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Coherent states and the classical limit of quantum mechanics

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Sommario

Si studia il limite classico della meccanica quantistica facendo uso di metodi ricavati dalla teoria degli stati coerenti impiegata in ottica quantistica. Si definiscono stati semiclassici dipendenti dalla costante di Planck ridotta $\hbar$ e da un punto dello spazio delle fasi classico e si mostra che i valori medi della posizione e dell'impulso di tali stati, nel limite formale $\hbar \to 0$, riproducono la dinamica classica secondo modalità che vengono illustrate in dettaglio.

Abstract

The classical limit of quantum mechanics is studied through the techniques deriving from the theory of coherent states which applies in the field of quantum optics. Semiclassical states, parameterized by the reduced Planck constant $\hbar$ and by a point in the classical phase space, are defined and it is shown that the expectation values of position and momentum of these states behave, in the formal limit $\hbar \to 0$, in accordance with classical dynamics in a way that is described in detail.

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Introduction

Right off its start, quantum mechanics had to deal with the problem of recovering classical mechanics in the form of the standard Hamilton-Jacobi theory. At the beginning of the 20th century, experimental results, together with the realisation of some theoretical inconsistencies, provided evidence that classical mechanics, which was until that moment regarded as a universal theory, was not possibly applicable at an atomic scale. The many problematic experiments performed – such as those of G. I. Taylor (1909), J. Frank and G. L. Hertz (1914), O. Stern and W. Gerlach (1922) and A. H. Compton (1924) – cast a light on the need for a new model which could fill the conceptual gaps left behind by classical mechanics in the attempt of explaining the obtained results. Regarding the theoretical issues, the Rayleigh-Jeans law – first derived by J. W. S. Rayleigh in the year 1900, who availed himself of the equipartition theorem, which is a fundamental result of statistical mechanics – is a paradigmatic example of the incompleteness and of the inconsistency of the classical mechanical model at the time. In the same year, M. Planck introduced the hypothesis that microscopic particles such as electrons could absorb and radiate electromagnetic energy only in finite quantities, which he called energy *quanta* (plural for energy *quantum*). From this single hypothesis, Einstein was able to explain the photoelectric effect, P. Debye, M. Born and T. Von Kármán managed to give a strong theoretical background to the Dulong-Petit law and Planck himself justified the Nernst theorem. Quantum physics set in and its formalism was then developed in its more modern form by E. Schroedinger, W. K. Heisenberg, W. Pauli, P. A. M. Dirac and others.

Classical mechanics was set aside but was not forgotten. Indeed, the epistemological problem of the possibility of identifying classical mechanics as a limit of quantum mechanics under specific conditions was always a core question in the understanding of the theory. Schroedinger himself (1926) more than once (see [4]) insisted on trying to persuade the scientific community that the similarity between Hamilton's and Fermat's principles was something physical before being a mathematical curiosity. Actually, on this basis, a first link between the classical and the quantum theory was again provided by Schroedinger, by pointing out that a statistical ensemble, that is a collection of a large number of copies of a classical particle, behaves as an undulatory system in some sort of geometrical optical regime. The undulatory system at issue is the Schroedinger undulatory system, which is found to describe the probabilistic behaviour of a quantum particle. It is then found that classical mechanics emerges from wave mechanics under the formal limit $\hbar \to 0$, which mathematically represents the fact that the action of a classical system is much larger than the action of its quantum equivalent.

With time, other procedures were introduced in order to study the semiclassical limit of the quantum theory. The WKB approximation is one of these and is perhaps one of the most widespread techniques in this field. It was developed by G. Wentzel, H. A. Kramers and L. Brillouin (see [9], [10], [11]) almost simultaneously to the Schroedinger's publications on wave mechanics. It essentially consists in writing the wavefunction $\psi(x)$ as

$$\psi(x) = \exp\left(i\hbar^{-1}S(x)\right)$$

and then expanding $S(x)$ as a power series of $\hbar$. The resulting expansion is substituted in the Schroedinger equation and a solution is searched.

In this thesis, the classical limit problem is tackled by following an alternative path. In particular, after a short introductory chapter on Weyl operators, where some technical aspects are dealt with, coherent states are introduced and their properties are studied in depth. Coherent states turn out to be relevant in the classical limit problem due to the fact that they are minimum-uncertainty states for position and momentum. In fact, the uncertainties of these observables in these states saturate the Heisenberg principle, which describes in mathematical terms the very basic distinction between classical and quantum physics. Then, in the last chapter, it is seen that a generalized version of coherent states, parameterized by the reduced Planck constant $\hbar$ and by a point in the classical phase space, enjoys really interesting properties when the formal limit $\hbar \to 0$ is taken. Indeed, in this limit, given a quantum system with a classical counterpart, having put appropriate smoothness conditions on the classical potential fields, the expectation values of position and momentum of these states are found to be centred, with very small uncertainty, on the solutions of the Hamilton equations of the classical system. In other words, these expectation values evolve in time along classical trajectories.

Chapter 1

A useful operatorial framework for quantum mechanics

It is our goal to discuss the properties of a special set of states which will be called coherent states. To this aim, it is useful to introduce two families of operators which will very often appear in the discussion that will be carried on in the next chapters. First, in section 1.1, some basic definitions and fundamental results about the so-called annihilation and creation operators and about their associated number operator are given in an informal way. The derivation of the propositions and theorems that are stated in this section will be purposely omitted, as these facts are very basic in the quantum theory and therefore well-known. A detailed analysis of the theory of these operators can be found in [6], on which 1.1 is based. Then, in section 1.2, the focus is set on the Weyl operators, which are introduced in a slightly more formal fashion. These operators play a key role in the definition of coherent states. For this reason, their properties are studied in depth and all the results of this section are shown. For further insight, see chapter 7 of [1], on which 1.2 is based.

1.1 Annihilation and creation operators

To start our review on the annihilation and creation operators, we give two basic definitions.

Definition 1.1.1 (Annihilation and creation operators)**.** *An operator $\hat{a}$ is said to be an annihilation operator if it obeys the following commutation rule with its adjoint operator $\hat{a}^+$:*

$$[\hat{a}, \hat{a}^+] = \hat{1}. \tag{1.1.1}$$

The adjoint operator $\hat{a}^+$ is then said to be a creation operator.

Definition 1.1.2 (Number operator)**.** *If $\hat{a}$ is an annihilation operator, the operator*

$$\hat{N} := \hat{a}^+\hat{a}. \tag{1.1.2}$$

is called number operator.

From these two definitions, one easily gets to the following results.

Proposition 1.1.1. *The number operator $\hat{N}$ is Hermitian.*

Proposition 1.1.2. *If $\hat{a}$ is an annihilation operator and $\hat{N}$ is its associated number operator, the following commutation relations hold:*

$$[\hat{N}, \hat{a}] = -\hat{a} \tag{1.1.3a}$$

$$[\hat{N}, \hat{a}^+] = \hat{a}^+. \tag{1.1.3b}$$

The commutation relation (1.1.1) has an important consequence regarding the spectrum of the number operator. This fact is summarized in the next theorem.

Theorem 1.1.1. *Given an annihilation operator $\hat{a}$, the spectrum of its associated number operator $\hat{N}$ corresponds to the set of all natural numbers $\mathbb{N}$. More precisely, given $n \in \mathbb{N}^{+}$, the ket*

$$|n\rangle := \hat{a}^{+n}|0\rangle(n!)^{-1/2}, \tag{1.1.4}$$

where $|0\rangle$ is the eigenket of $\hat{N}$ belonging to the 0 eigenvalue of $\hat{N}$, is such that

$$\hat{N}|n\rangle = |n\rangle n. \tag{1.1.5}$$

Example 1.1.1 (Harmonic oscillator)**.** Consider a harmonic oscillator of mass m and angular frequency ω. The following operators are defined:

$$\hat{a} := \frac{1}{(2m\hbar\omega)^{1/2}}(\hat{p} - im\omega\hat{q}) \tag{1.1.6a}$$

$$\hat{a}^{+} = \frac{1}{(2m\hbar\omega)^{1/2}}(\hat{p} + im\omega\hat{q}), \tag{1.1.6b}$$

where $\hat{q}$ is the position operator and $\hat{p}$ is the momentum operator of the harmonic oscillator. Using the canonical commutation relations, it can be shown that these operators satisfy the commutation relation (1.1.1). These consequently are annihilation and creation operators. From 1.1.2, the associated number operator has the following expression:

$$\hat{N} = \hat{a}^{+}\hat{a} = \frac{1}{\hbar\omega}\left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2\right) - \frac{\hat{1}}{2}. \tag{1.1.7}$$

The Hamiltonian operator of the harmonic oscillator can then be cast in the form:

$$\hat{H} = \left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2\right) = \hbar\omega\left(\hat{N} + \frac{\hat{1}}{2}\right). \tag{1.1.8}$$

The computation of the energy eigenvalues w_n of $\hat{H}$ is then straightforward and from (1.1.5) it is found that

$$\hat{H}|n\rangle = |n\rangle w_n \tag{1.1.9}$$

where

$$w_n = \hbar\omega\left(n + \frac{1}{2}\right). \tag{1.1.10}$$

It can be shown that $\hat{a}$ and $\hat{a}^{+}$ exhibit interesting features when acting upon the kets (1.1.4). This property earns $\hat{a}$ and $\hat{a}^{+}$, respectively, the name of lowering and raising operators. Incidentally, it explains the etymology of the terms "annihilation" and "creation".

Corollary 1.1.1 (Ladder property)**.** *Given an annihilation operator $\hat{a}$ and the eigenkets $|n\rangle$ of its number operator, one has that*

$$\hat{a}|0\rangle = 0 \tag{1.1.11a}$$

$$\hat{a}|n\rangle = |n-1\rangle n^{1/2} \qquad (n \in \mathbb{N}^{+}) \tag{1.1.11b}$$

$$\hat{a}^{+}|n\rangle = |n+1\rangle(n+1)^{1/2}. \tag{1.1.11c}$$

Basis of the Hilbert space of states $\mathcal{H}$ can be built from the eigenkets of $\hat{N}$ under certain conditions. In fact, one has the following corollary.

Corollary 1.1.2. *Given an annihilation operator $\hat{a}$, if its associated number operator $\hat{N}$ is selfadjoint and $|0\rangle$ is normalizable and unique (that is if $\hat{N}$ is injective), then $\{|n\rangle\}_{n\in\mathbb{N}}$ forms a complete orthonormal set of $\mathcal{H}$:*

$$\langle n'|n\rangle = \delta_{n',n} \tag{1.1.12a}$$

$$\hat{1} = \sum_{n=0}^{\infty} |n\rangle\langle n|. \tag{1.1.12b}$$

This property is extremely relevant in many physical applications. One of them is the quantum harmonic oscillator.

Example 1.1.2 (Heisenberg set for the harmonic oscillator)**.** The set $\{|n\rangle\}_{n\in\mathbb{N}}$ of the harmonic oscillator forms a complete orthonormal set. Indeed, noticing that $\hat{N}$ is selfadjoint (an operator function of selfadjoint operators is a selfadjoint operator) and that $|0\rangle$ is unique (this fact follows from the spectral structure theorem), its eigenkets constitute a complete orthonormal set and they form a Heisenberg representation of $\mathcal{H}$ through $\hat{H}$.

Proposition 1.1.3. *Given an annihilation operator $\hat{a}$, its creation operator $\hat{a}^+$ has no eigenstates.*

Proof. We demonstrate the claim by contradiction. Let's assume that there is $|\lambda\rangle \in \mathcal{H}$ such that

$$\hat{a}^+|\lambda\rangle = |\lambda\rangle\lambda, \quad \lambda \in \mathbb{C}. \tag{1.1.13}$$

Using (1.1.11b) and (1.1.13), we find

$$\langle n|\hat{a}^+|\lambda\rangle = n^{1/2}\langle n-1|\lambda\rangle = \langle n|\lambda\rangle\lambda \quad \text{for } n \in \mathbb{N}^+ \tag{1.1.14}$$

$$\langle 0|\hat{a}^+|\lambda\rangle = \langle 0|\lambda\rangle\lambda = 0. \tag{1.1.15}$$

Hence, from (1.1.15), one has two possibilities. In the first case, $\lambda = 0$, which, through (1.1.14), leads to $\langle\lambda|n-1\rangle = 0$, meaning that $|\lambda\rangle = 0$. In the second case, $\langle\lambda|0\rangle = 0$. Then, using n times (1.1.11b) on an arbitrary $\langle n|$, one finds $\langle n|\hat{a}^{+n}|\lambda\rangle = (n!)^{1/2}\langle 0|\lambda\rangle = 0$, but, through (1.1.13), $\langle n|\hat{a}^{+n}|\lambda\rangle = \langle n|\lambda\rangle\lambda^n$, meaning that $\langle n|\lambda\rangle = 0$, which implies that $|\lambda\rangle = 0$. Thus $|\lambda\rangle$ cannot be an eigenket of $\hat{a}$. □

Remark 1.1.1. The previous definitions and results can be generalized by introducing p pairs of annihilation and creation operators, $\hat{a}_i$ and $\hat{a}_i^+$ $(i = 1, \dots, p)$, with the following properties:

$$[\hat{a}_i, \hat{a}_i^+] = \delta_{i,j}\hat{1} \tag{1.1.16a}$$

$$[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^+, \hat{a}_j^+] = 0. \tag{1.1.16b}$$

The corresponding number operators can then be defined as

$$\hat{N}_i = \hat{a}_i^+\hat{a}_i. \tag{1.1.17}$$

This definition allows one to say that

$$[\hat{N}_i, \hat{N}_j] = 0. \tag{1.1.18}$$

From the properties of $\hat{N}$, these operators are found to be Hermitian and to have a spectrum consisting of non negative integers. Then, the joint spectrum of the $\hat{N}_i$ is formed by all the p-tuples $(n_1, \dots, n_p)$ of non negative integers. We denote with $|0, \dots, 0\rangle$ the common eigenket of the $\hat{N}_i$ belonging to the joint eigenvalue $(0, \dots, 0)$, that is the ket such that

$$\hat{N}_i|0, \dots, 0\rangle = 0 \qquad \forall\, i = 0, \dots, p. \tag{1.1.19}$$

Then, $\forall\, n_1, \dots, n_p$ non negative integers, the ket

$$|n_1, \dots, n_p\rangle = \hat{a}^{+n_1} \cdots \hat{a}^{+n_p}|0, \dots, 0\rangle(n_1! \cdots n_p!)^{-1/2} \tag{1.1.20}$$

is found to be the common eigenket of the $\hat{N}_i$ belonging to the joint eigenvalue $(n_1, \dots, n_p)$:

$$\hat{N}_i|n_1, \dots, n_p\rangle = |n_1, \dots, n_p\rangle n_i \qquad \forall\, i = 0, \dots, p. \tag{1.1.21}$$

Again, these kets show interesting features when acted upon by $\hat{a}_i$ and $\hat{a}_i^+$:

$$\hat{a}_i|0, \dots, 0\rangle = 0 \tag{1.1.22a}$$

$$\hat{a}_i|n_1, \dots, n_i, \dots, n_p\rangle = |n_1, \dots, n_i - 1, \dots, n_p\rangle n_i^{1/2} \qquad \text{(for } n_i > 0\text{)} \tag{1.1.22b}$$

$$\hat{a}_i^+|n_1, \dots, n_i, \dots, n_p\rangle = |n_1, \dots, n_i + 1, \dots, n_p\rangle(n_i + 1)^{1/2}. \tag{1.1.22c}$$

Equation (1.1.18) states that the $\hat{N}_i$ are commuting operators. If, in addition, the $\hat{N}_i$ are selfadjoint and if $|0, \dots, 0\rangle$ is normalizable and unique, then $\{|n_1, \dots, n_p\rangle\}_{n_i \in \mathbb{N}}$ consitutes an orthonormal basis of $\mathcal{H}$:

$$\langle n_1', \dots, n_p'|n_1, \dots, n_p\rangle = \delta_{n_1', n_1} \cdots \delta_{n_p', n_p} \tag{1.1.23a}$$

$$\hat{1} = \sum_{n_1, \dots, n_p = 0}^{\infty} |n_1, \dots, n_p\rangle\langle n_1, \dots, n_p|. \tag{1.1.23b}$$

1.2 Weyl operators

Weyl operators are fundamental in the construction of the explicit expression of the states we are wanting to study. We start by stating the definition of such operators and by introducing some useful results.

Definition 1.2.1 (Weyl operator)**.** *Given an annihilation operator $\hat{a}$ and given a complex number $\gamma \in \mathbb{C}$, a Weyl operator is an operator $\hat{W}(\gamma)$ defined as*

$$\hat{W}(\gamma) := \exp(\gamma\hat{a}^+ - \gamma^*\hat{a}). \tag{1.2.1}$$

One immediately has the following result.

Proposition 1.2.1. *A Weyl operator is an unitary operator.*

We now introduce two propositions which will be helpful to better characterize the Weyl operators. These results and their consequences will be extensively used in the next chapters.

Proposition 1.2.2. *Given two arbitrary operators $\hat{A}$ and $\hat{B}$, if these operators commute with their commutator, the following identity holds:*

$$\exp(\hat{A})\hat{B}\exp(-\hat{A}) = \hat{B} + [\hat{A}, \hat{B}]. \tag{1.2.2}$$

Proof. It is known that

$$\exp(\hat{A}) = \sum_{k=0}^{\infty} \frac{\hat{A}^k}{k!}, \tag{1.2.3}$$

which, in our case, leads to

$$\begin{aligned}\exp(\hat{A})\hat{B}\exp(-\hat{A}) &= \sum_{k=0}^{\infty} \frac{\hat{A}^k}{k!}\hat{B}\sum_{l=0}^{\infty}\frac{(-1)^l\hat{A}^l}{l!} \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^l}{k!l!}\hat{A}^k\hat{B}\hat{A}^l \\ &= \hat{B} - \hat{B}\hat{A} + \hat{A}\hat{B} + \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-1)^l}{k!l!}\hat{A}^k\hat{B}\hat{A}^l.\end{aligned} \tag{1.2.4}$$

The last term is the zero operator. Indeed, we consider the following identity,

$$\underbrace{[\hat{A}, [\hat{A}, [\ldots, [\hat{A}}_{n \text{ times}}, \hat{B}]] \ldots] = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \hat{A}^{n-k}\hat{B}\hat{A}^k, \tag{1.2.5}$$

and we prove it by induction. For $n = 1$, one has

$$\sum_{k=0}^{1} \binom{n}{k} (-1)^k \hat{A}^{n-k}\hat{B}\hat{A}^k = \hat{A}\hat{B} - \hat{B}\hat{A} = [\hat{A}, \hat{B}]. \tag{1.2.6}$$

Then, assuming (1.2.5) to hold for $n = m$, it can be shown to hold for $n = m + 1$:

$$\begin{aligned}\underbrace{[\hat{A}, [\hat{A}, [\ldots, [\hat{A}}_{m+1 \text{ times}}, \hat{B}]] \ldots] &= [\hat{A}, \sum_{k=0}^{m}\binom{m}{k}(-1)^k\hat{A}^{m-k}\hat{B}\hat{A}^k] \\ &= \sum_{k=0}^{m}\binom{m}{k}(-1)^k\hat{A}^{m-k+1}\hat{B}\hat{A}^k - \sum_{j=0}^{m}\binom{m}{j}(-1)^k\hat{A}^{m-j}\hat{B}\hat{A}^{j-1} \\ &= \sum_{k=0}^{m}\binom{m}{k}(-1)^k\hat{A}^{m-k+1}\hat{B}\hat{A}^k + \sum_{l=1}^{m+1}\binom{m}{l-1}(-1)^l\hat{A}^{m-l+1}\hat{B}\hat{A}^l \\ &= \hat{A}^{m+1}\hat{B} + \sum_{k=1}^{m}\left(\binom{m}{k} + \binom{m}{k-1}\right)(-1)^k\hat{A}^{m-k+1}\hat{B}\hat{A}^k + (-1)^{n+1}\hat{B}\hat{A}^{n+1} \\ &= \hat{A}^{m+1}\hat{B} + \sum_{k=1}^{m}\binom{m+1}{k}(-1)^k\hat{A}^{m-k+1}\hat{B}\hat{A}^k + (-1)^{n+1}\hat{B}\hat{A}^{n+1} \\ &= \sum_{k=0}^{m+1}\binom{m+1}{k}(-1)^k\hat{A}^{m+1-k}\hat{B}\hat{A}^k,\end{aligned} \tag{1.2.7}$$

as stated. Since $\hat{A}$ commutes with $[\hat{A}, \hat{B}]$, (1.2.5) is the zero operator. Considering the partial sums of the last term in (1.2.4) and reordering indices, it is straightforward to show that this term is the zero operator. □

Proposition 1.2.3 (Instance of the Baker-Campbell-Hausdorff formula)**.** *Given two operators $\hat{A}$ and $\hat{B}$ which commute with their commutator, the following identity is found:*

$$\exp(\hat{A} + \hat{B}) = \exp\left(-\frac{1}{2}[\hat{A}, \hat{B}]\right)\exp(\hat{A})\exp(\hat{B}). \tag{1.2.8}$$

Proof. This formula is a particular case of the so-called Baker-Campbell-Hausdorff formula. To show its validity, we start by writing that

$$\exp(\hat{B})\exp(\hat{A}) = \Big(\exp(\hat{B})\exp(\hat{A})\exp(-\hat{B})\Big)\exp(\hat{B}). \tag{1.2.9}$$

It can be seen that the sandwich between brackets can be written as

$$\Big(\exp(\hat{B})\exp(\hat{A})\exp(-\hat{B})\Big)\exp(\hat{B}) = \exp\Big(\exp(\hat{B})\hat{A}\exp(-\hat{B})\Big)\exp(\hat{B}). \tag{1.2.10}$$

In fact, by definition of the exponential operator,

$$\exp\Big(\exp(\hat{B})\hat{A}\exp(-\hat{B})\Big)\exp(\hat{B}) = \sum_{k=0}^{\infty}\frac{1}{k!}\Big(\exp(\hat{B})\hat{A}\exp(-\hat{B})\Big)^{k}. \tag{1.2.11}$$

The j-th term can be written as

$$\Big(\exp(\hat{B})\hat{A}\exp(-\hat{B})\Big)^{j} = \exp(\hat{B})\hat{A}^{j}\exp(-\hat{B}), \tag{1.2.12}$$

which shows (1.2.10). The expression appearing in the argument of the first exponential of the right side of (1.2.10) can be cast in the form (1.2.2), as $\hat{A}$ and $\hat{B}$ commute with their commutator:

$$\exp(\hat{B})\exp(\hat{A}) = \exp(\hat{A} + [\hat{B}, \hat{A}])\exp(\hat{B}). \tag{1.2.13}$$

It is known that, given two commutative linear operators $\hat{X}$ and $\hat{Y}$,

$$\exp(\hat{X})\exp(\hat{Y}) = \exp(\hat{X} + \hat{Y}), \tag{1.2.14}$$

which implies that, from (1.2.13),

$$\exp(\hat{B})\exp(\hat{A}) = \exp([\hat{B}, \hat{A}])\exp(\hat{A})\exp(\hat{B}). \tag{1.2.15}$$

This identity can be generalized in the following way: given $n \in \mathbb{N}$, one has

$$\Big(\exp(\hat{B})\exp(\hat{A})\Big)^{n} = \exp\left(\frac{n(n+1)}{2}[\hat{B}, \hat{A}]\right)\exp(n\hat{A})\exp(n\hat{B}). \tag{1.2.16}$$

Indeed, setting $\hat{C} = [\hat{B}, \hat{A}]$ for convenience,

$$\begin{aligned}
\Big(\exp(\hat{B})\exp(\hat{A})\Big)^{n} &= \exp(\hat{C})\exp(\hat{A})\exp(\hat{B})\Big(\exp(\hat{B})\exp(\hat{A})\Big)^{n-1} \\
&= \exp(\hat{C})\exp(\hat{A})\exp(\hat{B})\exp(\hat{B})\exp(\hat{A})\Big(\exp(\hat{B})\exp(\hat{A})\Big)^{n-2} \\
&= \exp(\hat{C})\exp(\hat{A})\exp(2\hat{B})\exp(\hat{A})\Big(\exp(\hat{B})\exp(\hat{A})\Big)^{n-2} \\
&= \exp(\hat{C})\exp(\hat{A})\exp(2\hat{C})\exp(\hat{A})\exp(2\hat{B})\Big(\exp(\hat{B})\exp(\hat{A})\Big)^{n-2} \\
&= \exp((1+2)\hat{C})\exp(2\hat{A})\exp(2\hat{B})\Big(\exp(\hat{B})\exp(\hat{A})\Big)^{n-2}.
\end{aligned} \tag{1.2.17}$$

We used repeatedly (1.2.15) and we used the fact that $\hat{C}$ and $\hat{A}$ commute to commute their exponentials (see (1.2.14)). Then, iterating $n-2$ times, we get

$$\Big(\exp(\hat{B})\exp(\hat{A})\Big)^{n} = \exp\left(\sum_{k=1}^{n} k\hat{C}\right)\exp(n\hat{A})\exp(n\hat{B}), \tag{1.2.18}$$

which, using Gauss sum formula, reduces to (1.2.16). Finally, recalling the Lie-Trotter product formula for two generic linear operators $\hat{X}$ and $\hat{Y}$,

$$\exp(\hat{X} + \hat{Y}) = \lim_{n\to\infty} \left(\exp(\hat{Y}/n)\exp(\hat{X}/n)\right)^n, \tag{1.2.19}$$

we finally obtain (1.2.8), from (1.2.16),

$$\begin{aligned}\exp(\hat{A} + \hat{B}) &= \lim_{n\to\infty} \left(\exp(\hat{B}/n)\exp(\hat{A}/n)\right)^n \\ &= \lim_{n\to\infty} \exp\left(\frac{(n+1)}{2n}[\hat{B}, \hat{A}]\right)\exp(\hat{A})\exp(\hat{B}) \\ &= \exp\left(\frac{1}{2}[\hat{B}, \hat{A}]\right)\exp(\hat{A})\exp(\hat{B}).\end{aligned} \tag{1.2.20}$$

□

The next results are readily obtained from the previous propositions.

Proposition 1.2.4 (Group property). *Given $\gamma_1, \gamma_2 \in \mathbb{C}$, one has that the following identity holds:*

$$\hat{W}(\gamma_1)\hat{W}(\gamma_2) = \exp\left(i\,\mathrm{Im}(\gamma_1\gamma_2^*)\right)\hat{W}(\gamma_1 + \gamma_2). \tag{1.2.21}$$

Proof. This can be easily checked from (1.1.1), (1.2.1) and (1.2.8). □

Remark 1.2.1. By introducing a suitable binary operation, this property of the Weyl operators can be used to turn the set of Weyl operators into a group.

Proposition 1.2.5 (Decomposition of the Weyl operator). *Let $\gamma \in \mathbb{C}$ be a generic complex number. Then,*

$$\hat{W}(\gamma) = \exp(-|\gamma|^2/2)\exp(\gamma\hat{a}^+)\exp(-\gamma^*\hat{a}). \tag{1.2.22}$$

Proof. Using (1.1.1), (1.2.1) and (1.2.8),

$$\exp(\gamma\hat{a}^+ - \gamma^*\hat{a}) = \exp\left(-\frac{1}{2}[\gamma\hat{a}^+, -\gamma^*\hat{a}]\right)\exp(\gamma\hat{a}^+)\exp(-\gamma^*\hat{a}), \tag{1.2.23}$$

which concludes the proof. □

Proposition 1.2.6 (Translation property). *Let $\hat{A}$ be a generic operator and let $\gamma \in \mathbb{C}$ be a generic complex number. If $\hat{A}$ and $\gamma\hat{a}^+ - \gamma^*\hat{a}$ commute with their commutator, then*

$$\hat{W}(\gamma)\hat{A}\hat{W}(\gamma)^{-1} = \hat{A} + [\hat{A}, \gamma^*\hat{a} - \gamma\hat{a}^+]. \tag{1.2.24}$$

Proof. This fact is a prompt consequence of (1.2.1) and (1.2.2). □

Accordingly, one can express the following important statement.

Proposition 1.2.7. *Given an annihilation operator* $\hat{a}$ *and* $\gamma \in \mathbb{C}$*, then*

$$\hat{a}|\gamma\rangle = |\gamma\rangle\gamma, \tag{1.2.25}$$

where $|\gamma\rangle := \hat{W}(\gamma)|0\rangle$ *and* $|0\rangle$ *is the eigenstate of the number operator associated to* $\hat{a}$ *belonging to the* 0 *eigenvalue.*

Proof. From (1.1.11a),

$$\hat{W}(\gamma)\hat{a}|0\rangle = 0. \tag{1.2.26}$$

Then, from 1.2.6,

$$\begin{aligned} 0 = \hat{W}(\gamma)\hat{a}|0\rangle &= \hat{W}(\gamma)\hat{a}\hat{W}(\gamma)^{-1}\hat{W}(\gamma)|0\rangle \\ &= \left(\hat{a} + [\hat{a}, \gamma^*\hat{a} - \gamma\hat{a}^+]\right)\hat{W}(\gamma)|0\rangle \\ &= (\hat{a} - \gamma\hat{1})\hat{W}(\gamma)|0\rangle. \end{aligned} \tag{1.2.27}$$

□

It is possible to show that $|\gamma\rangle$ has an explicit expression in terms of the $|n\rangle$ kets. This is a fundamental fact.

Proposition 1.2.8. *Given* $\gamma \in \mathbb{C}$*, the associated* $|\gamma\rangle := \hat{W}(\gamma)|0\rangle$ *enjoys the following expansion:*

$$|\gamma\rangle = \exp(-|\gamma|^2/2)\sum_{n=0}^{\infty}|n\rangle\frac{\gamma^n}{(n!)^{1/2}}. \tag{1.2.28}$$

Proof. By (1.1.11a), it is straighforward that

$$\exp(-\gamma^*\hat{a})|0\rangle = |0\rangle. \tag{1.2.29}$$

Then, by (1.2.22) and by the definition of the exponential operator,

$$\begin{aligned} |\gamma\rangle = \hat{W}(\gamma)|0\rangle &= \exp(-|\gamma|^2/2)\exp(\gamma\hat{a}^+)|0\rangle \\ &= \exp(-|\gamma|^2/2)\sum_{n=0}^{\infty}\hat{a}^{+n}|0\rangle\frac{\gamma^n}{n!} \end{aligned} \tag{1.2.30}$$

Using the definition (1.1.4) of the $|n\rangle$ kets, the claim is proven. □

Proposition 1.2.9. *Given p pairs of annihilation and creation operators $\hat{a}_i$ and $\hat{a}_i^+$ obeying the commutation relations* (1.1.16) *and given* $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_p) \in \mathbb{C}^p$*, then the operator defined as*

$$\hat{W}(\boldsymbol{\gamma}) := \prod_{j=1}^{p} \exp\left(\gamma_j \hat{a}_j^+ - \gamma_j^* \hat{a}_j^+\right) \tag{1.2.31}$$

is a Weyl operator.

Proof. By (1.2.14) and (1.1.16), $\hat{W}(\boldsymbol{\gamma})$ can be cast in the following form:

$$\hat{W}(\boldsymbol{\gamma}) = \exp\left(\sum_{j=1}^{p} \gamma_j \hat{a}_j^+ - \sum_{j=1}^{p} \gamma_j^* \hat{a}_j\right). \tag{1.2.32}$$

Then, we define $\gamma \in \mathbb{R}$ as

$$\gamma := \left(\sum_{i=1}^{p} |\gamma_i|^2\right)^{1/2}. \tag{1.2.33}$$

From (1.1.16), it can be easily checked that

$$\left[\sum_{l=1}^{p} \gamma_l^* \hat{a}_l, \sum_{m=1}^{p} \gamma_m \hat{a}_m^+\right] = \gamma^2 \hat{1}. \tag{1.2.34}$$

Hence, the operators

$$\boldsymbol{\gamma}_0^* \cdot \hat{\boldsymbol{a}} := \frac{1}{\gamma} \sum_{i=1}^{p} \gamma_i^* \hat{a}_i \tag{1.2.35a}$$

$$\boldsymbol{\gamma}_0 \cdot \hat{\boldsymbol{a}}^+ := \frac{1}{\gamma} \sum_{i=1}^{p} \gamma_i \hat{a}_i^+ \tag{1.2.35b}$$

form a pair of annihilation and creation operators. From (1.2.32), it follows that

$$\hat{W}(\boldsymbol{\gamma}) = \exp\left(\gamma \boldsymbol{\gamma}_0 \cdot \hat{\boldsymbol{a}}^+ - \gamma \boldsymbol{\gamma}_0^* \cdot \hat{\boldsymbol{a}}\right), \tag{1.2.36}$$

thus showing the statement. □

Chapter 2

Time evolution and pictures

A short digression on the quantum time evolution theory is made in order to introduce the concepts and the results which will be used when studying the classical limit of quantum mechanics in chapter 4. First, we shall take into consideration the evolution operator of a quantum system in an informal way, presenting and giving for granted the main results of its theory. Then, we shall proceed with an equally colloquial account on the theory of pictures. This chapter entirely relies on [6].

2.1 Evolution operator

The Hamiltonian operator of a certain system can sometimes be time-dependent. In this case, we can suppose that the Schroedinger equation still describes the dynamics of the system:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle. \tag{2.1.1}$$

A solution of this equation is unambiguously determined once an initial condition is set. By linearity, if $|\psi_1(t)\rangle$ and $|\psi_2(t)\rangle$ are solutions of (2.1.1), unambiguously determined by the respective initial conditions $|\psi_1(0)\rangle = |\psi_{10}\rangle$ and $|\psi_2(0)\rangle = |\psi_{20}\rangle$, then, given $c_1, c_2 \in \mathbb{C}$, the ket $|\psi(t)\rangle = |\psi_1(t)\rangle c_1 + |\psi_2(t)\rangle c_2$ is, in turn, a solution of (2.1.1) satisfying $|\psi(0)\rangle = |\psi_{10}\rangle c_1 + |\psi_{20}\rangle c_2$. This fact suggests to represent the time evolution of the state of the system through some linear operator $\hat{U}(t, 0)$ which should act on the initial condition $|\psi(0)\rangle$ in the following way:

$$|\psi(t)\rangle = \hat{U}(t, 0)|\psi(0)\rangle. \tag{2.1.2}$$

Thus, $\hat{U}(t, 0)$ takes the system from its initial state to its state at time t in accordance with (2.1.1). The choice of $t = 0$ as a starting point for the evolution of the system is quite arbitrary. Thus, there should be a linear operator $\hat{U}(t, s)$ which should act in a more general way on the states of the system:

$$|\psi(t)\rangle = \hat{U}(t, s)|\psi(s)\rangle, \tag{2.1.3}$$

where t, s are two arbitrary instants of time. (2.1.3) reduces to (2.1.2) when s is set to 0. The $\hat{U}(t, s)$ operator is called evolution operator. Pictorially speaking, the evolution operator has to guide the state along a suitable trajectory in $\mathcal{H}$ which satisfies the Schroedinger equation. Furthermore, as already remarked, time evolution should be independent from the choice of s. These observations find mathematical substance in the following equations

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t, s) = \hat{H}(t)\hat{U}(t, s) \tag{2.1.4a}$$

$$i\hbar\frac{\partial}{\partial s}\hat{U}(t, s) = -\hat{U}(t, s)\hat{H}(s), \tag{2.1.4b}$$

which must be obeyed by $\hat{U}(t,s)$ with the initial condition

$$\hat{U}(s,s) = \hat{1}. \tag{2.1.5}$$

Equations (2.1.4a) and (2.1.4b), with (2.1.5), follow from (2.1.1) and (2.1.3). To maintain its probabilistic interpretation, the state of the system must be normalized in each instant of time. This mathematically translates to the fact that the norm of $|\psi(t)\rangle$ must be preserved in time, that is to the fact that $\hat{U}(t,s)$ must be an unitary operator. The evolution operator must also obey some sort of composition law which should ensure that evolving the state from a time s to a certain time u and, then, evolving it from u to t is equivalent to taking the state directly from s to t. Finally, the operator which makes the state evolve from s to t should be the inverse of the operator which makes the state evolve from t to s. Luckily, all these physical requests mathematically follow from the evolution operator equations. In fact, from (2.1.4a) and (2.1.4b), one has that

$$\hat{U}(t,s)^{+} = \hat{U}(t,s)^{-1} \tag{2.1.6a}$$

$$\hat{U}(t,u)\hat{U}(u,s) = \hat{U}(t,s) \tag{2.1.6b}$$

$$\hat{U}(t,s) = \hat{U}(s,t)^{-1}. \tag{2.1.6c}$$

We see that solving the evolution operator equations is equivalent to solving the Schroedinger equation. In a particular case, this can be readily done. Indeed, when $\hat{H}(t)$ is actually time-independent, say $\hat{H}(t) = \hat{H}_0$, the evolution operator has the following expression:

$$\hat{U}(t,s) = \exp\left(-i\hbar^{-1}(t-s)\hat{H}_0\right), \tag{2.1.7}$$

which can be easily found to satisfy (2.1.4a) and (2.1.4b) with the initial condition (2.1.5).

2.2 Pictures

Solving the Schroedinger equation or finding an appropriate evolution operator is mostly hard work. Classical mechanics has taught us how there are certain transformations, which we call canonical transformations, which can simplify the problem of solving the equations of motion. The question which arises is whether there are, in quantum mechanics, transformations which simplify the calculation of the quantities which we are interested in, that is probabilities and expectation values. Given a certain state $|\psi\rangle$ which a quantum state lies in and given a certain observable $\hat{A}$, the expectation value $\langle A\rangle$ of the observable $\hat{A}$ in the state $|\psi\rangle$ is computed as

$$\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle \tag{2.2.1}$$

and the probability $P(\varkappa)$ that a certain measure of $\hat{A}$ produces a value laying in some range $\varkappa$ is evaluated as

$$P(\varkappa) = \int_{x(\xi)\in\varkappa} d\mu(\xi)\,|\langle\xi|\psi\rangle|^2, \tag{2.2.2}$$

where $x(\xi)$ are the eigenvalues of $\hat{A}$ and $|\xi\rangle$ are the associated eigenkets. The $|\xi\rangle$ kets form an orthonormal basis for $\mathcal{H}$. For some arbitrary unitary operator $\hat{T}$, the quantities $\langle A\rangle$ and $P(\varkappa)$ are invariant under the following transformations:

$$|\psi\rangle_T = \hat{T}^{-1}|\psi\rangle \tag{2.2.3a}$$

$$\hat{A}_T = \hat{T}^{+}\hat{A}\hat{T}. \tag{2.2.3b}$$

In fact, through the unitarity of $\hat{T}$, it can be easily checked that

$$|\langle\xi|\psi\rangle|^2 = |_T\langle\xi|\psi\rangle_T|^2 \tag{2.2.4a}$$
$$\langle\psi|\hat{A}|\psi\rangle = {}_T\langle\psi|\hat{A}_T|\psi\rangle_T, \tag{2.2.4b}$$

that is that $\langle A\rangle$ and $P(\varkappa)$ are invariant under (2.2.3a) and (2.2.3b). The fact that the fundamental quantum physical quantities are preserved by $\hat{T}$ suggests that, if $\hat{T}$ was to enjoy similar properties to those of the evolution operator, one could perform some sort of artificial time evolution of the state of the system without touching its main quantitative features and, if possible, simplify their calculation and the Schroedinger equation itself. Hence, chosen some selfadjoint time-dependent operator $\hat{K}(t)$, we suppose that $\hat{T}(t,s)$ is a time-dependent unitary operator obeying to the following equations:

$$i\hbar\frac{\partial}{\partial t}\hat{T}(t,s) = \hat{K}(t)\hat{T}(t,s) \tag{2.2.5a}$$
$$i\hbar\frac{\partial}{\partial s}\hat{T}(t,s) = -\hat{T}(t,s)\hat{K}(s) \tag{2.2.5b}$$

with the following initial condition:

$$\hat{T}(s,s) = \hat{1}. \tag{2.2.6}$$

Comparing (2.2.5a) with (2.1.4a) and (2.2.5b) with (2.2.5b), we see that the $\hat{T}(t,s)$ operator plays the role of an evolution operator for a system whose dynamics are governed by $\hat{K}(t)$, which plays the role of a Hamiltonian operator. These equations ensure that $\hat{T}(t,s)$ enjoys the same properties as the evolution operator itself (see (2.1.6)). For each instant of time, performing the transformation

$$|\psi(t)\rangle_K = \hat{T}(t,0)^{-1}|\psi(t)\rangle, \tag{2.2.7}$$

if an equally fit transformation is performed on $\hat{A}$, (2.2.1) and (2.2.2) are preserved and their calculation could be simplified, as well as the search for a solution of the Schroedinger equation, which, for the new $|\psi(t)\rangle_K$, reads as

$$i\hbar\frac{d}{dt}|\psi(t)\rangle_K = \hat{H}^{(K)}(t)|\psi(t)\rangle_K \tag{2.2.8}$$

where

$$\hat{H}^{(K)}(t) := \hat{T}(t,0)^+\left(\hat{H}(t) - \hat{K}(t)\right)\hat{T}(t,0). \tag{2.2.9}$$

This can be easily seen by calculating the derivative on the left side of (2.2.8) through (2.1.1), (2.2.7), (2.2.5a) and (2.2.5b). Once (2.2.8) is solved (an initial condition must be assigned), $|\psi(t)\rangle$ is retrieved through (2.2.7). We say that (2.2.7) defines a K-picture. A K-picture is then determined by the choice of $\hat{K}(t)$. Analogously to the standard Schroedinger problem, instead of solving directly the K-picture Schroedinger equation, one can define an evolution operator for the K-picture. This operator is defined as the solution $\hat{U}^{(K)}(t,s)$ of the following equations:

$$i\hbar\frac{\partial}{\partial t}\hat{U}^{(K)}(t,s) = \hat{K}(t)\hat{U}^{(K)}(t,s) \tag{2.2.10a}$$
$$i\hbar\frac{\partial}{\partial s}\hat{U}^{(K)}(t,s) = -\hat{U}^{(K)}(t,s)\hat{K}(s), \tag{2.2.10b}$$

with the initial condition

$$\hat{U}^{(K)}(s,s) = \hat{1}. \tag{2.2.11}$$

It can be seen that

$$\hat{U}^{(K)}(t,s) = \hat{T}(t,0)^{+}\hat{U}(t,s)\hat{T}(s,0) \tag{2.2.12}$$

solves the equations (2.2.10a) and (2.2.10b) with the initial condition (2.2.11). One can define the K-picture version of a generic selfadjoint time-dependent operator $\hat{A}(t)$ as

$$\hat{A}_K(t) := \hat{T}(t,0)^{+}\hat{A}(t)\hat{T}(t,0), \tag{2.2.13}$$

in order to comply with the conservation of (2.2.1) and (2.2.2). It is found that $\hat{A}_K(t)$ obeys the so-called Heisenberg equation:

$$i\hbar\frac{d}{dt}\hat{A}_K(t) = -[\hat{K}^{(K)}(t), \hat{A}_K(t)] + i\hbar\left(\frac{d}{dt}\hat{A}\right)_K(t), \tag{2.2.14}$$

where

$$\hat{K}^{(K)}(t) := \hat{T}(t,0)^{+}\hat{K}(t)\hat{T}(t,0) = \hat{K}_K(t) \tag{2.2.15}$$

is called the Heisenberg Hamiltonian. (2.2.14) is obtained by calculating its left term using (2.2.13) and (2.2.5a). There are some standard K-pictures. The choice

$$\hat{K}(t) = 0 \tag{2.2.16}$$

defines the so-called Schroedinger picture. It follows from (2.2.5a), (2.2.5b) and (2.2.6), that

$$\hat{T}(t,s) = \hat{1} \tag{2.2.17}$$

and, from (2.2.7), that

$$|\psi(t)\rangle_S = |\psi(t)\rangle. \tag{2.2.18}$$

In consequence of (2.2.17), from (2.2.9), (2.2.12), (2.2.13) and (2.2.15) one has that

$$\hat{H}^{(S)}(t) = \hat{H}(t) \tag{2.2.19a}$$

$$\hat{U}^{(S)}(t,s) = \hat{U}(t,s) \tag{2.2.19b}$$

$$\hat{A}_S(t) = \hat{A}(t) \tag{2.2.19c}$$

$$\hat{K}^{(S)}(t) = 0. \tag{2.2.19d}$$

The Schroedinger picture is nothing but the standard form of quantum mechanics. Instead, the choice

$$\hat{K}(t) = \hat{H}(t) \tag{2.2.20}$$

defines the so-called Heisenberg picture. From (2.2.5a), (2.2.5b) and (2.2.6), it can be seen that

$$\hat{T}(t,s) = \hat{U}(t,s), \tag{2.2.21}$$

where $\hat{U}(t,s)$ is the evolution operator of the Schroedinger picture. From (2.2.7) and (2.1.2), one has that

$$|\psi(t)\rangle_H = \hat{U}(t,0)^{-1}|\psi(t)\rangle = |\psi(0)\rangle. \tag{2.2.22}$$

The state of the system is time-independent in the Heisenberg picture. Again, as a consequence of (2.2.9), (2.2.12), (2.2.13) and (2.2.15), it follows that

$$\hat{H}^{(H)}(t) = 0 \tag{2.2.23a}$$

$$\hat{U}^{(H)}(t,s) = \hat{1} \tag{2.2.23b}$$

$$\hat{A}_H(t) = \hat{U}(t,0)^{+}\hat{A}(t)\hat{U}(t,0) \tag{2.2.23c}$$

$$\hat{K}^{(H)}(t) = \hat{U}(t,0)^{+}\hat{H}(t)\hat{U}(t,0) = \hat{H}_H(t). \tag{2.2.23d}$$

If the system is subject to a time-dependent perturbation energy $\hat{W}(t)$ and if the unperturbed Hamiltonian $\hat{H}_0$ is time-independent, one can write the perturbed Hamiltonian as

$$\hat{H}(t) = \hat{H}_0 + \hat{W}(t). \tag{2.2.24}$$

Then, the choice

$$\hat{K}(t) = \hat{H}_0 \tag{2.2.25}$$

defines the so-called Dirac picture. From (2.2.5a), (2.2.5b) and (2.2.6), one has that

$$\hat{T}(t,s) = \hat{U}_0(t,s) = \exp\left(-i\hbar^{-1}(t-s)\hat{H}_0\right). \tag{2.2.26}$$

By (2.2.7), it follows that

$$|\psi(t)\rangle_D = \hat{U}_0(t,0)^{-1}|\psi(t)\rangle. \tag{2.2.27}$$

Finally, following from (2.2.9), (2.2.12), (2.2.13) and (2.2.15) and from the fact that $\hat{H}_0$ commutes with $\hat{U}_0(t,s)$, it is straightforward that

$$\hat{H}^{(D)}(t) = \hat{U}_0(t,0)^{+}\hat{W}(t)\hat{U}_0(t,0) = \hat{W}_D(t) \tag{2.2.28a}$$

$$\hat{U}^{(D)}(t,s) = \hat{U}_0(t,0)^{+}\hat{U}(t,s)\hat{U}_0(s,0) \tag{2.2.28b}$$

$$\hat{A}_D(t) = \hat{U}_0(t,0)^{+}\hat{A}(t)\hat{U}_0(t,0) \tag{2.2.28c}$$

$$\hat{K}^{(D)}(t) = \hat{U}_0(t,0)^{+}\hat{H}_0\hat{U}_0(t,0) = \hat{H}_0. \tag{2.2.28d}$$

Chapter 3

Coherent states

The aim of this chapter is to introduce a class of special states which are called coherent states. To start with, we show a general form of the Heisenberg principle, which involves a generic pair of selfadjoint operators rather than just the position and momentum operators. From there, it will be found that there are some states which saturate the inequality appearing in the mathematical expression of such principle. These states, under physically loose conditions on the operators appearing in the generalized Heisenberg principle, are eigenstates of a distinctive class of annihilation operators. At this point, these states will be defined to be the coherent states of the system and their properties will be analysed in great detail. Even a general expression of this states will be found. What follows is based on chapter 7 of [1].

3.1 The Heisenberg principle

The Heisenberg principle is one of the main features of quantum mechanics. It quantitatively describes the fundamental difference between the macroscopic and microscopic worlds, that is the phenomenon of interference in the measurement of observables. For this reason, as it will be shown in this section, studying the Heisenberg principle should provide some clues about the form of the states which behave in a kind of classical manner. Indeed, classical observable quantities are defined in each state of the system: there is a one-to-one correspondence between states and observables, differently from the case of quantum mechanics. So, it seems sensible to suppose that the quantum states which we are interested in should minimize the uncertainty associated to the observable quantities of their system, thus minimizing the undefinedness of their expectation values and mimicking a classical behaviour.
The Heisenberg principle is mostly known in the form

$$\Delta q_s \Delta p_s \geq \frac{\hbar}{2}, \tag{3.1.1}$$

where Δq_s and Δp_s are the uncertainties associated to the position and momentum operators in a generic state $|s\rangle$ of the system. However, there is a more general form of this principle which is established by the geometry of the Hilbert space of states $\mathcal{H}$.

Theorem 3.1.1 (Generalized Heisenberg principle)**.** *Given a generic state $|s\rangle \in \mathcal{H}$ and two arbitrary selfadjoint operators $\hat{A}$ and $\hat{B}$, it can be shown that*

$$\frac{1}{2}\left|\langle s|[\hat{A},\hat{B}]|s\rangle\right| \leq \langle s|\hat{A}^2|s\rangle^{1/2}\langle s|\hat{B}^2|s\rangle^{1/2}. \tag{3.1.2}$$

Proof. By linearity and by the triangle inequality, one has

$$\left|\langle s|[\hat{A},\hat{B}]|s\rangle\right| = \left|\langle s|\hat{A}\hat{B}-\hat{B}\hat{A}|s\rangle\right| = \left|\langle s|\hat{A}\hat{B}|s\rangle - \langle s|\hat{B}\hat{A}|s\rangle\right| \leq \left|\langle s|\hat{A}\hat{B}|s\rangle\right| + \left|\langle s|\hat{B}\hat{A}|s\rangle\right|. \quad (3.1.3)$$

Without loss of generality, as it will turn out later, we can assume that

$$\left|\langle s|\hat{B}\hat{A}|s\rangle\right| \leq \left|\langle s|\hat{A}\hat{B}|s\rangle\right|, \quad (3.1.4)$$

which implies, by (3.1.3), that

$$\left|\langle s|[\hat{A},\hat{B}]|s\rangle\right| \leq 2\left|\langle s|\hat{A}\hat{B}|s\rangle\right|. \quad (3.1.5)$$

Using the Schwarz inequality and the Hermiticity of $\hat{A}$ and $\hat{B}$, it is easy to find the following upper bound for (3.1.5):

$$\left|\langle s|\hat{A}\hat{B}|s\rangle\right| \leq \|\hat{A}|s\rangle\|\|\hat{B}|s\rangle\| = \langle s|\hat{A}\hat{A}|s\rangle^{1/2}\langle s|\hat{B}\hat{B}|s\rangle^{1/2} = \langle s|\hat{A}^2|s\rangle^{1/2}\langle s|\hat{B}^2|s\rangle^{1/2}, \quad (3.1.6)$$

which completes the proof for (3.1.2). This last step highlights how the assumption (3.1.4) is completely generic and does not spoil the general nature of (3.1.2). □

The generalized Heisenberg principle works because of the Schwarz inequality and of the triangle inequality. Hence, its validity is only due to the geometry induced on $\mathcal{H}$ by the bra-ket product.

Example 3.1.1. As an example of how (3.1.2) works, we can obtain (3.1.1) from it. Let us define the following operators for a generic state $|s\rangle$ labelled by the letter s:

$$\widehat{\Delta p_s} := \hat{p} - \langle p\rangle_s \hat{1} \quad (3.1.7a)$$
$$\widehat{\Delta q_s} := \hat{q} - \langle q\rangle_s \hat{1}, \quad (3.1.7b)$$

where $\langle p\rangle_s = \langle s|\hat{p}|s\rangle$ and $\langle q\rangle_s = \langle s|\hat{q}|s\rangle$. It follows that

$$\widehat{\Delta p_s^2} := \widehat{\Delta p_s}\widehat{\Delta p_s} = \hat{p}^2 - 2\langle p\rangle_s\hat{p} + \langle p\rangle_s^2 \quad (3.1.8)$$
$$\widehat{\Delta q_s^2} := \widehat{\Delta q_s}\widehat{\Delta q_s} = \hat{q}^2 - 2\langle q\rangle_s\hat{q} + \langle q\rangle_s^2, \quad (3.1.9)$$

that is, by linearity and writing $\langle p^2\rangle_s = \langle s|\hat{p}^2|s\rangle$ and $\langle q\rangle_s = \langle s|\hat{q}^2|s\rangle$,

$$\Delta p_s^2 := \langle s|\widehat{\Delta p_s^2}|s\rangle = \langle p^2\rangle_s - 2\langle p\rangle_s\langle p\rangle_s + \langle p\rangle_s^2 = \langle p^2\rangle_s - \langle p\rangle_s^2 \quad (3.1.10)$$
$$\Delta q_s^2 := \langle s|\widehat{\Delta q_s^2}|s\rangle = \langle q^2\rangle_s - 2\langle q\rangle_s\langle q\rangle_s + \langle q\rangle_s^2 = \langle q^2\rangle_s - \langle q\rangle_s^2. \quad (3.1.11)$$

By the canonical commutation relations, by (3.1.7) and by (3.1.2), where we set $\hat{A} = \widehat{\Delta q_s}$ and $\hat{B} = \widehat{\Delta p_s}$, it follows promptly that

$$\frac{1}{2}\hbar = \frac{1}{2}\left|\langle s|[\hat{q},\hat{p}]|s\rangle\right| = \frac{1}{2}\left|\langle s|[\widehat{\Delta q_s},\widehat{\Delta p_s}]|s\rangle\right| \leq \langle s|\widehat{\Delta q_s^2}|s\rangle^{1/2}\langle s|\widehat{\Delta p_s^2}|s\rangle^{1/2}. \quad (3.1.12)$$

This inequality, using (3.1.10) and (3.1.11), translates to (3.1.1).

There are some states which can be safely said to saturate the (3.1.2) inequality. This fact is made more precise in the following basic statement, where a sufficient condition on $|s\rangle$ is found for (3.1.2) to become an equality.

Proposition 3.1.1. *Let $\hat{A}$ and $\hat{B}$ be two arbitrary selfadjoint operators and let $|s'\rangle \in \mathcal{H}$ be a state satisfying the equation*

$$\hat{A}|s'\rangle = -\hat{B}|s'\rangle ir, \tag{3.1.13}$$

where $r \in \mathbb{R}$. Then, (3.1.2) becomes an equality.

Proof. With the assumptions made, one has that

$$\langle s'|\hat{A}^2|s'\rangle^{1/2} = (ir\langle s'|\hat{B}\hat{B}|s'\rangle ir)^{1/2} = r\langle s'|\hat{B}^2|s'\rangle^{1/2}, \tag{3.1.14}$$

that is

$$\langle s'|\hat{A}^2|s'\rangle^{1/2}\langle s'|\hat{B}^2|s'\rangle^{1/2} = r\left|\langle s'|\hat{B}^2|s'\rangle\right|. \tag{3.1.15}$$

Moreover,

$$\left|\langle s'|\hat{A}\hat{B}|s'\rangle\right| = \left|ir\langle s'|\hat{B}^2|s'\rangle\right| = r\left|\langle s'|\hat{B}^2|s'\rangle\right|. \tag{3.1.16}$$

Finally, being r real,

$$\begin{aligned} \left|\langle s'|[\hat{A},\hat{B}]|s'\rangle\right| &= \left|\langle s'|\hat{A}\hat{B}|s'\rangle - \langle s'|\hat{B}\hat{A}|s'\rangle\right| \\ &= \left|\langle s'|\hat{B}^2|s'\rangle ir + ir\langle s'|\hat{B}^2|s'\rangle\right| = 2r\left|\langle s'|\hat{B}^2|s'\rangle\right|, \end{aligned} \tag{3.1.17}$$

which concludes the proof. □

Remark 3.1.1. If (3.1.13) is satisfied and if

$$[\hat{A},\hat{B}] = i\alpha\hat{1}, \tag{3.1.18}$$

with $\alpha \in \mathbb{R}$ (α and r must have the same sign), one can find an interesting property which will turn out to be useful later. In fact, dividing (3.1.13) by $(2r\alpha)^{1/2}$ and rearranging its terms, one has an eigenket equation

$$\frac{1}{(2r\alpha)^{1/2}}(\hat{A} + ir\hat{B})|s'\rangle = 0, \tag{3.1.19}$$

which can be solved to find the $|s'\rangle$ which saturates the Heisenberg principle. Furthermore, $(\hat{A}+ir\hat{B})/(2r\alpha)^{1/2}$ is an annihilation operator. This can be easily seen by checking the definition (1.1.1).

3.2 Coherent states: definition

Having identified a sufficient condition to find minimum-uncertainty states (see (3.1.13)), the focus is set on the possibility of finding a general expression for these states. A set of purpose-built states is now going to be introduced in order to solve this problem. The elements of this set satisfy all the properties which make them fit candidates for being considered as possibly, under suitable conditions, classically-behaving states. We call these states coherent states.

Definition 3.2.1 (Coherent states)**.** *Let $\hat{A}$ and $\hat{B}$ be two selfadjoint operators satisfying the commutation relation*

$$[\hat{A}, \hat{B}] = i\alpha\hat{1}, \tag{3.2.1}$$

with $\alpha \in \mathbb{R}$ and let $r \in \mathbb{R}$ be a generic real number such that $\operatorname{sgn}(r) = \operatorname{sgn}(\alpha)$. Let $\gamma \in \mathbb{C}$ be defined by

$$\operatorname{Re}(\gamma) = \frac{c_1}{(2r\alpha)^{1/2}} \tag{3.2.2a}$$

$$\operatorname{Im}(\gamma) = c_2 \left(\frac{r}{2\alpha}\right)^{1/2}, \tag{3.2.2b}$$

with $c_1, c_2 \in \mathbb{R}$. Then, we call coherent state the state $|\gamma\rangle$ defined as

$$|\gamma\rangle := \hat{W}(\gamma)|0\rangle, \tag{3.2.3}$$

where $\hat{W}(\gamma)$ is the Weyl operator associated to an annihilation operator of the form

$$\hat{a} = \frac{1}{(2r\alpha)^{1/2}}(\hat{A} + ir\hat{B}) \tag{3.2.4}$$

and where $|0\rangle$ is the ket belonging to the 0 eigenvalue of the number operator of $\hat{a}$.

We will say that the two operators and the annihilation operator appearing in the definition are the operators and the annihilation operator associated with the coherent state. From the definition, the main features of the coherent states immediately follow.

Proposition 3.2.1. *Coherent states are eigenstates of their associated annihilation operator:*

$$\hat{a}|\gamma\rangle = |\gamma\rangle\gamma. \tag{3.2.5}$$

Proof. This fact is an immediate consequence of proposition 1.2.7. □

Proposition 3.2.2. *Coherent states are minimum-uncertainty states with respect to their associated operators.*

Proof. The uncertainty operators associated with $\hat{A}$ and $\hat{B}$ can be generically defined as:

$$\widehat{\Delta A} := \hat{A} - c_1\hat{1} \tag{3.2.6a}$$

$$\widehat{\Delta B} := \hat{B} - c_2\hat{1}, \tag{3.2.6b}$$

where c_1 and c_2 are the constants introduced in definition 3.2.1. Then, (3.2.5) becomes

$$\frac{1}{(2r\alpha)^{1/2}}(\widehat{\Delta A} + ir\widehat{\Delta B})|\gamma\rangle = 0, \tag{3.2.7}$$

that is

$$\widehat{\Delta A}|\gamma\rangle = -\widehat{\Delta B}|\gamma\rangle ir. \tag{3.2.8}$$

From proposition 3.1.1, the statement is shown. □

The fact that we introduced c_1 and c_2 in the definition of $\widehat{\Delta A}$ and $\widehat{\Delta B}$ seems quite an arbitrary choice. This is not fortuitous. In fact, the following statement holds true.

Proposition 3.2.3. *Let $|\gamma\rangle$ be a coherent state. Let $\hat{A}$ and $\hat{B}$ be its associated operators, with r, α, c_1 and c_2 as above. Then,*

$$\langle A\rangle_\gamma = c_1 \tag{3.2.9a}$$
$$\langle B\rangle_\gamma = c_2 \tag{3.2.9b}$$
$$\langle \Delta A^2\rangle_\gamma = \frac{r\alpha}{2} \tag{3.2.9c}$$
$$\langle \Delta B^2\rangle_\gamma = \frac{\alpha}{2r}, \tag{3.2.9d}$$

where we defined $\langle X\rangle_\gamma := \langle\gamma|\hat{X}|\gamma\rangle$.

Proof. We start by calculating the expectation values.

Step 1. By definition,

$$\langle A\rangle_\gamma = \langle\gamma|\hat{A}|\gamma\rangle = \langle 0|\hat{W}(\gamma)^+\hat{A}\hat{W}(\gamma)|0\rangle. \tag{3.2.10}$$

Our goal is to compute the unitary sandwich $\hat{W}(\gamma)^+\hat{A}\hat{W}(\gamma)$. First, we evaluate the following expression through (3.2.1), (3.2.4) and (3.2.2a):

$$\begin{aligned}[\hat{A}, \gamma\hat{a}^+ - \gamma^*\hat{a}] &= \frac{\gamma}{(2r\alpha)^{1/2}}[\hat{A}, \hat{A} - ir\hat{B}] - \frac{\gamma^*}{(2r\alpha)^{1/2}}[\hat{A}, \hat{A} + ir\hat{B}] \\ &= \frac{-ir}{(2r\alpha)^{1/2}}(\gamma+\gamma^*)i\alpha\hat{1} = (2r\alpha)^{1/2}\,\mathrm{Re}(\gamma)\hat{1} \\ &= c_1\hat{1}.\end{aligned} \tag{3.2.11}$$

By proposition 1.2.6, the unitary sandwich we want to evaluate can be expressed as

$$\hat{W}(\gamma)^+\hat{A}\hat{W}(\gamma) = \hat{A} + c_1\hat{1}. \tag{3.2.12}$$

This leads to

$$\langle A\rangle_\gamma = \langle 0|(\hat{A} + c_1\hat{1})|0\rangle = \langle A\rangle_0 + c_1. \tag{3.2.13}$$

Using (3.2.4), it can be easily found that

$$\hat{A} = \left(\frac{r\alpha}{2}\right)^{1/2}(\hat{a} + \hat{a}^+). \tag{3.2.14}$$

Then, from (1.1.11a), it is straightforward that

$$\langle A\rangle_0 = 0, \tag{3.2.15}$$

showing (3.2.9a). Analogously, it is easy to find that

$$\hat{W}(\gamma)^+\hat{B}\hat{W}(\gamma) = \hat{B} + c_2\hat{1}. \tag{3.2.16}$$

From (3.2.4),

$$\hat{B} = -i\left(\frac{\alpha}{2r}\right)^{1/2}(\hat{a} - \hat{a}^+), \tag{3.2.17}$$

which leads to

$$\langle B \rangle_0 = 0, \tag{3.2.18}$$

that is (3.2.9b).

Step 2. We now compute the uncertainties. The definitions (3.2.6), have been somehow justified, since, generically speaking, the operator which gives the uncertainty of a certain observable for a given state is of the form:

$$\widehat{\Delta X}_s = \hat{X} - \langle X \rangle_s \hat{1}. \tag{3.2.19}$$

Then, taking (3.2.6) for granted, it easy to find that

$$\langle \Delta A^2 \rangle_\gamma = \langle A^2 \rangle_\gamma - \langle A \rangle_\gamma^2 = \langle A^2 \rangle_\gamma - c_1^2 \tag{3.2.20a}$$
$$\langle \Delta B^2 \rangle_\gamma = \langle B^2 \rangle_\gamma - \langle B \rangle_\gamma^2 = \langle B^2 \rangle_\gamma - c_2^2. \tag{3.2.20b}$$

Then, to evaluate these expressions, by definition, one has to compute the following unitary sandwiches:

$$\hat{W}(\gamma)^+ \hat{A}^2 \hat{W}(\gamma) \tag{3.2.21a}$$
$$\hat{W}(\gamma)^+ \hat{B}^2 \hat{W}(\gamma) \tag{3.2.21b}$$

For instance, we can evaluate (3.2.21a). Since $\hat{W}(\gamma)$ is an unitary operator (see proposition 1.2.1), one has that

$$\hat{W}(\gamma)^+ \hat{A}^2 \hat{W}(\gamma) = (\hat{W}(\gamma)^+ \hat{A} \hat{W}(\gamma))^2. \tag{3.2.22}$$

Then, by (3.2.12),

$$\hat{W}(\gamma)^+ \hat{A}^2 \hat{W}(\gamma) = \hat{A}^2 + 2c_1\hat{A} + c_1^2, \tag{3.2.23}$$

which, taking into account (3.2.15), leads to

$$\langle \Delta A^2 \rangle_\gamma = \langle A^2 \rangle_0. \tag{3.2.24}$$

Using (1.1.1), (1.1.11a) and (3.2.14), it is easy to find that

$$\langle A^2 \rangle_0 = \frac{r\alpha}{2}, \tag{3.2.25}$$

thus showing (3.2.9c). (3.2.9d) is analogously found. □

Remark 3.2.1. The previous result could also be used to show that a coherent state is a minimum-uncertainty state. Indeed,

$$\langle \Delta A^2 \rangle_\gamma^{1/2} \langle \Delta B^2 \rangle_\gamma^{1/2} = \frac{\alpha}{2} = \frac{1}{2} \langle [\widehat{\Delta A}, \widehat{\Delta B}] \rangle_\gamma. \tag{3.2.26}$$

Finally, a general form for the coherent states can be found. In fact, coherent states can be expanded in terms of the $|n\rangle$ with well-known expansion coefficients.

Proposition 3.2.4. *Let $|\gamma\rangle$ be a coerent state. Then,*

$$|\gamma\rangle = \exp(-|\gamma|^2/2) \sum_{n=0}^{\infty} |n\rangle \frac{\gamma^n}{(n!)^{1/2}}. \tag{3.2.27}$$

Proof. The statement readily follows from proposition 1.2.8. □

Example 3.2.1 (Relevant quantities derived from the coherent states of the harmonic oscillator)**.** To identify the coherent states of the harmonic oscillator of mass m and angular frequency ω, we have to find two operators of the sort (3.2.1) in order to define a proper annihilation operator. The position and momentum operators satisfy this request with $\alpha = \hbar$:

$$[\hat{q}, \hat{p}] = i\hbar\hat{1}. \tag{3.2.28}$$

We have that r must have the dimensions of a time divided a mass. Then, we set $r = 1/(m\omega)$. From (3.2.4), we define

$$\hat{a} = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\hat{q} + \frac{i}{m\omega}\hat{p}\right). \tag{3.2.29}$$

We choose two real numbers q and p and we set $c_1 = q$ and $c_2 = p$. From proposition 3.2.3, it immediately follows that

$$\langle q\rangle_\gamma = q = \left(\frac{2\hbar}{m\omega}\right)^{1/2}\operatorname{Re}(\gamma) \tag{3.2.30a}$$

$$\langle p\rangle_\gamma = p = (2\hbar m\omega)^{1/2}\operatorname{Im}(\gamma) \tag{3.2.30b}$$

$$\Delta q_\gamma = \left(\frac{\hbar}{2m\omega}\right)^{1/2} \tag{3.2.30c}$$

$$\Delta p_\gamma = \left(\frac{\hbar m\omega}{2}\right)^{1/2}. \tag{3.2.30d}$$

It follows that

$$\Delta q_\gamma \Delta p_\gamma = \frac{\hbar}{2}. \tag{3.2.30e}$$

Example 3.2.2 (Coherent states of the harmonic oscillator in the Schroedinger representation)**.** Consider an harmonic oscillator of mass m and frequency ω. Then, for $x \in \mathbb{R}$ in its configuration space, consider $|x\rangle$, i.e. the eigenket of the position operator $\hat{q}$ belonging to the x eigenvalue. For a given $n \in \mathbb{N}$, the expression of the normalized wavefunction of the harmonic oscillator reads as (see [6])

$$\phi_n(x) := \langle x|n\rangle = \frac{(-1)^n i^n}{\ell^{1/2}\pi^{1/4}(2^n n!)^{1/2}} H_n\left(\frac{x}{\ell}\right)\exp\left(-\frac{x^2}{2\ell^2}\right), \tag{3.2.31}$$

where $\ell := (\hbar/m\omega)^{1/2}$ and where $H_n(\xi)$ is the Hermite polynomial of order n. Through (3.2.27) it is possible to find that, for a given $\gamma \in \mathbb{C}$, the normalized wavefunction of the coherent states of the harmonic oscillator are

$$\phi_\gamma(x) := \langle x|\gamma\rangle = \frac{1}{\ell^{1/2}\pi^{1/4}}\exp\left(-\frac{1}{2}(\gamma^{*2} + |\gamma|^2)\right)\exp\left(-\frac{1}{2}\left(\frac{x}{\ell} + 2^{1/2} i\gamma\right)^2\right). \tag{3.2.32}$$

Indeed, combining (3.2.31) and (3.2.27),

$$\langle x|\gamma\rangle = \frac{1}{\ell^{1/2}\pi^{1/4}}\exp(-|\gamma|^2/2)\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i\gamma}{2^{1/2}}\right)^n H_n\left(\frac{x}{\ell}\right)\exp\left(-\frac{x^2}{2\ell^2}\right). \tag{3.2.33}$$

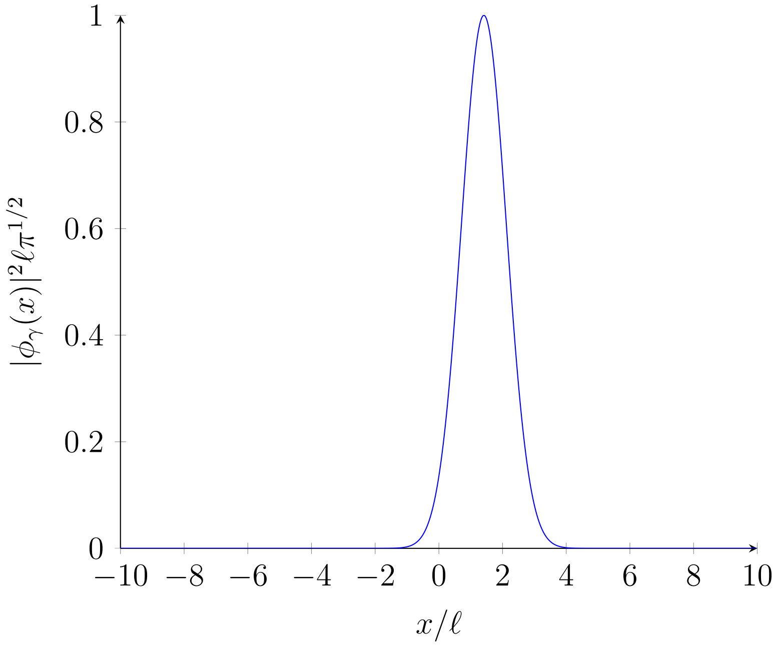

Figure 3.1: Plot of the squared modulus $|\phi_\gamma(x)|^2$ of $\phi_\gamma(x)$, with $\gamma = 1 + i$, as a function of x.

Then, expression (3.2.32) is obtained through the exponential generating function (see [8])

$$\exp(2\xi t - t^2) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(\xi), \tag{3.2.34}$$

with $\xi = x/\ell$ and $t = -i\gamma/2^{1/2}$. The squared modulus of $\phi_\gamma(x)$ has a Gaussian profile, as shown in figure 3.1.

Example 3.2.1 is one of the many applications of the fact that quantum systems described by canonically commuting operators admit coherent states. This is contained in definition 3.2.1. The coherent states of any system which is the quantum version of a classical system can always be defined through the operators $\hat{q}$ and $\hat{p}$ or through other equivalent canonically commuting operators. Furthermore, from Dirac's quantization principle, given two canonically conjugate phase functions f_1 and f_2,

$$\{f_1, f_2\} = 1 \quad \Longrightarrow \quad [\hat{f}_1, \hat{f}_2] = i\hbar\hat{1} + O(\hbar^2), \tag{3.2.35}$$

meaning that $\hat{f}_1$ and $\hat{f}_2$ can, under suitable conditions ($O(\hbar^2) = 0$), possibly satisfy (3.2.1) and therefore are fit candidates to define the coherent states of the associated quantum system. This connection between classical systems and coherent states strengthens the idea that the behaviour of these states should be studied in a semiclassical limit. This will be done in chapter 4 for a more general class of states which contains the set of coherent states. We now move on to a further characterization of these states.

3.3 Coherent states: general properties

The way coherent states were built allows one to find some other properties of these. For instance, it can be seen that coherent states form a non-orthogonal set of normalized states.

Proposition 3.3.1. *Let $|\gamma\rangle$ and $|\gamma'\rangle$ be two coherent states with $\gamma \neq \gamma'$. Then,*

$$\langle\gamma|\gamma\rangle = 1 \tag{3.3.1a}$$

$$\langle\gamma|\gamma'\rangle = \exp(-|\gamma - \gamma'|^2/2 + i\,\mathrm{Im}(\gamma^*\gamma')) \neq 0. \tag{3.3.1b}$$

Proof. The two relations will be proved in two separate steps.

Step 1. From the definition of $|\gamma\rangle$, from the fact that $\hat{W}(\gamma)$ is unitary and from the orthonormality relations (1.1.12a), we find

$$\langle\gamma|\gamma\rangle = \langle 0|\hat{W}(\gamma)^{-1}\hat{W}(\gamma)\rangle 0 = \langle 0|0\rangle = 1. \tag{3.3.2}$$

Step 2. Using the explicit expression (3.2.27) for $|\gamma\rangle$ and using the orthonormality relations (1.1.12a) which exist between the eigenkets of the number operator, we find

$$\begin{aligned}
\langle\gamma|\gamma'\rangle &= \exp(-|\gamma|^2/2 - |\gamma'|^2/2)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\gamma^{*m}}{(m!)^{1/2}}\langle m|n\rangle\frac{\gamma'^n}{(n!)^{1/2}} \\
&= \exp(-|\gamma|^2/2 - |\gamma'|^2/2)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\gamma^{*m}}{(m!)^{1/2}}\delta_{m,n}\frac{\gamma'^n}{(n!)^{1/2}} \\
&= \exp(-|\gamma|^2/2 - |\gamma'|^2/2)\sum_{n=0}^{\infty}\frac{(\gamma^*\gamma')^n}{n!} \\
&= \exp(-|\gamma|^2/2 - |\gamma'|^2/2)\exp(\gamma^*\gamma') \\
&= \exp(-|\gamma|^2/2 - |\gamma'|^2/2 + \gamma^*\gamma').
\end{aligned} \tag{3.3.3}$$

Hence,

$$\begin{aligned}
-|\gamma|^2/2 - |\gamma'|^2/2 + \gamma^*\gamma' &= -|\gamma|^2/2 - |\gamma'|^2/2 + \gamma^*\gamma'/2 + \gamma^*\gamma'/2 + \gamma\gamma'^*/2 - \gamma\gamma'^*/2 \\
&= -(|\gamma|^2 + |\gamma'|^2 - \gamma\gamma'^* - \gamma^*\gamma')/2 - (\gamma^*\gamma' - \gamma\gamma'^*)/2 \\
&= -\left(|\gamma|^2 + |\gamma'|^2 - 2\,\mathrm{Re}(\gamma\gamma'^*)\right)/2 + i\,\mathrm{Im}(\gamma^*\gamma') \\
&= -|\gamma - \gamma'|^2/2 + i\,\mathrm{Im}(\gamma^*\gamma'),
\end{aligned} \tag{3.3.4}$$

which concludes the proof. □

The following corollary is an immediate consequence of the previous relations.

Corollary 3.3.1. *Let $|\gamma\rangle$ and $|\gamma'\rangle$ be two coherent states. Then,*

$$0 < |\langle\gamma|\gamma'\rangle| \leq 1. \tag{3.3.5}$$

Given the fact that coherent states are parameterized by a complex number γ, if we choose $\gamma' \in \mathbb{C}$ close enough to γ in the Euclidean metric sense of this expression, it is natural to wonder whether we will get a $|\gamma'\rangle$ close enough to $|\gamma\rangle$ in the metric induced on $\mathcal{H}$ by the bra-ket product or not. The answer to this question is affirmative. In other words, coherent states display some sort of continuity.

Proposition 3.3.2. *For some fixed $\gamma \in \mathbb{C}$,*

$$\forall\,\epsilon > 0, \exists\,\delta > 0 \,:\, \forall\,\gamma' \in \mathbb{C},\, |\gamma - \gamma'| < \delta \implies \||\gamma\rangle - |\gamma'\rangle\| < \epsilon, \tag{3.3.6}$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{C}$ and $\|\cdot\|$ is the norm induced on $\mathcal{H}$ by the bra-ket product and where $|\gamma\rangle$ and $|\gamma'\rangle$ are coherent states.

Proof. By Proposition 3.3.1 and the definition of the norm, one gets

$$\begin{aligned}\||\gamma\rangle - |\gamma'\rangle\|^2 &= (\langle\gamma| - \langle\gamma'|)(|\gamma\rangle - |\gamma'\rangle) \\ &= \langle\gamma|\gamma\rangle + \langle\gamma'|\gamma'\rangle - \langle\gamma|\gamma'\rangle - \langle\gamma'|\gamma\rangle \\ &= 2\,(1 - \mathrm{Re}(\langle\gamma|\gamma'\rangle))\,. \end{aligned} \tag{3.3.7}$$

By continuity of the bra-ket product,

$$\lim_{\gamma'\to\gamma} \mathrm{Re}(\langle\gamma|\gamma'\rangle) = 1, \tag{3.3.8}$$

which concludes the proof. □

3.4 Overcompleteness of coherent states

For most quantum systems, it is customary to represent $\mathcal{H}$ through a complete orthonormal set consisting of eigenkets of selfadjoint operators representing observables of the system. Therefore, it might seem odd to represent $\mathcal{H}$ through an overcomplete set of eigenkets of non-Hermitian operators, such as the annihilation operator $\hat{a}$. The expressions (3.3.1a) and (3.3.1b) state that $G := \{|\gamma\rangle\}_{\gamma\in\mathbb{C}}$ consists of non-orthogonal normalized kets. However, it can be shown (see [12]) that the set

$$S := \{|\gamma\rangle \in G \,:\, \gamma = \pi^{1/2}(l + im),\, l, m \in \mathbb{Z}\} \subset G \tag{3.4.1}$$

does constitute a basis for $\mathcal{H}$. This means that the eigenkets of $\hat{a}$ form an overcomplete representation of $\mathcal{H}$. We will not show this result. However, we will show that G is complete. Indeed, one has the following result.

Proposition 3.4.1. *Let $G := \{|\gamma\rangle\}_{\gamma\in\mathbb{C}}$ be the set of the coherent states of a quantum system. Then,*

$$\hat{1} = \frac{1}{\pi}\int_{\mathbb{C}} d^2\gamma |\gamma\rangle\langle\gamma| \tag{3.4.2}$$

holds.

Proof. Showing (3.4.2) is equivalent to showing that, for any $|\lambda\rangle, |\psi\rangle \in \mathcal{H}$,

$$\langle\lambda|\psi\rangle = \frac{1}{\pi}\int_{\mathbb{C}} d^2\gamma \langle\lambda|\gamma\rangle\langle\gamma|\psi\rangle, \tag{3.4.3}$$

holds. Using (3.2.27), one gets

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{C}} d^2\gamma\langle\lambda|\gamma\rangle\langle\gamma|\psi\rangle &= \frac{1}{\pi}\int d^2\gamma \exp(-|\gamma|^2) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle\lambda|m\rangle\frac{\gamma^m\gamma^{*n}}{(m!n!)^{1/2}}\langle n|\psi\rangle \\ &= \frac{1}{\pi}\int_0^{\infty}\int_0^{2\pi} r dr d\theta \exp(-r^2)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle\lambda|m\rangle\frac{r^{m+n}\exp(i(m-n)\theta)}{(m!n!)^{1/2}}\langle n|\psi\rangle,\end{aligned} \tag{3.4.4}$$

where we changed the integration on $\mathbb{C}$ into an integration on the polar plane, setting $\gamma = r\exp(i\theta)$ and $d^2\gamma = rdrd\theta$. Writing that

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{C}} d^2\gamma\langle\lambda|\gamma\rangle\langle\gamma|\psi\rangle &= \frac{1}{\pi}\lim_{R\to\infty}\int_0^R\int_0^{2\pi} rdrd\theta\exp(-r^2)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle\lambda|m\rangle\frac{r^{m+n}\exp(i(m-n)\theta)}{(m!n!)^{1/2}}\langle n|\psi\rangle \\ &= \frac{1}{\pi}\lim_{R\to\infty} I_R, \end{aligned} \tag{3.4.5}$$

we can concentrate on I_R. I_R can be separated into a radial integration and an angular integration:

$$\begin{aligned} I_R &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle\lambda|m\rangle\frac{1}{(m!n!)^{1/2}}\langle n|\psi\rangle\left(\int_0^R drr^{m+n+1}\exp(-r^2)\right)\left(\int_0^{2\pi} d\theta\exp(i(m-n)\theta)\right) \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle\lambda|m\rangle\frac{1}{(m!n!)^{1/2}}\langle n|\psi\rangle I_r(R;m,n)I_\theta. \end{aligned} \tag{3.4.6}$$

It can be easily seen that

$$I_\theta = 2\pi\delta_{m,n}, \tag{3.4.7}$$

which leads to

$$I_R = 2\pi\sum_{n=0}^{\infty}\langle\lambda|n\rangle\frac{1}{n!}\langle n|\psi\rangle\int_0^R drr^{2n+1}\exp(-r^2). \tag{3.4.8}$$

Let's take a look at the integral in dr. Setting $r^2 = y$ and $2rdr = dy$, we get

$$\int_0^R drr^{2n+1}\exp(-r^2) = \frac{1}{2}\int_0^R dyy^n\exp(-y). \tag{3.4.9}$$

Then, we define

$$\Gamma_n(R) := \frac{1}{2n!}\int_0^R dyy^n\exp(-y). \tag{3.4.10}$$

By the fact that the integrand is non-negative, we have $\Gamma_n(R) > 0$ and that

$$\Gamma_n(R) < \frac{1}{n!}\int_0^\infty dyy^n\exp(-y) = \frac{1}{n!}\Gamma(n+1) = 1, \tag{3.4.11}$$

where Γ is the gamma function. Then, using (3.4.10), we obtain the following expression for I_R:

$$I_R = 2\pi\sum_{n=0}^{\infty}\langle\lambda|n\rangle\langle n|\psi\rangle\Gamma_n(R). \tag{3.4.12}$$

Since $\{|n\rangle\}_{n\in\mathbb{N}}$ is complete,

$$\langle\lambda|\psi\rangle = \sum_{n=0}^{\infty}\langle\lambda|n\rangle\langle n|\psi\rangle < \infty. \tag{3.4.13}$$

With the fact that $0 < \Gamma_n(R) < 1$, one has that I_R converges too and the limit (3.4.5) can then be interchanged with the limit which defines the series in n:

$$\frac{1}{\pi}\int_{\mathbb{C}} d^2\gamma\langle\lambda|\gamma\rangle\langle\gamma|\psi\rangle = 2\sum_{n=0}^{\infty}\langle\lambda|n\rangle\langle n|\psi\rangle\lim_{R\to\infty}\Gamma_n(R) = \sum_{n=0}^{\infty}\langle\lambda|n\rangle\langle n|\psi\rangle = \langle\lambda|\psi\rangle, \tag{3.4.14}$$

as, from (3.4.10), it can be easily seen that

$$\lim_{R\to\infty}\Gamma_n(R) = \frac{1}{2n!}\lim_{R\to\infty}\int_0^R dyy^n\exp(-y)$$
$$= \frac{1}{2n!}\int_0^\infty dyy^n\exp(-y) = \frac{\Gamma(n+1)}{2n!} = \frac{1}{2}. \tag{3.4.15}$$

□

There are some important properties which follow from (3.4.2). In particular, all kets $|\psi\rangle \in \mathcal{H}$ now enjoy the following decomposition in terms of coherent states with the complex coefficients $\langle\gamma|\psi\rangle$:

$$|\psi\rangle = \frac{1}{\pi}\int_{\mathbb{C}} d^2\gamma|\gamma\rangle\langle\gamma|\psi\rangle. \tag{3.4.16}$$

The coefficients $\langle\gamma|\psi\rangle$ display some interesting properties. In fact, they have an explicit expression and exhibit boundedness and analyticity (up to a gaussian factor) on $\mathbb{C}$. To begin with, we give these coefficients a name.

Definition 3.4.1 (Vector representative function)**.** *Let $|\psi\rangle \in \mathcal{H}$ be an arbitrary state of the Hilbert space. Then, the complex-valued function $\psi(\gamma)$ defined as*

$$\begin{aligned}\psi : \mathbb{C} &\longrightarrow \mathbb{C}\\ \gamma &\longmapsto \psi(\gamma) := \langle\gamma|\psi\rangle\end{aligned} \tag{3.4.17}$$

is called the vector representative function of $|\psi\rangle$.

Proposition 3.4.2. *Let $|\psi\rangle \in \mathcal{H}$ be an arbitrary state of the Hilbert space. Then, the vector representative function $\psi(\gamma)$ of $|\psi\rangle$ has the following explicit expression:*

$$\psi(\gamma) := \langle\gamma|\psi\rangle = \exp(-|\gamma|^2/2)\sum_{n=0}^{\infty}\frac{\gamma^{*n}}{(n!)^{1/2}}\langle n|\psi\rangle. \tag{3.4.18}$$

Proof. This follows immediately from proposition 3.2.4. □

Proposition 3.4.3. *Let $|\psi\rangle \in \mathcal{H}$ be an arbitrary state of the Hilbert space. Then, the vector representative function $\psi(\gamma)$ of $|\psi\rangle$ is bounded, continuous and*

$$f_\psi(\gamma) := \exp(|\gamma|^2/2)\psi(\gamma) \tag{3.4.19}$$

is analytic everywhere on $\mathbb{C}$.

Proof. This proof will be divided in two steps. In the first step, $\psi(\gamma)$ will be shown to be bounded and continuous. In the second step, $f_\psi(\gamma)$ will be shown to be entire.

Step 1. We need to show that

$$\exists\, C \in \mathbb{R} \,:\, \forall\, \gamma \in \mathbb{C} \implies |\psi(\gamma)| \leq C. \tag{3.4.20}$$

We start by fixing some $\gamma' \in \mathbb{C}$. Then, we compute $|\psi(\gamma')|$. Using the Cauchy-Schwarz inequality and (3.3.1a),

$$|\psi(\gamma')| = |\langle\gamma'|\psi\rangle| \leq \langle\gamma'|\gamma'\rangle^{1/2}\langle\psi|\psi\rangle^{1/2} = \langle\psi|\psi\rangle^{1/2} = \||\psi\rangle\| < \infty. \tag{3.4.21}$$

Hence, this first statement is proven just by choosing $C = \||\psi\rangle\|$. We now want to show that, fixed some $\gamma \in \mathbb{C}$,

$$\forall\, \epsilon > 0,\, \exists\, \delta > 0 \,:\, \forall \gamma' \in \mathbb{C},\, |\gamma - \gamma'| < \delta \implies |\psi(\gamma) - \psi(\gamma')| < \epsilon. \tag{3.4.22}$$

We start by computing $|\psi(\gamma) - \psi(\gamma')|$. By the linearity of the bra-ket product, by the Cauchy-Schwarz inequality,

$$\begin{aligned} |\psi(\gamma) - \psi(\gamma')| &= |\langle\gamma|\psi\rangle - \langle\gamma'|\psi\rangle| \\ &= |(\langle\gamma| - \langle\gamma'|)|\psi\rangle| \\ &\leq \|\langle\gamma| - \langle\gamma'|\|\, \||\psi\rangle\|. \end{aligned} \tag{3.4.23}$$

Hence, by proposition (3.3.2), (3.4.22) follows.

Step 2. By definition (see [7]), a complex-valued function f defined on an open set $D \subseteq \mathbb{C}$ is said to be analytic on D if, $\forall\, \gamma_0 \in D$, $\exists\, \rho(\gamma_0) \in \mathbb{R}, \rho(\gamma_0) > 0$, such that, $\forall\, \gamma \,:\, |\gamma - \gamma_0| < \rho(\gamma_0)$,

$$\exists\, a_0, \ldots, a_i, \ldots \,:\, f(\gamma) = \sum_{k=0}^{\infty} a_k(\gamma - \gamma_0)^k. \tag{3.4.24}$$

A complex-valued function f is said to be entire if it is analytic on $D = \mathbb{C}$. We now proceed to show that $f_\psi(\gamma)$ satisfies this condition. First, by the triangle and the Cauchy-Schwarz inequalities, we notice that

$$\left|\sum_{n=0}^{\infty} \frac{\gamma^{*n}}{(n!)^{1/2}}\langle n|\psi\rangle\right| \leq \sum_{n=0}^{\infty} \frac{|\gamma^*|^n}{(n!)^{1/2}}\, |\langle n|\psi\rangle| \leq \sum_{n=0}^{\infty} \frac{|\gamma^*|^n}{(n!)^{1/2}}\langle n|n\rangle^{1/2}\langle\psi|\psi\rangle^{1/2}. \tag{3.4.25}$$

In other words, from (1.1.12a),

$$\exp(|\gamma|^2/2)|\psi(\gamma)| = \left|\sum_{n=0}^{\infty} \frac{\gamma^{*n}}{(n!)^{1/2}}\langle n|\psi\rangle\right| \leq \||\psi\rangle\| \sum_{n=0}^{\infty} \frac{|\gamma^*|^n}{(n!)^{1/2}} < \infty, \tag{3.4.26}$$

for all $\gamma \in \mathbb{C}$, as can be easily seen by the ratio test for convergence. Then, the series expansion appearing in

$$f_\psi(\gamma) := \exp(|\gamma|^2/2)\psi(\gamma) = \sum_{n=0}^{\infty} \frac{\gamma^{*n}}{(n!)^{1/2}}\langle n|\psi\rangle \tag{3.4.27}$$

is convergent for all $\gamma \in \mathbb{C}$ and, set $\gamma_0 = 0$, $f_\psi(\gamma)$ is found to be analytic on $\mathbb{C}$. □

Remark 3.4.1. Being $\psi(\gamma)$ entire up to a a factor $\exp(-|\gamma|^2/2)$, it is known from complex analysis that it is infinitely differentiable in γ. In addition, it is worth pointing out that not every entire function is a fit candidate for $\psi(\gamma)$, as (3.4.26) imposes a growth restriction on $\psi(\gamma)$.

That being said, (3.4.16) can be written in terms of the vector representative function of $|\psi\rangle$:

$$|\psi\rangle = \frac{1}{\pi}\int_{\mathbb{C}} d^2\gamma |\gamma\rangle\psi(\gamma). \tag{3.4.28}$$

In an analogous fashion, operators too can be expanded in terms of the coherent states. Indeed, given a generic operator $\hat{B}$, multiplying on both sides by (3.4.2),

$$\hat{B} = \frac{1}{\pi^2}\int_{\mathbb{C}^2} d^2\gamma_1 d^2\gamma_2 |\gamma_1\rangle\langle\gamma_1|\hat{B}|\gamma_2\rangle\langle\gamma_2| = \frac{1}{\pi^2}\int_{\mathbb{C}^2} d^2\gamma_1 d^2\gamma_2 |\gamma_1\rangle(B\gamma_2)(\gamma_1)\langle\gamma_2|, \tag{3.4.29}$$

where $(B\gamma_2)$ is the vector representative function of $\hat{B}|\gamma_2\rangle$. There are further properties which the vector representative functions must satisfy.

Proposition 3.4.4 (Integral equation for the representative function)**.** *Let $|\psi\rangle \in \mathcal{H}$ be an arbitrary state of the Hilbert space. Then, its vector representative function solves the following integral equation:*

$$\psi(\gamma) = \frac{1}{\pi}\int_{\mathbb{C}} d^2\gamma' K(\gamma,\gamma')\psi(\gamma'), \tag{3.4.30}$$

where the integral kernel $K(\gamma,\gamma')$ is defined as

$$K(\gamma,\gamma') := \langle\gamma|\gamma'\rangle. \tag{3.4.31}$$

Proof. The claim is a prompt consequence of (3.4.16) and definition (3.4.1). □

Similar integral equations must be fulfilled by the representative functions for operators. Moreover, The integral kernel (3.4.31) satisfies an idempotence condition and acts like a projector.

Proposition 3.4.5 (Idempotence condition)**.** *Given $K(\gamma,\gamma'') := \langle\gamma|\gamma''\rangle$, where $|\gamma\rangle, |\gamma''\rangle \in G$, one has that*

$$K(\gamma,\gamma'') = \frac{1}{\pi}\int_{\mathbb{C}} d^2\gamma' K(\gamma,\gamma')K(\gamma',\gamma''). \tag{3.4.32}$$

Proof. This fact results from choosing $|\psi\rangle = |\gamma''\rangle$ in (3.4.30). □

To conclude this section, it is interesting to point out that the fact that G is overcomplete has some bizarre consequences. First, (3.4.2) is not unique. Indeed, one can create infinitely many decompositions for $\hat{1}$ and, consequently, for every ket and operator in the following manner:

$$\hat{1} = \hat{1}\hat{1} = \frac{1}{\pi^2}\int_{\mathbb{C}^2} d^2\gamma_1 d^2\gamma_2 |\gamma_1\rangle\langle\gamma_1|\gamma_2\rangle\langle\gamma_2| = \frac{1}{\pi^2}\int_{\mathbb{C}^2} d^2\gamma_1 d^2\gamma_2 |\gamma_1\rangle K(\gamma_1,\gamma_2)\langle\gamma_2|. \tag{3.4.33}$$

Another consequence of the overcompleteness of G is that there are some sort of linear dependencies between coherent states. For instance, simply by using (3.4.16) with $|\psi\rangle = |\gamma'\rangle \in G$, one can express $|\gamma'\rangle$ in terms of every element of G. A light can be cast on other forms of dependencies between coherent states.

Proposition 3.4.6. *Let $\gamma \in \mathbb{C}$ be a generic complex number and let $|\gamma\rangle \in G$ be its associated coherent state. Then, given an arbitrary $p \in \mathbb{N}^+$,*

$$\int_{\mathbb{C}} d^2\gamma |\gamma\rangle \gamma^p = 0. \tag{3.4.34}$$

Proof. We start by writing the polar form of γ, i.e. $\gamma = r\exp(i\theta)$, and by using proposition (3.2.4):

$$|\gamma\rangle = |r\exp(i\theta)\rangle = \exp(-r^2/2)\sum_{n=0}^{\infty}|n\rangle\frac{r^n\exp(in\theta)}{(n!)^{1/2}}. \tag{3.4.35}$$

Then, we multiply this expression by $\exp(ip\theta)$ and we integrate this product in θ from 0 to 2π:

$$\int_0^{2\pi} d\theta |r\exp(i\theta)\rangle \exp(ip\theta) = \exp(-r^2/2)\sum_{n=0}^{\infty}|n\rangle\frac{r^n}{(n!)^{1/2}}\int_0^{2\pi} d\theta \exp(i(p+n)\theta) = 0. \tag{3.4.36}$$

This follows from the fact that $p + n \neq 0\ \forall p$ (see (3.4.7)). Multiplying (3.4.36) by $r^p r$ and integrating in r from 0 to ∞, relation (3.4.34) is reached. □

Despite the fact (see proposition 3.4.6) that sets of infinitely many coherent states are linearly dependent, it can be shown that, when there is a finite number of them, they are linearly independent.

Proposition 3.4.7. *Let $N \in \mathbb{N}^+$, with $N < \infty$. Let $|\gamma_k\rangle \in G$, with $k = 1, \ldots, N$, be N coherent states. Then, given N arbitrary complex numers $c_k \in \mathbb{C}$, with $k = 1, \ldots, N$ the following statement holds true:*

$$\sum_{k=0}^{N}|\gamma_k\rangle c_k = 0 \quad \Longleftrightarrow \quad c_k = 0 \quad \forall\, k = 1, \ldots, N. \tag{3.4.37}$$

Proof. The ($\Longleftarrow$) implication is straightforward. Let's show ($\Longrightarrow$) by contradiction. We start by assuming that finite linear combinations of coherent states are in turn coherent states. This is equivalent to assuming that, in

$$\sum_{k=0}^{N}|\gamma_k\rangle c_k = 0 \tag{3.4.38}$$

there is a least one $m \in \{0, \ldots, N\}$ such that $c_m \neq 0$. Then, rearranging (3.4.38), we obtain

$$|\gamma_m\rangle = -\sum_{k=0,\, k\neq m}^{N}|\gamma_k\rangle\frac{c_k}{c_m}, \tag{3.4.39}$$

Choosen some $|\gamma_n\rangle$ with $n \in \{0, \ldots, N\}$, we define $|\gamma_\perp\rangle$ as

$$|\gamma_\perp\rangle := |\gamma_m\rangle - |\gamma_n\rangle\langle\gamma_n|\gamma_m\rangle = |\gamma_m\rangle + |\gamma_n\rangle\sum_{k=0,\, k\neq m}^{N}\langle\gamma_n|\gamma_k\rangle\frac{c_k}{c_m}. \tag{3.4.40}$$

We assumed that finite linear combinations of coherent states are coherent states. Then, $|\gamma_\perp\rangle$ is a coherent state. It is easy to check that

$$\langle\gamma_n|\gamma_\perp\rangle = 0. \tag{3.4.41}$$

This, however, contradicts (3.3.1b). The statement is thus shown. □

Chapter 4

Classical particle limit

In the quantum formalism, the experimental fact that, under certain conditions, quantum systems with a classical analogue display a classical behaviour is not immediately obvious, although it is something sensible to expect from the theory if it models accurately the phenomenology. Therefore, a link between Hamiltonian mechanics and quantum theory should exist and, under fit circumstances, it should be possible to retrieve the dynamics of a classical particle through the quantum formalism. Classical observables are in one-to-one correspondence with phase functions, while quantum observables are in one-to-one correspondence with selfadjoint operators. In a given state, the physically relevant quantities which can be obtained from selfadjoint operators are their expectation value and their uncertainty: these are significant quantities in the transition between the two theories. For this reason, the focus should be set on recovering classical equations from the expectation values and the uncertainties of the selfadjoint operators corresponding to classical observables. The problem essentially boils down to finding the suitable quantum states in which the computed quantities produce the desired result. The theory of coherent states which was studied in the previous chapter already suggested in some way a suitable form for these states.
In this chapter, we start by rephrasing the question in a more formal and precise manner. Then, ignoring time evolution, we first solve the problem for position and momentum. In the last section, time evolution is taken into consideration and, again, the problem is solved for position and momentum, bringing really interesting results. This chapter relies on the introduction of [3].

4.1 Statement of the problem

Let ν be the number of degrees of freedom of a certain quantum system which has a classical counterpart. In quantum mechanics, the states of the system are represented by the rays of a certain Hilbert space $\mathcal{H}$ and observable quantities of the system are represented by selfadjoint operators on $\mathcal{H}$. In classical mechanics, the state of the system is represented by a point in a 2ν-dimensional space $\mathcal{F}$ called phase space. Observable quantities are represented by phase functions, that is by real functions on $\mathcal{F}$.
One wants to obtain classical observables from the quantum ones. To this goal, given a $\hbar$-dependent selfadjoint operator $\hat{A}_\hbar$ and given its classical equivalent phase function $a(\boldsymbol{\zeta})$, with $\boldsymbol{\zeta} \in \mathcal{F}$, one must find the states $|\hbar, \boldsymbol{\zeta}\rangle$ such that

$$\langle A_\hbar \rangle_{|\hbar,\boldsymbol{\zeta}\rangle} \xrightarrow{\hbar\to 0} a(\boldsymbol{\zeta}) \tag{4.1.1a}$$

$$\langle \Delta A_\hbar \rangle_{|\hbar,\boldsymbol{\zeta}\rangle} \xrightarrow{\hbar\to 0} 0. \tag{4.1.1b}$$

Our aim is then to determine the form of the states satisfying (4.1.1).

Remark 4.1.1. In this context, the symbol $\hbar$ has to be regarded as a parameter that represents to which degree the quantum system is close to its classical counterpart. Accordingly, taking the formal limit $\hbar \to 0$ mathematically describes the process of taking the system to its classical form. This process must not be intended as an actual variation of $\hbar$, since the reduced Planck constant is unalterable and its value is measurable and well-known. The limit $\hbar \to 0$ must rather be regarded as a mathematical way to express the fact that, when the quantum system approaches its classical analogue, $\hbar$ becomes small compared to the action scale of the system.

4.2 Classical limit of the expectation values of observables

Finding the states $|\hbar, \boldsymbol{\zeta}\rangle$ is relatively easy when $\hat{A}_\hbar$ corresponds to the position or momentum operators. To this end, let $\hat{q}_1, \dots, \hat{q}_\nu, \hat{p}_1, \dots, \hat{p}_\nu$ be 2ν selfadjoint operators obeying the following commutation relations:

$$[\hat{q}_k, \hat{p}_l] = i\delta_{k,l}\hat{1} \tag{4.2.1a}$$
$$[\hat{q}_i, \hat{q}_j] = 0 \tag{4.2.1b}$$
$$[\hat{p}_i, \hat{p}_j] = 0, \tag{4.2.1c}$$

$\forall\, k, l, i, j \in \{1, \dots, \nu\}$. These operators can be arranged in the vectors operators

$$\hat{\boldsymbol{q}} = (\hat{q}_1, \dots, \hat{q}_\nu) \tag{4.2.2a}$$
$$\hat{\boldsymbol{p}} = (\hat{p}_1, \dots, \hat{p}_\nu)\,. \tag{4.2.2b}$$

Remark 4.2.1. The operators $\hat{q}_k$ and $\hat{p}_l$ are position and momentum operators. For a symmetry argument, the constant $\hbar$ which usually appears in the commutation relation (4.2.1a) is included inside $\hat{q}_k$ and $\hat{p}_l$.

Given two arbitrary elements in $\mathbb{R}^\nu$, say $\boldsymbol{\xi} = (\xi_1, \dots, \xi_\nu)$ and $\boldsymbol{\pi} = (\pi_1, \dots, \pi_\nu)$, the following operator can be defined

$$\hat{C}(\boldsymbol{\xi}, \boldsymbol{\pi}) := \exp\left(i\left(\boldsymbol{\pi} \cdot \hat{\boldsymbol{q}} - \boldsymbol{\xi} \cdot \hat{\boldsymbol{p}}\right)\right), \tag{4.2.3}$$

where $\cdot$ is a shorthand notation for

$$\boldsymbol{\pi} \cdot \hat{\boldsymbol{q}} := \sum_{j=0}^{\nu} \pi_j \hat{q}_j, \tag{4.2.4a}$$

$$\boldsymbol{\xi} \cdot \hat{\boldsymbol{p}} := \sum_{j=0}^{\nu} \xi_j \hat{p}_j. \tag{4.2.4b}$$

The $\hat{C}(\boldsymbol{\xi}, \boldsymbol{\pi})$ operator will be used to build some states which behave as (4.1.1) in the semiclassical limit of $\hbar \to 0$.

Lemma 4.2.1. *$\hat{C}(\xi, \pi)$ is a Weyl operator.*

Proof. By definition, one has, through the properties of the exponential operator and through (4.2.1),

$$\begin{aligned}\hat{C}(\boldsymbol{\xi}, \boldsymbol{\pi}) &:= \exp\left(i\left(\boldsymbol{\pi} \cdot \hat{\boldsymbol{q}} - \boldsymbol{\xi} \cdot \hat{\boldsymbol{p}}\right)\right) \\ &= \prod_{j=1}^{\nu} \exp\left(i\left(\pi_j \hat{q}_j - \xi_j \hat{p}_j\right)\right).\end{aligned} \tag{4.2.5}$$

Then, for each pair $\hat{q}_k, \hat{p}_k$, the following operators can be defined:

$$\hat{a}_k := \frac{1}{2^{1/2}} \left(\hat{q}_k + i\hat{p}_k\right) \tag{4.2.6}$$

$$\hat{a}_k^+ = \frac{1}{2^{1/2}} \left(\hat{q}_k - i\hat{p}_k\right), \tag{4.2.7}$$

which, through (4.2.1a), are found to be a pair of annihilation and creation operators. Setting

$$\gamma_k := \frac{1}{2^{1/2}} \left(\xi_k + i\pi_k\right), \tag{4.2.8}$$

(4.2.5) can be written as

$$\hat{C}(\boldsymbol{\xi}, \boldsymbol{\pi}) = \prod_{j=1}^{\nu} \exp\left(\gamma_j \hat{a}_j^+ - \gamma_j^* \hat{a}_j\right). \tag{4.2.9}$$

Thus, by proposition 1.2.9, the statement is readily shown. □

To deal in more concise terms with the vector operators $\hat{\boldsymbol{q}}$ and $\hat{\boldsymbol{p}}$ and with the ν-tuples $\boldsymbol{\xi}$ and $\boldsymbol{\pi}$, the following notation is introduced

$$\hat{\boldsymbol{z}} = (\hat{\boldsymbol{q}}, \hat{\boldsymbol{p}}) \tag{4.2.10a}$$

$$\boldsymbol{\zeta} = (\boldsymbol{\xi}, \boldsymbol{\pi}). \tag{4.2.10b}$$

From the properties of the Weyl operators, a translation property is found for $\hat{C}(\boldsymbol{\zeta})$.

Proposition 4.2.1 (Translation property). *Given an arbitrary $\boldsymbol{\zeta} \in \mathbb{R}^{2\nu}$, the identity*

$$\hat{C}(\boldsymbol{\zeta})^+ \hat{\boldsymbol{z}} \hat{C}(\boldsymbol{\zeta}) = \hat{\boldsymbol{z}} + \boldsymbol{\zeta}\hat{1} \tag{4.2.11}$$

holds true.

Proof. We start by considering one element of the 2ν-tuple $\hat{\boldsymbol{z}}$. For instance, let us consider $\hat{z}_j = \hat{q}_j$. Then, by the translation property 1.2.6 of the Weyl operators and by Lemma 4.2.1,

$$\hat{C}(\boldsymbol{\zeta})^+ \hat{q}_j \hat{C}(\boldsymbol{\zeta}) = \hat{q}_j + \left[\hat{q}_j, \sum_{k=1}^{\nu} \gamma_k^* \hat{a}_k - \sum_{k=1}^{\nu} \gamma_k \hat{a}_k^+\right], \tag{4.2.12}$$

where $\hat{a}_k, \hat{a}_k^+$ and γ_k are defined as in (4.2.6), (4.2.7) and (4.2.8). Using these definitions and the commutation relations (4.2.1), one is left with

$$\begin{aligned}
\hat{C}(\boldsymbol{\zeta})^+ \hat{q}_j \hat{C}(\boldsymbol{\zeta}) &= \hat{q}_j - [\hat{q}_j, \gamma_j^* \hat{a}_j - \gamma_j \hat{a}_j^+] \\
&= \hat{q}_j - \frac{1}{2}[\hat{q}_j, (\xi_j - i\pi_j)(\hat{q}_j + i\hat{p}_j) - (\xi_j + i\pi_j)(\hat{q}_j - i\hat{p}_j)] \\
&= \hat{q}_j - \frac{1}{2}[\hat{q}_j, (\xi_j - i\pi_j)\, i\hat{p}_j + (\xi_j + i\pi_j)\, i\hat{p}_j] \\
&= \hat{q}_j - \frac{1}{2}\left(-(\xi_j - i\pi_j)\hat{1} - (\xi_j + i\pi_j)\hat{1}\right) \\
&= \hat{q}_j + \xi_j \hat{1}.
\end{aligned} \tag{4.2.13}$$

Proceeding in the same fashion for a generic $\hat{z}_j = \hat{p}_j$ and putting the relations found in the vector notation, (4.2.11) follows. □

To parameterize the theory in function of the classicality degree $\hbar$, we set

$$\hat{\boldsymbol{z}}_\hbar := \hbar^{1/2}\hat{\boldsymbol{z}}. \tag{4.2.14}$$

From there, the form of the states $|\hbar, \boldsymbol{\zeta}\rangle$ is found by choosing an arbitrary normalized state $|\Omega\rangle$ and by setting

$$|\hbar, \boldsymbol{\zeta}\rangle := \hat{C}(\hbar^{-1/2}\boldsymbol{\zeta})|\Omega\rangle. \tag{4.2.15}$$

Indeed, one has the following result.

Proposition 4.2.2. *Given a generic normalized state $|\Omega\rangle$ and a generic $\boldsymbol{\zeta} \in \mathbb{R}^{2\nu}$, the expectation value and the uncertainty of $\hat{\boldsymbol{z}}_\hbar$ in the state $|\hbar, \boldsymbol{\zeta}\rangle = \hat{C}(\hbar^{-1/2}\boldsymbol{\zeta})|\Omega\rangle$ are given by the following expressions:*

$$\langle \boldsymbol{z}_\hbar \rangle_{|\hbar,\boldsymbol{\zeta}\rangle} = \boldsymbol{\zeta} + \hbar^{1/2}\langle \boldsymbol{z}\rangle_{|\Omega\rangle} \tag{4.2.16a}$$

$$\langle \Delta\boldsymbol{z}_\hbar \rangle_{|\hbar,\boldsymbol{\zeta}\rangle} = \hbar^{1/2}\langle \Delta\boldsymbol{z}\rangle_{|\Omega\rangle}. \tag{4.2.16b}$$

Proof. The first expression can be easily obtained from (4.2.11), (4.2.14) and by the fact that $|\Omega\rangle$ is normalized:

$$\begin{aligned}\langle \boldsymbol{z}_\hbar \rangle_{|\hbar,\boldsymbol{\zeta}\rangle} &:= \langle \Omega|\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta})^{+}\hat{\boldsymbol{z}}_\hbar\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta})|\Omega\rangle \\ &= \boldsymbol{\zeta}\langle\Omega|\Omega\rangle + \hbar^{1/2}\langle\Omega|\hat{\boldsymbol{z}}|\Omega\rangle \\ &= \boldsymbol{\zeta} + \hbar^{1/2}\langle\boldsymbol{z}\rangle_{|\Omega\rangle}.\end{aligned} \tag{4.2.17}$$

Similarly, using the fact that, for the i-th component of $\hat{\boldsymbol{z}}$,

$$\langle \Delta z_i\rangle^2_{|\hbar,\boldsymbol{\zeta}\rangle} := \langle z_i^2\rangle_{|\hbar,\boldsymbol{\zeta}\rangle} - \langle z_i\rangle^2_{|\hbar,\boldsymbol{\zeta}\rangle}, \tag{4.2.18}$$

we compute $\langle z_i^2\rangle_{|\hbar,\boldsymbol{\zeta}\rangle}$ through (4.2.11), (4.2.14) and through the fact that $|\Omega\rangle$ is normalized and that $\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta})$ is an unitary operator:

$$\begin{aligned}\langle z_i^2\rangle_{|\hbar,\boldsymbol{\zeta}\rangle} &= \langle\Omega|\left(\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta})^{+}\hat{z}_i\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta})\right)\left(\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta})^{+}\hat{z}_i\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta})\right)|\Omega\rangle \\ &= \langle\Omega|\left(\hat{z}_i + \hbar^{-1/2}\zeta_i\hat{1}\right)\left(\hat{z}_i + \hbar^{-1/2}\zeta_i\hat{1}\right)|\Omega\rangle \\ &= \langle\Omega|\left(\hat{z}_i^2 + \hbar^{-1}\zeta_i^2\hat{1} + 2\hbar^{-1/2}\zeta_i\hat{z}_i\right)|\Omega\rangle \\ &= \langle z_i^2\rangle_{|\Omega\rangle} + \hbar^{-1}\zeta_i^2 + 2\hbar^{-1/2}\zeta_i\langle z_i\rangle_{|\Omega\rangle}.\end{aligned} \tag{4.2.19}$$

On the other hand, for $\langle z_i\rangle^2_{|\hbar,\boldsymbol{\zeta}\rangle}$, it is found from the components of (4.2.16a) that

$$\begin{aligned}\langle z_i\rangle^2_{|\hbar,\boldsymbol{\zeta}\rangle} &= \left(\hbar^{-1/2}\zeta_i + \langle z_i\rangle_{|\Omega\rangle}\right)^2 \\ &= \hbar^{-1}\zeta_i^2 + \langle z_i\rangle^2_{|\Omega\rangle} + 2\hbar^{-1/2}\zeta_i\langle z_i\rangle_{|\Omega\rangle}.\end{aligned} \tag{4.2.20}$$

Hence, from (4.2.18),

$$\langle \Delta z_i\rangle^2_{|\hbar,\boldsymbol{\zeta}\rangle} = \langle \Delta z_i\rangle^2_{|\Omega\rangle}. \tag{4.2.21}$$

Going back to the vector notation and using (4.2.14), (4.2.16b) is found. □

Remark 4.2.2. The choice of $|\Omega\rangle$ in (4.2.15) is completely arbitrary. Then, the choice $|\Omega\rangle = |0,\dots,0\rangle$, where $|0,\dots,0\rangle$ is the common 0-eigenket of the number operators defined from the annihilation (4.2.6) and creation (4.2.7) operators, can be made. This, by (3.2.1), defines $|\hbar,\boldsymbol{\zeta}\rangle$ as a coherent state.

The previous proposition shows that the states $|\hbar,\boldsymbol{\zeta}\rangle = \hat{C}(\hbar^{-1/2}\boldsymbol{\zeta})|\Omega\rangle$ have the expected behaviour as $\hbar \to 0$. In fact, in the semiclassical regime, as an immediate consequence of (4.2.16),

$$\langle \boldsymbol{z}_\hbar\rangle_{|\hbar,\boldsymbol{\zeta}\rangle} \xrightarrow{\hbar\to 0} \boldsymbol{\zeta} \tag{4.2.22a}$$

$$\langle \Delta\boldsymbol{z}_\hbar\rangle_{|\hbar,\boldsymbol{\zeta}\rangle} \xrightarrow{\hbar\to 0} 0, \tag{4.2.22b}$$

which complies with (4.1.1). The search of the form of states having the properties (4.1.1) is then completed if the time evolution of the system is not taken into consideration.

4.3 Recovery of Hamiltonian mechanics from quantum mechanics

To deal with the time evolution of the system, its Hamiltonian operator must be taken into account. In fact, as our goal is to study the classical limit, we consider solely quantum systems which have a classical counterpart. This bounds the form of the Hamiltonian operator to the expression of the classical Hamiltonian phase function. More precisely, the quantum Hamiltonian operator $\hat{H}_\hbar$ must be an operator function of $\hat{\boldsymbol{z}}_\hbar$ of the form

$$\hat{H}_\hbar = \mathscr{H}(\hat{\boldsymbol{z}}_\hbar), \tag{4.3.1}$$

where $\mathscr{H}(\boldsymbol{\zeta})$ is the classical Hamiltonian phase function, which we assume to be time-independent and whose general expression reads as

$$\mathscr{H}(\boldsymbol{\zeta}) = \frac{1}{2}(\boldsymbol{\pi} - \boldsymbol{a}(\boldsymbol{\xi}))^2 + v(\boldsymbol{\xi}). \tag{4.3.2}$$

The phase functions $v(\boldsymbol{\xi})$ and $\boldsymbol{a}(\boldsymbol{\xi})$ are the scalar and vector potential fields of the classical system. By (2.1.7), the evolution operator of the quantum system is

$$\hat{U}_\hbar(t,s) = \exp\left(-i\hbar^{-1}(t-s)\hat{H}_\hbar\right). \tag{4.3.3}$$

Using the theory outlined in 2.2, we work on the problem of finding the states satisfying (4.1.1) in the Heisenberg picture. By (2.2.23c), the vector operator associated to $\hat{\boldsymbol{z}}_\hbar$ in the Heisenberg picture is

$$\hat{\boldsymbol{z}}_\hbar(t) := \hat{U}_\hbar(t,s)^+\hat{\boldsymbol{z}}_\hbar\hat{U}_\hbar(t,s). \tag{4.3.4}$$

Then, we have the following important result.

Proposition 4.3.1. *Let $\boldsymbol{\zeta}(t)$ be a generic element of $\mathbb{R}^{2\nu}$ with some kind of time dependence and with the condition that*

$$\boldsymbol{\zeta}(s) = \boldsymbol{\zeta}_0, \tag{4.3.5}$$

with $\boldsymbol{\zeta}_0 \in \mathbb{R}^{2\nu}$. Then, given a generic normalized state $|\Omega\rangle$, in the state $|\hbar,\boldsymbol{\zeta}_0\rangle$ defined as (4.2.15) with $\boldsymbol{\zeta} = \boldsymbol{\zeta}_0$, the expectation value and the uncertainty of $\hat{\boldsymbol{z}}_\hbar(t)$ read as

$$\langle \boldsymbol{z}_\hbar(t)\rangle_{|\hbar,\boldsymbol{\zeta}_0\rangle} = \boldsymbol{\zeta}(t) + \langle W_\hbar(t,s)^+\boldsymbol{z}_\hbar W_\hbar(t,s)\rangle_{|\Omega\rangle} \tag{4.3.6a}$$

$$\langle \Delta\boldsymbol{z}_\hbar(t)\rangle_{|\hbar,\boldsymbol{\zeta}_0\rangle} = \langle W_\hbar(t,s)^+(\boldsymbol{z}_\hbar + \boldsymbol{\zeta}(t) - \langle \boldsymbol{z}_\hbar(t)\rangle_{|\hbar,\boldsymbol{\zeta}_0\rangle})^2 W_\hbar(t,s)\rangle_{|\Omega\rangle}^{1/2}, \tag{4.3.6b}$$

where

$$\hat{W}_\hbar(t,s) := \hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))^+\hat{U}_\hbar(t,s)\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}_0). \tag{4.3.7}$$

Remark 4.3.1. By (2.2.22), in the Heisenberg picture,

$$|\hbar, \boldsymbol{\zeta}(t)\rangle_H = |\hbar, \boldsymbol{\zeta}_0\rangle. \tag{4.3.8}$$

For this reason, by (2.2.4b), $\langle \boldsymbol{z}_\hbar(t)\rangle_{|\hbar,\boldsymbol{\zeta}_0\rangle}$ and $\langle \Delta\boldsymbol{z}_\hbar(t)\rangle_{|\hbar,\boldsymbol{\zeta}_0\rangle}$ are nothing but

$$\langle \boldsymbol{z}_\hbar(t)\rangle_{|\hbar,\boldsymbol{\zeta}_0\rangle} = \langle \boldsymbol{z}_\hbar\rangle_{|\hbar,\boldsymbol{\zeta}(t)\rangle} \tag{4.3.9}$$
$$\langle \Delta\boldsymbol{z}_\hbar(t)\rangle_{|\hbar,\boldsymbol{\zeta}_0\rangle} = \langle \Delta\boldsymbol{z}_\hbar\rangle_{|\hbar,\boldsymbol{\zeta}(t)\rangle}. \tag{4.3.10}$$

Proof. We start by the calculation of $\langle \boldsymbol{z}_\hbar(t)\rangle_{|\hbar,\boldsymbol{\zeta}_0\rangle}$. Then, in the second step, $\langle \Delta\boldsymbol{z}_\hbar(t)\rangle_{|\hbar,\boldsymbol{\zeta}_0\rangle}$ is computed.

Step 1. By definition of $|\hbar, \boldsymbol{\zeta}_0\rangle$ and $\hat{\boldsymbol{z}}_\hbar(t)$,

$$\langle \boldsymbol{z}_\hbar(t)\rangle_{|\hbar,\boldsymbol{\zeta}_0\rangle} = \langle \Omega|\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}_0)^+\hat{U}_\hbar(t,s)^+\hat{\boldsymbol{z}}_\hbar\hat{U}_\hbar(t,s)\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}_0)|\Omega\rangle. \tag{4.3.11}$$

Using the fact that $\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))$ is an unitary operator,

$$\langle \boldsymbol{z}_\hbar(t)\rangle_{|\hbar,\boldsymbol{\zeta}_0\rangle} = \langle \Omega|\hat{W}_\hbar(t,s)^+\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))^+\hat{\boldsymbol{z}}_\hbar\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))\hat{W}_\hbar(t,s)|\Omega\rangle, \tag{4.3.12}$$

as, from (4.3.7), it is easy to find that

$$\hat{W}_\hbar(t,s)^+ = \hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}_0)^+\hat{U}_\hbar(t,s)^+\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t)). \tag{4.3.13}$$

By (4.2.11) and by the unitarity of the operators appearing in the expression,

$$\langle \boldsymbol{z}_\hbar(t)\rangle_{|\hbar,\boldsymbol{\zeta}_0\rangle} = \langle \Omega|\hat{W}_\hbar(t,s)^+\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))^+(\hat{\boldsymbol{z}}_\hbar + \boldsymbol{\zeta}(t))\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))\hat{W}_\hbar(t,s)|\Omega\rangle \tag{4.3.14}$$
$$= \boldsymbol{\zeta}(t) + \langle \Omega|\hat{W}_\hbar(t,s)^+\hat{\boldsymbol{z}}_\hbar\hat{W}_\hbar(t,s)|\Omega\rangle, \tag{4.3.15}$$

which proves the first claim.

Step 2. By definition, chosen the j-th component of $\hat{\boldsymbol{z}}_\hbar(t)$,

$$\langle \Delta z_{\hbar,j}(t)\rangle^2_{|\hbar,\boldsymbol{\zeta}_0\rangle} := \langle (z_{\hbar,j}(t) - \langle z_{\hbar,j}(t)\rangle_{|\hbar,\boldsymbol{\zeta}_0\rangle})^2\rangle_{|\hbar,\boldsymbol{\zeta}_0\rangle}. \tag{4.3.16}$$

Hence, by the fact that $\hat{U}_\hbar(t,s)$ and $\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))$ are unitary operators and using (4.3.7) and (4.2.11),

$$\langle z_{\hbar,j}(t)^2\rangle_{|\hbar,\boldsymbol{\zeta}_0\rangle} = \langle \Omega|\hat{W}_\hbar(t,s)^+\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))^+(\hat{z}_{\hbar,j} - \langle z_{\hbar,j}(t)\rangle_{|\hbar,\boldsymbol{\zeta}_0\rangle}\hat{1})^2\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))\hat{W}_\hbar(t,s)|\Omega\rangle$$
$$= \langle \Omega|\hat{W}_\hbar(t,s)^+(\hat{z}_{\hbar,j} + \zeta_j - \langle z_{\hbar,j}(t)\rangle_{|\hbar,\boldsymbol{\zeta}_0\rangle}\hat{1})^2\hat{W}_\hbar(t,s)|\Omega\rangle, \tag{4.3.17}$$

which, switching to the vector notation, brings to the second claim. □

We notice that the $\hat{W}_\hbar(t,s)$ operator is defined up to phase factors, as a phase factor can be introduced without compromising the previous result:

$$\hat{W}_\hbar(t,s) \to \hat{W}_\hbar(t,s)\exp(i\varphi), \tag{4.3.18}$$

for some $\varphi \in \mathbb{R}$. If $\mathscr{H}(\boldsymbol{\zeta})$ is the Hamiltonian function of the classical counterpart of a quantum system and if $\boldsymbol{\zeta}(t)$ is a solution of the canonical equations for $\mathscr{H}(\boldsymbol{\zeta})$ with initial condition

$$\boldsymbol{\zeta}(s) = \boldsymbol{\zeta}_0, \tag{4.3.19}$$

there is a certain choice of φ which brings interesting results, as it will be shown.

Proposition 4.3.2. *Let $\mathscr{H}(\boldsymbol{\zeta})$ and $\boldsymbol{\zeta}(t)$ be as above. If the $\hat{W}_\hbar(t,s)$ operator is redefined as*

$$\hat{W}_\hbar(t,s) := \hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))^+\hat{U}_\hbar(t,s)\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}_0)\exp(i\varphi_\hbar(t,s)), \tag{4.3.20}$$

with the choice

$$\varphi_\hbar(t,s) = \hbar^{-1}\int_s^t dr\left(\mathscr{H}(\boldsymbol{\zeta}(r)) - \frac{1}{2}\nabla_{\boldsymbol{\zeta}}\mathscr{H}(\boldsymbol{\zeta}(r))\cdot\boldsymbol{\zeta}(r)\right), \tag{4.3.21}$$

it obeys the following equations with initial condition $\hat{W}_\hbar(s,s) = \hat{1}$:

$$i\hbar\frac{\partial}{\partial t}\hat{W}_\hbar(t,s) = \hat{K}_\hbar(t)\hat{W}_\hbar(t,s) \tag{4.3.22a}$$

$$i\hbar\frac{\partial}{\partial s}\hat{W}_\hbar(t,s) = -\hat{W}_\hbar(t,s)\hat{K}_\hbar(s), \tag{4.3.22b}$$

with

$$\hat{K}_\hbar(r) = \mathscr{H}(\hat{\boldsymbol{z}}_\hbar + \boldsymbol{\zeta}(r)\hat{1}) - \mathscr{H}(\boldsymbol{\zeta}(r)\hat{1}) - \nabla_{\boldsymbol{\zeta}}\mathscr{H}(\boldsymbol{\zeta}(r))\cdot\hat{\boldsymbol{z}}_\hbar. \tag{4.3.23}$$

Proof. The aim of this proof is to formally compute the derivatives appearing on the left hand of (4.3.22). We start by computing the partial derivative of $\hat{W}_\hbar(t,s)$ with respect to t. By (4.3.7) and the Leibniz product rule,

$$\begin{aligned}\frac{\partial}{\partial t}\hat{W}_\hbar(t,s) &= \frac{\partial}{\partial t}\left(\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))^+\hat{U}_\hbar(t,s)\exp(i\varphi_\hbar(t,s))\right)\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}_0)\\ &= \hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))^+\left(\frac{\partial}{\partial t}\hat{U}_\hbar(t,s)\right)\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}_0)\exp(i\varphi_\hbar(t,s))\\ &\quad+\left(\frac{\partial}{\partial t}\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))^+\right)\hat{U}_\hbar(t,s)\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}_0)\exp(i\varphi_\hbar(t,s))\\ &\quad+\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))^+\hat{U}_\hbar(t,s)\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}_0)\left(\frac{\partial}{\partial t}\exp(i\varphi_\hbar(t,s))\right).\end{aligned} \tag{4.3.24}$$

The derivative appearing in the first term can be easily computed from (4.3.3):

$$\frac{\partial}{\partial t}\hat{U}_\hbar(t,s) = -i\hbar^{-1}\hat{H}_\hbar\hat{U}_\hbar(t,s). \tag{4.3.25}$$

Then, using the fact that $\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))$ is an unitary operator and using (4.3.1), (4.3.7) and (4.2.11), the first term can be written as

$$-i\hbar^{-1}\mathscr{H}(\hat{\boldsymbol{z}}_\hbar + \boldsymbol{\zeta}(t)\hat{1})\hat{W}_\hbar(t,s). \tag{4.3.26}$$

Through the Baker-Campbell-Hausdorff decomposition (1.2.8), the derivative appearing in the second term can be computed from (4.2.1), (4.2.3) and by using the Leibniz product rule:

$$\begin{aligned}\frac{\partial}{\partial t}\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))^+ &= \frac{\partial}{\partial t}\exp\left(-i\hbar^{-1/2}\left(\boldsymbol{\pi}(t)\cdot\hat{\boldsymbol{q}} - \boldsymbol{\xi}(t)\cdot\hat{\boldsymbol{p}}\right)\right)\\ &= \frac{\partial}{\partial t}\exp\left((2\hbar i)^{-1}\boldsymbol{\pi}(t)\cdot\boldsymbol{\xi}(t)\right)\exp\left(-i\hbar^{-1/2}\boldsymbol{\pi}(t)\cdot\hat{\boldsymbol{q}}\right)\exp\left(i\hbar^{-1/2}\boldsymbol{\xi}(t)\cdot\hat{\boldsymbol{p}}\right)\\ &= \left(\frac{\partial}{\partial t}\exp\left((2\hbar i)^{-1}\boldsymbol{\pi}(t)\cdot\boldsymbol{\xi}(t)\right)\right)\exp\left(-i\hbar^{-1/2}\boldsymbol{\pi}(t)\cdot\hat{\boldsymbol{q}}\right)\exp\left(i\hbar^{-1/2}\boldsymbol{\xi}(t)\cdot\hat{\boldsymbol{p}}\right)\\ &\quad+\exp\left((2\hbar i)^{-1}\boldsymbol{\pi}(t)\cdot\boldsymbol{\xi}(t)\right)\left(\frac{\partial}{\partial t}\exp\left(-i\hbar^{-1/2}\boldsymbol{\pi}(t)\cdot\hat{\boldsymbol{q}}\right)\right)\exp\left(i\hbar^{-1/2}\boldsymbol{\xi}(t)\cdot\hat{\boldsymbol{p}}\right)\\ &\quad+\exp\left((2\hbar i)^{-1}\boldsymbol{\pi}(t)\cdot\boldsymbol{\xi}(t)\right)\exp\left(-i\hbar^{-1/2}\boldsymbol{\pi}(t)\cdot\hat{\boldsymbol{q}}\right)\left(\frac{\partial}{\partial t}\exp\left(i\hbar^{-1/2}\boldsymbol{\xi}(t)\cdot\hat{\boldsymbol{p}}\right)\right),\end{aligned} \tag{4.3.27}$$

where it was calculated that

$$-2^{-1}[-i\hbar^{-1/2}\boldsymbol{\pi}(t)\cdot\hat{\boldsymbol{q}}, i\hbar^{-1/2}\boldsymbol{\xi}(t)\cdot\hat{\boldsymbol{p}}] = (2\hbar i)^{-1}\boldsymbol{\pi}(t)\cdot\boldsymbol{\xi}(t)\hat{1}. \tag{4.3.28}$$

It follows, using (4.2.14) and the fact that $\boldsymbol{\zeta}(t)$ satisfies the canonical equations, that

$$\begin{aligned}\frac{\partial}{\partial t}\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))^{+} = (2\hbar i)^{-1}&\left(-\nabla_{\boldsymbol{\xi}}\mathscr{H}(\boldsymbol{\zeta}(t))\cdot\boldsymbol{\xi}(t) + \nabla_{\boldsymbol{\pi}}\mathscr{H}(\boldsymbol{\zeta}(t))\cdot\boldsymbol{\pi}(t)\right)\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))^{+}\\ &+ i\hbar^{-1}\nabla_{\boldsymbol{\xi}}\mathscr{H}(\boldsymbol{\zeta}(t))\cdot\hat{\boldsymbol{q}}_{\hbar}\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))^{+}\\ &+ i\hbar^{-1}\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))^{+}\nabla_{\boldsymbol{\pi}}\mathscr{H}(\boldsymbol{\zeta}(t))\cdot\hat{\boldsymbol{p}}_{\hbar}.\end{aligned} \tag{4.3.29}$$

Then, by (4.3.20), using in (4.3.29) the fact that $\hat{C}(\hbar^{-1/2}\boldsymbol{\zeta}(t))$ is an unitary operator and the translation property (4.2.11), it turns out that the second term of (4.3.24) can be written as

$$i\hbar^{-1}\nabla_{\boldsymbol{\zeta}}\mathscr{H}(\boldsymbol{\zeta}(t))\cdot\left(\hat{\boldsymbol{z}}_{\hbar} + \frac{1}{2}\boldsymbol{\zeta}(t)\hat{1}\right)\hat{W}_{\hbar}(t,s). \tag{4.3.30}$$

The derivative appearing in the third term is immediately computed from (4.3.21):

$$\frac{\partial}{\partial t}\exp(i\varphi_{\hbar}(t,s)) = i\hbar^{-1}\left(\mathscr{H}(\boldsymbol{\zeta}(t)) - \frac{1}{2}\nabla_{\boldsymbol{\zeta}}\mathscr{H}(\boldsymbol{\zeta}(t))\cdot\boldsymbol{\zeta}(t)\right)\exp(i\varphi_{\hbar}(t,s)). \tag{4.3.31}$$

Then, using (4.3.7), the third term can be written as

$$i\hbar^{-1}\left(\mathscr{H}(\boldsymbol{\zeta}(t)) - \frac{1}{2}\nabla_{\boldsymbol{\zeta}}\mathscr{H}(\boldsymbol{\zeta}(t))\cdot\boldsymbol{\zeta}(t)\right)\hat{W}_{\hbar}(t,s). \tag{4.3.32}$$

Assembling the three terms (4.3.26), (4.3.30) and (4.3.32), we get

$$\frac{\partial}{\partial t}\hat{W}_{\hbar}(t,s) = i\hbar^{-1}\left(-\mathscr{H}(\hat{\boldsymbol{z}}_{\hbar} + \boldsymbol{\zeta}(t)\hat{1}) + \nabla_{\boldsymbol{\zeta}}\mathscr{H}(\boldsymbol{\zeta}(t))\cdot\hat{\boldsymbol{z}}_{\hbar} + \mathscr{H}(\boldsymbol{\zeta}(t))\hat{1}\right)\hat{W}_{\hbar}(t,s), \tag{4.3.33}$$

thus proving (4.3.22a). The computation of the derivative of $\hat{W}_{\hbar}(t,s)$ with respect to s proceeds in a totally analogous fashion. □

This result allows one to express $\hat{W}_{\hbar}(t,s)$ in terms of $\hat{K}_{\hbar}(t)$, by solving formally the pair of differential equations with the chosen initial condition. This will be helpful to evaluate the semiclassical limit $\hbar \to 0$. In fact, one has the following result.

Proposition 4.3.3. *The couple of differential equations* (4.3.22) *with the initial condition* $\hat{W}_{\hbar}(s,s) = \hat{1}$ *is formally solved by*

$$\hat{W}_{\hbar}(t,s) = \mathrm{Texp}\left(-i\hbar^{-1}\int_{s}^{t} dr\,\hat{K}_{\hbar}(r)\right), \tag{4.3.34}$$

where $\mathrm{Texp}(\,\cdot\,)$ *denotes the time-ordered exponential.*

Proof. We start by inspecting (4.3.22a) with the initial condition $\hat{W}_\hbar(s,s) = \hat{1}$. This differential equation is equivalent to the integral equation

$$\hat{W}_\hbar(t,s) = \hat{1} - i\hbar^{-1}\int_s^t dr \hat{K}_\hbar(r)\hat{W}_\hbar(r,s). \tag{4.3.35}$$

Indeed, from (4.3.22a),

$$\begin{aligned}\hat{W}_\hbar(t,s) - \hat{1} &= \hat{W}_\hbar(t,s) - \hat{W}_\hbar(s,s)\\ &= \int_s^t dr \frac{\partial}{\partial r}\hat{W}_\hbar(r,s) = -i\hbar^{-1}\int_s^t dr \hat{K}_\hbar(r)\hat{W}_\hbar(r,s).\end{aligned} \tag{4.3.36}$$

The integral equation (4.3.35) is solved formally by iteration. This brings to the so-called Dyson series in $\hat{K}_\hbar(u)$ for $\hat{W}_\hbar(t,s)$:

$$\hat{W}_\hbar(t,s) = \sum_{n=0}^{\infty}(-i\hbar^{-1})^n\int_s^t dr_1\int_s^{r_1} dr_2\cdots\int_s^{r_{n-1}} dr_n \hat{K}_\hbar(r_1)\hat{K}_\hbar(r_2)\cdots\hat{K}_\hbar(r_n). \tag{4.3.37}$$

By definition of the time-ordered exponential for a generic operator $\hat{X}$,

$$\mathrm{Texp}\left(\int_a^b du \hat{X}(u)\right) := \sum_{n=0}^{\infty}\int_a^b db_1\int_a^{b_1} db2\cdots\int_a^{b_{n-1}} db_n \hat{X}(b_1)\hat{X}(b_2)\cdots\hat{X}(b_n), \tag{4.3.38}$$

the claim is shown. The proof for (4.3.22b) runs in a totally analogous fashion. □

This is enough to start inspecting the properties of the states $|\hbar,\boldsymbol{\zeta}(t)\rangle$ as $\hbar\to 0$. Indeed, assuming that the potential fields which appear in (4.3.2) are sufficiently smooth, it can be shown that, in the semiclassical limit, the expected value and the uncertainty of $\hat{\boldsymbol{z}}_\hbar$ in these states display the wanted behaviour (4.1.1). In particular, the expected value of $\hat{\boldsymbol{z}}_\hbar$ converges to the classical phase orbit which solves the Hamilton equations for $\mathscr{H}(\boldsymbol{\zeta})$, while the uncertainty of $\hat{\boldsymbol{z}}_\hbar$ approaches 0. These facts are summarised in the following result.

Theorem 4.3.1. *Given a quantum system whose classical counterpart is characterized by the Hamiltonian function* $\mathscr{H}(\boldsymbol{\zeta})$,

$$\mathscr{H}(\boldsymbol{\zeta}) = \frac{1}{2}(\boldsymbol{\pi} - \boldsymbol{a}(\boldsymbol{\xi}))^2 + v(\boldsymbol{\xi}), \tag{4.3.39}$$

defined on the classical phase space $\mathcal{F}$, *with* $\boldsymbol{\zeta} = (\boldsymbol{\xi},\boldsymbol{\pi})\in\mathcal{F}$ *and where* $\boldsymbol{a}(\boldsymbol{\xi})\in C^\infty(\mathcal{F},\mathbb{R}^n)$ *and* $v(\boldsymbol{\xi})\in C^\infty(\mathcal{F},\mathbb{R})$. *Let* $\boldsymbol{\zeta}(t)$ *be a solution of the Hamilton equations with the initial condition*

$$\boldsymbol{\zeta}(s) = \boldsymbol{\zeta}_0, \tag{4.3.40}$$

where $\boldsymbol{\zeta}_0\in\mathbb{R}^{2\nu}$. *Then,*

$$\langle\boldsymbol{z}_\hbar\rangle_{|\hbar,\boldsymbol{\zeta}(t)\rangle} \xrightarrow{\hbar\to 0} \boldsymbol{\zeta}(t) \tag{4.3.41a}$$

$$\langle\Delta\boldsymbol{z}_\hbar\rangle_{|\hbar,\boldsymbol{\zeta}(t)\rangle} \xrightarrow{\hbar\to 0} 0. \tag{4.3.41b}$$

Remark 4.3.2. This result only concerns the semiclassical limit of the expectation values and the uncertainties of $\boldsymbol{z}_\hbar$ computed in the states $|\hbar,\boldsymbol{\zeta}(t)\rangle$. It does not pertain to the limit $\hbar\to 0$ for the states $|\hbar,\boldsymbol{\zeta}(t)\rangle$ themselves. Actually, from their definition (4.2.15), it is clear that these states do not have a limit for $\hbar\to 0$. This is closely related to the fact that $\hbar\to 0$ is a formal limit: $\hbar$ may be very small compared to the characteristic action scale of the system, but it is not zero.

Sketch of the proof. Only a swift overview of the proof will be provided. For a deeper insight, see [3]. We start by defining the operator $\hat{H}_2(r)$ as

$$\hat{H}_2(r) := \frac{1}{2} \left(\nabla_{\boldsymbol{\zeta}} \left(\nabla_{\boldsymbol{\zeta}} \mathscr{H}(\boldsymbol{\zeta}(r)) \cdot \hat{\boldsymbol{z}} \right) \right) \cdot \hat{\boldsymbol{z}}. \tag{4.3.42}$$

Then,

$$\hbar^{-1} \hat{K}_\hbar(r) \xrightarrow{\hbar \to 0} \hat{H}_2(r). \tag{4.3.43}$$

Indeed, by the smoothness hypothesis on the potential fields, one has that $\mathscr{H}(\boldsymbol{\zeta}) \in C^\infty(\mathcal{F}, \mathbb{R})$. Then, by the Taylor formula (see [2]),

$$\mathscr{H}(\boldsymbol{\zeta} + \boldsymbol{\epsilon}) = \mathscr{H}(\boldsymbol{\zeta}) + \nabla_{\boldsymbol{\zeta}} \mathscr{H}(\boldsymbol{\zeta}) \cdot \boldsymbol{\epsilon} + \frac{1}{2} \left(\nabla_{\boldsymbol{\zeta}} \left(\nabla_{\boldsymbol{\zeta}} \mathscr{H}(\boldsymbol{\zeta}(r)) \cdot \boldsymbol{\epsilon} \right) \right) \cdot \boldsymbol{\epsilon}, \tag{4.3.44}$$

for $\boldsymbol{\epsilon} \in \mathcal{F}$ such that $|\boldsymbol{\zeta} - \boldsymbol{\epsilon}| \ll 1$. Since functions of operators preserve the relationships existing between the functions and since this property must hold as well for $|\boldsymbol{\zeta} - \boldsymbol{\epsilon}| \ll 1$, (4.3.43) is shown. Now, we define $\hat{U}_2(t, s)$ as

$$\hat{U}_2(t, s) := \mathrm{Texp} \left(-i \int_s^t dr \hat{H}_2(r) \right). \tag{4.3.45}$$

By (4.3.34) and (4.3.43),

$$\hat{W}_\hbar(t, s) \xrightarrow{\hbar \to 0} \hat{U}_2(t, s). \tag{4.3.46}$$

In other terms, by (4.3.42), this implies that

$$[\hat{\boldsymbol{z}}_\hbar, \hat{W}_\hbar(t, s)] \xrightarrow{\hbar \to 0} 0, \tag{4.3.47}$$

thus proving the claim. □

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