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Path integral approach to heat kernel in massive gravity

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Abstract

This thesis aims to examine the behaviour of the Fierz-Pauli theory of massive gravity in a curved spacetime. Massive gravity has a long history, dating back to the 1930s when Wolfgang Pauli and Markus Fierz first developed a theory of a massive spin-2 field propagating on a flat spacetime background. The latter theory can be obtained by linearizing the Einstein-Hilbert action of general relativity, leading to the kinetic part of the Fierz-Pauli action, which alone describes massless gravity. Then, a mass term can be introduced, making the theory massive. The Fierz-Pauli theory is linear, i.e. without considering self-interactions. It was later realized in the 1970s that theories of a massive graviton generally suffer from dangerous pathologies, including a ghost mode and a discontinuity with general relativity in the limit where the graviton mass goes to zero. These problems arise whenever we try to formulate non-linear theories. The discontinuity problem in the massless limit was solved with the so-called Vainshtein mechanism in 1972. Solutions to the ghost's problem had existed for some time in three spacetime dimensions, but they were not found in four dimensions and higher until the work of Claudia de Rham, Gregory Gabadadze, and Andrew Tolley (dRGT model) in the 2010s.

In this thesis, we consider the Fierz-Pauli theory of linearized massive gravity in an Einstein space. The aim is to study the one-loop effective action of this theory employing the heat kernel method, which consists in a variety of perturbation methods applied to minimal second order operators on manifolds, which allow us to study asymptotic expansions and singularities of Green functions. It is a powerful technique in mathematical physics, with applications ranging from black hole entropy to mathematical finance. In the Fierz-Pauli model of massive gravity, the operator entering the heat kernel is non-minimal, so we need ways to relate it to minimal operators in order to avoid a rather tedious treatment of the heat kernel expansion in the presence of non-minimal operators, which can be analyzed either by means of covariant projectors [1] or by employing the reduction method suggested by Barvinsky and Vilkovisky [2]. Our approach is based on computing the path integral of the Fierz-Pauli action with the Faddeev-Popov procedure, using appropriate gauge-fixing functions. In fact, the addition of a mass term to the action for massless gravity breaks the gauge symmetry of the theory, which is the general coordinate invariance. Because of this, the Fierz-Pauli theory of massive gravity is not a gauge theory and, of course, a gauge-fixing cannot be performed. Nevertheless, by first introducing new fields in the theory with the so-called Stückelberg trick, we can restore a gauge symmetry to the theory. These manipulations allow us to perform the computation of the path integral and the evaluation of the heat kernel coefficients by using the well-known Seeley-DeWitt method.

Contents

Introduction

In this thesis, we use the heat kernel method to study the one-loop effective action of linearized massive gravity, the theory of a spin-2 massive particle in a curved spacetime. We specifically work in an Einstein space of dimension D with Euclidean signature.

In the first chapter we briefly introduce two quantum field theory concepts used later on in the thesis: the Faddeev-Popov procedure and the background field method.

In the second chapter we explain in details what the heat kernel is and the technique for calculation of heat kernel coefficients. We describe the Seeley-DeWitt method, valid for minimal differential operators of second order, which allows us to write the one-loop effective action of a quantum field theory as a power series of heat kernel coefficients, which is possible if the theory is massive.

In the third chapter we describe the theory of massive gravity in flat spacetime. Starting from the free Fierz-Pauli action, we can count the number of degrees of freedom in the theory and then examine the discontinuity which arises when we consider the massless limit of the theory. Finally, we describe the Stückelberg trick, used to introduce new fields in the theory in order to perform a correct massless limit without losing degrees of freedom. This chapter is the building block to extend the theory in curved spacetime.

The fourth and last chapter is the core of this thesis. Putting all the pieces together from the previous chapters, we start with the Fierz-Pauli action in a spacetime with a curved fixed background. In our case, we work in an Einstein spacetime. This action, written in its initial form, is not useful for the calculation of the heat kernel coefficients with the Seeley-DeWitt method as the kinetic operator is not minimal. So, we can perform the Stückelberg trick described in chapter 3, introducing two new fields, in order to restore a gauge symmetry to the theory. We can then choose appropriate gauge-fixing functions and finally rewrite the action in a diagonal form, with all the operators of second order and minimal. With the Faddeev-Popov procedure, described in chapter 1, we can compute the path integral of the theory and then, with the Seeley-DeWitt method, described in chapter 2, we can calculate the heat kernel coefficients and write down the one-loop effective action for linearized massive gravity in an Einstein spacetime up to the finite cubic terms in curvatures in $D = 4$.

1 Elements of quantum field theory

This chapter serves as a brief introduction. We quickly review two quantum field theory concepts we will use again in the next chapters of this thesis: the Faddeev-Popov procedure and the background field method. The main reference is [3].

1.1 Faddeev-Popov procedure

In this section we briefly describe the Faddeev-Popov procedure, used in the quantization of gauge theories, in order to construct a well-defined QFT.

In order to start with our treatment, let's consider the case of Maxwell theory, a gauge theory which enjoys a $U(1)$ local symmetry, in $D = 4$

$$S[A] = \int d^4x \Big(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \Big) \tag{1.1.1}$$

$$Z = \int DA \exp\left(iS[A]\right) \sim \infty \ , \tag{1.1.2}$$

with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. The path integral diverges because we are summing over an infinite number of gauge equivalent configurations

$$A_\mu(x) \to A^g_\mu(x) = A_\mu(x) + ig(x)\partial_\mu g^{-1}(x) \ , \quad g(x) = \exp\left(i\alpha(x)\right) \in U(1) \ , \tag{1.1.3}$$

which have the same value of the action, $S[A^g] = S[A]$. The field space decomposes into inequivalent gauge orbits.

We'd like to define the path integral in order to get a finite and gauge invariant result

$$Z = \int \frac{DA}{\mathrm{Vol(Gauge)}} \exp\left(iS[A]\right) \sim \text{finite} \ , \tag{1.1.4}$$

with Vol(Gauge) as the infinite volume of the gauge group. This definition can be implemented by using a gauge-fixing function à la Faddeev-Popov, where unphysical ghost fields are introduced to exponentiate a measure factor. The gauge-fixing function should pick just one representative from each gauge orbits, as shown in Figure 1.

Let's start the description of the Faddeev-Popov procedure. The idea is to use a gauge-fixing condition like in Figure 1 and insert the identity, written as

$$1 = \int df\, \delta(f) = \int dy\, \frac{\partial f(y)}{\partial y}\, \delta(f(y)) \ , \tag{1.1.5}$$

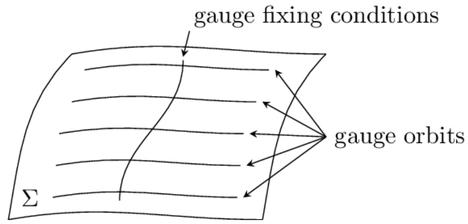

Figure 1: *Gauge fixing*

in the path integral to extract the volume of the gauge group. This can be generalized to n-dimensions as

$$1 = \int d^n f\, \delta^{(n)}(f) = \int d^n y \det\left(\frac{\partial f^i(y)}{\partial y^j}\right) \delta^{(n)}(f(y)) \ , \tag{1.1.6}$$

with f as a vector with n components. We can extend it also to functional integrals

$$1 = \int Dg\, \delta\big(f(A^g(x))\big) \operatorname{Det}\left(\frac{\delta f(A^g(x))}{\delta g(y)}\right) \ , \tag{1.1.7}$$

with Dg as a gauge invariant measure so that

$$\int Dg = \mathrm{Vol(Gauge)} \tag{1.1.8}$$

and Det as a functional determinant, known as Faddeev-Popov ($\Phi\Pi$) determinant. The "delta functional" $\delta(f(x))$ means that the whole function $f(x)$ is set to vanish.

Let's now compute (1.1.4), plugging in (1.1.7)

$$\begin{aligned}
Z &= \int \frac{DA}{\mathrm{Vol(Gauge)}} \exp\left(iS[A]\right) = \\
&= \int \frac{DA}{\mathrm{Vol(Gauge)}} \int Dg\, \delta\big(f(A^g(x))\big) \operatorname{Det}\left(\frac{\delta f(A^g(x))}{\delta g(y)}\right) \exp\left(iS[A]\right) = \\
&= \frac{1}{\mathrm{Vol(Gauge)}} \int DA^g \int Dg\, \delta\big(f(A^g(x))\big) \operatorname{Det}\left(\frac{\delta f(A^g(x))}{\delta g(y)}\right) \exp\left(iS[A^g]\right) = \\
&= \int \frac{Dg}{\mathrm{Vol(Gauge)}} \int DA\, \delta\big(f(A(x))\big) \operatorname{Det}\left(\frac{\delta f(A^g(x))}{\delta g(y)}\right)\bigg|_{g=1} \exp\left(iS[A]\right) = \\
&= \int DA\, \delta\big(f(A(x))\big) \operatorname{Det}\left(\frac{\delta f(A^g(x))}{\delta g(y)}\right)\bigg|_{g=1} \exp\left(iS[A]\right) \ .
\end{aligned} \tag{1.1.9}$$

In these manipulations we have first inserted the identity (1.1.7) and then used the fact that the action and the measure are both gauge invariant, namely $S[A^g] = S[A]$ and

$DA^g = DA$. This is certainly true for the classical action, but it is an assumption for the measure (related to the regularization methods used to make sense of the diverging Feynman diagrams of the perturbation expansion). Then we have changed variables from A^g to A, so that nothing depends on $g(x)$ anymore and the integration on Dg can be factorized out to cancel the infinite gauge volume Vol(Gauge). The final expression is correct, but can be written in a more useful form:
(1) we can introduce ghosts, i.e. anticommuting fields $c(x)$ and $\bar{c}(x)$ to exponentiate the $\Phi\Pi$ determinant.
(2) we can modify the gauge-fixing function to get rid of the delta functional from the integral. Instead of setting $f(A(x)) = 0$ we can set $f(A(x)) = h(x)$, with $h(x)$ as an arbitrary function, and then functionally average over the function $h(x)$ with the gaussian weight $\exp -\frac{i}{2\xi}\int h^2$, with ξ as a parameter. The physical quantities should not depend on ξ. If we perform this calculations, we find

$$\begin{aligned} Z = \int DA \int Dc \int D\bar{c} \int Dh\delta\Big(f(A(x)) - h(x)\Big)\times \\ \times \exp\left[i\left(S[A] + \int d^4x\, d^4y\, \bar{c}(x)\frac{\delta f(A^g(x))}{\delta g(y)}\bigg|_{g=1} c(y) - \frac{1}{2\xi}\int d^4x\, h^2(x)\right)\right] , \end{aligned} \tag{1.1.10}$$

which is simplified by path integrating over $h(x)$ to eliminate the delta functional and find the gauge-fixed total action S_{TOT} in the exponent

$$\begin{aligned} Z = \int DA \int Dc \int D\bar{c} \exp\Bigg[i\Bigg(S[A] + \int d^4x\, d^4y\, \bar{c}(x)\frac{\delta f(A^g(x))}{\delta g(y)}\bigg|_{g=1} c(y) + \\ - \frac{1}{2\xi}\int d^4x\, h^2(x)\Bigg)\Bigg] = \\ = \int DA \int Dc \int D\bar{c} \exp\Big(iS_{TOT}[A, c, \bar{c}]\Big) . \end{aligned} \tag{1.1.11}$$

Finally, the ghosts can be integrated out and eliminated, as, depending on the choice of the gauge-fixing function, they contribute at most to an overall normalization factor.

To exemplify the above construction, let's choose as gauge-fixing function

$$f(A) = \partial^\mu A_\mu \ , \tag{1.1.12}$$

which corresponds to the Lorenz gauge for the path integral in (1.1.9) and to a weighted Lorenz gauge, called also R_ξ for the path integral in (1.1.11). Under a gauge variation $\delta A_\mu = \partial_\mu \alpha$

$$\delta f(A) = \partial^\mu \delta A_\mu = \partial^\mu \partial_\mu \alpha \tag{1.1.13}$$

and so

$$\frac{\delta f(A^g(x))}{\delta g(y)}\bigg|_{g=1} \sim \frac{\delta f(A(x))}{\delta \alpha(y)} = \partial^\mu \partial_\mu \delta^4(x-y) \tag{1.1.14}$$

whose $\Phi\Pi$ determinant can be reproduced by path integrating over the ghost fields $c(x)$ and $\bar{c}(x)$. Thus, the gauge fixed action reads

$$S_{tot}[A, c, \bar{c}] = S[A] + \int d^4x \left[-\partial^\mu \bar{c}\,\partial_\mu c - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 \right] . \qquad (1.1.15)$$

The corresponding total lagrangian is

$$\mathcal{L}_{tot} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \partial^\mu \bar{c}\,\partial_\mu c - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 \qquad (1.1.16)$$

which, up to total derivatives, usually dropped, can be written as

$$\mathcal{L}_{tot} = -\frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu \bar{c}\,\partial_\mu c + \frac{1}{2}\left(1 - \frac{1}{\xi}\right)(\partial^\mu A_\mu)^2 \ . \qquad (1.1.17)$$

In the Feynman gauge ($\xi = 1$) we have

$$\mathcal{L}_{tot} = -\frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu \bar{c}\,\partial_\mu c \ . \qquad (1.1.18)$$

The path integral is now well-defined; we can add sources and compute propagators, which, in the Feynman gauge, read as

$$\langle A_\mu(x) A_\nu(x) \rangle = \int \frac{d^4p}{(2\pi)^4} \exp\Big(ip(x-y)\Big) \frac{-i\eta_{\mu\nu}}{p^2 - i\epsilon} \ , \qquad (1.1.19)$$

$$\langle c(x) \bar{c}(x) \rangle = \int \frac{d^4p}{(2\pi)^4} \exp\Big(ip(x-y)\Big) \frac{-i}{p^2 - i\epsilon} \ . \qquad (1.1.20)$$

The Faddeev-Popov procedure will be used in chapter 4 to compute the path integrals in the theory of massive gravity in an Einstein space.

1.2 Background field method

The effective action is the generator of one-particle irreducible (1PI) graphs. It is considered as a classical action that includes all quantum corrections. The background field method is a useful technique for computing the effective action. In this section we will present it for a scalar theory.

The various generating functionals for a field ϕ with action $S[\phi]$ are

$$Z[J] = \exp\Big(iW[J]\Big) = \int D\phi \exp\Big(iS[\phi] + iJ_i\phi^i\Big) \qquad (1.2.1)$$

and

$$\Gamma[\varphi] = \min_J \Big\{ W[J] - J_i \varphi^i \Big\} \ , \tag{1.2.2}$$

with $Z[J]$ as generating functional of correlation functions, $W[J]$ as the generating functional of connected correlation functions, J as an arbitrary function called "source" and $\Gamma[\varphi]$ as the effective action, defined as the Legendre transform of $W[J]$. In the background field method we first split the variable ϕ as

$$\phi(x) = \varphi(x) + \tilde{\phi}(x) \ , \tag{1.2.3}$$

with $\varphi(x)$ as an arbitrary fixed classical background and $\tilde{\phi}(x)$ as the quantum field to be quantized (path-integrated over in our case). $\varphi(x)$ is just an inert spectator in the quantization process.

Then, we define

$$Z_B[\tilde{J};\varphi] = \exp\Big(iW_B[\tilde{J};\varphi]\Big) = \int D\tilde{\phi} \exp\Big(iS[\tilde{\phi}+\varphi] + i\tilde{J}_i\tilde{\phi}^i\Big) \tag{1.2.4}$$

and

$$\Gamma_B[\tilde{\varphi},\varphi] = \max_{\tilde{J}} \Big\{ W_B[\tilde{J};\varphi] - \tilde{J}_i\tilde{\varphi}^i \Big\} \ . \tag{1.2.5}$$

By changing path integration variables $\tilde{\phi} \to \phi = \tilde{\phi} + \varphi$ in (1.2.4) and (1.2.5) and considering that the measure is invariant under translations, we find

$$Z_B[\tilde{J};\varphi] = Z[\tilde{J}] \exp\Big(-i\tilde{J}_i\varphi^i\Big) \tag{1.2.6}$$

so

$$W_B[\tilde{J};\varphi] = W[\tilde{J}] - \tilde{J}_i\varphi^i \tag{1.2.7}$$

and

$$\Gamma_B[\tilde{\varphi},\varphi] = \Gamma[\tilde{\varphi}+\varphi] \ . \tag{1.2.8}$$

Hence

$$\Gamma[\varphi] = \Gamma_B[0,\varphi] \ . \tag{1.2.9}$$

Therefore, the standard effective action $\Gamma[\varphi]$ can be computed as the sum of 1PI vacuum diagrams in presence of the background field φ.

Let's now review once more the generating functionals in QFT and check perturbatively that the effective action contains only 1PI diagrams. We use the Euclidean version of QFT.

The standard functionals in Euclidean QFT are defined by

$$Z[J] = \exp\left(\frac{1}{\hbar}W[J]\right) = \int D\phi \ \exp\left(-\frac{1}{\hbar}S[\phi] + \frac{1}{\hbar}J_i\phi^i\right) \tag{1.2.10}$$

and

$$\Gamma[\varphi] = J_i \varphi^i - W[J] \ , \quad \text{with} \quad \varphi^i = \frac{\delta W[J]}{\delta J_i} \tag{1.2.11}$$

with the effective action $\Gamma[\varphi]$ obtained by evaluating the right-hand side, using $J_i = J_i(\varphi)$, that inverts the defining relation $\varphi^i = \frac{\delta W[J]}{\delta J_i}$. Now we can invert the Legendre transformation, defining $\Gamma[\varphi]$ by

$$W[J] = J_i \varphi^i - \Gamma[\varphi] \ , \quad \text{with} \quad J_i = \frac{\delta \Gamma[J]}{\delta \varphi_i} \ . \tag{1.2.12}$$

With these relations, we find an equation for the effective action $\Gamma[\varphi]$

$$\exp\left[-\frac{1}{\hbar}\left(\Gamma[\varphi] - \frac{\delta\Gamma[\varphi]}{\delta\varphi^i}\varphi^i \right)\right] = \int D\phi \, \exp\left[-\frac{1}{\hbar}\left(S[\phi] - \frac{\delta\Gamma[\varphi]}{\delta\varphi^i}\phi^i \right)\right] \ . \tag{1.2.13}$$

With a change of variables that implements the shift $\phi \to \phi + \varphi$ in the path integral, we can write

$$\exp\left(-\frac{1}{\hbar}\Gamma[\varphi] \right) = \int D\phi \, \exp\left[-\frac{1}{\hbar}\left(S[\phi+\varphi] + \frac{1}{\hbar}\frac{\delta\Gamma[\varphi]}{\delta\varphi^i}\phi^i \right)\right] \ . \tag{1.2.14}$$

We use this equation to study the $\hbar$ expansion, i.e. the expansion in loops, which are counted by the parameter $\hbar$. We recognize the structure of the background field method.

Let's now expand the classical action in a Taylor series

$$S[\phi+\varphi] = \sum_{n=0}^{\infty} \frac{1}{n!} S_n[\varphi]\phi^n \ , \tag{1.2.15}$$

with

$$S_n[\varphi] = \frac{\delta^n S[\varphi]}{\delta\varphi^n} \ . \tag{1.2.16}$$

With a similar notation, we can write

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi} = \Gamma_1[\varphi] \ . \tag{1.2.17}$$

Rescaling $\phi \to \sqrt{\hbar}\phi$, we find

$$\begin{aligned} \exp\left(-\frac{1}{\hbar}\Gamma[\varphi] + \frac{1}{\hbar}S[\varphi] \right) = \int D\phi \exp\Bigg(& -\frac{1}{2}S_2[\varphi]\phi^2 - \sum_{n=3}^{\infty}\frac{\hbar^{\frac{n}{2}-1}}{n!}S_n[\varphi]\phi^n + \\ & + \frac{1}{\sqrt{\hbar}}\big(\Gamma_1[\varphi] - S_1[\varphi]\big)\phi \Bigg) \ , \end{aligned} \tag{1.2.18}$$

which depends only on $\bar{\Gamma}[\varphi] = \Gamma[\varphi] - S[\varphi]$. Expanding $\bar{\Gamma}[\varphi]$ in powers of $\hbar$

$$\bar{\Gamma}[\varphi] = \sum_{n=1}^{\infty} \hbar^n \bar{\Gamma}^{(n)}[\varphi] \tag{1.2.19}$$

one obtains the following master equation

$$\begin{aligned} \exp\Big(- \sum_{n=1}^{\infty} \hbar^{n-1} \bar{\Gamma}^{(n)}[\varphi] \Big) = \int D\phi \exp\Big(- \frac{1}{2} S_2[\varphi]\phi^2 - \sum_{n=3}^{\infty} \frac{\hbar^{\frac{n}{2}-1}}{n!} S_n[\varphi]\phi^n + \\ + \sum_{n=1}^{\infty} \hbar^{n-\frac{1}{2}} \bar{\Gamma}_1^{(n)}[\varphi]\phi \Big) \ , \end{aligned} \tag{1.2.20}$$

which we analyze by matching powers of $\hbar$ in the perturbative expansion. In the exponential, the first term $\frac{1}{2}S_2[\varphi]\phi^2$ corresponds to the propagator, the second term $\sum_{n=3}^{\infty} \frac{\hbar^{\frac{n}{2}-1}}{n!} S_n[\varphi]\phi^n$ to the vertices and the third one, $\sum_{n=1}^{\infty} \hbar^{n-\frac{1}{2}} \bar{\Gamma}_1^{(n)}[\varphi]\phi$, to the extra vertices that remove diagrams which are not 1PI.

At one loop ($n = 1$), we get from (1.2.20)

$$\begin{aligned} \exp\Big(- \bar{\Gamma}^{(1)}[\varphi] \Big) = \int D\phi \exp\Big(- \frac{1}{2} S_2[\varphi]\phi^2 + O(\hbar^{\frac{1}{2}}) \Big) = \Big(\mathrm{Det} S_2[\varphi] \Big)^{-\frac{1}{2}} = \\ = \exp\Big[- \frac{1}{2} \log\Big(\mathrm{Det} S_2[\varphi] \Big) \Big] \ , \end{aligned} \tag{1.2.21}$$

so that

$$\bar{\Gamma}^{(1)}[\varphi] = \frac{1}{2} \log\Big(\mathrm{Det} S_2[\varphi] \Big) = \frac{1}{2} \mathrm{Tr}\Big(\log S_2[\varphi] \Big) \ . \tag{1.2.22}$$

Hence, at one loop, the effective action is given by

$$\Gamma[\varphi] = S[\varphi] + \frac{\hbar}{2} \mathrm{Tr}\Big(\log S_2[\varphi] \Big) + O(\hbar^2) \ . \tag{1.2.23}$$

This approach will be generalized in chapter 2 for arbitrary fields in n dimensions and used in chapter 4 to compute the one loop effective action of linearized massive gravity.

2 Heat kernel

In this chapter we review the mathematical foundations of the heat kernel method and show its connection with the effective action defined in the background field method in chapter 1. We also show the technique for computing the Seeley-DeWitt coefficients. Our main reference is [4].

2.1 Generating functional, Green functions and effective action

Let's consider an arbitrary field $\varphi(x)$ on a n-dimensional spacetime given by its contravariant components $\varphi^A(x)$ that transform with respect to some representation of the diffeomorphism group. The field components $\varphi^A(x)$ can be of both bosonic and fermionic nature. The fermionic components are treated as anticommuting Grassmann variables [5]

$$\varphi^A \varphi^B = (-1)^{AB} \varphi^B \varphi^A \tag{2.1.1}$$

where the indices in the exponent of the (-1) are equal to 0 for bosonic indices and to 1 for the fermionic ones. In order to build a local action functional $S(\varphi)$ we also need a metric of the configuration space E_{AB}, i.e. a scalar product

$$(\varphi_1, \varphi_2) = \varphi_1^A E_{AB} \varphi_2^B \ , \tag{2.1.2}$$

which enables us to define the covariant field components

$$\varphi_A = \varphi^B E_{BA} \ , \quad \varphi^B = \varphi_A (E^{-1})^{AB} \ , \tag{2.1.3}$$

where $(E^{-1})^{AB}$ is the inverse matrix

$$(E^{-1})^{AB} E_{BC} = \delta^A{}_C \ , \quad E_{AC} (E^{-1})^{CB} = \delta_A{}^B \ . \tag{2.1.4}$$

The metric E_{AB} must be non-degenerate both in bose-bose and fermi-fermi sectors and satisfy the supersymmetry conditions

$$E_{AB} = (-1)^{A+B+AB} E_{BA} \ , \quad (E^{-1})^{AB} = (-1)^{AB} (E^{-1})^{BA} \ . \tag{2.1.5}$$

In the case of gauge-invariant field theories we assume that the corresponding ghosts are included in the set of the fields φ^A and the action $S(\varphi)$ is modified by the inclusion of the gauge-fixing and the ghosts. In the future we'll use the DeWitt notation $i \equiv (A, x)$ and $\varphi^i \equiv \varphi^A(x)$ [6, 7]. The combined summation-integration looks like

$$\varphi_{1,i}\, \varphi_2^i \equiv \int d^n x\, \varphi_{1,A}(x) \varphi_2^A(x) \ . \tag{2.1.6}$$

Now let's consider two causally connected in and out regions in the spacetime that lie in the past and in the future respectively relative to the region. Let's define the vacuum states $|\mathrm{in},\mathrm{vac}\rangle$ and $|\mathrm{out},\mathrm{vac}\rangle$ in these regions and consider the vacuum-vacuum transition amplitude

$$\langle \mathrm{out},\mathrm{vac}\,|\,\mathrm{in},\mathrm{vac}\rangle \equiv \exp\left\{\frac{i}{\hbar}W(J)\right\} \tag{2.1.7}$$

in presence of some classical background sources J_i vanishing in the in and out regions. The amplitude (2.1.7) can be expressed in form of a formal functional integral [8, 9, 10]

$$\exp\left\{\frac{i}{\hbar}W(J)\right\} = \int d\varphi\,\mathcal{M}(\varphi)\exp\left\{\frac{i}{\hbar}[S(\varphi)+J_i\varphi^i]\right\}\,, \tag{2.1.8}$$

with $\mathcal{M}(\varphi)$ as a measure functional, determined by the canonical quantization of the theory [11, 12]. $W(J)$ is the generating functional for the Schwinger averages

$$\left\langle\varphi^{i_1}\cdots\varphi^{i_k}\right\rangle = \exp\left\{-\frac{i}{\hbar}W(J)\right\}\left(\frac{\hbar}{i}\right)^k\frac{\delta_L^k}{\delta J_{i_1}\cdots\delta J_{i_k}}\exp\left\{\frac{i}{\hbar}W(J)\right\}\Bigg|_{J=0}\,, \tag{2.1.9}$$

with

$$\langle F(\varphi)\rangle \equiv \frac{\langle \mathrm{out},\mathrm{vac}|\,T(F(\varphi))\,|\mathrm{in},\mathrm{vac}\rangle}{\langle \mathrm{out},\mathrm{vac}\,|\,\mathrm{in},\mathrm{vac}\rangle}\,, \tag{2.1.10}$$

with δ_L as the left functional derivative and T as the operator of time ordering. The first derivative of the functional $W(J)$ gives the mean field

$$\left\langle\varphi^i\right\rangle \equiv \Phi^i(J) = \frac{\delta_L}{\delta J_i}W(J)\,. \tag{2.1.11}$$

The second derivative determines the propagator

$$\left\langle\varphi^i\varphi^j\right\rangle = \Phi^i\Phi^k + \frac{\hbar}{i}\mathcal{G}^{ik} \tag{2.1.12}$$

with

$$\mathcal{G}^{ik}(J) = \frac{\delta_L^2}{\delta J_i\delta J_k}W(J)\,. \tag{2.1.13}$$

The higher derivatives give the many-point Green functions

$$\mathcal{G}^{i_1\cdots i_k}(J) = \frac{\delta_L^k}{\delta J_{i_1}\cdots\delta J_{i_k}}W(J)\,. \tag{2.1.14}$$

The generating functional for the vertex functions, called the effective action $\Gamma(\Phi)$, is defined by the functional Legendre transform

$$\Gamma(\Phi) = W(J) - J_i\Phi^i\,, \tag{2.1.15}$$

where the sources are expressed in terms of the backgrond fields $J = J(\Phi)$, by the inversion of the functional equation $\Phi = \Phi(J)$ (2.1.11).

The first derivative of the effective action gives the sources

$$\frac{\delta_R}{\delta\Phi^i}\Gamma(\Phi) \equiv \Gamma_{,i}(\Phi) = -J_i(\Phi) \ . \tag{2.1.16}$$

The second derivative determines the propagator

$$\begin{gathered}\frac{\delta_L\delta_R}{\delta\Phi^i\delta\Phi^k}\Gamma(\Phi) \equiv \mathcal{D}_{ik}(\Phi) \ , \quad \text{with} \quad \mathcal{D}_{ik} = (-1)^{i+k+ik}\mathcal{D}_{ki} \ , \\ \mathcal{D}_{ik}\mathcal{G}^{kn} = -\delta_i{}^n \ , \end{gathered} \tag{2.1.17}$$

with δ_R as a right functional derivative.

The higher derivatives determine the vertex functions

$$\Gamma_{i_1\cdots i_k}(\Phi) = \frac{\delta_R^k}{\delta\Phi^{i_1}\ldots\delta\Phi^{i_k}}\Gamma(\Phi) \ . \tag{2.1.18}$$

From (2.1.15) and (2.1.8) it's easy to obtain the functional equation for the effective action

$$\exp\left\{\frac{i}{\hbar}\Gamma(\Phi)\right\} = \int d\varphi\,\mathcal{M}(\varphi)\exp\left\{\frac{i}{\hbar}\Big[S(\varphi) - \Gamma_{,i}(\Phi)(\varphi^i - \Phi^i)\Big]\right\} \ . \tag{2.1.19}$$

By differentiating (2.1.16) w.r.t. the sources, we can express all the many-point Green functions (2.1.14) in terms of the vertex functions (2.1.18) and the propagator (2.1.12). A many-point Green function is represented by all kinds of tree diagrams with a given number of external lines. Thus when using the effective action functional for the construction of the S-matrix we need only the tree diagrams, since all quantum corrections determined by the loops are already included in the full propagator and the full vertex functions. Therefore, the effective equations (2.1.16) in absence of classical sources ($J = 0$)

$$\Gamma_{,i}(\Phi) = 0 \ , \tag{2.1.20}$$

describe the dynamics of the background fields with regard to all quantum corrections.

Working directly with the effective action is an advantage. It contains all the informations needed to build the standard S-matrix [13, 14, 12] and it gives the effective equations (2.1.20) that enable us to take into account the influence of the quantum effects on the classical configurations of the background fields [15, 16].

The formal scheme of quantum field theory we have described has a concrete meaning in the framework of perturbation theory in the number of loops [8, 9, 10]:

$$\Gamma(\Phi) = S(\Phi) + \sum_{k\geq 1}\hbar^k\Gamma_{(k)}(\Phi) \ . \tag{2.1.21}$$

Plugging (2.1.21) into (2.1.19), shifting the integration variable in the functional integral $\varphi^i = \Phi^i + \sqrt{\hbar}\,\phi^i$, expanding the action $S(\varphi)$ and the measure $M(\varphi)$ in the quantum fields ϕ^i and equating the coefficients at equal powers of $\hbar$, we obtain the recurrence relations that uniquely define all the coefficients $\Gamma_{(k)}$. All the functional integrals are gaussian and can be calculated easily [17], as briefly described in the previous chapter. As the result, the diagrammatic technique for the effective action is reproduced. The elements of this technique are the bare one-point propagators, i.e., the Green functions of the differential operator

$$\Delta_{ik}(\varphi) = \frac{\delta_L \delta_r}{\delta\varphi^i \delta\varphi^k} S(\varphi) \tag{2.1.22}$$

and the local vertices, determined by the classical action $S(\varphi)$ and the measure $M(\varphi)$. In particular, the one-loop effective action has the form

$$\Gamma_{(1)}(\Phi) = -\frac{1}{2i} \log \frac{\mathrm{sDet}\Delta(\Phi)}{\mathcal{M}^2(\Phi)} \ , \tag{2.1.23}$$

with

$$\mathrm{sDet}\Delta = \exp(\mathrm{sTr}\log\Delta) \tag{2.1.24}$$

as the Berezin superdeterminant [5] and

$$\mathrm{sTr}F = (-1)^i F^i{}_i = \int d^n x (-1)^A F^A{}_A(x) \tag{2.1.25}$$

as the functional supertrace.

The local functional measure $M(\varphi)$ can be taken in the form of the superdeterminant of the metric of the configuration space

$$\mathcal{M} = (\mathrm{sDet}E_{ik}(\varphi))^{\frac{1}{2}} \ , \tag{2.1.26}$$

with

$$E_{ik}(\varphi) = E_{AB}(\varphi(x))\delta(x, x') \ . \tag{2.1.27}$$

In this case $d\varphi\mathcal{M}(\varphi)$ is the volume element of the configuration space that is invariant under the point transformations of the fields: $\varphi(x) \to F(\varphi(x))$. Using the multiplicativity of the superdeterminant, the one-loop effective action with the measure (2.1.26) can be rewritten as

$$\Gamma_{(1)}(\Phi) = -\frac{1}{2i} \log \mathrm{sDet}\hat{\Delta} \ , \tag{2.1.28}$$

with

$$\mathrm{sDet}\hat{\Delta}^i{}_k = (E^{-1})^{in}\Delta_{nk} \ . \tag{2.1.29}$$

The local measure $M(\varphi)$ can be also chosen so that the leading ultraviolet divergences in the theory, proportional to $\delta(0)$, vanish [18, 19].

2.2 Green functions of minimal differential operators

The construction of Green functions of arbitrary differential operators (2.1.22), (2.1.29) can be finally reduced to the construction of the Green functions of the minimal differential operators of second order [2] with the form

$$\Delta^i{}_k = \Big[\delta^A{}_B(\Box - m^2) + Q^A{}_B(x)\Big] g^{\frac{1}{2}}(x)\delta(x,x') \ , \tag{2.2.1}$$

with $g_{\mu\nu}$ as the metric of the background spacetime, g as the modulus of the determinant of the metric tensor, ∇_μ as the covariant derivative and $\Box \equiv g_{\mu\nu}\nabla^\mu\nabla^\nu$.

The Green functions $G^{A\prime}{}_B(x,x')$ of the differential operator (2.2.1) are two-point objects, which transform as the field $\varphi^A(x)$ under the transformations of coordinates at the point x and as the current $J_{B'}(x')$ under the coordinate transformations at the point x'. The indices belonging to the tangent space at the point x' are labeled with a prime.

Now we can construct the solutions of the equation for the Green functions

$$\Big[\delta^A{}_C(\Box - m^2) + Q^A{}_C\Big] G^C{}_{B'}(x,x') = -\delta^A{}_B g^{-\frac{1}{2}}(x)\delta(x,x') \ , \tag{2.2.2}$$

with appropriate boundary conditions, using the Fock-Schwinger-DeWitt proper time method [20, 21, 19, 6, 22] in form of a contour integral over an auxiliary variable s

$$G = \int_C i\, ds \exp\big(-ism^2\big) U(s) \ . \tag{2.2.3}$$

The evolution function, called also *heat kernel*, $U(s) \equiv U^A{}_{B'}(s|x,x')$ satisfies the equation

$$\frac{\partial}{\partial is} U(s) = \Big(\hat{1}\Box + Q\Big) U(s) \ , \quad \text{with} \quad \hat{1} \equiv \delta^A{}_B \ , \tag{2.2.4}$$

with the boundary condition

$$U^A{}_{B'}(s|x,x')\Big|_{\partial C} = -\delta^A{}_B g^{-\frac{1}{2}}(x)\delta(x,x') \ , \tag{2.2.5}$$

with ∂C as the boundary of the contour C. The evolution equation (2.2.4) is as difficult to solve as (2.2.2). However, the representation of the Green functions in form of the contour integrals over proper time, (2.2.3), is more convenient to use for the construction of the asymptotic expansion of the Green functions in inverse powers of the mass and for the study of the behaviour of the Green functions and their derivatives on the light cone $x \to x'$, as well as for the regularization and renormalization of the divergent vacuum expectation values of local variables. To obtain the causal Green function (Feynman propagator) one has to integrate over s from 0 to ∞ and add an infinitesimal negative imaginary part to the m^2 [6, 22].

Let's now single out in the evolution function a rapidly oscillating factor that reproduces the initial condition (2.2.5) at $s \to 0$

$$U(s) = i(4\pi s)^{-\frac{n}{2}}\,\Delta^{\frac{1}{2}}\exp\Big(-\frac{\sigma}{2is}\Big)\mathcal{P}\,\Omega(s)\ , \tag{2.2.6}$$

with the world function $\sigma(x, x')$ as half the square of the geodesic distance between the points x and x';

$$\Delta(x, x') = -g^{-\frac{1}{2}}(x)\det\Big(-\nabla_{\mu'}\nabla_{\nu}\,\sigma(x, x')\Big)g^{-\frac{1}{2}}(x') \tag{2.2.7}$$

is the Van Vleck-Morette determinant and $\mathcal{P} \equiv \mathcal{P}^{A}{}_{B'}(x, x')$ is the parallel displacement operator of the field along the geodesic from the point x' to the point x. The function $\Omega(s) \equiv \Omega^{A'}{}_{B'}(s|x, x')$ is called transfer function and transforms as a scalar at the point x and as a matrix at the point x'. This function is regular in s at the point $s = 0$, i.e.,

$$\Omega^{A'}{}_{B'}(0|x, x')\Big|_{x\to x'} = \delta^{A'}{}_{B'} \tag{2.2.8}$$

independently on the way how $x \to x'$. Using the equations [6, 22]

$$\sigma = \frac{1}{2}\sigma_{\mu}\sigma^{\mu} \qquad \sigma_{\mu} \equiv \nabla_{\mu}\sigma\ , \tag{2.2.9}$$

$$\sigma^{\mu}\nabla_{\mu}\mathcal{P} = 0 \qquad \mathcal{P}^{A}{}_{B'}(x, x') = \delta^{A'}{}_{B'}\ , \tag{2.2.10}$$

$$\sigma^{\mu}\nabla_{\mu}\log\Delta^{\frac{1}{2}} = \frac{1}{2}(n - \Box\sigma)\ , \tag{2.2.11}$$

we obtain from (2.2.4) and (2.2.6) the transfer equation for the function $\Omega(s)$

$$\Big(\frac{\partial}{\partial is} + \frac{1}{is}\sigma^{\mu}\nabla_{\mu}\Big)\Omega(s) = \mathcal{P}^{-1}\Big(\hat{1}\Delta^{-\frac{1}{2}}\Box\Delta^{\frac{1}{2}}\Big)\mathcal{P}\Omega(s) \tag{2.2.12}$$

If we solve this equation (2.2.12) in form of power series in the variable s

$$\Omega(s) = \sum_{k\geq 0}\frac{(is)^k}{k!}b_k\ , \tag{2.2.13}$$

we can write the asymptotic expansion of the heat kernel as

$$U(s) \sim i(4\pi s)^{-\frac{n}{2}}\sum_{k\geq 0}\frac{(is)^k}{k!}b_k\ , \tag{2.2.14}$$

known as Minakshisundaram-Pleijel equation and then, from (2.2.8) and (2.2.12), we get the recurrence relations for the b_k

$$\sigma^{\mu}\nabla_{\mu}b_0 = 0\ , \quad b_0^{A'}{}_{B'}(x', x') = \delta^{A'}{}_{B'} \tag{2.2.15}$$

$$\left(1+\frac{1}{k}\sigma^\mu\nabla_\mu\right)b_k=\mathcal{P}^{-1}\left(\hat{1}\Delta^{-\frac{1}{2}}\Box\Delta^{\frac{1}{2}}+Q\right)\mathcal{P}\,b_{k-1}\ . \tag{2.2.16}$$

The coefficients $b_k(x,x')$ are known as Seeley-DeWitt coefficients [23, 24, 25, 26].

We can now move for further convenience to Euclidean time $\beta\equiv is$ and exploit the operatorial identity

$$\log A=-\int_0^\infty\frac{d\beta}{\beta}\exp(-\beta A)\ . \tag{2.2.17}$$

By recalling the expressions (1.2.22) an (2.1.24) and setting $A\equiv\hat{\Delta}$, (2.1.28) becomes

$$\begin{aligned}\Gamma_{(1)}&=\frac{1}{2}\log\mathrm{sDet}\hat{\Delta}=-\frac{1}{2}\int_0^\infty\frac{d\beta}{\beta}\mathrm{sTr}\exp(-\beta\hat{\Delta})=\\&=-\frac{1}{2}\int_0^\infty\frac{d\beta}{\beta}\exp\Big(-\beta m^2\Big)\int d^nx\sqrt{g}\,\mathrm{str}\,U(\beta)\ ,\end{aligned} \tag{2.2.18}$$

where the new notation str for the supertrace indicates

$$\mathrm{sTr}A=\int d^nx\sqrt{g}\,\mathrm{str}A\ . \tag{2.2.19}$$

By using the expansion (2.2.14) in Euclidean time β, the action (2.2.18) becomes

$$\Gamma_{(1)}=-\frac{1}{2}\int_0^\infty\frac{d\beta}{\beta}\exp\Big(-\beta m^2\Big)\int\frac{d^nx\sqrt{g}}{(4\pi\beta)^{\frac{n}{2}}}\,\mathrm{str}\sum_{k=0}^\infty\frac{\beta^k}{k!}b_k(n;\,x)\ , \tag{2.2.20}$$

which is the general form of the effective action in terms of heat kernel coefficients. The n in $b_k(n;\,x)$ refers to the dimension of the spacetime.

2.3 Calculation of Seeley-DeWitt coefficients

From the recurrence relation (2.2.16) we obtain the formal solution [27, 28, 29, 30]

$$b_k=\left(1+\frac{1}{k}D\right)^{-1}F\left(1+\frac{1}{k-1}D\right)^{-1}F\cdots(1+D)^{-1}F\ , \tag{2.3.1}$$

with the operator D defined by

$$(D-2)\sigma=0\ ,\quad D\equiv\sigma^\mu\nabla_\mu \tag{2.3.2}$$

and F as

$$F=\mathcal{P}^{-1}\left(\hat{1}\Delta^{-\frac{1}{2}}\Box\Delta^{\frac{1}{2}}+Q\right)\mathcal{P}\ . \tag{2.3.3}$$

Let's also define the concept of coincidence limit of a function f

$$[f(x,x')] \equiv \lim_{x\to x'} f(x,x') \ . \tag{2.3.4}$$

In this section we'll develop a convenient covariant and effective method that gives a practical meaning to (2.3.1). It will suffice to calculate the coincidence limits of the Seeley-DeWitt coefficients b_k and their derivatives.

First of all, we suppose that there exist finite coincidence limits of the Seeley-DeWitt coefficients

$$[b_k] \equiv \lim_{x\to x'} b_k(x,x') \ , \tag{2.3.5}$$

that don't depend on the way how the points x and x' approach each other, i.e., the Seeley-DeWitt coefficients $b_k(x,x')$ are analytical functions of the coordinates of the point x near the point x'. It can be proven that the Seeley-DeWitt coefficients can be expressed in form of covariant Taylor series

$$b_k = \sum_{n\geq 0} |n\rangle \langle n \,|\, b_k\rangle \ \ , \tag{2.3.6}$$

with $|n\rangle$ as a complete set of eigenfunctions, By defining the inverse operator $\left(1+\frac{1}{k}D\right)^{-1}$ in form of the eigenfunctions expansion

$$\left(1+\frac{1}{k}D\right)^{-1} = \sum_{n\geq 0} \frac{k}{k+n} |n\rangle \langle n| \ \ , \tag{2.3.7}$$

from (2.3.1), we get

$$\begin{aligned} \langle n \,|\, b_k\rangle = \sum_{n_1,\cdots,n_{k-1}\geq 0} &\frac{k}{k+n}\cdot\frac{k-1}{k-1+n_{k-1}}\cdots\frac{1}{1+n_1}\times \\ &\times \langle n \,|\, F \,|\, n_{k-1}\rangle \langle n_{k-1} \,|\, F \,|\, n_{k-2}\rangle \cdots \langle n_1 \,|\, F \,|\, 0\rangle \ \ , \end{aligned} \tag{2.3.8}$$

where

$$\langle m \,|\, F \,|\, n\rangle = \left[\nabla_{(\mu_1}\cdots\nabla_{\mu_m)} F \frac{(-1)^n}{n!}\sigma^{\nu'_1}\cdots\sigma^{\nu'_n}\right] , \tag{2.3.9}$$

with

$$\sigma^{\mu'} = \eta^{\mu'}{}_{\nu}\sigma^{\nu} \ , \quad \eta^{\mu'}{}_{\nu} = \nabla_{\nu}\sigma^{\mu'} \ . \tag{2.3.10}$$

As the operator F is a differential operator of second order, the matrix elements (2.3.9) don't vanish only for $n \leq m+2$, so the summation (2.3.8) is always finite, and in particular $n_1 \geq 0$, $n_j \leq n_{j+1}+2$. The problem of computing Seeley-DeWitt coefficients is therefore reduced to compute the matrix elements (2.3.9).

The exact computation of the matrix elements is quite involved and will not be carried out here; the results obtained by Avramidi [4], however, are in accordance with the earlier ones by Gilkey [31] and De Witt [6, 32].

In order to show the outcome of these involved computations, it is useful to write the operator (2.2.1) in the following simplified form

$$H \equiv -\Box - V \ . \tag{2.3.11}$$

Now, we'll denote by $n = D$ the dimension of our spacetime manifold. The trace of the heat kernel coefficients (2.2.14) can be written as

$$\frac{1}{(4\pi s)^{\frac{D}{2}}} \operatorname{Tr}\left[\sum_{k=0}^{\infty} s^k a_k\right] \equiv \frac{1}{(4\pi s)^{\frac{D}{2}}} \operatorname{Tr}\left[\exp\left(\sum_{k=0}^{\infty} s^k \alpha_k\right)\right] , \tag{2.3.12}$$

where the functional trace Tr contains also a finite dimensional trace tr on the discrete indices of the fields, with the new notation analogous to (2.2.19)

$$\operatorname{Tr} A = \int d^n x \sqrt{g}\, \operatorname{tr} A \tag{2.3.13}$$

and the heat kernel coefficients are related as

$$a_k \equiv \frac{1}{k!} b_k \quad \text{with} \quad a_k = \alpha_k + \beta_k \ , \tag{2.3.14}$$

with

$$\begin{cases} \beta_0 = \beta_1 = 0 \\ \\ \beta_2 = \frac{1}{2}\alpha_1^2 \\ \\ \beta_3 = \frac{1}{6}\alpha_1^3 + \alpha_1 \alpha_2 \ . \end{cases} \tag{2.3.15}$$

By using (2.3.11) for the differential operator and by denoting the gauge field strength tensor as $\mathcal{R}_{\mu\nu} \equiv [\nabla_\mu, \nabla_\nu]$, the first four coefficients are given by

$$\alpha_0(x) = \mathbb{1} \tag{2.3.16}$$

$$\alpha_1(x) = \frac{1}{6} R\mathbb{1} + V \tag{2.3.17}$$

$$\alpha_2(x) = \frac{1}{6}\Box\left(\frac{1}{5}R\mathbb{1} + V\right) + \frac{1}{180}(R^2_{\mu\nu\rho\sigma} - R^2_{\mu\nu})\mathbb{1} + \frac{1}{12}\mathcal{R}^2_{\mu\nu} \tag{2.3.18}$$

$$\begin{aligned}\alpha_3(x) = \frac{1}{7!}\Big[& 18\Box^2 R + 17(\nabla_\mu R)^2 - 2(\nabla_\mu R_{\nu\sigma})^2 - 4\nabla_\mu R_{\nu\sigma}\nabla^\nu R^{\mu\sigma} + \\ & + 9(\nabla_\alpha R_{\mu\nu\rho\sigma})^2 - 8R_{\mu\nu}\Box R^{\mu\nu} + 24 R_{\mu\nu}\nabla^\nu\nabla_\sigma R^{\mu\sigma} + \\ & + 12 R_{\mu\nu\rho\sigma}\Box R^{\mu\nu\rho\sigma} - \frac{208}{9}R_\mu{}^\nu R_\nu{}^\sigma R_\sigma{}^\mu + \frac{64}{3}R_{\mu\nu}R_{\rho\sigma}R^{\mu\nu\rho\sigma} + \\ & - \frac{16}{3}R_{\mu\nu}R^\mu{}_{\rho\sigma\tau}R^{\nu\rho\sigma\tau} + \frac{44}{9}R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} + \\ & + \frac{80}{9}R_{\mu\nu\rho\sigma}R^{\mu\alpha\rho\beta}R^\nu{}_\alpha{}^\sigma{}_\beta\Big]\mathbb{1} + \\ + \frac{2}{6!}\Big[& 8(\nabla_\mu \mathcal{R}_{\nu\sigma})^2 + 2(\nabla^\mu\mathcal{R}_{\mu\nu})^2 + 12\mathcal{R}_{\mu\nu}\Box\mathcal{R}^{\mu\nu} - 12\mathcal{R}_\mu{}^\nu\mathcal{R}_\nu{}^\sigma\mathcal{R}_\sigma{}^\mu + \\ & + 6R_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu}\mathcal{R}^{\rho\sigma} - 4R_{\mu\nu}\mathcal{R}^{\mu\sigma}\mathcal{R}^\nu{}_\sigma + 6\Box^2 V + 30(\nabla_\mu V)^2 + \\ & + 4R_{\mu\nu}\nabla^\mu\nabla^\nu V + 12\nabla_\mu R\nabla^\mu V\Big] \,.\end{aligned} \tag{2.3.19}$$

The expressions above [33] will be used in chapter 4 to compute the heat kernel coefficients for massive gravity in curved spacetime.

3 Massive gravity in flat spacetime

In this chapter we review the main aspects of massive gravity in a flat space. We will first introduce the massless gravity theory and then see what happens when the graviton becomes massive, leading to the Fierz-Pauli action for massive gravity. In chapter 4 we will then formulate the massive gravity theory, but in a curved spacetime. Our main reference is [34].

3.1 Gauge symmetries

Considering a field theory, we know the degrees of freedom (d.o.f.) are the particles, classified by their spin. In case of long range macroscopic forces, these degrees of freedom are carried by bosonic fields, with integer spin $s = 0, 1, 2, \text{etc}$, due to the spin statistics theorem. A field ψ, which carries a particle of mass m, satisfies the Klein-Gordon equation $(\Box - m^2)\psi = 0$, whose solution a distance r from a localized source goes like $\sim \frac{\exp{(-mr)}}{r}$. Long range forces have no exponential suppression, so they are described by massless fields $m = 0$. Massless particles are described by how they transform under rotations transverse to their direction of motion. The transformation rule is characterized by an integer $h \geq 0$ called helicity. When $h = 2$, the required gauge symmetry to have a manifestly Lorentz covariant description is linearized general coordinate invariance. Asking for consistent self-interactions leads uniquely to General Relativity (GR) and full general coordinate invariance [35, 36, 37, 38, 39, 40, 41, 42].

Let's now focus briefly on the nature of gauge symmetries. They are not fundamental properties, but, to be more precise, redundancies in the description of a theory. In fact, we can always fix the gauge and eliminate the gauge symmetry without breaking the global symmetries, which are the true physical ones. Fixing a gauge doesn't change the physics of the system, but, in this case, the global symmetries and locality are not manifest. On the other hand, if we start from a system with no gauge invariance, it's always possible to introduce gauge symmetry by putting in redundant variables. We will perform this procedure by introducing Stückelberg fields in order to make any lagrangian invariant under general coordinate diffeomorphisms. This possibility suggests that general coordinate invariance is not a defining feature of GR. In fact, we can define GR as *the theory of a non-trivially interacting massless helicity 2 particle.* The other properties are consequences of this statement.

3.2 An historical overview

The theory of massive gravity propagates a massive spin 2 particle. A straightforward way to formulate this theory is adding a mass term to the Einstein-Hilbert action, by giving a mass m to the graviton, in order to recover GR when $m \to 0$. As GR is the unique theory of a massless spin 2 particle, we should consider that changing it means changing the degrees of freedom. The possibility of a graviton mass has been studied since 1939, when Fierz and Pauli [43] first wrote the action describing a free massive graviton.

However, introducing a mass produces new patologies in the theory, like non-linearities, which increase as the mass shrinks, ghost-like instabilities and a very low cutoff. Also, a new mechanism arises. The extra degrees of freedom carried by the massive graviton must decouple themselves as $m \to 0$ to restore GR. Unfortunately, this process is not as clean as we should expect. In fact, in 1970, when there was a flurry of renewed interest in quantum field theory, the linear theory coupled to a source was studied by van Dam, Veltman, and Zakharov [44, 45], who discovered the curious fact that the theory makes predictions different from those of linear GR even in the limit as the graviton mass goes to zero. In fact, massive gravity in the $m \to 0$ limit gives a prediction for light bending which is off by 25% from GR. This is known as *vDVZ discontinuity*.

This discontinuity is due to the fact that not all the d.o.f. decouple as the mass goes to zero. The massive graviton has 5 spin states, which in the massless limit become the 2 helicity states of a massless graviton, 2 helicity states of a massless vector and a massless scalar, which is a longitudinal graviton. So, the massless limit a massive graviton is not simply a massless graviton, but a massless graviton plus a coupled scalar, which is the reason of the vDVZ discontinuity.

If the linear theory is accurate, the vDVZ discontinuity represents a true physical discontinuity in predictions, violating our intuition that physics should be continuous in the parameters. Measuring the light bending in this theory would be a way to show that the graviton mass is mathematically zero rather than just very small.

The possible non-linearities of a real theory were studied by Vainshtein in 1972 [46], who found that the extra degree of freedom responsible for the vDVZ discontinuity gets screened by its own interactions, which dominate over the linear terms in the massless limit. Non-linearities of the theory become stronger and stronger as the mass of the graviton shrinks. What he found was that around any massive source of mass M, such as the Sun, there's a new length scale known as the *Vainshtein radius*, $r_V = \left(\frac{M}{m^4 M_P^2}\right)^{1/5}$. At distances $r \lesssim r_V$, non-linearities begin to dominate and the predictions of the linear theory cannot be trusted. The Vainshtein radius goes to infinity as $m \to 0$, so there's no radius at which the linear approximation tells us something trustworthy about the massless limit. This opens the possibility that the non-linear effects cure the discontinuity. To have some values in mind, if we take M as the mass of the Sun and m with a very small value, say the Hubble constant $m \sim 10^{-33} eV$, the scale at which we might want to modify gravity to explain the cosmological constant, we have $r_V \sim 10^{18} km$, about the size of the Milky Way.

Later on, in the same year, Boulware and Deser [47] studied some specific fully non-linear massive gravity theories and showed that they possess a ghost-like instability. Whereas the linear theory has 5 d.o.f., the non-linear theories they studied turned out to have 6 and the extra degree of freedom manifests itself around non-trivial backgrounds as a scalar field with a wrong sign kinetic term, known as the *Boulware-Deser ghost*.

Meanwhile, the ideas of effective field theory were being developed and it was realized that a non-renormalizable theory, even one with apparent instabilities such as massive gravity, can be made sense of as an effective field theory, valid only at energies below some UV cutoff scale. In 2003, Arkani-Hamed, Georgi and Schwartz [48] brought to attention a method of restoring gauge invariance to massive gravity in a way which makes it very simple to see what the effective field theory properties are. They showed that massive gravity generically has a maximum UV cutoff of $\Lambda_5^{-1} \sim 10^{11}$ km. This is a very small cutoff, parametrically smaller than the Planck mass and goes to zero as $m \to 0$. Around a massive source, the quantum effects become important at the radius $r_Q = (\frac{M}{M_P})^{1/3}\frac{1}{\Lambda_5}$, which is parametrically larger than the Vainshtein radius at which non-linearities enter. For the Sun, $r_Q \sim 10^{24}$ km. Without finding a UV completion or some other re-summation, there's no sense in which we can trust the solution inside this radius and the usefulness of massive gravity is limited. In particular, since the whole non-linear regime is below this radius, there's no hope to examine the continuity of physical quantities in m and explore the Vainshtein mechanism in a controlled way. On the other hand, it can be seen that the mass of the Boulware-Deser ghost drops below the cutoff only when $r \lesssim r_Q$, so the ghost is not really in the effective theory at all and can be consistently excluded.

Putting aside the issue of quantum corrections, there has been continued study of the Vainshtein mechanism in a purely classical context. It has been shown that classical non-linearities do indeed restore continuity with GR in certain circumstances. In fact, the ghost degree of freedom can play an essential role in this, by providing a repulsive force in the non-linear region to counteract the attractive force of the longitudinal scalar mode.

By adding higher order graviton self-interactions with appropriately tuned coefficients, it is in fact possible to raise the UV cutoff of the theory to $\Lambda_3 = (M_P m^2)^{1/3}$, corresponding to roughly $\Lambda_3^{-1} \sim 10$ km. In 2010, the complete action of this theory in a certain decoupling limit was worked out by de Rham and Gabadadze (dRGT theories) [49], and they show that, remarkably, it is free of the Boulware-Deser ghost. Recently, it was shown that the complete theory is free of the Boulware-Deser ghost. This Λ_3 theory is the best hope of realizing a useful and interesting massive gravity theory.

3.3 The free Fierz-Pauli theory on flat spacetime

We want to study the action for a single massive spin 2 particle of mass m in a flat D-dimensional spacetime (with $d = D-1$ space dimensions) with metric $g_{\mu\nu}$, with the graviton as a symmetric tensor field $h_{\mu\nu}$. We will treat only the case with no external sources, i.e. no couplings to other fields ($T_{\mu\nu} = 0$). This action is known as the *Fierz-Pauli action* [43, 50] and it can be expressed as a sum of two terms, a massless one and a massive one.

The massless term can be obtained by linearizing the Einstein-Hilbert action of general relativity

$$S_{EH}[g] = \frac{1}{2\kappa^2} \int d^D x \sqrt{g}\, R(g) \ , \tag{3.3.1}$$

with $\kappa \equiv M_P^{-\frac{D-2}{2}}$, M_P as the Planck mass and g as the modulus of the determinant of the metric. By expanding the metric as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \ , \tag{3.3.2}$$

with $\eta_{\mu\nu}$ as the Minkowski metric for flat spacetime and $h_{\mu\nu}(x)$ as a small quantum perturbation, i.e. the graviton, we can linearize the Einstein-Hilbert action. We will show explicitely how to perform the linearization in Appendix A. Hence, the massless term, at second order, has the following expression

$$S[h]_{m=0} = \int d^D x \left[-\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h \right] \tag{3.3.3}$$

and is invariant under the gauge-symmetry

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 2\partial_{(\mu}\xi_{\nu)} \ , \tag{3.3.4}$$

for a spacetime dependent parameter $\xi_\mu(x)$, which is broken by the following most general mass term for the $h_{\mu\nu}$ field

$$S[h]_{m_1,m_2} = -\frac{1}{2}\int d^D x \left[m_1^2\, h^{\mu\nu} h_{\mu\nu} + m_2^2\, h^2 \right] . \tag{3.3.5}$$

However, it can be shown [51] that the total action $S[h]_{m=0} + S[h]_{m_1,m_2}$ describes the propagation of 5 d.o.f. only if

$$m_1^2 + m_2^2 = 0 \ , \tag{3.3.6}$$

so we can use the notation $m \equiv m_1$ and the mass term reduces to

$$S[h]_m = -\frac{1}{2}\int d^D x \left[m^2 \Big(h^{\mu\nu} h_{\mu\nu} - h^2 \Big) \right] , \tag{3.3.7}$$

which corresponds to the Fierz-Pauli mass term. The coefficient -1 between the h^2 and $h_{\mu\nu}h^{\mu\nu}$ terms is called *Fierz-Pauli tuning* and it's not enforced by any known symmetry. If the condition (3.3.6) is not satisfied, a sixth ghost mode with negative energy appears [52], and the theory does not describe a massive graviton. This is a consequence of Ostrogradsky's theorem [53].

Hence, the total action can be written as

$$\begin{aligned}S[h] &= S[h]_{m=0} + S[h]_m = \\ &= \int d^D x \Big[-\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h + \\ &\qquad\qquad - \frac{1}{2}m^2\Big(h^{\mu\nu}h_{\mu\nu} - h^2\Big)\Big] \ .\end{aligned} \tag{3.3.8}$$

This action contains all the possible contractions of two powers of h, with up to two derivatives.

Now, by putting the action in the hamiltonian form, we can count the number degrees of freedom, which are $\frac{D(D-1)}{2} - 1$. We start by Legendre-transforming (3.3.8) only with respect to the spatial components h_{ij}. The canonical momenta are

$$\pi_{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \dot{h}_{ij} - \dot{h}_{kk}\delta_{ij} - 2\partial_{(i}h_{j)0} + 2\partial_k h_{0k}\delta_{ij} \ , \tag{3.3.9}$$

so

$$\dot{h}_{ij} = \pi_{ij} - \frac{1}{D-2}\pi_{kk}\delta_{ij} + 2\partial_{(i}h_{j)0} \ . \tag{3.3.10}$$

In terms of the hamiltonian variables, (3.3.8) becomes

$$S[h] = \int d^D x \Big[\pi_{ij}\dot{h}_{ij} - \mathcal{H} + 2h_{0i}(\partial_j \pi_{ij}) + m^2 h_{0i}^2 + h_{00}\Big(\nabla^2 h_{ii} - \partial_i\partial_j h_{ij} - m^2 h_{ii}\Big)\Big] \ , \tag{3.3.11}$$

with $\nabla^2 \equiv g_{ij}\partial^i\partial^j$, with $i, j = 1, 2, ..., d$ and

$$\begin{aligned}\mathcal{H} = &\frac{1}{2}\pi_{ij}^2 - \frac{1}{2}\frac{1}{D-2}\pi_{ii}^2 + \\ &+ \frac{1}{2}\partial_k h_{ij}\partial_k h_{ij} - \partial_i h_{jk}\partial_j h_{ik} + \partial_i h_{ij}\partial_j h_{kk} - \frac{1}{2}\partial_i h_{jj}\partial_i h_{kk} + \\ &+ \frac{1}{2}m^2\big(h_{ij}h_{ij} - h_{ii}^2\big)\end{aligned} \tag{3.3.12}$$

Let's first consider the $m = 0$ case. The time-like components h_{0i} and h_{00} appear linearly multiplied by terms with no time derivatives. By interpreting them as Lagrange multipliers, two constraints are enforced:

$$\partial_j \pi_{ij} = 0 \quad \text{and} \quad \nabla^2 h_{ii} - \partial_i\partial_j h_{ij} = 0 \ . \tag{3.3.13}$$

For $D = 4$, these constraints generate 4 gauge invariances, so the gauge orbits are 4 dimensional and the gauge invariant quotient by the orbits is 4 dimensional [54]. These are the two polarizations of the massless graviton, along with their conjugate momenta.

Now let's analyze the $m \neq 0$ case. The h_{0i} components appear quadratically and they are auxiliary variables instead of Lagrange multipliers. Their equation of motion yield

$$h_{0i} = -\frac{1}{m}\partial_j \pi_{ij} \ , \tag{3.3.14}$$

which can be plugged into (3.3.11) to obtain

$$S[h] = \int d^D x \Big[\pi_{ij}\dot{h}_{ij} - \mathcal{H} + h_{00}\Big(\nabla^2 h_{ii} - \partial_i\partial_j h_{ij} - m^2 h_{ii} \Big) \Big] \ , \tag{3.3.15}$$

with

$$\begin{aligned}\mathcal{H} = \frac{1}{2}\pi_{ij}^2 - \frac{1}{2}\frac{1}{D-2}\pi_{ii}^2 + \frac{1}{2}\partial_k h_{ij}\partial_k h_{ij} - \partial_i h_{jk}\partial_j h_{ik} + \partial_i h_{ij}\partial_j h_{kk} + \\ - \frac{1}{2}\partial_i h_{jj}\partial_i h_{kk} + \frac{1}{2}m^2\big(h_{ij}h_{ij} - h_{ii}^2\big) + \frac{1}{m^2}(\partial_j \pi_{ij})^2 \ .\end{aligned} \tag{3.3.16}$$

The component h_{00} remains a Lagrange multiplier enforcing a single constraint

$$\mathcal{C} = -\nabla^2 h_{ii} + \partial_i\partial_j h_{ij} + m^2 h_{ii} = 0 \ . \tag{3.3.17}$$

The Fierz-Pauli tuning is crucial to the appearence of h_{00} as a Lagrange multiplier. If it's violated, then h_{00} appears quadratically and it's an auxiliary variable.

A second constraint arises from the Poisson bracket

$$\{H, \mathcal{C}\}_{PB} = \frac{1}{D-2}m^2 \pi_{ii} + \partial_i\partial_j \pi_{ij} \tag{3.3.18}$$

with the Hamiltonian $H = \int d^d x \, \mathcal{H}$. For $D = 4$ we have 10 total degrees of freedom, corresponding to the 5 polarizations of the massive graviton and their conjugate momenta. If the Fierz-Pauli tuning is violated, then we have no constraints and 12 d.o.f. The 2 extra d.o.f. are the scalar ghost and its conjugate momentum.

From (3.3.8) we can also derive the equations of motion, which read

$$\begin{aligned}\frac{\delta S}{\delta h^{\mu\nu}} = \Box h_{\mu\nu} - \partial_\lambda\partial_\mu h^\lambda{}_\nu - \partial_\lambda\partial_\nu h^\lambda{}_\mu + \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h + \\ - m^2(h_{\mu\nu} - \eta_{\mu\nu}h) = 0 \ ,\end{aligned} \tag{3.3.19}$$

with $\Box \equiv \eta_{\mu\nu}\partial^\mu\partial^\nu$. If we act on (3.3.19) with ∂^μ, with $m \neq 0$, we find the constraint

$$\partial^\mu h_{\mu\nu} - \partial_\nu h = 0 \ , \tag{3.3.20}$$

which, plugged back in (3.3.19), gives

$$\Box h_{\mu\nu} - \partial_\mu\partial_\nu h - m^2(h_{\mu\nu} - \eta_{\mu\nu}h) = 0 \ . \tag{3.3.21}$$

By taking the trace of this we get $h = 0$, which implies $\partial^\mu h_{\mu\nu} = 0$. By applying this two conditions to (3.3.19), we have

$$(\Box - m^2) h_{\mu\nu} = 0. \tag{3.3.22}$$

Hence, the equations (3.3.19) are equivalent to the equations

$$\begin{cases} (\Box - m^2) h_{\mu\nu} = 0 \\ \partial^\mu h_{\mu\nu} = 0 \\ h = 0 \ . \end{cases} \tag{3.3.23}$$

This form allows us to count the number of d.o.f. more easily. In fact, in $D = 4$, the first expression is an evolution equation for the 10 components of the symmetric tensor $h_{\mu\nu}$ and the other two expressions are constraints. The first of the two gives 4 constraints, killing 4 real space d.o.f., while the second one determines the trace, killing 1 degree of freedom. In total, we have 5 real space degrees of freedom, in agreement with the analysis we have done.

Note that the tracelessness constraint $h = 0$ is not satisfied if the Fierz-Pauli tuning is not present, i.e. the condition (3.3.6) is not satisfied. In this case, one degree of freedom is not removed, leading to a theory with 6 d.o.f. in $D = 4$: a ghost-like scalar mode inside $h_{\mu\nu}$ becomes propagating. At the classical level a ghost is a field with negative kinetic energy which gives rise to an unbounded Hamiltonian and thus causes fatal instabilities; at the quantum level ghosts must be avoided in order to ensure unitarity. It is therefore crucial to work with the above Lagrangian with correct relative coefficient in the mass term. In $D = 4$, it describes the on-shell propagation of a traceless, transverse and symmetric tensor field $h_{\mu\nu}$ with 5 massive degrees of freedom. This allows us to identify $h_{\mu\nu}$ with a massive spin-2 field with helicities ± 2, ± 1, 0.

The first equation in (3.3.23) is the standard Klein-Gordon equation, with the general solution

$$h^{\mu\nu}(x) = \int \frac{d^d \boldsymbol{p}}{\sqrt{(2\pi)^d 2\omega_{\boldsymbol{p}}}} \Big(h^{\mu\nu}(\boldsymbol{p}) \exp\left(ip \cdot x\right) + h^{\mu\nu *}(\boldsymbol{p}) \exp\left(-ip \cdot x\right) \Big) \ , \tag{3.3.24}$$

with $\boldsymbol{p}$ as the spatial momenta, $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$ and the $D-$momenta p^μ on shell, so that $p^\mu = (\omega_{\boldsymbol{p}}, \boldsymbol{p})$. Next we expand the Fourier coefficients $h^{\mu\nu}(\boldsymbol{p})$ over some basis of symmetric tensors, indexed by λ,

$$h^{\mu\nu}(\boldsymbol{p}) = a_{\boldsymbol{p},\lambda}\, \bar{\epsilon}^{\mu\nu}(\boldsymbol{p}, \lambda). \tag{3.3.25}$$

We will fix the momentum dependence of the basis elements $\bar{\epsilon}^{\mu\nu}(\boldsymbol{p}, \lambda)$ by choosing some basis $\bar{\epsilon}^{\mu\nu}(\boldsymbol{k}, \lambda)$ at the standard momentum $k^\mu = (m, 0, 0, 0, \dots)$ and then acting with some standard boost $L(p)$, which takes k into p, $p^\mu = L(p)^\mu{}_\nu k^\nu$. This standard boost will choose for us the basis at $\boldsymbol{p}$, relative to that at $\boldsymbol{k}$. Thus we have

$$\bar{\epsilon}^{\mu\nu}(\boldsymbol{p}, \lambda) = L(p)^\mu{}_\alpha L(p)^\nu{}_\beta\, \bar{\epsilon}^{\alpha\beta}(\boldsymbol{k}, \lambda) \ . \tag{3.3.26}$$

Imposing the conditions $\partial_\mu h^{\mu\nu} = 0$ and $h = 0$ on (3.3.24) then reduces to imposing

$$k_\mu \, \bar{\epsilon}^{\mu\nu}(\boldsymbol{k}, \lambda) = 0, \quad \eta_{\mu\nu} \, \bar{\epsilon}^{\mu\nu}(\boldsymbol{k}, \lambda) = 0 \ . \tag{3.3.27}$$

The first says that $\bar{\epsilon}^{\mu\nu}(\boldsymbol{k}, \lambda)$ is purely spatial, i.e. $\bar{\epsilon}^{0\mu}(\boldsymbol{k}, \lambda) = 0$. The second says that it is traceless, so that $\bar{\epsilon}^{i}{}_{i}(\boldsymbol{k}, \lambda) = 0$ also. Thus, the basis need to be only of symmetric traceless spatial tensors, $\lambda = 1, \ldots, \frac{d(d+1)}{2} - 1$. We also demand the basis to be orthonormal

$$\bar{\epsilon}^{\mu\nu}(\boldsymbol{k}, \lambda) \, \bar{\epsilon}^{*}_{\mu\nu}(\boldsymbol{k}, \lambda') = \delta_{\lambda\lambda'} \ . \tag{3.3.28}$$

This basis forms the symmetric traceless representation of the rotation group $SO(d)$, which is the little group for a massive particle in D dimensions. If $R^{\mu}{}_{\nu}$ is a spatial rotation, we have

$$R^{\mu}{}_{\mu'} R^{\nu}{}_{\nu'} \, \bar{\epsilon}^{\mu\nu}(\boldsymbol{k}, \lambda') = R^{\lambda'}{}_{\lambda} \, \bar{\epsilon}^{\mu\nu}(\boldsymbol{k}, \lambda') \ , \tag{3.3.29}$$

where $R^{\lambda'}{}_{\lambda}$ is the symmetric traceless tensor representation of $R^{\mu}{}_{\mu'}$. We are free to use any other basis $\epsilon^{\mu\nu}(\boldsymbol{k}, \lambda)$, related to the $\bar{\epsilon}^{\mu\nu}(\boldsymbol{k}, \lambda)$ by

$$\bar{\epsilon}^{\mu\nu}(\boldsymbol{k}, \lambda) = B^{\lambda'}{}_{\lambda} \, \bar{\epsilon}^{\mu\nu}(\boldsymbol{k}, \lambda') \ , \tag{3.3.30}$$

where B is a unitary matrix.

Given a particular spatial direction, with the unit vector $\hat{k}^i$, there's an $SO(d-1)$ subgroup of the little group $SO(d)$ which leaves $\hat{k}^i$ invariant and the symmetric traceless rep of $SO(d)$ breaks up into three reps of $SO(d-1)$, a scalar, a vector and a symmetric traceless tensor. The scalar mode is called the longitudinal graviton and has spatial components

$$\epsilon^{ij}_L = \sqrt{\frac{d}{d-1}} \left(\hat{k}^i \hat{k}^j - \frac{1}{d} \delta^{ij} \right) \ . \tag{3.3.31}$$

After a large boost in the $\hat{k}^i$ direction, it goes like $\epsilon_L \sim p^2/m^2$. As we will see later, in the massless limit, or large boost limit, this mode is carried by a scalar field, which generally becomes strongly coupled once interactions are taken into account. The vector modes have spatial components

$$\epsilon^{ij}_{V,k} = \sqrt{2} \hat{k}^{(i} \delta^{j)}_k \tag{3.3.32}$$

and after a large boost in the $\hat{k}^i$ direction, they go like $\epsilon_L \sim p/m$. In the massless limit, these modes are carried by a vector field, which decouples from conserved sources. The remaining linearly independent modes are symmetric traceless tensors with no components in the $\hat{k}^i$ directions and form the symmetric traceless mode of $SO(d-1)$. They are invariant under a boost in the $\hat{k}^i$ direction and in the massless limit they are carried by a massless graviton. In the massless limit, we should therefore expect that the extra degrees of freedom of the massive graviton should organize themselves into a massless vector and a massless scalar.

Upon boosting to $\boldsymbol{p}$, the polarization tensors satisfy the following properties: they are transverse to p^μ and traceless,

$$p_\mu\, \epsilon^{\mu\nu}(\boldsymbol{p},\lambda) = 0, \quad \eta_{\mu\nu}\, \epsilon^{\mu\nu}(\boldsymbol{p},\lambda) = 0 \tag{3.3.33}$$

and they satisfy orthogonality and completeness relations

$$\epsilon^{\mu\nu}(\boldsymbol{p},\lambda)\, \epsilon^*_{\mu\nu}(\boldsymbol{p},\lambda') = \delta_{\lambda\lambda'} \ , \tag{3.3.34}$$

$$\sum_\lambda \epsilon^{\mu\nu}(\boldsymbol{p},\lambda)\, \epsilon^{*\alpha\beta}(\boldsymbol{p},\lambda) = \frac{1}{2}(P^{\mu\alpha}P^{\nu\beta} + P^{\mu\beta}P^{\nu\alpha}) - \frac{1}{D-1}P^{\mu\nu}P^{\alpha\beta} \ , \tag{3.3.35}$$

where $P^{\alpha\beta} \equiv \eta^{\alpha\beta} + \frac{p^\alpha p^\beta}{m^2}$. The right hand side of the completeness relation (3.3.35) is the projector onto the symmetric and transverse traceless subspace of tensors, i.e. the identity on this space. We also have the following symmetric properties in $\boldsymbol{p}$, which can be deduced from the form of the standard boost,

$$\epsilon^{ij}(-\boldsymbol{p},\lambda) = \epsilon^{ij}(\boldsymbol{p},\lambda) \ , \quad i,j = 1,2,\dots,d \tag{3.3.36}$$

$$\epsilon^{0i}(-\boldsymbol{p},\lambda) = -\epsilon^{0i}(\boldsymbol{p},\lambda) \ , \quad i = 1,2,\dots,d \tag{3.3.37}$$

$$\epsilon^{00}(-\boldsymbol{p},\lambda) = \epsilon^{00}(\boldsymbol{p},\lambda) \ . \tag{3.3.38}$$

The general solution to (3.3.19) thus reads

$$h^{\mu\nu}(x) = \int \frac{d^d\boldsymbol{p}}{\sqrt{(2\pi)^d 2\omega_{\boldsymbol{p}}}} \sum_\lambda \left[a_{\boldsymbol{p},\lambda}\, \epsilon^{\mu\nu}(\boldsymbol{p},\lambda) \exp{(ip\cdot x)} + a^*_{\boldsymbol{p},\lambda}\, \epsilon^{*\mu\nu}(\boldsymbol{p},\lambda) \exp{(-ip\cdot x)}\right] \ . \tag{3.3.39}$$

The solution is a general linear combination of the following mode functions and their conjugates

$$u^{\mu\nu}_{\boldsymbol{p},\lambda} \equiv \frac{1}{\sqrt{(2\pi)^d 2\omega_{\boldsymbol{p}}}}\, \epsilon^{\mu\nu}(\boldsymbol{p},\lambda) \exp{(ip\cdot x)}, \quad \lambda = 1,2,\dots,d. \tag{3.3.40}$$

These are the solutions representing gravitons and they have the following Poincarè transformation properties

$$u^{\mu\nu}_{\boldsymbol{p},\lambda}(x-a) = u^{\mu\nu}_{\boldsymbol{p},\lambda}(x) \exp{(-ip\cdot a)} \ , \tag{3.3.41}$$

$$\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\, u^{\mu'\nu'}_{\boldsymbol{p},\lambda}(\Lambda^{-1}x) = \sqrt{\frac{\omega_{\Lambda\boldsymbol{p}}}{\omega_{\boldsymbol{p}}}}\, W(\Lambda,p)_{\lambda'\lambda} u^{\mu\nu}_{\Lambda\boldsymbol{p},\lambda}(x) \tag{3.3.42}$$

where $W(\Lambda,p) = L^{-1}(\Lambda p)\Lambda L(p)$ is the Wigner rotation and $W(\Lambda,p)_{\lambda'\lambda}$ is its spin 2 representation $R^\mu{}_\nu \to (B^{-1}RB)_{\lambda'\lambda}$. Thus, the gravitons are spin 2 solutions.

In terms of the modes, the general solution reads

$$h^{\mu\nu}(x) = \int d^d\boldsymbol{p} \sum_\lambda \left[a_{\boldsymbol{p},\lambda} u^{\mu\nu}_{\boldsymbol{p},\lambda}(x) + a^*_{\boldsymbol{p},\lambda} u^{\mu\nu *}_{\boldsymbol{p},\lambda}(x)\right] \ . \tag{3.3.43}$$

The inner (symplectic) product on the space of solutions to the equations of motion is,

$$(h, h') = i \int d^d\boldsymbol{x}\, h^{\mu\nu *}(x) \overleftrightarrow{\partial_0} h'_{\mu\nu}(x) \Big|_{t=0} \ . \tag{3.3.44}$$

The u functions are orthonormal with respect to this product,

$$(u_{\boldsymbol{p},\lambda} u_{\boldsymbol{p}',\lambda'}) = \delta^d(\boldsymbol{p} - \boldsymbol{p'})\delta_{\lambda\lambda'} \ , \tag{3.3.45}$$

$$(u^*_{\boldsymbol{p},\lambda} u^*_{\boldsymbol{p}',\lambda'}) = -\delta^d(\boldsymbol{p} - \boldsymbol{p'})\delta_{\lambda\lambda'} \ , \tag{3.3.46}$$

$$(u_{\boldsymbol{p},\lambda} u^*_{\boldsymbol{p}',\lambda'}) = 0 \tag{3.3.47}$$

and we can use the product to extract the a and a^* coefficients from any solution $h_{\mu\nu}(x)$,

$$a_{\boldsymbol{p},\lambda} = (u_{\boldsymbol{p},\lambda}, h) \ , \tag{3.3.48}$$

$$a^*_{\boldsymbol{p},\lambda} = -(u^*_{\boldsymbol{p},\lambda}, h) \ . \tag{3.3.49}$$

In the quantum theory, the a and a^* become creation and annihilation operators which satisfy the usual commutation relations and produce massive spin 2 states. The fields h_{ij} and their canonical momenta π_{ij}, constructed from the a and a^* will then automatically satisfy the Dirac algebra and constraints of the Hamiltonian analysis, providing a quantization of the system. Once interactions are taken into account, external lines of Feynman diagrams will get a factor of the mode functions (3.3.40).

3.4 Propagator

We can rewrite (3.3.8) in the following form by integrating by parts

$$S[h] = \int d^Dx\, \frac{1}{2}\, h_{\mu\nu} \mathcal{O}^{\mu\nu}{}_{\alpha\beta}\, h^{\alpha\beta} \ , \tag{3.4.1}$$

with

$$\mathcal{O}^{\mu\nu}{}_{\alpha\beta} = \left(\eta^{(\mu}{}_{\alpha}\eta^{\nu)}{}_{\beta} - \eta^{\mu\nu}\eta_{\alpha\beta}\right)(\Box - m^2) - 2\partial^{(\mu}\partial_{(\alpha}\eta^{\nu)}{}_{\beta)} + \partial^\mu\partial^\nu\eta_{\alpha\beta} + \partial_\alpha\partial_\beta\eta^{\mu\nu} \ , \tag{3.4.2}$$

which satisfies

$$\mathcal{O}^{\mu\nu\alpha\beta} = \mathcal{O}^{\nu\mu\alpha\beta} = \mathcal{O}^{\mu\nu\beta\alpha} = \mathcal{O}^{\alpha\beta\mu\nu} \ , \tag{3.4.3}$$

with

$$\eta^{\mu(\alpha}\eta^{\beta)\nu} = \frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\beta\nu} + \eta^{\mu\beta}\eta^{\alpha\nu}\right) \ . \tag{3.4.4}$$

$$\partial^{(\alpha}\partial^{\beta)} = \frac{1}{2}(\partial^\alpha\partial^\beta + \partial^\beta\partial^\alpha) \tag{3.4.5}$$

Proof. The calculations are trivial and straightforward. We start with the total action (3.4.1) and rearrange the terms in order to write the action in diagonal form

$$
\begin{aligned}
S[h] &= \int d^D x \bigg[-\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h + \\
&\qquad - \frac{1}{2} m^2 \Big(h^{\mu\nu} h_{\mu\nu} - h^2 \Big) \bigg] = \\
&= \int d^D x \bigg[\frac{1}{2} h_{\mu\nu} \Box h^{\mu\nu} - h_{\nu\lambda}\partial_\mu \partial^\nu h^{\mu\lambda} + h^{\mu\nu}\partial_\mu\partial_\nu h - \frac{1}{2} h \Box h + \\
&\qquad - \frac{1}{2} m^2 \Big(h^{\mu\nu} h_{\mu\nu} - h^2 \Big) \bigg] = \\
&= \int d^D x \bigg[\frac{1}{2} h_{\mu\nu} \Box \, \eta^{(\mu}{}_\alpha \eta^{\nu)}{}_\beta h^{\alpha\beta} - h_{\mu\nu}\partial_\lambda \partial^\nu h^{\mu\lambda} + \\
&\qquad + h_{\mu\nu}\partial^\mu\partial^\nu \eta_{\alpha\beta} h^{\alpha\beta} - \frac{1}{2} h_{\mu\nu} \Box \eta^{\mu\nu}\eta_{\alpha\beta}\, h^{\alpha\beta} + \\
&\qquad - \frac{1}{2} m^2 \Big(h_{\mu\nu} \eta^{(\mu}{}_\alpha \eta^{\nu)}{}_\beta h^{\alpha\beta} - h_{\mu\nu}\eta^{\mu\nu}\eta_{\alpha\beta}\, h^{\alpha\beta} \Big) \bigg] = \\
&= \int d^D x \, \frac{1}{2} \, h_{\mu\nu} \bigg[\Big(\eta^{(\mu}{}_\alpha \eta^{\nu)}{}_\beta - \eta^{\mu\nu}\eta_{\alpha\beta} \Big)(\Box - m^2) + \\
&\qquad - 2\partial^{(\mu}\partial_{(\alpha}\eta^{\nu)}{}_{\beta)} + \partial^\mu\partial^\nu \eta_{\alpha\beta} + \partial_\alpha\partial_\beta \eta^{\mu\nu} \bigg] h_{\alpha\beta} \ .
\end{aligned}
\tag{3.4.6}
$$

It is then straightforward to verify the property (3.4.3).

■

The equations of motion (3.3.19) can be written as

$$
\frac{\delta S}{\delta h^{\mu\nu}} = \mathcal{O}^{\mu\nu\alpha\beta} h_{\alpha\beta} = 0 \ . \tag{3.4.7}
$$

Going to the momentum space, we can then derive the propagator $\mathcal{D}_{\alpha\beta\sigma\lambda}$, which has to satisfy

$$
\mathcal{O}^{\mu\nu\alpha\beta}\mathcal{D}_{\alpha\beta\sigma\lambda} = \frac{i}{2}\Big(\delta^\mu{}_\sigma \delta^\nu{}_\lambda + \delta^\nu{}_\sigma \delta^\mu{}_\lambda \Big) \ . \tag{3.4.8}
$$

By solving it, we get

$$
\mathcal{D}_{\alpha\beta\sigma\lambda} = \frac{-i}{p^2 + m^2} \bigg[\frac{1}{2}\Big(P_{\alpha\sigma}P_{\beta\lambda} + P_{\alpha\lambda}P_{\beta\sigma} \Big) - \frac{1}{D-1} P_{\alpha\beta}P_{\sigma\lambda} \bigg] \ , \tag{3.4.9}
$$

with

$$
P_{\alpha\beta} \equiv \eta_{\alpha\beta} + \frac{p_\alpha p_\beta}{m^2} \ . \tag{3.4.10}
$$

Let's now see what happens when $m = 0$. The massless action (3.3.3) can be written as

$$S[h]_{m=0} = \int d^D x \, \frac{1}{2} \, h_{\mu\nu} E^{\mu\nu}{}_{\alpha\beta} \, h^{\alpha\beta} \ , \tag{3.4.11}$$

with

$$E^{\mu\nu}{}_{\alpha\beta} = \mathcal{O}^{\mu\nu}{}_{\alpha\beta}|_{m=0} = \Big(\eta^{(\mu}{}_{\alpha}\eta^{\nu)}{}_{\beta} - \eta^{\mu\nu}\eta_{\alpha\beta}\Big)\Box - 2\partial^{(\mu}\partial_{(\alpha}\eta^{\eta)}{}_{\beta)} + \partial^{\mu}\partial^{\nu}\eta_{\alpha\beta} + \partial_{\alpha}\partial_{\beta}\eta^{\mu\nu} \tag{3.4.12}$$

and the symmetries (3.4.3).

If we act on a symmetric tensor $Y_{\alpha\beta}$ we have

$$E^{\mu\nu\alpha\beta}Y_{\alpha\beta} = \Box Y^{\mu\nu} - \eta^{\mu\nu}\Box Y - 2\partial^{(\mu}\partial_{\alpha}Y^{\nu)\alpha} + \partial^{\mu}\partial^{\nu}Y + \eta^{\mu\nu}\partial_{\alpha}\partial_{\beta}Y^{\alpha\beta} \ . \tag{3.4.13}$$

The massless action (3.3.3) is invariant under the gauge symmetry (3.3.4). This leads to the following conditions

$$\partial_{\mu}\Big(E^{\mu\nu\alpha\beta}Y_{\alpha\beta}\Big) = 0 \ , \quad E^{\mu\nu\alpha\beta}(\partial_{\alpha}\xi_{\beta} + \partial_{\beta}\xi_{\alpha}) = 0 \ . \tag{3.4.14}$$

Now, in order to find the propagator, we must fix the gauge freedom. Choosing the de Donder gauge corresponds to

$$\partial^{\mu}h_{\mu\nu} - \frac{1}{2}\partial_{\nu}h = 0 \ . \tag{3.4.15}$$

In this gauge the equations of motion simplify to

$$\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = 0 \tag{3.4.16}$$

and the solutions also satisfy (3.4.15). We can now add the gauge fixing term

$$\mathcal{L}_{GF} = -\Big(\partial^{\nu}h_{\mu\nu} - \frac{1}{2}\partial_{\mu}h\Big)^2 \tag{3.4.17}$$

to the lagrangian of (3.4.11). Quantum mechanically, this comes form the Faddeev-Popov gauge fixing procedure described in chapter 1. Hence, we have

$$\mathcal{L} + \mathcal{L}_{GF} = \frac{1}{2}h_{\mu\nu}\Box h^{\mu\nu} - \frac{1}{4}h\Box h = \frac{1}{2}h_{\mu\nu}\tilde{\mathcal{O}}^{\mu\nu,\alpha\beta}h_{\alpha\beta} \ , \tag{3.4.18}$$

with

$$\tilde{\mathcal{O}}^{\mu\nu\alpha\beta} = \frac{1}{2}\Big(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta}\Big)\Box \tag{3.4.19}$$

and (3.4.16) as the equations of motion.

Going to the momentum space we obtain the propagator

$$\mathcal{D}_{\alpha\beta\sigma\lambda} = \frac{-i}{p^2}\left[\frac{1}{2}\Big(\eta_{\alpha\sigma}\eta_{\beta\lambda} + \eta_{\alpha\lambda}\eta_{\beta\sigma}\Big) - \frac{1}{D-2}\eta_{\alpha\beta}\eta_{\sigma\lambda}\right] \ , \tag{3.4.20}$$

which satisfies (3.4.8) with $\tilde{\mathcal{O}}$ instead of $\mathcal{O}$.

Comparing (3.4.20) with $m \to 0$ and (3.4.9), there's a difference in the coefficient of the last term, which is $\frac{1}{D-1}$ w.r.t. $\frac{1}{D-2}$. This a sign of a discontinuity in the $m \to 0$ limit.

3.5 Linear response to sources and vDVZ discontinuity

We now add a fixed external source $T^{\mu\nu}(x)$ to the action (3.3.8)

$$S[h] = \int d^D x \bigg[-\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h + \\ -\frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) + \kappa h_{\mu\nu}T^{\mu\nu}\bigg] , \tag{3.5.1}$$

with $\kappa = M_P^{-\frac{D-2}{2}}$ as the coupling strength to the source. The equations of motion are now sourced by $T_{\mu\nu}$,

$$\frac{\delta S}{\delta h^{\mu\nu}} = \Box h_{\mu\nu} - \partial_\lambda\partial_\mu h^\lambda{}_\nu - \partial_\lambda\partial_\nu h^\lambda{}_\mu + \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h + \\ - m^2(h_{\mu\nu} - \eta_{\mu\nu}h) = -\kappa T_{\mu\nu} . \tag{3.5.2}$$

In the case $m = 0$, acting on the left with ∂^μ gives identically zero, so we must have the conservation condition $\partial^\mu T_{\mu\nu} = 0$ if there is to be a solution. For $m \neq 0$, there's no such condition.

We now want to find the retarded solution of (3.5.2), to which the homogeneous solutions of (3.3.8) can be added to obtain the general solution. Acting on the equations of motion (3.5.2) with ∂^μ leads to

$$\partial^\mu h_{\mu\nu} - \partial_\nu h = \frac{\kappa}{m^2}\partial^\mu T_{\mu\nu} . \tag{3.5.3}$$

By plugging this back into (3.5.2), we find

$$\Box h_{\mu\nu} - \partial_\mu\partial_\nu h - m^2(h_{\mu\nu} - \eta_{\mu\nu}h) = -\kappa T_{\mu\nu} + \frac{\kappa}{m^2}\Big[\partial^\lambda\partial_\mu T_{\nu\lambda} + \partial^\lambda\partial_\nu T_{\mu\lambda} - \eta_{\mu\nu}\partial\partial T\Big] , \tag{3.5.4}$$

with $\partial\partial T \equiv \partial_\mu\partial_\nu T^{\mu\nu}$. By taking the trace of this, we find

$$h = -\frac{\kappa}{m^2(D-1)}T - \frac{\kappa}{m^4}\frac{D-2}{D-1}\partial\partial T . \tag{3.5.5}$$

By applying this to (3.5.3), we find

$$\partial^\mu h_{\mu\nu} = -\frac{\kappa}{m^2(D-1)}\partial_\nu T + \frac{\kappa}{m^2}\partial^\mu T_{\mu\nu} - \frac{\kappa}{m^2}\partial^\mu T_{\mu\nu} - \frac{\kappa}{m^4}\frac{D-2}{D-1}\partial_\nu\partial\partial T , \tag{3.5.6}$$

which, when applied along with (3.5.5) to the equations of motion, gives

$$(\Box - m^2)h_{\mu\nu} = -\kappa\left[T_{\mu\nu} - \frac{1}{D-1}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2}\right)T\right] + \\ + \frac{\kappa}{m^2}\bigg[\partial^\lambda\partial_\mu T_{\nu\lambda} + \partial^\lambda\partial_\nu T_{\mu\lambda} + \\ - \frac{1}{D-1}\left(\eta_{\mu\nu} + (D-2)\frac{\partial_\mu\partial_\nu}{m^2}\right)\partial\partial T\bigg] . \tag{3.5.7}$$

Hence, the equations of motion imply the three equations (3.5.7), (3.5.6) and (3.5.5). Viceversa, these three equations imply the equations of motion (3.5.2).

Taking (3.5.7) and tracing, we find

$$(\Box - m^2)\left[h + \frac{\kappa}{m^2(D-1)}T + \frac{\kappa}{m^4}\frac{D-2}{D-1}\partial\partial T\right] = 0 \, . \tag{3.5.8}$$

Under the assumption that, for $(\Box - m^2)f = 0$, we have $f = 0$ for any function f, the equation (3.5.5) is implied. This will be the case with good boundary conditions, such as the retarded boundary conditions we impose when we are interested in the classical response to sources. The equation (3.5.6) can also be shown to follow under this assumption, so that we may obtain the solution by Fourier transforming only the equation (3.5.7). This solution can also be obtained by applying the propagator (3.4.9) to the Fourier transform of the source.

When the source is conserved, we are left with just the equation

$$(\Box - m^2)h_{\mu\nu} = -\kappa\left[T_{\mu\nu} - \frac{1}{D-1}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{m^2}\right)T\right] \, . \tag{3.5.9}$$

The general solution for a conserved source is then

$$h_{\mu\nu}(x) = \kappa \int \frac{d^D p}{(2\pi)^D}\frac{\exp{(ipx)}}{p^2 + m^2}\left[T_{\mu\nu}(p) - \frac{1}{D-1}\left(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right)T(p)\right] \, , \tag{3.5.10}$$

with $T^{\mu\nu}(p)$ as the Fourier transform of the source

$$T^{\mu\nu}(p) = \int d^D x \exp{(-ipx)} T^{\mu\nu}(x) \, . \tag{3.5.11}$$

To get the retarded field, we should integrate above the poles in the p^0 plane.

Now, we can study with more details two particular solutions: the first for a point source, the second for the massless graviton. They will be useful to study the vDVZ discontinuity.

We now specialize to 4 dimensions, so that $\kappa = 1/M_P$ and we consider as a source the stress tensor of a mass M point particle at rest at the origin

$$T^{\mu\nu}(x) = M\delta^\mu_0\delta^\nu_0\delta^3(\boldsymbol{x}), \quad T^{\mu\nu}(x) = 2\pi M\delta^\mu_0\delta^\nu_0\delta^3(p^0) \, . \tag{3.5.12}$$

Note that this source is conserved. The general solution (3.5.10) gives

$$\begin{aligned} h_{00}(x) &= \frac{2M}{3M_P}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\exp{(i\boldsymbol{p}\cdot\boldsymbol{x})}\frac{1}{\boldsymbol{p}^2 + m^2} \, , \\ h_{0i}(x) &= 0 \, , \\ h_{ij}(x) &= \frac{M}{3M_P}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\exp{(i\boldsymbol{p}\cdot\boldsymbol{x})}\frac{1}{\boldsymbol{p}^2 + m^2}\left(\delta_{ij} + \frac{p_i p_j}{m^2}\right) \, . \end{aligned} \tag{3.5.13}$$

Using the formulae

$$\int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \exp\left(i\boldsymbol{p}\cdot\boldsymbol{x}\right) \frac{1}{\boldsymbol{p}^2+m^2} = \frac{1}{4\pi}\frac{\exp\left(-mr\right)}{r} \ , \tag{3.5.14}$$

$$\begin{aligned}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \exp\left(i\boldsymbol{p}\cdot\boldsymbol{x}\right) \frac{p_i p_j}{\boldsymbol{p}^2+m^2} &= -\partial_i\partial_j \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \exp\left(i\boldsymbol{p}\cdot\boldsymbol{x}\right) \frac{1}{\boldsymbol{p}^2+m^2} = \\ &= \frac{1}{4\pi}\frac{\exp\left(-mr\right)}{r}\left[\frac{1}{r^2}(1+mr)\delta_{ij} + \right. \\ &\qquad \left. - \frac{1}{r^4}(3+3mr+m^2r^2)x_i x_j\right] \ ,\end{aligned} \tag{3.5.15}$$

with $r \equiv \sqrt{x_i x_i}$, we have

$$\begin{aligned} h_{00}(x) &= \frac{2M}{3M_P}\frac{1}{4\pi}\frac{\exp\left(-mr\right)}{r} \ , \\ h_{0i}(x) &= 0 \ , \\ h_{ij}(x) &= \frac{M}{3M_P}\frac{1}{4\pi}\frac{\exp\left(-mr\right)}{r}\left[\frac{1+mr+m^2r^2}{m^2r^2}(1+mr)\delta_{ij} + \right. \\ &\qquad \left. - \frac{1}{m^2r^4}(3+3mr+m^2r^2)x_i x_j\right] \ . \end{aligned} \tag{3.5.16}$$

Note the Yukawa suppression factors $\exp\left(-mr\right)$, characteristic of massive field.

We can also write these expressions in spherical coordinates for the spatial variables. Using

$$[F(r)\delta_{ij} + G(r)x_i x_j]dx^i dx^j = (F(r) + r^2 G(r))dr^2 + F(r)r^2 d\Omega^2 \ , \tag{3.5.17}$$

we find

$$h_{\mu\nu}dx^\mu dx^\nu = -B(r)dt^2 + C(r)dr^2 + A(r)r^2 d\Omega^2 \ , \tag{3.5.18}$$

where

$$\begin{aligned} B(r) &= -\frac{2M}{3M_P}\frac{1}{4\pi}\frac{\exp\left(-mr\right)}{r} \ , \\ C(r) &= -\frac{2M}{3M_P}\frac{1}{4\pi}\frac{\exp\left(-mr\right)}{r}\frac{1+mr}{m^2r^2} \ , \\ A(r) &= \frac{M}{3M_P}\frac{1}{4\pi}\frac{\exp\left(-mr\right)}{r}\frac{1+mr+m^2r^2}{m^2r^2} \ . \end{aligned} \tag{3.5.19}$$

In the limit $r \ll 1/m$, these reduce to

$$\begin{aligned} B(r) &= -\frac{2M}{3M_P}\frac{1}{4\pi r} \ , \\ C(r) &= -\frac{2M}{3M_P}\frac{1}{4\pi m^2 r^3} \ , \\ A(r) &= \frac{M}{3M_P}\frac{1}{4\pi m^2 r^3} \ . \end{aligned} \tag{3.5.20}$$

Corrections are suppressed by powers of mr.

In order to have a comparison, let's compute the point source solution for the massless case as well. We choose the Lorentz gauge (3.4.15). In this gauge the equations of motion simplify to

$$\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = -\kappa T_{\mu\nu} \ . \tag{3.5.21}$$

By taking the trace, we find

$$\Box h = \frac{2}{D-2}\kappa T \tag{3.5.22}$$

and upon substituting it back in (3.5.21), we get

$$\Box h_{\mu\nu} = -\kappa\left[T_{\mu\nu} - \frac{1}{D-2}\eta_{\mu\nu}T\right] \ . \tag{3.5.23}$$

This equation, along with the Lorenz gauge condition (3.4.15), is equivalent to the original equation of motion in Lorenz gauge.

By taking ∂^μ on (3.5.21) and on its trace, using the conservation of $T_{\mu\nu}$ and comparing, we have

$$\Box\Big(\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h\Big) = 0 \ , \tag{3.5.24}$$

so that the Lorentz condition is automatically satisfied when boundary conditions are satisfied with the property that $\Box f = 0$ implies $f = 0$ for any function f, as is the case when we impose retarded boundary conditions. We can then solve (3.5.21) by Fourier transforming

$$h_{\mu\nu}(x) = \kappa\int\frac{d^Dp}{(2\pi)^D}\exp\left(ip\cdot x\right)\frac{1}{p^2}\left[T_{\mu\nu}(p) - \frac{1}{D-2}\eta_{\mu\nu}T(p)\right] \ , \tag{3.5.25}$$

with $T^{\mu\nu}(p) = \int d^Dx \exp\left(-ip\cdot x\right)T^{\mu\nu}(x)$ as the Fourier transform of the source. In order to get the retarded field, we should integrate above the poles in the p^0 plane.

Now we specialize to $D = 4$ and we consider as a source the point particle of mass M at the origin (3.5.12). For this source, the general solution (3.5.25) gives

$$
\begin{aligned}
h_{00}(x) &= \frac{M}{2M_P} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \exp\left(i\boldsymbol{p}\cdot\boldsymbol{x}\right) \frac{1}{\boldsymbol{p}^2} = \frac{M}{2M_P}\frac{1}{4\pi r} \ , \\
h_{0i}(x) &= 0 \ , \\
h_{ij}(x) &= \frac{M}{2M_P} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \exp\left(i\boldsymbol{p}\cdot\boldsymbol{x}\right) \frac{1}{\boldsymbol{p}^2}\delta_{ij} = \frac{M}{2M_P}\frac{1}{4\pi r}\delta_{ij} \ .
\end{aligned}
\tag{3.5.26}
$$

We can also write these expressions in spherical coordinates for the spatial variables. Using

$$
[F(r)\delta_{ij} + G(r)x_i x_j]dx^i dx^j = (F(r) + r^2 G(r))dr^2 + F(r)r^2 d\Omega^2 \ , \tag{3.5.27}
$$

we find

$$
h_{\mu\nu}dx^\mu dx^\nu = -B(r)dt^2 + C(r)dr^2 + A(r)r^2 d\Omega^2 \ , \tag{3.5.28}
$$

where

$$
\begin{aligned}
B(r) &= -\frac{M}{2M_P}\frac{1}{4\pi r} \ , \\
C(r) &= \frac{M}{2M_P}\frac{1}{4\pi r} \ , \\
A(r) &= \frac{M}{2M_P}\frac{1}{4\pi r} \ .
\end{aligned}
\tag{3.5.29}
$$

Now, let's study the vDVZ discontinuity. We want to extract some physical predictions from the point source solution. We assume we have a test particle moving in this field and responding to $h_{\mu\nu}$ like a test particle in general relativity responding to the metric deviation $\delta g_{\mu\nu} = \frac{2}{M_P}h_{\mu\nu}$ We know that if $h_{\mu\nu}$ takes the form $2h_{00}/M_P = -2\phi$, $2h_{ij}/M_P = -2\psi\delta_{ij}$, $h_{0i} = 0$ for some functions $\phi(r)$ and $\psi(r)$, then the newtonian potential experienced by the particle is given by $\phi(r)$. Furthermore, if $\psi(r) = \gamma\phi(r)$ for some constant γ, called PPN parameter and if $\phi(r) = -\frac{k}{r}$ for some constant k, then the angle for the bending of light at impact parameter b around the heavy source is given by $\alpha = 2(1+\gamma)/b$. By looking at (3.5.26), the massless graviton gives us the values

$$
\phi = -\frac{GM}{r}, \quad \psi = -\frac{GM}{r} \ , \tag{3.5.30}
$$

using $\frac{1}{M_P^2} = 8\pi G$. The PPN parameter is therefore $\gamma = 1$ and the magnitude of the light bending angle for light incident at impact parameter b is

$$
\alpha = \frac{4GM}{b} \ , \tag{3.5.31}
$$

For the massive case, the metric (3.5.16) is not in the right form to read off the newtonian potential and light bending. In order to simplify things, we notice that while the massive gravity action is not gauge invariant, we have assumed that the coupling to the test particle is that of GR, so this coupling is gauge invariant. Thus, we can make a gauge transformation on the solution $h_{\mu\nu}$ and there will be no effect on the test particle. In order to simplify the metric (3.5.16), we go back to (3.5.13) and notice that $\frac{p_i p_j}{m^2}$ term in h_{ij} is pure gauge, so we can ignore this term. Thus, our metric is gauge equivalent to the metric

$$\begin{aligned} h_{00}(x) &= \frac{2M}{3M_P}\frac{1}{4\pi}\frac{\exp\left(-mr\right)}{r} \ , \\ h_{0i}(x) &= 0 \ , \\ h_{ij}(x) &= \frac{M}{3M_P}\frac{1}{4\pi}\frac{\exp\left(-mr\right)}{r}\delta_{ij} \ . \end{aligned} \tag{3.5.32}$$

Then, in the small mass limit

$$\phi = -\frac{4}{3}\frac{GM}{r}, \quad \psi = -\frac{2}{3}\frac{GM}{r}\delta_{ij} \ . \tag{3.5.33}$$

The newtonian potential is larger with respect to the massless case. The PPN parameter is $\gamma = 1/2$ and the magnitude of the light bending angle for light incident at impact parameter b is the same as in the massless case

$$\alpha = \frac{4GM}{b} \ . \tag{3.5.34}$$

If we want, we can make the newtonian potential agree with GR by scaling $G \to \frac{3}{4}G$. Then, the light bending would change to $\alpha = \frac{3GM}{b}$, off by 25% from GR.

Hence, linearized massive gravity, even in the limit of zero mass, gives predictions which are one order different from linearized GR. If nature we described by either one or the other of these theories, we would, by making a finite measurement, be able to tell whether the graviton mass is mathematically zero or not, in violation of our intuition that the physics of nature should be continuous in parameter. This is the vDVZ discontinuity (van Dam, Veltman, Zakharov) [55, 56, 57, 58]. It's present in other physical predictions as well, such as the emission of gravitational radiation [59].

3.6 The Stückelberg trick

In this section we'll study the origin of this discontinuity. We'll see that the correct massless limit of massive gravity is the massless gravity plus extra degrees of freedom, as expected since the gauge symmetry which kills the extra degrees of freedom only appears when the mass is exactly zero. The extra d.o.f. are a massless vector and a massless scalar which couples to the trace of the energy momentum tensor.

Taking the limit $m \to 0$ straight away in (3.5.1) doesn't yield to a smooth limit, because some d.o.f. are lost. In order to find the correct limit we have to introduce new fields and gauge symmetries in the massive theory without altering it. This is called *Stückelberg trick*.

First, we can show an example, related to the theory of a massive photon and later apply the same procedure to massless gravity.

Let's consider the theory of a massive photon A_μ coupled to a source J_μ, not necessarily conserved.

$$S[A] = \int d^D x \Big[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu + A_\mu J^\mu \Big] \ , \tag{3.6.1}$$

with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. The mass term breaks the would-be gauge invariance, $\delta A_\mu = \partial_\mu \xi$ and for $D = 4$ this theory describes the 3 degrees of freedom of a massive spin 1 particle. Recall that the propagator for massive vector is $\frac{-i}{p^2+m^2}\Big(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\Big)$, which goes like $\sim \frac{1}{m^2}$ for large momenta, invalidating the usual power counting arguments.

The limit $m \to 0$ is not a smooth limit because we lose a degree of freedom. For $m = 0$ we have Maxwell electromagnetism which in $D = 4$ propagates only 2 d.o.f., the two polarizations of a massless helicity 1 particle. Also, the limit doesn't exist unless the source is conserved as this is a consistency requirement in the massless case.

The Stückelberg trick consists of introducing a new scalar field φ so that the new action has a gauge symmetry, but it's still dynamically equivalent to the original action. It will expose a different $m \to 0$ limit which is smooth, so that no d.o.f are gained or lost. So, we make the replacement

$$A_\mu \to A_\mu + \partial_\mu \varphi \ , \tag{3.6.2}$$

following the pattern of the gauge symmetry we want to introduce [60]. This is *not* a change of field variables. It is *not* a decomposition of A_μ into transverse and longitudinal parts and it's *not* a gauge transformation as the lagrangian (3.6.1) is not gauge invariant. This is creating a new lagrangian from the old one, by the addition of a new field φ. $F_{\mu\nu}$ is invariant under this replacement, since the replacement looks like a gauge transformation and $F_{\mu\nu}$ is gauge invariant. The only thing that changes is the mass term and the coupling to the source

$$S[A,\varphi] = \int d^D x \Big[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 (A_\mu + \partial_\mu \varphi)^2 + A_\mu J^\mu - \varphi\, \partial_\mu J^\mu \Big] \ , \tag{3.6.3}$$

where we have integrated by parts in the coupling to the source. The new action now has the gauge symmetry

$$\delta A_\mu = \partial_\mu \xi \qquad \delta \varphi = -\xi \ . \tag{3.6.4}$$

By fixing the gauge $\varphi = 0$, called unitary gauge, we recover the original massive lagrangian (3.6.1), which means that (3.6.3) and (3.6.1) are equivalent theories. They both describe the 3 d.o.f of a massive spin 1 particle in $D = 4$. The Stückelberg trick is a terrific illustration of the fact that the gauge symmetry represents nothing more than a redundancy of the description. We can consider any theory and make it a gauge theory by introducing redundant variables. Also, given any gauge theory, we can always eliminate the gauge symmetry by eliminating the redundant degrees of freedom. However, removing redundancies is not always a smart thing to do. In Maxwell electromagnetism it's impossible to remove the redundancies and at the same time preserve manifest Lorentz invariance and locality. The theory is still Lorentz invariant and local, but not manifestly. With the Stückelberg trick presented here, on the other hand, we are adding and removing extra gauge symmetry in a rather simple way, which doesn't mess with the manifest Lorentz invariance and locality.

We see from (3.6.3) that φ has a kinetic term, in addition to cross terms. Rescaling $\varphi \to \frac{1}{m}\varphi$ in order to normalize the kinetic term, we have

$$\begin{aligned} S[A,\varphi] = \int d^D x \Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu - m A_\mu \partial^\mu \varphi - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \\ + A_\mu J^\mu + -\frac{1}{m}\varphi\,\partial_\mu J^\mu \Big] \end{aligned} \tag{3.6.5}$$

and the gauge symmetry reads as

$$\delta A_\mu = \partial_\mu \xi \qquad \delta\varphi = -m\xi \ . \tag{3.6.6}$$

Considering now the $m \to 0$ limit, assuming a conserved source, the action becomes

$$S[A,\varphi] = \int d^D x \Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi \Big] \tag{3.6.7}$$

and the gauge symmetry is

$$\delta A_\mu = \partial_\mu \xi \qquad \delta\varphi = 0. \tag{3.6.8}$$

It is now clear that the number of d.o.f. is preserved in the limit. For $D = 4$ two of the three d.o.f. go into the massless vector and one goes into the scalar.

Now, let's consider the Fierz-Pauli action (3.5.1) in the following form

$$S[h] = \int d^D x \Big[\mathcal{L}_{m=0} - \frac{1}{2}m^2\big(h_{\mu\nu}h^{\mu\nu} - h^2\big) + \kappa h_{\mu\nu}T^{\mu\nu} \Big] \ , \tag{3.6.9}$$

with $\mathcal{L}_{m=0}$ as the lagrangian of the massless graviton (3.3.3). We aim to preserve the gauge symmetry (3.3.4) in the massless case, so we introduce a Stückelberg vector field A_μ patterned after the gauge symmetry so that $h_{\mu\nu}$ is transformed in the following way

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu A_\nu + \partial_\nu A_\mu \ . \tag{3.6.10}$$

The $\mathcal{L}_{m=0}$ term remains invariant because it's gauge invariant and (3.6.10) looks like a gauge transformation, so all that changes is the mass term

$$\begin{aligned} S[h, A] = \int d^D x \Big[& \mathcal{L}_{m=0}(h) - \frac{1}{2} m^2 \big(h_{\mu\nu} h^{\mu\nu} - h^2 \big) - \frac{1}{2}\, m^2\, F_{\mu\nu} F^{\mu\nu} + \\ & - 2m^2 (h_{\mu\nu} \partial^\mu A^\nu - h \partial_\mu A^\mu) + \kappa h_{\mu\nu} T^{\mu\nu} - 2\kappa A_\mu \partial_\nu T^{\mu\nu} \Big] \,. \end{aligned} \tag{3.6.11}$$

Proof. Let's see how the term $-\frac{1}{2} m^2 \big(h_{\mu\nu} h^{\mu\nu} - h^2 \big)$ transforms under the introduction of the Stückelberg field A_μ. First, let's look at the terms $h_{\mu\nu} h^{\mu\nu}$ and h^2 separately

$$h_{\mu\nu} h^{\mu\nu} \to h_{\mu\nu} h^{\mu\nu} + 4 h_{\mu\nu} \partial^\mu A^\nu + 2 (\partial_\mu A_\nu \partial^\mu A^\nu + \partial_\mu A_\nu \partial^\nu A^\mu) \tag{3.6.12}$$

$$h^2 \to h^2 + 4 (\partial_\mu A^\mu)^2 + 4 h \partial_\mu A^\mu \ . \tag{3.6.13}$$

Now, as

$$\frac{1}{2} F_{\mu\nu} F^{\mu\nu} = \partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu \tag{3.6.14}$$

we can write

$$\begin{aligned} \partial_\mu A_\nu \partial^\mu A^\nu + \partial_\mu A_\nu \partial^\nu A^\mu &= \partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu + 2\, \partial_\mu A_\nu \partial^\nu A^\mu = \\ &= \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + 2 (\partial_\mu A^\mu)^2 \ , \end{aligned} \tag{3.6.15}$$

so (3.6.12) can be rewritten as

$$h_{\mu\nu} h^{\mu\nu} \to h_{\mu\nu} h^{\mu\nu} + 4 h_{\mu\nu} \partial^\mu A^\nu + F_{\mu\nu} F^{\mu\nu} + 4 (\partial_\mu A^\mu)^2 \ , \tag{3.6.16}$$

leading to

$$h_{\mu\nu} h^{\mu\nu} - h^2 \to h_{\mu\nu} h^{\mu\nu} - h^2 + F_{\mu\nu} F^{\mu\nu} + 4 (h_{\mu\nu} \partial^\mu A^\nu - h \partial_\mu A^\mu) \ . \tag{3.6.17}$$

The term $\kappa h_{\mu\nu} T^{\mu\nu}$ changes as follows

$$\kappa h_{\mu\nu} T^{\mu\nu} \to \kappa h_{\mu\nu} T^{\mu\nu} - 2\kappa A_\mu \partial_\nu T^{\mu\nu} \ , \tag{3.6.18}$$

by integrating by parts, leading to (3.6.11).

■

There's now a gauge symmetry

$$\delta h_{\mu\nu} = 2 \partial_{(\mu} \xi_{\nu)} \ , \quad \delta A_\mu = -\xi_\mu \ . \tag{3.6.19}$$

At this point we may consider rescaling $A_\mu \to \frac{1}{m} A_\mu$ to normalize the vector kinetic term then take the massless limit, but we would end up with a massless graviton and a massless photon, having only 4 d.o.f. out of 5 (in $D = 4$). The massless limit is still not smooth. We have to introduce another field, which is the scalar Stückelberg φ

$$A_\mu \to A_\mu + \partial_\mu \varphi \ . \tag{3.6.20}$$

(3.6.11) becomes

$$\begin{aligned} S[h, A, \varphi] = \int d^D x \Big[& \mathcal{L}_{m=0}(h) - \frac{1}{2} m^2 \big(h_{\mu\nu} h^{\mu\nu} - h^2 \big) - \frac{1}{2} m^2 \, F_{\mu\nu} F^{\mu\nu} + \\ & - 2m^2 (h_{\mu\nu} \partial^\mu A^\nu - h \partial_\mu A^\mu) - 2m^2 (h_{\mu\nu} \partial^\mu \partial^\nu \varphi - h \Box \varphi) + \\ & + \kappa h_{\mu\nu} T^{\mu\nu} - 2\kappa A_\mu \partial_\nu T^{\mu\nu} + 2\kappa \varphi \partial\partial T \Big] \ , \end{aligned} \tag{3.6.21}$$

with $\partial\partial T \equiv \partial_\mu \partial_\nu T^{\mu\nu}$, integrating by parts in the last term. The term $F_{\mu\nu} F^{\mu\nu}$ is invariant under $U(1)$ transformations like (3.6.20), so it is not modified and the rest of the computation is trivial. There are now two gauge symmetries:

$$\delta h_{\mu\nu} = 2\partial_{(\mu} \xi_{\nu)} \ , \quad \delta A_\mu = -\xi_\mu \ . \tag{3.6.22}$$

$$\delta A_\mu = \partial_\mu \Lambda \ , \quad \delta \varphi = -\Lambda \ . \tag{3.6.23}$$

Now we can rescale $A_\mu \to \frac{1}{m} A_\mu$, $\varphi \to \frac{1}{m} \varphi$, so

$$\begin{aligned} S[h, A, \varphi] = \int d^D x \Big[& \mathcal{L}_{m=0}(h) - \frac{1}{2} m^2 \big(h_{\mu\nu} h^{\mu\nu} - h^2 \big) - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \\ & - 2m (h_{\mu\nu} \partial^\mu A^\nu - h \partial_\mu A^\mu) - 2 (h_{\mu\nu} \partial^\mu \partial^\nu \varphi - h \Box \varphi) + \\ & + \kappa h_{\mu\nu} T^{\mu\nu} - \frac{2}{m} \kappa A_\mu \partial_\nu T^{\mu\nu} + \frac{2}{m^2} \kappa \varphi \partial\partial T \Big] \ , \end{aligned} \tag{3.6.24}$$

and the gauge symmetries become

$$\delta h_{\mu\nu} = 2\partial_{(\mu} \xi_{\nu)} \ , \quad \delta A_\mu = -m \xi_\mu \tag{3.6.25}$$

$$\delta A_\mu = \partial_\mu \Lambda \ , \quad \delta \varphi = -m \Lambda \ . \tag{3.6.26}$$

Now, in the $m \to 0$ limit, assuming a conserved source, otherwise φ and A_μ become strongly coupled to the divergence of the source, the theory takes the form

$$S[h, A, \varphi] = \int d^D x \Big[\mathcal{L}_{m=0}(h) - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - 2 (h_{\mu\nu} \partial^\mu \partial^\nu \varphi - h \Box \varphi) + \kappa h_{\mu\nu} T^{\mu\nu} \Big] \ , \tag{3.6.27}$$

with all 5 d.o.f. in $D = 4$. To see this, let's un-mix the scalar and the tensor with a field redefinition

$$h_{\mu\nu} = h'_{\mu\nu} + \pi \eta_{\mu\nu} , \tag{3.6.28}$$

where π is any scalar. The massless spin 2 part changes as follows

$$\mathcal{L}_{m=0}(h) = \mathcal{L}_{m=0}(h') + (D-2)\left[\partial_\mu \pi \partial^\mu h' - \partial_\mu \pi \partial_\nu h'^{\mu\nu} + \frac{1}{2}(D-1)\partial_\mu \pi \partial^\mu \pi\right] . \tag{3.6.29}$$

Proof. Considering the massless lagrangian

$$\mathcal{L}_{m=0}(h) = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h \ , \tag{3.6.30}$$

let's perform the field redefinition (3.6.28)

$$\begin{aligned}\mathcal{L}_{m=0}(h) = &-\frac{1}{2}\,\partial_\lambda(h'_{\mu\nu} + \eta_{\mu\nu}\pi)\,\partial^\lambda(h'^{\mu\nu} + \eta^{\mu\nu}\pi) + \\ &+ \partial_\mu\,(h'_{\nu\lambda} + \eta_{\nu\lambda}\pi)\partial^\nu\,(h'^{\mu\lambda} + \eta^{\mu\lambda}\pi) + \\ &- \partial_\mu\,(h'^{\mu\nu} + \eta^{\mu\nu}\pi)\,\partial_\nu(h' + D\pi) + \frac{1}{2}\partial_\lambda(h' + D\pi)\partial^\lambda(h' + D\pi) = \\ = &-\frac{1}{2}\partial_\lambda h'_{\mu\nu}\partial^\lambda h'^{\mu\nu} + \partial_\mu h'_{\nu\lambda}\partial^\nu h'^{\mu\lambda} - \partial_\mu h'^{\mu\nu}\partial_\nu h' + \frac{1}{2}\partial_\lambda h'\partial^\lambda h' + \\ &\quad + (D-2)\partial_\mu\pi\partial^\nu h'^{\mu\nu} + (D-2)\partial_\mu\pi\partial^\mu h' + \\ &\quad + \frac{1}{2}(D-2)(D-1)\partial_\mu\pi\partial^\mu\pi \ .\end{aligned} \tag{3.6.31}$$

which leads to (3.6.29).

■

By taking

$$\pi = \frac{2}{D-2}\varphi \tag{3.6.32}$$

in (3.6.28) we can cancel all the off-diagonal terms in (3.6.27) by trading in for a φ kinetic term. Hence, (3.6.27) takes the form

$$\begin{aligned}S[h', A, \varphi] = \int d^D x\Big[&\mathcal{L}_{m=0}(h') - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2\,\frac{D-1}{D-2}\,\partial_\mu\varphi\partial^\mu\varphi + \\ &+ k h'_{\mu\nu}T^{\mu\nu} + \frac{2}{D-2}\kappa\varphi T\Big]\end{aligned} \tag{3.6.33}$$

Proof. By (3.6.28) and (3.6.32), the following terms are modified as follows

$$\begin{aligned}-2(h_{\mu\nu}\partial^\mu\partial^\nu\varphi - h\Box\varphi) &= -2(h'_{\mu\nu}\partial^\mu\partial^\nu\varphi + \eta_{\mu\nu}\pi\partial^\mu\partial^\nu\varphi - h'\Box\varphi - D\pi\Box\varphi) = \\ &= -2(D-1)\partial_\mu\pi\,\partial^\mu\varphi = -4\,\frac{D-1}{D-2}\partial_\mu\varphi\,\partial^\mu\varphi \ ,\end{aligned} \tag{3.6.34}$$

$$
\begin{aligned}
\mathcal{L}_{m=0}(h) = \mathcal{L}_{m=0}(h') + 2\left[\partial_\mu\varphi\,\partial^\mu h' - \partial_\mu\varphi\,\partial_\nu h'^{\mu\nu} + \frac{D-1}{D-2}\,\partial_\mu\varphi\,\partial^\mu\varphi\right] = \\
= \mathcal{L}_{m=0}(h') + 2\,\frac{D-1}{D-2}\,\partial_\mu\varphi\,\partial^\mu\varphi\ ,
\end{aligned} \tag{3.6.35}
$$

$$
\kappa h_{\mu\nu}T^{\mu\nu} = \kappa h'_{\mu\nu}T^{\mu\nu} + \kappa\pi\eta_{\mu\nu}T^{\mu\nu} = \kappa h'_{\mu\nu}T^{\mu\nu} + \frac{2}{D-2}\kappa\varphi T\ , \tag{3.6.36}
$$

leading to (3.6.33) when plugged into (3.6.27).

■

The gauge transformations now read

$$
\delta h'_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)}\ , \quad \delta A_\mu = 0 \tag{3.6.37}
$$

$$
\delta A_\mu = \partial_\mu\Lambda\ , \quad \delta\varphi = 0\ . \tag{3.6.38}
$$

In $D = 4$ there are now manifestly 5 degrees of freedom, two in a canonical massless graviton, two in a canonical massless vector and one in a canonical massless scalar.

In chapter 4 we'll study the Fierz-Pauli action for massive gravity in a curved spacetime and perform the same procedure.

Note that the coupling of the scalar to the trace of the stress tensor survives the $m = 0$ limit. This is the origin of the vDVZ discontinuity. The extra scalar degree of freedom does not affect the bending of light by coupling to the trace of the stress tensor (for which $T = 0$), but it does that by affecting the newtonian potential. This extra scalar potential exactly accounts for the discrepancy between the massless limit of massive gravity and massless gravity.

As a side note, one can see from this Stückelberg trick that violating the Fierz-Pauli tuning for the mass term leads to a ghost. Any deviation from this form and the Stückelberg scalar will acquire a kinetic term with four derivatives $\sim (\Box\varphi)^2$, indicating that it carries 2 d.o.f., one of which is a ghost [61, 62]. The tuning is required to exactly cancel these terms, up to a total derivative.

Returning to the action (3.6.24) for $m \neq 0$ and a source not necessarily conserved, we now know how to apply the transformation $h_{\mu\nu} = h'_{\mu\nu} + \frac{2}{D-2}\varphi\eta_{\mu\nu}$, which yields

$$
\begin{aligned}
S[h', A, \varphi] = \int d^Dx\Big[\mathcal{L}_{m=0}(h') - \frac{1}{2}m^2(h'_{\mu\nu}h'^{\mu\nu} - h'^2) - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \\
+ 2\,\frac{D-1}{D-2}\varphi\left(\Box + \frac{D}{D-2}m^2\right) + \\
- 2m(h'_{\mu\nu}\partial^\mu A^\nu - h'\partial_\mu A^\mu) + \\
+ 2\,\frac{D-1}{D-2}(m^2 h'\varphi + 2m\varphi\partial_\mu A^\mu) + \kappa h'_{\mu\nu}T^{\mu\nu} + \\
+ \frac{2}{D-2}\kappa\varphi T - \frac{2}{m}\kappa A_\mu\partial_\nu T^{\mu\nu} + \frac{2}{m^2}\kappa\varphi\partial\partial T\Big]\ .
\end{aligned} \tag{3.6.39}
$$

Proof. Let's see how the terms in (3.6.24) are modified when the transformation $h_{\mu\nu} = h'_{\mu\nu} + \frac{2}{D-2}\varphi\eta_{\mu\nu}$ is applied

$$\mathcal{L}_{m=0}(h) = \mathcal{L}_{m=0}(h') - 2\,\frac{D-1}{D-2}\,\varphi\Box\varphi \ , \tag{3.6.40}$$

$$\kappa h_{\mu\nu}T^{\mu\nu} = \kappa h'_{\mu\nu}T^{\mu\nu} + \frac{2}{D-2}\kappa\varphi T \ , \tag{3.6.41}$$

$$h_{\mu\nu}h^{\mu\nu} = h'_{\mu\nu}h'^{\mu\nu} + \frac{4D}{(D-2)^2}\varphi^2 + \frac{4}{D-2}\varphi h' \ , \tag{3.6.42}$$

$$h^2 = h'^2 + \frac{4D^2}{(D-2)^2}\varphi^2 + \frac{4D}{D-2}\varphi h' \ , \tag{3.6.43}$$

$$-2(h_{\mu\nu}\partial^\mu\partial^\nu\varphi - h\Box\varphi) = -2(h'_{\mu\nu}\partial^\mu\partial^\nu\varphi - h'\Box\varphi) + 4\frac{D-1}{D-2}\varphi\Box\varphi \ , \tag{3.6.44}$$

$$-2m(h_{\mu\nu}\partial^\mu A^\nu - h\partial_\mu A^\mu) = -2m(h'_{\mu\nu}\partial^\mu A^\nu - h'\partial_\mu A^\mu) + 4m\frac{D-1}{D-2}\varphi\,\partial_\mu A^\mu \ . \tag{3.6.45}$$

These expressions lead to (3.6.39).

■

The gauge symmetries read

$$\delta h'_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + \frac{2}{D-2}m\Lambda\eta_{\mu\nu}, \quad \delta A_\mu = \partial_\mu\Lambda - m\xi_\mu, \quad \delta\varphi = -m\Lambda \ . \tag{3.6.46}$$

We can go to a Lorentz-like gauge, by imposing the gauge conditions [63, 64]

$$\partial^\nu h'_{\mu\nu} - \frac{1}{2}\partial_\mu h' + mA_\mu = 0 \ , \tag{3.6.47}$$

$$\partial_\mu A^\mu + m\left(\frac{1}{2}h' + 2\,\frac{D-1}{D-2}\varphi\right) = 0 \ . \tag{3.6.48}$$

The first condition fixes the ξ_μ symmetry up to a residual transformation satisfying the equations $(\Box - m^2)\xi_\mu = 0$. It is invariant under Λ transformations, so it fixes none of this symmetry. The second condition fixes the Λ symmetry up to a residual transformation satisfying $(\Box - m^2)\Lambda = 0$. It is invariant under ξ_μ transformations, so it fixes none of this symmetry. We add two corresponding gauge fixing terms to the action, resulting from either Faddeev-Popov gauge-fixing procedure or classical gauge-fixing,

$$S_{GF1}[h', A] = -\int d^Dx\left[\partial^\nu h'_{\mu\nu} - \frac{1}{2}\partial_\mu h' + mA_\mu\right]^2 \ , \tag{3.6.49}$$

$$S_{GF2}[h', A, \varphi] = -\int d^D x \left[\partial_\mu A^\mu + m\left(\frac{1}{2} h' + 2\,\frac{D-1}{D-2} \varphi \right) \right]^2 . \tag{3.6.50}$$

These have the effect of diagonalizing the action

$$\begin{aligned} S_{TOT}[h', A, \varphi] = S[h', A, \varphi] &+ S_{GF1}[h', A] + S_{GF2}[h', A, \varphi] = \\ = \int d^D x &\left[\frac{1}{2} h'_{\mu\nu} (\Box - m^2) h'^{\mu\nu} - \frac{1}{4} h' (\Box - m^2) h' + \right. \\ &+ A_\mu (\Box - m^2) A^\mu + 2\,\frac{D-1}{D-2} \varphi (\Box - m^2) \varphi + \\ &+ \kappa h'_{\mu\nu} T^{\mu\nu} + \frac{2}{D-2} k \varphi T - \frac{2}{m} \kappa A_\mu \partial_\nu T^{\mu\nu} + \\ &\left. + \frac{2}{m^2} \kappa \varphi \partial\partial T \right] . \end{aligned} \tag{3.6.51}$$

4 Heat kernel for linearized massive gravity

This chapter is the core of the thesis. We start with the Fierz-Pauli action in a spacetime with a curved fixed background. We perform the Stückelberg trick to restore a gauge symmetry to the theory. We can then choose proper gauge-fixing functions and rewrite the action in a diagonal form with all the operators of second order and minimal. With the Faddeev-Popov procedure, we can compute the path integral of the theory. These calculations allow us to use the Seeley-DeWitt method to compute the heat kernel coefficients and finally write down the one-loop effective action for linearized massive gravity in an Einstein spacetime up to the finite cubic terms in curvatures in $D = 4$.

4.1 Introduction

We want to study the Fierz-Pauli action describing the propagation of a massive graviton (represented by the symmetric tensor field $h_{\mu\nu}$), with mass m, on a fixed curved background with metric $g_{\mu\nu}$ and no external sources ($T_{\mu\nu} = 0$). We are working in an Einstein space of dimension D, defined by the conditions:

$$R_{\mu\nu} = \frac{R}{D} g_{\mu\nu} \ , \tag{4.1.1}$$

$$\Lambda = \left(\frac{D-2}{2D}\right) R \ , \tag{4.1.2}$$

with

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = g^{\kappa\lambda} R_{\kappa\mu\lambda\nu} \ , \tag{4.1.3}$$

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu} \ , \quad \text{with} \quad R > 0 \quad \text{on spheres,} \tag{4.1.4}$$

with $R_{\kappa\mu\lambda\nu}$, $R_{\mu\nu}$ and R as the Riemann tensor, Ricci tensor and Ricci scalar respectively and Λ as the cosmological constant.

Einstein manifolds are indeed a special class of Riemaniann manifolds whose metric satisfies Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \ . \tag{4.1.5}$$

By contracting (4.1.5) with $g^{\mu\nu}$ we get

$$R - \frac{1}{2} D R + D\Lambda = 0 \ , \tag{4.1.6}$$

which leads to (4.1.2). By plugging (4.1.2) back into (4.1.5), we also find (4.1.1), as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{D-2}{2D} g_{\mu\nu} R = 0 \ . \tag{4.1.7}$$

There's a reason why we are working in an Einstein space and not in a generic curved spacetime. A massless graviton can consistently propagate only in an Einstein background [65, 66]. On the other hand, in the case of massive gravity, we know from the literature that we can formulate consistent theories in an arbitrary background [67, 68, 69], but this seems to be possible only by linearizing the non-linear dRGT theory on the curved background. It's also possible to formulate alternative consistent theories of a massive graviton in maximally symmetric spaces [70], which are, however, a subcase of Einstein spaces, as the Riemann tensor is proportional to the Ricci tensor with the condition

$$R_{\mu\nu\rho\sigma} = \frac{R}{D(D-1)}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \ . \tag{4.1.8}$$

Moreover, we can formulate different consistent theories in Petrov-type D spaces [71], which are a class of Ricci symmetric spacetimes, i.e. spaces with a covariantly constant Ricci tensor, identified by the condition

$$\nabla_\rho R_{\mu\nu} = 0 \ , \tag{4.1.9}$$

but it's possible only for the partially massless case (which will be analyzed in Appendix B). However, in the case of the massive Fierz-Pauli theory, the one studied in this thesis, the only acceptable background which keeps the consistency is the Einstein one, as we shall explain below.

The spacetime we work on has a Euclidean signature. This is particularly important because the one-loop effective action (2.2.20) is written with the Euclidean time and we want to keep consistency.

4.2 Fierz-Pauli action on curved spacetime

The action for the Fierz-Pauli theory in a generic curved background with Euclidean signature is obtained by substituting in the action on flat spacetime (3.3.8) all the partial derivatives with the covariant ones and also adding all the possible non-minimal couplings to the background curvature with dimensionless coefficients c_i. As a result, we can write the most general action for a massive spin 2 field in curved spacetime, with second order terms, quadratic in derivatives and consistent with the flat limit.

$$\begin{aligned} S_{FP}[h]_{m=0} = \int d^D x\sqrt{g}\Big[&\frac{1}{2}\nabla^\rho h^{\mu\nu}\nabla_\rho h_{\mu\nu} - \frac{1}{2}\nabla^\mu h\nabla_\mu h - \nabla^\rho h^{\mu\nu}\nabla_\nu h_{\mu\rho} + \\ &+ \nabla^\mu h\nabla^\nu h_{\mu\nu} + c_1 R h^{\mu\nu}h_{\mu\nu} + c_2 R h^2 + \\ &+ c_3 R^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\beta} + c_4 R^{\alpha\beta}h_{\alpha\sigma}h_\beta{}^\sigma + c_5 R^{\mu\nu}h_{\mu\nu}h\Big] \ , \end{aligned} \tag{4.2.1}$$

with ∇_μ as the covariant derivative and g as the modulus of the determinant of the background $g_{\mu\nu}$. As already mentioned, the extension of the Fierz-Pauli theory to curved backgrounds leads to the appearence of ghosts. In order to get rid of them, we can consider all the couplings with the curvature of the spacetime and the resulting constraints, coming from the equations of motion. The values of the dimensionless coefficients can then be fixed in order to ensure the propagation of only 5 physical d.o.f., avoiding the presence of ghosts. We don't perform the calculations in this thesis as they are pretty cumbersome, but they can be found in [65]. After the computations, we see that the only consistent background is Einstein, defined by (4.1.1) and (4.1.2). The coefficients are set to

$$\begin{cases} c_1 = -\frac{1}{D} \\ \\ c_2 = \frac{1}{2D} \\ \\ c_3 = c_4 = c_5 = 0 \ , \end{cases} \tag{4.2.2}$$

leading to

$$\begin{aligned} S_{FP}[h]_{m=0} = \frac{1}{2}\int d^D x\sqrt{g}\Big[&\nabla^\rho h^{\mu\nu}\nabla_\rho h_{\mu\nu} - \nabla^\mu h\nabla_\mu h - 2\nabla^\rho h^{\mu\nu}\nabla_\nu h_{\mu\rho} + \\ &+ 2\nabla^\mu h\nabla^\nu h_{\mu\nu} - \frac{2R}{D}\Big(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2\Big)\Big] \ . \end{aligned} \tag{4.2.3}$$

By using the well known commutation relation between two covariant derivatives [72]

$$\big[\nabla_\mu, \nabla_\nu\big] h^\gamma{}_\alpha = R^\gamma{}_{\lambda\mu\nu} h^\lambda{}_\alpha + R^\lambda{}_{\alpha\nu\mu} h^\gamma{}_\lambda \ , \tag{4.2.4}$$

which, taking into account Einstein manifolds simplifications, leads to the following relation

$$-2\nabla^\rho h^{\mu\nu}\nabla_\nu h_{\mu\rho} = -2\nabla_\nu h^{\mu\nu}\nabla^\rho h_{\mu\rho} - 2R_{\mu\alpha\nu\beta}h^{\mu\nu}h^{\alpha\beta} + \frac{2R}{D}h^{\mu\nu}h_{\mu\nu} \ , \tag{4.2.5}$$

we can rewrite the action (4.2.3) in the following form

$$\begin{aligned} S_{FP}[h]_{m=0} = \frac{1}{2}\int d^D x\sqrt{g}\Big[&\nabla^\rho h^{\mu\nu}\nabla_\rho h_{\mu\nu} - \nabla^\mu h\nabla_\mu h - 2\nabla_\nu h^{\mu\nu}\nabla^\rho h_{\mu\rho} + \\ &- 2R_{\mu\alpha\nu\beta}h^{\mu\nu}h^{\alpha\beta} + \frac{R}{D}h^2 + 2\nabla^\mu h\nabla^\nu h_{\mu\nu}\Big] \ . \end{aligned} \tag{4.2.6}$$

The action above describes the propagation of a massless spin 2 particle (2 degrees of freedom) and it is invariant under the gauge symmetry

$$\delta h_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 2\nabla_{(\mu}\xi_{\nu)} \ . \tag{4.2.7}$$

In order to describe massive gravity, this symmetry must be broken by the introduction of the mass term

$$S[h]_m = \frac{1}{2}\int d^D x\sqrt{g}\Big[m^2\Big(h^{\mu\nu}h_{\mu\nu} - h^2\Big)\Big] \ . \tag{4.2.8}$$

Hence, the total action for massive gravity in an Einstein space with Euclidean signature is [34]

$$\begin{aligned} S_{FP}[h] = S_{FP}[h]_{m=0} + S[h]_m &= \\ = \frac{1}{2}\int d^D x\sqrt{g}\Big[&\nabla^\rho h^{\mu\nu}\nabla_\rho h_{\mu\nu} - \nabla^\mu h\nabla_\mu h - 2\nabla_\nu h^{\mu\nu}\nabla^\rho h_{\mu\rho} + \\ &- 2R_{\mu\alpha\nu\beta}h^{\mu\nu}h^{\alpha\beta} + \frac{R}{D}h^2 + 2\nabla^\mu h\nabla^\nu h_{\mu\nu} + \\ &+ m^2\Big(h^{\mu\nu}h_{\mu\nu} - h^2\Big)\Big] \ . \end{aligned} \tag{4.2.9}$$

As we will see in Appendix B, for $R = \frac{D(D-1)}{D-2}m^2$, $m \neq 0$, the action has a scalar gauge symmetry and propagates 4 d.o.f. in $D = 4$. For all other values of m^2 and R, it has no gauge symmetry and propagates 5 d.o.f in $D = 4$. The Fierz-Pauli action (4.2.9) can be put in the diagonal form:

$$S_{FP}[h] = \frac{1}{2}\int d^D x\sqrt{g}\, h_{\mu\nu}\chi^{\mu\nu\alpha\beta}h_{\alpha\beta} \ , \tag{4.2.10}$$

with the operator $\chi^{\mu\nu\alpha\beta}$ as

$$\begin{aligned} \chi^{\mu\nu\alpha\beta} = \Big(-\Box + m^2\Big)\Big[g^{\mu(\alpha}g^{\beta)\nu} - g^{\alpha\beta}g^{\mu\nu}\Big] - 2R^{\mu(\alpha|\nu|\beta)} + \frac{R}{D}g^{\mu\nu}g^{\alpha\beta} + \\ - 2g^{\alpha\beta}\nabla^{(\nu}\nabla^{\mu)} + 2g^{(\mu(\alpha}\nabla^{\nu)}\nabla^{\beta)} \ , \end{aligned} \tag{4.2.11}$$

with the notation

$$R^{\mu(\alpha|\nu|\beta)} = \frac{1}{2}\big(R^{\mu\alpha\nu\beta} + R^{\mu\beta\nu\alpha}\big) \ , \tag{4.2.12}$$

$$g^{\mu(\alpha}g^{\beta)\nu} = \frac{1}{2}\big(g^{\mu\alpha}g^{\beta\nu} + g^{\mu\beta}g^{\alpha\nu}\big) \ , \tag{4.2.13}$$

$$\nabla^{(\alpha}\nabla^{\beta)} = \frac{1}{2}(\nabla^\alpha\nabla^\beta + \nabla^\beta\nabla^\alpha) \ . \tag{4.2.14}$$

Proof. The calculations are trivial and straightforward, similar to the ones performed in chapter 3 to derive (3.4.1). We start with the total action (4.2.9) and rearrange the terms in order to write the action in diagonal form

$$
\begin{aligned}
S_{FP}[h] = \frac{1}{2}\int d^Dx\sqrt{g}\Big[&\nabla^\rho h^{\mu\nu}\nabla_\rho h_{\mu\nu} - \nabla^\mu h\nabla_\mu h - 2\nabla_\nu h^{\mu\nu}\nabla^\rho h_{\mu\rho} + \\
&- 2R_{\mu\alpha\nu\beta}h^{\mu\nu}h^{\alpha\beta} + \frac{R}{D}h^2 + 2\nabla^\mu h\nabla^\nu h_{\mu\nu} + \\
&+ m^2\Big(h^{\mu\nu}h_{\mu\nu} - h^2\Big)\Big] = \\
= \frac{1}{2}\int d^Dx\sqrt{g}\Big[&- h_{\mu\nu}\Box h^{\mu\nu} + h_{\mu\nu}g^{\mu\nu}\Box g^{\alpha\beta}h_{\alpha\beta} + 2h_{\mu\nu}\nabla^\nu\nabla^\beta h^\mu{}_\beta + \\
&+ 2h\nabla^\mu\nabla^\nu h_{\mu\nu} + h_{\mu\nu}\frac{R}{D}g^{\mu\nu}g^{\alpha\beta}h_{\alpha\beta} - 2h_{\mu\nu}R^{\mu(\alpha|\nu|\beta)}h_{\alpha\beta} + \\
&+ h_{\mu\nu}m^2h^{\mu\nu} - h_{\mu\nu}m^2g^{\mu\nu}g^{\alpha\beta}h_{\alpha\beta}\Big] = \\
= \frac{1}{2}\int d^Dx\sqrt{g}\, h_{\mu\nu}\Big\{&\Big(-\Box + m^2\Big)\Big[g^{\mu(\alpha}g^{\beta)\nu} - g^{\alpha\beta}g^{\mu\nu}\Big] - 2R^{\mu(\alpha|\nu|\beta)} + \\
&+ \frac{R}{D}g^{\mu\nu}g^{\alpha\beta} - 2g^{\alpha\beta}\nabla^{(\nu}\nabla^{\mu)} + 2g^{(\mu(\alpha}\nabla^{\nu)}\nabla^{\beta)}\Big\}\, h_{\alpha\beta}\ ,
\end{aligned}
\tag{4.2.15}
$$

leading to (4.2.10).

■

The terms $-2g^{\alpha\beta}\nabla^{(\nu}\nabla^{\mu)} + 2g^{(\mu(\alpha}\nabla^{\nu)}\nabla^{\beta)}$ make the operator non-minimal in the heat kernel sense. In fact, we can rewrite $\chi^{\mu\nu\alpha\beta}$ in the following form

$$
\chi^{\mu\nu\alpha\beta} = \tilde{\chi}^{\mu\nu\alpha\beta} - 2g^{\alpha\beta}\nabla^{(\nu}\nabla^{\mu)} + 2g^{(\mu(\alpha}\nabla^{\nu)}\nabla^{\beta)}\ , \tag{4.2.16}
$$

with

$$
\tilde{\chi}^{\mu\nu\alpha\beta} = \Big(-\Box + m^2\Big)\Big[g^{\mu(\alpha}g^{\beta)\nu} - g^{\alpha\beta}g^{\mu\nu}\Big] - 2R^{\mu(\alpha|\nu|\beta)} + \frac{R}{D}g^{\mu\nu}g^{\alpha\beta} \tag{4.2.17}
$$

being a minimal operator.

4.3 Comparison with linearized Einstein gravity

We now want to unveil the connection between the massless part of the action and the action of linearized Einstein gravity in the de Donder gauge. Let's rewrite the action (4.2.3) in the following form:

$$
\begin{aligned}
S_{FP}[h]_{m=0} = \int d^Dx\sqrt{g}\Big[&-\frac{1}{2}h^{\mu\nu}\Box h_{\mu\nu} + h^{\mu\nu}\nabla^\rho\nabla_\nu h_{\mu\rho} + \nabla^\mu h\nabla^\nu h_{\mu\nu} + \\
&-\frac{1}{2}\nabla^\mu h\nabla_\mu h\ - \frac{R}{D}\Big(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2\Big)\Big]\ .
\end{aligned}
\tag{4.3.1}
$$

We impose the de Donder gauge,

$$\nabla^\nu h_{\mu\nu} - \frac{1}{2}\nabla_\mu h = 0 \ , \tag{4.3.2}$$

by means of the gauge-fixing action

$$\begin{aligned} S_{dD}[h] &= \int d^Dx\sqrt{g}\Big[\nabla^\nu h_{\mu\nu} - \frac{1}{2}\nabla_\mu h\Big]^2 = \\ &= \int d^Dx\sqrt{g}\Big[\nabla^\nu h_{\mu\nu}\nabla_\rho h^{\mu\rho} + \frac{1}{4}\nabla_\mu h\nabla^\mu h - \nabla^\nu h_{\mu\nu}\nabla^\mu h\Big] \ . \end{aligned} \tag{4.3.3}$$

The action becomes

$$\begin{aligned} S_{lin}[h] &= S_{FP}[h]_{m=0} + S_{dD}[h] = \\ &= \int d^Dx\sqrt{g}\Big[-\frac{1}{2}h^{\mu\nu}\Box h_{\mu\nu} + h^{\mu\nu}\nabla^\rho\nabla_\nu h_{\mu\rho} + \frac{1}{4}h\Box h + \\ &\qquad - h^{\mu\rho}\nabla_\rho\nabla^\nu h_{\mu\nu} - \frac{R}{D}\Big(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2\Big)\Big] \ . \end{aligned} \tag{4.3.4}$$

With the identity

$$h^{\mu\nu}\nabla^\rho\nabla_\nu h_{\mu\rho} = h^{\mu\nu}\nabla_\nu\nabla^\rho h_{\mu\rho} + \frac{R}{D}h^{\mu\nu}h_{\mu\nu} - R_{\mu\lambda\nu\rho}h^{\mu\nu}h^{\rho\lambda} \ , \tag{4.3.5}$$

we finally obtain the action written in the following diagonal form

$$S_{lin}[h] = \int d^Dx\sqrt{g}\big(h_{\mu\nu}\mathfrak{E}^{\mu\nu\alpha\beta}h_{\alpha\beta}\big) \ , \tag{4.3.6}$$

with

$$\mathfrak{E}^{\mu\nu\alpha\beta} = \Box\Big(\frac{1}{4}g^{\mu\nu}g^{\alpha\beta} - \frac{1}{2}g^{\mu(\alpha}g^{\beta)\nu}\Big) + \frac{R}{2D}g^{\mu\nu}g^{\alpha\beta} - R^{\mu(\alpha|\nu|\beta)} \tag{4.3.7}$$

which is precisely the linearized Einstein gravity action considered in [73].

Proof. By plugging (4.3.5) into (4.3.4) we can reach the final form of the action and then diagonalize it like in the previous section.

$$\begin{aligned} S_{lin}[h] &= \int d^Dx\sqrt{g}\Big[-\frac{1}{2}h^{\mu\nu}\Box h_{\mu\nu} - R_{\mu\alpha\nu\beta}h^{\mu\nu}h^{\alpha\beta} + \frac{1}{4}h\Box h + \frac{R}{2D}h^2\Big] = \\ &= \int d^Dx\sqrt{g}\Big[-\frac{1}{2}h_{\mu\nu}g^{\mu(\alpha}g^{\beta)\nu}\Box h_{\alpha\beta} - h_{\mu\nu}R^{\mu(\alpha|\nu|\beta)}h_{\alpha\beta} + \\ &\qquad + \frac{1}{4}h_{\mu\nu}g^{\mu\nu}\Box g^{\alpha\beta}h_{\alpha\beta} + \frac{R}{2D}h_{\mu\nu}g^{\mu\nu}g^{\alpha\beta}h_{\alpha\beta}\Big] = \\ &= \int d^Dx\sqrt{g}\Big\{h_{\mu\nu}\Big[\Box\Big(\frac{1}{4}g^{\mu\nu}g^{\alpha\beta} - \frac{1}{2}g^{\mu(\alpha}g^{\beta)\nu}\Big) + \\ &\qquad + \frac{R}{2D}g^{\mu\nu}g^{\alpha\beta} - R^{\mu(\alpha|\nu|\beta)}\Big]h_{\alpha\beta}\Big\} \ . \end{aligned} \tag{4.3.8}$$

This leads to (4.3.6).

■

By fixing the gauge, we can immediately obtain an action with a minimal operator like (4.3.7), which is not possible in (4.2.10) because of the mass term which breaks the gauge symmetry (4.2.7).

4.4 Stückelberg vector field

The partition function is given by the path integration of the Fierz-Pauli action over all the possible field configurations:

$$Z[g_{\mu\nu}] = \int Dh \exp\Big(- S_{FP}[h; g_{\mu\nu}]\Big) \ . \tag{4.4.1}$$

Note that the issue of summing over gauge-equivalent configurations is absent since the gauge symmetry is broken by the mass term, which plays the double and unnatural role of both mass term and gauge fixing [74]. In the following, we aim to restore said gauge symmetry. To do so, we can introduce the Stückelberg vector field A_μ, patterned after the $m = 0$ gauge symmetry:

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{1}{m}\Big(\nabla_\mu A_\nu + \nabla_\nu A_\mu\Big) \ . \tag{4.4.2}$$

In fact, the massless $S_{FP}[h]_{m=0}$ term remains invariant. The mass term (4.2.8) changes as follows

$$S[h]_m \to S[h]_m + S_{STU}[h, A] \ , \tag{4.4.3}$$

with

$$S_{STU}[h, A] = \int d^D x\sqrt{g}\Big[\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{2R}{D}A^\mu A_\mu + 2m\big(h_{\mu\nu}\nabla^\mu A^\nu - h\nabla_\mu A^\mu\big)\Big] \ , \tag{4.4.4}$$

having defined

$$F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu \ . \tag{4.4.5}$$

Proof. As the spacetime is curved and the derivatives are covariant, we have to consider the relation

$$\nabla_\mu A_\nu \nabla^\nu A^\mu = (\nabla_\mu A^\mu)^2 - R_{\mu\nu}A^\mu A^\nu = (\nabla_\mu A^\mu)^2 - \frac{R}{D}A^\mu A_\mu \ , \tag{4.4.6}$$

which holds up to total derivatives and can be further simplified considering that we work in an Einstein space (4.1.1). This leads to the presence of the term $-\frac{2R}{D}A^\mu A_\mu$ in (4.4.6). Taking into account this point, the rest of the computation is trivial and analogous to the one performed in the previous chapter to derive the expression (3.6.11). Let's see how the term $\frac{1}{2}m^2\big(h_{\mu\nu}h^{\mu\nu} - h^2\big)$ transforms under the introduction of the Stückelberg field A_μ. First, let's look at the terms $h_{\mu\nu}h^{\mu\nu}$ and h^2 separately

$$h_{\mu\nu}h^{\mu\nu} \to h_{\mu\nu}h^{\mu\nu} + \frac{4}{m}h_{\mu\nu}\nabla^\mu A^\nu + \frac{2}{m^2}(\nabla_\mu A_\nu \nabla^\mu A^\nu + \nabla_\mu A_\nu \nabla^\nu A^\mu) \tag{4.4.7}$$

$$h^2 \to h^2 + \frac{4}{m^2}(\nabla_\mu A^\mu)^2 + \frac{4}{m}h\nabla_\mu A^\mu \ . \tag{4.4.8}$$

Now, as

$$\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \nabla_\mu A_\nu \nabla^\mu A^\nu - \nabla_\mu A_\nu \nabla^\nu A^\mu \tag{4.4.9}$$

we can write

$$\begin{aligned} \nabla_\mu A_\nu \nabla^\mu A^\nu + \nabla_\mu A_\nu \nabla^\nu A^\mu &= \nabla_\mu A_\nu \nabla^\mu A^\nu - \nabla_\mu A_\nu \nabla^\nu A^\mu + 2\,\nabla_\mu A_\nu \nabla^\nu A^\mu = \\ &= \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + 2(\nabla_\mu A^\mu)^2 - \frac{2R}{D}A^\mu A_\mu \ , \end{aligned} \tag{4.4.10}$$

so (4.4.7) can be rewritten as

$$h_{\mu\nu}h^{\mu\nu} \to h_{\mu\nu}h^{\mu\nu} + \frac{4}{m}h_{\mu\nu}\nabla^\mu A^\nu + \frac{1}{m^2}F_{\mu\nu}F^{\mu\nu} + \frac{4}{m^2}(\nabla_\mu A^\mu)^2 - \frac{4R}{m^2 D}A^\mu A_\mu \ , \tag{4.4.11}$$

leading to

$$h_{\mu\nu}h^{\mu\nu} - h^2 \to h_{\mu\nu}h^{\mu\nu} - h^2 + \frac{1}{m^2}F_{\mu\nu}F^{\mu\nu} + \frac{4}{m}(h_{\mu\nu}\nabla^\mu A^\nu - h\nabla_\mu A^\mu) - \frac{4R}{m^2 D}A^\mu A_\mu \tag{4.4.12}$$

and finally to (4.4.4).

■

Now the total action has a gauge symmetry:

$$\delta h_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)} \qquad \delta A_\mu = -m\xi_\mu \ . \tag{4.4.13}$$

Now, let's consider the following gauge-fixing condition:

$$\nabla^\nu h_{\mu\nu} - \frac{1}{2}\nabla_\mu h + mA_\mu = 0 \ , \tag{4.4.14}$$

which is implemented inside the path integral with the following gauge-fixing term

$$
\begin{aligned}
S_{gf}[h,A] &= \int d^Dx\sqrt{g}\Big[\nabla^\nu h_{\mu\nu} - \frac{1}{2}\nabla_\mu h + mA_\mu\Big]^2 = \\
&= \int d^Dx\sqrt{g}\Big[\nabla^\nu h_{\mu\nu}\nabla_\rho h^{\mu\rho} - \nabla^\nu h\nabla^\mu h_{\mu\nu} + 2mA^\mu\nabla^\nu h_{\mu\nu} + \\
&\qquad + \frac{1}{4}\nabla_\mu h\nabla^\mu h - mA^\mu\nabla_\mu h + m^2A^\mu A_\mu\Big] \ .
\end{aligned} \tag{4.4.15}
$$

So, we can compute $S_{TOT}[h,A] = S_{FP}[h] + S_{STU}[h,A] + S_{gf}[h,A]$

$$
\begin{aligned}
S_{TOT}[h,A] = \frac{1}{2}\int d^Dx\sqrt{g}\Big[&\nabla^\rho h^{\mu\nu}\nabla_\rho h_{\mu\nu} - \frac{1}{2}\nabla^\mu h\nabla_\mu h - 2R_{\mu\alpha\nu\beta}h^{\mu\nu}h^{\alpha\beta} + \\
&+ \frac{R}{D}h^2 + m^2\Big(h^{\mu\nu}h_{\mu\nu} - h^2\Big)\Big] + \\
&+ \int d^Dx\sqrt{g}\Big[\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \mu A^\mu A_\mu - mh\nabla_\mu A^\mu\Big] \ ,
\end{aligned} \tag{4.4.16}
$$

with

$$
\mu \equiv m^2 - \frac{2R}{D} \tag{4.4.17}
$$

as an "effective mass". Following the Faddeev-Popov ($\Phi\Pi$) procedure, evaluating the $\Phi\Pi$ determinant to be inserted in the path integral [75] as $\mathrm{Det}[\Upsilon^\mu{}_\nu]$, with:

$$
\Upsilon^\mu{}_\nu = \Big(-\Box + m^2 - \frac{R}{D}\Big)\delta^\mu{}_\nu \ , \tag{4.4.18}
$$

the resulting partition function reduces to

$$
Z[g_{\mu\nu}] = \int Dh\int DA\exp\Big(-S_{TOT}[h,A;g_{\mu\nu}]\Big)\,\mathrm{Det}[\Upsilon^\mu{}_\nu] \ . \tag{4.4.19}
$$

We have implicitely introduced the ghosts and then integrated them out to leave the Faddeev-Popov determinant.

Proof. Considering the gauge-fixing function

$$
f^\mu(h,A) = \nabla_\alpha h^{\mu\alpha} - \frac{1}{2}\nabla^\mu h + mA^\mu \tag{4.4.20}
$$

and the gauge symmetry (4.4.13), we can immediately compute

$$
\begin{aligned}
\Upsilon^\mu{}_\nu &= \frac{\delta f^\mu}{\delta\xi_\nu} = \nabla_\nu\nabla^\mu + \Box\,\delta^\mu{}_\nu - \frac{1}{2}\cdot 2\,\nabla^\mu\nabla_\nu - m^2\delta^\mu{}_\nu = \\
&= \big(-\Box + m^2\big)\delta^\mu{}_\nu + \nabla_\nu\nabla^\mu - \nabla^\mu\nabla_\nu = \Big(-\Box + m^2 - \frac{R}{D}\Big)\delta^\mu{}_\nu
\end{aligned} \tag{4.4.21}
$$

as

$$[\nabla^\alpha, \nabla_\mu]V_\alpha = R_{\mu\nu}V^\nu = \frac{R}{D}V_\mu \tag{4.4.22}$$

considering the Einstein space condition (4.1.1).

■

4.5 Redefinition of the Stückelberg vector field

In order to eliminate the mixed term $-mh\nabla_\mu A^\mu$ let's make the following change of variables for the vector field A_μ

$$A_\mu \to A_\mu + \alpha\nabla_\mu h \ , \tag{4.5.1}$$

with α as a real parameter to be determined.

The only part of S_{TOT} (4.4.16) which changes is:

$$\begin{aligned}
&\int d^Dx\sqrt{g}\Big[\mu A^\mu A_\mu - mh\nabla_\mu A^\mu\Big] \to \\
&\to \int d^Dx\sqrt{g}\Big[\mu A^\mu A_\mu - mh\nabla_\mu A^\mu - \frac{4R}{D}\alpha A_\mu\nabla^\mu h + 2m^2\alpha A_\mu\nabla^\mu h + \\
&\qquad - \frac{2R}{D}\alpha^2\nabla_\mu h\nabla^\mu h + m^2\alpha^2\nabla_\mu h\nabla^\mu h - m\alpha h\Box h\Big] = \\
&= \int d^Dx\sqrt{g}\Big[\mu A^\mu A_\mu + \Big(-m + \frac{4R}{D}\alpha - 2m^2\alpha\Big)h\nabla_\mu A^\mu + \\
&\qquad + \Big(-\frac{2R}{D}\alpha^2 + m\alpha + m^2\alpha^2\Big)\nabla_\mu h\nabla^\mu h\Big] \ .
\end{aligned} \tag{4.5.2}$$

Now, we can choose a value for α such that the term $\Big(-m+\frac{4R}{D}\alpha-2m^2\alpha\Big)h\nabla_\mu A^\mu$ vanishes.

We immediately see that, by choosing

$$\alpha = \frac{mD}{4R - 2Dm^2} \ , \tag{4.5.3}$$

the action (4.4.16) becomes

$$\begin{aligned}
S_{TOT}[h, A] = \frac{1}{2}\int d^Dx\sqrt{g}\Big[&\nabla^\rho h^{\mu\nu}\nabla_\rho h_{\mu\nu} - \mathfrak{X}\nabla^\mu h\nabla_\mu h - 2R_{\mu\alpha\nu\beta}h^{\mu\nu}h^{\alpha\beta} + \frac{R}{D}h^2 + \\
&+ m^2\Big(h^{\mu\nu}h_{\mu\nu} - h^2\Big)\Big] + \\
&+ \int d^Dx\sqrt{g}\Big[\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \mu A_\mu A^\mu\Big] \ ,
\end{aligned} \tag{4.5.4}$$

with

$$\mathfrak{X} \equiv \frac{Dm^2 - R}{Dm^2 - 2R} \ . \tag{4.5.5}$$

By looking at (4.5.3), we can see that α is not properly defined for $m^2 = \frac{2R}{D}$. We will study this case in Appendix C, where we will show that the shift (4.5.1) is a convenient but not crucial step to get to the final result.

4.6 Stückelberg scalar field

Now, by introducing the Stückelberg scalar field φ

$$A_\mu \to A_\mu + \frac{1}{m}\nabla_\mu \varphi \ , \tag{4.6.1}$$

the $\mu A_\mu A^\mu$ term changes as follows

$$\mu A_\mu A^\mu \to \mu A_\mu A^\mu + \mu\bigg(\frac{2}{m}A^\mu \nabla_\mu \varphi + \frac{1}{m^2}\nabla_\mu \varphi \nabla^\mu \varphi\bigg) \ . \tag{4.6.2}$$

The total action has an additional gauge symmetry

$$\delta A_\mu = \nabla_\mu \epsilon \qquad \delta\varphi = -m\epsilon \ . \tag{4.6.3}$$

Now, let's consider the following gauge-fixing condition:

$$\nabla^\mu A_\mu + \frac{\mu}{m}\varphi = 0 \ , \tag{4.6.4}$$

which is implemented inside the path integral with the following gauge-fixing term

$$\begin{aligned} S_{gf'}[A,\varphi] &= \int d^Dx\sqrt{g}\bigg[\nabla^\mu A_\mu + \frac{\mu}{m}\varphi\bigg]^2 = \\ &= \int d^Dx\sqrt{g}\bigg[(\nabla^\mu A_\mu)^2 + \frac{2\mu}{m}\varphi\nabla_\mu A^\mu + \Big(\frac{\mu}{m}\Big)^2\varphi^2\bigg] \ . \end{aligned} \tag{4.6.5}$$

We get:

$$\begin{aligned} S_{TOT}[h,A,\varphi] = \frac{1}{2}\int d^Dx\sqrt{g}\bigg[&\nabla^\rho h^{\mu\nu}\nabla_\rho h_{\mu\nu} - \mathfrak{X}\nabla^\mu h\nabla_\mu h - 2R_{\mu\alpha\nu\beta}h^{\mu\nu}h^{\alpha\beta} + \\ &+ \frac{R}{D}h^2 + m^2\Big(h^{\mu\nu}h_{\mu\nu} - h^2\Big)\bigg] + \\ + \int d^Dx\sqrt{g}\bigg[&\nabla_\nu A_\mu \nabla^\nu A^\mu + \Big(m^2 - \frac{R}{D}\Big)A^\mu A_\mu + \\ &+ \frac{\mu}{m^2}\nabla^\mu\varphi\nabla_\mu\varphi + \Big(\frac{\mu}{m}\Big)^2\varphi^2\bigg] \ . \end{aligned} \tag{4.6.6}$$

The associated $\Phi\Pi$ determinant corresponds to Det$[Y]$, with

$$Y = -\Box + \mu \ . \tag{4.6.7}$$

The resulting partition function reduces to

$$Z[g_{\mu\nu}] = \int Dh \int DA \int D\varphi \exp\Big(- S_{TOT}[h, A, \varphi; g_{\mu\nu}]\Big) \, \mathrm{Det}[\Upsilon^{\mu}{}_{\nu}] \, \mathrm{Det}[Y] \ . \tag{4.6.8}$$

Again, we have implicitely introduced the ghosts and integrated them out.

Proof. Considering the gauge-fixing function

$$f(A, \varphi) = -\nabla^{\mu} A_{\mu} - \frac{\mu}{m}\varphi \tag{4.6.9}$$

and the gauge symmetry (4.6.3), we can immediately compute

$$Y = \frac{\tilde{\delta} f}{\tilde{\delta}\epsilon} = -\Box + \mu \ . \tag{4.6.10}$$

■

Let's now rewrite S_{TOT} (4.6.6) in order to highlight the three different contributions of $h_{\mu\nu}, A_{\mu}, \varphi$

$$S_{TOT}[h, A, \varphi] = S_{gr}[h] + S_{vec}[A] + \frac{\mu}{m^2} S_{sca}[\varphi] \ , \tag{4.6.11}$$

with

$$\begin{aligned} S_{gr}[h] = \frac{1}{2} \int d^D x \sqrt{g} \Big[& \nabla^{\rho} h^{\mu\nu} \nabla_{\rho} h_{\mu\nu} - \mathfrak{X} \nabla^{\mu} h \nabla_{\mu} h - 2 R_{\mu\alpha\nu\beta} h^{\mu\nu} h^{\alpha\beta} + \frac{R}{D} h^2 + \\ & + m^2 \Big(h^{\mu\nu} h_{\mu\nu} - h^2 \Big) \Big] \ , \end{aligned} \tag{4.6.12}$$

$$S_{vec}[A] = \int d^D x \sqrt{g} \big(A_{\mu} \Upsilon^{\mu}{}_{\nu} A^{\nu} \big) \ , \tag{4.6.13}$$

$$S_{sca}[\varphi] = \int d^D x \sqrt{g} \big(\varphi \, Y \varphi \big) \ , \tag{4.6.14}$$

with $\Upsilon^{\mu}{}_{\nu}$ given by (4.4.18) and Y by (4.6.7).

4.7 Decomposition of the gravitational field

The gravitational term is not in a diagonal form, so let's rearrange it by decomposing the field into its traceless part $\Phi_{\mu\nu}$ and scalar part ϕ

$$h_{\mu\nu} = \Phi_{\mu\nu} + \frac{1}{D} g_{\mu\nu}\phi \ , \tag{4.7.1}$$

with

$$g^{\mu\nu}\Phi_{\mu\nu} = 0 \ , \qquad \phi = g^{\mu\nu}h_{\mu\nu} \ . \tag{4.7.2}$$

By doing this, we finally have

$$S_{gr}[h] = S_{gr_1}[\Phi] + k\, S_{gr_2}[\phi] \ , \tag{4.7.3}$$

with

$$S_{gr_1}[\Phi] = \frac{1}{2}\int d^Dx\sqrt{g}\big(\Phi^{\mu\nu}\mathcal{Y}_{\mu\nu}{}^{\alpha\beta}\Phi_{\alpha\beta}\big) \tag{4.7.4}$$

and

$$S_{gr_2}[\phi] = \int d^Dx\sqrt{g}\big(\phi\, Y \phi\big) \ , \tag{4.7.5}$$

with

$$\mathcal{Y}_{\mu\nu}{}^{\alpha\beta} = \big(-\Box + m^2\big)\delta^{(\alpha}{}_{\mu}\delta^{\beta)}{}_{\nu} - 2R_{\mu}{}^{(\alpha}{}_{\nu}{}^{\beta)} \tag{4.7.6}$$

and Y defined in (4.6.7), with

$$k \equiv \frac{m^2 D(D-1) + R(2-D)}{2D(Dm^2 - 2R)} \ . \tag{4.7.7}$$

We immediately see that $S_{sca}[\varphi]$ (4.6.14) and $S_{gr_2}[\phi]$ (4.7.5) have the same form.

Finally, this decomposition leads to

$$Z[g_{\mu\nu}] = \int D\Phi \int DA \int D\varphi \int D\phi \exp\Big(-S_{TOT}[\Phi, A, \varphi, \phi; g_{\mu\nu}]\Big)\mathrm{Det}[\Upsilon^{\mu}{}_{\nu}]\mathrm{Det}[Y] \ , \tag{4.7.8}$$

with $S_{TOT}[\Phi, A, \varphi, \phi; g_{\mu\nu}]$ given by

$$S_{TOT}[\Phi, A, \varphi, \phi; g_{\mu\nu}] = S_{gr_1}[\Phi] + S_{vec}[A] + \frac{\mu}{m^2}S_{sca}[\varphi] + k\, S_{gr_2}[\phi] \ . \tag{4.7.9}$$

4.8 Seeley-DeWitt coefficients

It is now possible to path integrate over all the field species, producing the following functional determinants

$$Z[g_{\mu\nu}] = \Big(\mathrm{Det}\big[\mathcal{Y}_{\mu\nu}{}^{\alpha\beta}\big]\Big)^{-\frac{1}{2}}\Big(\mathrm{Det}[\Upsilon^{\mu}{}_{\nu}]\Big)^{\frac{1}{2}} \ . \tag{4.8.1}$$

As the operator F appears in both $S_{sca}[\varphi]$ (4.6.14) and $S_{gr_2}[\phi]$ (4.7.5) up to some prefactors, which may be absorbed in the overall path integral normalization, its determinant is cancelled out with the one stemming from the $\Phi\Pi$ procedure [76, 77].

Recalling that the determinant of $\mathcal{Y}_{\mu\nu}{}^{\alpha\beta}$ refers to the traceless modes only, for our calculation it's useful to make the operator $\tilde{\mathcal{Y}}_{\mu\nu}{}^{\alpha\beta}$

$$\tilde{\mathcal{Y}}_{\mu\nu}{}^{\alpha\beta} = \big(-\Box + m^2\big)\delta^{(\alpha}{}_{\mu}\delta^{\beta)}{}_{\nu} - 2R_{\mu}{}^{(\alpha}{}_{\nu}{}^{\beta)} \tag{4.8.2}$$

appear, with the same form as $\mathcal{Y}_{\mu\nu}{}^{\alpha\beta}$ (4.7.6), for a full rank-2 tensor $\Psi_{\mu\nu}$, as discussed in [76, 77]. This is immediate with the following decomposition:

$$\Psi_{\mu\nu} = \Phi_{\mu\nu} + \frac{1}{D}g_{\mu\nu}\psi \ , \tag{4.8.3}$$

with

$$g^{\mu\nu}\Phi_{\mu\nu} = 0 \ , \qquad \psi = g^{\mu\nu}\Psi_{\mu\nu} \ , \tag{4.8.4}$$

which leads to the action

$$\begin{aligned} S[\Psi] &= \frac{1}{2}\int d^Dx\sqrt{g}\big(\Psi^{\mu\nu}\tilde{\mathcal{Y}}_{\mu\nu}{}^{\alpha\beta}\Psi_{\alpha\beta}\big) = \\ &= \frac{1}{2}\int d^Dx\sqrt{g}\big(\Phi^{\mu\nu}\mathcal{Y}_{\mu\nu}{}^{\alpha\beta}\Phi_{\alpha\beta}\big) + \frac{1}{2}\int d^Dx\sqrt{g}\big(\psi Y\psi\big) \end{aligned} \tag{4.8.5}$$

and then

$$\begin{aligned} Z_{\Psi}[g_{\mu\nu}] &= \int D\Psi \exp\big(-S[\Psi]\big) = \\ &= \Big(\mathrm{Det}[\tilde{\mathcal{Y}}_{\mu\nu}{}^{\alpha\beta}]\Big)^{-\frac{1}{2}} = \Big(\mathrm{Det}[\mathcal{Y}_{\mu\nu}{}^{\alpha\beta}]\Big)^{-\frac{1}{2}}\Big(\mathrm{Det}[Y]\Big)^{-\frac{1}{2}} \ . \end{aligned} \tag{4.8.6}$$

Therefore, the path integral of linearized massive gravity is given by

$$Z[g_{\mu\nu}] = \Big(\mathrm{Det}\big[\tilde{\mathcal{Y}}_{\mu\nu}{}^{\alpha\beta}\big]\Big)^{-\frac{1}{2}}\Big(\mathrm{Det}[\Upsilon^{\mu}{}_{\nu}]\Big)^{\frac{1}{2}}\Big(\mathrm{Det}[Y]\Big)^{\frac{1}{2}} \tag{4.8.7}$$

and finally, the one loop effective action is given by

$$\begin{aligned} \Gamma[g_{\mu\nu}] &= -\log[Z] = \\ &= \frac{1}{2}\log\Big(\mathrm{Det}\big[\tilde{\mathcal{Y}}_{\mu\nu}{}^{\alpha\beta}\big]\Big) - \frac{1}{2}\log\Big(\mathrm{Det}[\Upsilon^{\mu}{}_{\nu}]\Big) - \frac{1}{2}\log\Big(\mathrm{Det}[Y]\Big) \ . \end{aligned} \tag{4.8.8}$$

The latter equation highlights that it's possible to show that the Seeley-DeWitt coefficients are given by the contributions of the three different operators. Recall that the mass term is singled out in the Schwinger–De Witt parametrization, and transformed into an exponential term in front of the effective action. Indeed, the mass term acts as a cut-off for infrared divergences, making the integral over the proper time convergent at the upper limit [78, 79]. Therefore, it is possible to exploit the already known coefficients previously computed in the literature. In fact, we can write the one-loop effective action in terms of the Seeley-DeWitt coefficients $a_k(D;\, x)$, as in (2.2.20), with D as the spacetime dimensions and the coefficients defined in (2.3.14)

$$\Gamma_{(1)} = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\exp\Big(-\beta m^2\Big)\int \frac{d^D x\sqrt{g}}{(4\pi\beta)^{\frac{D}{2}}}\,\mathrm{str}\sum_{k=0}^{\infty}\beta^k a_k(D;\, x)\ , \tag{4.8.9}$$

with β as the Euclidean time and str as the supertrace defined in (2.2.19). The heat kernel expansion can then be obtained from (4.8.8) using the notions from chapter 2, giving:

$$\mathrm{str}[a_k(D;\, x)] = \mathrm{tr}[a_k^{gr}(D;\, x)] - \mathrm{tr}[a_k^{vec}(D;\, x)] - \mathrm{tr}[a_k^{sca}(D;\, x)]\ , \tag{4.8.10}$$

with $a_k^{gr}(D;\, x)$, $a_k^{vec}(D;\, x)$ and $a_k^{sca}(D;\, x)$ as the heat kernel coefficients for the operators $\tilde{\mathcal{Y}}_{\mu\nu}{}^{\alpha\beta}$ (4.8.2), $\Upsilon^\mu{}_\nu$ (4.4.18) and Y (4.6.7) respectively and tr as the trace defined in (2.3.13). These coefficients can be computed with the well-known method described in [4] and plugged in (4.8.9). The following results for the first four coefficients have been previously computed in [79].

Thanks to the second Bianchi identity

$$\nabla_\mu R_{\alpha\beta\nu\rho} + \nabla_\nu R_{\alpha\beta\rho\mu} + \nabla_\rho R_{\alpha\beta\mu\nu} = 0\ , \tag{4.8.11}$$

which can be contracted to

$$\nabla^\rho R_{\mu\nu\alpha\rho} = \nabla_\nu R_{\mu\alpha} - \nabla_\mu R_{\nu\alpha} \tag{4.8.12}$$

and to

$$\nabla^\nu R_{\mu\nu} = \frac{1}{2}\nabla_\mu R\ , \tag{4.8.13}$$

known as the contracted Bianchi identities, we can find $\nabla_\mu R = \nabla^\nu R_{\mu\nu} = 0$, by plugging (4.1.1) into (4.8.13), as $R_{\mu\nu} \propto R$. Moreover, by taking the covariant derivative of (4.1.1), we get $\nabla_\alpha R_{\mu\nu} = 0$, so that the equation (4.8.12) together with the last result implies that $\nabla^\rho R_{\mu\nu\alpha\rho} = 0$ as well. Therefore, on Einstein manifolds all covariant derivatives of the form $\nabla_\mu R$, $\nabla_\alpha R_{\mu\nu}$ and $\nabla^\sigma R_{\mu\nu\rho\sigma}$ vanish identically. Because of this, many terms in (2.3.19) vanish.

In order to express the heat kernel coefficients, let's first introduce the following notation

$$\mathcal{E}_1^3 \equiv R^3, \tag{4.8.14}$$

$$\mathcal{E}_2^3 \equiv R R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = R R^2_{\mu\nu\rho\sigma}, \tag{4.8.15}$$

$$\mathcal{E}_3^3 \equiv R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\alpha\beta} R_{\alpha\beta}{}^{\mu\nu}, \tag{4.8.16}$$

$$\mathcal{E}_4^3 \equiv R_{\alpha\mu\nu\beta} R^{\mu\rho\sigma\nu} R_\rho{}^{\alpha\beta}{}_\sigma \ . \tag{4.8.17}$$

For the operator Y we have

$$\mathrm{tr}[a_0^{sca}(D;\, x)] = 1 \ , \tag{4.8.18}$$

$$\mathrm{tr}[a_1^{sca}(D;\, x)] = \frac{2R}{3} \ , \tag{4.8.19}$$

$$\mathrm{tr}[a_2^{sca}(D;\, x)] = \frac{40D-1}{180D} R^2 + \frac{1}{180} R^2_{\mu\nu\rho\sigma} \ , \tag{4.8.20}$$

$$\mathrm{tr}[a_3^{sca}(D;\, x)] = \frac{140D^2 - 1890D - 1}{2835D^2} \mathcal{E}_1^3 + \frac{1+28D}{7560D} \mathcal{E}_2^3 + \frac{17}{45360} \mathcal{E}_3^3 + \frac{1}{1620} \mathcal{E}_4^3 \ . \tag{4.8.21}$$

For the operator $\Upsilon^\mu{}_\nu$ we have

$$\mathrm{tr}[a_0^{vec}(D;\, x)] = D \ , \tag{4.8.22}$$

$$\mathrm{tr}[a_1^{vec}(D;\, x)] = \left(\frac{D}{6} + 1\right) R \ , \tag{4.8.23}$$

$$\mathrm{tr}[a_2^{vec}(D;\, x)] = \frac{5D^2 + 58D + 180}{360D} R^2 + \frac{D-15}{180} R^2_{\mu\nu\rho\sigma} \ , \tag{4.8.24}$$

$$\begin{aligned}\mathrm{tr}[a_3^{vec}(D;\, x)] = {} & \frac{35D^3 + 588D^2 + 3512D + 7560}{45360D^2} \mathcal{E}_1^3 + \frac{7D^2 - 62D - 714}{7560D} \mathcal{E}_2^3 + \\ & + \frac{17D - 252}{45360} \mathcal{E}_3^3 + \frac{D-18}{1620} \mathcal{E}_4^3 \ .\end{aligned} \tag{4.8.25}$$

For the operator $\tilde{\mathcal{Y}}_{\mu\nu}{}^{\alpha\beta}$ we have

$$\mathrm{tr}[a_0^{gr}(D;\, x)] = \frac{D(D+1)}{2} \ , \tag{4.8.26}$$

$$\mathrm{tr}[a_1^{gr}(D;\, x)] = \frac{(D+4)(D-3)}{12} R \ , \tag{4.8.27}$$

$$\mathrm{tr}[a_2^{gr}(D;\, x)] = \frac{5D^2 + 3D - 122}{720} R^2 + \frac{D^2 - 29D + 480}{360} R^2_{\mu\nu\rho\sigma} \ , \tag{4.8.28}$$

$$\begin{aligned}\mathrm{tr}[a_3^{gr}(D;\, x)] = {} & \frac{35D^3 - 7D^2 - 1318D + 488}{90720D} \mathcal{E}_1^3 + \\ & + \frac{7D^3 - 202D^2 + 3109D + 9744}{15120D} \mathcal{E}_2^3 + \\ & + \frac{17D^2 - 487D - 16128}{90720} \mathcal{E}_3^3 + \frac{D^2 - 35D - 1152}{3240} \mathcal{E}_4^3 \ .\end{aligned} \tag{4.8.29}$$

Now, using (4.8.10), we can finally write down the total coefficients

$$\mathrm{str}[a_0(D;\, x)] = \frac{(D+1)(D-2)}{2}\ , \tag{4.8.30}$$

$$\mathrm{str}[a_1(D;\, x)] = \frac{D^2 - D - 32}{12} R\ , \tag{4.8.31}$$

$$\mathrm{str}[a_2(D;\, x)] = \frac{5D^3 - 7D^2 - 398D - 356}{720R} R^2 + \frac{D^2 - 31D + 508}{360} R^2_{\mu\nu\rho\sigma}\ , \tag{4.8.32}$$

$$\begin{aligned}\mathrm{str}[a_3(D;\, x)] = & \frac{35D^4 - 77D^3 - 6974D^2 + 53944D - 15088}{90720D^2} \mathcal{E}_1^3 + \\ & + \frac{7D^3 - 216D^2 + 3177D + 11170}{15120D} \mathcal{E}_2^3 + \\ & + \frac{17D^2 - 521D - 15658}{90720} \mathcal{E}_3^3 + \frac{D^2 - 37D - 1118}{3240} \mathcal{E}_4^3\ .\end{aligned} \tag{4.8.33}$$

The one-loop effective action can then be written as an expansion in powers of the Euclidean time β

$$\begin{aligned}\Gamma_{(1)} = -\frac{1}{2} \int_0^\infty \frac{d\beta}{\beta} \exp\Big(-\beta m^2\Big) \int \frac{d^D x \sqrt{g}}{(4\pi\beta)^{\frac{D}{2}}}\, \mathrm{str}\Big[& a_0(D;\, x) + a_1(D;\, x)\beta + a_2(D;\, x)\beta^2 + \\ & + a_3(D;\, x)\beta^3 + \mathcal{O}(\beta^4)\Big]\ .\end{aligned} \tag{4.8.34}$$

In four spacetime dimensions, the different powers of β give rise to the quartic, quadratic, and logarithmic divergences parametrized by a_0, a_1, a_2, respectively. Our results for them coincides precisely with those calculated in [80] in $D = 4$, with the path integral approach. More generally, the a_2 coefficient in D spacetime dimensions has been recently evaluated in [1] and is correctly reproduced by our results. The main result of this thesis is the newly computed coefficient $a_3(D;\, x)$, previously not fully known in the literature, which parametrizes a class of divergences that start to appear in $D \geq 6$ dimensions. More importantly, this coefficient identifies a finite contribution to the derivative expansion of the effective action in $D = 4$, as we shall describe shortly. It has been calculated in [81] with worldline techniques with the restriction to Ricci-flat spacetimes: our result are in agreement upon setting $R_{\mu\nu} = 0$. Therefore, our results for the Seeley-DeWitt coefficients could serve as a benchmark for verifying alternative approaches to massive gravity, and their precise expression should be known explicitly.

Coming back to the application mentioned above, let us derive the contribution of a_3 to the derivative expansion of the effective action in $D = 4$. We set $D = 4$ in (4.8.9) and extract from it the $k = 3$ term. Then, the integral in β can be performed, giving rise to

$$\Gamma_{(1)}^{(k=3)} = \int d^4 x \sqrt{g}\, \mathcal{L} \; , \tag{4.8.35}$$

with the Lagrangian $\mathcal{L}$ given by

$$\mathcal{L} = -\frac{1}{2} \frac{\mathrm{str}[a_3(4;\, x)]}{(4\pi)^2} \int_0^\infty \frac{d\beta}{\beta} \exp\Big(- \beta m^2 \Big) = -\frac{\mathrm{str}[a_3(4;\, x)]}{32\pi^2 m^2} \; , \tag{4.8.36}$$

as

$$\int_0^\infty \frac{d\beta}{\beta} \exp\Big(- \beta m^2 \Big) = \left[-\frac{\exp\Big(- \beta m^2 \Big)}{m^2} \right]_{\beta=0}^{\beta=\infty} = \frac{1}{m^2} \; , \tag{4.8.37}$$

with $\mathrm{str}[a_3(4;\, x)]$ obtained from (4.8.33) with $D = 4$

$$\begin{aligned} \mathrm{str}[a_3(4;\, x)] = \frac{5821}{90720} R^3 + \frac{2087}{6048} R R^2_{\mu\nu\rho\sigma} - \frac{1747}{9072} R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\alpha\beta} R_{\alpha\beta}{}^{\mu\nu} + \\ - \frac{125}{324} R_{\alpha\mu\nu\beta} R^{\mu\rho\sigma\nu} R_\rho{}^{\alpha\beta}{}_\sigma \; . \end{aligned} \tag{4.8.38}$$

The result (4.8.35) is very interesting as it is a finite term in the one-loop effective action. In fact, for massless fields, the derivative expansion of the effective action is not consistent, but for massive fields, when the energy is lower than the particle's mass, the expansion is consistent and often very useful.

Conclusions

The aim of this thesis was to compute the first four heat kernel coefficients a_0 , a_1, a_2 and a_3 for the theory of massive gravity in a curved spacetime, specifically an Einstein space of dimension D with Euclidean signature. The first part of the thesis, consisting of the first three chapters, is dedicated to an introduction of the topics: the Faddeev-Popov procedure, the background field method, the heat kernel method and the theory of massive gravity in flat spacetime, which is a preview of the same theory in curved spaces. The heat kernel method developed by Seeley-DeWitt is not directly applicable to the Fierz-Pauli action of massive gravity in curved spacetime as the operator entering the heat kernel is not minimal. Therefore, we first have to manipulate the action with the Stückelberg trick, by introducing two new fields (a vector and a scalar), which lead to new gauge symmetries. By fixing the gauge properly, we can then write a new action as a sum of four actions, all in diagonal form with minimal operators. In this way, the Faddeev-Popov procedure can be used to compute the path integral of the theory and the Seeley-DeWitt method can be applied to compute the first four heat kernel coefficients and finally express the one-loop effective action as a power series of heat kernel coefficients.

The results obtained are in agreement with the ones obtained by [80] and [1]. The main result of this thesis is the newly computed coefficient $a_3(D; x)$, previously not fully known in the literature, which parametrizes a class of divergences that start to appear in $D \geq 6$ dimensions and identifies a finite contribution to the derivative expansion of the effective action in $D = 4$. It has been calculated in [81], with worldline techniques, but with the restriction of working in a Ricci-flat spacetime.

A Appendix

The metric can be expanded as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \ . \tag{A.0.1}$$

For the moment, we absorb the coupling constant κ in $h_{\mu\nu}$. At linear order

$$g^{\mu\nu}(x) = \eta^{\mu\nu} - h^{\mu\nu}(x) \ , \quad g = \left|\,\mathrm{det} g_{\mu\nu}\right| = 1 + h \ , \quad \sqrt{g} = 1 + \frac{1}{2}h \ . \tag{A.0.2}$$

Indices are raised and lowered with the flat metric $\eta_{\mu\nu}$. Hence, the Christoffel symbols linearize as

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}\Big(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\Big) \ , \tag{A.0.3}$$

the Riemann tensor as

$$R_{\mu\nu}{}^{\rho}{}_{\sigma} = \frac{1}{2}\partial_\sigma\Big(\partial_\mu h_\nu{}^\rho - \partial_\nu h_\mu{}^\rho\Big) - \frac{1}{2}\partial^\rho\Big(\partial_\mu h_{\nu\sigma} - \partial_\nu h_{\mu\sigma}\Big) \tag{A.0.4}$$

and the Ricci tensor as

$$R_{\nu\sigma} = R_{\mu\nu}{}^{\mu}{}_{\sigma} = \frac{1}{2}\Big(\partial_\nu\partial^\mu h_{\sigma\mu} + \partial_\sigma\partial^\mu h_{\nu\mu} - \partial_\nu\partial_\sigma h - \Box h_{\nu\sigma}\Big) \ . \tag{A.0.5}$$

Now, in order to get the quadratic approximation we need to keep at least a linear order in the variation of the $\sqrt{g}g^{\mu\nu}$ part of the Einstein-Hilbert action, as at the quadratic level the Ricci tensor will not contribute. This is seen recalling that in a first order formalism, the action depends on the metric and Christoffel symbols independently

$$S_{EH}[g,\Gamma] = \frac{1}{2\kappa^2}\int d^D x\, \sqrt{g}\, g^{\mu\nu} R_{\mu\nu}(\Gamma) \ . \tag{A.0.6}$$

The equations of motion of $g_{\mu\nu}$ give

$$R_{\mu\nu}(\Gamma) - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}R_{\alpha\beta}(\Gamma) = 0 \tag{A.0.7}$$

while the equations of motion from varying $\Gamma^{\rho}{}_{\mu\nu}$ give algebraic equations whose solutions are precisely the ones defining the usual Christoffel symbols, that can be substituted back in the action and in (A.0.7). The latter giving the Einstein equation in its second order form

$$R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g) = 0 \tag{A.0.8}$$

which could as well be obtained from the action in the second order form, varying only the $\sqrt{g}g^{\mu\nu}$ part, the remaining $g_{\mu\nu}$ dependence does not need to be varied as that variation will be automatically vanishing (schematically $\frac{\delta R_{\mu\nu}}{\delta g} = \frac{\delta R_{\mu\nu}}{\delta \Gamma}\frac{\delta\Gamma}{\delta g}$, but $\frac{\delta R_{\mu\nu}}{\delta\Gamma}$ vanish so does $\frac{\delta R_{\mu\nu}}{\delta g}$).

Thus, at the linear order, the variation of the Einstein-Hilbert action in the second order formulation may be written as

$$\delta S_{EH}[g] = \frac{1}{2\kappa^2} \int d^D x \, \sqrt{g} \, \delta g^{\mu\nu} \left[R_{\mu\nu}(g) - \frac{1}{2} g_{\mu\nu} R(g) \right] . \tag{A.0.9}$$

For a second variation, needed to identify the quadratic approximation, one sees that only a linear variation R and $R_{\mu\nu}$ is needed. Therefore, at the quadratic order we consider only the terms that will contribute, i.e.

$$S_2[h] = \frac{1}{4\kappa^2} \int d^D x \left(1 + \frac{1}{2} h \right) \left(\eta^{\nu\sigma} - h^{\nu\sigma} \right) \left(\partial_\nu \partial^\mu h_{\sigma\mu} + \partial_\sigma \partial^\mu h_{\nu\mu} - \partial_\nu \partial_\sigma h - \Box h_{\nu\sigma} \right) , \tag{A.0.10}$$

leading, after some integrations by parts to collect similar terms, to

$$S_2[h] = \frac{1}{4\kappa^2} \int d^D x \left[h^{\mu\nu} \Box h_{\mu\nu} - h \Box h + 2h \partial^\mu \partial^\nu h_{\mu\nu} + 2(\partial^\mu h_{\mu\nu})^2 \right] . \tag{A.0.11}$$

By finally redefining $h_{\mu\nu} \to \kappa h_{\mu\nu}$, with the proper normalization, further integrations by parts and grouping some terms, we finally obtain (3.3.3).

B Appendix

Let's consider the Fierz-Pauli action for massive gravity in a curved spacetime (4.2.9) and introduce the Stückelberg vector field A_μ

$$h_{\mu\nu} \to h_{\mu\nu} + \nabla_\mu A_\nu + \nabla_\nu A_\mu \ , \tag{B.0.1}$$

leading to

$$\begin{aligned} S_{FP}[h,A] = \int d^D x \Big\{ \mathcal{L}_{m=0}(h) + \sqrt{g}\Big[\frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) + \frac{1}{2}m^2 F_{\mu\nu}F^{\mu\nu} + \\ - \frac{2m^2 R}{D}A^\mu A_\mu + 2m^2\big(h_{\mu\nu}\nabla^\mu A^\nu - h\nabla_\mu A^\mu\big)\Big]\Big\} \ , \end{aligned} \tag{B.0.2}$$

with $F_{\mu\nu}$ defined in (4.4.5). We can now introduce a Stückelberg scalar field φ

$$A_\mu \to A_\mu + \nabla_\mu \varphi \ , \tag{B.0.3}$$

leading to

$$\begin{aligned} S_{FP}[h,A,\varphi] = \int d^D x \Big\{ \mathcal{L}_{m=0}(h) + \sqrt{g}\Big[\frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) + \frac{1}{2}m^2 F_{\mu\nu}F^{\mu\nu} + \\ - \frac{2m^2 R}{D}A^\mu A_\mu + 2m^2\big(h_{\mu\nu}\nabla^\mu A^\nu - h\nabla_\mu A^\mu\big) + \\ - \frac{2m^2 R}{D}\nabla_\mu\varphi\nabla^\mu\varphi - \frac{4m^2 R}{D}A_\mu\nabla^\mu\varphi + \\ + 2m^2(h_{\mu\nu}\nabla^\mu\nabla^\nu\varphi - h\Box\varphi)\Big]\Big\} \ . \end{aligned} \tag{B.0.4}$$

Under the conformal transformation

$$h_{\mu\nu} = h'_{\mu\nu} + \pi g_{\mu\nu} \ , \tag{B.0.5}$$

with π as any scalar, the change in the massless part is

$$\begin{aligned} \mathcal{L}_{m=0}(h) = \mathcal{L}_{m=0}(h') + \sqrt{g}\Big[(D-2)\Big(\nabla_\mu\pi\nabla_\nu h'^{\mu\nu} - \nabla_\mu\pi\nabla^\mu h' - \frac{1}{2}(D-1)\nabla_\mu\pi\nabla^\mu\pi\Big) + \\ + R\frac{D-2}{D}\Big(h'\pi + \frac{D}{2}\pi^2\Big)\Big] \ . \end{aligned} \tag{B.0.6}$$

Proof. Considering the massless lagrangian

$$\begin{aligned}\mathcal{L}_{m=0}(h) = \sqrt{g}\Big[&\frac{1}{2}\nabla^\rho h^{\mu\nu}\nabla_\rho h_{\mu\nu} - \nabla^\rho h^{\mu\nu}\nabla_\nu h_{\mu\rho} + \nabla^\mu h \nabla^\nu h_{\mu\nu} - \frac{1}{2}\nabla^\mu h \nabla_\mu h + \\ &- \frac{R}{D}\left(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2\right)\Big] \ ,\end{aligned} \tag{B.0.7}$$

by applying (B.0.5) we obtain

$$\begin{aligned}\mathcal{L}_{m=0}(h) = \sqrt{g}\Big[&\frac{1}{2}\nabla^\rho (h'^{\mu\nu} + \pi g^{\mu\nu})\nabla_\rho (h'_{\mu\nu} + \pi g_{\mu\nu}) + \\ &- \nabla^\rho (h'^{\mu\nu} + \pi g^{\mu\nu})\nabla_\nu (h'_{\mu\rho} + \pi g_{\mu\rho}) + \\ &+ \nabla^\mu (h' + D\pi)\nabla^\nu (h'_{\mu\nu} + \pi g_{\mu\nu}) + \\ &- \frac{1}{2}\nabla^\mu (h' + D\pi)\nabla_\mu (h' + D\pi) + \\ &- \frac{R}{D}\left((h'^{\mu\nu} + \pi g^{\mu\nu})(h'_{\mu\nu} + \pi g_{\mu\nu}) - \frac{1}{2}(h' + D\pi)^2\right)\Big] = \\ = \sqrt{g}\Big[&\frac{1}{2}\nabla^\rho h'^{\mu\nu}\nabla_\rho h'_{\mu\nu} - \nabla^\rho h'^{\mu\nu}\nabla_\nu h'_{\mu\rho} + \nabla^\mu h' \nabla^\nu h'_{\mu\nu} - \frac{1}{2}\nabla^\mu h' \nabla_\mu h' + \\ &- \frac{R}{D}\left(h'^{\mu\nu}h'_{\mu\nu} - \frac{1}{2}h'^2\right)\Big] + \\ &+ \sqrt{g}\Big[(D-2)\left(\nabla_\mu \pi \nabla_\nu h'^{\mu\nu} - \nabla_\mu \pi \nabla^\mu h' - \frac{1}{2}(D-1)\nabla_\mu \pi \nabla^\mu \pi\right) + \\ &\qquad + \frac{R}{D}(D-2)\left(h'\pi + \frac{D}{2}\pi^2\right)\Big] = \\ = \mathcal{L}_{m=0}(h') + \sqrt{g}\Big[&(D-2)\left(\nabla_\mu \pi \nabla_\nu h'^{\mu\nu} - \nabla_\mu \pi \nabla^\mu h' - \frac{1}{2}(D-1)\nabla_\mu \pi \nabla^\mu \pi\right) + \\ &\qquad + \frac{R}{D}(D-2)\left(h'\pi + \frac{D}{2}\pi^2\right)\Big] \ ,\end{aligned} \tag{B.0.8}$$

■

If we apply $\pi = \frac{2}{D-2}m^2\varphi$ we obtain

$$\begin{aligned}S_{FP}[h, A, \varphi] = \int d^D x \Big\{&\mathcal{L}_{m=0}(h') + \sqrt{g}\Big[\frac{1}{2}m^2(h'_{\mu\nu}h'^{\mu\nu} - h'^2) + \frac{1}{2}m^2 F_{\mu\nu}F^{\mu\nu} + \\ &- \frac{2m^2 R}{D}A^\mu A_\mu + 2m^2(h'_{\mu\nu}\nabla^\mu A^\nu - h'\nabla_\mu A^\mu) + 2m^2\left(\frac{D-1}{D-2}m^2 - \frac{R}{D}\right)\times \\ &\times \left(\nabla_\mu \varphi \nabla^\mu \varphi - m^2\frac{D}{D-2}\varphi^2 - 2\varphi\nabla_\mu A^\mu - h'\varphi\right)\Big]\Big\} \ .\end{aligned} \tag{B.0.9}$$

Proof. Let's see how the action (B.0.6) is modified under the transformation (B.0.5), with $\pi = \frac{2}{D-2}m^2\varphi$. We consider the following terms

$$\mathcal{L}_{m=0}(h) = \mathcal{L}_{m=0}(h') + \sqrt{g}\Big[(D-2)\Big(\nabla_\mu\pi\nabla_\nu h'^{\mu\nu} - \nabla_\mu\pi\nabla^\mu h' - \frac{1}{2}(D-1)\nabla_\mu\pi\nabla^\mu\pi\Big) + \\ + R\frac{D-2}{D}\Big(h'\pi + \frac{D}{2}\pi^2\Big)\Big] = \\ = \mathcal{L}_{m=0}(h') - \sqrt{g}\frac{D-1}{D-2}2m^4\nabla_\mu\varphi\nabla^\mu\varphi + \sqrt{g}\frac{R}{D}2m^2\Big(h'\varphi + m^2\frac{D}{D-2}\varphi^2\Big) \ , \tag{B.0.10}$$

$$\frac{1}{2}(h_{\mu\nu}h^{\mu\nu} - h^2) = \frac{1}{2}(h'_{\mu\nu}h'^{\mu\nu} - h'^2) - 2m^6\frac{D(D-1)}{(D-2)^2}\varphi^2 - 2m^4\frac{D-1}{D-2}h'\varphi \ , \tag{B.0.11}$$

$$2m^2(h_{\mu\nu}\nabla^\mu A^\nu - h\nabla_\mu A^\mu) = 2m^2(h'_{\mu\nu}\nabla^\mu A^\nu - h'\nabla_\mu A^\mu) - 4m^4\frac{D-1}{D-2}\varphi\nabla_\mu A^\mu \ , \tag{B.0.12}$$

$$2m^2(h_{\mu\nu}\nabla^\mu\nabla^\nu\varphi - h\Box\varphi) = -4m^4\frac{D-1}{D-2}\varphi\Box\varphi = 4m^4\frac{D-1}{D-2}\nabla_\mu\varphi\nabla^\mu\varphi \ . \tag{B.0.13}$$

By plugging in these terms into (B.0.6), we obtain the action (B.0.9).

■

The gauge symmetry of the action reads

$$\delta h'_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + \frac{2}{D-2}\Lambda g_{\mu\nu} \ , \quad \delta A_\mu = -\xi_\mu \tag{B.0.14}$$

$$\delta A_\mu = \partial_\mu\Lambda \ , \quad \delta\varphi = -\Lambda \ . \tag{B.0.15}$$

Note that for the special value

$$R = \frac{D(D-1)}{D-2}m^2 \ , \tag{B.0.16}$$

the dependence on φ completely cancels out of (B.0.9). By setting the unitary gauge $A_\mu = 0$ and given the replacements (B.0.1), (B.0.3) and the conformal transformation (B.0.5), this implies that the original lagrangian (4.2.9) with the mass (B.0.16) has the gauge symmetry

$$\delta h_{\mu\nu} = \nabla_\mu\nabla_\nu\lambda + \frac{1}{D-2}m^2\lambda g_{\mu\nu} \ , \tag{B.0.17}$$

with $\lambda(x)$ as a scalar gauge parameter. At the value (B.0.16), the theory is called *partially massless* [82, 83, 84, 85, 86]. Due to the gauge symmetry (B.0.17), this theory propagates one fewer d.o.f. than usual so for $D = 4$ it carries 4 d.o.f. instead of 5. In addition, it marks a boundary in the R, m^2 plane between stable and unstable theories, see Figure 2.

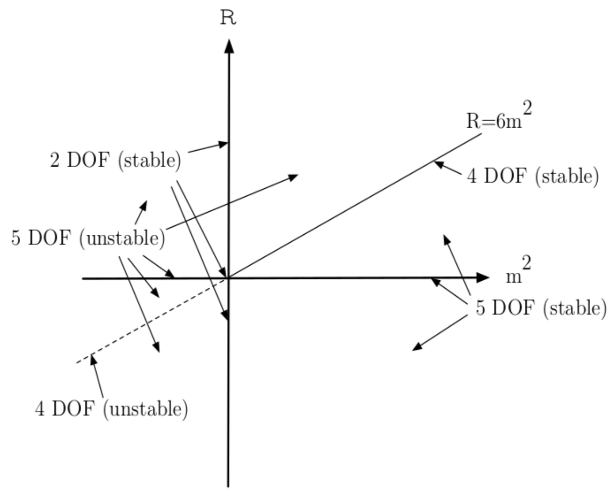

Figure 2: *Degrees of freedom and their stability for values in the R, m^2 plane for massive gravity on an Einstein space in $D = 4$. The line $R = 6m^2$, $m^2 \neq 0$ is where a scalar gauge symmetry appears, reducing the number of d.o.f. by one. The line $m^2 = 0$ is where the vector gauge symmetries appear, reducing the number of d.o.f. by three.*

C Appendix

In the redefinition of the vector field A_μ (4.5.1) we have introduced the parameter α, which is not properly defined for $m^2 = \frac{2R}{D}$. We now show a way to avoid the replacement, thus proving that our method still holds even in the aforementioned seemingly singular case. Indeed, it's possible to write the Fierz-Pauli action in diagonal form with only minimal operators, without having to change variables.

Let's start from (4.4.16)

$$S_{TOT}[h, A] = S_{gr}[h] + \int d^D x\sqrt{g}\left[\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \mu A^\mu A_\mu - mh\nabla_\mu A^\mu\right] , \tag{C.0.1}$$

with $F_{\mu\nu}$ and μ defined in (4.4.5) and (4.4.17) respectively and

$$\begin{aligned} S_{gr}[h] = \frac{1}{2}\int d^D x\sqrt{g}\Big[\nabla^\rho h^{\mu\nu}\nabla_\rho h_{\mu\nu} - \frac{1}{2}\nabla^\mu h\nabla_\mu h - 2R_{\mu\alpha\nu\beta}h^{\mu\nu}h^{\alpha\beta} + \\ + \frac{R}{D}h^2 + m^2\Big(h^{\mu\nu}h_{\mu\nu} - h^2\Big)\Big] . \end{aligned} \tag{C.0.2}$$

Now let's introduce the Stückelberg scalar field φ

$$A_\mu \to A_\mu + \frac{1}{m}\nabla_\mu\varphi . \tag{C.0.3}$$

The only part of S_{TOT} which changes is

$$\begin{aligned} &\int d^D x\sqrt{g}\Big[\mu A^\mu A_\mu - mh\nabla_\mu A^\mu\Big] \to \\ &\to \int d^D x\sqrt{g}\Big[\mu A^\mu A_\mu - mh\nabla_\mu A^\mu + \frac{\mu}{m^2}\nabla_\mu\varphi\nabla^\mu\varphi + \frac{2\mu}{m}A_\mu\nabla^\mu\varphi - h\Box\varphi\Big] . \end{aligned} \tag{C.0.4}$$

So the total action will look like

$$\begin{aligned} S_{TOT}[h, A, \varphi] = \frac{1}{2}\int d^D x\sqrt{g}\Big[&\nabla^\rho h^{\mu\nu}\nabla_\rho h_{\mu\nu} - \frac{1}{2}\nabla^\mu h\nabla_\mu h - 2R_{\mu\alpha\nu\beta}h^{\mu\nu}h^{\alpha\beta} + \\ &+ \frac{R}{D}h^2 + m^2\Big(h^{\mu\nu}h_{\mu\nu} - h^2\Big)\Big] + \\ &+ \int d^D x\sqrt{g}\Big[\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \mu A^\mu A_\mu - mh\nabla_\mu A^\mu \\ &\qquad + \frac{\mu}{m^2}\nabla_\mu\varphi\nabla^\mu\varphi + \frac{2\mu}{m}A_\mu\nabla^\mu\varphi - h\Box\varphi\Big] . \end{aligned} \tag{C.0.5}$$

Now, let's introduce the gauge-fixing

$$\nabla_\mu A^\mu + \frac{\mu}{m}\varphi + \frac{m}{2}h = 0 \ , \tag{C.0.6}$$

implemented in the action as

$$\begin{aligned} S_{gf}[h, A, \varphi] = \int d^D x\sqrt{g}\Big[\big(\nabla_\mu A^\mu\big)^2 + \frac{\mu^2}{m^2}\varphi^2 + \frac{m^2}{4}h^2 + \\ + \frac{2\mu}{m}\nabla_\mu A^\mu\varphi + \mu\varphi h + mh\nabla_\mu A^\mu\Big] \ . \end{aligned} \tag{C.0.7}$$

Now the final action looks like

$$\begin{aligned} S_{final}[h, A, \varphi] &= S_{TOT}[h, A, \varphi] + S_{gf}[h, A, \varphi] = \\ &= \frac{1}{2}\int d^D x\sqrt{g}\Big[\nabla^\rho h^{\mu\nu}\nabla_\rho h_{\mu\nu} - \frac{1}{2}\nabla^\mu h\nabla_\mu h - 2R_{\mu\alpha\nu\beta}h^{\mu\nu}h^{\alpha\beta} + \\ &\quad + \left(\frac{R}{D} + \frac{m^2}{2}\right)h^2 + m^2\Big(h^{\mu\nu}h_{\mu\nu} - h^2\Big)\Big] + \\ &\quad + \int d^D x\sqrt{g}\Big[A_\mu \Upsilon^\mu{}_\nu A^\nu\Big] + \\ &\quad + \int d^D x\sqrt{g}\Big[\frac{\mu}{m^2}\varphi Y\varphi + \nabla_\mu\varphi\nabla^\mu h + \mu\varphi h\Big] \ , \end{aligned} \tag{C.0.8}$$

with $\Upsilon^\mu{}_\nu$ (4.4.18) and Y (4.6.7).

The gravitational term is not in a diagonal form, so let's rearrange it by decomposing again the field into its traceless part $\Phi_{\mu\nu}$ and scalar part ϕ:

$$h_{\mu\nu} = \Phi_{\mu\nu} + \frac{1}{D}g_{\mu\nu}\phi \ , \tag{C.0.9}$$

with

$$g^{\mu\nu}\Phi_{\mu\nu} = 0 \ , \qquad \phi = g^{\mu\nu}h_{\mu\nu} \ . \tag{C.0.10}$$

By doing this, we finally have

$$S_{final}[\Phi, A, \phi, \varphi] = S_{gr_1}[\Phi] + S_{vec}[A] + S_{sca}[\phi, \varphi] \ , \tag{C.0.11}$$

with

$$S_{gr_1}[\Phi] = \frac{1}{2}\int d^D x\sqrt{g}\big(\Phi^{\mu\nu}\mathcal{Y}_{\mu\nu}{}^{\alpha\beta}\Phi_{\alpha\beta}\big) \ , \tag{C.0.12}$$

$$S_{vec}[A] = \int d^D x\sqrt{g}\big(A_\mu \Upsilon^\mu{}_\nu A^\nu\big) \ , \tag{C.0.13}$$

$$S_{sca}[\phi,\varphi] = \int d^D x\sqrt{g}\Bigg[\left(\frac{2-D}{4D}\right)\phi\, Y\phi + \frac{\mu}{m^2}\varphi\, Y\varphi + \varphi\, Y\phi\Bigg] \ , \tag{C.0.14}$$

with $\mathcal{Y}_{\mu\nu}{}^{\alpha\beta}$ (4.7.6). Let's now rewrite the scalar sector as

$$S_{sca}[\psi] = \int d^D x\sqrt{g}\Bigg[\psi^A M_{AB}\psi^B\Bigg] \ , \tag{C.0.15}$$

with

$$\psi^A \equiv \begin{bmatrix}\phi \\ \varphi\end{bmatrix} \ , \qquad M_{AB} \equiv \begin{bmatrix}\left(\frac{2-D}{4D}\right)Y & 0 \\ Y & \frac{\mu}{m^2}Y\end{bmatrix} \ . \tag{C.0.16}$$

Considering the related partition function

$$\begin{aligned} Z_{sca}[g_{\mu\nu}] &= \int D\phi\int D\varphi \exp\Big(-S_{sca}[\phi,\varphi]\Big) = \text{Det}[M_{AB}]^{-\frac{1}{2}} \sim \\ &\sim \left(\Big(\text{Det}[Y]\Big)^2\right)^{-\frac{1}{2}} = \Big(\text{Det}[Y]\Big)^{-1} \ , \end{aligned} \tag{C.0.17}$$

we can notice that the determinant of M_{AB} is proportional to $\big(\det[Y]\big)^2$ up to some constants we can absorb in the partition function with a proper normalization. So, the total partition function will be

$$\begin{aligned} Z[g_{\mu\nu}] &= \int D\Phi\int DA\int D\phi\int D\varphi \exp\Big(-S_{final}[\Phi,A,\phi,\varphi;g_{\mu\nu}]\Big)\times \\ &\qquad\quad \times \text{Det}[\Upsilon^\mu{}_\nu]\,\text{Det}[Y] = \\ &= \int D\Phi \exp\Big(-S_{gr_1}[\Phi;g_{\mu\nu}]\Big)\int DA \exp\Big(-S_{vec}[A;g_{\mu\nu}]\Big) Z_{sca}[g_{\mu\nu}]\times \\ &\qquad\quad \times \text{Det}\,[\Upsilon^\mu{}_\nu]\text{Det}[Y] = \\ &= \Big(\text{Det}\big[\mathcal{Y}_{\mu\nu}{}^{\alpha\beta}\big]\Big)^{-\frac{1}{2}}\Big(\text{Det}[\Upsilon^\mu{}_\nu]\Big)^{-\frac{1}{2}}\Big(\text{Det}[Y]\Big)^{-1}\text{Det}[\Upsilon^\mu{}_\nu]\text{Det}[Y] = \\ &= \Big(\text{Det}\big[\mathcal{Y}_{\mu\nu}{}^{\alpha\beta}\big]\Big)^{-\frac{1}{2}}\Big(\text{Det}[\Upsilon^\mu{}_\nu]\Big)^{\frac{1}{2}} \ , \end{aligned} \tag{C.0.18}$$

which is exactly (4.8.1).

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