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Weak existence for degenerate McKean-Vlasov equations

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To my Emma

Introduction

The purpose of this thesis is to establish existance results for weak solutions of a large class of degenerate McKean-Vlasov equations with rough coefficients by expanding upon techniques developed by A. Yu. Veretennikov in [11].

A McKean-Vlasov equation (MKV) is an object of the form

$$\mathrm{d}X_t = B(t, X_t, \mu_{X_t})\mathrm{d}t + \Sigma(t, X_t, \mu_{X_t})\mathrm{d}W_t, \qquad X_0 \sim \mu_0, \tag{1}$$

where W is a d-dimensional Wiener Process, μ_{X_t} is the law of the process X at time t, μ_0 is an element of $\mathcal{P}^2(\mathbb{R}^N)$ which is the space of the measures on $\mathbb{R}^N$ with finite second moment and the stochastic processes B, Σ are defined over the spaces $B : \Omega \times [0,T] \times \mathbb{R}^N \times \mathcal{P}^2(\mathbb{R}^N) \to \mathbb{R}^N$ and $\Sigma : \Omega \times [0,T] \times \mathbb{R}^N \times \mathcal{P}^2(\mathbb{R}^N) \to \mathbb{R}^{N\times d}$.

Historically Vlasov's idea, proposed originally in 1938 and present in the reprinted paper [12], called mean field interaction in mathematical psysics and in stochastic analysis, estimates that in a large system of many particles subject to long-range interaction forces between the particles the interaction of a particle on the others may be replaced by an averaged field. That is the case for example in electron gas or more in general ionized particles. Later M. Kac [3] proposed equations of the form (1) as a model for Vlasov's kinetic equation in plasma. Then it was McKean [6] that started a systematic study of equations of this type.

MKV equations also due to their historical origin are very effective to describe multi-agent systems and degenerate MKV equations are of particular interest since they naturally arise in mechanical systems subject to stochastical forces or noise, indeed that is studied in [2] for turbolent flows as an alternative approach to the more classical study of the Navier-Stokes equations.

It is a classical result that under Lipschitz conditions for the coefficients by the fixed point theorem we can find a pathwise unique strong solution to the equation. The problem arises when the coefficients are rough, in this case usually an hypothesis of non degeneracy of the diffusion coefficient is taken as in [7] but to work with degenerate coefficients that is obviously out of the question, thus the alternative assumption taken is that the covariance matrix $\Sigma\Sigma^*$ is a block diagonal matrix, in particular the chosen class of equations will be of the form

$$\begin{cases} \mathrm{d}X_{0,t} = B_0(t, X_t, \mu_{X_t})\mathrm{d}t, \\ \mathrm{d}X_{1,t} = B_1(t, X_t, \mu_{X_t})\mathrm{d}t + \Sigma_1(t, X_t, \mu_{X_t})\mathrm{d}W_t, \end{cases} \qquad X_0 \sim \mu_0,$$

where $X_t = (X_{0,t}, X_{1,t}) \in \mathbb{R}^{N-d} \times \mathbb{R}^d$ and Σ_1 non degenerate. A particularly interesting example that is encompassed by our framework is the following MKV-Langevin-type equation:

$$\begin{cases} \mathrm{d}X_{0,t} = X_{1,t}, \\ \mathrm{d}X_{1,t} = B_1(t, X_t, \mu_{X_t})\mathrm{d}t + \Sigma_1(t, X_t, \mu_{X_t})\mathrm{d}W_t. \end{cases}$$

Classically the Langevin model describes via its solution X_t the dynamics of a system of d particles with position $X_{0,t}$ and velocity $X_{1,t}$ at time t. The case with measurable coefficients is primarily driven by applications in control problems. In finance, SDEs of this type describe path-dependent contingent claims, such as Asian options or some local stochastic volatility model [9].

The thesis is organized as follows: in the first chapter notations and theorems that will be used in the following chapters are stated but the proof is omitted since the proofs are quite convoluted and outside the scope of the thesis. In the second chapter classical results for MKV equations are derived (for example existance results for Lipschitz coefficients) using classical results from Stochastic Calculus. Lastly in the third chapter, which amounts to the most part of the thesis, the main theorem is derived.

This last result was firstly established by Veretennikov [11] by building upon its previous paper [7] but with more strict assumptions, specifically requiring the coefficients to be bounded and the diffusion matrix Σ to be symmetric while here these hypothesis are dropped and the coefficients are only needed to be of sublinear growth. The main tools used in this proof are Krylov's bounds [4], Skorokhod's lemma to obtain weak convergence [10] and Nisio's approach to SDEs in [8] and [7].

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Chapter 1

Preliminary Notions

Definition 1.1. *In this thesis we will need a couple of interesting process spaces:*

- M^2 *is the space of processes* Y_t *progressively measurable w.r.t a filtration* $\mathcal{F}_t$ *such that the norm* $||Y||_{M^2} := \mathbb{E}\left[\int_0^T |Y_s|^2 \mathrm{d}s\right]^{\frac{1}{2}}$ *is finite.*
- S^2 *is the space of progressively measurable continuous processes* Y_t *such that the norm* $||Y||_{S^2} := \mathbb{E}\left[\sup_t |Y_t|^2\right]^{\frac{1}{2}}$ *is finite. It is possible to prove that* $(S^2, ||\cdot||_{S^2})$ *is a Banach space.*

Definition 1.2 (Solution of an SDE). *Given a probability space* Ω. *Given an SDE with coefficients* $b : \Omega \times [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ *and* $\sigma : \Omega \times [0,T] \times \mathbb{R}^N \to \mathbb{R}^{N\times d}$, *a filtration* $\mathcal{F}_t$ *and a d-dimensional Wiener process* W_t *with respect to the filtration. The* N*-dimensional process* $X = (X_t)_{t\in[t_0,T]}$ *defined over* Ω *is a solution if*

1. X *is adapted and continuous, which means that* $X_t \in m\mathcal{F}_t$, $\forall t \in [t_0, T]$.
2. *almost certainly we have*

$$X_t = X_{t_0} + \int_{t_0}^t b_s(X_s)\mathrm{d}s + \int_{t_0}^t \sigma_s(X_s)\mathrm{d}W_s, \qquad t \in [t_0, T].$$

And we may write $X \in SDE(b, \sigma, W, \mathcal{F}_t)$.

We say that X *is a* ***strong solution*** *to the SDE if given* $(\mathcal{F}_t, W_t)$ *and a random variable* $Z \in m\mathcal{F}_{t_0}$ *we have that the process* X *is a solution to the SDE, almost certainly* $X_{t_0} = Z$ *and, crucially,* X *is adapted to the completed filtration generated by* W *and* Z, *that is* $\mathcal{F}_t^{W,Z}$.

Definition 1.3 (Uniqueness of the solution of an SDE). *Given an SDE with coefficients* b *and* σ *we have uniqueness:*

- *in* ***strong sense***, *if* $X, Y \in SDE(b, \sigma, W, \mathcal{F}_t)$ *and* $X_{t_0} = Y_{t_0}$ *almost certainly implies that* X *and* Y *are pathwise equal.*

- *in* ***weak sense (or in law)****, if* $X \in SDE(b, \sigma, W, \mathcal{F}_t)$*,* $Y \in SDE(b, \sigma, B, \mathcal{G}_t)$ *and* $X_{t_0} = Y_{t_0}$ *in law implies that* $(X, W) = (Y, B)$ *in law.*

Definition 1.4 (**Explosion time**)**.** *Given a probability space* Ω*, an SDE with coefficients* $b : \Omega \times [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$ *and* $\sigma : \Omega \times [0, T] \times \mathbb{R}^N \to \mathbb{R}^{N \times d}$*, a filtration* $\mathcal{F}_t$ *and a d-dimensional Wiener process* W_t *with respect to the filtration. If* $X \in SDE(b, \sigma, W, \mathcal{F}_t)$ *up to time*

$$\tau_n = \inf(t \in [0, T],\ s.t.\ |X_t| > n)$$

for all $n \in \mathbb{N}$*. We may define*

$$\tau = lim_{n \to \infty} \tau_n$$

the ***explosion time*** *for the process* X*.*

A solution for an SDE is called explosive if $P(\tau < T) > 0$*.*

Definition 1.5 (**Wasserstein measure**)**.** *Let* E *be a complete, separable metric space with Borel* σ*-algebra* $\mathcal{B}$*. Let* $\mathcal{P}_r(E)$ *be the set of all Borel measures over* $(E, \mathcal{B})$ *with finite* r^{th} *moments. Let* $\mu, \nu \in \mathcal{P}_r(E)$*. We define the Wasserstein r-distance* $W_E^{(r)} : \mathcal{P}_r(E) \times \mathcal{P}_r(E) \to \mathbb{R}^+$ *as*

$$W_E^{(r)}(\mu, \nu) = \inf_{\gamma \in \mathcal{P}(E \times E)} \left(\int_{E \times E} d(x, y)^r \gamma(\mathrm{d}x, \mathrm{d}y) \right)^{\frac{1}{r}},$$

where γ *is a joint distribution over* $E \times E$ *with marginals* μ*,*ν*.*

It is possible to prove ([1]) that $\mathcal{P}_2(E)$ *with the Wasserstein* 2*-distance is complete and separable.*

Chapter 2

Preliminary Results

In this chapter we will build the tools necessary to tackle the main theorem for the next chapter. Let's start with this widely known estimates.

Lemma 2.1. *Let* $b_t(x)$ *and* $\sigma_t(x)$ *be stochastic processes defined over* $\Omega \times [0,T] \times \mathbb{R}^N$ *such that their growth is sublinear in* x *uniformly in the other variables, explicitly* $|b_t(x)| + |\sigma_t(x)| \leq M(1+|x|)$. *Let* $q \geq 1$, $X_0 \in L^{2q}(\Omega)$ *and, if it exists,* X_t *be the solution of*

$$X_t = X_0 + \int_0^t b_s(X_s)\mathrm{d}s + \int_0^t \sigma_s(X_s)\mathrm{d}W_s, \qquad t \in [0,T].$$

Then these inequalities are true:

- $\sup_{\substack{t,s\in[0,T]\\ |t-s|\leq h}} \mathbb{E}\left[|X_t - X_s|^{2q}\right] \leq C \cdot h^q$.
- $\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^{2q}\right] \leq C\left(1 + \mathbb{E}\left[|X_0|^{2q}\right]\right)$.

Where the constant C *only depends on* q, T *and* M.

Proof. Let's start from the first inequality and apply a small variation of Lemma 2.5.2 of [4] where instead of considering processes that start at time 0 with value 0 we consider the starting time s still with value 0.

Thus,

$$\mathbb{E}\left[|X_t - X_s|^{2q}\right] \leq \left(\frac{1}{\epsilon}\int_s^t e^{\lambda(t-r)}M^2\mathrm{d}r + 2(2q-1)\int_s^t e^{\lambda(t-r)}M^2\mathrm{d}r\right)^q \leq C \cdot |t-s|^q.$$

Where $\epsilon > 0$ and $\lambda = 4qM + \epsilon$. Now we conclude by passing to the $\sup_{|t-s|\leq h}$.

The second inequality is a classical result, we will still provide the proof for completeness' sake.

By using the inequality $(\sum_{i=1}^n x_i)^p \leq n^{p-1}\sum_{i=1}^n x_i^p$ we observe that

$$\mathbb{E}\left[\sup_{s\in[0,t]}|X_s|^{2q}\right] \leq 3^{2q-1}\left(\mathbb{E}\left[|X_0|^{2q}\right] + \mathbb{E}\left[\sup_{s\in[0,t]}\left|\int_0^s b_r(X_r)\mathrm{d}r\right|^{2q}\right] + \mathbb{E}\left[\sup_{s\in[0,t]}\left|\int_0^s \sigma_r(X_r)\mathrm{d}W_r\right|^{2q}\right]\right)$$

$$\overset{Holder+Doob}{\leq} 3^{2q-1}\left(\mathbb{E}\left[|X_0|^{2q}\right] + \mathbb{E}\left[\sup_{s\in[0,t]} s^{2q-1}\int_0^s |b_r(X_r)|^{2q}\mathrm{d}r\right] + C\mathbb{E}\left[\int_0^t |\sigma_r(X_r)|^{2q}\mathrm{d}r\right]\right)$$

$$\leq 3^{2q-1}\left(\mathbb{E}\left[|X_0|^{2q}\right] + M^{2q}(2T)^{2q-1}\mathbb{E}\left[\int_0^t 1+|X_r|^{2q}\mathrm{d}r\right] + CM^{2q}2^{2q-1}\mathbb{E}\left[\int_0^t 1+|X_r|^{2q}\mathrm{d}r\right]\right)$$

$$\leq C_{M,T,q}\left(1+\mathbb{E}\left[|X_0|^{2q}\right] + \int_0^t \mathbb{E}\left[\sup_{s\in[0,r]}|X_s|^{2q}\right]\mathrm{d}r\right).$$

This means that Gronwall's lemma may be used. Thus

$$\mathbb{E}\left[\sup_{s\in[0,T]}|X_s|^{2q}\right] \leq C_{M,T,q}\left(1+\mathbb{E}\left[|X_0|^{2q}\right]\right).$$

□

For the main theorem we will use the so called Krylov bounds, these are usually presented for bounded coefficients but can be extended for locally bounded coefficients uniformly in time. This is a very deep result derived by the study of parabolic PDEs, thus they may only be applied in the non degenerate case.

Lemma 2.2 (Lemma 3 of [7]). *Let Z_t be a non-explosive strong Markov process in $\mathbb{R}^d$ satisfying*

$$\mathrm{d}Z_t = b(t,Z_t)\mathrm{d}t + \sigma(t,Z_t)\mathrm{d}W_t, \qquad Z_0 = Z_0,$$

where $b(t,z) : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ and $\sigma(t,z) : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times d}$ are measurable non-random functions, σ uniformly non-degenerate:

$$\exists\nu > 0, \quad \inf_{s,z}\inf_{|\lambda|=1}\lambda^\top\sigma(s,z)\lambda \geq \nu > 0,$$

and both bounded locally in z uniformly in t:

$$\sup_{|z|\leq R}\sup_t(|b(t,z)| + |\sigma(t,z)|) < \infty, \qquad \forall R > 0.$$

Let $D \subseteq B_R$ for some $R > 0$ be a bounded domain in $\mathbb{R}^d$. Then for any $p \geq d$ there exists a constant N that only depends on p, R, d, the ellipticity constant of $\sigma\sigma^*$ and the local upper bounds of b and σ on B_R such that for any $g : \mathbb{R}^d \to \mathbb{R}$ and $f : \mathbb{R}^+\times\mathbb{R}^d \to \mathbb{R}$ measurable functions vanishing outside of D

$$\mathbb{E}\left[\int_0^T |g(Z_t)|\mathrm{d}t\right] \leq N||g||_{L^p(D)} \tag{2.1}$$

$$\mathbb{E}\left[\int_0^T |f(t,Z_t)|\mathrm{d}t\right] \leq N||f||_{L^{p+1}([0,T]\times D)}. \tag{2.2}$$

Remark 2.3. We can observe that the hypothesis for non-explosive processes is needed to have something meaningful in the estimates. Indeed if it weren't the case with positive probability the process will diverge in finite time which would mean that it couldn't be considered up to time T. This would make the integral ill-posed.

Proof. Since (2.1) is a special case of (2.2) we just prove the latter. Let D' be a bounded domain such that $\bar{D} \subseteq D' \subseteq B_{R+1}$. Without loss of generality $d(D,(D')^c) > 0$. We now define these two sequences of stopping times:

$$\begin{aligned} \tau^0 = 0, \qquad & T^1 = \inf(t \geq \tau^0 \ : \ Z_t \notin \bar{D}'), \\ \tau^k = \inf(t \geq T^k \ : \ Z_t \in D), \qquad & T^{k+1} = \inf(t \geq \tau^k \ : \ Z_t \notin \bar{D}') \qquad k \geq 1. \end{aligned}$$

We may now define

$$\hat{Z}^k_t = Z_{\tau^k} + \int_{\tau^k}^t \mathbb{1}_{(s<T^{k+1})} b(s,Z_s)\mathrm{d}s + \int_{\tau^k}^t (\mathbb{1}_{(s<T^{k+1})}\sigma(s,Z_s) + \mathbb{1}_{(s\geq T^{k+1})})\mathrm{d}W_s, \qquad k \geq 0,$$

for $t \geq \tau^k$. We may notice that for any $k \geq 0$ the processes $\hat{Z}^k_t$ are Ito processes with the same coefficients as Z_t in $\tau^k \leq t \leq T^{k+1}$. Since they also start from the same random variable Z_{τ^k} at time τ^k so they must be equal on that interval. We can now observe that for any $k \geq 0$ on the interval $[T^k, \tau^k]$ the process Z_t is outside D and so $f(t, Z_t)$ must be equal to 0. By using the theorem 2.2.4 of [4] (Krylov's Bounds for bounded coefficients) we get for any $p \geq d$,

$$\begin{aligned} \mathbb{E}\left[\int_0^T |f(s,Z_s)|\mathrm{d}s\right] &= \sum_{k=0}^{\infty} \mathbb{E}\left[\int_{\tau^k\wedge T}^{T^{k+1}\wedge T} |f(s,Z_s)|\mathrm{d}s\right] = \sum_{k=0}^{\infty} \mathbb{E}\left[\mathbb{1}_{(\tau^k\leq T)}\int_{\tau^k\wedge T}^{T^{k+1}\wedge T} |f(s,Z_s)|\mathrm{d}s\right] \\ &\leq \sum_{k=0}^{\infty} \mathbb{E}\left[\mathbb{1}_{(\tau^k\leq T)}\int_{\tau^k}^{T^{k+1}\wedge T} |f(s,Z_s)|\mathrm{d}s\right] \leq \sum_{k=0}^{\infty} \mathbb{E}\left[\mathbb{1}_{(\tau^k\leq T)}\int_{\tau^k}^{T^{k+1}} |f(s,Z_s)|\mathrm{d}s\right] \\ &\leq \sum_{k=0}^{\infty} \mathbb{E}\left[\mathbb{1}_{(\tau^k\leq T)}\mathbb{E}\left[\int_{\tau^k}^{T^{k+1}} |f(s,Z_s)|\mathrm{d}s \middle| \mathcal{F}_{\tau^k}\right]\right] \leq N||f||_{L^{p+1}([0,T]\times\mathbb{R}^d)} \sum_{k=0}^{\infty} \mathbb{P}(\tau^k \leq T). \end{aligned}$$

Lastly we can notice that $\mathbb{P}(\tau^k \leq T) \leq \mathbb{P}(T^k \leq T)$. A strong Markov process with positive probability to exit in any finite amount of time from the bounded domain D' due to the non degeneracy of $\sigma\sigma^*$ and the boundedness of the coefficients on $\bar{D}'$ will admit exponential bounds of the form

$$\mathbb{P}(T^k \leq T) \leq Cq^{k-1}.$$

For some $C < \infty$ and $q < 1$. But since f vanishes outside D we will obtain the final bound:

$$\mathbb{E}\left[\int_0^T |f(s,Z_s)|\mathrm{d}s\right] \leq N||f||_{L^{p+1}([0,T]\times\mathbb{R}^d)} \sum_{k=0}^{\infty} \mathbb{P}(\tau^k \leq T) \leq CN||f||_{L^{p+1}([0,T]\times D)}.$$

□

Definition 2.4. *A McKean-Vlasov stochastic differential equation is an SDE of the type*

$$dX_t = B_t(\omega, X_t, \mu_{X_t})dt + \Sigma_t(\omega, X_t, \mu_{X_t})dW_t, \qquad X_0 \sim \mu_0, \qquad t \in [0,T]$$

where $B : [0,T] \times \Omega \times \mathbb{R}^N \times \mathcal{P}_2(\mathbb{R}^N) \to \mathbb{R}^N$ *and* $\Sigma : [0,T] \times \Omega \times \mathbb{R}^N \times \mathcal{P}_2(\mathbb{R}^N) \to \mathbb{R}^{N\times d}$ *are measurable,* W_t *d-dimensional Wiener process ,* μ_{X_t} *the law of* X_t *and* μ_0 *an element of* $\mathcal{P}_2(\mathbb{R}^N)$*, the space of all measures on* $\mathbb{R}^N$ *with finite second momentum, equipped with the 2-distance of Wasserstein and* Ω *a probability space.*

This means that a MKV SDE has the coefficients that not only depend on the current time and the current state of the solution but also on the current law of the solution.

Theorem 2.5. *We consider the McKean-Vlasov stochastic differential equation*

$$dX_t = B_t(\omega, X_t, \mu_{X_t})dt + \Sigma_t(\omega, X_t, \mu_{X_t})dW_t, \qquad X_0 = \mu_0, \qquad t \in [0,T]$$

where:

- B *and* Σ *are progressively measurable, that is*

$$B_{|_{[0,t]\times\Omega\times\mathbb{R}^N\times\mathcal{P}_2}}, \Sigma_{|_{[0,t]\times\Omega\times\mathbb{R}^N\times\mathcal{P}_2}} \in \ m\mathbb{B} \otimes \mathcal{F}_t \otimes \mathcal{B}_{\mathbb{R}^N} \otimes \mathcal{B}_{\mathcal{P}_2}, \qquad \forall t \in [0,T]$$

- $\mu_0 \in L^2(\Omega, \mathbb{R}^N)$
- $\mathbb{E}\left[\left(\int_0^T |B_t(\cdot, 0, \delta_0)|\,\mathrm{d}t\right)^2\right], \quad \mathbb{E}\left[\left(\int_0^T |\Sigma_t(\cdot, 0, \delta_0)|\,\mathrm{d}t\right)^2\right] < \infty$
- $\exists L > 0$ *such that almost certantly for* $t \in [0,T]$ *and for* $\omega \in \Omega$, $\forall\, x, x' \in \mathbb{R}^N$, $\forall\, \mu, \mu' \in \mathcal{P}_2(\mathbb{R}^N)$

$$|B_t(\omega, x, \mu) - B_t(\omega, x', \mu')| + |\Sigma_t(\omega, x, \mu) - \Sigma_t(\omega, x', \mu')| \le L\dot{(}|x - x'| + W^{(2)}(\mu, \mu')). \quad (2.3)$$

then exists a unique (pathwise) process X *in* $S^2([0,T], \mathbb{R}^N)$*, strong solution of the SDE.*

Remark 2.6. This is the equivalent to the result for classical SDEs where the Lipschitz property and sublinear growth is enough to prove strong well posedness. Effectively the hypotheses are the same up to translating them to the space $\mathcal{P}^2(\mathbb{R}^N)$.

Proof. Since it doesn't affect the outcome of the proof we won't write explicitly the dependence in ω of B,Σ. Let's consider the operator $J : S^2 \to S^2$ such that

$$J(Y)_t = \mu_0 + \int_0^t B_s(Y_s, \mu_{Y_s})\mathrm{d}s + \int_0^t \Sigma_s(Y_s, \mu_{Y_s})\mathrm{d}W_s.$$

Firstly we will prove that the integrals are well-posed. We may notice that if $Y \in S^2$ then it is continuous; and also

$$W^{(2)}(\mu_{Y_t}, \mu_{Y_s})^2 \le \int_{\mathbb{R}^{2N}} |x-y|^2 \mu_{(Y_t,Y_s)}(\mathrm{d}x, \mathrm{d}y) = \mathbb{E}\left[|Y_t - Y_s|^2\right] \overset{t\to s}{\to} 0$$

by the dominated convergence theorem since it's evident that $||Y_t||_{L^2} \leq ||Y||_{S^2}$. This means that both $B_s(y,\mu)$ and $\Sigma_s(y,\mu)$ are continuous functions since they are lipschitz and we have continuity of the Wasserstein measure. Since they are also adapted this means that they are progressively measurable.

By definition of the 2-distance of Wasserstein

$$\begin{aligned}
\mathbb{E}\left[\int_0^T W^{(2)}(\mu_s,\nu_s)^2 \mathrm{d}s\right] &= \int_0^T W^{(2)}(\mu_s,\nu_s)^2 \mathrm{d}s \leq \int_0^T \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^2 \mu_s(\mathrm{d}x)\nu_s(\mathrm{d}y)\mathrm{d}s \\
&\leq 2\int_0^T \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x|^2 + |y|^2 \mu_s(\mathrm{d}x)\nu_s(\mathrm{d}y)\mathrm{d}s \\
&\leq 2\int_0^T \int_{\mathbb{R}^N} |x|^2 \mu_s(\mathrm{d}x)\mathrm{d}s + 2\int_0^T \int_{\mathbb{R}^N} |y|^2 \nu_s(\mathrm{d}y)\mathrm{d}s.
\end{aligned} \tag{2.4}$$

And with this and the Lipschitz property we can prove that

$$\begin{aligned}
\mathbb{E}\left[\int_0^T |\Sigma_s(Y_s,\mu_{Y_s})|^2 \mathrm{d}s\right] &\leq 2\mathbb{E}\left[\int_0^T |\Sigma_s(Y_s,\mu_{Y_s}) - \Sigma_s(0,\delta_0)|^2\right] + 2\mathbb{E}\left[\int_0^T |\Sigma_s(0,\delta_0)|^2\right] \\
&\leq 4L^2 \mathbb{E}\left[\int_0^T |Y_s|^2 + W^{(2)}(\mu_{Y_s},\delta_0)^2 \mathrm{d}s\right] + 2\mathbb{E}\left[\int_0^T |\Sigma_s(0,\delta_0)|^2\right] \\
&\leq 4L^2 ||Y||_{M^2} + 8L^2 ||Y||_{M^2} + 2\mathbb{E}\left[\int_0^T |\Sigma_s(0,\delta_0)|^2\right].
\end{aligned}$$

We may notice that $S^2 \hookrightarrow M^2$. Thus since $Y \in S^2$ we have

$$\mathbb{E}\left[\int_0^T |\Sigma_s(Y_s,\mu_{Y_s})|^2 \mathrm{d}s\right] \leq C||Y||_{S^2} + 2\mathbb{E}\left[\int_0^T |\Sigma_s(0,\delta_0)|^2\right] < \infty.$$

This proves that $\Sigma_s(Y_s,\mu_{Y_s}) \in M^2$. It is well known that stochastic integrals of M^2 processes are continuous martingales of summable square. In similar ways we can prove that the Lebesgue integrand is measurable and so the Lebesgue integral is continuous. For these reasons $J(Y)_t$ is a continuous and adapted process. Now we have to prove that $\mathbb{E}\left[\sup_{t\in[0,T]} |J(Y)_t|^2\right] < \infty$.

$$\begin{aligned}
\sup_{t\in[0,T]} |J(Y)_t|^2 &= \sup_{t\in[0,T]} \left|\mu_0 + \int_0^t B_s(Y_s,\mu_{Y_s})\mathrm{d}s + \int_0^t \Sigma_s(Y_s,\mu_{Y_s})\mathrm{d}W_s\right|^2 \\
&\leq 3|\mu_0|^2 + 3\sup_{t\in[0,T]}\left|\int_0^t B_s(Y_s,\mu_{Y_s})\mathrm{d}s\right|^2 + 3\sup_{t\in[0,T]}\left|\int_0^t \Sigma_s(Y_s,\mu_{Y_s})\mathrm{d}W_s\right|^2.
\end{aligned}$$

Consequently

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} |J(Y)_t|^2\right] \leq 3\mathbb{E}\left[|\mu_0|^2\right] &+ 3\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t B_s(Y_s,\mu_{Y_s})\mathrm{d}s\right|^2\right] \\
&+ 3\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \Sigma_s(Y_s,\mu_{Y_s})\mathrm{d}W_s\right|^2\right].
\end{aligned}$$

By applying Holder's and Doob's inequalities respectively on the Lebesgue integral and on the stochastic one we get

$$\mathbb{E}\left[\sup_{t\in[0,T]}|J(Y)_t|^2\right] \leq 3\mathbb{E}\left[|\mu_0|^2\right] + 3\mathbb{E}\left[\sup_{t\in[0,T]}\int_0^t \mathrm{d}s\int_0^t |B_s(Y_s,\mu_{Y_s})|^2\mathrm{d}s\right] + 12\mathbb{E}\left[\int_0^T |\Sigma_s(Y_s,\mu_{Y_s})|^2\mathrm{d}s\right].$$

We now apply Lipschitz's property by adding and subtracting in the integrals respectively $B_s(0,\delta_0)$ and $\Sigma_s(0,\delta_0)$:

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}|J(Y)_t|^2\right] \leq{}& 3\mathbb{E}\left[|\mu_0|^2\right] + 6TL^2\mathbb{E}\left[\int_0^T \left(|Y_s| + W^{(2)}(\mu_{Y_s},\delta_0)\right)^2\mathrm{d}s\right]\\ &+ 24L^2\mathbb{E}\left[\int_0^T \left(|Y_s| + W^{(2)}(\mu_{Y_s},\delta_0)\right)^2\mathrm{d}s\right]\\ &+ 6T\mathbb{E}\left[\int_0^T |B_s(0,\delta_0)|^2\right]\\ &+ 24\mathbb{E}\left[\int_0^T |\Sigma_s(0,\delta_0)|^2\right].\end{aligned} \tag{2.5}$$

By hypothesis the terms containing $B_s(0,\delta_0)$ and $\Sigma_s(0,\delta_0)$ are bounded. Since we proved (2.4) we notice that

$$\begin{aligned}\mathbb{E}\left[\int_0^T \left(|Y_s| + W^{(2)}(\mu_{Y_s},\delta_0)\right)^2\mathrm{d}s\right] &\leq 2\mathbb{E}\left[\int_0^T |Y_s|^2 + W^{(2)}(\mu_{Y_s},\delta_0)^2\mathrm{d}s\right]\\ \leq 2\mathbb{E}\left[\int_0^T |Y_s|^2\mathrm{d}s\right] + 2\int_0^T\int_{\mathbb{R}^d}|x|^2\mu_{Y_s}(\mathrm{d}x)\mathrm{d}s + 0 &= 4\mathbb{E}\left[\int_0^T |Y_s|^2\mathrm{d}s\right].\end{aligned}$$

Now, since $S^2 \hookrightarrow M^2$ and $Y \in S^2$, we obtain

$$\mathbb{E}\left[\int_0^T \left(|Y_s| + W^{(2)}(\mu_{Y_s},\delta_0)\right)^2\mathrm{d}s\right] \leq 4\mathbb{E}\left[\int_0^T |Y_s|^2\mathrm{d}s\right] \leq C\cdot\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t|^2\right] < \infty,$$

which guarantees the well posedness of J since we bounded (2.5).

Let's now prove that the operator J is a contraction if the interval $[0,T]$ is small. Let Y and $\tilde{Y}$ be elements of S^2 such that $Y_0 = \tilde{Y}_0 = \mu_0$; then,

$$||J(Y) - J(\tilde{Y})||^2_{S^2} =$$
$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\mu_0 - \mu_0 + \int_0^t B_s(Y_s,\mu_{Y_s}) - B_s(\tilde{Y}_s,\mu_{\tilde{Y}_s})\mathrm{d}s + \int_0^t \Sigma_s(Y_s,\mu_{Y_s}) - \Sigma_s(\tilde{Y}_s,\mu_{\tilde{Y}_s})\mathrm{d}W_s\right|^2\right].$$

We apply again first Holder's and Doob's inequalities and then Lischitz's property

$$
\begin{aligned}
&\leq 3T\mathbb{E}\left[\int_0^T |B_s(Y_s,\mu_{Y_s}) - B_s(\tilde{Y}_s,\mu_{\tilde{Y}_s})|^2 \mathrm{d}s\right] \\
&\quad + 12\mathbb{E}\left[\int_0^T |\Sigma_s(Y_s,\mu_{Y_s}) - \Sigma_s(\tilde{Y}_s,\mu_{\tilde{Y}_s})|^2 \mathrm{d}s\right] \\
&\leq 3TL^2\mathbb{E}\left[\int_0^T \left(|Y_s - \tilde{Y}_s| + W^{(2)}(\mu_{Y_s},\mu_{\tilde{Y}_s})\right)^2 \mathrm{d}s\right] \\
&\quad + 12L^2\mathbb{E}\left[\int_0^T \left(|Y_s - \tilde{Y}_s| + W^{(2)}(\mu_{Y_s},\mu_{\tilde{Y}_s})\right)^2 \mathrm{d}s\right].
\end{aligned}
$$

Which means that the following is true:

$$
||J(Y) - J(\tilde{Y})||_{S^2}^2 \leq 6L^2(T+4)\mathbb{E}\left[\int_0^T |Y_s - \tilde{Y}_s|^2 + W^{(2)}(\mu_{Y_s},\mu_{\tilde{Y}_s})^2 \mathrm{d}s\right].
$$

By applying the definition of the 2-distance of Wasserstein

$$
\begin{aligned}
||J(Y) - J(\tilde{Y})||_{S^2}^2 &\leq C\mathbb{E}\left[\int_0^T |Y_s - \tilde{Y}_s|^2 \mathrm{d}s\right] + C\int_0^T \int_{\mathbb{R}^{2N}} |x-y|^2 \mu_{(Y_s,Y_t)}(\mathrm{d}x,\mathrm{d}y)\mathrm{d}s \\
&\leq C\mathbb{E}\left[\int_0^T |Y_s - \tilde{Y}_s|^2 \mathrm{d}s\right] + C\mathbb{E}\left[\int_0^T |Y_s - \tilde{Y}_s|^2 \mathrm{d}s\right].
\end{aligned}
$$

But S^2 is embedded into M^2, therefore

$$
||J(Y) - J(\tilde{Y})||_{S^2}^2 \leq C\mathbb{E}\left[\int_0^T |Y_s - \tilde{Y}_s|^2 \mathrm{d}s\right] \leq \tilde{C}\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t - \tilde{Y}_t|^2\right] = \tilde{C}||Y - \tilde{Y}||_{S^2}^2,
$$

where $\tilde{C} = 12L^2(T+4)T$, thus it will be less than 1 if T is very small. If that's the case by the fixed-point theorem there will be a unique strong solution of the SDE.

In particular exists an $\epsilon > 0$ such that if $T < \epsilon$, then the above is true. Therefore exists $k \in \mathbb{N}$ such that $\frac{T}{k} < \epsilon$; we now define $t_j = j \cdot \frac{T}{k}, \quad j = 0 \cdots k$.

Since $|t_j - t_{j-1}| < \epsilon$ we can construct inductively and uniquely the solutions $X^{(j)}$ defined over $[t_{j-1}, t_j]$ with initial value $X_{t_{j-1}}^{(j-1)}$. Indeed all the hypotheses of the theorem are verified since $X^{(j-1)} \in S^2$. Thus we define

$$
X_t = \mu_0 \mathbb{1}_{\{0\}}(t) + \sum_{j=1}^k X_t^{(j)} \mathbb{1}_{]t_{j-1},t_j]}(t), \quad t \in [0,T].
$$

It's easy to prove inductively that X is the solution of the original SDE in $S^2([0,T],\mathbb{R}^N)$. □

Chapter 3

Degenerate MKV SDEs

In this chapther we will consider a particular case of MKV SDE with a degenerate covariance matrix, this makes necessary the use of finer techniques to find interesting results. Let's consider

$$\mathrm{d}X_t = B(t, X_t, \mu_{X_t})\mathrm{d}t + \Sigma(t, X_t, \mu_{X_t})\mathrm{d}W_t^x, \quad X_0 = \check{X}_0 \tag{3.1}$$

in the "true" form of the equation, where

$$B(t, x, \mu) = \int b(t, x, y)\mu(\mathrm{d}y), \quad \Sigma(t, x, \mu) = \int \sigma(t, x, y)\mu(\mathrm{d}y).$$

Where $x \in \mathbb{R}^N$, $y \in \mathbb{R}^N$, $b : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, $\sigma : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^{N\times d}$, W^x d-dimensional Wiener process, $\check{X}_0$ random variable in $\mathbb{R}^N$ and $d \leq N$.

We will use a lot the following lemma that can be seen as a probabilistic version of Ascoli-Arzelà's theorem:

Lemma 3.1 (Skorokhod)**.** *Suppose that the N-dimensional processes X_t^n, with $t \in [0, T]$ and $n \in \mathbb{N}$, are defined on some probability space, possibly different from one another. Assume that for any $\epsilon > 0$ we have*

$$\lim_{c\to\infty} \sup_n \sup_{t\in[0,T]} P(|X_t^n| > c) = 0$$

$$\lim_{h\to 0^+} \sup_n \sup_{\substack{t,s\in[0,T] \\ |t-s|\leq h}} P(|X_t^n - X_s^n| > \epsilon) = 0.$$

Then, it is possibile to choose a subsequence n', a probability space and random processes $\tilde{X}_t^{n'}$, $\tilde{X}_t$ defined over that probability space such that $\tilde{X}^{n'} = X^{n'}$ in law and $\tilde{X}_t^{n'} \xrightarrow{P} \tilde{X}_t$, $\forall t \in [0, T]$.

In particular the probability space is $[0, 1]$ with the Borel σ-algebra and the Lebesgue measure.

Proof. This formulation is in [4] as theorem 2.6.2 but the proof is in [10] as theorem 1.6 □

Corollary 3.2. *Under the hypotheses of Lemma 3.1 we also have that $\tilde{X}^{n'} \xrightarrow{d} \tilde{X}$*

Proof. To prove weak convergence of the processes we just need to prove weak convergence of the finite dimensional processes. Fix $t_1, ..., t_n \in [0, T]$. We know that convergence in probability of the components of a random variable implies convergence in probability of the whole random variable, thus by Skorokhod's Lemma

$$(\tilde{X}^{n'}_{t_1}, ..., \tilde{X}^{n'}_{t_n}) \xrightarrow{P} (\tilde{X}_{t_1}, ..., \tilde{X}_{t_n}),$$

which implies weak convergence of the finite dimensional processes. □

Remark 3.3. In the next theorem we will prove the existance of the weak solutions of a more general class of SDEs than the ones in (3.1). Indeed we will consider the case where the coefficients are in the form

$$\phi(t, x, \Sigma(t, x, \mu)) = \phi(t, x, \langle\sigma(t, x, \cdot), \mu\rangle) = \phi\left(t, x, \int \sigma(t, x, y)\mu(\mathrm{d}y)\right). \tag{3.2}$$

This is pretty useful for applications.

Theorem 3.4 (based on Veretennikov [11]). *Consider the SDE of McKean-Vlasov in true form defined in* (3.1) *with the extra terms given by* (3.2).

It is convenient to split in the SDE the degenerate and the diffusive parts. For $x \in \mathbb{R}^N$ we use the notation $x = (x_0, x_1) \in \mathbb{R}^{N-d} \times \mathbb{R}^d$ and call x_0 and x_1 the degenerate and the non-degenerate components respectively. We also set $X_t = (X_{0,t}, X_{1,t})$. Thus the SDE becomes

$$\begin{cases} \mathrm{d}X_{0,t} = \psi_0(t, X_t, B(t, X_t, \mu_{X_t}))\mathrm{d}t \\ \mathrm{d}X_{1,t} = \psi_1(t, X_t, B(t, X_t, \mu_{X_t}))\mathrm{d}t + \phi_1(t, X_t, \Sigma_1(t, X_t, \mu_{X_t}))\mathrm{d}W^x_t, \qquad X_0 = \check{X}_0, \end{cases}$$

where B, Σ_1 are defined as in

$$\Sigma_1(t, X_t, \mu_{X_t}) = \mathbb{E}\left[\sigma_1(t, x, X_t)\right]_{|x=X_t} = \int_{\mathbb{R}^N} \sigma_1(t, X_t, x)\mu_{X_t}(\mathrm{d}x).$$

Under the hypotheses

- $\mathbb{E}[|\check{X}_0|^4] < \infty$.
- *The matrix σ is in the block form*

$$\sigma = \begin{pmatrix} 0 \\ \sigma_1 \end{pmatrix}$$

  where 0 is a $N - d \times d$ block with null entries and σ_1 is a $d \times d$ uniformly non degenerate matrix, that means

$$\exists \nu > 0, \quad \inf_{s,z,\zeta} \inf_{|\lambda|=1} \lambda^\top \sigma_1(s, z, \zeta)\lambda \geq \nu > 0.$$

- *b and σ of sub-linear growth in x and y uniformly in t, this means*

$$|b(t, x, y)| + |\sigma(t, x, y)| \leq C(1 + |x| + |y|), \qquad \forall t, x, y \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N.$$

- *$b(t, x_0, x_1, y_0, y_1)$ and $\sigma_1(t, x_0, x_1, y_0, y_1)$ continuous with respect to (x_0, y_0) uniformly in (t, x_1, y_1) with modulus of continuity $\rho(\cdot)$, that means that the coefficients are continuous in the degenerate variables uniformly with respect to time and the non-degenerate variables.*

and the hypotheses upon ϕ_1, ψ

- *ϕ_1 and ψ of sub-linear growth in x, B or Σ respectively, uniformly in t, this means*

$$|\psi(t, x, B)| \leq C(1 + |x| + |B|), \qquad \forall t \in [0, T],\ \forall x \in \mathbb{R}^N,\ \forall B \in \mathbb{R}^N,$$
$$|\phi_1(t, x, \Sigma)| \leq C(1 + |x| + |\Sigma|), \qquad \forall t \in [0, T],\ \forall x \in \mathbb{R}^N,\ \forall \Sigma \in \mathbb{R}^{d\times d}.$$

- *ϕ_1 and ψ Lipschitz, in particular*

$$\exists C > 0 \ s.t. \ \forall s, t \in [0, T],\ \forall x, y \in \mathbb{R}^N,\ \forall B, B' \in \mathbb{R}^N,\ \forall \Sigma, \Sigma' \in \mathbb{R}^{d\times d}$$
$$|\psi(t, x, B) - \psi(s, y, B')| \leq C(|t - s| + |x - y| + |B - B'|),$$
$$|\phi_1(t, x, \Sigma) - \phi_1(s, y, \Sigma')| \leq C(|t - s| + |x - y| + |\Sigma - \Sigma'|).$$

- *ϕ_1 is a function that maps any space of uniformly definite positive matrices into another space of uniformly definite positive matrices, namely exists a function $\pi : \mathbb{R}^+ \to \mathbb{R}^+$ such that given*

$$\mathcal{S}_\nu := \left\{ \Sigma \in \mathbb{R}^{d\times d} \quad \inf_{|\lambda|=1} \lambda^\top \Sigma \lambda \geq \nu \right\},$$

then, uniformly in $x \in \mathbb{R}^N$ and $t \in [0, T]$,

$$\phi_1(t, x, \mathcal{S}_\nu) \subseteq \mathcal{S}_{\pi(\nu)},$$

in other terms

$$\phi_1([0, T], \mathbb{R}^N, \mathcal{S}_\nu) \subseteq \mathcal{S}_{\pi(\nu)}.$$

exists a weak solution to the SDE.

Remark 3.5. The hypothesis of L^4 summability upon the initial datum $\check{X}_0$ is very strong, and indeed it could be lessened to just be $L^{2+\epsilon}$ summability. Since this doesn't need any particular modification of the proof for simplicity's sake we tackle the L^4 case, but it actually works for $L^{2+\epsilon}$.

Remark 3.6. Under these hypotheses B and Σ are of sublinear growth uniformly in t with the same constant as b and σ. Indeed

$$|B(t, x, \mu)| \leq \int_{R^N} |b(t, x, y)|\mu(\mathrm{d}y) \leq C \int_{R^N} (1 + |x| + |y|)\mu(\mathrm{d}y) \leq C(1 + |x| + \mathbb{E}[|Y|]),$$

where Y is a random variable whose law is μ. Analogously for Σ.

Proof. The proof is quite involved, so for clarity's sake we will split it up in different steps:

1) We will start by constructing a variation of b and σ_1 in order to work with functions that converge over compact sets to b and σ_1 in some sense and such that they are uniformly continuous over some variables. Let $\Psi_n(x)$ be such that

$$\Psi_n(x) = \begin{cases} x, & \text{if } |x| \le n \\ n\frac{x}{|x|}, & \text{otherwise.} \end{cases} \tag{3.3}$$

By construction Ψ_n is a continuous function that maps the whole space onto the ball of radius n. With a slight abuse of notation we will apply Ψ_n both over $\mathbb{R}^d$ and over $\mathbb{R}^{N-d}$. We will now define

$$b^n(t,x,y) = b(t, \Psi_n(x_0), \Psi_n(x_1), \Psi_n(y_0), \Psi_n(y_1)) * \Phi_n(t,x,y),$$
$$\sigma_1^n(t,x,y) = \sigma_1(t, \Psi_n(x_0), \Psi_n(x_1), \Psi_n(y_0), \Psi_n(y_1)) * \Phi_n(t,x,y),$$

where Φ_n are the standard mollifiers[1] of radius $\frac{1}{n}$. We may observe that since the image of Ψ_n is bounded and the functions are of sublinear growth in x we have that the functions that get mollified in the definition of b^n and σ_1^n are L^∞. We will now prove that the mollification of an L^∞ function is globally Lipschitz. Indeed given $f \in L^\infty$ and ϕ_n a mollifier

$$\begin{aligned} |f * \phi_n(x) - f * \phi_n(y)| &\overset{\text{MeanValueThm}}{=} |\bigtriangledown (f * \phi_n)(z) \cdot (x-y)| \\ &\le ||\bigtriangledown (f * \phi_n)||_{L^\infty}|x-y| \overset{\text{Holder}}{\le} ||f||_{L^\infty}||\bigtriangledown \phi_n||_{L^1}|x-y|. \end{aligned} \tag{3.4}$$

It is easy to prove that b^n and σ_1^n are still of sublinear growth (actually they are bounded) but most importantly the constant of sublinear growth is uniform in n:

$$\begin{aligned} |b^n(t,x,y)| &= |b(t, \Psi_n(x_0), \Psi_n(x_1), \Psi_n(y_0), \Psi_n(y_1)) * \Phi_n(t,x,y)| \\ &\le \int_{|t-s|\vee|x-x'|\vee|y-y'|<1/n} |b(s, \Psi_n(x_0'), \Psi_n(x_1'), \Psi_n(y_0'), \Psi_n(y_1'))|\Phi_n(t-s, x-x', y-y')\mathrm{d}s\mathrm{d}x'\mathrm{d}y' \\ &\le \int C(1 + |(\Psi_n(x_0'), \Psi_n(x_1'))| + |(\Psi_n(y_0'), \Psi_n(y_1'))|)\check{\Phi}_n(s,x',y')\mathrm{d}s\mathrm{d}x'\mathrm{d}y' \\ &\le \int C(1 + |x'| + |y'|)\check{\Phi}_n(s,x',y')\mathrm{d}s\mathrm{d}x'\mathrm{d}y' \\ &\le C\left(1 + |x| + \frac{1}{n} + |y| + \frac{1}{n}\right)\int \check{\Phi}_n(s,x',y')\mathrm{d}s\mathrm{d}x'\mathrm{d}y' \\ &\le C(3 + |x| + |y|) \le \tilde{C}(1 + |x| + |y|), \end{aligned}$$

where $\tilde{\Phi}_n(s,x',y') = \Phi_n(t-s, x-x', y-y')|\mathbb{1}_{(|t-s|\vee|x-x'|\vee|y-y'|<1/n)}$; more in general the mollifier of a function of sublinear growth is a function of sublinear growth and if the mollifiers are chosen with radius $1/n$ the sublinear growth constant is uniform over n. Thus this also works for σ_1^n.

[1] b and σ_1 are not defined over $\mathbb{R}$ in t, therefore we must extend them in such a way to make convolutions with them: b will have value 0 outside $[0,T]$ and σ will be equal to the identity matrix (this way it will preserve the uniform nondegeneracy property).

We will now prove that b^n admits $\tilde{\rho}$ as modulus of continuity in (x_0, y_0) uniformly in (t, x_1, y_1), the proof for σ_1^n is analogous. It is easy to see that if a function admits a modulus of continuity then the mollification will admit the same modulus of continuity, we can then observe that

$$\begin{aligned}&|b_0(t, \Psi_n(x_0), \Psi_n(x_1), \Psi_n(y_0), \Psi_n(y_1)) - b_0(t, \Psi_n(x_0'), \Psi_n(x_1), \Psi_n(y_0'), \Psi_n(y_1))| \\ &\quad \le \rho\left(|(\Psi_n(x_0), \Psi_n(x_1), \Psi_n(y_0), \Psi_n(y_1)) - (\Psi_n(x_0'), \Psi_n(x_1), \Psi_n(y_0'), \Psi_n(y_1))|\right) \\ &\quad \le \rho(|(x_0, y_0) - (x_0', y_0')|).\end{aligned}$$

This means that the same modulus of continuity ρ works for b, σ_1, b^n and σ_1^n and is uniform in n. It is also evident that σ_1^n is still uniformly positive definite.

One could also observe that in the definition of b^n and σ_1^n the function to be mollified coincides with b and σ_1 respectively over $[0, T] \times Q_n$, where

$$Q_n = \left\{(x_0, x_1, y_0, y_1) \in \mathbb{R}^{2N},\ |x_0| \vee |x_1| \vee |y_0| \vee |y_1| \le n\right\}.$$

2) By applying the theorem 2.5 we show that for any $n \in \mathbb{N}$ exists X_t^n solution of the mollified SDE.

The hypotheses are indeed verified: by considering

$$B^n(t, x, \mu) = \psi(t, x, \int b^n(t, x, y)\mu(\mathrm{d}y)), \quad \Sigma^n(t, x, \mu) = \phi(t, x, \int \sigma^n(t, x, y)\mu(\mathrm{d}y))$$

it is known that if b^n and σ^n are measurable and integrable with respect to μ, then $\int b^n \mu$ and $\int \sigma^n \mu$ are measurable. Then since ϕ, ψ are continuous we have B^n and Σ^n measurable. Also we observe that if thats the case they will be continuous thanks to the mollification step and adapted. For this reason B^n and Σ^n are progressively measurable.

Due to Jensen's inequality we have $\mathbb{E}[\check{X}_0^2] < \infty$. Since b^n and σ^n are bounded and ϕ, ψ of sublinear growth we get

$$\mathbb{E}\left[\left(\int_0^T |B^n(t, 0, \delta_0)|\mathrm{d}t\right)^2\right] \le \mathbb{E}\left[\left(\int_0^T C(1 + |b^n(t, 0, 0)|\mathrm{d}t\right)^2\right] \le C_n^2 \cdot T^2 < \infty$$

and analogously Σ^n. Lastly

$$|B^n(t, x, \mu) - B^n(t, x', \mu')| = \left|\psi\left(t, x, \int b^n(t, x, y)\mu(\mathrm{d}y)\right) - \psi\left(t, x', \int b^n(t, x', y')\mu'(\mathrm{d}y')\right)\right|,$$

by the Lipschitz condition upon ψ we have

$$\le C\left(|x - x'| + \left|\int b^n(t, x, y)\mu(\mathrm{d}y) - \int b^n(t, x', y')\mu'(\mathrm{d}y')\right|\right)$$

If we just consider the difference of the integrals and take γ a joint distribution with marginals μ

and μ':

$$\begin{aligned}
&\left|\int b^n(t,x,y)\mu(\mathrm{d}y) - \int b^n(t,x',y')\mu'(\mathrm{d}y')\right| \\
&= \left|\int\int b^n(t,x,y)\gamma(\mathrm{d}y,\mathrm{d}y') - \int\int b^n(t,x',y')\gamma(\mathrm{d}y,\mathrm{d}y')\right| \\
&\leq \int\int |b^n(t,x,y) - b^n(t,x',y')|\,\gamma(\mathrm{d}y,\mathrm{d}y') \\
&\overset{MeanValueThm}{=} \int\int |\nabla b^n|_{L^\infty}\cdot|(x-x',y-y')|\,\gamma(\mathrm{d}y,\mathrm{d}y') \\
&\overset{(3.4)}{\leq} C_n\cdot||\nabla\Phi_n||_{L^1}\int\int |x-x'|+|y-y'|\gamma(\mathrm{d}y,\mathrm{d}y') \\
&= \tilde{C}_n\left(|x-x'| + \int\int |y-y'|\gamma(\mathrm{d}y,\mathrm{d}y')\right) \\
&\overset{Jensen}{\leq} \tilde{C}_n\left(|x-x'| + \left(\int\int |y-y'|^2\gamma(\mathrm{d}y,\mathrm{d}y')\right)^{\frac{1}{2}}\right).
\end{aligned}$$

We may now pass to the inf over all possible joint distributions γ, putting everthing together

$$\begin{aligned}
|B^n(t,x,\mu) - B^n(t,x',\mu')| &\leq C\left(|x-x'| + \tilde{C}_n\left(|x-x'| + W^{(2)}(\mu,\mu')\right)\right) \\
&\leq \check{C}_n\left(|x-x'| + W^{(2)}(\mu,\mu')\right) \qquad (3.5)
\end{aligned}$$

similarly for Σ^n. This proves uniform global Lipschitz's property; the hypotheses of theorem 2.5 are verified, thus exists the unique strong solution $X^n \in S^2([0,T],\mathbb{R}^N)$ for any $n \in \mathbb{N}$.

3) From now on with B we will mean the integral of b, analogously with Σ. We will now obtain some crucial inequalities: by fixing the solution X^n with its law μ_{X^n} we can construct the new coefficients

$$\hat{B}^n(t,x) = B^n(t,x,\mu_{X^n}), \qquad \hat{\Sigma}^n(t,x) = \Sigma^n(t,x,\mu_{X^n}),$$

thus X^n is a solution to the standard SDE where $W^{x,n}$ is a sequence of Wiener processes

$$\begin{cases}
\mathrm{d}X^n_{0,t} = \psi_0(t,X^n_t,\hat{B}^n(t,X^n_t))\mathrm{d}t \\
\mathrm{d}X^n_{1,t} = \psi_1(t,X^n_t,\hat{B}^n(t,X^n_t))\mathrm{d}t + \phi_1(t,X^n_t,\hat{\Sigma}^n_1(t,X_t))\mathrm{d}W^{x,n}_t, \quad X^n_0 = \check{X}_0,
\end{cases}$$

where all the hypotheses for classical results for SDE are verified. In particular since $\check{X}_0 \in L^4$ we may use the lemma 2.1 with $q = 2$. Since the constant of sublinear growth C of $\hat{B}^n$ and $\hat{\Sigma}^n$ is uniform in n the lemma gives a constant C_T that does not depend upon n such that:

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}|X^n_t|^4\right] &\leq C_T\cdot(1+\mathbb{E}\left[|\check{X}_0|^4\right]), \\
\sup_{\substack{s,t\in[0,T]\\|s-t|\leq h}} \mathbb{E}\left[|X^n_t - X^n_s|^4\right] &\leq C_T\cdot h^2.
\end{aligned} \qquad (3.6)$$

4) In the proof we will need to split up the process and the law of the process, to do that we now consider, possibly by enlarging the probability space, $W^{y,n}$ a Wiener process that is independent from $W^{x,n}$ and a distribution $\mu_{\check{Y}_0}$ equal in law to $\mu_{\check{X}_0}$ and independent, since we proved existence of strong solutions for the mollified SDE by theorem 2.5 we have $Y^n = (Y_0^n, Y_1^n)$ strong solution of the SDE with $W^{y,n}$:

$$\begin{cases} \mathrm{d}Y_{0,t}^n = \psi_0(t, Y_t^n, B^n(t, Y_t^n, \mu_{Y_t^n}))\mathrm{d}t \\ \mathrm{d}Y_{1,t}^n = \psi_1(t, Y_t^n, B^n(t, Y_t^n, \mu_{Y_t^n}))\mathrm{d}t + \phi_1(t, Y_t^n, \Sigma_1^n(t, Y_t^n, \mu_{Y_t^n}))\mathrm{d}W_t^{y,n}, \quad t \in [0,T], \end{cases}$$
$$\text{where} \quad \mu_{Y_0^n} = \mu_{\check{Y}_0};$$

Since X^n and Y^n are strong solutions and $W^{x,n}, \mu_{\check{X}_0}$ are independent from $W^{y,n}, \mu_{\check{Y}_0}$ we have that the pairs $(X^n, W^{x,n})$ and $(Y^n, W^{y,n})$ are independent.

We can observe that since in theorem 2.5 we also proved strong uniqueness for the solutions, we have uniqueness in law for the solutions of our mollified SDE: this means that $X^n = Y^n$ in law. Also since Y^n is a solution for an SDE we can use the lemma 2.1 to observe that inequalities (3.6) are true also for Y^n.

5) Now we want to copy the processes $X^n, Y^n, W^{x,n}, W^{y,n}$ in a probability space where we have some type of convergence. To do that we are now going to use Lemma 3.1. Indeed we can observe that:

- Uniformly in n we have

$$P\left(\sup_{t\in[0,T]} |X_t^n| > \lambda\right) \overset{\text{Markov}}{\leq} \frac{\mathbb{E}\left[\sup_{t\in[0,T]} |X_t^n|^4\right]}{\lambda^4} \overset{(3.6)}{\leq} \frac{C_T \cdot (1 + \mathbb{E}[|\check{X}_0|^4])}{\lambda^4},$$

  hence it is trivial that

$$\sup_n \sup_t P(|X_t^n| > \lambda) \leq \frac{c}{\lambda^4} \overset{\lambda\to+\infty}{\to} 0. \tag{3.7}$$

  These inequalities extend to Y^n, $W^{x,n}$ and $W^{y,n}$.

- Uniformly in n,t,s, where $|t-s| \leq h$, we have

$$P(|X_t^n - X_s^n| > \epsilon) \overset{Markov}{\leq} \frac{\mathbb{E}[|X_t^n - X_s^n|^4]}{\epsilon^4} \overset{(3.6)}{\leq} \frac{C_T \cdot h^2}{\epsilon^4}. \tag{3.8}$$

  Passing to the *sup* we may notice that

$$\sup_n \sup_{\substack{t,s\in[0,T] \\ |t-s|\leq h}} P(|X_t^n - X_s^n| > \epsilon) \leq \frac{C_T \cdot h^2}{\epsilon^4} \overset{h\to 0}{\to} 0. \tag{3.9}$$

  These inequalities too extend to Y^n, $W^{x,n}$ and $W^{y,n}$.

For this reason we can apply Skorokhod's Lemma that allows us to define a sequence n', some probability space and the random processes $\tilde{X}_t^{n'}$, $\tilde{Y}_t^{n'}$, $\tilde{W}_t^{x,n'}$ and $\tilde{W}_t^{y,n'}$; such that they are equivalent in law to the respective processes $X_t^{n'}$, $Y_t^{n'}$, $W_t^{x,n'}$, $W_t^{y,n'}$ and such that limit processes exist

$$(\tilde{X}_t^{n'}, \tilde{Y}_t^{n'}, \tilde{W}_t^{x,n'}, \tilde{W}_t^{y,n'}) \xrightarrow{P} (\tilde{X}_t^{\infty}, \tilde{Y}_t^{\infty}, \tilde{W}_t^{x,\infty}, \tilde{W}_t^{y,\infty}), \quad n' \to \infty, \quad \forall t \in [0,T]. \tag{3.10}$$

To simplify the notation we will continue to write n rather than n'. Due to the corollary 3.2 we also have weak convergence of the processes.

Since $(X^n, W^{x,n})$ and $(Y^n, W^{y,n})$ are independent we have $(\tilde{Y}^n, \tilde{W}^{y,n})$ indipendent from $(\tilde{X}^n, \tilde{W}^{x,n})$ for any n since they are equal in law. Now, for any finite-dimensional distribution of the joint process $(\tilde{X}^n, \tilde{Y}^n, \tilde{W}^{x,n}, \tilde{W}^{y,n})$ we have $\gamma^n_{t_1,\dots,t_k} = \mu_{(\tilde{Y}^n_{t_1,\dots,t_k}, \tilde{W}^{y,n}_{t_1,\dots,t_k})} \otimes \mu_{(\tilde{X}^n_{t_1,\dots,t_k}, \tilde{W}^{x,n}_{t_1,\dots,t_k})}$, due to convegence in law we can pass to the limit and obtain $\gamma^\infty_{t_1,\dots,t_k} = \mu_{(\tilde{Y}^\infty_{t_1,\dots,t_k}, \tilde{W}^{y,\infty}_{t_1,\dots,t_k})} \otimes \mu_{(\tilde{X}^\infty_{t_1,\dots,t_k}, \tilde{W}^{x,\infty}_{t_1,\dots,t_k})}$, which implies the indipendence of the limit processes, thus $(\tilde{Y}^\infty, \tilde{W}^{y,\infty})$ is indipendent from $(\tilde{X}^\infty, \tilde{W}^{x,\infty})$.

The copied processes are equivalent in law respectively to X^n, $W^{x,n}$, Y^n and $W^{y,n}$. Since for the original processes we have the inequalities (3.6), these inequalities extend to $\tilde{X}^n$, $\tilde{W}^{x,n}$, $\tilde{Y}^n$ and $\tilde{W}^{y,n}$:

$$\begin{aligned} \mathbb{E}\left[\sup_{t\in[0,T]} |\tilde{X}_t^n|^4\right] &\le C_T \cdot (1 + \mathbb{E}\left[|\check{X}_0|^4\right]), \\ \sup_{\substack{s,t\in[0,T] \\ |s-t|\le h}} \mathbb{E}\left[|\tilde{X}_t^n - \tilde{X}_s^n|^4\right] &\le C_T \cdot h^2. \end{aligned} \tag{3.11}$$

Now, thanks to Kolmogorov's continuity theorem and the inequalities of (3.11), $\tilde{X}^n$, $\tilde{W}^{x,n}$, $\tilde{Y}^n$ and $\tilde{W}^{y,n}$ are continuous.

Since it is a Wiener process the increments of $W^{x,n}$ are indipendent with respect to the filtration $\sigma(X_s^n, Y_s^n, W_s^{x,n}, s \le t)$, thus if we construct the filtration $\sigma(\tilde{X}_s^n, \tilde{Y}_s^n, \tilde{W}_s^{x,n}, s \le t)$ the processes $\tilde{W}^{x,n}$, $\tilde{Y}_s^n$ and $\tilde{X}^n$ will be obviously adapted to it, but since they are equal in law to X^n, Y^n and $W^{x,n}$ this means that also $\tilde{W}^{x,n}$ has indipendent increments with respect to the filtration $\sigma(\tilde{X}_s^n, \tilde{Y}_s^n, \tilde{W}_s^{x,n}, s \le t)$, this means that it is a Wiener process. We may even pass this property to the completion of the σ-algebra, that we will call $\mathcal{F}_t^{(n)}$, and prove that $\tilde{W}^{x,n}$ will still be a Wiener process. Analogously seeing as how X^n and Y^n are adapted to $\sigma(X_s^n, Y_s^n, W_s^{x,n}, s \le t)$, this property passes first to $\tilde{X}^n$, $\tilde{Y}^n$ and then to the completion: therefore $\tilde{X}^n$ and $\tilde{Y}^n$ are adapted to $\mathcal{F}_t^{(n)}$. This proves the well posedness of the stochastical integrals in $\tilde{X}^n, \tilde{W}^{x,n}, \tilde{Y}^n$.

Now we want to observe that the copied processes are still solutions to an SDE of type (3.1), this follows from a standard approximation argument. Let $[t]_m = \frac{\lfloor 2^m t \rfloor}{2^m}$ the dyadic approximation of t. We have

$$\mathbb{E}\left[\left|\tilde{X}_t^n - \tilde{X}_0^n - \int_0^t \psi(s, \tilde{X}_s^n, B^n(s, \tilde{X}_s^n, \mu_{\tilde{Y}_s^n}))\mathrm{d}s - \int_0^t \phi(s, \tilde{X}_s^n, \Sigma^n(s, \tilde{X}_s^n, \mu_{\tilde{Y}_s^n}))\mathrm{d}\tilde{W}_s^{x,n}\right|^2\right]$$
$$\le E_1^m + E_2^m + E_3^m,$$

where

$$E_1^m := \mathbb{E}\left[\left|\tilde{X}_t^n - \tilde{X}_0^n - \int_0^t \psi([s]_m, \tilde{X}_{[s]_m}^n, B^n([s]_m, \tilde{X}_{[s]_m}^n, \mu_{\tilde{Y}_{[s]_m}^n}))\mathrm{d}s \right.\right.$$
$$\left.\left. - \int_0^t \phi([s]_m, \tilde{X}_{[s]_m}^n, \Sigma^n([s]_m, \tilde{X}_{[s]_m}^n, \mu_{\tilde{Y}_{[s]_m}^n}))\mathrm{d}\tilde{W}_s^{x,n}\right|^2\right],$$
$$E_2^m := \mathbb{E}\left[\left|\int_0^t \psi(s, \tilde{X}_s^n, B^n(s, \tilde{X}_s^n, \mu_{\tilde{Y}_s^n})) - \psi([s]_m, \tilde{X}_{[s]_m}^n, B^n([s]_m, \tilde{X}_{[s]_m}^n, \mu_{\tilde{Y}_{[s]_m}^n}))\mathrm{d}s\right|^2\right],$$
$$E_3^m := \mathbb{E}\left[\left|\int_0^t \phi(s, \tilde{X}_s^n, \Sigma^n(s, \tilde{X}_s^n, \mu_{\tilde{Y}_s^n})) - \phi([s]_m, \tilde{X}_{[s]_m}^n, \Sigma^n([s]_m, \tilde{X}_{[s]_m}^n, \mu_{\tilde{Y}_{[s]_m}^n}))\mathrm{d}\tilde{W}_s^{x,n}\right|^2\right].$$

By the definition of Lebesgue and Stochastic integrals over step functions we have

$$E_1^m = \mathbb{E}[|\tilde{X}_t^n - \tilde{X}_0^n - \sum_{k2^{-m}\le t} \psi(k2^{-m}, \tilde{X}_{k2^{-m}}^n, B^n(k2^{-m}, \tilde{X}_{k2^{-m}}^n, \mu_{\tilde{Y}_{k2^{-m}}^n}))2^{-m}$$
$$- \sum_{k2^{-m}\le t} \phi(k2^{-m}, \tilde{X}_{k2^{-m}}^n, \Sigma^n(k2^{-m}, \tilde{X}_{k2^{-m}}^n, \mu_{\tilde{Y}_{k2^{-m}}^n}))\left(\tilde{W}_{(k+1)2^{-m}}^{x,n} - \tilde{W}_{k2^{-m}}^{x,n}\right)|^2]$$

for the equivalence in law of the copied processes and the originals we get

$$E_1^m = \mathbb{E}[|X_t^n - \check{X}_0 - \sum_{k2^{-m}\le t} \psi(2k^{-m}, X_{2k^{-m}}^n, B^n(k2^{-m}, X_{k2^{-m}}^n, \mu_{Y_{k2^{-m}}^n}))2^{-m}$$
$$- \sum_{k2^{-m}\le t} \phi(2k^{-m}, X_{2k^{-m}}^n, \Sigma^n(k2^{-m}, X_{k2^{-m}}^n, \mu_{Y_{k2^{-m}}^n}))\left(W_{(k+1)2^{-m}}^{x,n} - W_{k2^{-m}}^{x,n}\right)|^2]$$
$$= \mathbb{E}[|X_t^n - \check{X}_0 - \sum_{k2^{-m}\le t} \psi(2k^{-m}, X_{2k^{-m}}^n, \int b^n(k2^{-m}, X_{k2^{-m}}^n, y)\mu_{Y_{k2^{-m}}^n}(\mathrm{d}y))2^{-m}$$
$$- \sum_{k2^{-m}\le t} \phi(2k^{-m}, X_{2k^{-m}}^n, \int \sigma^n(k2^{-m}, X_{k2^{-m}}^n, y)\mu_{Y_{k2^{-m}}^n}(\mathrm{d}y))\left(W_{(k+1)2^{-m}}^{x,n} - W_{k2^{-m}}^{x,n}\right)|^2].$$

Thus since $X^n = Y^n$ in law, the fact that $(X^n, W^{x,n})$ solves the n-mollified SDE and by the continuity and boundedness of b^n and σ^n we have $E_1^m \overset{m\to\infty}{\to} 0$. We also have

$$\lim_{m\to\infty} E_2^m = \lim_{m\to\infty} E_3^m = 0$$

by the standard approximation of the integrals with bounded and continuous coefficients. This proves that $\tilde{X}_t^n$ and

$$\tilde{X}_0^n + \int_0^t \psi(s, \tilde{X}_s^n, B^n(s, \tilde{X}_s^n, \mu_{\tilde{Y}_s^n}))\mathrm{d}s + \int_0^t \phi(s, \tilde{X}_s^n, \Sigma^n(s, \tilde{X}_s^n, \mu_{\tilde{Y}_s^n}))\mathrm{d}\tilde{W}_s^{x,n}$$

are modifications, but since they are continuous (by Kolmogorov's continuity theorem) this proves

$$\begin{cases} \mathrm{d}\tilde{X}_t^n = \psi(s, \tilde{X}_s^n, B^n(s, \tilde{X}_s^n, \mu_{\tilde{Y}_s^n}))\mathrm{d}s + \phi(s, \tilde{X}_s^n, \Sigma^n(s, \tilde{X}_s^n, \mu_{\tilde{Y}_s^n}))\mathrm{d}\tilde{W}_s^{x,n}, \\ \tilde{X}_0^n = \check{X}_0. \end{cases} \tag{3.12}$$

We just proved that $(\tilde{X}^n, \tilde{W}^{x,n})$ is a weak solution, since $(\tilde{X}^n, \tilde{W}^{x,n})$ and $(\tilde{Y}^n, \tilde{W}^{y,n})$ form a simmetry, the same result is valid by exchanging X with Y and W^x with W^y.

6) Having (3.12) we hope to be able to pass that to the limit to obtain

$$\begin{cases} \mathrm{d}\tilde{X}^\infty_{0,t} = \psi_0(t, \tilde{X}^\infty_t, B(t, \tilde{X}^\infty_t, \mu_{\tilde{Y}^\infty_t}))\mathrm{d}t \\ \mathrm{d}\tilde{X}^\infty_{1,t} = \psi_1(t, \tilde{X}^\infty_t, B(t, \tilde{X}^\infty_t, \mu_{\tilde{Y}^\infty_t}))\mathrm{d}t + \phi_1(t, \tilde{X}^\infty_t, \Sigma_1(t, \tilde{X}^\infty_t, \mu_{\tilde{Y}^\infty_t}))\mathrm{d}\tilde{W}^{x,\infty}_t, \quad \tilde{X}^\infty_0 = \check{X}_0, \end{cases} \tag{3.13}$$

which will be possible to do up to a subsequence n' obtained by applying Skorokhod's Lemma.

But first to be able to write the limit SDE (3.13) we need the well posedness of the integrals. We may use the same arguments used in the previous point to prove the well-posedness of the limit SDE. First things first by using (3.11) we are able to apply Kolmogorov's continuity theorem to have, up to a modification, continuity of all processes. Then due to corollary 3.2 we have convergence in law of $(\tilde{X}^n, \tilde{Y}^n, \tilde{W}^{x,n})$ to $(\tilde{X}^\infty, \tilde{Y}^\infty, \tilde{W}^{x,\infty})$. This means that with respect to the completion of the filtration $\sigma(\tilde{X}^\infty_s, \tilde{Y}^\infty_s, \tilde{W}^{x,\infty}_s, s \le t)$ the limit processes are adapted and since the property of indipendence of the increments only depends on the law of the process this means that the process $\tilde{W}^{x,\infty}$ is a Wiener process with respect to the completion of the filtration $\sigma(\tilde{X}^\infty_s, \tilde{Y}^\infty_s, \tilde{W}^{x,\infty}_s, s \le t)$. This proves the well-posedness of the limit SDE.

To conclude the only thing that is left to prove is that for every $t \in [0, T]$

$$\int_0^t \psi_1(s, \tilde{X}^n_s, B^n(s, \tilde{X}^n_s, \tilde{Y}^n_s))\mathrm{d}s \xrightarrow{P} \int_0^t \psi_1(s, \tilde{X}^\infty_s, B(s, \tilde{X}^\infty_s, \tilde{Y}^\infty_s))\mathrm{d}s \tag{3.14}$$

$$\int_0^t \phi_1(s, \tilde{X}^n_s, \Sigma^n_1(s, \tilde{X}^n_s, \tilde{Y}^n_s))\mathrm{d}\tilde{W}^{x,n}_s \xrightarrow{P} \int_0^t \phi_1(s, \tilde{X}^\infty_s, \Sigma_1(s, \tilde{X}^\infty_s, \tilde{Y}^\infty_s))\mathrm{d}\tilde{W}^{x,\infty}_s. \tag{3.15}$$

7) We will now prove that the inequalities (3.11) can be passed to the limit to $\tilde{X}^\infty$:

- Let's fix $t, s \in [0, T]$. Up to a subsequence we know that $\tilde{X}^n_t \to \tilde{X}^\infty_t$ and $\tilde{X}^n_s \to \tilde{X}^\infty_s$ almost surely. Due to inequalities (3.11) and Fatou's lemma we know that

$$\mathbb{E}\left[|\tilde{X}^\infty_t - \tilde{X}^\infty_s|^4\right] = \mathbb{E}\left[\liminf_n |\tilde{X}^n_t - \tilde{X}^n_s|^4\right] \overset{Fatou}{\le} \liminf_n \mathbb{E}\left[|\tilde{X}^n_t - \tilde{X}^n_s|^4\right] \le C|t-s|^2.$$

  By passing to the $\sup_{|t-s|\le h}$ we obtain the wanted inequality

$$\sup_{|t-s|\le h} \mathbb{E}\left[|\tilde{X}^\infty_t - \tilde{X}^\infty_s|^4\right] \le Ch^2.$$

- We may consider D^N as the dyadics of the N-th degree, that means $D^N = \left\{\frac{k}{2^N}T \ s.t. \ 0 \le k \le 2^N\right\}$. Since D^N is finite we may find a subsequence n_{D^N} such that $\tilde{X}^{n_{D^N}}_t \to \tilde{X}^\infty_t$ almost certainly for any $t \in D^N$. By fixing $t \in D^N$ we have

$$|\tilde{X}^{n_{D^N}}_t|^4 \le \sup_{s\in[0,T]} |\tilde{X}^{n_{D^N}}_s|^4.$$

We may now pass both sides to the $\liminf_{n_{D^N}\to\infty}$, since the left side converges we have

$$|\tilde{X}^\infty_t|^4 \leq \liminf_{n_{D^N}\to\infty} \sup_{s\in[0,T]} |\tilde{X}^{n_{D^N}}_s|^4.$$

But the right hand side is uniform over the choice of $t \in D^N$, this means that

$$\sup_{t\in D^N} |\tilde{X}^\infty_t|^4 \leq \liminf_{n_{D^N}\to\infty} \sup_{s\in[0,T]} |\tilde{X}^{n_{D^N}}_s|^4.$$

But then by applying Fatou's lemma and the expected value we get

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in D^N} |\tilde{X}^\infty_t|^4\right] &\leq \mathbb{E}\left[\liminf_{n_{D^N}\to\infty} \sup_{s\in[0,T]} |\tilde{X}^{n_{D^N}}_s|^4\right] \\
&\overset{Fatou}{\leq} \liminf_{n_{D^N}\to\infty} \mathbb{E}\left[\sup_{s\in[0,T]} |\tilde{X}^{n_{D^N}}_s|^4\right] \\
&\overset{(3.11)}{\leq} \liminf_{n_{D^N}\to\infty} C_T\left(1+\mathbb{E}\left[|\check{X}_0|^4\right]\right) = C_T\left(1+\mathbb{E}\left[|\check{X}_0|^4\right]\right).
\end{aligned}$$

We may now observe that $\left(\sup_{t\in D^N} |\tilde{X}^\infty_t|^4\right)_{N\in\mathbb{N}}$ is a monotone increasing function, thus we are able to apply the monotone convergence theorem to prove that

$$\mathbb{E}\left[\sup_{t\in D} |\tilde{X}^\infty_t|^4\right] = \lim_N \mathbb{E}\left[\sup_{t\in D^N} |\tilde{X}^\infty_t|^4\right] \leq C_T\left(1+\mathbb{E}\left[|\check{X}_0|^4\right]\right),$$

where $D = \cup_{N\in\mathbb{N}} D^N$. Using the previous point and Kolmogorov's continuity theorem we have that up to a modification $\tilde{X}^\infty$ is a continuous process. Thus since D is dense in $[0,T]$ we have $\sup_{t\in[0,T]} |\tilde{X}^\infty_t|^4 = \sup_{t\in D} |\tilde{X}^\infty_t|^4$ which proves the wanted inequality.

It's evident by the construction of $\tilde{Y}^n$ that this point also works for $\tilde{Y}^n$ and $\tilde{Y}^\infty$.

8) We will now proceed with the construction of a very valuable instrument for our proof: the ϵ-net. Let ϵ and α be two positive fixed constants. By Ascoli-Arzela's theorem, for any $h \in \mathbb{N}$ the space

$$C^\alpha_h := \{\phi : C([0,T], \mathbb{R}^{N-d}) \mid |\phi(t)| \leq h,\ |\phi(t)-\phi(s)| \leq h|t-s|^\alpha,\ t,s \in [0,T]\}$$

is totally bounded. Hence, for any $\epsilon > 0$ there exists an ϵ-net that is a finite collection of functions $\phi^{(h)}_0, \ldots, \phi^{(h)}_{\kappa_h} \in C^\alpha_h$ such that

$$C^\alpha_h = \bigcup_{j=1}^{\kappa_\epsilon} Q_{\epsilon,h}(\phi^{(h)}_j), \qquad Q_{\epsilon,h}(\phi^{(h)}_j) := \{\phi \in C^\alpha_h \mid \sup_{t\in[0,T]} |\phi(t) - \phi^{(h)}_j(t)| < \epsilon\}. \tag{3.16}$$

The constant κ_ϵ depends only on α, ϵ, T, the dimension $N-d$ and $h \in \mathbb{N}$.

Now assume $\alpha < \frac{1}{2}$: due to (3.11) and Kolmogorov's continuity theorem we have $\tilde{X}^n_\cdot(\omega), \tilde{Y}^n_\cdot(\omega) \in C^\alpha([0,T], \mathbb{R}^N)$ for any $\omega \in \Omega$ and any $n \in \mathbb{N}\cup\{\infty\}$[2]. Moreover by (3.11) and Markov's inequality there exists $h \in \mathbb{N}$ such that

$$P\left(\left(\tilde{X}^n_{0,\cdot} \in C^\alpha_h\right) \cap \left(\tilde{Y}^n_{0,\cdot} \in C^\alpha_h\right)\right) \geq 1 - \frac{\epsilon}{2}, \qquad n \in \mathbb{N}\cup\{\infty\}.$$

[2]This works up to infinity by the previous point.

With h that depends on ϵ but not n. Thus we have

$$P(\mathcal{B}_{n,\epsilon}) = P\left(\bigcup_{j,k=0}^{\kappa_\epsilon}\left(\tilde{X}^n_{0,\cdot}\in Q_{\epsilon,h}(\phi^{(h)}_k)\right)\cap\left(\tilde{Y}^n_{0,\cdot}\in Q_{\epsilon,h}(\phi^{(h)}_j)\right)\right)\geq 1-\frac{\epsilon}{2}, \tag{3.17}$$

notice that κ_ϵ depends on ϵ, α, T and the dimension $N-d$ but not on $n\in\mathbb{N}\cup\{\infty\}$. This means that up to a small set of trajectories the trajectories are close to a finite subset of our ϵ-net that from now on we will call $\mathcal{N}_{\kappa_\epsilon,\epsilon}$.

We may observe that

$$\begin{aligned} P\left(\sup_{t\in[0,T]}|\tilde{X}^n_t|\vee|\tilde{Y}^n_t|>M\right) &= P\left(\sup_{t\in[0,T]}|\tilde{X}^n_t|\vee\sup_{t\in[0,T]}|\tilde{Y}^n_t|>M\right)\\ \stackrel{Markov}{\leq} \frac{\mathbb{E}\left[\left(\sup_{t\in[0,T]}|\tilde{X}^n_t|\vee\sup_{t\in[0,T]}|\tilde{Y}^n_t|\right)^4\right]}{M^4} &\leq \frac{\mathbb{E}\left[\sup_{t\in[0,T]}|\tilde{X}^n_t|^4\right]}{M^4}+\frac{\mathbb{E}\left[\sup_{t\in[0,T]}|\tilde{Y}^n_t|^4\right]}{M^4}\\ &\stackrel{(3.11)}{\leq}\frac{2C_T(1+\mathbb{E}[\check{X}^4_0])}{M^4}\stackrel{M\to\infty}{\to}0. \end{aligned} \tag{3.18}$$

Thus for any $\epsilon>0$, exists a $M_\epsilon>0$ uniform in $n\in\mathbb{N}$ such that $P\left(\sup_{t\in[0,T]}|\tilde{X}^n_t|\vee|\tilde{Y}^n_t|>M_\epsilon\right)\leq\frac{\epsilon}{2}$, consequently $P\left(\sup_{t\in[0,T]}|\tilde{X}^n_t|\vee|\tilde{Y}^n_t|\leq M_\epsilon\right)\geq 1-\frac{\epsilon}{2}$.

Finally, we introduce the events

$$\begin{aligned} \mathcal{Q}_{n,\epsilon} &= \left(\sup_{t\in[0,T]}|\tilde{X}^n_t|\vee|\tilde{Y}^n_t|\leq M_\epsilon\right)\\ D_{n,\epsilon} &= \mathcal{Q}_{n,\epsilon}\cap\mathcal{B}_{n,\epsilon}, &(3.19)\\ D^{k,j}_{n,\epsilon} &= \left(\tilde{X}^n_{0,\cdot}\in Q_{\epsilon,h}(\phi^{(h)}_k)\right)\cap\left(\tilde{Y}^n_{0,\cdot}\in Q_{\epsilon,h}(\phi^{(h)}_j)\right)\cap D_{n,\epsilon},\qquad 0\leq k,j\leq\kappa_\epsilon, &(3.20) \end{aligned}$$

with $Q_{\epsilon,h}(\phi)$ as in (3.16). Then, by (3.16), (3.17) and the fact that since the collection of events $(D^{k,j}_{n,\epsilon})_{k,j}$ is finite we may change a bit the definition to take them pairwise disjoint

$$D_{n,\epsilon} = \dot{\bigcup_{0\leq k,j\leq\kappa_\epsilon}} D^{k,j}_{n,\epsilon},\qquad n\in\mathbb{N}\cup\{\infty\}. \tag{3.21}$$

We also observe that $\mathbb{P}(D_{n,\epsilon})\geq 1-\epsilon$.

Lastly, before diving in the proof, given $\phi^k,\phi^j\in\mathcal{N}_{\kappa_\epsilon,\epsilon}$, we define

$$\begin{aligned} g^{n,n_0,k,j}(s,x_1,y_1) &= b^n_1(s,\phi^k_s,x_1,\phi^j_s,y_1)-b^{n_0}_1(s,\phi^k_s,x_1,\phi^j_s,y_1)\\ g^{n,k,j}(s,x_1,y_1) &= b^n_1(s,\phi^k_s,x_1,\phi^j_s,y_1)-b_1(s,\phi^k_s,x_1,\phi^j_s,y_1). \end{aligned} \tag{3.22}$$

Let's continue from (3.14), let $n_0 \in \mathbb{N}$ which will be later fixed:

$$
\begin{aligned}
& P\left(\left|\int_0^t \left(\psi(s,\tilde{X}^n_s,B^n(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s})) - \psi(s,\tilde{X}^\infty_s,B(s,\tilde{X}^\infty_s,\mu_{\tilde{Y}^\infty_s}))\right)\mathrm{d}s\right| > c\right) \le \\
I_1 \quad & \le P\left(\left|\int_0^t \left(\psi(s,\tilde{X}^n_s,B^n(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s})) - \psi(s,\tilde{X}^n_s,B^{n_0}(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s})\right))\mathrm{d}s\right| > \frac{c}{3}\right) \\
I_2 \quad & + P\left(\left|\int_0^t \left(\psi(s,\tilde{X}^n_s,B^{n_0}(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s})) - \psi(s,\tilde{X}^\infty_s,B^{n_0}(s,\tilde{X}^\infty_s,\mu_{\tilde{Y}^\infty_s}))\right)\mathrm{d}s\right| > \frac{c}{3}\right) \\
I_3 \quad & + P\left(\left|\int_0^t \left(\psi(s,\tilde{X}^\infty_s,B^{n_0}(s,\tilde{X}^\infty_s,\mu_{\tilde{Y}^\infty_s})) - \psi(s,\tilde{X}^\infty_s,B(s,\tilde{X}^\infty_s,\mu_{\tilde{Y}^\infty_s}))\right)\mathrm{d}s\right| > \frac{c}{3}\right),
\end{aligned}
$$

What we want to do now is ensure that the terms I_1,I_2 and I_3 go to 0 when n goes to infinity as long as n_0 is big enough.

9) Consider now the term I_1:

$$
I_1 = P\left(\left|\int_0^t \left(\psi(s,\tilde{X}^n_s,B^n(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s})) - \psi(s,\tilde{X}^n_s,B^{n_0}(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s}))\right)\mathrm{d}s\right| > \frac{c}{3}\right).
$$

Given $P((D_{n,\epsilon})^c) \le \epsilon$ we have

$$
\begin{aligned}
I_1 = & P\left((D_{n,\epsilon})^c \cap \left(\left|\int_0^t \left(\psi(s,\tilde{X}^n_s,B^n(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s})) - \psi(s,\tilde{X}^n_s,B^{n_0}(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s}))\right)\mathrm{d}s\right| > \frac{c}{3}\right)\right) \\
& + P\left(D_{n,\epsilon} \cap \left(\left|\int_0^t \left(\psi(s,\tilde{X}^n_s,B^n(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s})) - \psi(s,\tilde{X}^n_s,B^{n_0}(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s}))\right)\mathrm{d}s\right| > \frac{c}{3}\right)\right) \\
\le & \epsilon + P\left(\left|\int_0^t \mathbb{1}_{D_{n,\epsilon}} \left(\psi(s,\tilde{X}^n_s,B^n(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s})) - \psi(s,\tilde{X}^n_s,B^{n_0}(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s}))\right)\mathrm{d}s\right| > \frac{c}{3}\right) \\
\overset{Markov}{\le} & \epsilon + C\mathbb{E}\left[\left|\int_0^t \mathbb{1}_{D_{n,\epsilon}} \left(\psi(s,\tilde{X}^n_s,B^n(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s})) - \psi(s,\tilde{X}^n_s,B^{n_0}(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s}))\right)\mathrm{d}s\right|\right] \\
\le & \epsilon + C\mathbb{E}\left[\int_0^t \mathbb{1}_{D_{n,\epsilon}} \left|B^n(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s}) - B^{n_0}(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s})\right|\mathrm{d}s\right],
\end{aligned}
$$

the last inequality is due to the lipschitzianity of ψ.

Thus

$$
\begin{aligned}
I_1 & \le \epsilon + C\mathbb{E}\left[\int_0^t \mathbb{1}_{D_{n,\epsilon}} \left|B^n(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s}) - B^{n_0}(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s})\right|\mathrm{d}s\right] \\
& \le \epsilon + C\mathbb{E}\left[\int_0^t \mathbb{1}_{D_{n,\epsilon}} \mathbb{E}\left[\left|b^n(s,x,\tilde{Y}^n_s) - b^{n_0}(s,x,\tilde{Y}^n_s)\right|\right]_{|x=\tilde{X}^n_s}\mathrm{d}s\right] \\
& \le \epsilon + C\int_0^t \mathbb{E}\left[\mathbb{E}\left[\left|b^n(s,x,\tilde{Y}^n_s) - b^{n_0}(s,x,\tilde{Y}^n_s)\right|\right]_{|x=\tilde{X}^n_s}\right]\mathrm{d}s
\end{aligned}
$$

by applying freezing lemma which may be applied since $\tilde{X}^n$ and $\tilde{Y}^n$ are independent we get

$$
\begin{aligned}
& \le \epsilon + C\mathbb{E}\left[\int_0^t \left|b^n(s,\tilde{X}^n_s,\tilde{Y}^n_s) - b^{n_0}(s,\tilde{X}^n_s,\tilde{Y}^n_s)\right|\mathrm{d}s\right] \\
& \le \epsilon + C\mathbb{E}\left[\left(\mathbb{1}_{D_{n,\epsilon}} + \mathbb{1}_{D_{n,\epsilon}{}^c}\right)\int_0^t \left|b^n(s,\tilde{X}^n_s,\tilde{Y}^n_s) - b^{n_0}(s,\tilde{X}^n_s,\tilde{Y}^n_s)\right|\mathrm{d}s\right].
\end{aligned}
$$

We proved that b and b^n are of sublinear growth in x and y uniformly in t with constant of sublinear growth uniform in n. Therefore since $P({D_{n,\epsilon}}^c) \le \epsilon$ we get

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{{D_{n,\epsilon}}^c}\int_0^t \left|b^n(s,\tilde{X}^n_s,\tilde{Y}^n_s)-b^{n_0}(s,\tilde{X}^n_s,\tilde{Y}^n_s)\right|\mathrm{d}s\right]\\
&\le \int_0^t \mathbb{E}\left[\mathbb{1}_{{D_{n,\epsilon}}^c}C\left(1+|\tilde{X}^n_s|+|\tilde{Y}^n_s|\right)\right]\mathrm{d}s\\
&\overset{Holder}{\le} C\int_0^t P({D_{n,\epsilon}}^c)\mathrm{d}s + C\int_0^t P({D_{n,\epsilon}}^c)^{\frac{1}{2}}\mathbb{E}\left[|\tilde{X}^n_s|^2\right]^{\frac{1}{2}}\mathrm{d}s + C\int_0^t P({D_{n,\epsilon}}^c)^{\frac{1}{2}}\mathbb{E}\left[|\tilde{Y}^n_s|^2\right]^{\frac{1}{2}}\mathrm{d}s
\end{aligned}$$

by applying Jensen's inequality and (3.11) we get

$$\begin{aligned}
&\le C\int_0^t P({D_{n,\epsilon}}^c)\mathrm{d}s + \tilde{C}\int_0^t P({D_{n,\epsilon}}^c)^{\frac{1}{2}}\mathrm{d}s\\
&\le \bar{C}t\cdot(\epsilon+\epsilon^{\frac{1}{2}}) \le \bar{C}T\cdot(\epsilon+\epsilon^{\frac{1}{2}}). \qquad (3.23)
\end{aligned}$$

We now need to examine the other term; we have (3.21). Consequently

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{D_{n,\epsilon}}\int_0^t \left|b^n(s,\tilde{X}^n_s,\tilde{Y}^n_s)-b^{n_0}(s,\tilde{X}^n_s,\tilde{Y}^n_s)\right|\mathrm{d}s\right]\\
&=\sum_{k,j=0}^{\kappa_\epsilon}\mathbb{E}\left[\mathbb{1}_{D^{k,j}_{n,\epsilon}}\int_0^t \left|b^n(s,\tilde{X}^n_s,\tilde{Y}^n_s)-b^{n_0}(s,\tilde{X}^n_s,\tilde{Y}^n_s)\right|\mathrm{d}s\right],
\end{aligned} \qquad (3.24)$$

Where κ_ϵ is the cardinality of the ϵ-net. Now we will bound all the terms in this way:

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{D^{k,j}_{n,\epsilon}}\int_0^t \left|b^n(s,\tilde{X}^n_s,\tilde{Y}^n_s)-b^{n_0}(s,\tilde{X}^n_s,\tilde{Y}^n_s)\right|\mathrm{d}s\right]\le\\
&\le \mathbb{E}\left[\mathbb{1}_{D^{k,j}_{n,\epsilon}}\int_0^t \left|b^n(s,\tilde{X}^n_s,\tilde{Y}^n_s)-b^{n}(s,\phi^k_s,\tilde{X}^n_{1,s},\phi^j_s,\tilde{Y}^n_{1,s})\right|\mathrm{d}s\right]\\
&+\mathbb{E}\left[\mathbb{1}_{D^{k,j}_{n,\epsilon}}\int_0^t \left|b^{n_0}(s,\tilde{X}^n_s,\tilde{Y}^n_s)-b^{n_0}(s,\phi^k_s,\tilde{X}^n_{1,s},\phi^j_s,\tilde{Y}^n_{1,s})\right|\mathrm{d}s\right]\\
&+\mathbb{E}\left[\mathbb{1}_{D^{k,j}_{n,\epsilon}}\int_0^t \left|b^n(s,\phi^k_s,\tilde{X}^n_{1,s},\phi^j_s,\tilde{Y}^n_{1,s})-b^{n_0}(s,\phi^k_s,\tilde{X}^n_{1,s},\phi^j_s,\tilde{Y}^n_{1,s})\right|\mathrm{d}s\right].
\end{aligned}$$

But we proved that b^n has a modulus of continuity ρ that is uniform in n; if we recall the definition of $D^{k,j}_{n,\epsilon}$ since in the expected value there is the term $\mathbb{1}_{D^{k,j}_{n,\epsilon}}$ we have

$$\mathbb{1}_{D^{k,j}_{n,\epsilon}}\left|b^n(s,\tilde{X}^n_s,\tilde{Y}^n_s)-b^n(s,\phi^k_s,\tilde{X}^n_{1,s},\phi^j_s,\tilde{Y}^n_{1,s})\right|\le\rho(\epsilon)\mathbb{1}_{D^{k,j}_{n,\epsilon}}.$$

Thus we can bound the first two terms

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{D^{k,j}_{n,\epsilon}}\int_0^t \left|b^n(s,\tilde{X}^n_s,\tilde{Y}^n_s)-b^{n_0}(s,\tilde{X}^n_s,\tilde{Y}^n_s)\right|\mathrm{d}s\right]\le\\
&\le C\cdot\rho(\epsilon)P\left(D^{k,j}_{n,\epsilon}\right)+\mathbb{E}\left[\mathbb{1}_{D^{k,j}_{n,\epsilon}}\int_0^t \left|b^n(s,\phi^k_s,\tilde{X}^n_{1,s},\phi^j_s,\tilde{Y}^n_{1,s})-b^{n_0}(s,\phi^k_s,\tilde{X}^n_{1,s},\phi^j_s,\tilde{Y}^n_{1,s})\right|\mathrm{d}s\right]\\
&\le C\cdot\rho(\epsilon)P\left(D^{k,j}_{n,\epsilon}\right)+\mathbb{E}\left[\mathbb{1}_{D^{k,j}_{n,\epsilon}}\int_0^t \left|g^{n,n_0,k,j}(s,\tilde{X}^n_{1,s},\tilde{Y}^n_{1,s})\right|\mathrm{d}s\right]\\
&\le C\cdot\rho(\epsilon)P\left(D^{k,j}_{n,\epsilon}\right)+\mathbb{E}\left[\mathbb{1}_{D^{k,j}_{n,\epsilon}}\int_0^t \left|g^{n,k,j}(s,\tilde{X}^n_{1,s},\tilde{Y}^n_{1,s})\right|+\left|g^{n_0,k,j}(s,\tilde{X}^n_{1,s},\tilde{Y}^n_{1,s})\right|\mathrm{d}s\right].
\end{aligned}$$

We may notice that over $D^{k,j}_{n,\epsilon}$ we have that $sup_{t\in[0,T]}|\tilde{X}^n_t| \vee |\tilde{Y}^n_t| \leq M_\epsilon$. Therefore since

$$\mathbb{1}_{D^{k,j}_{n,\epsilon}} \mathbb{1}_{[0,T]\times B_{M_\epsilon}\times B_{M_\epsilon}}(s, \tilde{X}^n_{1,s}, \tilde{Y}^n_{1,s}) = \mathbb{1}_{D^{k,j}_{n,\epsilon}}$$

we have

$$\begin{aligned} \leq C\cdot\rho(\epsilon)P\left(D^{k,j}_{n,\epsilon}\right) &+ \mathbb{E}\left[\mathbb{1}_{D^{k,j}_{n,\epsilon}} \int_0^\infty \left|g^{n,k,j}(s,\tilde{X}^n_{1,s},\tilde{Y}^n_{1,s})\right| \cdot \mathbb{1}_{[0,T]\times B_{M_\epsilon}\times B_{M_\epsilon}}(s,\tilde{X}^n_{1,s},\tilde{Y}^n_{1,s})\mathrm{d}s\right] \\ &+ \mathbb{E}\left[\mathbb{1}_{D^{k,j}_{n,\epsilon}} \int_0^\infty \left|g^{n_0,k,j}(s,\tilde{X}^n_{1,s},\tilde{Y}^n_{1,s})\right| \cdot \mathbb{1}_{[0,T]\times B_{M_\epsilon}\times B_{M_\epsilon}}(s,\tilde{X}^n_{1,s},\tilde{Y}^n_{1,s})\mathrm{d}s\right]. \end{aligned}$$

Then we can use Krylov's Bounds from 2.2 to bound from above: indeed all the hypotheses are verified[3] by the fact that in $D^{k,j}_{n,\epsilon}$ both $\tilde{X}^n_{1,\cdot}$ and $\tilde{Y}^n_{1,\cdot}$ are bounded. Here we are able to see why we put all the effort of creating the ϵ-net: if we didn't the process $\tilde{X}^n_\cdot$ would have a degenerate diffusion coefficient and we could not use this result. Thus

$$\begin{aligned} &\mathbb{E}\left[\mathbb{1}_{D^{k,j}_{n,\epsilon}} \int_0^t \left|b^n(s,\tilde{X}^n_s,\tilde{Y}^n_s) - b^{n_0}(s,\tilde{X}^n_s,\tilde{Y}^n_s)\right| \mathrm{d}s\right] \leq \\ &\leq C\cdot\rho(\epsilon)P\left(D^{k,j}_{n,\epsilon}\right) + N_{M_\epsilon}\left(\int_0^T \int_{|x_1|\leq M_\epsilon} \int_{|y_1|\leq M_\epsilon} |g^{n,k,j}|^{2d+1}\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}s\right)^{\frac{1}{2d+1}} \\ &\qquad + N_{M_\epsilon}\left(\int_0^T \int_{|x_1|\leq M_\epsilon} \int_{|y_1|\leq M_\epsilon} |g^{n_0,k,j}|^{2d+1}\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}s\right)^{\frac{1}{2d+1}} \\ &\leq C\cdot\rho(\epsilon)P\left(D^{k,j}_{n,\epsilon}\right) + N_{M_\epsilon}||g^{n,k,j}||_{L^{2d+1}([0,T]\times B_{M_\epsilon}\times B_{M_\epsilon})} + N_{M_\epsilon}||g^{n_0,k,j}||_{L^{2d+1}([0,T]\times B_{M_\epsilon}\times B_{M_\epsilon})}. \end{aligned}$$

Where N_{M_ϵ} is independent from n. We can observe that since we fixed previously ϵ we also fixed M_ϵ, this way if n, n_0 are big enough in B_{M_ϵ} we will have that b^n coincides with the mollifier of b of radius $1/n$. Thanks to the properties of mollifiers there is convergence over L^{2d+1}_{loc} of b^n to b and in particular of b^n to b; however $[0,T]\times B_{M_\epsilon}\times B_{M_\epsilon}$ is compact and there b is bounded, therefore the L^{2d+1} norm of g converges to 0 by the definition of g (3.22).

Now putting everything together we have

$$\begin{aligned} I_1 \leq \epsilon + C(\epsilon+\epsilon^{\frac{1}{2}}) + \sum_{k,j=0}^{\kappa_\epsilon} C\rho(\epsilon)P\left(D^{k,j}_{n,\epsilon}\right) \\ \sum_{k,j=0}^{\kappa_\epsilon} N_{M_\epsilon}||g^{n,k,j}||_{L^{2d+1}([0,T]\times B_{M_\epsilon}\times B_{M_\epsilon})} + N_{M_\epsilon}||g^{n_0,k,j}||_{L^{2d+1}([0,T]\times B_{M_\epsilon}\times B_{M_\epsilon})}, \end{aligned}$$

since the sets $D^{k,j}_{n,\epsilon}$ are pairwise disjoint and the convergence of the norms of g^n to 0 we have

$$\lim_{n,n_0} I_1 \leq \epsilon + C(\epsilon+\epsilon^{\frac{1}{2}}) + C\rho(\epsilon).$$

This way we just proved that $I_1 \overset{n,n_0\to\infty}{\to} 0$ due to the arbitrarity of the choice of ϵ.

[3]indeed by (3.12) we can construct a standard SDE where the pair $\tilde{X}^n_{1,\cdot}, \tilde{Y}^n_{1,\cdot}$ is a strong solution and thus a strong Markov process.

10) We may proceed with I_2:

$$
\begin{aligned}
I_2 &= P\left(\left|\int_0^t \left(\psi(s,\tilde X_s^n, B^{n_0}(s,\tilde X_s^n,\mu_{\tilde Y_s^n})) - \psi(s,\tilde X_s^\infty, B^{n_0}(s,\tilde X_s^\infty,\mu_{\tilde Y_s^\infty}))\right)\mathrm{d}s\right| > \frac{c}{3}\right)\\
&\leq P\left(D_{n,\epsilon}\cap D_{\infty,\epsilon}\cap\left(\left|\int_0^t \left(\psi(s,\tilde X_s^n, B^{n_0}(s,\tilde X_s^n,\mu_{\tilde Y_s^n})) - \psi(s,\tilde X_s^\infty, B^{n_0}(s,\tilde X_s^\infty,\mu_{\tilde Y_s^\infty}))\right)\mathrm{d}s\right| > \frac{c}{3}\right)\right)\\
&+ P\left((D_{n,\epsilon}\cap D_{\infty,\epsilon})^c\cap\left(\left|\int_0^t \left(\psi(s,\tilde X_s^n, B^{n_0}(s,\tilde X_s^n,\mu_{\tilde Y_s^n})) - \psi(s,\tilde X_s^\infty, B^{n_0}(s,\tilde X_s^\infty,\mu_{\tilde Y_s^\infty}))\right)\mathrm{d}s\right| > \frac{c}{3}\right)\right)\\
&\leq 2\epsilon + P\left(\mathbb{1}_{D_{n,\epsilon}\cap D_{\infty,\epsilon}}\left|\int_0^t \left(\psi(s,\tilde X_s^n, B^{n_0}(s,\tilde X_s^n,\mu_{\tilde Y_s^n})) - \psi(s,\tilde X_s^\infty, B^{n_0}(s,\tilde X_s^\infty,\mu_{\tilde Y_s^\infty}))\right)\mathrm{d}s\right| > \frac{c}{3}\right)
\end{aligned}
$$

We may now apply Markov's inequality,

$$
\leq 2\epsilon + C\mathbb{E}\left[\mathbb{1}_{D_{n,\epsilon}\cap D_{\infty,\epsilon}}\int_0^t \left|\psi(s,\tilde X_s^n, B^{n_0}(s,\tilde X_s^n,\mu_{\tilde Y_s^n})) - \psi(s,\tilde X_s^\infty, B^{n_0}(s,\tilde X_s^\infty,\mu_{\tilde Y_s^\infty}))\right|\mathrm{d}s\right].
$$

Due to the lipschitzianity of ψ we have

$$
\leq 2\epsilon + C\mathbb{E}\left[\mathbb{1}_{D_{n,\epsilon}\cap D_{\infty,\epsilon}}\int_0^t \left(|\tilde X_s^n - \tilde X_s^\infty| + \left|B^{n_0}(s,\tilde X_s^n,\mu_{\tilde Y_s^n}) - B^{n_0}(s,\tilde X_s^\infty,\mu_{\tilde Y_s^\infty})\right|\right)\mathrm{d}s\right].
$$

Since over the set $D_{n,\epsilon}\cap D_{\infty,\epsilon}$ the processes are bounded the function $\mathbb{1}_{D_{n,\epsilon}\cap D_{\infty,\epsilon}}|\tilde X_s^\infty - \cdot|$ is bounded and continuous, thus by weak convergence for the expected value and Lebesgue's dominated convergence for the integral we have

$$
\mathbb{E}\left[\mathbb{1}_{D_{n,\epsilon}\cap D_{\infty,\epsilon}}\int_0^t \left|\tilde X_s^n - \tilde X_s^\infty\right|\mathrm{d}s\right] \overset{n\to\infty}{\to} 0.
$$

To summarise

$$
\begin{aligned}
I_2 &= P\left(\left|\int_0^t \left(\psi(s,\tilde X_s^n, B^{n_0}(s,\tilde X_s^n,\mu_{\tilde Y_s^n})) - \psi(s,\tilde X_s^\infty, B^{n_0}(s,\tilde X_s^\infty,\mu_{\tilde Y_s^\infty}))\right)\mathrm{d}s\right| > \frac{c}{3}\right)\\
&\leq 2\epsilon + o(1) + C\mathbb{E}\left[\int_0^t \left|\int b^{n_0}(s,\tilde X_s^n,y)\mu_{\tilde Y_s^n}(\mathrm{d}y) - \int b^{n_0}(s,\tilde X_s^\infty,\bar y)\mu_{\tilde Y_s^\infty}(\mathrm{d}\bar y)\right|\mathrm{d}s\right]\\
&\overset{Fubini}{\leq} 2\epsilon + o(1) + C\int_0^t \mathbb{E}\left[\left|\int b^{n_0}(s,\tilde X_s^n,y)\mu_{\tilde Y_s^n}(\mathrm{d}y) - \int b^{n_0}(s,\tilde X_s^\infty,\bar y)\mu_{\tilde Y_s^\infty}(\mathrm{d}\bar y)\right|\right]\mathrm{d}s.
\end{aligned}
$$

Let's concentrate on the last term: we sum and subtract in the integral $\int b^{n_0}(s,\tilde X_s^\infty,y)\mu_{\tilde Y_s^n}(\mathrm{d}y)$, fixed $\delta > 0$ we have

$$
\begin{aligned}
&\leq C\int_0^t \mathbb{E}\left[\left|\int b^{n_0}(s,\tilde X_s^\infty,y)\mu_{\tilde Y_s^n}(\mathrm{d}y) - \int b^{n_0}(s,\tilde X_s^\infty,\bar y)\mu_{\tilde Y_s^\infty}(\mathrm{d}\bar y)\right|\right]\mathrm{d}s\\
&+ C\int_0^t \mathbb{E}\left[\mathbb{1}_{|\tilde X_s^n - \tilde X_s^\infty|\leq\delta}\int \left|b^{n_0}(s,\tilde X_s^n,y) - b^{n_0}(s,\tilde X_s^\infty,y)\right|\mu_{\tilde Y_s^n}(\mathrm{d}y)\right]\mathrm{d}s\\
&+ C\int_0^t \mathbb{E}\left[\mathbb{1}_{|\tilde X_s^n - \tilde X_s^\infty|>\delta}\int \left|b^{n_0}(s,\tilde X_s^n,y) - b^{n_0}(s,\tilde X_s^\infty,y)\right|\mu_{\tilde Y_s^n}(\mathrm{d}y)\right]\mathrm{d}s
\end{aligned}
$$

since b^{n_0} is bounded and since $\tilde Y^n$ converges weakly to $\tilde Y^\infty$ the first integral converges to 0 as n goes to infinity by dominated and weak convergence. Since b^{n_0} is mollified it is also Lipschitz with constant L_{n_0} which makes us able to control the second integral; in the third integral the boundedness of b^{n_0} can make us write:

$$I_2 \leq 2\epsilon + o(1) + 2TL_{n_0}\delta + 2C_{n_0}\int_0^t P\left(|\tilde{X}_s^n - \tilde{X}_s^\infty| > \delta\right) \mathrm{d}s,$$

finally since we know that $\forall t \in [0,T]$ we have convergence in probability of $\tilde{X}_t^n$ to $\tilde{X}_t^\infty$ this means that the probability inside the integral goes to 0 as n goes to infinity. Thus by Lebesgue's dominated convergence theorem we have

$$I_2 \leq 2\epsilon + o(1) + 2TL_{n_0}\delta + 2C\int_0^t P\left(|\tilde{X}_s^n - \tilde{X}_s^\infty| > \delta\right) \mathrm{d}s \overset{n\to\infty}{\to} 2\epsilon + 2TL_{n_0}\delta.$$

This estimate is uniform over the choice of δ and then[4] of ϵ, thus $\lim_{n\to\infty} I_2 = 0$.

11) We still need to prove that I_3 goes to 0. The proof will be very similar to the one for I_1 with a bit of extra care. Since $\tilde{X}_s^\infty, \tilde{Y}_s^\infty$ are independent we can work as in the previous point to obtain

$$\begin{aligned}
I_3 &= P\left(\left|\int_0^t \left(\psi(s, \tilde{X}_s^\infty, B^{n_0}(s, \tilde{X}_s^\infty, \mu_{\tilde{Y}_s^\infty})) - \psi(s, \tilde{X}_s^\infty, B(s, \tilde{X}_s^\infty, \mu_{\tilde{Y}_s^\infty}))\right) \mathrm{d}s\right| > \frac{c}{3}\right) \\
&\leq \epsilon + C\mathbb{E}\left[\int_0^t \left|b^{n_0}(s, \tilde{X}_s^\infty, \tilde{Y}_s^\infty) - b(s, \tilde{X}_s^\infty, \tilde{Y}_s^\infty)\right| \mathrm{d}s\right] \\
&\leq \epsilon + C\mathbb{E}\left[\int_0^t \left(\mathbb{1}_{D_{\infty,\epsilon}{}^c} + \mathbb{1}_{D_{\infty,\epsilon}}\right) \left|b^{n_0}(s, \tilde{X}_s^\infty, \tilde{Y}_s^\infty) - b(s, \tilde{X}_s^\infty, \tilde{Y}_s^\infty)\right| \mathrm{d}s\right].
\end{aligned}$$

Let's concentrate on the last term. Analogously to I_1 we may follow the same reasoning as (3.23) since we proved in point (7) that (3.11) extends to the limiting processes. Thus

$$C\mathbb{E}\left[\int_0^t \mathbb{1}_{D_{\infty,\epsilon}{}^c} \left|b^{n_0}(s, \tilde{X}_s^\infty, \tilde{Y}_s^\infty) - b(s, \tilde{X}_s^\infty, \tilde{Y}_s^\infty)\right| \mathrm{d}s\right] \leq \bar{C}T \cdot (\epsilon + \epsilon^{\frac{1}{2}}).$$

We may go on by following what was done in (3.24):

$$I_3 \leq C(\epsilon + \epsilon^{\frac{1}{2}}) + \sum_{k,j=0}^{\kappa_\epsilon} C\mathbb{E}\left[\int_0^t \mathbb{1}_{D_{\infty,\epsilon}^{k,j}} \left|b^{n_0}(s, \tilde{X}_s^\infty, \tilde{Y}_s^\infty) - b(s, \tilde{X}_s^\infty, \tilde{Y}_s^\infty)\right| \mathrm{d}s\right].$$

Split the remaining term like this:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \mathbb{1}_{D_{\infty,\epsilon}^{k,j}} \left|b^{n_0}(s, \tilde{X}_s^\infty, \tilde{Y}_s^\infty) - b(s, \tilde{X}_s^\infty, \tilde{Y}_s^\infty)\right| \mathrm{d}s\right] \\
&\leq \mathbb{E}\left[\int_0^t \mathbb{1}_{D_{\infty,\epsilon}^{k,j}} \left|b^{n_0}(s, \tilde{X}_s^\infty, \tilde{Y}_s^\infty) - b^{n_0}(s, \phi_s^k, \tilde{X}_{1,s}^\infty, \phi_s^j, \tilde{Y}_{1,s}^\infty)\right| \mathrm{d}s\right] \\
&+\mathbb{E}\left[\int_0^t \mathbb{1}_{D_{\infty,\epsilon}^{k,j}} \left|b(s, \tilde{X}_s^\infty, \tilde{Y}_s^\infty) - b(s, \phi_s^k \tilde{X}_{1,s}^\infty, \phi_s^j, \tilde{Y}_{1,s}^\infty)\right| \mathrm{d}s\right] \\
&+\mathbb{E}\left[\int_0^t \mathbb{1}_{D_{\infty,\epsilon}^{k,j}} \left|b^{n_0}(s, \phi_s^k, \tilde{X}_{1,s}^\infty, \phi_s^j, \tilde{Y}_{1,s}^\infty) - b(s, \phi_s^k, \tilde{X}_{1,s}^\infty, \phi_s^j, \tilde{Y}_{1,s}^\infty)\right| \mathrm{d}s\right].
\end{aligned}$$

[4]The order is important since L_{n_0} depends on n_0 which must be bigger then M_ϵ which depends upon ϵ.

We previously noticed that in $D^{k,j}_{\infty,\epsilon}$ we have $|\tilde{X}^{\infty}_{0,\cdot} - \phi^k| < \epsilon$ and $|\tilde{Y}^{\infty}_{0,\cdot} - \phi^j| < \epsilon$; moreover b and b^{n_0} have modulus of continuity ρ:

$$\mathbb{1}_{D^{k,j}_{\infty,\epsilon}} |b^{n_0}(s, \tilde{X}^{\infty}_s, \tilde{Y}^{\infty}_s) - b^{n_0}(s, \phi^k_s, \tilde{Y}^{\infty}_s, \phi^j_s, \tilde{\eta}^{\infty}_s)| \leq \rho(\epsilon)\mathbb{1}_{D^{k,j}_{\infty,\epsilon}}$$
$$\mathbb{1}_{D^{k,j}_{\infty,\epsilon}} |b(s, \tilde{X}^{\infty}_s, \tilde{Y}^{\infty}_s) - b(s, \phi^k_s \tilde{Y}^{\infty}_s, \phi^j_s, \tilde{\eta}^{\infty}_s)| \leq \rho(\epsilon)\mathbb{1}_{D^{k,j}_{\infty,\epsilon}}$$

Thus we can easily bound the first two terms as in point (9) by using this inequality and the disjointness of the sets $D^{k,j}_{\infty,\epsilon}$. We still need to restrain the third term where we would like to use Krylov's bounds, however we cannot currently do that since a priori $\tilde{X}^{\infty}_{1,\cdot}$ and $\tilde{Y}^{\infty}_{1,\cdot}$ are not solutions of SDEs. Shortly we will extend the estimates for $\tilde{X}^{n}_{1,\cdot}$ and $\tilde{Y}^{n}_{1,\cdot}$ to the limit for n that goes to infinity. Thanks to Krylov's Bounds that we used to ensure the limit to 0 of I_1 we know that

$$\mathbb{E}\left[\int_0^T \left|\tilde{g}^{n_0,k,j}(s, \tilde{X}^n_{1,s}, \tilde{Y}^n_{1,s})\right| \mathrm{d}s\right] \leq N_{M_\epsilon} ||\tilde{g}^{n_0,k,j}||_{L^{2d+1}([0,T]\times B_{M_\epsilon}\times B_{M_\epsilon})}, \tag{3.25}$$

We want to prove the same bounds for the limit process, namely for any $g \in b\mathcal{B}$ non-negative, vanishing outside of $[0,T] \times B_{M_\epsilon} \times B_{M_\epsilon}$

$$\mathbb{E}\left[\int_0^T \left|g(s, \tilde{X}^{\infty}_{1,s}, \tilde{Y}^{\infty}_{1,s})\right| \mathrm{d}s\right] \leq N_{M_\epsilon} ||g||_{L^{2d+1}([0,T]\times B_{M_\epsilon}\times B_{M_\epsilon})}. \tag{3.26}$$

Firstly we may notice that the bound is pretty direct for continuous functions, indeed if $g \in bC$ by weak convergence of the processes

$$\begin{aligned}\mathbb{E}\left[\int_0^T \left|g(s, \tilde{X}^{\infty}_{1,s}, \tilde{Y}^{\infty}_{1,s})\right| \mathrm{d}s\right] &= \lim_n \mathbb{E}\left[\int_0^T \left|g(s, \tilde{X}^{n}_{1,s}, \tilde{Y}^{n}_{1,s})\right| \mathrm{d}s\right] \\ &\leq \lim_n N_{M_\epsilon} ||g||_{L^{2d+1}([0,T]\times B_{M_\epsilon}\times B_{M_\epsilon})} = N_{M_\epsilon} ||g||_{L^{2d+1}([0,T]\times B_{M_\epsilon}\times B_{M_\epsilon})},\end{aligned}$$

since N_{M_ϵ} is independent of n. Then if $g = \mathbb{1}_K$ where K is a compact set there exists a monotonically decreasing sequence of bounded and continuous functions $f_m \searrow g$. This convergence is pointwise everywhere, thus by Fatou's Lemma

$$\begin{aligned}\mathbb{E}\left[\int_0^T \left|g(s, \tilde{X}^{\infty}_{1,s}, \tilde{Y}^{\infty}_{1,s})\right| \mathrm{d}s\right] &= \mathbb{E}\left[\int_0^T \left|\lim_m f_m(s, \tilde{X}^{\infty}_{1,s}, \tilde{Y}^{\infty}_{1,s})\right| \mathrm{d}s\right] \\ &\overset{Fatou}{\leq} \liminf_m \mathbb{E}\left[\int_0^T \left|f_m(s, \tilde{X}^{\infty}_{1,s}, \tilde{Y}^{\infty}_{1,s})\right| \mathrm{d}s\right] \\ &\leq \liminf_m N_{M_\epsilon} ||f_m||_{L^{2d+1}([0,T]\times B_{M_\epsilon}\times B_{M_\epsilon})} \leq N_{M_\epsilon} ||g||_{L^{2d+1}([0,T]\times B_{M_\epsilon}\times B_{M_\epsilon})}.\end{aligned}$$

Now we must notice that if $g = \mathbb{1}_D$ where D is a measurable set in $[0,T] \times B_{M_\epsilon} \times B_{M_\epsilon}$ actually

$$\mathbb{E}\left[\int_0^T \left|\mathbb{1}_D(s, \tilde{X}^{\infty}_{1,s}, \tilde{Y}^{\infty}_{1,s})\right| \mathrm{d}s\right] = \nu(D)$$

where

$$\nu = \mathcal{L}_{[0,T]} \otimes \mu_{\tilde{X}^{\infty}_{1,s}} \otimes \mu_{\tilde{Y}^{\infty}_{1,s}},$$

since ν is regular (the product of regular measures is regular and both Lebegue and probability measures are regular) we have that, given a sequence of increasing compact sets K_i such that $K_i \nearrow D$

$$\nu(D) = \lim_i \nu(K_i).$$

Thus

$$\mathbb{E}\left[\int_0^T \left|\mathbb{1}_D(s, \tilde{X}^\infty_{1,s}, \tilde{Y}^\infty_{1,s})\right| \mathrm{d}s\right] = \nu(D) = \lim_i \nu(K_i)$$
$$\leq \lim_i N_{M_\epsilon} ||\mathbb{1}_{K_i}||_{L^{2d+1}([0,T]\times B_{M_\epsilon}\times B_{M_\epsilon})} = N_{M_\epsilon}||\mathbb{1}_D||_{L^{2d+1}([0,T]\times B_{M_\epsilon}\times B_{M_\epsilon})}.$$

Then by triangle inequality the result follows for any simple function $g = \sum_{i=1}^N c_i \mathbb{1}_{D_i}$ where D_i disjoint measurable sets and $c_i > 0$. Lastly any $g \in b\mathcal{B}$ non-negative, vanishing outside of $[0,T]\times B_{M_\epsilon}\times B_{M_\epsilon}$ may be seen as the pointwise everywhere limit of a sequence of simple functions as before which concludes the proof by monotone convergence

$$\mathbb{E}\left[\int_0^T \left|g(s, \tilde{X}^\infty_{1,s}, \tilde{Y}^\infty_{1,s})\right| \mathrm{d}s\right] = \lim_n \mathbb{E}\left[\int_0^T \left|\left(\sum_{i=1}^{N_n} c_{n,i}\mathbb{1}_{D_{n,i}}\right)(s, \tilde{X}^\infty_{1,s}, \tilde{Y}^\infty_{1,s})\right| \mathrm{d}s\right]$$
$$\leq \lim_n N_{M_\epsilon}||\sum_{i=1}^{N_n} c_{n,i}\mathbb{1}_{D_{n,i}}||_{L^{2d+1}([0,T]\times B_{M_\epsilon}\times B_{M_\epsilon})} \leq N_{M_\epsilon}||g||_{L^{2d+1}([0,T]\times B_{M_\epsilon}\times B_{M_\epsilon})}.$$

In this way we have extended Krylov's Bounds up to the limit.

To conclude we notice that b^{n_0} converges L^{2d+1}_{loc} to b due to mollifiers' properties. Therefore $||\tilde{g}||_{L^{2d+1}([0,T]\times B_{M_\epsilon}\times B_{M_\epsilon})} \to 0$ which gives us (3.14).

12) Now we just need to ensure (3.15) to close the proof. Effectively we will do the same things done to the Lebesgue integral; Ito's isometry and Markov's inequality will aid us to reduce the gap between the two tipes of integrals. Let's start with no further hesitation. We want to prove that for any $c > 0$,

$$\lim_{n\to\infty} P\left(\left|\int_0^t \phi_1(s, \tilde{X}^n_s, \Sigma^n_1(s, \tilde{X}^n_s, \tilde{Y}^n_s))\mathrm{d}\tilde{W}^{x,n}_s - \int_0^t \phi_1(s, \tilde{X}^\infty_s, \Sigma_1(s, \tilde{X}^\infty_s, \tilde{Y}^\infty_s))\mathrm{d}\tilde{W}^{x,\infty}_s\right| > c\right) = 0.$$

The main problem is pretty clear: we need to work with stochastic integrals with different Wiener processes. We start by splitting the problem

$$P\left(\left|\int_0^t \phi_1(s, \tilde{X}^n_s, \Sigma^n_1(s, \tilde{X}^n_s, \mu_{\tilde{Y}^n_s}))\mathrm{d}\tilde{W}^{x,n}_s - \int_0^t \phi_1(s, \tilde{X}^\infty_s, \Sigma_1(s, \tilde{X}^\infty_s, \mu_{\tilde{Y}^\infty_s}))\mathrm{d}\tilde{W}^{x,\infty}_s\right| > c\right) \leq$$
$$\leq P\left(\left|\int_0^t \phi_1(s, \tilde{X}^n_s, \Sigma^n_1(s, \tilde{X}^n_s, \mu_{\tilde{Y}^n_s}))\mathrm{d}\tilde{W}^{x,n}_s - \int_0^t \phi_1(s, \tilde{X}^n_s, \Sigma^{n_0}_1(s, \tilde{X}^n_s, \mu_{\tilde{Y}^n_s}))\mathrm{d}\tilde{W}^{x,n}_s\right| > \frac{c}{3}\right)$$
$$+ P\left(\left|\int_0^t \phi_1(s, \tilde{X}^n_s, \Sigma^{n_0}_1(s, \tilde{X}^n_s, \mu_{\tilde{Y}^n_s}))\mathrm{d}\tilde{W}^{x,n}_s - \int_0^t \phi_1(s, \tilde{X}^\infty_s, \Sigma^{n_0}_1(s, \tilde{X}^\infty_s, \mu_{\tilde{Y}^\infty_s}))\mathrm{d}\tilde{W}^{x,\infty}_s\right| > \frac{c}{3}\right)$$
$$+ P\left(\left|\int_0^t \phi_1(s, \tilde{X}^\infty_s, \Sigma^{n_0}_1(s, \tilde{X}^\infty_s, \mu_{\tilde{Y}^\infty_s}))\mathrm{d}\tilde{W}^{x,\infty}_s - \int_0^t \phi_1(s, \tilde{X}^\infty_s, \Sigma_1(s, \tilde{X}^\infty_s, \mu_{\tilde{Y}^\infty_s}))\mathrm{d}\tilde{W}^{x,\infty}_s\right| > \frac{c}{3}\right),$$

where n_0 is an integer that we will later fix. By applying Markov's inequality

$$
\begin{aligned}
&\overset{Markov}{\leq} C\mathbb{E}\left[\left|\int_0^t \left(\phi_1(s,\tilde{X}^n_s,\Sigma^n_1(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s})) - \phi_1(s,\tilde{X}^n_s,\Sigma^{n_0}_1(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s}))\right)\mathrm{d}\tilde{W}^{x,n}_s\right|^2\right]\\
&+P\left(\left|\int_0^t \phi_1(s,\tilde{X}^n_s,\Sigma^{n_0}_1(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s}))\mathrm{d}\tilde{W}^{x,n}_s - \int_0^t \phi_1(s,\tilde{X}^\infty_s,\Sigma^{n_0}_1(s,\tilde{X}^\infty_s,\mu_{\tilde{Y}^\infty_s}))\mathrm{d}\tilde{W}^{x,\infty}_s\right| > \frac{c}{3}\right)\\
&+C\mathbb{E}\left[\left|\int_0^t \left(\phi_1(s,\tilde{X}^\infty_s,\Sigma^{n_0}_1(s,\tilde{X}^\infty_s,\mu_{\tilde{Y}^\infty_s})) - \phi_1(s,\tilde{X}^\infty_s,\Sigma_1(s,\tilde{X}^\infty_s,\mu_{\tilde{Y}^\infty_s}))\right)\mathrm{d}\tilde{W}^{x,\infty}_s\right|^2\right]\\
&\overset{\substack{Ito's\\ Isometry}}{\leq} C\mathbb{E}\left[\int_0^t \left|\phi_1(s,\tilde{X}^n_s,\Sigma^n_1(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s})) - \phi_1(s,\tilde{X}^n_s,\Sigma^{n_0}_1(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s}))\right|^2\mathrm{d}s\right]\\
&+P\left(\left|\int_0^t \phi_1(s,\tilde{X}^n_s,\Sigma^{n_0}_1(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s}))\mathrm{d}\tilde{W}^{x,n}_s - \int_0^t \phi_1(s,\tilde{X}^\infty_s,\Sigma^{n_0}_1(s,\tilde{X}^\infty_s,\mu_{\tilde{Y}^\infty_s}))\mathrm{d}\tilde{W}^{x,\infty}_s\right| > \frac{c}{3}\right)\\
&+C\mathbb{E}\left[\int_0^t \left|\phi_1(s,\tilde{X}^\infty_s,\Sigma^{n_0}_1(s,\tilde{X}^\infty_s,\mu_{\tilde{Y}^\infty_s})) - \phi_1(s,\tilde{X}^\infty_s,\Sigma_1(s,\tilde{X}^\infty_s,\mu_{\tilde{Y}^\infty_s}))\right|^2\mathrm{d}s\right] =: J_1 + J_2 + J_3.
\end{aligned}
$$

We can deal with J_1 and J_3 just like I_1 and I_3[5]. The problem arises with J_2 where the stochastic integrals have different Wiener processes. To tackle J_2 firstly we split it up in easier chunks:

$$
\begin{aligned}
&P\left(\left|\int_0^t \phi_1(s,\tilde{X}^n_s,\Sigma^{n_0}_1(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s}))\mathrm{d}\tilde{W}^{x,n}_s - \int_0^t \phi_1(s,\tilde{X}^\infty_s,\Sigma^{n_0}_1(s,\tilde{X}^\infty_s,\mu_{\tilde{Y}^\infty_s}))\mathrm{d}\tilde{W}^{x,\infty}_s\right| > \frac{c}{3}\right) \leq\\
&\leq P\left(\left|\int_0^t \phi_1(s,\Psi_m(\tilde{X}^n_s),\Sigma^{n_0}_1(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s}))\mathrm{d}\tilde{W}^{x,n}_s - \int_0^t \phi_1(s,\tilde{X}^n_s,\Sigma^{n_0}_1(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s}))\mathrm{d}\tilde{W}^{x,n}_s\right| > \frac{c}{3}\right)\\
&+P\left(\left|\int_0^t \underbrace{\phi_1(s,\Psi_m(\tilde{X}^n_s),\Sigma^{n_0}_1(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s}))}_{f^n(s,\omega)}\mathrm{d}\tilde{W}^{x,n}_s - \int_0^t \underbrace{\phi_1(s,\Psi_m(\tilde{X}^\infty_s),\Sigma^{n_0}_1(s,\tilde{X}^\infty_s,\mu_{\tilde{Y}^\infty_s}))}_{f^\infty(s,\omega)}\mathrm{d}\tilde{W}^{x,\infty}_s\right| > \frac{c}{3}\right)\\
&+P\left(\left|\int_0^t \phi_1(s,\Psi_m(\tilde{X}^\infty_s),\Sigma^{n_0}_1(s,\tilde{X}^\infty_s,\mu_{\tilde{Y}^\infty_s}))\mathrm{d}\tilde{W}^{x,\infty}_s - \int_0^t \phi_1(s,\tilde{X}^\infty_s,\Sigma^{n_0}_1(s,\tilde{X}^\infty_s,\mu_{\tilde{Y}^\infty_s}))\mathrm{d}\tilde{W}^{x,\infty}_s\right| > \frac{c}{3}\right)\\
&\overset{\substack{Ito's\\ Isometry}}{\leq} C\mathbb{E}\left[\int_0^t \left|\phi_1(s,\Psi_m(\tilde{X}^n_s),\Sigma^{n_0}_1(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s})) - \phi_1(s,\tilde{X}^n_s,\Sigma^{n_0}_1(s,\tilde{X}^n_s,\mu_{\tilde{Y}^n_s}))\right|^2\mathrm{d}s\right]\\
&+P\left(\left|\int_0^t f^n(s,\omega)\mathrm{d}\tilde{W}^{x,n}_s - \int_0^t f^\infty(s,\omega)\mathrm{d}\tilde{W}^{x,\infty}_s\right| > \frac{c}{3}\right)\\
&+C\mathbb{E}\left[\int_0^t \left|\phi_1(s,\Psi_m(\tilde{X}^\infty_s),\Sigma^{n_0}_1(s,\tilde{X}^\infty_s,\mu_{\tilde{Y}^\infty_s})) - \phi_1(s,\tilde{X}^\infty_s,\Sigma^{n_0}_1(s,\tilde{X}^\infty_s,\mu_{\tilde{Y}^\infty_s}))\right|^2\mathrm{d}s\right]\\
&=: J_{2,1} + J_{2,2} + J_{2,3}.
\end{aligned}
$$

where Ψ_m is the function defined in (3.3). It's easy to control $J_{2,1}$ and $J_{2,3}$ via techniques similar

[5]The interesting thing in the calculations is that at this point we can see why we assume the L^4 summability of the law and not just L^2, since at this point we already have a square in the integral during the calculations we need just a bit more summability, $L^{2+\delta}$ would be enough, but for semplicity we decided to write L^4.

to the ones used of I_1, since they are bounded the same way we will proceed only with $J_{2,1}$:

$$\begin{aligned} J_{2,1} &= \mathbb{E}\left[\int_0^t \left|\phi_1(s,\Psi_m(\tilde{X}_s^n),\Sigma_1^{n_0}(s,\tilde{X}_s^n,\mu_{\tilde{Y}_s^n})) - \phi_1(s,\tilde{X}_s^n,\Sigma_1^{n_0}(s,\tilde{X}_s^n,\mu_{\tilde{Y}_s^n}))\right|^2 \mathrm{d}s\right] \\ &\overset{Lipschitz}{\leq} \mathbb{E}\left[\int_0^t \left|C|\Psi_m(\tilde{X}_s^n) - \tilde{X}_s^n|\right|^2 \mathrm{d}s\right] \\ &\leq C\int_0^t \mathbb{E}\left[\mathbb{1}_{D_{n,\epsilon}}|\Psi_m(\tilde{X}_s^n) - \tilde{X}_s^n|^2\right]\mathrm{d}s + C\int_0^t \mathbb{E}\left[\mathbb{1}_{(D_{n,\epsilon})^c}|\Psi_m(\tilde{X}_s^n) - \tilde{X}_s^n|^2\right]\mathrm{d}s. \end{aligned}$$

We may observe that fixed $\epsilon > 0$ we get $M_\epsilon > 0$ such that $P((D_{n,\epsilon})^c) < \epsilon$. Now if we fix $m \in \mathbb{N}$ bigger than M_ϵ the function Ψ_m will just be the identity over the ball of radius M_ϵ. Thus the first integral is just 0. In the second integral we apply Holder's inequality and the fact that the norm of $\Psi_m(\tilde{X}_s^n)$ is always less than $\tilde{X}_s^n$ to get

$$\leq 0 + \tilde{C}\int_0^t P((D_{n,\epsilon})^c)^{\frac{1}{2}}\mathbb{E}[|\tilde{X}_s^n|^4]^{\frac{1}{2}}\mathrm{d}s \overset{(3.11)}{\leq} TC\left(1 + \mathbb{E}[|\check{X}_0|^4]\right)^{\frac{1}{2}}\epsilon$$

since the last inequality is uniform over the choice of ϵ the limit of $J_{2,1}$ for n that goes to infinity must be 0. Analogous for $J_{2,3}$.

In $J_{2,2}$ to ensure convergence in probability we use the version of Skorokhod's lemma present in theorem (2.3) of [10]. Indeed if we consider f^n and $\tilde{W}^{x,n}$ we have that f^n is equibounded since σ^{n_0} is and Ψ_m bounds $\tilde{X}^n$, and is equicontinuous in probability due to Lipschitz's property of σ^{n_0} and ϕ_1, indeed

$$\begin{aligned} &\lim_{h\to 0}\sup_n \sup_{\substack{s,t\in[0,T]\\|s-t|\leq h}} P(|f^n(s) - f^n(t)| > \epsilon) \leq \lim_{h\to 0}\sup_n \sup_{\substack{s,t\in[0,T]\\|s-t|\leq h}} \frac{\mathbb{E}\left[|f^n(s) - f^n(t)|\right]}{\epsilon} \leq \\ &\overset{(3.5)}{\leq} \lim_{h\to 0}\sup_n \sup_{\substack{s,t\in[0,T]\\|s-t|\leq h}} C_{m,n_0}\frac{L_{n_0}}{\epsilon}|t-s| + C_{m,n_0}\frac{L_{n_0}}{\epsilon}\mathbb{E}\left[|\tilde{X}_s^n - \tilde{X}_t^n|\right] + C_{m,n_0}\frac{L_{n_0}}{\epsilon}W^{(2)}(\mu_{Y_s^n},\mu_{Y_t^n}), \end{aligned}$$

but now, due to the definition of the Wasserstein measure as an inf over the set of measures whose marginals are $\mu_{Y_s^n}$, $\mu_{Y_t^n}$ in particular it will be smaller than the evaluation with measure $\mu_{(Y_s^n,Y_t^n)}$, which means that

$$\begin{aligned} &\leq \lim_{h\to 0}\sup_n \sup_{\substack{s,t\in[0,T]\\|s-t|\leq h}} C_{m,n_0,\epsilon}|t-s| + C_{m,n_0,\epsilon}\mathbb{E}\left[|\tilde{X}_s^n - \tilde{X}_t^n|\right] + C_{m,n_0,\epsilon}\left(\int |y-\tilde{y}|^2\mu_{(Y_s^n,Y_t^n)}(\mathrm{d}y,\mathrm{d}\tilde{y})\right)^{\frac{1}{2}} \leq \\ &\leq \lim_{h\to 0}\sup_n \sup_{\substack{s,t\in[0,T]\\|s-t|\leq h}} C_{m,n_0,\epsilon}|t-s| + C_{m,n_0,\epsilon}\mathbb{E}\left[|\tilde{X}_s^n - \tilde{X}_t^n|\right] + C_{m,n_0,\epsilon}\mathbb{E}\left[|\tilde{Y}_s^n - \tilde{Y}_t^n|^2\right]^{\frac{1}{2}} \leq \\ &\leq \lim_{h\to 0}\sup_n \sup_{\substack{s,t\in[0,T]\\|s-t|\leq h}} C_{m,n_0,\epsilon}|t-s| + C_{m,n_0,\epsilon}\mathbb{E}\left[|\tilde{X}_s^n - \tilde{X}_t^n|^4\right]^{\frac{1}{4}} + C_{m,n_0,\epsilon}\mathbb{E}\left[|\tilde{Y}_s^n - \tilde{Y}_t^n|^4\right]^{\frac{1}{4}} \leq \\ &\overset{(3.11)}{\leq} \lim_{h\to 0}\sup_n \sup_{\substack{s,t\in[0,T]\\|s-t|\leq h}} C\cdot h + \tilde{C}h^{\frac{1}{2}} = 0. \end{aligned} \tag{3.27}$$

We already know that $\tilde{W}_s^{x,n}$ converges to $\tilde{W}_s^{x,\infty}$ for any s in probability. We can now notice that for any $s \in [0,T]$,

$$\begin{aligned}
&P\left(|f^n(s,\omega) - f^\infty(s,\omega)| > c\right) \\
&\quad \leq P\left(\left(|\tilde{X}_s^n - \tilde{X}_s^\infty| \leq \delta\right) \cap \left(|f^n(s,\omega) - f^\infty(s,\omega)| > c\right)\right) \\
&\qquad + P\left(\left(|\tilde{X}_s^n - \tilde{X}_s^\infty| > \delta\right) \cap \left(|f^n(s,\omega) - f^\infty(s,\omega)| > c\right)\right) \\
&\quad \leq P\left(|\tilde{X}_s^n - \tilde{X}_s^\infty| > \delta\right) \\
&\qquad + C\mathbb{E}\left[\mathbb{1}_{|\tilde{X}_s^n - \tilde{X}_s^\infty| \leq \delta}\left|\phi_1\left(s, \Psi_m(\tilde{X}_s^n,)\Sigma_1^{n_0}(s, \tilde{X}_s^n, \tilde{Y}_s^n)\right) - \phi_1\left(s, \Psi_m(\tilde{X}_s^\infty), \Sigma_1^{n_0}(s, \tilde{X}_s^\infty, \tilde{Y}_s^\infty)\right)\right|\right] \\
&\quad \overset{Lipschitz}{\leq} P\left(|\tilde{X}_s^n - \tilde{X}_s^\infty| > \delta\right) + C\mathbb{E}\left[\left|\int \sigma_1^{n_0}(s, \tilde{X}_s^\infty, y)\mu_{\tilde{Y}_s^n}(\mathrm{d}y) - \int \sigma_1^{n_0}(s, \tilde{X}_s^\infty, \bar{y})\mu_{\tilde{Y}_s^\infty}(\mathrm{d}\bar{y})\right|\right] \\
&\qquad + C\mathbb{E}\left[\mathbb{1}_{|\tilde{X}_s^n - \tilde{X}_s^\infty| \leq \delta}\int \left(|\tilde{X}_s^n - \tilde{X}_s^\infty| + \left|\sigma_1^{n_0}(s, \tilde{X}_s^n, y) - \sigma_1^{n_0}(s, \tilde{X}_s^\infty, y)\right|\right)\mu_{\tilde{Y}_s^n}(\mathrm{d}y)\right] \\
&\quad \leq o(1) + (L_{n_0}+1)\delta + P\left(|\tilde{X}_s^n - \tilde{X}_s^\infty| > \delta\right) \overset{n\to\infty}{\to} (L_{n_0}+1)\delta,
\end{aligned}$$

since we use weak convergence of $\tilde{Y}^n$ to $\tilde{Y}^\infty$, lipschitzianity and boundedness of σ^{n_0} convergence in probability of $\tilde{X}_s^n$ to $\tilde{X}_s^\infty$ and lastly that the norm of $\Psi_m(x) - \Psi_m(y)$ is bounded by the norm of $x-y$. Since this upper bound is uniform over the choice of δ this means that $f^n(s,\cdot)$ converges in probability to $f^\infty(s,\cdot)$. Lastly we use theorem (2.3) of [10]: the functions f^n are uniformly bounded and uniformly continuous in probability thanks to (3.27), moreover we just proved that $(f^n, \tilde{W}^{x,n}) \overset{P}{\to} (f^\infty, \tilde{W}^{x,\infty})$; therefore

$$\int_0^t f^n(s,\omega)\mathrm{d}\tilde{W}_s^{x,n} \overset{P}{\to} \int_0^t f^\infty(s,\omega)\mathrm{d}\tilde{W}_s^{x,\infty}.$$

This ensures that $J_2 \to 0$ for n that goes to infinity. This proves (3.15), now if we consider the SDE we have by passing to the limit up a subsequence that

$$\forall t \in [0,T], \quad \omega\text{−a.c.}$$

$$\begin{cases} \tilde{X}_{0,t}^\infty = \check{X}_{0,0} + \int_0^t \psi_0(s, \tilde{X}_s^\infty, B(s, \tilde{X}_s^\infty, \mu_{\tilde{Y}_s^\infty}))\mathrm{d}s \\ \tilde{X}_{1,t}^\infty = \check{X}_{1,0} + \int_0^t \psi_1(s, \tilde{X}_s^\infty, B(s, \tilde{X}_s^\infty, \mu_{\tilde{Y}_s^\infty}))\mathrm{d}s + \int_0^t \phi_1(s, \tilde{X}_s^\infty, \Sigma_1(s, \tilde{X}_s^\infty, \mu_{\tilde{Y}_s^\infty}))\mathrm{d}\tilde{W}_s^{x,\infty}. \end{cases}$$

Or, simply, the two sides are modifications. But one can easily prove the applicability of Kolmogorov's continuity theorem for the two sides; this means that up to a subsequence the two sides will be continuous modifications. This implies that they are almost surely equal as processes. Thus almost surely $\forall t \in [0,T]$,

$$\begin{cases} \tilde{X}_{0,t}^\infty = \check{X}_{0,0} + \int_0^t \psi_0(s, \tilde{X}_s^\infty, B(s, \tilde{X}_s^\infty, \mu_{\tilde{Y}_s^\infty}))\mathrm{d}s \\ \tilde{X}_{1,t}^\infty = \check{X}_{1,0} + \int_0^t \psi_1(s, \tilde{X}_s^\infty, B(s, \tilde{X}_s^\infty, \mu_{\tilde{Y}_s^\infty}))\mathrm{d}s + \int_0^t \phi_1(s, \tilde{X}_s^\infty, \Sigma_1(s, \tilde{X}_s^\infty, \mu_{\tilde{Y}_s^\infty}))\mathrm{d}\tilde{W}_s^{x,\infty}, \end{cases}$$

which concludes the proof. □

Bibliography

[1] F. Bolley, *Separability and completeness for the Wasserstein distance*, Lecture Notes in Math., vol. 1934, Springer, Berlin, 2008. MR2483740

[2] M. Bossy, J.-F. Jabir, D. Talay, *On conditional McKean Lagrangian stochastic models* Probab. Theory Related Fields 151, 1-2 (**2011**), 319-351.

[3] M. Kac, *Foundations of kinetic theory.* In Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954?1955, vol. III, pages 171-197. University of California Press, Berkeley and Los Angeles, **1956**.

[4] N.V. Krylov, *Controlled Diffusion Processes*, Springer, **1980**, ISBN: 9783540709138

[5] N.V. Krylov, *Introduction to the Theory of Random Processes*, American Mathematical Society, **2002**, ISBN: 0-8218-2985-8

[6] H. P. McKean, *A class of Markov processes associated with nonlinear parabolic equations*, Proc. Nat. Acad. Sci. U.S.A., 56:1907?1911, **1966**.

[7] Yu.S. Mishura & A.Yu. Veretennikov, *Existence and uniqueness theorems for solutions of McKean-Vlasov stochastic equations*, arXiv 1603.02212v12, **2020**

[8] M. Nisio, *On the existence of solutions of stochastic differential equations.* Osaka Math. J. 10 (**1973**), 185-208.

[9] A. Pascucci, *PDE and martingale methods in option pricing*, vol. 2 of Bocconi & Springer Series. Springer, Milan; Bocconi University Press, Milan, **2011**.

[10] A.V. Skorokhod, *Studies in the theory of random processes*, Addison-Wesley, **1965**

[11] A.Yu. Veretennikov, *On weak existence of solutions of degenerate McKean-Vlasov equations*, arXiv:2301.01532v1, **2023**

[12] A. A. Vlasov, *The vibrational properties of an electron gas*, Physics-Uspekhi, 10(6):721?733, **1968**.

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