Alma Mater Studiorum · Università di Bologna

FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI

Corso di Laurea Specialistica in Matematica

PROCESSO SQUARE ROOT

Relatore:
Chiar.mo Prof.
ANDREA PASCUCCI

Presentata da:
ANDREA MAZZON

Sessione I
Anno Accademico 2010/2011

Introduction

In this thesis we will study the unique strong solution of the Stochastic Differential Equation (SDE in the sequel)

$$dX_t = (bX_t + c)dt + \sqrt{2aX_t}dW_t, \qquad X_0 = \bar{x} > 0, \tag{1}$$

with $b, c \in \mathbb{R}$ and $a > 0$, and where W_t is a standard Brownian motion.
This process is known as *square root process*, on account of the fact that X_t appears in a square root in the diffusion term of the SDE, and it has been originally mentioned in the literature by Feller [7].
The most important applications in finance of the square root process are essentially two: the CIR model and the Heston model.
Before briefly describing them, we underline the fact that in both the models one take b negative and c positive, because in this case it is ensured the *mean reversion* property: the drift, that is the term that "pushes" the process towards a certain value, is positive if $x < -\frac{c}{b}$ and it is negative if $x > -\frac{c}{b}$. Therefore, the process X_t is pushed towards the value $-\frac{c}{b}$, that can be seen as a long-run mean.
The CIR model (that takes the name from John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross that introduced it in 1985 as an extension of the Vasicek model, see [3]) describes the evolution of the term structure of interest rates, and so it is used in the valuation of interest rates derivatives.
This model specifies that the interest rate follows the SDE

$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t, \tag{2}$$

where all the constants are taken positive and where W_t is a standard Brownian motion.
The unique strong solution of (2) is said CIR process, and it is straightforward to see that (2) is the equation of the square root process with $k = -b$ and $\theta = -\frac{c}{b}$. Here k is said the *mean reversion rate*, and θ the *mean reversion level*.
Under the no-arbitrage assumption, this model is useful to determine a price of a bond with expiry date T at time t, that is given by

$$B(t,T) = \exp\{A(t,T) + r(t)C(t,T)\},$$

where $A(t,T)$ and $C(t,T)$ are certain functions of the time.
The Heston model, named after Steven Heston that introduced it in 1993 (see [9]) is a model in which an underlying S_t follows a stochastic process with a stochastic variance $\nu(t)$ that follows a CIR process.
Hence, we have

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{\nu_t S_t} dW_t^1 \\ d\nu_t = k(\theta - r_t)dt + \sigma\sqrt{\nu_t} dW_t^2, \end{cases} \tag{3}$$

where W_t^1 and W_t^2 are standard Brownian motion with a certain correlation ρ.
Therefore, if we take the SDE (1), we notice that the term $\sqrt{2a}$ is the variance of the variance of the underlying of the Heston model, and that $-b$ (we remind that b is taken to be negative) and $-\frac{b}{c}$ are respectively the mean reversion rate and the mean reversion level of the variance.
For these two models c is taken to be positive because when the interest rate for the CIR model and the volatility of the underlying for the Heston model get close to zero, their evolution become dominated by the drift factor c, that, if it is positive, pushes them upwards: in other words, if c is positive the drift is positive for lower values of the process, so the process itself will be pushed upwards.
As we will see, if $c \geq a$ (and so, for the CIR process, if $2k\theta \geq \sigma^2$) it is also ensured that the process will never assume the value of zero: the interest rate

for the CIR model and the volatility for the Heston model remain strictly positive if this condition, known as *Feller condition*, is satisfied.
Unfortunately, the Feller condition is almost never satisfied for the smiles typically encountered in practice (see Table 6.3 in [2]) so one has to be concerned with violation of this condition.
Therefore, it becomes of great importance the study of the behaviour of the solution of equation (1) at the boundary $\{X = 0\}$, that we also call *the origin*.
There are three possible cases:

- boundary not attainable (that is the case that we have if the Feller condition is satisfied);
- boundary attainable and reflecting, that is the case in which a path that hits the origin is pushed away to positive regions;
- boundary attainable and absorbing, that is the case in which a path that hits the origin in the time t remains trapped here for all times $s > t$.

As we will see, the behaviour of the square root process at the origin depends on the parameter c: we will distinguish the three possible cases, $c \leq 0$, $0 < c < a$ and $c \geq a$, and we will examine the nature of the process for every one of this cases.
To do it, for each case we will also study the fundamental solutions of the forward Kolmogorov equation associated to the SDE (1), that is

$$\partial_t u(t,x) = \partial_{xx}\big(axu(t,x)\big) - \partial_x\big((bx+c)u(t,x)\big), \tag{4}$$

with a certain initial condition that we will see in the sequel.
For each case $c \leq 0$, $0 < c < a$ and $c \geq a$ we will determine if the process has a transition density, and which of the fundamental solution of (4) is the transition density itself. Obviously, we will also give the expression of the transition density of the process, that is well known.

Here is a summary of the thesis: in Chapter 1 we give an overview of the results that we will obtain, briefly considering also a particular case of the square root process, that is the Bessel process

$$dX_t = \delta dt + 2\sqrt{X}dW_t, \qquad X_0 = \bar{x} > 0,$$

that can be useful to price Asian Options.
In Chapter 2 we compute a general form for the Laplace transform of a fundamental solution of (4), following the method presented by Feller in [7], and we find the fundamental solution itself in the case $c \leq 0$, always following [7].
In Chapter 3 we determine the transition density of the process in the case $c > 0$, by a method shown by Dufresne in [5]: we compute the moments of the process, and so we find the moment generating function (MGF) $\mathbb{E}[e^{sX_t}] = \mathbb{E}\Big[\sum_{k=0}^{\infty} \frac{(sX_t)^k}{k!}\Big]$. In this way, we see that X_t is the sum of a Gamma variable X_t' and a compound Poisson/Exponential X_t''. Therefore, we find the transition density of the process from the known densities of X_t' and X_t''.
In chapter 4 we compute the derivative of the transition density, and we see that it is similar to the transition density itself; in Chapter 5 we study the behaviour of the transition density for large values of x, and we give an asymptotic expansion for the transition density itself and for its derivative.
In Chapter 6 we give the results found by Feller in [7] for $c > 0$, we compute the integrals with respect to x of the fundamental solutions of (4) and we discuss the relations between transition density and fundamental solutions: we determine in which cases a fundamental solution is also the transition density of the process and in which cases this is not true. We see that if $0 < c < a$ there is a fundamental solution that it is the transition density of the process and another fundamental solution that is not. Moreover, we give another method to determine the transition density itself.
In Chapter 7 we study the behaviour of the process at the origin, for $c \leq 0$, $0 < c < a$ and $c \geq a$: for each case we establish if the boundary is absorbing, reflecting or not attainable. To do it, we also compute the limit of the density for $x \to 0$. We see that for $0 < c < a$ the square root process is

reflected at the boundary whereas the fundamental solution that is not the transition density of the process is the density of the process itself stopped at $\{X = 0\}$, that therefore is absorbed at the origin. Moreover, when the boundary is absorbing, we study the probability that a path will be trapped at the origin within time t; taking the limit of this probability for $t \to \infty$, we will also have the probability that a path will be eventually trapped at the origin. In the case $0 < c < a$, the probability that the process stopped at the origin will be trapped here within time t is the probability that the square root process, unique solution of (1), will reach the origin within time t. We also give some graphics of these probabilities.

Contents

Chapter 1

Main results

1.1 Square root process

1.1.1 Fundamental solutions and transition density

We define the square root process as the unique strong solution of the SDE

$$dX_t = (bX_t + c)dt + \sqrt{2aX_t}dW_t, \qquad X_0 = \bar{x} > 0, \tag{1.1}$$

with $b, c \in \mathbb{R}$, $a > 0$, $X_t > 0$, $t > 0$.
In this chapter we give an overview of the main results that we will find about the density of this process and the fundamental solutions of the associated forward Kolmogorov equation, when $c \leq 0$, $0 < c < a$, $c \geq a$.
Moreover, we will see how one can applicate these results to the Bessel process, unique strong solution of the SDE

$$dX_t = \delta dt + 2\sqrt{X_t}dW_t, \qquad X_0 = \bar{x} > 0, \tag{1.2}$$

with $\delta > 0$, $X_t > 0$, $t > 0$.
Feller [7] studied fundamental solutions of the forward Kolmogorov equation associated to the SDE (1.1),

$$\partial_t u(t, x) = \partial_{xx}\big(axu(t, x)\big) - \partial_x\big((bx + c)u(t, x)\big), \tag{1.3}$$

with $x > 0$, $t > 0$; we give a resume of this work, that we will see in detail later.

Feller first of all shows that, if we require some integrability properties on a fundamental solution $p(t, x, \bar{x})$ with initial datum $\bar{x}$ and on its derivative with respect to t so that one can define the Laplace transform

$$\omega(t, s, \bar{x}) = \int_0^\infty e^{-sx} u(t, x) dx$$

and the derivative with respect to t of the Laplace transform itself, this Laplace transform has to satisfy the first-order partial differential equation

$$\frac{\partial \omega}{\partial t} + s(as - b)\frac{\partial \omega}{\partial t} = -cs\omega + f(t), \tag{1.4}$$

where

$$f(t) = -\lim_{x \to 0} \Big(\partial_x \big(axp(t, x, \bar{x}) \big) + (bx + c)p(t, x, \bar{x}) \Big)$$

is called *the flux of* p *at the origin.*

He solves (1.4) with the initial condition $\lim_{t \to 0} \omega(t, s, \bar{x}) = e^{-s\bar{x}}$ by the method of characteristic, and he finds

$$\begin{aligned} \omega(t, s, \bar{x}) = {} & \left(\frac{b}{as(e^{bt} - 1) + b} \right)^{c/a} exp \left\{ \frac{-\bar{x}sbe^{bt}}{sa(e^{bt} - 1) + b} \right\} + \\ & \int_0^t f(\tau) \left(\frac{b}{sa((e^{b(t-\tau)} - 1) + b)} \right)^{c/a}, \end{aligned} \tag{1.5}$$

where if $b = 0$ we take $\lim_{b \to 0} (\frac{b}{e^{bt} - 1}) = \frac{1}{t}$.

Now, for each case $c \le 0$, $0 < c < a$ and $c \ge a$, one can see how the flux at the origin f has to be to satisfy the conditions on p that we have already mentioned.

We have that:

- for $c \le 0$, the flux f is the unique solution of a certain equation, which we will see in detail later, so it is univocally determined. Therefore, (1.1) admits an unique fundamental solution which satisfies the integrability properties, and we can calculate this solution by inverting (1.5) in which we put the expression of f that we have found as the unique

solution of the given equation.
We find

$$p_1(t,x;\bar{x}) = \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\} \cdot \left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} I_{1-c/a}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right). \tag{1.6}$$

- When $0 < c < a$ we can find two fundamental solutions that satisfy the integrability conditions that we require: one corresponding to $f = 0$ and one that we obtain when $f \neq 0$. In the case $f \neq 0$, the integrability properties of p are satisfied if and only if the flux has the same form as for the case $c \leq 0$: the corresponding fundamental solution will be therefore the solution that we have found for $c \leq 0$. When $f = 0$ instead one can calculate the fundamental solution by inverting the Laplace transform (1.5) in which $f = 0$.
Feller does not invert this Laplace transform, but as we will see later we obtain

$$p_2(t,x;\bar{x}) = \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\} \cdot \left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} I_{c/a-1}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right). \tag{1.7}$$

- When $c \geq a$ the conditions on p are satisfied if and only if $f = 0$. In this case hence the unique fundamental solution of (1.3) will be (1.7).

Therefore we find that when c is negative and when $c \geq a$ we have one and only one fundamental solution of (1.3) which satisfies the conditions that we require for p, whereas when $0 < c < a$ these fundamental solutions are two. We will see that p_2 is the transition density of the process, while p_1 is not the transition density.
As we will show in the sequel, one can explicitly calculate the integral with respect to x of a fundamental solution also when he does not know the exact form of the fundamental solution itself. This integral is equal to one if $f = 0$,

whereas for the fundamental solution p_1 that we obtain when $f < 0$ we have

$$\int_0^\infty p_1(t, x; \bar{x}) dx = \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt} - 1)}\right). \tag{1.8}$$

This fundamental solution is said to be a *defective density*.

1.1.2 Behaviour of the process at the boundary

We have three possible cases:

- the boundary is attainable and absorbing (if a path hits the boundary it will be trapped in the boundary itself);
- the boundary is attainable and reflecting (if a path hits the boundary it will be pushed away in the region $\{x > 0\}$);
- the boundary is not attainable.

We will see that:

- when $c \le 0$ the boundary is attainable and absorbing; there exists a positive probability that a path of the process will be trapped at the origin within a certain time t, and this probability is given by

  $$1 - \int_0^\infty p_1(t, x; \bar{x}) dx = 1 - \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt} - 1)}\right).$$

  If we take the limit of this probability as the time goes to infinity, we obtain the probability that a path will be eventually trapped at the origin: this is

  $$1 - \lim_{t\to\infty} \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt} - 1)}\right) = 1 - \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}}{a}\right)$$

  if b is positive and one if b is negative. We will give some graphics about the behaviour of this limit for different values of the parameters a and b.

- When $0 < c < a$ the boundary is attainable; if we take the process that is the unique strong solution of (1.1), and that has density p_2, we see that it will be reflected at the boundary; we can also stop this process when it hits the boundary: this stopped process is solution of (1.1) only before it hits the boundary, and it has defective density p_1. The probability that a path of this process will be trapped at the boundary within time t has the same expression as for the case $c \leq 0$, and it is equal to the probability that a path of the process with transition density p_2 hits the boundary within time t.

- When $c > a$, the boundary is not attainable.

We can also calculate the limit of the fundamental solution $p_1(t, x, \bar{x})$ in (1.6) and $p_2(t, x; \bar{x})$ in (1.7) as $x \to 0$:we have that

$$\lim_{x\to 0} p_1(t, x, \bar{x}) = \frac{1}{\Gamma(2-\frac{c}{a})} \left(\frac{b}{a(e^{bt}-1)}\right)^{2-\frac{c}{a}} (\bar{x}e^{bt})^{1-\frac{c}{a}} \exp\left\{-\frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right\}$$

and

$$\lim_{x\to 0^+} p_2(t, x; \bar{x}) = \begin{cases} \infty & 0 < c < a \\ \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\} & c = a \\ 0 & c > a \end{cases} \tag{1.9}$$

This limit is useful to have a complete view of the behaviour of the process at the origin, according to the different values of the parameter c; we can summarize the situation as follow:

- if c is negative, we have only one appropriate fundamental solution of (1.3); this fundamental solution is not the transition density of the process, but it is a *defective density*: the distribution of the process is given by the sum of the defective density and the Dirac measure at zero.

  The boundary is accessible and absorbing (every path that hits the boundary will be trapped in the boundary itself). The probability that a path will not be trapped within time t is given by (1.8) and the probability that a path never be trapped at the origin is given by $\Gamma\left(1-\frac{c}{a};\frac{b\bar{x}}{a}\right)$ if b is positive and it is zero if b is negative.

- If c is positive but smaller than a there are two appropriate fundamental solutions, and they depend by the conditions that we give at the boundary: if we impose a zero flux at the boundary (that is $f=0$) we find the transition density of the process, that is (obviously) norm preserving and that corresponds to a reflecting boundary: if a path hits the origin, it will be pushed away.
  Instead, if we impose a flux at the origin that it is not zero, this flux has to take one particular form, and corresponds to a fundamental solution that is the same that we found for $c \leq 0$: therefore, just as for $c \leq 0$ the fundamental solution is not the transition density of the square root process but it is a *defective density*: the process that has this defective density is not the strong solution of (1.1), but it is this solution stopped at the origin: we can say that it is the square root process stopped at the origin. The probability that this process will be trapped at the origin is the same as for $c \leq 0$.
  At the end, we can therefore say that if $0 < c < a$, when a path hits the origin we can either impose a reflecting boundary (and in this case we find a transition density) or impose an absorbing boundary (and in this case we find a defective density).

- If $c \geq a$, we can only have the fundamental solution (1.7) that is also the transition density of the process. The boundary is not attainable, so the process never hits the origin.

1.1.3 Behaviour of the process at infinity

But we are not only interested in the behaviour of the process for $x \to 0$: we can also give an estimation of the transition density for big values of x. We will see that one can do this by exploiting a result on the behaviour of the Bessel modified function of the second kind $I_\nu(z)$ for large values of z.

We find

$$p_2(t,x;\bar{x}) \sim exp\Big\{\frac{b}{a(e^{bt}-1)}(2\sqrt{e^{bt}\bar{x}}\sqrt{x}-x)\Big\}x^{\frac{c}{2a}-\frac{3}{4}}.$$

We notice that this estimate is valid for all c, therefore the limit is the same not only for the transition density but also for all the fundamental solutions that we have found.

We can do this also for the derivative:

$$\frac{\partial p_2(t,x,\bar{x})}{\partial x} \sim \exp\left\{\frac{b}{a(e^{bt}-1)}\left(-x+2\sqrt{e^{bt}\bar{x}}\sqrt{x}\right)\right\}\left(x^{\frac{c}{2a}-\frac{7}{4}}+x^{\frac{c}{2a}-\frac{5}{4}}-x^{\frac{c}{2a}-\frac{3}{4}}\right).$$

1.2 Bessel process

We can easily apply these results to the Bessel process, that we define as the unique strong solution of the SDE

$$dX_t = \delta dt + 2\sqrt{X}dW_t, \qquad X_0 = \bar{x} > 0, \tag{1.10}$$

with $\delta \in \mathbb{R}$.

We note that one can see the Bessel process as a particular case of the *square root* process in which we put $a = 2$, $b = 0$, $c = \delta$.

We substitute these values of the parameters in (1.6) and in (1.7), and we find that:

- when δ is negative, there is only one fundamental solution of the forward Kolmogorov equation associated to the SDE (1.10) that satisfies the integrability condition that one gives in order to define the Laplace transform of the solution. This fundamental solution is

$$q_1(t,x;\bar{x}) = \frac{1}{2t}\left(\frac{x}{\bar{x}}\right)^{\frac{\delta/2-1}{2}} exp\Big\{-\frac{x+\bar{x}}{2t}\Big\}I_{-\frac{\delta}{2}+1}\left(\frac{\sqrt{x\bar{x}}}{t}\right). \tag{1.11}$$

- When $0 < \delta < 2$, there are two two fundamental solution that satisfy these conditions: one of the two is (1.11), the other is

$$q_2(t,x;\bar{x}) = \frac{1}{2t}\left(\frac{x}{\bar{x}}\right)^{\frac{\delta/2-1}{2}} exp\Big\{-\frac{x+\bar{x}}{2t}\Big\}I_{\frac{\delta}{2}-1}\left(\frac{\sqrt{x\bar{x}}}{t}\right); \tag{1.12}$$

- When $\delta \geq 2$ there is only one appropriate fundamental solution, and it is (1.12).

The fundamental solution (1.12) is also the transition density of the process; on the other hand, (1.11) has integral

$$\int_0^\infty q_1(t,x;\bar{x})dx = \Gamma\Big(1-\frac{\delta}{2},\frac{\bar{x}}{2t}\Big), \tag{1.13}$$

and it is not the transition density.
As for the case of the square root process, when $\delta < 0$ we have a distribution that is the sum of the defective density (1.11) and a Dirac measure at zero with weight

$$2\bigg(1-\Gamma\Big(-\frac{\delta}{2}+1,\frac{\bar{x}}{t}\Big)\bigg).$$

The integral (1.13) is also the probability that the process will not be trapped at the origin by time t, when δ is negative.
If we take the limit for $t \to \infty$, we find

$$\lim_{t\to\infty}\int_0^\infty q_1(t,x;\bar{x})dx = \lim_{t\to\infty}\Gamma\Big(1-\frac{c}{a};\frac{\bar{x}}{t}\Big) = \Gamma\Big(1-\frac{c}{a};0\Big) = 0.$$

So, if δ is negative, the process will be *always* trapped at $\{x=0\}$.
This probability is obviously zero when we take the fundamental solution (1.12), that is also the transition density of the process: the integral equals to one, because the transition density is *norm preserving*, so the probability that the process will be trapped at the origin by time t is one.
We can also calculate the limit of the fundamental solution for $x \to 0$, by substituting the values of the parameters b, c and a in (1.9): we find

$$\lim_{x\to 0^+} q_1(t,x;\bar{x}) = \frac{1}{\Gamma(1-\delta/2)}\Big(\frac{1}{2t}\Big)^{2-\frac{\delta}{2}} exp\Big\{-\frac{\bar{x}}{2t}\Big\} \tag{1.14}$$

and

$$\lim_{x\to 0^+} q_2(t,x;\bar{x}) = \begin{cases} \infty & 0<\delta<2 \text{ with } f=0 \\ \frac{1}{2t}exp\left\{-\frac{\bar{x}}{2t}\right\} & \delta = 2 \\ 0 & \delta > 2 \end{cases} \tag{1.15}$$

Therefore, we can state that:

- for $\delta \leq 0$ the boundary is attainable and absorbing: *every* path will be trapped at the origin;

- for $0 < \delta < 2$ the boundary is attainable, and when a path hits $\{x = 0\}$, we may either impose an absorbing boundary condition and end the process, or impose a reflecting boundary condition and return to $\{x = 0\}$: in the first case we do not have a transition density, but a defective density (that is the fundamental solution of the Kolmogorov equation); in the second case, we have a transition density;

- for $\delta \geq 2$ the boundary is not attainable; we have a transition density, and the limit of this density at the origin is 0.

We can also calculate the limit of the fundamental solutions for $x \to \infty$: this limit, as in the case of the *square root* process, is valid for all δ and therefore it is unique, for all the fundamental solutions:

$$\lim_{x\to\infty} q_1(t,x;\bar{x}) = \lim_{x\to\infty} q_2(t,x;\bar{x}) = exp\left\{\frac{1}{2t}(2\sqrt{\bar{x}}\sqrt{x} - x)\right\} x^{\frac{\delta-3}{4}}.$$

Chapter 2

Feller's calculation for $c \leq 0$

Feller considers the parabolic operator

$$Lu(t,x) := \partial_{xx}\big(axu(t,x)\big) - \partial_x\big((bx+c)u(t,x)\big) - \partial_t u(t,x). \tag{2.1}$$

where $x > 0$, $t > 0$, $a > 0$, $b, c \in \mathbb{R}$.
We give the following definition of a fundamental solution of (2.1):

Definition 2.0.1. *A fundamental solution of (2.1) with pole at $\bar{x}$ is a function $p(t,x,\bar{x})$ such that:*

i) $p(\cdot,\cdot,\bar{x}) \in C^2 \cap L^1\left(]0,T[\times\mathbb{R}_+\right)$ *for any* $T > 0$*;*

ii) $\partial_t p(t,\cdot,\bar{x}) \in L^1\left(\mathbb{R}_+\right)$ *for any* $t > 0$*;*

iii) $p(0,x,\bar{x}) = \delta_{\bar{x}}$ *in the sense that, for any* $\phi \in C_b\left(\mathbb{R}_+\right)$

$$\lim_{(t,x)\to(0,\bar{x})} \int_{\mathbb{R}_+} p(t,x,\bar{x})\phi(y)dx = \phi(\bar{x}), \tag{2.2}$$

and

$$Lp(t,x,\bar{x}) = 0, \qquad (t,x) \in \mathbb{R}^2_+. \tag{2.3}$$

In this case, $p(t,x,\bar{x})$ is also said a fundamental solution of the PDE

$$\partial_t u(t,x) = \partial_{xx}\big(axu(t,x)\big) - \partial_x\big((bx+c)u(t,x)\big) \tag{2.4}$$

with initial condition (2.2).

2.1 Laplace transform

If we admit the existence of a fundamental solution $p(t, x, \bar{x})$ of (2.4) as in Definition 2.0.1, we can define for $s > 0$ its Laplace transform

$$\omega(t, s, \bar{x}) = \int_0^\infty e^{-sx} p(t, x, \bar{x}) dx. \tag{2.5}$$

According to point ii) of the definition, we can also take

$$\partial_t \omega(t, s, \bar{x}) = \int_0^\infty e^{-sx} \partial_t p(t, x, \bar{x}) dx.$$

We want to determine a general form of the Laplace transform (2.5). Let's integrate on the left and on the right of (2.4) with respect to x from $y \in (0, 1)$ to 1.
We obtain

$$\int_y^1 \partial_t p(t, x, \bar{x}) dx = \int_y^1 \Big(\partial_{xx} \big(axp(t, x, \bar{x}) \big) - \partial_x \big((bx + c) p(t, x, \bar{x}) \big) \Big) dx =$$
$$\Big[\partial_x \big(axp(t, x, \bar{x}) \big) + (bx + c) p(t, x, \bar{x}) \Big]_y^1 .$$

We can observe that the integral on the left is well defined, because $\partial_t p(t, \cdot, \bar{x})$ is integrable.
Looking at the term on the right, we see that obviously there is not any problem for $x = 1$, consequently letting $y \to 0$, we see that $\lim_{x \to 0} \Big(\partial_x \big(axp(t, x, \bar{x}) \big) + (bx + c) p(t, x, \bar{x}) \Big)$ exists and it is bounded in every interval $[t_1, t_2]$.
Therefore we can put

$$f(t) = - \lim_{x \to 0} \Big(\partial_x \big(axp(t, x, \bar{x}) \big) + (bx + c) p(t, x, \bar{x}) \Big);$$

this function $f(t)$ is said *flux at the origin*.
Now, let's transform with Laplace on the left and on the right of (2.4): we

have

$$
\begin{aligned}
&\int_0^\infty e^{-sx}\partial_t p(t,x,\bar{x})dx = \partial_t\omega(t,s,\bar{x}) = \\
&\int_0^\infty e^{-sx}\Big(\partial_{xx}\big(axp(t,x,\bar{x})\big) - \partial_x\big((bx+c)p(t,x,\bar{x})\big)\Big)dx = \\
&\Big[e^{-sx}\Big(\partial_x\big(axp(t,x,\bar{x})\big) - (bx+c)p(t,x,\bar{x})\Big)\Big]_0^\infty + \\
&s\int_0^\infty e^{-sx}\Big(a\partial_x\big(xp(t,x,\bar{x})\big) - (bx+c)p(t,x,\bar{x})\Big)dx = \\
&f(t) + s\int_0^\infty e^{-sx}\Big(a\partial_x\big(xp(t,x,\bar{x})\big) - (bx+c)p(t,x,\bar{x})\Big)dx = \\
&f(t) + as\int_0^\infty e^{-sx}\partial_x\big(xp(t,x,\bar{x})\big)dx - bs\int_0^\infty e^{-sx}xp(t,x,\bar{x})dx - cs\int_0^\infty e^{-sx}p(t,x,\bar{x})dx = \\
&f(t) - as^2\Big[e^{-sx}(xp(t,x,\bar{x}))\Big]_0^\infty + as^2\int_0^\infty e^{-sx}(xp(t,x,\bar{x}))dx + bs\partial_s\omega(t,s,\bar{x}) - cs\omega(t,s,\bar{x}) = \\
&f(t) + (bs - as^2)\partial_s\omega(t,s,\bar{x}) - cs\omega(t,s,\bar{x}).
\end{aligned}
$$

Therefore, we obtain the following partial differential equation of the first order for $\omega(t,s,\bar{x})$:

$$\partial_t\omega(t,s) + s(as-b)\partial_s\omega(t,s,\bar{x}) = -cs\omega(t,s,\bar{x}) + f(t), \tag{2.6}$$

with the initial condition

$$\lim_{t\to 0}\omega(t,s,\bar{x}) = e^{-s\bar{x}} \tag{2.7}$$

This initial condition comes from the initial condition for the fundamental solution:

$$\lim_{t\to 0}\omega(t,s,\bar{x}) = \lim_{t\to 0}\int_0^\infty e^{-sx}p(t,x,\bar{x})dx = \int_0^\infty e^{-sx}\delta_{\bar{x}}(dx) = e^{-s\bar{x}}$$

Feller solves this equation by the method of characteristics. During the calculations we will suppose $b \neq 0$, and when we will finish we will see that there are no problems for $b = 0$.
The characteristic equations of (2.7) are

$$dt = \frac{ds}{s(as-b)} = \frac{d\omega}{f(t) - cs\omega}. \tag{2.8}$$

We find the general solution of the first equation, that is $C_1 = g(t,s)$ such that $\frac{dg}{dt} = -s(as-b)\frac{dg}{ds}$. We have

$$C_1 = e^{-bt}\frac{as-b}{s}.$$

Now we take C_1 as a constant, and we think at ω like a function of t. We want to find it as the general solution of

$$\frac{d\omega}{dt} = f(t) - cs\omega.$$

Therefore, we find

$$\omega = |C_1 - ae^{-bt}|^{c/a}\left(C_2 + \int_0^t \frac{f(\tau)}{|C_1 - ae^{-b\tau}|^{c/a}}d\tau\right), \tag{2.9}$$

because we have

$$\begin{aligned}\frac{d\omega}{dt} =& \frac{c}{a}|C_1 - ae^{-bt}|^{c/a-1}(abe^{-bt})\{\ldots\} + f(t) = \\ & \frac{\omega}{C_1 - ae^{-bt}}(cbe^{-bt}) + f(t) = \\ & \frac{\omega}{exp\{-bt(a-b/s-a)\}}(cbe^{-bt}) = cs\omega + f(t).\end{aligned}$$

C_2 in this expression is a constant, now we want to determine it as a function of C_1, putting $C_2 = A(C_1)$, and exploiting the initial condition $\lim_{t\to 0}\omega(t,s,\bar{x}) = e^{-s\bar{x}}$.

Afterwards, we will put the values of C_1 and C_2 in (2.9), so that we will obtain the general solution of (2.8).

We see that for $t = 0$ we have $C_1 = a - b/s$, so

$$\lim_{t\to 0}\omega(t,s,\bar{x}) = \left|\frac{b}{s}\right|^{c/a} A\left(a - \frac{b}{s}\right)$$

Therefore, we want to find $A(y)$ such that

$$e^{-s\bar{x}} = \left|\frac{b}{s}\right|^{c/a} A\left(a - \frac{b}{s}\right).$$

Putting $y = a - \frac{b}{s}$, we obtain

$$A(y) = |a-y|^{-c/a}e^{-\bar{x}(\frac{b}{a-y})}.$$

Consequently,

$$C_2 = A(C_1) = \Big|a - e^{-bt}\big(a - \frac{b}{s}\big)\Big|^{-c/a} exp\Big\{ - \frac{b\bar{x}}{a - e^{-bt}(a - \frac{b}{s})}\Big\}.$$

Putting the expression of C_1 e C_2 in (2.9), we have

$$\begin{aligned}\omega(t,s,\bar{x}) =& \Big|\frac{b}{s}e^{-bt}\Big|^{c/a}\Bigg(\Big|a - e^{-bt}\big(a - \frac{b}{s}\big)\Big|^{-c/a} exp\Big\{\frac{-b\bar{x}}{a - e^{-bt}(a - \frac{b}{s})}\Big\} + \\ & \int_0^t f(\tau)\Big|e^{-bt}(\frac{as-b}{s}) - ae^{-b\tau}\Big|^{-c/a} d\tau\Bigg) = \\ & \Big|\frac{be^{-bt}}{s(a - e^{-bt}(a - \frac{b}{s}))}\Big|^{c/a} exp\Big\{\frac{-b\bar{x}}{a - e^{-bt}(a - \frac{b}{s})}\Big\} + \\ & \int_0^t f(\tau)\Big(\frac{be^{-bt}}{se^{-bt}(\frac{as-b}{s}) - ae^{-b\tau}}\Big)^{c/a} d\tau = \\ & \Big(\frac{b}{as(e^{bt}-1)+b}\Big)^{c/a} exp\Big\{\frac{-\bar{x}sbe^{bt}}{sa(e^{bt}-1)+b}\Big\} + \int_0^t f(\tau)\Big(\frac{b}{sa(e^{b(t-\tau)}-1)+b}\Big)^{c/a}\end{aligned}$$

Therefore, we have find

$$\begin{aligned}\omega(t,s,\bar{x}) = & \left(\frac{b}{as(e^{bt}-1)+b}\right)^{c/a} exp\left\{\frac{-\bar{x}sbe^{bt}}{sa(e^{bt}-1)+b}\right\} + \\ & \int_0^t f(\tau)\left(\frac{b}{sa((e^{b(t-\tau)}-1)+b}\right)^{c/a}\end{aligned} \tag{2.10}$$

We see that for $b = 0$ there are no problems, because $\lim_{b\to 0}(\frac{b}{e^{bt}-1}) = \frac{1}{t}$.

2.2 Calculation of f in the case $c \leq 0$

Now we want to see, in the case $c \leq 0$, what expression has to take the flux f in order to make $\omega(t,x,\bar{x})$ in (2.10) the Laplace transform of a fundamental solution of (2.4) as in Definition (2.0.1).

Theorem 2.2.1. *If $c \leq 0$, $\omega(t,x,\bar{x})$ in (2.10) is the Laplace transform of a fundamental solution of (2.4) with pole at $\bar{x}$ only if $f(t)$ satisfies the following equation:*

$$exp\left\{\frac{-\bar{x}b}{a(1-e^{-bt})}\right\} + \int_0^t f(\tau)\left(\frac{e^{bt}-1}{e^{b(t-\tau)}-1}\right)^{c/a} = 0. \tag{2.11}$$

Proof. A fundamental solution of (2.4) with pole at $\bar{x}$ is such that $\omega(t,s,\bar{x}) \to 0$ when $s \to \infty$ (see [12] page 181). We have $c \leq 0$, therefore

$$\Big|\frac{sa(e^{bt}-1)+b}{b}\Big|^{c/a}\omega(t,s,\bar{x}) \to \infty,$$

because for large values of s the term $\frac{sa(e^{bt}-1)+b}{b}$ is greater than 1, and we rase it to a negative power.
Looking at the (2.10), we see that

$$\begin{aligned}
&\lim_{s\to\infty}\Big|\frac{sa(e^{bt}-1)+b}{b}\Big|^{c/a}\omega(t,s,\bar{x}) = \\
&\lim_{s\to\infty}\left(exp\Big\{-\frac{s\bar{x}be^{bt}}{sa(e^{bt}-1)+b}\Big\} + \int_0^t f(\tau)\Big(\frac{sa(e^{bt}-1)+b}{sa(e^{b(t-\tau)}-1)+b}\Big)^{c/a}d\tau\right) = \\
&exp\Big\{-\frac{\bar{x}be^{bt}}{a(e^{bt}-1)}\Big\} + \int_0^t f(\tau)\left(\frac{e^{bt}-1}{e^{b(t-\tau)}-1}\right)^{c/a}d\tau = \\
&exp\Big\{-\frac{\bar{x}b}{a(1-e^{-bt})}\Big\} + \int_0^t f(\tau)\left(\frac{e^{bt}-1}{e^{b(t-\tau)}-1}\right)^{c/a}d\tau.
\end{aligned}$$

□

We want to solve this equation. We give the following result:

Theorem 2.2.2. *The equation*

$$exp\left\{\frac{-\bar{x}b}{a(1-e^{-bt})}\right\} + \int_0^t f(\tau)\left(\frac{e^{bt}-1}{e^{b(t-\tau)}-1}\right)^{c/a}d\tau = 0.$$

has unique solution

$$f(t) = \frac{-b}{\Gamma(1-\frac{c}{a})}\frac{e^{-bt}}{1-e^{-bt}}\left(\frac{\bar{x}b}{a(1-e^{-bt})}\right)^{(-c+a)/a} exp\{-\frac{\bar{x}b}{a(1-e^{-bt})}\},$$

where

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt.$$

Proof. We rewrite the equation in the following way:

$$\int_0^t f(\tau)\left(\frac{1-e^{-bt}}{e^{-b\tau}-e^{-bt}}\right)^{c/a}d\tau = -exp\left(-\frac{\bar{x}b}{a(1-e^{-bt})}\right),$$

and we make the substitutions

$$z = (1 - e^{-bt})^{-1}, \zeta = (1 - e^{-b\tau})^{-1}.$$

We have

$$\tau = \frac{-\log(1 - 1/\zeta)}{b},$$

therefore

$$\frac{d\tau}{d\zeta} = -\frac{1}{b}\frac{1}{\zeta(\zeta - 1)}$$

Moreover, we see that $\lim_{\tau \to 0} \zeta = \infty$, whereas for $\tau = t$ we have $\zeta = z$. Consequently,

$$\begin{aligned}
&\int_0^t f(\tau)\left(\frac{1 - e^{-bt}}{e^{-b\tau} - e^{-bt}}\right)^{c/a} d\tau = \\
&\int_z^\infty \frac{1}{b} f\Big(-\frac{\log(1 - 1/\zeta)}{b}\Big)\left(\frac{z^{-1}}{z^{-1} - \zeta^{-1}}\right)^{c/a} \frac{1}{\zeta(\zeta - 1)} d\zeta = \\
&\int_z^\infty \frac{1}{b} f\Big(-\frac{\log(1 - 1/\zeta)}{b}\Big)\left(\frac{\zeta}{\zeta - z}\right)^{c/a} \frac{1}{\zeta(\zeta - 1)} d\zeta = \\
&\int_z^\infty g(\zeta)(\zeta - z)^{-c/a} d\zeta,
\end{aligned}$$

where

$$g(\zeta) = \frac{1}{b} f\Big(-\frac{\log(1 - 1/\zeta)}{b}\Big)\frac{\zeta^{c/a}}{\zeta(\zeta - 1)}. \tag{2.12}$$

We can the rewrite our equation as

$$\int_z^\infty g(\zeta)(\zeta - z)^{-c/a} d\zeta = -e^{-\frac{\bar{x}bz}{z}}. \tag{2.13}$$

Now we want to solve this equation for $g(\zeta)$, and so put the value that we have found equal to

$$g(\zeta) = \frac{1}{b} f(\tau)\frac{\zeta^{c/a}}{\zeta(\zeta - 1)},$$

to find in this way the value of f.
Putting

$$g(\zeta) = -\frac{1}{\Gamma(1 - \frac{c}{a})}\left(\frac{\bar{x}b}{a}\right)^{\frac{(-c+a)}{a}} exp\left\{-\frac{\bar{x}b\zeta}{a}\right\} \tag{2.14}$$

(2.13) is solved. In fact,

$$\begin{aligned}
&\int_z^\infty g(\zeta)(\zeta - z)^{-\frac{c}{a}} d\zeta = \\
&-\frac{1}{\Gamma(1-\frac{c}{a})}\left(\frac{\bar{x}b}{a}\right)^{\frac{(-c+a)}{a}} \int_z^\infty exp\Big\{-\frac{\bar{x}b\zeta}{a}\Big\}(\zeta - z)^{-\frac{c}{a}} d\zeta = \\
&-\frac{1}{\Gamma(1-\frac{c}{a})}\left(\frac{\bar{x}b}{a}\right)^{\frac{(-c+a)}{a}} \int_0^\infty exp\Big\{-\frac{\bar{x}b(t+z)}{a}\Big\}(t)^{-\frac{c}{a}} dt = \\
&-\frac{1}{\Gamma(1-\frac{c}{a})}\left(\frac{\bar{x}b}{a}\right)^{\frac{(-c+a)}{a}} exp\Big\{-\frac{\bar{x}bz}{a}\Big\}\int_0^\infty exp\Big\{-\frac{\bar{x}bt}{a}\Big\}(t)^{-\frac{c}{a}} dt = \\
&-\frac{1}{\Gamma(1-\frac{c}{a})}\frac{\bar{x}b}{a} exp\Big\{-\frac{\bar{x}bz}{a}\Big\}\int_0^\infty exp\Big\{-\frac{\bar{x}bt}{a}\Big\}\left(\frac{\bar{x}bt}{a}\right)^{-\frac{c}{a}} dt = \\
&-\frac{1}{\Gamma(1-\frac{c}{a})} exp\Big\{-\frac{\bar{x}bz}{a}\Big\}\int_0^\infty e^{-y}y^{-\frac{c}{a}} dy = \\
&-e^{-\frac{\bar{x}bz}{a}}
\end{aligned}$$

Equalizing (2.12) and (2.14) we find

$$\begin{aligned}
f(\tau) = -\frac{b}{\Gamma(1-\frac{c}{a})}\left(\frac{\bar{x}b\zeta}{a}\right)^{\frac{-c+a}{a}} exp\Big\{-\frac{\bar{x}b\zeta}{a}\Big\} = \\
\frac{-b}{\Gamma(1-\frac{c}{a})}\frac{e^{-b\tau}}{1-e^{-b\tau}}\left(\frac{\bar{x}b}{a(1-e^{-b\tau})}\right)^{(-c+a)/a} exp\Big\{-\frac{\bar{x}b}{a(1-e^{-b\tau})}\Big\}
\end{aligned}$$

□

2.3 Calculation of the Laplace transform

Theorem 2.3.1. *If $f(t)$ is the function of Theorem (2.2.2), then the expression (2.10) takes the form*

$$\begin{aligned}
\omega_1(t,s,\bar{x}) = \left(\frac{b}{sa(e^{bt}-1)+b}\right)^{c/a} exp\left\{\frac{-sb\bar{x}e^{bt}}{sa(e^{bt}-1)+b}\right\}\cdot \\
\Gamma\left(1-\frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\frac{b}{sa(e^{bt}-1)+b}\right),
\end{aligned} \tag{2.15}$$

where

$$\Gamma(n;z) = \frac{1}{\Gamma(n)}\int_0^z e^{-x}x^{n-1}dx.$$

Proof. We want to see that

$$\left(\frac{b}{sa(e^{bt}-1)+b}\right)^{c/a} exp\Big\{-\frac{sb\bar{x}e^{bt}}{sa(e^{bt}-1)+b}\Big\}\Gamma\left(1-c/a;\frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\frac{b}{sa(e^{bt}-1)+b}\right)=$$
$$\left(\frac{b}{sa(e^{bt}-1)+b}\right)^{c/a} exp\Big\{-\frac{sb\bar{x}e^{bt}}{sa(e^{bt}-1)+b}\Big\}+\int_0^t f(\tau)\left(\frac{b}{sa(e^{b(t-\tau)}-1)+b})\right)^{c/a} d\tau=$$
$$\left(\frac{b}{sa(e^{bt}-1)+b}\right)^{c/a} exp\Big\{-\frac{sb\bar{x}e^{bt}}{sa(e^{bt}-1)+b}\Big\}.$$
$$\left(1+\left(\frac{b}{sa(e^{bt}-1)+b}\right)^{-c/a} exp\Big\{\frac{sb\bar{x}e^{bt}}{sa(e^{bt}-1)+b}\Big\}\int_0^t f(\tau)\left(\frac{b}{sa(e^{b(t-\tau)}-1)+b}\right)^{c/a} d\tau\right).$$

Therefore, we want to prove that

$$\Gamma\left(1-c/a;\frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\frac{b}{sa(e^{bt}-1)+b}\right)=$$
$$1+\left(\frac{b}{sa(e^{bt}-1)+b}\right)^{-c/a} exp\Big\{\frac{sb\bar{x}e^{bt}}{sa(e^{bt}-1)+b}\Big\}\cdot\int_0^t f(\tau)\left(\frac{b}{sa(e^{b(t-\tau)}-1)+b}\right)^{c/a} d\tau.$$

We observe that

$$\begin{aligned}\Gamma(n,z)=&\frac{1}{\Gamma(n)}\int_0^z e^{-y}y^{n-1}dy=\\ &\frac{1}{\Gamma(n)}\Big(\int_0^\infty e^{-y}y^{n-1}dy-\int_z^\infty e^{-y}y^{n-1}dy\Big)=\\ &1-\frac{1}{\Gamma(n)}\int_z^\infty e^{-y}y^{n-1}dy.\end{aligned}$$

Consequently, we want to show that

$$\left(\frac{b}{sa(e^{bt}-1)+b}\right)^{-c/a} exp\left\{\frac{sb\bar{x}e^{bt}}{sa(e^{bt}-1)+b}\right\}\int_0^t f(\tau)\left(\frac{b}{sa(e^{b(t-\tau)}-1)+b}\right)^{c/a} d\tau=$$
$$-\frac{1}{\Gamma(1-\frac{c}{a})}\int_k^\infty e^{-y}y^{-\frac{c}{a}}dy,\neq$$

where

$$k=\frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\frac{b}{sa(e^{bt}-1)+b}.$$

To do it, we take just as we have done before

$$z=\frac{1}{1-e^{-bt}},\zeta=\frac{1}{1-e^{-b\tau}},$$

and we see that if we take the function $f(t)$ defined by Theorem (2.2.2) we find

$$f(\zeta) = -\frac{b}{\Gamma(1-\frac{c}{a})}(\zeta-1)\Big(\frac{\bar{x}b\zeta}{a}\Big)^{\frac{-c+a}{a}} exp\Big\{-\frac{\bar{x}b\zeta}{a}\Big\} =$$
$$-\frac{b}{\Gamma(1-\frac{c}{a})}\zeta(\zeta-1)\Big(\frac{\bar{x}b}{a}\Big)^{\frac{-c+a}{a}} \zeta^{-c/a} exp\Big\{-\frac{\bar{x}b\zeta}{a}\Big\}$$

Consequently,

$$\left(\frac{b}{sa(e^{bt}-1)+b}\right)^{-c/a} exp\Big\{\frac{sb\bar{x}e^{bt}}{sa(e^{bt}-1)+b}\Big\}\int_0^t f(\tau)\left(\frac{b}{sa(e^{b(t-\tau)}-1)+b}\right)^{c/a} d\tau =$$
$$exp\Big\{\frac{zsb\bar{x}}{sa+b(z-1)}\Big\}\left(\frac{b(z-1)}{sa+b(z-1)}\right)^{-c/a}\int_z^\infty f(\zeta)\left(\frac{\zeta b(z-1)}{sa(\zeta-z)+b\zeta(z-1)}^{c/a}\right)\frac{1}{b\zeta(\zeta-1)}d\zeta =$$
$$-\,exp\Big\{\frac{zsb\bar{x}}{sa+b(z-1)}\Big\}\left(\frac{b(z-1)}{sa+b(z-1)}\right)^{-c/a}\frac{1}{\Gamma(1-c/a)}\Big(\frac{\bar{x}b}{a}\Big)^{\frac{-c+a}{a}}.$$
$$\int_z^\infty exp\Big\{-\frac{\bar{x}b\zeta}{a}\Big\}\left(\frac{b(z-1)}{sa(\zeta-z)+b\zeta(z-1)}\right)^{c/a} d\zeta =$$
$$-\frac{1}{\Gamma(1-c/a)}\Big(\frac{\bar{x}b}{a}\Big)\int_0^\infty exp\Big\{\frac{zsb\bar{x}}{sa+b(z-1)}-\frac{\bar{x}b\zeta}{a}\Big\}\left(\frac{sa+b(z-1)}{sa(\zeta-z)+b\zeta(z-1)}\frac{a}{\bar{x}b}\right)^{c/a} d\zeta =$$
$$-\frac{1}{\Gamma(1-c/a)}\Big(\frac{\bar{x}b}{a}\Big)\int_0^\infty exp\Big\{-\Big(\frac{\bar{x}b\zeta}{a}-\frac{zsb\bar{x}}{sa+b(z-1)}\Big)\Big\}.$$
$$\left(\frac{\bar{x}b\zeta}{a}-\frac{zsb\bar{x}}{sa+b(z-1)}\right)^{-c/a} d\zeta.$$

Making the substitutions

$$y = \frac{\bar{x}b\zeta}{a} - \frac{zsb\bar{x}}{sa+b(z-1)},$$

we obtain

$$\left(\frac{b}{sa(e^{bt}-1)+b}\right)^{-c/a} exp\Big\{\frac{sb\bar{x}e^{bt}}{sa(e^{bt}-1)+b}\Big\}\int_0^t f(\tau)\left(\frac{b}{sa(e^{b(t-\tau)}-1)+b}\right)^{c/a} d\tau =$$
$$-\frac{1}{\Gamma(1-\frac{c}{a})}\int_k^\infty e^{-y}y^{-\frac{c}{a}}dy,$$

□

So, we know the expression of the Laplace transform of the (unique) fundamental solution of (2.4) for $c \leq 0$.
Knowing it, we can compute the integral of the fundamental solution even if we do not invert the Laplace transform:

Theorem 2.3.2. *The unique fundamental solution $p_1(t, x, \bar{x})$ of (2.4) with $c \leq 0$ is such that*

$$\int_0^\infty p_1(t, x, \bar{x})dx = \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt} - 1)}\right)$$

Proof. Observing the Laplace transform of $p_1(t, x, \bar{x})$ in (2.15) we see that

$$\int_0^\infty p_1(t, x, \bar{x})dx = \omega_1(t, 0, \bar{x}) = \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt} - 1)}\right)$$

□

This result is very important, because as we will see with more precision later, this integral provides the probability that the square root process with $c \leq 0$ will be trapped at the origin within time t.
Moreover, this integral will show that if $c \leq 0$ the process has not a transition density, but a distribution that is the sum of a Dirac measure at zero and a defective density.

2.4 Calculation of the fundamental solution

Now we will invert the Laplace transform (2.15) and so we will find the unique fundamental solution of (2.4) for $c \leq 0$. We call this fundamental solution $p_1(t, x, \bar{x})$.
To do it, we want to see first of all that putting

$$A = \frac{b\bar{x}}{a(1 - e^{-bt})}, \qquad z = \frac{sa(e^{bt} - 1) + b}{b},$$

we have

$$\omega_1(t, s, \bar{x}) = \frac{e^{-A}A^{1-c/a}}{\Gamma(1 - c/a)} \int_0^1 (1 - v)^{-c/a} e^{\frac{Av}{z}} z^{-1} dv.$$

Actually,

$$
\begin{aligned}
&\omega_1(t,s,\bar{x}) = \\
&\quad \left(\frac{b}{sa(e^{bt}-1)+b}\right)^{c/a} exp\Big\{\frac{-sb\bar{x}e^{bt}}{sa(e^{bt}-1)+b}\Big\}\Gamma\left(1-\frac{c}{a};\frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\frac{b}{sa(e^{bt}-1)+b}\right) = \\
&\quad z^{-c/a} exp\Big\{-\frac{s\bar{x}e^{bt}}{z}\Big\}\Gamma\left(1-\frac{c}{a},\frac{A}{z}\right) = \\
&\quad z^{-c/a} exp\Big\{-\frac{A(z-1)}{z}\Big\}\frac{1}{\Gamma(1-c/a)}\int_0^{\frac{A}{z}} x^{-c/a}e^{-x}dx = \\
&\quad z^{-c/a}e^{\frac{A}{z}-A}\frac{1}{\Gamma(1-c/a)}\left(\int_0^1\left(\frac{Aw}{z}\right)^{-c/a}e^{-\frac{Aw}{z}}dw\right)\frac{A}{z} = \\
&\quad e^{\frac{A}{z}-A}\frac{1}{\Gamma(1-c/a)}A^{1-c/a}\int_0^1 w^{-c/a}e^{-\frac{Aw}{z}}z^{-1}dw = \\
&\quad e^{\frac{A}{z}-A}\frac{1}{\Gamma(1-c/a)}A^{1-c/a}\int_0^1 (1-v)^{-c/a}e^{-\frac{A(1-v)}{z}}z^{-1}dw = \\
&\quad \frac{e^{-A}A^{1-c/a}}{\Gamma(1-c/a)}\int_0^1 (1-v)^{-c/a}e^{\frac{Av}{z}}z^{-1}dv
\end{aligned}
$$

We have the following lemma:

Lemma 2.4.1. *Given $A = \frac{b\bar{x}}{a(1-e^{-bt})}$, the function $q(z) = e^{\frac{Av}{z}}z^{-1}$is the Laplace transform of $I_0(2\sqrt{Avx})$, where*

$$I_\nu(y) = \sum_{n=0}^{\infty}\frac{(y/2)^{2n+k}}{n!\Gamma(n+k+1)}$$

is the modified Bessel function of the first kind.

Proof. We directly compute the Laplace transform of $I_0(2\sqrt{Avx})$ as

$$q(z) = \int_0^\infty e^{-zx}I_0(2\sqrt{Avx})dx,$$

and we have

$$\int_0^\infty e^{-zx} I_0(2\sqrt{Avx})dx = \int_0^\infty e^{-zx} \sum_0^\infty \frac{(Avx)^n}{n!\Gamma(n+1)}dx =$$
$$\sum_0^\infty \frac{(Av)^n}{n!\Gamma(n+1)} \int_0^\infty e^{-zx} x^n dx =$$
$$\sum_0^\infty \frac{(Av)^n}{n!\Gamma(n+1)} \left(\frac{1}{z} \int_0^\infty e^{-y} (\frac{y}{z})^n dy \right) =$$
$$\sum_0^\infty \frac{(Av)^n}{n!\Gamma(n+1)} \Big(\frac{1}{z} \frac{\Gamma(n+1)}{z^n} \Big) =$$
$$\frac{1}{z} e^{\frac{Av}{z}}$$

$\square$

By Lemma (2.4.1) we know that

$$\omega_1(t,s,\bar{x}) = \widetilde{\omega}(t,z) = \int_0^\infty e^{-zx} \widetilde{g}(x)dx,$$

where

$$\widetilde{g}(x) = \frac{e^{-A}A^{1-c/a}}{\Gamma(1-c/a)} \int_0^1 (1-v)^{-c/a} I_0(2\sqrt{Avx})dv$$

and

$$z = \frac{sa(e^{bt}-1)+b}{b}.$$

Therefore,

$$\omega_1(t,s,\bar{x}) = \widetilde{\omega_1}(t,z) = \int_0^\infty e^{-zx} \widetilde{g}(x)dx =$$
$$\int_0^\infty exp\Big\{ -\frac{sa(e^{bt}-1)}{b} x \Big\} e^{-x} \widetilde{g}(x)dx =$$
$$\frac{b}{a(e^{bt}-1)} \int_0^\infty e^{-sy} exp\Big\{ -\frac{b}{a(e^{bt}-1)} y \Big\} \widetilde{g} \left(\frac{b}{a(e^{bt}-1)} y \right) dy.$$

Consequently,

$$\begin{aligned}
p_1(t,x,\bar{x}) =& \frac{b}{a(e^{bt}-1)} exp\Big\{-\frac{b}{a(e^{bt}-1)}x\Big\}\widetilde{g}\left(\frac{b}{a(e^{bt}-1)}x\right) = \\
& \frac{1}{\Gamma(a-c/a)}\frac{b}{a(e^{bt}-1)}\left(\frac{b\bar{x}}{a(1-e^{-bt})}\right)^{1-c/a} exp\Big\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\Big\}\cdot \\
& \int_0^1 (1-v)^{-c/a} I_0\left(2\sqrt{\frac{b}{a(e^{bt}-1)}x\frac{b\bar{x}v}{a(1-e^{-bt})}}\right)dv = \\
& \frac{1}{\Gamma(a-c/a)}\frac{b}{a(e^{bt}-1)}\left(\frac{b\bar{x}}{a(1-e^{-bt})}\right)^{1-c/a} exp\Big\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\Big\}\cdot \\
& \int_0^1 (1-v)^{-c/a} I_0\left(\frac{2b}{a(1-e^{-bt})}\sqrt{\bar{x}e^{-bt}vx}\right)dv.
\end{aligned}$$

So, we have seen that

$$\begin{aligned}
p_1(t,x,\bar{x}) =& \frac{1}{\Gamma(a-c/a)}\frac{b}{a(e^{bt}-1)}\left(\frac{b\bar{x}}{a(1-e^{-bt})}\right)^{1-c/a} exp\Big\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\Big\}\cdot \\
& \int_0^1 (1-v)^{-c/a} I_0\left(\frac{2b}{a(1-e^{-bt})}\sqrt{\bar{x}e^{-bt}vx}\right)dv
\end{aligned} \tag{2.16}$$

Now we want to see that, expanding $p(t,x,\bar{x})$ in (2.16) into a power series in x, we find

$$p_1(t,x,\bar{x}) = \frac{b}{a(e^{bt}-1)} exp\Big\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\Big\}\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} I_{1-c/a}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right).$$

We have to see that

$$\begin{aligned}
&\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} I_{1-c/a}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right) = \\
&\frac{1}{\Gamma(1-c/a)}\left(\frac{b\bar{x}}{a(1-e^{-bt})}\right)^{1-c/a}\int_0^1 (1-v)^{-c/a} I_0\left(\frac{2b}{a(1-e^{-bt})}\sqrt{\bar{x}e^{-bt}vx}\right)dv.
\end{aligned}$$

Actually,

$$
\begin{aligned}
&\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} I_{1-c/a}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right) = \\
&\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} \sum_{r=0}^{\infty} \frac{\left(\frac{b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right)^{2r+1-c/a}}{r!\Gamma(r+2-c/a)} = \\
&\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} (e^{-bt}x\bar{x})^{\frac{a-c}{2a}} \left(\frac{b}{a(1-e^{-bt})}\right)^{1-c/a} \sum_{r=0}^{\infty} \frac{\left(\frac{b}{a(1-e^{-bt})}\right)^{2r}(e^{-bt}x\bar{x})^{r}}{r!\Gamma(r+2-c/a)} = \\
&\left(\frac{b\bar{x}}{a(1-e^{-bt})}\right)^{1-c/a} \sum_{r=0}^{\infty} \frac{\left(\frac{b}{a(1-e^{-bt})}\right)^{2r}(e^{-bt}x\bar{x})^{r}}{r!\Gamma(r+2-c/a)} = \\
&\frac{1}{\Gamma(1-c/a)}\left(\frac{b\bar{x}}{a(1-e^{-bt})}\right)^{1-c/a} \int_0^1 (1-v)^{-c/a} I_0\left(\frac{2b}{a(1-e^{-bt})}\sqrt{\bar{x}e^{-bt}vx}\right) dv,
\end{aligned}
$$

because

$$
\begin{aligned}
&\int_0^1 (1-v)^{-c/a} I_0\left(\frac{2b}{a(1-e^{-bt})}\sqrt{\bar{x}e^{-bt}vx}\right) dv = \\
&\sum_{r=0}^{\infty} \int_0^1 (1-v)^{-c/a} \left(\frac{b}{a(1-e^{-bt})}\right)^{2r} (\bar{x}e^{-bt}x)^r v^r dv \frac{1}{r!\Gamma(1+r)} = \\
&\sum_{r=0}^{\infty} \left(\frac{b}{a(1-e^{-bt})}\right)^{2r} (\bar{x}e^{-bt}x)^r \frac{1}{r!\Gamma(1+r)} \int_0^1 (1-v)^{-c/a} v^r dv = \\
&\sum_{r=0}^{\infty} \left(\frac{b}{a(1-e^{-bt})}\right)^{2r} (\bar{x}e^{-bt}x)^r \frac{1}{r!\Gamma(1+r)} \frac{\Gamma(1-c/a)\Gamma(1+r)}{\Gamma(2+r-c/a)} = \\
&\sum_{r=0}^{\infty} \frac{\left(\frac{b}{a(1-e^{-bt})}\right)^{2r}(e^{-bt}x\bar{x})^{r}}{r!\Gamma(r+2-c/a)} \Gamma(1-c/a)
\end{aligned}
\tag{2.17}
$$

Consequently,

$$
\begin{aligned}
&\frac{1}{\Gamma(1-c/a)}\left(\frac{b\bar{x}}{a(1-e^{-bt})}\right)^{1-c/a} \int_0^1 (1-v)^{-c/a} I_0\left(\frac{2b}{a(1-e^{-bt})}\sqrt{\bar{x}e^{-bt}vx}\right) dv = \\
&\left(\frac{b\bar{x}}{a(1-e^{-bt})}\right)^{1-c/a} \sum_{r=0}^{\infty} \frac{\left(\frac{b}{a(1-e^{-bt})}\right)^{2r}(e^{-bt}x\bar{x})^{r}}{r!\Gamma(r+2-c/a)}
\end{aligned}
$$

Therefore, we have proved that when $c \leq 0$ the unique fundamental solution of (2.4) as in Definition 2.0.1 is

$$p_1(t,x,\bar{x}) = \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} I_{1-c/a}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right).$$

Chapter 3

Calculation of the transition density

We want to determine the transition density of the square root process, that we define as the unique strong solution of the SDE

$$dX_t = (bX_t + c)dt + \sqrt{2aX_t}dW_t, \qquad X_0 = \bar{x} > 0 \tag{3.1}$$

with $a > 0$, $b, c \in \mathbb{R}$, W standard Brownian motion.
We report a calculation shown by Dufresne in [5], in which he take $b \neq 0$, $c > 0$.
We will see how Dufresne obtain the transition density of the process by computing the moments of X_t as solutions of a sequence of ordinary equations. These moments allow the determination of the moment generating function (MGF) and so of the density of the process.

3.1 Calculations of the moments of X_t

We want to compute the moments of X_t, that are $m_k(t) = \mathbb{E}[X_t^k], k \geq 1$. Dufresne takes $\alpha = b$, $\beta = c$ and $\gamma = \sqrt{2a}$ in (3.1): this notation is more convenient for the following calculations, so we will use it.

Let's apply Ito's formula to

$$dX_t = (\alpha X_t + \beta)dt + \gamma\sqrt{X_t}dW_t, \qquad X_0 = \bar{x} > 0 \tag{3.2}$$

with $f(x) = x^k$,

We find

$$\begin{aligned} dX^k =& \Big(kX^{k-1}(\alpha X + \beta) + k(k-1)X^{k-2}(\frac{1}{2}\gamma^2 X)\Big)dt + \Big((kX^{k-1})\gamma\sqrt{X}\Big)dW_t = \\ & (a_k X^k + b_k X^{k-1})dt + \gamma k X^{k-\frac{1}{2}} dW_t, \end{aligned} \tag{3.3}$$

with

$$a_k = \alpha k, \qquad b_k = \beta k + \frac{1}{2}\gamma^2 k(k-1). \tag{3.4}$$

We can write (3.3) as

$$X_t^k = \bar{x}^k + \int_0^t (a_k X_s^k + b_k X_s^{k-1})ds + \int_0^t \gamma k X_s^{k-\frac{1}{2}} dW_s. \tag{3.5}$$

We observe that $b(x) = \alpha x + \beta$ and $\sigma(x) = \gamma\sqrt{x}$ in (3.1) are linear, therefore for all $p \geq 1$ we have

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |X_t|^p\Big] < \infty.$$

Consequently, $X_t^k \in \mathbb{L}^2$ $\forall k \leq 0$, so the Ito integral above has expected value 0.

Therefore, taking the expected value of each side of (3.5), if we put $m_k(t) = \mathbb{E}[X_t^k]$ for $k \geq 1$ we find

$$m_k(t) = \bar{x}^k + \int_0^t (a_k m_k(s) + b_k m_{k-1}(s))ds.$$

Differentiating with respect to t we obtain

$$m_k'(t) = a_k m_k(t) + b_k m_{k-1}(t), \qquad k \geq 1, t > 0 \tag{3.6}$$

(since $\alpha_0 = \beta_0 = 0$) with initial condition

$$m_k(0) = \bar{x}^k. \tag{3.7}$$

We will solve this sequence of equations recursively, finding $\mathbb{E}[X_t^k]$ for all $k \geq 1$.

Theorem 3.1.1. *Let $a_0 = 0$, and suppose that the numbers $\{a_0, a_1, \dots, a_K\}$ are distincts. Then the solution of* (3.6) *with initial condition* (3.7) *and with $m_0(t) = 1$ is given by*

$$m_k(t) = \sum_{j=0}^{k} d_{kj} e^{a_j t}, \qquad k = 1, \dots K, \tag{3.8}$$

where

$$d_{kj} = \sum_{i=0}^{j} \bar{x}^i \left(\prod_{m=i+1}^{k} b_m \right) \prod_{h=i, h\neq j}^{k} \frac{1}{a_j - a_h}, \qquad j = 0, \dots, k \tag{3.9}$$

Proof. First of all, we want to prove that for all $k \geq 0$ we obtain $m_k(t)$ as a linear combination of $e^{a_r t}$, with $r \leq k$: we will demonstrate by induction on k that we can write

$$m_k(t) = \sum_{j=0}^{k} d_{kj} e^{a_j t} \tag{3.10}$$

as in (3.8) (when the coefficients d_{kj} are still unknown).
We see that this is true for $k = 0$, $m_0(t) = 1$.
We take $k = 1$, we multiply on the left and on the right of (3.6) by $e^{-a_1 t}$ and so we differentiate: we find

$$e^{-a_1 t} m_1'(t) = a_1 e^{-a_1 t} m_1(t) + b_1 e^{-a_1 t} m_1(t),$$

and so

$$\int_0^t e^{-a_1 s} m_1'(s) ds = a_1 \int_0^t e^{-a_1 s} m_1(s) ds + b_1 \int_0^t e^{-a_1 s} ds. \tag{3.11}$$

Moreover, we observe that

$$\begin{aligned} &\int_0^t e^{-a_1 s} m_1'(s) ds = \left[e^{-a_1 s} m_1(s) \right]_0^t + a_1 \int_0^t e^{-a_1 s} m_1(s) ds = \\ &e^{-a_1 t} m_1(t) - \bar{x} + a_1 \int_0^t e^{-a_1 s} m_1(s) ds. \end{aligned} \tag{3.12}$$

Therefore, comparing (3.11) and (3.12) we see that

$$e^{-a_1 t} m_1(t) = \bar{x} + b_1 \int_0^t e^{-a_1 s} ds,$$

and therefore that

$$m_1(t) = e^{a_1 t}\bar{x} + b_1 e^{a_1 t}\int_0^t e^{-a_1 s}ds = e^{a_1 t}\bar{x} + b_1\frac{e^{a_1 t}-1}{a_1} = -\frac{b_1}{a_1} + (\frac{b_1}{a_1}+\bar{x})e^{a_1 t}.$$

Consequently, (3.8) is valid for $k = 0$ and for $k = 1$.
We assume that it is true for $j = 0, 1, \ldots, k-1$; in this case, $m_k(t)$ will be the solution of an inhomogeneous ordinary differential equation with a forcing term equal to a combination of exponentials:

$$m_k'(t) - a_k m_k(t) = \sum_{j=0}^{k-1} C_j e^{a_j t}. \tag{3.13}$$

Multiplying by $e^{-a_k t}$ an then integrating, we obtain

$$\begin{aligned} m_k(t) =& \bar{x}^k e^{a_k t} + e^{a_k t}\int_0^t e^{-a_k s}\sum_{j=0}^{k-1} C_j e^{a_j s}ds = \\ & \bar{x}^k e^{a_k t} + e^{a_k t}\sum_{j=0}^{k-1} C_j \int_0^t e^{(a_j-a_k)s}ds = \\ & \bar{x}^k e^{a_k t} + e^{a_k t}\sum_{j=0}^{k-1}\frac{C_j}{a_j-a_k}(e^{(a_j-a_k)t}-1) = \\ & \bar{x}^k e^{a_k t} + \sum_{j=0}^{k-1}\frac{C_j}{a_j-a_k}(e^{a_j t}-e^{a_k t}). \end{aligned} \tag{3.14}$$

There are no problems at the denominator because the numbers $\{a_0, \ldots, a_K\}$ are distinct; therefore, we can say that $m_k(t)$ has the form

$$d_{k0} + d_{k1}e^{a_1 t} + \ldots + d_{kk}e^{a_k t}. \tag{3.15}$$

We have proved by induction that this is true for $k = 0, \ldots, K$, as we wanted.
Now, we want to see that the terms d_{kj} are as in (3.9).
We insert (3.8) in (3.6), and we obtain

$$\sum_{j=0}^{k} a_j d_{kj} e^{a_j t} = \sum_{j=0}^{k} a_k d_{kj} e^{a_j t} + \sum_{j=0}^{k-1} b_k d_{k-1,j} e^{a_j t},$$

which implies

$$d_{kj} = \frac{b_k}{a_j - a_k} d_{k-1,j} = \frac{b_k}{a_j - a_k} \cdots \frac{b_{j+1}}{a_j - a_{j+1}} d_{jj}, \qquad 0 \leq j \leq k. \tag{3.16}$$

Now we consider $m_k(t, \bar{x})$ as a function of the initial point $\bar{x}$, and we observe that $m_k(t)$ is a polynomial of degree k in $\bar{x}$.

$m_k(t)$ is of the form (3.15), so d_{kj} can not be a polynomial of degree bigger than k in $\bar{x}$, but from (3.16) we know that it has degree equal to j, since it is a constant multiplied by d_{jj}.

Therefore, we can write it as

$$d_{kj} = \sum_{i=0}^{j} d_{k,j,i} \bar{x}^i.$$

We know from (3.16) that

$$d_{k,j,i} = \frac{b_k}{a_j - a_k} d_{k-1,j,i} = \frac{b_k}{a_j - a_k} \cdots \frac{b_{j+1}}{a_j - a_{j+1}} d_{j,j,i}, \qquad 0 \leq i \leq j \leq k.$$

We also know that $d_{k,k,k} = 1$, since the leading coefficient of $m_k(t)$ as a polynomial in $\bar{x}$ is $e^{a_k t}$. The missing constants are $d_{k,k,i}$, with $i < k, k \geq 1$. The problem is thus the same for every power of $\bar{x}$: for all $i = 0, 1, \ldots,$ we have

$$\begin{aligned} & d_{i,i,i} = 1 \\ & d_{k,j,i} = \frac{b_k}{a_j - a_k} d_{k-1,j,i}, \qquad i \leq j \leq k \\ & \sum_{j=i}^{k} d_{k,j,i} = 0, \qquad i \leq k. \end{aligned} \tag{3.17}$$

The third equality comes from the initial condition $m_k(0) = \bar{x}^k$:

$$\bar{x}^k = m_k(0) = \sum_{j=0}^{k} d_{kj} = \sum_{j=0}^{k} \sum_{i=0}^{j} d_{k,j,i} \bar{x}^i = \sum_{i=0}^{k} \sum_{j=i}^{k} d_{k,j,i} \bar{x}^i = \sum_{i=0}^{k-1} \sum_{j=i}^{k} d_{k,j,i} \bar{x}^i + \bar{x}^k,$$

therefore $\sum_{j=i}^{k} d_{kji} = 0$.

We want to prove by induction that

$$d_{k,j,i} = \left(\prod_{m=i+1}^{k} b_m \right) \prod_{h=i, h \neq j}^{k} \frac{1}{a_j - a_h}, \qquad i \leq j \leq k. \tag{3.18}$$

If we do this, we demonstrate the equality (3.9) (since $d_{kj} = \sum_{i=0}^{j} d_{k,j,i}\bar{x}^i$). First of all, we prove this for $\{d_{k,j,0}, 0 \leq j \leq k\}$, then we will apply the same method to find $d_{k,j,i}$ when $i \neq 0$.

We use the following

Lemma 3.1.2.

$$\sum_{j=0}^{k} \prod_{h=0,h\neq j}^{k} \frac{1}{a_j - a_h} = 0, \tag{3.19}$$

where $a - 1, \ldots, a_k$ are all distinct.

Proof. We prove this equation by exploiting Lagrange's formula for partial fractions decomposition of a rational functions, that is

$$\frac{P(x)}{Q(x)} = \sum_{j=0}^{k} \frac{1}{x - a_j} \frac{P(a_j)}{Q'(a_j)},$$

where P is a polynomial with degree less or equal to k, an Q is a polynomial with $k+1$ zeros $\{a_0, \ldots, a_k\}$, all distinct.
If we take

$$Q(x) = \prod_{j=0}^{k}(x - a_j), \qquad P(x) = \frac{y - x}{Q(y)},$$

we find that

$$P(x) = \frac{P(x)}{Q(x)} Q(x) = \sum_{j=0}^{k} \frac{y - a_j}{(x - a_j)Q(y)Q'(a_y)} Q(x)$$

Letting $y \to x$ yields

$$0 = P(y) = \sum_{j=0}^{k} \frac{1}{Q'(a_y)} = \sum_{j=0}^{k} \prod_{h=0,h\neq j}^{k} \frac{1}{a_j - a_h},$$

as we wanted. □

Now, from the first and the third equalities of (3.17) we obtain

$$d_{k,0,0} = \left(\prod_{m=1}^{k} b_m\right) \prod_{h=0,h\neq 0}^{k} \frac{1}{a_0 - a_h}, \qquad k \geq 1.$$

Moreover, $d_{1,1,0} = -d_{1,0,0} = \frac{b_1}{(a_1-a_0)}$ (we obtain the first equality from $\sum_{j=i}^{k} d_{k,j,i} = 0$ in (3.17)) hence

$$d_{k,1,0} = \frac{b_k \cdots b_2}{(a_1 - a_k)\cdots(a_1 - a_2)} d_{1,1,0} = \left(\prod_{m=1}^{k} b_m\right) \prod_{h=0,h\neq 1}^{k} \frac{1}{a_1 - a_h}, \qquad k \geq 1.$$

We have seen that (3.18) is true for $k \geq 0$, $j = 0, 1$, $i = 0$.
For some integer $J > 0$ suppose that

$$d_{k,j,0} = \left(\prod_{m=1}^{k} b_m\right) \prod_{h=0,h\neq j}^{k} \frac{1}{a_j - a_h} \qquad k \geq j, \qquad j = 0, \ldots, J-1. \quad (3.20)$$

We obtain from the third equality of (3.17) that $\sum_{j=0}^{J} d_{J,j,0} = 0$, hence

$$d_{J,J,0} = -\sum_{j=0}^{J-1} d_{J,j,0} = -\sum_{j=0}^{J-1} \left\{ \left(\prod_{m=1}^{J} b_m\right) \prod_{h=0,h\neq j}^{J} \frac{1}{a_j - a_h} \right\} =$$
$$-\left(\prod_{m=1}^{J} b_m\right)\left(\sum_{j=0}^{J-1} \prod_{h=0,h\neq j}^{J-1} \frac{1}{a_j - a_h} - \prod_{h=0}^{J-1} \frac{1}{a_J - a_h}\right) =$$
$$\left(\prod_{m=1}^{J} b_m\right) \prod_{h=0,h\neq J}^{J} \frac{1}{a_J - a_h},$$

since the (3.19) implies

$$\sum_{j=0}^{J-1} \prod_{h=0,h\neq j}^{J-1} \frac{1}{a_j - a_h} = 0.$$

Consequently, for $k > J$ we obtain

$$d_{k,J,0} = \frac{b_k \cdots b_{J+1}}{(a_J - a_k)\cdots(a_J - a_{J+1})} d_{J,J,0} = \left(\prod_{m=1}^{k} b_m\right) \prod_{h=0,h\neq J}^{k} \frac{1}{a_J - a_h}. \quad (3.21)$$

Thus, we have seen by induction that (3.18) is true for all $0 \leq j \leq k$, for $i = 0$. Now we want to prove that it is true for all $i \leq j$.
To do it, we repeat more or less the previous passages, replacing k with $k+i$ j with $j+i$.
We want to show that

$$d_{k+i,j+i,i} = \left(\prod_{m=i+1}^{k+i} b_m\right) \prod_{h=i,h\neq j+i}^{k+i} \frac{1}{a_{j+i} - a_h}, \qquad j \leq k, i \geq 0. \qquad (3.22)$$

Let's see that it is true for $j = 0, 1$.

$$d_{k+i,i,i} = \left(\prod_{m=i+1}^{k+i} b_m\right) \prod_{h=i,h\neq i}^{k} \frac{1}{a_i - a_h}, \qquad k \geq 1, i \geq 0.$$

Moreover, $d_{i+1,i+1,i} = -d_{i+1,i,i} = \frac{b_{i+1}}{(a_{i+1}-a_i)}$, therefore

$$\begin{aligned} d_{k+i,i+1,i} =& \frac{b_{k+i}b_{k+i+1}\cdots b_{i+2}}{(a_{i+1}-a_{k+i})\cdots(a_{i+1}-a_{i+2})} d_{i+1,i+1,i} = \\ & \left(\prod_{m=i+1}^{k+i} b_m\right) \prod_{h=i,h\neq i+1}^{k+i} \frac{1}{a_{i+1} - a_h}, \qquad k \geq 1, i \geq 0. \end{aligned}$$

Let's follow the previous passages, and let's suppose for some integer J that

$$d_{k+i,j+i,i} = \left(\prod_{m=i+1}^{k+i} b_m\right) \prod_{h=i,h\neq j+i}^{k+i} \frac{1}{a_{j+i} - a_h} \qquad k \geq j, i \geq 0, j = 0, \ldots, J-1.$$

We see that

$$d_{J+i,J+i,i} = -\sum_{j=0}^{J-1} d_{J+i,j+i,i} = \left(\prod_{m=i+1}^{J+i} b_m\right) \prod_{h=0,h\neq J+i}^{J+i} \frac{1}{a_{J+i} - a_h},$$

and so that

$$d_{k+j,J+i,i} = \frac{b_{k+i}\cdots b_{i+J+1}}{(a_{J+i}-a_{k+i})\cdots(a_{J+i}-a_{J+i+1})} d_{J+i,J+i,i} = \left(\prod_{m=i+1}^{k+i} b_m\right) \prod_{h=i,h\neq J+i}^{k+i} \frac{1}{a_{J+i} - a_h},$$

as we wanted.

Thus, we have shown the following equality for $j \geq k$, $k \geq 1$, $i \geq 0$:

$$d_{k+i,j+i,i} = \left(\prod_{m=i+1}^{k+i} b_m\right) \prod_{h=i,h\neq j+i}^{k+i} \frac{1}{a_{j+i} - a_h}.$$

Now we return to k and j, with $k \geq j \geq i$, and we obtain

$$d_{k,j,i} = \left(\prod_{m=i+1}^{k} b_m\right) \prod_{h=i,h\neq j}^{k} \frac{1}{a_j - a_h}, \qquad i \leq j \leq k.$$

Therefore, we have proved the Theorem. □

Now we see another way to writing the moments of X_t, that will be useful to compute the MGF of the process itself.

Theorem 3.1.3. *Suppose $\alpha \neq 0$; if X_t satisfies (3.1) then its moments are*

$$\mathbb{E}[X_t^k] = \sum_{j=0}^{k} \theta_{kj} e^{a_j t}, \qquad k = 0, 1, \dots \tag{3.23}$$

where

$$\begin{aligned}
\theta_{kj} &= \sum_{i=0}^{j} \bar{x}^i \frac{k!(-1)^{k-j}\bar{u}^{k-i}}{i!(j-i)!(k-j)!} \frac{(\bar{v})_k}{(\bar{v})_i}, \qquad 0 \leq j \leq k \\
\bar{u} &= \frac{\gamma^2}{2\alpha}, \qquad \bar{v} = \frac{2\beta}{\gamma^2} \\
(y)_k &= \frac{(y+k-1)!}{(y-1)!}, \qquad k \geq 0.
\end{aligned}$$

Proof. We know that

$$\mathbb{E}[X_t^k] = \sum_{j=0}^{k} d_{kj} e^{a_j t}, \qquad k = 1, \dots K,$$

with

$$d_{kj} = \sum_{i=0}^{j} \bar{x}^i \left(\prod_{m=i+1}^{k} b_m \right) \prod_{h=i, h\neq j}^{k} \frac{1}{a_j - a_l}, \qquad j = 0, \dots, k.$$

Thus, we want to see that $\theta_{kj} = d_{kj}$.
First of all, $b_m = \beta m + \frac{1}{2}\gamma^2 m(m-1)$, so

$$\begin{aligned}
\prod_{m=1}^{k} b_m = \prod_{m=1}^{k} m\Big(\beta + \frac{1}{2}\gamma^2 m - \frac{\gamma^2}{2}\Big) &= \\
k!\Big(\beta + \frac{\gamma^2}{2}k - \frac{\gamma^2}{2}\Big)\Big(\beta + \frac{\gamma^2}{2}(k-1) - \frac{\gamma^2}{2}\Big) \cdots \Big(\beta + \frac{\gamma^2}{2}\Big)\beta &= \\
k!\Big(\frac{\gamma^2}{2}\Big)^k \Big(\frac{2\beta}{\gamma^2} + k - 1\Big)\Big(\frac{2\beta}{\gamma^2} + k - 2\Big) \cdots \Big(\frac{2\beta}{\gamma^2} + 1\Big)\Big(\frac{2\beta}{\gamma^2}\Big) &= \\
k!\Big(\frac{\gamma^2}{2}\Big)^k \Big(\frac{2\beta}{\gamma^2}\Big)_k &= \\
k!\Big(\frac{\gamma^2}{2}\Big)^k (\bar{v})_k.
\end{aligned}$$

Consequently, we have

$$\prod_{m=1}^{k} b_m = k!\Big(\frac{\gamma^2}{2}\Big)^k (\bar{v})_k. \tag{3.24}$$

Moreover,

$$\prod_{h=i,h\neq j}^{k} (a_j - a_h) = a^{k-i}\prod_{h=i}^{j-1}(j-h)\prod_{h=j+1}^{k}(j-h) = (-1)^{k-j}\alpha^{k-i}(j-i)!(k-j)!$$

hence

$$\begin{aligned}
d_{kj} =&\sum_{i=0}^{j}\bar{x}^i\Big(\prod_{m=i+1}^{k} b_m\Big)\Big(\prod_{h=i,h\neq j}^{k}\frac{1}{a_j-a_h}\Big) = \\
&\sum_{i=0}^{j}\bar{x}^i\frac{\Big(\prod_{m=1}^{k} b_m\Big)}{\Big(\prod_{m=1}^{i} b_m\Big)\Big(\prod_{h=i,h\neq j}^{k}(a_j-a_h)\Big)} = \\
&\sum_{i=0}^{j}\bar{x}^i\frac{k!(-1)^{k-j}(\frac{\gamma^2}{2\alpha})^{k-i}(\bar{v})_k}{i!(j-i)!(k-j)!(\bar{v})_i} = \\
&\sum_{i=0}^{j}\bar{x}^i\frac{k!(-1)^{k-j}\bar{u}^{k-i}}{i!(j-i)!(k-j)!}\frac{(\bar{v})_k}{(\bar{v})_i}
\end{aligned}$$

□

3.2 Calculation of the MGF

We want to compute the moment generating function of X_t, that is

$$\begin{aligned}
\mathbb{E}[e^{sX_t}] =&\mathbb{E}\Big[\sum_{k=0}^{\infty}\frac{(sX_t)^k}{k!}\Big] = \sum_{k=0}^{\infty}\frac{s^k}{k!}\mathbb{E}[X_t^k] = \\
&\sum_{k=0}^{\infty}\frac{s^k}{k!}\sum_{j=0}^{k}e^{a_jt}\sum_{i=0}^{j}\bar{x}^i\frac{k!(-1)^{k-j}\bar{u}^{k-i}}{i!(j-i)!(k-j)!}\frac{(\bar{v})_k}{(\bar{v})_i}.
\end{aligned} \tag{3.25}$$

We will evaluate this sum summing first over k, then over j and finally over i. To do it, we have to see that this sum converges; we will do it later.

We will use the formula

$$(1-y)^{-c} = \sum_{n=0}^{\infty} (c)_n \frac{y^n}{n!}, \qquad c \in \mathbb{R}, |y| < 1. \tag{3.26}$$

The equality (3.26) is valid because

$$\sum_{n=0}^{\infty} (c)_n \frac{y^n}{n!} = \sum_{n=0}^{\infty} \frac{(c+n-1)!}{(c-1)!} \frac{y^n}{n!} = \sum_{n=0}^{\infty} \binom{c+n-1}{n} y^n,$$

which is the expansion on Taylor series of $(1-y)^{-c}$ for $|y| < 1$.
We will apply this formula to

$$\sum_{k=j}^{\infty} \frac{(-s\bar{u})^{k-j}}{(k-j)!} (\bar{v}+j)_{k-j} = \sum_{n=0}^{\infty} \frac{(-s\bar{u})^n}{n!} (\bar{v}+j)_n = (1+s\bar{u})^{-\bar{v}-j}$$

if $|su| < 1$, and thus if $|s| < 1/|u|$, and to

$$\sum_{j=i}^{\infty} \frac{(\bar{v}+i)_{j-i}}{(j-i)!} \left(\frac{s\bar{u}e^{\alpha t}}{1+s\bar{u}} \right)^{j-i} = \left(1 - \frac{s\bar{u}e^{\alpha t}}{1+s\bar{u}} \right)^{-\bar{v}-i}$$

if $\left|\frac{s\bar{u}e^{\alpha t}}{1+s\bar{u}}\right| < 1$ and thus if $|s| < \frac{1}{\bar{u}(e^{\alpha t}-1)}$.
Hence, if $|s| < \frac{1}{\bar{u}(e^{\alpha t}-1)}$ we obtain

$$\begin{aligned}
&\sum_{i=0}^{\infty}\sum_{j=i}^{\infty}\sum_{k=j}^{\infty} s^k \frac{\bar{x}^i e^{a_j t} k!(-1)^{k-j}\bar{u}^{k-i}(\bar{v})_k}{k!i!(j-i)!(k-j)!(\bar{v})_i} = \\
&\sum_{i=0}^{\infty}\sum_{j=i}^{\infty} \bar{x}^i \frac{e^{\alpha j t} s^j \bar{u}^{j-i}(\bar{v})_j}{i!(j-i)!(\bar{v})_i} \sum_{k=j}^{\infty} \frac{(-su)^{k-j}}{(k-j)!}(\bar{v}+j)_{k-j} = \\
&\sum_{i=0}^{\infty}\sum_{j=i}^{\infty} \bar{x}^i \frac{e^{\alpha j t} s^j \bar{u}^{j-i}(\bar{v})_j}{i!(j-i)!(\bar{v})_i}(1+s\bar{u})^{-\bar{v}-j} = \\
&\sum_{i=0}^{\infty} \frac{\bar{x}^i s^i e^{\alpha i t}}{i!}(1+s\bar{u})^{-\bar{v}-i} \sum_{j=i}^{\infty} \frac{(\bar{v}+i)_{j-i}}{(j-i)!}\left(\frac{s\bar{u}e^{\alpha t}}{1+s\bar{u}}\right)^{j-i} = \\
&\sum_{i=0}^{\infty} \frac{\bar{x}^i s^i e^{\alpha i t}}{i!}(1+s\bar{u})^{-\bar{v}-i}\left(1-\frac{s\bar{u}e^{\alpha t}}{1+s\bar{u}}\right)^{-\bar{v}-i} = \\
&\sum_{i=0}^{\infty} \frac{\bar{x}^i s^i e^{\alpha i t}}{i!}(1-s\bar{u}(e^{\alpha t}-1))^{-\bar{v}-i} = \\
&\Big(1-s\bar{u}(e^{\alpha t}-1)\Big)^{-\bar{v}} \sum_{i=0}^{\infty}\left(\frac{s\bar{x}e^{\alpha t}}{1-s\bar{u}(e^{\alpha t}-1)}\right)^i \frac{1}{i!} = \\
&\Big(1-s\bar{u}(e^{\alpha t}-1)\Big)^{-\bar{v}} exp\left\{\frac{s\bar{x}e^{\alpha t}}{1-s\bar{u}(e^{\alpha t}-1)}\right\}
\end{aligned} \tag{3.27}$$

Therefore, we have shown that (if the sum in (3.25) converges) for $|s| < \frac{1}{\bar{u}(e^{\alpha t}-1)}$ we have

$$\mathbb{E}[e^{sX_t}] = \Big(1-s\bar{u}(e^{\alpha t}-1)\Big)^{-\bar{v}} exp\left\{\frac{s\bar{x}e^{\alpha t}}{1-s\bar{u}(e^{\alpha t}-1)}\right\}, \tag{3.28}$$

Now we want to see that (3.25) converges.
We have summed the terms

$$c_{ijk} = s^k \frac{\bar{x}^i e^{a_j t}(-1)^{k-j}\bar{u}^{k-i}(\bar{v})_k}{i!(j-i)!(k-j)!(\bar{v})_i}, \qquad 0 \le i \le j \le k < \infty,$$

and the passages that we have just seen show that

$$\sum_{0\le i\le j\le k<\infty} |c_{ijk}| = \Big(1-s\bar{u}(1+e^{\alpha t})\Big)^{-\bar{v}} exp\left\{\frac{s\bar{x}e^{\alpha t}}{1-s\bar{u}(1+e^{\alpha t})}\right\}.$$

for $|s| < \frac{1}{\bar{u}(e^{\alpha t}+1)}$.
Actually, the only difference here is that taking the absolut value the term $(-1)^{k-j}$ vanishes, and so we have

$$\sum_{k=j}^{\infty} \frac{(s\bar{u})^{k-j}}{(k-j)!}(\bar{v}+j)_{k-j} = (1-s\bar{u})^{-\bar{v}-j}$$

if $|su| < 1$ and then if $|s| < 1/|u|$ and

$$\sum_{j=i}^{\infty} \frac{(\bar{v}+i)_{j-i}}{(j-i)!}\left(\frac{s\bar{u}e^{\alpha t}}{1-s\bar{u}}\right)^{j-i} = \left(1-\frac{s\bar{u}e^{\alpha t}}{1-s\bar{u}}\right)^{-\bar{v}-i}$$

if $\left|\frac{s\bar{u}e^{\alpha t}}{1-s\bar{u}}\right| < 1$ and then if $|s| < \frac{1}{\bar{u}(e^{\alpha t}+1)}$.
Hence, if $|s| < \frac{1}{\bar{u}(e^{\alpha t}+1)}$ we obtain that:

- the sum (3.25) is convergent;
- the order of summation of the $c_{i,j,k}$ is irrelevant;
- the result is an analytic function of s, at least for $|s| < \frac{1}{\bar{u}(e^{\alpha t}+1)}$.
  Expression (3.28) is therefore correct for $s < \frac{1}{\bar{u}(e^{\alpha t}-1)}$, by analytic continuation: we can see the function $G(s) = \left(1-s\bar{u}(e^{\alpha t}-1)\right)^{-\bar{v}} exp\left\{\frac{s\bar{x}e^{\alpha t}}{1-s\bar{u}(e^{\alpha t}-1)}\right\}$ as the analytic continuation of $\mathbb{E}[e^{sX_t}]$ in $\{s < \frac{1}{\bar{u}(e^{\alpha t}-1)}\}$, because $G(s) = \mathbb{E}[e^{sX_t}]$ in $\{|s| < \frac{1}{\bar{u}(e^{\alpha t}+1)}\}$ and $G(s)$ is an analytic function defined in $\{s < \frac{1}{\bar{u}(e^{\alpha t}-1)}\}$.

Now, let's see that if we put

$$\mu_t = \frac{\gamma^2}{2}\left(\frac{e^{\alpha t}-1}{\alpha}\right) = \bar{u}(e^{\alpha t}-1), \qquad \lambda_t = \frac{2\alpha\bar{x}}{\gamma^2(1-e^{-\alpha t})}, \qquad \phi(s) = (1-s\mu_t)^{-1},$$

we have

$$\mathbb{E}[e^{sX_t}] = \phi(s)^{\bar{v}} exp\{\lambda_t(\phi(s)-1)\}. \tag{3.29}$$

Actually,

$$\mu_t\lambda_t = \frac{\bar{x}(e^{\alpha t}-1)}{1-e^{-\alpha t}} = \frac{\bar{x}(e^{\alpha t}-1)}{e^{-\alpha t}(e^{\alpha t}-1)} = \bar{x}e^{\alpha t},$$

so

$$\lambda_t(\phi(s)-1) = \frac{s\mu_t\lambda_t}{1-s\mu_t} = \frac{s\bar{x}e^{\alpha t}}{1-s\bar{u}(e^{\alpha t}-1)}.$$

Therefore, we have found that the MGF of the square root process is

$$\mathbb{E}[e^{sX_t}] = \phi(s)^{\bar{v}} exp\{\lambda_t(\phi(s)-1)\}.$$

Now, we note that:

- if $\bar{v} > 0$, $\phi(s)^{\bar{v}}$ is the MGF of a random variable with Gamma distribution of parameters $(\bar{v}, 1/\mu_t)$: we know that the MGF of $Y \sim \Gamma(\alpha, \lambda)$ is $\psi(s) = \left(\frac{\lambda}{\lambda-s}\right)^{\alpha}$, hence the MGF of a random variable $Z' \sim \Gamma(\bar{v}, 1/\mu_t)$ is
$$\left(\frac{1/\mu_t}{1/\mu_t - s}\right)^{\bar{v}} = \left(\frac{1}{1-s\mu_t}\right)^{\bar{v}} = \phi(s)^{\bar{v}}.$$
- $exp\{\lambda(\phi(s)-1)\}$ is the MGF of a compound Poisson/Exponential, that is a random variable
$$Z'' = \sum_{i=1}^{N} U_i,$$
where $N \sim$ **Poisson**(λ_t), $U_i \sim$ **Exp**$(1/\mu_t)$.

The MGF of the sum of two independent random variables is the product of the two MGFs, therefore we can give the following result:

Proposition 3.2.1. *The distribution of X_t is the distribution of the sum of two independent random variables X_t' and X_t'', where*

$$X_t' \sim \Gamma(\bar{v}, 1/\mu_t), \qquad X_t'' = \sum_{i=1}^{N} U_i,$$

*with $N \sim$ **Poisson**(λ_t), $U_i \sim$ **Exp**$(1/\mu_t)$.*
Therefore, the distribution of X_t is the convolution of the distributions of X_t' and X_t''.

We underline that we have asked that $\bar{v} = \frac{2\beta}{\gamma^2}$ is positive: therefore, it is here that we ask $\beta = c > 0$.
If $\beta = c$ is negative, the expression of the MGF is the same, but we can not say that is the sum of the two variables, one Gamma and the other compound Poisson/Exponential.

3.3 Calculation of the density

Now we want to compute the density of $X_t = X_t' + X_t''$, from the densities of X_t' and X_t''.

We indicate with

$$f_{\alpha,\lambda}(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

the density of a Gamma variable with parameters (α, λ).
Therefore, X_t' has density

$$f_{X_t'}(x) = f_{\bar{v},1/\mu_t}(x)$$

whereas X_t'' has density

$$f_{X_t''}(x) = \sum_{n=0}^{\infty} P(N = n) f_{n,1/\mu_t}(x) = \sum_{n=0}^{\infty} e^{-\lambda_t} \frac{\lambda_t^n}{n!} f_{n,1/\mu_t}(x),$$

since $N \sim \textbf{Poisson}(\lambda_t)$ and so $P(N = n) = e^{-\lambda_t} \frac{\lambda_t^n}{n!}$.
Hence, if we call $f_{X_t}(x)$ the density of X_t, we obtain

$$\begin{aligned} f_{X_t}(x) =& (f_{X_t'} * f_{X_t''})(y) = \Big(f_{\bar{v},1/\mu_t} * \sum_{n=0}^{\infty} e^{-\lambda_t} \frac{\lambda_t^n}{n!} f_{n,1/\mu_t} \Big)(x) = \\ & \sum_{n=0}^{\infty} e^{-\lambda_t} \frac{\lambda_t^n}{n!} (f_{\bar{v},1/\mu_t} * f_{n,1/\mu_t})(x) = \\ & \sum_{n=0}^{\infty} e^{-\lambda_t} \frac{\lambda_t^n}{n!} f_{\bar{v}+n,1/\mu_t}(x). \end{aligned}$$

Consequently,

$$f_{X_t}(x) = \sum_{n=0}^{\infty} e^{-\lambda_t} \frac{\lambda_t^n}{n!} f_{\bar{v}+n,1/\mu_t}(x). \tag{3.30}$$

To simplify the calculations, we put $U_t = X_t/\mu_t$: we know that (see [11]) if $Y$is a random variable with density f and $F \in C^1(\mathbb{R})$ is a monotonically increasing function, then the stochastic variable $Z = F(Y)$ has density

$$g(x) = f(G(x))G'(x),$$

where G is the inverse function of F.

In our case, we have that if Z is a Gamma variable with parameters $(\bar{v}+n, 1/\mu)$, if we put $F(y)=y/\mu$ we obtain

$$f_{F(Z)} = \frac{(1/\mu)^{\bar{v}+n}}{\Gamma(\bar{v}+n)}(\mu y)^{\bar{v}+n-1}e^{-\frac{\mu y}{\mu}}\mu = \frac{1}{\Gamma(\bar{v}+n)}y^{\bar{v}+n-1}e^{-y},$$

and then $F(Z)$ has Gamma distribution of parameters $(\bar{v}+n,1)$.

Therefore, if we put $U_t = X_t/\mu_t$, the density of U_t will be

$$\begin{aligned} f_{U_t}(x) = &\sum_{n=0}^{\infty} e^{-\lambda_t}\frac{\lambda_t^n}{n!}f_{\bar{v}+n,1}(x) = \\ &\sum_{n=0}^{\infty} e^{-\lambda_t}\frac{\lambda_t^n}{n!}\frac{x^{\bar{v}+n-1}}{\Gamma(\bar{v}+n)}e^{-x}\mathbb{1}_{\{x>0\}} = \\ &\left(\frac{x}{\lambda_t}\right)^{\frac{\bar{v}-1}{2}} e^{-\lambda_t - x}\sum_{n=0}^{\infty}\frac{(\sqrt{\lambda_t x})^{\bar{v}-1+2n}}{n!\Gamma(\bar{v}+n)}\mathbb{1}_{\{x>0\}} = \\ &\left(\frac{x}{\lambda_t}\right)^{\frac{\bar{v}-1}{2}} e^{-\lambda_t - x} I_{\bar{v}-1}(2\sqrt{\lambda_t x})\mathbb{1}_{\{x>0\}}, \end{aligned}$$

where

$$I_\nu(z) = \sum_{n=0}^{\infty}\frac{(z/2)^{\nu+2n}}{n!\Gamma(n+\nu+1)},$$

is the modified Bessel function of the first kind.

The density of X_t therefore is

$$\begin{aligned} f_{X_t}(x) = &\frac{1}{\mu_t}\left(\frac{x}{\mu_t\lambda_t}\right)^{\frac{\bar{v}-1}{2}} e^{-\lambda_t - \frac{x}{\mu_t}} I_{\bar{v}-1}\left(2\sqrt{\frac{\lambda_t}{\mu_t}x}\right) = \\ &\frac{1}{\mu_t}\left(\frac{xe^{-\alpha t}}{\bar{x}}\right)^{\frac{\bar{v}-1}{2}} e^{-\lambda_t - \frac{x}{\mu_t}} I_{\bar{v}-1}\left(\frac{4\alpha}{\gamma^2(e^{\alpha t}-1)}\sqrt{\bar{x}xe^{\alpha t}}\right), \end{aligned}$$

where the last equality comes from $\mu_t = \frac{\gamma^2}{2\alpha}(e^{\alpha t}-1)$, $\lambda_t = \frac{2\alpha}{\gamma^2}\frac{\bar{x}}{1-e^{-\alpha t}}$,

$$\mu_t\lambda_t = e^{\alpha t}\bar{x}, \qquad \frac{\lambda_t}{\mu_t} = \left(\frac{2\alpha}{\gamma^2}\right)^2\frac{\bar{x}e^{\alpha t}}{(e^{\alpha t}-1)^2}.$$

Thus, we have proved that

$$f_{X_t}(x) = \frac{1}{\mu_t}\left(\frac{xe^{-\alpha t}}{\bar{x}}\right)^{\frac{\bar{v}-1}{2}} e^{-\lambda_t - \frac{x}{\mu_t}} I_{\bar{v}-1}\left(\frac{4\alpha}{\gamma^2(e^{\alpha t}-1)}\sqrt{\bar{x}xe^{\alpha t}}\right).$$

Using Feller's notation, and so putting $a = \frac{\gamma^2}{2}, b = \alpha, c = \beta, \bar{x} = \bar{x}$, we find

$$\mu_t = \frac{a}{b}(e^{bt} - 1), \qquad \lambda_t = \frac{b\bar{x}}{a(1 - e^{-bt})}, \qquad \bar{v} = \frac{2\beta}{\gamma^2} = \frac{c}{a}.$$

Therefore, calling $p_2(t, x, \bar{x})$ the density $f_{X_t}(x)$ of X_t, we have

$$p_2(t, x, \bar{x}) = \frac{b}{a(e^{bt} - 1)} exp\left\{ -\frac{b(x + \bar{x}e^{bt})}{a(e^{bt} - 1)} \right\} \left(\frac{e^{-bt}x}{\bar{x}} \right)^{\frac{c-a}{2a}} I_{c/a-1}\left(\frac{2b}{a(1 - e^{-bt})} \sqrt{e^{-bt}x\bar{x}} \right).$$

This is the transition density of the square root process.

3.4 Non-central chi-squared distribution

It is well known that the density of a random variable with non-central chi-square distribution with k degrees of freedom and non-centrality parameter λ is

$$f_{k,\lambda}(x) = \frac{1}{2} e^{-\frac{x+\lambda}{2}} \left(\frac{x}{\lambda} \right)^{\frac{k}{4} - \frac{1}{2}} I_{\frac{k}{2}-1}(\sqrt{\lambda x}).$$

One can immediately note that this is the same form of the transition density p_2, for $c > 0$; actually, if we take $\lambda' = \frac{2\bar{x}be^{bt}}{a(e^{bt}-1)}$ and $k' = \frac{2c}{a}$, we have that

$$\begin{aligned}
& f_{k',\lambda'}\left(\frac{2bx}{a(e^{bt} - 1)} \right) = \\
& \frac{1}{2} \exp\left\{ -\frac{bx}{a(e^{bt} - 1)} - \frac{\bar{x}be^{bt}}{a(e^{bt} - 1)} \right\} \left(\frac{x}{\bar{x}e^{bt}} \right)^{\frac{c}{2a} - \frac{1}{2}} I_{c/a-1}\left(\sqrt{\frac{4b^2 x\bar{x}e^{bt}}{a^2(e^{2(bt-1)})}} \right) = \\
& exp\left\{ -\frac{b(x + \bar{x}e^{bt})}{a(e^{bt} - 1)} \right\} \left(\frac{e^{-bt}x}{\bar{x}} \right)^{\frac{c-a}{2a}} I_{c/a-1}\left(\frac{2b}{a(1 - e^{-bt})} \sqrt{e^{-bt}x\bar{x}} \right) = \\
& \frac{a(e^{bt} - 1)}{2b} p_2(t, x, \bar{x}).
\end{aligned}$$

Consequently, for $c > 0$

$$\begin{aligned}
P(X_t < z | X_0 = \bar{x}) = & \int_0^z p_2(t, x, \bar{x}) dx = \frac{2b}{a(e^{bt} - 1)} \int_0^z f_{k',\lambda'}\left(\frac{2bx}{a(e^{bt} - 1)} \right) dx = \\
& \int_0^y f_{k',\lambda'}(x) dx
\end{aligned}$$

with $y = \frac{2bz}{a(e^{bt}-1)}$.
Therefore we have

$$F\left(\frac{2bz}{a(e^{bt}-1)}, \frac{2c}{a}, \frac{2\bar{x}be^{bt}}{a(e^{bt}-1)}\right),$$

where $F(z, k, \lambda)$ is the well known cumulative distribution function for the non-central chi-square distribution with k degrees of freedom and the non-centrality parameter k.

Chapter 4

Calculation of the derivative

Now we take the transition density

$$p_2(t,x,\bar{x}) = \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} I_{c/a-1}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right).$$

and we compute the derivative.
Let's write p_2 as a function of x only, $p_2(x) = F(x)G(x)$, where

$$F(x) = \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}}$$

and

$$\begin{aligned} G(x) =& I_{c/a-1}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right) = \\ & \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+c/a)}\left(\frac{b}{a(1-e^{-bt})}\sqrt{e^{-bt}\bar{x}}\right)^{2n+c/a-1} x^{n+\frac{c}{2a}-\frac{1}{2}}. \end{aligned}$$

Let us derive $F(x)$ and then $G(x)$:

$$F'(x) = \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}}\left(\frac{b}{a(1-e^{bt})} + \frac{c/(2a)-1/2}{x}\right)$$

and

$$G'(x) = \sum_{n=0}^{\infty}\left(\frac{n+c/(2a)-1/2}{x}\right)\frac{1}{n!\Gamma(n+c/a)}\left(\frac{b}{a(1-e^{-bt})}\sqrt{e^{-bt}\bar{x}}\right)^{2n+c/a-1} x^{n+\frac{c}{2a}-\frac{1}{2}}.$$

Therefore, we have that

$$p_2'(x) =$$
$$F(x)G'(x) + F'(x)G(x) =$$
$$\frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} \cdot$$
$$\left(\sum_{n=0}^{\infty}\left(\frac{n+c/(2a)-1/2}{x}\right)\frac{1}{n!\Gamma(n+c/a)}\left(\frac{b}{a(1-e^{-bt})}\sqrt{e^{-bt}\bar{x}}\right)^{2n+c/a-1} x^{n+\frac{c}{2a}-\frac{1}{2}}+\right.$$
$$\left.\sum_{n=0}^{\infty}\left(\frac{b}{a(1-e^{bt})}+\frac{c/(2a)-1/2}{x}\right)\frac{1}{n!\Gamma(n+c/a)}\left(\frac{b}{a(1-e^{-bt})}\sqrt{e^{-bt}\bar{x}}\right)^{2n+c/a-1} x^{n+\frac{c}{2a}-\frac{1}{2}}\right) =$$
$$\frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} \cdot$$
$$\sum_{n=0}^{\infty}\left(\left(\frac{b}{a(1-e^{bt})}+\frac{n+c/a-1}{x}\right)\frac{1}{n!\Gamma(n+c/a)}\left(\frac{b}{a(1-e^{-bt})}\sqrt{e^{-bt}\bar{x}}\right)^{2n+c/a-1} x^{n+\frac{c}{2a}-\frac{1}{2}}\right).$$

Therefore, we have seen that the derivative of the transition density with respect to x is

$$\partial_x p_2(t,x,\bar{x}) = \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} \cdot$$
$$\sum_{n=0}^{\infty}\left(\left(\frac{b}{a(1-e^{bt})}+\frac{n+c/a-1}{x}\right)\frac{1}{n!\Gamma(n+c/a)}\left(\frac{b}{a(1-e^{-bt})}\sqrt{e^{-bt}\bar{x}}\right)^{2n+c/a-1} x^{n+\frac{c}{2a}-\frac{1}{2}}\right).$$

We can notice that it differs from the expression of the transition density itself only for the term $\left(\frac{b}{a(1-e^{bt})}+\frac{n+c/a-1}{x}\right)$ in the sum.

Chapter 5

Behaviour for large values of x

We want to study the behaviour of the density

$$p_2(t,x,\bar{x}) = \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} I_{c/a-1}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right)$$

for large values of x.

Obviously we expected that the limit of $p_2(t,x,\bar{x})$ for $x \to \infty$ will be zero, but we want to verify this and to see with which rate $p_2(t,x,\bar{x})$ tends to zero. We give the following fundamental result about the behaviour of the modified Bessel function of the first kind $I_\nu(z)$ for large values of z (see for example pp 355 of [1]).

Lemma 5.0.1. *The modified Bessel function of the first kind*

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{\nu+2n}}{n!\Gamma(n+\nu+1)}$$

has an asymptotic expansion for large values of z

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}\left(1+\frac{\mu-1}{8z}+\frac{(\mu-1)(\mu-9)}{2!(8z)^2}-\frac{(\mu-1)(\mu-9)(\mu-24)}{3!(8z)^3}+\dots\right)$$

Now, we can find the following result:

Theorem 5.0.2. *The transition density $p_2(t,x,\bar{x})$ has the following asymptotic expansion for large values of x:*

$$p_2(t,x,\bar{x}) \sim x^{\frac{c}{2a}-\frac{3}{4}} \exp\left\{\frac{b}{a(e^{bt}-1)}\left(-x+2\sqrt{e^{bt}\bar{x}}\sqrt{x}\right)\right\} \tag{5.1}$$

Proof. We demonstrate the Theorem by exploiting the previous result, and by an easy calculation:

$$
\begin{aligned}
p_2(t,x,\bar{x}) =& \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} I_{c/a-1}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right) \sim \\
& x^{\frac{c-a}{2a}} \exp\left\{-\frac{bx}{a(e^{bt}-1)}\right\}\exp\left\{\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right\}\left(\frac{4\pi b}{a(1-e^{-bt})}\sqrt{e^{-bt}\bar{x}}\sqrt{x}\right)^{-1/2} \sim \\
& x^{\frac{c}{2a}-\frac{3}{4}} \exp\left\{\frac{b}{a(e^{bt}-1)}\left(-x+2\sqrt{e^{bt}\bar{x}}\sqrt{x}\right)\right\}
\end{aligned}
$$

□

We can also write

$$
p_2(t,x,\bar{x}) \sim x^{\frac{c}{2a}-\frac{3}{4}} \exp\left\{-\frac{b}{a(e^{bt}-1)}\sqrt{x}\left(\sqrt{x}-2\sqrt{e^{bt}\bar{x}}\right)\right\}.
$$

Consequently, we can say that the transition density $p_2(t,x,\bar{x})$ tends to zero when $x \to \infty$ faster than a negative exponential of the square root of x, and slower then a negative exponential of x.
There is also a term $x^{\frac{c}{2a}-\frac{3}{4}}$, that obviously tends to zero if $c < \frac{3}{2}$ and goes to infinity if $c > \frac{3}{2}$, but however it is dominated by the exponential.
Therefore, an important result that we find is the following:

Corollary 5.0.3. *For all $n \in \mathbb{N}$*

$$
\lim_{x\to\infty} x^n p_2(t,x,\bar{x}) = 0.
$$

Now, looking (5.1), we want to study how the parameters a, b, c, $\bar{x}$ and t influence the convergence of p_2 at zero for $x \to \infty$:

- the leading term is obviously the exponential, hence we can say that c does not affect so much p_2 for large values of x: this is not surprising, because for large values of x the drift $bx + c$ is much more affected by b;

- the leading term of the exponential is $-\frac{b}{a(e^{bt}-1)}x$, therefore we can say that when a, b and t are big the density tends to zero slower. In

particular, b and t appear in this term with an exponential e^{-bt}, so they affect in a marked way the convergence of p_2.
Actually, this is reasonable, because it is more probable that the process reaches big values when it is passed a lot of time, and if the drift is high it will push the process in regions $\{x >> 0\}$.

- $\bar{x}$ appears in a square root in the term that multiply the square root of x, so when it is big the density tends to zero less quickly. This does not surprise ourselves, because it is the starting point, and it is quite obvious that there is an higher probability that a process will be in a region far from the origin if it starts from a point far from the origin. However, we see that it do not affect too much the behaviour of the density, because it is not in the leading term.

Therefore, we have seen that for large values of x the parameters that mainly affect the behaviour of the density are b and the time t, more then the starting point.

5.1 Behaviour of the derivative

We can also examine the behaviour of the derivative:

$$\begin{aligned}\frac{\partial p_2(t,x,\bar{x})}{\partial x} \sim & \frac{\partial}{\partial x}\left(x^{\frac{c}{2a}-\frac{3}{4}} \exp\left\{\frac{b}{a(e^{bt}-1)}\left(-x+2\sqrt{e^{bt}\bar{x}}\sqrt{x}\right)\right\}\right) = \\ & \exp\left\{\frac{b}{a(e^{bt}-1)}\left(-x+2\sqrt{e^{bt}\bar{x}}\sqrt{x}\right)\right\}\left[\left(-\frac{3}{4}+\frac{c}{a}\right)x^{\frac{c}{2a}-\frac{7}{4}}+\right. \\ & \left.\frac{b}{a(e^{bt}-1)}\left(\sqrt{\bar{x}e^{bt}}x^{\frac{c}{2a}-\frac{5}{4}}-x^{\frac{c}{2a}-\frac{3}{4}}\right)\right] \sim \\ & \exp\left\{\frac{b}{a(e^{bt}-1)}\left(-x+2\sqrt{e^{bt}\bar{x}}\sqrt{x}\right)\right\}\left(x^{\frac{c}{2a}-\frac{7}{4}}+x^{\frac{c}{2a}-\frac{5}{4}}-x^{\frac{c}{2a}-\frac{3}{4}}\right)\end{aligned}$$

Therefore, we can say that for all $n \in \mathbb{N}$

$$\lim_{x\to\infty} \frac{\partial}{\partial x}\left(x^n p_2(t,x,\bar{x})\right) = 0.$$

5.2 Behaviour of the MGF

We remind that Dufresne has found the expression of the MGF of the process $\mathbb{E}[e^{sX_t}]$ when $s < \frac{b}{a(e^{bt}-1)}$, that is for the values of s for which the sum $\sum_{k=0}^{\infty} \frac{s^k}{k!}\mathbb{E}[X_t^k] = \mathbb{E}[e^{sX_t}]$ converges.
Now, we see that our result on the asymptotic behaviour of the transition density is exactly in accordance with the calculation of Dufresne: if we give a look to

$$\mathbb{E}[e^{sX_t}] = \int_0^\infty e^{sx} p_2(t,x,\bar{x})dx,$$

we see that $e^{sx}p_2(t,x,\bar{x})$ has an asymptotic expansion

$$\begin{aligned} e^{sx}p_2(t,x,\bar{x}) \sim & x^{\frac{c}{2a}-\frac{3}{4}} \exp\left\{ \frac{b}{a(e^{bt}-1)} \left(-x + 2\sqrt{e^{bt}\bar{x}}\sqrt{x} \right) + sx \right\} = \\ & x^{\frac{c}{2a}-\frac{3}{4}} \exp\left\{ x\left(s - \frac{b}{a(e^{bt}-1)} \right) + \frac{2b\sqrt{e^{bt}\bar{x}}}{a(e^{bt}-1)}\sqrt{x} \right\}, \end{aligned}$$

that tends to zero if and only if $s < \frac{b}{a(e^{bt}-1)}$.
Finally, we notice that the asymptotic behaviour of $p_2(t,x,\bar{x})$ does not depend by the index of the Bessel function, so we obtain the same results for p_1, that is identical to p_2 except that for the index of the Bessel function itself.

Chapter 6

Transition density and defective density

We have seen how Feller finds the unique fundamental solution of the forward Kolmogorov equation associated to the SDE (1.1) in the case $c \leq 0$, and we have proved that it is

$$p_1(t,\bar{x},x) = \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} I_{1-c/a}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right).$$

We emphasize that we do not know *a priori* if this is the transition density of the square root process.
In the case $c > 0$, instead, we do know the transition density of the process: as we have already seen, with the calculation proposed by Dufresne, it is

$$p_2(t,x,\bar{x}) = \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} I_{c/a-1}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right).$$

6.1 Feller's results for $c > 0$

Now we give an overview of the results found by Feller on the fundamental solutions of (1.3) in the case $c > 0$.
We return to the point in which Feller finds that the Laplace transform of a

fundamental solution of (2.4) must be of the form

$$\begin{aligned}\omega(t,s,\bar{x}) = &\left(\frac{b}{as(e^{bt}-1)+b}\right)^{c/a} exp\left\{\frac{-\bar{x}sbe^{bt}}{sa(e^{bt}-1)+b}\right\}+\\ &\int_0^t f(\tau)\left(\frac{b}{sa(e^{b(t-\tau)}-1)+b}\right)^{c/a},\end{aligned} \tag{6.1}$$

where $f(t) = -\lim_{x\to 0}\Big(\partial_x\big(axu(t,x)\big) + (bx+c)u(t,x)\Big)$ is the flux at the origin. We have found (6.1) as solution of the problem

$$\begin{cases}\partial_t\omega(t,s) + s(as-b)\partial_s\omega(t,s,\bar{x}) = -cs\omega(t,s,\bar{x}) + f(t),\\ \omega(0,s) = e^{-s\bar{x}}.\end{cases} \tag{6.2}$$

As we have already seen, Feller proves that for $c \leq 0$ the flux has to satisfy equation (2.11), that has unique solution; he solves this equation, he find therefore the Laplace transform of the solution and he invert it.
On the other hand, when c is positive Feller distinguishes two cases: $0 < c < a$ and $c \geq a$.
We have the following situation:

- when $0 < c < a$ there are two solutions ω_1 and ω_2 of problem (6.7) that are Laplace transforms of a fundamental solution of (1.3), and they correspond to $f \neq 0$ and $f = 0$ respectively. If $f \neq 0$, f has to satisfy (2.11), as in the case $c \leq 0$ (and so we know that is $f < 0$). Hence, the inverse of ω_1 will be obviously the fundamental solution found for $c \leq 0$, that is $p_1(t,x,\bar{x})$.

- When $c \geq a$, there is as unique solution of (6.7) that is a Laplace transform of a fundamental solution of (1.3), and correspond to $f = 0$: it is ω_2.

When $f = 0$ the Laplace transform of the corresponding fundamental solution of (1.3) is

$$\omega_2(t,s,\bar{x}) = \left(\frac{b}{as(e^{bt}-1)+b}\right)^{c/a} exp\left\{\frac{-s\bar{x}be^{bt}}{sa(e^{bt}-1)+b}\right\}, \tag{6.3}$$

as we can easily check by putting $f = 0$ in (6.9), but Feller does not invert it.

We can see that ω_2 is the Laplace transform of the transition density of the square root process, that is p_2. We can directly see this by integrating p_2 multiplied by e^{-sx}, but we can also considerate that as we have seen Dufresne finds that

$$\mathbb{E}[e^{sX_t}] = \int_0^\infty e^{sx} p_2(t,x,\bar{x})dx = \Big(1 - s\bar{u}(e^{\alpha t}-1)\Big)^{-\bar{v}} exp\Big\{\frac{s\bar{x}e^{\alpha t}}{1 - s\bar{u}(e^{\alpha t}-1)}\Big\},$$

with $\alpha = b, \bar{u} = \frac{a}{b}, \bar{v} = \frac{c}{a}$, when $|s| < \frac{1}{\bar{u}(e^{\alpha t}-1)} = \frac{b}{a(e^{bt}-1)}$
Therefore, we find

$$\int_0^\infty e^{sx} p_2(t,x,\bar{x})dx = \Big(-s\frac{a}{b}(e^{bt}-1)\Big)^{-c/a} exp\Big\{\frac{s\bar{x}e^{bt}}{1 - s\frac{a}{b}(e^{bt}-1)}\Big\} = \left(\frac{b}{-as(e^{bt}-1)+b}\right)^{c/a} \exp\left\{\frac{s\bar{x}be^{bt}}{-as(e^{bt}-1)+b}\right\}.$$

Consequently, changing the sign of s we have

$$\omega(t,s,\bar{x}) = \int_0^\infty e^{-sx} p_2(t,\bar{x},x)dx = \left(\frac{b}{as(e^{bt}-1)+b}\right)^{c/a} \exp\left\{\frac{-s\bar{x}be^{bt}}{sa(e^{bt}-1)+b}\right\},$$

therefore we have $\omega(t,s,\bar{x}) = \omega_2(t,s,\bar{x})$.
Hence, the transition density found with Dufresne's calculation is the fundamental solution of (2.4) with Laplace transform $\omega_2(t,s,\bar{x})$, and so it is the fundamental solution that corresponds to $f = 0$.
Or, to say this in another way: when the flux at the origin is zero (and it can be zero only for $c > 0$) the fundamental solution of the forward Kolmogorov equation (2.4) is also the transition density of the process, and it has the form

$$p_2(t,x,\bar{x}) = \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} I_{c/a-1}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right).$$

6.2 A relation between fundamental solutions and transition density

We know that when $c > 0$ and we take a zero flux at the origin we find a fundamental solution of the forward Kolmogorov equation (1.3) that is also the transition density of the process.
But we still do not know if the fundamental solution $p_1(t, x, \bar{x})$, that we find when c is negative and when $0 < c < a$ and $f < 0$ is or not the transition density of the process.
Let's see the following result (one can find it in [4]):

Proposition 6.2.1. *The function $w(t, s) = \mathbb{E}[e^{-sX_t}]$ with $s > 0$ and $t \geq 0$ is the unique solution of the first-order PDE*

$$\begin{cases} \frac{\partial w}{\partial t} + (as^2 - bs)\, \frac{\partial w}{\partial s} + csw = 0, \\ w(0, s) = e^{-s\bar{x}}. \end{cases} \tag{6.4}$$

Proof. The Ito formula gives

$$e^{-sX_t} = e^{-sx} + \int_0^t e^{-sX_\tau}(as^2 X_\tau - s(bX_\tau + c))d\tau - sM_t,$$

where $M_t = \int_0^t e^{-sX_\tau}\sqrt{2aX_\tau}dW_\tau$ is a local martingale.
But M_t is also a martingale, because

$$\mathbb{E}\left[\int_0^t (\sqrt{X_\tau}e^{-sX_\tau})^2 d\tau\right] = \mathbb{E}\left[\int_0^t X_\tau e^{-2sX_\tau} d\tau\right] \leq \mathbb{E}\left[\int_0^t \frac{d\tau}{2es}\right] < \infty.$$

Consequently $\mathbb{E}[M_t] = M_0 = 0$, and so

$$\mathbb{E}[e^{-sX_t}] - as^2 \int_0^t \mathbb{E}[X_\tau e^{-sX_\tau}]d\tau - e^{-sx} + s\int_0^t \mathbb{E}[(bX_\tau + c)e^{-sX_\tau}]d\tau = 0, \tag{6.5}$$

therefore we have

$$\mathbb{E}[e^{-sX_t}] + (sb - as^2)\int_0^t \mathbb{E}[X_\tau e^{-sX_\tau}]d\tau - e^{-sx} + cs\int_0^t \mathbb{E}[e^{-sX_\tau}]d\tau = 0$$

We note that $\mathbb{E}[X_\tau e^{-sX_\tau}] = -\frac{\partial}{\partial s}\mathbb{E}[e^{-sX_\tau}]$.
Differentiation of (6.5) to t gives (6.8).

Now we want to prove the uniqueness of the solution: $(bx+c)e^{-sx}$ is bounded and measurable for $s > 0$, so $\mathbb{E}[(bX_t + c)]e^{-sX_t}$ is continuous in t, and for each fixed t it is analytic in s. The coefficients of the PDE are analytic, and the initial datum is analytic too. Therefore, the uniqueness of the solution follows from the Cauchy-Kovalevskaya Theorem for first-order systems. □

This result tells us a lot about the problem of establish the condition under which a fundamental solution of the forward Kolmogorov equation associated to (1.1) is also the transition density of the process.
In fact, we notice that the PDE

$$\begin{cases} \frac{\partial w}{\partial t} + (as^2 - bs)\,\frac{\partial w}{\partial s} + csw = 0, \\ w(0,s) = e^{-s\bar{x}}, \end{cases} \tag{6.6}$$

which has $w(t,s) = \mathbb{E}[e^{-sX_t}]$ as unique solution, is the PDE found by Feller for the Laplace transform of a fundamental solution,

$$\begin{cases} \frac{\partial w}{\partial t} + (as^2 - bs)\,\frac{\partial w}{\partial s} + csw = f(t), \\ w(0,s) = e^{-s\bar{x}}, \end{cases} \tag{6.7}$$

in which $f = 0$.
Therefore, we have the following situation:

- one can state that the fundamental solution with $f = 0$ is the transition density of the process, even if he do not know the expression of its Laplace transform and the expression of the Laplace transform of the density (that is the way by which we proved that this fundamental solution is the density found by Dufresne): let's call $p(t, x, \bar{x})$ the fundamental solution correspondent to $f = 0$; both $\omega(t, s, \bar{x}) = \int_0^\infty e^{-sx}p(t, x, \bar{x})dx$ and $\mathbb{E}[e^{-sX_t}]$ solve (6.8), and the solution is unique. Therefore,
$$\mathbb{E}[e^{-sX_t}] = \int_0^\infty e^{-sx}p(t, x, \bar{x})dx,$$
so $p(t, x, \bar{x})$ is the transition density of the process.
This is another way to prove that the fundamental solution correspon-

dent to $f = 0$ is the transition density of the process, and perhaps it is finer than the one illustrated before.

- if $f < 0$, the fundamental solution that we find, that is $p_1(t, x, \bar{x})$, is not the transition density of the process: $\omega_1(t, s) = \int_0^\infty e^{-sx} p_1(t, x, \bar{x}) dx$ solves the PDE (6.7) with $f < 0$, so it can not solve (6.8), which on other hand is solved by $\mathbb{E}[e^{-sX_t}]$.
  Therefore,
  $$\mathbb{E}[e^{-sX_t}] \neq \int_0^\infty e^{-sx} p_1(t, x, \bar{x}) dx,$$
  and so $p_1(t, x, \bar{x})$ is not the transition density of the process.

6.3 Another way to find the density

Proposition 6.2.1 suggests another method to find the transition density of the square root process, for $c > 0$.
We know that for $s > 0$, $t \geq 0$, $\mathbb{E}[e^{-sX_t}]$ is the unique solution of the problem

$$\begin{cases} \frac{\partial w}{\partial t} + (as^2 - bs) \frac{\partial w}{\partial s} + csw = 0, \\ w(0, s) = e^{-s\bar{x}}, \end{cases} \tag{6.8}$$

so we can solve (6.8) by the method of characteristics, that is exactly what Feller has done for the PDE (6.7), finding

$$\begin{aligned} \omega(t, s, \bar{x}) = & \left(\frac{b}{as(e^{bt} - 1) + b} \right)^{c/a} exp \left\{ \frac{-\bar{x} s b e^{bt}}{sa(e^{bt} - 1) + b} \right\} + \\ & \int_0^t f(\tau) \left(\frac{b}{sa(e^{b(t-\tau)} - 1) + b} \right)^{c/a}. \end{aligned} \tag{6.9}$$

Here $f = 0$, so we will find

$$\omega_2(t, s, \bar{x}) = \left(\frac{b}{as(e^{bt} - 1) + b} \right)^{c/a} exp \left\{ \frac{-\bar{x} s b e^{bt}}{sa(e^{bt} - 1) + b} \right\}.$$

We can invert this transform by the method shown by Dufresne, and illustrated in Chapter 3: changing the sign of s in (6.9) we find the MGF of the process,

$$\mathbb{E}[e^{sX_t}] = \left(\frac{b}{-as(e^{bt}-1)+b}\right)^{c/a} exp\left\{\frac{\bar{x}sbe^{bt}}{-sa(e^{bt}-1)+b}\right\} = (\phi(s))^{\bar{v}}\, e^{\lambda_t(\phi(s)-1)},$$

and we calculate the density noticing that this is the MGF of the sum of a Gamma variable and a compound Poisson.
We remind that Dufresne computes the transition density of the process for $b \neq 0$, whereas this method works also for $b = 0$.
Nevertheless, there are two points that we have to underline:

- Dufresne finds the expression of the MGF of the process for $s < \frac{b}{a(e^{bt}-1)}$, that is for all s such that the sum by which one compute the MGF itself converges.
  Whereas, by this last method we find the MGF only for negative values of s, because we find $\mathbb{E}[e^{-sX_t}]$ as the unique solution of (6.8) only when s is positive. This is due to the hypothesis that we take in Proposition 6.2.1, that is $s > 0$; if s is negative, in the proof of the proposition we have some problems to demonstrate that $M_t = \int_0^t e^{-sX_\tau}\sqrt{2aX_\tau}dW_\tau$ is a martingale, because the inequality $\mathbb{E}\left[\int_0^t X_\tau e^{-2sX_\tau}d\tau\right] \leq \mathbb{E}\left[\int_0^t \frac{d\tau}{2es}\right]$ does not hold anymore.

- Dufresne's calculation of the moments of the process is not only useful to determine the transition density of the square root process, but it is also propaedeutic to the determination of the moments of the integral of the square root process itself (see [5]).

6.4 The defective density $p_1(t, x, \bar{x})$

6.4.1 The Dirac measure at the origin

Let's now concentrate ourselves on $p_1(t, x, \bar{x})$, the fundamental solution that is not the transition density of the process.
In Chapter 2, when we were illustrating Feller's calculation, we have seen

that, even if we do not know the exact form of $p_1(t,x,\bar{x})$, we can calculate its integral over x, putting $s=0$ in the expression of $\omega_2(t,s,\bar{x})$:

$$\omega_2(t,0)=\int_0^\infty p_2(t,x,\bar{x})dx=\Gamma\left(1-\frac{c}{a},\frac{b\xi e^{bt}}{a(e^{bt}-1)}\right)<1.$$

We know that the integral over the space x of the transition density of a process has to be equal to one, so this is another way to see that $p_1(t,x,\bar{x})$ can not be the transition density of the process.
We call $p_1(t,x,\bar{x})$ a *defective density*: let's take the case $c\leq 0$, in which p_1 is the unique fundamental solution of the Kolmogorov forward equation associated to the SDE of the square root process: in this case, the unique strong solution of this SDE has not a transition density at all, but it has a distribution that is the sum of the Dirac measure at zero, with a certain weight that we are going to calculate, and the defective density p_1.
The weight of the Dirac measure δ_0 must be such that

$$\int_0^\infty (d_0(x)+p_1(t,x,\bar{x}))dx=1$$

(see [10]).
We have already seen that

$$\int_0^\infty p_2(t,x,\bar{x})dx=\Gamma\left(1-\frac{c}{a},\frac{b\xi e^{bt}}{a(e^{bt}-1)}\right)<1,$$

and on the other hand

$$\int_0^\infty \delta_0(x)dx=1/2,$$

therefore the mass in the origin must have a weight

$$2\left[1-\Gamma\left(1-\frac{c}{a};\frac{b\xi e^{bt}}{a(e^{bt}-1)}\right)\right].$$

6.4.2 The flux as the derivative of the integral

We have the following result on the flux at the origin $f(t)$:

Proposition 6.4.1. *The flux at the origin*

$$f(t) = -\lim_{x\to 0}\Big(\partial_x\big(axp(t,x,\bar{x})\big) + (bx+c)p(t,x,\bar{x})\Big)$$

where $p(t,x,\bar{x})$ is a fundamental solution of (1.3) (so it can be p_1 or p_2) is the derivative with respect to time t of the integral of $p(t,x,\bar{x})$ over x:

$$\partial_t \int_0^\infty p(t,x,\bar{x}) = f(t).$$

Proof. We integrate on the left and on the right of equation (1.3), and we find

$$\begin{aligned}\int_0^\infty \partial_t p(t,x,\bar{x})dx =& \partial_t \int_0^\infty p(t,x,\bar{x})dx = \\ & \int_0^\infty \partial_{xx}\big(axp(t,x,\bar{x})\big) - \partial_x\big((bx+c)p(t,x,\bar{x})\big)dx = \\ & \lim_{x\to\infty}\{\partial_x\big(axp(t,x,\bar{x})\big) - (bx+c)p(t,x,\bar{x})\} + f(t).\end{aligned}$$

We want to see that the first term

$$\begin{aligned}&\lim_{x\to\infty}\{\partial_x\big(axp(t,x,\bar{x})\big) - (bx+c)p(t,x,\bar{x})\} = \\ &\lim_{x\to\infty}\{ax\partial_x p(t,x,\bar{x}) - bxp(t,x,\bar{x}) + (a-c)p(t,x,\bar{x})\}\end{aligned}$$

equals to zero.
We have studied the behaviour of $p_1(t,x,\bar{x})$ and $p_2(t,x,\bar{x})$ for large values of x, and we have seen that

$$\lim_{x\to\infty} x^n p_i(t,x,\bar{x}) = 0 \qquad \forall n\in\mathbb{N}, \qquad i=1,2$$

Moreover, we know that this is true also for the derivative of $p_1(t,x,\bar{x})$ and $p_2(t,x,\bar{x})$ with respect to x:

$$\lim_{x\to\infty} \partial_x(x^n p_i(t,x,\bar{x})) = 0 \qquad \forall n\in\mathbb{N}, \qquad i=1,2.$$

Therefore,

$$\lim_{x\to\infty}\{ax\partial_x p(t,x,\bar{x}) - bxp(t,x,\bar{x}) + (a-c)p(t,x,\bar{x})\} = 0,$$

so we have proved that

$$f(t) = \partial_t \int_0^\infty p(t, x, \bar{x}).$$

□

This result shows the real nature of the flux, and points out the fact that the flux at the origin for p_1 and p_2, $f(t) = -\lim_{x\to 0}\Big(\partial_x\big(axp_i(t,x,\bar{x})\big) + (bx + c)p_i(t,x,\bar{x})\Big)$, $i = 1, 2$ defines the nature of the two fundamental solutions: if this flux equals to zero one will find a fundamental solution that is norm preserving, and that as we have already seen is the transition density of the process, if this flux is negative one will find a fundamental solution that is norm decreasing, and that it is not the transition density.
So, this is a very intuitive way to understand the reason why a zero flux carries out a transition density and a negative flux do not.
We have seen that for the defective density p_1 the flux is

$$f(t) = \frac{-b}{\Gamma(1-\frac{c}{a})}\frac{e^{-bt}}{1-e^{-bt}}\left(\frac{\bar{x}b}{a(1-e^{-bt})}\right)^{(-c+a)/a}\exp\Big\{-\frac{\bar{x}b}{a(1-e^{-bt})}\Big\},$$

so studying this function we will have a measure of the rate of decrease during time of its integral.

6.5 A recapitulation

Now we can sum up what we have seen about the relations between fundamental solution of (1.3) and the transition density of the process, in reference to the flux at the origin:

- If $c \le 0$ there is only one fundamental solution of (1.3), so that no boundary conditions at $x = 0$ can be imposed: the flux must be of the form

$$f(t) = \frac{-b}{\Gamma(1-\frac{c}{a})}\frac{e^{-bt}}{1-e^{-bt}}\left(\frac{\bar{x}b}{a(1-e^{-bt})}\right)^{(-c+a)/a}\exp\Big\{-\frac{\bar{x}b}{a(1-e^{-bt})}\Big\},$$

and it is negative.
The fundamental solution that we find is

$$p_1(t,x,\bar{x}) = \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} I_{1-c/a}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right),$$

and it is not the transition density of the process.
Actually, the flux represents the derivative with respect to time of its integral over x, and it is negative: f_1 is norm decreasing, and its integral is

$$\int_0^{\infty} p_1(t,x,\bar{x})dx = \Gamma\left(1-\frac{c}{a}, \frac{b\xi e^{bt}}{a(e^{bt}-1)}\right) < 1.$$

Therefore, p_1 is a *defective density*, and the distribution of the process is the sum of the defective density $p_2(t,x,\bar{x})$ and a Dirac measure at zero with weight

$$2\left[1-\Gamma\left(1-\frac{c}{a}; \frac{b\xi e^{bt}}{a(e^{bt}-1)}\right)\right].$$

- If $0 < c < a$, we can impose two boundary conditions: a zero flux at the origin or a negative flux at the origin.
  If we choose a zero flux, we find a fundamental solution

  $$p_2(t,x,\bar{x}) = \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} I_{c/a-1}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right)$$

  that is also the transition density of the process.
  If we impose a negative flux, we have that this flux must be equal to the flux that found for $c \leq 0$, therefore we find again p_1, that is not the transition density of the process.

- If $c \geq a$ there is only one fundamental solution, so that one can not impose boundary conditions: the flux must be zero, therefore the fundamental solution will be p_2, that is the transition density of the process.

Chapter 7

Behaviour of the process at the boundary

7.1 A result by Ekström and Tysk

When one wants to study the transition density of a stochastic process that is the unique strong solution of a certain SDE, he can do it by studying the fundamental solutions of the associated forward Kolmogorov equation. This is what we have done in this thesis for the square root process.
Ekström and Tysk in [6] give a symmetry relation for the density that transforms this forward equation into a backward equation: take a process Y_t that is the unique strong solution of the SDE

$$dY_t = \beta(Y_t)dt + \sigma(Y_t)dW_t, \qquad Y_0 = \bar{y} > 0. \tag{7.1}$$

In this section, we will indicate the density of the process with the notation $p(x, y, t) = P(Y_t \in (y, y + dy)/dy | Y_0 = x)$: x is the initial point, as $\bar{x}$ in the rest of the thesis. Here we use this notation because it is more practical to indicate the passage from a forward equation to a backward equation.
So, let's see the following result:

Theorem 7.1.1. *Assume that the drift $\beta : [0, \infty) \to \mathbb{R}$ in (7.1) is continuously differentiable with a bounded derivative and satisfies $\beta(0) \geq 0$, and that* *the volatility* $\sigma : [0, \infty) \to [0, \infty)$ *satisfies* $\sigma(0) = 0$, $\sigma(x) > 0$ *for* $x > 0$ *and the linear bound*

$$|\sigma(x)| \le C(1 + x),$$

where C *is a positive constant. Moreover, suppose that* $\alpha := \frac{\sigma^2}{2}$ *is continuously differentiable on* $[0, \infty)$.
Then the corresponding density $p(x, y, t)$ *exists and is continuous, and it satisfies the symmetry relation*

$$m(x)p(x, y, t) = m(y)p(y, x, t), \tag{7.2}$$

where

$$m(x) = \frac{1}{\alpha(x)} \exp\left\{\int_1^x \frac{\beta(z)}{\alpha(z)} dz\right\},$$

where $\alpha(x) = \sigma^2(x)/2$.

This theorem suggests to study the density p solving the backward equation in the first variable, rather then the forward equation in the second variable. The advantage of this procedure is that the density p is known to be well behaved at the origin as a function of the first variable, whereas (as we will see for the square root process) it may explode as a function of the second variable.
By this theorem, Ekström and Tysk deduce the asymptotic behaviour of the density for small values of y:

Theorem 7.1.2. *Take a process with drift and volatility that satisfy the hypothesis of Theorem 7.1.1, and with density* $p(x, y, t)$. *For fixed* $x > 0$, $p(x, y, t)$ *is such that*

$$\lim_{y \to 0^+} \frac{p(x, y, t)}{m(y)} = \frac{p(0, x, t)}{m(x)} \ge 0.$$

If we put $C(t) = \frac{p(0,x,t)}{m(x)}$, *we have that:*

(i) if $\beta(0) > 0$, *then* $C(t)$ *is strictly positive for* $t > 0$.

(ii) if $\beta(0) = 0$, then $C(t) = 0$. Define

$$D(t) = \lim_{y \to 0} \frac{p(x, y, t)}{y m(y)}.$$

- *If there exists a constant $\epsilon > 0$ such that $\sigma(x) \geq \epsilon x^{1-\epsilon}$ for $0 < x < \epsilon$, then $D(t)$ is strictly positive for $t > 0$.*
- *If there exists $\epsilon > 0$ such that $\sigma(x) \leq \epsilon^{-1} x$ for $x \in (0, \epsilon)$, then $D = 0$.*

We can apply this theorem to the case of the square root process, for which we have $\beta(x) = bx + c$, $\sigma(x) = \sqrt{2ax}$, $\alpha(x) = \sigma^2(x)/2 = ax$.
We can easily check that if we take $c > 0$ these two functions satisfy the hypothesis of Theorem 7.1.1.
We compute

$$m(x) = \frac{1}{ax} \exp\left\{ \int_1^x \frac{bz + c}{az} dz \right\} = \frac{1}{ax} \exp\left\{ \frac{b}{a}(x-1) + \frac{c}{a}\ln(x) \right\} = \frac{1}{a} x^{\frac{c}{a}-1} e^{\frac{b}{a}(x-1)}.$$

This employs the relation

$$p(x, y, t) = m(y) \frac{p(y, x, t)}{m(x)} = \left[\frac{1}{a} y^{\frac{c}{a}-1} e^{\frac{b}{a}(y-1)} \right] \frac{p(y, x, t)}{m(x)}. \tag{7.3}$$

By (i) of Theorem 7.1.2 we find that for fixed x and t, the density $p(x, y, t)$ behaves like a positive multiple of $y^{\frac{c}{a}-1}$ for small y.
Therefore, we can state that the density at the origin explodes if $c < a$, it tends to zero if $c > a$ and it tends to a positive number if $c = a$.
Using the notation of the rest of the thesis, the relation takes the form

$$p_2(t, x, \bar{x}) = \left[\frac{1}{a} x^{\frac{c}{a}-1} e^{\frac{b}{a}(x-1)} \right] \frac{p(t, \bar{x}, x)}{m(\bar{x})}, \tag{7.4}$$

and we have that

$$\lim_{x \to 0} p_2(t, x, \bar{x}) = \lim_{x \to 0} x^{\frac{c}{a}-1} \frac{p(t, \bar{x}, 0)}{m(\bar{x})},$$

where $\frac{p(t, \bar{x}, 0)}{m(\bar{x})}$ is a positive number.

7.2 A direct calculation

We can refine these results taking the expression of the transition density of the process

$$p_2(t,x,\bar{x}) = \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} I_{c/a-1}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right).$$

We will also give the limit for $x \to 0$ of the fundamental solution of (2.4) $p_1(t,x,\bar{x})$, that is not the transition density of the process.
We use the explicit form of the modified Bessel function of the first kind

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{\nu+2n}}{n!\Gamma(n+\nu+1)}.$$

Let's start from the the calculation of the limit for $x \to 0$ of

$$p_1(t,x,\bar{x}) = \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} I_{1-c/a}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right).$$

Taking

$$C_1^{a,b,c,t,\bar{x}} = \frac{b}{a(e^{bt}-1)}\left(\frac{e^{-bt}}{\bar{x}}\right)^{\frac{c-a}{2a}} exp\left\{-\frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right\}, \qquad C_2^{a,b,t,\bar{x}} = \frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}\bar{x}},$$

we have

$$\lim_{x\to 0} p_1(t,x,\bar{x}) = \lim_{x\to 0} C_1^{a,b,c,t,\bar{x}} x^{\frac{c-a}{2a}} \sum_{n=0}^{\infty} \frac{(C_2^{a,b,t,\bar{x}})^{2n+1-\frac{c}{a}} x^{n+\frac{a-c}{2a}}}{n!\Gamma\left(n+2-\frac{c}{a}\right)} =$$

$$\lim_{x\to 0} C_1^{a,b,t,\bar{x}} (C_2^{a,b,t,\bar{x}})^{1-\frac{c}{a}} \sum_{n=0}^{\infty} \frac{(C_2^{a,b,t,\bar{x}})^{2n} x^n}{n!\Gamma(n+2-\frac{c}{a})} = C_1^{a,b,c,t,\bar{x}} (C_2^{a,b,t,\bar{x}})^{1-\frac{c}{a}} \frac{1}{\Gamma(2-\frac{c}{a})} =$$

$$\frac{1}{\Gamma(2-\frac{c}{a})} \frac{b}{a(e^{bt}-1)}\left(\frac{e^{-bt}}{\bar{x}}\right)^{\frac{c-a}{2a}} exp\left\{-\frac{b(\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}\bar{x}}\right)^{1-\frac{c}{a}}$$

Therefore, we have seen that

$$\lim_{x\to 0} p_1(t,x,\bar{x}) = \frac{1}{\Gamma(2-\frac{c}{a})}\left(\frac{b}{a(e^{bt}-1)}\right)^{2-\frac{c}{a}} (\bar{x}e^{bt})^{1-\frac{c}{a}} \exp\left\{-\frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right\}:$$

the limit of the defective density when x tends to zero is a positive number. We remind that here we take $c < a$, because when $c \geq a$ the unique fundamental solution is the transition density of the process. As we can see, the expression of the limit is the same for $c \leq 0$ and $c > 0$.
Let's now compute the limit for the transition density

$$p_2(t,x,\bar{x}) = \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}\left(\frac{e^{-bt}x}{\bar{x}}\right)^{\frac{c-a}{2a}} I_{c/a-1}\left(\frac{2b}{a(1-e^{-bt})}\sqrt{e^{-bt}x\bar{x}}\right).$$

We take the same constants $C_1^{a,b,c,t,\bar{x}}$ and $C_2^{a,b,t,\bar{x}}$ of the previous calculation, and we have

$$\begin{aligned}
&\lim_{x\to 0} p_2(t,x,\bar{x}) = \lim_{x\to 0} C_1^{a,b,t,\bar{x}} x^{\frac{c-a}{2a}} \sum_{n=0}^{\infty} \frac{(C_2^{a,b,t,\bar{x}})^{2n+\frac{c}{a}-1} x^{n+\frac{c-a}{2a}}}{n!\Gamma(n+\frac{c}{a})} = \\
&\lim_{x\to 0} C_1^{a,b,t,\bar{x}} x^{\frac{c-a}{a}} (C_2^{a,b,t,\bar{x}})^{\frac{c}{a}-1} \sum_{n=0}^{\infty} \frac{(C_2^{a,b,t,\bar{x}})^{2n} x^n}{n!\Gamma(n+\frac{c}{a})} = \qquad (7.5)\\
&\left[\frac{C_1^{a,b,t,\bar{x}}(C_2^{a,b,t,\bar{x}})^{\frac{c}{a}-1}}{\Gamma\left(\frac{c}{a}\right)}\right] x^{\frac{c}{a}-1}.
\end{aligned}$$

We can see this expression as a refinement of the relation (7.4): now we know the multiplying factor of $x^{\frac{c}{a}-1}$, and therefore we have the exact form of the limit when $c = a$; putting $c = a$ in (7.5) we find

$$\lim_{x\to 0} p_2(t,x,\bar{x}) = C_1^{a=c,b,t,\bar{x}} = \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(\bar{x}e^{bt})}{a(e^{bt}-1)}\right\}.$$

Therefore, to sum up what we have seen in this section, we have that

$$\lim_{x\to 0} p_1(t,x,\bar{x}) = \frac{1}{\Gamma(2-\frac{c}{a})}\left(\frac{b}{a(e^{bt}-1)}\right)^{2-\frac{c}{a}} (\bar{x}e^{bt})^{1-\frac{c}{a}} \exp\left\{-\frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right\}$$

and

$$\lim_{x\to 0^+} p_2(t,x;\bar{x}) = \begin{cases} \infty & 0 < c < a \\ \frac{b}{a(e^{bt}-1)} exp\left\{-\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)}\right\} & c = a \\ 0 & c > a \end{cases} \qquad (7.6)$$

Now we show this situation by three graphics: we fix the parameters $a = 2$, $b = 4$, $t = 2$, $\bar{x} = 2$, and in the first graphic we take $c = 1 < a$, in the

second one $c = 3 > a$ and in the third one $c = 2 = a$.
We observe the behaviour of the density p_2 for small values of x.
According to the theory (see (7.6)) we expect that the density at the origin tends to infinity for $c = 1$, to zero for $c = 3$ and to a positive number for $c = 2$.
For $c = 1$ the theory is confirmed: we see how the density explodes when x is very closed to the origin.

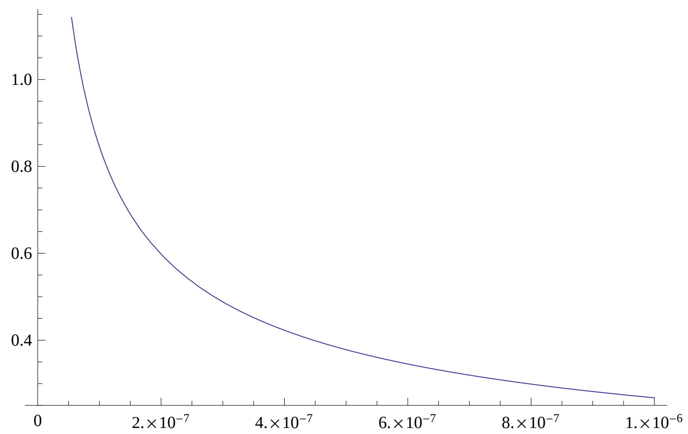

Figure 7.1: the transition density of the square root process for small values of x with $c < a$

In Figure 6.2 and in Figure 6.3, on the other hand, we can see the situation when we take $c = 3$ and $c = 2$ respectively: if $c = 3$ the density tends to zero for $x \to 0$, whereas if $c = 2 = a$ the density at zero is the positive number $\frac{b}{a(e^{bt}-1)} exp \left\{ -\frac{b(x+\bar{x}e^{bt})}{a(e^{bt}-1)} \right\}$.

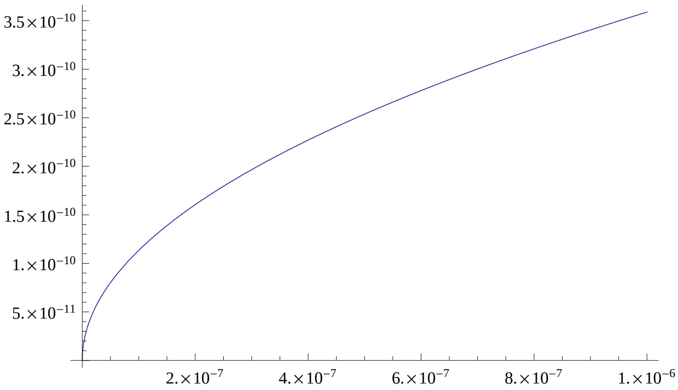

Figure 7.2: the transition density of the square root process for small values of x with $c > a$

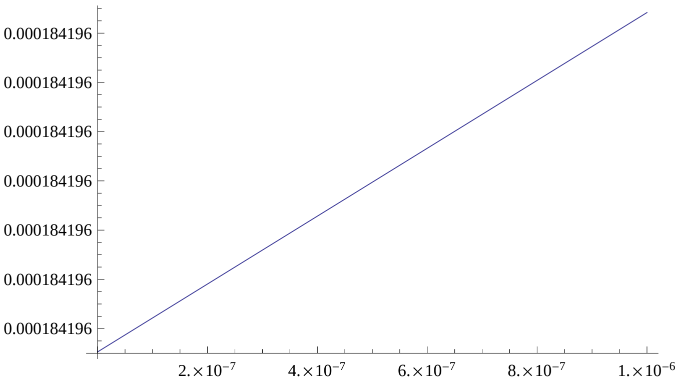

Figure 7.3: the transition density of the square root process for small values of x with $c = a$

7.3 Unattainable boundary

Now we prove the fact that the boundary is unattainable if $c \geq a$, that is if it is satisfied the Feller condition: we want to prove that, in this case, the process remains strictly positive for all times t.
We do this by following the demonstration given in Chapter 6 of [2]. First of all, we give the following theorem, shown by Gorovoi ad Linetsky in [8].

Theorem 7.3.1. *Define the diffusion*

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t,$$

where W_t is a standard Brownian motion, μ, μ', σ and σ' are continuous over $(0,\infty)$ and σ is strictly positive in $(0,\infty)$.
Take the scale and speed measures,

$$s(x) = \exp\left\{-\int_1^\infty \frac{2\mu(y)}{\sigma^2(y)}dy\right\}, \tag{7.7}$$
$$m(x) = \frac{2}{\sigma^2(x)s(x)}, \tag{7.8}$$

with the associate scale function

$$S(x) = \int_{x_0}^x s(y)dy. \tag{7.9}$$

If for any ϵ in a neighbourhood of $x=0$

$$\int_0^\epsilon m(x)dx < \infty,$$

and the integrals

$$I = \int_0^\epsilon s(x)dx \tag{7.10}$$

and

$$J = \int_0^\epsilon S(x)m(x)dx \tag{7.11}$$

are such that $I=\infty$ and $J<\infty$, the boundary is unattainable.
On the other hand, if $I=\infty$ and $J=\infty$, the boundary is attainable.

We have already compute the expressions of the scale and speed measures for the square root process, that are

$$s(x) = x^{-\frac{c}{a}}e^{-\frac{b}{a}(x-1)}$$

and

$$m(x) = \frac{1}{a}x^{\frac{c}{a}-1}e^{-\frac{b}{a}(x-1)}.$$

Let's check the condition

$$\int_0^\epsilon m(x)dx = \frac{e^{\frac{b}{a}}}{a}\int_0^\epsilon x^{\frac{c}{a}-1}e^{-\frac{b}{a}x}dx < \infty :$$

if $b > 0$, $\int_0^\epsilon x^{\frac{c}{a}-1}e^{-\frac{b}{a}x}dx \le \int_0^\epsilon x^{\frac{c}{a}-1}dx$, and if b is negative $\int_0^\epsilon x^{\frac{c}{a}-1}e^{-\frac{b}{a}x}dx \le e^{-\frac{b}{a}\epsilon}\int_0^\epsilon x^{\frac{c}{a}-1}dx$.
On the other hand, $\int x^{\frac{c}{a}-1}dx = \frac{x^{\frac{c}{a}}}{c/a} + cost$ for $c \neq 0$, so the integral is finite.
Now, let's examine the two integrals I and J in turn.
Consider $I = \int_0^\epsilon s(x)dx = e^{\frac{b}{a}}\int_0^\epsilon x^{-\frac{c}{a}}e^{-\frac{b}{a}x}$. For $x \in [0,\epsilon]$, if b is positive we have $Cx^{-\frac{c}{a}} \le x^{-\frac{c}{a}}e^{-\frac{b}{a}x} \le x^{-\frac{c}{a}}$, for $C = e^{-\frac{b}{a}\epsilon}$, and if b is negative we have $x^{-\frac{c}{a}} \le x^{-\frac{c}{a}}e^{-\frac{b}{a}x} \le Cx^{-\frac{c}{a}}$.
From elementary calculus we have that $\int x^{-\frac{c}{a}}dx = x^{1-\frac{c}{a}}/(1-\frac{c}{a}) + cost$ for $\frac{c}{a} \ge 1$, and that $\int x^{-\frac{c}{a}}dx = \ln x + cost$ for $c/a = 1$.
From this we see that for $c/a \ge 1$ the integral $\int_0^\epsilon x^{-\frac{c}{a}}dx$, is infinite, and therefore $I = \int_0^\epsilon s(x)dx = \infty$ if $c/a \ge 1$ also.
It remains to show that $J < \infty$ for $c/a \ge 1$. Take

$$J = \frac{e^{\frac{2b}{a}}}{a}\int_0^\epsilon x^{\frac{c}{a}-1}e^{-\frac{bx}{a}}\int_{x_0}^x y^{-\frac{c}{a}}e^{-\frac{by}{a}}dydx.$$

We have $J \le C'\int_0^\epsilon x^{\frac{c}{a}-1}\int_{x_0}^x y^{-\frac{c}{a}}dydx$ for a suitably chosen finite constant C'.
Clearly, for $c/a > 1$

$$\int_{x_0}^x y^{-\frac{c}{a}}dy = \left[\frac{y^{1-\frac{c}{a}}}{1-\frac{c}{a}}\right]_{x_0}^x = \frac{1}{1-\frac{c}{a}}(x^{1-\frac{c}{a}} - x_0^{1-\frac{c}{a}}),$$

and therefore we have

$$J \le \frac{C'}{1-\frac{c}{a}}\int_0^\epsilon x^{\frac{c}{a}-1}(x^{1-\frac{c}{a}} - x_0^{1-\frac{c}{a}})dx = \frac{C'}{1-\frac{c}{a}}\int_0^\epsilon (1 - x^{\frac{c}{a}-1}x_0^{1-\frac{c}{a}})dx.$$

This integral is finite for $c/a > 1$, and so for $c > a$.
In the $c = a$ case, we have

$$J = \frac{e^{\frac{2b}{a}}}{a}\int_0^\epsilon e^{-\frac{bx}{a}}\int_{x_0}^x y^{-1}e^{-\frac{by}{a}}dydx.$$

For brevity we assume $a = -b$ (the proof of finiteness of J follows similarly for other values of b and a subject to extra algebraic manipulations).
We have

$$\begin{aligned} J =& \frac{e^{\frac{2b}{a}}}{a}\int_0^\epsilon e^{-x}\int_{x_0}^x y^{-1}e^{y}dydx = \\ & \frac{e^{\frac{2b}{a}}}{a}\int_0^\epsilon e^{-x}(Ei(x) - Ei(x_0))dx, \end{aligned}$$

where $Ei(x) = \int_{-\infty}^{x}(e^u/u)du$.
It can be shown that this integral is finite: if we enter the expression '$Integrate[Ei[x]* Exp[-x], \{x, 0, 1\}]$' into Mathematica we can look that this is so.
Therefore, exploiting Theorem (7.3.1) we can say that the boundary is unattainable for $c \geq a$.

7.4 Absorbing and reflecting boundary

We consider the case $c < a$, when the boundary is attainable, and we want to discuss the cases in which the boundary is absorbing (that is, a path that hits the origin will be trapped at the origin) and reflecting (when a path hits the origin, it will be pushed away in the positive region).
In practice, one can say that the boundary is reflecting if the probability that a process will be trapped at the origin is zero, and that it is absorbing if this probability is not zero.
We have that $\int_0^\infty p_1(t, x, \bar{x})dx$ and $\int_0^\infty p_2(t, x, \bar{x})dx$ are respectively the probabilities that the process with the defective density p_1 and the process with the transition density p_2 are not zero at time t.
We do know the values of these integrals:

$$\int_0^\infty p_1(t, x, \bar{x})dx = \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right) < 1 \qquad (7.12)$$

and, obviously,

$$\int_0^\infty p_2(t, x, \bar{x})dx = 1. \qquad (7.13)$$

We notice that if we put $\bar{x} = 0$ in the first expression, the result is zero ($\Gamma(z, 0) = 0\ \forall z$); this means that when a path hits the origin at time s remains in the origin for all times $t > s$ (remind that $\bar{x}$ is the starting point of the process, but obviously we can also suppose $X_s = \bar{x}$, scaling the axis of times): the boundary is absorbing, and (7.12) is the probability that the process will be trapped at the origin within time t.
On the other hand, the second integral is one: for t fixed, the process is positive with probability one; we underline that this means that the process

will never be trapped at the origin (so the boundary is absorbing) but there can be a positive probability that the process hits the origin within a certain time T: in other words, the fact that

$$P(X_t > 0, X_0 = \bar{x}) = 1 \qquad \forall t > 0$$

does not implies that

$$P(X_t > 0 \ \ \forall t > 0, X_0 = \bar{x}) = 1.$$

However, we can state that:

- when $c \leq 0$ the boundary is attainable and absorbing, and the process will be trapped at the origin within time t with probability
$$P(X_t = 0|X_0 = \bar{x}) = 1 - \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt} - 1)}\right).$$
- when $0 < c < a$ the boundary is attainable, and the fact that the drift is positive when $X = 0$ means that a process hitting $\{X = 0\}$ will be naturally pushed back into the region $\{X > 0\}$: the boundary is reflecting.
On the other hand, one can impose an absorbing boundary and end the process; in effect, this is equivalent to choose the fundamental solution p_1, and so it is equivalent to impose a negative flux at the origin as a boundary condition for the PDE (1.3): if we stop the process when it hits the boundary, this process will be not a strong solution of the SDE of square root process, because this strong solution is unique, but it is a stopped process with defective density p_1.
Therefore, to summing up, we can schematize the situation as follows:
boundary condition $f(t) = 0 \longrightarrow$ fundamental solution p_2, reflecting boundary, process with transition density p_2;
boundary condition $f(t) < 0 \longrightarrow$ fundamental solution p_1, absorbing boundary, process stopped when it hits at the origin that is not solution of (1.1), defective density p_1.
- when $c \geq a$ the boundary is not attainable.

7.5 Processes trapped at the origin

7.5.1 Probability that the process will be trapped within time t

Now we concentrate ourselves on the case $c < a$ (and $f < 0$) when, as we have already seen, there is a positive probability that the process will be trapped at the origin within a certain time t.
We have seen that this probability is

$$P(X_t = 0|X_0 = \bar{x}) = 1 - P(X_t > 0|X_0 = \bar{x}) = 1 - \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt} - 1)}\right).$$

We want to study this probability in relation to the constants a, b, c and $\bar{x}$.
We have to study the behaviour of the Gamma function

$$\Gamma(h, z) = \frac{1}{\Gamma(h)} \int_0^z y^{h-1} e^{-y} dy = \frac{\int_0^z y^{h-1} e^{-y} dy}{\int_0^\infty y^{h-1} e^{-y} dy}.$$

As a function of the first variable it is monotonically decreasing, and as a function of the second variable it is monotonically increasing.
In our case, $h = 1 - \frac{c}{a}$ and $z = \frac{b\bar{x}}{a}$, so we can state that:

- if c is positive, $\Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right)$ is monotonically decreasing as a function of the variable a: $h = 1 - \frac{c}{a}$ increases when a increases, and $\Gamma(h, z)$ as we have already seen is decreasing as a function of h; $z = \frac{b\bar{x}}{a}$ decreases when a increases, and $\Gamma(h, z)$ is monotonically increasing as a function of z.

  If $c < 0$, the situation is much more complicated: both $h = 1 - \frac{c}{a}$ and $z = \frac{b\bar{x}}{a}$ decrease when a increases, and as we have just emphasize $\Gamma(h, z)$ is monotonically decreasing in h and monotonically increasing in z. In this case, it is very difficult to study the derivative of $g(a) = \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right)$, because its expression is very involved, therefore we can study the behaviour of this function only in an empirical way.

  We notice that when b and $\bar{x}$ are big and $|c|$ is small the function is monotonically decreasing. When $|c|$ is big the function grows for small values of a and decreases for large values of a. We will illustrate this behaviour by the support of some graphics.

- $\Gamma\left(1-\frac{c}{a};\frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right)$ is monotonically increasing as a function of the variable $c<a$: $h=1-\frac{c}{a}$ decreases when c increases, and $\Gamma(h,z)$ is decreasing as a function of h.

- $\Gamma\left(1-\frac{c}{a};\frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right)$ is monotonically increasing as a function of $\bar{x}$: $z=\frac{b\bar{x}}{a}$ increases when $\bar{x}$ increases, and $\Gamma(h,z)$ is monotonically increasing as a function of z.
  We have not to be surprised by this, because $\bar{x}$ is the starting point, so it is obvious that if we move it away from the origin the probability that the process will not hit the origin itself increases.

- $\Gamma\left(1-\frac{c}{a};\frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right)$ is monotonically increasing as a function of b: $z=\frac{b\bar{x}}{a}$ increases when b increases, and $\Gamma(h,z)$ is monotonically increasing as a function of z. Actually, the bigger is b the bigger is the drift, that if b is positive pushes the process away from $\{X=0\}$; if b is negative, the drift will push the process towards the origin.

So, we can state that the probability $1-\Gamma\left(1-\frac{c}{a};\frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right)$ that the process will be trapped at the origin grows when:

- the parameters b, c, $\bar{x}$ decrease;

- the parameter a increases, if c is positive or if c is negative and b and $\bar{x}$ are big and $|c|$ is small;

Obviously, the probability that the process will be trapped at the origin within time t grows with the time itself, and in fact we have seen that

$$f(t)=\partial_t\int_0^\infty p_1(t,x,\bar{x})dx<0.$$

Therefore, the flux $f(t)$ indicates the rate of increase of the probability that the process will be trapped at the origin within time t.

One can study $f(t) = \frac{-b}{\Gamma(1-\frac{c}{a})} \frac{e^{-bt}}{1-e^{-bt}} \left(\frac{\bar{x}b}{a(1-e^{-bt})}\right)^{(-c+a)/a} exp\Big\{ - \frac{\bar{x}b}{a(1-e^{-bt})}\Big\}$ as a function of the parameters a, b, c and $\bar{x}$, and so find how they influence this rate of increase of the probability $P(X_t = 0|X_0 = \bar{x})$.
The bigger is the absolute value of the flux, the more the probability increases during time.
We concentrate ourselves on b and $\bar{x}$, and we note that for both the parameters we have that the absolute value of $f(t)$ decreases when they increase (this is because they appear in the numerator of the negative exponential).
In particular, if b is big the probability that the process will be trapped within time t increases less quickly: this is due again to the fact that if the drift is positive it pushes the process away from the origin.
We underline that in the case $0 < c < a$, the probability that the process stopped at $\{X = 0\}$ will be trapped at the origin within time t corresponds to the probability that the square root process will reach the origin within time t, and that it will be reflected.
In other words, if we call $\bar{X}_t$ the process stopped at the origin, we have that

$$P(\bar{X}_t = 0, X_0 = \bar{x}) = P(\exists\, s \leq t | X_s = 0, X_0 = \bar{x}).$$

Therefore, for the square root process X_t in the case $0 < c < a$,

$$P(\exists\, s \leq t | X_s = 0, X_0 = \bar{x}) = 1 - \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right).$$

7.5.2 Probability that the process will never be trapped

Now, if we compute the limit for $t \to \infty$ of the probability $P(\bar{X}_t > 0|X_0 = \bar{x})$ (in this case for simplicity we call $\bar{X}_t$ also the process with $c \leq 0$) we will have the probability that the process will *never* be trapped at the origin.
It is straightforward to check that if b is positive

$$\lim_{t\to\infty} P(\bar{X}_t > 0|X_0 = \bar{x}) = \lim_{t\to\infty} \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right) = \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}}{a}\right).$$

and if b is negative

$$\lim_{t\to\infty} P(\bar{X}_t > 0|\bar{X}_0 = \bar{x}) = \lim_{t\to\infty} \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right) = \Gamma\left(1 - \frac{c}{a}; 0\right) = 0.$$

Therefore, when $b < 0$ every process will be eventually trapped in the origin: the drift push the process towards {x=0}, because the parameter b is negative. We can see that this probability is zero also when $b = 0$:

$$\lim_{b\to 0} P(\bar{X}_t > 0|\bar{X}_0 = \bar{x}) = \lim_{b\to 0} \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right) = \Gamma\left(1 - \frac{c}{a}, \frac{\bar{x}}{at}\right),$$

and it tends to zero to $t \to \infty$.
For $0 < c < a$ this is the probability that the square root process never hits the origin: therefore, we know that if $b \leq 0$ the process eventually hits the origin, with probability one.

7.5.3 Some graphics

Now we illustrate this situation by some graphics, made with Mathematica.
We start fixing the parameters $a = 10$, $c = 5$, $\bar{x} = 2$, and we observe the graphics of the probability $P(\bar{X}_t = 0|\bar{X}_0 = \bar{x})$ for $0 \leq \ t \leq 5$ when $b = -6, 0, 6$.
Notice that, as we have already seen, this is also the probability that the square root process will reach the origin within time t.

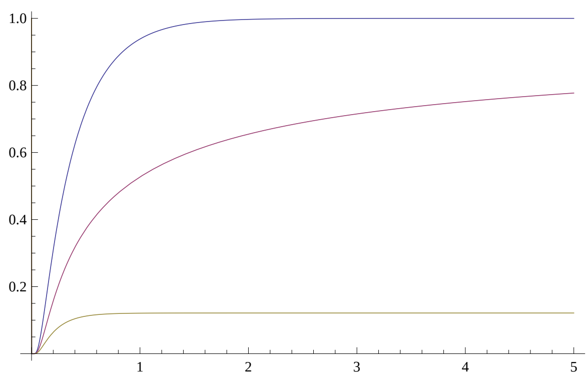

Figure 7.4: the probability $1-\Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right)$ as a function of t for different values of b

The blue curve corresponds to the probability with $b = -6$, the red one to the probability with $b = 0$ and the yellow one to the probability with $b = 6$. How we have analytically checked, the probability grows for lower values of b.
Moreover, we see that the difference is very marked: when b is negative the probability converges to 1 very quickly: the drift pushes the process towards small values of x.
When b is positive, on the other hand, the positive drift pushes the process far from the origin, so the probability that the process will be trapped is quite small.
We can notice that the red curve, which is the curve of the probability for $b = 0$, has to converge to 1, but it does not converge to this value very quickly.
Now we fix $b = 6$, and we take $a = 3, 9, 27$.

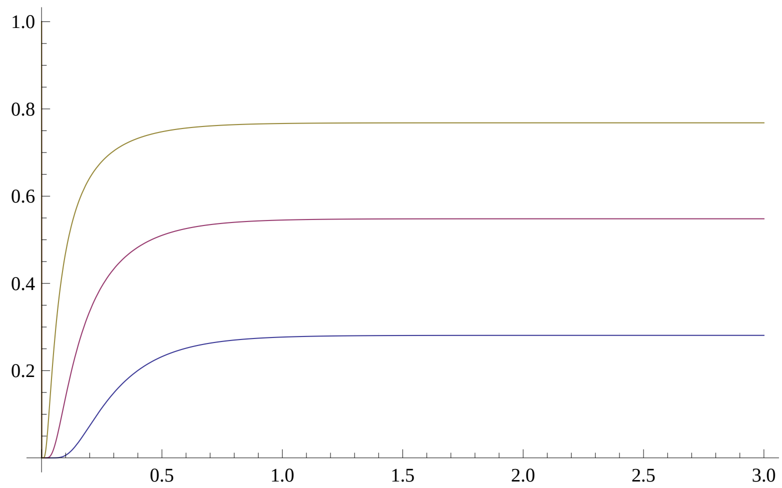

Figure 7.5: the probability $1-\Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right)$ a s function of t for different values of a,with $c = -5$

Here the blue curve is the probability that the process will be trapped within time t with $a = 3$, an the red and the yellow ones corresponds to the proba-

bilities with $a = 9$ and $a = 27$ respectively.
We see that the probability grows with higher values of a: therefore, we can expect that for these values of b, c and $\bar{x}$ the function $1-\Gamma\left(1-\frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right)$ is monotonically increasing. This is true, as we can see in the graphic of Figure 6.6, in which we take $t = 2$: we see that the probability that the process will be trapped in the origin within time $t = 2$ grows very quickly from values close to zero when a is *small* to higher values when a increases. We can also notice that the curve seems to converge to a value close to 1.

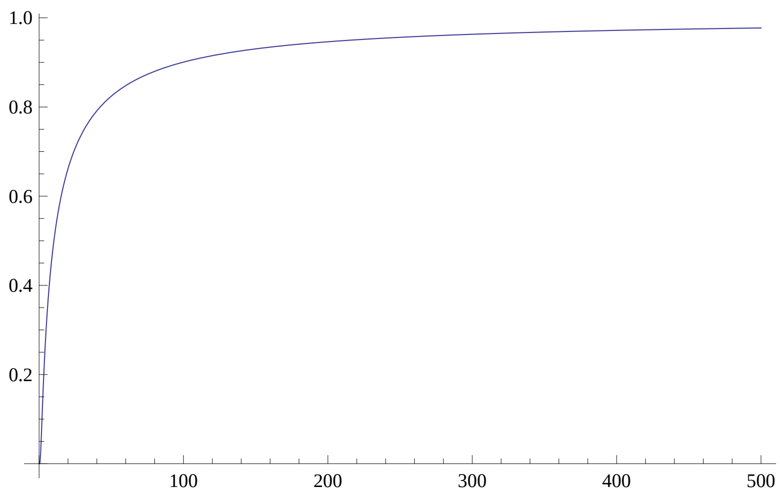

Figure 7.6: the probability $1-\Gamma\left(1-\frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right)$ as a function of a with $c = -5$

On the other hand, if we put $c = -50$, we can see in the graphic of Figure 7.7 that the behaviour of the probability as a function of a is quite different: we notice first of all that the value of the probability is very close to 1, and this behaviour do not surprise ourselves because we have already seen that the probability grows for lower values of the parameter c: here is $c = -50$, so the drift $b\bar{X}_t + c$ pushes the process towards the origin with much more strength then in the previous case, when $c = -5$.
Moreover, we see that the value of the probability decreases for the first

values of a, and increases when a become bigger.

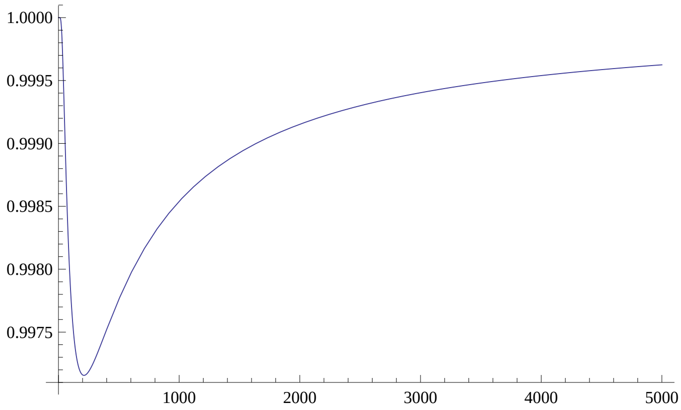

Figure 7.7: the probability $1 - \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right)$ as a function of a with $c = -50$

Now we propose the graphic of the behaviour of the probability $P(\bar{X}_t = 0|\bar{X}_0 = \bar{x})$ for $0 \leq\ t \leq 3$ with $b = 6$, $\bar{x} = 2$ and $a = 3, 9, 27$ as in the previous case, but with $c = -50$.

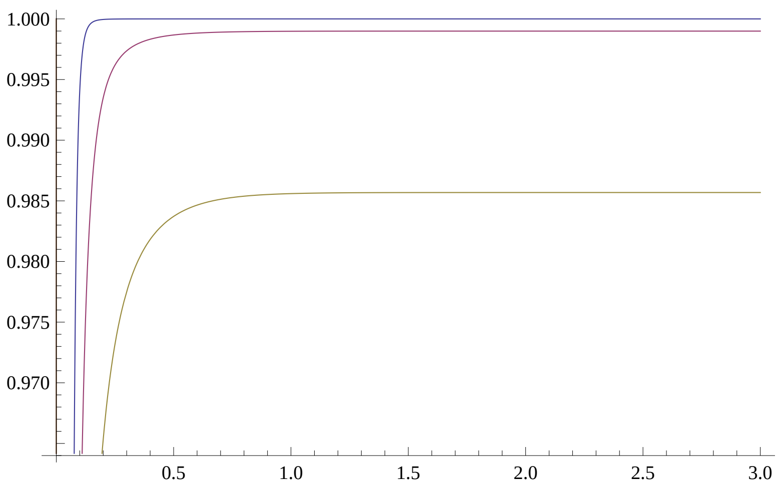

Figure 7.8: the probability $1-\Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right)$ a s function of t for different values of a,with $c = -50$

As in the previous graphic, the blue curve corresponds to $a = 3$, the red one to $a = 9$ and the yellow one to $a = 27$. Here the situation is reversed: we have the greatest values when a is small.
Now, to conclude, we give the graphics of $P(\bar{X}_t|\bar{X}_0 = \bar{x})$ for $0 \leq t \leq 3$ fixing $a = 10$ and $b = 6$ and letting variate $c = -5, 0, 5$ (and so $c/a = -0.5,\ 0,\ 0.5$) and $\bar{x} = 1, 3, 5$.

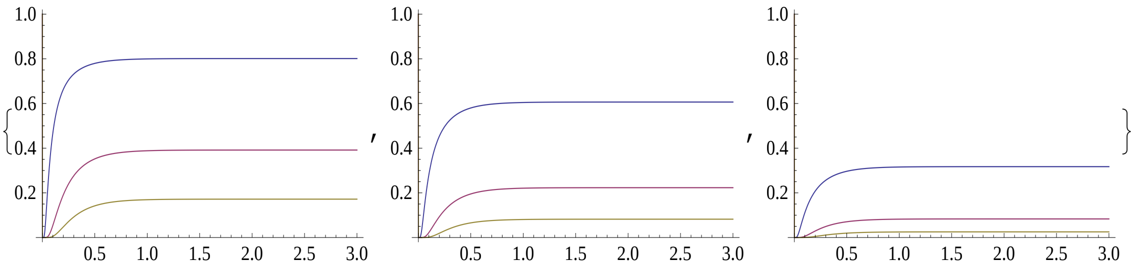

Figure 7.9: the probability $1-\Gamma\left(1-\frac{c}{a}; \frac{b\bar{x}e^{bt}}{a(e^{bt}-1)}\right)$ a s function of t for different values of a, and $\bar{x}$

Here, the graphics on the left corresponds to $c/a = -0.5$, the graphic in the center corresponds to $c/a = 0$ and the graphic on the right corresponds to $c/a = 0.5$.
In each graphic, the blue curve represents $\bar{x} = 1$, the red one $\bar{x} = 3$ and the yellow one $\bar{x} = 5$.
Also in this situation the theory is confirmed: the probabilities become lower when c and $\bar{x}$ increase: in particular, when $c = 5$ and $\bar{x} = 5$ the probability that the process will be trapped at the origin is almost zero.
Now we take b positive and we examine the limit

$$\lim_{t\to\infty} P(\bar{X}_t = 0|\bar{X}_0 = \bar{x}) = 1 - \lim_{t\to\infty} P(\bar{X}_t > 0|\bar{X}_0 = \bar{x}) = 1 - \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}}{a}\right).$$

Obviously, this is the probability that a process with starting point $\bar{x}$ will be sooner or later trapped at the origin.

We give two graphics about this probability: the first one shows the trend of this probability when we fix $c = -5$ and $\bar{x} = 2$ and we take $0.1 < a, b \leq 20$:

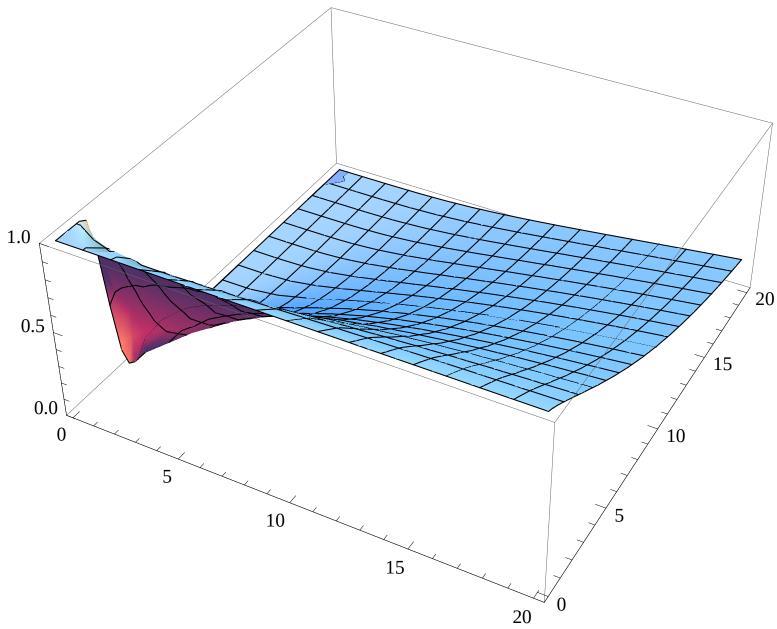

Figure 7.10: the probability $1 - \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}}{a}\right)$ as a function of a and b

We can notice that this probability is minimum, and it is almost zero, when a is close to zero and when b is maximum: this confirm that the probability grows when b decreases and when a increases (here c is negative but it has a small absolute value). It is maximum when b is very close to zero, for every value of a. This show that the drift is of great importance to determine the probability that the process will be trapped at the origin, and when the term b is close to zero and c is negative the process will be almost surely dragged towards the origin.
On the other hand, when b starts to take higher values the probability decreases very quickly, and it become close to zero when b takes the final values, if the value a is not too much high.
Actually, we can see that the parameter a affects the value of the probability

less than b: this behaviour of the probability does not surprise ourselves, because as we have already say when c is negative there is also the term $1 - \frac{c}{a}$ in $\Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}}{a}\right)$ that increases when a increases.
We do not consider the case $b < 0$ because we know that when b is negative every process will be eventually trapped, so we yet know that the probability is one.
Now we fix $a = 10$ and $b = 6$ and we take $-10 \le c < a$ and $0 < \bar{x} \le 4$. We have the following situation:

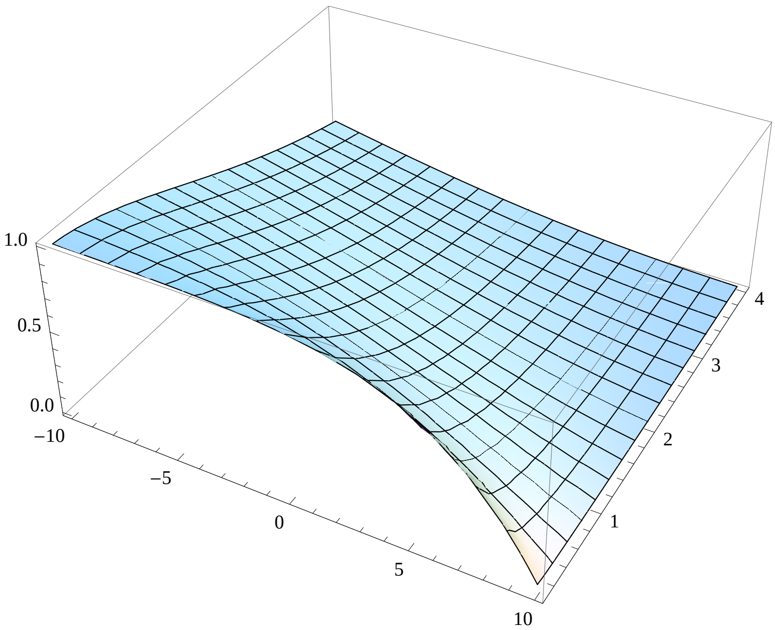

Figure 7.11: the probability $1 - \Gamma\left(1 - \frac{c}{a}; \frac{b\bar{x}}{a}\right)$ as a function of $\bar{x}$ and c

Here we notice that a process with starting point $\bar{x}$ very close to zero will be almost surely trapped at the origin itself when c is negative; if c is positive, the probability decreases with the growth of c, and become lower then 0.5 when the value of c is close to 10: this is again the effect of the importance of the drift, that if it is positive pushes the process far from the origin, as we have already seen.

However, we see that the probability decreases very quickly when the starting point moves away from the origin, also if c is negative.

Bibliography

[1] Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, New York, ninth dover printing, tenth gpo printing edition, 1964. ISBN 978-0486612720.

[2] Iain J. Clark. *Foreign Exchange Option Pricing: A Practitioners Guide.* Wiley Finance Series, 2010. ISBN 978-0-470-68368-2.

[3] John C. Cox, Jonathan E. Ingersoll, Jr., and Stephen A. Ross. A theory of the term structure of interest rates. *Econometrica*, 53 (2):385–407, 1985. ISSN 0012-9682. doi: 10.2307/1911242. URL `http://dx.doi.org/10.2307/1911242`.

[4] Mark Craddock. Fundamental solutions, transition densities and the integration of Lie symmetries. *J. Differential Equations*, 246(6):2538–2560, 2009. ISSN 0022-0396. doi: 10.1016/j.jde.2008.10.017. URL `http://dx.doi.org/10.1016/j.jde.2008.10.017`.

[5] Daniel Dufresne. The integrated square-root process. *Technical report, University of Montreal*, 2001. URL `http://eprints.infodiv.unimelb.edu.au/archive/00000157/`.

[6] Erik Ekström and Johan Tysk. Boundary behaviour of densities for non-negative diffusions. *preprint*, 2011.

[7] William Feller. Two singular diffusion problems. *Ann. of Math. (2)*, 54: 173–182, 1951. ISSN 0003-486X.

[8] V. Gorovoi and V. Linetsky. Black's model of interest rates as options, eigenfunction expansions and japanese interest rates. *Mathematical Finance*, 14(1):49–78, 2004.

[9] S. L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Finan. Stud.*, 6: 327–343, 1993.

[10] Alan Lindsay and Dominic Brecher. Results on the CEV Process, Past and Present. *SSRN eLibrary*, 2010.

[11] Andrea Pascucci. *Calcolo stocastico per la finanza.* UNITEXT, 2007.

[12] D. V. Widder. *The Laplace Transform*, volume 6 of *Princeton Mathematical Series.* Princeton University Press, 1941.