Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
Corso di Laurea (Triennale) in Matematica

Poset topology and Complex shellability

Relatore:
Chiar.mo Prof.
Giovanni Paolini

Presentata da:
Jacopo Del Gobbo

Anno accademico 2023-2024

A mia mamma

Abstract

Se ci viene dato un insieme parzialmente ordinato, c'è un modo "naturale" di associargli una topologia?

Vedremo che preso un poset P si può considerare il suo complesso d'ordine, un particolare complesso simpliciale la cui realizzazione geometrica sarà lo spazio topologico a cui pensiamo quando parliamo di topologia di P.

Concentreremo la trattazione sull'omologia e coomologia di complessi simpliciali, trovando anche basi esplicite. Ci aiuterà in questo la nozione di *shellability*, che ci permetterà di dire che alcune classi importanti di complessi simpliciali sono omotopicamente equivalenti ad un wedge di sfere.

Introdurremo un modo di etichettare le relazioni tra elementi di un poset, detto *edge-lexicographical labeling* (EL-labeling) e dimostreremo che, sotto certe condizioni, avere un EL-labeling equivale, per il poset, ad essere *shellable*.

Introduction

The theory of poset topology establishes a deep connection between combinatorics and other branches of mathematics. Just see some of the different fields which played an important role in the development of the theory:

- commutative algebra (proof of the upper bound conjecture)
- group theory and representation theory (p-subgroups posets and group actions on the homology of posets)
- combinatorics (Björner's work on poset shellability and the extension of the notion of shellability to non-pure complexes, performed with Wachs, an interesting aspect we will spend some time on)
- topology (in particular the theory of hyperplane arrangements, that we will introduce)
- complexity theory (evasiveness conjecture)
- knot theory and graph connectivity

So it is clear that poset topology is a very interdisciplinar topic, but what is it about? We will see that given a poset one can associate a certain simplicial complex to it, and one regards the topology of the geometric realization of the complex as the topology of the poset.

We will focus on the property of *shellability* of a complex, which reveals everything about its homotopy type. We will see how simple is the co(homology) of a shellable complex and, since research in poset topology is driven by the study of specific examples (both from inside and outside of combinatorics) we will touch with hand the theory we develop, through various interesting examples.

Contents

Introduction **i**

1 Basic notions **1**
1.1 Order complexes and face posets 1
1.2 Hyperplane and subspace arrangements 5
1.3 Poset homology and cohomology 7

2 Shellability and EL-shellability **9**
2.1 Shellability of simplicial complexes 9
2.2 Lexicographic shellability 13

3 Examples and applications **17**
3.1 The Boolean lattice B_n 17
3.2 The partition lattice Π_n. 18
3.3 Geometric lattices 20

Bibliography **23**

Acknowledgements **25**

Chapter 1

Basic notions

1.1 Order complexes and face posets

We begin by defining the order complex of a poset and the face poset of a simplicial complex. These construction enable us to view posets and simplicial complexes as essentially the same topological object. We will work all the time with finite posets and simplicial complexes.

Definition 1. An *abstract simplicial complex* Δ on a finite vertex set V is a nonempty collection of subsets of V such that

- $v \in \Delta$ for all $v \in V$
- if $G \in \Delta$ and $F \subseteq G$ then $F \in \Delta$.

The elements of Δ are called *faces* (or *simplices*) of Δ and the maximal faces are called *facets*. We say that a face F has dimension d and write $\dim F = d$ if $d = |F| - 1$. Faces of dimension d are referred to as *d-faces*. The *dimension* $\dim \Delta$ of Δ is defined to be $\max_{F \in \Delta} \dim F$. We also allow the (-1)-dimensional complex $\{\emptyset\}$, which we refer to as the *empty simplicial complex*. It will be convenient to refer to the empty set $\emptyset$ as the *degenerate empty complex* and say that it has dimension -2 (even though we don't really consider it to be simplicial complex). If all facets of Δ have the same dimension then Δ is said to be *pure*.

Definition 2. A d-dimensional *geometric simplex* in $\mathbb{R}^n$ is defined to be the convex hull of $d + 1$ affinely independent points in $\mathbb{R}^n$ called *vertices*. The convex hull of any subset of the vertices is called a *face* of the geometric simplex.

A *geometric simplicial complex* K in $\mathbb{R}^n$ is a nonempty collection of geometric simplices in $\mathbb{R}^n$ such that

- Every face of a simplex in K is in K
- The intersection of any two simplices of K is a face of both of them.

One should ask why we use the adjectives "abstract" and "geometric" for simplicial complexes. Well, observe that from a geometric simplicial complex K we can obtain an abstract simplicial complex $\Delta(K)$ by letting the faces of $\Delta(K)$ be the vertex sets of the simplices of K. Every abstract simplicial complex Δ can be obtained in this way, i.e., there is a geometric simplicial complex K such that $\Delta(K) = \Delta$. Although K is not unique, the underlying topological space, obtained by taking the union of the simplices of K (under the Euclidean topology of $\mathbb{R}^n$) is unique up to homeomorphism. We refer to this space as the *geometric realization* of Δ and denote it by $\|\Delta\|$. From now on we will drop the $\| \ \|$ and let Δ denote an abstract simplicial complex as well as its geometric realization.

Now we shall see how these concepts find place in poset theory. First of all, recall that *poset* stands for "partially ordered set", so we are dealing with a set in which exists a binary relation which is reflexive, transitive and antisymmetric. In other words, for certain pairs of elements you can tell if one is greater than the other (if you could do this for every pair, then the order would be *total*.

Definition 3. To every poset P we can associate an abstract simplicial complex $\Delta(P)$ called the *order complex* of P. The vertices of $\Delta(P)$ are the elements of P and the faces of $\Delta(P)$ are the chains (i.e., totally ordered subsets) of P. (The order complex of the empty poset is the empty simplicial complex $\{\emptyset\}$).

For example, you can see below in Figure 1.1 in the Hasse diagram of a poset P and its order complex.

Figure 1.1: The poset $P = \bar{B}_3$, where if an edge connects two nodes it means that the upper node is greater than the lower node, and its order complex

Definition 4. To every simplicial complex Δ, we can associate a poset $P(\Delta)$ called the *face poset* of Δ, which is defined to be the poset of nonempty faces ordered by inclusion.

The *face lattice* $L(\Delta)$ is $P(\Delta)$ with a smallest element $\hat{0}$ and a largest element $\hat{1}$ attached. See Figure 1.3 (but also Figure 1.2).

If we have a simplicial complex Δ, take its face poset $P(\Delta)$, and then take the order complex $\Delta(P(\Delta))$ we get a simplicial complex known as the *baricentric subdivision* of Δ.

The geometric realizations are always homeomorphic: $\Delta \cong \Delta(P(\Delta))$. See Figure 1.2.

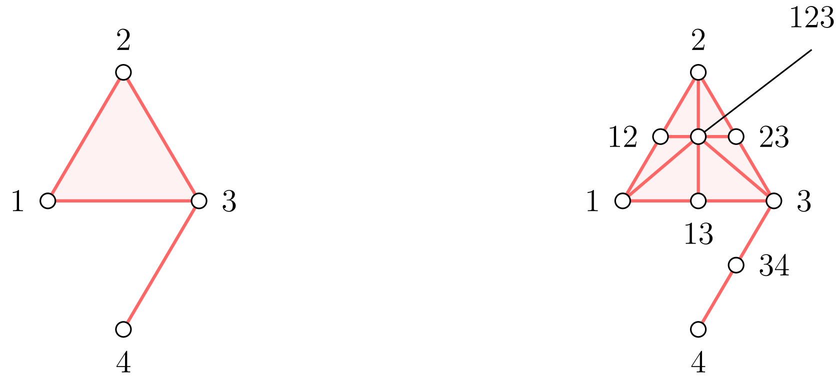

Figure 1.2: A simplicial complex Δ and its barycentric subdivision: note the homeomorphism between the two

When we attribute a topological property to a poset, we mean that the geometric realization of the order complex of the poset has that property.

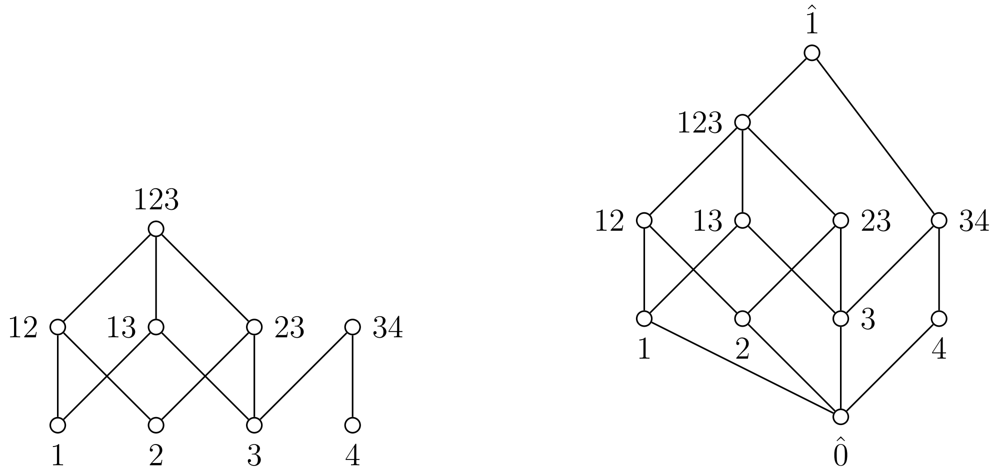

Figure 1.3: The face poset of Δ (the same of Figure 1.2), and the face lattice $L(\Delta)$

For instance, if we say that the poset P has the homotopy type of a wedge of spheres, we mean that $\|\Delta(P)\|$ has the homotopy type of a wedge of spheres.

Let us review some basic poset terminology. An m-chain of a poset P is a totally ordered subset $c = \{x_1 < x_2 < \ldots < x_{m+1}\}$ of P. We say the length $l(c)$ of c is m. We consider the empty chain to be a (-1)-chain. The length $l(P)$ of P is defined to be

$$l(P) := \max\{l(c) : c \text{ is a chain of } P\}.$$

Thus, $l(P) = \dim \Delta(P)$ and $l(P(\Delta)) = \dim \Delta$.

A chain of P is said to be *maximal* if it is inclusion-wise maximal. Observe that from this follows that the set $M(P)$ of maximal chains of P is the set of facets of $\Delta(P)$. A poset P is said to be *pure* (or ranked, or graded) if all maximal chains have the same length. Thus, P is pure if and only if $\Delta(P)$ is pure. Also a simplicial complex Δ is pure if and only if its face poset $P(\Delta)$ is pure.

For $x \leq y$ in P, let (x, y) denote the open interval $\{z \in P : x < z < y\}$ and let $[x, y]$ denote the closed interval $\{z \in P : x \leq z \leq y\}$. Half open intervals $(x, y]$ and $[x, y)$ are defined similarly.

If P has a unique minimum element, it is conventionally denoted by $\hat{0}$ and refer to it as the *bottom* element. Analogously, if P has a unique maximum element, we denote it by $\hat{1}$ and call it the *top* element. Now observe that if P has a bottom element $\hat{0}$ or a top element $\hat{1}$ then $\Delta(P)$ is a cone, hence it is contractible. This is the reason why we usually remove the bottom and top elements and study the more interesting topology of the remaining poset. Define the *proper part* of a poset P, for which $|P| > 1$, to be

$$\bar{P} := P - \{\hat{0}, \hat{1}\}$$

In the case that $|P| = 1$ it will be convenient to define $\Delta(\bar{P})$ to be the degenerate empty complex $\emptyset$. We will also say $\Delta((x, x)) = \emptyset$ and $l((x, x)) = -2$.

For posets with a bottom element $\hat{0}$, the elements that cover $\hat{0}$ are called *atoms*. For posets with a top element $\hat{1}$, the elements that are covered by $\hat{1}$ are called *coatoms*.

A poset P is said to be *bounded* if it has a top element $\hat{1}$ and a bottom element $\hat{0}$. Given a poset P, we define the bounded extension

$$\hat{P} := P \cup \{\hat{0}, \hat{1}\}$$

where new elements $\hat{0}$ and $\hat{1}$ are adjoined, even if P already has a bottom or top element.

A poset P is said to be a *meet semilattice* if every pair of elements $x, y \in P$ has a meet $x \wedge y$, i.e. an element less than or equal to both x and y that is greater than all other such elements. A poset P is said to be a *join semilattice* if every pair of elements $x, y \in P$ has a join $x \vee y$, i.e. a unique element greater than or equal to both x and y that is less than all other such elements. If P is both a join semilattice and a meet semilattice then P is said to be a *lattice*. It is a basic fact of lattice theory that any finite meet (join) semilattice with a top (bottom) element is a lattice.

Example 1 (The Boolean lattice)**.** Let B_n denote the lattice of subsets of $[n] := \{1, 2, \ldots, n\}$ ordered by containment, and let $\bar{B}_n := B_n - \{\emptyset, [n]\}$. Then

$$\bar{B}_n \cong \mathbb{S}^{n-2}$$

because $\Delta(B_n)$ is the barycentric subdivision of the boundary of the $(n-1)$-simplex. You can see the case $n = 3$ in Figure 1.1 a few pages ago.

1.2 Hyperplane and subspace arrangements

A *hyperplane arrangement* A is a finite collection of (affine) hyperplanes in some vector space V. We will consider only real hyperplane arrangements

($V = \mathbb{R}^n$) here. Real hyperplane arrangements divide $\mathbb{R}^n$ into regions. A remarkable formula for the number of regions was given by Zaslavsky in 1975, but we will not treat it. However, the formula involves an important basic notion that interests us, namely the intersection semilattice of a hyperplane arrangement.

Definition 5. The intersection semilattice $L(A)$ of a hyperplane arrangement A is defined to be the meet semilattice of nonempty intersections of hyperplanes in A ordered by reverse inclusion.

You can see in Figure 3a hyperplane arrangementin $\mathbb{R}^2$ and its intersection semilattice.

Note that we include the entire space V (seen as the intersection over the empty set) which is the bottom element $\hat{0}$ of $L(A)$. Now observe that $L(A)$ has a top element if and only if $\bigcap A \neq \emptyset$. Such an arrangement is called a *central arrangement*. Hence for central arrangements A, the intersection semilattice $L(A)$ is actually a lattice.

Example 2 (The (type A) coordinate hyperplane arrangement and the Boolean lattice B_n)**.** The coordinate hyperplane arrangement is the central hyperplane arrangement consisting of the coordinate hyperplanes $x_i = 0$ in $\mathbb{R}^n$. It is easy to see that the intersection lattice of this arrangement is isomorphic to the subset lattice B_n. Indeed, the intersection

$$\{\mathbf{x} \in \mathbb{R}^n : x_{i_1} = x_{i_2} = \cdots = x_{i_k} = 0\},$$

where $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, corresponds to the subset $\{i_1, i_2, \ldots, i_k\}$. This correspondence is an isomorphism from the intersection lattice to B_n.

Example 3 (The type A braid arrangement and the partition lattice Π_n)**.** For $1 \leq i < j \leq n$, let

$$H_{i,j} = \{\mathbf{x} \in \mathbb{R}^n : x_i = x_j\}.$$

The hyperplane arrangement

$$A_{n-1} := \{H_{i,j} : 1 \leq i < j \leq n\}$$

is known as the *braid arrangement* or the *type A Coxeter arrangement*. The intersection lattice $L(A_{n-1})$ is isomorphic to Π_n, the lattice of partitions of the set $[n]$ ordered by refinement. Indeed, for each partition $\pi \in \Pi_n$, let l_π be the linear subspace of $\mathbb{R}^n$ consisting of all points $(x_1, \ldots, x_n)$ such that $x_i = x_j$ whenever i and j are in the same block of π. The map $\pi \to l_\pi$ is a poset isomorphism from Π_n to $L(A_{n-1})$. You can see the intersection between the braid arrangement A_2 and the plane $\{x_1 + x_2 + x_3 = 0\}$ at left and the partition lattice Π_3 in Figure 3.

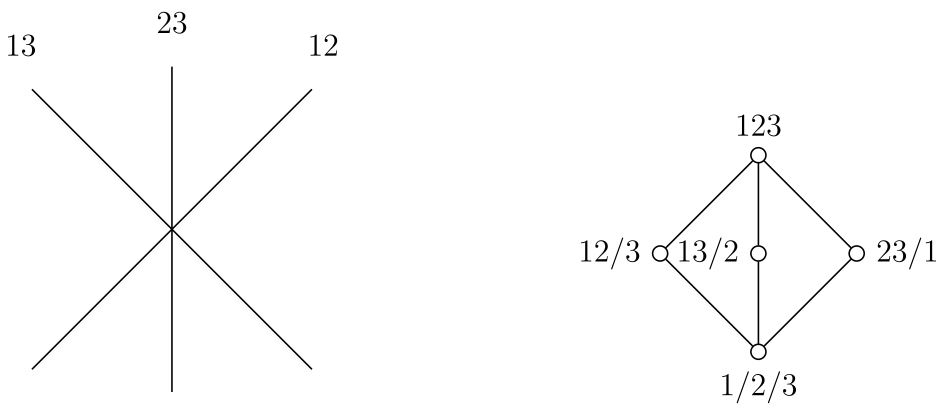

Figure 1.4: A_2 (where ij means $x_i = x_j$ for $i, j = 1, 2, 3$) and $L(A_2) = \Pi_3$

1.3 Poset homology and cohomology

By (co)homology of a poset, we mean the reduced simplicial (co)homology of its order complex. We will briefly introduce simplicial homology and cohomology in terms of chains of the poset.
For each poset P and integer j, define the chain space

$$C_j(P, \mathbf{k}) := \mathbf{k}\text{-module freely generated by } j\text{-chains of } P,$$

where $\mathbf{k}$ is a field or $\mathbb{Z}$.
The boundary map $\partial_j : C_j(P; \mathbf{k}) \to C_{j-1}(P; \mathbf{k})$ is defined by

$$\partial_j(x_1 < \cdots < x_{j+1}) = \sum_{i=1}^{j+1} (-1)^i (x_1 < \cdots < \hat{x}_i < \cdots < x_{j+1}),$$

where the hat denotes deletion. Since $\partial_{j-1}\partial_j = 0$ we have that $C_j(P; \mathbf{k})$ is an algebraic complex. Define the cycle space $Z_j(P; \mathbf{k}) := \ker \partial_j$ and the boundary space $B_j(P; \mathbf{k}) := \operatorname{im} \partial_{j+1}$. The homology of the poset P in dimension j

is defined by

$$\tilde{H}_j(P;\mathbf{k}) := Z_j(P;\mathbf{k})/B_j(P;\mathbf{k}).$$

The coboundary map $\delta_j : C_j(P;\mathbf{k}) \to C_{j+1}(P;\mathbf{k})$ is defined by

$$\langle \delta_j(\alpha), \beta\rangle = \langle \alpha, \partial_{j+1}(\beta)\rangle$$

where $\alpha \in C_j(P;\mathbf{k})$, $\beta \in C_{j+1}(P;\mathbf{k})$ and $\langle\cdot,\cdot\rangle$ is the bilinear form on $\bigoplus_{j\geq -1} C_j(P;\mathbf{k})$ for which the chains of P form an orthonormal basis. This is equivalent to saying

$$\delta_j(x_1 < \cdots < x_j) = \sum_{i=1}^{j+1}(-1)^i \sum_{x\in(x_{i-1},x_i)} (x_1 < \cdots < x_{i-1} < x < x_i < \cdots < x_j),$$

for all chains $x_1 < \cdots < x_j$, where x_0 is the bottom element of $\hat{P}$ and x_{j+1} is the top element of $\hat{P}$. Define the cocycle space to be $Z^j(P;\mathbf{k}) := \ker \delta_j$ and the coboundary space to be $B^j(P;\mathbf{k}) := \operatorname{im} \delta_{j-1}$. The cohomology of the poset P in dimension j is defined to be

$$\tilde{H}^j(P;\mathbf{k}) := Z^j(P;\mathbf{k})/B^j(P;\mathbf{k}).$$

Chapter 2

Shellability and EL-shellability

2.1 Shellability of simplicial complexes

Shellability is a combinatorial property of simplicial and more general cell complexes, with strong topological and algebraic consequences. Imagine having in front of you all the facets of a simplicial complex Δ. If you were asked to build Δ you could begin choosing a certain facet. Then you may pick up another facet and, along a codimension-one face, you could try to attach it to the first one. When you try to fit in the third facet, you could make it touch one or both the preceding facets, again along codimension-one faces. Well, if you are able (and lucky enough) to repeat the process until there are no facets left, you will have more or less found a shelling of Δ.

Definition 6. A simplicial complex Δ is said to be shellable if there exists a linear ordering of its facets $F_1, F_2, ..., F_t$ such that the subcomplex $(\bigcup_{i=1}^{k-1} \langle F_i \rangle) \cap \langle F_k \rangle$ is pure and $(\dim F_k - 1)$-dimensional for all $k = 2, \ldots, t$, where $\langle F \rangle$ denotes the subcomplex generated by F. Such an ordering of the facets it is called a *shelling*.

Let us see an example.

Figure 2.1: A shellable and a non-shellable complex

Example 4. In Figure 2.1 you can see a shellable complex on the left and a non-shellable complex on the right. In particular, observe that in in the first case whe can choose as F_1 the vertical segment, and as F_2 the isolated point. In this way we only have to check the dimension of the intersection between $\langle F_1 \rangle$ and F_2, but $\dim(\langle F_1 \rangle \cap F_2) = \dim \emptyset = -1 = \dim F_2 - 1$. In the second example, it does not matter how we enumerate the facets, we will have, at the first check, $\dim(\langle F_1 \rangle \cap \langle F_1 \rangle) = \dim \emptyset = -1 \neq \dim(\langle F_1 \rangle) - 1 = 2 - 1$.

The topology of a shellable complex is very nice and straightforward, as shown by the following:

Theorem 1 (Björner and Wachs [4])**.** A shellable simplicial complex has the homotopy type of a wedge of spheres (in varying dimensions), where for each i, the number of i-spheres is the number of i-facets whose entire boundary is contained in the union of the earlier facets. Such facets are usually called *homology facets.*

Before seeing the details, we give the idea for the proof.

Let Δ be a shellable simplicial complex. We first observe that any shelling of Δ can be rearranged to produce a shelling in which the homology facets come last. So Δ has a shelling $F_1, F_2, \ldots, F_k$ where $F_1, ..., F_j$ are not homology facets and $F_{j+1}, ..., F_k$ are, where $1 \leq j \leq k$. The basic idea of the proof is that as we attach the first j facets, we construct a contractible simplicial complex at each step. Each of the homology facets creates a sphere since the entire boundary of the facet can be contracted to a point.

The approach we will be adopting in order to proof this result is inspired by [4] (Björner and Wachs).

The following is a useful restatement of the definition of shellability:

Lemma 1. An order $F_1, F_2, ..., F_t$ of the facets of Δ is a shelling if and only if for every i and k with $1 \leq i < k \leq t$ there is a j with $1 \leq j < k$ and an $x \in F_k$ such that $F_i \cap F_k \subseteq F_j \cap F_k \ = F_k - \{x\}$.

Lemma 2. Let F be a facet of a complex Δ, $R \subseteq F$, and let Δ' be the subcomplex generated by the other facets of Δ. Then the following statements are equivalent:

1) $\langle F \rangle - \Delta' = [R, F]$,

2) $\langle F\rangle \cap \Delta' = \bigcup_{x\in R}\langle F-\{x\}\rangle$,

where $[R,F]=\{C \mid R\subseteq C\subseteq F\}$.

Proof. $\langle F\rangle$ is the disjoint union of $[R,F]$ and $\bigcup_{x\in R}\langle F-\{x\}\rangle$. □

Definition 7. Given a shelling $F_1,...,F_t$ of Δ, with successively generated subcomplexes $\Delta_j=\bigcup_{i=1}^{j}\langle F_i\rangle$, define the *restriction* of the facet F_k by

$$R(F_k)=\{x\in F_k \mid F_k-\{x\}\in \Delta_{k-1}\}.$$

Lemma 3. Let $F_1,F_2,...,F_t$ be a shelling of Δ with restriction map R. Let $F_{i_1},F_{i_2},...,F_{i_t}$ be the rearrangement obtained by taking first all facets F such that $R(F)\neq F$ in the induced order, and then all remaining facets in arbitrary order. Then this rearrangement is also a shelling with the same restriction map R.

Proof. Follows from the definition of shellability. □

Proof of Theorem 1. For some fixed shelling of Δ, let $\Gamma=\{$facets of F such that $R(F)=F\}$ and $\Delta^*=\Delta-\Gamma$. Then by Lemma 3 the induced order of the remaining facets is a shelling of Δ^*, whose restriction map is R. Let F_k be the k-th facet of Δ^* and set $\Delta^*_k=\bigcup_{i=1}^{k}\langle F_i\rangle$. The facet F_k has a free face in Δ^*_k (i.e. a face contained in no other facet), namely $R(F_k)$. This is a proper face since $R(F_k)\neq F_k$. Thus, removing $R(F_k)$ and all faces containing it collapses Δ^*_k to $\Delta^*_k-[R(F_k),F_k]=\Delta^*_{k-1}$. It follows that Δ^*_{k-1} and Δ^*_k are homotopy equivalent (in fact, Δ^*_{k-1} is a strong deformation retract of Δ^*_k). Since Δ^*_1 is a simplex, we conclude that Δ^* is contractible.
Now we use the fact that passing to the quotient space modulo a contractible subspace does not alter homotopy type (see for instance [1, Proposition 0.17]). The space Δ is obtained from Δ^* by attaching the cells (simplices) in Γ, each one along its entire boundary. Thus when Δ^* is contracted to a point, each i-cell is deformed into a i-sphere with a distinguished point Δ^*/Δ^*.

Observe that outside this point the resulting spheres are topologically independent, so this is a wedge of spheres with the right dimensions. □

This result is interesting per se, but it also tells everything about the (co)homology of a shellable complex:

Corollary 1. If Δ is shellable then for all i,

$$\tilde{H}_i(\Delta;\mathbb{Z}) \cong \tilde{H}^i(\Delta;\mathbb{Z}) \cong \mathbb{Z}^{r_i} \tag{2.1}$$

where R_i is the number of homology i-facets of Δ.

Homology facets tell us the Betti numbers of Δ; good to know. But there is more, they induce dual bases for homology and cohomology.

Let us call Γ the set of homology facets. Now let $\Gamma_j = \{F \in \Gamma \colon |F| = j\}$. For each $F \in \Gamma_j$ define a $(j-1)$-cochain σ^F in terms of its values on the basis elements $A \in \Delta_{j-1}$=$\{(j-1) - \text{faces of } \Delta\}$ as follows:

$$\sigma^F(A) = \begin{cases} 1, & A = F \\ 0, & A \neq F \end{cases}$$

Since F is a facet of Δ one sees that σ^F is a cocycle. Hence it determines a cohomology class $[\sigma^F] \in \tilde{H}^{j-1}(\Delta,\mathbb{Z})$.

Theorem 2 (Björner and Wachs [4])**.** Let Γ_j be the set of homology facets of size j induced by a shelling of Δ. Then the classes $[\sigma^F]$, for $F \in \Gamma_j$, are a basis of $\tilde{H}^{j-1}(\Delta,\mathbb{Z})$.

Proof. Let ρ be any cocycle in $C^{j-1}(\Delta)$. Consider the cocycle

$$\tau = \rho - \sum_{F \in \Gamma_j} \rho(F)\sigma^F.$$

We have that $\tau(F) = 0$ for all $F \in \Gamma_j$, so τ is in fact a cocycle in $C^{j-1}(\Delta^*)$. But $\tilde{H}^{j-1}(\Delta^*,\mathbb{Z}) = 0$, since Δ^* is contractible. Hence $\tau = \delta_{\Delta^*}(\tau')$ for some $\tau' \in C^{j-2}(\Delta^*)$. This implies that $\delta_\Delta(\tau') = \tau + \sum_{F\in\Gamma_j} a_F\sigma^F$, and hence $[\rho] = \sum_{F\in\Gamma_j}(\rho(F) - a_F)[\sigma^F]$.
We know from before that $\tilde{H}^{j-1}(\Delta,\mathbb{Z})$ is a free abelian group of rank r_j and we have just shown that it is generated by the $|\Gamma_j| = r_j$ elements $[\sigma^F]$. Hence, these elements form a basis. □

Shellable complexes arise naturally in mathematics. An interesting example is the following:

Theorem 3 (Shareshian [7])**.** The lattice of subgroups of a finite group G is shellable if and only if G is solvable.

In order to prove the pure version of this statement, which states that *the lattice of subgroups of a finite group G is pure shellable if and only if G is supersolvable*, Björner ([2]) introduced in the late 70's a technique called *lexicographic shellability*, and this will be our next topic, in the more general context of nonpure complexes.

2.2 Lexicographic shellability

Definition 8. An *edge labeling* of a bounded poset P is a map $\lambda\colon E(P) \to \Lambda$, where $E(P)$ is the set of edges of the Hasse diagram of P, i.e., the covering relations $x \lessdot y$ of P, and Λ is some poset (usually $\mathbb{Z}$ with its natural total order relation).

Definition 9. Given an edge labeling $\lambda\colon E(P) \to \Lambda$ we associate a word

$$\lambda(c) = \lambda(\hat{0}, x_1)\lambda(x_1, x_2) \cdots \lambda(x_t, \hat{1})$$

with each maximal chain $c = (\hat{0} \lessdot x_1 \lessdot \cdots \lessdot x_t \lessdot \hat{1})$.

We say that c is *increasing* if the associated word is *strictly* increasing. In other words, c is increasing if $\lambda(c) = \lambda(\hat{0}, x_1) < \lambda(x_1, x_2) < \cdots < \lambda(x_t, \hat{1})$.

We say that c is *decreasing* if the associated word $\lambda(c)$ is *weakly* increasing.

We are now able to order the maximal chains lexicographically by using the lexicographic order on the corresponding words.

Note that every edge labeling of P restricts to an edge labeling of any closed interval $[x, y]$ of P, so we can talk about increasing and decreasing maximal chains of $[x, y]$, and lexicographic order of maximal chains of $[x, y]$.

Also remember that we are working with *bounded* posets, so the maximal chains of P will begin with $\hat{1}$ and end with $\hat{0}$.

Definition 10. Let P be a bounded poset. An *edge-lexicogrphical labeling* (EL-labeling) of P is an edge labeling such that in each closed interval $[x, y]$ of P there is a unique increasing maximal chain, which lexicographically precedes all the other maximal chains of $[x, y]$.

Now comes the interesting part: a bounded poset that admits an EL-labeling is said to be *edge-lexicographic shellable* (EL-shellable). There has to be some connection between EL-labelings and shellability...

Theorem 4 (Björner an Wachs [5]). Let P be a bounded poset with an EL-labeling. Then the lexicographic order of the maximal chains of P is a shelling of $\Delta(P)$.

Björner proved this result in the pure case, but we will see the proof for non-pure posets since there are just a few additional details to pay attention to.

The proof uses the notion of *rooted interval*, which is just a pair $([x, y], c)$ in P where $[x, y]$ is a closed interval and c is a saturated chain from $\hat{1}$ to y.

Proof. We are going to show that any linear ordering of the set M of maximal chains which extends the lexicographic ordering of the edges is a shelling order. So assign a linear order, denoted "$\lessdot$" to M such that $\lambda(m) <_L \lambda(m')$ implies $m \lessdot m'$. We have to prove that if $k \lessdot m$ for $k, m \in M$ then there exists an $h \in M$ such that $h \lessdot m, (k \cap m) \subseteq (h \cap m)$ and $|h \cap m| = |m| - 1$.

Consider two maximal chains in P, $k : \hat{1} = k_0 \to k_1 \to \cdots \to k_r = \hat{0}$ and $m : \hat{1} = m_0 \to m_1 \to \cdots \to m_t = \hat{0}$, and suppose that $k \lessdot m$. Let d be the greatest integer such that $m_i = k_i$ for $i = 0, \cdots, d$, and let g be the least integer such that $d < g$ and $k_g = m_g$. Then $g - d \geq 2$ and $d < i < g$ implies that $k_i \neq m_i$. Now consider the rooted interval, $([m_g, m_d], m_0 \to m_1 \to \cdots \to m_d)$. The chain $m_d \to m_{d+1} \to \cdots \to m_g$ cannot be the unique maximal chain of this interval with increasing labels because, from the definition of EL-labeling, it would then be $\lambda(m) <_L \lambda(k)$ contrary to the assumption that $k \lessdot m$. Consequently, the label $\lambda(m)$ must have a descent $\lambda_e(m) > \lambda_{e+1}(m)$ for some e with $d < e < g$. Then in the rooted interval $([m_{e+1}, m_{e-1}], m_0 \to m_1 \to \cdots \to m_{e-1})$ the chain $m_{e-1} \to m_e \to m_{e+1}$ has a decreasing label so there is a chain $m_{e-1} \to x_{i_1} \to x_{i_2} \to \cdots \to x_{i_r} \to m_{e+1}$ whose label comes earlier in the lexicographic order. If we let $h : \hat{1} = m_0 \to m_1 \to \cdots \to m_{e-1} \to x_{i_1} \to x_{i_2} \to \cdots \to x_{i_r} \to m_{e+1} \to m_{e+2} \to \cdots \to \hat{0}$ it follows that $\lambda(h) <_L \lambda(m)$, hence $h \lessdot m$, and since m_e cannot be in

k because $d < e < g$ and by definition of g, the construction shows that $h \cap m = m - m_e \supseteq k \cap m$.

□

Chapter 3

Examples and applications

3.1 The Boolean lattice B_n

There is a very natural EL-labeling of the Boolean lattice B_n; simply label the covering relation $A_1 \lessdot A_2$ with the unique element in the singleton set $A_2 - A_1$. The maximal chains correspond to the permutations in $\mathfrak{S}_n$. It is easy to see that each interval has a unique increasing chain that is lexicographically first. There is only one decreasing chain, which is consistent with the fact that $\Delta(B_n)$ is a sphere.

For each $k \leq n$, define the truncated Boolean algebra B_n^k to be the subposets of B_n given by

$$B_n^k = \{A \subseteq [n] : |A| \geq k\}.$$

Define an edge labeling λ of $B_n^k \cup \{\hat{0}\}$ as follows:

$$\lambda(A_1, A_2) = \begin{cases} \max A_2 & \text{if } A_1 = \hat{0} \text{ and } |A_2| = k \\ a & \text{if } A_2 - A_1 = \{a\} \end{cases}$$

It easy to check that this is an EL-labeling. The decreasing chains correspond to permutations with descent set $\{k, k+1, \cdots, n-1\}$.

Recall that in the symmetric group $\mathfrak{S}_n$ a permutation $w = a_1 a_2 \ldots a_n$ has descent set $D(w) = \{i : a_i > a_{i+1}\}$.

Hence $\dim \tilde{H}^{n-k-1}(\bar{B}_n^k)$ equals the number of permutations in $\mathfrak{S}_n$ with descent set $\{k, \cdots, n-1\}$.

3.2 The partition lattice Π_n.

We will now give two different EL-labelings of the partition lattice, due to Gessel (you can find it in Björner's work [2]) and Wachs [8] respectively. If $x \lessdot y$ in Π_n then y is obtained from x by merging two blocks, say B_1 and B_2. For the first edge labeling λ_1, let

$$\lambda_1(x,y) = \max\{\min B_1, \min B_2\}$$

and for the second edge labeling λ_2, let

$$\lambda_2(x,y) = \max B_1 \cup B_2.$$

The increasing chain from $\hat{0}$ to $\hat{1}$ is the same for both labelings; it consists of partitions with only one nonsingleton block. More precisely, the chain is given by

$$\hat{0} \lessdot \{1,2\} \lessdot \{1,2,3\} \lessdot \cdots \lessdot \hat{1},$$

where we have written only the nonsingleton block of each partition in the chain.

The decreasing maximal chains for λ_1 and λ_2 are not the same. If you look at λ_2, for instance, they consist of partitions with only one nonsingleton block and are of the form

$$c_\sigma := (\hat{0} \lessdot \{\sigma(n), \sigma(n-1)\} \lessdot \{\sigma(n), \sigma(n-1), \sigma(n-2)\} \lessdot \cdots$$

$$\lessdot \{\sigma(n), \sigma(n-1), \sigma(n-2), \cdots, \sigma(1)\}),$$

where $\sigma \in \mathfrak{S}_n$ and $\sigma(n) = n$. We conclude that the homotopy type of $\bar{\Pi}_n$ is given by

$$\bar{\Pi}_n \simeq \bigvee_{(n-1)!} \mathbb{S}^{n-3}$$

and that the chains $\bar{c}_\sigma$, where $\sigma \in \mathfrak{S}_n$ and $\sigma(n) = n$, form a basis for $\tilde{H}^{n-3}(\bar{\Pi}_n; \mathbb{Z})$. We can also describe a nice basis for the homology of $\bar{\Pi}_n$ that is dual to the decreasing chain basis for λ_2. But first we have to define what does it mean to *split* a permutation $\sigma \in \mathfrak{S}_n$ at positions $j_1 < j_2 < \cdots < j_k$ in $[n-1]$: it simply means that we form the partition

$$\sigma(1)\sigma(2)\cdots\sigma(j_1)|\sigma(j_1+1)\sigma(j_1+2)\cdots\sigma(j_2)|\cdots|$$

$$\sigma(j_k+1)\sigma(j_k+2)\cdots\sigma(n)$$

of $[n]$. For each $\sigma \in \mathfrak{S}_n$, let Π_σ be the induced subposet of the partition lattice Π_n consisting of the partitions obtained by splitting the permutations

σ at any set of positions in $[n-1]$. Each poset Π_σ is isomorphic to the subset lattice B_{n-1}. Therefore $\Delta(\bar{\Pi}_\sigma)$ is an $(n-3)$-sphere embedded in $\Delta(\bar{\Pi}_n)$, and hence it determines a fundamental cycle $\rho_\sigma \in \tilde{H}_{n-3}(\bar{\Pi}_n;\mathbb{Z})$.

We have the following:

Theorem 5 (Wachs [8])**.** *$\{\rho_\sigma : \sigma \in \mathfrak{S}_n, \sigma(n) = n\}$ form a basis for $\tilde{H}_{n-3}(\bar{\Pi}_n;\mathbb{Z})$ dual to the decreasing chain basis $\{\bar{c}_\sigma : \sigma \in \mathfrak{S}_n, \sigma(n) = n\}$ for cohomology. The homology basis is called the* splitting basis*.*

Now we describe the decreasing chain basis for cohomology for the EL-labeling λ_1 and its dual basis for homology. If we have a rooted nonplanar tree T (which simply means that children of a node are unordered) on node set $[n]$, by removing any set of edges of T we form a partition of $[n]$ whose blocks are the node sets of the connected components of the resulting graph. Let Π_T be the induced subposet of the partition lattice Π_n consisting of partitions obtained by removing edges of T. Note that if T is a linear tree then Π_T is the same as Π_σ, where σ is the permutation obtained by reading the nodes of the three from the root down. Each poset Π_T is isomorphic to the subset lattice B_{n-1}. We let ρ_T be the fundamental cycle of the spherical complex $\Delta(\bar{\Pi}_T)$. Let T be an increasing tree on node set $[n]$, i.e., a rooted nonplanar tree on node set $[n]$ in which each node i is greater than its parent $p(i)$. We form the chain c_T in Π_T, from top down, by removing the edges $\{i, p(i)\}$, one at time, in increasing order of i. One can show that the c_T, where T is an increasing tree on node set $[n]$, are the decreasing chains of λ_1.

Theorem 6. *Let T_n be the set of increasing trees on node set $[n]$. The set $\{\rho_T : T \in T_n\}$ forms a basis for $\tilde{H}_{n-3}(\bar{\Pi}_n;\mathbb{Z})$ dual to the decreasing chain basis $\{\bar{c}_T : T \in T_n\}$ for cohomology. This homology basis is called the* tree splitting basis*.*

See Wachs [9] for further details. There is also a geometric interpretation of the splitting basis, in which the fundamental cycles correspond to bounded regions in an affine slice of the real braid arrangement, but we will not see the details of it.

These EL-labelings and their corresponding bases are special cases of more general constructions. For instance, the first EL-labeling comes from constructions for geometric lattices, which will be our next and last topic.

3.3 Geometric lattices

Definition 11. A *geometric lattice* is a lattice L that is semimodular and atomic. In other words, for all $x, y \in L$, the join $x \wedge y$ covers y whenever x covers the meet $x \vee y$, and every element of L is the join of atoms.

We have already encountered 2 specific examples: the subset lattice B_n and the partition lattice Π_n; more in general, the intersection lattice $L(A)$ of any central hyperplane arrangement is a geometric lattice.

Now we shall describe an edge labeling for geometric lattices. Fix an ordering $a_1, a_2, \cdots, a_k$ of the atoms of the geometric lattice L. Then label each edge $x \lessdot y$ of the Hasse diagram with the smallest i for which $x \wedge a_i = y$.

Observe that if we order the atoms of B_n like $\{1\}, \{2\}, \cdots, \{n\}$, then this labeling coincides with the one given a few pages ago.

Let us focus on the decreasing chains of the geometric lattice EL-labeling (see Björner's work [3]). In order to do this, we have to introduce something from what is called *matroid theory*.

Definition 12. Let A be a set of atoms in a geometric lattice L. A is *independent* if $r(\bigvee A) = |A|$. A *circuit* is a set of atoms *minimally dependent*, which means that every proper subset is independent. A *broken circuit* is an independent set of atoms that can be obtained from a circuit by removing its smallest element (recall that we fixed an ordering of the atoms of L).

A maximal independent set of atoms is called an NBC base if contains no broken circuits. Observe that there is a natural bijection between NBC bases of L and decreasing chains of L: an NBC base $A = \{a_{i_1}, \cdots, a_{i_r}\}$ (where $1 \leq i_1 < i_2 < \cdots < i_r \leq k$ corresponds to the maximal chain

$$c_A \colon = (\hat{0} < a_{i_r} < a_{i_r} \vee a_{i_{r-1}} < \cdots < a_{i_r} \vee a_{i_{r-1}} \vee \cdots \vee a_{i_1} = \hat{1}).$$

One can see that the label sequence of c_A is $(i_r, i_{r-1}, \cdots, i_1)$, which is decreasing. The map $A \mapsto c_A$ the bijection from the NBC bases of L to the decreasing chains of L, so the elements $\{\bar{c}_A \colon A \text{ is an NBC base of } L\}$ form a basis for top cohomology of L.

The dual basis of this one is a basis for the homology (see Björner [3]). If we have an independent set of atoms A in a geometric lattice and we take joins, we generate a Boolean algebra L_A embedded in the geometric lattice. Let ρ_A be the fundamental cycle of $\bar{L}_A$.

Theorem 7 (Björner [3])**.** Fix an ordering of the set of atoms of a geometric lattice L. The set

$$\{\rho_A \colon A \text{ is an NBC base of } L\}$$

is a basis for top homology of $\bar{L}$, which is dual to the decreasing chain basis

$$\{\bar{c}_A \colon A \text{ is an NBC base of } L\}$$

for the top cohomology.

We conclude by mentioning that geometric lattices are exactly the lattices of flat of matroids:

Theorem 8. [6, Theorem 1.7.15] A lattice L is geometric if and only if it is the lattice of flats of a matroid.

Bibliography

[1] A. Hatcher, *Algebraic topology.*

[2] A. Björner, Shellable and Cohen-Macaulay partially ordered sets, Trans. AMS

[3] A. Björner, On the homology of geometric lattices, Alg. Univ. 14 (1982), 107–128.

[4] A. Björner and M.L. Wachs, Nonpure shellable complexes and posets I, Trans. AMS 348 (1996), 1299–1327.

[5] A. Björner and M. Wachs, Bruhat order of Coxeter groups and shellability, Advances in Math. 43 (1982), 87–100. MR 83i:20043.

[6] J.Oxley, *Matroid theory.*

[7] J. Shareshian, On the shellability of the order complex of the subgroup lattice of a finite group, Trans. AMS 353 (2001), 2689–2703.

[8] M. L. Wachs, A basis for the homology of the d-divisible partition lattice, Advances in Math. 117 (1996), 294–318.

[9] M. L. Wachs, *Poset topology: tools and applications.*

Acknowledgements

Se dovessi davvero ringraziare chi mi ha aiutato durante questi due anni, la lista sarebbe interminabile: fortunatamente queste persone lo sanno, e sanno il poco peso che do alle formalità, quindi penso che non rimarrà male nessuno.

In ogni caso, parto dal ringraziare il mio relatore, il prof. Giovanni Paolini, per la pazienza, disponibilità e gentilezza che lo contraddistinguono. Inoltre ringrazio i prof. Andrea Bonfiglioli, Nicoletta Cantarini e Luca Migliorini per essersi battuti per me quando ho chiesto la possibilità di laurearmi con un anno di anticipo. Non posso non ringraziare Alice, pilastro portante del Dipartimento. Ma ringrazio soprattutto chi in un modo o nell'altro ha cercato di mettermi i bastoni tra le ruote senza conoscermi e ha ottenuto l'effetto opposto.

Durante tutto il percorso è stato fondamentale l'aiuto dei miei compagni del Falansterio, che si sono mostrati sempre gentili e accoglienti nei miei confronti, rispettando i miei ritmi che mi hanno portato a condividere con loro meno di quanto avrei voluto. bastoni Ringrazio poi i miei genitori, i miei fratelli Mattia e Davide, e tutta la mia famiglia. In particolare mando un abbraccio a nonna Renata.

Ora sarebbe il turno dei miei migliori amici e amiche, che sono un po' una famiglia allargata... qui la lista sarebbe davvero lunga e cito giusto i Capi per avere l'anima in pace.

Infine, ringrazio la mia fidanzata bellissima per tutto quello che già sa.