{-4}	2.07×10^{-4}	1.08×10^{-4}	9.47×10^{-5}	1.79×10^{-4}	5.96×10^{-4}
	8.46×10^{-3}	8.82×10^{-4}	1.77×10^{-4}	8.52×10^{-5}	8.04×10^{-5}	1.82×10^{-4}	6.72×10^{-4}
2	2.14×10^{-1}	6.78×10^{-3}	8.01×10^{-2}	8.29×10^{-2}	4.84×10^{-2}	4.77×10^{-3}	6.53×10^{-2}
	1.75	7.83×10^{-1}	3.74×10^{-1}	2.5×10^{-1}	2.6×10^{-1}	3.53×10^{-1}	4.86×10^{-1}
	6.44×10^{-3}	7.56×10^{-4}	5.64×10^{-4}	8.98×10^{-4}	1.94×10^{-4}	1.61×10^{-4}	9.94×10^{-4}
	1.65×10^{-1}	3.12×10^{-2}	7.27×10^{-3}	3.76×10^{-3}	$4.\times10^{-3}$	8.63×10^{-3}	2.02×10^{-2}
	2.65×10^{-2}	2.66×10^{-3}	5.26×10^{-4}	2.57×10^{-4}	2.13×10^{-4}	4.29×10^{-4}	1.42×10^{-3}
	3.26×10^{-2}	2.85×10^{-3}	4.6×10^{-4}	2.09×10^{-4}	1.86×10^{-4}	4.52×10^{-4}	1.63×10^{-3}
3	5.45×10^{-1}	1.65×10^{-3}	1.23×10^{-1}	1.24×10^{-1}	6.43×10^{-2}	1.76×10^{-2}	1.04×10^{-1}
	3.98	1.4	6.22×10^{-1}	3.76×10^{-1}	3.79×10^{-1}	5.11×10^{-1}	6.93×10^{-1}
	3.6×10^{-2}	2.35×10^{-3}	1.11×10^{-3}	1.98×10^{-3}	2.8×10^{-4}	2.23×10^{-4}	2.7×10^{-3}
	7.8×10^{-1}	9.59×10^{-2}	1.96×10^{-2}	8.5×10^{-3}	8.67×10^{-3}	1.92×10^{-2}	4.42×10^{-2}
	2.5×10^{-1}	1.38×10^{-2}	2.24×10^{-3}	8.82×10^{-4}	6.64×10^{-4}	1.4×10^{-3}	4.67×10^{-3}
	3.48×10^{-1}	1.57×10^{-2}	2.07×10^{-3}	7.46×10^{-4}	6.05×10^{-4}	1.54×10^{-3}	5.54×10^{-3}
5	2.41	5.21×10^{-2}	2.18×10^{-1}	2.05×10^{-1}	8.93×10^{-2}	4.45×10^{-2}	1.72×10^{-1}
	1.47×10^{1}	3.14	1.15	6.29×10^{-1}	5.99×10^{-1}	7.89×10^{-1}	1.04
	1.28×10^{-1}	2.44×10^{-2}	1.72×10^{-3}	5.09×10^{-3}	2.97×10^{-4}	1.15×10^{-4}	7.95×10^{-3}
	8.35	4.52×10^{-1}	6.58×10^{-2}	2.39×10^{-2}	2.24×10^{-2}	4.94×10^{-2}	1.09×10^{-1}
	7.06	1.19×10^{-1}	1.26×10^{-2}	4.14×10^{-3}	2.68×10^{-3}	5.76×10^{-3}	1.85×10^{-2}
	1.37×10^{1}	1.74×10^{-1}	1.43×10^{-2}	3.89×10^{-3}	2.6×10^{-3}	6.66×10^{-3}	2.32×10^{-2}

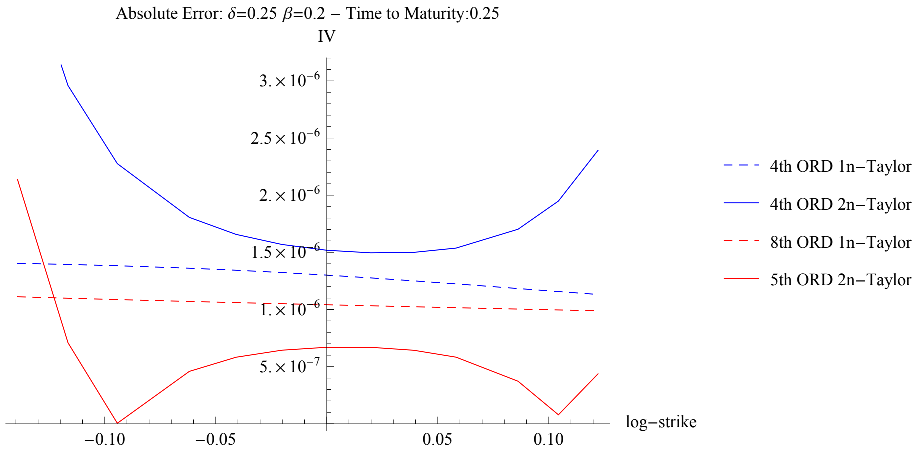
	1.57×10^{1}	3.61×10^{-1}	4.03×10^{-1}	3.78×10^{-1}	1.22×10^{-1}	1.14×10^{-1}	3.1×10^{-1}
	8.35×10^{1}	7.52	2.41	1.29	1.08	1.35	1.69
10	9.32	5.42×10^{-1}	6.77×10^{-2}	6.07×10^{-5}	2.54×10^{-3}	1.75×10^{-3}	2.91×10^{-2}
	9.67×10^{1}	2.33	3.41×10^{-1}	1.15×10^{-1}	7.87×10^{-2}	1.63×10^{-1}	3.33×10^{-1}
	6.41×10^{2}	1.09	6.15×10^{-2}	2.2×10^{-2}	1.54×10^{-2}	3.48×10^{-2}	1.04×10^{-1}
	1.33×10^{3}	2.13	2.01×10^{-1}	5.06×10^{-2}	1.89×10^{-2}	4.43×10^{-2}	1.43×10^{-1}

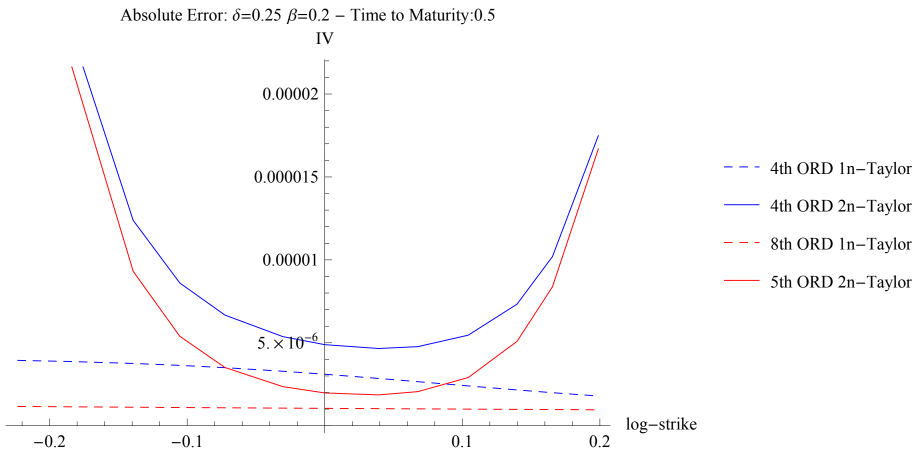
then we make a comparison with our 1n-Taylor approximation

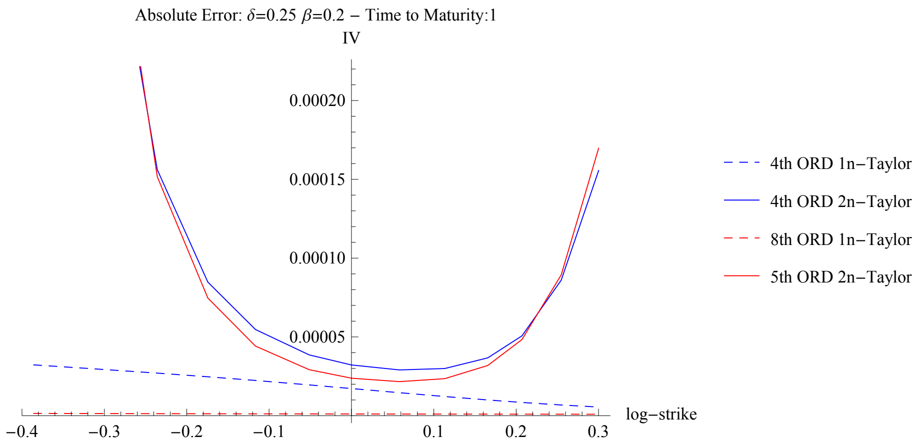
	1		7		13	
	1.38×10^{-3}	1.38×10^{-3}	1.04×10^{-2}	1.04×10^{-2}	$5.\times10^{-4}$	$5.\times10^{-4}$
	1.38×10^{-3}	7.8×10^{-2}	1.04×10^{-2}	3.13×10^{-2}	$5.\times10^{-4}$	5.54×10^{-2}
	3.2×10^{-5}	$5.\times10^{-6}$	3.61×10^{-5}	1.53×10^{-5}	1.96×10^{-5}	2.91×10^{-6}
	3.2×10^{-5}	3.5×10^{-4}	3.61×10^{-5}	5.76×10^{-5}	1.96×10^{-5}	1.95×10^{-4}
0.25	1.4×10^{-6}	4.28×10^{-6}	1.3×10^{-6}	1.52×10^{-6}	1.13×10^{-6}	2.39×10^{-6}
	1.4×10^{-6}	2.14×10^{-6}	1.3×10^{-6}	6.69×10^{-7}	1.13×10^{-6}	4.37×10^{-7}
	1.12×10^{-6}	0	1.04×10^{-6}	0	9.9×10^{-7}	0
	1.12×10^{-6}	0	1.04×10^{-6}	0	9.9×10^{-7}	0
	1.11×10^{-6}	0	1.04×10^{-6}	0	9.89×10^{-7}	0
	9.18×10^{-3}	9.18×10^{-3}	2.08×10^{-2}	2.08×10^{-2}	7.99×10^{-3}	7.99×10^{-3}
	9.18×10^{-3}	1.91×10^{-1}	2.08×10^{-2}	6.25×10^{-2}	7.99×10^{-3}	1.22×10^{-1}
	1.37×10^{-4}	1.01×10^{-5}	1.41×10^{-4}	5.81×10^{-5}	5.76×10^{-5}	4.45×10^{-6}
	1.37×10^{-4}	2.11×10^{-3}	1.41×10^{-4}	2.34×10^{-4}	5.76×10^{-5}	1.05×10^{-3}
0.5	3.94×10^{-6}	4.52×10^{-5}	3.1×10^{-6}	4.89×10^{-6}	1.79×10^{-6}	1.75×10^{-5}
	3.94×10^{-6}	4.55×10^{-5}	3.1×10^{-6}	1.98×10^{-6}	1.79×10^{-6}	1.67×10^{-5}
	1.24×10^{-6}	0	1.09×10^{-6}	0	9.78×10^{-7}	0
	1.24×10^{-6}	0	1.09×10^{-6}	0	9.78×10^{-7}	0
	1.16×10^{-6}	0	1.05×10^{-6}	0	9.6×10^{-7}	0
	5.02×10^{-2}	5.02×10^{-2}	4.16×10^{-2}	4.16×10^{-2}	2.44×10^{-2}	2.44×10^{-2}
	5.02×10^{-2}	5.46×10^{-1}	4.16×10^{-2}	1.25×10^{-1}	2.44×10^{-2}	2.47×10^{-1}
	6.8×10^{-4}	3.27×10^{-4}	5.55×10^{-4}	2.28×10^{-4}	1.73×10^{-4}	1.1×10^{-4}
	6.8×10^{-4}	1.72×10^{-2}	5.55×10^{-4}	9.38×10^{-4}	1.73×10^{-4}	4.66×10^{-3}
1	3.23×10^{-5}	9.57×10^{-4}	1.72×10^{-5}	3.22×10^{-5}	5.5×10^{-6}	1.56×10^{-4}
	3.23×10^{-5}	1.07×10^{-3}	1.72×10^{-5}	2.38×10^{-5}	5.5×10^{-6}	1.7×10^{-4}
	3.43×10^{-6}	0	1.81×10^{-6}	0	1.11×10^{-6}	0
	3.43×10^{-6}	0	1.81×10^{-6}	0	1.11×10^{-6}	0
	1.46×10^{-6}	0	1.1×10^{-6}	0	9.33×10^{-7}	0
	1.28×10^{-1}	1.28×10^{-1}	6.23×10^{-2}	6.23×10^{-2}	4.58×10^{-2}	4.58×10^{-2}
	1.28×10^{-1}	1.1	6.23×10^{-2}	1.88×10^{-1}	4.58×10^{-2}	3.73×10^{-1}
	1.99×10^{-3}	2.4×10^{-3}	1.24×10^{-3}	5.09×10^{-4}	3.17×10^{-4}	4.62×10^{-4}
	1.99×10^{-3}	6.79×10^{-2}	1.24×10^{-3}	2.11×10^{-3}	3.17×10^{-4}	1.14×10^{-2}
1.5	1.48×10^{-4}	7.18×10^{-3}	5.47×10^{-5}	1.08×10^{-4}	1.21×10^{-5}	5.96×10^{-4}
	1.48×10^{-4}	8.46×10^{-3}	5.47×10^{-5}	8.52×10^{-5}	1.21×10^{-5}	6.72×10^{-4}
	1.87×10^{-5}	0	4.77×10^{-6}	0	1.55×10^{-6}	0
	1.87×10^{-5}	0	4.77×10^{-6}	0	1.55×10^{-6}	0
	4.05×10^{-6}	0	1.41×10^{-6}	0	9.43×10^{-7}	0

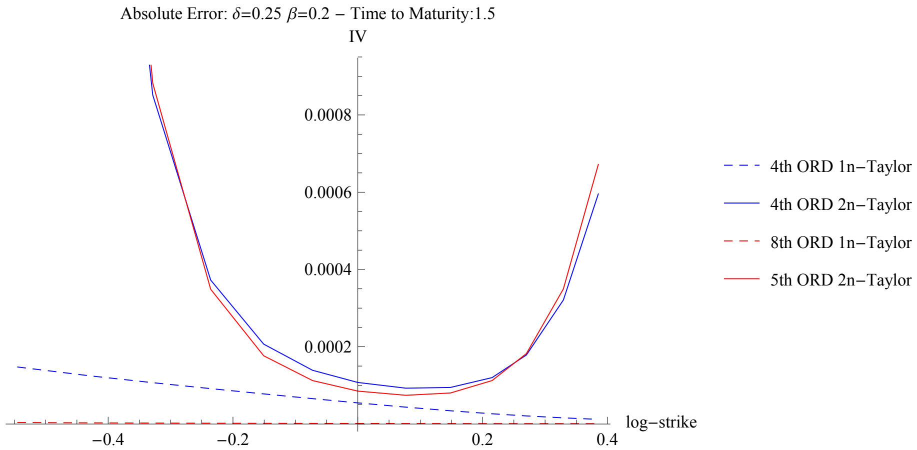
	2.14×10^{-1}	2.14×10^{-1}	8.29×10^{-2}	8.29×10^{-2}	6.53×10^{-2}	6.53×10^{-2}
	2.14×10^{-1}	1.75	8.29×10^{-2}	2.5×10^{-1}	6.53×10^{-2}	4.86×10^{-1}
	4.43×10^{-3}	6.44×10^{-3}	2.18×10^{-3}	8.98×10^{-4}	4.9×10^{-4}	9.94×10^{-4}
	4.43×10^{-3}	1.65×10^{-1}	2.18×10^{-3}	3.76×10^{-3}	4.9×10^{-4}	2.02×10^{-2}
2	4.58×10^{-4}	2.65×10^{-2}	1.26×10^{-4}	2.57×10^{-4}	2.19×10^{-5}	1.42×10^{-3}
	4.58×10^{-4}	3.26×10^{-2}	1.26×10^{-4}	2.09×10^{-4}	2.19×10^{-5}	1.63×10^{-3}
	7.74×10^{-5}	0	1.24×10^{-5}	0	2.42×10^{-6}	0
	7.74×10^{-5}	0	1.24×10^{-5}	0	2.42×10^{-6}	0
	1.74×10^{-5}	0	2.49×10^{-6}	0	1.02×10^{-6}	0
	5.45×10^{-1}	5.45×10^{-1}	1.24×10^{-1}	1.24×10^{-1}	1.04×10^{-1}	1.04×10^{-1}
	5.45×10^{-1}	3.98	1.24×10^{-1}	3.76×10^{-1}	1.04×10^{-1}	6.93×10^{-1}
	1.51×10^{-2}	3.6×10^{-2}	4.79×10^{-3}	1.98×10^{-3}	8.98×10^{-4}	2.7×10^{-3}
	1.51×10^{-2}	7.8×10^{-1}	4.79×10^{-3}	8.5×10^{-3}	8.98×10^{-4}	4.42×10^{-2}
3	1.53×10^{-3}	2.5×10^{-1}	4.03×10^{-4}	8.82×10^{-4}	4.92×10^{-5}	4.67×10^{-3}
	1.53×10^{-3}	3.48×10^{-1}	4.03×10^{-4}	7.46×10^{-4}	4.92×10^{-5}	5.54×10^{-3}
	2.65×10^{-4}	0	5.41×10^{-5}	0	5.73×10^{-6}	0
	2.65×10^{-4}	0	5.41×10^{-5}	0	5.73×10^{-6}	0
	6.53×10^{-4}	0	1.06×10^{-5}	0	1.49×10^{-6}	0
	2.41	2.41	2.05×10^{-1}	2.05×10^{-1}	1.72×10^{-1}	1.72×10^{-1}
	2.41	1.47×10^{1}	2.05×10^{-1}	6.29×10^{-1}	1.72×10^{-1}	1.04
	1.51×10^{-1}	1.28×10^{-1}	1.26×10^{-2}	5.09×10^{-3}	1.89×10^{-3}	7.95×10^{-3}
	1.51×10^{-1}	8.35	1.26×10^{-2}	2.39×10^{-2}	1.89×10^{-3}	1.09×10^{-1}
5	$2.\times10^{-1}$	7.06	1.57×10^{-3}	4.14×10^{-3}	1.29×10^{-4}	1.85×10^{-2}
	$2.\times10^{-1}$	1.37×10^{1}	1.57×10^{-3}	3.89×10^{-3}	1.29×10^{-4}	2.32×10^{-2}
	1.62×10^{-1}	0	2.59×10^{-4}	0	1.95×10^{-5}	0
	1.62×10^{-1}	0	2.59×10^{-4}	0	1.95×10^{-5}	0
	9.27×10^{-2}	0	2.28×10^{-5}	0	4.33×10^{-6}	0
	1.57×10^{1}	1.57×10^{1}	3.78×10^{-1}	3.78×10^{-1}	3.1×10^{-1}	3.1×10^{-1}
	1.57×10^{1}	8.35×10^{1}	3.78×10^{-1}	1.29	3.1×10^{-1}	1.69
	1.11	9.32	2.64×10^{-2}	6.07×10^{-5}	4.93×10^{-3}	2.91×10^{-2}
	1.11	9.67×10^{1}	2.64×10^{-2}	1.15×10^{-1}	4.93×10^{-3}	3.33×10^{-1}
10	1.02×10^{1}	6.41×10^{2}	5.76×10^{-3}	2.2×10^{-2}	3.7×10^{-4}	1.04×10^{-1}
	1.02×10^{1}	1.33×10^{3}	5.76×10^{-3}	5.06×10^{-2}	3.7×10^{-4}	1.43×10^{-1}
	3.28×10^{1}	0	1.06×10^{-2}	0	$7.\times10^{-5}$	0
	3.28×10^{1}	0	1.06×10^{-2}	0	$7.\times10^{-5}$	0
	5.21×10^{1}	0	1.09×10^{-2}	0	9.64×10^{-6}	0

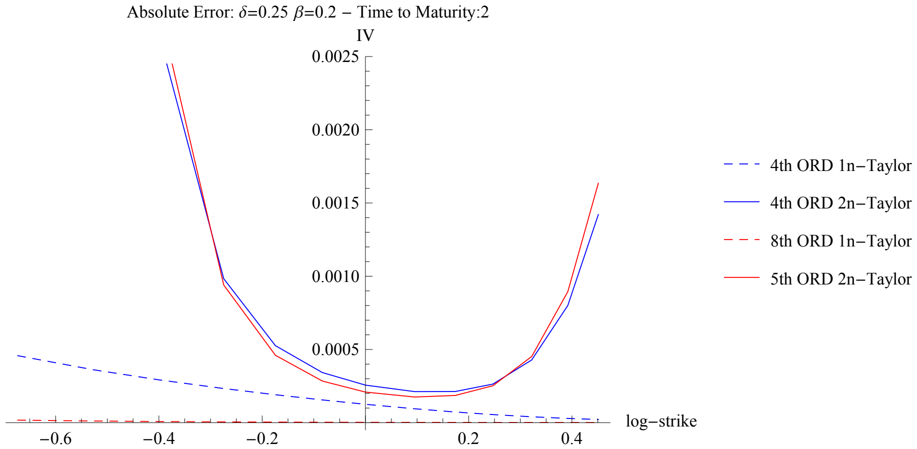
Also in this case we see that the 2n-Taylor succedes in producing comparable values to 1n-Taylor only for short maturities, while for long maturities produces acceptable results but significantly worse than those of 1n-Taylor. For maturities longer than 5 years in the ITM region the 1n-Taylor approximation explodes as we have already seen, but the deterioration of the 2n-Taylor approx. is more intense.

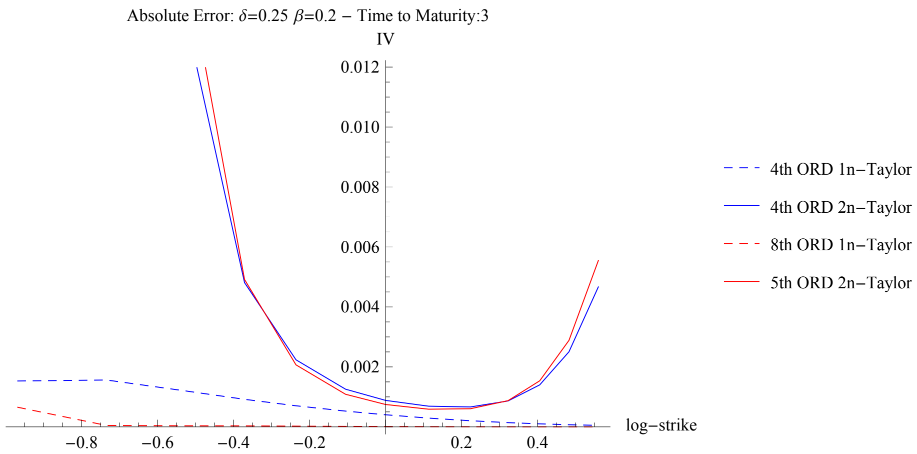

Absolute Error: δ=0.25 β=0.2 – Time to Maturity:0.25
IV
3. × 10^−6
2.5 × 10^−6
2. × 10^−6
1.5 × 10^−6
1. × 10^−6
5. × 10^−7
−0.10
−0.05
0.05
0.10
log−strike
4th ORD 1n−Taylor
4th ORD 2n−Taylor
8th ORD 1n−Taylor
5th ORD 2n−Taylor

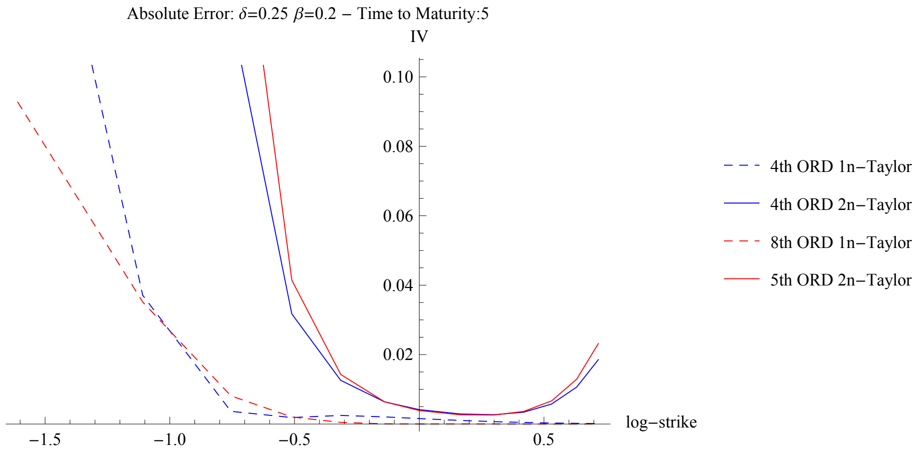

Absolute Error: δ=0.25 β=0.2 – Time to Maturity:0.5
IV
0.00002
0.000015
0.00001
5. × 10^−6
−0.2
−0.1
0.1
0.2
log−strike
4th ORD 1n−Taylor
4th ORD 2n−Taylor
8th ORD 1n−Taylor
5th ORD 2n−Taylor

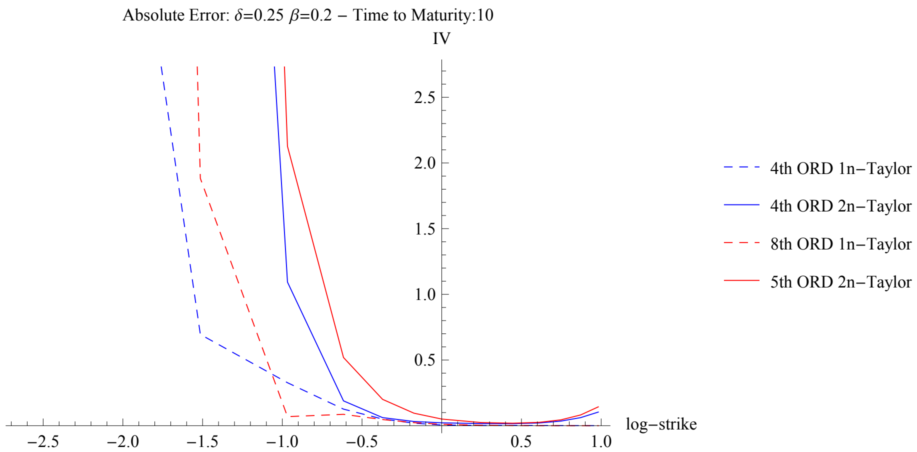

Absolute Error: δ=0.25 β=0.2 – Time to Maturity:1
IV
0.00020
0.00015
0.00010
0.00005
−0.4
−0.3
−0.2
−0.1
0.1
0.2
0.3
log−strike
4th ORD 1n−Taylor
4th ORD 2n−Taylor
8th ORD 1n−Taylor
5th ORD 2n−Taylor


Absolute Error: δ=0.25 β=0.2 – Time to Maturity:1.5
IV
0.0008
0.0006
0.0004
0.0002
log–strike
–0.4
–0.2
0.2
0.4
4th ORD 1n–Taylor
4th ORD 2n–Taylor
8th ORD 1n–Taylor
5th ORD 2n–Taylor


Absolute Error: δ=0.25 β=0.2 – Time to Maturity:2
IV
0.0025
0.0020
0.0015
0.0010
0.0005
log–strike
–0.6
–0.4
–0.2
0.2
0.4
4th ORD 1n–Taylor
4th ORD 2n–Taylor
8th ORD 1n–Taylor
5th ORD 2n–Taylor


Absolute Error: δ=0.25 β=0.2 – Time to Maturity:3
IV
0.012
0.010
0.008
0.006
0.004
0.002
log–strike
–0.8
–0.6
–0.4
–0.2
0.2
0.4
4th ORD 1n–Taylor
4th ORD 2n–Taylor
8th ORD 1n–Taylor
5th ORD 2n–Taylor





9.1.3 Third Set: $\delta = 0.4$, $\beta = 0.5$

In the third set we have ($\delta = 0.4$, $\beta = 0.5$), and we consider as in 8.1.3 the following table of strikes expressed in bp

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.25	77	81	85	88	92	96	100	104	109	114	118	122	127
0.5	65	71	77	82	88	94	100	108	115	122	130	138	145
1	49	58	66	74	83	92	100	113	125	138	151	163	176
1.5	39	49	59	70	80	90	100	117	134	151	168	185	202
2	31	42	54	66	77	88	100	121	142	164	185	206	227
3	20	33	47	60	73	87	100	129	158	186	215	244	273
5	9	24	39	54	70	85	100	143	187	230	273	317	360
10	3	19	35	52	68	84	100	177	254	330	407	484	561

which correspond to the following values of the exact implied volatility

	1	3	5	7	9	11	13
0.25	42.6906	41.6659	40.8572	40.0166	39.1599	38.3823	37.6713
0.5	44.5073	42.7104	41.3281	40.0329	38.6483	37.4607	36.4242
1	47.6641	44.3863	41.9665	40.0648	37.8650	36.0679	34.6551
1.5	50.3351	45.6469	42.3853	40.0958	37.2218	35.1011	33.4365
2	53.1210	46.6698	42.8216	40.1256	36.6930	34.2426	32.4286
3	58.6938	48.3208	43.4544	40.1818	35.7310	32.9364	30.8803
5	68.9979	50.5961	44.0136	40.2770	34.2488	30.9319	28.6602
10	78.0078	51.4828	44.3241	40.3608	31.6487	27.7744	25.3465

Now we compute absolute errors committed by our 2n-Taylor approximation

	1	3	5	7	9	11	13
0.25	1.03×10^{-2}	7.22×10^{-3}	1.47×10^{-2}	1.66×10^{-2}	1.25×10^{-2}	3.67×10^{-3}	8.59×10^{-3}
	1.52×10^{-1}	9.1×10^{-2}	6.22×10^{-2}	5.01×10^{-2}	5.61×10^{-2}	7.71×10^{-2}	1.09×10^{-1}
	4.88×10^{-6}	4.59×10^{-6}	1.51×10^{-5}	2.38×10^{-5}	1.12×10^{-5}	4.56×10^{-6}	6.87×10^{-6}
	8.75×10^{-4}	2.78×10^{-4}	1.3×10^{-4}	9.21×10^{-5}	1.1×10^{-4}	2.21×10^{-4}	5.18×10^{-4}
	1.11×10^{-5}	3.92×10^{-6}	2.76×10^{-6}	2.43×10^{-6}	2.44×10^{-6}	3.09×10^{-6}	6.03×10^{-6}
	8.09×10^{-6}	2.93×10^{-7}	8.36×10^{-7}	1.07×10^{-6}	9.58×10^{-7}	1.55×10^{-7}	3.21×10^{-6}
0.5	4.1×10^{-2}	9.47×10^{-3}	2.91×10^{-2}	3.29×10^{-2}	2.18×10^{-2}	1.48×10^{-4}	2.75×10^{-2}
	3.97×10^{-1}	2.14×10^{-1}	1.32×10^{-1}	$1.\times10^{-1}$	1.14×10^{-1}	1.63×10^{-1}	2.33×10^{-1}
	1.51×10^{-4}	4.47×10^{-5}	4.83×10^{-5}	8.72×10^{-5}	2.38×10^{-5}	2.5×10^{-5}	9.5×10^{-5}
	$6.\times10^{-3}$	1.53×10^{-3}	5.78×10^{-4}	3.74×10^{-4}	4.6×10^{-4}	1.06×10^{-3}	2.59×10^{-3}
	1.61×10^{-4}	2.75×10^{-5}	1.12×10^{-5}	7.83×10^{-6}	8.08×10^{-6}	1.6×10^{-5}	5.12×10^{-5}
	1.73×10^{-4}	2.31×10^{-5}	6.15×10^{-6}	3.18×10^{-6}	3.85×10^{-6}	1.3×10^{-5}	5.29×10^{-5}
1	1.45×10^{-1}	7.7×10^{-3}	5.91×10^{-2}	6.48×10^{-2}	3.53×10^{-2}	1.62×10^{-2}	7.31×10^{-2}
	1.11	5.16×10^{-1}	2.78×10^{-1}	2.02×10^{-1}	2.3×10^{-1}	3.4×10^{-1}	4.79×10^{-1}
	1.94×10^{-3}	2.91×10^{-4}	1.61×10^{-4}	3.17×10^{-4}	2.05×10^{-5}	4.6×10^{-5}	7.53×10^{-4}
	4.5×10^{-2}	8.73×10^{-3}	2.52×10^{-3}	1.5×10^{-3}	1.89×10^{-3}	5.01×10^{-3}	1.21×10^{-2}
	3.14×10^{-3}	3.3×10^{-4}	8.79×10^{-5}	5.16×10^{-5}	5.2×10^{-5}	1.43×10^{-4}	5.1×10^{-4}
	3.59×10^{-3}	3.3×10^{-4}	7.09×10^{-5}	3.83×10^{-5}	4.26×10^{-5}	1.49×10^{-4}	5.72×10^{-4}
1.5	2.82×10^{-1}	6.8×10^{-3}	9.04×10^{-2}	9.58×10^{-2}	4.4×10^{-2}	3.33×10^{-2}	1.16×10^{-1}
	2.03	8.52×10^{-1}	4.26×10^{-1}	3.04×10^{-1}	3.44×10^{-1}	5.01×10^{-1}	6.97×10^{-1}
	6.66×10^{-3}	8.81×10^{-4}	3.15×10^{-4}	6.49×10^{-4}	5.11×10^{-5}	1.1×10^{-5}	1.98×10^{-3}
	1.43×10^{-1}	2.31×10^{-2}	5.86×10^{-3}	3.4×10^{-3}	4.31×10^{-3}	1.14×10^{-2}	2.7×10^{-2}
	1.73×10^{-2}	1.39×10^{-3}	3.11×10^{-4}	1.72×10^{-4}	1.67×10^{-4}	4.89×10^{-4}	1.74×10^{-3}
	2.05×10^{-2}	1.43×10^{-3}	2.64×10^{-4}	1.38×10^{-4}	1.48×10^{-4}	5.28×10^{-4}	$2.\times10^{-3}$
2	4.86×10^{-1}	7.97×10^{-3}	1.21×10^{-1}	1.26×10^{-1}	5.03×10^{-2}	5.53×10^{-2}	1.59×10^{-1}
	3.3	1.21	5.89×10^{-1}	4.08×10^{-1}	4.54×10^{-1}	6.62×10^{-1}	9.04×10^{-1}
	1.78×10^{-2}	2.04×10^{-3}	3.85×10^{-4}	1.03×10^{-3}	1.96×10^{-4}	1.69×10^{-4}	3.86×10^{-3}
	3.6×10^{-1}	4.49×10^{-2}	1.1×10^{-2}	6.06×10^{-3}	7.63×10^{-3}	2.08×10^{-2}	4.75×10^{-2}
	6.68×10^{-2}	3.78×10^{-3}	7.95×10^{-4}	4.09×10^{-4}	3.81×10^{-4}	1.2×10^{-3}	4.11×10^{-3}
	8.33×10^{-2}	3.96×10^{-3}	6.97×10^{-4}	3.39×10^{-4}	3.5×10^{-4}	1.33×10^{-3}	4.81×10^{-3}
3	1.12	1.09×10^{-2}	1.8×10^{-1}	1.82×10^{-1}	5.34×10^{-2}	9.68×10^{-2}	2.38×10^{-1}
	7.02	1.97	9.22×10^{-1}	6.18×10^{-1}	6.7×10^{-1}	9.51×10^{-1}	1.27
	6.53×10^{-2}	7.62×10^{-3}	4.6×10^{-5}	1.72×10^{-3}	7.77×10^{-4}	8.36×10^{-4}	9.08×10^{-3}
	1.45	1.13×10^{-1}	2.65×10^{-2}	1.38×10^{-2}	1.72×10^{-2}	4.56×10^{-2}	$1.\times10^{-1}$
	5.05×10^{-1}	1.5×10^{-2}	2.92×10^{-3}	1.39×10^{-3}	1.22×10^{-3}	3.91×10^{-3}	1.28×10^{-2}
	7.05×10^{-1}	1.66×10^{-2}	2.75×10^{-3}	1.22×10^{-3}	1.18×10^{-3}	4.46×10^{-3}	1.55×10^{-2}

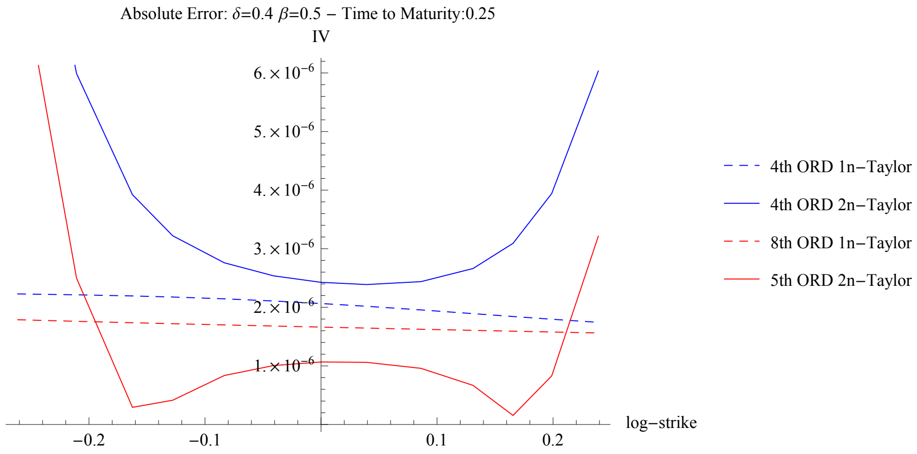
5	4.03	2.06×10^{-2}	2.83×10^{-1}	2.77×10^{-1}	4.3×10^{-2}	1.87×10^{-1}	3.79×10^{-1}
	2.1×10^{1}	3.66	1.58	1.06	1.07	1.47	1.88
	1.92×10^{-1}	5.48×10^{-2}	7.85×10^{-3}	5.09×10^{-4}	3.01×10^{-3}	4.06×10^{-3}	2.46×10^{-2}
	9.49	3.59×10^{-1}	7.75×10^{-2}	4.03×10^{-2}	4.61×10^{-2}	1.19×10^{-1}	2.42×10^{-1}
	8.03	7.68×10^{-2}	1.27×10^{-2}	6.1×10^{-3}	5.03×10^{-3}	1.68×10^{-2}	4.98×10^{-2}
	1.34×10^{1}	1.05×10^{-1}	1.58×10^{-2}	6.68×10^{-3}	5.26×10^{-3}	2.02×10^{-2}	6.38×10^{-2}
10	1.81×10^{1}	5.22×10^{-1}	2.75×10^{-1}	3.61×10^{-1}	3.61×10^{-2}	3.87×10^{-1}	6.44×10^{-1}
	7.97×10^{1}	7.58	3.42	2.31	1.94	2.47	2.99
	1.89×10^{1}	4.78×10^{-1}	1.53×10^{-1}	6.4×10^{-2}	1.52×10^{-2}	2.03×10^{-2}	7.91×10^{-2}
	3.26×10^{1}	1.25	3.78×10^{-1}	2.03×10^{-1}	1.66×10^{-1}	3.87×10^{-1}	7.01×10^{-1}
	2.36×10^{2}	6.8×10^{-1}	9.04×10^{-2}	3.64×10^{-2}	3.09×10^{-2}	1.02×10^{-1}	2.59×10^{-1}
	7.25×10^{2}	6.64×10^{-1}	1.6×10^{-1}	7.62×10^{-2}	3.81×10^{-2}	1.35×10^{-1}	3.68×10^{-1}

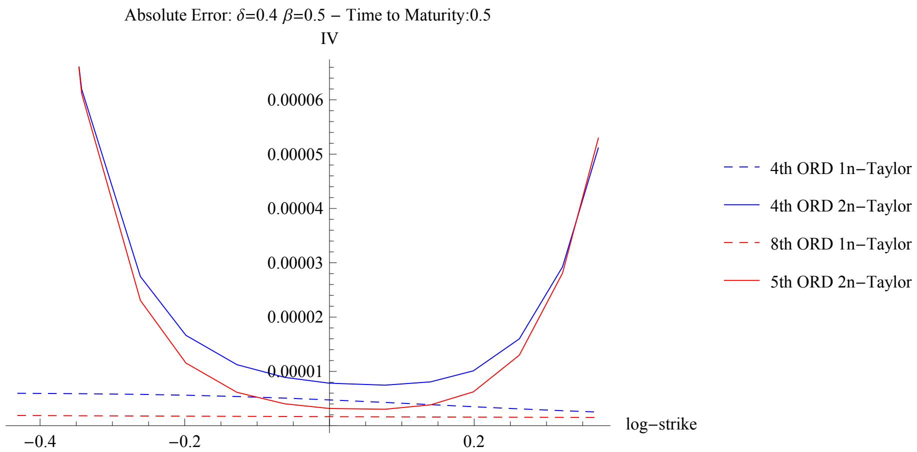
then we compare them with those obtained with the 1n-Taylor

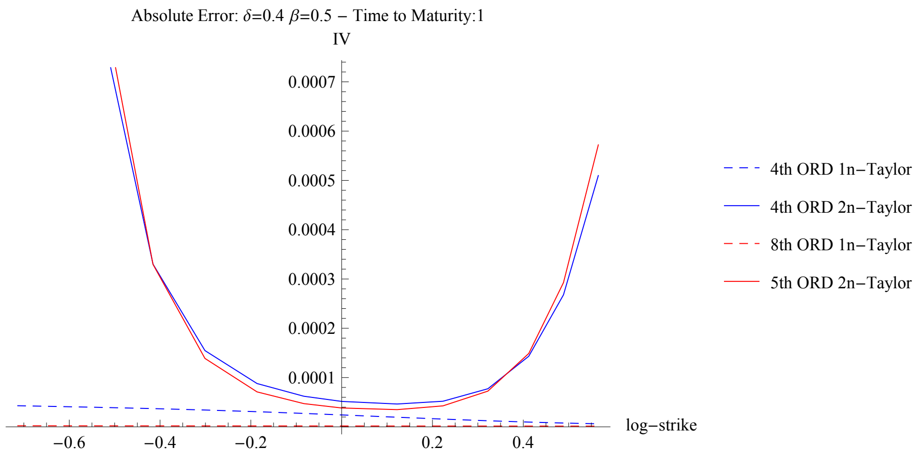
	1		7		13	
0.25	1.03×10^{-2}	1.03×10^{-2}	1.66×10^{-2}	1.66×10^{-2}	8.59×10^{-3}	8.59×10^{-3}
	1.03×10^{-2}	1.52×10^{-1}	1.66×10^{-2}	5.01×10^{-2}	8.59×10^{-3}	1.09×10^{-1}
	4.73×10^{-5}	4.88×10^{-6}	5.67×10^{-5}	2.38×10^{-5}	2.55×10^{-5}	6.87×10^{-6}
	4.73×10^{-5}	8.75×10^{-4}	5.67×10^{-5}	9.21×10^{-5}	2.55×10^{-5}	5.18×10^{-4}
	2.23×10^{-6}	1.11×10^{-5}	2.06×10^{-6}	2.43×10^{-6}	1.74×10^{-6}	6.03×10^{-6}
	2.23×10^{-6}	8.09×10^{-6}	2.06×10^{-6}	1.07×10^{-6}	1.74×10^{-6}	3.21×10^{-6}
	1.79×10^{-6}	0	1.67×10^{-6}	0	1.56×10^{-6}	0
	1.79×10^{-6}	0	1.67×10^{-6}	0	1.56×10^{-6}	0
	1.79×10^{-6}	0	1.66×10^{-6}	0	1.56×10^{-6}	0
0.5	4.1×10^{-2}	4.1×10^{-2}	3.29×10^{-2}	3.29×10^{-2}	2.75×10^{-2}	2.75×10^{-2}
	4.1×10^{-2}	3.97×10^{-1}	3.29×10^{-2}	$1.\times10^{-1}$	2.75×10^{-2}	2.33×10^{-1}
	2.05×10^{-4}	1.51×10^{-4}	2.17×10^{-4}	8.72×10^{-5}	7.64×10^{-5}	9.5×10^{-5}
	2.05×10^{-4}	$6.\times10^{-3}$	2.17×10^{-4}	3.74×10^{-4}	7.64×10^{-5}	2.59×10^{-3}
	5.97×10^{-6}	1.61×10^{-4}	4.75×10^{-6}	7.83×10^{-6}	2.51×10^{-6}	5.12×10^{-5}
	5.97×10^{-6}	1.73×10^{-4}	4.75×10^{-6}	3.18×10^{-6}	2.51×10^{-6}	5.29×10^{-5}
	2.01×10^{-6}	0	1.74×10^{-6}	0	1.53×10^{-6}	0
	2.01×10^{-6}	0	1.74×10^{-6}	0	1.53×10^{-6}	0
	1.88×10^{-6}	0	1.67×10^{-6}	0	1.51×10^{-6}	0
1	1.45×10^{-1}	1.45×10^{-1}	6.48×10^{-2}	6.48×10^{-2}	7.31×10^{-2}	7.31×10^{-2}
	1.45×10^{-1}	1.11	6.48×10^{-2}	2.02×10^{-1}	7.31×10^{-2}	4.79×10^{-1}
	1.04×10^{-3}	1.94×10^{-3}	8.22×10^{-4}	3.17×10^{-4}	2.35×10^{-4}	7.53×10^{-4}
	1.04×10^{-3}	4.5×10^{-2}	8.22×10^{-4}	1.5×10^{-3}	2.35×10^{-4}	1.21×10^{-2}
	4.29×10^{-5}	3.14×10^{-3}	2.42×10^{-5}	5.16×10^{-5}	6.17×10^{-6}	5.1×10^{-4}
	4.29×10^{-5}	3.59×10^{-3}	2.42×10^{-5}	3.83×10^{-5}	6.17×10^{-6}	5.72×10^{-4}
	4.74×10^{-6}	0	2.65×10^{-6}	0	1.62×10^{-6}	0
	4.74×10^{-6}	0	2.65×10^{-6}	0	1.62×10^{-6}	0
	2.26×10^{-6}	0	1.73×10^{-6}	0	1.45×10^{-6}	0

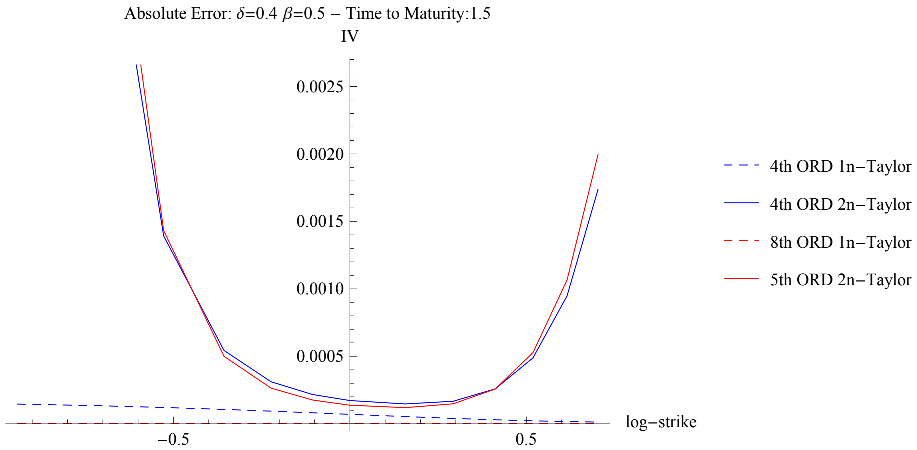
	2.82×10^{-1}	2.82×10^{-1}	9.58×10^{-2}	9.58×10^{-2}	1.16×10^{-1}	1.16×10^{-1}
	2.82×10^{-1}	2.03	9.58×10^{-2}	3.04×10^{-1}	1.16×10^{-1}	6.97×10^{-1}
	2.75×10^{-3}	6.66×10^{-3}	1.75×10^{-3}	6.49×10^{-4}	4.57×10^{-4}	1.98×10^{-3}
	2.75×10^{-3}	1.43×10^{-1}	1.75×10^{-3}	3.4×10^{-3}	4.57×10^{-4}	2.7×10^{-2}
1.5	1.46×10^{-4}	1.73×10^{-2}	$7.\times10^{-5}$	1.72×10^{-4}	1.24×10^{-5}	1.74×10^{-3}
	1.46×10^{-4}	2.05×10^{-2}	$7.\times10^{-5}$	1.38×10^{-4}	1.24×10^{-5}	$2.\times10^{-3}$
	1.42×10^{-5}	0	5.81×10^{-6}	0	1.98×10^{-6}	0
	1.42×10^{-5}	0	5.81×10^{-6}	0	1.98×10^{-6}	0
	2.98×10^{-6}	0	2.01×10^{-6}	0	1.43×10^{-6}	0
	4.86×10^{-1}	4.86×10^{-1}	1.26×10^{-1}	1.26×10^{-1}	1.59×10^{-1}	1.59×10^{-1}
	4.86×10^{-1}	3.3	1.26×10^{-1}	4.08×10^{-1}	1.59×10^{-1}	9.04×10^{-1}
	5.39×10^{-3}	1.78×10^{-2}	2.94×10^{-3}	1.03×10^{-3}	7.34×10^{-4}	3.86×10^{-3}
	5.39×10^{-3}	3.6×10^{-1}	2.94×10^{-3}	6.06×10^{-3}	7.34×10^{-4}	4.75×10^{-2}
2	2.14×10^{-4}	6.68×10^{-2}	1.45×10^{-4}	4.09×10^{-4}	2.02×10^{-5}	4.11×10^{-3}
	2.14×10^{-4}	8.33×10^{-2}	1.45×10^{-4}	3.39×10^{-4}	2.02×10^{-5}	4.81×10^{-3}
	1.77×10^{-5}	0	1.2×10^{-5}	0	2.57×10^{-6}	0
	1.77×10^{-5}	0	1.2×10^{-5}	0	2.57×10^{-6}	0
	2.14×10^{-5}	0	2.61×10^{-6}	0	1.45×10^{-6}	0
	1.12	1.12	1.82×10^{-1}	1.82×10^{-1}	2.38×10^{-1}	2.38×10^{-1}
	1.12	7.02	1.82×10^{-1}	6.18×10^{-1}	2.38×10^{-1}	1.27
	4.48×10^{-3}	6.53×10^{-2}	5.76×10^{-3}	1.72×10^{-3}	1.42×10^{-3}	9.08×10^{-3}
	4.48×10^{-3}	1.45	5.76×10^{-3}	1.38×10^{-2}	1.42×10^{-3}	$1.\times10^{-1}$
3	6.23×10^{-3}	5.05×10^{-1}	3.44×10^{-4}	1.39×10^{-3}	3.81×10^{-5}	1.28×10^{-2}
	6.23×10^{-3}	7.05×10^{-1}	3.44×10^{-4}	1.22×10^{-3}	3.81×10^{-5}	1.55×10^{-2}
	4.2×10^{-3}	0	2.37×10^{-5}	0	4.28×10^{-6}	0
	4.2×10^{-3}	0	2.37×10^{-5}	0	4.28×10^{-6}	0
	2.72×10^{-3}	0	2.23×10^{-7}	0	1.58×10^{-6}	0
	4.03	4.03	2.77×10^{-1}	2.77×10^{-1}	3.79×10^{-1}	3.79×10^{-1}
	4.03	2.1×10^{1}	2.77×10^{-1}	1.06	3.79×10^{-1}	1.88
	2.97×10^{-1}	1.92×10^{-1}	1.03×10^{-2}	5.09×10^{-4}	3.2×10^{-3}	2.46×10^{-2}
	2.97×10^{-1}	9.49	1.03×10^{-2}	4.03×10^{-2}	3.2×10^{-3}	2.42×10^{-1}
5	1.87×10^{-1}	8.03	6.86×10^{-5}	6.1×10^{-3}	6.46×10^{-5}	4.98×10^{-2}
	1.87×10^{-1}	1.34×10^{1}	6.86×10^{-5}	6.68×10^{-3}	6.46×10^{-5}	6.38×10^{-2}
	1.76×10^{-2}	0	4.71×10^{-4}	0	6.2×10^{-6}	0
	1.76×10^{-2}	0	4.71×10^{-4}	0	6.2×10^{-6}	0
	1.37×10^{-1}	0	3.27×10^{-4}	0	1.16×10^{-6}	0
	1.81×10^{1}	1.81×10^{1}	3.61×10^{-1}	3.61×10^{-1}	6.44×10^{-1}	6.44×10^{-1}
	1.81×10^{1}	7.97×10^{1}	3.61×10^{-1}	2.31	6.44×10^{-1}	2.99
	6.32	1.89×10^{1}	3.92×10^{-2}	6.4×10^{-2}	8.91×10^{-3}	7.91×10^{-2}
	6.32	3.26×10^{1}	3.92×10^{-2}	2.03×10^{-1}	8.91×10^{-3}	7.01×10^{-1}
10	6.18	2.36×10^{2}	3.8×10^{-2}	3.64×10^{-2}	6.96×10^{-5}	2.59×10^{-1}
	6.18	7.25×10^{2}	3.8×10^{-2}	7.62×10^{-2}	6.96×10^{-5}	3.68×10^{-1}
	4.61×10^{-1}	0	2.39×10^{-2}	0	6.44×10^{-5}	0
	4.61×10^{-1}	0	2.39×10^{-2}	0	6.44×10^{-5}	0
	5.07×10^{1}	0	1.28×10^{-2}	0	4.1×10^{-5}	0

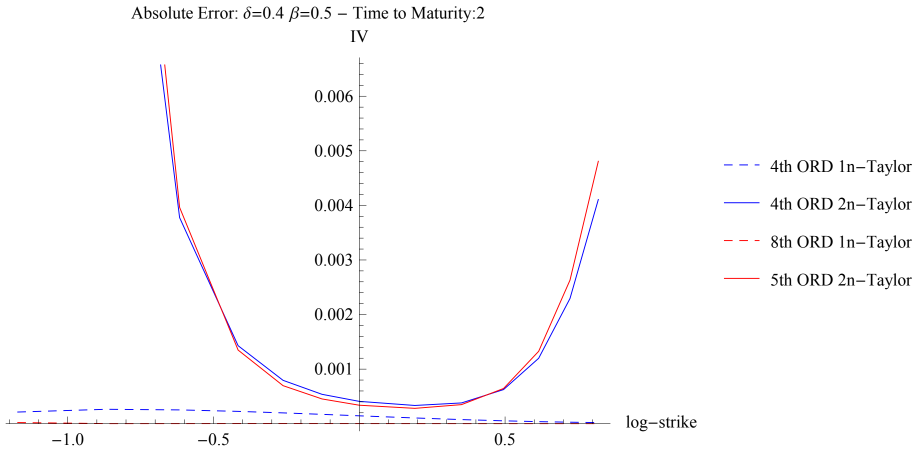
and also in this case we see that the 2n-Taylor method does not produce better results.

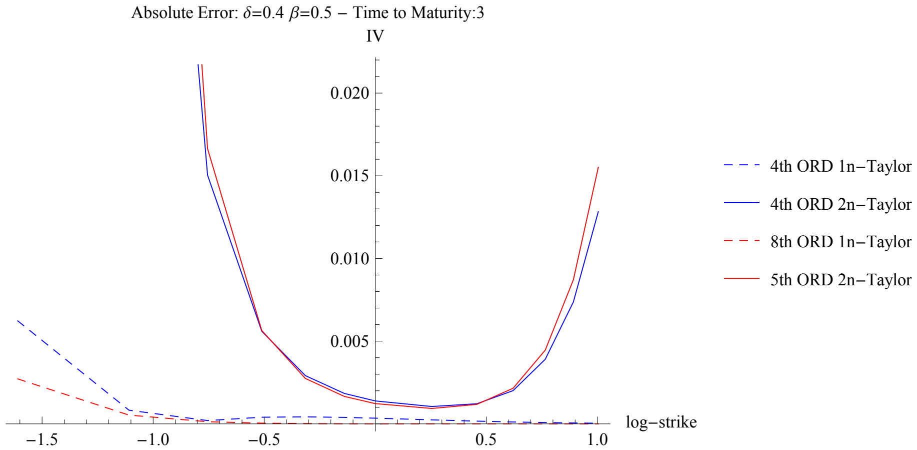

Absolute Error: δ=0.4 β=0.5 – Time to Maturity:0.25
IV
$6.\times10^{-6}$
$5.\times10^{-6}$
$4.\times10^{-6}$
$3.\times10^{-6}$
$2.\times10^{-6}$
$1.\times10^{-6}$
−0.2
−0.1
0.1
0.2
log−strike
4th ORD 1n−Taylor
4th ORD 2n−Taylor
8th ORD 1n−Taylor
5th ORD 2n−Taylor

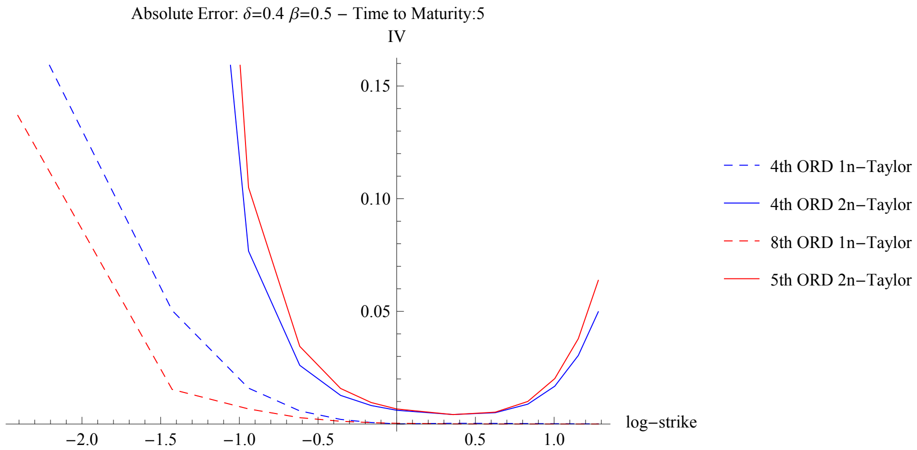

Absolute Error: δ=0.4 β=0.5 – Time to Maturity:0.5
IV
0.00006
0.00005
0.00004
0.00003
0.00002
0.00001
−0.4
−0.2
0.2
log−strike
4th ORD 1n−Taylor
4th ORD 2n−Taylor
8th ORD 1n−Taylor
5th ORD 2n−Taylor

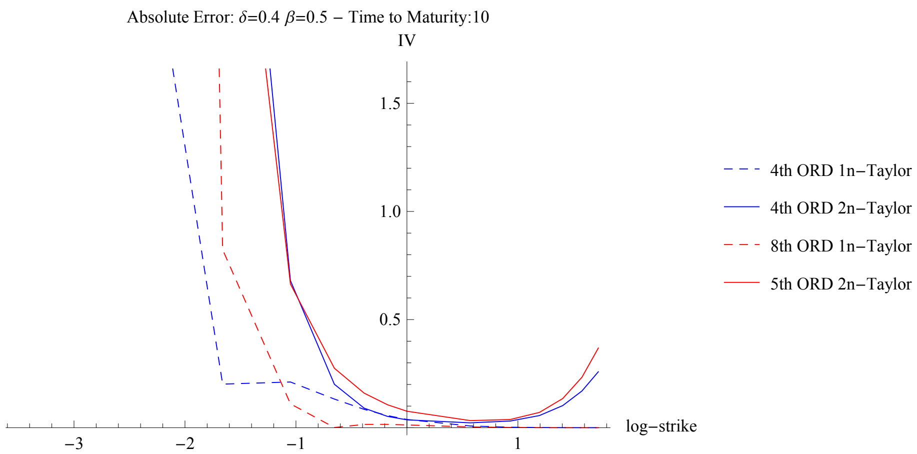

Absolute Error: δ=0.4 β=0.5 – Time to Maturity:1
IV
0.0007
0.0006
0.0005
0.0004
0.0003
0.0002
0.0001
−0.6
−0.4
−0.2
0.2
0.4
log−strike
4th ORD 1n−Taylor
4th ORD 2n−Taylor
8th ORD 1n−Taylor
5th ORD 2n−Taylor


Absolute Error: δ=0.4 β=0.5 – Time to Maturity:1.5
IV
0.0025
0.0020
0.0015
0.0010
0.0005
log–strike
–0.5
0.5
4th ORD 1n–Taylor
4th ORD 2n–Taylor
8th ORD 1n–Taylor
5th ORD 2n–Taylor


Absolute Error: δ=0.4 β=0.5 – Time to Maturity:2
IV
0.006
0.005
0.004
0.003
0.002
0.001
log–strike
–1.0
–0.5
0.5
4th ORD 1n–Taylor
4th ORD 2n–Taylor
8th ORD 1n–Taylor
5th ORD 2n–Taylor


Absolute Error: δ=0.4 β=0.5 – Time to Maturity:3
IV
0.020
0.015
0.010
0.005
log–strike
–1.5
–1.0
–0.5
0.5
1.0
4th ORD 1n–Taylor
4th ORD 2n–Taylor
8th ORD 1n–Taylor
5th ORD 2n–Taylor





9.2 Quadratic local volatility model

For the quadratic model we have computed our approximation with the Enhanced Taylor approach till fifth order, and we present here only the order 0 and 1 because they are very long.

$$\sigma_0 = e^{-\bar{x}} \sqrt{\frac{\left(-e^{ll} + e^{\bar{x}}\right)^2 \left(-e^{uu} + e^{\bar{x}}\right)^2}{\left(e^{ll} - e^{uu}\right)^2}}\ \delta$$

$$\sigma_1 = \frac{1}{12 \left(e^{ll} - e^{uu}\right)^2 \sqrt{\frac{\left(e^{ll}-e^{\bar{x}}\right)^2 \left(e^{uu}-e^{\bar{x}}\right)^2}{\left(e^{ll}-e^{uu}\right)^2}}} e^{-\bar{x}}\, \delta \left(-6 \left(e^{ll} - e^{\bar{x}}\right) \left(-e^{uu} + e^{\bar{x}}\right) \left(e^{ll+uu} - e^{2\bar{x}}\right) (-k + x) + \right.$$

$$
\begin{aligned}
&2\left(2\, e^{2\,(ll+uu)} + 2\, e^{4\,\bar{x}} - e^{2\,ll+uu+\bar{x}} - e^{ll+2\,uu+\bar{x}} - e^{ll+3\,\bar{x}} - e^{uu+3\,\bar{x}}\right)(-k+x)^2 - \frac{1}{\left(e^{ll} - e^{uu}\right)^2} \\
&e^{-2\,\bar{x}}\left(e^{ll} - e^{\bar{x}}\right)^2\left(e^{uu} - e^{\bar{x}}\right)^2\left(-2\, e^{2\,(ll+uu)} - 2\, e^{4\,\bar{x}} + e^{2\,ll+uu+\bar{x}} + e^{ll+2\,uu+\bar{x}} + e^{ll+3\,\bar{x}} + e^{uu+3\,\bar{x}}\right) t\,\delta^2 + \\
&12\left(e^{ll} - e^{\bar{x}}\right)\left(-e^{uu} + e^{\bar{x}}\right)\left(e^{ll+uu} - e^{2\,\bar{x}}\right)(x-\bar{x}) - \\
&6\left(2\, e^{2\,(ll+uu)} + 2\, e^{4\,\bar{x}} - e^{2\,ll+uu+\bar{x}} - e^{ll+2\,uu+\bar{x}} - e^{ll+3\,\bar{x}} - e^{uu+3\,\bar{x}}\right)(-k+x)\,(x-\bar{x}) + \\
&\left.6\left(2\, e^{2\,(ll+uu)} + 2\, e^{4\,\bar{x}} - e^{2\,ll+uu+\bar{x}} - e^{ll+2\,uu+\bar{x}} - e^{ll+3\,\bar{x}} - e^{uu+3\,\bar{x}}\right)(x-\bar{x})^2\right)
\end{aligned}
$$

9.2.1 First Set: $\delta = 0.02$, uu = 15, ll = 2.2

As in 8.2.1 we consider ($\delta = 0.02$, uu = 15, ll = 2.2) and the table of strikes

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.25	94	95	96	97	98	99	100	101	102	103	104	105	106
0.5	89	91	93	94	96	98	100	102	104	106	107	109	111
1	81	84	87	90	94	97	100	103	106	109	112	115	118
1.5	75	79	83	88	92	96	100	104	108	112	116	120	124
2	69	74	79	84	90	95	100	105	110	114	119	124	129
3	60	67	73	80	87	93	100	106	113	119	125	132	138
5	43	52	62	72	81	90	100	109	117	126	135	143	152
10	10	25	40	55	70	85	100	113	126	140	153	166	179

and the corresponding table of exact implied volatilities

	1	3	5	7	9	11	13
0.25	16.6182	16.4252	16.2373	16.0543	15.8760	15.7021	15.5325
0.5	17.1300	16.7216	16.4298	16.0586	15.7061	15.4530	15.1296
1	18.0345	17.3539	16.6324	16.0671	15.5440	15.0581	14.6050
1.5	18.7964	17.8123	16.8411	16.0756	15.3856	14.7592	14.1869
2	19.6426	18.3010	17.0563	16.0841	15.2307	14.5450	13.8587
3	21.1195	19.1067	17.3972	16.1013	15.0081	14.1361	13.3079
5	24.9134	20.8504	18.1322	16.1358	14.7306	13.5097	12.5402
10	47.2059	26.1319	19.7494	16.2236	14.1491	12.5276	11.2940

Now we compute absolute errors of our 2n-Taylor approximation

	1	3	5	7	9	11	13
	1.03×10^{-3}	2.96×10^{-3}	$4.\times10^{-3}$	4.24×10^{-3}	3.74×10^{-3}	2.54×10^{-3}	7.19×10^{-4}
	1.03×10^{-3}	2.96×10^{-3}	$4.\times10^{-3}$	4.24×10^{-3}	3.74×10^{-3}	2.54×10^{-3}	7.19×10^{-4}
	1.02×10^{-5}	1.11×10^{-5}	1.14×10^{-5}	1.11×10^{-5}	1.03×10^{-5}	8.93×10^{-6}	7.32×10^{-6}
	1.02×10^{-5}	1.11×10^{-5}	1.14×10^{-5}	1.11×10^{-5}	1.03×10^{-5}	8.93×10^{-6}	7.32×10^{-6}
0.25	6.55×10^{-8}	6.63×10^{-8}	6.49×10^{-8}	6.13×10^{-8}	5.58×10^{-8}	4.9×10^{-8}	4.15×10^{-8}
	6.55×10^{-8}	6.63×10^{-8}	6.49×10^{-8}	6.13×10^{-8}	5.58×10^{-8}	4.9×10^{-8}	4.15×10^{-8}
	5.84×10^{-10}	6.15×10^{-10}	5.8×10^{-10}	5.34×10^{-10}	4.78×10^{-10}	4.18×10^{-10}	3.55×10^{-10}
	5.84×10^{-10}	6.15×10^{-10}	5.8×10^{-10}	5.34×10^{-10}	4.78×10^{-10}	4.18×10^{-10}	3.55×10^{-10}
	4.3×10^{-11}	8.36×10^{-12}	7.01×10^{-12}	6.3×10^{-12}	5.44×10^{-12}	5.05×10^{-12}	4.01×10^{-12}

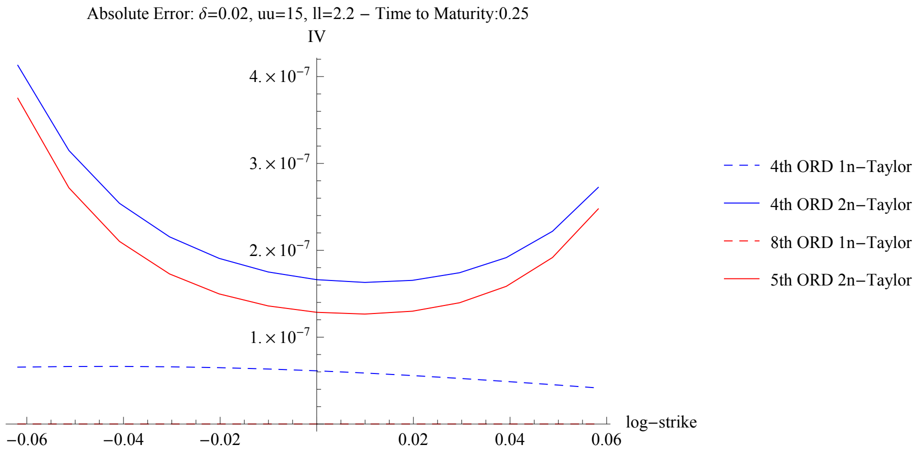
	3.03×10^{-3}	4.51×10^{-3}	7.51×10^{-3}	8.49×10^{-3}	6.51×10^{-3}	3.37×10^{-3}	2.74×10^{-3}
	3.03×10^{-3}	4.51×10^{-3}	7.51×10^{-3}	8.49×10^{-3}	6.51×10^{-3}	3.37×10^{-3}	2.74×10^{-3}
	4.02×10^{-5}	4.57×10^{-5}	4.71×10^{-5}	4.46×10^{-5}	3.78×10^{-5}	3.1×10^{-5}	2.15×10^{-5}
	4.02×10^{-5}	4.57×10^{-5}	4.71×10^{-5}	4.46×10^{-5}	3.78×10^{-5}	3.1×10^{-5}	2.15×10^{-5}
0.5	5.78×10^{-7}	5.75×10^{-7}	5.5×10^{-7}	4.9×10^{-7}	4.07×10^{-7}	3.38×10^{-7}	2.48×10^{-7}
	5.78×10^{-7}	5.75×10^{-7}	5.5×10^{-7}	4.9×10^{-7}	4.07×10^{-7}	3.38×10^{-7}	2.48×10^{-7}
	1.22×10^{-8}	1.11×10^{-8}	1.01×10^{-8}	8.53×10^{-9}	6.86×10^{-9}	5.64×10^{-9}	4.15×10^{-9}
	1.22×10^{-8}	1.11×10^{-8}	1.01×10^{-8}	8.53×10^{-9}	6.86×10^{-9}	5.64×10^{-9}	4.15×10^{-9}
	3.46×10^{-10}	2.94×10^{-10}	2.56×10^{-10}	2.06×10^{-10}	1.59×10^{-10}	1.29×10^{-10}	8.53×10^{-11}
	2.12×10^{-2}	2.23×10^{-3}	1.52×10^{-2}	1.7×10^{-2}	1.23×10^{-2}	2.41×10^{-3}	1.14×10^{-2}
	2.12×10^{-2}	2.23×10^{-3}	1.52×10^{-2}	1.7×10^{-2}	1.23×10^{-2}	2.41×10^{-3}	1.14×10^{-2}
	1.74×10^{-4}	1.94×10^{-4}	1.98×10^{-4}	1.78×10^{-4}	1.43×10^{-4}	$1.\times10^{-4}$	6.26×10^{-5}
	1.74×10^{-4}	1.94×10^{-4}	1.98×10^{-4}	1.78×10^{-4}	1.43×10^{-4}	$1.\times10^{-4}$	6.26×10^{-5}
1	5.83×10^{-6}	5.46×10^{-6}	4.74×10^{-6}	3.92×10^{-6}	3.01×10^{-6}	2.15×10^{-6}	1.42×10^{-6}
	5.83×10^{-6}	5.46×10^{-6}	4.74×10^{-6}	3.92×10^{-6}	3.01×10^{-6}	2.15×10^{-6}	1.42×10^{-6}
	2.78×10^{-7}	2.31×10^{-7}	1.78×10^{-7}	1.36×10^{-7}	9.96×10^{-8}	6.91×10^{-8}	4.57×10^{-8}
	2.78×10^{-7}	2.31×10^{-7}	1.78×10^{-7}	1.36×10^{-7}	9.96×10^{-8}	6.91×10^{-8}	4.57×10^{-8}
	1.77×10^{-8}	1.32×10^{-8}	9.2×10^{-9}	6.56×10^{-9}	4.54×10^{-9}	2.96×10^{-9}	1.95×10^{-9}
	4.6×10^{-2}	2.62×10^{-4}	2.26×10^{-2}	2.55×10^{-2}	1.69×10^{-2}	1.57×10^{-4}	2.25×10^{-2}
	4.6×10^{-2}	2.62×10^{-4}	2.26×10^{-2}	2.55×10^{-2}	1.69×10^{-2}	1.57×10^{-4}	2.25×10^{-2}
	4.52×10^{-4}	4.77×10^{-4}	4.67×10^{-4}	4.02×10^{-4}	3.01×10^{-4}	1.95×10^{-4}	1.1×10^{-4}
	4.52×10^{-4}	4.77×10^{-4}	4.67×10^{-4}	4.02×10^{-4}	3.01×10^{-4}	1.95×10^{-4}	1.1×10^{-4}
1.5	2.49×10^{-5}	2.14×10^{-5}	1.72×10^{-5}	1.32×10^{-5}	9.39×10^{-6}	6.14×10^{-6}	3.69×10^{-6}
	2.49×10^{-5}	2.14×10^{-5}	1.72×10^{-5}	1.32×10^{-5}	9.39×10^{-6}	6.14×10^{-6}	3.69×10^{-6}
	1.95×10^{-6}	1.43×10^{-6}	9.9×10^{-7}	6.88×10^{-7}	4.57×10^{-7}	2.88×10^{-7}	1.71×10^{-7}
	1.95×10^{-6}	1.43×10^{-6}	9.9×10^{-7}	6.88×10^{-7}	4.57×10^{-7}	2.88×10^{-7}	1.71×10^{-7}
	2.03×10^{-7}	1.3×10^{-7}	7.86×10^{-8}	4.96×10^{-8}	3.06×10^{-8}	1.83×10^{-8}	1.05×10^{-8}
	8.72×10^{-2}	6.89×10^{-3}	2.98×10^{-2}	3.41×10^{-2}	2.07×10^{-2}	1.47×10^{-3}	3.35×10^{-2}
	8.72×10^{-2}	6.89×10^{-3}	2.98×10^{-2}	3.41×10^{-2}	2.07×10^{-2}	1.47×10^{-3}	3.35×10^{-2}
	9.51×10^{-4}	9.29×10^{-4}	8.71×10^{-4}	7.14×10^{-4}	5.03×10^{-4}	3.17×10^{-4}	1.65×10^{-4}
	9.51×10^{-4}	9.29×10^{-4}	8.71×10^{-4}	7.14×10^{-4}	5.03×10^{-4}	3.17×10^{-4}	1.65×10^{-4}
2	7.61×10^{-5}	5.93×10^{-5}	4.39×10^{-5}	3.13×10^{-5}	2.06×10^{-5}	1.3×10^{-5}	7.19×10^{-6}
	7.61×10^{-5}	5.93×10^{-5}	4.39×10^{-5}	3.13×10^{-5}	2.06×10^{-5}	1.3×10^{-5}	7.19×10^{-6}
	8.77×10^{-6}	5.59×10^{-6}	3.45×10^{-6}	2.17×10^{-6}	1.31×10^{-6}	7.94×10^{-7}	4.3×10^{-7}
	8.77×10^{-6}	5.59×10^{-6}	3.45×10^{-6}	2.17×10^{-6}	1.31×10^{-6}	7.94×10^{-7}	4.3×10^{-7}
	1.34×10^{-6}	7.13×10^{-7}	3.73×10^{-7}	2.08×10^{-7}	1.15×10^{-7}	6.55×10^{-8}	3.4×10^{-8}
	1.83×10^{-1}	1.93×10^{-2}	4.55×10^{-2}	5.12×10^{-2}	2.8×10^{-2}	8.4×10^{-3}	5.74×10^{-2}
	1.83×10^{-1}	1.93×10^{-2}	4.55×10^{-2}	5.12×10^{-2}	2.8×10^{-2}	8.4×10^{-3}	5.74×10^{-2}
	3.05×10^{-3}	2.51×10^{-3}	2.13×10^{-3}	1.61×10^{-3}	1.04×10^{-3}	5.89×10^{-4}	2.8×10^{-4}
	3.05×10^{-3}	2.51×10^{-3}	2.13×10^{-3}	1.61×10^{-3}	1.04×10^{-3}	5.89×10^{-4}	2.8×10^{-4}
3	4.15×10^{-4}	2.62×10^{-4}	1.66×10^{-4}	1.05×10^{-4}	6.2×10^{-5}	3.51×10^{-5}	1.73×10^{-5}
	4.15×10^{-4}	2.62×10^{-4}	1.66×10^{-4}	1.05×10^{-4}	6.2×10^{-5}	3.51×10^{-5}	1.73×10^{-5}
	8.31×10^{-5}	4.04×10^{-5}	2.03×10^{-5}	1.09×10^{-5}	5.76×10^{-6}	3.07×10^{-6}	1.47×10^{-6}
	8.31×10^{-5}	4.04×10^{-5}	2.03×10^{-5}	1.09×10^{-5}	5.76×10^{-6}	3.07×10^{-6}	1.47×10^{-6}
	2.2×10^{-5}	8.37×10^{-6}	3.41×10^{-6}	1.57×10^{-6}	7.35×10^{-7}	3.62×10^{-7}	1.63×10^{-7}

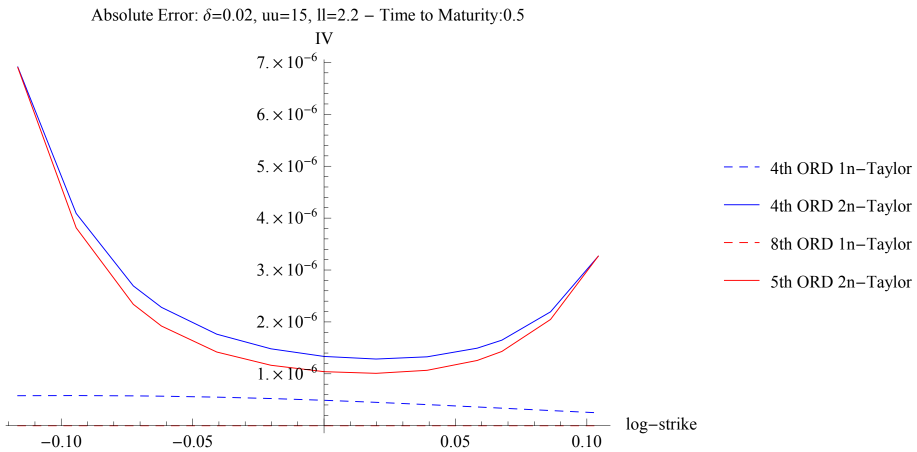
5	6.13×10^{-1}	7.32×10^{-2}	7.66×10^{-2}	8.58×10^{-2}	4.33×10^{-2}	2.53×10^{-2}	$1.\times10^{-1}$
	6.13×10^{-1}	7.32×10^{-2}	7.66×10^{-2}	8.58×10^{-2}	4.33×10^{-2}	2.53×10^{-2}	$1.\times10^{-1}$
	2.07×10^{-2}	1.02×10^{-2}	6.98×10^{-3}	4.47×10^{-3}	2.61×10^{-3}	1.22×10^{-3}	5.43×10^{-4}
	2.07×10^{-2}	1.02×10^{-2}	6.98×10^{-3}	4.47×10^{-3}	2.61×10^{-3}	1.22×10^{-3}	5.43×10^{-4}
	6.09×10^{-3}	2.13×10^{-3}	9.85×10^{-4}	4.86×10^{-4}	2.5×10^{-4}	1.15×10^{-4}	5.09×10^{-5}
	6.09×10^{-3}	2.13×10^{-3}	9.85×10^{-4}	4.86×10^{-4}	2.5×10^{-4}	1.15×10^{-4}	5.09×10^{-5}
	2.85×10^{-3}	6.45×10^{-4}	2.15×10^{-4}	8.35×10^{-5}	3.72×10^{-5}	1.56×10^{-5}	6.58×10^{-6}
	2.85×10^{-3}	6.45×10^{-4}	2.15×10^{-4}	8.35×10^{-5}	3.72×10^{-5}	1.56×10^{-5}	6.58×10^{-6}
	1.75×10^{-3}	2.63×10^{-4}	6.44×10^{-5}	1.99×10^{-5}	7.59×10^{-6}	2.83×10^{-6}	1.11×10^{-6}
10	7.87	4.08×10^{-1}	1.75×10^{-1}	1.74×10^{-1}	6.89×10^{-2}	6.5×10^{-2}	1.97×10^{-1}
	7.87	4.08×10^{-1}	1.75×10^{-1}	1.74×10^{-1}	6.89×10^{-2}	6.5×10^{-2}	1.97×10^{-1}
	4.37	1.25×10^{-1}	4.03×10^{-2}	1.79×10^{-2}	8.37×10^{-3}	3.18×10^{-3}	1.23×10^{-3}
	4.37	1.25×10^{-1}	4.03×10^{-2}	1.79×10^{-2}	8.37×10^{-3}	3.18×10^{-3}	1.23×10^{-3}
	7.66	7.88×10^{-2}	1.31×10^{-2}	3.85×10^{-3}	1.48×10^{-3}	5.36×10^{-4}	1.9×10^{-4}
	7.66	7.88×10^{-2}	1.31×10^{-2}	3.85×10^{-3}	1.48×10^{-3}	5.36×10^{-4}	1.9×10^{-4}
	2.28×10^{1}	7.23×10^{-2}	6.54×10^{-3}	1.3×10^{-3}	4.03×10^{-4}	1.28×10^{-4}	4.19×10^{-5}
	2.28×10^{1}	7.23×10^{-2}	6.54×10^{-3}	1.3×10^{-3}	4.03×10^{-4}	1.28×10^{-4}	4.19×10^{-5}
	8.03×10^{1}	8.82×10^{-2}	4.46×10^{-3}	6.11×10^{-4}	1.51×10^{-4}	4.05×10^{-5}	1.2×10^{-5}

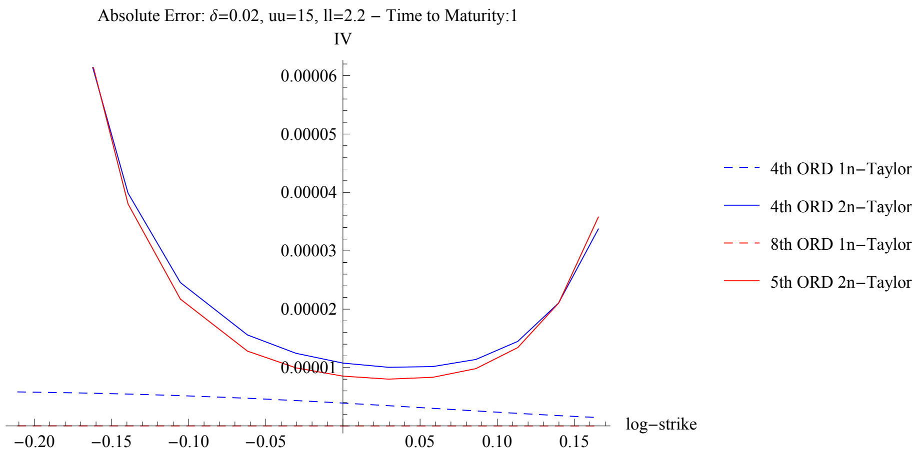
and in the next table we compare the 1n-Taylor approximation with the 2n-Taylor

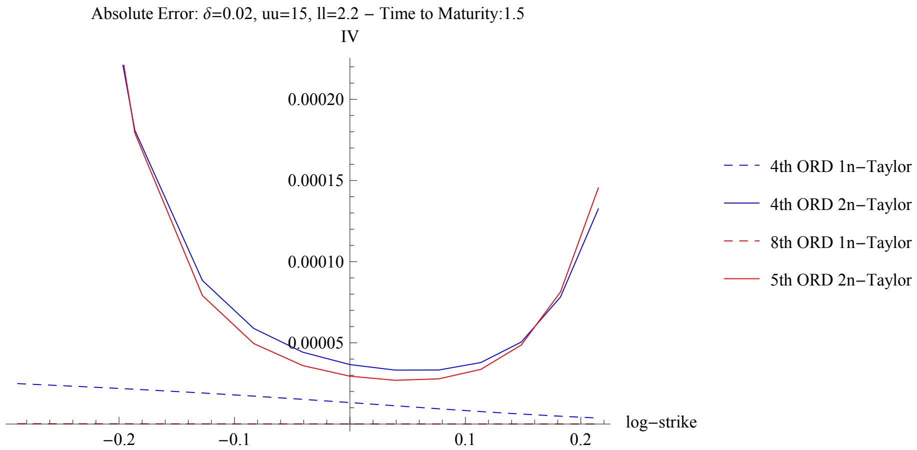
	1		7		13	
0.25	1.03×10^{-3}	1.03×10^{-3}	4.24×10^{-3}	4.24×10^{-3}	7.19×10^{-4}	7.19×10^{-4}
	1.03×10^{-3}	2.8×10^{-2}	4.24×10^{-3}	1.63×10^{-2}	7.19×10^{-4}	2.33×10^{-2}
	1.02×10^{-5}	2.67×10^{-6}	1.11×10^{-5}	7.24×10^{-6}	7.32×10^{-6}	2.08×10^{-6}
	1.02×10^{-5}	6.59×10^{-5}	1.11×10^{-5}	2.5×10^{-5}	7.32×10^{-6}	4.9×10^{-5}
	6.55×10^{-8}	4.13×10^{-7}	6.13×10^{-8}	1.66×10^{-7}	4.15×10^{-8}	2.72×10^{-7}
	6.55×10^{-8}	3.75×10^{-7}	6.13×10^{-8}	1.29×10^{-7}	4.15×10^{-8}	2.48×10^{-7}
	5.84×10^{-10}	0	5.34×10^{-10}	0	3.55×10^{-10}	0
	5.84×10^{-10}	0	5.34×10^{-10}	0	3.55×10^{-10}	0
	4.3×10^{-11}	0	6.3×10^{-12}	0	4.01×10^{-12}	0
0.5	3.03×10^{-3}	3.03×10^{-3}	8.49×10^{-3}	8.49×10^{-3}	2.74×10^{-3}	2.74×10^{-3}
	3.03×10^{-3}	7.55×10^{-2}	8.49×10^{-3}	3.27×10^{-2}	2.74×10^{-3}	5.42×10^{-2}
	4.02×10^{-5}	1.02×10^{-5}	4.46×10^{-5}	2.9×10^{-5}	2.15×10^{-5}	7.35×10^{-6}
	4.02×10^{-5}	4.93×10^{-4}	4.46×10^{-5}	$1.\times10^{-4}$	2.15×10^{-5}	2.89×10^{-4}
	5.78×10^{-7}	6.91×10^{-6}	4.9×10^{-7}	1.33×10^{-6}	2.48×10^{-7}	3.27×10^{-6}
	5.78×10^{-7}	6.9×10^{-6}	4.9×10^{-7}	1.04×10^{-6}	2.48×10^{-7}	3.27×10^{-6}
	1.22×10^{-8}	0	8.53×10^{-9}	0	4.15×10^{-9}	0
	1.22×10^{-8}	0	8.53×10^{-9}	0	4.15×10^{-9}	0
	3.46×10^{-10}	0	2.06×10^{-10}	0	8.53×10^{-11}	0
1	2.12×10^{-2}	2.12×10^{-2}	1.7×10^{-2}	1.7×10^{-2}	1.14×10^{-2}	1.14×10^{-2}
	2.12×10^{-2}	2.15×10^{-1}	1.7×10^{-2}	6.53×10^{-2}	1.14×10^{-2}	1.15×10^{-1}
	1.74×10^{-4}	1.23×10^{-4}	1.78×10^{-4}	1.17×10^{-4}	6.26×10^{-5}	4.79×10^{-5}
	1.74×10^{-4}	4.06×10^{-3}	1.78×10^{-4}	$4.\times10^{-4}$	6.26×10^{-5}	1.43×10^{-3}
	5.83×10^{-6}	1.44×10^{-4}	3.92×10^{-6}	1.08×10^{-5}	1.42×10^{-6}	3.37×10^{-5}
	5.83×10^{-6}	1.56×10^{-4}	3.92×10^{-6}	8.53×10^{-6}	1.42×10^{-6}	3.58×10^{-5}
	2.78×10^{-7}	0	1.36×10^{-7}	0	4.57×10^{-8}	0
	2.78×10^{-7}	0	1.36×10^{-7}	0	4.57×10^{-8}	0
	1.77×10^{-8}	0	6.56×10^{-9}	0	1.95×10^{-9}	0

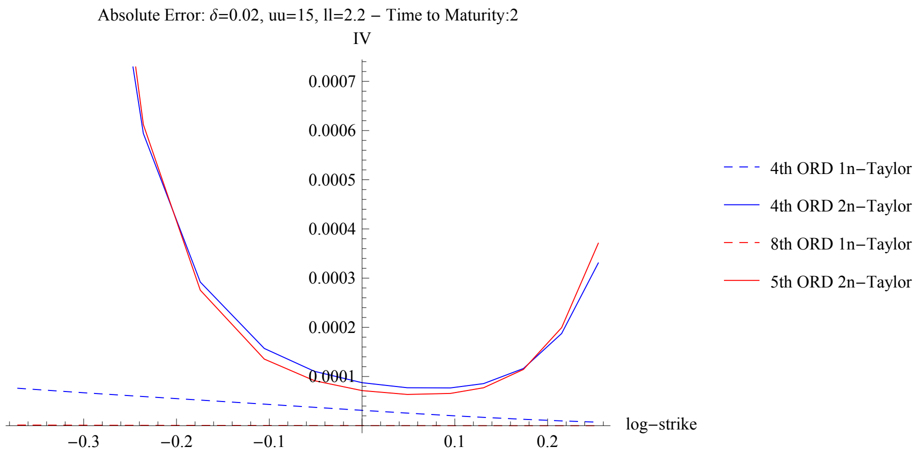
	4.6×10^{-2}	4.6×10^{-2}	2.55×10^{-2}	2.55×10^{-2}	2.25×10^{-2}	2.25×10^{-2}
	4.6×10^{-2}	3.92×10^{-1}	2.55×10^{-2}	9.8×10^{-2}	2.25×10^{-2}	1.77×10^{-1}
	4.52×10^{-4}	4.97×10^{-4}	4.02×10^{-4}	2.64×10^{-4}	1.1×10^{-4}	1.55×10^{-4}
	4.52×10^{-4}	1.34×10^{-2}	4.02×10^{-4}	9.02×10^{-4}	1.1×10^{-4}	3.6×10^{-3}
1.5	2.49×10^{-5}	8.25×10^{-4}	1.32×10^{-5}	3.66×10^{-5}	3.69×10^{-6}	1.33×10^{-4}
	2.49×10^{-5}	9.26×10^{-4}	1.32×10^{-5}	2.95×10^{-5}	3.69×10^{-6}	1.45×10^{-4}
	1.95×10^{-6}	0	6.88×10^{-7}	0	1.71×10^{-7}	0
	1.95×10^{-6}	0	6.88×10^{-7}	0	1.71×10^{-7}	0
	2.03×10^{-7}	0	4.96×10^{-8}	0	1.05×10^{-8}	0
	8.72×10^{-2}	8.72×10^{-2}	3.41×10^{-2}	3.41×10^{-2}	3.35×10^{-2}	3.35×10^{-2}
	8.72×10^{-2}	6.45×10^{-1}	3.41×10^{-2}	1.31×10^{-1}	3.35×10^{-2}	2.35×10^{-1}
	9.51×10^{-4}	1.62×10^{-3}	7.14×10^{-4}	4.72×10^{-4}	1.65×10^{-4}	3.27×10^{-4}
	9.51×10^{-4}	3.57×10^{-2}	7.14×10^{-4}	1.61×10^{-3}	1.65×10^{-4}	6.64×10^{-3}
2	7.61×10^{-5}	3.45×10^{-3}	3.13×10^{-5}	8.75×10^{-5}	7.19×10^{-6}	3.31×10^{-4}
	7.61×10^{-5}	4.03×10^{-3}	3.13×10^{-5}	7.14×10^{-5}	7.19×10^{-6}	3.71×10^{-4}
	8.77×10^{-6}	0	2.17×10^{-6}	0	4.3×10^{-7}	0
	8.77×10^{-6}	0	2.17×10^{-6}	0	4.3×10^{-7}	0
	1.34×10^{-6}	0	2.08×10^{-7}	0	3.4×10^{-8}	0
	1.83×10^{-1}	1.83×10^{-1}	5.12×10^{-2}	5.12×10^{-2}	5.74×10^{-2}	5.74×10^{-2}
	1.83×10^{-1}	1.27	5.12×10^{-2}	1.96×10^{-1}	5.74×10^{-2}	3.48×10^{-1}
	3.05×10^{-3}	6.6×10^{-3}	1.61×10^{-3}	1.07×10^{-3}	2.8×10^{-4}	9.28×10^{-4}
	3.05×10^{-3}	1.31×10^{-1}	1.61×10^{-3}	3.62×10^{-3}	2.8×10^{-4}	1.56×10^{-2}
3	4.15×10^{-4}	2.29×10^{-2}	1.05×10^{-4}	$3.\times10^{-4}$	1.73×10^{-5}	1.19×10^{-3}
	4.15×10^{-4}	2.86×10^{-2}	1.05×10^{-4}	2.51×10^{-4}	1.73×10^{-5}	1.38×10^{-3}
	8.31×10^{-5}	0	1.09×10^{-5}	0	1.47×10^{-6}	0
	8.31×10^{-5}	0	1.09×10^{-5}	0	1.47×10^{-6}	0
	2.2×10^{-5}	0	1.57×10^{-6}	0	1.63×10^{-7}	0
	6.13×10^{-1}	6.13×10^{-1}	8.58×10^{-2}	8.58×10^{-2}	$1.\times10^{-1}$	$1.\times10^{-1}$
	6.13×10^{-1}	3.86	8.58×10^{-2}	3.26×10^{-1}	$1.\times10^{-1}$	5.42×10^{-1}
	2.07×10^{-2}	6.38×10^{-2}	4.47×10^{-3}	3.04×10^{-3}	5.43×10^{-4}	2.87×10^{-3}
	2.07×10^{-2}	1.07	4.47×10^{-3}	1.01×10^{-2}	5.43×10^{-4}	4.12×10^{-2}
5	6.09×10^{-3}	4.75×10^{-1}	4.86×10^{-4}	1.43×10^{-3}	5.09×10^{-5}	5.2×10^{-3}
	6.09×10^{-3}	7.17×10^{-1}	4.86×10^{-4}	1.26×10^{-3}	5.09×10^{-5}	6.34×10^{-3}
	2.85×10^{-3}	0	8.35×10^{-5}	0	6.58×10^{-6}	0
	2.85×10^{-3}	0	8.35×10^{-5}	0	6.58×10^{-6}	0
	1.75×10^{-3}	0	1.99×10^{-5}	0	1.11×10^{-6}	0
	7.87	7.87	1.74×10^{-1}	1.74×10^{-1}	1.97×10^{-1}	1.97×10^{-1}
	7.87	6.29×10^{1}	1.74×10^{-1}	6.5×10^{-1}	1.97×10^{-1}	9.34×10^{-1}
	4.37	1.04×10^{1}	1.79×10^{-2}	1.27×10^{-2}	1.23×10^{-3}	1.16×10^{-2}
	4.37	1.35×10^{2}	1.79×10^{-2}	4.09×10^{-2}	1.23×10^{-3}	1.41×10^{-1}
10	7.66	5.73×10^{2}	3.85×10^{-3}	1.23×10^{-2}	1.9×10^{-4}	3.38×10^{-2}
	7.66	1.73×10^{3}	3.85×10^{-3}	1.19×10^{-2}	1.9×10^{-4}	4.53×10^{-2}
	2.28×10^{1}	0	1.3×10^{-3}	0	4.19×10^{-5}	0
	2.28×10^{1}	0	1.3×10^{-3}	0	4.19×10^{-5}	0
	8.03×10^{1}	0	6.11×10^{-4}	0	1.2×10^{-5}	0

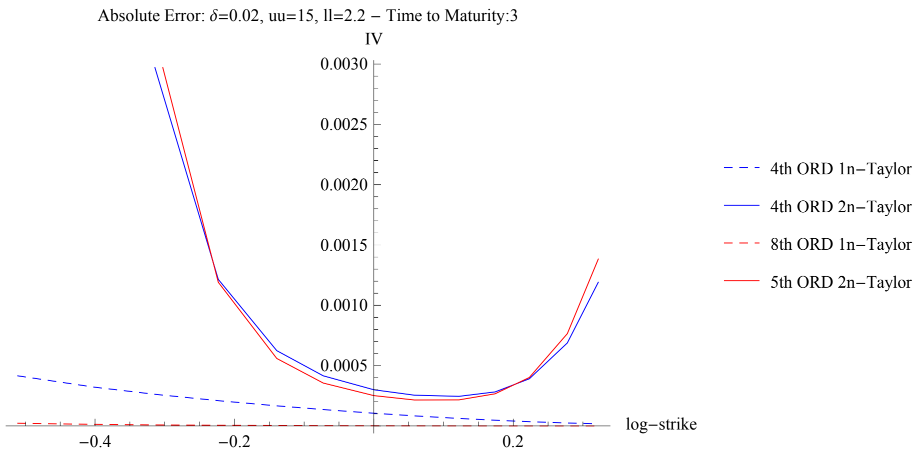

Absolute Error: δ=0.02, uu=15, ll=2.2 – Time to Maturity:0.25
IV
log–strike
4th ORD 1n–Taylor
4th ORD 2n–Taylor
8th ORD 1n–Taylor
5th ORD 2n–Taylor

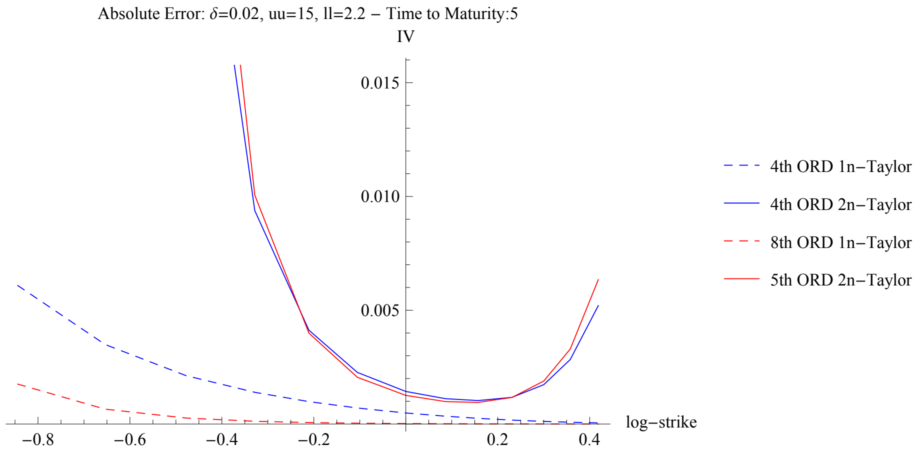

Absolute Error: δ=0.02, uu=15, ll=2.2 – Time to Maturity:0.5
IV
log–strike
4th ORD 1n–Taylor
4th ORD 2n–Taylor
8th ORD 1n–Taylor
5th ORD 2n–Taylor

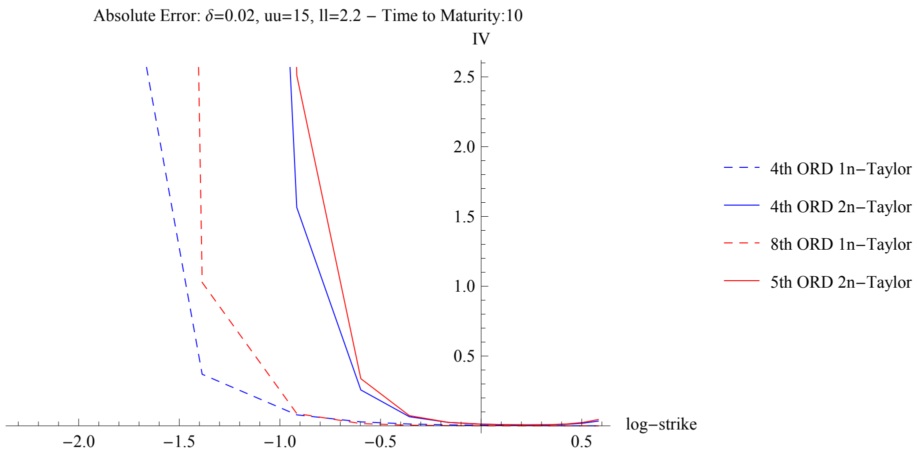

Absolute Error: δ=0.02, uu=15, ll=2.2 – Time to Maturity:1
IV
log–strike
4th ORD 1n–Taylor
4th ORD 2n–Taylor
8th ORD 1n–Taylor
5th ORD 2n–Taylor


Absolute Error: δ=0.02, uu=15, ll=2.2 – Time to Maturity:1.5
IV
0.00020
0.00015
0.00010
0.00005
log–strike
–0.2
–0.1
0.1
0.2
4th ORD 1n–Taylor
4th ORD 2n–Taylor
8th ORD 1n–Taylor
5th ORD 2n–Taylor


Absolute Error: δ=0.02, uu=15, ll=2.2 – Time to Maturity:2
IV
0.0007
0.0006
0.0005
0.0004
0.0003
0.0002
0.0001
log–strike
–0.3
–0.2
–0.1
0.1
0.2
4th ORD 1n–Taylor
4th ORD 2n–Taylor
8th ORD 1n–Taylor
5th ORD 2n–Taylor


Absolute Error: δ=0.02, uu=15, ll=2.2 – Time to Maturity:3
IV
0.0030
0.0025
0.0020
0.0015
0.0010
0.0005
log–strike
–0.4
–0.2
0.2
4th ORD 1n–Taylor
4th ORD 2n–Taylor
8th ORD 1n–Taylor
5th ORD 2n–Taylor





We note that the 2n-Taylor approximation produces good results for maturities up to 5 years, and explodes ITM for a 10-years maturity, as well as the 1n-Taylor approximation. However comparing the values, the results of the 1n-Taylor approximation are significantly better both for orders comparable from a computational point of view, both for the same order in the two approximations.

Therefore also for the quadratic model the normal Taylor approach is the best choice for a practical use.

9.2.2 Second Set: $\delta = 0.05$, uu = 2.2, ll = 2

As in 8.2.2 we consider ($\delta = 0.05$, uu = 2.2, ll = 2) and the table of strikes

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.25	10	25	40	55	70	85	100	115	130	145	160	175	190
0.5	10	25	40	55	70	85	100	115	130	145	160	175	190
1	10	25	40	55	70	85	100	115	130	145	160	175	190
1.5	10	25	40	55	70	85	100	115	130	145	160	175	190
2	10	25	40	55	70	85	100	115	130	145	160	175	190
3	10	25	40	55	70	85	100	115	130	145	160	175	190
5	10	25	40	55	70	85	100	115	130	145	160	175	190
10	10	25	40	55	70	85	100	115	130	145	160	175	190

and the corresponding table of exact implied volatilities

	1	3	5	7	9	11	13
0.25	1516.161	280.797	202.474	160.939	133.709	113.911	98.593
0.5	3541.393	735.526	212.555	165.707	136.358	115.517	99.620
1	2100.000	9480.363	242.331	177.464	142.419	119.043	101.818
1.5	2100.000	10798.438	317.469	194.134	149.845	123.075	104.231
2	2100.000	12493.8663	13514.119	221.982	159.295	127.719	106.862
3	2100.000	1431.074	2540.429	3242.972	191.738	139.540	112.764
5	2100.000	2505.786	8834.002	1835.552	3683.198	206.870	128.666
10	2100.000	6514.533	2261.573	919.668	547.501	4046.632	5644.426

Now we compute absolute errors of our 2n-Taylor approximation

	1	3	5	7	9	11	13
0.25	9.21×10^{2} 9.21×10^{2} 1.43×10^{3} 1.43×10^{3} 1.84×10^{3} 1.84×10^{3} $2.\times10^{3}$ $2.\times10^{3}$ 1.06×10^{5}	6.77 6.77 1.39×10^{1} 1.39×10^{1} 2.09×10^{1} 2.09×10^{1} 4.56×10^{1} 4.56×10^{1} 1.3×10^{2}	6.48 6.48 3.33 3.33 2.83 2.83 3.67 3.67 6.47	4.23 4.23 1.24 1.24 7.19×10^{-1} 7.19×10^{-1} 6.46×10^{-1} 6.46×10^{-1} 7.97×10^{-1}	1.63 1.63 5.1×10^{-1} 5.1×10^{-1} 2.3×10^{-1} 2.3×10^{-1} 1.59×10^{-1} 1.59×10^{-1} 1.5×10^{-1}	9.18×10^{-1} 9.18×10^{-1} 1.98×10^{-1} 1.98×10^{-1} 7.83×10^{-2} 7.83×10^{-2} 4.51×10^{-2} 4.51×10^{-2} 3.47×10^{-2}	3.32 3.32 6.82×10^{-2} 6.82×10^{-2} 2.57×10^{-2} 2.57×10^{-2} 1.32×10^{-2} 1.32×10^{-2} 8.74×10^{-3}
0.5	2.95×10^{3} 2.95×10^{3} 4.95×10^{3} 4.95×10^{3} 1.56×10^{4} 1.56×10^{4} 1.09×10^{5} 1.09×10^{5} 3.08×10^{4}	4.62×10^{2} 4.62×10^{2} 4.88×10^{2} 4.88×10^{2} 5.71×10^{2} 5.71×10^{2} 8.74×10^{2} 8.74×10^{2} 1.92×10^{3}	1.66×10^{1} 1.66×10^{1} 1.41×10^{1} 1.41×10^{1} 2.12×10^{1} 2.12×10^{1} 4.75×10^{1} 4.75×10^{1} 1.41×10^{2}	9. 9. 5.03 5.03 5.44 5.44 8.98 8.98 2.01×10^{1}	4.28 4.28 2.14 2.14 1.82 1.82 2.36 2.36 4.14	6.88×10^{-1} 6.88×10^{-1} 9.61×10^{-1} 9.61×10^{-1} 6.9×10^{-1} 6.9×10^{-1} 7.38×10^{-1} 7.38×10^{-1} 1.06	2.29 2.29 4.22×10^{-1} 4.22×10^{-1} 2.73×10^{-1} 2.73×10^{-1} 2.53×10^{-1} 2.53×10^{-1} 3.11×10^{-1}
1	1.5×10^{3} 1.5×10^{3} 9.89×10^{3} 9.89×10^{3} 5.89×10^{5} 5.89×10^{5} 5.86×10^{7} 5.86×10^{7} 6.38×10^{9}	9.21×10^{3} 9.21×10^{3} 9.36×10^{3} 9.36×10^{3} 9.73×10^{3} 9.73×10^{3} 9.27×10^{3} 9.27×10^{3} 2.26×10^{4}	4.63×10^{1} 4.63×10^{1} 6.24×10^{1} 6.24×10^{1} 1.41×10^{2} 1.41×10^{2} 4.3×10^{2} 4.3×10^{2} 1.36×10^{3}	2.08×10^{1} 2.08×10^{1} 2.09×10^{1} 2.09×10^{1} 3.85×10^{1} 3.85×10^{1} 1.04×10^{2} 1.04×10^{2} 3.6×10^{2}	1.03×10^{1} 1.03×10^{1} 8.94 8.94 1.36×10^{1} 1.36×10^{1} 3.07×10^{1} 3.07×10^{1} 9.17×10^{1}	4.21 4.21 4.26 4.26 5.5 5.5 1.06×10^{1} 1.06×10^{1} 2.7×10^{1}	9.08×10^{-2} 9.08×10^{-2} 2.13 2.13 2.42 2.42 4.03 4.03 8.9
1.5	1.5×10^{3} 1.5×10^{3} 2.08×10^{4} 2.08×10^{4} 3.51×10^{6} 3.51×10^{6} 9.43×10^{8} 9.43×10^{8} 2.93×10^{11}	1.05×10^{4} 1.05×10^{4} 1.09×10^{4} 1.09×10^{4} 1.06×10^{4} 1.06×10^{4} 2.9×10^{3} 2.9×10^{3} 1.82×10^{5}	1.21×10^{2} 1.21×10^{2} 1.79×10^{2} 1.79×10^{2} 3.94×10^{2} 3.94×10^{2} 8.79×10^{2} 8.79×10^{2} 3.32×10^{3}	3.74×10^{1} 3.74×10^{1} 4.98×10^{1} 4.98×10^{1} 1.14×10^{2} 1.14×10^{2} 3.54×10^{2} 3.54×10^{2} 1.15×10^{3}	1.78×10^{1} 1.78×10^{1} 2.08×10^{1} 2.08×10^{1} 4.19×10^{1} 4.19×10^{1} 1.21×10^{2} 1.21×10^{2} 4.37×10^{2}	8.25 8.25 $1.\times10^{1}$ $1.\times10^{1}$ 1.77×10^{1} 1.77×10^{1} 4.54×10^{1} 4.54×10^{1} 1.51×10^{2}	2.32 2.32 5.18 5.18 8.11 8.11 1.85×10^{1} 1.85×10^{1} 5.51×10^{1}

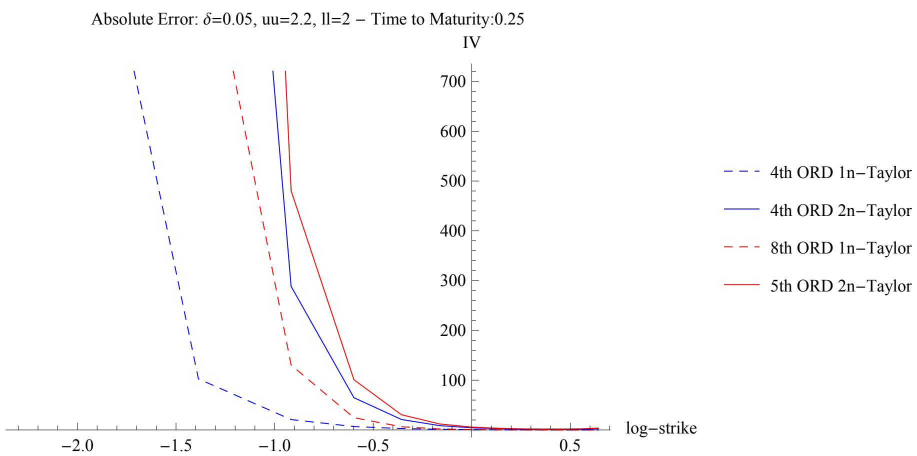
	1.5×10^{3}	1.22×10^{4}	1.33×10^{4}	6.53×10^{1}	2.72×10^{1}	1.29×10^{1}	4.95
	1.5×10^{3}	1.22×10^{4}	1.33×10^{4}	6.53×10^{1}	2.72×10^{1}	1.29×10^{1}	4.95
	3.63×10^{4}	1.29×10^{4}	1.34×10^{4}	9.79×10^{1}	3.83×10^{1}	1.83×10^{1}	9.6
	3.63×10^{4}	1.29×10^{4}	1.34×10^{4}	9.79×10^{1}	3.83×10^{1}	1.83×10^{1}	9.6
2	1.19×10^{7}	8.18×10^{3}	1.38×10^{4}	2.34×10^{2}	$9.\times10^{1}$	3.92×10^{1}	1.86×10^{1}
	1.19×10^{7}	8.18×10^{3}	1.38×10^{4}	2.34×10^{2}	$9.\times10^{1}$	3.92×10^{1}	1.86×10^{1}
	6.16×10^{9}	3.68×10^{3}	1.28×10^{4}	6.57×10^{2}	2.86×10^{2}	1.18×10^{2}	5.13×10^{1}
	6.16×10^{9}	3.68×10^{3}	1.28×10^{4}	6.57×10^{2}	2.86×10^{2}	1.18×10^{2}	5.13×10^{1}
	3.76×10^{12}	1.14×10^{5}	2.43×10^{4}	2.18×10^{2}	9.64×10^{2}	4.35×10^{2}	1.8×10^{2}
	1.5×10^{3}	1.16×10^{3}	2.34×10^{3}	3.09×10^{3}	5.97×10^{1}	2.47×10^{1}	1.09×10^{1}
	1.5×10^{3}	1.16×10^{3}	2.34×10^{3}	3.09×10^{3}	5.97×10^{1}	2.47×10^{1}	1.09×10^{1}
	8.1×10^{4}	2.89×10^{3}	2.66×10^{3}	3.18×10^{3}	9.81×10^{1}	4.34×10^{1}	2.25×10^{1}
	8.1×10^{4}	2.89×10^{3}	2.66×10^{3}	3.18×10^{3}	9.81×10^{1}	4.34×10^{1}	2.25×10^{1}
3	6.47×10^{7}	4.22×10^{4}	2.19×10^{3}	3.45×10^{3}	2.43×10^{2}	1.13×10^{2}	5.61×10^{1}
	6.47×10^{7}	4.22×10^{4}	2.19×10^{3}	3.45×10^{3}	2.43×10^{2}	1.13×10^{2}	5.61×10^{1}
	8.06×10^{10}	1.28×10^{6}	1.15×10^{4}	2.79×10^{3}	6.24×10^{2}	3.67×10^{2}	1.87×10^{2}
	8.06×10^{10}	1.28×10^{6}	1.15×10^{4}	2.79×10^{3}	6.24×10^{2}	3.67×10^{2}	1.87×10^{2}
	1.21×10^{14}	3.39×10^{7}	1.66×10^{5}	2.77×10^{4}	1.56×10^{3}	9.47×10^{2}	6.79×10^{2}
	1.5×10^{3}	2.23×10^{3}	8.64×10^{3}	1.68×10^{3}	3.55×10^{3}	9.2×10^{1}	2.68×10^{1}
	1.5×10^{3}	2.23×10^{3}	8.64×10^{3}	1.68×10^{3}	3.55×10^{3}	9.2×10^{1}	2.68×10^{1}
	2.25×10^{5}	7.32×10^{3}	9.62×10^{3}	$2.\times10^{3}$	3.69×10^{3}	1.61×10^{2}	6.6×10^{1}
	2.25×10^{5}	7.32×10^{3}	9.62×10^{3}	$2.\times10^{3}$	3.69×10^{3}	1.61×10^{2}	6.6×10^{1}
5	5.27×10^{8}	4.77×10^{5}	8.26×10^{3}	9.79×10^{2}	3.87×10^{3}	3.57×10^{2}	1.9×10^{2}
	5.27×10^{8}	4.77×10^{5}	8.26×10^{3}	9.79×10^{2}	3.87×10^{3}	3.57×10^{2}	1.9×10^{2}
	1.92×10^{12}	6.45×10^{7}	2.91×10^{5}	1.44×10^{4}	4.23×10^{2}	1.83×10^{2}	4.84×10^{2}
	1.92×10^{12}	6.45×10^{7}	2.91×10^{5}	1.44×10^{4}	4.23×10^{2}	1.83×10^{2}	4.84×10^{2}
	8.5×10^{15}	9.68×10^{9}	3.14×10^{6}	1.26×10^{5}	7.42×10^{4}	1.76×10^{4}	2.51×10^{3}
	1.5×10^{3}	6.24×10^{3}	2.07×10^{3}	7.63×10^{2}	4.15×10^{2}	3.93×10^{3}	5.54×10^{3}
	1.5×10^{3}	6.24×10^{3}	2.07×10^{3}	7.63×10^{2}	4.15×10^{2}	3.93×10^{3}	5.54×10^{3}
	9.03×10^{5}	2.74×10^{4}	6.31×10^{3}	2.22×10^{3}	1.06×10^{3}	4.26×10^{3}	5.73×10^{3}
	9.03×10^{5}	2.74×10^{4}	6.31×10^{3}	2.22×10^{3}	1.06×10^{3}	4.26×10^{3}	5.73×10^{3}
10	8.75×10^{9}	9.45×10^{6}	4.64×10^{5}	5.59×10^{4}	9.28×10^{3}	2.25×10^{3}	5.47×10^{3}
	8.75×10^{9}	9.45×10^{6}	4.64×10^{5}	5.59×10^{4}	9.28×10^{3}	2.25×10^{3}	5.47×10^{3}
	1.32×10^{14}	6.38×10^{9}	7.34×10^{7}	2.82×10^{6}	1.21×10^{5}	1.02×10^{4}	3.66×10^{3}
	1.32×10^{14}	6.38×10^{9}	7.34×10^{7}	2.82×10^{6}	1.21×10^{5}	1.02×10^{4}	3.66×10^{3}
	2.45×10^{18}	5.11×10^{12}	1.3×10^{10}	1.39×10^{8}	5.44×10^{5}	1.91×10^{4}	1.25×10^{5}

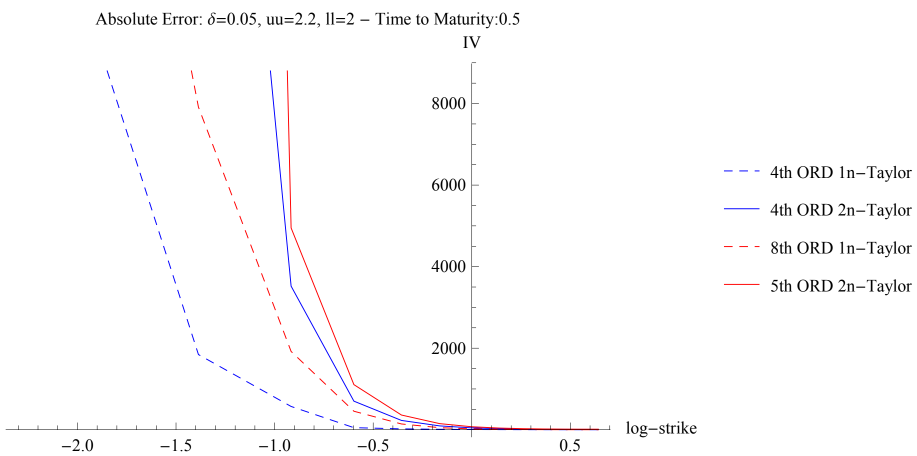
and in the next table we compare the 1n-Taylor approximation with the 2n-Taylor

	1		7		13	
	9.21×10^{2}	9.21×10^{2}	4.23	4.23	3.32	3.32
	9.21×10^{2}	3.66×10^{2}	4.23	1.95×10^{1}	3.32	1.71×10^{1}
	1.43×10^{3}	2.61×10^{3}	1.24	1.47	6.82×10^{-2}	4.37×10^{-1}
	1.43×10^{3}	4.18×10^{3}	1.24	4.12	6.82×10^{-2}	4.88
0.25	1.84×10^{3}	1.25×10^{5}	7.19×10^{-1}	4.32	2.57×10^{-2}	2.4
	1.84×10^{3}	1.02×10^{6}	7.19×10^{-1}	5.81	2.57×10^{-2}	3.69
	$2.\times10^{3}$	0	6.46×10^{-1}	0	1.32×10^{-2}	0
	$2.\times10^{3}$	0	6.46×10^{-1}	0	1.32×10^{-2}	0
	1.06×10^{5}	0	7.97×10^{-1}	0	8.74×10^{-3}	0

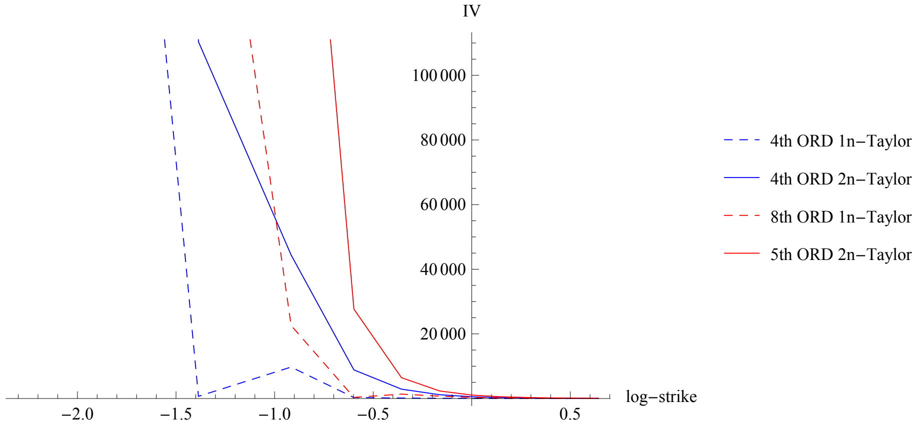
	2.95×10^{3}	2.95×10^{3}	9.	9.	2.29	2.29
	2.95×10^{3}	6.69×10^{2}	9.	3.85×10^{1}	2.29	2.37×10^{1}
	4.95×10^{3}	1.57×10^{4}	5.03	6.84	4.22×10^{-1}	9.06×10^{-1}
	4.95×10^{3}	4.85×10^{4}	5.03	1.7×10^{1}	4.22×10^{-1}	8.71
0.5	1.56×10^{4}	7.81×10^{5}	5.44	4.54×10^{1}	2.73×10^{-1}	7.11
	1.56×10^{4}	7.07×10^{7}	5.44	7.28×10^{1}	2.73×10^{-1}	1.23×10^{1}
	1.09×10^{5}	0	8.98	0	2.53×10^{-1}	0
	1.09×10^{5}	0	8.98	0	2.53×10^{-1}	0
	3.08×10^{4}	0	2.01×10^{1}	0	3.11×10^{-1}	0
	1.5×10^{3}	1.5×10^{3}	2.08×10^{1}	2.08×10^{1}	9.08×10^{-2}	9.08×10^{-2}
	1.5×10^{3}	2.75×10^{3}	2.08×10^{1}	7.42×10^{1}	9.08×10^{-2}	3.69×10^{1}
	9.89×10^{3}	$1.\times10^{5}$	2.09×10^{1}	3.52×10^{1}	2.13	3.86
	9.89×10^{3}	1.3×10^{6}	2.09×10^{1}	6.27×10^{1}	2.13	2.2×10^{1}
1	5.89×10^{5}	1.02×10^{8}	3.85×10^{1}	5.5×10^{2}	2.42	4.18×10^{1}
	5.89×10^{5}	3.95×10^{9}	3.85×10^{1}	1.07×10^{3}	2.42	$9.\times10^{1}$
	5.86×10^{7}	0	1.04×10^{2}	0	4.03	0
	5.86×10^{7}	0	1.04×10^{2}	0	4.03	0
	6.38×10^{9}	0	3.6×10^{2}	0	8.9	0
	1.5×10^{3}	1.5×10^{3}	3.74×10^{1}	3.74×10^{1}	2.32	2.32
	1.5×10^{3}	4.73×10^{3}	3.74×10^{1}	1.05×10^{2}	2.32	4.98×10^{1}
	2.08×10^{4}	3.3×10^{5}	4.98×10^{1}	9.78×10^{1}	5.18	1.01×10^{1}
	2.08×10^{4}	1.73×10^{7}	4.98×10^{1}	1.04×10^{2}	5.18	4.27×10^{1}
1.5	3.51×10^{6}	3.38×10^{8}	1.14×10^{2}	2.48×10^{3}	8.11	1.49×10^{2}
	3.51×10^{6}	1.89×10^{11}	1.14×10^{2}	6.53×10^{3}	8.11	3.7×10^{2}
	9.43×10^{8}	0	3.54×10^{2}	0	1.85×10^{1}	0
	9.43×10^{8}	0	3.54×10^{2}	0	1.85×10^{1}	0
	2.93×10^{11}	0	1.15×10^{3}	0	5.51×10^{1}	0
	1.5×10^{3}	1.5×10^{3}	6.53×10^{1}	6.53×10^{1}	4.95	4.95
	1.5×10^{3}	6.71×10^{3}	6.53×10^{1}	1.25×10^{2}	4.95	6.25×10^{1}
	3.63×10^{4}	7.74×10^{5}	9.79×10^{1}	2.12×10^{2}	9.6	2.04×10^{1}
	3.63×10^{4}	8.86×10^{7}	9.79×10^{1}	8.26×10^{1}	9.6	7.01×10^{1}
2	1.19×10^{7}	4.16×10^{9}	2.34×10^{2}	7.21×10^{3}	1.86×10^{1}	3.96×10^{2}
	1.19×10^{7}	1.51×10^{12}	2.34×10^{2}	2.89×10^{4}	1.86×10^{1}	1.1×10^{3}
	6.16×10^{9}	0	6.57×10^{2}	0	5.13×10^{1}	0
	6.16×10^{9}	0	6.57×10^{2}	0	5.13×10^{1}	0
	3.76×10^{12}	0	2.18×10^{2}	0	1.8×10^{2}	0
	1.5×10^{3}	1.5×10^{3}	3.09×10^{3}	3.09×10^{3}	1.09×10^{1}	1.09×10^{1}
	1.5×10^{3}	1.07×10^{4}	3.09×10^{3}	2.8×10^{3}	1.09×10^{1}	8.73×10^{1}
	8.1×10^{4}	2.58×10^{6}	3.18×10^{3}	3.57×10^{3}	2.25×10^{1}	5.59×10^{1}
	8.1×10^{4}	7.9×10^{8}	3.18×10^{3}	3.29×10^{3}	2.25×10^{1}	1.37×10^{2}
3	6.47×10^{7}	1.99×10^{11}	3.45×10^{3}	3.27×10^{4}	5.61×10^{1}	1.68×10^{3}
	6.47×10^{7}	1.28×10^{13}	3.45×10^{3}	3.05×10^{5}	5.61×10^{1}	5.98×10^{3}
	8.06×10^{10}	0	2.79×10^{3}	0	1.87×10^{2}	0
	8.06×10^{10}	0	2.79×10^{3}	0	1.87×10^{2}	0
	1.21×10^{14}	0	2.77×10^{4}	0	6.79×10^{2}	0

	1.5×10^{3}	1.5×10^{3}	1.68×10^{3}	1.68×10^{3}	2.68×10^{1}	2.68×10^{1}
	1.5×10^{3}	1.86×10^{4}	1.68×10^{3}	1.2×10^{3}	2.68×10^{1}	1.33×10^{2}
	2.25×10^{5}	1.19×10^{7}	$2.\times10^{3}$	3.77×10^{3}	6.6×10^{1}	2.08×10^{2}
	2.25×10^{5}	1.13×10^{10}	$2.\times10^{3}$	4.19×10^{3}	6.6×10^{1}	2.36×10^{2}
5	5.27×10^{8}	1.11×10^{13}	9.79×10^{2}	7.9×10^{4}	1.9×10^{2}	1.08×10^{4}
	5.27×10^{8}	9.2×10^{15}	9.79×10^{2}	6.84×10^{6}	1.9×10^{2}	7.27×10^{4}
	1.92×10^{12}	0	1.44×10^{4}	0	4.84×10^{2}	0
	1.92×10^{12}	0	1.44×10^{4}	0	4.84×10^{2}	0
	8.5×10^{15}	0	1.26×10^{5}	0	2.51×10^{3}	0
	1.5×10^{3}	1.5×10^{3}	7.63×10^{2}	7.63×10^{2}	5.54×10^{3}	5.54×10^{3}
	1.5×10^{3}	3.84×10^{4}	7.63×10^{2}	1.87×10^{2}	5.54×10^{3}	5.23×10^{3}
	9.03×10^{5}	9.41×10^{7}	2.22×10^{3}	1.64×10^{4}	5.73×10^{3}	6.79×10^{3}
	9.03×10^{5}	3.92×10^{11}	2.22×10^{3}	1.05×10^{5}	5.73×10^{3}	5.8×10^{3}
10	8.75×10^{9}	1.81×10^{15}	5.59×10^{4}	7.45×10^{6}	5.47×10^{3}	9.4×10^{4}
	8.75×10^{9}	8.43×10^{18}	5.59×10^{4}	8.92×10^{7}	5.47×10^{3}	3.82×10^{6}
	1.32×10^{14}	0	2.82×10^{6}	0	3.66×10^{3}	0
	1.32×10^{14}	0	2.82×10^{6}	0	3.66×10^{3}	0
	2.45×10^{18}	0	1.39×10^{8}	0	1.25×10^{5}	0

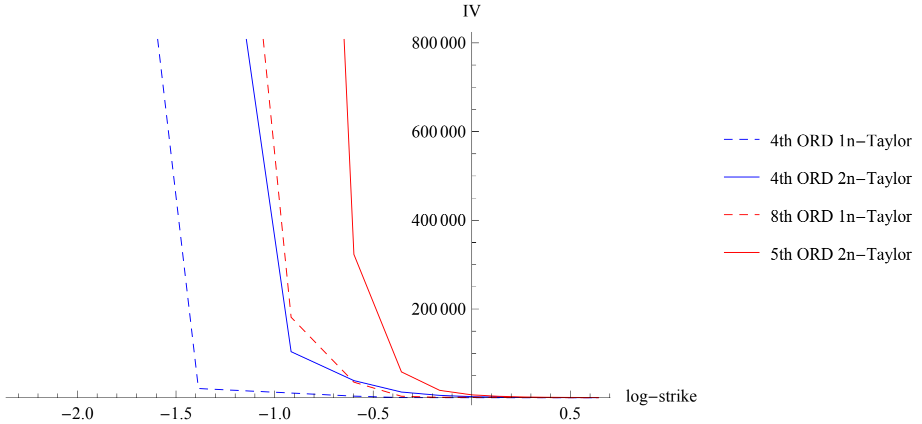




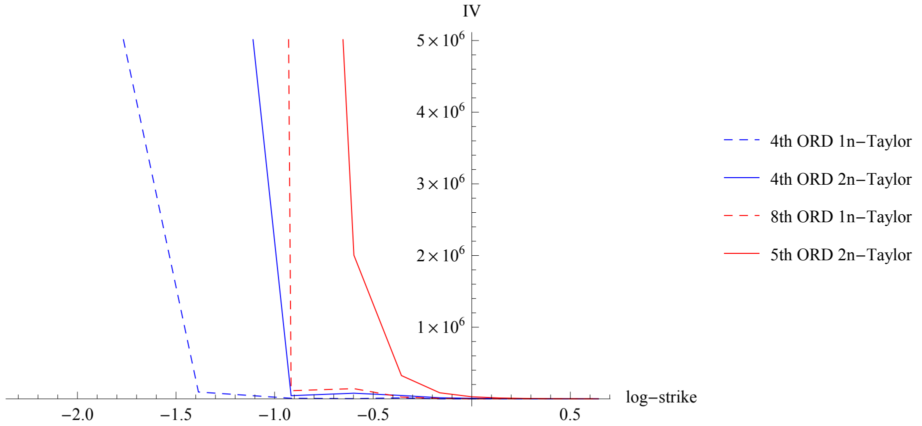
Absolute Error: δ=0.05, uu=2.2, ll=2 – Time to Maturity:1

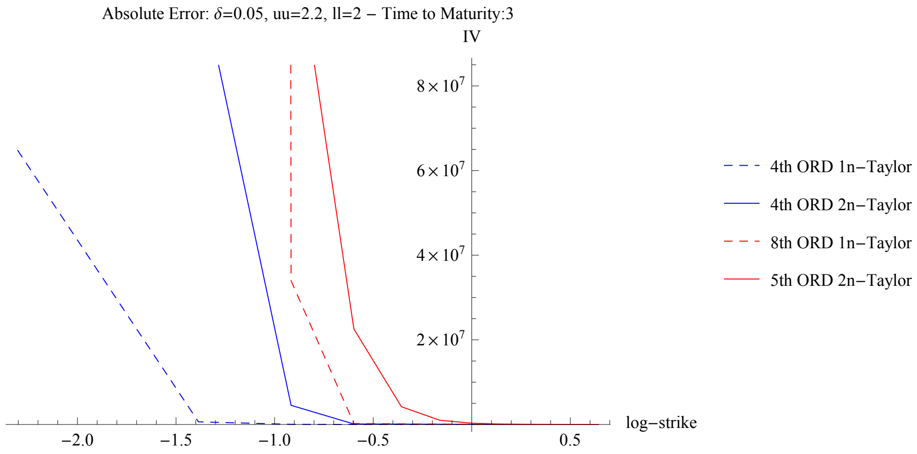


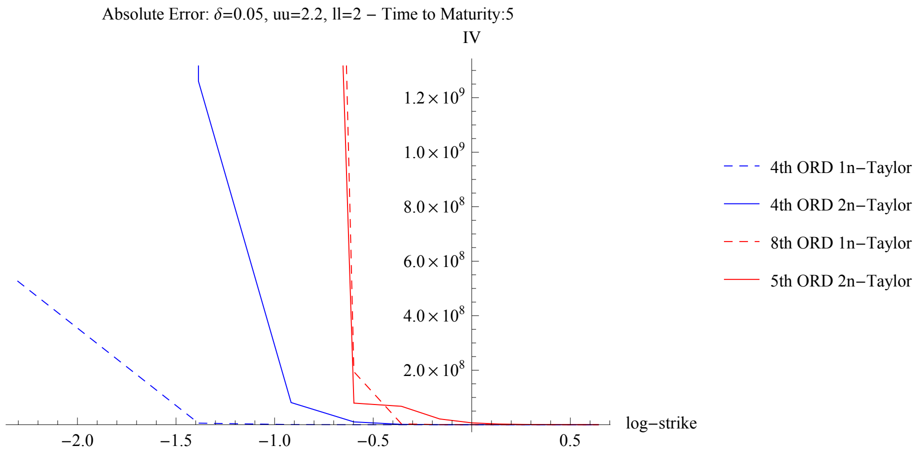
Absolute Error: δ=0.05, uu=2.2, ll=2 – Time to Maturity:1.5

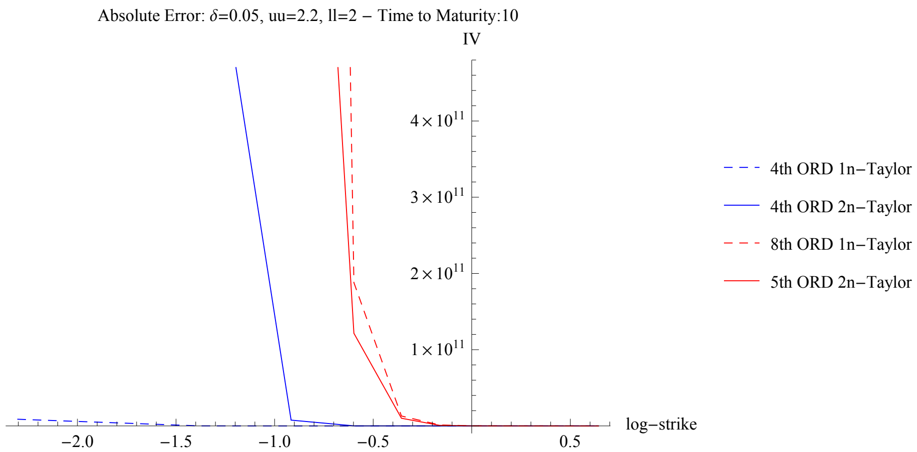


Absolute Error: δ=0.05, uu=2.2, ll=2 – Time to Maturity:2




Absolute Error: δ=0.05, uu=2.2, ll=2 – Time to Maturity:3
IV
8×10^7
6×10^7
4×10^7
2×10^7
log–strike
–2.0
–1.5
–1.0
–0.5
0.5
4th ORD 1n–Taylor
4th ORD 2n–Taylor
8th ORD 1n–Taylor
5th ORD 2n–Taylor


Absolute Error: δ=0.05, uu=2.2, ll=2 – Time to Maturity:5
IV
1.2×10^9
1.0×10^9
8.0×10^8
6.0×10^8
4.0×10^8
2.0×10^8
log–strike
–2.0
–1.5
–1.0
–0.5
0.5
4th ORD 1n–Taylor
4th ORD 2n–Taylor
8th ORD 1n–Taylor
5th ORD 2n–Taylor


Absolute Error: δ=0.05, uu=2.2, ll=2 – Time to Maturity:10
IV
4×10^{11}
3×10^{11}
2×10^{11}
1×10^{11}
log–strike
–2.0
–1.5
–1.0
–0.5
0.5
4th ORD 1n–Taylor
4th ORD 2n–Taylor
8th ORD 1n–Taylor
5th ORD 2n–Taylor

9.3 Heston stochastic volatility model

For the Heston model we compute formulas of the 2n-Taylor appoximation till the 3rd order, and we present here the first ones

$$\sigma_0 = e^{\frac{\bar{y}}{2}}$$

$$\begin{aligned}\sigma_1 = \frac{1}{48} e^{-\frac{3\bar{y}}{2}} \Big(& \left(2\,(-k + x)\,\delta\,\rho + t\left(\delta^2 - 2\,\theta\,\kappa + 2\,r\,\delta\,\rho\right)\right)^2 + \\ & e^{2\bar{y}}\left(t\,(2\,\kappa - \delta\,\rho)\,(-6 + 2\,t\,\kappa - t\,\delta\,\rho) + 6\,(4 - 2\,t\,\kappa + t\,\delta\,\rho)\,(y - \bar{y}) + 12\,(y - \bar{y})^2\right) - \\ & 2\,e^{\bar{y}}\Big(t^2\,(-2\,\kappa + \delta\,\rho)\left(\delta^2 - 2\,\theta\,\kappa + 2\,r\,\delta\,\rho\right) + t\Big(2\,(-3\,\theta\,\kappa + \delta\,\rho\,(3\,r - 2\,(-k + x)\,\kappa + \delta\,\rho + (-k + x)\,\delta\,\rho)) + \\ & \qquad 3\left(\delta^2 - 2\,\theta\,\kappa + 2\,r\,\delta\,\rho\right)(y - \bar{y})\Big) + 6\,(-k + x)\,\delta\,\rho\,(1 + y - \bar{y})\Big)\Big)\end{aligned}$$

For every set of parameters we consider only the choice $\bar{x} = x$ and $\bar{y} = y$ because as we have seen in 8.3, produces better results with an inferior computational cost.

9.3.1 First Set by Ribeiro and Poulsen

As in 8.3.1, we consider parameters as given by Ribeiro and Poulsen in ([RP13]), so we have ($\kappa = 1$, $\theta = 0.08$, $\delta = 0.39$, $\rho = -0.93$, $y = 2\,\mathrm{Log}(0.245)$).

Then we consider the same table of strikes

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.1	94	95	96	97	98	99	100	101	102	102	103	104	105
0.25	86	88	91	93	95	98	100	102	103	105	107	108	110
0.5	76	80	84	88	92	96	100	103	105	108	111	113	116
1	63	69	75	82	88	94	100	104	108	112	116	120	124
1.5	55	62	70	78	85	92	100	105	111	116	121	127	132
2	48	57	65	74	83	91	100	107	113	120	127	133	140
3	38	48	59	69	79	90	100	109	119	128	137	147	156
5	38	48	59	69	79	90	100	115	129	144	159	173	188

and corresponding exact implied volatilities

	1	3	5	7	9	11	13
0.1	26.4519	25.7834	25.0984	24.3942	23.6673	23.2942	22.5258
0.25	28.6691	27.1509	25.8952	24.2514	23.2114	21.7408	20.5581
0.5	30.9509	28.7281	26.4527	24.0610	22.4702	20.4238	18.5902
1	33.1609	30.1114	26.8893	23.8763	21.8010	19.6558	17.4762
1.5	33.9733	30.4313	27.0892	23.8396	21.5608	19.4796	17.1115
2	34.7029	30.8063	26.9724	23.9888	21.7937	19.3489	16.9145
3	36.1483	29.9554	27.1799	24.5620	22.0946	19.6375	16.6877
5	28.4428	28.8170	27.5302	25.6980	23.0257	20.2249	16.9747

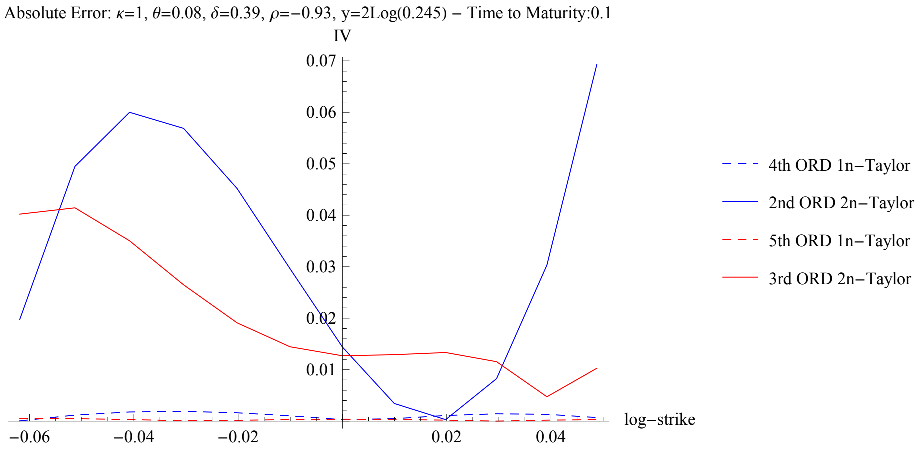
Now we compute absolute errors of our 2n-Taylor approximation from exact values

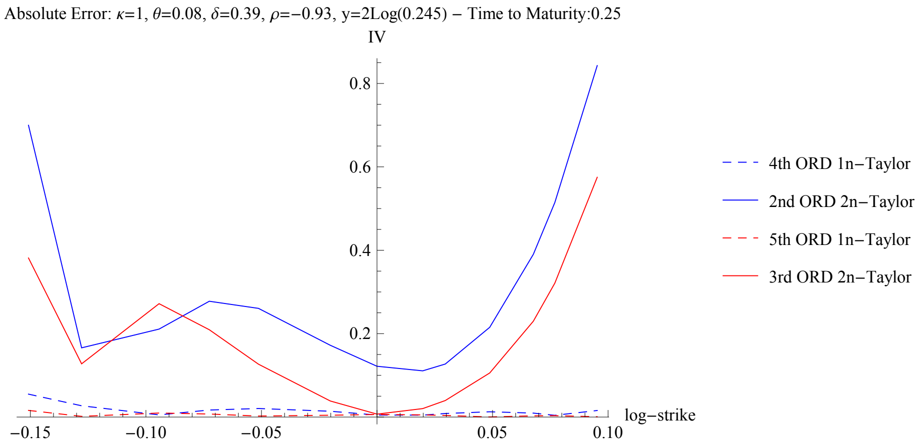
	1	3	5	7	9	11	13
0.1	1.95 1.24×10^{-1} 1.98×10^{-2} 4.02×10^{-2}	1.28 $9.\times10^{-2}$ $6.\times10^{-2}$ 3.5×10^{-2}	5.98×10^{-1} 2.05×10^{-1} 4.52×10^{-2} 1.91×10^{-2}	1.06×10^{-1} 2.23×10^{-1} 1.44×10^{-2} 1.27×10^{-2}	8.33×10^{-1} 1.46×10^{-1} 2.87×10^{-4} 1.33×10^{-2}	1.21 7.1×10^{-2} 8.25×10^{-3} 1.16×10^{-2}	1.97 1.51×10^{-1} 6.93×10^{-2} 1.03×10^{-2}
0.25	4.17 1.34 $7.\times10^{-1}$ 3.81×10^{-1}	2.65 1.26×10^{-1} 2.11×10^{-1} 2.72×10^{-1}	1.4 3.82×10^{-1} 2.61×10^{-1} 1.27×10^{-1}	2.49×10^{-1} 4.79×10^{-1} 1.22×10^{-1} $7.\times10^{-3}$	1.29 2.65×10^{-1} 1.27×10^{-1} 3.96×10^{-2}	2.76 3.34×10^{-1} 3.9×10^{-1} 2.3×10^{-1}	3.94 1.02 8.43×10^{-1} 5.75×10^{-1}
0.5	6.45 4.41 5.45 9.65	4.23 9.53×10^{-1} 2.49×10^{-1} 1.01	1.95 6.34×10^{-1} 8.8×10^{-1} 2.3×10^{-1}	4.39×10^{-1} 6.99×10^{-1} 6.95×10^{-1} 6.87×10^{-1}	2.03 6.54×10^{-2} 1.04 1.34	4.08 1.32 2.24 2.69	5.91 2.96 3.93 4.42
1	8.66 1.06×10^{1} 2.48×10^{1} 9.08×10^{1}	5.61 2.28 5.71×10^{-1} 3.09	2.39 9.65×10^{-1} 3.25 3.45	6.24×10^{-1} 3.83×10^{-1} 4.45 1.18×10^{1}	2.7 1.4 7.07 1.75×10^{1}	4.84 4.07 1.11×10^{1} 2.24×10^{1}	7.02 7.43 1.58×10^{1} 2.51×10^{1}
1.5	9.47 1.5×10^{1} 4.5×10^{1} 2.22×10^{2}	5.93 2.42 3.3 3.96	2.59 9.19×10^{-1} 8.04 2.45×10^{1}	6.6×10^{-1} 1.06 1.39×10^{1} 5.62×10^{1}	2.94 4.59 2.15×10^{1} 7.11×10^{1}	5.02 8.94 2.92×10^{1} 7.19×10^{1}	7.39 1.47×10^{1} 3.64×10^{1} 5.93×10^{1}
2	1.02×10^{1} 1.98×10^{1} 7.51×10^{1} 4.81×10^{2}	6.31 2.83 6.82 4.26	2.47 1.77×10^{-1} 1.72×10^{1} 9.08×10^{1}	5.11×10^{-1} 3.58 3.15×10^{1} 1.67×10^{2}	2.71 9.09 4.56×10^{1} 1.82×10^{2}	5.15 1.68×10^{1} 5.79×10^{1} 1.45×10^{2}	7.59 2.52×10^{1} 6.21×10^{1} 8.94×10^{1}
3	1.16×10^{1} 2.63×10^{1} 1.36×10^{2} 1.2×10^{3}	5.46 3.48 2.44×10^{1} 1.37×10^{2}	2.68 2.95 5.57×10^{1} 5.3×10^{2}	6.2×10^{-2} 1.22×10^{1} 1.01×10^{2} 7.55×10^{2}	2.41 2.48×10^{1} 1.33×10^{2} 6.01×10^{2}	4.86 3.88×10^{1} 1.35×10^{2} 2.86×10^{2}	7.81 5.5×10^{1} 9.85×10^{1} 1.79×10^{2}
5	3.94 1.43×10^{1} 4.59×10^{1} 7.2×10^{2}	4.32 9.13 1.82×10^{2} 3.44×10^{3}	3.03 2.32×10^{1} 3.32×10^{2} 5.22×10^{3}	1.2 4.46×10^{1} 4.49×10^{2} 4.72×10^{3}	1.47 7.77×10^{1} 4.54×10^{2} 1.64×10^{3}	4.28 1.13×10^{2} 2.34×10^{2} 3.94×10^{2}	7.53 1.47×10^{2} 1.87×10^{2} 6.21×10^{3}

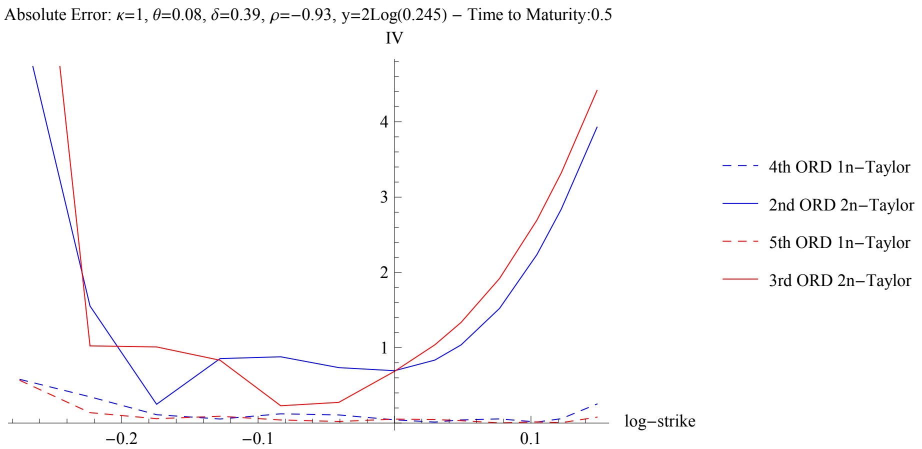
and then we compare 1n-Taylor approximation with 2n-Taylor approximation

	1		7		13	
0.1	1.95	1.95	1.06×10^{-1}	1.06×10^{-1}	1.97	1.97
	3.45×10^{-1}	1.24×10^{-1}	5.77×10^{-1}	2.23×10^{-1}	5.15×10^{-1}	1.51×10^{-1}
	6.99×10^{-2}	1.98×10^{-2}	1.31×10^{-2}	1.44×10^{-2}	5.64×10^{-2}	6.93×10^{-2}
	1.15×10^{-2}	4.02×10^{-2}	6.94×10^{-3}	1.27×10^{-2}	5.98×10^{-3}	1.03×10^{-2}
	4.66×10^{-5}	0	2.86×10^{-4}	0	6.93×10^{-4}	0
	4.98×10^{-4}	0	3.74×10^{-4}	0	2.89×10^{-4}	0

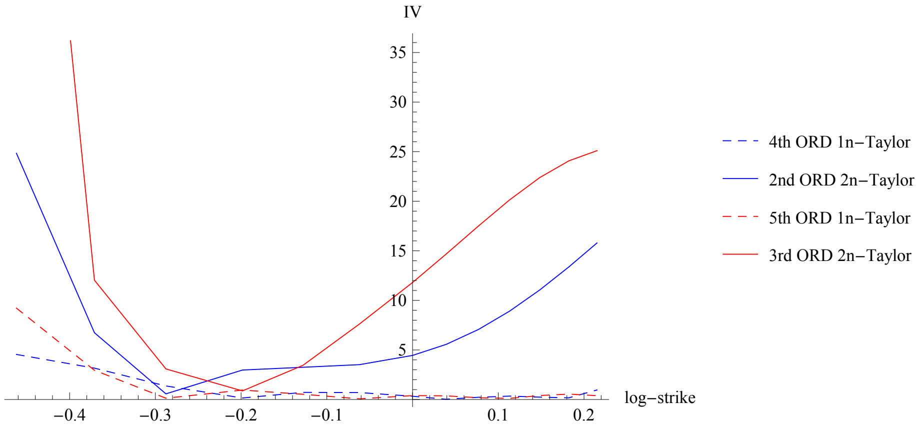
0.25	4.17	4.17	2.49×10^{-1}	2.49×10^{-1}	3.94	3.94
	2.95×10^{-1}	1.34	1.46	4.79×10^{-1}	1.29	1.02
	2.05×10^{-1}	$7.\times10^{-1}$	8.14×10^{-2}	1.22×10^{-1}	2.7×10^{-1}	8.43×10^{-1}
	1.29×10^{-1}	3.81×10^{-1}	4.03×10^{-2}	$7.\times10^{-3}$	9.01×10^{-2}	5.75×10^{-1}
	5.49×10^{-2}	0	4.82×10^{-3}	0	1.57×10^{-2}	0
	1.59×10^{-2}	0	6.13×10^{-3}	0	6.98×10^{-4}	0
0.5	6.45	6.45	4.39×10^{-1}	4.39×10^{-1}	5.91	5.91
	2.9×10^{-1}	4.41	2.98	6.99×10^{-1}	3.	2.96
	1.76×10^{-1}	5.45	3.2×10^{-1}	6.95×10^{-1}	1.	3.93
	4.83×10^{-1}	9.65	1.37×10^{-1}	6.87×10^{-1}	6.15×10^{-1}	4.42
	5.79×10^{-1}	0	4.25×10^{-2}	0	2.53×10^{-1}	0
	5.66×10^{-1}	0	5.22×10^{-2}	0	7.81×10^{-2}	0
1	8.66	8.66	6.24×10^{-1}	6.24×10^{-1}	7.02	7.02
	1.61	1.06×10^{1}	6.21	3.83×10^{-1}	7.77	7.43
	4.14×10^{-1}	2.48×10^{1}	1.25	4.45	3.2	1.58×10^{1}
	1.57	9.08×10^{1}	3.71×10^{-1}	1.18×10^{1}	2.59	2.51×10^{1}
	4.55	0	3.32×10^{-1}	0	9.65×10^{-1}	0
	9.22	0	3.95×10^{-1}	0	3.89×10^{-1}	0
1.5	9.47	9.47	6.6×10^{-1}	6.6×10^{-1}	7.39	7.39
	2.4	1.5×10^{1}	9.59	1.06	1.31×10^{1}	1.47×10^{1}
	6.62×10^{-1}	4.5×10^{1}	2.89	1.39×10^{1}	5.61	3.64×10^{1}
	4.2	2.22×10^{2}	5.87×10^{-1}	5.62×10^{1}	5.49	5.93×10^{1}
	1.51×10^{1}	0	9.26×10^{-1}	0	1.37	0
	3.84×10^{1}	0	1.09	0	3.65	0
2	1.02×10^{1}	1.02×10^{1}	5.11×10^{-1}	5.11×10^{-1}	7.59	7.59
	3.3	1.98×10^{1}	1.32×10^{1}	3.58	1.85×10^{1}	2.52×10^{1}
	1.46	7.51×10^{1}	5.21	3.15×10^{1}	7.61	6.21×10^{1}
	8.17	4.81×10^{2}	5.27×10^{-1}	1.67×10^{2}	9.03	8.94×10^{1}
	3.69×10^{1}	0	1.98	0	1.02	0
	1.17×10^{2}	0	2.25	0	1.19×10^{1}	0
3	1.16×10^{1}	1.16×10^{1}	6.2×10^{-2}	6.2×10^{-2}	7.81	7.81
	3.66	2.63×10^{1}	2.06×10^{1}	1.22×10^{1}	2.91×10^{1}	5.5×10^{1}
	2.88	1.36×10^{2}	1.22×10^{1}	1.01×10^{2}	9.84	9.85×10^{1}
	2.47×10^{1}	1.2×10^{3}	1.01	7.55×10^{2}	1.87×10^{1}	1.79×10^{2}
	1.33×10^{2}	0	5.45	0	1.83	0
	5.41×10^{2}	0	5.53	0	5.05×10^{1}	0
5	3.94	3.94	1.2	1.2	7.53	7.53
	2.3	1.43×10^{1}	3.54×10^{1}	4.46×10^{1}	$5.\times10^{1}$	1.47×10^{2}
	3.53×10^{1}	4.59×10^{1}	3.64×10^{1}	4.49×10^{2}	6.49	1.87×10^{2}
	1.24×10^{2}	7.2×10^{2}	1.48×10^{1}	4.72×10^{3}	5.38×10^{1}	6.21×10^{3}
	3.63×10^{2}	0	1.62×10^{1}	0	1.75×10^{1}	0
	5.31×10^{2}	0	8.17	0	2.75×10^{2}	0


Absolute Error: κ=1, θ=0.08, δ=0.39, ρ=−0.93, y=2Log(0.245) − Time to Maturity:0.1
IV
0.07
0.06
0.05
0.04
0.03
0.02
0.01
log−strike
−0.06
−0.04
−0.02
0.02
0.04
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor

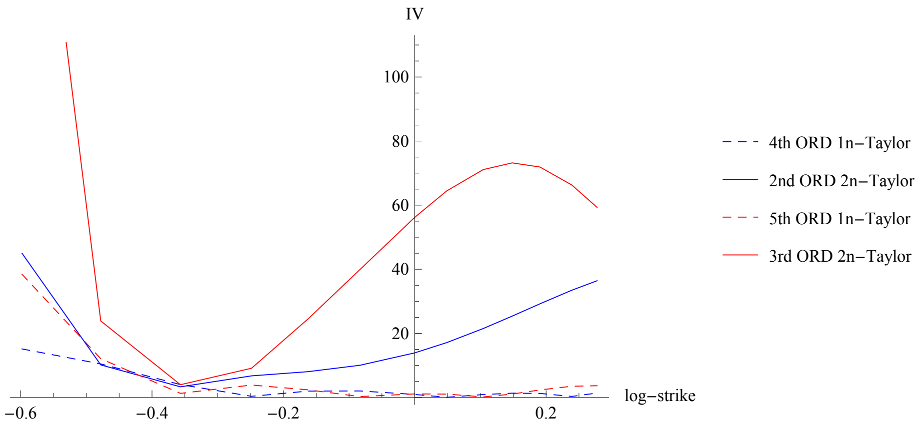

Absolute Error: κ=1, θ=0.08, δ=0.39, ρ=−0.93, y=2Log(0.245) − Time to Maturity:0.25
IV
0.8
0.6
0.4
0.2
log−strike
−0.15
−0.10
−0.05
0.05
0.10
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor


Absolute Error: κ=1, θ=0.08, δ=0.39, ρ=−0.93, y=2Log(0.245) − Time to Maturity:0.5
IV
4
3
2
1
log−strike
−0.2
−0.1
0.1
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor

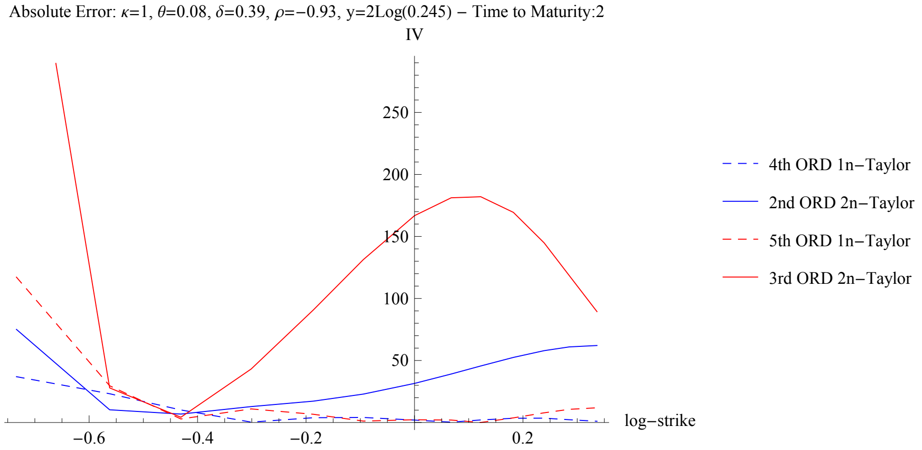
Absolute Error: κ=1, θ=0.08, δ=0.39, ρ=−0.93, y=2Log(0.245) − Time to Maturity:1

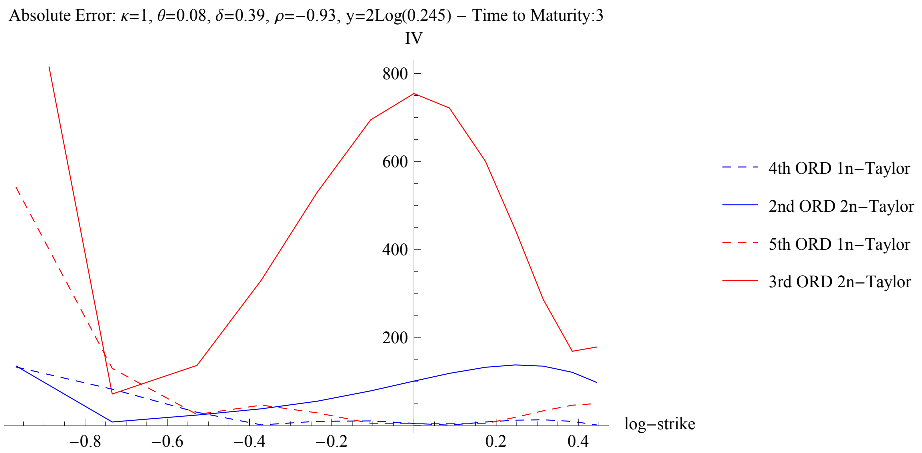


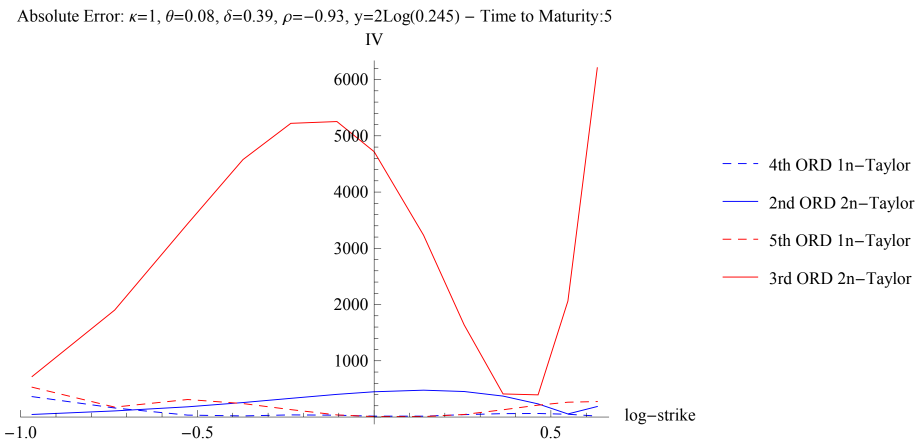
Absolute Error: κ=1, θ=0.08, δ=0.39, ρ=−0.93, y=2Log(0.245) − Time to Maturity:1.5



Absolute Error: κ=1, θ=0.08, δ=0.39, ρ=−0.93, y=2Log(0.245) − Time to Maturity:2







9.3.2 Second Set by Pascucci

The second set of parameters is proposed by Pascucci, and we have $(\kappa = 1, \theta = 0.3,\ \delta = 0.8,\ \rho = -0.7,\ x = 0,\ y = \mathrm{Log}(\theta)\ )$.

As in 8.3.2 we consider the following table of strikes expressed in bp

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.1	80	83	87	90	93	97	100	103	107	110	113	117	120
0.25	65	71	77	82	88	94	100	106	112	118	125	131	137
0.5	51	59	67	76	84	92	100	110	119	129	139	148	158
1	35	46	57	68	78	89	100	116	132	148	163	179	195
1.5	27	39	51	64	76	88	100	122	144	166	188	210	232
2	23	36	49	62	74	87	100	128	157	185	213	242	270
3	17	31	45	58	72	86	100	133	167	200	233	267	300
5	11	26	41	56	70	85	100	133	167	200	233	267	300

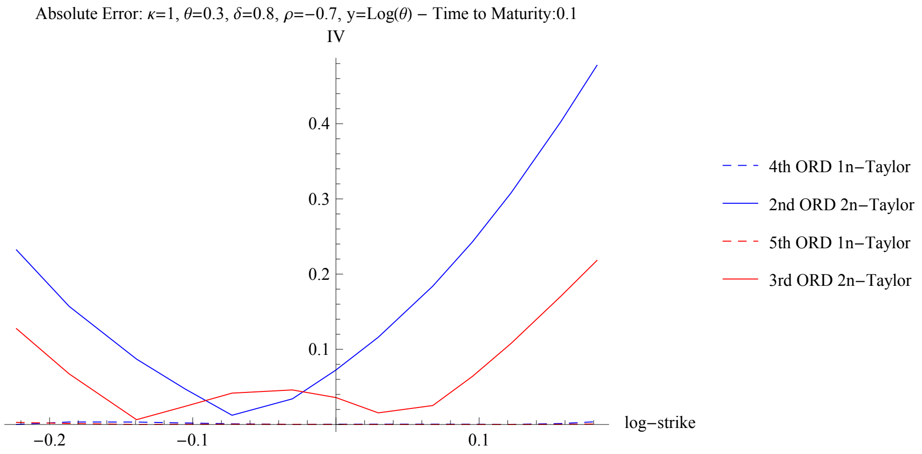
and the corresponding exact implied volatilities

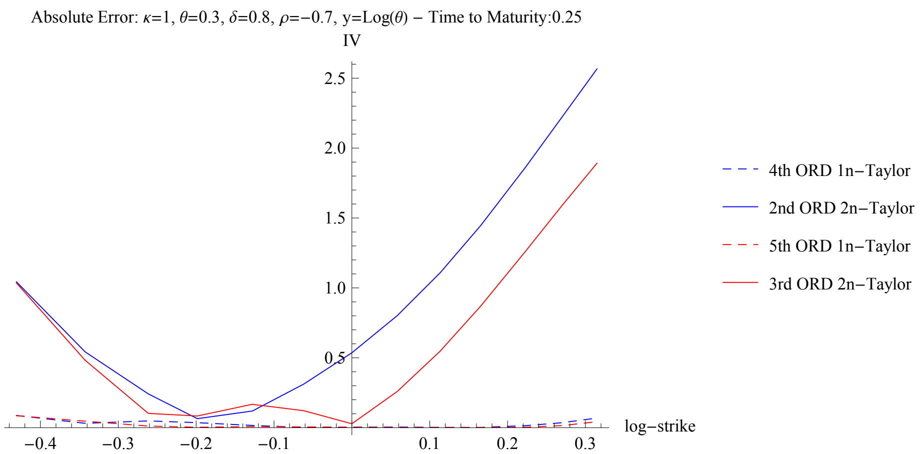
	1	3	5	7	9	11	13
0.1	59.3749	57.4122	55.7995	53.9965	52.2807	50.8866	49.3637
0.25	62.6016	59.0201	56.0034	52.9565	50.1661	47.4973	45.4576
0.5	65.1244	60.0348	55.3859	51.5276	47.5726	44.3076	42.1863
1	67.2568	59.7090	54.2581	49.6378	44.5101	41.2433	39.3597
1.5	67.3294	58.8485	52.9059	48.6054	43.0630	39.8029	38.1237
2	66.3222	57.4101	52.0719	48.0366	42.2414	39.1230	37.5144
3	64.8343	55.6842	50.9278	47.5392	42.4361	39.5896	37.8969
5	63.4838	53.5445	50.0072	47.4109	43.7258	41.5763	40.1461

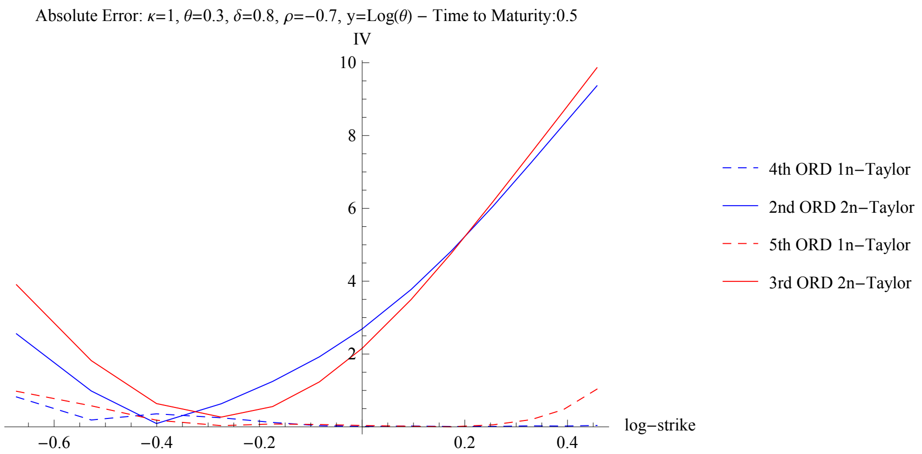
Now we compute absolute errors of our 2n-Taylor approximation

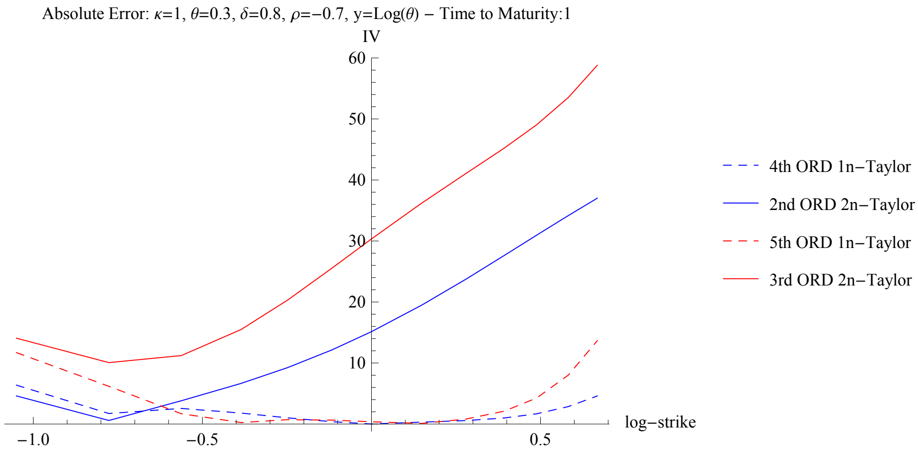
	1	3	5	7	9	11	13
0.1	4.6 6.03×10^{-1} 2.32×10^{-1} 1.27×10^{-1}	2.64 1.31×10^{-1} 8.7×10^{-2} 6.4×10^{-3}	1.03 3.29×10^{-2} 1.23×10^{-2} 4.17×10^{-2}	7.76×10^{-1} 2.02×10^{-3} 7.24×10^{-2} 3.58×10^{-2}	2.49 2.12×10^{-1} 1.84×10^{-1} 2.52×10^{-2}	3.89 5.02×10^{-1} 3.08×10^{-1} 1.08×10^{-1}	5.41 9.18×10^{-1} 4.78×10^{-1} 2.18×10^{-1}
0.25	7.83 2.03 1.04 1.04	4.25 3.86×10^{-1} 2.42×10^{-1} 1.02×10^{-1}	1.23 7.11×10^{-2} 1.19×10^{-1} 1.67×10^{-1}	1.82 1.82×10^{-1} 5.36×10^{-1} 2.83×10^{-2}	4.61 9.3×10^{-1} 1.11 5.47×10^{-1}	7.27 2.01 1.86 1.26	9.31 3.02 2.57 1.89
0.5	1.04×10^{1} 4.11 2.56 3.9	5.26 6.96×10^{-1} 9.1×10^{-2} 6.39×10^{-1}	6.14×10^{-1} 9.26×10^{-2} 1.25 5.57×10^{-1}	3.24 1.01 2.69 2.16	7.2 3. 4.83 4.77	1.05×10^{1} 5.32 7.26 7.53	1.26×10^{1} 7.24 9.37 9.86
1	1.25×10^{1} 7.46 4.61 1.41×10^{1}	4.94 1.23 3.82 1.12×10^{1}	5.14×10^{-1} 1.82 9.24 2.03×10^{1}	5.13 4.81 1.51×10^{1} 3.03×10^{1}	1.03×10^{1} 1.04×10^{1} 2.37×10^{1} 4.1×10^{1}	1.35×10^{1} 1.57×10^{1} 3.1×10^{1} 4.9×10^{1}	1.54×10^{1} 2.04×10^{1} 3.7×10^{1} 5.88×10^{1}
1.5	1.26×10^{1} 8.82 2.07 3.94×10^{1}	4.08 2.88 1.67×10^{1} 7.51×10^{1}	1.87 6.35 3.16×10^{1} 1.14×10^{2}	6.17 1.19×10^{1} 4.42×10^{1} 1.32×10^{2}	1.17×10^{1} 2.28×10^{1} 6.22×10^{1} 1.44×10^{2}	1.5×10^{1} 3.26×10^{1} 7.41×10^{1} 1.58×10^{2}	1.66×10^{1} 4.11×10^{1} 8.1×10^{1} 1.96×10^{2}
2	1.15×10^{1} 8.82 9.19 1.32×10^{2}	2.64 6.85 4.82×10^{1} 2.94×10^{2}	2.7 1.39×10^{1} 7.56×10^{1} 3.63×10^{2}	6.74 2.26×10^{1} 9.66×10^{1} 3.74×10^{2}	1.25×10^{1} 4.08×10^{1} 1.24×10^{2} 3.53×10^{2}	1.56×10^{1} 5.6×10^{1} 1.34×10^{2} 3.94×10^{2}	1.73×10^{1} 6.9×10^{1} 1.33×10^{2} 5.42×10^{2}
3	1.01×10^{1} 1.19×10^{1} 7.19×10^{1} 1.02×10^{3}	9.12×10^{-1} 2.33×10^{1} 1.99×10^{2} 1.81×10^{3}	3.84 $4.\times10^{1}$ 2.64×10^{2} 1.79×10^{3}	7.23 5.59×10^{1} $3.\times10^{2}$ 1.6×10^{3}	1.23×10^{1} 8.74×10^{1} 3.22×10^{2} 1.3×10^{3}	1.52×10^{1} 1.12×10^{2} $3.\times10^{2}$ 1.52×10^{3}	1.69×10^{1} 1.32×10^{2} 2.57×10^{2} 2.24×10^{3}
5	8.71 3.58×10^{1} 6.25×10^{2} 1.47×10^{4}	1.23 9.83×10^{1} 1.17×10^{3} 1.59×10^{4}	4.77 1.39×10^{2} 1.29×10^{3} 1.23×10^{4}	7.36 1.71×10^{2} 1.3×10^{3} 9.53×10^{3}	1.1×10^{1} 2.25×10^{2} 1.16×10^{3} 7.17×10^{3}	1.32×10^{1} 2.64×10^{2} 9.56×10^{2} 8.67×10^{3}	1.46×10^{1} 2.96×10^{2} 7.24×10^{2} 1.27×10^{4}

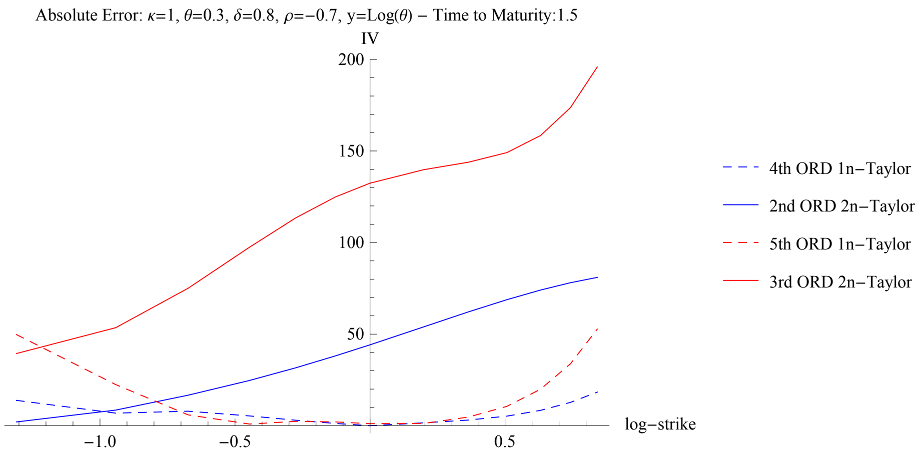
and we compare errors of 1n-Taylor approximation with those of 2n-Taylor.

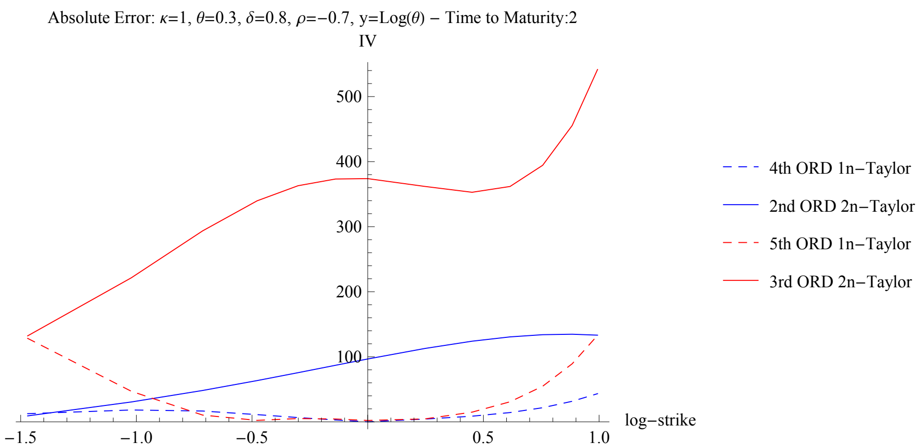

Absolute Error: κ=1, θ=0.3, δ=0.8, ρ=−0.7, y=Log(θ) − Time to Maturity:0.1
IV
0.4
0.3
0.2
0.1
log−strike
−0.2
−0.1
0.1
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor

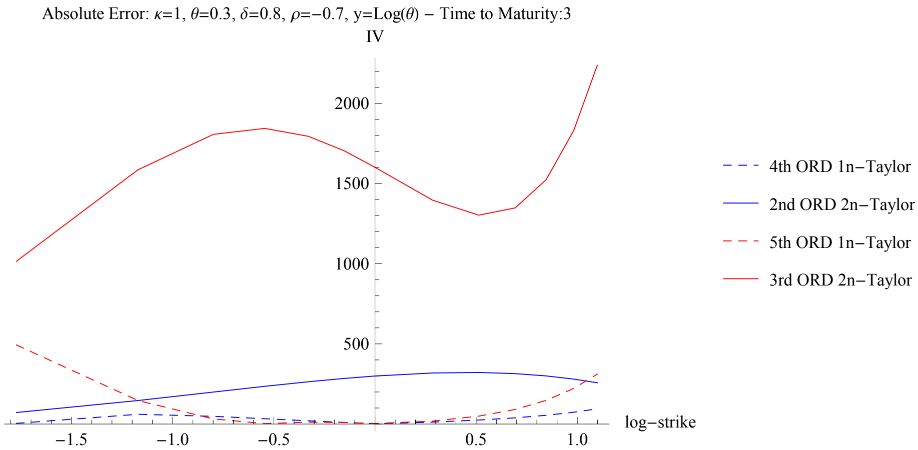

Absolute Error: κ=1, θ=0.3, δ=0.8, ρ=−0.7, y=Log(θ) − Time to Maturity:0.25
IV
2.5
2.0
1.5
1.0
0.5
log−strike
−0.4
−0.3
−0.2
−0.1
0.1
0.2
0.3
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor

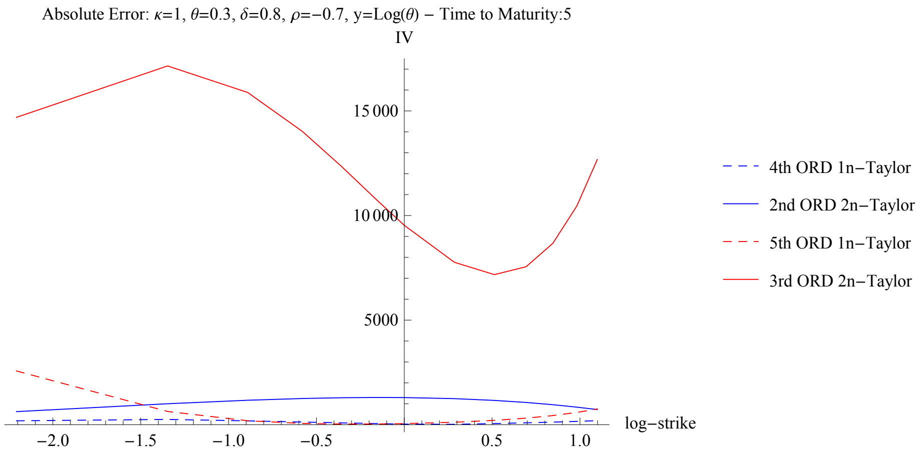

Absolute Error: κ=1, θ=0.3, δ=0.8, ρ=−0.7, y=Log(θ) − Time to Maturity:0.5
IV
10
8
6
4
2
log−strike
−0.6
−0.4
−0.2
0.2
0.4
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor


Absolute Error: κ=1, θ=0.3, δ=0.8, ρ=−0.7, y=Log(θ) − Time to Maturity:1
IV
60
50
40
30
20
10
log−strike
−1.0
−0.5
0.5
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor


Absolute Error: κ=1, θ=0.3, δ=0.8, ρ=−0.7, y=Log(θ) − Time to Maturity:1.5
IV
200
150
100
50
log−strike
−1.0
−0.5
0.5
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor


Absolute Error: κ=1, θ=0.3, δ=0.8, ρ=−0.7, y=Log(θ) − Time to Maturity:2
IV
500
400
300
200
100
log−strike
−1.5
−1.0
−0.5
0.5
1.0
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor





	1		7		13	
0.1	4.6	4.6	7.76×10^{-1}	7.76×10^{-1}	5.41	5.41
	7.43×10^{-1}	6.03×10^{-1}	1.07	2.02×10^{-3}	1.1	9.18×10^{-1}
	2.31×10^{-1}	2.32×10^{-1}	1.04×10^{-2}	7.24×10^{-2}	2.01×10^{-1}	4.78×10^{-1}
	4.56×10^{-2}	1.27×10^{-1}	5.35×10^{-3}	3.58×10^{-2}	$4.\times10^{-2}$	2.18×10^{-1}
	6.53×10^{-5}	0	3.21×10^{-4}	0	3.73×10^{-3}	0
	2.8×10^{-3}	0	8.67×10^{-5}	0	6.62×10^{-4}	0
0.25	7.83	7.83	1.82	1.82	9.31	9.31
	1.43	2.03	2.79	1.82×10^{-1}	3.34	3.02
	1.13	1.04	7.24×10^{-2}	5.36×10^{-1}	1.11	2.57
	5.4×10^{-1}	1.04	3.45×10^{-2}	2.83×10^{-2}	4.19×10^{-1}	1.89
	8.76×10^{-2}	0	3.7×10^{-3}	0	6.92×10^{-2}	0
	8.57×10^{-2}	0	2.66×10^{-3}	0	4.31×10^{-2}	0

0.5	1.04×10^{1}	1.04×10^{1}	3.24	3.24	1.26×10^{1}	1.26×10^{1}
	2.36	4.11	5.98	1.01	8.33	7.24
	3.44	2.56	3.25×10^{-1}	2.69	3.58	9.37
	2.73	3.9	1.31×10^{-1}	2.16	1.7	9.86
	8.22×10^{-1}	0	1.24×10^{-2}	0	3.83×10^{-2}	0
	9.78×10^{-1}	0	3.79×10^{-2}	0	1.04	0
1	1.25×10^{1}	1.25×10^{1}	5.13	5.13	1.54×10^{1}	1.54×10^{1}
	4.09	7.46	1.33×10^{1}	4.81	2.01×10^{1}	2.04×10^{1}
	1.04×10^{1}	4.61	1.56	1.51×10^{1}	9.15	3.7×10^{1}
	1.32×10^{1}	1.41×10^{1}	4.56×10^{-1}	3.03×10^{1}	4.31	5.88×10^{1}
	6.37	0	5.74×10^{-3}	0	4.6	0
	1.17×10^{1}	0	3.7×10^{-1}	0	1.37×10^{1}	0
1.5	1.26×10^{1}	1.26×10^{1}	6.17	6.17	1.66×10^{1}	1.66×10^{1}
	6.75	8.82	2.15×10^{1}	1.19×10^{1}	3.25×10^{1}	4.11×10^{1}
	1.94×10^{1}	2.07	4.21	4.42×10^{1}	1.37×10^{1}	8.1×10^{1}
	2.96×10^{1}	3.94×10^{1}	9.75×10^{-1}	1.32×10^{2}	5.74	1.96×10^{2}
	1.38×10^{1}	0	1.43×10^{-3}	0	1.84×10^{1}	0
	4.97×10^{1}	0	1.11	0	5.28×10^{1}	0
2	1.15×10^{1}	1.15×10^{1}	6.74	6.74	1.73×10^{1}	1.73×10^{1}
	1.09×10^{1}	8.82	3.01×10^{1}	2.26×10^{1}	4.5×10^{1}	6.9×10^{1}
	3.01×10^{1}	9.19	8.65	9.66×10^{1}	1.66×10^{1}	1.33×10^{2}
	4.74×10^{1}	1.32×10^{2}	1.58	3.74×10^{2}	5.96	5.42×10^{2}
	1.26×10^{1}	0	1.22×10^{-1}	0	4.33×10^{1}	0
	1.28×10^{2}	0	2.17	0	1.33×10^{2}	0
3	1.01×10^{1}	1.01×10^{1}	7.23	7.23	1.69×10^{1}	1.69×10^{1}
	2.01×10^{1}	1.19×10^{1}	4.81×10^{1}	5.59×10^{1}	6.65×10^{1}	1.32×10^{2}
	5.85×10^{1}	7.19×10^{1}	2.37×10^{1}	$3.\times10^{2}$	1.05×10^{1}	2.57×10^{2}
	1.01×10^{2}	1.02×10^{3}	1.84	1.6×10^{3}	3.51	2.24×10^{3}
	4.17	0	2.11	0	9.47×10^{1}	0
	4.93×10^{2}	0	3.35	0	3.11×10^{2}	0
5	8.71	8.71	7.36	7.36	1.46×10^{1}	1.46×10^{1}
	4.45×10^{1}	3.58×10^{1}	8.48×10^{1}	1.71×10^{2}	1.06×10^{2}	2.96×10^{2}
	1.41×10^{2}	6.25×10^{2}	8.01×10^{1}	1.3×10^{3}	3.6×10^{1}	7.24×10^{2}
	2.75×10^{2}	1.47×10^{4}	1.37×10^{1}	9.53×10^{3}	6.15	1.27×10^{4}
	1.87×10^{2}	0	2.8×10^{1}	0	1.95×10^{2}	0
	2.57×10^{3}	0	5.23×10^{1}	0	7.52×10^{2}	0

9.3.3 Third Set by Bakshi, Cao and Chen

The third set of parameters considered is as given by Bakshi, Cao and Chen in ([BCC97]), so we have $\left(\kappa = 1.15, \theta = \frac{0.04}{1.15}, \delta = 0.39, \rho = -0.64, x = 0, y = 2\,\mathrm{Log}(0.2)\right)$. Now we create as in 8.3.3 the matrix of strikes expressed in bp

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.1	96	97	97	98	99	99	100	101	101	102	103	103	104
0.25	90	92	93	95	97	98	100	101	102	104	105	106	107
0.5	83	86	89	92	94	97	100	102	104	106	108	110	112
1	74	78	83	87	91	96	100	103	106	109	112	115	118
1.5	67	72	78	84	89	94	100	104	108	112	116	120	124
2	62	68	75	81	87	94	100	105	110	115	120	125	130
3	55	62	70	78	85	92	100	107	114	120	127	134	141
5	39	49	59	70	80	90	100	111	121	132	143	153	164

and the corresponding exact implied volatilities

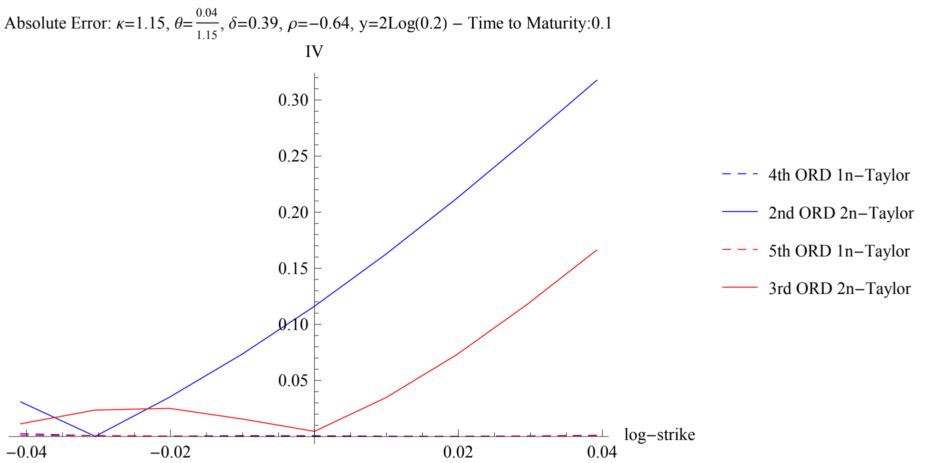
	1	3	5	7	9	11	13
0.1	20.8514	20.5375	19.9098	19.5976	19.2877	18.6812	18.3885
0.25	22.1212	21.2130	19.9853	19.0641	18.4626	17.6127	17.1079
0.5	23.2380	21.5344	20.0918	18.3722	17.3012	16.3866	15.7154
1	24.0045	21.7211	19.7221	17.5843	16.3363	15.3634	14.7517
1.5	24.3455	21.8086	19.4109	17.2243	15.8844	14.8918	14.3152
2	24.3397	21.5905	19.2765	17.0562	15.6568	14.6525	14.0915
3	24.0332	21.2940	18.9430	16.9522	15.4781	14.5175	13.9322
5	27.6529	21.7978	17.4399	16.8007	16.6548	15.9243	14.0942

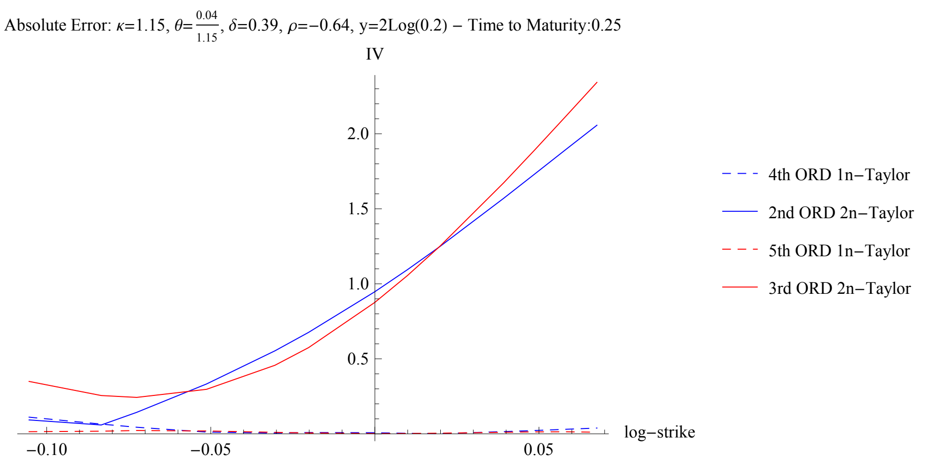
Noe we compute absolute errors of our 2n-Taylor approximation

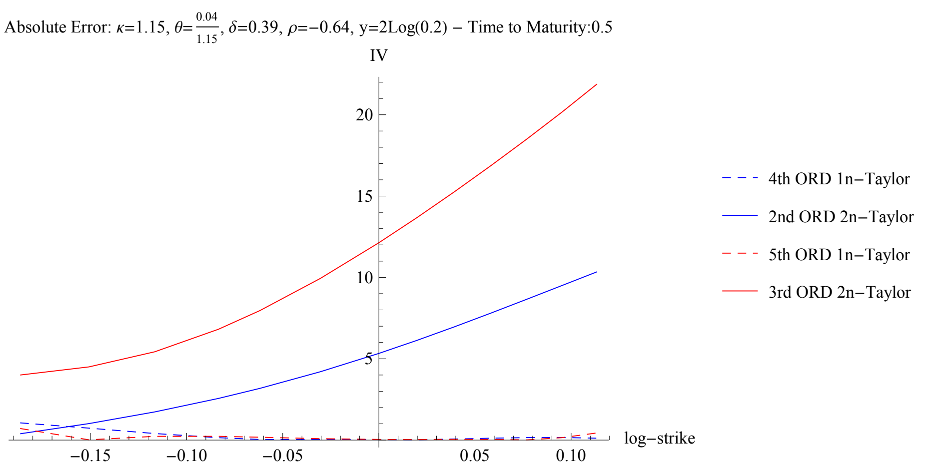
	1	3	5	7	9	11	13
0.1	8.51×10^{-1} 2.76×10^{-2} 3.09×10^{-2} 1.13×10^{-2}	5.38×10^{-1} 1.71×10^{-2} 1.71×10^{-4} 2.36×10^{-2}	9.02×10^{-2} 4.67×10^{-2} 7.34×10^{-2} 1.57×10^{-2}	4.02×10^{-1} 8.44×10^{-2} 1.16×10^{-1} 4.62×10^{-3}	7.12×10^{-1} 1.35×10^{-1} 1.63×10^{-1} 3.47×10^{-2}	1.32 2.69×10^{-1} 2.65×10^{-1} 1.18×10^{-1}	1.61 3.47×10^{-1} 3.18×10^{-1} 1.66×10^{-1}
0.25	2.12 1.95×10^{-1} 9.28×10^{-2} 3.49×10^{-1}	1.21 7.26×10^{-2} 1.43×10^{-1} 2.43×10^{-1}	1.47×10^{-2} 1.81×10^{-1} 5.53×10^{-1} 4.57×10^{-1}	9.36×10^{-1} 4.37×10^{-1} 9.47×10^{-1} 8.76×10^{-1}	1.54 6.7×10^{-1} 1.25 1.25	2.39 1.07 1.73 1.89	2.89 1.34 2.06 2.34
0.5	3.24 $6.\times10^{-1}$ 3.92×10^{-1} 4.	1.53 3.34×10^{-1} 1.73 5.43	9.18×10^{-2} 6.75×10^{-1} 3.18 7.96	1.63 1.62 5.33 1.21×10^{1}	2.7 2.45 6.96 1.53×10^{1}	3.61 3.33 8.66 1.85×10^{1}	4.28 4.14 1.03×10^{1} 2.19×10^{1}
1	4. 1.57 7.27 6.62×10^{1}	1.72 1.83 1.56×10^{1} 1.04×10^{2}	2.78×10^{-1} 3.45 2.43×10^{1} 1.39×10^{2}	2.42 6.34 3.51×10^{1} 1.73×10^{2}	3.66 8.63 4.25×10^{1} 1.95×10^{2}	4.64 1.1×10^{1} 4.98×10^{1} 2.17×10^{2}	5.25 1.33×10^{1} 5.69×10^{1} 2.43×10^{2}
1.5	4.35 3.17 $3.\times10^{1}$ 4.05×10^{2}	1.81 4.88 5.67×10^{1} 6.14×10^{2}	5.89×10^{-1} 9. 8.55×10^{1} 7.74×10^{2}	2.78 1.46×10^{1} 1.14×10^{2} 8.88×10^{2}	4.12 1.91×10^{1} 1.33×10^{2} 9.62×10^{2}	5.11 2.38×10^{1} 1.51×10^{2} 1.05×10^{3}	5.68 2.84×10^{1} 1.68×10^{2} 1.18×10^{3}
2	4.34 5.76 8.6×10^{1} 1.6×10^{3}	1.59 1.03×10^{1} 1.53×10^{2} 2.3×10^{3}	7.24×10^{-1} 1.73×10^{1} 2.13×10^{2} 2.7×10^{3}	2.94 2.66×10^{1} 2.71×10^{2} 2.96×10^{3}	4.34 3.42×10^{1} 3.1×10^{2} 3.17×10^{3}	5.35 4.19×10^{1} 3.44×10^{2} 3.46×10^{3}	5.91 4.96×10^{1} 3.72×10^{2} 3.92×10^{3}
3	4.03 1.58×10^{1} 3.99×10^{2} 1.2×10^{4}	1.29 2.88×10^{1} 6.27×10^{2} 1.53×10^{4}	1.06 4.49×10^{1} 8.17×10^{2} 1.67×10^{4}	3.05 6.22×10^{1} 9.64×10^{2} 1.75×10^{4}	4.52 7.85×10^{1} 1.07×10^{3} 1.86×10^{4}	5.48 9.33×10^{1} 1.14×10^{3} 2.04×10^{4}	6.07 1.09×10^{2} 1.19×10^{3} 2.37×10^{4}
5	7.65 4.36×10^{1} 2.33×10^{3} 1.46×10^{5}	1.8 9.08×10^{1} 3.7×10^{3} 1.79×10^{5}	2.56 1.4×10^{2} 4.62×10^{3} 1.84×10^{5}	3.2 1.81×10^{2} 5.14×10^{3} 1.88×10^{5}	3.35 2.2×10^{2} 5.45×10^{3} 2.01×10^{5}	4.08 2.6×10^{2} 5.57×10^{3} 2.31×10^{5}	5.91 2.96×10^{2} 5.56×10^{3} 2.75×10^{5}

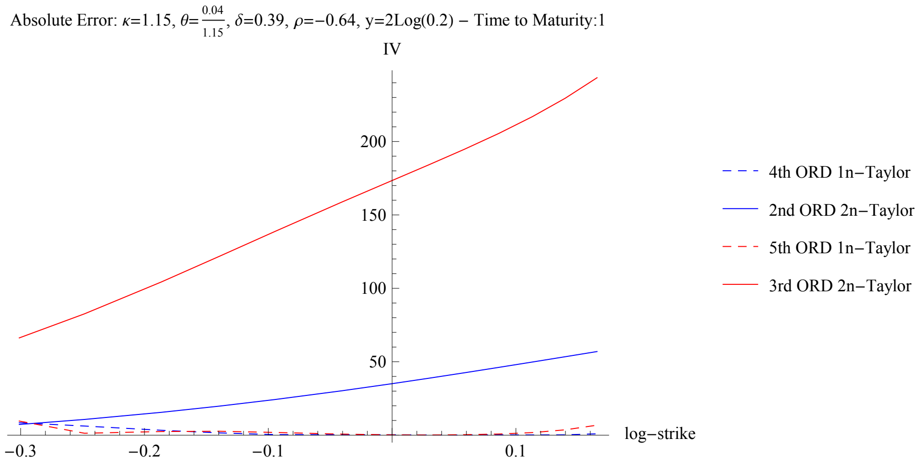
then we compare them with 1n-Taylor errors

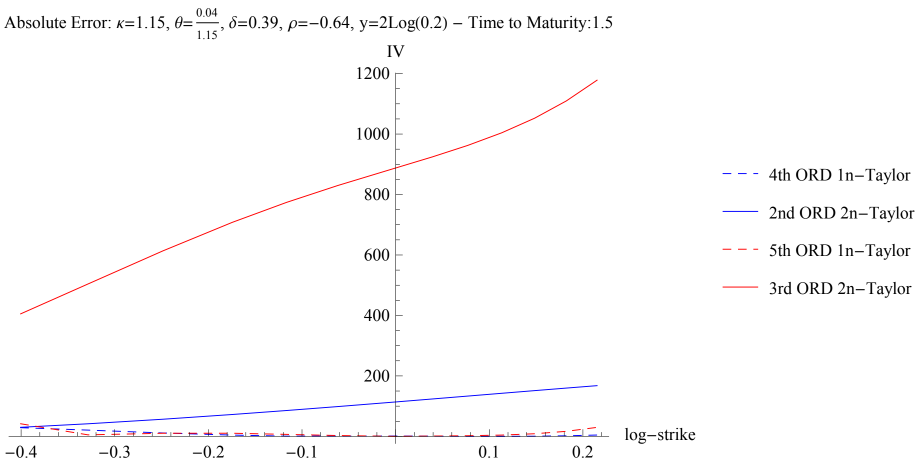
	1		7		13	
0.1	8.51×10^{-1}	8.51×10^{-1}	4.02×10^{-1}	4.02×10^{-1}	1.61	1.61
	6.66×10^{-1}	2.76×10^{-2}	6.86×10^{-1}	8.44×10^{-2}	$7.\times10^{-1}$	3.47×10^{-1}
	7.5×10^{-2}	3.09×10^{-2}	2.96×10^{-3}	1.16×10^{-1}	7.95×10^{-2}	3.18×10^{-1}
	7.94×10^{-3}	1.13×10^{-2}	4.19×10^{-3}	4.62×10^{-3}	9.52×10^{-3}	1.66×10^{-1}
	2.56×10^{-3}	0	6.73×10^{-4}	0	1.1×10^{-3}	0
	1.02×10^{-3}	0	3.58×10^{-5}	0	6.44×10^{-4}	0
0.25	2.12	2.12	9.36×10^{-1}	9.36×10^{-1}	2.89	2.89
	1.55	1.95×10^{-1}	1.78	4.37×10^{-1}	1.94	1.34
	5.89×10^{-1}	9.28×10^{-2}	1.61×10^{-2}	9.47×10^{-1}	4.34×10^{-1}	2.06
	1.57×10^{-2}	3.49×10^{-1}	2.35×10^{-2}	8.76×10^{-1}	1.44×10^{-1}	2.34
	1.11×10^{-1}	0	7.42×10^{-3}	0	3.86×10^{-2}	0
	1.43×10^{-2}	0	1.97×10^{-3}	0	1.08×10^{-2}	0
0.5	3.24	3.24	1.63	1.63	4.28	4.28
	2.86	$6.\times10^{-1}$	3.81	1.62	4.69	4.14
	2.21	3.92×10^{-1}	7.09×10^{-2}	5.33	1.83	1.03×10^{1}
	4.38×10^{-1}	4.	5.34×10^{-2}	1.21×10^{1}	9.44×10^{-1}	2.19×10^{1}
	1.06	0	2.8×10^{-2}	0	1.16×10^{-1}	0
	7.07×10^{-1}	0	3.97×10^{-2}	0	4.37×10^{-1}	0
1	4.	4.	2.42	2.42	5.25	5.25
	5.49	1.57	8.46	6.34	1.08×10^{1}	1.33×10^{1}
	7.17	7.27	1.61×10^{-1}	3.51×10^{1}	5.03	5.69×10^{1}
	2.59	6.62×10^{1}	6.41×10^{-2}	1.73×10^{2}	3.38	2.43×10^{2}
	8.45	0	2.68×10^{-1}	0	9.36×10^{-1}	0
	9.59	0	1.66×10^{-1}	0	6.8	0
1.5	4.35	4.35	2.78	2.78	5.68	5.68
	8.17	3.17	1.35×10^{1}	1.46×10^{1}	1.73×10^{1}	2.84×10^{1}
	1.42×10^{1}	$3.\times10^{1}$	1.43×10^{-1}	1.14×10^{2}	8.6	1.68×10^{2}
	6.95	4.05×10^{2}	3.58×10^{-2}	8.88×10^{2}	7.3	1.18×10^{3}
	2.87×10^{1}	0	1.16	0	4.42	0
	4.17×10^{1}	0	1.1×10^{-2}	0	2.98×10^{1}	0
2	4.34	4.34	2.94	2.94	5.91	5.91
	1.12×10^{1}	5.76	1.88×10^{1}	2.66×10^{1}	2.4×10^{1}	4.96×10^{1}
	2.26×10^{1}	8.6×10^{1}	1.07	2.71×10^{2}	1.22×10^{1}	3.72×10^{2}
	1.17×10^{1}	1.6×10^{3}	2.04×10^{-1}	2.96×10^{3}	1.34×10^{1}	3.92×10^{3}
	6.62×10^{1}	0	2.96	0	1.01×10^{1}	0
	9.46×10^{1}	0	8.56×10^{-1}	0	8.28×10^{1}	0
3	4.03	4.03	3.05	3.05	6.07	6.07
	1.8×10^{1}	1.58×10^{1}	2.96×10^{1}	6.22×10^{1}	3.73×10^{1}	1.09×10^{2}
	4.3×10^{1}	3.99×10^{2}	5.17	9.64×10^{2}	1.82×10^{1}	1.19×10^{3}
	$2.\times10^{1}$	1.2×10^{4}	2.61	1.75×10^{4}	3.47×10^{1}	2.37×10^{4}
	2.03×10^{2}	0	9.48	0	1.93×10^{1}	0
	1.94×10^{2}	0	5.5	0	3.12×10^{2}	0
5	7.65	7.65	3.2	3.2	5.91	5.91
	3.27×10^{1}	4.36×10^{1}	5.12×10^{1}	1.81×10^{2}	6.39×10^{1}	2.96×10^{2}
	1.11×10^{2}	2.33×10^{3}	2.32×10^{1}	5.14×10^{3}	2.72×10^{1}	5.56×10^{3}
	1.07×10^{2}	1.46×10^{5}	2.57×10^{1}	1.88×10^{5}	1.39×10^{2}	2.75×10^{5}
	1.14×10^{3}	0	3.31×10^{1}	0	4.64×10^{1}	0
	2.03×10^{3}	0	2.21×10^{1}	0	1.57×10^{3}	0

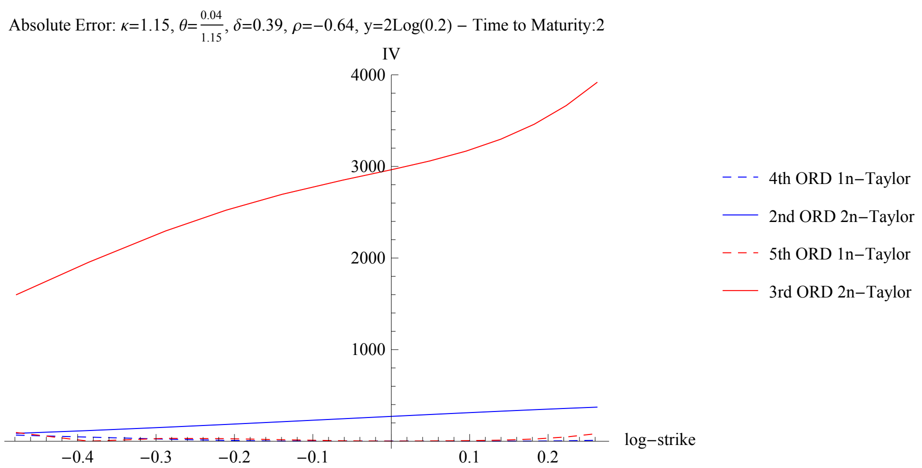

Absolute Error: κ=1.15, $\theta=\frac{0.04}{1.15}$, δ=0.39, ρ=−0.64, y=2Log(0.2) − Time to Maturity:0.1
IV
0.30
0.25
0.20
0.15
0.10
0.05
−0.04
−0.02
0.02
0.04
log−strike
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor

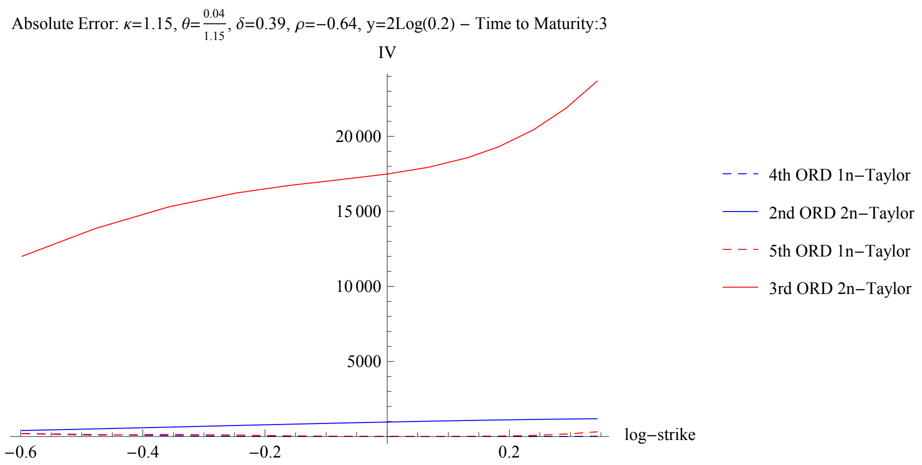

Absolute Error: κ=1.15, $\theta=\frac{0.04}{1.15}$, δ=0.39, ρ=−0.64, y=2Log(0.2) − Time to Maturity:0.25
IV
2.0
1.5
1.0
0.5
−0.10
−0.05
0.05
log−strike
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor

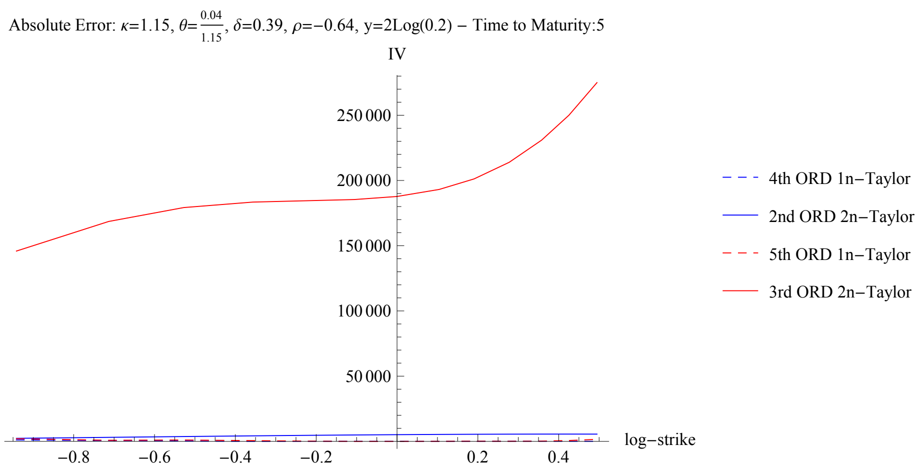

Absolute Error: κ=1.15, $\theta=\frac{0.04}{1.15}$, δ=0.39, ρ=−0.64, y=2Log(0.2) − Time to Maturity:0.5
IV
20
15
10
5
−0.15
−0.10
−0.05
0.05
0.10
log−strike
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor


Absolute Error: κ=1.15, θ=0.04/1.15, δ=0.39, ρ=−0.64, y=2Log(0.2) − Time to Maturity:1
IV
200
150
100
50
log−strike
−0.3
−0.2
−0.1
0.1
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor


Absolute Error: κ=1.15, θ=0.04/1.15, δ=0.39, ρ=−0.64, y=2Log(0.2) − Time to Maturity:1.5
IV
1200
1000
800
600
400
200
log−strike
−0.4
−0.3
−0.2
−0.1
0.1
0.2
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor


Absolute Error: κ=1.15, θ=0.04/1.15, δ=0.39, ρ=−0.64, y=2Log(0.2) − Time to Maturity:2
IV
4000
3000
2000
1000
log−strike
−0.4
−0.3
−0.2
−0.1
0.1
0.2
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor





9.4 Three Halves stochastic volatility model

For the 3/2 model we compute our 2n-Taylor approximation till third order but we present here only first formulas because they are too long:

$$\sigma_0 = e^{\frac{\bar{y}}{2}}$$

$$\begin{aligned}\sigma_1 = \frac{1}{48} e^{\frac{\bar{y}}{2}} \Big(& e^{2\bar{y}} t^2 \left(\delta^2 + 2\kappa - \delta\rho\right)^2 + \\ & 4 \left(t^2 (\theta\kappa - r\delta\rho)^2 + (-k + x)\,\delta\rho\,(-3 + (-k + x)\,\delta\rho) - t\,(\theta\kappa - r\delta\rho)\,(-3 + 2\,(-k + x)\,\delta\rho)\right) - \\ & 2 e^{\bar{y}} t \big(4 t\theta\kappa^2 + \delta\rho\,(-3 - 2\,(-k + x)\,\delta\,(\delta - \rho) + 2\delta\rho + 2 r t\delta\,(-\delta + \rho)) + \\ & \quad 2\kappa\,(3 - 2\,(-k + x)\,\delta\rho + t\delta\,(\delta\theta - (2r + \theta)\,\rho))\big) - \\ & 6 \left(e^{\bar{y}} t \left(\delta^2 + 2\kappa - \delta\rho\right) + 2\,(-2 - t\theta\kappa + r t\delta\rho + (-k + x)\,\delta\rho)\right) (y - \bar{y}) + 12\,(y - \bar{y})^2 \Big)\end{aligned}$$

$$\sigma_2 = \frac{1}{23\,040} e^{\frac{\bar{y}}{2}} \left(e^{4\bar{y}} t^4 \left(\delta^2 + 2\kappa - \delta\rho\right)^2 \left(25\delta^4 + 100\kappa^2 - 50\delta^3\rho - 100\delta\kappa\rho + \delta^2 \left(-24 + 100\kappa + 45\rho^2\right)\right) - \right.$$

$$16 \left(45 t^2 \theta^2 \kappa^2 - t^4 \theta^4 \kappa^4 - 6 t (-k + x) \delta \theta \kappa \left(25 + 10 t \theta \kappa + 2 t^2 \theta^2 \kappa^2\right) \rho + r^4 t^4 \delta^4 \rho^2 \left(-24 + 35 \rho^2\right) +\right.$$
$$(-k + x)^4 \delta^4 \rho^2 \left(-24 + 35 \rho^2\right) + 3 (-k + x)^3 \delta^3 \rho \left(25 - 40 \rho^2 - 4 t \theta \kappa \left(-4 + 7 \rho^2\right)\right) +$$
$$(-k + x)^2 \delta^2 \left(30 \left(-2 + 5 \rho^2\right) + 15 t \theta \kappa \left(-5 + 12 \rho^2\right) + 2 t^2 \theta^2 \kappa^2 \left(-12 + 31 \rho^2\right)\right) +$$
$$r^3 t^3 \delta^3 \rho \left(75 - 120 \rho^2 + 12 t \theta \kappa \left(4 - 7 \rho^2\right) + 4 (-k + x) \delta \rho \left(-24 + 35 \rho^2\right)\right) -$$
$$r t \delta \left(6 t \theta \kappa \left(25 + 10 t \theta \kappa + 2 t^2 \theta^2 \kappa^2\right) \rho + 4 (-k + x)^3 \delta^3 \rho^2 \left(24 - 35 \rho^2\right) +\right.$$
$$9 (-k + x)^2 \delta^2 \rho \left(4 t \theta \kappa \left(-4 + 7 \rho^2\right) + 5 \left(-5 + 8 \rho^2\right)\right) -$$
$$\left. 2 (-k + x) \delta \left(30 \left(-2 + 5 \rho^2\right) + 15 t \theta \kappa \left(-5 + 12 \rho^2\right) + 2 t^2 \theta^2 \kappa^2 \left(-12 + 31 \rho^2\right)\right)\right) +$$
$$r^2 t^2 \delta^2 \left(2 t^2 \theta^2 \kappa^2 \left(-12 + 31 \rho^2\right) - 3 t \theta \kappa \left(25 - 60 \rho^2 + 12 (-k + x) \delta \rho \left(-4 + 7 \rho^2\right)\right) +\right.$$
$$\left.\left. 3 \left(-20 + 50 \rho^2 - 15 (-k + x) \delta \rho \left(-5 + 8 \rho^2\right) + 2 (-k + x)^2 \delta^2 \rho^2 \left(-24 + 35 \rho^2\right)\right)\right)\right) -$$
$$8 e^{\bar{y}} t \left(4 t \theta \kappa^2 \left(15 + 60 t \theta \kappa + 14 t^2 \theta^2 \kappa^2\right) - 2 \delta \kappa \left(6 r t \left(-5 + 20 t \theta \kappa + 6 t^2 \theta^2 \kappa^2\right) +\right.\right.$$
$$\left. 6 (-k + x) \left(-5 + 20 t \theta \kappa + 6 t^2 \theta^2 \kappa^2\right) + t \theta \left(15 + 60 t \theta \kappa + 14 t^2 \theta^2 \kappa^2\right)\right) \rho +$$
$$12 (-k + r t + x)^3 \delta^5 \rho \left(-4 + 5 \rho^2\right) - (-k + r t + x)^2 \delta^4$$
$$\left(-45 - 48 (1 - k + x) \rho^2 + 20 (4 + 3 (-k + x)) \rho^4 + t \left(12 r \rho^2 \left(-4 + 5 \rho^2\right) + 4 \theta \kappa \left(-12 + 13 \rho^2\right)\right)\right) +$$
$$(-k + r t + x) \delta^3 \rho \left(24 r^2 t^2 \kappa \left(-4 + 5 \rho^2\right) + 24 (-k + x)^2 \kappa \left(-4 + 5 \rho^2\right) -\right.$$
$$6 \left(-5 + 6 t^2 \theta^2 \kappa^2 + t \theta \kappa \left(23 - 4 \rho^2\right)\right) + (-k + x) \left(-75 + 60 \rho^2 + 4 t \theta \kappa \left(-12 + 13 \rho^2\right)\right) +$$
$$\left. r t \left(-75 + 60 \rho^2 + 48 (-k + x) \kappa \left(-4 + 5 \rho^2\right) + 4 t \theta \kappa \left(-12 + 13 \rho^2\right)\right)\right) -$$
$$2 \delta^2 \left(-60 + 45 t \theta \kappa - 39 t^2 \theta^2 \kappa^2 - 14 t^3 \theta^3 \kappa^3 + 15 \rho^2 - 60 t \theta \kappa \rho^2 - 18 t^2 \theta^2 \kappa^2 \rho^2 -\right.$$
$$3 (-k + x) \left(-5 + 20 t \theta \kappa + 6 t^2 \theta^2 \kappa^2\right) \rho^2 + r^2 t^2 \kappa \left(-75 + 60 \rho^2 + 4 t \theta \kappa \left(-12 + 13 \rho^2\right)\right) +$$
$$(-k + x)^2 \kappa \left(-75 + 60 \rho^2 + 4 t \theta \kappa \left(-12 + 13 \rho^2\right)\right) +$$
$$\left.\left. r t \left(-3 \left(-5 + 20 t \theta \kappa + 6 t^2 \theta^2 \kappa^2\right) \rho^2 + 2 (-k + x) \kappa \left(-75 + 60 \rho^2 + 4 t \theta \kappa \left(-12 + 13 \rho^2\right)\right)\right)\right)\right) +$$
$$4 e^{2 \bar{y}} t^2 \left(12 \kappa^2 \left(25 + 80 t \theta \kappa + 26 t^2 \theta^2 \kappa^2\right) - 12 \delta \kappa \left(25 + 80 t \theta \kappa + 26 t^2 \theta^2 \kappa^2 +\right.\right.$$
$$\left. 12 r t \kappa (5 + 3 t \theta \kappa) + 12 (-k + x) \kappa (5 + 3 t \theta \kappa)\right) \rho + 2 (-k + r t + x)^2 \delta^6 \left(12 + 5 \rho^2\right) -$$
$$4 (-k + r t + x) \delta^5 \rho \left(-3 + 27 t \theta \kappa + 30 \rho^2 + r t \left(12 + 5 \rho^2\right) + (-k + x) \left(12 + 5 \rho^2\right)\right) +$$
$$2 \delta^4 \left(12 (-k + x) \rho^2 \left(7 + 9 t \theta \kappa + 5 \rho^2\right) + 3 \left(-45 + 13 t^2 \theta^2 \kappa^2 + 28 \rho^2 + 10 \rho^4 + 9 t \theta \kappa \left(-1 + 4 \rho^2\right)\right) +\right.$$
$$r^2 t^2 \left(15 \rho^4 + 4 \kappa \left(12 + 5 \rho^2\right)\right) + (-k + x)^2 \left(15 \rho^4 + 4 \kappa \left(12 + 5 \rho^2\right)\right) +$$
$$\left. 2 r t \left(6 \rho^2 \left(7 + 9 t \theta \kappa + 5 \rho^2\right) + (-k + x) \left(48 \kappa + 20 \kappa \rho^2 + 15 \rho^4\right)\right)\right) +$$
$$\delta^2 \left(312 t^2 \theta^2 \kappa^3 + 60 \left(-1 + 2 \rho^2\right) + 15 \kappa \left(-44 + 48 (1 - k + x) \rho^2 + t \left(48 r \rho^2 + 5 \theta \left(-1 + 4 \rho^2\right)\right)\right) +\right.$$
$$2 \kappa^2 \left(216 t (-k + x) \theta \rho^2 + 4 r^2 t^2 \left(12 + 5 \rho^2\right) + 4 (-k + x)^2 \left(12 + 5 \rho^2\right) +\right.$$
$$\left.\left. t \theta \left(186 - 12 t \theta + 216 \rho^2 + 49 t \theta \rho^2\right) + 8 r t \left(27 t \theta \rho^2 + (-k + x) \left(12 + 5 \rho^2\right)\right)\right)\right) -$$
$$\delta^3 \rho \left(8 r^2 t^2 \kappa \left(12 + 5 \rho^2\right) + 8 (-k + x)^2 \kappa \left(12 + 5 \rho^2\right) +\right.$$
$$6 \left(-55 + 26 t^2 \theta^2 \kappa^2 + 60 \rho^2 + t \theta \kappa \left(31 + 36 \rho^2\right)\right) +$$
$$(-k + x) \left(-75 + 432 t \theta \kappa^2 + 240 \rho^2 + 4 \kappa \left(84 + 60 \rho^2 + t \theta \left(-12 + 37 \rho^2\right)\right)\right) + r t$$
$$\left.\left. \left(-75 + 432 t \theta \kappa^2 + 240 \rho^2 + 4 \kappa \left(84 + 60 \rho^2 + 4 (-k + x) \left(12 + 5 \rho^2\right) + t \theta \left(-12 + 37 \rho^2\right)\right)\right)\right)\right) +$$
$$2 e^{3 \bar{y}} t^3 \left(-32 \kappa^3 (30 + 19 t \theta \kappa) + 60 (-k + r t + x) \delta^7 \rho + 48 \delta \kappa^2\right.$$
$$(30 + 10 r t \kappa + 10 (-k + x) \kappa + 19 t \theta \kappa) \rho +$$
$$4 \delta^5 \rho \left(-36 + 57 t \theta \kappa + 90 \rho^2 + r t \left(-12 + 90 \kappa + 55 \rho^2\right) + (-k + x) \left(-12 + 90 \kappa + 55 \rho^2\right)\right) -$$
$$4 \delta^6 \left(-33 + 45 (1 - k + x) \rho^2 + t \left(19 \theta \kappa + 45 r \rho^2\right)\right) +$$
$$\delta^3 \rho \left(-75 + 48 (15 r t + 15 (-k + x) + 19 t \theta) \kappa^2 + 180 \rho^2 +\right.$$
$$\left. 4 \kappa \left(108 - 24 (-k + x) - 12 t \theta + 180 \rho^2 + 110 (-k + x) \rho^2 + 29 t \theta \rho^2 + 2 r t \left(-12 + 55 \rho^2\right)\right)\right) -$$
$$2 \delta^2 \kappa \left(-75 + 456 t \theta \kappa^2 + 420 \rho^2 + 4 \kappa \left(18 \left(3 + 5 (1 - k + x) \rho^2\right) + t \left(90 r \rho^2 + \theta \left(-12 + 67 \rho^2\right)\right)\right)\right) -$$
$$\delta^4 \left(-45 + 456 t \theta \kappa^2 - 24 (-5 + 2 r t + 2 (-k + x)) \rho^2 + 100 (2 - k + r t + x) \rho^4 +\right.$$
$$\left.\left. 4 \kappa \left(36 \left(-2 + 5 (1 - k + x) \rho^2\right) + t \left(180 r \rho^2 + \theta \left(-12 + 67 \rho^2\right)\right)\right)\right)\right) -$$
$$30 \left(e^{3 \bar{y}} t^3 \left(8 \delta^6 + 64 \kappa^3 - 24 \delta^5 \rho - 96 \delta \kappa^2 \rho + \delta^3 \rho \left(5 - 96 \kappa - 12 \rho^2\right) + \delta^4 \left(-5 + 48 \kappa + 28 \rho^2\right) +\right.\right.$$
$$\left. 2 \delta^2 \kappa \left(-5 + 48 \kappa + 28 \rho^2\right)\right) - 4 e^{\bar{y}} t \left(-8 \kappa \left(3 + 7 t \theta \kappa + 2 t^2 \theta^2 \kappa^2\right) + 4 \delta \left(3 + 7 t \theta \kappa + 2 t^2 \theta^2 \kappa^2 +\right.\right.$$
$$\left. 2 r t \kappa (5 + 2 t \theta \kappa) + 2 (-k + x) \kappa (5 + 2 t \theta \kappa)\right) \rho + (-k + r t + x)^2 \delta^4 \left(-5 + 4 \rho^2\right) -$$
$$(-k + r t + x) \delta^3 \rho \left(-20 - 5 (-k + x) - 8 t \theta \kappa + 4 (-k + x) \rho^2 + r t \left(-5 + 4 \rho^2\right)\right) +$$
$$2 \delta^2 \left(-2 (-k + x) (5 + 2 t \theta \kappa) \rho^2 + r^2 t^2 \kappa \left(-5 + 4 \rho^2\right) + (-k + x)^2 \kappa \left(-5 + 4 \rho^2\right) - 2 r t\right.$$

$$\left((5 + 2 t \theta \kappa) \rho^2 + (-k + x) \kappa \left(5 - 4 \rho^2\right)\right) - 2 \left(-2 + 2 t^2 \theta^2 \kappa^2 + 5 \rho^2 + t \theta \kappa \left(5 + 2 \rho^2\right)\right)\Big)\Big) -$$

$$8 \Big(-2 t \theta \kappa (6 + t \theta \kappa) + 4 (-k + x) \delta \left(3 + 3 t \theta \kappa + t^2 \theta^2 \kappa^2\right) \rho + r^3 t^3 \delta^3 \rho \left(-5 + 8 \rho^2\right) +$$

$$(-k + x)^3 \delta^3 \rho \left(-5 + 8 \rho^2\right) + (-k + x)^2 \delta^2 \left(8 - 16 \rho^2 + t \theta \kappa \left(5 - 12 \rho^2\right)\right) + r^2 t^2 \delta^2$$

$$\left(8 - 16 \rho^2 + t \theta \kappa \left(5 - 12 \rho^2\right) + 3 (-k + x) \delta \rho \left(-5 + 8 \rho^2\right)\right) + r t \delta \Big(4 \left(3 + 3 t \theta \kappa + t^2 \theta^2 \kappa^2\right) \rho +$$

$$3 (-k + x)^2 \delta^2 \rho \left(-5 + 8 \rho^2\right) - 2 (-k + x) \delta \left(-8 + 16 \rho^2 + t \theta \kappa \left(-5 + 12 \rho^2\right)\right)\Big)\Big) + 2 e^{2 \overline{y}} t^2$$

$$\Big(-8 \kappa^2 (15 + 8 t \theta \kappa) + 12 (-k + r t + x) \delta^5 \rho + 8 \delta \kappa (15 + 6 r t \kappa + 6 (-k + x) \kappa + 8 t \theta \kappa) \rho +$$

$$\delta^3 \rho \left(4 \left(7 + 8 t \theta \kappa + 6 \rho^2\right) + r t \left(-5 + 48 \kappa + 16 \rho^2\right) + (-k + x) \left(-5 + 48 \kappa + 16 \rho^2\right)\right) -$$

$$2 \delta^4 \left(-1 + 12 (1 - k + x) \rho^2 + 4 t \left(2 \theta \kappa + 3 r \rho^2\right)\right) -$$

$$\delta^2 \left(-8 + 64 t \theta \kappa^2 + 36 \rho^2 + \kappa \left(8 \left(7 + 6 (1 - k + x) \rho^2\right) + t \left(48 r \rho^2 + 5 \theta \left(-1 + 4 \rho^2\right)\right)\right)\right)\Big)\Big)$$

$$(y - \overline{y}) + 60 \Big(e^{2 \overline{y}} t^2 \left(15 \delta^4 + 60 \kappa^2 - 30 \delta^3 \rho - 60 \delta \kappa \rho + 2 \delta^2 \left(-2 + 30 \kappa + 9 \rho^2\right)\right) -$$

$$4 \Big(12 + 6 t \theta \kappa + t^2 \theta^2 \kappa^2 - 6 (-k + x) \delta (1 + t \theta \kappa) \rho + 4 r^2 t^2 \delta^2 \left(-1 + 2 \rho^2\right) +$$

$$4 (-k + x)^2 \delta^2 \left(-1 + 2 \rho^2\right) - 2 r t \delta \left(3 (\rho + t \theta \kappa \rho) + (-k + x) \delta \left(4 - 8 \rho^2\right)\right)\Big) +$$

$$4 e^{\overline{y}} t \Big(-2 \kappa (15 + 7 t \theta \kappa) + 5 (-k + r t + x) \delta^3 \rho + \delta (15 + 10 r t \kappa + 10 (-k + x) \kappa + 7 t \theta \kappa) \rho -$$

$$\delta^2 \left(t \left(7 \theta \kappa + 5 r \rho^2\right) + 5 \left(2 + (1 - k + x) \rho^2\right)\right)\Big)\Big) (y - \overline{y})^2 -$$

$$240 \left(4 + 2 t \theta \kappa - 2 r t \delta \rho - 2 (-k + x) \delta \rho + 5 e^{\overline{y}} t \left(\delta^2 + 2 \kappa - \delta \rho\right)\right)$$

$$(y - \overline{y})^3 - 240$$

$$(y - \overline{y})^4\Big)$$

For every set of parameters we consider only the choice $\overline{x} = x$ and $\overline{y} = y$ because as we have seen in 8.4, produces better results with an inferior computational cost.

9.4.1 First Set by Baldeaux and Badran modified #1

In the first case we use parameters values as proposed by Baldeaux and Badran in ([BB12]), so we have ($\kappa = 30.84$, $\theta = 0.48^2$, $\rho = -0.55$, $y = \mathrm{Log}\left(0.19^2\right)$), but we set $\delta = 10$ instead of 70.56.

We consider as in 8.4.1 the table of strikes

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.08	96	97	97	98	99	99	100	100	101	102	102	102	103
0.17	91	92	94	96	97	98	100	101	103	104	105	107	108
0.25	87	89	91	94	96	98	100	102	104	106	107	109	111
0.5	77	81	85	88	92	96	100	104	107	111	115	118	122
0.75	70	75	80	85	90	95	100	105	111	116	121	127	132
1	65	71	77	82	88	94	100	107	114	121	128	135	142
1.5	57	64	71	78	86	93	100	110	120	130	141	151	161
2	52	60	68	76	84	92	100	114	127	140	154	168	181

and the corresponding exact implied volatilities

	1	3	5	7	9	11	13
0.08	22.0039	21.6903	21.1088	20.8403	20.5862	20.3464	20.1206
0.17	25.2075	24.2234	23.3414	22.5611	21.8812	21.4828	20.9660
0.25	27.2799	26.0280	24.6489	23.6973	22.8813	22.3576	21.7751
0.5	30.4996	28.6050	27.1587	25.7483	24.7255	23.7922	23.1711
0.75	31.5292	29.6597	28.0672	26.7300	25.5337	24.6767	23.9623
1	31.8317	29.9839	28.5685	27.2827	26.0781	25.1487	24.4530
1.5	31.9912	30.3392	28.9322	27.8720	26.6811	25.7462	25.0853
2	31.8333	30.2994	29.1185	28.1791	26.9741	26.1019	25.4604

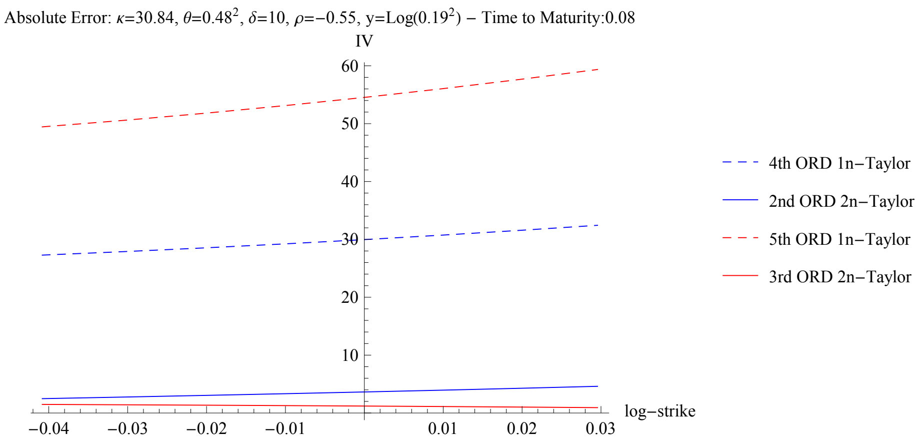
Now we compute abosolute errors of the 2n-Taylor approximation till 3rd order

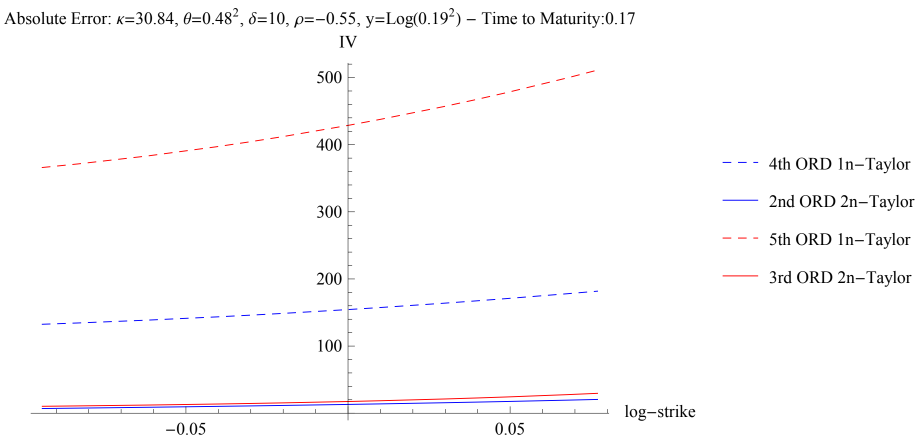
	1	3	5	7	9	11	13
0.08	1.45×10^{1}	1.48×10^{1}	1.54×10^{1}	1.57×10^{1}	1.59×10^{1}	1.62×10^{1}	1.64×10^{1}
	1.04×10^{1}	1.09×10^{1}	1.2×10^{1}	1.25×10^{1}	1.3×10^{1}	1.35×10^{1}	1.41×10^{1}
	2.48	2.75	3.33	3.63	3.95	4.27	4.61
	1.48	1.42	1.28	1.2	1.11	1.02	9.25×10^{-1}
0.17	1.13×10^{1}	1.23×10^{1}	1.32×10^{1}	1.39×10^{1}	1.46×10^{1}	1.5×10^{1}	1.55×10^{1}
	1.18×10^{1}	1.36×10^{1}	1.54×10^{1}	1.73×10^{1}	1.92×10^{1}	2.04×10^{1}	2.24×10^{1}
	6.95	8.81	1.08×10^{1}	1.31×10^{1}	1.56×10^{1}	1.74×10^{1}	2.05×10^{1}
	1.03×10^{1}	1.21×10^{1}	1.45×10^{1}	1.74×10^{1}	2.12×10^{1}	2.42×10^{1}	2.96×10^{1}
0.25	9.22	1.05×10^{1}	1.19×10^{1}	1.28×10^{1}	1.36×10^{1}	1.41×10^{1}	1.47×10^{1}
	1.39×10^{1}	1.64×10^{1}	1.98×10^{1}	2.26×10^{1}	2.55×10^{1}	2.77×10^{1}	3.06×10^{1}
	1.83×10^{1}	2.32×10^{1}	2.99×10^{1}	3.6×10^{1}	4.28×10^{1}	4.86×10^{1}	5.72×10^{1}
	7.72×10^{1}	8.86×10^{1}	1.06×10^{2}	1.24×10^{2}	1.47×10^{2}	1.67×10^{2}	1.99×10^{2}
0.5	6.	7.9	9.34	1.08×10^{1}	1.18×10^{1}	1.27×10^{1}	1.33×10^{1}
	2.46×10^{1}	3.09×10^{1}	3.71×10^{1}	4.48×10^{1}	5.18×10^{1}	6.01×10^{1}	6.74×10^{1}
	1.21×10^{2}	1.66×10^{2}	2.08×10^{2}	2.62×10^{2}	3.16×10^{2}	3.86×10^{2}	4.56×10^{2}
	1.59×10^{3}	1.94×10^{3}	2.31×10^{3}	2.83×10^{3}	3.4×10^{3}	4.19×10^{3}	5.03×10^{3}
0.75	4.97	6.84	8.44	9.77	1.1×10^{1}	1.18×10^{1}	1.25×10^{1}
	3.95×10^{1}	4.98×10^{1}	6.16×10^{1}	7.42×10^{1}	8.85×10^{1}	1.02×10^{2}	1.16×10^{2}
	3.95×10^{2}	5.47×10^{2}	7.09×10^{2}	8.87×10^{2}	1.1×10^{3}	1.33×10^{3}	1.6×10^{3}
	9.21×10^{3}	1.15×10^{4}	1.42×10^{4}	1.75×10^{4}	2.19×10^{4}	2.67×10^{4}	3.3×10^{4}
1	4.67	6.52	7.93	9.22	1.04×10^{1}	1.14×10^{1}	1.21×10^{1}
	5.8×10^{1}	7.45×10^{1}	9.12×10^{1}	1.1×10^{2}	1.33×10^{2}	1.55×10^{2}	1.77×10^{2}
	9.68×10^{2}	1.35×10^{3}	1.72×10^{3}	2.15×10^{3}	2.71×10^{3}	3.32×10^{3}	3.99×10^{3}
	3.34×10^{4}	4.28×10^{4}	5.27×10^{4}	6.53×10^{4}	8.29×10^{4}	1.04×10^{5}	1.29×10^{5}
1.5	4.51	6.16	7.57	8.63	9.82	1.08×10^{1}	1.14×10^{1}
	1.04×10^{2}	1.35×10^{2}	1.7×10^{2}	2.01×10^{2}	2.45×10^{2}	2.89×10^{2}	3.29×10^{2}
	3.57×10^{3}	4.93×10^{3}	6.4×10^{3}	7.82×10^{3}	9.96×10^{3}	1.24×10^{4}	1.48×10^{4}
	2.16×10^{5}	2.81×10^{5}	3.58×10^{5}	4.38×10^{5}	5.69×10^{5}	7.28×10^{5}	$9.\times10^{5}$
2	4.67	6.2	7.38	8.32	9.53	1.04×10^{1}	1.1×10^{1}
	1.67×10^{2}	2.19×10^{2}	2.7×10^{2}	3.18×10^{2}	3.92×10^{2}	4.6×10^{2}	5.22×10^{2}
	9.59×10^{3}	1.31×10^{4}	1.66×10^{4}	2.01×10^{4}	2.6×10^{4}	3.21×10^{4}	3.84×10^{4}
	8.75×10^{5}	1.15×10^{6}	1.44×10^{6}	1.76×10^{6}	2.35×10^{6}	$3.\times10^{6}$	3.71×10^{6}

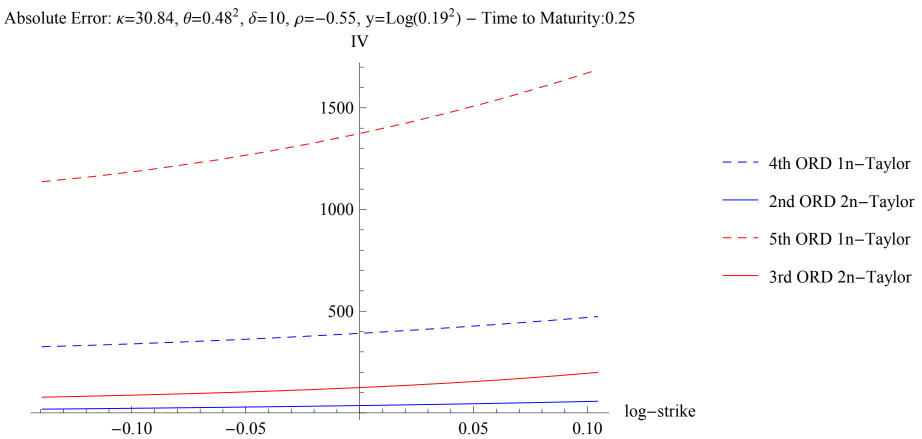
then in the next table we compare the 1n-Taylor approximation with the 2n-Taylor approximation

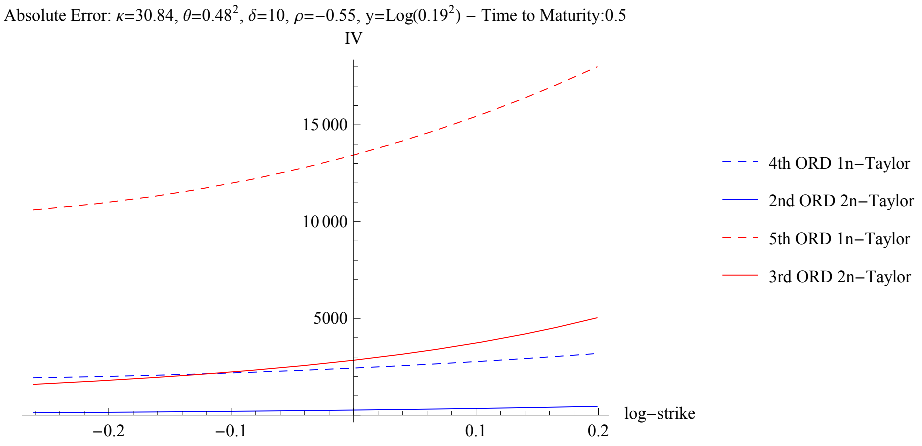
	1		7		13	
0.08	1.45×10^{1}	1.45×10^{1}	1.57×10^{1}	1.57×10^{1}	1.64×10^{1}	1.64×10^{1}
	1.02×10^{1}	1.04×10^{1}	1.11×10^{1}	1.25×10^{1}	1.19×10^{1}	1.41×10^{1}
	1.16×10^{1}	2.48	1.27×10^{1}	3.63	1.35×10^{1}	4.61
	1.66×10^{1}	1.48	1.82×10^{1}	1.2	1.96×10^{1}	9.25×10^{-1}
	2.73×10^{1}	0	$3.\times10^{1}$	0	3.24×10^{1}	0
	4.94×10^{1}	0	5.45×10^{1}	0	5.94×10^{1}	0

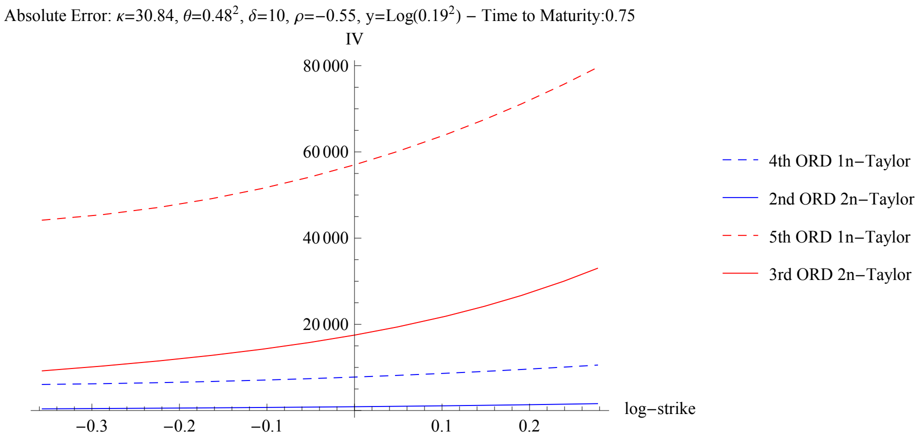
0.17	1.13×10^{1}	1.13×10^{1}	1.39×10^{1}	1.39×10^{1}	1.55×10^{1}	1.55×10^{1}
	1.41×10^{1}	1.18×10^{1}	1.61×10^{1}	1.73×10^{1}	1.84×10^{1}	2.24×10^{1}
	2.45×10^{1}	6.95	2.82×10^{1}	1.31×10^{1}	3.23×10^{1}	2.05×10^{1}
	5.31×10^{1}	1.03×10^{1}	6.15×10^{1}	1.74×10^{1}	7.15×10^{1}	2.96×10^{1}
	1.32×10^{2}	0	1.54×10^{2}	0	1.82×10^{2}	0
	3.66×10^{2}	0	4.29×10^{2}	0	5.11×10^{2}	0
0.25	9.22	9.22	1.28×10^{1}	1.28×10^{1}	1.47×10^{1}	1.47×10^{1}
	1.68×10^{1}	1.39×10^{1}	2.02×10^{1}	2.26×10^{1}	2.35×10^{1}	3.06×10^{1}
	3.77×10^{1}	1.83×10^{1}	4.46×10^{1}	3.6×10^{1}	5.23×10^{1}	5.72×10^{1}
	1.03×10^{2}	7.72×10^{1}	1.23×10^{2}	1.24×10^{2}	1.47×10^{2}	1.99×10^{2}
	3.25×10^{2}	0	3.91×10^{2}	0	4.73×10^{2}	0
	1.14×10^{3}	0	1.37×10^{3}	0	1.69×10^{3}	0
0.5	6.	6.	1.08×10^{1}	1.08×10^{1}	1.33×10^{1}	1.33×10^{1}
	2.31×10^{1}	2.46×10^{1}	3.15×10^{1}	4.48×10^{1}	3.89×10^{1}	6.74×10^{1}
	8.9×10^{1}	1.21×10^{2}	1.11×10^{2}	2.62×10^{2}	1.39×10^{2}	4.56×10^{2}
	3.88×10^{2}	1.59×10^{3}	4.85×10^{2}	2.83×10^{3}	6.2×10^{2}	5.03×10^{3}
	1.93×10^{3}	0	2.43×10^{3}	0	3.18×10^{3}	0
	1.06×10^{4}	0	1.34×10^{4}	0	1.8×10^{4}	0
0.75	4.97	4.97	9.77	9.77	1.25×10^{1}	1.25×10^{1}
	2.86×10^{1}	3.95×10^{1}	4.17×10^{1}	7.42×10^{1}	5.28×10^{1}	1.16×10^{2}
	1.54×10^{2}	3.95×10^{2}	$2.\times10^{2}$	8.87×10^{2}	2.57×10^{2}	1.6×10^{3}
	9.14×10^{2}	9.21×10^{3}	1.16×10^{3}	1.75×10^{4}	1.54×10^{3}	3.3×10^{4}
	6.05×10^{3}	0	7.76×10^{3}	0	1.05×10^{4}	0
	4.42×10^{4}	0	5.7×10^{4}	0	7.96×10^{4}	0
1	4.67	4.67	9.22	9.22	1.21×10^{1}	1.21×10^{1}
	3.43×10^{1}	5.8×10^{1}	5.14×10^{1}	1.1×10^{2}	6.62×10^{1}	1.77×10^{2}
	2.35×10^{2}	9.68×10^{2}	3.1×10^{2}	2.15×10^{3}	4.06×10^{2}	3.99×10^{3}
	1.74×10^{3}	3.34×10^{4}	2.23×10^{3}	6.53×10^{4}	3.01×10^{3}	1.29×10^{5}
	1.43×10^{4}	0	1.84×10^{4}	0	2.56×10^{4}	0
	1.28×10^{5}	0	1.67×10^{5}	0	2.4×10^{5}	0
1.5	4.51	4.51	8.63	8.63	1.14×10^{1}	1.14×10^{1}
	4.63×10^{1}	1.04×10^{2}	7.04×10^{1}	2.01×10^{2}	9.15×10^{1}	3.29×10^{2}
	4.4×10^{2}	3.57×10^{3}	5.93×10^{2}	7.82×10^{3}	7.86×10^{2}	1.48×10^{4}
	4.49×10^{3}	2.16×10^{5}	5.86×10^{3}	4.38×10^{5}	8.03×10^{3}	$9.\times10^{5}$
	5.05×10^{4}	0	6.59×10^{4}	0	9.37×10^{4}	0
	6.18×10^{5}	0	8.12×10^{5}	0	1.2×10^{6}	0
2	4.67	4.67	8.32	8.32	1.1×10^{1}	1.1×10^{1}
	5.99×10^{1}	1.67×10^{2}	8.91×10^{1}	3.18×10^{2}	1.16×10^{2}	5.22×10^{2}
	7.13×10^{2}	9.59×10^{3}	9.6×10^{2}	2.01×10^{4}	1.28×10^{3}	3.84×10^{4}
	9.16×10^{3}	8.75×10^{5}	1.2×10^{4}	1.76×10^{6}	1.65×10^{4}	3.71×10^{6}
	1.29×10^{5}	0	1.69×10^{5}	0	2.43×10^{5}	0
	1.98×10^{6}	0	2.61×10^{6}	0	3.9×10^{6}	0

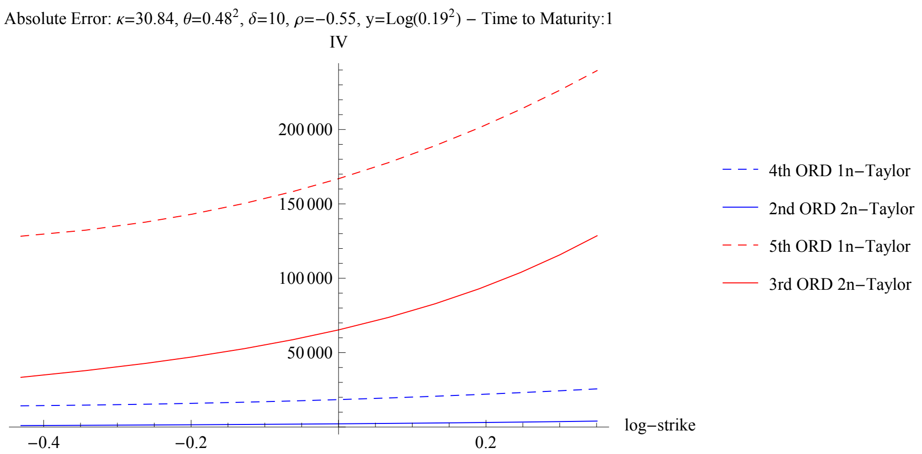

Absolute Error: κ=30.84, θ=0.48², δ=10, ρ=−0.55, y=Log(0.19²) − Time to Maturity:0.08
IV
60
50
40
30
20
10
log−strike
−0.04
−0.03
−0.02
−0.01
0.01
0.02
0.03
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor

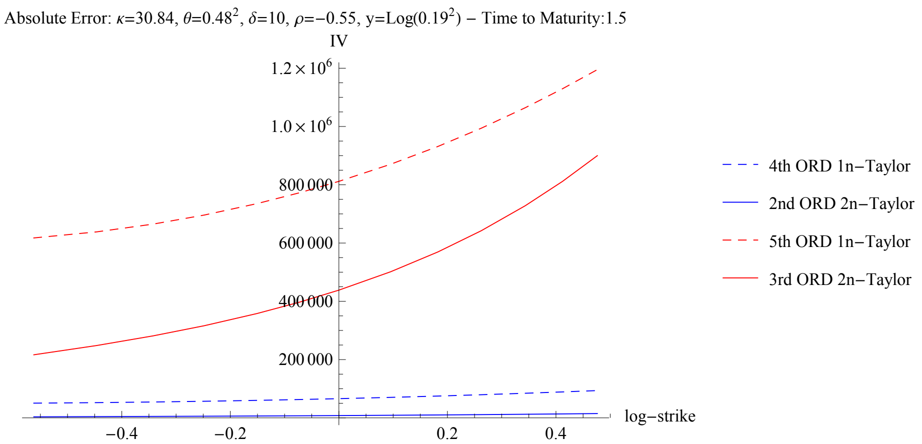

Absolute Error: κ=30.84, θ=0.48², δ=10, ρ=−0.55, y=Log(0.19²) − Time to Maturity:0.17
IV
500
400
300
200
100
log−strike
−0.05
0.05
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor

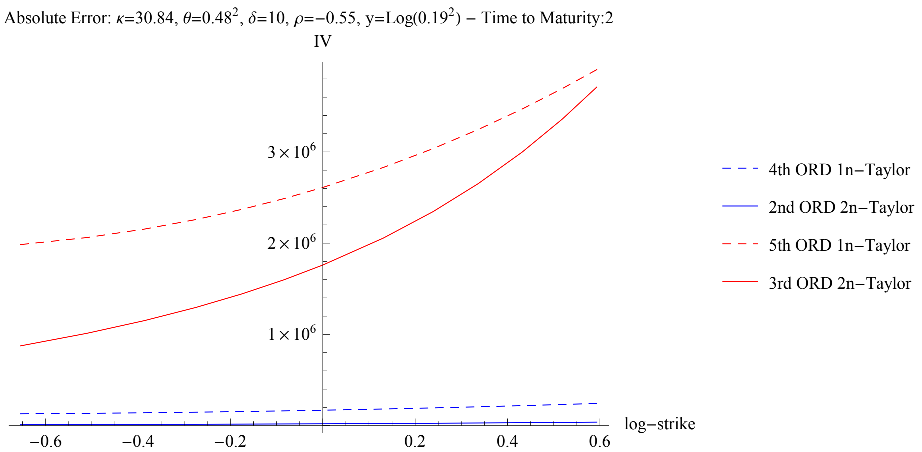

Absolute Error: κ=30.84, θ=0.48², δ=10, ρ=−0.55, y=Log(0.19²) − Time to Maturity:0.25
IV
1500
1000
500
log−strike
−0.10
−0.05
0.05
0.10
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor


Absolute Error: κ=30.84, θ=0.48², δ=10, ρ=−0.55, y=Log(0.19²) − Time to Maturity:0.5
IV
15 000
10 000
5000
−0.2
−0.1
0.1
0.2
log−strike
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor


Absolute Error: κ=30.84, θ=0.48², δ=10, ρ=−0.55, y=Log(0.19²) − Time to Maturity:0.75
IV
80 000
60 000
40 000
20 000
−0.3
−0.2
−0.1
0.1
0.2
log−strike
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor


Absolute Error: κ=30.84, θ=0.48², δ=10, ρ=−0.55, y=Log(0.19²) − Time to Maturity:1
IV
200 000
150 000
100 000
50 000
−0.4
−0.2
0.2
log−strike
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor





9.4.2 Second Set by Baldeaux and Badran modified #2

In the second case we use parameters values as proposed by Baldeaux and Badran in ([BB12]), so we have ($\kappa = 30.84$, $\theta = 0.48^2$, $\rho = -0.55$, $y = \text{Log}\left(0.19^2\right)$), but we set $\delta = 30$ instead of 70.56.

We consider as in 8.4.2 the table of strikes

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.08	97	98	98	98	99	100	100	100	101	101	101	102	102
0.17	95	96	97	98	98	99	100	101	101	102	103	103	104
0.25	94	95	96	97	98	99	100	101	102	102	103	104	105
0.5	90	92	93	95	97	98	100	101	103	104	105	107	108
0.75	88	90	92	94	96	98	100	102	103	105	107	108	110
1	86	88	91	93	95	98	100	102	104	106	108	110	112
1.5	82	85	88	91	94	97	100	103	106	108	111	114	117
2	79	82	86	90	93	96	100	104	107	110	114	118	121

and the corresponding exact implied volatilities

	1	3	5	7	9	11	13
0.08	18.6240	18.0494	17.5131	17.0188	16.5698	16.5698	16.1694
0.17	17.4098	16.5817	16.1932	15.4764	15.1518	14.5789	14.3335
0.25	16.5623	15.8865	15.2548	14.6763	14.1600	13.9280	13.5207
0.5	15.5677	14.8887	14.0475	13.4768	12.9689	12.6700	12.2868
0.75	14.8754	14.1968	13.5652	12.9899	12.6014	12.1481	11.8609
1	14.5150	13.8301	13.3212	12.7407	12.3265	11.9617	11.6494
1.5	14.1603	13.5527	12.9971	12.4965	12.0542	11.7324	11.4037
2	13.9120	13.3466	12.8356	12.3785	11.9756	11.6266	11.3308

Now we compute abosolute errors of the 2n-Taylor approximation till 3rd order

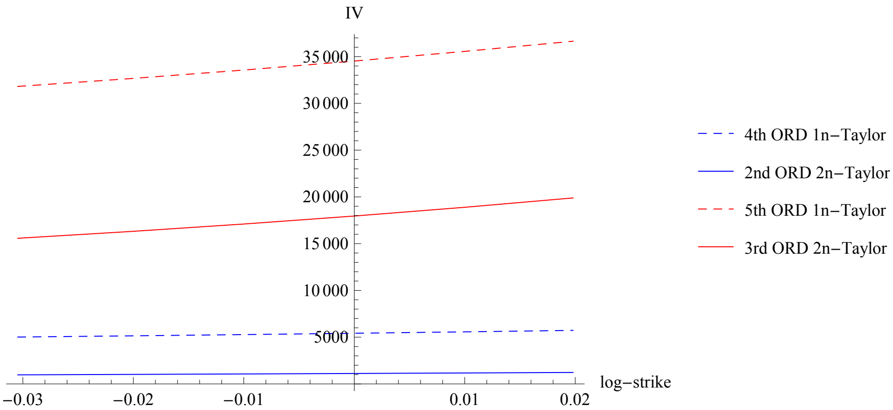
	1	3	5	7	9	11	13
0.08	1.79×10^{1}	1.85×10^{1}	1.9×10^{1}	1.95×10^{1}	1.99×10^{1}	1.99×10^{1}	2.03×10^{1}
	1.08×10^{2}	1.14×10^{2}	1.19×10^{2}	1.25×10^{2}	1.3×10^{2}	1.3×10^{2}	1.36×10^{2}
	9.72×10^{2}	1.02×10^{3}	1.07×10^{3}	1.12×10^{3}	1.17×10^{3}	1.17×10^{3}	1.23×10^{3}
	1.56×10^{4}	1.63×10^{4}	1.71×10^{4}	1.8×10^{4}	1.89×10^{4}	1.89×10^{4}	1.99×10^{4}
0.17	1.91×10^{1}	1.99×10^{1}	2.03×10^{1}	2.1×10^{1}	2.14×10^{1}	2.19×10^{1}	2.22×10^{1}
	3.58×10^{2}	3.79×10^{2}	3.9×10^{2}	4.11×10^{2}	4.22×10^{2}	4.43×10^{2}	4.54×10^{2}
	1.19×10^{4}	1.26×10^{4}	1.3×10^{4}	1.37×10^{4}	1.41×10^{4}	1.49×10^{4}	1.53×10^{4}
	6.21×10^{5}	6.58×10^{5}	6.78×10^{5}	7.2×10^{5}	7.42×10^{5}	7.9×10^{5}	8.15×10^{5}
0.25	1.99×10^{1}	2.06×10^{1}	2.12×10^{1}	2.18×10^{1}	2.23×10^{1}	2.26×10^{1}	2.3×10^{1}
	7.13×10^{2}	7.43×10^{2}	7.74×10^{2}	8.04×10^{2}	8.35×10^{2}	8.5×10^{2}	8.81×10^{2}
	4.68×10^{4}	4.89×10^{4}	5.1×10^{4}	5.31×10^{4}	5.54×10^{4}	5.65×10^{4}	5.89×10^{4}
	4.58×10^{6}	4.79×10^{6}	5.02×10^{6}	5.25×10^{6}	5.5×10^{6}	5.63×10^{6}	5.91×10^{6}
0.5	2.09×10^{1}	2.16×10^{1}	2.25×10^{1}	2.3×10^{1}	2.35×10^{1}	2.38×10^{1}	2.42×10^{1}
	2.57×10^{3}	2.67×10^{3}	2.79×10^{3}	2.88×10^{3}	2.97×10^{3}	3.03×10^{3}	3.11×10^{3}
	5.95×10^{5}	6.18×10^{5}	6.49×10^{5}	6.72×10^{5}	6.96×10^{5}	7.12×10^{5}	7.37×10^{5}
	1.91×10^{8}	1.99×10^{8}	2.1×10^{8}	2.19×10^{8}	2.28×10^{8}	2.34×10^{8}	2.44×10^{8}
0.75	2.16×10^{1}	2.23×10^{1}	2.29×10^{1}	2.35×10^{1}	2.39×10^{1}	2.44×10^{1}	2.46×10^{1}
	5.68×10^{3}	5.87×10^{3}	6.05×10^{3}	6.23×10^{3}	6.37×10^{3}	6.54×10^{3}	6.66×10^{3}
	2.8×10^{6}	2.91×10^{6}	3.01×10^{6}	3.11×10^{6}	3.19×10^{6}	3.29×10^{6}	3.37×10^{6}
	1.87×10^{9}	1.94×10^{9}	2.02×10^{9}	2.1×10^{9}	2.16×10^{9}	2.25×10^{9}	2.31×10^{9}
1	2.2×10^{1}	2.27×10^{1}	2.32×10^{1}	2.38×10^{1}	2.42×10^{1}	2.45×10^{1}	2.49×10^{1}
	$1.\times10^{4}$	1.03×10^{4}	1.06×10^{4}	1.09×10^{4}	1.11×10^{4}	1.13×10^{4}	1.15×10^{4}
	8.53×10^{6}	8.84×10^{6}	9.08×10^{6}	9.38×10^{6}	9.62×10^{6}	9.86×10^{6}	1.01×10^{7}
	9.64×10^{9}	$1.\times10^{10}$	1.04×10^{10}	1.08×10^{10}	1.11×10^{10}	1.14×10^{10}	1.17×10^{10}
1.5	2.23×10^{1}	2.3×10^{1}	2.35×10^{1}	2.4×10^{1}	2.44×10^{1}	2.48×10^{1}	2.51×10^{1}
	2.23×10^{4}	2.29×10^{4}	2.34×10^{4}	2.4×10^{4}	2.45×10^{4}	2.49×10^{4}	2.54×10^{4}
	4.15×10^{7}	4.28×10^{7}	4.4×10^{7}	4.52×10^{7}	4.64×10^{7}	4.73×10^{7}	4.85×10^{7}
	$1.\times10^{11}$	1.04×10^{11}	1.08×10^{11}	1.11×10^{11}	1.15×10^{11}	1.17×10^{11}	1.21×10^{11}
2	2.26×10^{1}	2.32×10^{1}	2.37×10^{1}	2.41×10^{1}	2.45×10^{1}	2.49×10^{1}	2.52×10^{1}
	3.95×10^{4}	4.05×10^{4}	4.14×10^{4}	4.22×10^{4}	4.3×10^{4}	4.38×10^{4}	4.45×10^{4}
	1.29×10^{8}	1.33×10^{8}	1.36×10^{8}	1.39×10^{8}	1.43×10^{8}	1.46×10^{8}	1.49×10^{8}
	5.41×10^{11}	5.59×10^{11}	5.76×10^{11}	5.93×10^{11}	6.1×10^{11}	6.26×10^{11}	6.41×10^{11}

then in the next table we compare the 1n-Taylor approximation with the 2n-Taylor approximation

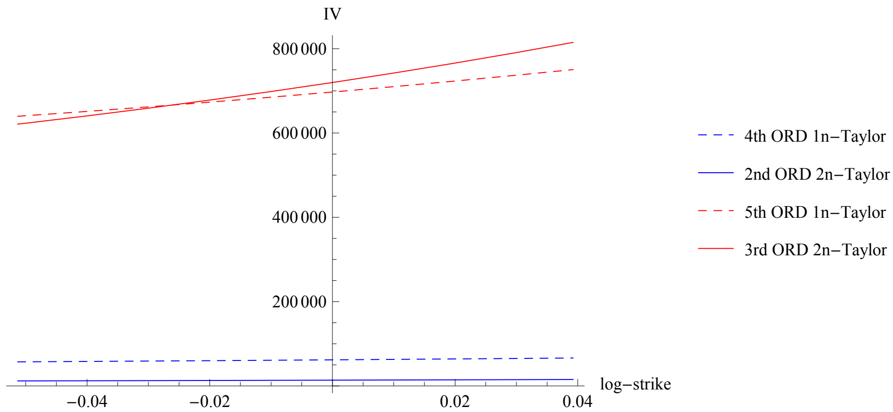
	1		7		13	
0.08	1.79×10^{1}	1.79×10^{1}	1.95×10^{1}	1.95×10^{1}	2.03×10^{1}	2.03×10^{1}
	4.38×10^{1}	1.08×10^{2}	4.67×10^{1}	1.25×10^{2}	4.89×10^{1}	1.36×10^{2}
	1.75×10^{2}	9.72×10^{2}	1.87×10^{2}	1.12×10^{3}	1.96×10^{2}	1.23×10^{3}
	8.73×10^{2}	1.56×10^{4}	9.39×10^{2}	1.8×10^{4}	9.88×10^{2}	1.99×10^{4}
	5.01×10^{3}	0	5.42×10^{3}	0	5.73×10^{3}	0
	3.18×10^{4}	0	3.45×10^{4}	0	3.66×10^{4}	0
0.17	1.91×10^{1}	1.91×10^{1}	2.1×10^{1}	2.1×10^{1}	2.22×10^{1}	2.22×10^{1}
	8.71×10^{1}	3.58×10^{2}	9.29×10^{1}	4.11×10^{2}	9.77×10^{1}	4.54×10^{2}
	6.25×10^{2}	1.19×10^{4}	6.69×10^{2}	1.37×10^{4}	7.06×10^{2}	1.53×10^{4}
	5.6×10^{3}	6.21×10^{5}	6.03×10^{3}	7.2×10^{5}	6.4×10^{3}	8.15×10^{5}
	5.72×10^{4}	0	6.19×10^{4}	0	6.63×10^{4}	0
	6.4×10^{5}	0	6.97×10^{5}	0	7.5×10^{5}	0
0.25	1.99×10^{1}	1.99×10^{1}	2.18×10^{1}	2.18×10^{1}	2.3×10^{1}	2.3×10^{1}
	1.27×10^{2}	7.13×10^{2}	1.34×10^{2}	8.04×10^{2}	1.41×10^{2}	8.81×10^{2}
	1.26×10^{3}	4.68×10^{4}	1.34×10^{3}	5.31×10^{4}	1.41×10^{3}	5.89×10^{4}
	1.56×10^{4}	4.58×10^{6}	1.67×10^{4}	5.25×10^{6}	1.76×10^{4}	5.91×10^{6}
	2.19×10^{5}	0	2.35×10^{5}	0	2.5×10^{5}	0
	3.34×10^{6}	0	3.62×10^{6}	0	3.88×10^{6}	0
0.5	2.09×10^{1}	2.09×10^{1}	2.3×10^{1}	2.3×10^{1}	2.42×10^{1}	2.42×10^{1}
	2.52×10^{2}	2.57×10^{3}	2.66×10^{2}	2.88×10^{3}	2.76×10^{2}	3.11×10^{3}
	4.65×10^{3}	5.95×10^{5}	4.92×10^{3}	6.72×10^{5}	5.13×10^{3}	7.37×10^{5}
	1.06×10^{5}	1.91×10^{8}	1.13×10^{5}	2.19×10^{8}	1.19×10^{5}	2.44×10^{8}
	2.71×10^{6}	0	2.91×10^{6}	0	3.08×10^{6}	0
	7.51×10^{7}	0	8.13×10^{7}	0	8.66×10^{7}	0
0.75	2.16×10^{1}	2.16×10^{1}	2.35×10^{1}	2.35×10^{1}	2.46×10^{1}	2.46×10^{1}
	3.8×10^{2}	5.68×10^{3}	3.98×10^{2}	6.23×10^{3}	4.11×10^{2}	6.66×10^{3}
	1.02×10^{4}	2.8×10^{6}	1.07×10^{4}	3.11×10^{6}	1.11×10^{4}	3.37×10^{6}
	3.39×10^{5}	1.87×10^{9}	3.58×10^{5}	2.1×10^{9}	3.74×10^{5}	2.31×10^{9}
	1.26×10^{7}	0	1.34×10^{7}	0	1.41×10^{7}	0
	5.04×10^{8}	0	5.39×10^{8}	0	5.7×10^{8}	0
1	2.2×10^{1}	2.2×10^{1}	2.38×10^{1}	2.38×10^{1}	2.49×10^{1}	2.49×10^{1}
	5.09×10^{2}	$1.\times10^{4}$	5.3×10^{2}	1.09×10^{4}	5.46×10^{2}	1.15×10^{4}
	1.8×10^{4}	8.53×10^{6}	1.88×10^{4}	9.38×10^{6}	1.94×10^{4}	1.01×10^{7}
	7.83×10^{5}	9.64×10^{9}	8.22×10^{5}	1.08×10^{10}	8.55×10^{5}	1.17×10^{10}
	3.8×10^{7}	0	4.03×10^{7}	0	4.21×10^{7}	0
	1.99×10^{9}	0	2.12×10^{9}	0	2.22×10^{9}	0
1.5	2.23×10^{1}	2.23×10^{1}	2.4×10^{1}	2.4×10^{1}	2.51×10^{1}	2.51×10^{1}
	7.66×10^{2}	2.23×10^{4}	7.95×10^{2}	2.4×10^{4}	8.17×10^{2}	2.54×10^{4}
	$4.\times10^{4}$	4.15×10^{7}	4.16×10^{4}	4.52×10^{7}	4.29×10^{4}	4.85×10^{7}
	2.57×10^{6}	$1.\times10^{11}$	2.69×10^{6}	1.11×10^{11}	2.79×10^{6}	1.21×10^{11}
	1.84×10^{8}	0	1.93×10^{8}	0	2.02×10^{8}	0
	1.41×10^{10}	0	1.49×10^{10}	0	1.56×10^{10}	0

2	2.26×10^{1}	2.26×10^{1}	2.41×10^{1}	2.41×10^{1}	2.52×10^{1}	2.52×10^{1}
	1.03×10^{3}	3.95×10^{4}	1.06×10^{3}	4.22×10^{4}	1.09×10^{3}	4.45×10^{4}
	7.09×10^{4}	1.29×10^{8}	7.34×10^{4}	1.39×10^{8}	7.55×10^{4}	1.49×10^{8}
	6.02×10^{6}	5.41×10^{11}	6.26×10^{6}	5.93×10^{11}	6.48×10^{6}	6.41×10^{11}
	5.68×10^{8}	0	5.95×10^{8}	0	6.19×10^{8}	0
	5.73×10^{10}	0	6.04×10^{10}	0	6.32×10^{10}	0

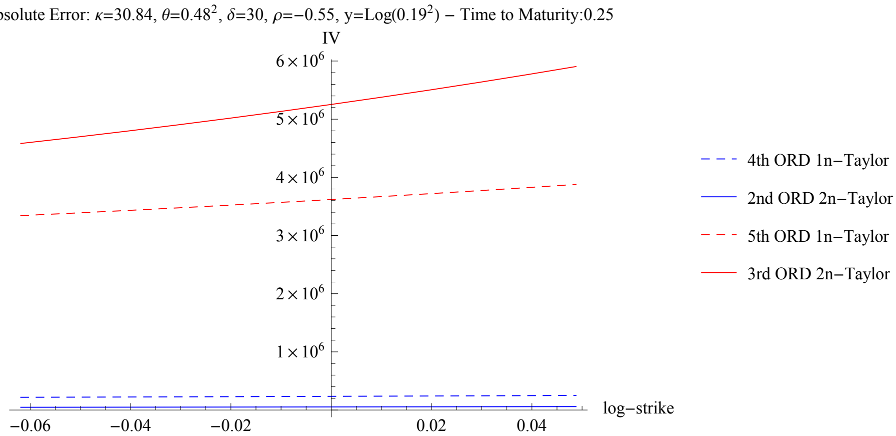
Absolute Error: κ=30.84, θ=0.48^2, δ=30, ρ=−0.55, y=Log(0.19^2) − Time to Maturity:0.08

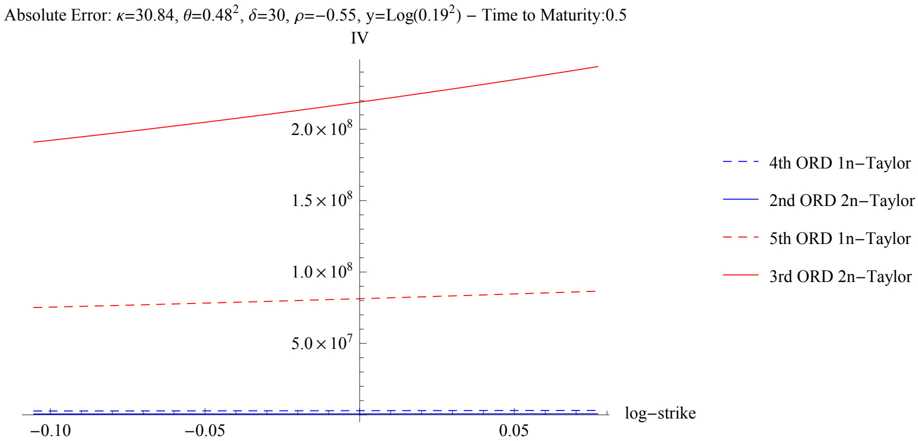


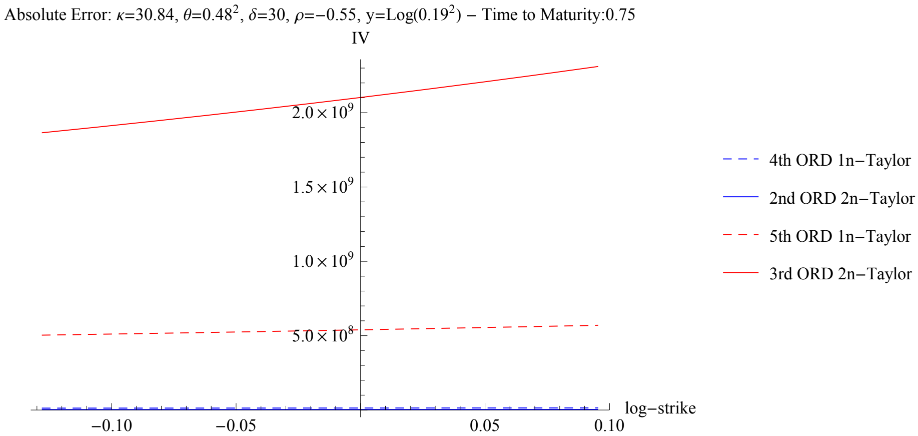
Absolute Error: κ=30.84, θ=0.48^2, δ=30, ρ=−0.55, y=Log(0.19^2) − Time to Maturity:0.17

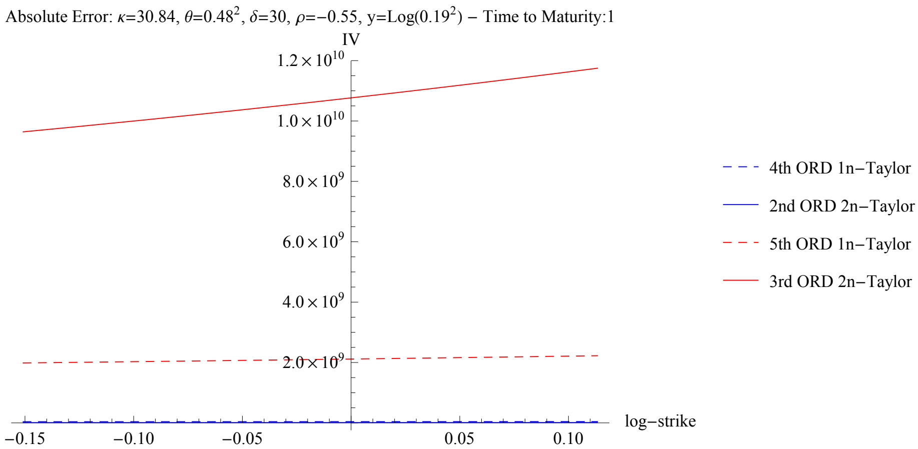


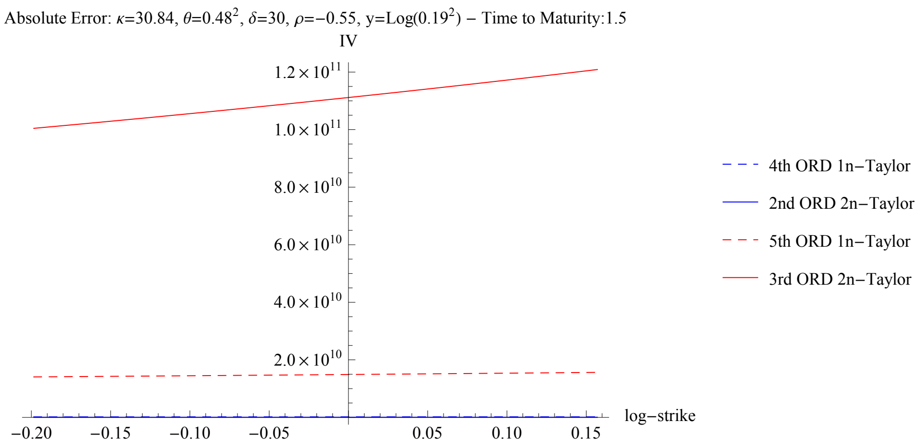
Absolute Error: κ=30.84, θ=0.48^2, δ=30, ρ=−0.55, y=Log(0.19^2) − Time to Maturity:0.25

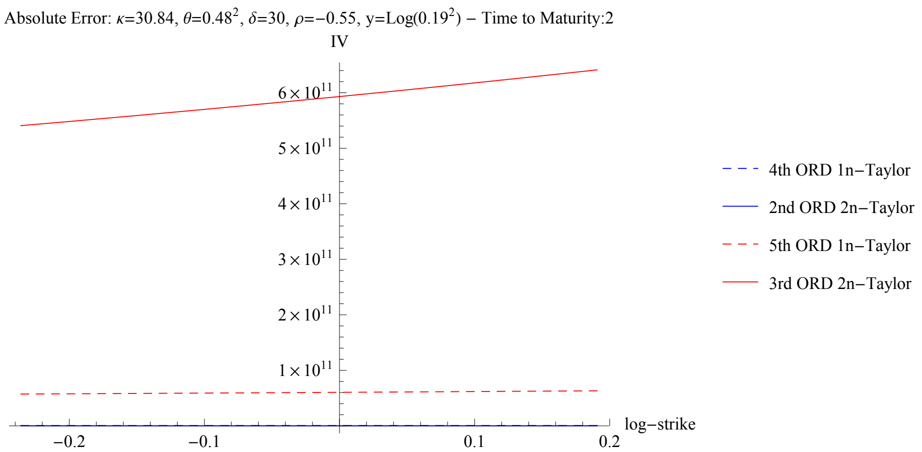



Absolute Error: κ=30.84, θ=0.48², δ=30, ρ=−0.55, y=Log(0.19²) − Time to Maturity:0.5
IV
2.0 × 10^8
1.5 × 10^8
1.0 × 10^8
5.0 × 10^7
−0.10
−0.05
0.05
log−strike
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor


Absolute Error: κ=30.84, θ=0.48², δ=30, ρ=−0.55, y=Log(0.19²) − Time to Maturity:0.75
IV
2.0 × 10^9
1.5 × 10^9
1.0 × 10^9
5.0 × 10^8
−0.10
−0.05
0.05
0.10
log−strike
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor


Absolute Error: κ=30.84, θ=0.48², δ=30, ρ=−0.55, y=Log(0.19²) − Time to Maturity:1
IV
1.2 × 10^10
1.0 × 10^10
8.0 × 10^9
6.0 × 10^9
4.0 × 10^9
2.0 × 10^9
−0.15
−0.10
−0.05
0.05
0.10
log−strike
4th ORD 1n−Taylor
2nd ORD 2n−Taylor
5th ORD 1n−Taylor
3rd ORD 2n−Taylor





9.4.3 Third Set by Drimus modified

In this third case we use parameters values as proposed by Drimus in ([Dri12]), so we have ($\kappa = 22.84$, $\theta = 0.4669^2$, $\rho = -0.99$, $y = \mathrm{Log}\left(0.245^2\right)$) but we consider $\delta = 20$ instead of 8.56.

We consider as in 8.4.3 the table of strikes

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.08	95	96	97	98	98	99	100	100	101	102	102	102	103
0.17	91	92	94	96	97	98	100	101	102	102	103	104	105
0.25	89	91	93	94	96	98	100	101	102	103	104	105	106
0.5	84	87	89	92	95	97	100	102	103	104	106	108	109
0.75	81	84	87	90	94	97	100	102	104	106	108	110	112
1	78	82	85	89	93	96	100	102	105	107	109	112	114
1.5	74	78	83	87	91	96	100	103	106	110	113	116	119
2	70	75	80	85	90	95	100	104	108	112	116	120	124

and the corresponding exact implied volatilities

	1	3	5	7	9	11	13
0.08	25.5616	23.8769	23.0516	21.4391	20.6535	19.8828	19.1277
0.17	25.0293	23.1715	21.3420	19.5490	18.3791	17.8032	16.6717
0.25	24.0517	22.0165	20.5063	18.5211	17.5441	16.5802	15.6320
0.5	22.3008	20.5806	18.5395	16.8566	15.8557	14.8631	13.8809
0.75	21.1187	19.5120	17.6695	16.1111	15.0815	14.0594	13.0463
1	20.4941	18.9347	17.1988	15.7095	14.6588	13.8250	12.7912
1.5	19.5507	18.0247	16.7245	15.3089	14.3866	13.3277	12.4323
2	19.0818	17.6828	16.3685	15.1177	14.1521	13.2110	12.2897

Now we compute abosolute errors of the 2n-Taylor approximation till 3rd order

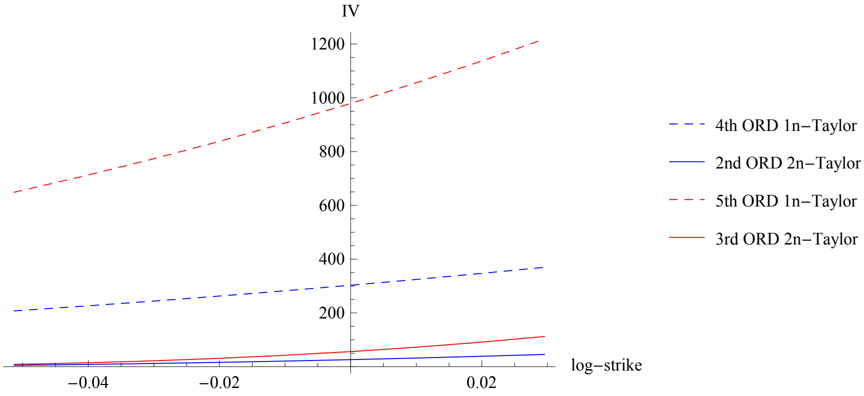
	1	3	5	7	9	11	13
0.08	1.17×10^{1}	1.34×10^{1}	1.42×10^{1}	1.58×10^{1}	1.66×10^{1}	1.74×10^{1}	1.82×10^{1}
	1.24×10^{1}	1.7×10^{1}	1.96×10^{1}	2.56×10^{1}	2.89×10^{1}	3.24×10^{1}	3.6×10^{1}
	5.2	1.18×10^{1}	1.61×10^{1}	2.62×10^{1}	3.22×10^{1}	3.86×10^{1}	4.56×10^{1}
	8.6	2.15×10^{1}	3.09×10^{1}	5.62×10^{1}	7.24×10^{1}	9.1×10^{1}	1.12×10^{2}
0.17	1.23×10^{1}	1.41×10^{1}	1.59×10^{1}	1.77×10^{1}	1.89×10^{1}	1.95×10^{1}	2.06×10^{1}
	3.84×10^{1}	5.17×10^{1}	6.71×10^{1}	8.43×10^{1}	9.67×10^{1}	1.03×10^{2}	1.17×10^{2}
	1.7×10^{2}	2.57×10^{2}	3.64×10^{2}	4.89×10^{2}	5.81×10^{2}	6.29×10^{2}	7.31×10^{2}
	1.59×10^{3}	2.5×10^{3}	3.67×10^{3}	5.14×10^{3}	6.28×10^{3}	6.9×10^{3}	8.24×10^{3}
0.25	1.32×10^{1}	1.53×10^{1}	1.68×10^{1}	1.88×10^{1}	1.97×10^{1}	2.07×10^{1}	2.17×10^{1}
	8.38×10^{1}	1.11×10^{2}	1.34×10^{2}	1.67×10^{2}	1.85×10^{2}	2.03×10^{2}	2.22×10^{2}
	8.92×10^{2}	1.28×10^{3}	1.63×10^{3}	2.13×10^{3}	2.41×10^{3}	2.69×10^{3}	2.99×10^{3}
	1.65×10^{4}	2.49×10^{4}	3.26×10^{4}	4.46×10^{4}	5.14×10^{4}	5.86×10^{4}	6.64×10^{4}
0.5	1.5×10^{1}	1.67×10^{1}	1.87×10^{1}	2.04×10^{1}	2.14×10^{1}	2.24×10^{1}	2.34×10^{1}
	3.61×10^{2}	4.38×10^{2}	5.35×10^{2}	6.18×10^{2}	6.69×10^{2}	7.2×10^{2}	7.72×10^{2}
	1.59×10^{4}	2.02×10^{4}	2.58×10^{4}	3.08×10^{4}	3.38×10^{4}	3.7×10^{4}	4.02×10^{4}
	1.03×10^{6}	1.36×10^{6}	1.81×10^{6}	2.22×10^{6}	2.49×10^{6}	2.77×10^{6}	3.06×10^{6}
0.75	1.62×10^{1}	1.78×10^{1}	1.96×10^{1}	2.12×10^{1}	2.22×10^{1}	2.32×10^{1}	2.42×10^{1}
	8.81×10^{2}	1.03×10^{3}	1.21×10^{3}	1.36×10^{3}	1.46×10^{3}	1.56×10^{3}	1.66×10^{3}
	8.77×10^{4}	1.06×10^{5}	1.29×10^{5}	1.49×10^{5}	1.63×10^{5}	1.76×10^{5}	1.9×10^{5}
	1.21×10^{7}	1.52×10^{7}	1.91×10^{7}	2.27×10^{7}	2.52×10^{7}	2.77×10^{7}	3.03×10^{7}
1	1.68×10^{1}	1.83×10^{1}	2.01×10^{1}	2.16×10^{1}	2.26×10^{1}	2.35×10^{1}	2.45×10^{1}
	1.63×10^{3}	1.88×10^{3}	2.15×10^{3}	2.39×10^{3}	2.56×10^{3}	2.69×10^{3}	2.85×10^{3}
	2.88×10^{5}	3.42×10^{5}	4.06×10^{5}	4.61×10^{5}	$5.\times10^{5}$	5.32×10^{5}	5.7×10^{5}
	6.83×10^{7}	8.4×10^{7}	1.03×10^{8}	1.2×10^{8}	1.32×10^{8}	1.42×10^{8}	1.55×10^{8}
1.5	1.77×10^{1}	1.93×10^{1}	2.06×10^{1}	2.2×10^{1}	2.29×10^{1}	2.4×10^{1}	2.49×10^{1}
	3.92×10^{3}	4.43×10^{3}	4.86×10^{3}	5.32×10^{3}	5.62×10^{3}	5.96×10^{3}	6.24×10^{3}
	1.56×10^{6}	1.81×10^{6}	2.04×10^{6}	2.28×10^{6}	2.43×10^{6}	2.61×10^{6}	2.76×10^{6}
	8.1×10^{8}	9.72×10^{8}	1.12×10^{9}	1.28×10^{9}	1.38×10^{9}	1.51×10^{9}	1.61×10^{9}
2	1.82×10^{1}	1.96×10^{1}	2.09×10^{1}	2.22×10^{1}	2.31×10^{1}	2.41×10^{1}	2.5×10^{1}
	7.18×10^{3}	7.99×10^{3}	8.73×10^{3}	9.42×10^{3}	9.95×10^{3}	1.04×10^{4}	1.09×10^{4}
	5.07×10^{6}	5.79×10^{6}	6.47×10^{6}	7.11×10^{6}	7.59×10^{6}	8.06×10^{6}	8.51×10^{6}
	4.61×10^{9}	5.41×10^{9}	6.18×10^{9}	6.93×10^{9}	7.51×10^{9}	8.08×10^{9}	8.64×10^{9}

then in the next table we compare the 1n-Taylor approximation with the 2n-Taylor approximation

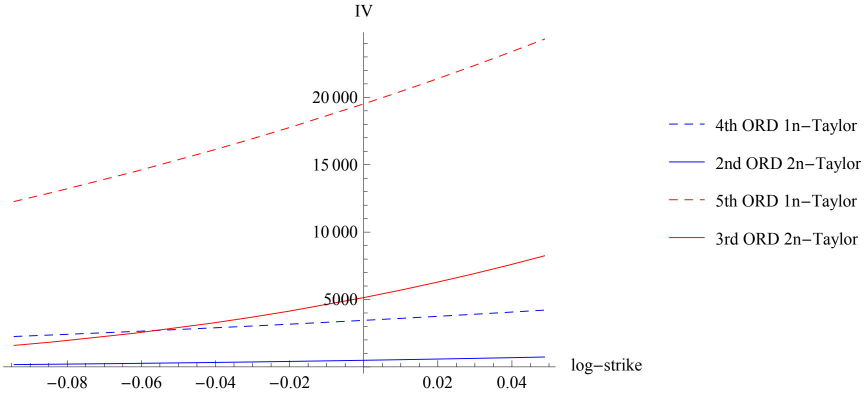
	1		7		13	
0.08	1.17×10^{1}	1.17×10^{1}	1.58×10^{1}	1.58×10^{1}	1.82×10^{1}	1.82×10^{1}
	1.49×10^{1}	1.24×10^{1}	2.02×10^{1}	2.56×10^{1}	2.34×10^{1}	3.6×10^{1}
	2.96×10^{1}	5.2	4.08×10^{1}	2.62×10^{1}	4.79×10^{1}	4.56×10^{1}
	7.33×10^{1}	8.6	1.04×10^{2}	5.62×10^{1}	1.24×10^{2}	1.12×10^{2}
	2.08×10^{2}	0	3.03×10^{2}	0	3.7×10^{2}	0
	6.49×10^{2}	0	9.79×10^{2}	0	1.22×10^{3}	0
0.17	1.23×10^{1}	1.23×10^{1}	1.77×10^{1}	1.77×10^{1}	2.06×10^{1}	2.06×10^{1}
	2.94×10^{1}	3.84×10^{1}	4.13×10^{1}	8.43×10^{1}	4.74×10^{1}	1.17×10^{2}
	1.04×10^{2}	1.7×10^{2}	1.49×10^{2}	4.89×10^{2}	1.74×10^{2}	7.31×10^{2}
	4.55×10^{2}	1.59×10^{3}	6.7×10^{2}	5.14×10^{3}	8.01×10^{2}	8.24×10^{3}
	2.26×10^{3}	0	3.45×10^{3}	0	4.21×10^{3}	0
	1.23×10^{4}	0	1.95×10^{4}	0	2.43×10^{4}	0
0.25	1.32×10^{1}	1.32×10^{1}	1.88×10^{1}	1.88×10^{1}	2.17×10^{1}	2.17×10^{1}
	4.47×10^{1}	8.38×10^{1}	6.07×10^{1}	1.67×10^{2}	6.85×10^{1}	2.22×10^{2}
	2.18×10^{2}	8.92×10^{2}	3.02×10^{2}	2.13×10^{3}	3.47×10^{2}	2.99×10^{3}
	1.3×10^{3}	1.65×10^{4}	1.86×10^{3}	4.46×10^{4}	2.18×10^{3}	6.64×10^{4}
	8.82×10^{3}	0	1.31×10^{4}	0	1.57×10^{4}	0
	6.53×10^{4}	0	1.01×10^{5}	0	1.23×10^{5}	0
0.5	1.5×10^{1}	1.5×10^{1}	2.04×10^{1}	2.04×10^{1}	2.34×10^{1}	2.34×10^{1}
	9.61×10^{1}	3.61×10^{2}	1.23×10^{2}	6.18×10^{2}	1.36×10^{2}	7.72×10^{2}
	8.64×10^{2}	1.59×10^{4}	1.12×10^{3}	3.08×10^{4}	1.26×10^{3}	4.02×10^{4}
	9.42×10^{3}	1.03×10^{6}	1.27×10^{4}	2.22×10^{6}	1.45×10^{4}	3.06×10^{6}
	1.16×10^{5}	0	1.61×10^{5}	0	1.88×10^{5}	0
	1.54×10^{6}	0	2.23×10^{6}	0	2.64×10^{6}	0
0.75	1.62×10^{1}	1.62×10^{1}	2.12×10^{1}	2.12×10^{1}	2.42×10^{1}	2.42×10^{1}
	1.52×10^{2}	8.81×10^{2}	1.86×10^{2}	1.36×10^{3}	2.04×10^{2}	1.66×10^{3}
	1.99×10^{3}	8.77×10^{4}	2.47×10^{3}	1.49×10^{5}	2.74×10^{3}	1.9×10^{5}
	3.15×10^{4}	1.21×10^{7}	4.04×10^{4}	2.27×10^{7}	4.57×10^{4}	3.03×10^{7}
	5.6×10^{5}	0	7.42×10^{5}	0	8.53×10^{5}	0
	1.07×10^{7}	0	1.47×10^{7}	0	1.72×10^{7}	0
1	1.68×10^{1}	1.68×10^{1}	2.16×10^{1}	2.16×10^{1}	2.45×10^{1}	2.45×10^{1}
	2.08×10^{2}	1.63×10^{3}	2.49×10^{2}	2.39×10^{3}	2.71×10^{2}	2.85×10^{3}
	3.58×10^{3}	2.88×10^{5}	4.35×10^{3}	4.61×10^{5}	4.77×10^{3}	5.7×10^{5}
	7.43×10^{4}	6.83×10^{7}	9.3×10^{4}	1.2×10^{8}	1.04×10^{5}	1.55×10^{8}
	1.73×10^{6}	0	2.23×10^{6}	0	2.52×10^{6}	0
	4.31×10^{7}	0	5.74×10^{7}	0	6.61×10^{7}	0
1.5	1.77×10^{1}	1.77×10^{1}	2.2×10^{1}	2.2×10^{1}	2.49×10^{1}	2.49×10^{1}
	3.25×10^{2}	3.92×10^{3}	3.76×10^{2}	5.32×10^{3}	4.06×10^{2}	6.24×10^{3}
	8.26×10^{3}	1.56×10^{6}	9.67×10^{3}	2.28×10^{6}	1.05×10^{4}	2.76×10^{6}
	2.53×10^{5}	8.1×10^{8}	3.04×10^{5}	1.28×10^{9}	3.36×10^{5}	1.61×10^{9}
	8.66×10^{6}	0	1.07×10^{7}	0	1.2×10^{7}	0
	3.18×10^{8}	0	4.02×10^{8}	0	4.58×10^{8}	0

2	1.82×10^1	1.82×10^1	2.22×10^1	2.22×10^1	2.5×10^1	2.5×10^1
	4.42×10^2	7.18×10^3	5.04×10^2	9.42×10^3	5.41×10^2	1.09×10^4
	1.48×10^4	5.07×10^6	1.71×10^4	7.11×10^6	1.85×10^4	8.51×10^6
	6.03×10^5	4.61×10^9	7.1×10^5	6.93×10^9	7.79×10^5	8.64×10^9
	2.72×10^7	0	3.29×10^7	0	3.66×10^7	0
	1.31×10^9	0	1.63×10^9	0	1.84×10^9	0

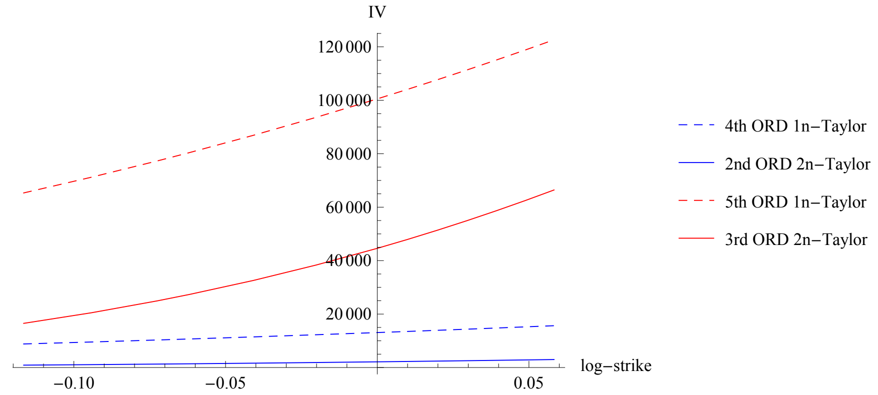
Absolute Error: κ=22.84, θ=0.4669^2, δ=20, ρ=−0.99, y=Log(0.245^2) − Time to Maturity:0.08



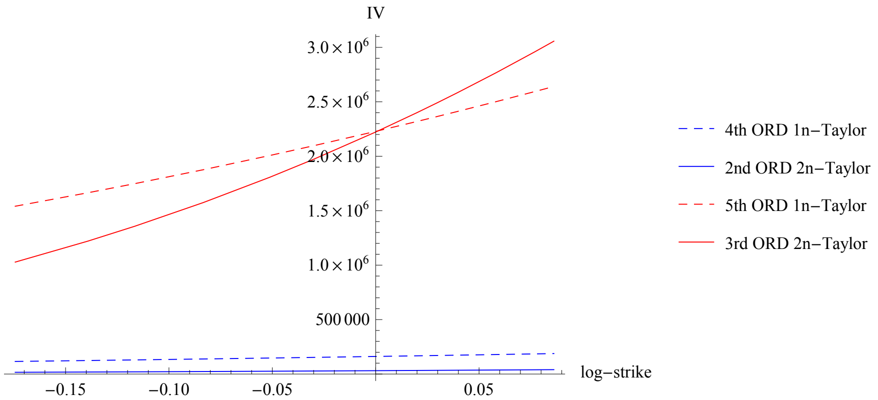
Absolute Error: κ=22.84, θ=0.4669^2, δ=20, ρ=−0.99, y=Log(0.245^2) − Time to Maturity:0.17



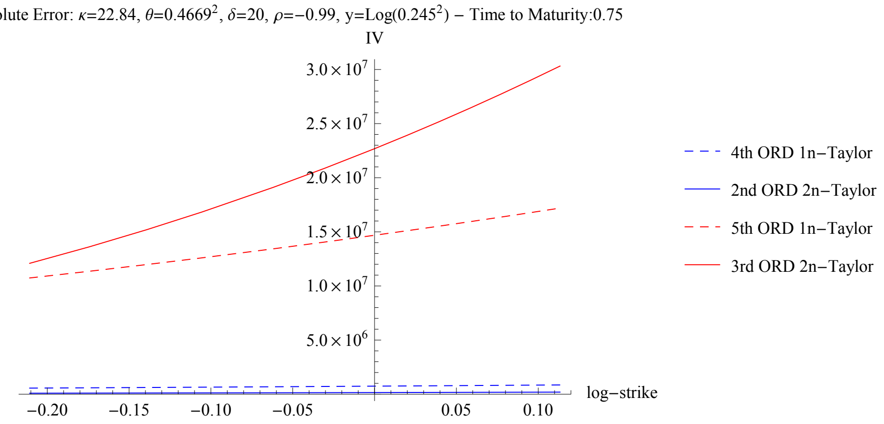
Absolute Error: κ=22.84, θ=0.4669^2, δ=20, ρ=−0.99, y=Log(0.245^2) − Time to Maturity:0.25



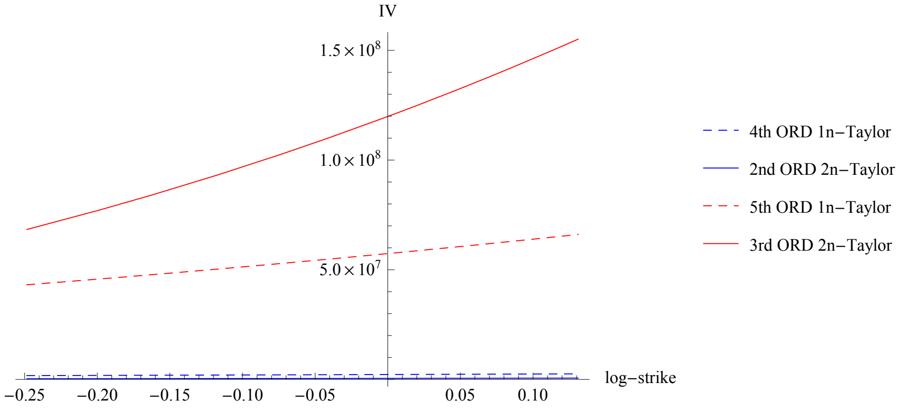
Absolute Error: κ=22.84, θ=0.4669^2, δ=20, ρ=−0.99, y=Log(0.245^2) − Time to Maturity:0.5



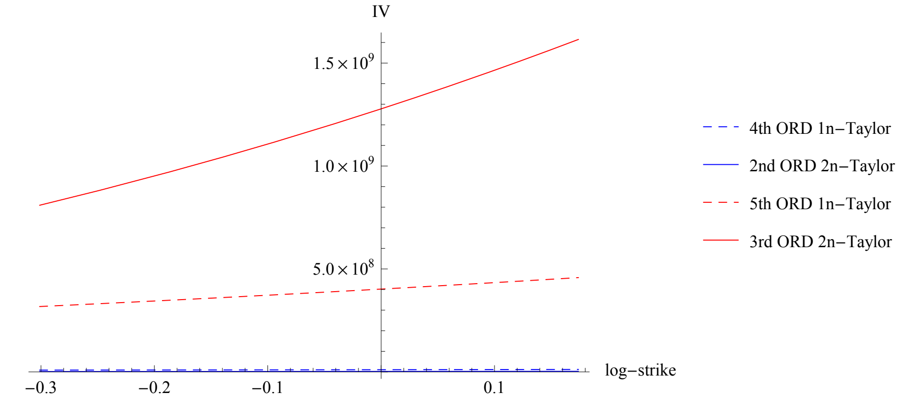
Absolute Error: κ=22.84, θ=0.4669^2, δ=20, ρ=−0.99, y=Log(0.245^2) − Time to Maturity:0.75



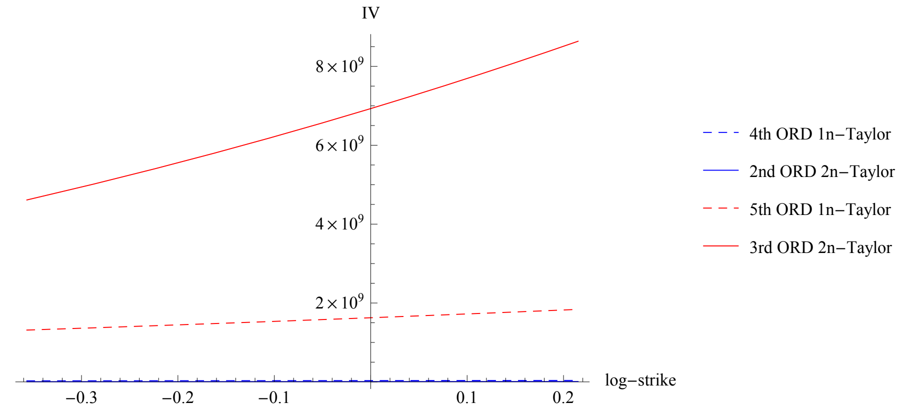
Absolute Error: κ=22.84, θ=0.4669^2, δ=20, ρ=−0.99, y=Log(0.245^2) − Time to Maturity:1



Absolute Error: κ=22.84, θ=0.4669^2, δ=20, ρ=−0.99, y=Log(0.245^2) − Time to Maturity:1.5



Absolute Error: κ=22.84, θ=0.4669^2, δ=20, ρ=−0.99, y=Log(0.245^2) − Time to Maturity:2



10 Change of Model for Heston

In the second set of parameters of Heston model proposed by Pascucci, the idea was to eliminate the drift of the process Z, and in fact it had improved our results. Following this idea and following what proposed by Bompis in ([Bom13]), we provide a model change.

We consider an Heston dynamic as previously defined:

$$d S_t = \sqrt{Z_t}\, S_t\, d W_t, \qquad S_0 = s > 0,$$

$$d Z_t = \kappa(\theta - Z_t)\, d t + \delta \sqrt{Z_t}\, d B_t, \quad Z_0 = z > 0,$$

$$d \langle W.B \rangle_t = \rho\, d t.$$

Now set $V_t = e^{\kappa t}\, Y_t$. A direct application of the Ito formula leads to:

$$d V_t = e^{\kappa t}\, \kappa\, \theta\, d t + \delta\, e^{\kappa \frac{t}{2}} \sqrt{V_t}\, d B_t, \;\; V_0 = v > 0$$

In log notation $(X, \Psi) := (\operatorname{Log} S, \operatorname{Log} V)$ we have the following dynamics

$$d X_t = -\frac{1}{2}\, e^{\psi_t - \kappa t}\, d t + e^{\frac{1}{2}(\Psi_t - \kappa t)}\, d W_t, \qquad X_0 = x := \operatorname{Log} s,$$

$$d \Psi_t = \left(\kappa\, \theta - \frac{1}{2}\, \delta^2\right) e^{-\Psi_t + \kappa t}\, d t + \delta e^{-\frac{1}{2}(\Psi_t - \kappa t)}\, d B_t, \;\; \Psi_0 = \psi := \operatorname{Log} v,$$

$$d \langle W.B \rangle_t = \rho\, d t.$$

So we indentify

$$a(t, x, \psi) = \frac{1}{2}\, e^{\psi - \kappa t}, \quad b(t, x, \psi) = \frac{1}{2}\, \delta^2\, e^{-\psi + \kappa t}, \quad c(t, x, \psi) = \rho\, \delta,$$

$$\alpha(t, x, \psi) = \left(\kappa\, \theta - \frac{1}{2}\, \delta^2\right) e^{-\psi + \kappa t}.$$

Thus the coefficients of the our approximation are no longer time homogeneous, so we had to change our code to fit the integration of time dependent coefficients; due to this change we had a significant increase of the computational complexity, and thus we only reached the second order of our approximation, which however is sufficient to make a comparison with previous methods.

Here we present only the first two formulas, beacuse the second order formula is too much long, ever respect to those of previous methods.

$$\sigma_0 = \sqrt{\frac{e^{\bar{y}}\,(1 - e^{-t\kappa})}{t\,\kappa}}$$

$$\sigma_1 = \frac{(-k + x)\,\delta\,(-1 + e^{t\kappa} - t\,\kappa)\,\rho}{2\,(1 - e^{t\kappa})\,\sqrt{e^{\bar{y}}\,(1 - e^{-t\kappa})\,t}\,\sqrt{\kappa}} + \frac{1}{4\,\sqrt{e^{\bar{y}}\,(1 - e^{-t\kappa})\,t}\,\kappa^{3/2}}\left(\left(-1 + e^{-t\kappa} + t\,\kappa\right)\left(-\delta^2 + 2\,\theta\,\kappa\right) - \frac{e^{\bar{y}}\,\delta\,(-1 + e^{t\kappa} - t\,\kappa)\,\rho}{1 - e^{t\kappa}} + \frac{e^{-t\kappa+\bar{y}}\,\delta\,(-1 + e^{t\kappa} - t\,\kappa)\,\rho}{1 - e^{t\kappa}} + \frac{2\,r\,t\,\delta\,\kappa\,(-1 + e^{t\kappa} - t\,\kappa)\,\rho}{1 - e^{t\kappa}} + 2\,e^{\bar{y}}\left(1 - e^{-t\kappa}\right)\kappa\,(y - \bar{y})\right)$$

10.1 Heston stochastic volatility model

10.1.1 First Set by Ribeiro and Poulsen

As in 8.3.1, we consider parameters as given by Ribeiro and Poulsen in ([RP13]), so we have ($\kappa = 1$, $\theta = 0.08$, $\delta = 0.39$, $\rho = -0.93$, $y = 2\,\mathrm{Log}(0.245)$).

Then we consider the same table of strikes

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.1	94	95	96	97	98	99	100	101	102	102	103	104	105
0.25	86	88	91	93	95	98	100	102	103	105	107	108	110
0.5	76	80	84	88	92	96	100	103	105	108	111	113	116
1	63	69	75	82	88	94	100	104	108	112	116	120	124
1.5	55	62	70	78	85	92	100	105	111	116	121	127	132
2	48	57	65	74	83	91	100	107	113	120	127	133	140
3	38	48	59	69	79	90	100	109	119	128	137	147	156
5	38	48	59	69	79	90	100	115	129	144	159	173	188

and corresponding exact implied volatilities

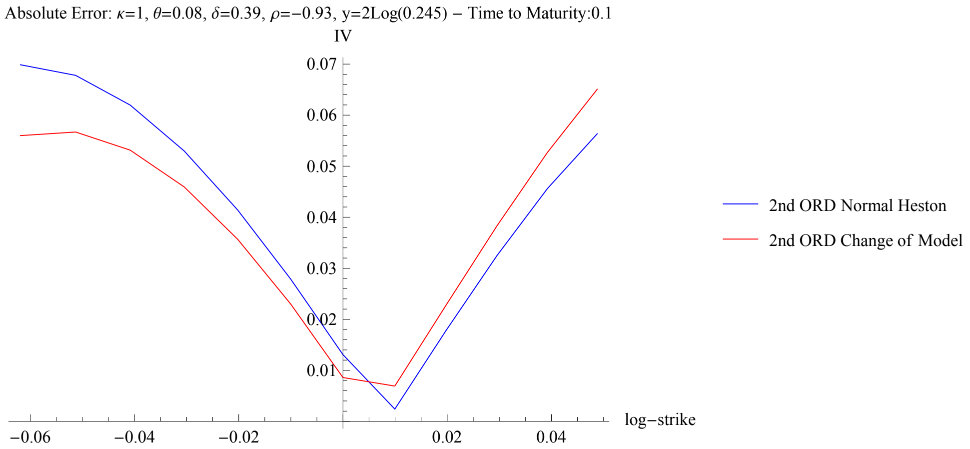
	1	3	5	7	9	11	13
0.1	26.4519	25.7834	25.0984	24.3942	23.6673	23.2942	22.5258
0.25	28.6691	27.1509	25.8952	24.2514	23.2114	21.7408	20.5581
0.5	30.9509	28.7281	26.4527	24.0610	22.4702	20.4238	18.5902
1	33.1609	30.1114	26.8893	23.8763	21.8010	19.6558	17.4762
1.5	33.9733	30.4313	27.0892	23.8396	21.5608	19.4796	17.1115
2	34.7029	30.8063	26.9724	23.9888	21.7937	19.3489	16.9145
3	36.1483	29.9554	27.1799	24.5620	22.0946	19.6375	16.6877
5	28.4428	28.8170	27.5302	25.6980	23.0257	20.2249	16.9747

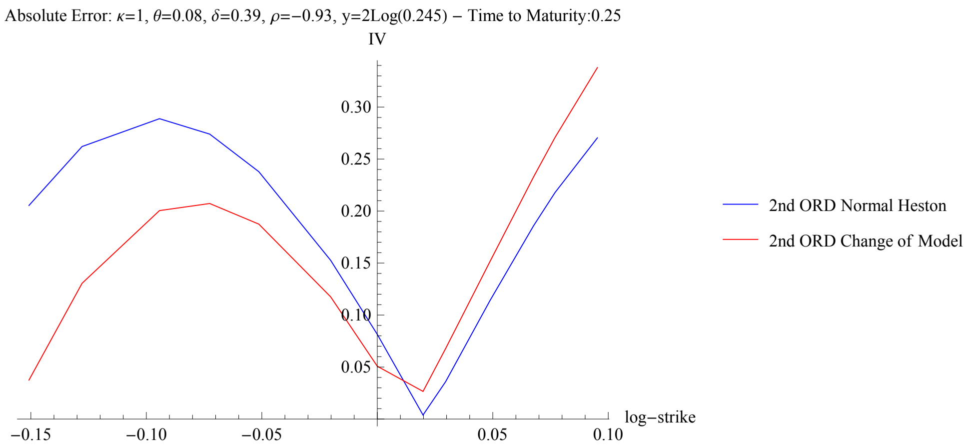
Now we compute our approximation with the change of model till 2nd order

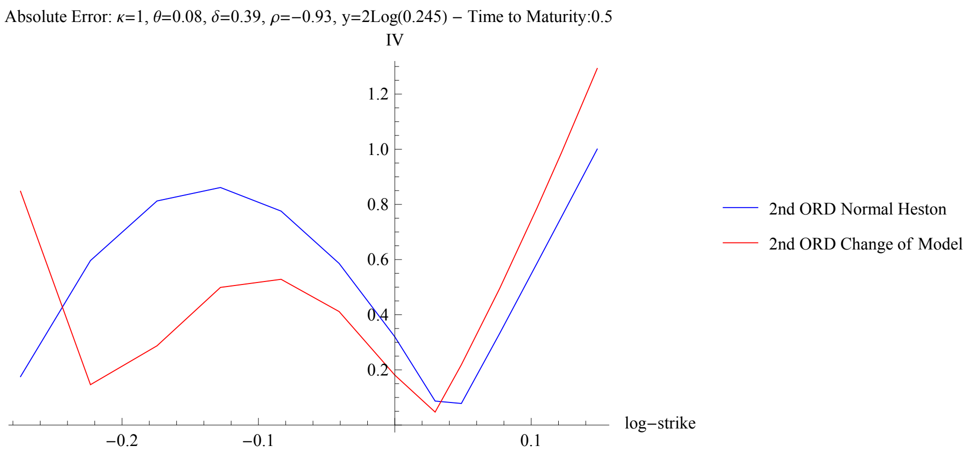
	1	3	5	7	9	11	13
0.1	2.55 3.1×10^{-1} 5.6×10^{-2}	1.88 4.27×10^{-1} 5.31×10^{-2}	1.2 5.11×10^{-1} 3.57×10^{-2}	4.94×10^{-1} 5.61×10^{-1} 8.59×10^{-3}	2.33×10^{-1} 5.73×10^{-1} 2.28×10^{-2}	6.06×10^{-1} 5.63×10^{-1} 3.83×10^{-2}	1.37 5.13×10^{-1} 6.51×10^{-2}
0.25	5.62 8.79×10^{-2} 3.75×10^{-2}	4.11 7.01×10^{-1} $2.\times10^{-1}$	2.85 1.07 1.87×10^{-1}	1.21 1.36 5.08×10^{-2}	1.66×10^{-1} 1.43 6.77×10^{-2}	1.3 1.4 2.33×10^{-1}	2.49 1.26 3.38×10^{-1}
0.5	9.22 1.02 8.48×10^{-1}	6.99 5.82×10^{-1} 2.87×10^{-1}	4.72 1.79 5.28×10^{-1}	2.33 2.59 1.81×10^{-1}	7.36×10^{-1} 2.86 2.19×10^{-1}	1.31 2.94 7.88×10^{-1}	3.14 2.79 1.29
1	1.37×10^{1} 3.93 5.7	1.06×10^{1} 1.98×10^{-1} 3.84×10^{-1}	7.41 2.8 1.29	4.4 4.76 5.77×10^{-1}	2.32 5.68 6.59×10^{-1}	1.77×10^{-1} 6.32 2.31	2. 6.73 4.29
1.5	1.63×10^{1} 6.62 1.27×10^{1}	1.28×10^{1} 7.51×10^{-1} 1.42	9.46 3.48 2.02	6.21 6.58 1.17	3.93 8.37 1.2	1.85 9.66 4.15	5.2×10^{-1} 1.07×10^{1} 7.94
2	1.86×10^{1} 9.48 2.24×10^{1}	1.47×10^{1} 1.65 3.3	1.09×10^{1} 3.97 2.92	7.88 8.19 1.83	5.68 1.07×10^{1} 1.42	3.24 1.28×10^{1} 6.23	8.05×10^{-1} 1.41×10^{1} 1.14×10^{1}
3	2.24×10^{1} 1.33×10^{1} 4.16×10^{1}	1.62×10^{1} 3.25 4.3	1.34×10^{1} 4.76 4.5	1.08×10^{1} 1.08×10^{1} 3.45	8.31 1.48×10^{1} 1.67	5.85 1.76×10^{1} 8.39	2.9 1.94×10^{1} 1.63×10^{1}
5	1.75×10^{1} 1.4×10^{1} 1.84×10^{1}	1.79×10^{1} 4.74×10^{-1} 3.65	1.66×10^{1} 8.55 9.12	1.48×10^{1} 1.43×10^{1} 8.11	1.21×10^{1} 1.98×10^{1} 1.47	9.31 2.37×10^{1} 8.22	6.05 2.58×10^{1} 1.81×10^{1}

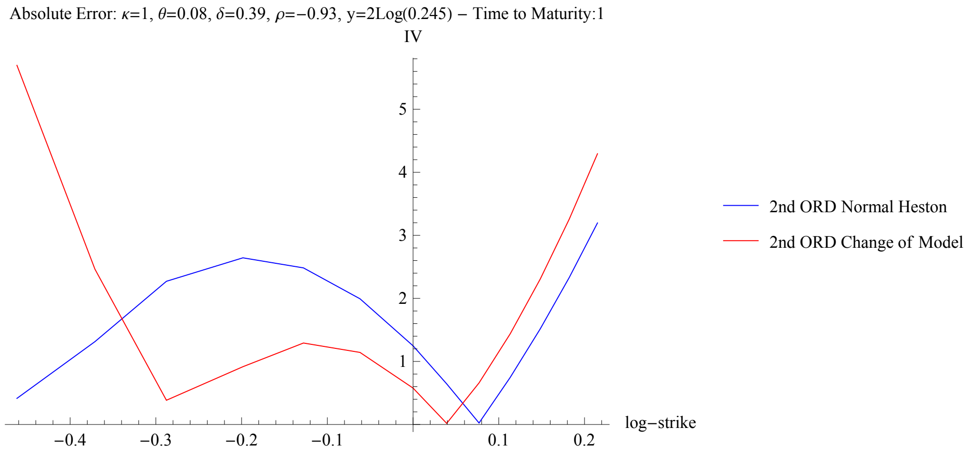
then we compare these values with those obtained from the normal Heston model.

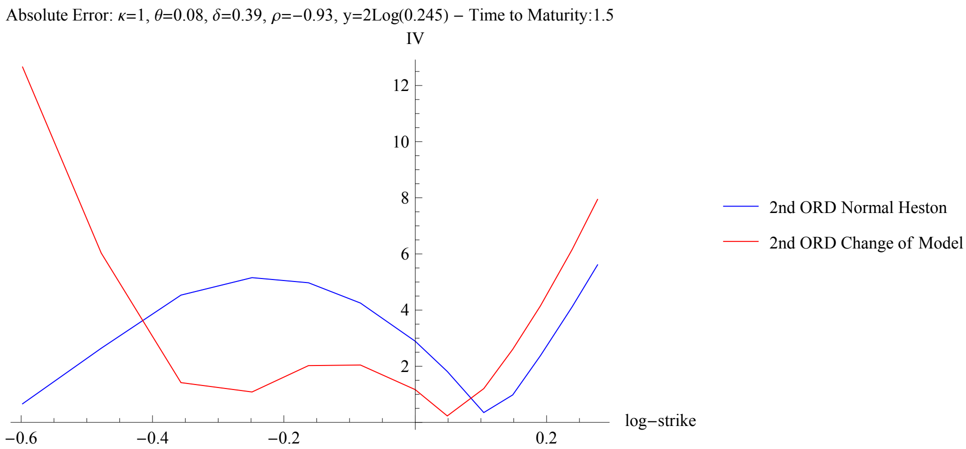
	1		7		13	
0.1	1.95	2.55	1.06×10^{-1}	4.94×10^{-1}	1.97	1.37
	3.45×10^{-1}	3.1×10^{-1}	5.77×10^{-1}	5.61×10^{-1}	5.15×10^{-1}	5.13×10^{-1}
	6.99×10^{-2}	5.6×10^{-2}	1.31×10^{-2}	8.59×10^{-3}	5.64×10^{-2}	6.51×10^{-2}
	1.15×10^{-2}	0	6.94×10^{-3}	0	5.98×10^{-3}	0
	4.66×10^{-5}	0	2.86×10^{-4}	0	6.93×10^{-4}	0
	4.98×10^{-4}	0	3.74×10^{-4}	0	2.89×10^{-4}	0
0.25	4.17	5.62	2.49×10^{-1}	1.21	3.94	2.49
	2.95×10^{-1}	8.79×10^{-2}	1.46	1.36	1.29	1.26
	2.05×10^{-1}	3.75×10^{-2}	8.14×10^{-2}	5.08×10^{-2}	2.7×10^{-1}	3.38×10^{-1}
	1.29×10^{-1}	0	4.03×10^{-2}	0	9.01×10^{-2}	0
	5.49×10^{-2}	0	4.82×10^{-3}	0	1.57×10^{-2}	0
	1.59×10^{-2}	0	6.13×10^{-3}	0	6.98×10^{-4}	0
0.5	6.45	9.22	4.39×10^{-1}	2.33	5.91	3.14
	2.9×10^{-1}	1.02	2.98	2.59	3.	2.79
	1.76×10^{-1}	8.48×10^{-1}	3.2×10^{-1}	1.81×10^{-1}	1.	1.29
	4.83×10^{-1}	0	1.37×10^{-1}	0	6.15×10^{-1}	0
	5.79×10^{-1}	0	4.25×10^{-2}	0	2.53×10^{-1}	0
	5.66×10^{-1}	0	5.22×10^{-2}	0	7.81×10^{-2}	0
1	8.66	1.37×10^{1}	6.24×10^{-1}	4.4	7.02	2.
	1.61	3.93	6.21	4.76	7.77	6.73
	4.14×10^{-1}	5.7	1.25	5.77×10^{-1}	3.2	4.29
	1.57	0	3.71×10^{-1}	0	2.59	0
	4.55	0	3.32×10^{-1}	0	9.65×10^{-1}	0
	9.22	0	3.95×10^{-1}	0	3.89×10^{-1}	0
1.5	9.47	1.63×10^{1}	6.6×10^{-1}	6.21	7.39	5.2×10^{-1}
	2.4	6.62	9.59	6.58	1.31×10^{1}	1.07×10^{1}
	6.62×10^{-1}	1.27×10^{1}	2.89	1.17	5.61	7.94
	4.2	0	5.87×10^{-1}	0	5.49	0
	1.51×10^{1}	0	9.26×10^{-1}	0	1.37	0
	3.84×10^{1}	0	1.09	0	3.65	0
2	1.02×10^{1}	1.86×10^{1}	5.11×10^{-1}	7.88	7.59	8.05×10^{-1}
	3.3	9.48	1.32×10^{1}	8.19	1.85×10^{1}	1.41×10^{1}
	1.46	2.24×10^{1}	5.21	1.83	7.61	1.14×10^{1}
	8.17	0	5.27×10^{-1}	0	9.03	0
	3.69×10^{1}	0	1.98	0	1.02	0
	1.17×10^{2}	0	2.25	0	1.19×10^{1}	0
3	1.16×10^{1}	2.24×10^{1}	6.2×10^{-2}	1.08×10^{1}	7.81	2.9
	3.66	1.33×10^{1}	2.06×10^{1}	1.08×10^{1}	2.91×10^{1}	1.94×10^{1}
	2.88	4.16×10^{1}	1.22×10^{1}	3.45	9.84	1.63×10^{1}
	2.47×10^{1}	0	1.01	0	1.87×10^{1}	0
	1.33×10^{2}	0	5.45	0	1.83	0
	5.41×10^{2}	0	5.53	0	5.05×10^{1}	0
5	3.94	1.75×10^{1}	1.2	1.48×10^{1}	7.53	6.05
	2.3	1.4×10^{1}	3.54×10^{1}	1.43×10^{1}	$5.\times10^{1}$	2.58×10^{1}
	3.53×10^{1}	1.84×10^{1}	3.64×10^{1}	8.11	6.49	1.81×10^{1}
	1.24×10^{2}	0	1.48×10^{1}	0	5.38×10^{1}	0
	3.63×10^{2}	0	1.62×10^{1}	0	1.75×10^{1}	0
	5.31×10^{2}	0	8.17	0	2.75×10^{2}	0

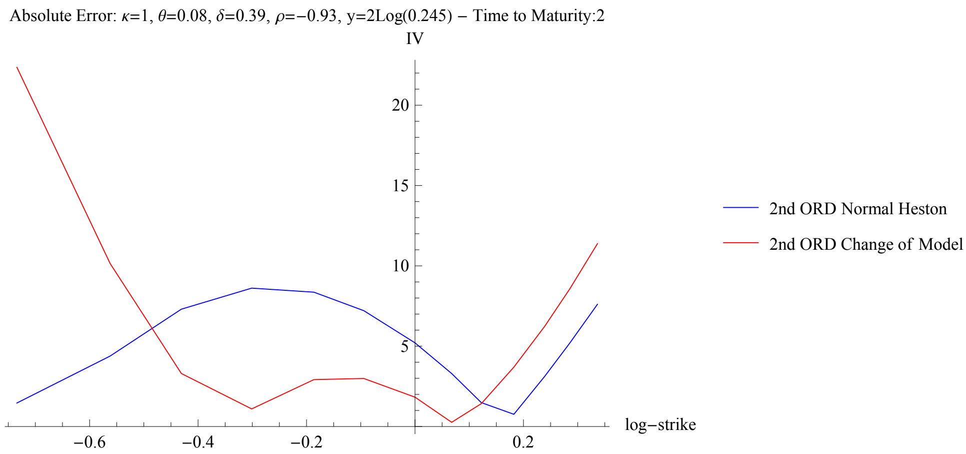

Absolute Error: κ=1, θ=0.08, δ=0.39, ρ=−0.93, y=2Log(0.245) − Time to Maturity:0.1
IV
0.07
0.06
0.05
0.04
0.03
0.02
0.01
log−strike
−0.06
−0.04
−0.02
0.02
0.04
2nd ORD Normal Heston
2nd ORD Change of Model

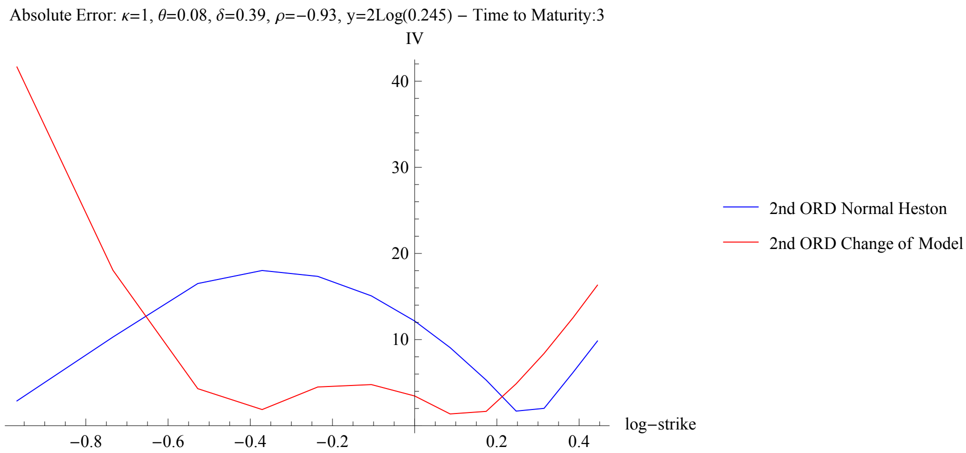

Absolute Error: κ=1, θ=0.08, δ=0.39, ρ=−0.93, y=2Log(0.245) − Time to Maturity:0.25
IV
0.30
0.25
0.20
0.15
0.10
0.05
log−strike
−0.15
−0.10
−0.05
0.05
0.10
2nd ORD Normal Heston
2nd ORD Change of Model

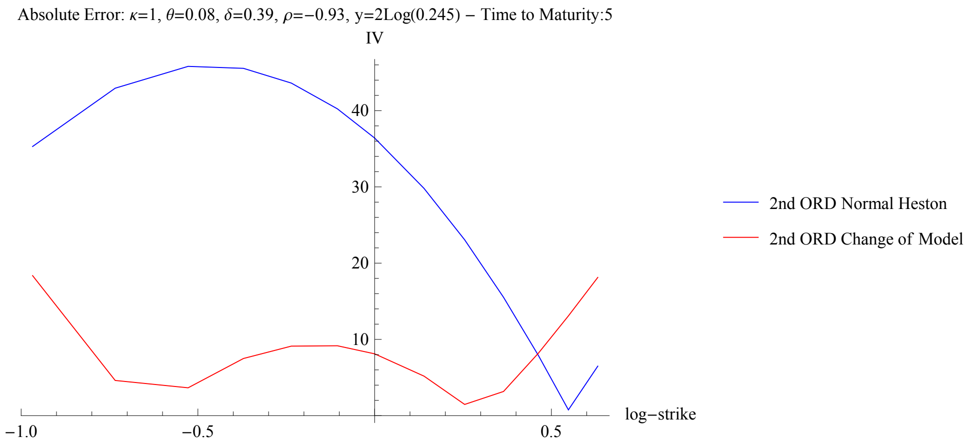

Absolute Error: κ=1, θ=0.08, δ=0.39, ρ=−0.93, y=2Log(0.245) − Time to Maturity:0.5
IV
1.2
1.0
0.8
0.6
0.4
0.2
log−strike
−0.2
−0.1
0.1
2nd ORD Normal Heston
2nd ORD Change of Model


Absolute Error: κ=1, θ=0.08, δ=0.39, ρ=−0.93, y=2Log(0.245) − Time to Maturity:1
IV
log−strike
2nd ORD Normal Heston
2nd ORD Change of Model


Absolute Error: κ=1, θ=0.08, δ=0.39, ρ=−0.93, y=2Log(0.245) − Time to Maturity:1.5
IV
log−strike
2nd ORD Normal Heston
2nd ORD Change of Model


Absolute Error: κ=1, θ=0.08, δ=0.39, ρ=−0.93, y=2Log(0.245) − Time to Maturity:2
IV
log−strike
2nd ORD Normal Heston
2nd ORD Change of Model





For short maturities the values of the two approximations are very close, even though the change of model performs better ATM; in addiction, for long maturities the new approximation seems to be more stable because it does not explode as the approximation of normal Heston.

10.1.2 Second Set by Pascucci

The second set of parameters is proposed by Pascucci, and we have $(\kappa = 1, \theta = 0.3,\ \delta = 0.8,\ \rho = -0.7,\ x = 0, y = \mathrm{Log}(\theta))$.

As in 8.3.2 we consider the following table of strikes expressed in bp

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.1	80	83	87	90	93	97	100	103	107	110	113	117	120
0.25	65	71	77	82	88	94	100	106	112	118	125	131	137
0.5	51	59	67	76	84	92	100	110	119	129	139	148	158
1	35	46	57	68	78	89	100	116	132	148	163	179	195
1.5	27	39	51	64	76	88	100	122	144	166	188	210	232
2	23	36	49	62	74	87	100	128	157	185	213	242	270
3	17	31	45	58	72	86	100	133	167	200	233	267	300
5	11	26	41	56	70	85	100	133	167	200	233	267	300

and the corresponding exact implied volatilities

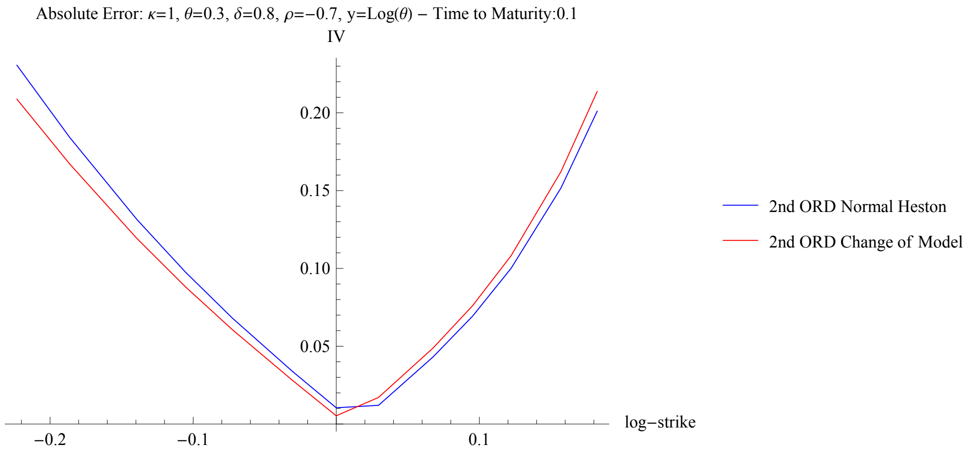
	1	3	5	7	9	11	13
0.1	59.3749	57.4122	55.7995	53.9965	52.2807	50.8866	49.3637
0.25	62.6016	59.0201	56.0034	52.9565	50.1661	47.4973	45.4576
0.5	65.1244	60.0348	55.3859	51.5276	47.5726	44.3076	42.1863
1	67.2568	59.7090	54.2581	49.6378	44.5101	41.2433	39.3597
1.5	67.3294	58.8485	52.9059	48.6054	43.0630	39.8029	38.1237
2	66.3222	57.4101	52.0719	48.0366	42.2414	39.1230	37.5144
3	64.8343	55.6842	50.9278	47.5392	42.4361	39.5896	37.8969
5	63.4838	53.5445	50.0072	47.4109	43.7258	41.5763	40.1461

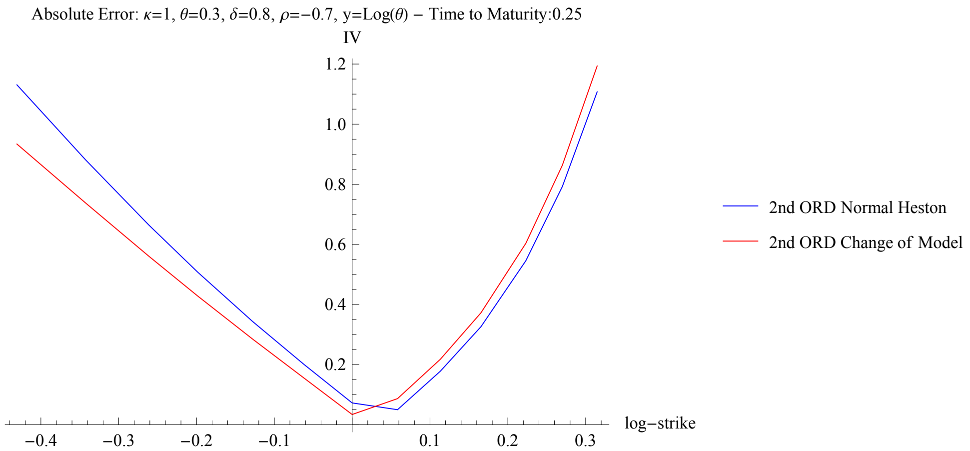
Now we compute absolute errors of our approximation with change of model

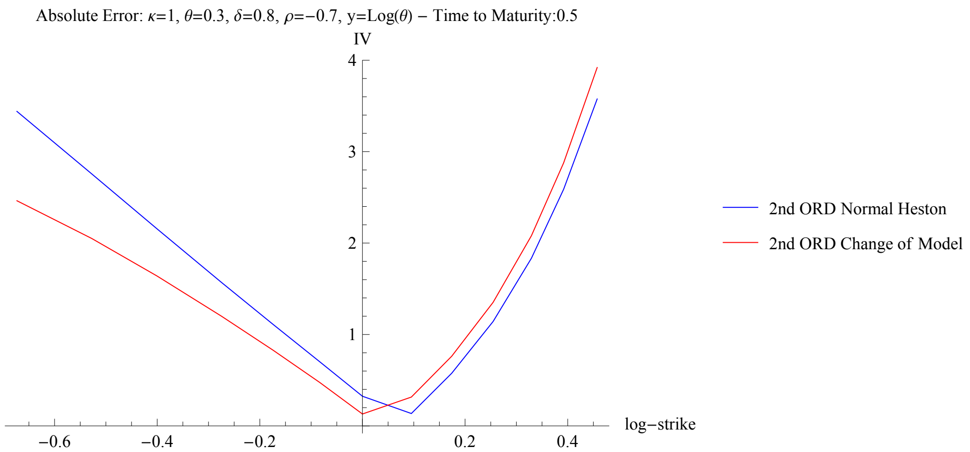
	1	3	5	7	9	11	13
0.1	5.94 6.53×10^{-1} 2.09×10^{-1}	3.98 8.51×10^{-1} 1.19×10^{-1}	2.37 9.57×10^{-1} 6.06×10^{-2}	5.65×10^{-1} 1.02 5.28×10^{-3}	1.15 1.05 4.89×10^{-2}	2.54 1.06 1.08×10^{-1}	4.07 1.09 2.14×10^{-1}
0.25	1.11×10^{1} 9.5×10^{-1} 9.34×10^{-1}	7.5 1.78 5.61×10^{-1}	4.48 2.24 2.85×10^{-1}	1.44 2.52 3.4×10^{-2}	1.35 2.68 2.18×10^{-1}	4.02 2.88 6.03×10^{-1}	6.06 3.22 1.19
0.5	1.65×10^{1} 7.43×10^{-1} 2.46	1.14×10^{1} 2.86 1.64	6.8 4.19 8.29×10^{-1}	2.94 4.94 1.3×10^{-1}	1.02 5.58 7.62×10^{-1}	4.28 6.42 2.08	6.4 7.68 3.92
1	2.37×10^{1} 1.11 5.64	1.62×10^{1} 4.45 4.22	1.07×10^{1} 7.43 2.42	6.09 9.48 4.79×10^{-1}	9.63×10^{-1} 1.18×10^{1} 2.59	2.3 1.42×10^{1} 6.16	4.19 1.72×10^{1} 1.05×10^{1}
1.5	2.79×10^{1} 3.02 8.59	1.94×10^{1} 5.65 7.07	1.35×10^{1} 1.05×10^{1} 4.13	9.19 1.36×10^{1} 1.16	3.65 1.78×10^{1} 4.44	3.85×10^{-1} 2.18×10^{1} 1.04×10^{1}	1.29 2.58×10^{1} 1.66×10^{1}
2	3.03×10^{1} 3.9 1.19×10^{1}	2.14×10^{1} 7.39 9.8	1.61×10^{1} 1.31×10^{1} 6.13	1.2×10^{1} 1.71×10^{1} 2.25	6.23 2.33×10^{1} 5.92	3.11 2.83×10^{1} 1.37×10^{1}	1.5 3.31×10^{1} 2.13×10^{1}
3	3.4×10^{1} 5.36 2.08×10^{1}	2.49×10^{1} 1.03×10^{1} 1.59×10^{1}	2.01×10^{1} 1.76×10^{1} 1.06×10^{1}	1.67×10^{1} 2.26×10^{1} 5.59	1.16×10^{1} 3.06×10^{1} 4.51	8.76 3.62×10^{1} 1.29×10^{1}	7.07 4.1×10^{1} 2.05×10^{1}
5	3.91×10^{1} 3.26 4.36×10^{1}	2.91×10^{1} 1.6×10^{1} 2.89×10^{1}	2.56×10^{1} 2.43×10^{1} 2.06×10^{1}	2.3×10^{1} 2.96×10^{1} 1.49×10^{1}	1.93×10^{1} 3.73×10^{1} 5.76	1.72×10^{1} 4.25×10^{1} 8.83×10^{-1}	1.57×10^{1} 4.67×10^{1} 6.38

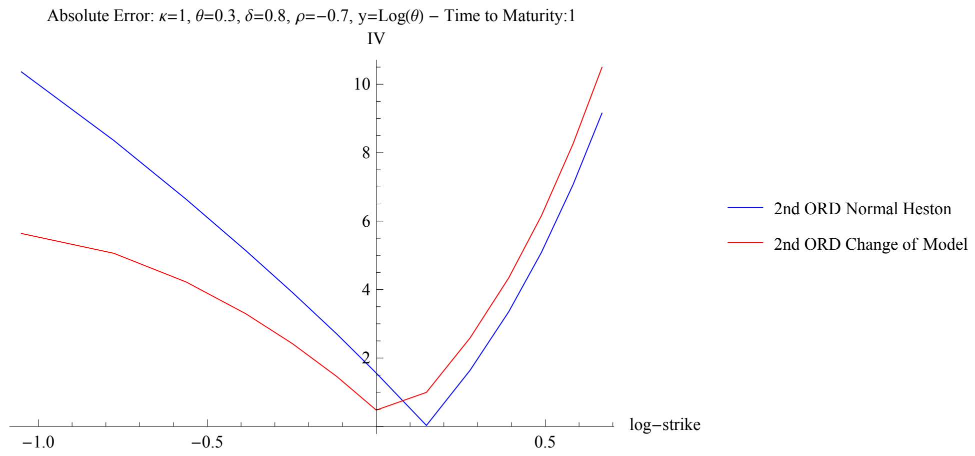
and we compare these values with those obtained with the normal Heston model.

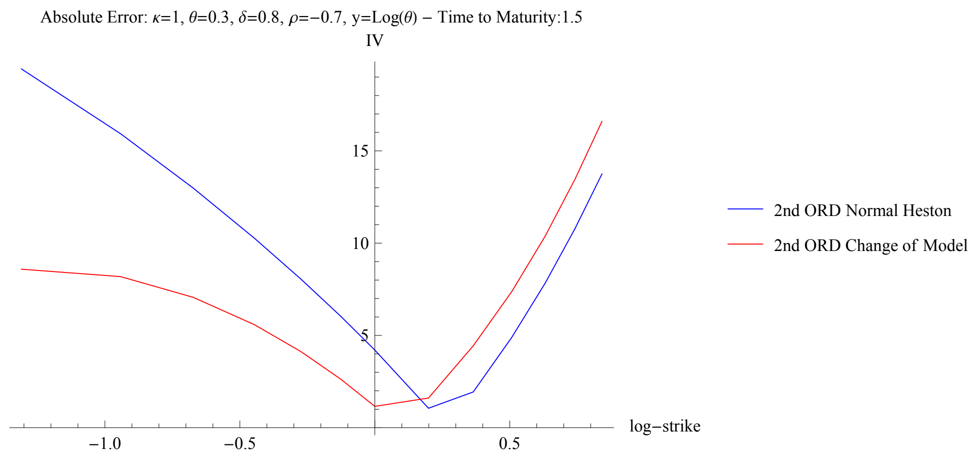
	1		7		13	
0.1	4.6	5.94	7.76×10^{-1}	5.65×10^{-1}	5.41	4.07
	7.43×10^{-1}	6.53×10^{-1}	1.07	1.02	1.1	1.09
	2.31×10^{-1}	2.09×10^{-1}	1.04×10^{-2}	5.28×10^{-3}	2.01×10^{-1}	2.14×10^{-1}
	4.56×10^{-2}	0	5.35×10^{-3}	0	$4.\times10^{-2}$	0
	6.53×10^{-5}	0	3.21×10^{-4}	0	3.73×10^{-3}	0
	2.8×10^{-3}	0	8.67×10^{-5}	0	6.62×10^{-4}	0
0.25	7.83	1.11×10^{1}	1.82	1.44	9.31	6.06
	1.43	9.5×10^{-1}	2.79	2.52	3.34	3.22
	1.13	9.34×10^{-1}	7.24×10^{-2}	3.4×10^{-2}	1.11	1.19
	5.4×10^{-1}	0	3.45×10^{-2}	0	4.19×10^{-1}	0
	8.76×10^{-2}	0	3.7×10^{-3}	0	6.92×10^{-2}	0
	8.57×10^{-2}	0	2.66×10^{-3}	0	4.31×10^{-2}	0
0.5	1.04×10^{1}	1.65×10^{1}	3.24	2.94	1.26×10^{1}	6.4
	2.36	7.43×10^{-1}	5.98	4.94	8.33	7.68
	3.44	2.46	3.25×10^{-1}	1.3×10^{-1}	3.58	3.92
	2.73	0	1.31×10^{-1}	0	1.7	0
	8.22×10^{-1}	0	1.24×10^{-2}	0	3.83×10^{-2}	0
	9.78×10^{-1}	0	3.79×10^{-2}	0	1.04	0
1	1.25×10^{1}	2.37×10^{1}	5.13	6.09	1.54×10^{1}	4.19
	4.09	1.11	1.33×10^{1}	9.48	2.01×10^{1}	1.72×10^{1}
	1.04×10^{1}	5.64	1.56	4.79×10^{-1}	9.15	1.05×10^{1}
	1.32×10^{1}	0	4.56×10^{-1}	0	4.31	0
	6.37	0	5.74×10^{-3}	0	4.6	0
	1.17×10^{1}	0	3.7×10^{-1}	0	1.37×10^{1}	0
1.5	1.26×10^{1}	2.79×10^{1}	6.17	9.19	1.66×10^{1}	1.29
	6.75	3.02	2.15×10^{1}	1.36×10^{1}	3.25×10^{1}	2.58×10^{1}
	1.94×10^{1}	8.59	4.21	1.16	1.37×10^{1}	1.66×10^{1}
	2.96×10^{1}	0	9.75×10^{-1}	0	5.74	0
	1.38×10^{1}	0	1.43×10^{-3}	0	1.84×10^{1}	0
	4.97×10^{1}	0	1.11	0	5.28×10^{1}	0
2	1.15×10^{1}	3.03×10^{1}	6.74	1.2×10^{1}	1.73×10^{1}	1.5
	1.09×10^{1}	3.9	3.01×10^{1}	1.71×10^{1}	4.5×10^{1}	3.31×10^{1}
	3.01×10^{1}	1.19×10^{1}	8.65	2.25	1.66×10^{1}	2.13×10^{1}
	4.74×10^{1}	0	1.58	0	5.96	0
	1.26×10^{1}	0	1.22×10^{-1}	0	4.33×10^{1}	0
	1.28×10^{2}	0	2.17	0	1.33×10^{2}	0
3	1.01×10^{1}	3.4×10^{1}	7.23	1.67×10^{1}	1.69×10^{1}	7.07
	2.01×10^{1}	5.36	4.81×10^{1}	2.26×10^{1}	6.65×10^{1}	4.1×10^{1}
	5.85×10^{1}	2.08×10^{1}	2.37×10^{1}	5.59	1.05×10^{1}	2.05×10^{1}
	1.01×10^{2}	0	1.84	0	3.51	0
	4.17	0	2.11	0	9.47×10^{1}	0
	4.93×10^{2}	0	3.35	0	3.11×10^{2}	0
5	8.71	3.91×10^{1}	7.36	2.3×10^{1}	1.46×10^{1}	1.57×10^{1}
	4.45×10^{1}	3.26	8.48×10^{1}	2.96×10^{1}	1.06×10^{2}	4.67×10^{1}
	1.41×10^{2}	4.36×10^{1}	8.01×10^{1}	1.49×10^{1}	3.6×10^{1}	6.38
	2.75×10^{2}	0	1.37×10^{1}	0	6.15	0
	1.87×10^{2}	0	2.8×10^{1}	0	1.95×10^{2}	0
	2.57×10^{3}	0	5.23×10^{1}	0	7.52×10^{2}	0

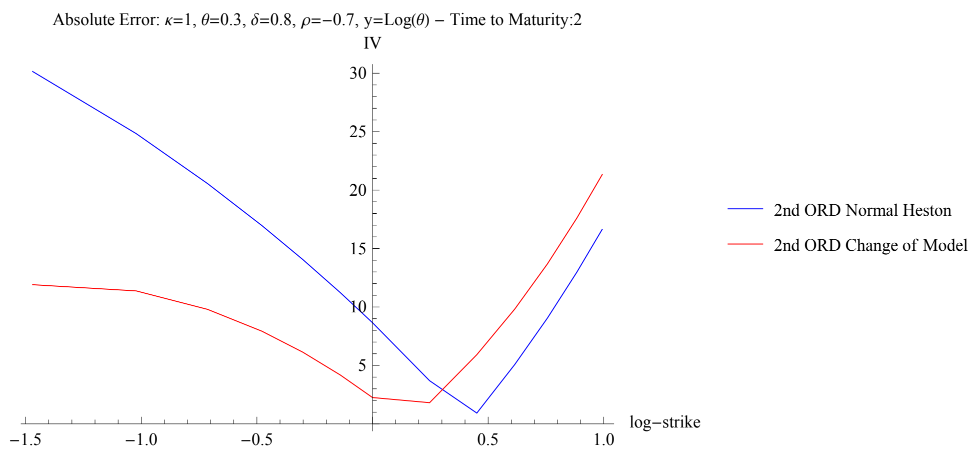

Absolute Error: κ=1, θ=0.3, δ=0.8, ρ=−0.7, y=Log(θ) − Time to Maturity:0.1
IV
0.20
0.15
0.10
0.05
log−strike
−0.2
−0.1
0.1
2nd ORD Normal Heston
2nd ORD Change of Model

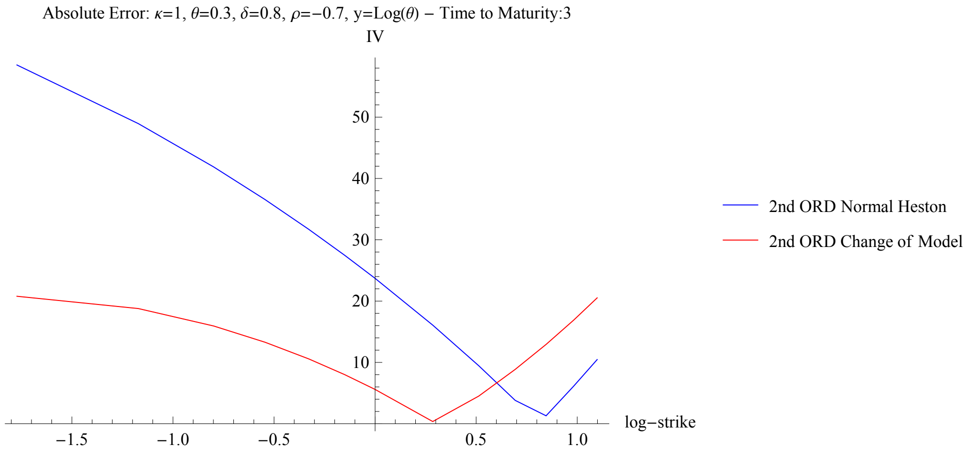

Absolute Error: κ=1, θ=0.3, δ=0.8, ρ=−0.7, y=Log(θ) − Time to Maturity:0.25
IV
1.2
1.0
0.8
0.6
0.4
0.2
log−strike
−0.4
−0.3
−0.2
−0.1
0.1
0.2
0.3
2nd ORD Normal Heston
2nd ORD Change of Model

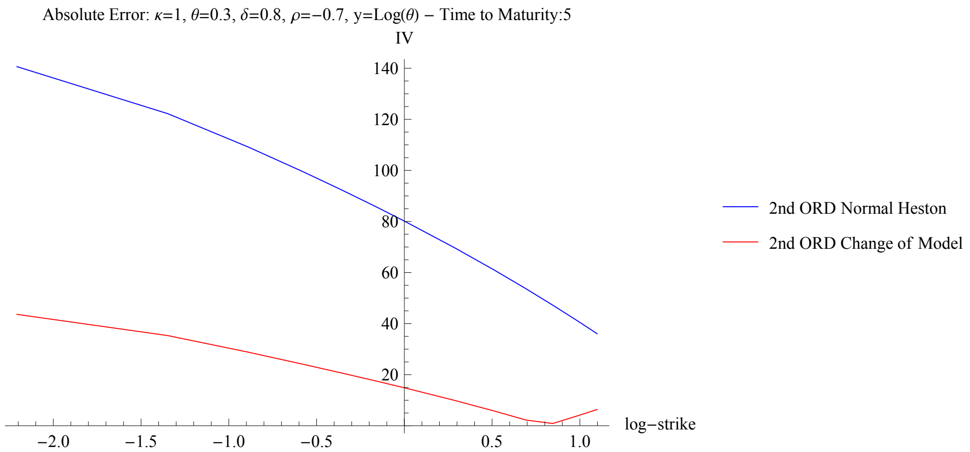

Absolute Error: κ=1, θ=0.3, δ=0.8, ρ=−0.7, y=Log(θ) − Time to Maturity:0.5
IV
4
3
2
1
log−strike
−0.6
−0.4
−0.2
0.2
0.4
2nd ORD Normal Heston
2nd ORD Change of Model


Absolute Error: κ=1, θ=0.3, δ=0.8, ρ=−0.7, y=Log(θ) − Time to Maturity:1
IV
log−strike
2nd ORD Normal Heston
2nd ORD Change of Model


Absolute Error: κ=1, θ=0.3, δ=0.8, ρ=−0.7, y=Log(θ) − Time to Maturity:1.5
IV
log−strike
2nd ORD Normal Heston
2nd ORD Change of Model


Absolute Error: κ=1, θ=0.3, δ=0.8, ρ=−0.7, y=Log(θ) − Time to Maturity:2
IV
log−strike
2nd ORD Normal Heston
2nd ORD Change of Model





We notice that the change of model produces better results specially for long maturities.

10.1.3 Third Set by Bakshi, Cao and Chen

The third set of parameters considered is as given by Bakshi, Cao and Chen in ([BCC97]), so we have $\left(\kappa = 1.15, \theta = \frac{0.04}{1.15}, \ \delta = 0.39, \ \rho = -0.64, \ x = 0, \ y = 2\,\mathrm{Log}(0.2)\right)$.

Now we create as in 8.3.3 the matrix of strikes expressed in bp

	1	2	3	4	5	6	7	8	9	10	11	12	13
0.1	96	97	97	98	99	99	100	101	101	102	103	103	104
0.25	90	92	93	95	97	98	100	101	102	104	105	106	107
0.5	83	86	89	92	94	97	100	102	104	106	108	110	112
1	74	78	83	87	91	96	100	103	106	109	112	115	118
1.5	67	72	78	84	89	94	100	104	108	112	116	120	124
2	62	68	75	81	87	94	100	105	110	115	120	125	130
3	55	62	70	78	85	92	100	107	114	120	127	134	141
5	39	49	59	70	80	90	100	111	121	132	143	153	164

and the corresponding exact implied volatilities

	1	3	5	7	9	11	13
0.1	20.8514	20.5375	19.9098	19.5976	19.2877	18.6812	18.3885
0.25	22.1212	21.2130	19.9853	19.0641	18.4626	17.6127	17.1079
0.5	23.2380	21.5344	20.0918	18.3722	17.3012	16.3866	15.7154
1	24.0045	21.7211	19.7221	17.5843	16.3363	15.3634	14.7517
1.5	24.3455	21.8086	19.4109	17.2243	15.8844	14.8918	14.3152
2	24.3397	21.5905	19.2765	17.0562	15.6568	14.6525	14.0915
3	24.0332	21.2940	18.9430	16.9522	15.4781	14.5175	13.9322
5	27.6529	21.7978	17.4399	16.8007	16.6548	15.9243	14.0942

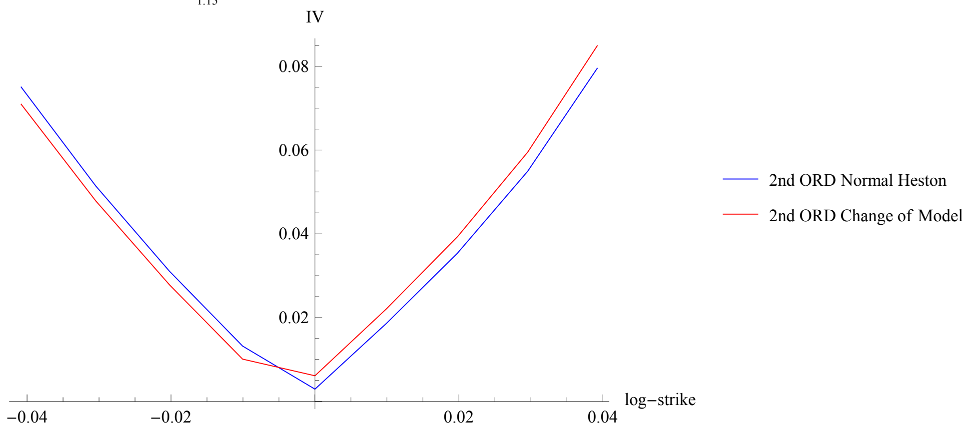
Now we compute our approximation with the change of model and we calculate absolute errors from exact values: as explained before we consider only the choice $\overline{x} = x$ and $\overline{y} = y$.

	1	3	5	7	9	11	13
0.1	5.94 6.53×10^{-1} 2.09×10^{-1}	3.98 8.51×10^{-1} 1.19×10^{-1}	2.37 9.57×10^{-1} 6.06×10^{-2}	5.65×10^{-1} 1.02 5.28×10^{-3}	1.15 1.05 4.89×10^{-2}	2.54 1.06 1.08×10^{-1}	4.07 1.09 2.14×10^{-1}
0.25	1.11×10^{1} 9.5×10^{-1} 9.34×10^{-1}	7.5 1.78 5.61×10^{-1}	4.48 2.24 2.85×10^{-1}	1.44 2.52 3.4×10^{-2}	1.35 2.68 2.18×10^{-1}	4.02 2.88 6.03×10^{-1}	6.06 3.22 1.19
0.5	1.65×10^{1} 7.43×10^{-1} 2.46	1.14×10^{1} 2.86 1.64	6.8 4.19 8.29×10^{-1}	2.94 4.94 1.3×10^{-1}	1.02 5.58 7.62×10^{-1}	4.28 6.42 2.08	6.4 7.68 3.92
1	2.37×10^{1} 1.11 5.64	1.62×10^{1} 4.45 4.22	1.07×10^{1} 7.43 2.42	6.09 9.48 4.79×10^{-1}	9.63×10^{-1} 1.18×10^{1} 2.59	2.3 1.42×10^{1} 6.16	4.19 1.72×10^{1} 1.05×10^{1}
1.5	2.79×10^{1} 3.02 8.59	1.94×10^{1} 5.65 7.07	1.35×10^{1} 1.05×10^{1} 4.13	9.19 1.36×10^{1} 1.16	3.65 1.78×10^{1} 4.44	3.85×10^{-1} 2.18×10^{1} 1.04×10^{1}	1.29 2.58×10^{1} 1.66×10^{1}
2	3.03×10^{1} 3.9 1.19×10^{1}	2.14×10^{1} 7.39 9.8	1.61×10^{1} 1.31×10^{1} 6.13	1.2×10^{1} 1.71×10^{1} 2.25	6.23 2.33×10^{1} 5.92	3.11 2.83×10^{1} 1.37×10^{1}	1.5 3.31×10^{1} 2.13×10^{1}
3	3.4×10^{1} 5.36 2.08×10^{1}	2.49×10^{1} 1.03×10^{1} 1.59×10^{1}	2.01×10^{1} 1.76×10^{1} 1.06×10^{1}	1.67×10^{1} 2.26×10^{1} 5.59	1.16×10^{1} 3.06×10^{1} 4.51	8.76 3.62×10^{1} 1.29×10^{1}	7.07 4.1×10^{1} 2.05×10^{1}
5	3.91×10^{1} 3.26 4.36×10^{1}	2.91×10^{1} 1.6×10^{1} 2.89×10^{1}	2.56×10^{1} 2.43×10^{1} 2.06×10^{1}	2.3×10^{1} 2.96×10^{1} 1.49×10^{1}	1.93×10^{1} 3.73×10^{1} 5.76	1.72×10^{1} 4.25×10^{1} 8.83×10^{-1}	1.57×10^{1} 4.67×10^{1} 6.38

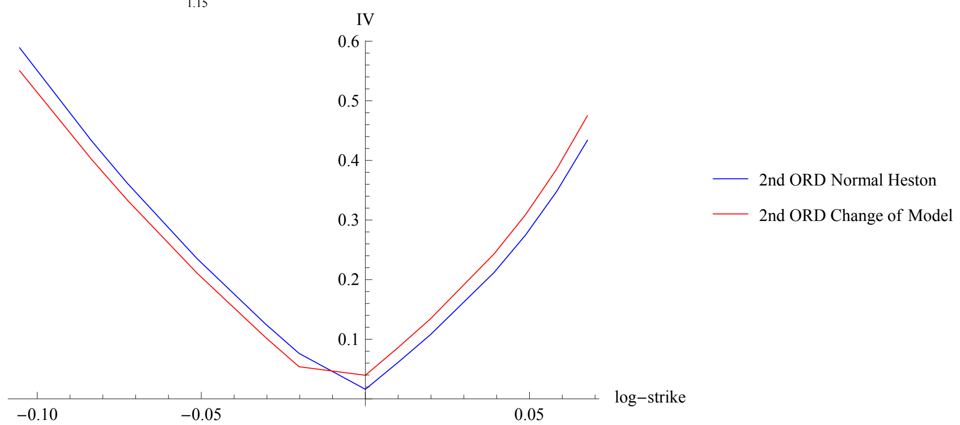
In the next table we propose a comparison between approximation of the normal Heston dynamics, and the approximation with the change of model.

	1		7		13	
0.1	4.6	5.94	7.76×10^{-1}	5.65×10^{-1}	5.41	4.07
	7.43×10^{-1}	6.53×10^{-1}	1.07	1.02	1.1	1.09
	2.31×10^{-1}	2.09×10^{-1}	1.04×10^{-2}	5.28×10^{-3}	2.01×10^{-1}	2.14×10^{-1}
	4.56×10^{-2}	0	5.35×10^{-3}	0	$4.\times10^{-2}$	0
	6.53×10^{-5}	0	3.21×10^{-4}	0	3.73×10^{-3}	0
	2.8×10^{-3}	0	8.67×10^{-5}	0	6.62×10^{-4}	0
0.25	7.83	1.11×10^{1}	1.82	1.44	9.31	6.06
	1.43	9.5×10^{-1}	2.79	2.52	3.34	3.22
	1.13	9.34×10^{-1}	7.24×10^{-2}	3.4×10^{-2}	1.11	1.19
	5.4×10^{-1}	0	3.45×10^{-2}	0	4.19×10^{-1}	0
	8.76×10^{-2}	0	3.7×10^{-3}	0	6.92×10^{-2}	0
	8.57×10^{-2}	0	2.66×10^{-3}	0	4.31×10^{-2}	0
0.5	1.04×10^{1}	1.65×10^{1}	3.24	2.94	1.26×10^{1}	6.4
	2.36	7.43×10^{-1}	5.98	4.94	8.33	7.68
	3.44	2.46	3.25×10^{-1}	1.3×10^{-1}	3.58	3.92
	2.73	0	1.31×10^{-1}	0	1.7	0
	8.22×10^{-1}	0	1.24×10^{-2}	0	3.83×10^{-2}	0
	9.78×10^{-1}	0	3.79×10^{-2}	0	1.04	0
1	1.25×10^{1}	2.37×10^{1}	5.13	6.09	1.54×10^{1}	4.19
	4.09	1.11	1.33×10^{1}	9.48	2.01×10^{1}	1.72×10^{1}
	1.04×10^{1}	5.64	1.56	4.79×10^{-1}	9.15	1.05×10^{1}
	1.32×10^{1}	0	4.56×10^{-1}	0	4.31	0
	6.37	0	5.74×10^{-3}	0	4.6	0
	1.17×10^{1}	0	3.7×10^{-1}	0	1.37×10^{1}	0
1.5	1.26×10^{1}	2.79×10^{1}	6.17	9.19	1.66×10^{1}	1.29
	6.75	3.02	2.15×10^{1}	1.36×10^{1}	3.25×10^{1}	2.58×10^{1}
	1.94×10^{1}	8.59	4.21	1.16	1.37×10^{1}	1.66×10^{1}
	2.96×10^{1}	0	9.75×10^{-1}	0	5.74	0
	1.38×10^{1}	0	1.43×10^{-3}	0	1.84×10^{1}	0
	4.97×10^{1}	0	1.11	0	5.28×10^{1}	0
2	1.15×10^{1}	3.03×10^{1}	6.74	1.2×10^{1}	1.73×10^{1}	1.5
	1.09×10^{1}	3.9	3.01×10^{1}	1.71×10^{1}	4.5×10^{1}	3.31×10^{1}
	3.01×10^{1}	1.19×10^{1}	8.65	2.25	1.66×10^{1}	2.13×10^{1}
	4.74×10^{1}	0	1.58	0	5.96	0
	1.26×10^{1}	0	1.22×10^{-1}	0	4.33×10^{1}	0
	1.28×10^{2}	0	2.17	0	1.33×10^{2}	0
3	1.01×10^{1}	3.4×10^{1}	7.23	1.67×10^{1}	1.69×10^{1}	7.07
	2.01×10^{1}	5.36	4.81×10^{1}	2.26×10^{1}	6.65×10^{1}	4.1×10^{1}
	5.85×10^{1}	2.08×10^{1}	2.37×10^{1}	5.59	1.05×10^{1}	2.05×10^{1}
	1.01×10^{2}	0	1.84	0	3.51	0
	4.17	0	2.11	0	9.47×10^{1}	0
	4.93×10^{2}	0	3.35	0	3.11×10^{2}	0
5	8.71	3.91×10^{1}	7.36	2.3×10^{1}	1.46×10^{1}	1.57×10^{1}
	4.45×10^{1}	3.26	8.48×10^{1}	2.96×10^{1}	1.06×10^{2}	4.67×10^{1}
	1.41×10^{2}	4.36×10^{1}	8.01×10^{1}	1.49×10^{1}	3.6×10^{1}	6.38
	2.75×10^{2}	0	1.37×10^{1}	0	6.15	0
	1.87×10^{2}	0	2.8×10^{1}	0	1.95×10^{2}	0
	2.57×10^{3}	0	5.23×10^{1}	0	7.52×10^{2}	0

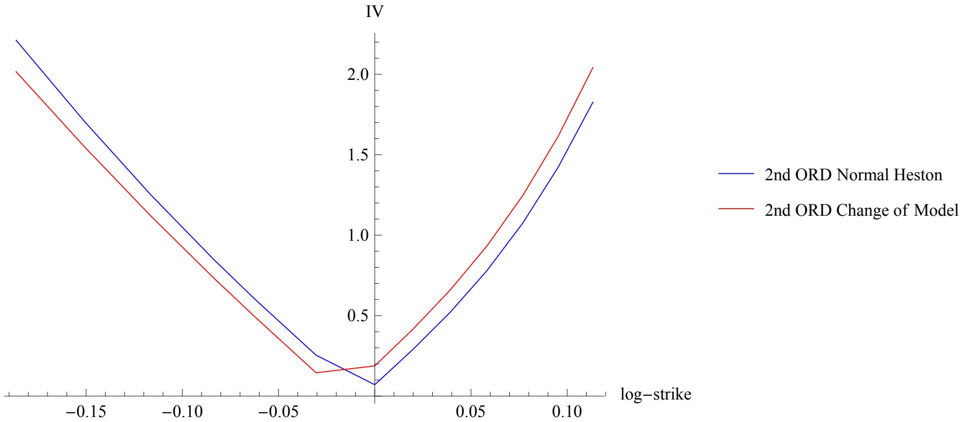
Absolute Error: κ=1.15, $\theta=\frac{0.04}{1.15}$, δ=0.39, ρ=−0.64, y=2Log(0.2) − Time to Maturity:0.1



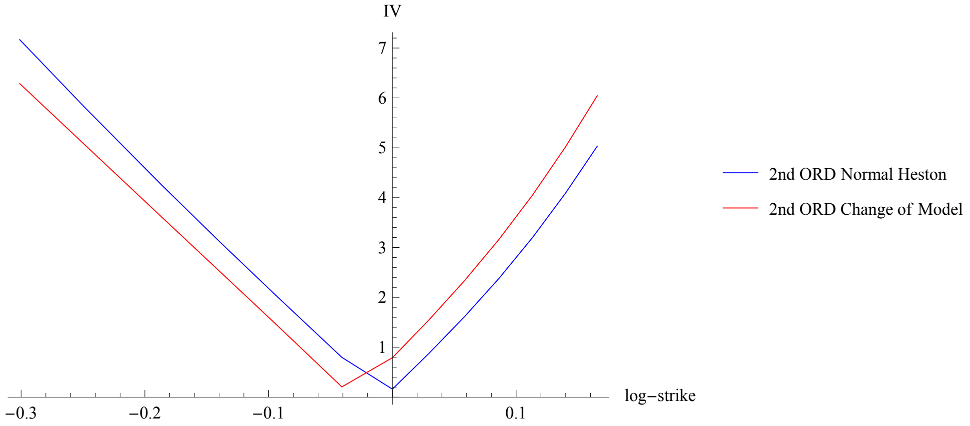
Absolute Error: κ=1.15, $\theta=\frac{0.04}{1.15}$, δ=0.39, ρ=−0.64, y=2Log(0.2) − Time to Maturity:0.25



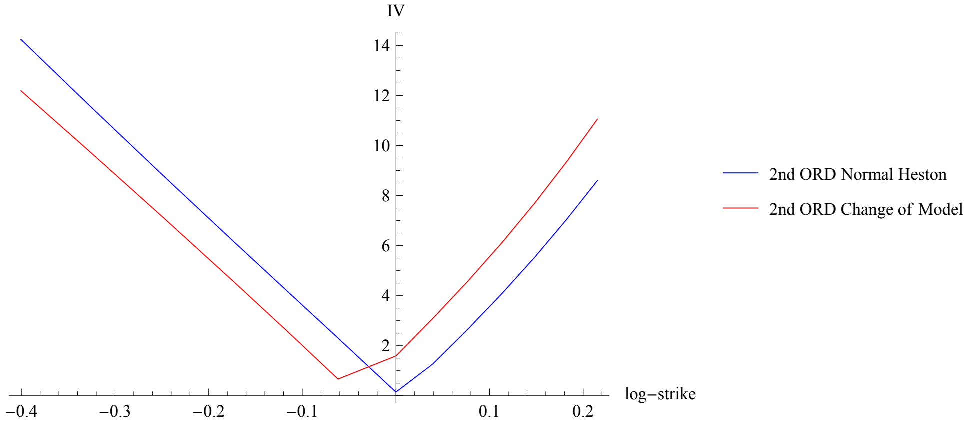
Absolute Error: κ=1.15, $\theta=\frac{0.04}{1.15}$, δ=0.39, ρ=−0.64, y=2Log(0.2) − Time to Maturity:0.5



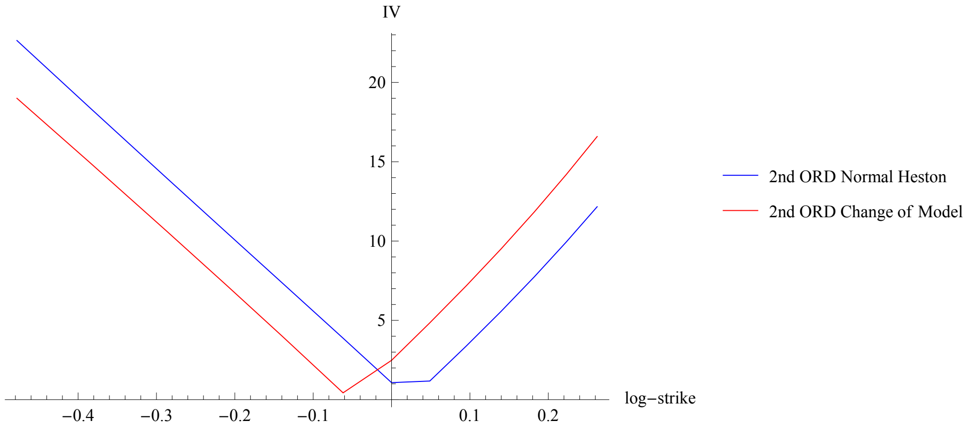
Absolute Error: κ=1.15, $\theta=\frac{0.04}{1.15}$, δ=0.39, ρ=−0.64, y=2Log(0.2) − Time to Maturity:1

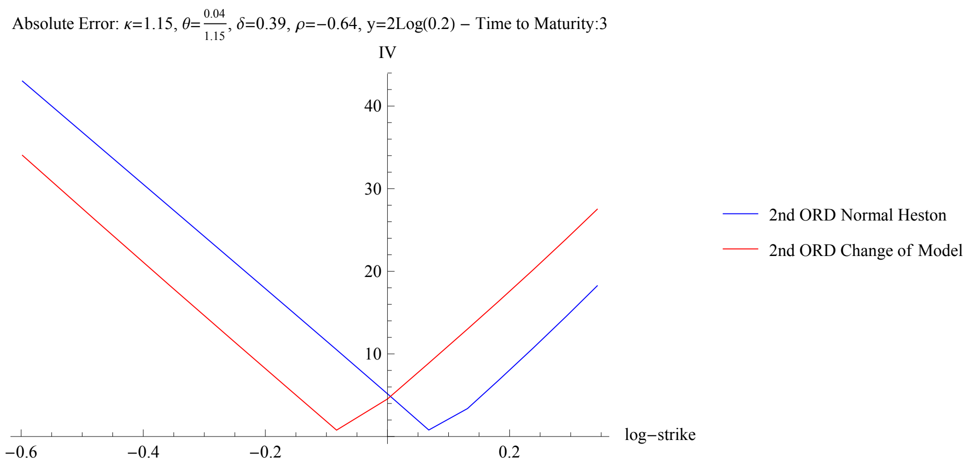


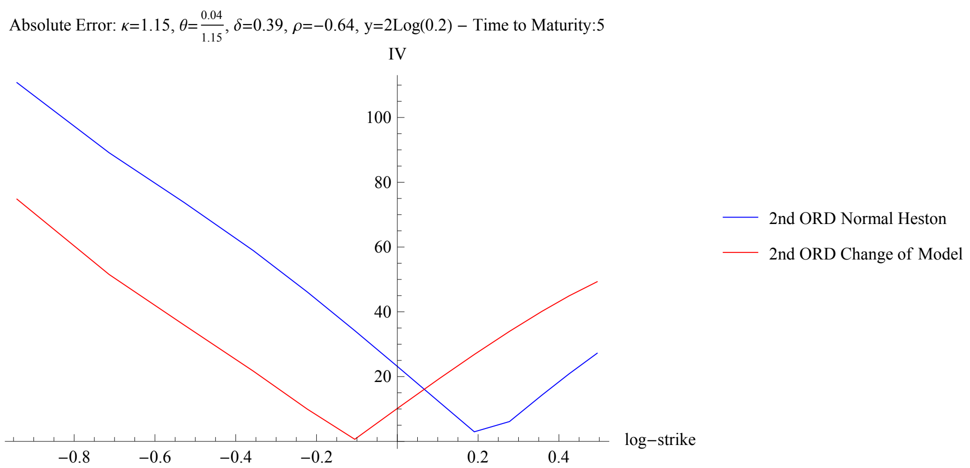
Absolute Error: κ=1.15, $\theta=\frac{0.04}{1.15}$, δ=0.39, ρ=−0.64, y=2Log(0.2) − Time to Maturity:1.5



Absolute Error: κ=1.15, $\theta=\frac{0.04}{1.15}$, δ=0.39, ρ=−0.64, y=2Log(0.2) − Time to Maturity:2







In this set of paramaters, as in previous sets, we note that for short maturities the two approximations are comparable even if the normal one is better OTM while the modified one is better ITM. However, in this set the approximation with the change of model is less stable for long maturities as in previous case.

CHAPTER 4

Forward Implied Volatility

In this Chapter, we provide analytical approximations to effciently price forward start option on equity. We use a conditional argument to represent the price as an expectation of a Black-Scholes formula computed with a stochastic implied volatility depending on the value of the equity at the forward date. Then we perform a volatility expansion to derive analytical approximations of the forward implied volatility. We also illustrate the accuracy of our formulas with some numerical experiments.

11 Forward Implied Volatility Expansions

11.1 Introduction

11.1.1 The forward volatility risk and associated derivative products

The volatility allows to quantify the risk associated to the return of an underlying asset. Many products are actively traded on financial markets to manage the observed volatility smile and skew whereas a lot of models have emerged in the two last decades to try to reproduce these phenomena. Thus practitioners and researchers began to have a good intuition of the implied volatility behaviour associated to the pricing of plain vanilla options with the Black-Scholes formula ([BS73]). Despite the significant research on implied volatility asymptotics, only a few studies have been carried out on the asymptotic of the forward smile. The forward volatility risk is harder to manage and forward skew and smile shapes are still open to research.

Recently a large class of new exotic options have emerged in order to take a bet or to hedge its exposure on the behaviour of the forward volatility surface. Among them we count the well-known forward start options. Basically it can be considered as a forward on an option. More precisely this is an option which begins at some specified future date $t_i > 0$, the forward date and with an expiration further in the future T with $T > t_i$, the premium being paid in advance at the initial date $t_0 = 0$. Denoting by S_t the price at time t of the underlying asset, we can distinguish two types of payoffs:

- (type A): $\left(\frac{S_T}{S_{t_i}} - K\right)^+$ for a given strike $K > 0$. It is essentially an option on the return of the asset between the dates t_i and T;

• (type B): $(S_T - K\,S_{t_i})^+$ with K > 0, which can be view as an option with a stochastic strike which will be determined at the forward date t_i. This looks like a spread option with the same underlying but considered at different dates.

From these payoffs one can build more complex structures of derivatives products. For instance a serie of consecutive forward start option creates a Cliquet option with payoffs of the forms:

$$\sum_{i=1}^{n}\left(\frac{S_{T_i}}{S_{t_i}} - K_i\right)^+ \text{ or } \sum_{i=1}^{n}(S_{T_i} - K_i\,S_{t_i})^+,$$

the valuation of such products being easily obtained summing the value of every legs.

11.1.2 Literature review on the forward start options pricing

Regarding the pricing of forward start options, many approaches could be considered as in the plain vanilla case. Basically one has to chose the mathematical modelling employed for the underlying asset (local volatility model, stochastic volatility model etc) and the analytical approximation methodology to be performed, closed formulas being available only in some very particular cases like in Gaussian or log-normal models. However it seems that many authors to have been interested by the pricing of forward start options have mainly considered the case of models with stochastic volatility like the Heston model: see for instance ([Luc03]), ([Hon04]) or ([KN05]). Brigo and Mercurio consider in the context of interest rates the Hull-White model in ([BM06]).

In all these works, owing to the properties of the affine models, it is possible if the model parameters are time-homogeneous to compute the forward characteristic function using the tower property for conditional expectations. Thus one can derive, up to numerical integration, (semi) analytical formulas. We also cite the work of Glasserman and Wu ([GW11]) where the authors investigate the notion of forward implied volatility in the framework of stochastic volatility models applied to the currency markets. Then using the analytical approximation of the implied volatility in the SABR model ([HKLW02]) and the asymptotic expansion for the bivariate density of both the underlying and its stochastic volatility developed in ([Wu10]), they provide tools for fast computation of the conditional expectations arising in the estimation of the forward implied volatility.

An alternative modeling is the use of Lévy processes proposed for instance in ([BK08]). If the simple exponential Lévy model induce the same forward volatility curve for all futures times, a non trivial subordinator changes its dynamic. The authors derive the forward characteristic function and employ a Fourier transform machinery to obtain analytical pricing formulas for forward start options in various models including the Variance Gamma model and the NIG model subordinated by a CIR process.

We also cite the work of Keller-Ressel and Kilin ([KRK08]) who derive a semi-analytical formula for the pricing of forward start option in the Barndoff-Nielsen-Stephard ([BNS01]) model using its affine property.

We finally mention the very recent work of Jacquier and Roome ([JR12]) in which is provided an expansion formula of the forward implied volatility using calculations based on the forward characteristic function and large deviations techniques. Remarkably their results can be applied for both small and large maturities in a large class of models, from the Heston model passing to the time-changed exponential Lévy processes.

Such an enthusiasm for the stochastic volatility models or more generally for two or more factors models in the literature can be explained by at least two reasons:

- At first glance the use of local volatility models, in which the volatility is a deterministic function of the random asset price, could seems inadequate to price forward start options which values appear to depend specifically on the random nature of the volatility itself. In addition it is well known that Skew / Smile generated by the non-constant local volatility function flattens for long maturities. As the forward start option depends on $\sigma(t, S_t)_{t\in[t_i,T]}$, we can expect it to be almost constant in S for large forward date t_i.
- The stochastic volatility models seem to induce more forward smile on $[t_i, T]$, which depends on the time-averaged stochastic volatility on $[t_i, T]$, than the implied volatility curve on $[0, T]$. Besides the availability of closed formulas is very attractive for practical uses.

However it seems to us that there is a theoretical and a practical interest to provide analytical formulas for the forward implied volatility generated by local volatility models. First this is a challenge because as previously mentioned there is no closed formula and only few authors have been focused on the question. Second there is a risk of underestimation of the forward smile which can adversely affect when pricing forward start options too cheap as mentioned in ([Gat03]).

11.1.3 Formulation of the problem and contribution of our study

As in previous numerical tests, we consider financial products in a world with null interest rate written w.r.t. a single asset which price at time t denoted by S_t assumed to pay no dividend. We consider a linear Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ where $(\mathcal{F}_t)_{t\geq 0}$ is the completion of the natural filtration of W. We suppose that S follows the local volatility model under the measure $\mathbb{P}$, assuming positive values; we define the log-asset X by posing $X = \operatorname{Log} S$ which satisfies the following SDE:

$$
\begin{aligned}
dX_t &= \mu(t, X_t, Y_t)\, dt + \sigma(t, X_t, Y_t)\, dW_t, \quad X_0 = x \in \mathbb{R},\\
dY_t &= \alpha(t, X_t, Y_t)\, dt + \beta(t, X_t, Y_t)\, dB_t, \quad Y_0 = y \in \mathbb{R},\\
d\langle W, B\rangle_t &= \rho(X_t, Y_t)\, dt \qquad |\rho| < 1,
\end{aligned}
$$

We are interested by the price at time 0 of a forward start call option of type A and B written as:

$$\begin{cases} \mathrm{Call}^{\mathrm{FS},A}(S_0, t_i, T, K) = \mathbb{E}\left[\left(\frac{S_T}{S_{t_i}} - K\right)^+\right] = \mathbb{E}\left[\left(e^{X_T - X_{t_i}} - e^k\right)^+\right], \\ \mathrm{Call}^{\mathrm{FS},B}(S_0, t_i, T, K) = \mathbb{E}[(S_T - K\, S_{t_i})^+] = \mathbb{E}\left[\left(e^{X_T} - e^{k + X_{t_i}}\right)^+\right], \end{cases} \tag{11.1}$$

where $t_i > 0$ is the forward date, $T > 0$ the forward maturity and $K = e^k > 0$ the strike and $\mathbb{E}$ stands as usual for the expectation operator. Notice that if S follows a log-normal model with fixed volatility σ, we have an analytical formula for the price of the forward start Call option of type A. The price is given in term of the Black-Scholes formula by:

$$\mathbb{E}\left[\left(\frac{S_T}{S_{t_i}} - K\right)^+\right] = \mathrm{Call}^{\mathrm{BS}}(0, \sigma, (T - t_i), k), \tag{11.2}$$

where $\mathrm{Call}^{\mathrm{BS}}(x, y, t, z)$ denotes the Black-Scholes Call price function depending on log-spot x, total variance $y^2\, t$ and log-strike z as defined in (1.20).

For forward start options of type B in the Black-Scholes framework, we readily have using the tower property for the expectations, the Markov property of S and the independence of $\frac{S_T}{S_i}$ and S_{t_i}:

$$\begin{gathered} \mathbb{E}[(S_T - K\, S_{t_i})^+] = \mathbb{E}\left[S_{t_i}\left(\frac{S_T}{S_{t_i}} - K\right)^+\right] = \mathbb{E}\left[S_{t_i}\,\mathbb{E}\left[\left(\frac{S_T}{S_{t_i}} - K\right)^+ \middle| S_{t_i}\right]\right] = \\ \mathbb{E}[S_{t_i}]\, \mathrm{Call}^{\mathrm{BS}}(0, \sigma, (T - t_i), k) = S_0\, \mathrm{Call}^{\mathrm{BS}}(0, \sigma, (T - t_i), k) = \\ \mathrm{Call}^{\mathrm{BS}}(x, \sigma, (T - t_i), x + k) \end{gathered} \tag{11.3}$$

As a conclusion, the log-normal assumption on S leads to analytical formulas using the Black-Scholes pricer with the constant volatility on $[t_i, T]$. If the choice of a local volatility model allows to take into account the implied volatility skew usually observed on the equity market, no closed formulas are available in general even for the plain vanilla case (i.e. $t_i = 0$). Instead of resorting to time-costing (especially for large forward maturity T) numerical method like PDE techniques or Monte Carlo simulations, we aim at providing an accurate analytical approximation involving the same computational time than the application of the Black-Scholes formula.

11.2 A Taylor series approach to type A

Our expansion method is quite similar to that we used in Chapter 2: as a first step we compute a price expansion, then we obtain an implied volatility expansion using formulas defined in Section 7.1.

Our approximation formula of the price is defined for a generic payoff, but considering as the payoff the Dirac delta, one can sees that this approximation it also works for the density function of the underlying; considering then the expected value of a forward option of type A, we can apply our approximation to the density of the process.

11.2.1 Price Expansions

Our price approximation for LSV model is based on a series of operator $\mathcal{L}_n$ which are essentially polynomial in ∂_x, ∂_y, $(x-\overline{x})$, $(y-\overline{y})$ and $(t-t_0)$, so we can approximate a density function as a sum in the multi-index α for some coefficients C_α

$$\Gamma(t_0, x, y, t, \xi, \omega) = \sum_\alpha C_\alpha (x-\overline{x})^{\alpha_3} (y-\overline{y})^{\alpha_4} (t-t_0)^{\alpha_5} \partial_x^{\alpha_1} \partial_y^{\alpha_2} \Gamma_0(t_0, x, y, t, \xi, \omega)$$

where $\Gamma_0(s, \xi, \omega, t, x, y)$ is the multinormal Gaussian density function defined in (6.7) for the two-dimensional case.

Theorem (13)

Given a forward option with a payoff of type A, we can compute price expansions

$$\mathbb{E}\left[\left(\frac{S_T}{S_{t_i}} - K\right)^+\right] = \left(1 + \sum_n \tilde{\mathcal{L}}_n\right) \mathrm{Call}^{\mathrm{BS}}(x_0, \sigma, (T-t_i), k),$$

with $x_0 = 0$ and where differential operators are of the form

$$\tilde{\mathcal{L}}_n = \sum_\gamma C_\gamma (x-\overline{x})^{\gamma_2} (y-\overline{y})^{\gamma_3} (t_i - t)^{\gamma_4} (T-t_i)^{\gamma_5} \partial_{x_0}^{\gamma_1}$$

The proof of this theorem is postponed in Appendix B.

11.2.2 Implied Volatility Expansions

Once we have obtained price expansions, we can compute implied volatility expansions using formula defined in (7.1) considering every Black-Scholes price as

$$\mathrm{Call}^{\mathrm{BS}}\left(x_0, \sqrt{2\, a_{0,0}}, (T-t_i), k\right)$$

with $x_0 = 0$.

Unfortunately, we can't use semplifications of section (7.2), because in this case $U_n = u_n / (\partial_\sigma u_0)$ are of the form

$$U_n = \frac{c_{n,1}\, u_0 + c_{n,2}\, \partial_{x_0} u_0}{\partial_\sigma u_0}$$

where $\mathrm{Call}^{\mathrm{BS}}\left(x_0, \sqrt{2\, a_{0,0}}, (T-t_i), k\right)$ with $x_0 = 0$.

11.3 A Taylor series approach to type B

All formulas found in section 11.2 for payoffs of the type A can be found in an analgous way even for type B payoffs, so we have the following theorem

Theorem (14)

Given a forward option with a payoff of type B, we can compute price expansions

$$\mathbb{E}[(S_T - K\, S_{t_i})^+] = \left(1 + \sum_n \tilde{\mathcal{L}}_n\right) e^x \,\mathrm{Call}^{\mathrm{BS}}(x_0, \sigma, (T - t_i), k)$$

$$= \left(1 + \sum_n \tilde{\mathcal{L}}_n\right) \mathrm{Call}^{\mathrm{BS}}(x + x_0, \sigma, (T - t_i), x + k),$$

with $x_0 = 0$ and where differential operators are of the form

$$\tilde{\mathcal{L}}_n = \sum_\gamma C_\gamma (x - \overline{x})^{\gamma_2} (y - \overline{y})^{\gamma_3} (t_i - t)^{\gamma_4} (T - t_i)^{\gamma_5} \partial_{x_0}^{\gamma_1}$$

The proof of this theorem is omitted as it is similar to the proof of Theorem (13).

12 Numerical Experiments

In this section we perform some numerical tests on our approximation of the forward implied volatility on the Heston model.

During the calculation of our approximation we have reached the second order, ie the differential operator which contains the Taylor series up to the second order, but we want to stress that these formulas are valid for any LSV model, because they are expressed in terms of functions a, b, c, α defined in section 8.

In the tests, we consider the same sets of parameters of the previous chapter, and first of all we compare the approximation of the forward volatility with our approximation of the implied volatility of normal setting $t_i = 0$, which means that the option starts at the current time.

Finally we change the start date of the option t_i and observe the evolution of the forward volatility surface.

12.1 Heston stochastic volatility model

12.1.1 First Set by Ribeiro and Poulsen

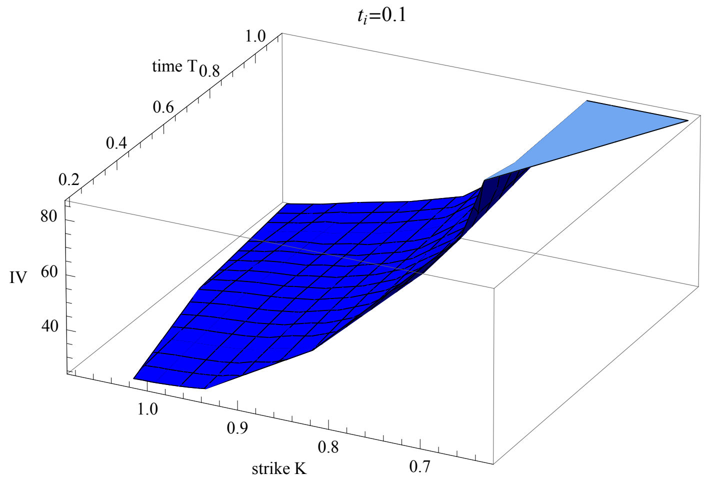
The first set of parameters considered is as given by Ribeiro and Poulsen in ([RP13]), so we have $(\kappa = 1, \theta = 0.08, \delta = 0.39, \rho = -0.93, x = 0, y = 2\,\mathrm{Log}(0.245))$.

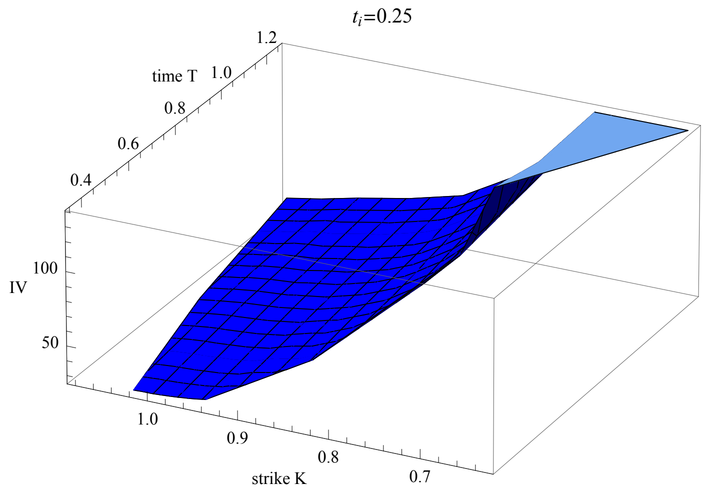
First of all we set $t_i = 0$, then we consider the same times to maturity and the same table of strikes as in 8.3.1 and 9.3.1, and consequently also the corresponding exact implied volatilities. We compute absolute errors committed by our approximation and we make a comparison between the normal implied volatility approximation and the one obtained by forward volatility formulas.

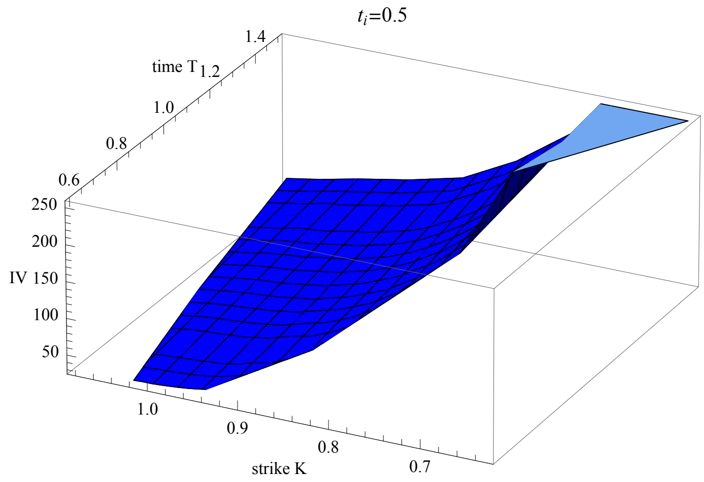
	1		7		13	
0.1	1.95	1.95	1.06×10^{-1}	1.06×10^{-1}	1.97	1.97
	3.45×10^{-1}	6.07×10^{-1}	5.77×10^{-1}	8.33×10^{-1}	5.15×10^{-1}	2.48
	6.99×10^{-2}	2.1×10^{-1}	1.31×10^{-2}	9.08×10^{-1}	5.64×10^{-2}	2.49
	1.15×10^{-2}	0	6.94×10^{-3}	0	5.98×10^{-3}	0
	4.66×10^{-5}	0	2.86×10^{-4}	0	6.93×10^{-4}	0
	4.98×10^{-4}	0	3.74×10^{-4}	0	2.89×10^{-4}	0
0.25	4.17	4.17	2.49×10^{-1}	2.49×10^{-1}	3.94	3.94
	2.95×10^{-1}	3.37×10^{-1}	1.46	2.59	1.29	5.57
	2.05×10^{-1}	6.83	8.14×10^{-2}	2.76	2.7×10^{-1}	5.28
	1.29×10^{-1}	0	4.03×10^{-2}	0	9.01×10^{-2}	0
	5.49×10^{-2}	0	4.82×10^{-3}	0	1.57×10^{-2}	0
	1.59×10^{-2}	0	6.13×10^{-3}	0	6.98×10^{-4}	0
0.5	6.45	6.45	4.39×10^{-1}	4.39×10^{-1}	5.91	5.91
	2.9×10^{-1}	1.7×10^{1}	2.98	5.39	3.	9.7
	1.76×10^{-1}	6.69×10^{1}	3.2×10^{-1}	5.2	1.	7.3
	4.83×10^{-1}	0	1.37×10^{-1}	0	6.15×10^{-1}	0
	5.79×10^{-1}	0	4.25×10^{-2}	0	2.53×10^{-1}	0
	5.66×10^{-1}	0	5.22×10^{-2}	0	7.81×10^{-2}	0
1	8.66	8.66	6.24×10^{-1}	6.24×10^{-1}	7.02	7.02
	1.61	3.11×10^{2}	6.21	7.87	7.77	1.51×10^{1}
	4.14×10^{-1}	5.58×10^{3}	1.25	5.52	3.2	1.33
	1.57	0	3.71×10^{-1}	0	2.59	0
	4.55	0	3.32×10^{-1}	0	9.65×10^{-1}	0
	9.22	0	3.95×10^{-1}	0	3.89×10^{-1}	0
1.5	9.47	9.47	6.6×10^{-1}	6.6×10^{-1}	7.39	7.39
	2.4	1.12×10^{3}	9.59	4.44	1.31×10^{1}	1.86×10^{1}
	6.62×10^{-1}	9.52×10^{4}	2.89	3.98	5.61	3.11×10^{1}
	4.2	0	5.87×10^{-1}	0	5.49	0
	1.51×10^{1}	0	9.26×10^{-1}	0	1.37	0
	3.84×10^{1}	0	1.09	0	3.65	0
2	1.02×10^{1}	1.02×10^{1}	5.11×10^{-1}	5.11×10^{-1}	7.59	7.59
	3.3	3.11×10^{3}	1.32×10^{1}	6.08	1.85×10^{1}	2.05×10^{1}
	1.46	8.58×10^{5}	5.21	1.38×10^{1}	7.61	8.85×10^{1}
	8.17	0	5.27×10^{-1}	0	9.03	0
	3.69×10^{1}	0	1.98	0	1.02	0
	1.17×10^{2}	0	2.25	0	1.19×10^{1}	0
3	1.16×10^{1}	1.16×10^{1}	6.2×10^{-2}	6.2×10^{-2}	7.81	7.81
	3.66	1.23×10^{4}	2.06×10^{1}	5.06×10^{1}	2.91×10^{1}	1.97×10^{1}
	2.88	1.57×10^{7}	1.22×10^{1}	1.69×10^{2}	9.84	3.08×10^{2}
	2.47×10^{1}	0	1.01	0	1.87×10^{1}	0
	1.33×10^{2}	0	5.45	0	1.83	0
	5.41×10^{2}	0	5.53	0	5.05×10^{1}	0
5	3.94	3.94	1.2	1.2	7.53	7.53
	2.3	9.65×10^{3}	3.54×10^{1}	2.43×10^{2}	$5.\times10^{1}$	2.33
	3.53×10^{1}	5.6×10^{6}	3.64×10^{1}	2.36×10^{3}	6.49	1.29×10^{3}
	1.24×10^{2}	0	1.48×10^{1}	0	5.38×10^{1}	0
	3.63×10^{2}	0	1.62×10^{1}	0	1.75×10^{1}	0
	5.31×10^{2}	0	8.17	0	2.75×10^{2}	0

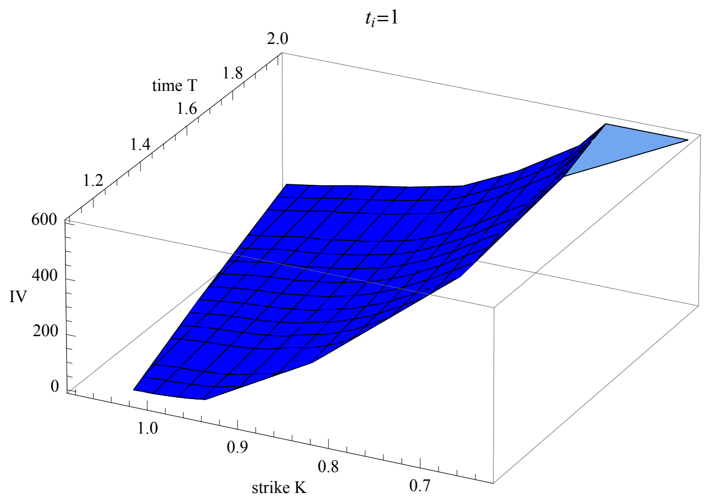
From this table we see that the approximation of the forward volatility produces absolute errors significantly higher than those of our approximation presented in Chapter 7; in addiction, when are considered maturities longer than a few months, there is an explosion of these errors. All of this leads us to think to an inadequacy of the method proposed.

In the following graphs we vary the starting time of the option t_i up to 1 year, while maintaining the same duration of the options (ie the difference between the starting date and the expiry date).









We see that the surfaces have a smooth flow and maintain the same degree of convexity, even if the values are too high, such as to suggest an explosion of the method.

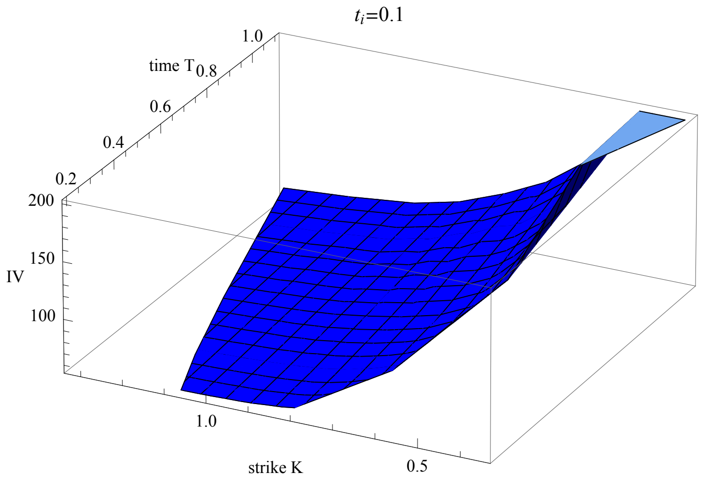
12.1.2 Second Set by Pascucci

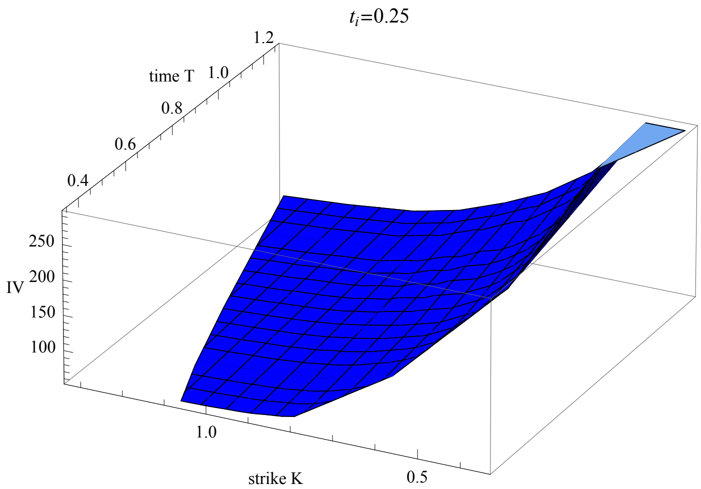
The second set of parameters is proposed by Pascucci, and we have $(\kappa = 1, \theta = 0.3,\ \delta = 0.8,\ \rho = -0.7,\ x = 0,\ y = \mathrm{Log}(\theta)\,)$;

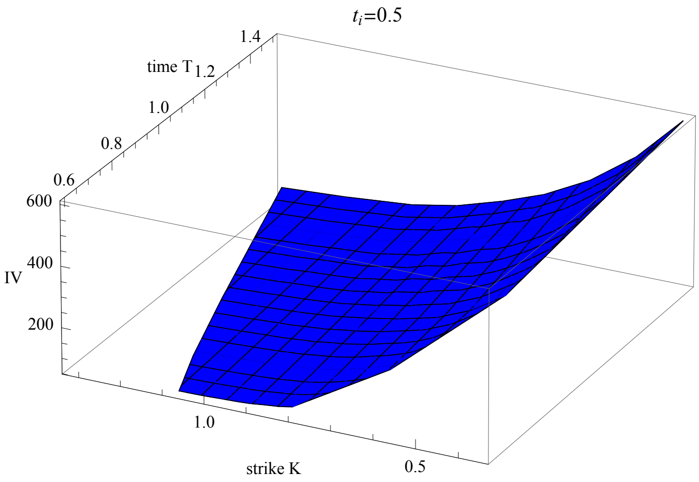
We consider the same times to maturity and strikes as in 8.3.2 and 9.3.2, then we compute the corresponding exact implied volatilities. Finally we compute our approximations using formulas obtained with the method presented in chapter 7 and 8, and with formulas obtained with the method presented in this chapter for the forward implied volatility. In the nex table we make a comparison between the two methods.

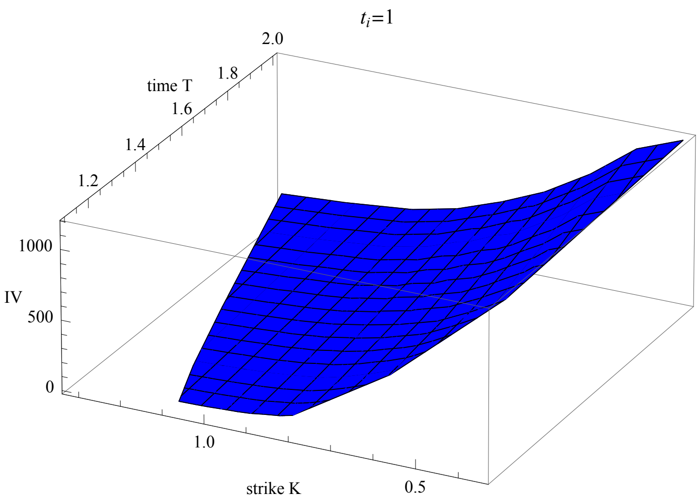
	1		7		13	
0.1	4.6	4.6	7.76×10^{-1}	7.76×10^{-1}	5.41	5.41
	7.43×10^{-1}	1.63	1.07	1.67	1.1	5.9
	2.31×10^{-1}	2.29×10^{-1}	1.04×10^{-2}	1.87	2.01×10^{-1}	5.94
	4.56×10^{-2}	0	5.35×10^{-3}	0	$4.\times10^{-2}$	0
	6.53×10^{-5}	0	3.21×10^{-4}	0	3.73×10^{-3}	0
	2.8×10^{-3}	0	8.67×10^{-5}	0	6.62×10^{-4}	0
0.25	7.83	7.83	1.82	1.82	9.31	9.31
	1.43	2.74	2.79	4.89	3.34	1.1×10^{1}
	1.13	2.54×10^{1}	7.24×10^{-2}	5.93	1.11	1.11×10^{1}
	5.4×10^{-1}	0	3.45×10^{-2}	0	4.19×10^{-1}	0
	8.76×10^{-2}	0	3.7×10^{-3}	0	6.92×10^{-2}	0
	8.57×10^{-2}	0	2.66×10^{-3}	0	4.31×10^{-2}	0
0.5	1.04×10^{1}	1.04×10^{1}	3.24	3.24	1.26×10^{1}	1.26×10^{1}
	2.36	5.32	5.98	1.04×10^{1}	8.33	1.7×10^{1}
	3.44	1.74×10^{2}	3.25×10^{-1}	1.44×10^{1}	3.58	1.63×10^{1}
	2.73	0	1.31×10^{-1}	0	1.7	0
	8.22×10^{-1}	0	1.24×10^{-2}	0	3.83×10^{-2}	0
	9.78×10^{-1}	0	3.79×10^{-2}	0	1.04	0
1	1.25×10^{1}	1.25×10^{1}	5.13	5.13	1.54×10^{1}	1.54×10^{1}
	4.09	2.12×10^{2}	1.33×10^{1}	1.95×10^{1}	2.01×10^{1}	2.59×10^{1}
	1.04×10^{1}	1.14×10^{2}	1.56	3.76×10^{1}	9.15	1.91×10^{1}
	1.32×10^{1}	0	4.56×10^{-1}	0	4.31	0
	6.37	0	5.74×10^{-3}	0	4.6	0
	1.17×10^{1}	0	3.7×10^{-1}	0	1.37×10^{1}	0
1.5	1.26×10^{1}	1.26×10^{1}	6.17	6.17	1.66×10^{1}	1.66×10^{1}
	6.75	7.32×10^{2}	2.15×10^{1}	2.4×10^{1}	3.25×10^{1}	3.34×10^{1}
	1.94×10^{1}	1.26×10^{4}	4.21	7.58×10^{1}	1.37×10^{1}	1.13×10^{1}
	2.96×10^{1}	0	9.75×10^{-1}	0	5.74	0
	1.38×10^{1}	0	1.43×10^{-3}	0	1.84×10^{1}	0
	4.97×10^{1}	0	1.11	0	5.28×10^{1}	0
2	1.15×10^{1}	1.15×10^{1}	6.74	6.74	1.73×10^{1}	1.73×10^{1}
	1.09×10^{1}	1.32×10^{3}	3.01×10^{1}	2.31×10^{1}	4.5×10^{1}	3.99×10^{1}
	3.01×10^{1}	4.42×10^{4}	8.65	1.41×10^{2}	1.66×10^{1}	9.79
	4.74×10^{1}	0	1.58	0	5.96	0
	1.26×10^{1}	0	1.22×10^{-1}	0	4.33×10^{1}	0
	1.28×10^{2}	0	2.17	0	1.33×10^{2}	0
3	1.01×10^{1}	1.01×10^{1}	7.23	7.23	1.69×10^{1}	1.69×10^{1}
	2.01×10^{1}	3.4×10^{3}	4.81×10^{1}	2.18	6.65×10^{1}	4.89×10^{1}
	5.85×10^{1}	3.1×10^{5}	2.37×10^{1}	4.22×10^{2}	1.05×10^{1}	8.11×10^{1}
	1.01×10^{2}	0	1.84	0	3.51	0
	4.17	0	2.11	0	9.47×10^{1}	0
	4.93×10^{2}	0	3.35	0	3.11×10^{2}	0
5	8.71	8.71	7.36	7.36	1.46×10^{1}	1.46×10^{1}
	4.45×10^{1}	1.03×10^{4}	8.48×10^{1}	1.31×10^{2}	1.06×10^{2}	3.41×10^{1}
	1.41×10^{2}	2.63×10^{6}	8.01×10^{1}	2.4×10^{3}	3.6×10^{1}	1.81×10^{2}
	2.75×10^{2}	0	1.37×10^{1}	0	6.15	0
	1.87×10^{2}	0	2.8×10^{1}	0	1.95×10^{2}	0
	2.57×10^{3}	0	5.23×10^{1}	0	7.52×10^{2}	0

Now we plot the implied volatility surface changing the value of the starting date t_i.


t_i=0.1
time T
1.0
0.8
0.6
0.4
0.2
200
150
IV
100
1.0
strike K
0.5


t_i=0.25
time T
1.2
1.0
0.8
0.6
0.4
250
200
IV
150
100
1.0
strike K
0.5





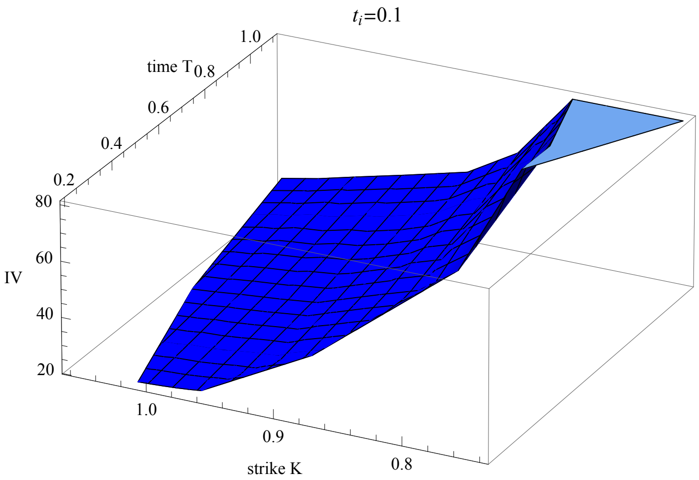
12.1.3 Third Set by Bakshi, Cao and Chen

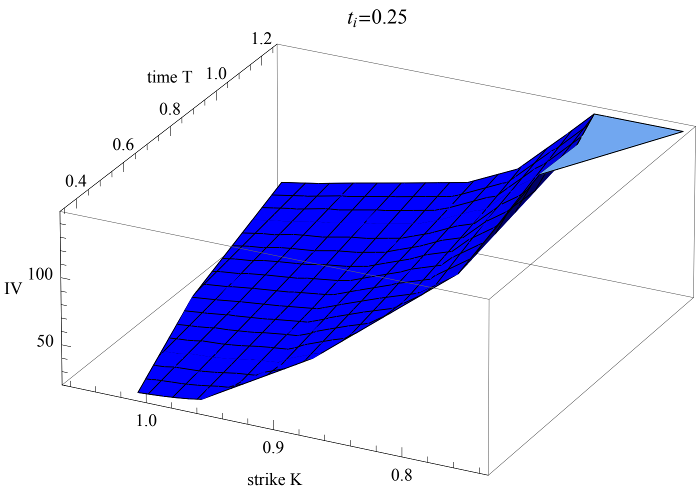
The third set of parameters considered is as given by Bakshi, Cao and Chen in ([BCC97]), so we have $\left(\kappa = 1.15, \theta = \frac{0.04}{1.15},\ \delta = 0.39,\ \rho = -0.64,\ x = 0,\right.$ $y = 2\,\mathrm{Log}(0.2)$).

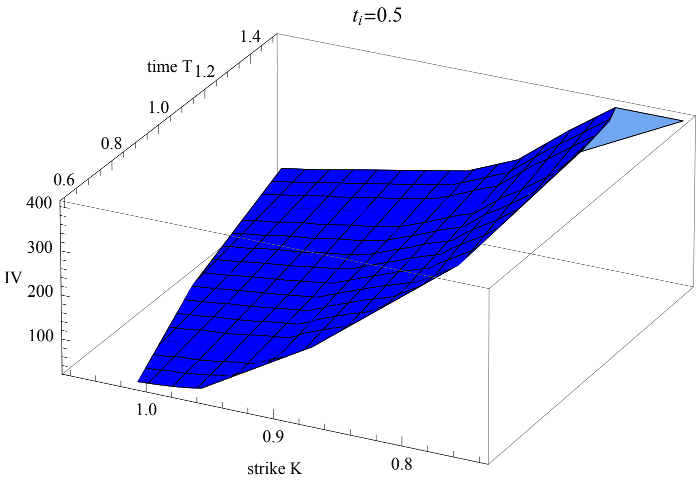
Proceding as in 12.1.1 and 12.1.2 we make a comparison between absolute errors committed by our approximation for the implied volatility, and those committed by our approximation for the forward implied volatility.

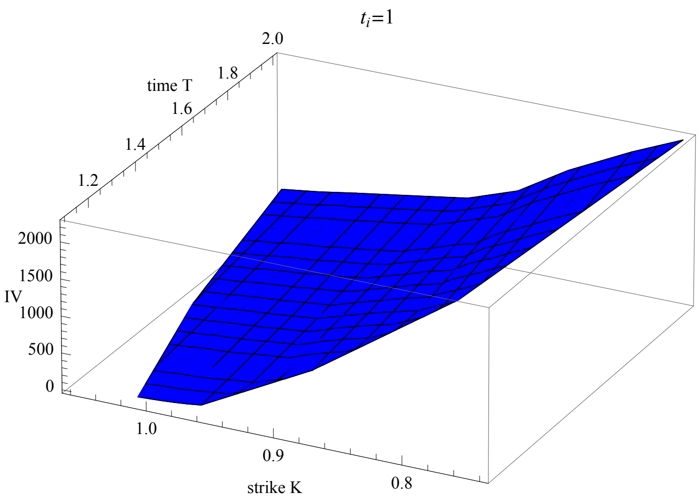
	1		7		13	
0.1	8.51×10^{-1}	8.51×10^{-1}	4.02×10^{-1}	4.02×10^{-1}	1.61	1.61
	6.66×10^{-1}	3.51×10^{-1}	6.86×10^{-1}	7.35×10^{-1}	$7.\times10^{-1}$	1.85
	7.5×10^{-2}	1.86×10^{-2}	2.96×10^{-3}	8.57×10^{-1}	7.95×10^{-2}	1.9
	7.94×10^{-3}	0	4.19×10^{-3}	0	9.52×10^{-3}	0
	2.56×10^{-3}	0	6.73×10^{-4}	0	1.1×10^{-3}	0
	1.02×10^{-3}	0	3.58×10^{-5}	0	6.44×10^{-4}	0
0.25	2.12	2.12	9.36×10^{-1}	9.36×10^{-1}	2.89	2.89
	1.55	1.31	1.78	1.88	1.94	3.64
	5.89×10^{-1}	4.26	1.61×10^{-2}	2.57	4.34×10^{-1}	3.79
	1.57×10^{-2}	0	2.35×10^{-2}	0	1.44×10^{-1}	0
	1.11×10^{-1}	0	7.42×10^{-3}	0	3.86×10^{-2}	0
	1.43×10^{-2}	0	1.97×10^{-3}	0	1.08×10^{-2}	0
0.5	3.24	3.24	1.63	1.63	4.28	4.28
	2.86	1.24×10^{1}	3.81	3.1	4.69	5.83
	2.21	3.47×10^{1}	7.09×10^{-2}	6.13	1.83	5.9
	4.38×10^{-1}	0	5.34×10^{-2}	0	9.44×10^{-1}	0
	1.06	0	2.8×10^{-2}	0	1.16×10^{-1}	0
	7.07×10^{-1}	0	3.97×10^{-2}	0	4.37×10^{-1}	0
1	4.	4.	2.42	2.42	5.25	5.25
	5.49	1.08×10^{2}	8.46	1.82	1.08×10^{1}	7.67
	7.17	2.2×10^{2}	1.61×10^{-1}	1.99×10^{1}	5.03	7.25
	2.59	0	6.41×10^{-2}	0	3.38	0
	8.45	0	2.68×10^{-1}	0	9.36×10^{-1}	0
	9.59	0	1.66×10^{-1}	0	6.8	0
1.5	4.35	4.35	2.78	2.78	5.68	5.68
	8.17	3.77×10^{2}	1.35×10^{1}	5.05	1.73×10^{1}	7.66
	1.42×10^{1}	7.42×10^{3}	1.43×10^{-1}	5.62×10^{1}	8.6	6.67
	6.95	0	3.58×10^{-2}	0	7.3	0
	2.87×10^{1}	0	1.16	0	4.42	0
	4.17×10^{1}	0	1.1×10^{-2}	0	2.98×10^{1}	0
2	4.34	4.34	2.94	2.94	5.91	5.91
	1.12×10^{1}	8.23×10^{2}	1.88×10^{1}	1.79×10^{1}	2.4×10^{1}	6.06
	2.26×10^{1}	4.23×10^{4}	1.07	1.38×10^{2}	1.22×10^{1}	5.75
	1.17×10^{1}	0	2.04×10^{-1}	0	1.34×10^{1}	0
	6.62×10^{1}	0	2.96	0	1.01×10^{1}	0
	9.46×10^{1}	0	8.56×10^{-1}	0	8.28×10^{1}	0
3	4.03	4.03	3.05	3.05	6.07	6.07
	1.8×10^{1}	2.19×10^{3}	2.96×10^{1}	6.23×10^{1}	3.73×10^{1}	1.98
	4.3×10^{1}	3.3×10^{5}	5.17	5.63×10^{2}	1.82×10^{1}	1.56×10^{1}
	$2.\times10^{1}$	0	2.61	0	3.47×10^{1}	0
	2.03×10^{2}	0	9.48	0	1.93×10^{1}	0
	1.94×10^{2}	0	5.5	0	3.12×10^{2}	0
5	7.65	7.65	3.2	3.2	5.91	5.91
	3.27×10^{1}	1.74×10^{4}	5.12×10^{1}	2.31×10^{2}	6.39×10^{1}	3.5×10^{1}
	1.11×10^{2}	3.28×10^{7}	2.32×10^{1}	3.8×10^{3}	2.72×10^{1}	1.73×10^{2}
	1.07×10^{2}	0	2.57×10^{1}	0	1.39×10^{2}	0
	1.14×10^{3}	0	3.31×10^{1}	0	4.64×10^{1}	0
	2.03×10^{3}	0	2.21×10^{1}	0	1.57×10^{3}	0

Now we plot forward implied volatility surfaces varying the starting date t_i.


t_i=0.1
time T
1.0
0.8
0.6
0.4
0.2
80
60
IV
40
20
1.0
0.9
strike K
0.8


t_i=0.25
time T
1.2
1.0
0.8
0.6
0.4
IV
100
50
1.0
0.9
strike K
0.8


t_i=0.5
time T
1.4
1.2
1.0
0.8
0.6
400
300
IV
200
100
1.0
0.9
strike K
0.8


t_i=1
time T
2.0
1.8
1.6
1.4
1.2
2000
1500
IV
1000
500
0
1.0
0.9
strike K
0.8

Appendix A - Proof of Theorem (6)

Proof. The proof is based on some symmetry properties of the Gaussian fundamental solution $\Gamma = \Gamma(t, x; s, \xi)$ as it is defined in (A.7) and (6.7), combined with an extensive use of other classical relations such as the Chapman-Kolmogorov equation and the Duhamel's principle.

We start by recalling the operator

$$\mathcal{M}(t, s) = \mathcal{M}^x(t, s) = x + \boldsymbol{m}(t, s) + \boldsymbol{C}(t, s)\,\nabla_x .$$

as it is defined in (6.13). Above, and throughout this Appendix, we use the superscript x to explicitly indicate the variables on which the operator acts. Furthermore, we define the operatore

$$\overline{\mathcal{M}}(t, s) = \overline{\mathcal{M}}^y(t, s) = y - \boldsymbol{m}(t, s) + \boldsymbol{C}(t, s)\,\nabla_y . \tag{A.1}$$

Next lemma shows how to the differentiation and multiplication operators on $\Gamma(t, x; T, y)$ transform when switching from the x to the y variables.

Lemma (a1)

For any $t < s$ and $x, y \in \mathbb{R}^d$, we have

$$\nabla_x \Gamma(t, x; s, y) = -\nabla_y \Gamma(t, x; s, y)\,, \tag{A.2}$$

and

$$y\,\Gamma(t, x; s, y) = \mathcal{M}^x(t, s)\,\Gamma(t, x; s, y)\,, \tag{A.3}$$

$$x\,\Gamma(t, x; s, y) = \overline{\mathcal{M}}^y(t, s)\,\Gamma(t, x; s, y)\,, \tag{A.4}$$

Proof. The proof is based on some properties of the Fourier transform

$$\mathcal{F}_x f(\xi) = \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} e^{i x \xi} f(x)\, dx\,. \tag{A.5}$$

First, we recall the

$$i\,\xi\,\mathcal{F}_x(f) = \mathcal{F}_x(-\nabla_x f), \qquad \mathcal{F}_x(x\,f) = -i\,\nabla_\xi \mathcal{F}_x f\,, \tag{A.6}$$

and

$$\mathcal{F}_x\,\Gamma(t,\ \cdot\ ;\ T,\ y)\,(\xi) = \frac{1}{\sqrt{(2\,\pi)^d\ |\boldsymbol{C}(t,\ T)|}}\ e^{i\,\xi(y-\boldsymbol{m}(t,T))-\frac{1}{2}\langle \boldsymbol{C}(t,T)\,\xi,\xi\rangle}, \tag{A.7}$$

$$\mathcal{F}_y\,\Gamma(t,\ x;\ T,\ \cdot\ )\,(\eta) = \frac{1}{\sqrt{(2\,\pi)^d\ |\boldsymbol{C}(t,\ T)|}}\ e^{i\,\eta(x+m(t,T))-\frac{1}{2}\langle \boldsymbol{C}(t,T)\,\eta,\eta\rangle}. \tag{A.8}$$

To obtain (A.2) we use (A.6) and (A.7) to compute

$$\mathcal{F}_x(\nabla_y\,\Gamma) = \nabla_y\,\mathcal{F}_x(\Gamma) = i\,\xi\,\mathcal{F}_x(\Gamma) = \mathcal{F}_x(-\nabla_x\,\Gamma) \tag{A.9}$$

where, as always, $\Gamma \equiv \Gamma(t,\ x;\ s,\ y)$. Analogously, we can proceed for (A.3):

$$\begin{aligned}\mathcal{F}_y(y\,\Gamma) &= -i\,\nabla_\eta\,\mathcal{F}_y(\Gamma)\\ &= (x+\boldsymbol{m}(t,\ T)+\boldsymbol{C}(t,\ T)\ i\,\eta)\,\mathcal{F}_y(\Gamma)\\ &= \mathcal{F}_y((x+\boldsymbol{m}(t,\ T)-\boldsymbol{C}(t,\ T)\,\nabla_y)\,\Gamma)\\ &= \mathcal{F}_y(\mathcal{M}^x(t,\ s)\,\Gamma),\end{aligned} \tag{A.10}$$

where we use in order (A.6), (A.8), (A.6),(A.2). The proof of identity (A.4) is completely analogous.

Corollary (a2)

For any $t < s,\ s_1 \in [0,\ T]$ and $x,\ y \in \mathbb{R}^d$ we have

$$\mathbb{a}_n^\alpha(s_1,\ y)\,\Gamma(t,\ x;\ s,\ y) = \mathbb{a}_n^\alpha(s_1,\ \mathcal{M}^x(t,\ s))\,\Gamma(t,\ x;\ s,\ y)\,, \tag{A.11}$$

$$\mathbb{a}_n^\alpha(s_1,\ x)\,\Gamma(t,\ x;\ s,\ y) = \mathbb{a}_n^\alpha\big(s_1,\ \overline{\mathcal{M}}^y(t,\ s)\big)\,\Gamma(t,\ x;\ s,\ y)\,, \tag{A.12}$$

Proof. First we note that the components $\mathcal{M}_i^x(t,\ s)$, $i = 1,\ ...,\ d$, of the operator $\mathcal{M}^x(t,\ s)$ commute when applied to $\Gamma \equiv \Gamma\,(t,\ x;\ s,\ y)$ and to its derivatives, but clearly not in general when applied to a generic function.

Indeed, for any multi-index β, we have

$$\begin{aligned}\mathcal{M}_i^x(t,\ s)\,\mathcal{M}_j^x(t,\ s)\,\mathcal{D}_x^\beta\,\Gamma &= (-1)^{|\beta|}\,\mathcal{M}_i^x(t,\ s)\,\mathcal{M}_j^x(t,\ s)\,\mathcal{D}_y^\beta\,\Gamma\\ &= (-1)^{|\beta|}\,\mathcal{D}_y^\beta\,\mathcal{M}_i^x(t,\ s)\,\mathcal{M}_j^x(t,\ s)\,\Gamma\\ &= (-1)^{|\beta|}\,\mathcal{D}_y^\beta\,\mathcal{M}_i^x(t,\ s)\,y_j\,\Gamma\\ &= (-1)^{|\beta|}\,\mathcal{D}_y^\beta\,y_j\,\mathcal{M}_i^x(t,\ s)\,\Gamma\\ &= (-1)^{|\beta|}\,\mathcal{D}_y^\beta\,y_j\,y_i\,\Gamma\\ &= \mathcal{M}_j^x(t,\ s)\,\mathcal{M}_i^x(t,\ s)\,\mathcal{D}_y^\beta\,\Gamma\end{aligned} \tag{A.13}$$

where the last equation holds reversing the steps above.

Therefore, and since $\mathbb{a}_n^\alpha(s_1,\ \cdot\,)$ is a polynomial by construction, we have that the operators $\mathbb{a}_n^\alpha(s_1,\ \mathcal{M}^x(t,\ s))$ are defined unambiguosly when applied to $\Gamma(t,\ x;\ s,\ y)$

and to its derivatives. Moreover, clearly (A.11) is now a straightforward consequence of (A.3). An analogous argument shows the validity of (A.12).

□

We now recall the operators

$$\mathcal{A}_n^x(s) = \sum_{|\alpha|\le 2} \mathbb{a}_n^\alpha(s, x)\, D_x^\alpha, \quad \mathcal{G}_n^x(t, s) = \sum_{|\alpha|\le 2} \mathbb{a}_n^\alpha(s, \mathcal{M}^x(t, s))\, D_x^\alpha, \quad n \ge 0, \tag{A.14}$$

as they are defined in (6.1) and (6.12), and we introduce the operator

$$\overline{\mathcal{G}}_n^y(t, s) = \sum_{|\alpha|\le 2} (-1)^{|\alpha|}\, D_y^\alpha \mathbb{a}_n^\alpha\big(t, \overline{\mathcal{M}}^y(t, s)\big), \qquad n \ge 0, \tag{A.15}$$

with $\overline{\mathcal{M}}^y$ as in (A.1). We remark explicitly that, by Corollary (a2), operators $\mathcal{G}_n^x(t, s)$ and $\overline{\mathcal{G}}_n^y(t, s)$ are defined unambiguously when applied to $\Gamma = \Gamma(t,\ x;\ s,\ y)$, to its derivatives and, more generally, by the representation formula (6.6), to solutions of the Cauchy problem (6.3).

The next proposition and its remarkable corollaries are the key of the proof of Theorem (6).

Proposition (a3)

For any $t < s < T,\ x, y \in \mathbb{R}^d$ and $n \ge 1$ we have

$$\int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi)\, \mathcal{A}_n^\xi(s)\, f(\xi)\, d\xi = \mathcal{G}_n^x(t, s) \int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi)\, f(\xi)\, d\xi, \tag{A.16}$$

$$\int_{\mathbb{R}^d} \mathcal{A}_n^\xi(s)\, f(\xi)\, \Gamma(s,\ \xi;\ T,\ y)\, d\xi = \overline{\mathcal{G}}_n^y(s, T) \int_{\mathbb{R}^d} f(\xi)\, \Gamma(s,\ \xi;\ T,\ y)\, d\xi, \tag{A.17}$$

for any $f \in C_0^2\big(\mathbb{R}^d\big)$. Furthermore, the following relation holds:

$$\mathcal{G}_n^x(t, s)\, \Gamma(t,\ x;\ T, y) = \overline{\mathcal{G}}_n^y(s, T)\, \Gamma(t,\ x;\ T,\ y) \tag{A.18}$$

Proof. We first prove (A.16). By definition of $\mathcal{A}_n^\xi$ we have

$$\int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi)\, \mathcal{A}_n^\xi(s)\, f(\xi)\, d\xi =$$

$$\sum_{|\alpha|\le 2} \int_{\mathbb{R}^d} \mathbb{a}_n^\alpha(s, \xi)\, \Gamma(t,\ x;\ s,\ \xi)\, D_\xi^\alpha f(\xi)\, d\xi$$

$$= \sum_{|\alpha|\le 2} \mathbb{a}_n^\alpha(s, \mathcal{M}^x(t, s)) \int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi)\, D_\xi^\alpha f(\xi)\, d\xi$$

$$= \sum_{|\alpha|\le 2} \mathbb{a}_n^\alpha(s, \mathcal{M}^x(t, s))\, (-1)^{|\alpha|} \int_{\mathbb{R}^d} D_\xi^\alpha\, \Gamma(t,\ x;\ s,\ \xi)\, f(\xi)\, d\xi$$

by (A.2)

$$= \sum_{|\alpha|\le 2} \mathbb{a}_n^\alpha(s, \mathcal{M}^x(t,s))\, \mathcal{D}_x^\alpha \int_{\mathbb{R}^d} \Gamma(t, x; s, \xi)\, f(\xi)\, d\xi$$

by (A.14)

$$= \mathcal{G}_n^x(t,s) \int_{\mathbb{R}^d} \Gamma(t, x; s, \xi)\, f(\xi)\, d\xi.$$

The proof of (A.17) is analogous. Identity (A.18) follows from (A.16) and (A.17). Indeed, using the Chapman-Kolmogorov equation we have

$$\mathcal{G}_n^x(t,s)\, \Gamma(t, x; T, y) = \mathcal{G}_n^x(t,s) \int_{\mathbb{R}^d} \Gamma(t, x; s, \xi)\, \Gamma(s, \xi; T, y)\, d\xi$$

$$= \int_{\mathbb{R}^d} \Gamma(t, x; s, \xi)\, \mathcal{A}_n^\xi(s)\, \Gamma(s, \xi; T, y)\, d\xi$$

$$= \overline{\mathcal{G}}_n^y(s,T) \int_{\mathbb{R}^d} \Gamma(t, x; s, \xi)\, \Gamma(s, \xi; T, y)\, d\xi$$

$$= \overline{\mathcal{G}}_n^y(s,T)\, \Gamma(t, x; T, y)$$

□

Corollary (a4)

For any $t < s < T$, $x, y \in \mathbb{R}$, $n \ge 1$, we have

$$\int_{\mathbb{R}^d} \Gamma(t, x; s, \xi)\, \mathcal{G}_{i_1}^\xi(s, s_1) \ldots \mathcal{G}_{i_n}^\xi(s, s_n)\, \Gamma(s, \xi; T, y)\, d\xi$$

$$= \mathcal{G}_{i_1}^x(t, s_1) \ldots \mathcal{G}_{i_n}^x(t, s_n)\, \Gamma(t, x; T, y)$$

for any $i \in \mathbb{N}^n$ and $s < s_1 < \ldots < s_n < T$.

Proof. We first prove by induction on n. For $n = 1$, and for any $i_1 \ge 1$, $t < s_1 < T$, we have

$$\int_{\mathbb{R}^d} \Gamma(t, x; s, \xi)\, \mathcal{G}_{i_1}^\xi(s, s_1)\, \Gamma(s, \xi; T, y)\, d\xi$$

$$= \overline{\mathcal{G}}_{i_1}^y(s_1, T) \int_{\mathbb{R}^d} \Gamma(t, x; s, \xi)\, \Gamma(s, \xi; T, y)\, d\xi$$

$$= \overline{\mathcal{G}}_{i_1}^y(s_1, T)\, \Gamma(t, x; T, y)$$

$$= \mathcal{G}_{i_1}^x(t, s_1)\, \Gamma(t, x; T, y)$$

We assume now the thesis to be true for $n \ge 1$ and for any $i \in \mathbb{N}^n$,

$s < s_1 < ... < s_n < T$. Then, for any $i_{n+1} \geq 1$, $s_n < s_{n++1} < T$ we have

$$\int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi)\, \mathcal{G}^{\xi}_{i_1}(s, s_1) \ldots \mathcal{G}^{\xi}_{i_n}(s, s_n)\, \mathcal{G}^{\xi}_{i_{n+1}}(s, s_{n+1})\, \Gamma(s,\ \xi;\ T,\ y)\, d\xi$$

by (A.18)

$$\overline{\mathcal{G}}^{y}_{i_{n+1}}(s_{n+1}, T) \int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi)\, \mathcal{G}^{\xi}_{i_1}(s, s_1) \ldots \mathcal{G}^{\xi}_{i_n}(s, s_n)\, \Gamma(s,\ \xi;\ T,\ y)\, d\xi$$

by inductive hypothesis

$$\overline{\mathcal{G}}^{y}_{i_{n+1}}(s_{n+1}, T)\, \mathcal{G}^{x}_{i_1}(t, s_1) \ldots \mathcal{G}^{x}_{i_n}(t, s_n)\, \Gamma(t,\ x;\ T,\ y)$$

$$\mathcal{G}^{x}_{i_1}(t, s_1) \ldots \mathcal{G}^{x}_{i_n}(t, s_n)\, \overline{\mathcal{G}}^{y}_{i_{n+1}}(s_{n+1}, T)\, \Gamma(t,\ x;\ T,\ y)$$

by (A.18)

$$\mathcal{G}^{x}_{i_1}(t, s_1) \ldots \mathcal{G}^{x}_{i_n}(t, s_n)\, \mathcal{G}^{x}_{i_{n+1}}(t, s_{n+1})\, \Gamma(t,\ x;\ T,\ y)$$

which concludes the proof.

□

Corollary (a5)

Let u_0 be as in (6.6) with $\gamma=0$. For any $t < s < T$, $x, y \in \mathbb{R}$, $n \geq 1$, we have

$$\int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi)\, \mathcal{G}^{\xi}_{i_1}(s, s_1) \ldots \mathcal{G}^{\xi}_{i_n}(s, s_n)\, u_0(s, \xi)\, d\xi$$

$$= \mathcal{G}^{x}_{i_1}(t, s_1) \ldots \mathcal{G}^{x}_{i_n}(t, s_n)\, u_0(t, x)$$

for any $i \in \mathbb{N}^n$ and $s < s_1 < ... < s_n < T$.

Proof. By (6.6) we have

$$\int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi)\, \mathcal{G}^{\xi}_{i_1}(s, s_1) \ldots \mathcal{G}^{\xi}_{i_n}(s, s_n)\, u_0(s, \xi)\, d\xi$$

$$= \int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi)\, \mathcal{G}^{\xi}_{i_1}(s, s_1) \ldots$$

$$\mathcal{G}^{\xi}_{i_n}(s, s_n) \int_{\mathbb{R}^d} \Gamma(s,\ \xi;\ T,\ y)\, h(y)\, dy\, d\xi$$

by Fubini's theorem

=

$$\int_{\mathbb{R}^d} h(y) \int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi)\, \mathcal{G}^{\xi}_{i_1}(s, s_1) \ldots \mathcal{G}^{\xi}_{i_n}(s, s_n)\, \Gamma(s,\ \xi;\ T,\ y)\, d\xi\, dy$$

by Corollary (a5)

$$= \int_{\mathbb{R}^d} h(y)\, \mathcal{G}_{i_1}^x(t, s_1) \dots \mathcal{G}_{i_n}^x(t, s_n)\, \Gamma(t,\ x;\ T,\ y)\, d y$$

by (6.6)

$$= \mathcal{G}_{i_1}^x(t, s_1) \dots \mathcal{G}_{i_n}^x(t, s_n)\, u_0(s, \xi)$$

which conludes the proof.

□

We are now in position to prove Theorem (6). For simplicity we only give the proof for $\gamma \equiv 0$. However, the general case with $\gamma \neq 0$ is completely analogous. Proceeding by induction on n, we first prove the case $n = 1$. By definition, u_1 is the unique solution of the non-homogeneous Cauchy problem with n = 1.

Then, by the Duhamel's principle we have

$$u_1(t, x) = \int_t^T \int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi)\, \mathcal{A}_1^{\xi}(s)\, u_0(s, \xi)\, d\xi\, ds$$

$$= \int_t^T \mathcal{G}_1^x(t, s) \int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi)\, u_0(s, \xi)\, d\xi\, ds$$

$$= \int_t^T \mathcal{G}_1^x(t, s) \int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi) \int_{\mathbb{R}^d} \Gamma(s,\ \xi;\ T,\ y)\, h(y)\, d y\, d\xi\, ds$$

$$= \int_t^T \mathcal{G}_1^x(t, s) \int_{\mathbb{R}^d} h(y) \int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi)\, \Gamma(s,\ \xi;\ T,\ y)\, d\xi\, d y\, ds$$

$$= \int_t^T \mathcal{G}_1^x(t, s)\, ds\, u_0(t, x)$$

$$= \mathcal{L}_1^x(t, T)\, u_0(t, x)$$

For the general case, let us assumeto be true the Theorem (6) for n ≥ 1, and prove it holds for $n + 1$. By definition, u_{n+1} is the unique solution of the non-homogeneous Cauchy problem. Thus, by the Duhamel’s principle, we have

$$u_{n+1}(t, x) = \int_t^T \int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi) \sum_{k=1}^{n+1} \mathcal{A}_k^{\xi}(s)\, u_{n+1-k}(s, \xi)\, d\xi\, ds$$

$$= \sum_{k=1}^{n+1} \int_t^T \mathcal{G}_k^x(t, s) \int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi)\, u_{n+1-k}(s, \xi)\, d\xi\, ds$$

by induction hypothesis

$$= \sum_{k=1}^{n+1} \int_t^T \mathcal{G}_k^x(t, s) \int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi)\, \mathcal{L}_{n+1-k}^{\xi}(s, T)\, u_0(s, \xi)\, d\xi\, ds \tag{A.19}$$

Now by definition we have

$$\int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi)\, \mathcal{L}_{n+1-k}^{\xi}(s, T)\, u_0(s, \xi)\, d\xi$$

$$= \sum_{h=1}^{n+1-k} \int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi) \int_s^T ds_1 \ldots \int_{s_{h-1}}^T ds_h \sum_{i \in I_{n+1-k,h}} \mathcal{G}_{i_1}^{\xi}(s, s_1) \ldots$$

$$\mathcal{G}_{i_h}^{\xi}(s, s_h)\, u_0(s, \xi)\, d\xi$$

$$= \sum_{h=1}^{n+1-k} \int_s^T ds_1 \ldots \int_{s_{h-1}}^T ds_h \sum_{i \in I_{n+1-k,h}} \int_{\mathbb{R}^d} \Gamma(t,\ x;\ s,\ \xi)\, \mathcal{G}_{i_1}^{\xi}(s, s_1) \ldots$$

$$\mathcal{G}_{i_h}^{\xi}(s, s_h)\, u_0(s, \xi)\, d\xi$$

$$= \sum_{h=1}^{n+1-k} \int_s^T ds_1 \ldots \int_{s_{h-1}}^T ds_h \sum_{i \in I_{n+1-k,h}} \mathcal{G}_{i_1}^{x}(t, s_1) \ldots \mathcal{G}_{i_h}^{x}(t, s_h)\, u_0(s, x) \tag{A.20}$$

Now by inserting (A.20) into (A.19) we obtain

$$u_{n+1}(t, x) = \tilde{\mathcal{L}}_n^x(t, T)\, u_0(t, x),$$

where

$$\tilde{\mathcal{L}}_n^x(s_0, T) =$$

$$\sum_{k=1}^{n+1} \sum_{h=1}^{n+1-k} \int_{s_0}^T ds \int_s^T ds_1 \ldots \int_{s_{h-1}}^T ds_h \sum_{i \in I_{n+1-k,h}} \mathcal{G}_k^x(s_0, s)\, \mathcal{G}_{i_1}^x(s_0, s_1) \ldots$$

$$\mathcal{G}_{i_h}^x(s_0, s_h)$$

In order to conclude the proof, it is enough to check that $\tilde{\mathcal{L}}_n^x(s_0, T) = \mathcal{L}_{n+1}^x(s_0, T)$, by exchanging the indexes in the sums. We left it as an exercise to the reader.

Appendix B - Proof of Theorem (13)

Remark (b1)

Let $\Gamma_0(t, x, y, s, \xi, \omega)$ be the multinormal Gaussian density function defined in (6.7) for the two-dimensional case. A direct computations reveals

$$\partial_\xi^n \partial_\omega^m \left((\xi - \overline{x})^h (\omega - \overline{y})^k \Gamma_0(t, x, y, s, \xi, \omega)\right) =$$
$$= (-1)^{n+m} \left(\mathcal{M}_1{}^{(x,y)}(t, s)\right)^h \left(\mathcal{M}_2{}^{(x,y)}(t, s)\right)^k \partial_x^n \partial_y^m \Gamma_0(t, x, y, s, \xi, \omega)$$

with

$$\mathcal{M}_1{}^{(x,y)}(t, s) = (x - \overline{x}) - \int_t^s a_{0,0}(\tau)\, d\tau + 2 \int_t^s a_{0,0}(\tau)\, d\tau\, \partial_x + \int_t^s c_{0,0}(\tau)\, d\tau\, \partial_y\,,$$
$$\mathcal{M}_2{}^{(x,y)}(t, s) = (y - \overline{y}) + \int_t^s \alpha_{0,0}(\tau)\, d\tau + 2 \int_t^s b_{0,0}(\tau)\, d\tau\, \partial_y + \int_t^s c_{0,0}(\tau)\, d\tau\, \partial_x\,.$$

Proof. of Theorem (13)

$$\mathbb{E}\left[\left(\frac{S_T}{S_{t_i}} - K\right)^+\right] = \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\left(e^{\nu-\xi} - e^k\right)^+ \Gamma(t_i, \xi, \omega, T, \nu, \eta)\, d\nu\, d\eta\right)\Gamma(t, x, y, t_i, \xi, \omega)\, d\xi\, d\omega = \tag{B.1}$$

applying our density expansions

$$\begin{aligned}= \int_{\mathbb{R}^2}\Bigg(\int_{\mathbb{R}^2}\left(e^{\nu-\xi} - e^k\right)^+ \sum_\alpha C_\alpha(\xi - \overline{x})^{\alpha_3} (\omega - \overline{y})^{\alpha_4} (T - t_i)^{\alpha_5} \partial_\xi^{\alpha_1} \partial_\omega^{\alpha_2} \Gamma_0(t_i, \xi, \omega, T, \nu, \eta)\, d\nu\, d\eta\Bigg)\\ \sum_\beta C_\beta(x - \overline{x})^{\beta_3} (y - \overline{y})^{\beta_4} (t_i - t)^{\beta_5} \partial_x^{\beta_1} \partial_y^{\beta_2} \Gamma_0(t, x, y, t_i, \xi, \omega)\, d\xi\, d\omega =\end{aligned} \tag{B.2}$$

$$\begin{aligned}= \sum_\beta C_\beta(x - \overline{x})^{\beta_3} (y - \overline{y})^{\beta_4} (t_i - t)^{\beta_5}\\ \partial_x^{\beta_1} \partial_y^{\beta_2} \sum_\alpha C_\alpha(T - t_i)^{\alpha_5} \int_{\mathbb{R}^2}\Bigg(\int_{\mathbb{R}^2}\left(e^{\nu-\xi} - e^k\right)^+ (\xi - \overline{x})^{\alpha_3} (\omega - \overline{y})^{\alpha_4} \partial_\xi^{\alpha_1} \partial_\omega^{\alpha_2} \Gamma_0(t_i, \xi, \omega, T, \nu, \eta)\\ d\nu\, d\eta\Bigg)\Gamma_0(t, x, y, t_i, \xi, \omega)\, d\xi\, d\omega =\end{aligned} \tag{B.3}$$

for the Fubini-Tonelli theorem

$$\begin{aligned}= \sum_\beta C_\beta(x - \overline{x})^{\beta_3} (y - \overline{y})^{\beta_4} (t_i - t)^{\beta_5}\\ \partial_x^{\beta_1} \partial_y^{\beta_2} \sum_\alpha C_\alpha(T - t_i)^{\alpha_5} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(e^{\nu-\xi} - e^k\right)^+ (\xi - \overline{x})^{\alpha_3} (\omega - \overline{y})^{\alpha_4} \Gamma_0(t, x, y, t_i, \xi, \omega)\\ \partial_\xi^{\alpha_1} \partial_\omega^{\alpha_2} \Gamma_0(t_i, \xi, \omega, T, \nu, \eta)\, d\xi\, d\omega\, d\nu\, d\eta =\end{aligned} \tag{B.4}$$

by Remark (b1)

$$= \sum_{\beta} C_{\beta}(x-\overline{x})^{\beta_3}(y-\overline{y})^{\beta_4}(t_i-t)^{\beta_5}\,\partial_x^{\beta_1}\partial_y^{\beta_2}\sum_{\alpha} C_{\alpha}(T-t_i)^{\alpha_5}\left(\mathcal{M}_1{}^{(x,y)}(t,t_i)\right)^{\alpha_3}\left(\mathcal{M}_2{}^{(x,y)}(t,t_i)\right)^{\alpha_4}$$
$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(e^{\nu-\xi}-e^{k}\right)^{+}\Gamma_0(t,x,y,t_i,\xi,\omega)\,\partial_{\xi}^{\alpha_1}\partial_{\omega}^{\alpha_2}\Gamma_0(t_i,\xi,\omega,T,\nu,\eta)\,d\xi\,d\omega\,d\nu\,d\eta = \tag{B.5}$$

again by Remark (b1)

$$= \sum_{\beta} C_{\beta}(x-\overline{x})^{\beta_3}(y-\overline{y})^{\beta_4}(t_i-t)^{\beta_5}$$
$$\partial_x^{\beta_1}\partial_y^{\beta_2}\sum_{\alpha} C_{\alpha}(T-t_i)^{\alpha_5}\left(\mathcal{M}_1{}^{(x,y)}(t,t_i)\right)^{\alpha_3}\left(\mathcal{M}_2{}^{(x,y)}(t,t_i)\right)^{\alpha_4}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\left(e^{\nu-\xi}-e^{k}\right)^{+}(-1)^{\alpha_1+\alpha_2} \tag{B.6}$$
$$\partial_{\nu}^{\alpha_1}\partial_{\eta}^{\alpha_2}\Gamma_0(t_i,\xi,\omega,T,\nu,\eta)\,d\nu\,d\eta\right)\Gamma_0(t,x,y,t_i,\xi,\omega)\,d\xi\,d\omega =$$

integrating by parts, we have null conditions at extremes, so the partial derivative respect to η disappears, so we must consider only $\alpha_2 = 0$

$$= \sum_{\beta} C_{\beta}(x-\overline{x})^{\beta_3}(y-\overline{y})^{\beta_4}(t_i-t)^{\beta_5}\,\partial_x^{\beta_1}\partial_y^{\beta_2}\sum_{\alpha} C_{\alpha}(T-t_i)^{\alpha_5}\left(\mathcal{M}_1{}^{(x,y)}(t,t_i)\right)^{\alpha_3}\left(\mathcal{M}_2{}^{(x,y)}(t,t_i)\right)^{\alpha_4}$$
$$\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\left(\partial_{\nu}^{\alpha_1}\left(e^{\nu-\xi}-e^{k}\right)^{+}\right)\Gamma_0(t_i,\xi,\omega,T,\nu,\eta)\,d\nu\,d\eta\right)\Gamma_0(t,x,y,t_i,\xi,\omega)\,d\xi\,d\omega = \tag{B.7}$$

change of variables ($\varphi = \nu-\xi,\ \rho = \eta-\omega,\ \xi = \xi,\ \omega = \omega$), with Jacobian equal to 1

$$= \sum_{\beta} C_{\beta}(x-\overline{x})^{\beta_3}(y-\overline{y})^{\beta_4}(t_i-t)^{\beta_5}\,\partial_x^{\beta_1}\partial_y^{\beta_2}\sum_{\alpha} C_{\alpha}(T-t_i)^{\alpha_5}\left(\mathcal{M}_1{}^{(x,y)}(t,t_i)\right)^{\alpha_3}\left(\mathcal{M}_2{}^{(x,y)}(t,t_i)\right)^{\alpha_4}$$
$$\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\left(\partial_{\varphi}^{\alpha_1}\left(e^{\varphi}-e^{k}\right)^{+}\right)\Gamma_0(t_i,0,0,T,\varphi,\rho)\,d\varphi\,d\rho\right)\Gamma_0(t,x,y,t_i,\xi,\omega)\,d\xi\,d\omega = \tag{B.8}$$

$$= \sum_{\beta} C_{\beta}(x-\overline{x})^{\beta_3}(y-\overline{y})^{\beta_4}(t_i-t)^{\beta_5}\,\partial_x^{\beta_1}\partial_y^{\beta_2}\sum_{\alpha} C_{\alpha}(T-t_i)^{\alpha_5}\left(\mathcal{M}_1{}^{(x,y)}(t,t_i)\right)^{\alpha_3}\left(\mathcal{M}_2{}^{(x,y)}(t,t_i)\right)^{\alpha_4}$$
$$\left(\int_{\mathbb{R}^2}\left(\partial_{\varphi}^{\alpha_1}\left(e^{\varphi}-e^{k}\right)^{+}\right)\Gamma_0(t_i,0,0,T,\varphi,\rho)\,d\varphi\,d\rho\right)\int_{\mathbb{R}^2}\Gamma_0(t,x,y,t_i,\xi,\omega)\,d\xi\,d\omega = \tag{B.9}$$

$$= \sum_{\beta} C_{\beta}(x-\overline{x})^{\beta_3}(y-\overline{y})^{\beta_4}(t_i-t)^{\beta_5}\,\partial_x^{\beta_1}\partial_y^{\beta_2}\sum_{\alpha} C_{\alpha}(T-t_i)^{\alpha_5}\left(\mathcal{M}_1{}^{(x,y)}(t,t_i)\right)^{\alpha_3}$$
$$\left(\mathcal{M}_2{}^{(x,y)}(t,t_i)\right)^{\alpha_4}\int_{\mathbb{R}^2}\left(\partial_{\varphi}^{\alpha_1}\left(e^{\varphi}-e^{k}\right)^{+}\right)\Gamma_0(t_i,0,0,T,\varphi,\rho)\,d\varphi\,d\rho\,. \tag{B.10}$$

First we notice that

$$\sum_{\beta} C_{\beta}(x-\overline{x})^{\beta_3}(y-\overline{y})^{\beta_4}(t_i-t)^{\beta_5}\,\partial_x^{\beta_1}\partial_y^{\beta_2} = 1 + \sum_{n}\mathcal{L}_n$$

where $\mathcal{L}_n$ are operators as defined in Theorem (6).

Now we consider the Black-Scholes Call price function $\mathrm{Call}^{\mathrm{BS}}(x, y, t, z)$, and we notice that

$$\int_{\mathbb{R}^2} \left(\partial_\varphi^{\alpha_1} \left(e^\varphi - e^k\right)^+\right) \Gamma_0(t_i, 0, 0, T, \varphi, \rho)\, d\varphi\, d\rho =$$

$$\begin{cases} \mathrm{Call}^{\mathrm{BS}}\left(x_0, \sqrt{2\, a_{0,0}}, (T - t_i), k\right) & \alpha_1 = 0 \\ \partial_{x_0} \mathrm{Call}^{\mathrm{BS}}\left(x_0, \sqrt{2\, a_{0,0}}, (T - t_i), k\right) & \alpha_1 > 0 \end{cases}$$

with $x_0 = 0$. So we define operators $\mathcal{L}_n^*$ such that

$$1 + \sum_n \mathcal{L}_n^* = \sum_\alpha C_\alpha (T - t_i)^{\alpha_5} \left(\mathcal{M}_1{}^{(x,y)}(t, t_i)\right)^{\alpha_3} \left(\mathcal{M}_2{}^{(x,y)}(t, t_i)\right)^{\alpha_4} \partial_{x_0}^{\alpha_1^*}$$

$$\alpha_1^* = \begin{cases} 0 & \alpha_1 = 0 \\ 1 & \alpha_1 > 0 \end{cases}.$$

Using these operators we can rewrite (B.10)

$$\left(1 + \sum_n \mathcal{L}_n\right)\left(1 + \sum_n \mathcal{L}_n^*\right) \mathrm{Call}^{\mathrm{BS}}\left(x_0, \sqrt{2\, a_{0,0}}, (T - t_i), k\right)$$

with $x_0 = 0$.

Now we choose to collect these operators defining a new class of differential operator $\tilde{\mathcal{L}}$ such that

$$\tilde{\mathcal{L}}_0 = 1, \quad \tilde{\mathcal{L}}_n = \left(1 + \sum_{k=1}^n \mathcal{L}_k\right)\left(1 + \sum_{k=1}^n \mathcal{L}_k^*\right) - \tilde{\mathcal{L}}_{n-1}$$

so we have

$$\begin{aligned} &\tilde{\mathcal{L}}_0 = 1, \\ &\tilde{\mathcal{L}}_1 = \mathcal{L}_1 + \mathcal{L}_1^* + \mathcal{L}_1\, \mathcal{L}_1^* \\ &\tilde{\mathcal{L}}_2 = \mathcal{L}_2 + \mathcal{L}_2^* + \mathcal{L}_2\, \mathcal{L}_1^* + \mathcal{L}_1\, \mathcal{L}_2^* + \mathcal{L}_2\, \mathcal{L}_2^* \end{aligned}$$

and so on.

A direct computation shows that $\tilde{\mathcal{L}}_n$ are of the form

$$\tilde{\mathcal{L}}_n = \sum_\gamma C_\gamma (x - \overline{x})^{\gamma_2} (y - \overline{y})^{\gamma_3} (t_i - t)^{\gamma_4} (T - t_i)^{\gamma_5} \partial_{x_0}^{\gamma_1}$$

where index γ_1 assumes only values 0 or 1, and this proves Theorem (13).

□

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