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Pseudoconvessità e Curvatura

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Introduzione

Il concetto di pseudoconvessità viene introdotto per cercare di dare una caratterizzazione ai *domini di olomorfia*, quei sottoinsiemi aperti $U \subseteq \mathbb{C}^n$ tali che non esistono due insiemi non vuoti U_1 e U_2, con U_2 connesso e $U_2 \nsubseteq U$, $U_1 \subseteq U_2 \cap U$, tali che per ogni funzione olomorfa h definita su U, esiste un'altra funzione olomorfa h_2 definita su U_2, tale che $h = h_2$ su U_1.
Nel caso in cui $n = 1$ ogni sottoinsieme aperto di $\mathbb{C}$ risulta essere un dominio di olomorfia. La situazione quando si considera piú di una variabile complessa è decisamente diversa e la loro caratterizzazione risulta essere particolarmente sottile, nello specifico si ha che non tutti i domini sono domini di olomorfia.
La nozione di pseudoconvessità deriva direttamente da quella di convessità nel caso reale, insieme alla definizione, a priori del tutto formale, della *forma di Levi*.
Risulta poi che ogni dominio convesso è un dominio di olomorfia. Si ha però che la convessità non è preservata sotto l'azione di mappe biolomorfe ed è quindi necessaria una condizione geometrica meno stringente per lo studio dei domini di olomorfia: questa condizione è proprio la pseudoconvessità.
Si avrà infatti che condizione necessaria e sufficiente per un insieme per essere un dominio di olomorfia è che sia pseudoconvesso.
Infine, come si vedrà, la forma di Levi e il concetto di *curvatura di Levi* da essa derivante, introdotti in un contesto puramente formale, hanno un significato geometrico profondo, strettamente legato alla struttura dell'insieme su cui sono definite. Si dimostrerà infatti una stima isoperimetrica che lega

la curvatura di Levi alla misura dell'insieme.

Questo lavoro partirà da un'estensione analitica del concetto di convessità geometrica, dimostrandone l'equivalenza e introducendo la forma di Levi, per poi arrivare alla definizione di pseudoconvessità.
Nel secondo capitolo di introdurrà il concetto di curvatura di Levi, dandone alcune caratterizzazioni ed esempi, fino a dimostrare la stima isoperimetrica che lega curvatura di Levi e misura di un insieme.
Nell'ultimo capitolo si definiranno una serie di operatori di curvatura, in relazione con la forma di Levi, che permetteranno di dimostrare alcuni teoremi di confronto.

Chapter 1

Convexity and Pseudoconvexity

1.1 Notions of convexity

The classical definition of convexity is given by:
a subset $\Omega \in \mathbb{R}^n$ *is said to be convex if for any* $p, q \in \Omega$ *and any* $\lambda \in [0, 1]$ *the combination* $(1 - \lambda)p + \lambda q \in \Omega$.
Later we shall refer to a set satisfying this condition as *geometrically convex*. We know that the most useful definitions are the ones written as differential conditions. Thus our wish is to find a differential characterization of convexity. We shall begin with some notion we will use in the remainder of the chapter.

Definition 1.1. Let $\Omega \subseteq \mathbb{R}^n$ be a connected open set with boundary $\partial\Omega$. We say that Ω has C^k *boundary*, $k \geq 1$, if exists a function $\varrho : \mathbb{R}^n \to \mathbb{R}$ belonging to $C^k(U)$, where U is an open neighborhood of the boundary of Ω, such that:

(i) $\Omega \cap U = \{x \in U | \varrho(x) < 0\}$

(ii) $\nabla\varrho(x) \neq 0$, $\forall x \in \partial\Omega$

We call this function ϱ a *defining function* for Ω.

Definition 1.2. Let $\Omega \subseteq \mathbb{R}^n$ be a connected open set with C^1 boundary and let ϱ be a C^1 defining function for Ω. Fixed a point $p \in \partial\Omega$ we say that $w = (w_1, \dots, w_n)$ a *tangent vector* to $\partial\Omega$ at p if

$$\sum_{j=1}^{n} \frac{\partial \varrho}{\partial x_j}(p) \cdot w_j = 0$$

and we write $w \in T_p(\partial\Omega)$.

Remark 1. To make sense this definition must be independent from the choice of the function ϱ, before showing this we give this Lemma.

Lemma 1.1. *Let ϱ_1 and ϱ_2 be two defining function for a connected open set $\Omega \subseteq \mathbb{R}^n$ and let p be a point in $\partial\Omega$. We suppose $\varrho_1, \varrho_2 \in C^k(U)$ where U is a neighborhood of p. Then there exists a positive function $h \in C^{k-1}(U)$ such that*

$$\varrho_1 = h\varrho_2 \qquad \text{on } U \tag{1.1}$$

Proof. Due to the conditions required for h it can be uniquely determined as

$$h = \frac{\varrho_1}{\varrho_2}.$$

and it is positive and of class C^k on $U \setminus \partial\Omega$. We fix now a point $q \in U \cap \partial\Omega$, after a C^k local change of coordinates, we may assume $q = 0$,

$$U \cap \partial\Omega = \left\{x \in U \;\middle|\; x_n = 0\right\}$$

and $\varrho_2(x) = x_n$. For $x' = (x_1, \dots, x_{n-1})$ near zero, we have $\varrho_1(x', 0) = 0$, and by fundamental theorem of calculus we obtain

$$\varrho_1(x', x_n) = \varrho_1(x', x_n) - \varrho_1(x', 0) = x_n \int_0^1 \frac{\partial \varrho_1}{\partial x_n}(x', tx_n)dt$$

the integral on the right side of the equation is a function of class C^{k-1} near 0. This last statement is independent of the choice of C^k-coordinates, then (1.1) holds with $h \in C^k$. $\square$

Remark 2. We now take another defining function η for Ω and there exists another function h, not vanishing on a neighborhood of $\partial\Omega$, such that $\eta(x) = h(x)\varrho(x)$. In this case, for $p \in \partial\Omega$:

$$\begin{aligned}\sum_{j=1}^{n} \frac{\partial \eta}{\partial x_j}(p) \cdot w_j &= \sum_{j=1}^{n} \frac{\partial (h\varrho)}{\partial x_j}(p) \cdot w_j \\ &= h(p) \sum_{j=1}^{n} \frac{\partial \varrho}{\partial x_j}(p) \cdot w_j + \varrho(p) \sum_{j=1}^{n} \frac{\partial h}{\partial x_j}(p) \cdot w_j \\ &= h(p) \sum_{j=1}^{n} \frac{\partial \varrho}{\partial x_j}(p) \cdot w_j\end{aligned}$$

because $\varrho(p) = 0$. So w is a tangent vector at P with respect to ϱ if and only if it is a tangent vector at P with respect to η.

Definition 1.3. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary and defining function ϱ. Let p be fixed in $\partial\Omega$. We say that $\partial\Omega$ is *(weakly) convex* at p if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varrho}{\partial x_i \partial x_j}(p) \cdot w_i w_j \geq 0$$

for every $w \in T_p(\partial\Omega)$.
We say that $\partial\Omega$ is *strongly convex* if this inequality holds strictly whenever $w \neq 0$.
A set Ω is said to be *convex* (*strongly convex*) if $\partial\Omega$ is convex (strongly convex) at each of its point.

Remark 3. The quadratic form

$$\left(\frac{\partial^2 \varrho}{\partial x_i \partial x_j}(p) \cdot w_i w_j \right)_{j,k=1,\ldots,n}$$

is called the *real Hessian* of the function ϱ.

Lemma 1.2. *Let $\Omega \subseteq \mathbb{R}^n$ be strongly convex. Then there is a constant $C > 0$ and a defining function η for Ω such that*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \eta}{\partial x_i \partial x_j}(p) w_i w_j \geq C|w|^2 \tag{1.2}$$

for every $p \in \partial\Omega$ and $w \in \mathbb{R}^n$.

Proof. Let ϱ be a C^2 defining function for Ω and we set, for $\lambda > 0$:

$$\varrho_\lambda = \frac{e^{\lambda\varrho(x)} - 1}{\lambda}$$

Let $p \in \partial\Omega$ and define

$$X = X_p = \left\{ w \in \mathbb{R}^n \;\middle|\; |w| = 1,\; \sum_{i,j=1}^{n} \frac{\partial^2 \varrho}{\partial x_i \partial x_j}(p) w_i w_j \leq 0 \right\}$$

no element in X_p could belong to $T_p(\partial\Omega)$, moreover

$$X_p \subseteq \left\{ w \in \mathbb{R}^n \;\middle|\; |w| = 1,\; \sum_{i=1}^{n} \frac{\partial \varrho}{\partial x_i}(p) w_i \neq 0 \right\}.$$

Since X_p has been defined by an nonstrict inequality, it's closed and also bounded, then it's compact and we can consider

$$\mu = \min_{w \in X_p} \left\{ \left| \sum_{i=1}^{n} \frac{\partial \varrho}{\partial x_i}(p) w_i \right| \right\}$$

that is nonzero.

Then we define

$$\lambda = \frac{-\min_{w \in X_p} \left\{ \sum_{i,j=1}^{n} \frac{\partial^2 \varrho}{\partial x_i \partial x_j}(p) w_i w_j \right\}}{\mu^2}$$

and set $\eta = \varrho_\lambda$. Since $e^{\varrho(p)} = 1$, we have, for any $w \in \mathbb{R}^n$ with $|w| = 1$, that

$$\sum_{i,j=1}^{n} \frac{\partial^2 \eta}{\partial x_i \partial x_j}(p) w_i w_j = \sum_{i,j=1}^{n} \left(\frac{\partial^2 \varrho}{\partial x_i \partial x_j}(p) + \lambda \frac{\partial \varrho}{\partial x_i}(p) \frac{\partial \varrho}{\partial x_j}(p) \right) w_i w_j$$

$$= \sum_{i,j=1}^{n} \frac{\partial^2 \varrho}{\partial x_i \partial x_j}(p) w_i w_j + \lambda \left| \sum_{i=1}^{n} \frac{\partial \varrho}{\partial x_i}(p) w_i \right|^2$$

If $w \notin X_p$ then this expression is positive. If $w \in X_p$ the expression is positive by the choice of λ.
Since the set $S = \{w \in \mathbb{R}^n \big| \ |w| = 1\}$ is compact, we can find a $M > 0$ such that

$$\sum_{j,k=1}^{k} \frac{\partial \eta}{\partial x_i \partial x_j}(p) w_i w_j \geq M \ , \qquad \forall w \in S$$

This give us the inequality (1.6) for $p \in \partial\Omega$ and $w \in \mathbb{R}^n$ such that $|w| = 1$.
If w is an arbitrary point in $\mathbb{R}^n$, we set $w = |w|\widetilde{w}$, with $\widetilde{w} \in S$. Then (1.6) holds for $\widetilde{w} \in S$ and multiplying both side of inequality (with $\widetilde{w} \in S$) by $|w|^2$ we can obtain the result for an arbitrary $w \in \mathbb{R}^n$.
Finally, this estimates hold uniformly in a neighborhood of p, contained in $\partial\Omega$, so, since $\partial\Omega$ is compact, we can choose M uniformly over all boundary points of Ω. □

Proposition 1.3. *If Ω is a bounded strongly convex domain, then Ω is geometrically convex.*

Proof. We consider the set $\Omega \times \Omega$ and is subset defined here:

$$S := \{(\omega_1, \omega_2) \in \Omega \times \Omega \mid (1-\lambda)\omega_1 + \lambda\omega_2 \in \Omega \ , \ for \ \lambda \in \,]0;1[\ \}$$

S is open and non empty.
We prove now it is also closed. We fix a defining function η for Ω such that Lemma 1.2 holds for η.
By contradiction we suppose that S is not closed as subset of $\Omega \times \Omega$. Then there exists a sequence $(\omega_1^j, \omega_2^j) \in S$ which converge to a point $(\omega_1, \omega_2) \in \Omega \times \Omega$ but not in S. By the defintion of S and of defining function, for every j the function $\eta((1-t)\omega_1^j + t\omega_2^j) < 0$, $t \in [0;1]$.
Taking the limit for $j \to \infty$ we obtain $\eta((1-t)\omega_1 + t\omega_2) \leq 0$. So there exists an interior point $\tau \in [0;1]$ such that $eta((1-\tau)\omega_1 + \tau\omega_2) \leq 0$.
This is an interior maximum point on [0; 1] and this fact contradicts the positive definition of the real Hessian of η, so S is also closed. □

Proposition 1.4. *Let Ω be a (weakly) convex set, then Ω is geometrically convex.*

Proof. To simplify the proof we assume that $\partial\Omega$ is, at least, C^3. Moreover we can assume $n \geq 2$ and $0 \in \Omega$ without losing generality.
Then for every $\varepsilon > 0$ and for $M \in \mathbb{N}$ we define

$$\varrho_\varepsilon(x) = \varrho(x) + \frac{\varepsilon|x|^{2M}}{M}$$

where ϱ is a defining function for Ω, and we define

$$\Omega_\varepsilon = \left\{x \in \Omega \;\middle|\; \varrho_\varepsilon(x) < 0\right\}$$

then we have $\Omega_\varepsilon \subset \Omega_{\varepsilon'}$ when $\varepsilon' < \varepsilon$ and $\bigcup_\varepsilon \Omega_\varepsilon = \Omega$.
If we consider M large and ε small then Ω_ε is strongly convex. It follows from previous proposition than every Ω_ε is geometrically covex, so it is also Ω. □

Proposition 1.5. *Let $\Omega \subset \mathbb{R}^n$ have C^2 boundary and be geometrically convex. Then Ω is (weakly) convex.*

Proof. Fixed a defining function ϱ for Ω, we suppose, by contradiction, there exist $p \in \partial\Omega$ and $w \in T_p(\partial\Omega)$ such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varrho}{\partial x_i \partial x_j}(p) w_i w_j = -2K > 0$$

Without losing generality we can assume that the $\mathbb{R}^n$ coordinates are such that $p = 0$ and $(0, 0, \ldots, 0, 1)$ is the outward normal vector to $\partial\Omega$ at p. We may also normalize the defining function so that $\frac{\partial \varrho}{\partial x_n}(p) = 1$
We define $Q = Q^t = tw + \varepsilon(0, 0, \ldots, 0, 1)$, where $\varepsilon > 0$ and $t \in \mathbb{R}$. By Taylor

expansion we have:

$$\begin{aligned}\varrho(Q) =&\varrho(0) + \sum_{i=1}^{n} \frac{\partial \varrho}{\partial x_i}(0)Q_i + \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 \varrho}{\partial x_i \partial x_j}(0)Q_iQ_j + o(|Q|^2)\\
=&\varepsilon \frac{\partial \varrho}{\partial x_n}(0) + \frac{t^2}{2}\sum_{j,k=1}^{n} \frac{\partial^2 \varrho}{\partial x_i \partial x_j}(0)w_iw_j + \mathcal{O}(\varepsilon^2) + o(\varepsilon^2 + t^2)\\
=&\varepsilon - Kt^2 + \mathcal{O}(\varepsilon^2) + o(\varepsilon^2 + t^2).\end{aligned}$$

If $t = 0$ and $\varepsilon > 0$ is small enough we have that $\varrho(Q) > 0$. However, if we consider $|t| > \sqrt{\frac{2\varepsilon}{K}}$ then $\varrho(Q) < 0$. This contradicts the definition of geometric convexity.
Infact if we consider $t_1, t_2 \in \mathbb{R}$ such that $t_1 < -\sqrt{\frac{2\varepsilon}{K}}$ and $t_2 > \sqrt{\frac{2\varepsilon}{K}}$. In this case $\varrho(Q^{t_1}) < 0$ and $\varrho(Q^{t_2}) < 0$ that means $Q^{t_1},\ Q^{t_2} \in \Omega$ and for geometric convexity of Ω also $\lambda Q^{t_1} + (1-\lambda)Q^{t_2}$ belongs to Ω, for every $\lambda \in [0;1]$.
Explicitly we have $\lambda Q^{t_1} + (1-\lambda)Q^{t_2} = (\lambda t_1 + (1-\lambda)t_2)w + \varepsilon(0, 0, \dots, 0, 1)$.
Exploiting the Taylor expansion written before and the previous result, for $\lambda' = \frac{t_2}{t_2 - t_1}$ (< 1), we have $\lambda' t_1 + (1-\lambda')t_2) = 0$ and for $\varepsilon > 0$ small enough we have $\varrho(\lambda' Q^{t_1} + (1-\lambda')Q^{t_2}) > 0$, i.e. $\lambda' Q^{t_1} + (1-\lambda')Q^{t_2} \notin \Omega$. □

The next step is to express the differential condition for convexity in complex notation. If $z \in \mathbb{C}^n$, then the complex coordinates for z are

$$z = (z_1, \dots, z_n) = (x_1 + iy_1, \dots, x_n + iy_n)$$

with $x_i, y_i \in \mathbb{R} \forall i = 1, \dots, n$. Obviusly we have a natural identification between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, given by:

$$(x_1 + iy_1, \dots, x_n + iy_n) \longmapsto (x_1, y_1, \dots, x_n, y_n)$$

Now, fixed a open set $\Omega \subset \mathbb{C}^n$ with C^2 boundary and assumed $\partial\Omega$ is convex at p, if ϱ is a defining function for Ω which is C^2 near p, then the condition that $w = (\xi_1 + i\eta_1, \ldots, \xi_n + \eta_n)$ belongs to $T_p(\partial\Omega)$ is given by:

$$\sum_{i=1}^{n} \frac{\partial \varrho}{\partial x_i}(p)\xi_i + \frac{\partial \varrho}{\partial y_i}(p)\eta_i = 0$$

where $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$.
In complex notation we recall we have:

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\Big) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\Big)$$

by adding this two term we obtain

$$\frac{\partial}{\partial x_j} = \frac{\partial}{\partial z_j} + \frac{\partial}{\partial \overline{z}_j}$$

and subtracting

$$\frac{\partial}{\partial y_j} = \frac{1}{i}\Big(\frac{\partial}{\partial \overline{z}_j} - \frac{\partial}{\partial z_j}\Big)$$

moreover, we have

$$\xi_i = \frac{w_j + \overline{w}_j}{2} \qquad \text{and} \qquad \eta_j = \frac{w_j - \overline{w}_j}{2i}$$

putting all this things together we can rewrite the equation as:

$$\frac{1}{2}\sum_{j=1}^{n}\Big[\Big(\frac{\partial}{\partial z_j}+\frac{\partial}{\partial \overline{z}_j}\Big)\varrho(p)\Big](w_j+\overline{w}_j)+\frac{1}{2}\sum_{j=1}^{n}\Big[\frac{1}{i}\Big(\frac{\partial}{\partial \overline{z}_j}-\frac{\partial}{\partial z_j}\Big)\varrho(p)\Big]\frac{1}{i}(w_j-\overline{w}_j) = 0$$

then by direct calculation we have

$$\frac{1}{2}\sum_{j=1}^{n}\Big[\Big(\frac{\partial}{\partial z_j}+\frac{\partial}{\partial \overline{z}_j}\Big)\varrho(p)\Big](w_j+\overline{w}_j)+\frac{1}{2}\sum_{j=1}^{n}\Big[\frac{1}{i}\Big(\frac{\partial}{\partial \overline{z}_j}-\frac{\partial}{\partial z_j}\Big)\varrho(p)\Big]\frac{1}{i}(w_j-\overline{w}_j)=$$

$$=\frac{1}{2}\sum_{j=1}^{n}\bigg(\frac{\partial\varrho}{\partial z_j}(p)w_j+\frac{\partial\varrho}{\partial z_j}(p)\overline{w}_j+\frac{\partial\varrho}{\partial \overline{z}_j}(p)w_j+\frac{\partial\varrho}{\partial \overline{z}_j}(p)\overline{w}_j$$

$$-\frac{\partial\varrho}{\partial \overline{z}_j}(p)w_j+\frac{\partial\varrho}{\partial \overline{z}_j}(p)\overline{w}_j-\frac{\partial\varrho}{\partial z_j}(p)\overline{w}_j+\frac{\partial\varrho}{\partial z_j}(p)w_j\bigg)$$

$$=\sum_{j=1}^{n}\frac{\partial\varrho}{\partial z_j}(p)w_j+\frac{\partial\varrho}{\partial \overline{z}_j}(p)\overline{w}_j$$

$$=\sum_{j=1}^{n}2Re\bigg(\frac{\partial\varrho}{\partial z_j}(p)w_j\bigg)=2Re\bigg(\sum_{i=1}^{n}\frac{\partial\varrho}{\partial z_i}(p)w_i\bigg)$$

which is equivalent to:

$$2Re\bigg(\sum_{i=1}^{n}\frac{\partial\varrho}{\partial z_i}(p)w_i\bigg)=0.$$

The space of the vector that satisfy this last equation is not closed under multiplication by i, so one prefers to study a slightly different tangent space defined as follow.

Definition 1.4. Let $\Omega\subseteq\mathbb{C}^n$ be a connected open set with C^2 boundary and let ϱ be a C^2 defining function for Ω. Fixed a point $p\in\partial\Omega$ we say that $w\in\mathbb{C}^n$ belongs to the *complex tangent space* to $\partial\Omega$ at p if

$$\sum_{j=1}^{n}\frac{\partial\varrho}{\partial z_j}(p)\cdot w_j=0$$

and we write $w\in\mathcal{T}_p(\partial\Omega)$.

Remark 4. It is quite obvious that $\mathcal{T}_p(\partial\Omega)$ is a linear subspace of $T_p(\partial\Omega)$.

Before going forward with convexity conditions we give a couple of definition that will be useful during the following dissertation.

Definition 1.5. Let $\Omega \subseteq \mathbb{C}$ be an open set and $f : \Omega \longrightarrow \mathbb{C}$ a complex function. Then f is said to be *holomorphic* in $z_0 \in \Omega$ if exists the limit:

$$\lim_{\zeta \to 0} \frac{f(z_0 + \zeta) - f(z_0)}{\zeta} = f'(z_0) \qquad \zeta \neq 0 \ , \ \zeta \in \mathbb{C}$$

We say f is *holomorphic on* Ω if it is holomorphic in every point of Ω.

Remark 5. We recall, taken $z = x + iy$, a function f, considered as a real function, is holomorphic if and only if

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}$$

then it said that f satisfies *the Cauchy-Riemann equation.*

Definition 1.6. Let now be $\Omega \subseteq \mathbb{C}^n$ a domain. A function $f : \Omega \longrightarrow \mathbb{C}^m$ is said to be *holomorphic* if it is locally expandable in Ω as a convergent power series.

Remark 6. The definition can be restated as a function f is holomorphic if and only if satisfies the Cauchy-Riemann equation in each variable separately and it is locally square-integrable.

Definition 1.7. Let Ω_1 and Ω_2 be two open set in $\mathbb{C}^n$, then a function $f : \Omega_1 \longrightarrow \Omega_2$ is said to be *biholomorphic* if it is holomorphic, bijective and its inverse is also holomorphic.

Now we take a look at the convexity condition on tangent vectors, rewriting it in complex notation. If $w \in \mathcal{T}_p(\partial\Omega)$, then

$$0 \leq \sum_{i,j=1}^{n} \frac{\partial^2 \varrho}{\partial x_i \partial x_j}(p)\xi_i\xi_k + 2\sum_{i,j=1}^{n} \frac{\partial^2 \varrho}{\partial x_i \partial y_j}(p)\xi_i\eta_k + \sum_{i,j=1}^{n} \frac{\partial^2 \varrho}{\partial y_i \partial y_j}(p)\eta_i\eta_k$$

$$= \frac{1}{4}\sum_{j,k=1}^{n} \Big(\frac{\partial}{\partial z_i} + \frac{\partial}{\partial \overline{z}_i}\Big)\Big(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \overline{z}_j}\Big)\varrho(p)(w_i + \overline{w}_i)(w_j + \overline{w}_j)$$

$$+ 2 \cdot \frac{1}{4}\sum_{j,k=1}^{n} \Big(\frac{\partial}{\partial z_i} + \frac{\partial}{\partial \overline{z}_i}\Big)\frac{1}{i}\Big(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial \overline{z}_j}\Big)\varrho(p)(w_i + \overline{w}_i)\frac{1}{i}(w_j - \overline{w}_j) \tag{1.3}$$

$$+ \frac{1}{4}\sum_{j,k=1}^{n} \frac{1}{i}\Big(\frac{\partial}{\partial z_i} + \frac{\partial}{\partial \overline{z}_i}\Big)\frac{1}{i}\Big(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial \overline{z}_j}\Big)\varrho(p)\frac{1}{i}(w_i - \overline{w}_i)\frac{1}{i}(w_j - \overline{w}_j)$$

$$= 2Re\Bigg(\sum_{j,k=1}^{n} \frac{\partial^2 \varrho}{\partial z_i \partial z_j}(p)w_i w_j\Bigg) + 2\sum_{j,k=1}^{n} \frac{\partial^2 \varrho}{\partial z_i \partial \overline{z}_j}(p)w_i\overline{w}_j$$

So we can see that the real Hessian, once we write it in complex coordinates, decomposes into two parts.

Our aim now is to prove that the second part does transform canonically under biholomorphyc mappings. We will call it *the complex Hessian* or the *Levi form* of Ω.

Let $\Omega \subset \mathbb{C}^n$ be a convex connected open set with C^2 boundary, let U be a neighborhood of $\overline{\Omega}$ and $\varrho : U \longrightarrow \mathbb{R}$ a defining funcion for Ω. Suppose that $\zeta : U \longrightarrow \zeta(U)$ is biholomorphic and set $\Omega' = \zeta(U)$. Then $\varrho' \equiv \varrho \circ \zeta^{-1}$ is a defining function for Ω' (this result is a consequence of Hopf's lemma). Then fix a point $p \in \partial\Omega$ and its corrisponding $p' = \zeta(p) \in \partial\Omega'$, finally, if $w \in \mathcal{T}_p(\partial\Omega)$, then

$$w' = \Bigg(\sum_{j=1}^{n} \frac{\partial \zeta_1(p)}{\partial z_j}w_j, \ldots, \sum_{j=1}^{n} \frac{\partial \zeta_n(p)}{\partial z_j}w_j\Bigg) \in \mathcal{T}_{p'}(\partial\Omega').$$

Now let fix the complex coordinates on $\zeta(U)$ as $z'_1, \dots, z'_n$, we want to write down the expression (1.3) determining convexity in terms of this coordinates and w'.
We have

$$\frac{\partial^2 \varrho}{\partial z_j \partial z_k}(p) = \frac{\partial}{\partial z_j} \sum_{l=1}^{n} \frac{\partial \varrho'}{\partial z'_l} \frac{\partial \zeta_l}{\partial z_k} = \sum_{l,m=1}^{n} \frac{\partial^2 \varrho'}{\partial z'_m \partial z'_l} \frac{\partial \zeta_m}{\partial z_j} \frac{\partial \zeta_l}{\partial z_k} + \sum_{l=1}^{n} \frac{\partial \varrho'}{\partial z'_l} \frac{\partial^2 \zeta_l}{\partial z_j \partial z_k}$$

$$\frac{\partial^2 \varrho}{\partial z_j \partial \overline{z}_k}(p) = \frac{\partial}{\partial z_j} \sum_{l=1}^{n} \frac{\partial \varrho'}{\partial \overline{z}'_l} \frac{\partial \overline{\zeta}_l}{\partial \overline{z}_k} = \sum_{l,m=1}^{n} \frac{\partial^2 \varrho'}{\partial z'_m \partial \overline{z}'_l} \frac{\partial \zeta_m}{\partial z_j} \frac{\partial \overline{\zeta}_l}{\partial \overline{z}_k}$$

Replacing this result in expression (1.3) we obtain:

$$2Re\left(\sum_{j,k=1}^{n} \frac{\partial^2 \varrho}{\partial z_j \partial z_k}(p) w_j w_k \right) + 2 \sum_{j,k=1}^{n} \frac{\partial^2 \varrho}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k$$

$$= 2Re\left(\sum_{j,k=1}^{n} \sum_{l,m=1}^{n} \frac{\partial^2 \varrho'}{\partial z'_m \partial z'_l}(p') \frac{\partial \zeta_m}{\partial z_j}(p) \frac{\partial \zeta_l}{\partial z_k}(p) w_j w_k + \sum_{j,k=1}^{n} \sum_{l=1}^{n} \frac{\partial \varrho'}{\partial z'_l}(p') \frac{\partial^2 \zeta_l}{\partial z_j \partial z_k}(p) w_j w_k \right)$$

$$+ 2 \sum_{j,k=1}^{n} \sum_{l,m=1}^{n} \frac{\partial^2 \varrho'}{\partial z'_m \partial \overline{z}'_l}(p') \frac{\partial \zeta_m}{\partial z_j}(p) \frac{\partial \overline{\zeta}_l}{\partial \overline{z}_k}(p) w_j \overline{w}_k$$

$$= 2Re\left(\sum_{l,m=1}^{n} \frac{\partial^2 \varrho'}{\partial z'_m \partial z'_l}(p') w'_m w'_l + \sum_{j,k=1}^{n} \sum_{l=1}^{n} \frac{\partial \varrho'}{\partial z'_l}(p') \frac{\partial^2 \zeta_l}{\partial z_j \partial z_k}(p) w_j w_k \right)$$

$$+ 2 \sum_{l,m=1}^{n} \frac{\partial^2 \varrho'}{\partial z'_m \partial \overline{z}'_l}(p') w'_m \overline{w}'_l \tag{1.4}$$

So *the last part of the expression characterizing convexity is preserved under biholomorphic mappings.*

1.2 Pseudoconvexity

Definition 1.8. Let $\Omega \subseteq \mathbb{C}^n$ be a connected open set with C^2 boundary and let ϱ be a C^2 defining function for $\partial\Omega$. Taken a $p \in \partial\Omega$ we say that Ω is *Levi pseudoconvex* at P if

$$\sum_{j,k} \frac{\partial^2 \varrho}{\partial z_j \partial z_k}(p) w_j \overline{w}_k \geq 0 \ , \qquad \forall w \in \mathcal{T}_p(\partial\Omega). \tag{1.5}$$

We say that the point p is *strongly (or strictly) Levi pseudoconvex* if this inequality holds strictly whenever $w \neq 0$.
A set Ω is said to be *Levi pseudoconvex* (*strongly Levi pseudoconvex*) if every $p \in \partial\Omega$ is Levi pseudoconvex (strongly Levi pseudoconvex).

Proposition 1.6. *Let $\Omega \subseteq \mathbb{C}^n$ be a connected open set with C^2 boundary and $p \in \partial\Omega$ one of its point of convexity, then p is also a point of pseudoconvexity.*

Proof. Let ϱ be a defining function for Ω, consider $w \in \mathcal{T}_p(\partial\Omega)$, it follows that also iw belongs to $\mathcal{T}_p(\partial\Omega)$. The hypothesis that p is a convexity point for Ω gives us the expression, in complex notation:

$$2Re\left(\sum_{j,k=1}^{n} \frac{\partial^2 \varrho}{\partial z_i \partial z_j}(p) w_i w_j\right) + 2 \sum_{j,k=1}^{n} \frac{\partial^2 \varrho}{\partial z_i \partial \overline{z}_j}(p) w_i \overline{w}_j \geq 0$$

when we consider w, and:

$$-2Re\left(\sum_{j,k=1}^{n} \frac{\partial^2 \varrho}{\partial z_i \partial z_j}(p) w_i w_j\right) + 2 \sum_{j,k=1}^{n} \frac{\partial^2 \varrho}{\partial z_i \partial \overline{z}_j}(p) w_i \overline{w}_j \geq 0$$

when we take iw. If we add the two inequality we find that

$$4 \sum_{j,k=1}^{n} \frac{\partial^2 \varrho}{\partial z_i \partial \overline{z}_j}(p) w_i \overline{w}_j \geq 0$$

So if w is a point of convexity, it is also a point of pseudoconvexity. $\square$

Lemma 1.7. *Let $\Omega \subseteq \mathbb{R}^n$ be strongly pseudoconvex. Then there is a constant $C > 0$ and a defining function η for Ω such that*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \eta}{\partial z_i \partial \overline{z}_j}(p) w_i w_j \geq C|w|^2 \tag{1.6}$$

for every $p \in \partial\Omega$ and $w \in \mathbb{C}^n$.

Proof. The proof of this Lemma is completely similar to the one of Lemma 1.2. □

Example 1.1. Disks are convex sets then they are also pseudoconvex. Explicity we take the unit disk

$$\Omega = \left\{ z \in \mathbb{C} \;\middle|\; |z| < 1 \right\}$$

where, if we denote $z = x + iy$, $|z| = \sqrt{x^2 + y^2}$. The function $\varrho = |z| - 1$ is a defining function for Ω.
Applying the *Levi form* to ϱ, we get for $p \in \partial\Omega$:

$$\frac{d\varrho}{dz d\overline{z}}(p) w\overline{w} = \frac{1}{4|p|^2}\left(2|p| - \frac{\overline{p}p}{|p|}\right)|w|^2 = \frac{1}{4|p|^2}(2|p| - |p|)|w|^2$$

$$= \frac{1}{4|p|}|w|^2 = \frac{1}{4}|w|^2.$$

So Ω is Levi pseudoconvex.

Example 1.2. Let us consider the set $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^4 < 1\}$. Then the function $\varrho := |z_1|^2 + |z_2|^4 - 1$ is a defining function for Ω. So we apply the *Levi form* to ϱ considering the point $(w_1, w_2) \in \mathbb{C}^2$:

$$\sum_{j,k=1}^{2} \frac{\partial^2 \varrho}{\partial z_i \partial \overline{z}_j}(p) w_i w_j = \frac{\partial}{\partial z_1}\left(\frac{\partial \varrho}{\partial \overline{z}_1}(p) w_1 \overline{w}_1 + \frac{\partial \varrho}{\partial \overline{z}_2}(p) w_1 \overline{w}_2\right)$$

$$+ \frac{\partial}{\partial z_2}\left(\frac{\partial \varrho}{\partial \overline{z}_1}(p) w_2 \overline{w}_1 + \frac{\partial \varrho}{\partial \overline{z}_2}(p) w_2 \overline{w}_2\right)$$

Now if we consider $z_j = x_j + iy_j$ for $j = 1, 2$ and we remember that

$$|z_j| = \sqrt{x_j^2 + y_j^2}$$

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$$

$$\frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

we obtain

$$\begin{aligned}\varrho(z_1, z_2) = \varrho(x_1, y_1, x_2, y_2) =& x_1^2 + y_1^2 + (x_2^2 + y_2^2)^2 - 1\\ & x_1^2 + y_1^2 + x_2^4 + y_2^4 + 2x_2^2y_2^2 - 1\end{aligned}$$

and

$$\frac{\partial \varrho}{\partial \overline{z}_1} = \frac{1}{2}\left(\frac{\partial \varrho}{\partial x_1} + i\frac{\partial \varrho}{\partial y_1}\right) = \frac{1}{2}\Big(2x_1 + 2iy_1\Big) = x_1 + iy_1 = z_1$$

$$\frac{\partial}{\partial \overline{z}_2} = \frac{1}{2}\left(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial y_2}\right) = \frac{1}{2}\Big(4x_2^3 + 4x_2y_2^2 + 4iy_2^3 + 4ix_2^2y_2\Big)$$

then the *Levi form* is given by:

$$\begin{aligned}\sum_{j,k=1}^{2} \frac{\partial^2 \varrho}{\partial z_i \partial \overline{z}_j}(p) w_i w_j = w_1\overline{w}_1 + \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)\left(\frac{\partial \varrho}{\partial \overline{z}_2}\right) w_2\overline{w}_2\\ =& w_1\overline{w}_1 + \frac{1}{4}\Big(12x_2^2 + 4y_2^2 + 8ix_2y_2 - 8ix_2y_2 + 12y_2^2 + 4x_2^2\Big) w_2\overline{w}_2\\ =& w_1\overline{w}_1 \frac{1}{4}\Big(16x_2^2 + 16y_2^2\Big) w_2\overline{w}_2 = |w_1|^2 + 4(x_2^2 + y_2^2)|w_2|^2\\ =& |w_1|^2 + 4|z_2|^2|w_2|^2\end{aligned}$$

So Ω is Levi pseudoconvex. Moreover $\partial\Omega$ is strongly pseudoconvex except at the boundary points where $|z_2|^2 = 0$ and the tangent vectors satisfy $w_1 = 0$. These point are of the form $(e^{i\theta}, 0)$.

Lemma 1.8 (*Narasimhan*).
Let $\Omega \subseteq \mathbb{C}^n$ be a connected open set with C^2 boundary and p a point of strong pseudoconvexity. Then there exists a neighborhood of p, $U \subseteq \mathbb{C}^n$ such that and a biholomorphic function ζ on U, such that $\zeta(U \cap \partial\Omega)$ is strongly convex.

Proof. Thanks to the previous proposition we know there exists a defining function η of Ω such that:

$$\sum_{j,k=1}^{n} \frac{\partial^2 \eta}{\partial x_i \partial x_j}(p) w_i w_j \geq C|w|^2 \qquad \forall w \in \mathbb{C}^n$$

We may assume that $P = 0$ and the unit outward normal vector to $\partial\Omega$ in p $\nu = (1, 0, \ldots, 0)$, this assumption can be obtained through rotations and translations of coordinates; all the given definitions are invariant under biholomorphic transformations, so they are invariant under translations or unitary complex transformation.
We consider now Taylor expansion of η near $p = 0$:

$$\begin{aligned}
\eta(w) = \eta(0) &+ \sum_{j=1}^{n} \frac{\partial \eta}{\partial z_j}(0) w_j + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \eta}{\partial z_j \partial z_k}(0) w_j w_k \\
&+ \sum_{j=1}^{n} \frac{\partial \eta}{\partial \overline{z}_j}(0) \overline{w}_j + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \eta}{\partial \overline{z}_j \partial \overline{z}_k}(0) \overline{w}_j \overline{w}_k \\
&+ \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \eta}{\partial z_j \partial \overline{z}_k}(0) w_j \overline{w}_k + o(|w|^2) \qquad (1.7) \\
&= 2Re\left(\sum_{j=1}^{n} \frac{\partial \eta}{\partial z_j}(0) w_j + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \eta}{\partial z_j \partial z_k}(0) w_j w_k \right) \\
&+ \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \eta}{\partial z_j \partial \overline{z}_k}(0) w_j \overline{w}_k + o(|w|^2)
\end{aligned}$$

$$= 2Re\left(w_1 + \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 \eta}{\partial z_j \partial z_k}(0) w_j w_k \right)$$
$$+ \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 \eta}{\partial z_j \partial \overline{z}_k}(0) w_j \overline{w}_k + o(|w|^2) \tag{1.8}$$

We now define the map ζ as follow:

$$\begin{aligned} w_1' =& \zeta_1(w) = w_1 + \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 \eta}{\partial z_j \partial z_k}(0) w_j w_k \\ w_2' =& \zeta_2(w) = w_2 \\ \vdots \quad & \quad \vdots \\ w_n' =& \zeta_n(w) = w_n. \end{aligned}$$

By the implicit function theorem, we have that for small enough w this is a well-posed invertible holomorphic map on a neighborhood of p. Through the equation (1.8), we can express the defining function in terms of the coordinates w':

$$\tilde{\eta}(w') = 2Re(w_1) + \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 \eta}{\partial z_j' \partial \overline{z}_k'}(0) w_j' \overline{w}_k' + o(|w|^2).$$

So the real Hessian at p of $\tilde{\eta}$ is the Levi form and this one is definite positive by hypothesis.
Then the boundary of $\zeta(W \cap \Omega)$ is strictly convex at $\zeta(p)$ and by continuity of the second derivatives of η, we may conclude that the boundary of $\zeta(W \cap \Omega)$ is strictly convex in a neighborhood V of $\zeta(p)$. To complete the proof we choose a neighborhood $U \subseteq W$ of p such that $\zeta(U) \subseteq V$. $\square$

Chapter 2

Levi Curvature

The purpose of this chapter is to introduce the Levi Curvature and to understand its geometrical content. We will start with some notations and with the very definition and we will finish up with an isoperimetric estimates, which bond together Levi curvature and set's measure.

Let's start with some notations. Hereafter we shall denote with Ω a connected open set such that $\Omega := \{z \in \mathbb{C}^{n+1} \mid f(z) < 0\}$, where $f \in C^2$ is its defining function and $\partial\Omega := \{z \in \mathbb{C}^n + 1 \mid f(z) = 0\}$ is a Real manifold.
We will write

$$f_j = f_{z_j} = \frac{\partial f}{\partial z_j}$$

in our hypothesis f is a real value function and $\partial_p f := (f_1(p), \dots, f_{n+1}(p)) \neq 0$ at any point $p \in \partial\Omega$.
We shall also denote by $\mathcal{T}_p(\partial\Omega)$ the complex tangent space to $\partial\Omega$ at point p

$$\mathcal{T}_p(\partial\Omega) = \Big\{ w \in \mathbb{C}^n + 1 \;\Big|\; \sum_{j=1}^{n+1} \frac{\partial f}{\partial z_j} w_j = 0 \Big\}.$$

We finally recall the *Levi form* defined by

$$L_p(f, w) := \langle \mathscr{H}_p^t(f) w, w \rangle = \sum_{j,k=1}^{n+1} \frac{\partial^2 f}{\partial z_i \partial \overline{z}_j}(p) w_i \overline{w}_j$$

Levi form, restricted to $\mathcal{T}_p(\partial\Omega)$, is invariant under biholomorphic maps and recall that a domain Ω is *Levi pseudoconvex* if Levi form of f is strictly positive definite ad any point $p \in \partial\Omega$.

2.1 Levi Total Curvature

Definition 2.1. Let $B = \{u_1, \dots, u_n\}$ be an othonormal basis of $\mathcal{T}_p(\partial\Omega)$. We define the *B-normalized Levi form* of $\partial\Omega$ at p the matrix

$$L_p(f, B) := \frac{1}{||\nabla_p f||}\Big(\langle \mathcal{H}_p^t(f)u_j, u_k\rangle\Big)_{j,k=1,\dots,n}$$

and we will denote by $\lambda_1(p), \dots, \lambda_n(p)$ the eigenvalues of this matrix.

Proposition 2.1. *The eigenvalues of the normalized Levi form don't depend on the defining function f and the basis B. The only depend on the domain Ω.*

Proof. Let $B = \{u_1, \dots, u_n\}$ and $B' = \{v_1, \dots, v_n\}$ be two othornormal basis of $\mathcal{T}_p(\partial\Omega)$ and f and f' two defining function for Ω.
We shall denote by U the matrix with columns the vector of B:

$$U = [u_1, \dots, u_n]$$

and by $A(U)$ the matrix $(\langle \mathscr{H}_p^t(f)u_j, u_k\rangle)_{j,k=1,\dots,n}$. Then we have

$$A^t(U) = U^* \mathscr{H}_p^t(f) U$$

where $U^* = [\overline{u}_1, \dots, \overline{u}_n]$.
Let now consider the matrix $V = [v_1, \dots, v_n]$ related to the other othonormal basis B' of $\mathcal{T}_p(\partial\Omega)$, there exists a $n \times n$ othornormal matrix R such that $V = UR$, then

$$A^t(V) = R^* U^* \mathscr{H}_p^t(f) U R = R * A^t(U) R$$

so that $A(V)$ and $A(U)$ have the same eigenvalues.
Since f and f' are defining function of Ω, there exists a function $h \in C^1$, strictly positive in a neighborhood of p, such that $f' = hf$. It follows that

$$f'_{\bar{j},k}(p) = h(p) f_{\bar{j},k}(p) + h_{\bar{j}} f_h(p) + h_k(p) f_{\bar{j}}(p).$$

Hence, for every $\zeta \in \mathcal{T}_p(\partial\Omega)$

$$\begin{aligned}\langle \mathscr{H}_p^t(f')\zeta, \zeta\rangle =& h(p)\langle \mathscr{H}_p^t(f)\zeta,\zeta\rangle + 2Re\big(\langle \zeta, \overline{\nabla}_p f\rangle \langle \overline{\nabla}_p h, \zeta\rangle\big)\\ \\ =& h(p)\langle \mathscr{H}_p^t(f')\zeta,\zeta\rangle.\end{aligned}$$

But we have $\nabla_p f' = h(p)\nabla_p f$, then

$$\frac{1}{||\partial_p f'||}\langle \mathscr{H}_p^t(f')\zeta,\zeta\rangle = \frac{1}{||\partial_p f||}\langle \mathscr{H}_p^t(f)\zeta,\zeta\rangle$$

for every $\zeta \in \mathcal{T}_p(\partial\Omega)$. □

Definition 2.2. Let $\Omega \subseteq \mathbb{C}^{n+1}$ be a connected open set, it's said *q-pseudoconvex* if $\forall j \in \{1, \ldots, q\}$

$$\sigma^{(j)}(\lambda_1(p), \ldots, \lambda_{n+1}(p)) := \sum_{1 \le i_1 < \ldots < i_j \le n+1} \lambda_{i_1} \ldots \lambda_{i_j} > 0$$

at every point $p \in \partial\Omega$.
We call the function $\sigma^{(j)}$ *jth elementary symmetric function.*

Definition 2.3. For every $q \in \{1, \ldots, n+1\}$ we define *q-curvature* of Ω

$$K^{(q)}_{\partial\Omega}(p) = \frac{1}{\begin{pmatrix} n \\ j \end{pmatrix}}\, \sigma^{(q)}(\lambda)$$

When $q = n$ we have

$$K^{(n)}_{\partial\Omega}(p) = \prod_{j=1}^{n+1} \lambda_j(p)$$

and in this case we call it *Levi total curvature.*

Remark 7. Levi total curvature can be considered as the complex analogous of the Gauss curvature.

Example 2.1. Let us consider the ball $B_R = \{z \in \mathbb{C}^n \mid |z|^2 < R^2\}$ and let $f(z) = |z|^2 - R^2$ be its defining function, then we have

$$L_p(f, B) = \frac{1}{R} I_n \qquad \forall p \in \partial B_R,$$

for any orthonormal basis of $\mathcal{T}_p(\partial\Omega)$. So, all the eigenvalues of the normalized Levi form are equal to $\frac{1}{R}$ and

$$K^{(q)}_{\partial B_R}(p) = \left(\frac{1}{R}\right)^q \tag{2.1}$$

for every $p \in \partial B_R$.

Remark 8. If Ω is bounded domain of $\mathbb{C}^{n+1}$ with boundary a C^2 real hypersurface and f is its defining function, then the *j-th Levi curvature* of $\partial\Omega$ in $z = (z_1, \ldots, z_{n+1}) \in \partial\Omega$ is given by

$$K^{(q)}_{\partial\Omega}(z) = \frac{1}{\begin{pmatrix} n \\ j \end{pmatrix}} \frac{1}{||\partial f||^{j+2}} \sum_{1 \le i_1 < \ldots < i_{j+1} \le n+1} \Delta_{(i_1, \ldots, i_{j+1})}(f)$$

for every $j = 1, \ldots, n+1$, where

$$||\partial f|| = \sqrt{\sum_{j=1}^{n} |f_j|^2},$$

$$\Delta_{(i_1, \ldots, i_{j+1})}(f) = \det \begin{pmatrix} 0 & f_{\overline{i_1}} & \cdots & f_{\overline{i_{j+1}}} \\ f_{i_1} & f_{i_1, \overline{i_1}} & \cdots & f_{i_1, \overline{i_{j+1}}} \\ \vdots & \vdots & \ddots & \vdots \\ f_{i_{j+1}} & f_{i_{j+1}, \overline{i_1}} & \cdots & f_{i_{j+1}, \overline{i_{j+1}}} \end{pmatrix}$$

with $f_{\bar{j}} = \overline{f_j}$ and $f_{j\bar{l}} = \frac{\partial^2 f}{\partial z_j \partial \overline{z_l}}$.

If we consider the Levi total curvature we obtain

$$K^{(n)}_{\partial\Omega}(p) = -\frac{1}{||\nabla_p f||^{n+2}} \det \begin{pmatrix} 0 & f_{\bar{1}} & \dots & f_{\overline{n+1}} \\ f_1 & f_{1,\bar{1}} & \dots & f_{1,\overline{n+1}} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n+1} & f_{n+1,\overline{n+1}} & \dots & f_{n+1,\overline{n+1}} \end{pmatrix}. \tag{2.2}$$

Example 2.2. If we now consider the cylinder

$$C_R = \left\{ (z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \;\middle|\; \sum_{j=1}^{n+1} |z_j|^2 < R^2 \right\}$$

from (2.2) we get

$$K^{(n)}_{\partial C_R}(p) = 0$$

for every $p \in \partial C_R$.

There exists some cylinder-like domains whose boundaries have strictly positive Levi total curvature, for instance, if we take

$$C^*_R = \left\{ (z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} \;\middle|\; \sum_{j=1}^{n+1} \left(\frac{z_j + \bar{z}_j}{2}\right)^2 < R^2 \right\}$$

then we have

$$K^{(n)}_{\partial C^*_R}(p) = \left(\frac{1}{2R}\right)^n$$

for every $p \in \partial C^*_R$.

We'll now see some integral formulas for compact hypersurfaces which relate elementary symmetric functions in the eigenvalues of the complex Hessian matrix of the defining function and the Levi curvatures of the hypersufaces. We consider H the $(n+1) \times (n+1)$ Hessian matrix, with eigenvalues $\lambda_1, \dots, \lambda_{n+1}$ and let $\sigma^{(j)}(H)$ be the jth elementary symmetric function in the eigenvalues of H:

$$\sigma^{(j)}(H) = \sum_{1 \le i_1 < \dots < i_j \le n+1} \lambda_{i_1} \dots \lambda_{i_j}$$

if we denote $H = (h_{l\bar{k}}) = \partial\overline{\partial}f$ and by $\frac{\partial\sigma^{(j)}(H)}{\partial h_{l\bar{k}}}$ the partial derivative of the function $\sigma^{(j)}$ with respect to the term of place $l\bar{k}$, we have, for all $j = 0, 1, \dots, n$:

$$\sum_{l=1}^{n+1} \partial_l \left(\frac{\partial\sigma^{(j+1)}(\partial\overline{\partial}f)}{\partial h_{l\bar{k}}} \right) = 0 \ , \qquad \forall k = 1, \dots, n+1$$

we also know, by [6], that

$$\sum_{l=1}^{n+1} \partial_l \left(\frac{\partial\sigma^{(j+1)}(\partial\overline{\partial}f)}{\partial h_{l\bar{k}}} \right) = \sum_{l=1}^{n+1} \partial_l \, \frac{1}{j!} \sum \delta \begin{pmatrix} i_1 & \dots & i_j & l \\ j_1 & \dots & j_j & k \end{pmatrix} f_{i_1\overline{j_1}} \cdots f_{i_j\overline{j_j}}$$

$$= \sum_{l=1}^{n+1} \frac{1}{j!} \sum \delta \begin{pmatrix} i_1 & \dots & i_j & l \\ j_1 & \dots & j_j & k \end{pmatrix} \left(j f_{i_1\overline{j_1}} \cdots f_{i_j\overline{j_j}l} \right)$$

where $1 \leq i_1, \dots, i_j, j_1, \dots, j_j \leq n+1$ and the Kronecker symbol δ assumes value 1 (-1 respectively) if $(i_1, \dots, i_j, l)$ are distinct and $(j_1, \dots, j_j, k)$ is a even permutation (an odd permutation respectively) of $(i_1, \dots, i_j, l)$, otherwise it has value 0. We also note that $f_{i_j\overline{j_j}l}$ is symmetric with respect to i_j, l if the Kronecker symbol is skew symmetric in those indices. So this sum is equal to zero.

Lemma 2.2. *For every $f \in C^2$ and for every $j = 1, \dots, n+1$, we have*

$$\sum_{l,k=1}^{n+1} \frac{\partial\sigma^{(j+1)}}{\partial h_{l\bar{k}}}(\partial\overline{\partial}f) \ f_l f_{\bar{k}} = - \sum_{1 \leq i_1 < \dots < i_j \leq n+1} \Delta_{(1, i_1, \dots, i_{j+1})}(f).$$

Proof. Writing down explicitly $\sigma^{(j)}$ we obtain

$$\sigma^{(j)}(\partial\overline{\partial}f) = \sum_{1 \leq i_1 < \dots < i_j + 1 \leq n+1} \begin{vmatrix} f_{i_1,\overline{i_1}} & \dots & f_{i_1,\overline{i_{j+1}}} \\ \vdots & \ddots & \vdots \\ f_{i_{j+1},\overline{i_1}} & \dots & f_{i_{j+1},\overline{i_{j+1}}} \end{vmatrix}$$

then we can rewrite the right hand side of the equality

$$\sum_{l,k=1}^{n+1} \frac{\partial \sigma^{(j+1}}{\partial h_{l\overline{k}}}(\partial\overline{\partial} f) f_l f_{\overline{k}} = \sum_{l,k=1}^{n+1} f_l f_{\overline{k}} \sum_{1\le i_1<\ldots<i_j\le n+1} \frac{\partial}{\partial h_{l\overline{k}}} \begin{vmatrix} f_{i_1,\overline{i_1}} & \cdots & f_{i_1,\overline{i_{j+1}}} \\ \vdots & \ddots & \vdots \\ f_{i_{j+1},\overline{i_1}} & \cdots & f_{i_{j+1},\overline{i_{j+1}}} \end{vmatrix}$$

$$= \sum_{1\le i_1<\ldots<i_j\le n+1} \quad \sum_{l,k\in\{i_1,\ldots,i_{j+1}\}} \frac{\partial}{\partial h_{l\overline{k}}} \begin{vmatrix} f_{i_1,\overline{i_1}} & \cdots & f_{i_1,\overline{i_{j+1}}} \\ \vdots & \ddots & \vdots \\ f_{i_{j+1},\overline{i_1}} & \cdots & f_{i_{j+1},\overline{i_{j+1}}} \end{vmatrix} f_l f_{\overline{k}}. \qquad (2.3)$$

On the other hand, if we call $F(\partial f, \overline{\partial} f, \partial\overline{\partial} f) = -\Delta_{(1,i_1,\ldots,i_{j+1})}(f)$, we have

$$\frac{\partial F}{\partial f_{i_l}} = (-1)^{l+1} \begin{vmatrix} f_{i_1} & \cdots & f_{\overline{i_{j+1}}} \\ f_{i_1,\overline{i_1}} & \cdots & f_{i_1,\overline{i_{j+1}}} \\ \vdots & \ddots & \vdots \\ f_{i_{l-1},\overline{i_1}} & \cdots & f_{i_{l-1},\overline{i_{j+1}}} \\ f_{i_{l+1},\overline{i_1}} & \cdots & f_{i_{l+1},\overline{i_{j+1}}} \\ \vdots & \ddots & \vdots \\ f_{i_{j+1},\overline{i_1}} & \cdots & f_{i_{j+1},\overline{i_{j+1}}} \end{vmatrix}$$

$$\frac{\partial^2 F}{\partial f_{i_l} \partial f_{\overline{i_k}}} = (-1)^{l+k} \begin{vmatrix} f_{i_1,\overline{i_1}} & \cdots & f_{i_1,\overline{i_{k-1}}} & f_{i_1,\overline{i_{k+1}}} & \cdots & f_{i_1,\overline{i_{j+1}}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{i_{l-1},\overline{i_1}} & \cdots & f_{i_{l-1},\overline{i_{k-1}}} & f_{i_{l-1},\overline{i_{k+1}}} & \cdots & f_{i_{l-1},\overline{i_{j+1}}} \\ f_{i_{l+1},\overline{i_1}} & \cdots & f_{i_{l+1},\overline{i_{k-1}}} & f_{i_{l+1},\overline{i_{k+1}}} & \cdots & f_{i_{l+1},\overline{i_{j+1}}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{i_{j+1},\overline{i_1}} & \cdots & f_{i_{j+1},\overline{i_{k-1}}} & f_{i_{j+1},\overline{i_{k+1}}} & \cdots & f_{i_{j+1},\overline{i_{j+1}}} \end{vmatrix}$$

$$= \frac{\partial}{\partial h_{l\overline{k}}} \begin{vmatrix} f_{i_1,\overline{i_1}} & \cdots & f_{i_1,\overline{i_{j+1}}} \\ \vdots & \ddots & \vdots \\ f_{i_{j+1},\overline{i_1}} & \cdots & f_{i_{j+1},\overline{i_{j+1}}} \end{vmatrix}.$$

Furthermore we have

$$\begin{aligned}
F(\partial f, \overline{\partial} f, \partial\overline{\partial} f) =& F(\partial f, \overline{\partial} f, \partial\overline{\partial} f) - F(0, \overline{\partial} f, \partial\overline{\partial} f) \\
=& \int_0^1 \frac{dF}{ds}(s\partial f, \overline{\partial} f, \partial\overline{\partial} f) ds \\
=& \int_0^1 \sum_{l\in\{i_1,\dots,i_{j+1}\}} \frac{\partial F}{\partial f_l}(\partial f, \overline{\partial} f, \partial\overline{\partial} f) f_l ds \\
=& \sum_{l\in\{i_1,\dots,i_{j+1}\}} \frac{\partial F}{\partial f_l}(\partial f, \overline{\partial} f, \partial\overline{\partial} f) f_l \int_0^1 ds \\
=& \sum_{l\in\{i_1,\dots,i_{j+1}\}} \frac{\partial F}{\partial f_l}(\partial f, \overline{\partial} f, \partial\overline{\partial} f) f_l
\end{aligned}$$

using the same argument and the previous results, we obtain

$$\begin{aligned}
F(\partial f, \overline{\partial} f, \partial\overline{\partial} f) &= \sum_{l\in\{i_1,\dots,i_{j+1}\}} \frac{\partial^2 F}{\partial f_l \partial f_{\overline{k}}}(\partial f, \overline{\partial} f, \partial\overline{\partial} f) f_l f_{\overline{k}} \\
&= \sum_{l\in\{i_1,\dots,i_{j+1}\}} \frac{\partial}{\partial h_{l\overline{k}}} \begin{vmatrix} f_{i_1,\overline{i_1}} & \dots & f_{i_1,\overline{i_{j+1}}} \\ \vdots & \ddots & \vdots \\ f_{i_{j+1},\overline{i_1}} & \dots & f_{i_{j+1},\overline{i_{j+1}}} \end{vmatrix} f_l f_{\overline{k}}.
\end{aligned}$$

Substituing this result into (2.3), we have proved the Lemma.

□

We also have the following theorem.

Theorem 2.3. *Let Ω be a bounded domain of $\mathbb{C}^{n+1}$ with boundary a real C^2 hypersurface. For every $f \in C^2$ that is a defining function for Ω and for* *every $j = 1, \ldots, n$, we have*

$$\int_{\Omega} \sigma^{(j+1)}(\partial\overline{\partial}f)dx = \binom{n+1}{j+1} \frac{1}{2(n+1)} \int_{\partial\Omega} K^{(j)}_{\partial\Omega}(z)||\partial f||^{j+1} d\sigma(x) \tag{2.4}$$

where $K^{(j)}_{\partial\Omega}$ is the jth Levi curvature of $\partial\Omega$.

Proof. $\sigma^{(j)}$ is an homogenous function of degree j, this means that for every $t \in \mathbb{R}$ $\sigma^{(j)}(tA) = t^j \sigma^{(j)}(A)$, due to this property we get

$$\sigma^{(j+1)}(\partial\overline{\partial}f) = \frac{1}{j+1} \sum_{l,k=1}^{n+1} \frac{\partial \sigma^{(j+1)(\partial\overline{\partial}f}}{\partial h_{l\overline{k}}} f_{l\overline{k}}.$$

We now call $\nu_l = \frac{\partial_l f}{||\partial f||}$ and we identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2(n+1)}$, then through the previous equations and the classical divergence theorem we obtain

$$\begin{aligned}
\int_{\Omega} \sigma^{(j+1)}(\partial\overline{\partial}f)dx =& \frac{1}{j+1} \int_{\Omega} \sum_{l,k}^{n+1} \partial_l \left(\frac{\partial \sigma^{(j+1)}}{\partial h_{l,\overline{k}}} (\partial\overline{\partial}f) f_{\overline{k}} \right) dx \\
=& \frac{1}{2(j+1)} \int_{\partial\Omega} \sum_{l,k}^{n+1} \left(\frac{\partial \sigma^{(j+1)}}{\partial h_{l,\overline{k}}} (\partial\overline{\partial}f) f_{\overline{k}} \nu_l \right) d\sigma(x) \\
=& \frac{1}{2(j+1)} \int_{\partial\Omega} \sum_{l,k}^{n+1} \frac{\left(\frac{\partial \sigma^{(j+1)}}{\partial h_{l,\overline{k}}} (\partial\overline{\partial}f) f_{\overline{k}} \nu_l \right)}{||\partial f||} d\sigma(x) \\
=& -\frac{1}{2(j+1)} \int_{\partial\Omega} \frac{\sum_{1 \le i_1 < \ldots < i_{j+1} \le n+1} \Delta_{(i_1, \ldots, i_j+1)}(f)}{||\partial f||} d\sigma(x) \\
=& -\binom{n}{j} \frac{1}{2(j+1)} \int_{\partial\Omega} K^{(j)}_{\partial\Omega}(z)||\partial f||^{j+1} d\sigma(x) \\
=& -\binom{n+1}{j+1} \frac{1}{2(n+1)} \int_{\partial\Omega} K^{(j)}_{\partial\Omega}(z)||\partial f||^{j+1} d\sigma(x).
\end{aligned}$$

□

We will use this integral formula to get an estimate of the jth Levi curvature and to prove the following theorem.

Theorem 2.4 (Isoperimetric estimates)**.**
Let $\Omega \in \mathbb{C}^{n+1}$ be a strictly bounded domain with boundary a real C^∞-hypersurface. If $K^{(j)}_{\partial\Omega}(z)$ is non negative at any point $z \in \partial\Omega$, then

$$\int_{\partial\Omega} \left(\frac{1}{K^{(j)}_{\partial\Omega}(\zeta)}\right)^{\frac{1}{j}} d\sigma(\zeta) \geq 2(n+1)|\Omega|$$

where $|\Omega|$ stands for the Lebesgue measure of Ω.
We have the equality, for $K^{(j)}_{\partial\Omega}$ constant, if and only if Ω is a ball of radius $\left(\frac{1}{K^{(j)}_{\partial\Omega}(z)}\right)^{\frac{1}{j}}$.

Proof. If $\int_{\partial\Omega}\left(\frac{1}{K^{(j)}_{\partial\Omega}(\zeta)}\right)^{\frac{1}{j}} d\sigma(\zeta) = +\infty$ we have nothing to prove, so we assume that $\int_{\partial\Omega}\left(\frac{1}{K^{(j)}_{\partial\Omega}(\zeta)}\right)^{\frac{1}{j}} d\sigma(\zeta) < +\infty$.
Let now $f : \overline{\Omega} \longrightarrow \mathbb{R}$ be the $C^2(\Omega)$ solution of the Dirichlet problem

$$\begin{cases} tr(\partial\overline{\partial} f) = 1 & \text{in } \Omega; \\ f = 0 & \text{in } \partial\Omega. \end{cases} \tag{2.5}$$

we recall that $tr(\partial\overline{\partial}) = \frac{1}{4}\Delta$, where Δ is the usual Laplace operator over $\mathbb{R}^{2n+2}$. If $\partial\Omega$ is $C^{2,\alpha}$, then this Dirichlet problem has a unique solution $f \in C^2(\overline{\Omega})$. We also recall that for every $(n+1)\times(n+1)$ Hermitian matrix A the Newton inequality holds

$$\sigma_j \leq \binom{n+1}{j}\left(\frac{tr(A)}{n+1}\right)^j$$

Moreover, this inequality holds if and only if the matrix A is proportional to the identity matrix. Applying this inequality to the complex Hessian matrix of f, where f is a solution of the Dirichlet problem (2.5), we obtain an estimate of the left side of (2.4)

$$\int_\Omega \sigma_{j+1}(\partial\overline{\partial} f)dx \leq \binom{n+1}{j}\frac{1}{(n+1)^{j+1}}\int_\Omega \left(tr(\partial\overline{\partial} f)\right)^{j+1}dx$$
$$= \binom{n+1}{j}\frac{|\Omega|}{(n+1)^{j+1}}$$

Applying again the divergence theorem and calling N the unit outward unit normal vector, we get

$$\int_{\partial\Omega}|\partial f|d\sigma(x) = \frac{1}{2}\int_{\partial\Omega}\langle\nabla f, N\rangle d\sigma(x) = \frac{1}{2}\int_\Omega \Delta f dx = 2|\Omega|$$

and using the Cauchy-Schwarz inequality in the right side of (2.4), we obtain

$$\binom{n+1}{j+1}\frac{1}{2(n+1)}\int_{\partial\Omega}K^{(j)}_{\partial\Omega}|\partial f|^{j+1}d\sigma(x) \geq \frac{\binom{n+1}{j+1}\left(\int_{\partial\Omega}|\partial f|d\sigma(x)\right)^{j+1}}{2(n+1)\left(\int_{\partial\Omega}\left(\frac{1}{K^{(j)}_{\partial\Omega}}\right)^{\frac{1}{j}}d\sigma(x)\right)^j}$$

$$= \frac{\binom{n+1}{j+1}(2|\Omega|)^{j+1}}{2(n+1)\left(\int_{\partial\Omega}\left(\frac{1}{K^{(j)}_{\partial\Omega}}\right)^{\frac{1}{j}}d\sigma(x)\right)^j}$$

the equality holds if and only if $|\partial f|$ is proportional to $\left(\frac{1}{K^{(j)}_{\partial\Omega}}\right)$.By equality (2.4) and by those two inequality we infer

$$\frac{(2|\Omega|)^j}{2(n+1)\left(\int_{\partial\Omega}\left(\frac{1}{K^{(j)}_{\partial\Omega}}\right)^{\frac{1}{j}}d\sigma(x)\right)^j} \leq \frac{1}{(n+1)^j}$$

and we obtain

$$\int_{\partial\Omega}\left(\frac{1}{K^{(j)}_{\partial\Omega}}\right)^{\frac{1}{j}}d\sigma(x) \geq 2(n+1)|\Omega|.$$

We have to prove now that the equality holds, for $K^{(j)}_{\partial\Omega}$ constant, if and only if Ω is a ball of radius $\left(\frac{1}{K^{(j)}_{\partial\Omega}(z)}\right)^{\frac{1}{j}}$.

We know that we have the equality if and only if the complex Hessian matrix of f is proportional to the identity matrix. The defining function for Ω has been chosen such that $tr(\partial\overline{\partial}f) = 1$, then $\partial\overline{\partial}f = \frac{1}{n+1}I$ in $\overline{\Omega}$ and by the characterization of Levi Curvature we have

$$\left(K^{(j)}_{\partial\Omega}\right)^{\frac{1}{j}} = \frac{1}{(n+1)|\partial f|} \quad \text{on } \partial\Omega \tag{2.6}$$

this equality is not enough to conclude that Ω is a ball. In fact, for every pluriharmonic [1] function h and for every constant R, the function

$$f(z) = -R^2 + \frac{1}{n+1}|z|^2 + h(z)$$

satisfies $\partial\overline{\partial}f = \frac{1}{n+1}I$. If we take

$$h(z_1, \dots, z_{n+1}) = \frac{Re(z_1^2 + \ldots + z_{n+1}^2)}{2(n+1)}$$

then h is plurisubharmonic and the set of the zeroes of the function

$$f(z_1, \dots, z_{n+1}) = -R^2 + \frac{3}{2(n+1)}\sum_{j=1}^{n+1}\left(Re(z_j)\right)^2 + \frac{1}{2(n+1)}\sum_{j=1}^{n+1}\left(Im(z_j)\right)^2$$

is not a sphere, it's an ellipsoid for every $R \neq 0$.
However, if $K^{(j)}_{\partial\Omega}$ is constant for some j, then by (2.6) $|\partial f|$ should be constant on $\partial\Omega$. It follows that the Dirichlet problem (2.5) is over determinate and by Serrin's theorem [7] we can conclude that Ω is a ball and $\partial\Omega$ is a sphere. □

Remark 9. If $K^{(j)}_{\partial\Omega}$ is constant, then we have

$$\left(K^{(j)}_{\partial\Omega}\right)^{\frac{1}{j}} \leq \frac{|\partial\Omega|}{2(n+1)|\Omega|} \tag{2.7}$$

Giving for known the definition of Euclidean mean curvature we end this section with a quite important symmetry theorem.

[1]Let $f : \Omega \to \mathbb{C}$ a C^2 function. f is said to be *pluriharmonic* if for every complex line $l = \{a + b\zeta\}$ the function $\zeta \to f(a + b\zeta)$ is harmonic on the set $\Omega_l = \{\zeta \in \mathbb{C} \mid a + b\zeta \in \Omega\}$.

Remark 10. Let H be the Euclidean mean curvature of $\partial\Omega$. We recall the Minkowski formula

$$\int_{\partial\Omega} d\sigma = \int_{\partial\Omega} H(x)\langle\nu, x\rangle d\sigma(x) \tag{2.8}$$

where ν is the outward unit normal.

Theorem 2.5. *Let $\Omega \subseteq \mathbb{C}^{n+1}$ be a bounded star-shaped domain with boundary a smooth real hyper surface. If the j-Levi curvature is a constant $K^{(j)} > 0$ at every point of $\partial\Omega$, then the maximum of the Euclidean mean curvature of $\partial\Omega$ is bounded from below by $\left(K^{(j)}\right)^{\frac{1}{j}}$. Moreover, if the mean curvature of $\partial\Omega$ is bounded from above by $\left(K^{(j)}\right)^{\frac{1}{j}}$, then $\partial\Omega$ is a sphere e Ω is a ball.*

Proof. If Ω is star-shaped with respect to a point, using the divergence theorem and by (2.8) we have

$$\begin{aligned} |\partial\Omega| &= \int_{\partial\Omega} d\sigma \le \max_{\partial\Omega} H \int_{\partial\Omega} \langle\nu, x\rangle d\sigma(x) \\ &= \max_{\partial\Omega} H \int_{\Omega}\left(\sum_{j=1}^{2(n+1)} \frac{\partial x_j}{\partial x_j}\right) dx = 2(n+1)|\Omega| \max_{\partial\Omega} H. \end{aligned} \tag{2.9}$$

Then by (2.9) we obtain

$$\max_{\partial\Omega} H \ge \frac{|\partial\Omega|}{2(n+1)|\Omega|}$$

since $K^{(j)}$ is a positive constant, by (2.7) we have

$$\left(K^{(j)}\right)^{\frac{1}{j}} \le \frac{|\partial\Omega|}{2(n+1)|\Omega|} \le \max_{\partial\Omega} H.$$

Moreover, if $\max_{\partial\Omega} H \le \left(K^{(j)}\right)^{\frac{1}{j}}$ then

$$\left(K^{(j)}_{\partial\Omega}\right)^{\frac{1}{j}} = \frac{|\partial\Omega|}{2(n+1)|\Omega|}$$

and by Theorem 2.4 we can conclude that Ω is a ball. $\square$

Chapter 3

Comparison Theorems

Definition 3.1. Let U be a subset of $\mathbb{R}^n$ and $s : U \longrightarrow \mathbb{R}$. The application s is said *generalized symmetric function in* $\mathbb{R}^n$ if:

(i) U and s are invariant with respect to one-to-one rearrangements of $\lambda_1, \dots, \lambda_n$.
Moreover, U is an open cone contained in the half-space

$$\left\{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \;\middle|\; \sum_{j=1}^{n} \lambda_j > 0\right\}$$

and if $\lambda(A)$, $\lambda(B) \in U$, then $\lambda(tA + (1-t)B) \in U$, for every $t \in [0, 1]$;

(ii) s is smooth and

$$\frac{\partial s}{\partial \lambda_j}(\lambda) > 0 \qquad \forall \lambda \in U \;\; \forall j = 1, \dots, n;$$

(iii) for every $n \times n$ Hermitian matrix A, the function $A \longrightarrow S(A)$, defined by

$$S(A) = s(\lambda(A)),$$

is smooth and $S(A) \to 0$ as $A \to 0$.

For brevity hereafter we shall denote $\lambda_p(\partial\Omega)$ the set of the eigenvalues of the B-normalized Levi for of $\partial\Omega$ at p, $\lambda_p(L_p(f, B))$.

Remark 11. Given a generalized symmetric funcion $s : U \longrightarrow \mathbb{R}$, $U \subseteq \mathbb{R}^n$, the real-value map $p \longmapsto s(\lambda_p(\partial\Omega))$, $p \in \partial\Omega$, can be seen as a *geometric feature* of $\partial\Omega$.

Definition 3.2. A domain Ω will be said to be *s-admissible* if $\lambda_p(\partial\Omega) \subseteq U$, for every $p \in \partial\Omega$.
Ω is said *s-pseudoconvex* if is s-admissible and $s(\lambda_p(\partial)) > 0$, for every $p \in \partial\Omega$.
A defining function f of a domain Ω is said *s-admissible* if Ω is s-admissible.
Finally, the real number

$$S_p(\partial\Omega) := s(\lambda_p(\partial\Omega))$$

will be called the *s-pseudocurvature* of $\partial\Omega$ at p.

Remark 12. Notion of s-pseudoconvexity and s-pseudocurvature are independent from the choice of the defining function f of Ω.

Definition 3.3 (Mean Levi Curvature)**.**
When $q = 1$ the Levi Curvature $K^{(1)}_{\partial\Omega}(p)$ is said *mean Levi curvature*, indeed we have

$$K^{(1)}_{\partial\Omega}(p) = \frac{\lambda_1 + \ldots + \lambda_n}{n}.$$

Remark 13. All the previous definitions can be "localized", then we can extend the notion of s-pseudoconvexity to all the graphs of functions defined in a open subset of $\mathbb{R}^{2n+1}$.

Definition 3.4. Let $\Omega \in \mathbb{R}^{2n+1}$ an open set and let u be a $C^2(\Omega, \mathbb{R})$ function. We denote

$$\Gamma(u) = \big\{(\xi, \tau) \in \Omega \times \mathbb{R} \bigm| u(\xi) < \tau\big\}$$

$$\gamma(u) = \big\{(\xi, u(\xi)) \in \Omega \times \mathbb{R} \bigm| \xi \in \Omega\big\}.$$

Remark 14. Identifying $\mathbb{R}^{2n+2}$ with $\mathbb{C}^{n+1}$, we can consider $\Gamma(u)$ and $\gamma(u)$ as subsets of $\mathbb{C}^{n+1}$.
A function u is said s-pseudoconvex if $\Gamma(u)$ is s-pseudoconvex in every point of $\gamma(u)$.

3.1 Curvature Operators

In this section we will denote by $\Omega = \{z \in \mathbb{C}^{n+1} \mid f(z) < 0\}$ a domain of $\mathbb{C}^{n+1}$ with defining function $f \in C^2$, such that $\nabla_p f \neq 0$ when $f(p) = 0$ and $\partial\Omega = \{z \in \mathbb{C}^{n+1} \mid f(z) = 0\}$. As first thing we want to show explicitly a basis of $\mathcal{T}_p(\partial\Omega)$. Since $\nabla_p f \neq 0$, we may assume $f_{n+1}(p) \neq 0$ and define

$$h_l = e_l - \alpha_l e_{n+1}$$

for $l = 1, \dots, n$, where $(e_1, \dots, e_{n+1})$ is the canonical basis for $\mathbb{C}^{n+1}$, and

$$\alpha_l = \alpha_l(p) := \frac{f_l(p)}{f_{n+1}(p)}. \tag{3.1}$$

Then $V = \{h_l \mid l = 1, \dots n\}$ is a basis for $\mathcal{T}_p(\partial\Omega)$, in fact

$$\langle h_l, \overline{\nabla}_p f\rangle = \sum_{j=1}^{n+1} \langle e_l - \alpha_l e_{n+1}, f_{\bar{j}}(p) e_j\rangle = f_l(p) - \alpha_l f_{n+1}(p) = 0.$$

Hereafter we will identify h_l with the complex differential operator

$$Z_l = \partial_{z_l} - \alpha_l \partial_{z_{n+1}} \qquad l = 1, \dots, n. \tag{3.2}$$

If we consider a point $p \in \partial\Omega$, we have

$$Z_l(f) = \langle h_l, \overline{\nabla}_p f\rangle = 0 \tag{3.3}$$

for every $l = 1, \dots, n$. We also put

$$\alpha_{\bar{l}} = \overline{\alpha}_l$$

and

$$Z_{\overline{l}} = \partial_{\overline{z}_l} - \alpha_{\overline{l}} \partial_{\overline{z}_{n+1}}.$$

Finally, we define for any $j, k \in \{1, \dots, n\}$

$$A_{j,\overline{k}} = A_{j,\overline{k}}(p) := \langle \mathscr{H}_p^t(f) h_j, h_k \rangle.$$

then we have

$$\begin{aligned} A_{j,\overline{k}} =& \langle \mathscr{H}_p^t(f)(e_j - \alpha_j e_{n+1}), e_k - \alpha_k e_{n+1} \rangle \\ =& f_{j,\overline{k}} - \alpha_{\overline{k}} f_{j,\overline{n+1}} - \alpha_j f_{n+1.\overline{k}} + \alpha_j \alpha_{\overline{k}} f_{n+1,\overline{n+1}}. \end{aligned}$$

replacing in the right-hand side of this equation the definitions of α_j and $\alpha_{\overline{k}}$, we obtain

$$A_{j,\overline{k}} = -\frac{1}{||f_{n+1}||^2} \det \begin{pmatrix} 0 & f_{\overline{k}} & f_{\overline{n+1}} \\ f_j & f_{j,\overline{k}} & f_{j,\overline{n+1}} \\ f_{n+1} & f_{n+1,\overline{k}} & f_{n+1,\overline{n+1}} \end{pmatrix}. \tag{3.4}$$

We will denote this matrix

$$A(f) = \Big(A_{j,\overline{k}}(f) \Big)_{j,k=1,\dots,n}.$$

Proposition 3.1. *The eigenvalues of the normalized Levi form of $\partial\Omega$ at the point $p \in \partial\Omega$ are eigenvalues of the matrix*

$$C(f) := \frac{1}{||\nabla_p f||} A(f) H(f) \tag{3.5}$$

where

$$H(f) = I_n - \frac{\alpha\alpha^*}{1 + ||\alpha||^2}$$

with $\alpha\alpha^*$ products of $\alpha = (\alpha_1, \dots, \alpha_n)^t$ and $\alpha^* = (\overline{\alpha}_1, \dots, \overline{\alpha}_n)$.

Proof. We shall denote as V the $(n+1) \times n$ matrix with columns $h_1, \dots, h_n$,

$$v = [h_1, \dots, h_n] \qquad h_l = e_l + \alpha_l e_{n+1}$$

then, taken $U = [u_1, \dots, u_n]$ with $u_1, \dots, u_n$ orthonormal basis of $\mathcal{T}_p(\partial\Omega)$ there exists an $n \times n$ matrix N such that

$$V = UN^t.$$

Since $A^t = V * \mathscr{H}_p^t(f)V$, we have $A^t(f) = \overline{N}(U^* \mathscr{H}_p^t(f)U)N^t$ and

$$\frac{1}{||\nabla_p f||} A(f) = N L_p(f, U) N^*$$

where $L_p(f, U)$ is the U-normalized Levi matrix, then follows that the matrix

$$L_p(f, U) = \frac{1}{||\nabla_p f||} N^{-1} A(f) (N^*)^{-1} \tag{3.6}$$

has the same eigenvalues of the matrix

$$\frac{1}{||\nabla_p f||} A(f)(NN^*)^{-1}.$$

On the other hand, since U is orthogonal, $(NN^*)^t = \overline{N}U^*UN^t = V * V$, and by direct calculation we have

$$(V^*V)^t = I_n + \alpha\alpha^*.$$

Finally, by Sherman-Morrison formula

$$(I_n + \alpha\alpha^*) = I_n - \frac{\alpha\alpha^*}{1 + ||\alpha||^2}$$

□

Remark 15. Levi total curvature and Levi mean curvature can be expressed in terms of the matrix $A(f)$ as follows

$$K_{\partial\Omega}^{(n)}(p) = \frac{||f_{n+1}||^2}{||\nabla_p f||^{n+2}} \det A(f)$$

and

$$K_{\partial\Omega}^{(1)}(p) = \frac{1}{||\nabla_p f||^{n+2}} tr\Big(\Big(I_n - \frac{\alpha\alpha^*}{1 + ||\alpha||^2}\Big) A(f)\Big)$$

Remark 16. By direct calculation we obtain

$$Z_j(\alpha_k) = -\frac{1}{f_{n+1}^3} \det \begin{pmatrix} 0 & f_k & f_{n+1} \\ f_j & f_{j,k} & f_{j,n+1} \\ f_{n+1} & f_{n+1,k} & f_{n+1,n+1} \end{pmatrix} \tag{3.7}$$

$$Z_j(\alpha_k) = -\frac{1}{||f_{n+1}||^2 f_{\overline{n+1}}} \det \begin{pmatrix} 0 & f_{\overline{k}} & f_{\overline{n+1}} \\ f_j & f_{j,\overline{k}} & f_{j,\overline{n+1}} \\ f_{n+1} & f_{n+1,\overline{k}} & f_{n+1,\overline{n+1}} \end{pmatrix} = \frac{1}{f_{\overline{n+1}}} A_{j,\overline{k}}. \tag{3.8}$$

As a consequence we have the identities

$$Z_j(\alpha_k) = Z_k(\alpha_j) \tag{3.9}$$

and

$$Z_{\overline{j}}(\alpha_k) = \overline{Z_j(\alpha_{\overline{k}}} = \frac{1}{f_{n+1}} \overline{A_{j,\overline{k}}} = \frac{1}{f_{n+1}} A_{k,\overline{j}}. \tag{3.10}$$

Proposition 3.2. *We have, for every $j, k = 1, \dots, n$,*

(i) $[Z_j, Z_k] = 0$*,*

(ii) $[Z_j, Z_{\overline{k}}] = A_{j,\overline{k}}(f)\left(\frac{1}{f_{n+1}}\partial_{z_{n+1}} - \frac{1}{f_{\overline{n+1}}}\partial_{z_{\overline{n+1}}}\right)$*.*

Proof. We have

$$[Z_j, Z_k] = \big(Z_k(\alpha_j) - Z_j(\alpha_k)\big)\partial_{z_{n+1}}$$

from (3.9) follows $[Z_j, Z_k] = 0$. As before we have

$$[Z_j, Z_{\overline{k}}] = \big(Z_{\overline{k}}(\alpha_j)\big)\partial_{z_{n+1}} - \big(Z_j(\alpha_{\overline{k}})\big)\partial_{z_{\overline{n+1}}}$$

then, by (3.8) and (3.13), we obtain

$$[Z_j, Z_{\overline{k}}] = \frac{1}{f_{n+1}} A_{j,\overline{k}}(f)\partial_{z_{n+1}} - \frac{1}{f_{\overline{n+1}}} A_{j,\overline{k}}(f)\partial_{z_{\overline{n+1}}}$$

. □

Proposition 3.3. *Let s be a generalized symmetric function. Let Ω be an s-admissible domain. Then the s-pseudocurvature of $\partial\Omega$ at $p \in \partial\Omega$ can be written as*

$$S_p(\partial\Omega) = \sum_{j,k=1}^{n} a_{j,\bar{k}} A_{j,\bar{k}}$$

where $a_{j,\bar{k}} = \overline{a_{k,\bar{j}}}$ smoothly depends on $\partial_z f$, $\partial_{\bar{z}} f$, $\partial_z\partial_{\bar{z}} f$ and

$$\sum_{j,k=1}^{n} a_{j,\bar{k}} \zeta_j \bar{\zeta}_k \geq m||\zeta||^2 \qquad \forall \zeta \in \mathbb{C}^n$$

with $m > 0$ which depends continuously on p and f.

Proof. By definition we have

$$S_p(\partial\Omega) = S(L_p(f,B)) = s(\lambda_1, \dots, \lambda_n)$$

where $L_p(f,B)$ is the B-normalized Levi form and $\lambda_1, \dots, \lambda_n$ its eigenvalues. Moreover, if we consider the set of the Hermitian admissible matrix $C = (c_{l,\bar{k}})_{l,k=1,\dots,n}$, the function $C \longmapsto S(C)$ is smooth. We will denote by $S_{l,\bar{k}}(C)$ the derivatives of S with respect to $c_{l,\bar{k}}$. Since $L = L_p(f,B)$ is admissible, also $L + C$ is admissible, for every Hermitian nonnegative matrix C small enough. Then we have

$$S(L+C) - S(L) = s(\eta_1, \dots, \eta_n) - s(\lambda_1, \dots, \lambda_n)$$

where $\eta_1, \dots, \eta_n$ are the eigenvalues of $L + C$. Since $C \geq 0$, we have $\eta_j \geq \lambda_j$, $\forall j = 1, \dots, n$, and by Definition 3.1-(ii)

$$\delta = \delta(L) = \frac{1}{2} \min \left\{ \frac{\partial s}{\partial \lambda_j}(\lambda_1, \dots, \lambda_n) \mid j = 1, \dots, n \right\} > 0$$

Hence, for C small enough

$$\begin{aligned} S(L+C) - S(L) =& \int_0^1 \frac{ds}{d\tau}(\lambda + \tau(\eta - \lambda)) d\tau \\ =& \sum_{j=1}^{n} \int_0^1 \frac{\partial s}{\partial \lambda_j} s(\lambda + \tau(\eta - \lambda)) d\tau (\eta_j - \lambda_j) \\ \geq& \delta \sum j = 1^n (\eta_j - \eta_j) = \delta\big(tr(L+C) - tr(L)\big) \\ =& \delta tr(C). \end{aligned}$$

We now apply this result to the matrix $C = t\zeta\zeta^*$, with $\zeta \in \mathbb{C}^n$ and $t > 0$ small enough, obtaining

$$S(L + t\zeta\zeta^*) - S(L) \geq \delta tr(C) = \delta t||\zeta||^2. \tag{3.11}$$

On the other hand

$$\frac{dS}{dt}(L + t\zeta\zeta^*)\Big|_{t=0} = \sum_{l,k=1}^{n} S_{l,\overline{k}}(L)\zeta_l\overline{\zeta}_k.$$

It follows from inequality (3.11) that

$$\sum_{l,k=1}^{n} S_{l,\overline{k}}(L)\zeta_l\overline{\zeta}_k \geq \delta||\zeta||^2 \qquad \forall\zeta \in \mathbb{C}^n. \tag{3.12}$$

We will denote by ∇S the matrix $(S_{l,\overline{k}})_{l,k=1,\dots,n}$. L is admissible so also tL is admissible for $0 < t \leq 1$, then

$$S(L) = \int_0^1 \frac{dS}{dt}(tL)dt = \int_0^1 tr(\nabla S(tL)L)dt$$

$$= \int_0^1 tr\Big(\nabla S(tL)\frac{1}{||\nabla_p f||}N^{-1}A(f)(N^*)^{-1}\Big)dt$$

$$= \int_0^1 tr\Big(\frac{(N^*)^{-1}\nabla S(tL)N^{-1}}{||\nabla_p f||}A(f)\Big)dt.$$

Denoting by $(a_{j,\overline{k}})_{j,k=1,\dots,n}$ the matrix

$$\int_0^1 tr\Big(\frac{(N^*)^{-1}\nabla S(tL)N^{-1}}{||\nabla_p f||}dt$$

we obtain

$$S(L) = \sum_{j,k=1}^{n} a_{j,\overline{k}}A_{j,\overline{k}}.$$

On the other hand, by (3.12)

$$\sum j,k = 1^n a_{j,\overline{k}}\zeta_j\overline{\zeta}_k = \int_0^1 \langle\nabla S(tL)N^{-1}\zeta, N^{-1}\zeta\rangle\frac{1}{||\nabla_p f||}dt$$

$$\geq \frac{1}{||\nabla_p f||}||N^{-1}\zeta||^2\int_0^1 \delta(tL)dt \geq m||\zeta||^2$$

where

$$m := \inf_{||\zeta||=1}\left(\frac{1}{||\nabla_p f||}||N^{-1}\zeta||^2\int_0^1 \delta(tL)dt\right)$$

is strictly positive and continuously depending on p and f. $\square$

Now we want to analyze the structure of the curvature operators when applied to the graph of a function u

$$\gamma(u) = \big\{(\xi, u(\xi)) \in \Omega\times\mathbb{R} \;\big|\; \xi \in \Omega\big\}.$$

we consider $\gamma(u)$ as (a subset of) the boundary of the domain

$$\Gamma(u) = \big\{(\xi, \tau) \in \Omega\times\mathbb{R} \;\big|\; u(\xi) < \tau\big\}$$

for which we will take the defining function $f(\xi,\tau) = u(\xi) - \tau$. We will also identify $\mathbb{R}^{2n+1}\times\mathbb{R}$ con $\mathbb{C}^{n+1}$ and we wil denote a point of $\mathbb{R}^{2n+1}$ by $\xi = (x_1, y_1, \ldots, x_n, y_n, t)$, while we will denote a point in $\mathbb{C}^{n+1}$ by $z = (z_1, \ldots, z_n)$ where $z_j = x_j + iy_j$, $\forall j = 1,\ldots,n$ and $z_{n+1} = t + i\tau$. Recalling the previous definitions u is s-admissible if f is s-admissible, u is said s-psudoconvex at a point $\xi\in\Omega$ if $\Gamma(u)$ is s-pseudoconvex at the point $(\xi, u(\xi)) \in \gamma(u)$. If u is s-pseudoconvex at any point we will say u is s-pseudoconvex.
Let now be $\xi \in \Omega$ and $p = (\xi, u(\xi)) \in \gamma(u)$, let f the defining function for $\Gamma(u)$, by (3.3) we have $0 = Z_l(u) - Z_l(\tau)$, so it follows

$$Z_l(u) = \frac{i}{2}\alpha_l \tag{3.13}$$

where

$$\alpha_l = \frac{f_l}{f_{n+1}} = \frac{\partial_{x_l}u - i\partial_{y_l}}{\partial_t u + i}.$$

We remark that for a function v independent of τ

$$Z_l(v) = \big(\partial_{z_l} - \frac{1}{2}\alpha_l\partial_t\big)(v),$$

we call W_l the complex vector field

$$W_l = \partial_{z_l} - \frac{1}{2}\alpha_l\partial_t$$

so that $Z_l(v) = W_l(v)$. we will also denote

$$W_{\bar{l}} = \partial_{\bar{z}_l} - \frac{1}{2}\alpha_{\bar{l}}\partial_t$$

with this notation

$$W_l(u) = \frac{i}{2}\alpha_l$$

and, finally, we put

$$B_{j,\bar{k}}(u) = A_{j,\bar{k}}(u-\tau) \tag{3.14}$$

Proposition 3.4. *At any point of* Ω *we have*

(i) $\frac{1}{2}(W_j W_{\bar{k}} + W_{\bar{k}} W_j)(u) = \frac{B_{j,\bar{k}}(u)}{1+u_t^2}$,

(ii) $[W_j, W_{\bar{k}}] = -4i\frac{B_{j,\bar{k}}(u)}{1+u_t^2}\partial_t$.

Proof. By identities (3.8) and (3.9) and by the independence of α_j from τ, we have

$$\begin{aligned} W_{\bar{k}} W_j(u) =& \frac{i}{2} W_{\bar{k}}(\alpha_j) = \frac{i}{2} Z_{\bar{k}}(\alpha_j) = i\frac{A_{j,\bar{k}}(u-\tau)}{(\partial_t - i\partial_\tau)(u-\tau)} \\ =& i\frac{B_{j,\bar{k}}(u)}{\partial_t u + i}. \end{aligned}$$

So

$$W_j W_{\bar{k}}(u) = \overline{W_{\bar{j}} W_k(u)} = -i\frac{\overline{B_{k,\bar{j}}}(u)}{\partial_t u - i} = -i\frac{B_{j,\bar{k}}(u)}{\partial_t u - i}$$

and

$$(W_j W_{\bar{k}} + W_{\bar{k}} W_j)(u) = iB_{j,\bar{k}}(u)\Big(\frac{1}{\partial_t u + 1} - \frac{1}{\partial_t u - 1}\Big) = 2\frac{B_{j,\bar{k}}(u)}{1+u_t^2},$$

this proves (i).

We remark that

$$[W_j, W_{\bar{k}}] = -(W_j(\alpha_{\bar{k}} - W_{\bar{k}}(\alpha_j))\partial_t.$$

Then, since

$$W_j(\alpha_{\bar{k}}) = 2iW_j W_{\bar{k}}(u), \qquad W_{\bar{k}}(\alpha_j) = -2iW_{\bar{k}} W_j(u),$$

(ii) follows from (i). $\square$

Corollary 3.5. *Let $u : \Omega \longrightarrow \mathbb{R}$ be an s-admissible function. Then*

$$\dim\Big(Span_{\mathbb{C}}\big\{W_j, [W_j, W_{\overline{k}}] \mid j, k = 1, \dots, n\big\}\Big) = n + 1 \tag{3.15}$$

at any point of Ω.

Proof. Let $\xi \in \Omega$ be fixed and $p = (\xi, u(\xi))$. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the normalized Levi form of $\gamma(u)$ at the point p. Then $\lambda_1 + \ldots + \lambda_n > 0$. As a consequence of Proposition (3.1) the matrix $(B_{j,\overline{k}})_{j,k=1,\dots,n}$ is not vanishing, so there exists a couple (l, m) such that $B_{l,\overline{m}} \neq 0$ and by Proposition (3.4) $W_1, \dots, W_n, [W_l, W_{\overline{m}}]$ are linearly independent in $\mathbb{C}^{n+1}$. □

Definition 3.5. Let K be a function:

$$K : \Omega \times \mathbb{R} \times \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}.$$

We say that u has *assigned s-Levi curvature K in Ω* if

$$S_p(\gamma(u)) = K(\xi, u, \nabla u)$$

where $p = (\xi, u(\xi))$, for every $\xi \in \Omega$. ∇u stands for the Euclidean gradient of u in $\mathbb{R}^{2n+1}$.

Proposition 3.6. *Let $u \in C^2(\Omega, \mathbb{R})$ be an s-admissible function. If u has the assigned s-Levi curvature K in Ω, the it satisfy*

$$\mathscr{L} u = K(\xi, u, \nabla u)$$

for $\xi \in \Omega$, where $\mathscr{L}$ is a fully nonlinear operator:

$$\mathscr{L} = \mathscr{L}_u := \sum_{j,k=1}^{n} b_{j,\overline{k}} \frac{W_j W_{\overline{k}} + W_{\overline{k}} W_j}{2} \tag{3.16}$$

and $b_{j,\overline{k}} = \overline{b_{k,\overline{j}}} = b_{j,\overline{k}}(\nabla u, \mathscr{H} u)$ smootly depends on ∇u and the real Hessian matrix $\mathscr{H} u$. Moreover, for every compact set $C \subseteq \Omega$ there exists $m > 0$ such that

$$\sum_{j,k=1}^{n} a_{j,\overline{k}}(\nabla u(\xi), \mathscr{H} u(\xi)) \zeta_j \overline{\zeta}_k \geq m ||\zeta||^2 \qquad \forall \zeta \in \mathbb{C}^n$$

for every $\xi \in C$.

Proof. By Proposition (3.3) we have

$$\sum_{j,k=1}^{n} a_{j,\bar{k}} B_{j,\bar{k}} = K(\xi, u, \nabla u) \quad \text{in } \Omega.$$

then by Proposition (3.4) we get the result with

$$b_{j,\bar{k}} = \frac{a_{j,\bar{k}}}{1+u_t^2}.$$

□

We now introduce the 'real' form for those curvature operator, let us take

$$X_j = 2Re(W_j) \qquad Y_j = -2Im(W_j)$$

and

$$a_j = -Re(\alpha_j) \qquad b_j = Im(\alpha_j).$$

for every $j = 1, \dots, n$. We recall that $W_j = \partial_{z_j} - \frac{\alpha_j}{2}\partial_t$, then we have

$$X_j = \partial_{x_j} + a_j \partial_t \qquad X_j = \partial_{y_j} + b_j \partial_t \tag{3.17}$$

for every $j = 1, \dots, n$. With this new notations we can rewrite (3.13)

$$(X_j - iY_j)(u) = -i\alpha_j = -b_j - ia_j$$

so

$$X_j(u) = -b_j \qquad Y_j(u) = a_j.$$

This relations together with (3.17) let us rewrite a_j and b_j

$$a_j = \frac{u_{y_j} - u_{x_j} u_t}{1+u_t^2},$$

$$b_j = \frac{-u_{x_j} - u_{y_j} u_t}{1+u_t^2}.$$

We now consider the matrix $B = (b_{j,\bar{k}})_{j,k=1,\dots,n}$, we put

$$B_1 = Re(B) \qquad B_2 = Im(B).$$

and we define the matrix $C = (c_{j,k}$ as the following $2n \times 2n$ block matrix

$$C = \frac{1}{4} \begin{pmatrix} B_1 & B_2 \\ -B_2 & B_1 \end{pmatrix}.$$

Renaming the vector fields X_j and Y_j

$$X_j = V_j \qquad Y_j = V_{n+j}$$

for $j = 1, \ldots, n$, we can rewrite the curvature operator

$$\mathscr{L} = \mathscr{L}_u = \sum_{j,k=1}^{2n} c_{j,k}(\xi) V_j V_k \tag{3.18}$$

with $c_{j,k}(\xi) = c_{j,k}(\nabla u(\xi), \mathscr{H} u(\xi))$. Moreover, by Proposition (3.6), for every compact set $C \subseteq \Omega$

$$\sum_{j,k=1}^{2n} c_{j,k}(\xi) \eta_j \eta_k \geq \frac{m}{4} \sum_{j=1}^{2n} \eta_j^2 \qquad \forall \eta \in \mathbb{R}^{2n} \ , \ \forall \xi \in C.$$

Hence, the operator $\mathscr{L}$ is *'elliptic' only along 2n linearly independent directions* and it is not elliptic at any point.
The missing ellipticity direction can be recovered by commutation, indeed the commutator

$$[V_j, V_k] = v_{j,k} \partial_t$$

for a suitable function $v_{j,k}$ in Ω. By Corollary (3.5), for every $\xi \in \Omega$ there exists (j,k) such that $v_j, k(\xi) \neq 0$, so

$$\dim \Big(Span_{\mathbb{R}} \big\{ V_j, [V_j, V_k] \bigm| j,k = 1, \ldots, 2n \big\} \Big) = 2n + 1$$

at any point of Ω.
This property will be crucial in the proof of strong maximum and comparison theorem.

3.2 Strong Maximum and Comparison Principle

Hereafter we will take $\Omega \subseteq \mathbb{R}^{2n+1}$ an open set and $X_1, \dots, X_{2n}$ linear C^1 vector fields in Ω such that

$$\dim \Big(Span_{\mathbb{R}} \big\{ X_j(\xi), [X_j, X_k](\xi) \bigm| j,k = 1, \dots, 2n \big\} \Big) = 2n+1$$

for every $\xi \in \Omega$. We consider the partial differential operator:

$$\mathscr{M} = \sum_{j,k=1}^{2n} \beta_{j,k}(\xi) X_j X_k + \langle \beta, \nabla \rangle + c$$

where $\beta = (\beta_1, \dots, \beta_{2n})$ and c are real continuos function in Ω. We finally assume that for every compact set $C \subset \Omega$ there exists a constant $m = m(C) > 0$ such that

$$\sum_{j,k=1}^{2n} \beta_{j,k}(\xi) \eta_j \eta_k \geq m||\xi||^2 \qquad \forall \xi \in C, \forall \eta \in \mathbb{R}^{2n}$$

.

Theorem 3.7 (Strong Maximum Principle).
Let $\Omega_0 \subseteq \Omega$ be an open and connected set. Let ω be a $C^2(\Omega_0, \mathbb{R})$ function such that

$$\begin{cases} \mathscr{M}\omega \geq 0 & in\ \Omega_0 \\ \omega \leq 0 & in\ \partial\Omega_0 \end{cases}$$

Then $\omega \leq 0$ in Ω_0 or $\omega \equiv 0$ in Ω_0.

Proof. For a detailed proof see [4] □

Theorem 3.8 (Comparison Principle).
Let $\Omega \subset \mathbb{R}^{2n+1}$ be a connected open set. Let $u, v \in C^2(\Omega, \mathbb{R})$ be two s-pseudoconvex functions. If $u \leq v$ in Ω and

$$\mathscr{L}u - K(\xi, u, \nabla u) \geq \mathscr{L}v - K(\xi, v, \nabla v)$$

in Ω for some smooth function $K : \Omega \times \mathbb{R} \times \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}$, then $u \equiv v$ in Ω.

Proof. First of all, we will denote by $L(u)$ and $L(v)$ the normalized Levi matrix of u and v, given by replacing in (3.5) the defining function f with $u-\tau$ and $v-\tau$ respectly.
The functions u and v are s-admissible, so the eigenvalues of $\varrho L(u)+(1-\varrho)L(v)$ belong to U for every $0\leq\varrho\leq 1$, where U is the domain of s.
Let us consider $w=u-v$, we have

$$\begin{aligned}\mathscr{L}u-\mathscr{L}v =&\mathscr{L}_u u-\mathscr{L}_v u=S(L(u))-S(L(v))\\ =&\int_0^1\frac{dS}{d\varrho}(\varrho L(u)+(1-\varrho)L(v))d\varrho\\ =&\int_0^1 tr(\nabla S(\varrho L(u)+(1-\varrho)L(v))\cdot(L(u)-L(v)))d\varrho\\ =&tr(I\cdot(L(u)-L(v)))\end{aligned}\tag{3.19}$$

where

$$I:=\int_0^1\nabla S(\varrho L(u)+(1-\varrho)L(v))d\varrho$$

which is a positive definite matrix by (3.12). Considering (3.6) we also have

$$tr(I\cdot(L(u)-L(v)))=tr(\widetilde{I}\cdot(B(u)-B(v)))+\langle\beta,\nabla w\rangle$$

where β is a continuous function and

$$\widetilde{I}=\frac{2}{\sqrt{1+||\nabla u||^2}}(N^{-1}(u-\tau)\cdot I\cdot(N^*)^{-1}(u-\tau))$$

is positive, Hermitian and have continuous coefficients.
Moreover, by Proposition 3.4 and by (3.14), we obtain

$$\begin{aligned}B(u)-B(v)=&\frac{1}{2}\big((1+u_t^2)(W_jW_{\overline{k}}+W_{\overline{k}}W_j)u-(1+v_t^2)(W_jW_{\overline{k}}+W_{\overline{k}}W_j)v\big)_{j,k=1}^n\\ =&\frac{1+u_t^2}{2}\big((W_jW_{\overline{k}}+W_{\overline{k}}W_j)u-(W_jW_{\overline{k}}+W_{\overline{k}}W_j)v\big)_{j,k=1}^n\\ &+\text{ first order derivatives of } w.\end{aligned}$$

We want to write this last term as a second-order operator acting on w, so we denote $W_j[u](w) = (\partial_{z_j} - \frac{1}{2}\alpha_j(u)\partial_t)(w)$, we also have $W_j(u) = W_j[u](u)$. Then, we obtain

$$(W_l W_{\overline{m}} + W_{\overline{m}} W_l)(u) - (W_l W_{\overline{m}} + W_{\overline{m}} W_l)(v) = (W_l[u]W_{\overline{m}}[u] + W_{\overline{m}}[u]W_l[u])(w) + \text{first order derivatives of } w. \tag{3.20}$$

Hence, denoting $V_j[u] = 2Re(W_j[u])$ and $V_{n+j}[u] = -2Im(W_j[u])$, we can write

$$\mathscr{L}_u u - \mathscr{L}_v v = \sum_{j,k=1}^{2n} c_{j,k} V_j[u] V_k[u](w) \ + \ \text{first order derivatives of } w. \tag{3.21}$$

where $C = (c_{j.k})$ is a positive, non symmetric matrix, defined by

$$C = \frac{1}{4}\begin{pmatrix} Re\widetilde{I} & Im\widetilde{I} \\ -Im\widetilde{I} & Re\widetilde{I} \end{pmatrix}.$$

Moreover

$$K(\xi, u, \nabla u) - K(\xi, v, \nabla v) = \text{ first order derivatives of } w \ + c_1 w, \tag{3.22}$$

and, by (3.20) and (3.22), we have

$$\mathscr{L} u - K(\xi, u, \nabla u) - \mathscr{L} v + K(\xi, v, \nabla v) = \mathscr{M} w.$$

Hence, $\mathscr{M} w \geq 0$ in Ω and $w \leq 0$ in $\partial\Omega$, so the thesis follows from Theorem 3.7. □

Theorem 3.9. *Let Ω and Ω' be s-pseudoconvex domains of $\mathbb{C}^{n+1}$ with connected boundaries. Let suppose the following conditions are satisfied*

(i) $\Omega' \subseteq \Omega$ *and* $\partial\Omega \cap \partial\Omega' \neq \emptyset$*,*

(ii) $S_{p'}(\partial\Omega') \leq S_p(\partial\Omega)$ *for every* $p \in \partial\Omega$ *and* $p' \in \partial\Omega'$*.*

Then $\Omega' = \Omega$*.*

Proof. We want to prove that for every fixed $p \in \partial\Omega \cap \partial\Omega'$ there exists an open set $U \subseteq \mathbb{C}^{n+1}$, $p \in U$ such that $U \cap \partial\Omega = U \cap \partial\Omega'$. Then, thanks to the connectedness of $\partial\Omega$ and $\partial\Omega'$, it will follow that $\partial\Omega = \partial\Omega'$. This, together with the inclusion $\Omega' \subseteq \Omega$ give us the equality $\Omega' = \Omega$.
So, let $p \in \partial\Omega \cap \partial\Omega'$. Without losing generality, we assume $p = (\xi_0, \tau_0)$ with $\xi_0 \in \mathbb{R}^{2n+1}$ and $\tau_0 \in \mathbb{R}$ and the existence of an open set $D \subseteq \mathbb{R}^{2n+1}$ and a connected open set $U \subseteq \mathbb{C}^{n+1} = \mathbb{R}^{2n+1} \times \mathbb{R}$ such that:

(i) $p \in U$ and $\xi_0 \in D$,

(ii) there exists $u,\ v \in C^2(D, \mathbb{R})$, such that

$$\Omega \cap U = \Gamma(u) \cap U, \qquad \partial\Omega \cap U = \gamma(u) \cap U,$$

$$\Omega' \cap U = \Gamma(v) \cap U, \qquad \partial\Omega' \cap U = \gamma(v) \cap U.$$

So, we have $\Omega' \subseteq \Omega$ and $p \in \partial\Omega \cap \partial\Omega' \cap U$, then $u \leq v$ in Ω and $u(\xi_0) = v(\xi_0)$. Moreover, u and v are s-pseudoconvex and by the second hypotesis of the theorem

$$(L)u \geq (L)v \qquad \text{in } \Omega.$$

By Theorem 3.8 we can conclude that $u \equiv v$ in Ω, then $\partial\Omega' \cap U = \partial\Omega \cap U$ and this concludes the proof. □

We conclude the section with some interesting corollaries of this theorem.

Corollary 3.10. *Let $\Omega \subseteq \mathbb{C}^{n+1}$ be a q-pseudoconvex domain with connected boundary, $1 \leq q \leq n$. Let $B_R(z_0) \subseteq \Omega$ be a ball tangent to $\partial\Omega$ in a point of $\partial\Omega$. Then, if*

$$K^{(q)}_{\partial\Omega}(p) \geq \left(\frac{1}{R}\right)^q \qquad \forall p \in \partial\Omega,$$

we have $\Omega = B_R(z_0)$.

Proof. The proof follows directly from Theorem 3.9 and identity (2.1) from Example 2.1. □

Corollary 3.11. *Let $u : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}$ be a q-pseudoconvex C^2 function, where*

$$B_R = \{\xi \in \mathbb{R}^{2n+1} \mid |\xi| < R\}.$$

Then

$$R \leq \sup_{\xi \in B_R} \left(\frac{1}{K^{(q)}(\xi, u)} \right)^{\frac{1}{q}}.$$

Proof. By contradiction we assume this inequality false. Then there exists $r > 0$, $r < R$ such that

$$K^{(q)}(\xi, u) > \left(\frac{1}{r}\right)^q \qquad \forall \xi \in B_r. \tag{3.23}$$

On the other hand, there exists $B_r(\alpha) \subset \mathbb{R}^{2n+1}$, contained in $\Gamma(u)$ such that touches $\gamma(u)$ at a point $p_0 = (\xi_0 u(\xi_0))$. We now consider $v : B_r(\beta) \longrightarrow \mathbb{R}$, whose graph $\gamma(v)$ is the lower hemisphere of $\partial B_r(\alpha)$. It follows from (3.23) and identity (2.1) from Example 2.1, that

$$K^{(q)}(\xi, u) > K^{(q)}(\xi, v) \qquad \forall \xi \in B_r(\beta),$$

Furthermore, $u \leq v$ in $B_r(\beta)$ and $u(\xi_0) = v(\xi_0)$, then by Theorem 3.8 $u \equiv v$ in $B_r(\beta)$. This contradict the fact that the gradient of u is bounded in $B_r(\beta)$ while the one of v is not. $\square$

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