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An introduction to MacDowell-Mansouri gravity

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There is no royal road to Geometry.

– Euclid (fl. 300 BC)

Abstract

Gauge theories for fundamental interactions suppose one, or more, fields for electromagnetic, weak and strong interactions which make matter particles interact with each others. For each interaction field there is an associated boson which is represented from a differential geometric perspective through a connection 1-form on a principal bundle with structure group named gauge group. Similarly, every matter particle has an associated fermion represented by a section of a vector bundle associated to the principal one via the Lie algebra representation of the gauge group, so that, in some sense, it can interact with the gauge boson. This thesis focuses on the first of this two properties and we want to give an insight into MacDowell-Mansouri formalism, a possible gauge theory for gravity with Cartan connections as connections on principal frame bundles. In the first two chapters I report the basics about differential topology and Lie groups. We need them to comprehend principal bundles formalism, which is the main topic we deal with in chapter three, together with topological gauge theory. Last two chapters concern geometric aspects of Klein and Cartan geometry and the physical interpretation of MacDowell-Mansouri theory. Here, Cartan connections describe gravitational interactions and the variational principle justifies physical meaning of this theory, indeed equations of motion return Einstein's field equation in Palatini formalism style.

Abstract

Le teorie di gauge per le interazioni fondamentali presuppongono che esistano una, o più, tipologie di campi mediatori associati all'interazione elettromagnetica, forte e debole. Ad ogni campo mediatore è associata una particella bosonica che dal punto di vista geometrico è rappresentata da una connessione 1-forma su un fibrato principale con gruppo di struttura detto gruppo di gauge. La materia, invece, è costituita da campi fermionici che sono rappresentati matematicamente da sezioni di un fibrato vettoriale associato a tale fibrato principale attraverso la rappresentazione dell'algebra di Lie del gruppo di gauge, perciò, in questo senso, possono interagire con il corrispondente bosone di gauge. Questa tesi si concentra sul primo di questi due aspetti fornendo un'introduzione al formalismo di MacDowell-Mansouri, una possibile teoria di gauge per la gravità con connessioni di Cartan al posto di generiche connessioni su fibrati principali. Nei primi due capitoli vengono riportate le conoscenze basilari di topologia differenziale e di teoria dei gruppi di Lie. Serviranno a comprendere le basi della teoria dei fibrati principali, ossia l'argomento cardine del terzo capitolo, insieme alle teorie di gauge topologiche. Gli ultimi due capitoli trattano gli aspetti a noi più utili delle geometrie di Klein e di Cartan e l'interpretazione fisica della teoria di MacDowell-Mansouri. In questo contesto le connessioni di Cartan descrivono l'interazione gravitazionale e il principio variazionale giustifica il siginficato fisico di questa teoria, infatti le equazioni del moto restituiscono l'equazione di campo di Einstein nel formalismo di Palatini.

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Chapter 1

Elements of differential topology

1.1 Smooth manifolds

Definition 1.1.1. Let M be a paracompact[1] Hausdorff space. We call M an *n-dimensional topological manifold* if for each point $p \in M$ there is an open set U in M so that U contains p and it is homeomorphic to an open subset of $\mathbb{R}^n$ by some homeomorphism[2] ϕ. The pair (U, ϕ) is called *local coordinate system* or *chart* on M topological manifold. On the other hand ϕ^{-1} is called *local parameterization* of M, often both ϕ and ϕ^{-1} are referred to as coordinate systems:

$$\begin{aligned} \phi : U &\to \mathbb{R}^n \\ p &\mapsto \phi(p) = \mathbf{x} = (x^1, ..., x^n) \ . \end{aligned} \tag{1.1}$$

We want now to add more structures to our manifolds in order to define functions, derivative of functions or more in general the concept of smoothness, in a consistent way.

Definition 1.1.2. Let M be a topological manifold, we define a *smooth function* at $p \in M$

$$f : M \to \mathbb{R} \ \in \ C^\infty(M, \mathbb{R})$$

if for a specific chart (U, ϕ) the following composition is smooth

$$\hat{f}(\hat{p}) := f \circ \phi^{-1}(\hat{p}) \ \in \ C^\infty(\mathbb{R}^n, \mathbb{R}) \tag{1.2}$$

which means that it needs to be smooth at $\hat{p} = \phi(p)$ as a real vector valued function. In particular, considering two coordinate charts (U_i, ϕ_i) and (U_j, ϕ_j) where U_i and U_j are two joint neighborhoods

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[1] This is equivalent to each component of M having a countable basis for its topology.

[2] A homeomorphism is map between two topological spaces which is continuous (in topological sense), bijective with its inverse continuous.

of $p \in U_i \cap U_j$ then we will have the restrictions of $\hat{f}$ defined as

$$\hat{f}|_{U_i} = f \circ \phi_i^{-1}$$

$$\hat{f}|_{U_i \cap U_j} = f \circ \phi_j^{-1} = (f \circ \phi_i^{-1}) \circ (\phi_i \circ \phi_j^{-1})$$

where the map $f \circ \phi_i^{-1}$ is smooth as starting condition, while this is not necessarily true for the other one $\phi_i \circ \phi_j^{-1}$. Hence the *transition map* $\phi_i \circ \phi_j^{-1}$ need to be smooth otherwise the definition of smoothness of f would not be unique but would depend on the chose local coordinate. In general, a map

$$f : M \to N \ \in \ C^\infty(M, N) \tag{1.4}$$

between smooth manifolds is called *smooth* if it is continuous and for each point $p \in M$ there is a chart (U, ϕ) on M with $p \in U$ and a chart (V, ψ) on N with $f(p) \in V$ such that the composition Φ is smooth

$$\Phi = \psi \circ f \circ \phi^{-1} \ \in C^\infty(\mathbb{R}^m, \mathbb{R}^n) \tag{1.5}$$

Φ is called *coordinate expression* for f and the map f is called *diffeomorphism* if it is smooth and bijective with smooth inverse. It is clear that the smoothness of a map is independent of the choice of charts. In case of topological manifolds the transition functions are obviously homeomorphisms, since they are compositions of two continuous functions. In particular, if M is a topological manifold

Definition 1.1.3. We define the collection of charts $\mathcal{A} = \{(U_i, \phi_i)\}$ a *smooth atlas* on M if:

i. $\{U_i\}$ form an open covering of M, meaning that $M \subseteq \cup\, U_i$, and

ii. for each pair (U_i, ϕ_i) and (U_j, ϕ_j) in $\mathcal{A}$ the map Φ_{ij} is a smooth map between open sets in $\mathbb{R}^n$:

$$\Phi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j) \ . \tag{1.6}$$

A *smooth structure* on a topological manifold is an equivalent class of atlases, and a *smooth manifold* is a topological manifold with a specific smooth structure. In particular we call *canonical smooth structure* a smooth structure with the equivalence class of atlases provided by the n-sphere respect to the standard scalar product in Euclidean space, explicitly:

$$S^n = \{ \ x \in \mathbb{R}^{n+1} \mid x \cdot x = 1 \ \} \ . \tag{1.7}$$

Example 1.1.1. The fundamental example of a smooth n-manifold is of course $\mathbb{R}^n$ itself with its map consisting of the identity map alone. More generally, any finite dimensional vector space V carries a canonical smooth structure in the following manner. If $\dim(V) = n$, we take the atlas consisting of linear isomorphism $\phi : V \to \mathbb{R}^n$. The collection of such maps is an atlas since for any two ϕ and ψ, the change of coordinate is a linear map $\phi \circ \psi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ and hence smooth. It turns out that the canonical structure on V is the unique smooth structure up to diffeomorphism.

Definition 1.1.4. Let (U_i, ϕ_i) and (U_j, ϕ_j) be two charts for the smooth manifold M. We say these charts *compatible* on $W \in U_i \cap U_j$ if the change of coordinate mapping $\Phi_{ij} := \phi_i \circ \phi_j^{-1}$ shows

$$\det J_{\Phi_{ij}}(\hat{p}) > 0 \quad \forall\ \hat{p} \in \phi_j(W) \tag{1.8}$$

so restricted to $\phi_j(U_i \cap U_j)$ it has positive Jacobian determinant at each point of $\phi_j(W)$. In particular we say them *compatible* if they are compatible on $U_i \cap U_j$. We define an atlas for M *oriented* if any two charts in it are compatible. In particular, if the manifold has boundary ∂M, then an orientation of M is defined to be an orientation of its interior and it induces an orientation of ∂M. Indeed, suppose that an orientation of M is fixed. Let $U \to \mathbb{R}^n$ be a chart at a boundary point of M which, when restricted to the interior of M, is in the chosen oriented atlas. The restriction of this chart to ∂M is a chart and such charts form an oriented atlas for ∂M.

Definition 1.1.5. The *rank* of a smooth map f between smooth manifolds at a point $p \in M$ is denoted $\mathrm{rank}_p(f)$ and it is given by the rank of the Jacobian matrix:

$$\Phi'(\phi(p)) = (\psi f\ \phi^{-1})'(\phi(p))\ , \tag{1.9}$$

where ϕ and ψ are charts containing p and $f(p)$ respectively. Varying the choice of the charts merely left and right composes Φ with local diffeomorphisms of Cartesian vector space which, by the chain rule, left and right multiplies $\Phi'(\phi(P))$ by invertible matrices, and this will not change the rank. In particular, if $f : M \to N$ is a smooth bijection between smooth manifolds of the same dimension m, then f is a diffeomorphism if and only if its rank at each point of M is m. See [11] page 12.

1.2 Tangent and Cotangent Space

Definition 1.2.1. Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ and a vector $\mathbf{v}_p \in \mathbb{R}^n_p$ applied at a point $p \in \mathbb{R}^n$ then we define the *tangent vector at a point* p with respect to chart coordinate $\{x^i\}$ an application

$$\tilde{\mathbf{v}}_p : C^\infty(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$$

which is given by the directional derivative of f in the $\mathbf{v}_p$ direction[3]

$$\tilde{\mathbf{v}}_p \cdot f := \lim_{t \to 0} \frac{f(p + t\mathbf{v}_p) - f(p)}{t} = v^i \frac{\partial}{\partial x^i}\bigg|_p f \tag{1.10}$$

In addition to this, we define a *derivation at* $p \in M$ smooth manifold a linear map

$$X_p : C^\infty(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R} \tag{1.11}$$

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[3] The equivalence (1.10) is guaranteed by the total derivative theorem or chain rule in $\mathbb{R}^n$.

which satisfies Leibniz rule for given real-valued smooth functions f and g

$$X_p(f \cdot g) = X_p(f) \cdot g(p) + f(p) \cdot X_p(g) \tag{1.12}$$

In particular, if c is a real-valued constant function then it derivation $X_p(c) = 0$. $T_p\mathbb{R}^n$ is defined as the vector space of derivation at a point $p \in \mathbb{R}^n$, its elements are $\tilde{\mathbf{v}}_p$ and it satisfies the isomorphism $\mathbb{R}^n \simeq T_p\mathbb{R}^n$ as it is shown in [9] page 11. Thus partial derivatives can be used as a basis for $T_p\mathbb{R}^n$.

Definition 1.2.2. A *smooth vector field* X on a open set $U \subseteq \mathbb{R}^n$ is smooth map

$$\begin{gathered} X : U \to T_p\mathbb{R}^n \\ p \mapsto X_p \end{gathered} \tag{1.13}$$

where X is an element of the tangent space of $\mathbb{R}^n$ and it can be written with respect to a basis $\{\partial_i\}$ in a generic point $p \in U$ as a derivation of f such that

$$(X \cdot f)(p) = X_p \cdot f = a^i \frac{\partial}{\partial x^i}\bigg|_p \cdot f \tag{1.14}$$

Definition 1.2.3. A *push-forward* of a linear map $F : M \to N$ between finite-dimensional smooth manifolds is a linear application between their tangent space

$$\begin{aligned} F_* :\ T_pM &\to\ T_{F(p)}N \\ X_p &\mapsto F_*(X_p) \end{aligned} \tag{1.15}$$

thus for a real-valued smooth function $f : N \to \mathbb{R}$ it satisfies

$$F_*(X_p)f = X_p(f \circ F) \ . \tag{1.16}$$

Here we have the main properties of the push-forward operator: $(G \circ F)_* = G_* \circ F_*$ for any pair of smooth function on manifolds, if F is a diffeomorphism then F_* too, $\mathrm{d}F_p = F_*|_p$ is a frequently used notation where F_* is calculated in $p \in M$.

Partial derivatives on smooth manifolds

If $\phi : U \to \mathbb{R}^n$ is a diffeomorphism on an open set of a smooth manifold $U \subseteq M$ and (U, ϕ) is the associated chart. The push-forward map in a point $p \in U$ with $\hat{p} = \phi(p)$ is given by

$$\phi_* : T_pM \to T_{\hat{p}}\ \mathbb{R}^n$$

and we define the partial derivative at $p \in M$ respect the coordinate x^i as

$$\boxed{\partial_i|_p := (\phi^{-1})_*\ \partial_i|_{\hat{p}}} \tag{1.17}$$

If $\{\partial_i|_{\hat{p}}\}$ is the canonical basis for $T_p\mathbb{R}^n$ then $\{\partial_i|_p\}$ is the canonical basis for T_pM indeed for a real valued smooth function f on M with $\hat{f} \equiv f \circ \phi^{-1}$, we have:

$$\partial_i|_p f = (\phi^{-1})_* \, \partial_i|_{\hat{p}} f = \; \partial_i|_{\hat{p}}(f \circ \phi^{-1}) \equiv \partial_i|_{\hat{p}}(\hat{f}) = \left.\frac{\partial \hat{f}}{\partial x^i}\right|_{\hat{p}} . \tag{1.18}$$

Definition 1.2.4. We define $X \in C^\infty(U)$ a smooth vector field defined over a smooth manifold M

$$\begin{aligned} X : U &\to T_pM \\ p \; &\mapsto X_p \; , \end{aligned}$$

if its composition with a generic coordinate chart is a smooth function over $V \subseteq \mathbb{R}^n$

$$X \circ \phi \in C^\infty(V) \; .$$

In particular we denote $\mathfrak{X}(M)$ the set of all smooth vector fields defined over the smooth manifold M.

Lemma 1.2.1. In every coordinate chart, the component functions of $X \in \mathfrak{X}(M)$ are smooth functions over $\mathbb{R}^n$. Moreover, for any $f \in C^\infty(M)$ a vector field naturally define the function $X \cdot f : M \to \mathbb{R}$ defined by $(X \cdot f)(p) = X_p \cdot f$, which is smooth too.

Let's proceed with the description of push-forward of the previous basis. Consider two charts (U, ϕ) and (V, ψ) for the respective smooth manifolds M and N and denote the coordinates in the domain by $\hat{p} = \phi(p) = (x^1, ..., x^n) \equiv x^i$ and $\hat{q} = \psi(q) = (y^1, ..., y^m) \equiv y^j$. Chosen $f \in C^\infty(N, \mathbb{R})$ then:

$$\begin{aligned} (F_*(\partial_{x^i}|_p))\, f &= (\partial_{x^i}|_p)(f \circ F) \\ &= (\partial_{x^i}|_{\hat{p}})(f \circ F \circ \phi^{-1}) \\ &= (\partial_{x^i}|_{\hat{p}})[(f \circ \psi^{-1}) \circ (\psi \circ F \circ \phi^{-1})] \\ &= (\partial_{x^i}|_{\hat{p}})(\hat{f} \circ \hat{F}) \\ &= \left.\frac{\partial \hat{f}}{\partial y^j}\right|_{\hat{F}(\hat{p})} \cdot \left.\frac{\partial \hat{F}^j}{\partial x^i}\right|_{\hat{p}} \quad \text{by chain rule} \\ &= \left.\frac{\partial f}{\partial y^j}\right|_{F(p)} \cdot \left.\frac{\partial \hat{F}^j}{\partial x^i}\right|_{\hat{p}} \quad \text{by } \; \hat{f} \circ \hat{F}(\hat{p}) = f \circ F(p) \end{aligned}$$

It has been shown how to push-forward a generic tangent basis through a smooth function F between smooth manifolds and the equation we obtain is the following

$$\boxed{F_*(\partial_{x^i}|p) = \left.\frac{\partial \hat{F}^j}{\partial x^i}\right|_{\hat{p}} \cdot \left.\frac{\partial}{\partial y^j}\right|_{F(p)}} \tag{1.21}$$

Coordinate chart transformation

In a given coordinate chart (U, ϕ) with coordinates function of ϕ given by $(x^1, ..., x^n)$ it has been reported that the natural basis for T_pM is given by $\{\partial_i|_p\}$. The dual space of T_pM is called *cotangent space* and it is denoted T^*_pM. In the given chart of open subset U and coordinate functions $(x^1, ..., x^n)$ every element of the tangent space of $p \in M$ can be expressed as

$$X_p = a^i \partial_i|_p = a^i \left.\frac{\partial}{\partial x^i}\right|_p$$

thus every element of the cotangent space of $p \in M$ is given by

$$\omega_p = \omega_i \ \mathrm{d}x^i|_p \ , \quad \omega_i = \omega_p(\partial_i|_p) \ ;$$

$\{\partial_i|_p\}$ forms a tangent basis for T_pM and $\{\mathrm{d}x^i|_p\}$ forms a dual basis for T^*_pM and they satisfy the orthonormal relation:

$$\mathrm{d}x^i|_p \ (\partial_j|_p) = \delta^i_j \ .$$

If we change chart (U, ϕ) with coordinate function x^i, to (V, ψ) with coordinate function $\tilde{x}^i$ such that the open neighborhoods of p sets U and V intersect each other:

$$(x^1, ..., x^n) \mapsto (\tilde{x}^1, .., \tilde{x^n}) \ , \tag{1.22}$$

the transition map is given by

$$\psi \circ \phi^{-1} = (\tilde{x}^1(\mathbf{x}), ..., \tilde{x}^n(\mathbf{x})) = \tilde{\mathbf{x}}(\mathbf{x}) \quad .$$

The tangent vectors at $p \in M$ can be written now as

$$X_p = \tilde{a}^i \tilde{\partial}_i|_p = \tilde{a}^i \left.\frac{\partial}{\partial \tilde{x}^i}\right|_p \quad , \tag{1.23}$$

We can summarize the procedure in the subsequent expression:

$$\partial_i|_p = \left.\frac{\partial \tilde{x}^j}{\partial x^i}\right|_{\phi(p)} \tilde{\partial}_j|_p \quad , \tag{1.24}$$

and analogously for the components of the vector

$$a^i = \left.\frac{\partial x^j}{\partial \tilde{x}^i}\right|_{\phi(p)} \tilde{a}^j \tag{1.25}$$

In an analogous way we can demonstrate that the elements of the dual space transform under the change of coordinate from the chart (U, ϕ) to (V, ψ), previously defined, according to these relations:

$$\mathrm{d}\tilde{x}^i|_p = \left.\frac{\partial \tilde{x}^i}{\partial x^j}\right|_{\psi(p)} \mathrm{d}x^i|_p \tag{1.26}$$

and for the covariant component of the dual vector at p

$$\tilde{\omega}_i = \left.\frac{\partial x^j}{\partial \tilde{x}^i}\right|_{\phi(p)} \omega_j \ . \tag{1.27}$$

where the relations for the change of coordinate chart can be derived as follows:

$$\begin{aligned} \partial_i|_p f &\equiv ((\phi^{-1})_* \ \partial_i|_{\phi(p)})f \\ &= (\partial_i|_{\phi(p)})f \circ \phi^{-1} \quad \text{where} \quad f \circ \phi^{-1} = \hat{f}(\mathbf{x}) \\ &= (\partial_i|_{\phi(p)})f \circ \psi^{-1} \circ \psi \circ \phi^{-1} \quad \text{where} \quad f \circ \psi^{-1} = \hat{f}(\tilde{\mathbf{x}}) \\ &= \left(\left.\frac{\partial \tilde{x}^j}{\partial x^i}\right|_{\phi(p)} \cdot \left.\frac{\partial}{\partial \tilde{x}^j}\right|_{\psi(p)} \right) \hat{f}(\tilde{\mathbf{x}}) \quad \text{by chain rule} \\ &\equiv \left(\left.\frac{\partial \tilde{x}^j}{\partial x^i}\right|_{\phi(p)} \cdot \left.\frac{\partial}{\partial \tilde{x}^j}\right|_{p} \right) f \end{aligned}$$

Differentials and pull-back application

Definition 1.2.5. Given a smooth real-valued function f, we can naturally produce a covector field $\mathrm{d}f$ called the *differential* of f and $\mathrm{d}f$ is defined as a covector by the property:

$$\mathrm{d}f_p \ (X_p) = X_p(f) \ , \tag{1.29}$$

and in a given coordinate chart x^i we get the previous expression in components

$$\mathrm{d}f_p \ (\partial_i|_p) = (\partial_i f)(p) \ . \tag{1.30}$$

Applying to both sides of the previous equation the i-th element of the dual basis $\mathrm{d}x^i$, we get the local expression of the differential of f as reported below:

$$\mathrm{d}f_p = (\partial_i f(p)) \ \mathrm{d}x^i_p \ .$$

Definition 1.2.6. Given the smooth map $F : M \to N$ we define its *pull-back* application

$$F^* : \ T^*_{F(p)}N \ \to \ T^*_p M$$

such that for a vector field X_p defined at $p \in M$ and a 1-form $\omega_{F(p)} \in T^*_{F(p)} \ N$:

$$(F^* \omega_{F(p)}) \ X_p = \omega_{F(p)} \ (F_* \ X_p) \ . \tag{1.31}$$

It is easy to check in a coordinate system the following observation. Consider the identity map from M with two given coordinate chart smoothly compatible. The pull-back can be then used to compute how a given covector changes under a change of coordinates. In particular, if $F : M \to N$ and $\omega \in \Omega^1(N)$ then $F^*\omega \in \Omega^1(M)$.

Corollary. Let $F : M \to N$ a smooth map, $f \in C^\infty(N)$ and $\omega \in \Omega^1(N)$ then its pull-back satisfies:

i. $F^*\ \mathrm{d}f = \mathrm{d}(f \circ F)$,

ii. $F^*\ (f\omega) = (f \circ F)F^*\omega$.

Proof. Let's demonstrate the first property. Taken a smooth vector field X then

$$(F^*\mathrm{d}f|_{F(p)})|_p\ X_p = \mathrm{d}f|_{F(p)}\ (F_*X_p)_{F(p)} = (F_*\ X_p)|_{F(p)} \cdot f = X_p\ (f \cdot F)|_p = \mathrm{d}(f \circ F)|_p\ X_p\ .$$

The second statement can be proved with the following argument:

$$F^*((f\omega)|_{F(p)}) \equiv F^*(f(F(p)) \cdot \omega(F(p))) = (f \circ F)|_p\ (F^*\omega_{F(p)})\ .$$

□

1.3 Symmetry Generators and Integral Curves

Proposition 1.3.1. We define a *smooth curve* on a smooth manifold M the following application

$$\gamma : J \subset \mathbb{R} \to M\ , \tag{1.33}$$

such that *the tangent vector to the curve* γ at a point $p \in U$ open set of M can be expressed as

$$\dot{\gamma}(p) \equiv \dot{\gamma}^i(t)\frac{\partial}{\partial \gamma^i}\bigg|_p\ . \tag{1.34}$$

Proof. We proceed to verify the previous statement for a real-valued smooth function on M

$$\begin{aligned}(\gamma_*\partial_t|_{t_0})f &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t_0}(f \circ \gamma)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t_0}[(f \circ \phi^{-1}) \circ (\phi \circ \gamma)]\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t_0}(\hat{f} \circ \hat{\gamma}) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t_0}\hat{f}(\gamma^1(t), ..., \gamma^n(t))\end{aligned}$$

where we have fixed the following condition

$$\hat{f} = f \circ \phi^{-1}\ ,\ \hat{\gamma} = \phi \circ \gamma\ ,\ \dot{\gamma}^i \equiv \partial_t\gamma^i$$

In basis $\{\partial_i\}$ and $\{\hat{\partial}_i\}$ we get:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}(f\circ\gamma)=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}(\hat{f}\circ\hat{\gamma})\ ,\quad \dot{\gamma}^i\frac{\partial f}{\partial\gamma^i}(\gamma(t_0))=\dot{\gamma}^i\frac{\partial\hat{f}}{\partial\hat{\gamma^i}}(\gamma(t_0))$$

□

Corollary. Let $p\in M$ then $x_p\in T_pM$ is a tangent vector of some curve γ passing at p.

Proof. Consider a coordinate chart (U,ϕ) centered at p, that is $\phi(p)=\mathbf{0}$, and the vector $X_p=a^i\partial_i|_p$. We want to construct a curve $\gamma:(-\epsilon,+\epsilon)\to U$ for small real values of ϵ such that $\dot{\gamma}^i(0)=a^i$ and $\gamma(0)=p$. To this aim we construct $\hat{\gamma}=(ta^1,...,ta^n)$ from which we have

$$\gamma_{(t=0)}=\phi^{-1}(\hat{\gamma}_{(t=0)})=\phi^{-1}(\mathbf{0})=p\ ,\quad \dot{\gamma}|_{\gamma(0)}=\dot{\gamma}^i(0)\ \partial_i|_{\gamma(0)}=a^i\partial_i|_p$$

□

Definition 1.3.1. Given X a smooth vector field acting upon M, an integral curve of X is a curve $\gamma:I\subseteq\mathbb{R}\to M$ on some open interval I such that:

$$X(\gamma(t_0))\equiv X_{\gamma(t_0)}=\dot{\gamma}(t_0)\ . \tag{1.35}$$

We say that the curve starts at a generic point p_0, or that p_0 is the initial point for γ, whenever holds:

$$p_0=\gamma(t=0)\ . \tag{1.36}$$

From this point of view we can consider an integral curve a system of ordinary differential equations in the variable γ differentiated over the real parameter t. Choosing a coordinate chart (U,ϕ) over the smooth manifold M such that $\gamma(J)\subseteq U$ for $J\subseteq I$ then we get the Cauchy's problem:

$$\begin{cases} a^i_{\gamma(t)}=\dot{\gamma}^i_{(t)} \\ \gamma^i(0)=x^i(p_0) \quad , \end{cases} \tag{1.37}$$

where we used $X_{\gamma(t)}=a^i(\gamma(t))\ \partial_i|_{\gamma(t)}$, $\dot{\gamma}(t)=\dot{\gamma}^i\ \partial_i$ and $x^i(p_0)=\phi(p_0)$. Thus we derive a formal solution which describe the vector field along a neighborhood of the point p_0 along the curve:

$$\dot{\gamma}^i(t)=\hat{a}(\gamma^1(t),...,\gamma^n(t))\ . \tag{1.38}$$

Theorem 1.3.1. In general the theorem of existence and uniqueness of the solution for a boundary value problem holds and it states that the previous system (3.65) of ODEs satisfies:

i. *(uniqueness)* Every two solutions $\gamma_1:I_1\to U$ and $\gamma_2:I_2\to U$ with $t_0\in i_1\cap I_2$ so that $I_1(t_0)=I_2(t_0)$ agree on $i_1\cap I_2$.

ii. *(existence)* For every $t_0 \in I$, $p \in U$ and $V \subseteq U$ neighborhood of p, then exists a curve on V which satisfies the previous boundary value problem.

iii. *(smoothness)* The solution of the previous system of ODEs depends smoothly on the choice of the initial point $(t, p) \in I \times U$.

Definition 1.3.2. A vector field X on a manifold M is *complete* if, at each point $p \in M$, the integral curve that passes through p can be extended to an integral curve for X that is defined for all $t \in \mathbb{R}$.

Definition 1.3.3. Given the flow domain

$$D \equiv \{(t, p) \in \mathbb{R} \times M \mid t \in I_p\} \ \subseteq \ \mathbb{R} \times M \ ,$$

we define the *flow of a vector field* X of integral curve $\gamma = \gamma(t)$, over a smooth manifold M, the map:

$$\mathrm{Fl}^X : \ D \to M \tag{1.39a}$$

$$(t, p) \mapsto \mathrm{Fl}^X(t, p) \equiv \mathrm{Fl}^X_t(p) = \gamma_p(t) \ . \tag{1.39b}$$

The vector field X is called *infinitesimal generator* of the flow and it is strictly related to the concept of symmetries of a manifold. In particular, fixed $t \in \mathbb{R}$ we can define the following domain

$$M_t = \{p \in M \mid (t, p) \in D\} \ ,$$

then there is a well defined map called *local 1-parameter group* of local diffeomorphism generated by X

$$\Phi_t : M_t \to M \ .$$

Theorem 1.3.2. If X is a smooth vector field, γ its integral curve and Fl^X the flow of X over γ then

i. Fl^X is a smooth map.

ii. The flow respect the composition law $(\mathrm{Fl}^X_t \circ \mathrm{Fl}^X_s)(p) = \mathrm{Fl}^X_{t+s}(p)$.

iii. The flow at the origin generates the identity map $\mathrm{Fl}^X_0 = \mathrm{id}_M$.

Proof. See reference [9]. □

Example 1.3.1. Let's consider the vector field $X = x\partial_x + y\partial_y$ defined on $\mathbb{R}^2$, its flow generates the dilations in x and y coordinate:

$$\mathrm{Fl}^X_t(x_0, y_0) = (e^t x_0, e^t y_0) \ .$$

Analogously the vector field $X = \partial_x$ defined over $\mathbb{R}^2$ has a vector flow which generates the translations along the x axis:

$$\mathrm{Fl}^X_t(x_0, y_0) = (x_0 + t, y_0) \ .$$

Lie derivatives as a measure of symmetry

Definition 1.3.4. Given a function $f \in C^\infty(M, \mathbb{R})$ over a smooth manifold we define *Lie derivative of the smooth function* f respect the flow generated by a smooth vector field X at a point $p \in M$

$$(\mathcal{L}_X f)(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left(f \circ \mathrm{Fl}_t^X(p)\right) \ , \tag{1.40}$$

in an analogous way we define the *Lie derivative of a smooth vector field* Y respect the flow generated by a smooth vector field X at a point $p \in M$

$$(\mathcal{L}_X Y)(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left[(\mathrm{Fl}_t^X)_* \ Y_{\mathrm{Fl}_t^X(p)}\right] \ . \tag{1.41}$$

In general for a tensor τ of (n, m)-type, its Lie derivative at a point $p \in M$ respect the flow generated by the vector field X can be expressed as follows:

$$(\mathcal{L}_X \tau)(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left[(\mathrm{Fl}_t^X)^* \ \tau \circ \mathrm{Fl}_t^X(p)\right] \ , \tag{1.42}$$

applying it to n 1-form and m vectors $(\omega^1, ..., \omega^n, Y_1, ..., Y_m)$ we get a scalar whose Lie derivative is:

$$\begin{aligned}(\mathcal{L}_X\tau)(\omega^1, ..., \omega^n, Y_1, ..., Y_m)(p) = {} & X \cdot \tau(\omega^1, ..., \omega^n, Y_1, ..., Y_m)(p) + \\ & - \tau(\mathcal{L}_X\omega^1, ..., \omega^n, Y_1, ..., Y_m)(p) - ... - \tau(\omega^1, ..., \mathcal{L}_X\omega^n, Y_1, ..., Y_m)(p) \\ & - \tau(\omega^1, ..., \omega^n, \mathcal{L}_X Y_1, ..., Y_m)(p) - ... - \tau(\omega^1, ..., \omega^n, Y_1, ..., \mathcal{L}_X Y_m)(p) \ .\end{aligned}$$

In order to better understand the Lie bracket of a vector field we take a introduce a new vector field called *Lie bracket* and denoted:

$$[X, Y]_p \ f \equiv X_p \cdot (Y \cdot f) - Y_p \cdot (X \cdot f) \tag{1.44}$$

Proposition 1.3.2. If X and Y are two smooth vector field defined over a smooth manifold M then for a real-value smooth function f holds:

i. The Lie bracket of two vector field is a vector field too $[X, Y]_p \ \in \ T_pM$.

ii. The assignment $p \mapsto [X, Y]_p$ defines a smooth vector field $[X, Y]$ satisfying

$$[X, Y]f = X \cdot Y \cdot f - Y \cdot X \cdot f \ .$$

iii. In given coordinate $X = a^i\partial_i$ and $Y = b^j\partial_j$ we have:

$$[X, Y] = (a^i\partial_i b^j - b^i\partial_j a^i) \ \partial_i \ . \tag{1.45}$$

In addition to this if $\mathfrak{X}(M)$ is the set of all of the tangent space T_pM for each $p \in M$, then we can think Lie bracket as an algebraic operator which respect the following properties:

$$[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \tag{1.46}$$

i. $[\cdot,\cdot]$ is a bilinear over $\mathbb{R}$ field.

ii. $[\cdot,\cdot]$ is skew symmetric.

iii. Satisfies the Jacobi identity for three vector fields $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$.

Proposition 1.3.3. Given a real-valued smooth function f and a smooth vector field Y over M smooth manifold, their Lie derivative over the flow of a vector field X can be calculated as:

i. $(\mathcal{L}_X f)(p) = X_p \cdot f$ for the function f.

ii. $(\mathcal{L}_X Y)(p) = [X,Y]_p$ for the vector field Y.

Proof. The first statement can be proven in the following way

$$\begin{aligned}
(\mathcal{L}_X f)(p) &:= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \Big(f \circ \mathrm{Fl}_t^X(p)\Big) \\
&= \left[\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 f\right] \Big(\mathrm{Fl}_t^X(p)\Big) \\
&= \Big[X \cdot f(\mathrm{Fl}_0^X(p))\Big] \\
&= X \cdot f(p) \\
&:= X_p \cdot f
\end{aligned}$$

The second is proven given below, where we have used $\tau = -t$, $\mathrm{Fl}_t^X(p) = p'$ and next $\mathrm{Fl}_{-t}^X(p') = p$

$$\begin{aligned}
(\mathcal{L}_X Y)(p) &:= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left[\Big(\mathrm{Fl}_{-t}^X\Big)_* Y_{\mathrm{Fl}_t^X(p)}\right] \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \Big[Y \circ \mathrm{Fl}_{-t}^X(\mathrm{Fl}_t^X(p))\Big] \\
&= \left[\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 Y\right] \Big(\mathrm{Fl}_{-t}^X\Big(\mathrm{Fl}_t^X(p)\Big)\Big) + Y\left[\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \mathrm{Fl}_{-t}^X\right]\Big(\mathrm{Fl}_t^X(p)\Big) \\
&= \left[\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 Y\right](p) + Y\left[\frac{\mathrm{d}\tau}{\mathrm{d}t} \cdot \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_0 \mathrm{Fl}_\tau^X\right](p') \\
&= X \cdot Y(p) - Y \cdot \left[\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_0 \mathrm{Fl}_\tau^X(p')\right] \\
&= X \cdot Y(p) - Y \cdot X(p) \\
&:= [X,Y]_p
\end{aligned}$$

□

Definition 1.3.5. We define a subset S of a n-dimensional manifold M a *regular submanifold* of dimension k if there is a covering $\{U_\alpha\}$ of $\bar{S}$ by open sets of M such that the components of $U_\alpha \cap S$ are all flat plaques of dimension k. Let (U, ϕ) be a chart on M n-dimensional smooth manifold and S a subset of M. The components of $S \cap U$ are called *plaques* of S in this *chart*. The chart (U, ϕ) on M is said to *straighten out* a plaque W if ϕ restricts to a homeomorphism between W and an open set in some m-dimensional affine subspace[4] $A \subset \mathbb{R}^n$. In this case, the plaque is a *flat plaque* of dimension m and the restriction $\phi|_W : W \to A$ is called a *plaque chart.*

Definition 1.3.6. A symmetry of a smooth function f on a manifold M is a diffeomorphism preserving the function f. We will call it local if it applies on a submanifold on M only.

A possible example is the rotation map $R \in \mathrm{SO}(2)$

$$R \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

taken a function depending on $r^2 = x^2 + y^2$ like

$$f(x, y) = f(r(x, y))$$

r is invariant under rotations such that:

$$(R^* f)(r(x, y)) = f(r(R(x, y))) = f(r(x, y)) \ .$$

Definition 1.3.7. An *infinitesimal symmetry* of a smooth function f over a smooth manifold M is a vector field $X \in \mathfrak{X}(M)$ whose flow is a local symmetry for f. A *symmetry of a vector field* Y on a manifold M is a diffeomorphism $\Phi : M \to M$ preserving Y

$$\Phi_* Y = Y \ .$$

Analogously a *local symmetry* of Y is a diffeomorphism between open submanifolds of M. An *infinitesimal symmetry* of a vector field Y is another vector field X on whose flow induce a local symmetry of Y.

Proposition 1.3.4. For a real-valued smooth function f and two smooth vector fields X and Y hold

i. X is a local symmetry for $f \iff X \cdot f = 0$.

ii. X is a local symmetry for $Y \iff [X, Y] = 0$.

Proof. See reference [9]. □

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[4] An affine subspace of $\mathbb{R}^n$ is a set of the form $\{v + a \in \mathbb{R}^n \mid v \in V\}$ where V is a vector subspace of $\mathbb{R}^n$ and $a \in \mathbb{R}^n$

Chapter 2

Theory of Lie groups

2.1 Formal Group Theory

Definition 2.1.1. Let G be a set with $G \neq \emptyset$. $(G, \cdot)$ is a pair where G is defined *group endowed with law of multiplication* $(\cdot)$ such that:

$$\begin{aligned} (\cdot):\ & G \times G \to G \\ & (g_1, g_2) \mapsto g_1 \cdot g_2 \quad , \end{aligned}$$

i. Associative property holds for each element $g_1,\ g_2,\ g_3 \in G$:

$$g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3 \quad \forall\ g_1,\ g_2,\ g_3\ \in G\ ,$$

ii. There is always a distinguished element $e \in G$ called neutral element, such that:

$$g \cdot e = e \cdot g = g\ ,$$

iii. For each element $g \in G$ there is always its inverse $g^{-1} \in G$ such that:

$$g \cdot g^{-1} = g^{-1} \cdot g = e\ .$$

iv. The group is said *Abelian* if it satisfies the commutative relation:

$$g_1 \cdot g_2 = g_2 \cdot g_1\ .$$

It's important to notice that the last condition is not always required and in general a multiplication law $(\cdot)$ associated to a group G is not always commutative.

Definition 2.1.2. The direct product $G = G_1 \times ... \times G_n$ of the groups G_i with $i = 1, ..n$, is the Cartesian product of G_i equipped for the following multiplication, inversion operation and neutral element. The product of two elements $g = (g_1, ..., g_n)$ and $g' = (g'_1, ..., g'_n)$ of G is the element of G given by

$$g \cdot g' = (g_1 \cdot g'_1, ..., g_n \cdot g'_n) \ .$$

The inverse of the element $g \in G$ is $g^{-1} \in G$ given by

$$g^{-1} = (g_1^{-1}, ..., g_n^{-1}) \ .$$

The neutral element e of G is the element

$$e = (e_1, ..., e_n) \ .$$

Proposition 2.1.1. The direct product $G = \prod_{i=1}^{n} G_i$ of the groups G_i is a group.

Proof. See page 95 reference [13]. ☐

Let G be a group and let $H \subseteq G$ with $H \neq \emptyset$.

Definition 2.1.3. H is a *subgroup* of G if it has the following properties:

i. H is closed under the multiplication of the group G: $h_1 \cdot h_2 \in H \quad \forall\, h_1, h_2 \in H$.

ii. H is closed under the inversion of G: $h^{-1} \in H \quad \forall\, h \in H$.

iii. H contains the neutral element of G: $e \in H$.

Corollary. If H is a subgroup of G, then H is a group, the mappings $H \times H \to H$, $(h_1, h_2) \mapsto h_1 \cdot h_2$, and $H \to H$, $h \mapsto h^{-1}$, and the distinguished element $e \in H$ being its multiplication, inversion, and neutral element.

Definition 2.1.4. N is a *normal subgroup* of the group G if for all $g \in G$ and $n \in N$: $g \cdot n \cdot g^{-1} \in N$. The usual notation for N normal subgroup of G is $N \triangleleft G$. In particular, the group is said *simple* if the only normal subgroups it posses are the trivial ones, which are $\{e\}$ and G. It is important to point out that $g \cdot n \cdot g^{-1}$ needs not to be equal n but only belong to the subgroup N.

Corollary. The subgroups $\{e\}$ and G of G are normal subgroups. If G is an Abelian group and H is a subgroup of G, then H is normal.

Example 2.1.1. Let U be an open subset of $\mathbb{R}^n$ (or in general an open set in any finite-dimensional vector space). Then the inclusion $U \subset \mathbb{R}^n$ is an atlas with one chart that provides a smooth structure on M. In particular

$$\mathrm{GL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\} \tag{2.2}$$

is an open set in the vector space $M_n(\mathbb{R})$ since $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is a continuous map (in fact it is a polynomial map). Thus $\mathrm{GL}_n(\mathbb{R})$ is, canonically, a smooth manifold and matrix multiplication μ and inversion $\imath$ are continuous maps

$$\begin{aligned} \mu : Gl_n(\mathbb{R}) \times Gl_n(\mathbb{R}) &\to Gl_n(\mathbb{R}) \\ \imath : Gl_n(\mathbb{R}) &\to Gl_n(\mathbb{R}) \end{aligned} \tag{2.3}$$

Thus $\mathrm{GL}_n(\mathbb{R})$ is an example of a *topological group*, that is a group which is also a topological space.

Example 2.1.2. Another example of group is given by the local Flow Fl^X generate by a smooth vector field X over a smooth manifold. This respect in particular the composition law

$$\mathrm{Fl}^X_t \circ \mathrm{Fl}^X_s = \mathrm{Fl}^X_{t+s} \quad \text{for each } t, s \in \mathbb{R} \ , \tag{2.4}$$

and the identity $\mathrm{Fl}^X_0 = \mathrm{id}_M$, for this reasons it is considered a one-parameter group.

Definition 2.1.5. Let G, G' be groups, a mapping $\phi : G \to G'$ is a *group homomorphism* of G into G' if it preserves the group operations:

i. ϕ preserves group multiplication for all $g_1,\ g_2\ \in G$:

$$\phi(g_1 \cdot g_2) = \phi(g_1) \cdot \phi(g_2) \ .$$

ii. ϕ preserves group inversion for all $g \in G$:

$$\phi(g^{-1}) = \phi(g)^{-1} \ .$$

iii. ϕ preserves the neutral element:

$$\phi(e) = e' \ .$$

We denote $\mathrm{Hom}(G, G')$ the set of all group homomorphisms of G into G' and it is always non-empty

$$\mathrm{Hom}(G, G') \neq \emptyset$$

as it is shown at page 124 reference [13]. In particular if the mapping $\phi : G \to G$ such that $G' = G$ then we call ϕ *group endomorphism* on G into G. The set of all group endomorphism is denoted by $\mathrm{End}(G)$.

Definition 2.1.6. Let $\phi \in \mathrm{Hom}(G, G')$ then we define *kernel* of ϕ the subset of G given by

$$\ker\phi = \{g \mid g \in G, \phi(g) = e\} \ ,$$

and the *range* of ϕ the subset of G' such that:

$$\mathrm{ran}\phi = \{g' \mid g' \in G', \exists\ g \in G \text{ s.t. } g' = \phi(g)\} \ .$$

Proposition 2.1.2. $\ker\phi$ is a normal subgroup of G and $\mathrm{ran}\phi$ is a subgroup of G'.

Proof. See page 126 reference [13]. ☐

Let G, G' be groups then we introduce the concept of isomorphism between groups.

Definition 2.1.7. $\phi : G \to G'$ is a *group isomorphism* of G into G' if $\phi \in \mathrm{Hom}(G, G')$, ϕ is invertible and $\phi^{-1} \in \mathrm{Hom}(G', G)$. We denote the set of all group isomorphisms of G and G' by $\mathrm{Iso}(G, G')$ and in case it's non-empty we say them *isomorphic* $G \simeq G'$. In particular, we call *group automorphism* of G a group isomorphism $\phi : G \to G$ and the set of all group automorphism is denoted by $\mathrm{Aut}(G)$.

The group isomorphism relation $\simeq$ is an equivalence relation on the set of all groups. To show $\simeq$ is an equivalence relation we need to prove that $\simeq$ is reflexive, symmetric and transitive as it is shown at page 131 reference [13].

Definition 2.1.8. The equivalence class $[G]_{\simeq}$ is the *abstract group* associated with G. Where $[G]_{\simeq}$ is the set of all groups related by the equivalence relation $\simeq$ of isomorphic groups.

Definition 2.1.9. Let A be an non empty set, we define $F(A)$ the set of all functions $f : A \to A$. A function f is said *invertible* if there is $f' \in F(A)$ such that

$$f \circ f' = f' \circ f = \mathrm{id}_A \ .$$

We denote by $S(A)$ the subset of $F(A)$ of all invertible functions. The crucial property of $S(A)$ is that is closed under the operations of composition and inversion.

Group actions basics

Definition 2.1.10. Let G be a group and A a non-empty set then we define a *G-action on A* an invertible homomorphism

$$\kappa : G \times A \to A \quad \text{or} \quad \kappa \in \mathrm{Hom}(G, S(A)) \ .$$

For convenience, we shall write κ_g rather than $\kappa(g)$ for $g \in G$. In particular we define *isotropy subgroup* of x under the G-action κ the subgroup of G for $x \in A$ and it is denoted

$$G_\kappa(x) = \{g \mid g \in G,\ \kappa_g(x) = x\} \ .$$

In particular, given a topological manifold M and a G-action κ on M we define *orbit* of $p \in M$ over G the isotropy subgroup whose G-action is given by the multiplication law of G

$$\mathcal{O}_p \equiv G \cdot p = \{g \cdot p \mid g \in G\} \ . \tag{2.5}$$

Definition 2.1.11. The G-action κ is said *free* if

$$G_\kappa(x) = \{e\} \quad \forall\ x \in A \ , \tag{2.6}$$

the G-action κ is said *effective* if

$$\ker(\kappa) = \{e\} \ , \tag{2.7}$$

the G-action κ is said *proper* if the following map is proper, which means its inverse images of compact subsets are compact too.

$$\begin{aligned} G \times A &\to A \times A \\ (g, x) &\mapsto (x, \kappa_g(x)) \end{aligned} \tag{2.8}$$

Proposition 2.1.3. The following statements hold:

i. κ is free $\iff$ for all $g \in G$ with $g \neq e$ and all $x \in A$, $\kappa_g(x) \neq x$.

ii. κ is effective $\iff$ for all $g \in G$ with $g \neq e$, there is $x \in A$ such that $\kappa_g(x) \neq x$.

iii. If κ is free then κ is effective.

Proof. See page 149 reference [13]. ☐

Definition 2.1.12. Let $B \subseteq A$ not-empty set. B is said *invariant under the* G-action κ on A if

$$\forall \ g \in G \ , \ \kappa_g(B) \subseteq B \implies (\kappa|_B)_g = \kappa_g|_B \ .$$

In particular the restriction of κ to B is a G-action on B.

Definition 2.1.13. For any $x_1, x_2 \in A$, we say that x_1 is κ*-related to* x_2 and write $x_1 \sim_\kappa x_2$ if

$$\exists \ g \in G \mid x_2 = \kappa_g(x_1)$$

in particular $\sim_\kappa$ is an equivalence relation on A as it is proven at page 155 reference [13] and it has a canonically associated partition of A constituted by the distinct equivalence classes of $A/\sim_\kappa$. For $x \in A$ one has

$$[x]_{\sim_\kappa} = \{x' \mid x' \in A, \ \text{there is } g \in G \text{ with } x' = \kappa_g(x)\}$$

since given $x' \in [x]_{\sim_\kappa}$ then $x \sim_\kappa x'$. So, there is $g \in G$ such that $x' = \kappa_g(x)$. Conversely, let $x' \in A$ such that $x' = \kappa_g(x)$ for some $g \in G$, then $x \sim_\kappa x'$. So $x' \in [x]_{\sim_\kappa}$.

Definition 2.1.14. The G-action κ is said *transitive* if

$$[x]_{\sim_\kappa} = A \quad \forall \ x \in A \ . \tag{2.9}$$

Since κ is transitive if and only if for every $x \in A$ counts $[x]_{\sim_\kappa} = A$, that is, if and only if for every $x_1, x_2 \in A$, $x_2 \in [x_1]_{\sim_\kappa}$, we can define

$$\kappa \text{ transitive} \iff \forall \ x_1, \ x_2 \in A \ \exists \ g \in G$$

Let G be a group and H be a subgroup of G, with $h \in H$.

Definition 2.1.15. The *left translation* by h is the mapping $L_{Hh} : G \to G$ defined by the expression:

$$L_{Hh}(g) = h \cdot g \quad \text{for } g \in G \ . \tag{2.10}$$

The *right translation* by h is the mapping $R_{Hh} : G \to G$ defined by the expression:

$$R_{Hh}(g) = g \cdot h \quad \text{for } g \in G \ . \tag{2.11}$$

The *adjoint action* by h on G is the mapping $\mathrm{Ad}_h : G \to G$ defined by the expression:

$$\mathrm{Ad}_h(g) \equiv h \cdot g \cdot h^{-1} \quad \text{for } g \in G. \tag{2.12}$$

L_H R_H, Ad are H-actions on G and satisfy

$$\mathrm{Ad}_h = L_h R_{h^{-1}} \quad \forall \ h \in H \ . \tag{2.13}$$

By the previous findings H-action L_H, R_H on G have associated equivalence relations we define

$$_H\sim \; \equiv \; \sim_{L_H} \qquad \text{and} \qquad \sim_H \; \equiv \; \sim_{R_H}$$

Definition 2.1.16. The *left coset space of* G by H is the quotient set

$$H \backslash G = G/_H \sim \ . \tag{2.14}$$

The *left coset of* $g \in G$ by H is the equivalence class

$$Hg = [g]_{_H\sim} \ . \tag{2.15}$$

The *right coset space of* G by H is the quotient set

$$G/H = G/\sim_H \ . \tag{2.16}$$

The *right coset of* $g \in G$ by H is the equivalence class

$$gH = [g]_{\sim_H} \ . \tag{2.17}$$

Definition 2.1.17. Given two groups H and K and an action $\phi : H \to \mathrm{Aut}(K)$, the corresponding *semidirect product* is given by $K \rtimes_\phi H$ that is a direct product

$$K \times H = \{(k, h) \mid h \in H, k \in K\} \ , \tag{2.18}$$

which respect group law $(k, h) \cdot (k', h') = (k \cdot \phi_h(k'), h \cdot h')$.

Corollary. If $G = N \rtimes H$ then there is a group homomorphism $\phi : H \to \mathrm{Aut}(N)$ given by

$$\phi_h(n) = h \cdot n \cdot h^{-1} = \mathrm{Ad}_h(n) \ . \tag{2.19}$$

Example 2.1.3. In case the group G is given by the group of affine transformation

$$f(\mathbf{x}) = \mathbf{y} + A\mathbf{x} \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \ , \ A \in \mathrm{GL}(n, \mathbb{R}) \tag{2.20}$$

It is associated to a group of matrices named *affine group* which is given by the semidirect product:

$$\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R}) \tag{2.21}$$

In particular $\mathrm{Aff}(\mathbb{R}) = \mathbb{R} \rtimes \mathrm{GL}(\mathbb{R}) = N \rtimes H$ because $G = H \times K$ and $H \cap K$ is trivial, where:

$$N = \left\{ \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \ \Big| \ y \in \mathbb{R} \right\} \quad \text{translations in } \mathbb{R} \ , \quad H = \left\{ \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \ \Big| \ x \in \mathbb{R} \right\} \quad \text{rotations in } \mathbb{R}$$

Group representation basics

Let G be a group. Let V be a finite dimensional vector space over the field $\mathbb{K}$ we shall assume $\mathbb{R}, \mathbb{C}$.

Definition 2.1.18. A *representation of the group* G in V is any group homomorphism $D : G \to \mathrm{GL}(V)$ where V is called *representation space* and $\mathrm{GL}(V)$ is the set of invertible endomorphism of V:

$$D \in \mathrm{Hom}(G, \mathrm{GL}(V)) \ .$$

Corollary. A mapping $D : G \to \mathrm{GL}(V)$ is a representation of G if and only if one has

i. $D(g \cdot h) = D(g) \cdot D(h) \quad \forall \ g, h \in G$,

ii. $D(g^{-1}) = D(g)^{-1} \quad \forall \ g \in G$,

iii. $D(e) = 1_V$ returns the identity of V.

Since these are the necessary and sufficient conditions for D being an homomorphism of G into $\mathrm{GL}(V)$.

Example 2.1.4. Here it is a standard example. Consider the group $\mathrm{SU}(n)$ of the special unitary transformation over $\mathbb{R}^n$ such that:

$$\mathrm{SU}(n) \equiv \{U \mid U \in GL(\mathbb{R}^n), \ U^\dagger = U^{-1} \ , \ \det U = 1\} \ . \tag{2.22}$$

It is characterized by two basic representations D, $\bar{D} : \mathrm{SU}(n) \to \mathrm{GL}(n, \mathbb{C})$ defined by

$$D(U) = U \ , \quad \bar{D}(U) = U^* \ . \tag{2.23}$$

These representations are called the *fundamental* and *antifundamental* ones of $\mathrm{SU}(n)$, respectively.

2.2 Basics of Lie Groups

Lie algebras

The notion of Lie algebra plays a fundamental role in Lie group theory. They describe Lie groups is an infinitesimal neighborhood of the neutral element.

Definition 2.2.1. A *Lie algebra* over the field $\mathbb{K}$ is a vector space $\mathfrak{a}$ over $\mathbb{K}$ equipped with a bilinear map $[\cdot,\cdot]$ with the following properties:

$$[\cdot,\cdot] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a} \ ,$$

i. *Skew-symmetry* for X,$Y \in \mathfrak{a}$ $\ [X,Y] + [Y,X] = 0 \ \ ,$

ii. *Jacobi identity* for X,Y, $Z \in \mathfrak{a}$ $\ [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \ \ .$

In particular, a Lie algebra $\mathfrak{a}$ over $\mathbb{K}$ is said *Abelian* if:

$$[X,Y] = 0 \quad X,Y \ \in \mathfrak{a} \ .$$

A Lie algebra $\mathfrak{a}$ is said *simple* if it is non-Abelian and it does not contains nonzero proper ideals, which means it must not contain any vector subspace $\mathfrak{i} \subseteq \mathfrak{a}$ such that $[I,X] \in \mathfrak{i}$ for each $I \in \mathfrak{i}$, $X \in \mathfrak{a}$. Furthermore, a Lie algebra $\mathfrak{a}$ is *semisimple* if it can be expressed as the finite direct sum of simple Lie algebras $\mathfrak{a}_i$ for $i = 1,...,n$, meaning that

$$\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus ... \oplus \mathfrak{a}_n \ . \tag{2.24}$$

In addition to this, as reported at page 253 reference [13], if $\{t_a \mid a = 1,..,n\}$ is a basis of a n-dimensional Lie algebra $\mathfrak{a}$ over $\mathbb{K}$, there exist $f^c_{ab} \in \mathbb{K}$ named *structure constants* of $\mathfrak{a}$ with respect to the basis $\{t_a\}$ with

$$[t_a,t_b] = \sum_{c=1}^{n} f^c_{ab} t_c \quad \text{for } a,b,c = 1,...,n \ . \tag{2.25}$$

Definition 2.2.2. $\mathfrak{b}$ is a *Lie subalgebra* of $\mathfrak{a}$ if it has the following properties:

i. $\mathfrak{b}$ is a linear subspace of $\mathfrak{a}$.

ii. $\mathfrak{b}$ is closed under the Lie brackets of $\mathfrak{a}$: for $X,Y \in \mathfrak{b}$ then $[X,Y] \in \mathfrak{b}$.

Let $\mathfrak{a}$, $\mathfrak{b}$ be Lie algebras.

Definition 2.2.3. A *Lie algebra homomorphism* of $\mathfrak{a}$ into $\mathfrak{b}$ is a mapping $l : \mathfrak{a} \to \mathfrak{b}$ such that:

i. l is linear.

ii. l respects the Lie brackets. For X, $Y \in \mathfrak{a}$: $l([X,Y]) = [l(X), l(Y)]$.

We denote by $\mathrm{Hom}(\mathfrak{a},\mathfrak{b})$ the set of all Lie algebra homomorphism of $\mathfrak{a}$ into $\mathfrak{b}$.

Definition 2.2.4. A mapping $l : \mathfrak{a} \to \mathfrak{b}$ is a *Lie algebra isomorphism* of $\mathfrak{a}$ into $\mathbf{b}$ if $l \in \mathrm{Hom}(\mathfrak{a}, \mathfrak{b})$, l is invertible and $l^{-1} \in \mathrm{Hom}(\mathfrak{b}, \mathfrak{a})$. We denote $\mathrm{Iso}(\mathfrak{a}, \mathfrak{b})$ the set of all Lie algebra isomorphisms of $\mathfrak{a}$ and $\mathfrak{b}$. The Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$ are said *isomorphic* if $\mathrm{Iso}(\mathfrak{a}, \mathfrak{b}) \neq \emptyset$ and we write $\mathfrak{a} \simeq \mathfrak{b}$.

Definition 2.2.5. Let $\mathfrak{a}$ be a Lie algebra over $\mathbb{K}$. Let V be a vector space over a subset of $\mathbb{K}$. A *Lie representation* of $\mathfrak{a}$ in V is any Lie algebra homomorphism

$$d : \mathfrak{a} \to \mathfrak{gl}(V) \quad \text{or} \quad d \in \mathrm{Hom}(\mathfrak{a}, \mathfrak{gl}(V))$$

where $\mathfrak{gl}(V)$ is the general linear Lie algebra of V and it is the vector space of endomorphisms of V

$$\mathfrak{gl}(V) = \mathrm{End}(V)$$

endowed with the commutator map given by endomorphism $[\cdot, \cdot] : \mathfrak{gl}(V) \times \mathfrak{gl}(V) \to \mathfrak{gl}(V)$, such that:

$$[X, Y] = XY - YX \quad \text{for all } X, Y \in V \ . \tag{2.26}$$

Definition 2.2.6. For X, $Y \in \mathfrak{a}$, ad is called *adjoint representation* of $\mathfrak{a}$ and it is defined by:

$$\mathrm{ad}(X)Y := [X, Y] \ . \tag{2.27}$$

Proposition 2.2.1. The adjoint representation ad satisfies the followings:

i. For any X, $Y \in \mathfrak{a}$ then $\mathrm{ad}(X)Y \in \mathfrak{a}$.

ii. For any $X \in \mathfrak{a}$ then $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{a})$.

iii. ad is a representation of $\mathfrak{a}$ in the vector space $\mathfrak{a}$.

Proof. See page 266 reference [13] □

Lie Groups

The key idea of a Lie group is that it is a group in the usual sense, but with the additional property that it is also a differentiable manifold, and in such a way that the group operations are 'smooth' with respect to this structure. A good example is the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ of all complex numbers of modulus one. This is clearly a group under the action of multiplication, but it is also a manifold.

Definition 2.2.7. A real *Lie group* G is a set that is

i. a group in the usual algebraic sense;

ii. a differentiable manifold with the properties that taking the product of two group elements,

$$\begin{aligned} \mu : G \times G \to G \quad & \text{multiplication} \\ (g_1, g_2) \mapsto g_1 \cdot g_2 \ , \end{aligned}$$

and taking the inverse of a group element,

$$\begin{aligned} \imath : G &\to G \quad \text{inversion} \\ g &\mapsto g^{-1} \ , \end{aligned}$$

are smooth operations, specifically, these maps are both of C^{∞} class over G.

In particular a *Lie subgroup* of G is a subset H of G that is a subgroup of the group G in the algebraic sense and a submanifold of the smooth manifold G.

Definition 2.2.8. Let M be a smooth manifold, H a Lie group and a smooth action[1] $M \times H \to M$. We define the smooth action *proper* if given two compact set $A, B \in M$ then

$$\{g \in G \mid Ag \cap B \neq \emptyset\} \ \text{ is compact.}$$

Proposition 2.2.2. Let G be a Lie group, $H \subset G$ a closed subgroup. Then the action $G \times H \to G$ given by multiplication is free and proper.

Proof. See page 145 reference [11]. ☐

Lemma 2.2.1. Let G be a Lie group with $H \subset G$ be both a subgroup and a smooth submanifold of G, then H is a Lie group.

Proof. See page 7 reference [3]. ☐

It is customary to call a map ϕ between two Lie groups a Lie group homomorphism if ϕ is a group homomorphism and ϕ is smooth, whereas we have required only that ϕ be continuous. Thus group homomorphisms come in only two varieties: the discontinuous ones and the smooth ones. There simply are not any intermediate ones.

Proposition 2.2.3. Let G and H be Lie groups, and ϕ a group homomorphism from G to H. Then if ϕ is continuous it is also smooth.

Proof. See page 22 reference [7]. ☐

Two Lie groups which are isomorphic should be thought of as being essentially the “same group”. In addition to this, we want to point out that if $\phi : G \to H$ is a Lie group isomorphism, then ϕ is smooth, bijective and its inverse is smooth as well. Hence, ϕ is a diffeomorphism over smooth manifold given by Lie groups.

---

[1] A smooth action is a group action on a smooth manifold M of type $G \times M \to M$

Matrix Lie Groups

Recall that the general linear group over the reals, denoted $\mathrm{GL}(n;\mathbb{R})$, is the group of all $n \times n$ invertible matrices with real entries. We may similarly define $\mathrm{GL}(n;\mathbb{C})$ to be the group of all $n \times n$ invertible matrices with complex entries. Of course, $\mathrm{GL}(n;\mathbb{R})$ is contained in $\mathrm{GL}(n;\mathbb{C})$.

Definition 2.2.9. Let A_n be a sequence of complex matrices. We say that A_n *converges to a matrix* A if each entry of A_n converges to the corresponding entry of A, i.e., if $(A_n)_{ij}$ converges to A_{ij} for all $1 \leq i, j \leq n$.

Definition 2.2.10. A *matrix Lie group* is any subgroup H of $\mathrm{GL}(n;\mathbb{C})$ with the following property: if A_n is any sequence of matrices in H, and A_n converges to some matrix A, then either $A \in H$, or A is not invertible.

Theorem 2.2.2. Every matrix Lie group is a Lie group.

There is the notable example provided by the general linear group $\mathrm{GL}(n;\mathbb{R})$ which is booth a smooth manifold and a group as it was shown in the previous section, then it is obvious it can be identified as a Lie group. To prove that every matrix Lie group is a Lie group, it suffices to show that a closed subgroup of a Lie group is a Lie group. This is shown in Theorem 3.11, Chapter 1 reference [4].

Definition 2.2.11. Let G and H be matrix Lie groups. A map ϕ from G to H is called a *Lie group homomorphism* if it is a group homomorphism and it is continuous. If in addition, ϕ is one-to-one and onto, and the inverse map ϕ^{-1} is continuous, then ϕ is called a *Lie group isomorphism.*

Example 2.2.1. For a matrix Lie group G with a Lie algebra $\mathfrak{g}$, which is the tangent space of G at the identity, as we will see later. For each $X \in G$ the adjoin action takes the form of a linear map $\mathrm{Ad}X : \mathfrak{g} \to \mathfrak{g}$ such that $\mathrm{Ad}X(Y) = XYX^{-1}$, according to the definition for a generic group.

Example 2.2.2. Here is a description of some of the most common matrix Lie groups. These examples are linear groups, which means that each occurs as a subgroup of $\mathrm{GL}_n(\mathbb{R})$, this time we use the notation with dimension as subscript. We also list their Lie algebras, named by the corresponding Gothic letters. The Positive General Linear Group

$$\begin{aligned}\mathrm{GL}_n^+(\mathbb{R}) &= \{A \in \mathrm{GL}_n(\mathbb{R}) \mid \det A > 0\}\,,\\ \mathfrak{gl}_n^+(\mathbb{R}) &= M_n(\mathbb{R}).\end{aligned}$$

The Orthogonal Group

$$\begin{aligned}\mathrm{O}_n(\mathbb{R}) &= \left\{A \in \mathrm{GL}_n(\mathbb{R}) \mid AA^t = 1\right\}\\ \mathfrak{o}_n(\mathbb{R}) &= \left\{A \in M_n(\mathbb{R}) \mid A + A^t = 0\right\}.\end{aligned}$$

The Orthogonal Group of signature p, q

$$\begin{aligned}\mathrm{O}_{p,q}(\mathbb{R}) &= \left\{A \in \mathrm{GL}_{p+q}(\mathbb{R}) \mid A\Sigma_{p,q}A^t = \Sigma_{p,q}\right\}\\ \mathfrak{o}_{p,q}(\mathbb{R}) &= \left\{A \in M_{p+q}(\mathbb{R}) \mid A\Sigma_{p,q} + \Sigma_{p,q}A^t = 0\right\}\end{aligned}$$

The Special Orthogonal Group

$$\begin{aligned}\mathrm{SO}_n(\mathbb{R}) &= \left\{A \in \mathrm{GL}_n(\mathbb{R}) \mid AA^t = 1, \det A = 1\right\},\\ \mathfrak{so}_n(\mathbb{R}) &= \left\{A \in M_n(\mathbb{R}) \mid A + A^t = 0\right\} = \mathfrak{o}_n(\mathbb{R}).\end{aligned}$$

The Special Orthogonal Group of signature p, q

$$\begin{aligned}\mathrm{SO}_{p,q}(\mathbb{R}) &= \left\{A \in \mathrm{GL}_{p+q}(\mathbb{R}) \mid A\Sigma_{p,q}A^t = \Sigma_{p,q}, \det A = 1\right\},\\ \mathfrak{so}_{p,q}(\mathbb{R}) &= \left\{A \in M_{p+q}(\mathbb{R}) \mid A\Sigma_{p,q} + \Sigma_{p,q}A^t = 0\right\},\end{aligned}$$

The Euclidean Group

$$\begin{aligned}\mathrm{ISO}_n(\mathbb{R}) &= \left\{\begin{bmatrix} 1 & 0 \\ v & A \end{bmatrix} \in \mathrm{GL}_{n+1}(\mathbb{R}) \,\middle|\, v \in \mathbb{R}^n,\ A \in \mathrm{SO}_n(\mathbb{R})\right\}\\ \mathfrak{iso}_n(\mathbb{R}) &= \left\{\begin{bmatrix} 1 & 0 \\ v & A \end{bmatrix} \in M_{n+1}(\mathbb{R}) \,\middle|\, v \in \mathbb{R}^n,\ A \in \mathfrak{so}_n(\mathbb{R})\right\}.\end{aligned}$$

The Positive Affine Group

$$\begin{aligned}\mathrm{Aff}_n^+(\mathbb{R}) &= \left\{\begin{bmatrix} 1 & 0 \\ v & A \end{bmatrix} \in \mathrm{GL}_{n+1}^+(\mathbb{R}) \,\middle|\, v \in \mathbb{R}^n, A \in \mathrm{GL}_n^+(\mathbb{R})\right\},\\ \mathfrak{aff}_n^+(\mathbb{R}) &= \left\{\begin{bmatrix} 1 & 0 \\ v & A \end{bmatrix} \in M_{n+1}(\mathbb{R}) \,\middle|\, v \in \mathbb{R}^n,\ A \in \mathrm{GL}_n(\mathbb{R})\right\}.\end{aligned}$$

where $\Sigma_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$ with n-dimensional identity matrix I_n.

Left and right invariant vector fields

As reported in the last example, the Lie algebra of a Lie group can be constructed starting from the vector space T_eG, which is nothing less than the tangent space to the Lie group G at its neutral element e. The following section is going into detail of this fact.

Definition 2.2.12. The *right* and *left translation* of a Lie group G are diffeomorphism of G labelled by the elements $g \in G$ and defined by

$$\begin{aligned} r_g : G &\to G & \qquad l_g : G &\to G \\ g' &\mapsto g' \cdot g & g' &\mapsto g \cdot g' \end{aligned}$$

The left and right translations satisfy the relations:

$$l_{g_1} \circ l_{g_2} = l_{g_1 \cdot g_2}\ , \quad r_{g_1} \circ r_{g_2} = r_{g_2 \cdot g_1}\ , \tag{2.30}$$

the maps $g \mapsto l_g$ and $g \mapsto r_g$ are both injective and hence define an isomorphism from G to $\text{Diff}(G)^2$.

Definition 2.2.13. Let $h : M \to N$ and let X and Y be vector fields on manifolds M and N respectively. Then X and Y are said to be *h-related* if, at all points $p \in M$:

$$h_*(X_p) = Y_{h(p)} \quad \text{and we write} \quad Y = h_* X \ . \tag{2.31}$$

As pointed out by [8], this definition arises from the problem of pushing-forward a vector field. Indeed, the induced vector field $h_* X \equiv (h_* X)_{h(p)}$ on N brings two issues with him. Firstly, if there are two points $p_1, p_2 \in M$ such that $h(p_1) = h(p_2)$ then the induced vector field will be ambiguous if $h_*(X_{p_1}) \neq h_*(X_{p_2})$. Secondly, if h is not surjective the induce vector field is not well-defined over the range of h.

Definition 2.2.14. A vector field X on a Lie group G is left-invariant if it is l_g-related to itself for all $g \in G$, i.e.,

$$l_{g*} X = X \quad \text{for all } g \in G$$

or, equivalently,

$$l_{g*}\left(X_{g'}\right) = X_{gg'} \quad \text{for all } g, g' \in G.$$

Similarly, X is right-invariant if it is r_g-related to itself for all $g \in G$, i.e.,

$$r_{g*} X = X \quad \text{for all } g \in G$$

or, equivalently,

$$r_{g^*}\left(X_{g'}\right) = X_{g'g} \quad \text{for all } g, g' \in G.$$

Proposition 2.2.4. Here we have some important properties of left invariant vector fields.

i. The set of all left-invariant vector fields on a Lie group G is denoted $\mathfrak{X}_L(G)$ and it is a real vector space, indeed, it is a vector subspace of $\mathfrak{X}(G)$ real vector space of all smooth vector fields in G.

ii. The mapping $\mathfrak{X}_L(G) \to \mathfrak{g}$, $X \mapsto X_e$ is an isomorphism.

iii. If vector fields X_1 and X_2 on a manifold M are h related to vector fields Y_1 and Y_2 respectively on a manifold N, where $h : M \to N$, then their commutators are h-related:

$$h_* \left[X_1, X_2\right]_p = \left[Y_1, Y_2\right]_{h(p)} \quad \text{for each } p \in M \ . \tag{2.32}$$

Proof. See proposition 2.3 at page 102 reference [11]. □

In particular, for left-invariant vector fields X_1 and X_2 over G

$$l_{g^*}\left[X_1, X_2\right] = \left[l_{g_*} X_1, l_{g_*} X_2\right] = \left[X_1, X_2\right] \ ,$$

---

[2] Diff(G) is the group of diffeomorphism of G onto itself

and given $g \in G$ we have for all $g' \in G$:

$$
\begin{aligned}
&l_{g^*}\left[X_1, X_2\right]_{g'} = \left[X_1, X_2\right]_{gg'} \quad \text{in case } g' = e \ , \\
&l_{g^*}\left[X_1, X_2\right]_e = \left[X_1, X_2\right]_{ge} = \left[X_1, X_2\right]_g \ .
\end{aligned}
$$

In some sense left-invariant vector fields (i.e. the Lie bracket $[\cdot,\cdot]$ in this case) do not depend on the point of the manifold where they are calculate. In particular T_eG can be considered a Lie algebra because it is a vector space as it is a tangent space to the point of a smooth manifold and it is endowed with a commutator $[\cdot,\cdot]$ given by the Lie brackets (expressed by the same symbol) applied to left invariant vector fields. We require the Lie brackets apply only on left invariant vector fields because they are well-defined only for vector fields, **not** for vector fields calculate at a fixed point on the manifold (in this case the neutral element e). This allows us to calculate $[\cdot,\cdot]$ only in T_eG because they have the same value all over the manifold G if they are applied to left invariant vector fields.

Corollary. Let G be a Lie group then its tangent space at the identity, equipped with a commutator map given by the following expression

$$
\mathfrak{X}_L(G) \to \mathfrak{X}_L(G) \ , \quad (X, Y) \mapsto [X, Y] = (\mathrm{ad}X)Y
$$

is a Lie algebra and it is denoted as follows and we refer to it as *Lie algebra* of the Lie group G.

$$
T_eG = \mathfrak{g} \ . \tag{2.34}
$$

This argument can be repeated as well with right-invariant vector fields leading to same results.

Lemma 2.2.3. The adjoint action of G Lie group on $\mathfrak{g}$ is given by $\mathrm{Ad}_g = (L_g)_*(R_{g^{-1}})_*$.

Proof. For left invariant vector fields X defined over Lie group G we get $X_e \in \mathfrak{g}$ then

$$
\begin{aligned}
&(R_{g^{-1}})_* X_e f = X_e(f \circ R_{g^{-1}}) = X(f \circ R_{g^{-1}}(e)) = X(f(eg^{-1})) = X(f(g^{-1})) = X_{g^{-1}}(f) \ , \\
&(L_g)_*(R_{g^{-1}})_* X_e f = (L_g)_* X_{g^{-1}}(f) = X_{g^{-1}}(f \circ L_g) = X(f \circ L_g(g^{-1})) = X(f(e)) = X_e(f) \ , \\
&\mathrm{Ad}_g X_e f = \mathrm{Ad}_g(Xf)(e) = (Xf)(geg^{-1}) = (Xf)(e) = X_e(f) \ .
\end{aligned}
$$

It is necessary $\mathrm{Ad}_g = (L_g)_*(R_{g^{-1}})_*$. ☐

Exponential map

It is know that for a matrix $X \in \mathrm{GL}(n, \mathbb{K})$ the exponential map is defined according to:

$$
e^X = \sum_{k=0}^{+\infty} \frac{X^k}{k!} \ , \tag{2.36}
$$

the power series converges and defines an analytic map $\mathfrak{gl}(n,\mathbb{K}) \to \mathfrak{gl}(n,\mathbb{K})$. In a similar way, we define the logarithmic map as reported below:

$$\log(I+X) = \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}X^k}{k} \ . \tag{2.37}$$

It is an analytic map defined in a neighborhood of $I \in \mathfrak{gl}(n,\mathbb{K})$. Our first goal now is to generalize this application on Lie groups, not only for the matrix groups. We cannot use power series in this case because multiplication is not well-defined in $\mathfrak{g}$. To do so we will use the notion of one-parameter subgroup intended as local flow generated by a vector $A \in \mathfrak{g}$.

Definition 2.2.15. Given a Lie group G, we call the vector space $\mathfrak{g}$ Lie algebra of G and the *exponential map* of G is defined as:

$$\begin{aligned} \exp : \mathfrak{g} \ &\to G \\ A \ &\mapsto \exp A \equiv \gamma_A(t)|_{t=1} \ , \end{aligned} \tag{2.38}$$

where γ_A is the maximal integral curve with initial point e of the left-invariant vector field X on G determined by $X(e) = A$.

Example 2.2.3. For $G \subset \mathrm{GL}(n,\mathbb{K})$ the definition agrees with the exponential map for a $n \times n$ real (or complex) invertible matrix that means $\exp(X) = e^X$ for each $X \in M_n(\mathbb{K})$. In case $G = \mathbb{R}$, so that $\mathfrak{g} = \mathbb{R}$, then for any $a \in \mathfrak{g}$, the corresponding one-parameter subgroup is $\gamma_A(t) = ta$, so that the exponential is given by $\exp(a) = a$. In case $G = S^1 = \{z \mid |z| = 1\}$ sphere of unitary module in $\mathbb{C}$, we have $\mathfrak{g} = \mathbb{R}$ and the exponential map is given by $\exp(a) = e^{2\pi i a}$.

The general linear group case is the one we are interested in more and the one we will deal with, for this reason we proceed listing some of its main properties.

Proposition 2.2.5. Let X be a $n \times n$ complex matrix then it satisfies for $t \in \mathbb{R}$:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 e^{tX} = e^{tX}X \quad \text{in particular} \quad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 e^X = X \ . \tag{2.39}$$

Proof. Differentiate the power series for e^{tX} term-by-term. We might check whether this is valid, but don't need it because for each i, j, $(e^{tX})_{ij}$ is given by a convergent power series in t, and it is a standard theorem that you can differentiate power series term-by-term. □

Proposition 2.2.6. Let G be a matrix Lie group. Then the Lie algebra of G, denoted $\mathfrak{g}$, is the set of all matrices X such that $e^{tX} \in G$ for each $t \in \mathbb{R}$ and we call it *matrix Lie algebra.*

Proof. Known exponential map maps $\mathfrak{g} \to G$, where G is a Lie matrix group. If $e^{tX} \in G \ \forall \ X \in \mathfrak{g}$ then $\mathfrak{g}$ needs to be the Lie algebra of G because it represents the domain of the exponential map $\forall \ t \in \mathbb{R}$. □

Proposition 2.2.7. Let G be a matrix Lie group, with Lie algebra $\mathfrak{g}$. Let X be an element of $\mathfrak{g}$, and A an element of G. Then AXA^{-1} is in $\mathfrak{g}$.

Proof. It is immediate because $e^{t(AXA^{-1})} = Ae^{tX}A^{-1}$ and $Ae^{tX}A^{-1} \in G$ because it is a product of $A \in G$, $A^{-1} \in G$ and $e^{tX} \in G$, which are all elements of G and every group is closed under multiplication operation. □

Theorem 2.2.4. Let G and H be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Suppose that $\phi : G \to H$ be a Lie group homomorphism. Then there exists a unique real linear map $\widetilde{\phi} : \mathfrak{g} \to \mathfrak{h}$ such that

$$\phi\left(e^X\right) = e^{\widetilde{\phi}(X)} \quad \forall\ X \in \mathfrak{g}\ .$$

The map $\widetilde{\phi}$ respect the following relation and the proof of this result is given at page 42 reference [7]:

$$\widetilde{\phi}(X) = \left.\frac{d}{dt}\right|_{t=0} \phi\left(e^{tX}\right) \quad \forall\ X \in \mathfrak{g}\ . \tag{2.40}$$

Proposition 2.2.8. Let G be a matrix Lie group, let $\mathfrak{g}$ its Lie algebra, and let $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ be the Lie group homomorphism defined above. Let $\tilde{\mathrm{Ad}} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be the associated Lie algebra map. Then

$$\tilde{\mathrm{Ad}}(X)(Y) = [X, Y] := \mathrm{ad}(X)Y \quad \forall\ X, Y \in \mathfrak{g}\ . \tag{2.41}$$

Which can be expressed in a general form for a Lie group G with the adjoint representation

$$\begin{aligned} \mathrm{ad}_g = (\mathrm{Ad}_g)_* &: \mathfrak{g} \to \mathfrak{g} \\ X \mapsto \mathrm{ad_g}\, X = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} &(g \exp(tX) g^{-1}) := \tilde{\mathrm{Ad}}_g(X)\ , \end{aligned} \tag{2.42}$$

in case G is a matrix Lie group the adjoint action is given by $\mathrm{Ad}_g(X) = gXg^{-1}$.

2.3 Maurer-Cartan Form

Definition 2.3.1. Let G be a Lie group with tangent space TG given by the disjoint union of all of the tangent spaces T_gG at each point $g \in G$. Given $v \in T_gG$ we define *Maurer-Cartan form* as

$$\begin{aligned} \omega_G &: TG \to \mathfrak{g} \\ v \mapsto \omega_G(v) &:= L_{g^{-1}}{}_*(v)\ . \end{aligned} \tag{2.43}$$

The term left-invariant refers to the fact that ω_G is invariant under left translation

$$\left(L_{h*}\omega_G\right)v = \omega_G\left(L_{h*}(v)\right) = L_{(hg)^{-1}*}\left(L_{h*}(v)\right) = L_{g^{-1}*}(v) = \omega_G(v)\ .$$

Furthermore, it is important to point out that Maurer-Cartan form is a $\mathfrak{g}$-valued (*vector-valued*) 1-form, this means that it is a differential form that returns an element of $\mathfrak{g}$, the vector space identified with the tangent space to the Lie group G at the identity e

$$\omega_G \in \Omega^1(G, \mathfrak{g})\ .$$

Example 2.3.1. If $G = \mathbb{R}$. The Maurer-Cartan form is exactly the form $\mathrm{d}x$, differential of $x \in \mathbb{R}$. If $G = (\mathbb{R}^+, \cdot)$. Here $e = 1$ and $\omega(x, v) = L_{x^{-1}*}(v) = (1/x)v = (1/x)\mathrm{d}x(v)$. Thus $\omega = (1/x)\mathrm{d}x$, where $\mathrm{d}x$ is as above but restricted to $\mathbb{R}^+$.

Example 2.3.2. If $G = S^1 = \{z \in \mathbf{C} || z \mid = 1\}$, so $T\left(S^1\right) = \left\{\left(e^{i\omega}, ire^{i\omega}\right) \mid r, \omega \in \mathbb{R}\right\}$. In particular, $\mathfrak{g} = T_e\left(S^1\right) = \{(1, ri) \mid r \in \mathbb{R}\}$. The left action of S^1 on $T\left(S^1\right)$ is given by

$$S^1 \times T\left(S^1\right) \rightarrow T\left(S^1\right).$$
$$\left(e^{i\varphi}, \left(e^{i\omega}, rie^{i\omega}\right)\right) \mapsto \left(e^{i(\omega+\varphi)}, rie^{i(\omega+\varphi)}\right).$$

Let us calculate the Maurer-Cartan form. We have

$$\omega_G\left(e^{i\omega}, ire^{i\omega}\right) = L_{e^{-i\omega}*}\left(e^{i\omega}, ire^{i\omega}\right) = (1, ir).$$

Example 2.3.3. If $G = \mathrm{GL}_n(\mathbb{R})$. The Maurer-Cartan form at a point $v \in T_g(G)$ is $\omega(v) = L_{g^{-1}*}(g, v) = \left(e, g^{-1}v\right)$. Or, identifying the $T_e(G)$ with $M_n(\mathbb{R})$, we have $\omega(v) = g^{-1}v$ for $v \in T_g(G)$. The classical way of writing the Maurer-Cartan form on $\mathrm{GL}_n(\mathbb{R})$

$$\omega_G = g^{-1}\mathrm{d}g\ , \tag{2.44}$$

where the factor g^{-1} is an abbreviation for $L_{g^{-1}*}$. If $v \in T_g\left(\mathrm{GL}_n(\mathbb{R})\right)$, then $g^{-1}\mathrm{d}g(g, v) = g^{-1}(g, v) = \left(e, g^{-1}v\right)$; namely, $g^{-1}\mathrm{d}g$ is the Maurer-Cartan form.

The relations between left-invariant vector field and Maurer-Cartan form is described below.

Lemma 2.3.1. Let G be a Lie group and X a vector field on G, the followings are equivalent:

i. $\omega_G(X)$ is constant as a $\mathfrak{g}$-valued function on G.

ii. X is a left-invariant vector field.

This is due to the fact that $\omega_G(X)$ is constant if and only if $L_{a^{-1}*}(X_a) = L_{(ga)^{-1}*}(X_{ga})$ for every $a, g \in G$, applying L_{ga_*} to both sides: $L_{g_*}(X_a) = X_{ga}$ which means X is left-invariant over G.

Lemma 2.3.2. Given a right action R_g on a Lie group G, then Maurer-Cartan form respects

$$R_g^*\, \omega_G = \mathrm{Ad}_{g^{-1}} \circ \omega_G \quad \forall\ g \in G\ . \tag{2.45}$$

Proof. In the last passage we use $\mathrm{Ad} = (L_g)_*(R_{g^{-1}})_*$ then it is clear

$$\begin{aligned} R_g^*\, \omega_{hg} &= \omega_{hg}\left(R_g\right)_* \\ &= \left(L_{(hg)^{-1}}\right)_*\left(R_g\right)_* \\ &= \left(L_{g^{-1}}\right)_*\left(L_{h^{-1}}\right)_*\left(R_g\right)_* \\ &= \left(L_{g^{-1}}\right)_*\left(R_g\right)_*\left(L_{h^{-1}}\right)_* \\ &= \mathrm{Ad}_{g^{-1}}\, \omega_h \end{aligned}$$

□

Structural equation

Given 1-forms of the form $\omega = f\mathrm{d}g$. Every 1-form is locally expressible as an exterior product of two real-valued smooth function f,g defined over a smooth manifold M:

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}f \wedge \mathrm{d}g \quad \text{where by definition} \\ &:= \mathrm{d}f \otimes \mathrm{d}g - \mathrm{d}g \otimes \mathrm{d}f \ , \end{aligned}$$

applying to both sides a pair of two smooth vector field (X, Y) we get

$$\begin{aligned} \mathrm{d}\omega(X,Y) &= \mathrm{d}f \otimes \mathrm{d}g(X,Y) - \mathrm{d}g \otimes \mathrm{d}f(X,Y) \\ &= \mathrm{d}f(X)\mathrm{d}g(Y) - \mathrm{d}g(X)\mathrm{d}f(Y) \quad \text{where by definition of vector field} \\ &:= X(f)Y(g) - Y(g)X(f) \\ &= X(f)Y(g) + f[X,Y](g) - Y(f)X(g) - f[X,Y](g) \\ &= X(f)Y(g) + fXY(g) - Y(f)X(g) - fYX(g) - f[X,Y](g) \\ &= X(fY(g)) - Y(fX(g)) - f[X,Y](g) \\ &= X(fdg(Y)) - Y(fdg(X)) - fdg([X,Y]) \\ &= X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \end{aligned}$$

Lemma 2.3.3. Let ω be a V-valued 1-form on the smooth manifold M, and let X and Y be two vector fields on M. Then $\mathrm{d}\omega(X,Y) = X(\omega(Y))- Y(\omega(X)) - \omega([X,Y])$.

In case we consider the Maurer-Cartan form ω_G as 1-form, a Lie group G as smooth manifold and we choose $(X,Y) \in \mathfrak{X}_L(M)$, so that they are left-invariant and smooth over G, by a previous lemma $\omega_G(X)$, $\omega_G(Y)$ need to be constant over G. This implies that if we replace these terms in the previous equation:

$$X(\omega_G(Y)) = 0 \quad \text{and} \quad Y(\omega_G(X)) = 0 \ . \tag{2.48}$$

Given $g \in G$ such that X, Y, $[X,Y] \in T_gG$ then

$$\begin{aligned} \mathrm{d}\omega_G(X,Y) &= X(\omega_G(Y)) - Y(\omega_G(X)) - \omega_G([X,Y]) \\ &= -L_{g^{-1}}{}_*([X,Y]) = -[X,Y]L_{g^{-1}} \\ &= -[X,Y]_e \quad \text{where } X_e, Y_e, [X,Y]_e \in T_eG \\ &= -[X_e, Y_e] \quad \text{where } X_e = X \cdot L_{g^{-1}} = L_{g^{-1}}{}_* X = \omega_G(X) \\ &= -[\omega_G(X), \omega_G(Y)] \end{aligned}$$

This yields to the following equation, know as *structural equation*, this is satisfied for a Maurer-Cartan form defined over a Lie group G and a pair of smooth vector fields X, Y in the tangent bundle of G:

$$\boxed{\mathrm{d}\omega_G(X,Y) + [\omega_G(X), \omega_G(Y)] = 0} \tag{2.50}$$

This is an equation for $\mathfrak{g}$-valued 2-forms indeed both terms are elements of $\Omega^2(M, \mathfrak{g})$, which is the set of tensor fields defined over the smooth manifold M given by $\mathfrak{g}$-valued 2-forms.

Lemma 2.3.4. Given two 1-forms ω_1, ω_2 then their commutator is a 2-form that respect for every pair of vector fields X, Y:

$$[\omega_1, \omega_2](X,Y) = [\omega_1(X), \omega_2(Y)] + [\omega_2(X), \omega_1(Y)] \ , \tag{2.51}$$

in particular if $\omega_1 = \omega_2$ we get

$$[\omega(X), \omega(Y)] = \frac{1}{2}[\omega, \omega](X,Y) \tag{2.52}$$

Proof. known $\Omega^p(M, \mathfrak{g}) := \Omega^p(M) \otimes \mathfrak{g}$ is the set of the $\mathfrak{g}$-valued p-forms defined over a smooth manifold M, which are p-rank covariant completely anti-symmetric tensors, and the exterior derivative d is an application which respects

$$\mathrm{d} : \Omega^p(M, \mathfrak{g}) \to \Omega^{p+1}(M, \mathfrak{g}) \tag{2.53}$$

such that, for instance, $x \mapsto \mathrm{d}\,x$ for real x and $\mathrm{d}\,x \in \Omega^1(\mathbb{R})$ and $\omega_G \mapsto \mathrm{d}\,\omega_G$ for Maurer-Cartan 1-form $\omega_G \in \Omega^1(M, \mathfrak{g})$ and $\mathrm{d}\,\omega_G \in \Omega^2(M, \mathfrak{g})$. Then we define the multiplication given by the Lie bracket of two vector in a finite-dimensional real vector space V

$$\begin{aligned} \mathrm{m} \equiv [\cdot, \cdot] : V \otimes V \to V \\ X \otimes Y \mapsto m(X \otimes Y) \equiv [X, Y] \ , \end{aligned} \tag{2.54}$$

In our case $V = \mathfrak{g}$ matrix Lie algebra of the matrix Lie group G we have for a $\mathfrak{g}$-valued one-form field over M the generators $\mathfrak{t}_i \in \mathfrak{g}$ and $A^i \in \Omega^2(M, \mathfrak{g})$ such that $A = A^i \mathfrak{t}_i$, then

$$\begin{aligned} \mathrm{m} \equiv [\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \\ \mathfrak{t}_i \otimes \mathfrak{t}_j \mapsto \mathrm{m}(\mathfrak{t}_i \otimes \mathfrak{t}_j) \equiv [\mathfrak{t}_i, \mathfrak{t}_j] \ , \end{aligned} \tag{2.55}$$

which is well-defined because the Lie bracket of two vector fields is a vector field too. Now, we introduce the wedge product given a generci finite-dimensional vector space V as

$$\wedge : \Omega^p(M, V) \times \Omega^q(M, V) \to \Omega^{p+q}(M, V) \ , \tag{2.56}$$

which becomes in our case

$$\begin{aligned} \wedge : \Omega^1(M, \mathfrak{g}) \times \Omega^1(M, \mathfrak{g}) \to \Omega^2(M, \mathfrak{g}) \\ (A, B) \mapsto A \wedge B \\ (A^i \mathfrak{t}_i, B^j \mathfrak{t}_j) \mapsto A^i \wedge B^j \mathfrak{t}_i \mathfrak{t}_j \ . \end{aligned} \tag{2.57}$$

At this point we can define a multiplication $\mathrm{m}_* \equiv [\cdot,\cdot]$ that allows us to use Lie bracket for p-forms

$$\begin{aligned} \mathrm{m}_* : \Omega^p(M,V) &\to \Omega^p(M,V) \\ \omega &\mapsto \mathrm{m}_*(w) \end{aligned} \tag{2.58}$$

such that its action on a set of p vectors $\{X_1,...,X_p\} \in V$ gives:

$$(\mathrm{m}_*\,\omega)(X_1,...,X_p) = \mathrm{m}(\omega(X_1,...,X_p)) \ , \tag{2.59}$$

in particular these operators satisfies the composition

$$\Omega^p(M,V) \times \Omega^q(M,V) \xrightarrow{\wedge} \Omega^{p+q}(M,V) \xrightarrow{\mathrm{m}_*} \Omega^{p+q}(M,V) \ . \tag{2.60}$$

In our case, where $V = \mathfrak{g}$ we get:

$$\begin{aligned} \mathrm{m}_* : \Omega^2(M,\mathfrak{g}) &\to \Omega^2(M,\mathfrak{g}) \\ A \wedge B &\mapsto \mathrm{m}_*(A \wedge B) := [A,B] \end{aligned} \tag{2.61}$$

such that $\Omega^1(M,\mathfrak{g}) \times \Omega^1(M,\mathfrak{g}) \xrightarrow{\wedge} \Omega^2(M,\mathfrak{g}\otimes\mathfrak{g}) \xrightarrow{\mathrm{m}_*} \Omega^2(M,\mathfrak{g})$. By the following calculations we derive the relation we claimed in this lemma, where given ω_1, $\omega_2 \in \Omega^1(M,\mathfrak{g})$, $[\omega_1,\omega_2] \in \Omega^2(M,\mathfrak{g})$ it implies

$$\begin{aligned} [\omega_1,\omega_2](X,Y) &= (\mathrm{m}_*(\omega_1\wedge\omega_2))(X,Y) \quad \text{by definition of } \mathrm{m}_* \\ &= \mathrm{m}((\omega_1\wedge\omega_2)(X,Y)) \quad \text{by definition of } \mathrm{m} \\ &= \mathrm{m}((\omega_1\otimes\omega_2 - \omega_2\otimes\omega_1)(X,Y)) \\ &= \mathrm{m}(\omega_1(X)\otimes\omega_2(Y) - \omega_2(X)\otimes\omega_1(Y)) \\ &= \mathrm{m}(\omega_1(X)\otimes\omega_2(Y)) - \mathrm{m}(\omega_1(X)\otimes\omega_2(Y)) \quad \text{due to linearity of } \mathrm{m} \equiv [\cdot,\cdot] \\ &= [\omega_1(X),\omega_2(Y)] - [\omega_2(X),\omega_1(Y)] \\ &= [\omega_1(X),\omega_2(Y)] + [\omega_1(Y),\omega_2(X)] \quad \text{due to skew-symmetry of } [\cdot,\cdot] \ . \end{aligned}$$

□

For this reason, another way to express the structural equation is:

$$\boxed{\mathrm{d}\omega_G + \frac{1}{2}[\omega_G,\omega_G] = 0} \tag{2.63}$$

and in case G is an Abelian group, the second term of the equation vanishes

$$\mathrm{d}\omega_G = 0 \ . \tag{2.64}$$

Chapter 3

Gauge theory on fiber bundles

3.1 Fiber bundles

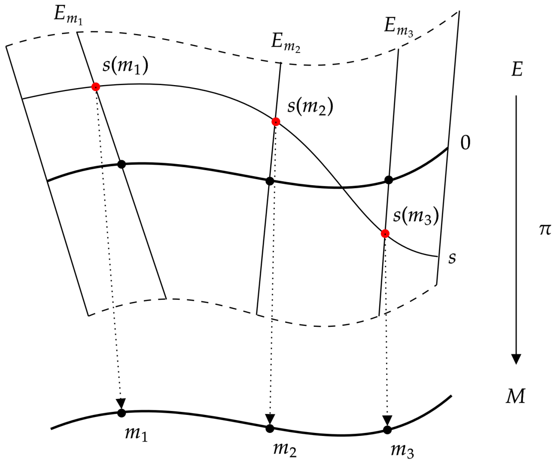

Figure 3.1: Graphic example of a fiber bundle $\pi : E \to M$ with abstract fiber F, fibers E_{m_i} over the total space E and sections $s(m_i)$ taken by reference [1].

Topological and smooth bundles

Definition 3.1.1. Let F be a topological space and $\pi : E \to B$ a continuous and surjective map between manifolds. We call the quadruple $\xi = (E, B, \pi, F)$ a *locally trivial fiber bundle* with *abstract fiber* F if it satisfies the condition of *local triviality*. This means that for each point $b \in B$, there is an open set $U \subset B$ containing b such that $\pi^{-1}(U)$ is homeomorphic to $U \times F$ by a homeomorphism ϕ, named *local trivialisation*, such that the following diagram commutes.

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "proj_1"] \\
& U &
\end{tikzcd} \tag{3.1}$$

The pair (U, ϕ) is called a *chart* or *local bundle coordinate system.* The fibers over B are denoted by $\pi^{-1}(b) = E_B$ for each $b \in B$ they are submanifolds of B, in particular E_B is homeomorphic to F for all $b \in B$ thanks to the local trivialization ϕ. B is called the *base space* and E the *total space* of the bundle. Given a bundle ξ, we sometimes denote by $B(\xi), E(\xi)$, etc., its corresponding parts. If the base, fiber, and total spaces are smooth manifolds, π is a smooth map, and the local trivialization is a diffeomorphism, then we have a *smooth fiber bundle.*

Lemma 3.1.1. Given two trivializations (U, ϕ) and (V, ψ) of a smooth fiber bundle with $U \cap V \neq \emptyset$, then we can find a smooth map named *transition function* h such that for every $(p, f) \in (U \cap V) \times F$

$$h := \phi \circ \psi^{-1} : U \cap V \to \mathrm{Diff}(F) \tag{3.2a}$$

$$(p, f) \mapsto h(p, f) = \phi \circ \psi^{-1}(p, f) = (p, h(p) \cdot f) \ , \tag{3.2b}$$

where we denoted by $h(p) \cdot f$ the action of an element of $\mathrm{Diff}(F)$ on an arbitrary element $f \in F$.

Proof. We know that $\mathrm{pr}_1 \circ \phi = \pi = \mathrm{pr}_1 \circ \psi$ thus the following diagram commutes

$$\begin{tikzcd}
(U \cap V) \times F \arrow[r, "\psi^{-1}"] \arrow[dr, "\mathrm{pr}_1"'] & \pi^{-1}(U \cap V) \arrow[r, "\phi"] \arrow[d, "\pi"] & (U \cap V) \times F \arrow[dl, "\mathrm{pr}_1"] \\
& U \cap V &
\end{tikzcd} \tag{3.3}$$

then we can conclude for some $\nu : (U \cap V) \times F \to F$ smooth by construction

$$\mathrm{pr}_1 \circ \phi \circ \psi^{-1} = \mathrm{pr}_1 \longrightarrow \phi \circ \psi^{-1}(p, f) = (p, \nu(p, f)) \ .$$

Fixed $p \in M$ we know that the trivialization map is a linear isomorphism thus is $\nu(p, f) = h(p) \cdot f$ and $f(p) \in \mathrm{Diff}(F)$. Smoothness of ν easily implies smoothness of h. $\square$

Definition 3.1.2. Let $\xi = (E, B, \pi, F)$ be a fiber bundle. A *local section* over $U \subset B$ is a continuous (or smooth in the case of a smooth bundle) map σ

$$\sigma : U \to E \quad \text{such that} \quad \pi \circ \sigma = \mathrm{id}_B \ .$$

The *global section* $\Gamma(\eta)$ is defined as the set of all (smooth) sections of a fiber bundle η and it represents the case when $U = B$. In addition to this, it is common to identify a section of a fiber bundle ξ at a point $p \in B$, in a given trivialization ϕ, with an element of its abstract fiber F, that is

$$\sigma^{\phi}(p) := \phi \circ \sigma(p) = (p, f) \sim f \in F \ . \tag{3.5}$$

Since the trivialization point-wise is an isomorphism, we can conclude local frames $\sigma(p)$ are in $1-1$ correspondence with the trivialization ϕ.

Definition 3.1.3. Let ξ_1 endowed with $\pi_1 : E_1 \to M$, ξ_2 endowed with $\pi_2 = E_2 \to N$ be two fiber bundles over the same fiber F. A *bundle map* is a pair (φ, ψ) of smooth maps between manifolds

$\varphi : M \to N$, $\psi = E_1 \to E_2$ such that the following diagram commutes and φ is fiber preserving.

$$\begin{tikzcd} E_1 \arrow[r, "\psi"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd} \quad \text{and} \quad \psi \circ \pi_2 = \varphi \circ \pi_1 \ .$$

Since π_1 is surjective, φ is uniquely determined by ψ, which is said to *cover* φ.

Definition 3.1.4. Two smooth bundles ξ_1 and ξ_2 over the same base B are said to be *isomorphic* if there is a diffeomorphism φ between the total spaces such that the following diagram commutes.

$$\begin{tikzcd} E_1 \arrow[rr, "\varphi", "\simeq"'] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & B & \end{tikzcd}$$

φ is called a *bundle isomorphism* or, in the case when $\xi_1 = \xi_2$ a *bundle automorphism.*

Example 3.1.1. The simplest example of a bundle is the product bundle $E = B \times F$ with $\pi = \mathrm{pr}_1$ (projection on the base space). This bundle is also called the *trivial bundle.* If E is a locally compact Hausdorff space, then a necessary and sufficient condition that a bundle $\xi = (E, B, \pi, F)$ be isomorphic to the trivial bundle is the existence of a continuous (smooth) map $t : E \to F$ that induces a homeomorphism (diffeomorphism) upon restriction to each fiber $\pi^{-1}(b)$. In this case the *trivialization* map $\phi = (p, t) : E \to B \times F$ is a homeomorphism (diffeomorphism) commuting with the canonical projection to B.

Example 3.1.2. Perhaps the simplest example of a nontrivial bundle is the *Möbius band* E pictured below. It is clear that if we remove any point from the base B, we get an open set U homeomorphic to the interval $(0, 1)$ and $\pi^{-1}(U)$ is homeomorphic to $(0, 1) \times [0, 1]$. Thus it is a bundle over $B = S^1$ with fiber $I = [0, 1]$. But it is not a trivial bundle; if it were, the boundary would consist of two components, whereas in fact it has only one.

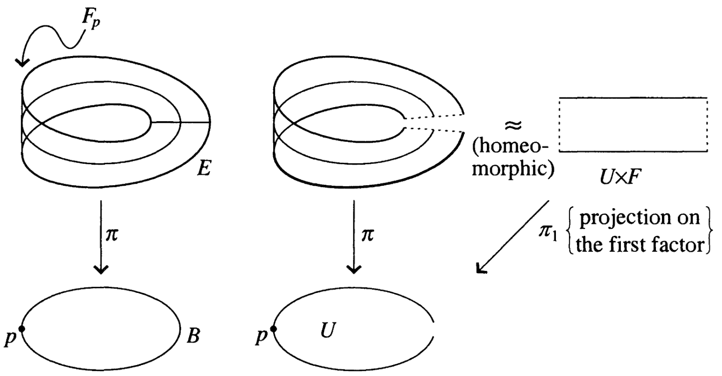

Figure 3.2: Graphic representation of a Möbius-type bundle take by [11] page 30, the total space E is identified with the Möbius band and the fibers over it are called F_p

G-Bundles

Now we consider the following, refined, notion of fiber bundle where a Lie group G appears.

Definition 3.1.5. Let $\xi = (E, B, \pi, F)$ be a smooth fiber bundle, and suppose that G is a Lie group that acts smoothly on F as a group of diffeomorphisms. A *G-atlas* for ξ is a collection $\mathcal{A} = \{(U_i, \varphi_i)\}$ of charts for ξ such that

i. the U_i cover B,

ii. for each pair of charts (U_i, φ_i) and (U_j, φ_j) in $\mathcal{A}$, the *coordinate change* $\Phi_{ij} := \varphi_i \circ \varphi_j^{-1}$ takes a form dependent from the *transition function* $h_{ij} : U_i \cap U_j \to G$ which is a smooth map such that:

$$\begin{aligned} \Phi_{ij} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F \\ (u, f) \mapsto \Phi_{ji}(u, f) = (u, h_{ij}(u) f) \ . \end{aligned} \tag{3.6}$$

Note that in the case when the homomorphism $G \to \mathrm{Diff}(F)$ has a kernel H, then H is a closed normal subgroup and the action factors through G/H, so the bundle may also be regarded as a G/H bundle.[1] If we make the restriction that G acts effectively on F, namely, that $H = \{e\}$ neutral element singlet, then it implies $\mathrm{Diff}(F) = G$. In this case we may speak of an *effective* G-bundle. Just as in the case of manifolds, we call two G-atlases *equivalent* if their union is also a G-atlas.

Definition 3.1.6. A *G-structure* $[G]_{\sim_\xi}$ on a smooth fiber bundle ξ is an equivalence class of G atlases on ξ where the equivalence relation $\sim_\xi$ is given by the G-atlases referred to the same fiber bundle ξ . Meanwhile, a *G-bundle* is a smooth fiber bundle with a specified G-structure i.e. a pair $(\xi, [G]_{\sim_\xi})$. In particular, a G-bundle is *flat* if all the transition functions $h : U \cap V \to G$ are constant.

Example 3.1.3. Let $\xi = (E, N, \pi_E, F)$ be a smooth G-bundle and let $f : M \to N$ be a smooth map. We define $f^*(\xi) = (f^*E, M, \pi, F)$ *induced* or *pullback bundle* with projection $\pi(p, e)$ and total space:

$$f^*E = \{(p, e) \in M \times E \mid f(p) = \pi_E(e)\} \ . \tag{3.7}$$

Restricting the canonical projections from $M \times E$ to f^*E we get the following maps $\pi : f^*E \to M$ and $\Phi : f^*E \to E$ making this diagram commutes:

$$\begin{tikzcd} f^*E \arrow[r, "\Phi"] \arrow[d, "\pi"'] & E \arrow[d, "\pi_E"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Taking $p \in M$, and (V, ψ) a local trivialisation for $E \to N$ with $f(p) \in V$, then $(f^{-1}(V), \varphi)$ with $\varphi : \pi^{-1}(f^{-1}(V)) \to f^{-1}(V) \times F$ defined by $\varphi(b, e) = (b, \mathrm{pr}_1(\psi(e)))$ which is a local trivialization for $f^*E \to M$. This show (f^*E, M, π, F) is a smooth fiber bundles with fibers $(f^*E)_p = E_{f(p)}$.

---

[1] H and G/H are again Lie groups

Construction of bundles

We return on the definition of G-bundle to see what ingredients are required for the construction of it. From the definition, given an effective G-bundle we have necessarily

i. a Lie group G acting smoothly on a smooth manifold F,

ii. an open covering $\{U_\alpha\}$ of a manifold B,

iii. smooth maps $h_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ satisfying *cocycle condition*, which means that if the intersection $U_\alpha \cap U_\beta \cap U_\gamma \neq 0$ then holds $h_{\gamma\beta} h_{\beta\alpha} = h_{\gamma\alpha}$.

The latter property is a consequence of the existence of the effective G-bundle in hypothesis and may be seen as follows. We may assume that the open sets U_α are index by the trivializations

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F \ .$$

The coordinate changes are $\varphi_\beta \circ \varphi_\alpha^{-1} \equiv \Phi_{\beta\alpha}$ such that

$$\begin{aligned} \Phi_{\beta\alpha} : (U_\alpha \cap U_\beta) \times F &\to (U_\alpha \cap U_\beta) \times F \\ (u, f) &\mapsto \Phi_{\beta\alpha}(u, f) = (u, h_{\alpha\beta} f) \end{aligned}$$

Thus, the obvious identity $\Phi_{\gamma\beta}\Phi_{\beta\alpha} = \Phi_{\gamma\alpha}$ is equivalent to the cocycle condition $h_{\gamma\beta}h_{\beta\alpha} = h_{\gamma\alpha}$.

Remark. This procedure can be reversed such that, if we are given the three data listed above, we can construct a G-bundle by forming the disjoint union of the products $\{U_\alpha \times F\}$ and dividing by the equivalence relation generated by making the identification

$$\begin{aligned} \{U_\alpha \times F\} \cup \{(U_\alpha \cap U_\beta) \times F\} &\to \{U_\beta \times F\} \cup \{(U_\alpha \cap U_\beta) \times F\} \\ (u, f) &\mapsto (u, h_{\alpha\beta} f) \ . \end{aligned}$$

This procedure yields a smooth G-bundle over B with fiber F given by

$$(E, B, \pi, F) \quad \text{with total space} \quad E = \sqcup_\alpha (U_\alpha \times F)/ \sim \ , \tag{3.9}$$

where the equivalence relation is given by the previous identification $(u, f) \sim (u, h_{\alpha\beta} f)$.

Principal bundles

A special type of G-bundle is the one where the group G and the fiber F are related by a diffeomorphism $\varphi : G \to F$, sending $g \mapsto g \cdot f_0$ for any $f_0 \in F$. It follows that G-bundle is effective, so the transition maps $h : U \cap V \to G$ are determined by the bundle. Note that the diffeomorphism $G \to F$ does not yield a canonical identification of G with the abstract fiber F, because the bijection will vary with the choice of $f_0 \in F$. When we have chosen such an identification, we can reconstruct the bundle using G itself as the abstract fiber together with the left action of G on itself, as reported in the commutative diagram

$$\begin{tikzcd}
U \times F \supset & (u, gf_0) \arrow[rrr, mapsto, "\text{change of coordinate } \Phi"] & & & (u, h(u)gf_0) & \subset V \times F \\
& & (U \cap V) \times F \arrow[r, "\Phi"] & (U \cap V) \times F & & \\
& & (U \cap V) \times G \arrow[r, "\Phi"'] \arrow[u, "\mathrm{id} \times \varphi"] & (U \cap V) \times G \arrow[u, "\mathrm{id} \times \varphi"'] & & \\
U \times G \supset & (u, g) \arrow[rrr, mapsto, "\text{change of coordinate } \Phi"'] \arrow[uuu, mapsto, "\mathrm{id} \times \varphi"] & & & (u, h(u)g) \arrow[uuu, mapsto, "\mathrm{id} \times \varphi"'] & \subset V \times G
\end{tikzcd}$$

This diagram further implies that a principal G-bundle has a smooth right G-action, as may be seen by comparing the coordinate changes before and after we identify the fiber with G. The right G-action commutes with coordinate changes, which themselves involve only the left G-action.

Definition 3.1.7. A *principal* G-bundle is a fiber bundle $\xi = (P, B, \pi, F)$ together with a continuous right action of a topological group G on the total space P, such that

$$\begin{aligned} r : P \times G &\to P \\ (p, g) &\mapsto r_g(p) = p \cdot g \end{aligned} \tag{3.10}$$

it is fiber preserving $\pi \circ r_g = \pi$, which means

$$(\pi \circ r_g)(p) = \pi(r_g(p)) = \pi(p) \quad \forall\, p \in P \ ; \tag{3.11}$$

it acts simply transitively on each fiber $P_b \equiv \pi^{-1}(b)$, which means the right action is transitive

$$\forall\, p, q \in P\ \exists\, g \in G : p \cdot g = q$$

and free, so that the stabilizer of the group is trivial, which implies effectiveness too:

$$G_r(p) = \{p \mid p \in P,\ r_g(p) = p\} = \{e\} \implies \ker(r) = \{e\} \ .$$

One often abbreviates the above data by saying that a principal bundle is given by notation:

$$\begin{tikzcd} P \arrow[d, "\pi"] & \text{or} & G \arrow[r] & P \arrow[d, "\pi"] \\ B & & & B \end{tikzcd}$$

The definition above is for arbitrary topological spaces. One can also define principal G-bundles over smooth manifolds. Here $\pi : P \to B$ is required to be a smooth map between smooth manifolds, G is required to be a Lie group and the corresponding action on P should be smooth, which determines ξ to be a smooth fiber bundle. In the following chapters we will adopt this convention.

Example 3.1.4. Since the fibers of a principal G-bundle look just like G, they inherit a Maurer-Cartan form in a natural way. Explicitly, the action of G on a principal right G-bundle P is such that, if P_x is any fiber and $y \in P_x$, the map

$$G \to P_x$$

$$g \mapsto yg$$

is invertible. The inverse map lets us pull the Maurer-Cartan form back to P_x in a unique way:

$$T_{xg}P_x \to T_gG \to \mathfrak{g} \ .$$

Because of this canonical construction, the 1-form thus obtained on P is a Maurer-Cartan form ω_G.

Definition 3.1.8. Given an open cover $\{U_\alpha\}$ of the base space B, we define a G-equivariant map $g_\alpha : \pi^{-1}(U_\alpha) \to G$ as a fiberwise[2] diffeomorphism for which given $p \in \pi^{-1}(U_\alpha)$:

$$g_\alpha(pg) = g_\alpha(p)g \quad \forall\ g \in G. \tag{3.12}$$

Theorem 3.1.2. (*Theorem of characterization of principal bundles*) let P be a smooth manifold, G a Lie group and $\mu : P \times G \to P$ a smooth, free and proper right action then

i. P/G with the quotient topology is a topological manifold,

ii. P/G has a unique smooth structure,

iii. $\xi = (P, P/G, \pi, G)$ is a smooth principal right G-bundle.

Proof. Given in appendix E of reference [11]. □

Corollary. Here we have listed some of the main properties of a principal G-bundles.

i. A principal G-bundle posses a canonical effective G-bundle structure.

ii. Fibers of a principal G-bundles are G-orbits of the continuous right action r_g.

iii. The coset space P/G is homeomorphic to the base space B, such that $P/G \cong B$.

iv. Local triviality implies that the local trivializations $\phi : \pi^{-1}(U) \to G$ are G-equivariant, which means that for some G-equivariant fiberwise diffeomorphism $g : \pi^{-1}(U) \to G$

$$\phi(p) = (\pi(p), g(p)) = (\pi, g)(p) \ . \tag{3.13}$$

Proposition 3.1.1. A principal G-bundle is said *trivial* if there is a G-equivariant diffeomorphism

$$\psi : P \to B \times G \tag{3.14}$$

and principal G-bundle (P, B, π, F) admits global sections if and only if it is trivial.

---

[2] i.e. an application defined over the fibers $P_b = \pi^{-1}(b)$ for each $b \in B$.

Proof. If $\pi : P \to B$ is trivial, $\psi : P \to B \times G$ defines a section

$$\begin{aligned} \sigma : B &\to P \\ b &\mapsto \sigma(b) = \psi^{-1}(b, e) \end{aligned}$$

Conversely, if σ is a section, it defines ψ by $\psi(p) = (\pi(p), \chi(p))$ where $\chi(p)$ is uniquely defined by $p = \sigma(\pi(p))\chi(p)$, then $p \cdot g = \sigma(\pi(p))\chi(p) \cdot g = \sigma(\pi(pg))\chi(p)g$ so $\chi(pg) = \chi(p)g$. ☐

Proposition 3.1.2. Given two open sets U_α, U_β of an open cover of the base space B for a principal G-bundle with neutral element e for the group structure, such that there is a non-empty overlaps $U_{\alpha\beta} = U_\alpha \cup U_\beta$. Then, the transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ obey the cocylce conditions:

$$g_{\alpha\beta}(p) = g_{\beta\alpha}(p) = e \quad \forall\, p \in U_{\alpha\beta} \ , \tag{3.15a}$$

$$g_{\alpha\beta}(p)g_{\beta\gamma}(p)g_{\gamma\alpha}(p) = e \quad \forall\, p \in U_{\alpha\beta} \ . \tag{3.15b}$$

Proof. Since principal fibre bundles are locally trivial, they admit local sections. Let $\{(U_\alpha, \varphi_\alpha)\}$ be a trivialising atlas for $G \to P \xrightarrow{\pi} M$. The canonical local sections $\sigma_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$ are given by

$$\sigma_\alpha(a) = \varphi_\alpha^{-1}(a, e) \ , \tag{3.16}$$

where e is the identity element and on $U_{\alpha\beta}$ we have sections $\sigma_\alpha, \sigma_\beta$. Writing $\varphi_\alpha(p) = (\pi(p), g_\alpha(p))$ for $g_\alpha : U_\alpha \to G$ equivariant we have that for $p \in \pi^{-1}(U_{\alpha\beta})$:

$$\begin{aligned} (\pi(p), g_\alpha(p)) = \varphi_\alpha(p) = (\varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\beta)(p) = (\varphi_\alpha \circ \varphi_\beta^{-1})(\pi(p), g_\beta(p)) \\ \Rightarrow \quad (\pi(p), g_\alpha(p)g_\beta^{-1}(p)g_\beta(p)) = (\varphi_\alpha \circ \varphi_\beta^{-1})(\pi(p), g_\beta(p)) \end{aligned}$$

Note that by G-equivariance $\hat{g}_{\alpha\beta}(pg) = g_\alpha(pg)g_\beta^{-1}(pg) = g_\alpha(p)gg^{-1}g_\beta^{-1}(p) = \hat{g}_{\alpha\beta}(p)$ and so it is constant along the fibres. Hence $\exists\ g_{\alpha\beta} : U_{\alpha\beta} \to G$ such that $\hat{g}_{\alpha\beta} = \pi^* g_{\alpha\beta}$ and $(\varphi_\alpha \circ \varphi_\beta^{-1})(a, g) = (a, g_{\alpha\beta}(a)g)$. It follows that the $g_{\alpha\beta}$ obey the cocycle conditions. ☐

Lemma 3.1.3. On double overlaps $U_{\alpha\beta}$, the canonical sections σ_α are related by

$$\sigma_\beta(a) = \sigma_\alpha(a)g_{\alpha\beta}(a) \tag{3.18}$$

Proof. Now note $g_\alpha \circ \sigma_\alpha : U_\alpha \to G$ is a constant map taking value e, and so letting $p = \sigma_\beta(a)$

$$\begin{aligned} g_\alpha(p) = \hat{g}_{\alpha\beta}(p)g_\beta(p) \quad \Rightarrow \quad g_\alpha\left(\sigma_\beta(a)\right) &= g_{\alpha\beta}(a)\left(g_\beta \circ \sigma_\beta\right)(a) \\ &= \left(g_\alpha \circ \sigma_\alpha\right)(a)g_{\alpha\beta}(a) \\ &= g_\alpha\left(\sigma_\alpha(a)g_{\alpha\beta}(a)\right) \end{aligned}$$

then $\sigma_\beta(a) = \sigma_\alpha(a)g_{\alpha\beta}(a)$ as g_α is a diffeomorphism. ☐

Vector bundles

A special case of a G-bundle is a *real vector bundle.* In this case the fiber is a real finite-dimensional vector space V and the group is the general linear group $G = \mathrm{GL}(V)$,

$$\xi = (E, B, \pi, V) \quad \text{endowed with} \quad [\mathrm{GL}(V)]_{\sim_\xi} \quad \text{as } G\text{-structure} \ .$$

An *isomorphism of vector bundles* is an isomorphism of bundles that is linear on the fibers. Since the action of G on V is transitive and effective, it follows that we may pass back and forth between the vector bundle and the principal G-bundle ξ. In addition to this, given any smooth linear representation $\rho : G \to \mathrm{GL}(W)$, where W is another finite dimensional vector space, we can pass from the original vector bundle to the associated principal bundle ξ and then to the G-bundle with fiber W, denote $\xi \times_\rho W$. In fact, the fibers of the canonical map

$$\xi \times W \to \xi \times_\rho W \ , \tag{3.19}$$

are just the orbits of the left G-action $G \times \xi \times W \to \xi \times W$, given by

$$g \cdot (x, w) = (xg^{-1}, \rho(g)w) \ . \tag{3.20}$$

Proposition 3.1.3. A vector bundle always admits sections, they are global sections named *zero sections.*

Proof. Every fiber E_b has the structure of a vector space and therefore has a zero element 0_b. Sending an element $b \in U_\alpha$ base space to the zero vector 0_b through each trivialization $\phi_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$ defines the continuous zero section

$$\sigma(b) = \phi_\alpha^{-1}(b, 0) \overset{!}{=} 0_b \ . \tag{3.21}$$

This argument cannot be applied to principal G-bundles because their fibers don't have necessarily a vector space structure, this is the reason why we require their triviality to admit sections. □

Definition 3.1.9. Given M, E a pair of smooth manifolds and a smooth surjective map $\pi : E \to M$ if $E_p \equiv \pi^{-1}(p)$ fibers over the points $p \in M$ are real vector space of k-dimension and for every $p \in M$ there is a neighborhood U of p and a diffeomorphism ϕ such that

$$\phi|_{\pi^{-1}(p)} : E_p \to \{p\} \times \mathbb{R}^k \tag{3.22}$$

is a linear isomorphism and $\mathrm{pr}_1 \circ \phi = \pi$ with a commutative diagram

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\phi"] \arrow[d, "\pi"'] & U \times \mathbb{R}^k \arrow[dl, "\mathrm{pr}_1"] \\
U &
\end{tikzcd}$$

The diffeomorphism ϕ satisfying these requirements is called *local trivialization* and the set of data $(E, M, \pi, \mathbb{R}^k)$ *smooth vector bundle* of rank k; usually we refer to it as $\pi : E \to M$ or directly as E.

Definition 3.1.10. A *local (global) frame* is a set of local (global) sections $\{\sigma_1, ..., \sigma_k\}$ such that for each point $p \in U$ (or every point in M), the set $\{\sigma_1(p), ..., \sigma_k(p)\}$ is a basis for E_p . In addition to this, we define *frame bundle* the vector bundle where the total space is given by the disjoint union of the set of all local frames F_p for each $p \in M$

$$(FM, M, \pi, V) \quad \text{with total space} \quad FM = \sqcup_p F_p M \ .$$

It is a vector bundle of rank k, indeed FM is a smooth manifold, π is a smooth surjective map and it can be equipped with a canonical set of local trivializations in $\mathbb{R}^k$ finite-dimensional vector space and abstract fiber of the bundle.

Remark. As we have seen in the general case, it is common to identify a section of a rank k real vector bundle at a point $p \in M$, in a given trivialization, with an element of $\mathbb{R}^k$, that is

$$\sigma^{\phi}(p) := \phi \circ \sigma(p) = (p, \mathbf{v}) \sim \mathbf{v} \in \mathbb{R}^k \ . \tag{3.23}$$

Since the trivialization point-wise is an isomorphism, we can conclude local frames $\sigma(p)$ are in $1-1$ correspondence with the trivialization ϕ.

As we have seen in the fiber bundle construction theorem, the disjoint union of products $U_\alpha \times F$ returns a G-bundle. Another example of this is the so-called tangent bundle.

Definition 3.1.11. A *tangent bundle* is a smooth vector bundle whose total space is given by the disjoint union of the tangent spaces at each point $p \in M$ on a smooth manifold:

$$(TM,\ M,\ \pi,\ \{(U, \phi)\}) \quad \text{with total space} \quad TM = \sqcup_p\ T_p M \ . \tag{3.24}$$

In particular, we can associate a frame bundle to the tangent bundle, in this case the frame bundle of a tangent bundle is said *tangent frame bundle* defined as previously but with total space given by the disjoint union of all F_p sets of frames of the tangent spaces T_pM to each point $p \in M$. In this case we have used the disjoint union of the point-wise fibers T_pM, instead of $U_\alpha \times F$, because they are isomorphic to each other thanks to the trivialization ϕ and, in addition to this, it is a vector bundle of rank k because it satisfies these three points: TM is a smooth manifold, $\pi : TM \to M$ is a smooth surjective map, it can be equipped with a canonical set of local trivializations $\{(U, \phi)\}$ mapping U in $\mathbb{R}^k$.

Remark. From this perspective, a *smooth vector field* is a section of TM. Sometimes one might be interested in maps from M to TM not necessary smooth or continuous but here, and afterwards, we will work only with smooth maps, thus smooth sections, of TM.

$$\sigma := X : M \to TM \tag{3.25a}$$

$$p \mapsto X_p \tag{3.25b}$$

The standard notation for the global section of M is $\Gamma(TM)$ and it represents the set of all smooth vector field defined over a smooth manifold M seen as base space of the tangent bundle:

$$\Gamma(TM) = \mathfrak{X}(M) \ . \tag{3.26}$$

Definition 3.1.12. Likewise for the tangent bundle, we define T^*M *cotangent bundle* the smooth vector bundle whose total space is given by the disjoint union of the cotangent spaces T^*_pM at each point

$$(T^*M,\ M,\ \pi,\ \{(U,\phi)\}) \quad \text{with total space} \quad T^*M := \sqcup_p\, T^*_pM \ . \tag{3.27}$$

As previously, we can demonstrate it has a natural structure of vector bundle of rank k over M. Furthermore, we define *cotangent frame bundle* the frame bundle associated to the cotangent bundle, with total space given by the disjoint union of all F_p sets of frames of all the cotangent spaces T^*_pM.

Remark. From this perspective, a one-form field $\omega \in \Omega(M,\mathbb{R})$ defined over the smooth manifold M is a section of the cotangent bundle T^*M, such that:

$$\begin{aligned} \sigma := \omega : M &\to T^*M \\ p &\mapsto \omega_p \end{aligned} \tag{3.28}$$

and the global section of the cotangent bundle can be identified with the set of all one-form field

$$\Gamma(T^*M) = \Omega(M) \ . \tag{3.29}$$

We list afterwards other examples of vector bundles related to tensor fields.

Definition 3.1.13. Covariant tensor bundle and covariant tensor field of rank k

$$T^kM := \sqcup_p\, T^k\left(T_pM\right) \ , \quad \mathcal{T}^k(M) := \Gamma\left(T^kM\right) \ .$$

Controvariat tensor bundle and controvariant tensor field of rank r

$$T_rM := \sqcup_p\, T_r\left(T_pM\right) \ , \quad \mathcal{T}_r(M) := \Gamma\left(T_rM\right) \ .$$

Mixed tensor bundle and mixed tensor field of rank (k, r)

$$T^k_rM := \sqcup_p\, T^k_r\left(T_pM\right) \ , \quad \mathcal{T}^k_r(M) := \Gamma\left(T^k_rM\right) \ .$$

In a given local coordinate system with coordinate functions x^i any tensor can be written as

$$T = T^{j_1 \dots j_r}{}_{i_1 \dots i_k} \mathrm{d}x^{i_1} \otimes \dots \otimes \mathrm{d}x^{i_k} \otimes \partial_{j_1} \otimes \dots \otimes \partial_{j_r} \ \in \mathcal{T}^k_r(M) \ .$$

As usual we will denote by $T_p \in T^k_r\left(T_pM\right)$ the value of a tensor field at a point namely

$$T_p\left(X_{1p}, \dots, X_{kp}, \omega_{1p}, \dots, \omega_{rp}\right) \equiv T\left(X_1, \dots, X_k, \omega_1, \dots, \omega_r\right)(p) \ .$$

Last but not least, the vector bundle of differential forms of rank k that is $\Lambda^k M$ and it is given by the disjoint union of alternating covariant k tensor field over the tangent spaces of the smooth manifold M

$$\Lambda^k M := \sqcup_p \Lambda^k(T_p M) \tag{3.30}$$

The set of the sections of $\Lambda^k M$ is denoted by $\Omega^k(M)$.

Associated bundles

Given an effective G-bundle namely (E, B, π, F) as previously introduced, we define the following concepts.

Definition 3.1.14. We define *associated G-bundle* (E', B, π', F') with abstract fiber F', same base space B and coordinate change dependent by the same transition function $g_{ij} : U_i \cap U_j \to G$

$$\begin{aligned} \Phi'_{ij} : (U_i \cap U_j) \times F' &\to (U_i \cap U_j) \times F' \\ (u, f') &\mapsto \Phi'_{ji}(u, f') = (u, g_{ij}(u)f') \ , \end{aligned} \tag{3.31}$$

this is a fiber bundle according to fiber bundle construction theorem. In particular, if the final bundle (E', B, π', F') is a principal G-bundle then we call it *associated principal G-bundle.* While, in case the final bundle is a vector fiber bundle we will call it *associated vector G-bundle.* Let G be a Lie group which acts on an abstract fiber F of a principal G-bundle and let $\rho : G \to \mathrm{Aut}(F)$ be the corresponding representation, for example, F could be a vector space and $\mathrm{Aut}(F) = \mathrm{GL}(F)$, or F could be a smooth manifold and $\mathrm{Aut}(F) = \mathrm{Diff}(F)$.

Definition 3.1.15. The data defining the principal fiber bundle $\pi : P \to B$ allows to define a G-bundle called *associated G-bundle* to P via ρ as follows

$$\xi \times_G F \equiv (P \times_G F,\ B,\ \pi_F,\ F) \quad \text{with total space} \quad P \times_G F := (P \times F)/G \ ,$$

where the G-action works as follows and since G acts freely on P, it acts freely on $P \times F$ hence P/G is a smooth manifold, so is $P \times_G F$.

$$\begin{aligned} G \times (P \times_G F) &\to P \times_G F \\ (g, (p, f)) &\mapsto (p, f)g = (pg, \rho(g^{-1})f) \ . \end{aligned} \tag{3.32}$$

Moreover the projection $\pi : P \to B$ induces a projection on the associated G-bundle which is well defined because $\pi(pg) = \pi(p)$

$$\begin{aligned} \pi_F : P \times_G F &\to B \\ (p, f) &\mapsto \pi_F(p, f) = \pi(p) \ . \end{aligned} \tag{3.33}$$

For example, taking the adjoint representations $\mathrm{ad} : G \to \mathrm{GL}(\mathfrak{g})$ and $\mathrm{Ad} : G \to \mathrm{Diff}(G)$ in turn, we arrive at the associated vector bundle $\mathrm{ad}\, P$ and the associated fiber bundle $\mathrm{Ad}\, P$ as:

$$\mathrm{ad}\, P = P \times_G \mathfrak{g} \ , \quad \mathrm{Ad}\, P = P \times_G G \ .$$

Remark. The associated bundle $\xi \times_G F$ can be also constructed locally from the data defining P, namely the open cover $\{U_\alpha\}$ and the transition functions $\{g_{\alpha\beta}\}$ on double overlaps. The total space is:

$$P \times_G F = \sqcup_\alpha \, (U_\alpha \times F)/\sim \quad \text{with}$$
$$(m, f) \sim (m, \rho(g_{\alpha\beta})f) \quad \forall \, m \in U_{\alpha\beta} \ , \ f \in F \ .$$

Sections of $P \times_G F$ are represented by functions $f : P \to F$ with the equivariance condition:

$$R_g^* f = \rho(g^{-1}) \circ f \ ,$$

or equivalently by a family of functions $f_\alpha : U_\alpha \to F$ such that

$$f_\alpha(m) = \rho(g_{\alpha\beta}(m)) f_\beta(m) \ .$$

The case of a principal G-bundle is fundamental because of the inverse constructions that allow us to pass back and forth between effective G-bundles with abstract fiber F and principal G-bundles. An effective G-bundle with abstract fiber F determines the transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$, and hence, as above, we can construct an *associated* principal G-bundle. Conversely given a principal G-bundle ξ we can construct a G-bundle with abstract fiber F denoted $\xi \times_G F$. These two constructions are inverse to each other. Therefore, the abstract fiber F of an effective G-bundle may be regarded as a variable for which we substitute any manifold on which G acts effectively.

$$\text{effective G-bundle} \quad \longrightarrow \quad \textit{associated} \text{ principal G-bundle}$$
$$\text{principal G-bundle} \quad \longrightarrow \quad \textit{associated} \text{ G-bundle } \ \xi \times_G F$$

3.2 Connections on bundles

The concept of connection arises from a differential operator called covariant derivative, which allows us to study how sections $\sigma : M \to E$ of smooth vector bundles change when we move along a curve $\gamma(t)$ on M starting at p. At the end of this part we will formalize the concepts of linear connection for vector bundles, also named *Kozul connection*, and for principal bundles, also named *Ehresmann connection.*

Definition 3.2.1. Given a smooth fiber bundle $= (E, M, \pi, F)$, we define *vertical vector space* at $P \in E$ over $p \in M$ where $\pi(P) = p$ as follows

$$V_P E := \{\mathbf{A} \in T_P E \mid \pi_* \mathbf{A} = \mathbf{0}\} = \ker(\pi_*) \tag{3.36}$$

We call *vertical bundle* the associated smooth fiber bundle with total space given by the disjoint union of the singles vertical vector spaces

$$(VE, M, \pi, F) \quad \text{with total space} \quad VE := \sqcup_{P \in E} \, V_P E \ .$$

Remark. Note that $V_P E = T_P E_p$ so the vertical vector space at $P \in E$ coincide with the tangent vector space at P of the fibers E_p of the point $p \in M$. From another point of view, given a basis with coordinate $\{x^1, ..., x^n\}$ for the tangent space $T_P E$ such that its vectors $\mathbf{A}_P = A^i \partial_i|_P$ and a generic smooth function $f \in C^\infty(M, \mathbb{R})$, then, we can read the vertical vector space as the space of vectors $\mathbf{A}_P$ which makes the composition $f \circ \pi$ constant along the fibers E_p of the point $p \in M$

$$(\pi_* \mathbf{A}_P) f = \mathbf{A}_P (f \circ \pi) = 0 \quad \forall \ P \in \pi^{-1}(p) = E_p \ . \tag{3.37}$$

A vector field $X \in \mathfrak{X}(E)$ is said *vertical* if $X(P) \in V_P E$ for all $P \in E$, the Lie bracket of two vertical vector fields is again a vertical vector field over E and in absence of any extra structures there is no natural complement to $V_P E$ in $T_P E$, in this sense this is what a connection provides.

Definition 3.2.2. We define *horizontal vector space* $H_P E$ at $P \in E$ over $p \in M$ for a smooth fiber bundle (E, M, π, F) one of the possible complements of the vertical vector space $V_P E$ at the same $P \in E$

$$T_P E = V_P E \oplus H_P E \tag{3.38}$$

in addition to this we introduce the *horizontal bundle* HE as the smooth fiber bundle with total space given by the disjoint union of each horizontal vector space $H_P E$ with $\pi(P) = p$

$$(HE, M, \pi, F) \quad \text{with total space} \quad HE := \sqcup_{P \in E} H_P E \ .$$

It is important to point out that the choice of $H_P E$ is not canonical because we don't have in general on $T_P E$ a notion of a scalar product.

Definition 3.2.3. Given the push-forward $\pi_* : T_P E \to T_p M$ with $V_P E$ as kernel, it induces an isomorphism that we denote *horizontal lift* at $P \in E$

$$\begin{aligned} \mathrm{Hor}_P : T_p M &\xrightarrow{\sim} H_P E \subset T_P E \\ X_p &\mapsto \tilde{X}_P := \mathrm{Hor}_P(X_p) \ . \end{aligned} \tag{3.39}$$

Then, given γ_p the maximal integral curve of X at $p \in M$ we construct its lift at $P \in M$ called $\tilde{\gamma}_P$ requiring that for each $P \in E$ such that $p = \pi(P)$:

$$\pi(\tilde{\gamma}_P) = \gamma_p \ , \quad \tilde{\gamma}_P(0) = P \ , \quad \dot{\tilde{\gamma}}_P|_{\tilde{\gamma}_P(t)} \in H_{\tilde{\gamma}_P(t)} E \ .$$

Definition 3.2.4. Given a smooth fiber bundle $\xi = (E, M, \pi, F)$ with vertical subspace $V_P E$ we define a linear *connection* a smooth choice of complementary subspace $H_P E \in T_P E$ such that

$$T_P E = V_P E \oplus H_P E \ . \tag{3.40}$$

This is equivalent to say a connection is linear horizontal lift Hor_P .

Connections on vector bundles

As we have seen just above, working with sections of different fibers gives problems because they belong to different vector spaces. In order to solve this, we need an isomorphism between smooth vector bundles which is also linear, this is due to the fact that covariant derivative (and connections in general) have property of linearity, for this reason we list below these notable facts.

Lemma 3.2.1. Given the horizontal lift of a vector field defined over the base space of a smooth vector bundle satisfying a linearity property, such that for each real a and vector $\mathbf{v} \in E_p$ fibers over p

$$\mathrm{Hor}_{a\mathbf{v}}(X_p) = a\,\mathrm{Hor}_{\mathbf{v}}(X_p) \ , \tag{3.41}$$

then the flow of the horizontal lift at $\mathbf{v} \in E$ over $p \in M$ respects:

$$\mathrm{Fl}_t^{\tilde{X}}(a\mathbf{v}) = a\,\mathrm{Fl}_t^{\tilde{X}}(\mathbf{v}) \ . \tag{3.42}$$

Proof. Consider $\tilde{\gamma}_{\mathbf{v}}$ and $\tilde{\gamma}_{a\mathbf{v}}$ for real a, then one has

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(a\tilde{\gamma}_{\mathbf{v}}) = a\,\mathrm{Hor}_{\mathbf{v}}(X_p) = \mathrm{Hor}_{a\mathbf{v}}(X_p) = \left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0}(\tilde{\gamma}_{a\mathbf{v}})$$

□

We use now this observation within the context of the following proposition.

Lemma 3.2.2. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ with $F(a\mathbf{x}) = aF(\mathbf{x})$ then F is linear.

Proof. The proof relies on the Taylor expansion

$$F\left(\mathbf{x} + \mathbf{x}_0\right) = F\left(\mathbf{x}_0\right) + \left.\partial_{\mathbf{x}} F\right|_{\mathbf{x}_0} \mathbf{x} + R(\mathbf{x})$$

Now we must have $R(a\mathbf{x}) = aR(x)$ thus the rest R is such that $R(\mathbf{x}) = 0$, indeed

$$\lim_{a\to 0} \frac{|R(a\mathbf{x})|}{|a\mathbf{x}|} = \frac{|R(\mathbf{x})|}{|\mathbf{x}|} \quad \text{where} \quad \frac{|R(\mathbf{x})|}{|\mathbf{x}|} \to 0 \ \text{ when } \ x \to 0$$

□

Because of the last two lemmas, the proposition below counts.

Proposition 3.2.1. The flow $\mathrm{Fl}_t^{\tilde{X}} : E_p \to E_{\gamma_p(t)}$ of the horizontal lift $\tilde{X}_P$ of the vector field $X_p \in T_pM$ induces a linear isomorphism among fibers E_p of a smooth vector bundle.

We have now all the ingredients needed to study how a section σ changes along the integral curve of a smooth vector fields X over the base space M.

Definition 3.2.5. We define *covariant derivative of a section* $\sigma : M \to E$ over the integral curve generated by the smooth vector field $X \in \mathfrak{X}(M)$ at the point $p \in M$

$$(\nabla_X \sigma)(p) := \left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0} \left(\mathrm{Fl}^{\tilde{X}}_{-t} \circ \sigma \circ \mathrm{Fl}^{X}_{t}\right)(p) = -\tilde{X}_{\sigma(p)} + \sigma_* X_p \ . \tag{3.43}$$

We report explicit calculations further, in the second passage we applied Leibniz rule for derivations, then we changed variable in the first term $\tau = -t$ for convenience and in the last part we made use of the derivative of the one-parameter flows to express $X_p \in T_pM$ and its horizontal lift $\tilde{X}_P \in H_PE$ as

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \mathrm{Fl}^{X}_{t}(p) = X_p \ , \quad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \mathrm{Fl}^{\tilde{X}}_{t}(p) = \tilde{X}_P \ . \tag{3.44}$$

n particular $\pi_* \tilde{X}_{\sigma(p)} = X_p$ by construction and $\pi_* \circ \sigma_* X_p = (\pi \circ \sigma)_* X_p = X_p$ thus the right hand side of the previous equation is an element of vertical vector space $V_{\sigma(p)}E$, that we identify with E_p.

$$\begin{aligned}
(\nabla_X \sigma)(p) := {} & \left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0} \left(\mathrm{Fl}^{\tilde{X}}_{-t} \circ \sigma \circ \mathrm{Fl}^{X}_{t}\right)(p) \\
= {} & \left[\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (\mathrm{Fl}^{\tilde{X}}_{-t})\right] \left(\sigma \circ \mathrm{Fl}^{X}_{0}\right)(p) + \mathrm{Fl}^{\tilde{X}}_{0} \left[\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (\sigma \circ \mathrm{Fl}^{X}_{t})\right](p) \\
= {} & \left[\frac{\mathrm{d}\tau}{\mathrm{d}t} \cdot \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_0 (\mathrm{Fl}^{\tilde{X}}_{-t})\right] \left(\sigma \circ \mathrm{Fl}^{X}_{0}\right)(p) + \mathrm{Fl}^{\tilde{X}}_{0} \left[\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (\sigma \circ \mathrm{Fl}^{X}_{t})\right](p) \\
= {} & -\left[\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_0 (\mathrm{Fl}^{\tilde{X}}_{-t})\right] \left(\sigma \circ \mathrm{Fl}^{X}_{0}\right)(p) + \mathrm{Fl}^{\tilde{X}}_{0} \left[\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (\sigma \circ \mathrm{Fl}^{X}_{t})\right](p) \\
= {} & -\left[\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_0 \mathrm{Fl}^{\tilde{X}}_{-t}\right] \left(\sigma \circ \mathrm{Fl}^{X}_{0}(p)\right) + \mathrm{Fl}^{\tilde{X}}_{0} \left[\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (\sigma \circ \mathrm{Fl}^{X}_{t})\right](p) \\
= {} & -\left[\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_0 \mathrm{Fl}^{\tilde{X}}_{-t}\right] (\sigma(p)) + \sigma_* \left[\mathrm{Fl}^{\tilde{X}}_{0} \circ X \circ \mathrm{Fl}^{X}\right](p) \\
= {} & -\tilde{X}_{\sigma(p)} + \sigma_* X_p
\end{aligned}$$

Taking into account the smoothness of all the ingredients this observation implies that ∇ can be viewed as a map defined as follows and covariant derivative it is a particular case of Kozul connections.

Definition 3.2.6. Let (E, M, π, F) be a smooth vector bundle, we define a *Kozul connection* on E an $\mathbb{R}$-bilinear map

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \Gamma(E) &\to \Gamma(E) \\ (X, \sigma) &\mapsto \nabla_X \sigma \end{aligned} \tag{3.45}$$

which satisfies the following properties for all $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(E)$:

i. *(Homogeneity)* $\nabla_{fX}\sigma = f\ \nabla_X \sigma$,

ii. *(Leibniz Rule)* $\nabla_X(f\sigma) = f\ \nabla_X\sigma + (X \cdot f)\sigma$,

iii. *(Bi-linearity)* $\nabla_{X_1+X_2}\sigma = \nabla_{X_1}\sigma + \nabla_{X_2}\sigma$, $\nabla_X(\sigma_1 + \sigma_2) = \nabla_X\sigma_1 + \nabla_X\sigma_2$.

In case the smooth vector bundle we are dealing with is the tangent bundle $E = TM$, then we call *affine connection* the Kozul connection over the tangent bundle.

Connections on principal bundles

If we work with principal G-bundles (P, M, π, F) the action of G on P is defined by a Lie group G and a linear connection on the total space P is a smooth choice of horizontal subspace $H_pP \in T_pP$, where the subscript p is a generic point on the total space, even though it was previously used as a generic point of the base space. From here on we will adopt this notation to introduce the following concepts.

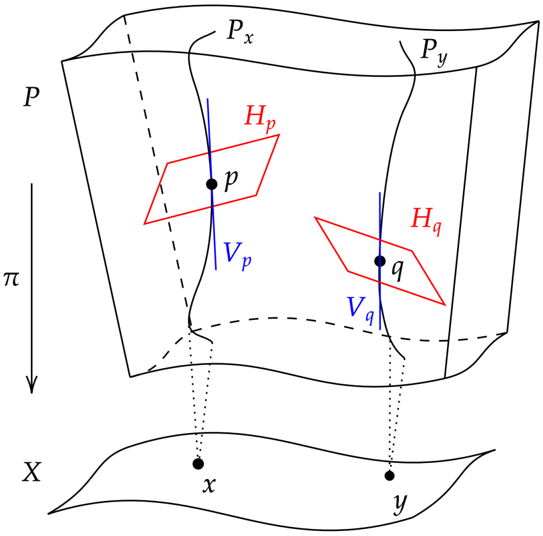

Figure 3.3: Graphic representation of a principal bundle $\pi : P \to X$ with P_x as fibers over the total space, H_p, V_p as horizontal and vertical subspace of the fibers, picture taken by reference [2].

Definition 3.2.7. We define an *Ehresmann connection* a linear connection on a principal G-bundle (P, M, π, F) which is G-invariant. This means that given a linear connection $H \subset TP$ tangent bundle of P and a right G-action $R_g : G \times HP \to HP$ the horizontal subspace $H_p \subset T_pP$ respects:

$$(R_g)_* H_p = H_{pg} \ . \tag{3.46}$$

Definition 3.2.8. Given a principal G-bundle (P, M, π, F) the right-action of G on P defines a map ξ named *fundamental vector field* over $\mathfrak{g}$ Lie algebra of G

$$\begin{aligned} \xi : \mathfrak{g} &\to \mathfrak{X}(P) \\ X &\mapsto \xi(X) \end{aligned} \tag{3.47}$$

whose value at $p \in P$ is given by

$$\xi_p(X) := \left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0} (p \cdot \exp{(tX)}) \tag{3.48}$$

Here $\exp{(tX)} \in G$, $p \cdot \exp{(tX)} \in P$ because of the right action $G \times P \to P$ and the fundamental vector field calculated at a point in the total space $\xi_p(X) \in T_pP$. In particular, notice that π does not depend on the real variable t thus $\pi_*\xi_p(X) = 0$, indeed

$$\pi_*\xi_p(X) = \left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_0 \pi(p \cdot e^{tX}) = \left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_0 \pi(p) = 0 \ ,$$

where we used Leibniz rule at the third passage. Then, the fundamental vector field $\xi(X)$ is an element of $\ker(\pi_*)$ and for this reason, by definition, it is a vertical vector field $\xi_p(X) \in V_pP$. In particular, since G acts freely, the fundamental vector field is an isomorphism

$$\xi_p : \mathfrak{g} \xrightarrow{\simeq} V_pP \quad \forall p \in P \ . \tag{3.49}$$

Lemma 3.2.3. Given the fundamental vector field ξ for a principal G-bundle

$$(R_g)_*\xi(X) = \xi(\mathrm{ad}_{g^{-1}} X) \ . \tag{3.50}$$

In particular, given $g_\alpha : \pi^{-1}U_\alpha \to G$ maps defined by the local trivialisation we observe that

$$(g_\alpha)_*\, \xi_p(X) = \left(L_{g_\alpha(p)}\right)_* X \ . \tag{3.51}$$

Proof. By definition, we have at $p \in P$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} R_g \left(pe^{tX}\right)\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\left(pe^{tX}g\right)\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\left(pgg^{-1}e^{tX}g\right)\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\left(pge^{t\mathrm{ad}_{g^{-1}}X}\right)\right|_{t=0}$$

$$(R_g)_*\, \xi_p(X) = \xi_{pg}\left(\mathrm{ad}_{g^{-1}} X\right) \ ,$$

for what concerns the second statement, if we use Leibniz rule at the third passage we get

$$(g_\alpha)_*\xi_p(X) = (g_\alpha)_* \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (pe^{tX}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 [g_\alpha(pe^{tX})] = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 g_\alpha(p) = XL_{g_\alpha(p)} = (L_{g_\alpha(p)})_*X \ .$$

□

The horizontal subspace $H_pP \subset T_pP$, being a linear subspace, is generated by $k = \dim G$ linear equations $T_pP \to \mathbb{R}$. In other words, H_pP is the kernel of k one-forms at $p \in P$, the components of a one-form ω at $p \in P$ with values in a k-dimensional vector space. There is a natural such vector space, namely the Lie algebra $\mathfrak{g}$ of G, and since ω annihilates horizontal vectors it is defined by what it does to the vertical vectors, and we do have a natural map $\mathbf{V} \to \mathfrak{g}$ given by the inverse of ξ_p.

Definition 3.2.9. Given a smooth vector field $\mathbf{V} \in \mathfrak{X}(P)$, the connection one-form of a connection in the horizontal bundle $HE \subset TP$ is the $\mathfrak{g}$-valued one-form $\omega \in \Omega^1(P; \mathfrak{g})$ defined by

$$\omega(\mathbf{V}) = \begin{cases} X & \text{if } \ \mathbf{V} = \xi(X) \\ 0 & \text{if } \ \mathbf{V} \in HP \end{cases} \tag{3.52}$$

In particular, we say that a one-form ω on P is *horizontal* if it annihilates the vertical vectors

$$\omega(\mathbf{V}) = 0 \quad \forall\, \mathbf{V} \in VP \, . \tag{3.53}$$

Notice that if ω and ω' are connection one-forms for two connections H and H' on P, their difference $\omega - \omega' \in \Omega^1(P; \mathrm{g})$ is horizontal. We will see later that this means that it defines a section through a bundle on M associated to P.

Proposition 3.2.2. The connection one-form in case $\mathbf{V} = \xi(X)$ obeys

$$R_g^* \; \omega = \mathrm{ad}_{g^{-1}} \circ X \; . \tag{3.54}$$

Proof. Let $\mathbf{V} \in H_pP$ thus $\omega(\mathbf{V}) = 0$. In addition to this $(R_g)_* \mathbf{V} \in H_{pg}P$ by G-equivariance of H, then $R_g^*\omega$ also annihilates $\mathbf{V}$ and the identity is trivially satisfied

$$R_g^*\omega = 0 \; . \tag{3.55}$$

Now let $\mathbf{V} = \xi_p(X)$ for some $X \in \mathfrak{g}$. Then, using the previous lemma in the second equivalence,

$$R_g^*\omega(\xi(X)) = \omega\left((R_g)_* \, \xi(X)\right) = \omega\left(\xi\left(\mathrm{ad}_{g^{-1}} X\right)\right) = \mathrm{ad}_{g^{-1}} X \; .$$

Conversely, given a one-form $\omega \in \Omega^1(P; \mathfrak{g})$ satisfying the identity in the proposition before and such that $\omega(\xi(X)) = X$, the distribution $H = \ker \omega$ defines a connection on P. □

3.3 Gauge theory on principal bundles

Finally, as advertised, we make contact with the more familiar notion of gauge fields as used in Physics, which live on the base space M, which will be used as spacetime further on, instead of the total space P.

Definition 3.3.1. Given local sections $\sigma_\alpha : \mathrm{U}_\alpha \to \pi^{-1}(\mathrm{U}_\alpha)$ associated canonically to the trivialisation of the principal G-bundle P, along which we can pull-back the connection one-form ω. We define *gauge fields* the following $\mathfrak{g}$-valued one-forms on U_α:

$$\boxed{\mathrm{A}_\alpha := \sigma_\alpha^* \; \omega \;\; \in \Omega^1\left(\mathrm{U}_\alpha; \mathfrak{g}\right)} \tag{3.56}$$

The curvature of a connection 1-form ω is given by the $\mathfrak{g} \otimes \mathfrak{g}$-valued 2-form on P denoted as Ω which is the structural equation left-handed term of the connection 1-form ω:

$$\Omega := \mathrm{d}\omega + \frac{1}{2}[\omega, \omega] \; . \tag{3.57}$$

Pulling back Ω via the canonical sections $\sigma_\alpha : \mathrm{U}_\alpha \to \mathrm{P}$ it yields the gauge field-strength

$$\boxed{\mathrm{F}_\alpha := \sigma_\alpha^* \ \Omega \ \in \Omega^2 \left(U_\alpha ; \mathfrak{g}\right)} \tag{3.58}$$

We may sometimes write F_A if we want to make the dependence on the gauge fields manifest, in particular, the $\{\mathrm{F}_\alpha\}$ define a global 2-form $\mathrm{F} \in \Omega^2(\mathrm{M}; \mathrm{ad}P)$ with values in $\mathrm{ad}P$. Another notation is:

$$\mathrm{F}_\alpha = \mathrm{dA}_\alpha + \frac{1}{2} \left[\mathrm{A}_\alpha, \mathrm{A}_\alpha\right] \ . \tag{3.59}$$

Proposition 3.3.1. The restriction of the connection one-form ω to $\pi^{-1}(\mathrm{U}_\alpha)$ agrees with the next equation, where θ is the Maurer-Cartan one-form

$$\omega_\alpha = \mathrm{ad}_{g_\alpha^{-1}} \circ \pi^* \ \mathrm{A}_\alpha + g_\alpha^* \ \theta \ . \tag{3.60}$$

Proof. We will prove this result in two steps. First we show that ω_α and ω agree on the image of σ_α. Indeed, let $m \in \mathrm{U}_\alpha$ and $p = \sigma_\alpha(m)$. We have a direct sum decomposition

$$\mathrm{T}_p\mathrm{P} = \mathrm{Im} \left(\sigma_\alpha \circ \pi\right)_* \oplus \mathrm{V}_p \ , \tag{3.61}$$

so that every $\mathbf{V} \in \mathrm{T}_p\mathrm{P}$ can be written uniquely as $\mathbf{V} = \left(\sigma_\alpha\right)_* \pi_*(\mathbf{V}) + \bar{\mathbf{V}}$, for a unique vertical vector $\bar{\mathbf{V}}$. Applying ω_α on $\mathbf{V}$, we obtain the next result, since $g_\alpha\left(\sigma_\alpha(m)\right) = e$:

$$\begin{aligned}
\omega_\alpha(\mathbf{V}) &= \left(\pi^* \sigma_\alpha^* \omega\right)(\mathbf{V}) + \left(g_\alpha^* \theta_e\right)(\mathbf{V}) \\
&= \omega\left(\left(\sigma_\alpha\right)_* \pi_* \mathbf{V}\right) + \theta_e\left(\left(g_\alpha\right)_* \mathbf{V}\right) \\
&= \omega\left(\left(\sigma_\alpha\right)_* \pi_* \mathbf{V}\right) + \theta_e\left(\left(g_\alpha\right)_* \bar{\mathbf{V}}\right) \quad \text{since } (g_\alpha \circ \sigma_\alpha)_* = 0 \\
&= \omega\left(\left(\sigma_\alpha\right)_* \pi_* \mathbf{V}\right) + \omega(\bar{\mathbf{V}}) \\
&= \omega(\mathbf{V}) \ .
\end{aligned}$$

Next we show that they transform in the same way under the right action of G:

$$\begin{aligned}
\mathrm{R}_g^* \left(\omega_\alpha\right)_{pg} &= \mathrm{ad}_{g_\alpha(pg)^{-1}} \circ \mathrm{R}_g^* \pi^* \sigma_\alpha^* \omega + \mathrm{R}_g^* g_\alpha^* \theta \\
&= \mathrm{ad}_{(g_\alpha(p)g)^{-1}} \circ \mathrm{R}_g^* \pi^* \sigma_\alpha^* \omega + g_\alpha^* \mathrm{R}_g^* \theta \quad \text{equivariance of } g_\alpha \\
&= \mathrm{ad}_{g^{-1} g_\alpha(p)^{-1}} \circ \pi^* \sigma_\alpha^* \omega + g_\alpha^* (\mathrm{ad}_{g^{-1}} \circ \theta) \quad \text{since } \pi \circ \mathrm{R}_g = \pi \\
&= \mathrm{ad}_{g^{-1}} \circ \left(\mathrm{ad}_{g_\alpha(p)^{-1}} \circ \pi^* \sigma_\alpha^* \omega + g_\alpha^* \theta\right) \\
&= \mathrm{ad}_{g^{-1}} \circ \left(\omega_\alpha\right)_p \ .
\end{aligned}$$

Therefore they agree everywhere on $\pi^{-1}(U_\alpha)$. □

Now since ω is defined globally, we have that $\omega_\alpha = \omega_\beta$ on $\pi^{-1}(U_{\alpha\beta})$. This allows us to relate A_α and A_β on $U_{\alpha\beta}$. Indeed on $U_{\alpha\beta}$, using $g_\beta \circ \sigma_\alpha = g_{\beta\alpha}$ we have

$$\begin{aligned}
\mathrm{A}_\alpha &= \sigma_\alpha^* \ \omega_\alpha \\
&= \sigma_\alpha^* \ \omega_\beta \\
&= \sigma_\alpha^* \left(\mathrm{ad}_{g_\beta(\sigma_\alpha)^{-1}} \circ \pi^* \mathrm{A}_\beta + g_\beta^* \theta \right) \\
&= \mathrm{ad}_{g_{\alpha\beta}} \circ \mathrm{A}_\beta + g_{\beta\alpha}^* \theta \ ,
\end{aligned}$$

or the equivalent expression: $\mathrm{A}_\alpha = \mathrm{ad}_{g_{\alpha\beta}} \circ (\mathrm{A}_\beta - g_{\alpha\beta}^* \theta)$ using $\mathrm{ad}_{g_{\alpha\beta}} \circ \ g_{\alpha\beta}^* \theta = -g_{\beta\alpha}^* \theta$. In summary, the relation between gauge fields defined over different open sets U_α, U_β are given by the following expressions. where the second one is the case with group G, related to the principal G-bundle we are dealing with, is a matrix Lie group

$$\boxed{\mathrm{A}_\alpha = \mathrm{ad}_{g_{\alpha\beta}} \circ \mathrm{A}_\beta + g_{\beta\alpha}^* \theta \ , \quad \mathrm{A}_\alpha = g_{\alpha\beta} \mathrm{A}_\beta g_{\alpha\beta}^{-1} - \mathrm{d} g_{\alpha\beta} g_{\alpha\beta}^{-1}} \tag{3.65}$$

this is due to the fact that for matrix Lie groups the Maurer-Cartan form is expressed as

$$\theta_g = g^{-1} \mathrm{d} g \quad , \quad g_{\beta\alpha}^* \theta = g_{\beta\alpha^{-1}} \mathrm{d} g_{\alpha\beta} = - \mathrm{d} g_{\alpha\beta} g_{\alpha\beta}^{-1} \ .$$

Given a family of gauge fields $A_\alpha \in \Omega^1 \left(U_\alpha ; \mathfrak{g}\right)$ satisfying one of these three relations above on overlaps $U_{\alpha\beta}$, we can construct a globally defined $\omega \in \Omega^1(\mathrm{P}; \mathfrak{g})$ by the previous formula, which relates connection one-form, gauge field and Maurer-Cartan form. Then ω is the connection one-form of a connection on P.

As usual, the natural question to ask is how do gauge strength fields F_α and F_β differ on $U_{\alpha\beta}$. From gauge fields transformation equation, using the Maurer-Cartan structure equation and simplifying,

$$\begin{aligned}
F_\alpha &= \mathrm{d} A_\alpha + \tfrac{1}{2} [A_\alpha, A_\alpha] \\
&= \mathrm{d} \left(\mathrm{ad}_{g_{\alpha\beta}} \circ (A_\beta - g_{\alpha\beta}^* \theta) \right) + \tfrac{1}{2} \left[\mathrm{ad}_{g_{\alpha\beta}} \circ (A_\beta - g_{\alpha\beta}^* \theta), \mathrm{ad}_{g_{\alpha\beta}} \circ (A_\beta - g_{\alpha\beta}^* \theta) \right] \\
&= \mathrm{ad}_{g_{\alpha\beta}} \circ (\mathrm{d} A_\beta - \mathrm{d} g_{\alpha\beta}^* \theta + \tfrac{1}{2} [A_\beta - g_{\alpha\beta}^* \theta, A_\beta - g_{\alpha\beta}^* \theta]) \\
&= \mathrm{ad}_{g_{\alpha\beta}} \circ (\mathrm{d} A_\beta - \mathrm{d} g_{\alpha\beta}^* \theta + \tfrac{1}{2} [A_\beta, A_\beta] - \tfrac{1}{2} [g_{\alpha\beta}^* \theta, g_{\alpha\beta}^* \theta]) \\
&= \mathrm{ad}_{g_{\alpha\beta}} \circ (\mathrm{d} A_\beta + \tfrac{1}{2} [A_\beta, A_\beta]) - \mathrm{ad}_{g_{\alpha\beta}} \circ (g_{\alpha\beta}^* \circ \mathrm{d} \theta + \tfrac{1}{2} g_{\alpha\beta}^* \circ [\theta \theta]) \\
&= \mathrm{ad}_{g_{\alpha\beta}} \circ F_\beta - \mathrm{ad}_{g_{\alpha\beta}} \circ g_{\alpha\beta}^* \circ (\mathrm{d} \theta + \tfrac{1}{2} [\theta, \theta]) \\
&= \mathrm{ad}_{g_{\alpha\beta}} \circ F_\beta
\end{aligned}$$

we find the following two expressions; where the second is the specific case for matrix Lie groups.

$$\boxed{\mathrm{F}_\alpha = \mathrm{ad}_{g_{\alpha\beta}} \circ \mathrm{F}_\beta \ , \quad \mathrm{F}_\alpha = g_{\alpha\beta} \mathrm{F}_\beta g_{\alpha\beta}^{-1}} \tag{3.66}$$

Remark. Eventually, we have three equivalent descriptions of a connection on a principal G-bundle P, each one has its virtues and we will use one or another as convenience:

1. a G-invariant horizontal distribution $H \subset TP$,
2. a one-form $\omega \in \Omega^1(P;\mathfrak{g})$ satisfying next relations given the fundamental vector field ξ

$$\omega(\xi(X)) = X \quad \text{and} \quad (R_g)^*\omega = \mathrm{ad}_{g^{-1}} \circ X \ , \tag{3.67}$$

3. a family of one-forms named gauge fields $A_\alpha \in \Omega^1 \left(U_\alpha;\mathfrak{g}\right)$ satisfying on open overlaps $U_{\alpha\beta}$

$$\mathrm{A}_\alpha = \mathrm{ad}_{g_{\alpha\beta}} \circ \mathrm{A}_\beta + g^*_{\beta\alpha}\theta \ . \tag{3.68}$$

Gauge transformations

Definition 3.3.2. A *gauge transformation* of a principal fibre bundle $\pi : P \to M$ is defined as a G-equivariant diffeomorphism $\Phi : \mathrm{P} \to \mathrm{P}$ making the following diagram commute

$$\begin{tikzcd} P \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & P \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

In particular, Φ maps fibers to themselves and equivariance means that $\Phi(pg) = \Phi(p)g$ for each $g \in G$ Lie group related to the principal bundle. Composition makes gauge transformations into a group named *group of gauge transformation* $\mathcal{G}$.

Proposition 3.3.2. The group of gauge transformation is a set of smooth maps defined over the base space of the principal G-bundle $\pi : P \to M$, with values in its associate fiber bundle $\mathrm{Ad}P$

$$\mathcal{G} = C^\infty(M;\mathrm{Ad}P) \tag{3.69}$$

Proof. We can describe $\mathcal{G}$ in terms of a trivialisation. Since it maps fibres to themselves, a gauge transformation Φ restricts to a gauge transformation of the trivial bundle $\pi^{-1}(U_\alpha)$ over U_α. Applying the trivialisation map $\psi_\alpha(\Phi(p)) = (\pi(p), g_\alpha(\Phi(p)))$, which lets us define

$$\begin{aligned} \bar{\phi}_\alpha : \pi^{-1}(\mathrm{U}_\alpha) \to \mathrm{G} \\ p \mapsto \bar{\phi}_\alpha(p) = g_\alpha(\Phi(p))g_\alpha(p)^{-1} \ , \end{aligned} \tag{3.70}$$

where the equivariance of g_α and Φ implies $\bar{\phi}$ is constant on the fibers for each $g \in G$

$$\bar{\phi}_\alpha(pg) = g_\alpha(\Phi(pg))g_\alpha(pg)^{-1} = g_\alpha(\Phi(p)g)g_\alpha(pg)^{-1} = g_\alpha(\Phi(p))gg_\alpha(p)^{-1}g^{-1} = \bar{\phi}_\alpha(p)$$

whence $\bar{\phi}_\alpha(p) = \phi_\alpha(\pi(p))$ for some function $\phi_\alpha : \mathrm{U}_\alpha \to \mathrm{G}$. For $m \in \mathrm{U}_{\alpha\beta}$, and $p \in \pi^{-1}(m)$ we have

$$\begin{aligned}
\phi_\alpha(m) &= g_\alpha(\Phi(p))\ g_\alpha(p)^{-1} \\
&= g_\alpha(\Phi(p))\ [g_\beta(\Phi(p))^{-1} g_\beta(\Phi(p))]\ [g_\beta(p)^{-1} g_\beta(p)]\ g_\alpha(p)^{-1} \quad \text{since } \pi(\Phi(p)) = m \\
&= g_{\alpha\beta}(m)\ \phi_\beta(m)\ g_{\alpha\beta}(m)^{-1} \\
&= \mathrm{Ad}_{g_{\alpha\beta}(m)}\ \phi_\beta(m)
\end{aligned}$$

$\{\phi_\alpha\}$ define a section of the associated fiber bundle $\mathrm{Ad}P$ and $\mathcal{G} = C^\infty(M; \mathrm{Ad}P)$, since $\{\phi_\alpha\}$ determine the gauge transformation Φ uniquely and viceversa. □

The action of the gauge group on affine space

The group $\mathcal{G}$ of gauge transformations acts naturally on the space $\mathcal{A}$ of connections. We can see this in several different ways.

Definition 3.3.3. Let $\mathrm{H} \subset \mathrm{TP}$ be a connection and let $\Phi : \mathrm{P} \to \mathrm{P}$ be a gauge transformation, we define

$$\mathrm{H}^\Phi := \Phi_* \mathrm{H}\ . \tag{3.72}$$

Lemma 3.3.1. H^Φ is an Ehresmann connection on P.

Proof. The equivariance of Φ makes $\mathrm{H}^\Phi \subset \mathrm{TP}$ into a G-invariant distribution as can be shown here

$$\begin{aligned}
(\mathrm{R}_g)_*\, \mathrm{H}^\Phi_{\Phi(p)} &= (\mathrm{R}_g)_*\, \Phi_* \mathrm{H}_p \\
&= \Phi_*\, (\mathrm{R}_g)_*\, \mathrm{H}_p \quad \text{by equivariance of } \Phi \\
&= \Phi_* \mathrm{H}_{pg} \quad \text{by invariance of H} \\
&= \mathrm{H}^\Phi_{\Phi(pg)} \quad \text{by definition of } \mathrm{H}^\Phi \\
&= \mathrm{H}^\Phi_{\Phi(p)g} \quad \text{by equivariance of } \Phi\ .
\end{aligned}$$

H^Φ is still complementary to V because Φ_* is an isomorphism which preserves the vertical subspace. □

Proposition 3.3.3. The fundamental vector fields $\xi(X)$ given by the G-action on a principal G-bundle are *gauge invariant*, which means given $p \in P$

$$\Phi_*\ \xi_p(X) = \xi_{\Phi(p)}(X) \quad \forall\ \Phi \in \mathcal{G}\ . \tag{3.74}$$

Proof. The gauge transformation Φ is G-equivariant, then given $p \in P$ in the third passage we know $e^{tX} \in G\ \forall\ X \in \mathfrak{g}$ and in the fourth we fix $\Phi(p) \equiv p'$:

$$\Phi_*\ \xi_p(X) = \Phi_* \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 (p \cdot e^{tX}) \right) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \left(\Phi(p \cdot e^{tX})\right) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \left(\Phi(p) \cdot e^{tX}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \left(p' \cdot e^{tX}\right) = \xi_{p'}(X)$$

□

Proposition 3.3.4. If ω is the connection one-form for a connection H then the connection one-form for H^{Φ} is given by the pull-back of the connection one-form ω through a gauge transformation Φ:

$$\omega^{\Phi} = (\Phi^{*})^{-1}\omega \ . \tag{3.76}$$

Proof. The connection H^{Φ} is defined by the push-forward Φ_* of a gauge transformation, such that given a vector $\mathbf{V} \in T_pP$ this map returns new vector $\mathbf{V}' \in T_{\Phi(p)}P$ which belongs to the tangent space of a different point $p' = \Phi(p)$ in the total space P:

$$\begin{aligned} \Phi_* : T_pP &\to T_{\Phi(p)}P \\ \mathbf{V} &\mapsto \Phi_*(\mathbf{V}) = \mathbf{V}' \end{aligned}$$

Then, the $\mathfrak{g}$-valued one-form $\omega^{\Phi} := \Phi^*\omega$ respects for each $\mathbf{V} \in TP$:

$$\omega^{\Phi}(\mathbf{V}) = \Phi^*\omega(\mathbf{V}) = \omega(\Phi_*(\mathbf{V})) = \omega(\mathbf{V}') = \begin{cases} X & \text{if } \ \mathbf{V}' = \xi_{p'}(X) \\ 0 & \text{if } \ \mathbf{V}' \in \mathrm{HP} \end{cases}$$

which is the previous one-form connection applied to a different tangent vector $\mathbf{V}' = \Phi(\mathbf{V})$. □

Gauge transformations for gauge fields

Finally we work out the effect of gauge transformations on a gauge field. Let $m \in \mathrm{U}_\alpha$ and $p \in \pi^{-1}(m)$. Let A_α and A_α^{Φ} be the gauge fields on U_α corresponding to the connections H and H^{Φ} are given by

$$\mathrm{A}_\alpha = \sigma_\alpha^* \ \omega \ , \quad \mathrm{A}_\alpha^{\Phi} = \sigma_\alpha^* \ \omega^{\Phi} \ . \tag{3.77}$$

and the connection one-forms ω and ω^{Φ} associated to the connections H and H^{Φ} are given at p by

$$\omega_p = \mathrm{ad}_{g_\alpha(p)^{-1}} \circ \pi^*\mathrm{A}_\alpha + g_\alpha^*\theta \ , \quad \omega_p^{\Phi} = \mathrm{ad}_{g_\alpha(p)^{-1}} \circ \pi^*\mathrm{A}_\alpha^{\Phi} + g_\alpha^*\theta \ . \tag{3.78}$$

On the other hand $\omega^{\Phi} = \left(\Phi^{-1}\right)^*\omega$, from where we can obtain a relation between A_α and A_α^{Φ}. Indeed, using $q = \Phi^{-1}(p)$, then functoriality of pull-back in the second passage, $\pi \circ \Phi^{-1} = \pi$ and $g_\alpha(p) = \bar{\phi}_\alpha(p)g_\alpha(q)$ we have:

$$\begin{aligned} \omega_p^{\Phi} = \left(\Phi^{-1}\right)^*\omega_q &= \mathrm{ad}_{g_\alpha(q)^{-1}} \circ \left(\Phi^{-1}\right)^*\pi^*\mathrm{A}_\alpha + \left(\Phi^{-1}\right)^* g_\alpha^*\theta \\ &= \mathrm{ad}_{g_\alpha(q)^{-1}} \circ \left(\pi \circ \Phi^{-1}\right)^*\mathrm{A}_\alpha + \left(g_\alpha \circ \Phi^{-1}\right)^*\theta \\ &= \mathrm{ad}_{g_\alpha(q)^{-1}} \circ \pi^* \ \mathrm{A}_\alpha + \left(g_\alpha \circ \Phi^{-1}\right)^*\theta \\ &= \mathrm{ad}_{g_\alpha(p)^{-1}\bar{\phi}_\alpha(p)} \circ \pi^* \ \mathrm{A}_\alpha + \left(g_\alpha \circ \Phi^{-1}\right)^*\theta \ . \end{aligned}$$

Now, $\left(g_\alpha \circ \Phi^{-1}\right)(p) = g_\alpha(q) = \bar{\phi}_\alpha(p)^{-1}g_\alpha(p)$, from where

$$\left(g_\alpha \circ \Phi^{-1}\right)^*\theta = g_\alpha^*\theta - \mathrm{ad}_{g_\alpha(p)^{-1}\bar{\phi}_\alpha(p)} \, \bar{\phi}_\alpha^*\theta \ ,$$

in case of matrix Lie groups it becomes

$$\left(g_\alpha \circ \Phi^{-1}\right)^* \theta = g_\alpha(p)^{-1}\ \bar{\phi}(p)\ \mathrm{d}\left(\bar{\phi}(p)^{-1} g_\alpha(p)\right)\ .$$

Now we put everything together using that $\bar{\phi}_\alpha = \phi_\alpha \circ \pi$ to arrive at

$$\omega_p^\Phi = \mathrm{ad}_{g_\alpha(p)^{-1}\phi_\alpha(m)} \circ \pi^*\left(\mathrm{A}_\alpha - \phi_\alpha^*\theta\right) + g_\alpha^*\theta\ ,$$

comparing with the previous expression of ω_p^Φ and we use the fact that the adjoint representation factorizes according to $\mathrm{ad}_{hg} = \mathrm{ad}_h \mathrm{ad}_g$ for all $h, g \in \mathfrak{g}$:

$$\begin{aligned}
\mathrm{ad}_{g_\alpha(p)^{-1}} \circ \pi * \mathrm{A}_\alpha^\Phi + g_\alpha^* &= \mathrm{ad}_{g_\alpha(p)^{-1}\phi_\alpha(m)} \circ \pi^*(\mathrm{A}_\alpha - \phi_\alpha^*\theta) + g_\alpha^*\theta \\
\mathrm{ad}_{g_\alpha(p)^{-1}} \circ \left[\pi^*\mathrm{A}_\alpha^\Phi\right] &= \mathrm{ad}_{g_\alpha(p)^{-1}}\left[\mathrm{ad}_{\phi_{\alpha(m)}} \circ \pi^*\mathrm{A}_\alpha - \mathrm{ad}_{\phi_\alpha(m)} \circ \pi^*\phi_\alpha^*\theta\right] \\
\pi^*\mathrm{A}_\alpha^\Phi &= \mathrm{ad}_{\phi_\alpha(m)} \circ \pi^*\left[\mathrm{A}_\alpha - \phi_\alpha^*\theta\right] \\
(\pi^*)_* \circ \mathrm{A}_\alpha^\Phi &= (\pi^*)_* \circ \mathrm{ad}_{\phi_\alpha} \circ \left[\mathrm{A}_\alpha - \phi_\alpha^*\theta\right] \\
\mathrm{A}_\alpha^\Phi &= \mathrm{ad}_{\phi_\alpha} \circ (\mathrm{A}_\alpha - \phi_\alpha^*\theta)
\end{aligned}$$

Comparing with equation (3.65), we see that in overlaps gauge fields change by a local gauge transformation defined on the overlap. Gauge fields change under action of gauge transformations as follows; the second equation is for matrix Lie groups case:

$$\boxed{\mathrm{A}_\alpha^\Phi = \mathrm{ad}_{\phi_\alpha} \circ (\mathrm{A}_\alpha - \phi_\alpha^*\theta)\ \ ,\quad \mathrm{A}_\alpha^\Phi = \phi_\alpha \mathrm{A}_\alpha \phi_\alpha^{-1} - \mathrm{d}\phi_\alpha \phi_\alpha^{-1}} \tag{3.81}$$

Making use of well-known facts like commutation of pullback and exterior derivative and structural equation we can compute gauge field strength expression as follows

$$\begin{aligned}
F_\alpha^\Phi &= \mathrm{d}A_\alpha^\Phi + \tfrac{1}{2}[A_\alpha^\Phi, A_\alpha^\Phi] \\
&= \mathrm{d}\left(\mathrm{ad}_{\phi_\alpha} \circ (A_\alpha - \phi_\alpha^*\theta)\right) + \tfrac{1}{2}\left[\mathrm{ad}_{\phi_\alpha} \circ (A - \phi_\alpha^*\theta), \mathrm{ad}_{\phi_\alpha} \circ (A - \phi_\alpha^*\theta)\right] \\
&= \mathrm{ad}_{\phi_\alpha} \circ (\mathrm{d}A - \mathrm{d}\phi_\alpha^*\theta + \tfrac{1}{2}[A - \phi_\alpha^*\theta, A - \phi_\alpha^*\theta]) \\
&= \mathrm{ad}_{\phi_\alpha} \circ (\mathrm{d}A - \mathrm{d}\phi_\alpha^*\theta + \tfrac{1}{2}[A, A] - \tfrac{1}{2}[\phi_\alpha^*\theta, \phi_\alpha^*\theta]) \\
&= \mathrm{ad}_{\phi_\alpha} \circ (\mathrm{d}A + \tfrac{1}{2}[A, A]) - \mathrm{ad}_{\phi_\alpha} \circ (\phi_\alpha^* \circ \mathrm{d}\theta + \tfrac{1}{2}\phi_\alpha^* \circ [\theta\theta]) \\
&= \mathrm{ad}_{\phi_\alpha} \circ F_\alpha - \mathrm{ad}_{\phi_\alpha} \circ \phi_\alpha^* \circ (\mathrm{d}\theta + \tfrac{1}{2}[\theta, \theta]) \\
&= \mathrm{ad}_{\phi_\alpha} \circ F_\alpha
\end{aligned}$$

Here we have gauge-transformed field-strength equations; the second is for matrix Lie groups case:

$$\boxed{\mathrm{F}_\alpha^\Phi = \mathrm{ad}_{\phi_\alpha} \circ \mathrm{F}_\alpha\ ,\quad \mathrm{F}_\alpha^\Phi = \phi_\alpha \mathrm{F}_\alpha \phi_\alpha^{-1}} \tag{3.82}$$

These equations mean that any gauge-invariant object which is constructed out of gauge fields will be well-defined globally on the base space M of the principal bundle with respect to any kind of diffeomorphism ϕ_α defined over an open neighborhood $U_\alpha \subset M$ as

$$\begin{aligned} \phi_\alpha : \; U_\alpha \;&\rightarrow\; G \\ \pi(p) \;&\mapsto\; g_\alpha(\Phi(p)) \cdot g_\alpha(p)^{-1} \end{aligned} \tag{3.83}$$

where g_α is a G-equivariant map defined over the open neighborhood U_α as defined previously in (3.12).

Chapter 4

High fantastical shapes

The idea behind a Cartan geometry, roughly speaking, is to give a generalization of Riemannian geometry obtained by replacing linear tangent spaces with general homogeneous spaces which will be defined at the beginning of this chapter; this provides an intuitive picture of Cartan geometry indeed. As Sharpe explains in the preface of his textbook [11], Cartan geometry is the common generalization of Riemannian and Klein geometries.

$$\begin{tikzcd}
\text{Euclidean Geometry} \arrow[r, "\text{allow curvature}"] \arrow[d, "\text{generalize symmetry group}"'] & \text{Riemannian Geometry} \arrow[d, "\text{generalize tangent space geometry}"] \\
\text{Klein Geometry} \arrow[r, "\text{allow curvature}"] & \text{Cartan Geometry}
\end{tikzcd}$$

4.1 Klein geometry

Definition 4.1.1. A *homogeneous space* (G, X) is a topological manifold[1] X together with a group G of transformations of X where G acts transitively on it:

$$\forall\ x, y \in X\ \exists\ g \in G \quad \text{such that} \quad gx = y\ .$$

For our immediate purposes, the most important cases are when X has at least the structure of a smooth manifold and G needs to be a Lie group which acts as diffeomorphisms.

Example 4.1.1. A homogeneous space can be identified with the total space of a principal G-bundle with base space given by one point. Indeed, in a principal G-bundle (P, B, π, F) the group G acts transitively

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[1] In some case we can find a a deliberate ambiguity here about what sort of space X should be. In different applications, X might be a mere discrete set, a topological space, a Riemannian manifold, etc.

and freely on each of the fibers P_x. Of course the actions on fibers give an action on the total space P, but this action is not transitive unless B has only one point x_0, which implies the total space is given by a single fiber P_{x_0}. The action needs to be fairly restricted by the property that points in P are only mapped to other points in the same fiber.

Main tools for exploring a homogeneous space (G, X) are subgroups $H \subset G$ which preserve, or "*stabilize*", interesting "*features*" of the geometry. What constitutes an interesting feature of course depends on the geometry, for example, Euclidean geometry $(\mathbb{R}^n, \mathrm{ISO}(n))$, has points, lines, planes, polyhedra, and so on... One can study subgroups of the Euclidean group $\mathrm{ISO}(n)$ which preserve any of these. "*Features*" in other homogeneous spaces may be thought of as generalizations of these notions but we can also work backwards, defining a feature of a geometry abstractly as that which is preserved by a given subgroup. If H is the subgroup preserving a given feature, then the space of all such features of X may be identified with the coset space G/H, which can be seen as

$$G/H = \{gH : g \in G\} = \text{ space of "\textit{features of type } H\text{"}}$$

where gH is the left coset space of H over g. Let us illustrate why this is true using the most basic of features, the feature of points.

Definition 4.1.2. Given a point $x \in X$, the subgroup of all *symmetries* $g \in G$ which fix x is called the *stabilizer*, or *isotropy subgroup* of x, and will be denoted H_x such that

$$H_x = \{g \in G \mid g \cdot x = x\} \subseteq G \tag{4.1}$$

then it is said x is a *fixed point* of g or that g *fixes* x and it is significant to point out that H_x is not necessarily a normal subgroup of G.

Lemma 4.1.1. Here we lists some of the main properties of isotropy subgroup:

i. given two symmetries $g, g' \in H_x$ for $x \in X$, then $g^{-1}g' \in H_x$;

ii. given a symmetry $g^{-1}g' \in H_x$ and $g \cdot x = y \in X$ then $g' \cdot x = y$.

Proof. The first one can be proven as follows. From hypothesis $gx = y$, $g'x = y$ and applying g^{-1} we get $g^{-1}(gx) = g^{-1}(y) = g^{-1}(g'x)$, comparing it to $g^{-1}(gx) = x$ it implies $g^{-1}(g'x) = x$ then $g^{-1}g'$ stabilizes x. The second one follows from $g'x = (gg^{-1})g'x = g(g^{-1}g')x = gx = y$ because of the previous property. ☐

Proposition 4.1.1. Fixing $x \in X$ respect to $g \in G$ we can identify each $y \in X$ with the set of all $g \in G$ such that $gx = y$, meaning that X and G/H_x are isomorphic homogeneous spaces:

$$X \simeq G/H_x \ .$$

Proof. By the previous lemma the two symmetries g, g' move x to the same point if and only if $gH_x = g'H_x$. The points of X are thus in one-to-one correspondence with cosets of H_x in G. Better yet, the map $f : X \to G/H_x$ induced by this correspondence is G-equivariant:

$$f(gy) = g \cdot f(y) \quad \forall\, g \in G, y \in X \ . \tag{4.2}$$

Then X and G/H_x are isomorphic to each other as homogeneous spaces. □

Proposition 4.1.2. If $g \in G$ such that $gx' = x$ for $x,\ x' \in X$ homogeneous space and $H_x,\ H'_x$ are the stabilizers of $x,\ x'$ then they are conjugate subgroup:

$$H_x = gH_{x'}g^{-1} \ . \tag{4.3}$$

Proof. Known X is an homogeneous space then $gx' = x$ and simultaneously $gx = x$ because H_x is the stabilizer of x, comparing them and applying g^{-1} to both terms then we have $x = gx'g^{-1}$. Working with coset space we get $H_x = gH_{x'}g^{-1}$. □

Since these conjugate subgroups of G are all isomorphic, it is common to simply speak of "*the point*" stabilizer H, even though fixing a particular one of these conjugate subgroups gives implicit significance to the points of X fixed by H. By the same looseness of vocabulary, the term "*homogeneous space*" often refers to the coset space G/H itself.

Example 4.1.2. To see the power of the Kleinian point of view, consider a familiar example of a homogeneous space: $(n+1)$-dimensional Minkowski spacetime $\mathbb{R}^{n,1}$. While this is most obviously thought of as the space of events, there are other interesting "features" of Minkowski spacetime, and each of the corresponding homogeneous spaces tell us something about the geometry of special relativity.

i. $\mathrm{ISO}(n,1)$ connected *Poincaré group* is the group of symmetries preserving orientation and time orientation, it is given by the non homogeneous Lorentz transformation and it includes the generators of translations P_i, of rotations J_i and of boosts B_i for each $i = 1, ..., n+1$.

$$x^\mu \ \mapsto\ x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu + a^\mu \qquad \Lambda \in \mathrm{SO}(n,1)\ ,\ \mathbf{x} = (x^\mu)\ ,\ \mathbf{a} = (a^\mu) \in \mathbb{R}^{n,1}$$

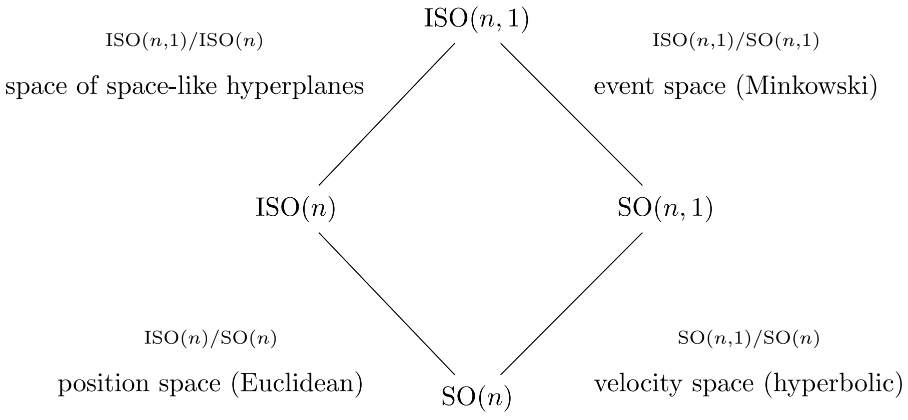
ii. $\mathrm{SO}(n,1)$ connected *Lorentz group* is the group stabilizer of an event in Minkowski spacetime and it consists of boosts and rotations given by the set of Lorentz transformations and it includes only generators of boosts B_i and rotations J_i for each $i = 1, ..., n+1$.

$$x^\mu \ \mapsto\ x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu \qquad \Lambda \in \mathrm{SO}(n,1)\ ,\ \mathbf{x} = (x^\mu) \in \mathbb{R}^{n,1}$$

iii. $\mathrm{ISO}(n)$ group of Euclidean transformations is the stabilizer of a space-like hyperplane and $\mathrm{SO}(n)$ group of spacial rotations around an event and it is the stabilizer of an event and a velocity.

$$\mathrm{ISO}(n,1) = \mathrm{SO}(n,1) \ltimes \mathbb{R}^{n,1} \tag{4.4}$$

This gives us a piece of the lattice of subgroups of the Poincaré group, with corresponding homogeneous spaces given by the associated coset spaces:



For the purposes of this treatise "Klein geometries" will be certain types of homogeneous spaces. The geometries we are interested in are all "smooth" geometries so we require the symmetry group G being a Lie group. In addition to this the subgroup H needs to be a closed subgroup of G, this is obviously necessary if we want the quotient G/H to have a topology where 1-point subsets are closed sets. In fact, the condition that H be closed in G suffices to guarantee H is a Lie subgroup and G/H is a smooth homogeneous manifold. We also want Klein geometries to be connected.

Definition 4.1.3. A (smooth, connected) *Klein geometry* (G, H) consists of a Lie group G with closed subgroup H, such that the left coset space X is connected

$$X := G/H \ .$$

G acts transitively on the homogeneous space X and we may think $H \subset G$ as the stabilizer subgroup of a point in X. In particular, we say that a Klein geometry is *reductive* if there is an $\mathrm{Ad}(H)$-module decomposition given by $\mathfrak{p}$ such that:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} \ . \tag{4.5}$$

Example 4.1.3. For $G = \mathrm{ISO}(n)$ n-dimensional Euclidean group; $H = \mathrm{SO}(n)$ the orthogonal group, then their homogeneous space is an n-dimensional Cartesian space:

$$X \simeq \mathbb{R}^n \ . \tag{4.6}$$

Analogously for the Poincaré group $G = \mathrm{ISO}(n, 1)$ and the Lorentz group $H = \mathrm{SO}(n, 1)$ their homogeneous space is an $(n + 1)$-dimensional Minkowski space:

$$X \simeq \mathbb{R}^{n,1} \ . \tag{4.7}$$

In particular the relation between the Lie algebras of Poincaré and Lorentz group is given by the vector spaces direct sum reported in next expression:

$$\mathfrak{iso}(n,1) \simeq \mathfrak{so}(n,1) \oplus \mathbb{R}^{n,1} \ . \tag{4.8}$$

Moreover, we can see a Klein geometry (G, H) as the principal right H-bundle $G \to G/H$, this is a principal bundle since the fibers are simply the left cosets of H by elements of G, and these cosets are isomorphic to H as right homogeneous sets. To be precise, a Klein geometry (G, H) is a principal right H-bundle (P, X, π, H) which is isomorphic to the principal H-bundle $G \to G/H$ as it is expressed

$$\begin{tikzcd}
P \arrow[r, "\sim"] \arrow[d, "\pi"'] & G \arrow[d] \\
X \arrow[r, "\sim"] & G/H
\end{tikzcd}$$

The isomorphism in question is not canonical. For our purposes, however, it will actually be convenient to have an obvious base-point in the Klein geometry. Since we are interested in approximating the local geometry of a manifold by placing a Klein geometry tangent to it, the preferred base-point $H \in G/H$ will serve naturally as the "point of tangency".

Definition 4.1.4. Given a finite n-dimensional Lie algebra $\mathfrak{g}$ over real field and its adjoint representation we define the *Killing form on* $\mathfrak{g}$ as the bilinear symmetric 2-form

$$\begin{aligned} B : \mathfrak{g} \times \mathfrak{g} &\to \mathbb{R} \\ (v, w) &\mapsto B(v, w) := \mathrm{tr}(\mathrm{ad}_v \circ \mathrm{ad}_w) \end{aligned} \tag{4.9}$$

In a given basis $\{e_1, .., e_n\}$ of $\mathfrak{g}$ and its dual $\{e^1, ..., e^n\}$ of $\mathfrak{g}^*$ we can express it as

$$B = B_{ij} e^i \otimes e^j = c^n_{jm} c^m_{jn} e^i \otimes e^j \tag{4.10}$$

where the structure constants of $\mathfrak{g}$ are given by

$$\mathrm{ad}(e_i) \circ \mathrm{ad}(e_j) e_k = [e_i, [e_j, e_k]] = c^n_{jm} c^m_{jk} e_n \tag{4.11}$$

In addition to this, as shown at page 163 reference [11], the total space $T(G/H)$ of tangent bundle of $\pi : G \to G/H$ is isomorphic to the associated vector G-bundle to G/H via adjoint representation $G \times_H \mathfrak{g}/\mathfrak{h}$ as shown in the commutative diagram below.

$$\begin{tikzcd}
T(G/H) \arrow[r, "\sim"] \arrow[dr] & G \times_H \mathfrak{g}/\mathfrak{h} \arrow[d, "\pi"] \\
 & G/H
\end{tikzcd}$$

The space of tangent vectors at any point in the Klein geometry G/H may be identified with $\mathfrak{g}/\mathfrak{h}$, and an $\mathrm{Ad}(H)$-invariant metric on $\mathfrak{g}/\mathfrak{h}$ induces an homogeneous metric on the tangent bundle $T(G/H)$. In physically interesting examples, this metric will generally be non-degenerate of Riemannian or Lorentzian signature. One way to obtain such a metric is to use the Killing form on $\mathfrak{g}$, which is invariant under $\mathrm{Ad}(G)$, hence under $\mathrm{Ad}(H)$, and passes to a metric on $\mathfrak{g}/\mathfrak{h}$. When $\mathfrak{g}$ is semisimple the Killing form is non-degenerate. But even when $\mathfrak{g}$ is not semisimple, it may be possible to find a non-degenerate H-invariant metric on $\mathfrak{g}/\mathfrak{h}$, hence on $T(G/H)$. This leads us to define:

Definition 4.1.5. A *metric Klein geometry* (G, H, η) is a Klein geometry (G, H) equipped with a possibly degenerate metric η on $\mathfrak{g}/\mathfrak{h}$

$$\begin{aligned} \eta : \mathfrak{g}/\mathfrak{h} \times \mathfrak{g}/\mathfrak{h} &\to \mathbb{R} \\ (v, w) &\mapsto \eta(v, w) \end{aligned} \tag{4.12}$$

which is $\mathrm{Ad}(H)$-invariant for each $X \in \mathfrak{h}$

$$\eta(\mathrm{ad}_X\, v, w) + \eta(v, \mathrm{ad}_X\, w) = 0 \tag{4.13}$$

Notice that any Klein geometry can be made into a metric Klein geometry in a trivial way by setting $\eta = 0$. In cases of physical interest, it is usually possible to choose η to be non-degenerate.

Homogeneous model spacetime

For MacDowell-Mansouri gravity, there are 4 homogeneous spacetimes we are most interested in, which corresponds to Lorentzian or Riemannian gravity with cosmological constant either positive or negative. These are the de-Sitter, anti de-Sitter, spherical, and hyperbolic models. We can also consider the $\Lambda \to 0$ limits of these, the Minkowski and Euclidean models. This gives us six homogeneous model spacetimes, each of which can be described as a Klein geometry G/H where G is the symmetry group and H the stabilizer subgroup:

	$\Lambda < 0$	$\Lambda = 0$	$\Lambda > 0$
Lorentzian	anti de-Sitter $\mathrm{SO}(3,2)/\mathrm{SO}(3,1)$	Minkowski $\mathrm{ISO}(3,1)/\mathrm{SO}(3,1)$	de-Sitter $\mathrm{SO}(4,1)/\mathrm{SO}(3,1)$
Riemannian	hyperbolic $\mathrm{SO}(4,1)/\mathrm{SO}(4)$	Euclidean $\mathrm{ISO}(4)/\mathrm{SO}(4)$	spherical $\mathrm{SO}(5)/\mathrm{SO}(4)$

For many purposes, these spacetimes can be dealt with simultaneously, with the cosmological constant as a parameter. Let us focus on the three Lorentzian cases in what follows; their Riemannian counterparts can be handled in the same way. In their fundamental representations, the Lie algebras $\mathfrak{so}(4,1)$, $\mathfrak{iso}(3,1)$ and $\mathfrak{so}(3,2)$ consist of 5×5 matrices of the form:

$$\begin{bmatrix} 0 & b^1 & b^2 & b^3 & p^0/\ell \\ b^1 & 0 & j^3 & -j^2 & p^1/\ell \\ b^2 & -j^3 & 0 & j^1 & p^2/\ell \\ b^3 & j^2 & -j^1 & 0 & p^3/\ell \\ \epsilon p^0/\ell & -\epsilon p^1/\ell & -\epsilon p^2/\ell & -\epsilon p^3/\ell & 0 \end{bmatrix} = j^i J_i + b^i B_i + \frac{1}{\ell}\, p^a P_a .$$

Here J_i, B_i are generators of rotations and boosts, $P_a = (P_0, P_i)$ are generators of translations while ϵ is the sign of the cosmological constant:

$$\epsilon = \begin{cases} 1 & \text{if} \quad \mathfrak{g} = \mathfrak{so}(4,1) \\ 0 & \text{if} \quad \mathfrak{g} = \mathfrak{iso}(3,1) \\ -1 & \text{if} \quad \mathfrak{g} = \mathfrak{so}(3,2) \end{cases}$$

and we have introduced a length scale ℓ so that p^a may be identified with the components of a "translation vector" on the homogeneous spacetime. Commutation relations are of course dependent on ϵ :

$$\begin{array}{lll} [J_i, J_j] = -\varepsilon_{ijk} J^k & & \\ [B_i, J_j] = \varepsilon_{ijk} B^k & [B_i, B_j] = \varepsilon_{ijk} J^k & \\ [P_i, J_j] = -\varepsilon_{ijk} P^k & [P_i, B_j] = -P_0 \delta_{ij} & [P_0, J_i] = 0 \\ [P_i, P_j] = -\epsilon \varepsilon_{ijk} J^k & [P_0, P_i] = -\epsilon B_i & [P_0, B_i] = -P_i \end{array}$$

All of these model spacetimes are naturally non-degenerate metric Klein geometries. For the cases with $\epsilon \neq 0$, we can equip the simple Lie algebra $\mathfrak{g}$ with a non-degenerate invariant metric which is proportional to the Killing form:

$$\langle \xi, \zeta \rangle = -\frac{\epsilon}{2} \operatorname{tr}(\xi \zeta) \tag{4.14}$$

with respect to this metric, we have the orthogonal, Ad(SO(3, 1))-invariant direct sum decomposition:

$$\mathfrak{g} = \mathfrak{so}(3,1) \oplus \mathfrak{p} \ , \quad \mathfrak{p} \simeq \mathfrak{g}/\mathfrak{so}(3,1) \simeq \mathbb{R}^{3,1} \tag{4.15}$$

where the subalgebra $\mathfrak{so}(3,1)$ is spanned by the rotation and boost generators J_i, B_i, and the complement $\mathfrak{p}$ is spanned by the P_a. The restriction of the metric to $\mathfrak{p}$ is the Minkowski metric with signature $(-+++)$, and we obtain a metric Klein geometry, as defined in the previous section, simply by translating this metric around the homogeneous space.

The choice of scale ℓ (and $\epsilon = \pm 1$) effectively selects the value of the cosmological constant to be

$$\Lambda = \frac{3\epsilon}{\ell^2} \ . \tag{4.16}$$

To see this, let us take a closer look at the de Sitter case, where $\Lambda > 0$. De-Sitter spacetime is most easily pictured as the 4-dimensional submanifold of 5d Minkowski space given by the hyperboloid

$$M_{\mathrm{dS}} = \left\{ (t, w, x, y, z) \in \mathbb{R}^{4,1} \;\middle|\; -t^2 + w^2 + x^2 + y^2 + z^2 = \frac{3}{\Lambda} \right\} \ .$$

The group $G = \mathrm{SO}(4,1)$ acts in the usual way on the ambient (4+1)-dimensional space, and the subgroup $H \simeq \mathrm{SO}(3,1)$ in the upper 4×4 block is the stabilizer of the point

$$x_o = (0, \ldots, 0, \sqrt{3/\Lambda}).$$

The intention of introducing the length scale ℓ is that the element $\frac{1}{\ell}p^\mu P_\mu \in \mathfrak{p}$ should be identified with $p^\mu \partial_\mu \in T_{x_o}M_{\mathrm{dS}}$ in the Klein geometry, via the exponential map. That is,

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{t=0} \exp\left(\frac{s}{\ell}\; p^\mu\;\, P_\mu\right) x_o = \frac{1}{\ell}\;\sqrt{\frac{3}{\Lambda}}\; p^\mu \partial_\mu$$

should equal $p^\mu \partial_\mu$, and hence we should take the form $\Lambda = 3/\ell^2$. The expression $\eta_{ab}v^a w^b$ may then be interpreted either as the metric (4.14) applied to $v, w \in \mathfrak{p}$ or as the metric of de Sitter space applied to the counterparts of v, w tangent to M_{dS} at x_o. The argument for the $\Lambda < 0$ case is the same except for a sign, and in either case we obtain the claimed value (4.16) for the cosmological constant.

When the cosmological constant vanishes, the situation is a bit more subtle. The isometry subgroup of Minkowski spacetime, $\mathrm{ISO}(3,1)$, does not have a non-degenerate adjoint-invariant metric on its Lie algebra. In fact, the metric induced by the trace vanishes on the subspace corresponding to $\mathbb{R}^{3,1}$. However, we require a metric on this subspace to be invariant only under $\mathrm{SO}(3,1)$, not under the full Poincaré group. Such metric is given by the direct sum of the Poincaré Lie algebra $\mathfrak{iso}(3,1) \simeq \mathfrak{so}(3,1) \oplus \mathbb{R}^{3,1}$. Using the trace on $\mathfrak{so}(3,1)$ together with the usual Minkowski metric on $\mathbb{R}^{3,1}$ gives an non-degenerate $\mathrm{SO}(3,1)$-invariant metric on the entire Poincaré Lie algebra. In particular, the metric on the $\mathbb{R}^{3,1}$ part makes $\mathrm{ISO}(3,1)/\mathrm{SO}(2)$ into a non-degenerate metric Klein geometry. Notice that in the Minkowski case, as in de-Sitter or anti de-Sitter, identifying the "translation" subspace of $\mathfrak{iso}(3,1)$ with spacetime tangent vectors still involves choosing a length ℓ by which to scale vectors. But now this choice is not constrained by the value of the cosmological constant. This points out a key difference between the $\Lambda = 0$ and $\Lambda \neq 0$ cases: Minkowski spacetime has an extra "rescaling" symmetry that is broken as the cosmological constant becomes nonzero.

When M is one of our homogeneous model spacetimes, one can, of course, calculate the Riemann curvature for the metric $g_{\mu\nu}$ on TM induced by the metric η_{ab} on $\mathfrak{g}/\mathfrak{h}$. The result is:

$$R_{\mu\nu\rho\sigma} = \frac{\Lambda}{3}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \tag{4.17}$$

Equivalently, if e^a_μ is a coframe field, locally identifying each T_xM with $\mathfrak{g}/\mathfrak{h} \simeq \mathbb{R}^{3,1}$, our homogeneous spacetimes satisfy:

$$R^{ab}{}_{\mu\nu} = \frac{\Lambda}{3}\left(e^a_\mu e^b_\nu - e^a_\nu e^b_\mu\right) \tag{4.18}$$

In this work Wise use form notation rather than spacetime indices; his conventions are given by:

$$R^{ab} = \frac{1}{2}R^{ab}{}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{\Lambda}{3}\left(e^a_\mu\,\mathrm{d}x^\mu\right) \wedge \left(e^b_\nu\,\mathrm{d}x^\nu\right) = \frac{\Lambda}{3}\; e^a \wedge e^b\ . \tag{4.19}$$

Wise often suppress internal indices as well, so the local condition for spacetime to be homogeneous with cosmological constant Λ may thus be written simply

$$R = \frac{\Lambda}{3} e \wedge e\ . \tag{4.20}$$

4.2 Cartan geometry

While the utility of Klein's perspective on geometry is widely recognized, the spacetime we live in is clearly not homogeneous. This does not mean, however, that Kleinian geometry offers no insight into actual spacetime geometry. Cartan discovered a beautiful generalization of Klein geometry - a way of modeling in-homogeneous spaces as 'infinitesimally Kleinian'. The goal of this section is to explain this idea as it relates to spacetime geometry. Ehresmann connections can be defined in several equivalent ways, here we will use the following as it suits better our purposes.

Definition 4.2.1. An *Ehresmann connection* on a principal right H-bundle $\pi : P \to M$ is an $\mathfrak{h}$-valued 1-form $\omega : TP \to \mathfrak{h}$ defined on P

$$\omega \in \Omega(P, \mathfrak{h}) \tag{4.21}$$

which satisfies the following two properties:

i. $R_h^* \omega = \mathrm{Ad}\left(h^{-1}\right)\omega$ for all $h \in H$;

ii. ω restricts to the Maurer-Cartan form on fibers of P

$$\omega|_{P_x} = \omega_H : TP_x \to \mathfrak{h} \quad \forall\, x \in M \tag{4.22}$$

Here $R_h^*\omega$ denotes the pullback of ω by the right action $R_h : P \to P$, $p \mapsto ph$ of $h \in H$ on P.

Definition 4.2.2. The *curvature* of an Ehresmann connection ω is given by the familiar formula

$$\Omega[\omega] = d\omega + \frac{1}{2}[\omega, \omega] \ \in \ \Omega^2(P, \mathfrak{h})$$

where the bracket of $\mathfrak{h}$-valued forms is defined using the Lie bracket on Lie algebra parts and the wedge product on form parts, which means this is an equation of $\Omega^2(P, \mathfrak{h})$ terms.

Proposition 4.2.1. The *Bianchi identity* for an Ehresmann connection ω with curvature Ω is given by:

$$\mathrm{d}\Omega = [\Omega, \omega] \qquad \Longleftrightarrow \qquad \mathrm{d}_\omega \Omega = 0 \tag{4.23}$$

where we have use the notation $\mathrm{d}_\omega\Omega \equiv \mathrm{d}\Omega + [\omega, \Omega]$.

Proof. Taking the exterior derivative of $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$, we get

$$\begin{aligned}
\mathrm{d}\Omega &= 0 + \tfrac{1}{2}\{[\mathrm{d}\omega, \omega] - [\omega, \mathrm{d}\omega]\} \\
&= \tfrac{1}{2}\left\{\left[\Omega - \tfrac{1}{2}[\omega, \omega], \omega\right] - \left[\omega, \Omega - \tfrac{1}{2}[\omega, \omega]\right]\right\} \\
&= \tfrac{1}{2}\left\{[\Omega, \omega] - \tfrac{1}{2}[[\omega, \omega], \omega] - [\omega, \Omega] + \tfrac{1}{2}[\omega, [\omega, \omega]]\right\} \\
&= \tfrac{1}{2}\{[\Omega, \omega] - [\omega, \Omega]\} \\
&= [\Omega, \omega] \quad \text{by graded commutativity.}
\end{aligned}$$

□

Definition 4.2.3. A *Cartan geometry* $(\pi : P \to M, A)$ modeled on the Klein Geometry (G, H) is a principal right H-bundle $P \xrightarrow{\pi} M$ equipped with a $\mathfrak{g}$-valued 1-form A on P

$$A : TP \to \mathfrak{g} \tag{4.24}$$

called *Cartan connection*, satisfying three properties:

i. For each $p \in P, A_p : T_pP \to \mathfrak{g}$ is a linear isomorphism;

ii. $(R_h)^* A = \mathrm{Ad}\left(h^{-1}\right) A \quad \forall\, h \in H;$

iii. A takes values in the subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ on vertical vectors, and in fact restricts to the Maurer-Cartan form $\omega_H : TP_x \to \mathfrak{h}$ on fibers of P.

Compare this definition to the definition of Ehresmann connection. The most obvious difference is that the Cartan connection on P takes values not in the Lie algebra $\mathfrak{h}$ of the gauge group of the bundle, but in the larger algebra $\mathfrak{g}$. The addition of the first requirement in the above definition has important consequences. G must be chosen to have the same dimension as T_pP, so the Klein geometry G/H must have the same dimension as M. In this way Cartan connections have a more concrete relationship to the base manifold than Ehresmann connections, which have no such dimensional restrictions.

Cartan geometries also inherit any additional structures on the tangent spaces of their model Klein geometries. In particular, when G/H is a metric Klein geometry, i.e. when it is equipped with an H-invariant metric on $\mathfrak{g}/\mathfrak{h}$, M inherits a metric of the same signature via the isomorphism $T_xM \simeq \mathfrak{g}/\mathfrak{h}$, which comes from the isomorphism $T_pP \simeq \mathfrak{g}$. The isomorphisms $A : T_pP \to \mathfrak{g}$ may be inverted at each point to give an injection X_A so any element of $\mathfrak{g}$ gives a vector field on P denoted:

$$X_A : \mathfrak{g} \to \mathrm{Vect}(P)\ . \tag{4.25}$$

The restriction of X_A to the subalgebra $\mathfrak{h}$ gives vertical vector fields on P, while the restriction of X_A to a complement $\mathfrak{p}$ of $\mathfrak{h}$ gives vector fields on the base manifold M.

$$X_A|_{\mathfrak{h}} : \mathfrak{h} \to V_pP \quad \text{and} \quad X_A|_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{X}(P)\ .$$

Definition 4.2.4. The *curvature* of a Cartan connection is given by the same formula as in the case of Ehresmann connections, which is the one below:

$$F[A] = \mathrm{d}A + \frac{1}{2}[A, A] \ \in\ \Omega^2(P, \mathfrak{g}). \tag{4.26}$$

This curvature is a 2-form valued in the Lie algebra $\mathfrak{g}$, in case the curvature of a Cartan connection vanishes for every point $p \in P$ then we call it *flat* Cartan geometry. It can be composed with the canonical projection onto $\mathfrak{g}/\mathfrak{h}$ and the composite T is called *torsion*:

$$\begin{tikzcd}\Lambda^2(TP) \arrow[r, "F"] \arrow[rr, bend right, "T"'] & \mathfrak{g} \arrow[r] & \mathfrak{g}/\mathfrak{h}\end{tikzcd} \tag{4.27}$$

We can consider it as the natural generalization of the sort of "torsion" from Riemannian geometry.

Example 4.2.1. The simplest examples of Cartan geometries are Klein geometries. Indeed, if G is a Lie group, then its Maurer-Cartan form ω_G is a canonical Cartan connection for the Klein geometry $G \to G/H$, for any closed subgroup $H \subseteq G$. The structural equation for the Maurer-Cartan form,

$$\mathrm{d}\omega_G = -\frac{1}{2}\left[\omega_G, \omega_G\right]$$

is interpreted in this context as the statement of vanishing Cartan curvature.

Reductive Cartan geometry

The most important special case of Cartan geometry for our purposes is the "reductive" case. Since $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$, we can always write as vector spaces: $\mathfrak{g} \simeq \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$.

Definition 4.2.5. A Cartan geometry is said to be *reductive* if the quotient $\mathfrak{g}/\mathfrak{h}$ may be identified with an $\mathrm{Ad}(H)$-invariant subspace of $\mathfrak{g}$, in other words, when the geometry is reductive, the above direct sum is a direct sum of $\mathrm{Ad}(H)$-representations. In the case of a *reductive Cartan connection* A we can write:

$$A = \omega + e \quad \text{with} \quad \omega \in \Omega^1(P, \mathfrak{h}) \ , \ e \in \Omega^1(P, \mathfrak{g}/\mathfrak{h}) \ . \tag{4.28}$$

Diagrammatically, the $\mathfrak{h}$-valued form ω is an Ehresmann connection on P and we interpret the $\mathfrak{g}/\mathfrak{h}$-valued form e as a *generalized coframe field* defined on P since for homogeneous space $TX \simeq \mathfrak{g}/\mathfrak{h}$ point-wise.

$$\begin{tikzcd} & & \mathfrak{h} \\ TP \arrow[r, "A"] \arrow[urr, bend left, "\omega"] \arrow[drr, bend right, "e"'] & \mathfrak{g} \arrow[ur] \arrow[dr] & \\ & & \mathfrak{g}/\mathfrak{h}\end{tikzcd}$$

The concept of a reductive Cartan connection provides MacDowell-Mansouri action a geometric foundation. In particular, it gives global meaning to the trick of combining the local connection and coframe field 1-forms of general relativity into a connection valued in a larger Lie algebra. Physically, for theories like MacDowell-Mansouri, the reductive case is most important because gauge transformations of the principal H-bundle act on $\mathfrak{g}$-valued forms via the adjoint action. The $\mathrm{Ad}(H)$-invariance of the decomposition says gauge transformations do not mix up the "connection" part with the "coframe" part of a reductive Cartan connection.

The curvature F of a reductive Cartan connection A is given by

$$\begin{aligned} F[A] &= \mathrm{d}A + \tfrac{1}{2}[A,A] \\ &= (\mathrm{d}\omega + \tfrac{1}{2}[\omega,\omega]) + \tfrac{1}{2}[e,e] + (\mathrm{d}e + [\omega,e]) \\ &= R[\omega] + e \wedge e + \mathrm{d}_\omega e \\ &\equiv \widehat{F} + T \ . \end{aligned}$$

One can of course use the $\mathrm{Ad}(H)$-invariant decomposition of $\mathfrak{g}$ to split any other $\mathfrak{g}$-valued differential form into $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ parts and we can split the curvature of the Cartan connection:

$$\begin{aligned} \widehat{F} &\equiv R[\omega] + e \wedge e \ \in \ \Omega^2(P,\mathfrak{h}) \\ T &\equiv \mathrm{d}_\omega e = \mathrm{d}e + [\omega,e] \ \in \ \Omega^2(P,\mathfrak{g}/\mathfrak{h}) \end{aligned}$$

The $\mathfrak{g}/\mathfrak{h}$ part T is the *torsion* while the $\mathfrak{h}$ part $\widehat{F}$ is related to the curvature of the Ehresmann connection ω, but there is an important difference, indeed, the *corrected curvature* $\widehat{F}$ is the Ehresmann curvature modified in such a way that it vanishes when the geometry is locally that of G/H.

$$\begin{tikzcd} & & \mathfrak{h} \\ \Lambda^2(TP) \arrow[rru, bend left, "\widehat{F}"] \arrow[r, "F"] \arrow[rrd, bend right, "T"'] & \mathfrak{g} \arrow[ru] \arrow[rd] & \\ & & \mathfrak{g}/\mathfrak{h} \end{tikzcd}$$

It's notable to point out that the Cartan connection A gives M a geometry locally the same as the Klein geometry G/H if $\widehat{F} = 0$ in order to have non-vanishing torsion T, the generalized coframe field e then is non-degenerate, meaning $e : T_xM \to \mathfrak{p}$ is invertible for all $x \in M$.

Example 4.2.2. To understand this claim, consider the most basic example of a "flat" Cartan geometry: the Klein geometry G/H itself, whose canonical Cartan connection is the Maurer-Cartan form ω_G. When the geometry is reductive, we can write $\omega_G = \omega_H + e$ such that:

$$\begin{tikzcd} & & \mathfrak{h} \\ TG \arrow[rru, bend left, "\omega_H"] \arrow[r, "\omega_G"] \arrow[rrd, bend right, "e"'] & \mathfrak{g} \arrow[ru] \arrow[rd] & \\ & & \mathfrak{g}/\mathfrak{h} \end{tikzcd}$$

where ω_H is the Maurer-Cartan form on the fibers of $G \to G/H$. By the structural equation, G/H is "flat" in the Cartanian sense. In particular, this means both $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ parts of the curvature must vanish even though the geometry certainly does not need to be "*flat*" in the Riemannian sense.

Reductive Klein model spacetimes

To make these ideas more concrete, let us work out the components of the curvature in the cases most relevant to gravity. The six Klein model spacetimes listed before: de Sitter, Minkowski, anti de Sitter are reductive and for any of these models, the Cartan connection is an $\mathfrak{g}$-valued 1-form $\mathcal{A}$ on a principal H-bundle, which we take to be the frame bundle FM on spacetime

$$\mathcal{A} \in \Omega(FM, \mathfrak{g}) \ . \tag{4.29}$$

As purposed by O' Farrill [5] in gauge theory formulation, we can work on the base space M by pulling back the Cartan connection one form of the given principal bundle, through a chosen section σ of it, in order to introduce new objects we will call gauge fields in analogy with the Ehresmann connection case:

$$A := \sigma^*\mathcal{A} \ \in \Omega(M, \mathfrak{g}) \ . \tag{4.30}$$

Here the connection one forms are Cartan connections thus gauge fields are Cartan connections as well. From now on we will adopt this notation: Cartan connections will be denoted by tilde notation $\mathcal{A}$ defined over the total space FM, while we will write A its gauge field defined over the base space M. In case of a reductive Cartan connection, thanks to linearity of pullback application, we can write

$$A = \sigma^*\mathcal{A} = \sigma^*(\mathcal{W} + \mathcal{E}) := \omega + e$$

We call e *coframe field* the pullback of the generalized coframe field $\mathcal{E}$, which was the previous $\mathfrak{g}/\mathfrak{h}$ part of the reductive Cartan connection and ω the pullback of the Ehresmann connection $\mathcal{W}$

$$e := \sigma^*\mathcal{E} \ \in \Omega(M, \mathfrak{g}/\mathfrak{h}) \ , \quad \omega := \sigma^*\mathcal{W} \ \in \Omega(M, \mathfrak{h}) \ . \tag{4.31}$$

This kind of construction allows us to pass from vector valued one form defined over the total space P of the principal bundle to the base space M, which we will further interpret as spacetime of our theory. In index notation Latin letters $a, b = \{0, 1, 2, 3\}$ represents homogeneous spacetime models Lie algebra $\mathfrak{g}$ indices, if we identify $\mathfrak{g}/\mathfrak{h}$ with 4d-Minkowski spacetime $\mathbb{R}^{3,1}$ by picking a unit of length ℓ we can write the two parts of the Cartan connection

$$A^a_b = \omega^a_b \quad , \quad A^a_4 = \frac{-\epsilon}{\ell} e^a \qquad \Longrightarrow \qquad A^4_b = \frac{1}{\ell} e_b \tag{4.32}$$

This result can be expressed as well in matrix formalism as 5×5 real matrix Lie algebra $\mathfrak{g}$, indeed when $\epsilon = \pm 1$ we are dealing with $\mathfrak{so}(4,1)$ or $\mathfrak{so}(3,2)$ respectively thus we have that the matrix

$$A^I_J = \begin{bmatrix} A^a_b & A^4_b \\ A^a_4 & 0 \end{bmatrix} = \begin{bmatrix} \omega^a_b & \frac{1}{\ell} e_b \\ \frac{-\epsilon}{\ell} e^a & 0 \end{bmatrix} \tag{4.33}$$

It satisfies SO-group defining relation, reported below, where appears the metric tensor G_{IJ}

$$AG + GA^T = 0 \ , \quad G_{IJ} = \begin{bmatrix} \eta_{ab} & 0 \\ 0 & \epsilon \end{bmatrix} \tag{4.34}$$

with capital letters indices $I, J = \{0, 1, 2, 3, 4\}$. In particular we have

$$\begin{aligned} {A^a}_4 G_{ab} + {A^a}_b G_{a4} &= 0 \\ \left(-\frac{\epsilon}{\ell} e^a\right) G_{ab} + A_{4b} &= 0 \\ -\frac{\epsilon}{\ell} e_b + A_{4b} &= 0 \end{aligned}$$

applying G to both sides we get ${A^4}_b$ last part of the connection

$$G^{44} A_{4b} = G^{44} \left(\frac{\epsilon}{\ell} e_b\right) \implies {A^4}_b = \frac{\epsilon^2}{\ell} e_b = \frac{1}{\ell} e_b$$

since the squared value of cosmological constant sign is given by

$$\epsilon^2 = \epsilon \cdot \epsilon = \begin{cases} +1 & \text{if } \mathfrak{g} = \mathfrak{so}(4,1) \text{ or } \mathfrak{so}(3,2) \\ 0 & \text{if } \mathfrak{g} = \mathfrak{iso}(3,1) \end{cases}$$

The arguments we have just applied for Cartan connections $\mathcal{A}$ count for their curvature forms $\mathcal{F}$ too, then curvature of a Cartan connection will be interpreted as gauge field strength $F[A]$

$$F[A] = \sigma^* \mathcal{F}[\mathcal{A}] = \widehat{F} + T \ \in \Omega^2(M, \mathfrak{g}) \tag{4.35}$$

and its decomposition for reductive case is

$$\begin{aligned} \widehat{F} &= R - \frac{\epsilon}{\ell^2} e \wedge e \ \in \ \Omega^2(M, \mathfrak{h}) \ , \\ T &= \frac{1}{\ell} \ \mathrm{d}_\omega e \ \in \ \Omega^2(M, \mathfrak{g}/\mathfrak{h}) \ . \end{aligned} \tag{4.36}$$

In matrix formalism the curvature of the Cartan connection, interpreted as gauge field strength, will be given by the following expression:

$$F^I_J = \mathrm{d}A^I_J + A^I_K \wedge A^K_J = \begin{bmatrix} F^a_b & F^4_b \\ F^a_4 & 0 \end{bmatrix} = \begin{bmatrix} R^a_b - \frac{\epsilon}{\ell^2} e^a \wedge e_b & \frac{1}{\ell} \mathrm{d}_\omega e_b \\ \frac{-\epsilon}{\ell} \mathrm{d}_\omega e^a & 0 \end{bmatrix} \tag{4.37}$$

indeed, for the $\mathfrak{so}(3,1)$ part we get

$$\begin{aligned} F^a_b &= \mathrm{d}A^a_b + A^a_c \wedge A^c_b + A^a_4 \wedge A^4_b \\ &= \mathrm{d}\omega^a_b + \omega^a_c \wedge \omega^c_b - \frac{\epsilon}{\ell^2} e^a \wedge e_b \\ &= R^a_b - \frac{\epsilon}{\ell^2} e^a \wedge e_b \ , \end{aligned} \tag{4.38}$$

where R is the pull-back of the curvature of $\mathrm{SO}(3,1)$-Ehresmann connection ω, while for the $\mathbb{R}^{3,1}$ part:

$$\begin{aligned} F^a_4 &= \mathrm{d}A^a_4 + A^a_b \wedge A^b_4 \\ &= -\frac{\epsilon}{\ell}\left(\mathrm{d}e^a + \omega^a_b \wedge e^b\right) \\ &= -\frac{\epsilon}{\ell}\ \mathrm{d}_\omega e^a\ , \end{aligned} \tag{4.39}$$

then we have necessarily the third component

$$\begin{aligned} F^4_b &= \mathrm{d}A^4_b + A^4_a \wedge A^a_b \\ &= \frac{1}{\ell}\left(\mathrm{d}e_b + e_a \wedge \omega^a_b\right) \\ &= \frac{1}{\ell}\ \mathrm{d}_\omega e_b\ . \end{aligned} \tag{4.40}$$

The same calculations hold formally in the Riemannian analogs as well, the only difference being that indices are lowered with δ_{ij} rather than η_{ij}. If we agree to use a model spacetime with cosmological constant Λ, the parts of a reductive connection and its curvature can be summarized diagrammatically in the three Lorentzian cases as

$$\begin{tikzcd} & & \mathfrak{so}(3,1) \\ TM \arrow[r, "A"] \arrow[urr, bend left, "\omega"] \arrow[drr, bend right, "e/\ell"'] & \mathfrak{g} \arrow[ur] \arrow[dr] & \\ & & \mathbb{R}^{3,1} \end{tikzcd} \qquad \begin{tikzcd} & & \mathfrak{so}(3,1) \\ \Lambda^2(TM) \arrow[r, "F"] \arrow[urr, bend left, "R-\frac{\Lambda}{3}e\wedge e"] \arrow[drr, bend right, "\frac{1}{\ell}\mathrm{d}_\omega e"'] & \mathfrak{g} \arrow[ur] \arrow[dr] & \\ & & \mathbb{R}^{3,1} \end{tikzcd}$$

We now see clearly what it means for a reductive Cartan connection A to be flat $F = 0$

$$R = \frac{\epsilon}{\ell^2}\ e \wedge e\ ,\ \ \mathrm{d}_\omega e = 0\ . \tag{4.41}$$

Comparing to (4.20), these are precisely the local equations characterizing the torsion free Levi-Civita connection for a homogeneous spacetime, provided we take the cosmological constant $\Lambda = 3\epsilon/\ell^2$. Naturally, this is the cosmological constant (4.16) of the model itself, for $\Lambda = \epsilon = 0$ the value of ℓ^2 is not constrained by Λ, so there is an additional scaling symmetry in Cartan geometry modelled on Minkowski or Euclidean spacetime. It is helpful to think of the cosmological constant (4.16) of the model homogeneous spacetime as a sort of *internal cosmological constant.* The criterion for a spacetime to be flat in the Cartanian sense is then that spacetime have purely cosmological curvature where the spacetime cosmological constant matches the internal one. As a final note on reductive Cartan geometry, we report *Bianchi identity* for a reductive Cartan connection pull-back A with curvature F splits up into:

$$\mathrm{d}_A F = 0 \qquad \Longleftrightarrow \qquad \mathrm{d}_\omega R = 0\ ,\ \ \mathrm{d}^2_\omega e = R \wedge e\ . \tag{4.42}$$

Chapter 5

MacDowell-Mansouri gravity

Part of the case Derek Wise [12] wish to make is that gravity, particularly MacDowell–Mansouri formulation, should be seen as based on a type of gauge theory where the connection is not an Ehresmann connection but a Cartan connection. Indeed, they give a concrete correspondence between spacetime and a Kleinian model, in a way that is ideally suited to a geometric theory like gravity. In this section, I discuss in first place BF theory as a gauge topological theory which will be used as build block to formalize Cartan-type gauge theories and then MacDowell-Mansouri one. In the middle there is a passage focused on Palatini formalism which help us to translate Einstein-like approach for general relativity to Cartan's one.

5.1 BF Theory and flat connections

As an example of a gauge theory with Ehresmann connection we report a topological gauge theory known as BF-theory, which we will use as starting point to introduce MacDowell-Mansouri approach.

G – gauge group as a Lie group with stabilizer H
M – spacetime as n-dimensional smooth manifold
(P, M, π, H) – smooth principal H-bundle with base space M

The Lie group H has an associated Lie algebra $\mathfrak{h}$ equipped with a non-degenerate $\mathrm{Ad}(\mathfrak{h})$-invariant bilinear form $\eta : \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$. There are two gauge fields which describe the dynamic of this particular situation: a pull-back of an Ehresmann connection A on the principal H-bundle P

$$A : TM \to \mathfrak{h} \ , \tag{5.1}$$

an auxiliary $\mathrm{Ad}(P)$-valued $(n-2)$-form B

$$B \in \Omega^{n-2}(M, \mathrm{Ad}(P)) \ , \tag{5.2}$$

where $\mathrm{Ad}(P)$ is the vector bundle associated to P via the adjoint representation of H on its Lie algebra, also known as adjoint vector bundle $\mathrm{Ad}(P) = P \times_H \mathfrak{h}$. The curvature F of the pull-back of Ehresmann connection A is an $\mathrm{Ad}(P)$-valued $(n-2)$-form B as well and the BF theory action is given by the integral of the BF theory Lagrangian over spacetime M

$$S_{BF} = \int_M \mathrm{tr}(B \wedge F) \ , \tag{5.3}$$

where tr denotes the operation that wedges differential form parts and, at each point, applies the bilinear form η to Lie algebra parts. The bilinear form in question is often the Killing form:

$$\mathrm{tr}(\cdot \wedge \cdot) : \Omega^{n-2}(M, \mathrm{Ad}(P)) \times \Omega^2(M, \mathrm{Ad}(P)) \to \Omega^n(M, \mathbb{R}) \ . \tag{5.4}$$

Taking the variation of the action it gives:

$$\begin{aligned} \delta S_{BF} &= \int_M \mathrm{tr}(\delta B \wedge F + B \wedge \delta F) \\ &= \int_M \mathrm{tr}(\delta B \wedge F + B \wedge \mathrm{d}_A(\delta A)) \\ &= \int_M \mathrm{tr}(\delta B \wedge F + (-1)^n \mathrm{d}_A(B \wedge \delta A) - (-1)^n \mathrm{d}_A B \wedge \delta A) \\ &= (-1)^n \int_{\partial M} \mathrm{tr}(B \wedge \delta A) + \int_M \mathrm{tr}\left(\delta B \wedge F - (-1)^n \mathrm{d}_A B \wedge \delta A\right) \end{aligned}$$

at the third passage we have used the following identity, proven by using the bracket of $\mathfrak{g}$-valued forms:

$$\delta F = \delta\left(\mathrm{d}A + \frac{1}{2}[A, A]\right) = \mathrm{d}\delta A + \frac{1}{2}[\delta A, A] + \frac{1}{2}[A, \delta A] = \mathrm{d}\delta A + [A, \delta A] = \mathrm{d}_A \delta A \ . \tag{5.5}$$

The forth equivalence is provided by the following property of exterior derivative d for a generic p-form ω defined over a smooth manifold M and another arbitrary differential form ρ

$$\mathrm{d}(\omega \wedge \rho) = \mathrm{d}\omega \wedge \rho + (-1)^p \omega \wedge \mathrm{d}\rho \tag{5.6}$$

$$\mathrm{d}_A(B \wedge \delta A) = \mathrm{d}_A B \wedge \delta A + (-1)^{n-2} B \wedge \mathrm{d}_A(\delta A) \tag{5.7}$$

The final passage is provided by Stokes theorem for differential forms defined over a smooth manifold, it states that given a smooth $(n-1)$-form ω on an oriented, n-dimensional manifold M with boundary ∂M with induced orientation satisfies the following relation

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega \ . \tag{5.8}$$

Integrating by parts, when the boundary term vanishes, the field equations for BF theory just say

$$F = 0 \quad \text{flat Ehresmann connection } A \tag{5.9a}$$

$$\mathrm{d}_A B = 0 \quad \text{covariantly closed field } B \tag{5.9b}$$

Cartan-type gauge theory

We wish to copy this picture as much as possible using the pull-back A of a Cartan connection of type G/H instead of the Ehresmann H connection. Doing so requires, first of all, the principal H-bundle (P, M, π, H) to be a Cartan geometry modeled on a Klein geometry (G, H) with homogeneous space G/H of the same dimension as M. The gauge field A is then a Cartan connection defined over M as

$$A \in \Omega^1(M, \mathfrak{g}) \ , \tag{5.10}$$

for what concerns B field, the obvious analog is an $(n-2)$-form with values in the vector bundle

$$\mathrm{Ad}_{\mathfrak{g}}(P) := P \times_H \mathfrak{g}$$

associated to the principal H-bundle $\pi : P \to M$ where H acts on $\mathfrak{g}$ via the restriction of the adjoint representation of G. While in BF theory it's not necessarily used a reductive model, here we choose to do so, the Cartan geometry we deal with is reductive and A decomposes in a coframe field as $\mathfrak{g}/\mathfrak{h}$ part and in an Ehresmann connection ω as $\mathfrak{h}$ part:

$$e \in \Omega^1(M, \mathfrak{g}/\mathfrak{h}) \ , \quad \omega \in \Omega^1(M, \mathfrak{h}) \ . \tag{5.11}$$

Formally, we obtain the same equations of motion but, now, they must be interpreted in the Cartan geometric context, in particular, the first equation states the Cartan connection is flat. In other words, "rolling" the tangent Klein geometry on spacetime is trivial, giving an isometric identification between any neighborhood in spacetime and a neighborhood of the model geometry G/H.

$$\begin{aligned} F = 0 &\quad \text{flat Cartan connection } A \\ \mathrm{d}_A B = 0 &\quad \text{covariantly closed field B} \end{aligned}$$

Let us work out a more explicit example Cartan-type BF theory based on one of the homogeneous reductive models discussed before like de-Sitter one. Each of these geometries is reductive, so we can decompose our $\mathfrak{g}$-valued fields A, F, and B into $\mathfrak{so}(3,1)$ and $\mathbb{R}^{3,1}$ parts. We have already seen:

$$A^a_b = \omega^a_b \ , \ A^a{}_4 = -\frac{\epsilon}{\ell} e^a \ , \tag{5.12}$$

$$F^a_b = R^a_b - \frac{\epsilon}{\ell^2} e^a \wedge e_b \ , \ F^a_4 = -\frac{\epsilon}{\ell} \mathrm{d}_\omega e^a \tag{5.13}$$

For what concerns the auxiliary field we choose the frame bundle as principal $\mathrm{SO}(3,1)$-bundle

$$B = \beta + b \ \in \Omega^2(M, \mathrm{Ad}_{\mathfrak{g}}(FM)) \tag{5.14}$$

$$\beta = \widehat{B} \in \Omega^2(M, \mathrm{Ad}_{\mathfrak{so}(3,1)}(FM)) \ , \ b \in \Omega^2(M, \mathrm{Ad}_{\mathbb{R}^{3,1}}(FM))$$

so that the components of the matrix expression are

$$B^a_b = \beta^a_b \ , \ B^a_4 = -\frac{\epsilon}{\ell} b^a \ , \ B^4_b = \frac{1}{\ell} b_b \tag{5.15}$$

which can be expressed as well in a 5 × 5 real matrix with ϵ the sign of the cosmological constant

$$B^I_J = \begin{bmatrix} B^a_b & B^4_b \\ B^a_4 & 0 \end{bmatrix} = \begin{bmatrix} \beta^a_b & \frac{1}{\ell} b_b \\ -\frac{\epsilon}{\ell} b^a & 0 \end{bmatrix} \tag{5.16}$$

We also need to know how to write $\mathrm{d}_A B$ in terms of these component fields, we know that

$$\begin{aligned} \mathrm{d}_A B^{IJ} &:= \mathrm{d}B^{IJ} + [A,B]^{IJ} \\ &= \mathrm{d}B^{IJ} + A^I{}_K \wedge B^{KJ} - B^I{}_K \wedge A^{KJ} \ , \end{aligned} \tag{5.17}$$

so for both internal indices between 0 and 3 we have

$$\begin{aligned} \mathrm{d}_A B^{ab} &:= \mathrm{d}B^{ab} + A^a{}_c \wedge B^{cb} - B^a{}_c \wedge A^{cb} + A^a{}_4 \wedge B^{4b} - B^a{}_4 \wedge A^{4b} \\ &= \mathrm{d}_\omega \beta^{ab} - \frac{\epsilon}{\ell} e^a \wedge b^b + \frac{\epsilon}{\ell} b^a \wedge e^b \end{aligned} \tag{5.18}$$

and for an internal index 4,

$$\begin{aligned} \mathrm{d}_A B^{a4} = \mathrm{d}_A \beta^{a4} &= \mathrm{d}B^{a4} + A^a{}_b \wedge B^{b4} - B^a{}_b \wedge A^{b4} \\ &= \mathrm{d}_\omega b^a - \frac{1}{\ell} \beta^a{}_b \wedge e^b \ . \end{aligned} \tag{5.19}$$

The equations for BF theory with Cartan connection based on de Sitter, Minkowski, or anti de Sitter model geometry are thus

$$R - \frac{\epsilon}{\ell^2} e \wedge e = 0 \tag{5.20a}$$

$$\mathrm{d}_\omega e = 0 \tag{5.20b}$$

$$\mathrm{d}_\omega \beta + \frac{\epsilon}{\ell^2}(b \wedge e - e \wedge b) = 0 \tag{5.20c}$$

$$\mathrm{d}_\omega b - \frac{1}{\ell} \beta \wedge e = 0 \tag{5.20d}$$

In terms of the constituent fields of the reductive geometry, classical Cartan-type BF theory is thus described by the Levi-Civita connection on a spacetime of purely cosmological curvature, with constant $\Lambda = 3\epsilon/\ell^2$, together with an pair of auxiliary fields β and b, satisfying two equations. We shall encounter equations very similar to these in the BF reformulation of MacDowell-Mansouri gravity.

5.2 From Palatini to MacDowell-Mansouri

Before describing the MacDowell-Mansouri approach, I briefly recall in this section the better known Palatini formalism. The main purpose in doing this is to establish the global differential geometric setting of the Palatini approach, in order to compare to that of MacDowell-Mansouri. In its modern form, Palatini formalism downplays the metric g on spacetime, which takes a subordinate role to the coframe field e, which, in some sense, deserves to be called "*gravitational field*" in general relativity.

Definition 5.2.1. The *fake tangent bundle* $(\mathcal{T}, M, \pi, \mathbb{R}^{3,1})$ is defined as the trivial vector bundle over M with Minkowski spacetime $\mathbb{R}^{3,1}$ as abstract fiber such that it is isomorphic to the tangent bundle

$$\mathcal{T} \simeq TM$$

it is equipped with a fixed metric

$$\eta : \mathcal{T} \times \mathcal{T} \to \mathbb{R} \tag{5.21}$$

and it satisfies the following vector bundle morphism:

$$\begin{tikzcd} TM \arrow[r, "e"] \arrow[dr] & \mathcal{T} \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

The name coframe field comes from the case where TM is trivializable and a possible trivialization is given by the choice of $\mathcal{T}$ as the direct product of M and $\mathbb{R}^{3,1}$ like here

$$e : TM \to \mathcal{T} = M \times \mathbb{R}^{3,1} \tag{5.22}$$

in case e restricts to its value at a point $p \in M$ we get a coframe on each tangent space $e_p : T_pM \to \mathbb{R}^{3,1}$ since $\mathcal{T}$ is locally trivializable we can treat e locally as an $\mathbb{R}^{3,1}$-valued 1-form.

$$e_p \in \Omega^1(M, \mathbb{R}^{3,1}) \tag{5.23}$$

In addition to this, we can see tangent bundle acquires a metric by pulling back the metric on $\mathcal{T}$

$$g(\cdot,\cdot) : \; T_pM \times T_pM \to \mathbb{R} \tag{5.24}$$

such that for any two vectors in the same tangent space T_pM

$$g(v, w) := \eta(ev, ew) \tag{5.25}$$

In index notation coframe field shows both Lorentzian internal index, denoted by Latin letters a, b, and spacetime index, denoted by Greek letters α, β

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \tag{5.26}$$

In the case where the metric g corresponds to a classical solution of general relativity $e : TM \to \mathcal{T}$ is an isomorphism, so that g is non-degenerate. The formalism makes sense when e is any bundle morphism.

Kozul connection in Palatini formalism

When e is an isomorphism, we can also pull a connection ω on the vector bundle $\mathcal{T}$ back to a connection on TM as follows. Given a covariant derivative over the fake tangent bundle

$$D : \mathfrak{X}(M) \times \Gamma(\mathcal{T}) \to \Gamma(\mathcal{T})$$

a linear connection over the fake tangent space $\omega : \mathcal{T} \to \mathfrak{g}$ and a section of the tangent bundle that is a vector field $w : M \to TM$, then the following expression defines a Kozul connection on TM:

$$\nabla_v w := e^{-1} D_v e w \ . \tag{5.27}$$

Indeed, the covariant derivative of a local section s of $\mathcal{T}$ working in coordinates is

$$(D_\mu s)^a = \partial_\mu s^a + \omega^a_{\mu b} s^b \ . \tag{5.28}$$

When e is an isomorphism, we can use D to differentiate a section w of TM in the obvious way: use e to turn w into a section of $\mathcal{T}$, differentiate this section, and use e^{-1} to turn the result back into a section of TM. This defines the connection $\nabla_v w$ on TM for any vector field v. In particular, if $v = \partial_\mu, \nabla_\mu := \nabla_{\partial_\mu}$, we get, in index notation:

$$\begin{aligned}
(\nabla_\mu w)^\alpha &= e^\alpha_a \left(D_\mu e_\beta w^\beta\right)^a \\
&= e^\alpha_a \left(\partial_\mu \left(e^a_\beta w^\beta\right) + \omega^a_{\mu b} e^b_\beta w^\beta\right) \\
&= e^\alpha_a \left(e^a_\beta \partial_\mu w^\beta + \left(\partial_\mu e^a_\beta\right) w^\beta + \omega^a_{\mu b} e^b_\beta w^\beta\right) \\
&= \partial_\mu w^\alpha + \left(e^\alpha_a \partial_\mu e^a_\beta + e^\alpha_a \omega^a_{\mu b} e^b_\beta\right) w^\beta
\end{aligned}$$

Known that for a vector w the covariant derivative respects the expression for affine connection:

$$(\nabla_\mu w)^\alpha = \partial_\mu w^\alpha + \Gamma^\alpha_{\mu\beta} w^\beta$$

The relation between connection ω, coframe fields e^a and affine connection is then:

$$\Gamma^\alpha_{\mu\beta} := e^\alpha_a \left(\delta^a_b \partial_\mu + \omega^a_{\mu b}\right) e^b_\beta \tag{5.29}$$

In an analogous way it is possible to calculate the transformation of the curvature F of ω with a lengthy calculation in the index notation, the result is the following:

$$R^\alpha_{\mu\nu\beta} = e^\alpha_a F^a_{\mu\nu b} e^b_\beta \ . \tag{5.30}$$

The simpler way to see that this formula is true is to compare to the way the curvature in gauge theory transforms under gauge transformations.

Palatini formalism for general relativity

In Palatini formalism we consider a n-dimensional smooth manifold M (with $n = 4$ for our purposes) and a Lie group G as structure group. In this case we deal with the fake tangent bundle $(\mathcal{T}, M, \pi, \mathbb{R}^{3,1})$, which is a smooth vector bundle this implies several things: there are not gauge fields as connections over principal bundles and coframe needs to be defined in a different way since there isn't any reductive Klein geometry. The connection ω over $\mathcal{T}$ defined in the last paragraph here for coherence of notations will be denoted A and its curvature F. The coframe field here is simply a a $\mathcal{T}$-valued vector 1-form

$$e : TM \to \mathcal{T} \tag{5.31}$$

according to the definition of fake tangent bundle we previously gave. The B field in this case is the $\mathfrak{g}$-valued $(n-2)$-form (with $n = 4$) given by $(n-2)$-times exterior product of coframe fields e:

$$e \wedge e \in \Omega^2(M, \mathfrak{g}) \ . \tag{5.32}$$

The Palatini Lagrangian for general relativity, without source terms, is then

$$\operatorname{tr}(e \wedge e \wedge F) \ ,$$

here the wedge product applies both to differential form parts and to the vector bundles in which the forms take values. That is, the wedge product is the obvious bilinear map

$$\wedge : \Omega^k\left(M, \Lambda^s \mathcal{T}\right) \otimes \Omega^\ell\left(M, \Lambda^t \mathcal{T}\right) \to \Omega^{k+\ell}\left(M, \Lambda^{s+t} \mathcal{T}\right)$$

given by anti-symmetrizing both spacetime and internal indices. Thus, since e is a $\mathcal{T}$-valued 1 -form, the $(n-2)$-fold wedge product $e \wedge \cdots \wedge e$ is an $(n-2)$-form with values in the vector bundle $\Lambda^{n-2}\mathcal{T}$. The curvature F is a 2 -form with values in $\Lambda^2\mathcal{T}$. Hence, the expression in parentheses in is a $\Lambda^n\mathcal{T}$-valued n-form on M, and the trace is really a map that turns such a form into an ordinary real-valued form:

$$\operatorname{tr} : \Omega\left(M, \Lambda^n \mathcal{T}\right) \to \Omega(M, \mathbb{R})$$

The Palatini action in our 4-dimensional gravity model is

$$S_{\mathrm{Pal}}(\omega, e) = \frac{1}{2G} \int_M \star \left(e \wedge e \wedge R - \frac{\Lambda}{6} e \wedge e \wedge e \wedge e \right) \tag{5.33}$$

where R is the curvature of ω and the wedge product $\wedge$ acts on both spacetime indices α, β and internal Lorentz indices a, b. Compatibility with the metric η forces the curvature R to take values in $\Lambda^2\mathcal{T}$. Hence, the expression in parentheses is a $\Lambda^4\mathcal{T}$-valued 4-form on M. The $\star$ is the internal Hodge star operator, which turns such a form into an ordinary real-valued 1-form using the volume form and orientation:

$$\star : \Omega\left(M, \Lambda^4 \mathcal{T}\right) \to \Omega(M, \mathbb{R}) \tag{5.34}$$

and it satisfies the following relation with the Killing form for each $\xi, \varsigma \in \mathfrak{g}$ matrix Lie algebra

$$\xi \wedge \star\, \varsigma = \frac{\epsilon}{2}\ \mathrm{tr}(\xi\varsigma)\ \varepsilon_{abcd} = \langle \xi, \varsigma \rangle\ \varepsilon_{abcd}\ . \tag{5.35}$$

With internal indices written explicitly, this action is:

$$S_{\mathrm{Pal}}(\omega, e) = \frac{1}{2G}\int_M \left(e^a \wedge e^b \wedge R^{cd} - \frac{\Lambda}{6} e^a \wedge e^b \wedge e^c \wedge e^d \right) \varepsilon_{abcd} \tag{5.36}$$

Taking the variation of the action, it gives

$$\begin{aligned} \delta S &= \int_M \mathrm{tr}\left(2\delta e \wedge e \wedge R + e \wedge e \wedge \delta R + \frac{2\Lambda}{3} \delta e \wedge e \wedge e \wedge e \right) \\ &= \int_M \mathrm{tr}\left(2\delta e \wedge \left(e \wedge R + \frac{\Lambda}{3} e \wedge e \wedge e \right) + e \wedge e \wedge \mathrm{d}_\omega \delta\omega \right) \\ &= \int_M \mathrm{tr}\left(2\delta e \wedge \left(e \wedge R + \frac{\Lambda}{3} e \wedge e \wedge e \right) \pm \mathrm{d}_\omega (e \wedge e) \wedge \delta\omega \right) \end{aligned}$$

where we used the identity $\delta R = \mathrm{d}_\omega \delta\omega$ and performed an integration by parts. The variations of ω and e give us the respective equations of motion

$$\mathrm{d}_\omega (e \wedge e) = 0\ , \tag{5.37a}$$

$$e \wedge R - \frac{\Lambda}{3} e \wedge e \wedge e = 0\ . \tag{5.37b}$$

In the classical case where e is an isomorphism, the first of these equations is equivalent to

$$\mathrm{d}_\omega e = 0$$

which says precisely that the induced connection on TM is torsion free, hence that $\Gamma^\alpha_{\mu\beta}$ is the Christoffel symbol for the Levi-Civita connection. The other equation of motion, rewritten in terms of the metric and Levi-Civita connection, is Einstein's equation.

Equivalence of formulations

We see that $e \wedge \mathrm{d}_A e = 0$ if and only if $\mathrm{d}_A e = 0$, since

$$\mathrm{d}_A (e \wedge e) = (\mathrm{d}_A e) \wedge e - e \wedge (\mathrm{d}_A e) = 2\,(\mathrm{d}_A e) \wedge e\ .$$

The equation $\mathrm{d}_A e = 0$ says precisely that ∇ is torsion free. To see this, it is easiest to work locally

$$0 = (\mathrm{d}_A e)^a_{\mu\nu} = (\mathrm{d}e)^a_{\mu\nu} + (A \wedge e)^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + A^a_{\mu b} e^b_\nu - A^a_{\nu b} e^b_\mu$$

Applying e^α_a to this we get Levi-Civita connection or torsion-free condition

$$e^\alpha_a \partial_\mu e^a_\nu + e^\alpha_a A^a_{\mu b} e^b_\nu = e^\alpha_a \partial_\nu e^a_\mu + e^\alpha_a A^a_{\nu b} e^b_\mu \implies \Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$$

which is the usual index-based way of saying that the connection ∇ with Christoffel symbols Γ is torsion free. In fact, ∇ is also metric preserving, since D is, so that ∇ is the Levi-Civita connection on spacetime.

The other equation of motion is Einstein's equation in disguise. In index notation we have

$$\epsilon_{ijkl}\left(e^i_\lambda \wedge R^{jk}{}_{\mu\nu} - \frac{2\Lambda}{3} e^i_\lambda \wedge e^j_\mu \wedge e^k_\nu\right) = 0 \ .$$

Since the curvature tensor is anti-symmetric in spacetime indices and internal indices, we may write

$$R^{jk}{}_{\mu\nu} = R^{jk}{}_{mn} e^m_\mu \wedge e^n_\nu \ .$$

Using this and applying the Hodge star operator on spacetime indices we get

$$\begin{aligned}
0 &= \varepsilon^{\lambda\mu\nu}{}_{\pi}\ \varepsilon_{ijk\ell}\left(R^{jk}{}_{mn} e^i_\lambda \wedge e^m_\mu \wedge e^n_\nu - \frac{2\Lambda}{3} e^i_\lambda \wedge e^j_\mu \wedge e^k_\nu\right) \\
&= \varepsilon_{ijk\ell}\left(R^{jk}{}_{mn}\varepsilon^{imn}{}_{p} - \frac{2\Lambda}{3}\varepsilon^{ijk}{}_{p}\right) e^p_\pi \\
&= \varepsilon_{ijk\ell}\left(R^{jk}{}_{mn}\varepsilon^{imnq} - \frac{2\Lambda}{3}\varepsilon^{ijkq}\right) \eta_{pq} e^p_\pi \\
&= \left(-3!\delta^m_{[j}\delta^n_k\delta^q_{\ell]} R^{jk}{}_{mn} + 4\Lambda\delta^{q\ell}\right) \eta_{pq} e^p_\pi
\end{aligned}$$

where in the second equality we have used

$$\varepsilon^{ijk}{}_{\lambda} = \varepsilon^{ijk}{}_{\ell} e^\ell_\lambda = \varepsilon^{\mu\nu\rho}{}_{\lambda} e^i_\mu e^j_\nu e^k_\rho \tag{5.38}$$

and in the fourth we've used standard contraction identities for Levi-Civita symbols

$$\varepsilon_{ijk\ell}\varepsilon^{imnp} = -1!3!\delta^m_{[j}\delta^n_k\delta^p_{\ell]} \tag{5.39}$$

with the convention for anti-symmetrization over indices which includes the factor $(1/p!)$. The anti-symmetrized δ in the first term serve to contract the curvature into the internal version of the Einstein tensor, as follows:

$$\begin{aligned}
3!\delta^m_{[j}\delta^n_k\delta^q_{\ell]} R^{jk}{}_{mn} &= +\delta^m_j\delta^n_k\delta^q_\ell R^{jk}{}_{mn} + \delta^m_k\delta^n_\ell\delta^q_j R^{jk}{}_{mn} + \delta^m_\ell\delta^n_j\delta^q_k R^{jk}{}_{mn} \\
&\quad - \delta^m_k\delta^n_j\delta^q_\ell R^{jk}{}_{mn} - \delta^m_j\delta^n_\ell\delta^q_k R^{jk}{}_{mn} - \delta^m_\ell\delta^n_k\delta^q_j R^{jk}{}_{mn} \\
&= R^{mn}{}_{mn}\delta^q_\ell + R^{qm}{}_{mn}\delta^n_\ell + R^{nq}{}_{mn}\delta^m_\ell - R^{nm}{}_{mn}\delta^q_\ell - R^{mq}{}_{mn}\delta^n_\ell - R^{qn}{}_{mn}\delta^m_\ell \\
&= 2R\delta^q_\ell - 4R^q{}_\ell = -4G^q_\ell
\end{aligned}$$

where the internal Einstein tensor is given by $G_{mn} = R_{mn} - \frac{1}{2}R\eta_{mn}$. Using this result we get

$$G_{mn} + \Lambda\eta_{mn} = 0 \tag{5.40}$$

or, applying the coframe field to turn internal indices to spacetime indices:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \ . \tag{5.41}$$

5.3 MacDowell-Mansouri gravity

In MacDowell-Mansouri theory [10] we set up a framework which makes Palatini formalism agree with Cartan-type gauge theories, in order to do so we introduce:

$$\begin{aligned} G \quad &- \quad \text{gauge group with closed subgroup } H \text{ as stabilizer} \\ M \quad &- \quad \text{spacetime as 4-dimensional smooth manifold} \\ (\pi : FM \to M, A) \quad &- \quad \text{Cartan geometry modeled on a metric Klein Geometry } (G, H, \eta) \end{aligned}$$

The gauge group G has as stabilizer $H = \mathrm{SO}(3,1)$ and it depends on the sign of the cosmological constant as reported in the expression below:

$$G = \begin{cases} \mathrm{SO}(4,1) & \text{if } \Lambda > 0 \\ \mathrm{SO}(3,2) & \text{if } \Lambda < 0 \end{cases}$$

We will focus on the physically favored $\Lambda > 0$ case, where $G = \mathrm{SO}(4,1)$ and its Lie algebra $\mathfrak{g} = \mathfrak{so}(4,1)$ has a $\mathrm{SO}(3,1)$-invariant Killing-orthogonal splitting as vector spaces

$$\mathfrak{so}(4,1) \simeq \mathfrak{so}(3,1) \oplus \mathbb{R}^{3,1} \ .$$

There is a gauge field A defined over base space M as a reductive $\mathrm{SO}(4,1)$-Cartan connection on the Cartan geometry $(\pi : FM \to M, A)$ given by

$$A = \omega + \frac{1}{\ell} e \ \in \ \Omega^1(M, \mathfrak{so}(4,1)) \tag{5.42}$$

where ℓ is a unit length, ω represents the Ehresmann connection which gives the $\mathfrak{so}(3,1)$ part of the reduction and e the coframe field $\mathfrak{so}(4,1)/\mathfrak{so}(3,1) \simeq \mathbb{R}^{3,1}$ part of the decomposition such that:

$$\omega \in \Omega^1(M, \mathfrak{so}(3,1)) \ , \quad e \in \Omega^1(M, \mathbb{R}^{3,1}) \ .$$

Its gauge field strength $F[A]$, a $\mathfrak{so}(4,1)$-valued 2-form defined over spacetime M, breaks up into $\mathfrak{so}(3,1)$ part given by $R[\omega]$ plus a cosmological constant term and $\mathbb{R}^{3,1}$ part given by the torsion term:

$$F = \widehat{F} + T = \left(R + \frac{\Lambda}{3} e \wedge e \right) + \frac{1}{\ell} \, \mathrm{d}_\omega e \ \in \ \Omega^2(M, \mathfrak{so}(4,1)) \tag{5.43}$$

MacDowell-Mansouri action is given by

$$\boxed{S_{\mathrm{MM}} = \frac{-3}{2G\Lambda} \int_M \mathrm{tr} \left(\widehat{F} \wedge \star \widehat{F} \right)} \tag{5.44}$$

BF reformulation of MacDowell-Mansouri formalism

In BF reformulation of MacDowell-Mansouri gravity introduced by Freidel and Starodubtsev [6], the MacDowell–Mansouri connection A is supplemented by an independent locally $\mathfrak{so}(4,1)$-valued 2-form B whose meaning emerges by equations of motion and the action is

$$S = \int_M \mathrm{tr}\left(B \wedge F - \frac{G\Lambda}{6}\ \widehat{B} \wedge \star \widehat{B} \right)\ . \tag{5.45}$$

We derive the classical field equations from this action for $\alpha = G\Lambda/3$, if we explicit calculations

$$\begin{aligned}
\delta\left(B \wedge F - \frac{\alpha}{2}\widehat{B} \wedge \star\widehat{B}\right) &= \delta B \wedge F + B \wedge \delta F - \frac{\alpha}{2}\left[\delta\widehat{B} \wedge \star\widehat{B} + \widehat{B} \wedge \star(\delta B)\right] \\
&= \delta B \wedge F + B \wedge \delta F - \frac{\alpha}{2}\left[\delta B \wedge \star\widehat{B} + (-1)^4 \star \widehat{B} \wedge \widehat{B}\right] \\
&= \delta B \wedge F + B \wedge \delta F - \frac{\alpha}{2}\left[\delta B \wedge \star\widehat{B} + \delta\widehat{B} \wedge \star\widehat{B}\right] \\
&= \delta B \wedge F + B \wedge \delta F - \frac{\alpha}{2}\left[\delta B \wedge \star\widehat{B} + \delta B \wedge \star\widehat{B}\right] \\
&= \delta B \wedge F + B \wedge \delta F - \alpha\delta B \wedge \star\widehat{B}
\end{aligned}$$

in the first and the third passage we used the identity $\widehat{B} \wedge \star\widehat{B} = B \wedge \star\widehat{B}$ which holds as shown next, in particular multiplying both sides by the volume 4-form $\mathrm{vol} = e^a \wedge e^b \wedge e^c \wedge e^d \wedge = \varepsilon^{abcd} = \star(1)$ we get

$$\mathrm{tr}(B\beta) = \mathrm{tr}\left\{ \begin{bmatrix} \beta^a_b & \frac{1}{\ell}b_b \\ -\frac{\epsilon}{\ell}b^a & 0 \end{bmatrix} \cdot \begin{bmatrix} \beta^c_d & 0 \\ 0 & 0 \end{bmatrix} \right\} = \mathrm{tr}\begin{bmatrix} \beta^a_b\beta^c_d & 0 \\ -\frac{\epsilon}{\ell}b^a\beta^c_d & 0 \end{bmatrix} = \mathrm{tr}(\beta\beta)$$

$$-\frac{\epsilon}{2}\ \mathrm{tr}(B\beta)\ \mathrm{vol} = -\frac{\epsilon}{2}\ \mathrm{tr}(\beta\beta)\ \mathrm{vol}$$

$$B \wedge \star\beta = \beta \wedge \star\beta\ .$$

In the second step we use the identity $\delta F = \mathrm{d}_A\delta A$ and integration by parts according to

$$B \wedge \delta F = B \wedge \mathrm{d}_A\delta A = \mathrm{d}_A(B \wedge \delta A) - \mathrm{d}_A B \wedge \delta A\ .$$

Computing the variation, then we get

$$\begin{aligned}
\delta S &= \int_M \mathrm{tr}\left(\delta\left\{B \wedge F - \frac{\alpha}{2}\widehat{B} \wedge \star\widehat{B}\right\}\right) \\
&= \int_M \mathrm{tr}\left(\delta B \wedge (F - \alpha \star \widehat{B}) + B \wedge \delta F\right) \\
&= \int_M \mathrm{tr}\left(\delta B \wedge (F - \alpha \star \widehat{B}) - \mathrm{d}_A B \wedge \delta A\right)
\end{aligned}$$

The equations of motion resulting from the variations of B and A are thus, respectively,

$$F = \alpha \star \widehat{B} \tag{5.46a}$$

$$\mathrm{d}_A B = 0 \tag{5.46b}$$

where the first equation tells us A has non vanishing Cartan curvature and the second that B is a closed differential form. Equivalently, we can decompose F and B fields into reductive components, and rewrite these equations of motion as:

$$R - \frac{\epsilon}{\ell^2} e \wedge e = \frac{G\Lambda}{3} \star \beta \tag{5.47a}$$

$$\mathrm{d}_\omega e = 0 \tag{5.47b}$$

$$\mathrm{d}_\omega \beta + \frac{\epsilon}{\ell^2}(b \wedge e - e \wedge b) = 0 \tag{5.47c}$$

$$\mathrm{d}_\omega b - \frac{1}{\ell} \beta \wedge e = 0 \tag{5.47d}$$

If we set $G = 0$, these are identical to the equations for Cartan-type BF theory obtained in the previous section, this means that 4d Cartan-type BF theory becomes physical when we upgrade its equation of motion introducing Newton's gravitational constant. At this point, we may wonder how these are the equations of general relativity, Freidel and Starodubtsev approach this question indirectly, by substituting (5.46) for B into the Lagrangian we obtain precisely MacDowell-Mansouri action

$$\begin{aligned}
S &= \int_M \mathrm{tr}\left(B \wedge F - \frac{\alpha}{2} \widehat{B} \wedge \star \widehat{B}\right) \\
&= \int_M \mathrm{tr}\left(\alpha B \wedge \star \widehat{B} - \frac{\alpha}{2} \widehat{B} \wedge \star \widehat{B}\right) \\
&= \int_M \mathrm{tr}\left(\alpha \widehat{B} \wedge \star \widehat{B} - \frac{\alpha}{2} \widehat{B} \wedge \star \widehat{B}\right) \\
&= \frac{\alpha}{2} \int_M \mathrm{tr}\left(\widehat{B} \wedge \star \widehat{B}\right) \\
&= -\frac{1}{2\alpha} \int_M \mathrm{tr}\left(\star F \wedge F\right) \\
&= -\frac{1}{2\alpha} \int_M \mathrm{tr}\left(F \wedge \star F\right) \\
&= -\frac{1}{2\alpha} \int_M \mathrm{tr}\left(\widehat{F} \wedge \star \widehat{F}\right)
\end{aligned}$$

thus writing α explicitly in terms of the cosmological constant we get

$$S_{\mathrm{MM}} = -\frac{3}{2G\Lambda} \int_M \mathrm{tr}\left(\widehat{F} \wedge \star \widehat{F}\right) \; .$$

In particular in the second passage we used the identity $B \wedge \star \widehat{B} = \widehat{B} \wedge \star \widehat{B}$ while in the third one we substituted the following expression, which holds since $\star^2 = -1$ and by equation of motion (5.46):

$$\begin{aligned}
\widehat{B} \wedge \star \widehat{B} &= \left(-\frac{1}{\alpha} \star F\right) \wedge \left(-\frac{1}{\alpha} \star F\right) \\
&= \frac{1}{\alpha^2} \star F \wedge \star^2 F \\
&= \frac{1}{\alpha^2} \star F \wedge (-F) \\
&= -\frac{1}{\alpha^2} \star F \wedge F \; .
\end{aligned}$$

In the last passage we have $F = \widehat{F}$ because the gauge field strength 2-form has vanishing curvature $\mathrm{d}_\omega e = 0$ by equation of motion (5.46). However, it is ought to see how Einstein's equation comes directly from the equations of motion reported above. For this, let us use the equivalent equations in terms of constituent fields. Taking the covariant differential of the first equation shows, by the Bianchi identity $\mathrm{d}_\omega R = 0$ and the second equation of motion the vanishing of the torsion $\mathrm{d}_\omega e$, that

$$\mathrm{d}_\omega \star \beta = 0 \implies \mathrm{d}_\omega \beta = 0$$

because this covariant differential passes through the Hodge star operator and hence. This reduces the third equation of motion to

$$e^a \wedge b^b = e^b \wedge b^a$$

the matrix part of the form $e \wedge b$ is thus a symmetric matrix which lives in $\Lambda^2 \mathbb{R}^4$, hence is zero. When the coframe field e is invertible, we therefore get

$$b = 0 \implies \beta \wedge e = 0 \implies e \wedge \star\beta = 0$$

by the fourth equation of motion. So wedging the first equation of motion with e, using the appropriate cosmological constant $\Lambda = 3\epsilon/\ell^2$, it gives Einstein's field equation in vacuum:

$$\boxed{e \wedge \left(R - \frac{\epsilon}{\ell^2} e \wedge e\right) = 0} \tag{5.49}$$

which is a remarkable expression since it shows only a dependence by coframe field, in particular it is not required any other derived structure such as metric tensor $g_{\mu\nu}$, in fact holds:

$$\epsilon_{ijkl}\left(R^{jk}{}_{mn} e^i_\lambda \wedge e^m_\mu \wedge e^n_\nu - \frac{2\Lambda}{3} e^i_\lambda \wedge e^j_\mu \wedge e^k_\nu\right) = 0 \ . \tag{5.50}$$

Furthermore it highlights the difference between generic local Riemann tensor and its corresponding cosmological form, hence it gives us a measure of how much our spacetime manifold M differs locally from a homogeneous model spacetime G/H. Indeed, when these two curvatures match we will have constant Riemann curvature in vacuum and we will recover a flat Cartan geometry which is identifiable with a metric Klein Geometry associated to an homogeneous model spacetime:

$$R = \frac{\epsilon}{\ell^2} e \wedge e \ , \ \mathrm{d}_\omega e = 0 \implies F[A] = 0 \ .$$

Bibliography

[1] *A graphic representation of a fiber bundle.* URL: `https://en.wikipedia.org/wiki/Vector_bundle#/media/File:Vector_bundle.png`.

[2] *A graphic representation of an Ehresmann connection.* URL: `https://en.wikipedia.org/wiki/Ehresmann_connection#/media/File:Ehresmann_connection.png`.

[3] E. P. van den Ban. *Lie Groups.* University of Utrecht, 2010.

[4] T. Bröcker and T. Dieck. *Representations of Compact Lie Groups.* Graduate Texts in Mathematics. Springer Berlin Heidelberg, 2003. ISBN: 9783540136781. URL: `https://books.google.it/books?id=AfBzWL5bIIQC`.

[5] José Figueroa-O'Farrill. *Notes on gauge theory.* University of Edinburgh, 2006.

[6] Laurent Freidel and Artem Starodubtsev. *Quantum gravity in terms of topological observables.* 2005. arXiv: `hep-th/0501191 [hep-th]`.

[7] Brian C. Hall. *An elementary inroduction to groups and representations.* University of Notre Dame, Departement of Mathematics, 2000.

[8] Chris J Isham. *Modern Differential Geometry for Physicists.* 2nd. WORLD SCIENTIFIC, 1999. DOI: `10.1142/3867`. eprint: `https://www.worldscientific.com/doi/pdf/10.1142/3867`. URL: `https://www.worldscientific.com/doi/abs/10.1142/3867`.

[9] Emanuele Latini. *Lectures on Riemannian geometry.* University of Bologna.

[10] S. W. MacDowell and F. Mansouri. "Unified Geometric Theory of Gravity and Supergravity". In: *Phys. Rev. Lett.* 38 (14 Apr. 1977), pp. 739–742. DOI: `10.1103/PhysRevLett.38.739`. URL: `https://link.aps.org/doi/10.1103/PhysRevLett.38.739`.

[11] R. W. Sharpe. *Differential geometry: Cartan 's generalization of Klein's Erlagen program.* Springer, 1997.

[12] Derek K Wise. "MacDowell–Mansouri gravity and Cartan geometry". In: *Classical and Quantum Gravity* 27.15 (June 2010), p. 155010. DOI: `10.1088/0264-9381/27/15/155010`. URL: `https://dx.doi.org/10.1088/0264-9381/27/15/155010`.

[13] Roberto Zucchini. *Lectures on group theory.* University of Bologna.