Alma Mater Studiorum · University of Bologna

School of Science
Department of Physics and Astronomy
Master Degree in Physics

O(d,d) covariant cosmology in isotropic and anisotropic spaces

Supervisor:
Prof. Francisco Gil Pedro

Submitted by:
Jorge Buces Sáez

Academic Year 2023/2024

Abstract

In this thesis we analyse the O(d,d) covariant cosmological theory presented by Hohm and Zwiebach motivated by the quest for de Sitter solutions in the Einstein 4-dimensional frame. For this purpose, we develop the theory for an anisotropic metric and search for the fixed points of the dynamical system in the string frame, later translating these stable solutions into the Einstein frame, we find Minkowski and exact de Sitter solutions as well as another family of solutions that can be constrained to have an accelerated expansion and a positive Hubble parameter asymptotically decreasing towards zero. These constraints establish possible intervals for our physical degrees of freedom. Finally, we perform the compactification of the action with all the α' corrections, encoded in the F function, for our anisotropic theory. Therefore, we have the possibility to examine the shape of the F function with all its corrections in the Einstein frame. We obtain an Einstein-Hilbert action minimally coupled to two scalars at order zero in α', and a perturbative potential given by F in this new frame which no longer depends only on the Hubble parameter, but mixes all the degrees of freedom of our action.

Contents

1 Introduction

The study of our firmament has always been a fundamental element in our civilisation. This study has evolved from an agricultural function to a more profound and curious one, with the aim of understanding the universe in which we live. We have always enlarged the horizon of our view of the world, until it reached the study of the universe as a whole. Since 1916, with the formulation of Einstein's theory of General Relativity (GR), many physicists such as Einstein himself (1917), de Sitter (1917) or Friedmann (1922), have tried to find a cosmological model. Initially, the dynamical state of the universe was not very clear. The first idea was that it was static, but it was not until 1929, when Edwin Hubble published his work on the recessional velocity of galaxies that increased with distance, that the paradigm changed. After this publication, the expansion of the universe became an integral part of the standard cosmological model. Another major shift of our understanding of the universe relevant for this thesis came in 1998. The study of type Ia supernovae pointed towards the fact that not only is the universe expanding, but the expansion is accelerating. This state of affairs can be accommodated within GR by the inclusion of a minute and positive cosmological constant.

Since then, one of the main goals of cosmology has been to search for solutions that give rise to an accelerating expanding universe. This search is closely related to, and is in fact part of, the so-called cosmological constant problem. Many theories have attempted to solve it, so far without complete success. Given that the problem involves gravity and quantum physics, it is natural to address it within a quantum theory of gravity like string theory. In this context it is conceivable that the O(D,D) covariance of the theory could play a role both in constraining the space of solutions and in going beyond the perturbative regime in α'. It appears that string theory does not possess a de Sitter solution to first order in α'. The problem is the difficulty in performing a systematic treatment of all these α' corrections. Recently, in 2019, Hohm and Zwiebach [8] managed to classify all α' corrections in a very simple way, obtaining an O(d,d) invariant action by considering a cosmological ansatz (only time dependent fields) with the metric tensor $g_{\mu\nu}$, the $b_{\mu\nu}$ field and the dilaton ϕ. When considering a vanishing $b_{\mu\nu}$ field and a Friedmann-Lemaître-Robertson-Walker (FLRW) metric in D dimensions, the action and the equations of motion (EOM) can be written in terms of the dilaton and a single function $F(H)$, where H is the usual Hubble factor, which encodes all the α' corrections. With this formalism de Sitter solutions can be found [9], and can be shown to be stable in the string frame. However, it is important to recall that these results are expressed in the D-dimensional string frame, and in order to make contact with observations we should go to the Einstein frame.

In this work we will first review the cosmological constant problem in section 2, given that it is the physics to which we want to apply the Hohm and Zwiebach formalism.

We continue by presenting the O(d,d) covariant theory of Hohm and Zwiebach [8] to all orders in α' in section 3, where we also introduce matter as done by Bernardo et al. in [2]. Subsequently in section 4, we break the O(d,d) covariance by developing the action for an anisotropic metric and vanishing $b_{\mu\nu}$ field. The motivations for this is two fold. On the one hand we want to apply the general formalism to less symmetric setups, on the other hand we would like to see what the O(d,d) formalism can teach us about compactification. Encouraged by the search for de Sitter solutions, we then study the fixed points (constant Hubble parameter in the string frame) of isotropic and anisotropic systems both with matter and in vacuum in section 5. In this section we are also interested in the aspect of these constant solutions in the Einstein frame. We continue in section 6 by looking at the structure of the function determining the structure of the theory, $F(H)$ and its possible interpretations. In order to have a clearer view of this function which encodes all the α' corrections, we perform a compactification to the 4-dimensional Einstein frame. We finish with a discussion in section 7.

2 The cosmological constant problem

In 1917, following the development of his General Relativity theory (1915–1916), Einstein attempted to use his equations to obtain a cosmological model by postulating an isotropic and homogeneous universe. He was certain that the Universe was static, but in order to obtain such a solution he had to introduce Λ, a free parameter in the field equations, which was not fixed by the theory:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1}$$

As explained in [5], such a constant introduces both a length and a timescale:

$$r_\Lambda = c t_\Lambda = \sqrt{3/|\Lambda|}. \tag{2}$$

Thus, a cosmological constant affects space time at scales larger than r_Λ and t_Λ. In order to understand the role of Λ, we now want to compare such a term with the vacuum energy density term issued from the energy-momentum tensor. By general covariance, any free falling inertial observer would see the same vacuum. Accordingly, this also applies to the vacuum energy-momentum tensor [16], that takes the following form:

$$T^V_{\mu\nu} = -\rho_V g_{\mu\nu}. \tag{3}$$

A free-falling observer can locally take the metric to be Minkowski. Comparing a general isotropic and homogeneous energy-momentum tensor $T_{\mu\nu} = \text{diag}(\rho, p, p, p)$ where ρ is the energy density of the fluid, and p is the pressure, with Eq. (3) we get the equation of state for the vacuum energy: $p_V = -\rho_V$. As we can observe, it has a negative pressure. By separating this vacuum contribution from the rest of the energy-momentum tensor we realise in Eq. (1) that we can either reabsorb the vacuum energy term into the cosmological constant term, or vice-versa, i.e.:

$$\Lambda_{\text{eff}} = \Lambda + 8\pi G \rho_V \quad , \quad \rho_\Lambda = \rho_V + \frac{\Lambda}{8\pi G}. \tag{4}$$

Therefore, anything that contributes to the vacuum energy density acts like a cosmological constant.

We can now take a look at some bounds on the total effective values. There was no need for precision measurement methods to see that the total effective cosmological constant has a very tiny value. We follow the approach in [5], where we start by supposing an empty universe. For $\Lambda > 0$, the only isotropic solution to Einstein's equations is de Sitter space, with a cosmological horizon r_Λ. This is the largest observable distance scale, and the presence of matter only shrinks this horizon. Thus $r_\Lambda \geq r$, being r the maximum scale distance up to which we can observe. On the contrary, if we had $\Lambda < 0$ the universe would recollapse on a timescale t_Λ. This means that $t_\Lambda > t$ for t the maximum timescale observed, i.e. the age of the universe. Using Eq. (2) we can get an upper and a lower bound on our cosmological constant:

$$-3t^{-2} \leq \Lambda_{\text{eff}} \leq 3r^{-2}, \tag{5}$$

where we can approximate $r > 10^{60} M_{Pl}^{-1} \approx 10^{25}\,\text{m}$, and $t > 10^{60} M_{Pl}^{-1} \approx 10^{16}\,\text{s}$, where we introduced the Planck mass $M_{Pl} = 1.22 \cdot 10^{19}$ GeV, we get:

$$|\Lambda_{\text{eff}}| \leq 3 \cdot 10^{-120} M_{Pl}^2 \quad , \quad |\rho_\Lambda| \leq 10^{-121} M_{Pl}^4. \tag{6}$$

This bound can also be imposed by anthropic considerations as was done by Weinberg [19]. There are various formulations of the anthropic principle, from the very strong, bordering on the religious, to the weaker, bordering on the trivial. He uses a moderate version known as the weak anthropic principle. Based on our existence, he claims that the universe has to be old enough for some stars to have had the time to finish their main sequence era, and then produce heavy elements necessary for our existence. But, at the same time, it must be young enough for stars to still produce energy. For a large cosmological constant Λ_{eff}, the universe enters very rapidly in an expanding de Sitter phase, preventing the formation of gravitationally bound structures like galaxies. We suppose that at a redshift of around $z \geq 4$ gravitational condensation has already begun in our universe. At that time, the matter energy density would be $\rho_M = (1+z)^3 \rho_{M_0} = 125 \rho_{M_0}$ being ρ_{M_0} the current matter energy density. The argument is that a cosmological constant would prevent gravitational condensation if it was bigger than $100\rho_{M_0}$. Moreover, if it was much smaller, the formation of structures would have taken place earlier. So if the anthropic principle explains the smallness of the cosmological constant, we would expect it to be slightly bigger than the matter energy density nowadays.

We should look at what information quantum mechanical considerations bring us about vacuum energy, since we must recall that one of the contributions to the effective cosmological constant is given by the vacuum energy density. Following [15], we now compute the zero point energies of standard model particles. Quantum field theory tells us that the vacuum has an energy, and we compute it by loop diagrams for each particle species. By using dimensional regularisation, for a canonical scalar ϕ of mass m we get:

$$\begin{aligned} \bigcirc &\approx -\frac{1}{2}\int d^4x \int \frac{d^4k}{(2\pi)^4} \log(k^2 + m^2) \\ &\approx -\frac{m^4}{(8\pi)^2}\left[-\frac{2}{\epsilon} + \log\left(\frac{m^2}{4\pi\mu^2}\right) + \gamma - \frac{3}{2}\right]\int d^4x \\ &\subset -\rho_{\text{V}} \int d^4x, \end{aligned} \tag{7}$$

with γ being the Euler-Mascharoni constant, and μ the mass scale introduced by dimensional regularisation. In order to eliminate divergences we introduce counterterms. Hence, for the total zero-point energy we expect:

$$\rho_{\rm V} \approx \sum_{\rm particles} \mathcal{O}(1) m^4_{\rm particle}. \tag{8}$$

But in the absence of gravity, the zero-point contributions do not affect dynamics, whereas when coupled to classical gravity they do. The equivalence principle states that:

$$\rho_{\rm V} \int d^4x \to \rho_{\rm V} \int d^4x \sqrt{-g}, \tag{9}$$

being g the determinant of the metric. In terms of Feynmann diagrams, this coupling of the determinant of the metric corresponds to attaching external graviton legs to our one-loop diagram. By using our effective description of quantum matter coupled to classical gravity, the vacuum curvature is given by $H_V^2 = \frac{\rho_V}{3M_{\rm Pl}^2}$. For a positive vacuum energy density we get a late-time accelerated de Sitter expansion compatible with current observations for $H_V^2 \leq H_0^2 \approx ({\rm meV})^4/M_{\rm Pl}^2$. The problem is that, as first noted by Dirac, if we take an ultraconservative approach and we only consider the contribution of the electron, giving $\rho_{\rm V} \approx ({\rm MeV})^4$, the cosmological horizon r_H would lie at a distance inferior to the Earth-Moon distance:

$$r_H \leq \frac{1}{H_V} = M_{\rm Pl}\sqrt{\frac{3}{({\rm MeV})^4}} \approx 10^6 {\rm km}. \tag{10}$$

In brief, considering the contributions from the other standard model particles, we would get a vacuum energy density of the order of $\rho_V \approx ({\rm TeV})^4 = 10^{60}({\rm meV})^4$, whereas observationally we have $\rho_\Lambda \approx ({\rm meV})^4$. We then need a fine-tuning of 10^{-60} in the parameter Λ.

Considering the huge gap between the observational bound on the effective cosmological constant ρ_Λ, and the result for the vacuum energy density ρ_V from QFT coupled to GR, we can ask ourselves, is this the cosmological constant problem? Why is the value of the cosmological constant so tiny? Or why do we need such a fine-tuned Λ parameter? As emphasised by Padilla in [15] this is only the beginning, since we can afford a fine-tuned free parameter to match observations. The problem arises when considering the description of matter to two-loops, the cancellation imposed at first order is completely spoiled! Consequently, we should constantly fine-tune our parameter depending on the cut-off of our effective theory.

Quite a lot of arguments and theories have been created in order to try to solve this radiative instability of the cosmological constant. To name a few of them, we have t'Hooft symmetry arguments, supersymmetry, long distance modifications of gravity such as the sequester model[15], anthropic considerations [19], or string theory as reviewed in [5], among many others.

3 O(d,d) invariant cosmology

Classical string theory is one of the potential solutions to the cosmological constant problem. The theory offers duality invariance, the symmetry that sends the scale factor $a(t)$ to $1/a(t)$. Moreover, it also offers infinitely many higher-derivative α' corrections, which could explain some features of early universe cosmology.

Recently Hohm and Zwiebach [8] have managed to classify all the α' corrections, by considering a $D = d + 1$ dimensional duality covariant theory. This was motivated by the work of Meissner and Veneziano [13] and Sen [17], who managed to put the duality covariant first order string action in an O(d,d) covariant way. Hohm and Zwiebach start by considering the more general O(d,d) covariant action with all order α' corrections. Afterwards, they suppose some specific ansatz on the metric and the b field, and all fields to be only time dependent, relevant for cosmology. Along with this, they manage to classify all the higher-derivative corrections to all orders in α', which are compatible with O(d,d) invariance. The space of our theory is then O(d,d) covariant theories, and string theory should occupy some point of this space. This O(d,d) symmetry contains the scale factor duality.

3.1 Vacuum

3.1.1 Two-derivative theory

The two derivative theory, without α' corrections, has the presence of the following degrees of freedom: the D-dimensional metric $g_{\mu\nu}$ and its associated Ricci scalar R, the b-field $b_{\mu\nu}$ and its 3-form $H_{\mu\nu\rho} = 3\partial_{[\mu}b_{\nu\rho]}$, and the dilaton ϕ. The string action at first order then reads:

$$I_0 = \int d^D x\sqrt{-g}e^{-2\phi}(R + 4(\partial\phi)^2 - \frac{1}{12}H^2). \tag{11}$$

By restricting the fields to the form:

$$g_{\mu\nu} = \begin{pmatrix} -n^2(t) & 0 \\ 0 & g_{ij}(t) \end{pmatrix} \quad , \quad b_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & b_{ij}(t) \end{pmatrix} \quad , \quad \phi = \phi(t), \tag{12}$$

the action takes the O(d,d) invariant form [13]:

$$I_0 = \int dt\, n e^{-\Phi} \left(-(\mathcal{D}\Phi)^2 - \frac{1}{8}tr((\mathcal{D}S)^2) \right), \tag{13}$$

where Φ is the O(d,d) invariant dilaton defined as $e^{-\Phi} = \sqrt{det\, g_{ij}}e^{-2\phi}$, $\mathcal{D} = \frac{1}{n(t)}\frac{\partial}{\partial t}$ is the covariant time derivative, and the O(d,d) valued matrix S is given by:

$$S = \eta\mathcal{H} = \begin{pmatrix} bg^{-1} & g - bg^{-1}b \\ g^{-1} & -g^{-1}b \end{pmatrix} \quad , \quad \eta = \begin{pmatrix} 0 & \mathbb{1}_d \\ \mathbb{1}_d & 0 \end{pmatrix}, \tag{14}$$

where η is an O(d,d) metric, invariant under O(d,d) transformations. If $\mathcal{H}$ transforms as a tensor under $\Omega \in O(d,d)$, then:

$$\eta \to \Omega^T \eta \Omega = \eta \quad , \quad \mathcal{H} \to \Omega^T \mathcal{H} \Omega. \tag{15}$$

This implies the following transformation law for S:

$$S \to \Omega^{-1} S \Omega. \tag{16}$$

This action is invariant under diffeomorphisms. We have the lapse function that transforms as a density under time reparametrisations $t \to t - \lambda(t)$ as $\delta_\lambda n = \partial_t(\lambda n)$. On the other hand, the b field, the shifted dilaton Φ and the metric g transform as scalars $\delta_\lambda g = \lambda \partial_t g$.

Firstly, we get the EOM at first order. By varying the action with respect to the dilaton we get:

$$\begin{aligned} \delta_\Phi I_0 &= \int dt n e^{-\Phi} \delta\Phi E_\Phi, \\ E_\Phi &= 2\mathcal{D}^2\Phi - (\mathcal{D}\Phi)^2 + \frac{1}{8} tr((\mathcal{D}S)^2). \end{aligned} \tag{17}$$

For the lapse function we write the variation as:

$$\begin{aligned} \delta_n I_0 &= \int dt n e^{-\Phi} \frac{\delta n}{n} E_n, \\ E_n &= (\mathcal{D}\Phi)^2 - \frac{1}{8} tr((\mathcal{D}S)^2). \end{aligned} \tag{18}$$

The form in which we write the variation $\delta n/n$ is due to the fact that δn transforms as a density under time reparametrisations, so that the ratio transforms as a scalar. Defining E_n this way, it will also be a scalar.
Now for the S matrix we must note that it is a constrained field, i.e. $S^2 = \mathbb{1}$, so that $\delta S = -S(\delta S)S$ is a constrained variation. By varying the action with respect to S we obtain:

$$\begin{aligned} \delta_S I_0 &= \int dt n e^{-\Phi} tr(\delta S F_S), \\ F_S &= \frac{1}{4}(\mathcal{D}^2 S - \mathcal{D}\Phi \mathcal{D}S). \end{aligned} \tag{19}$$

Since δS is a constrained variation, the vanishing of F_S does not give the correct EOM [8]. We can rewrite the variation δS in terms of an unconstrained variation δK:

$$\delta S = \frac{1}{2}(\delta K - S \delta K S), \tag{20}$$

which introduced into Eq. (19) gives us:

$$\begin{aligned} \delta_S I_0 =& \int dtne^{-\Phi}tr(\delta K E_S), \\ E_S =& \frac{1}{2}(F_S - SF_S S), \end{aligned} \tag{21}$$

where now the vanishing of E_S gives the correct EOM. We can also note that $tr(\delta S E_S) = tr(\delta S F_S)$, so that the variation of the action can now be expressed in terms of the off-shell EOM. We simplify the equation for E_S by using the anticonmutativity property of S and $\mathcal{D}S$ since $\mathcal{D}(S^2) = (\mathcal{D}S)S + S\mathcal{D}S = 0$. By deriving again this equation we obtain the relation:

$$S(\mathcal{D}^2 S)S = -\mathcal{D}^2 S - 2S(\mathcal{D})^2. \tag{22}$$

By introducing both equations and Eq. (19) into (21) we get the EOM given by the variation of S:

$$\begin{aligned} \delta_S I_0 =& \int dtne^{-\Phi}tr(\delta S E_S), \\ E_S =& \frac{1}{4}(\mathcal{D}^2 S + S(\mathcal{D}S)^2 - \mathcal{D}\Phi\mathcal{D}S). \end{aligned} \tag{23}$$

3.1.2 O(d,d) invariant α' corrections

We must now highlight the work of Hohm and Zwiebach [8]. By introducing a cosmological ansatz for the fields, and by using field redefinitions with the EOM at first order, they recursively get a very simplified form of the action. We first suppose at each order in α' the most general action, where we have $2k+2$ time derivatives by dimensional analysis. We can write the total action as:

$$I = \sum_{k=0}^{\infty} I_k \quad , \quad I_k = \sum_{p=0}^{k} I_{k,p} \quad , \quad I_{k,p} = \alpha'^k \int dtne^{-\Phi} X_p(\{\mathcal{D}\Phi\}, \{S\}), \tag{24}$$

where the brackets mean that the function X_p depends on the values in brackets and higher covariant derivative terms. According to the dilaton theorem [1], the dependence on non-derivative terms of Φ is in the exponential. At each order we will have as many $I_{k,p}$ as combinations of covariant derivative terms in Φ and S are possible, always satisfying the condition that imposes the total number of derivatives to be $2k+2$. The demonstration consists in getting to:

$$I_k = \alpha'^k \int dtne^{-\Phi} X(\mathcal{D}S), \tag{25}$$

by induction. We suppose that all I_k are written in this form up to order $k-1$, and we attempt to prove the same form at order k. We will be using the properties of the traces of the S field given in [10]:

$$tr(S) = tr(\mathcal{D}S) = tr(\mathcal{D}^2 S) = ... = 0 \ , \tag{26a}$$

$$tr((\mathcal{D}S)^{2k+1}) = 0 \quad , \quad \text{for} \quad k = 0, 1, 2... \ , \tag{26b}$$

$$tr(S(\mathcal{D}S)^k) = 0 \quad , \quad \text{for} \quad k = 0, 1, 2... \ , \tag{26c}$$

We will also consider field redefinitions with corrections of order α'^k:

$$\Phi \rightarrow \Phi + \alpha'^k \delta\Phi \quad , \quad S \rightarrow S + \alpha'^k \delta S. \tag{27}$$

Using these field redefinitions we will only need to vary I_0 at each field redefinition, since it will generate terms of order k. By varying the higher terms of the action we will obtain $\mathcal{O}(\alpha'^{k+1})$, that will be corrected afterwards. With such field redefinitions, the variation of I_0 is given by:

$$\delta I_0 = \int dt\, n e^{-\Phi}(\delta\Phi E_\Phi + tr(\delta K E_S)). \tag{28}$$

The demonstration is split in 5 parts:

1. A factor $\mathcal{D}^2\Phi$ in an action can be replaced by a factor of Q_Φ with only first derivatives. We examine a term from the action with these characteristics:

$$Z_k = \alpha'^k \int dt n e^{-\Phi} X(\{\mathcal{D}\Phi\}, \{S\}) \mathcal{D}^2\Phi. \tag{29}$$

   Considering only shifted dilaton redefinitions and using the EOM at first order for the dilaton, we get that the variation of the action is given by Eq. (17):

$$\delta I_0 = \int dt\, n e^{-\Phi} 2\delta\Phi \left(\mathcal{D}^2\Phi - \frac{1}{2}(\mathcal{D}\Phi)^2 + \frac{1}{16} tr((\mathcal{D}S)^2) \right), \tag{30}$$

   which by a proper choice of $2\delta\Phi = -X(\{\mathcal{D}\Phi\}, \{S\})$, and summed to Eq. (29) will eliminate the second derivative term in Eq. (29). This means that at order k we will have a replacement of second derivatives of Φ by an expression of just first derivatives:

$$\mathcal{D}^2\Phi \rightarrow Q_\Phi = \frac{1}{2}(\mathcal{D}\Phi)^2 - \frac{1}{16} tr((\mathcal{D}S)^2). \tag{31}$$

2. A factor $\mathcal{D}^2 S$ in an action can be replaced by a factor of Q_S with only first derivatives. As we did before, we use the EOM at first order and a redefinition of the S

fields (by means of δK). The term that we want to cancel is the generic term with a second derivative of S:

$$Z_k = \alpha'^k \int dtne^{-\Phi} X(\{\mathcal{D}\Phi\}, \{S\}) tr(\mathcal{G}\mathcal{D}^2 S), \tag{32}$$

where $\mathcal{G}$ is a matrix which is a function of S such as $\mathcal{G}(\{S\})$. By varying the action at first order we get from Eq. (23):

$$\delta I_0 = \int dt\, ne^{-\Phi} tr \left(\frac{1}{4} \delta K (\mathcal{D}^2 S + S(\mathcal{D}S)^2 - \mathcal{D}\Phi\mathcal{D}S) \right). \tag{33}$$

Now, if we choose the variation of the matrix to be:

$$\frac{1}{4}\delta K = -X(\{\mathcal{D}\Phi\}, \{S\})\mathcal{G}, \tag{34}$$

the net effect of the variation of the action at first order, will be the replacement in the term Z'_k:

$$\mathcal{D}^2 S \to Q_S = -S(\mathcal{D}S)^2 + \mathcal{D}\Phi\mathcal{D}S. \tag{35}$$

3. Any action can be reduced so that it only has first time derivatives of Φ. Such general terms can be expressed as:

$$Z_k = \alpha'^k \int dtne^{-\Phi} X(\{\mathcal{D}\Phi\}, \{S\}) \mathcal{D}^{m+2}\Phi \quad , \quad 0 \le m \le 2k. \tag{36}$$

The $\mathcal{D}^{m+2}\Phi$ term can be written as $\mathcal{D}^m(\mathcal{D}^2\Phi)$. We integrate by parts the m derivatives and then replace $\mathcal{D}^2\Phi$ by Q_Φ, so that the transformed Eq. (36) at order k reads:

$$Z'_k = \alpha'^k \int dtn(-\mathcal{D}^m)(e^{-\Phi}X)(\frac{1}{2}(\mathcal{D}\Phi)^2 - \frac{1}{16}tr((\mathcal{D}S)^2)). \tag{37}$$

We now integrate back all the m derivatives one by one. It is done one by one since at each step we will generate second derivatives of Φ and S, which with the first 2 properties can be reduced to first derivative terms. Therefore, the most general action at order k can now be written as:

$$I_k = \alpha'^k \int dtne^{-\Phi} X(\mathcal{D}\Phi, \{S\}). \tag{38}$$

4. Any action can be reduced so that it only has first time derivatives of S. Again, we proceed the same way we did in the dilaton case. Now such term will have the form:

$$Z_k = \alpha'^k \int dtne^{-\Phi} X(\mathcal{D}\Phi, \{S\}) tr(\mathcal{G}\mathcal{D}^{m+2}S) \quad , \quad 0 \le m \le 2k, \tag{39}$$

with $\mathcal{G}$ again being a matrix in terms of S and its derivatives $\mathcal{G}(\{S\})$. As X is a scalar, in order to facilitate the notation we can write:

$$\mathcal{F}(\mathcal{D}\Phi, \{S\}) = e^{-\Phi} X(\mathcal{D}\Phi, \{S\}) \mathcal{G}(\{S\}). \tag{40}$$

We now integrate by parts the m derivatives and then replace D^2S by Q_S, getting a variation of the Z_k term:

$$Z'_k = \alpha'^k \int dtntr((-\mathcal{D}^m)\mathcal{F}Q_S). \tag{41}$$

Once again we integrate back by parts, by one derivative at a time. The second derivative terms generated by $\mathcal{D}Q_S$ will be replaced by the expressions of Q_S and Q_Φ found in properties 1 and 2. Only terms like $\mathcal{D}\Phi$, S, and $\mathcal{D}S$ can appear. We now recall Eq. (26c), which implies that terms with an S will not appear, neither quadratic or cubic terms since it is a constrained field $S^2 = \mathbb{1}$. At order k the action takes now the simplified form:

$$I_k = \alpha'^k \int dtne^{-\Phi} X(\mathcal{D}\Phi, \mathcal{D}S). \tag{42}$$

5. Any action I_k with $k > 1$ is equivalent to one without any appearance of $\mathcal{D}\Phi$. Now that we only have first derivatives of the dilaton, we can write the pieces of the action as:

$$I_{k,p} = \alpha'^k \int dtne^{-\Phi} (\mathcal{D}\Phi)^p X_l(\mathcal{D}S), \tag{43}$$

where X_l is a generic invariant function of traces of $\mathcal{D}S$ and powers of it:

$$X_l(\mathcal{D}S) = tr((\mathcal{D}S)^{l_1}...(\mathcal{D}S)^{l_n}) \quad , \quad l = l_1 + ... + l_n. \tag{44}$$

These functions satisfy: $l + p = 2k + 2$. Now we can rearrange Eq. (43), and then integrate by parts:

$$\begin{aligned} I_{k,p} &= -\int dtn\mathcal{D}(e^{-\Phi})(\mathcal{D}\Phi)^{p-1} X_l \\ &= \int dtne^{-\Phi} \left((p-1)(\mathcal{D}\Phi)^{p-2}\mathcal{D}^2\Phi X_l + (\mathcal{D}\Phi)^{p-1}\mathcal{D}X_l\right). \end{aligned} \tag{45}$$

We now replace $\mathcal{D}^2\Phi$ by Q_Φ. Moreover, the covariant derivative acting on X_l will give second derivative terms of S inside the traces, which are replaced by $Q_S = -S(\mathcal{D}S)^2 + \mathcal{D}\Phi\mathcal{D}S$. Looking at Eq. (44) we see that the first part of the replacement vanishes since it will give terms such as $tr(-S(\mathcal{D}S)^{l_1+2})$ that vanish identically (26c), so the net change inside X_l is $\mathcal{D}^2S \to \mathcal{D}\Phi\mathcal{D}S$ and we get:

$$\begin{aligned} \mathcal{D}X_l &= \mathcal{D}\Phi \left(l_1 tr((\mathcal{D}S)^{l_1}...(\mathcal{D}S)^{l_n} + ... + l_n tr((\mathcal{D}S)^{l_1}...(\mathcal{D}S)^{l_n}\right) \\ &= l\mathcal{D}\Phi X_l(\mathcal{D}S). \end{aligned} \tag{46}$$

With both replacements Eq. (45) reads:

$$\begin{aligned} I_{k,p} &= \int dtne^{-\Phi}\left((p-1)(\mathcal{D}\Phi)^{p-2}\left(\frac{1}{2}(\mathcal{D}\Phi)^2 - \frac{1}{16}tr((\mathcal{D}S)^2)\right)X_l + l(\mathcal{D}\Phi)^p X_l\right) \\ &= \int dtne^{-\Phi}\left(\frac{1}{2}(p+2l-1)(\mathcal{D}\Phi)^p X_l - \frac{1}{16}(p-1)(\mathcal{D}\Phi)^{p-2}tr((\mathcal{D}S)^2)X_l\right). \end{aligned} \tag{47}$$

We note that the first term is proportional to $(\mathcal{D}\Phi)^p X_l$ as the original Eq. (43). We can equate them obtaining:

$$\int dtne^{-\Phi}(\mathcal{D}\Phi)^p X_l = \frac{1}{8}\frac{(p-1)}{(p+2l-3)}\int dtne^{-\Phi}(\mathcal{D}\Phi)^{p-2}tr((\mathcal{D}S)^2)X_l. \tag{48}$$

The denominator does not vanish since $p+l=2k+2$, then the denominator reads $2k+l-1 \geq 1$ for $k \geq 1$. We can recursively use Eq. (48) in order to reduce the powers of $\mathcal{D}\Phi$ by two, by also increasing l, the powers of $\mathcal{D}\Phi$ by two. We have two cases. For the case of even powers of $\mathcal{D}\Phi$, we can use Eq. (48) recursively until we get to a vanishing power of dilaton derivatives. For the case of odd powers, we can get to a single $\mathcal{D}\Phi$ term, then looking at Eq. (48), we would obtain $p=1$ and the prefactor on the right hand side makes it vanish. We have finally achieved our objective, any I_k term can be written as:

$$I_k = \alpha'^k \int dtne^{-\Phi}X(\mathcal{D}S). \tag{49}$$

By imposing O(d,d) invariance and using covariant redefinitions of S and Φ we get to this much simpler action. But, we can further constrain the form of X_l by using redefinitions of the lapse function. Up to now, the total action has the form:

$$I = \int dtne^{-\Phi}(L_0 + \alpha' L_1 + \alpha'^2 L_2 + ...), \tag{50}$$

where L_k posseses $2k+2$ derivatives of S in all possible combinations. Taking into account that traces of odd powers vanish (Eq. (26b)), L_1, L_2... take the form:

$$L_1 = a_1 tr((\mathcal{D}S)^4) + a_2(tr((\mathcal{D}S)^2))^2, \tag{51}$$

$$L_2 = b_1 tr((\mathcal{D}S)^6) + b_2 tr((\mathcal{D}S)^4)tr((\mathcal{D}S)^2) + b_3(tr((\mathcal{D}S)^2))^3. \tag{52}$$

Inspired in the previous demonstrations, we can still use time lapse redefinitions in order to remove all the $tr((\mathcal{D}S)^2)$ terms. Once again, we use a variation of order α'^k, this time of the lapse function, which produces a variation in the first order action:

$$n \to n + \alpha'^k \delta n, \tag{53}$$

$$\delta_n I_0 = \alpha'^k \int dtne^{-\Phi}\frac{\delta n}{n}((\mathcal{D}\Phi)^2 - \frac{1}{8}tr((\mathcal{D}S)^2)). \tag{54}$$

We now want to eliminate terms like:

$$I_{k,p} = \alpha'^k \int dtne^{-\Phi} X_{2k}(\mathcal{D}S)tr(\mathcal{D}S)^2. \tag{55}$$

By using a further Φ redefinition, in order to replace $(\mathcal{D}\Phi)^2$ by $\frac{1}{8(4k-1)}tr((\mathcal{D}S)^2)$ (from Eq. (48)), we can eliminate such terms by choosing:

$$\frac{\delta n}{n} = \beta X_{2k}(\mathcal{D}S), \tag{56}$$

with β a constant to be determined. It is clear that we can cancel the $I_{k,p}$ with the variations of the first order action by imposing $\frac{\beta k}{2(4k-1)} = -1$. So any term with a $tr(\mathcal{D}S)^2$ term can be removed by further dilaton and lapse function covariant redefinitions.

3.1.3 Equations of motion

For a Friedman-Lemaître-Robertson-Walker (FLRW) metric, only single-trace terms in S are necessary, as explained in section 3.1.4. By neglecting multi-trace terms, we get to the final action:

$$I = \int dt\, ne^{-\Phi} \left(-(\mathcal{D}\Phi)^2 + \sum_{k=1}^{\infty} \alpha'^{k-1} c_k\, tr(\mathcal{D}S)^{2k} \right). \tag{57}$$

We get an O(d,d) covariant action where only first derivatives appear, a huge and unexpected simplification. In the same way as the first order case, we vary the action with respect to Φ, S, and n. The derivation of the EOM for Φ and n is similar as in the first order case. We should look at the EOM for S in more detail. We fix the gauge $n = 1$. The general case will be recovered in section 3.2, for it we only have to replace the dots by the covariant derivative. The variation of the action with respect to S reads:

$$\begin{aligned} \delta_S I &= \sum_{k=1}^{\infty} \alpha'^{k-1} c_k \int dt e^{-\Phi} (2k) tr\left((\delta \frac{dS}{dt}) \dot{S}^{2k-1} \right) \\ &= -\sum_{k=1}^{\infty} \alpha'^{k-1} c_k \int dt (2k) tr\left(\delta S \frac{d}{dt}(e^{-\Phi} \dot{S}^{2k-1}) \right) \\ &= \int dt e^{-\Phi} tr\left(\delta S \sum_{k=1}^{\infty} \alpha'^{k-1} c_k (2k) \left[\dot{\Phi} \dot{S}^{2k-1} - \frac{d}{dt} \dot{S}^{2k-1} \right] \right). \end{aligned} \tag{58}$$

This relation gives $tr(\delta S F_S)$, as before, we get E_S with Eq. (21):

$$E_S = \frac{1}{2} \sum_{k=1}^{\infty} \alpha'^{k-1} c_k (2k) \left(\left[\dot{\Phi} \dot{S}^{2k-1} - \frac{d}{dt} \dot{S}^{2k-1} \right] - S \left[\dot{\Phi} \dot{S}^{2k-1} - \frac{d}{dt} \dot{S}^{2k-1} \right] S \right). \tag{59}$$

In order to simplify it we must recall a few properties of the S matrices. Firstly, as we saw S and $\dot{S}$ anticonmute, so the same happens for S and an odd number of $\dot{S}$. Secondly, if we differentiate again the equation of anticonmutativity we get:

$$S\ddot{S} = -\ddot{S}S - 2\dot{S}^2. \tag{60}$$

With all that we can look for an expression for $S\frac{d}{dt}\dot{S}^{2k-1}S$:

$$\begin{aligned}
S\frac{d}{dt}\dot{S}^{2k-1}S =& S\frac{d}{dt}(\underbrace{\dot{S}\dot{S}...\dot{S}}_{2k-1})S \\
=& \underbrace{S(\ddot{S}\dot{S}...\dot{S})S + S(\dot{S}\ddot{S}...\dot{S})S + ... + S(\dot{S}\dot{S}...\ddot{S})S}_{2k-1} \\
=& -2\dot{S}^{2k}S - \ddot{S}\dot{S}...\dot{S} + 2\dot{S}^{2k}S - \dot{S}\ddot{S}...\dot{S} - ... - 2\dot{S}^{2k}S - \dot{S}\dot{S}...\ddot{S} \\
=& -\frac{d}{dt}\dot{S}^{2k} - 2S\dot{S}^{2k}.
\end{aligned} \tag{61}$$

By the anticonmutativity property of S and $\dot{S}$ we will have alternance in the sign of $2S\dot{S}^{2k}$ and as we have an odd number of them, we are left with only one of those terms with a negative sign. Taking Eqs. (59) and (61) into account, we get the final EOM for S. We write the three EOM with $n = 1$:

$$E_\Phi = 2\ddot{\Phi} - \dot{\Phi}^2 - \sum_{k=1}^{\infty} \alpha'^{k-1} c_k tr(\dot{S}^{2k}), \tag{62a}$$

$$E_S = -2\sum_{k=1}^{\infty} \alpha'^{k-1} k\, c_k \left(\frac{d}{dt}\dot{S}^{2k-1} - \dot{\Phi}\dot{S}^{2k-1} + S\dot{S}^{2k}\right), \tag{62b}$$

$$E_n = \dot{\Phi}^2 - \sum_{k=1}^{\infty} \alpha'^{k-1}(2k-1) c_k tr(\dot{S}^{2k}). \tag{62c}$$

With a general variation of the action given by:

$$\delta I = \int dt\, n e^{-\Phi}\left(\delta\Phi E_\Phi + tr(\delta S E_S) + \frac{\delta n}{n}E_n\right). \tag{63}$$

We can further note that by imposing time reparametrisation invariance of the action under:

$$\delta_\xi S = \xi\dot{S} \quad , \quad \delta_\xi \Phi = \xi\dot{\Phi} \quad , \quad \delta_\xi n = \partial_t(\xi n), \tag{64}$$

we obtain a Bianchi identity that reads:

$$\dot{\Phi}^2(E_\Phi + E_n) + tr(\dot{S}E_S) = \frac{d}{dt}E_n. \tag{65}$$

This equation tells us that it is enough to solve $E_\Phi + E_n = 0$ and $E_S = 0$, so that $E_n = 0$ is a Hamiltonian constraint that has to be satisfied for the initial conditions, and will hold at all times.

3.1.4 FLRW

We will focus on the case of the FLRW metric, with vanishing b field, such that:

$$S = \begin{pmatrix} 0 & a^2(t) \cdot \mathbb{1}_d \\ a^{-2}(t) \cdot \mathbb{1}_d & 0 \end{pmatrix}, \tag{66}$$

where $a(t)$ is the scale factor for the d spatial dimensions, and H is the usual Hubble parameter $H = \dot{a}/a$. The covariant dilaton and its first derivative are now defined as:

$$e^{-\Phi} = (a(t))^d e^{-2\phi} \quad \rightarrow \quad \dot{\Phi} = -dH + 2\dot{\phi}. \tag{67}$$

Let us evaluate the terms appearing in Eqs. (62). The derivatives of S are given by:

$$\dot{S} = 2H \begin{pmatrix} 0 & a^2(t) \cdot \mathbb{1}_d \\ -a^{-2}(t) \cdot \mathbb{1}_d & 0 \end{pmatrix} = 2HJ \quad \rightarrow \quad \ddot{S} = 2\dot{H}J + 4H^2 S, \tag{68}$$

where we introduced the J matrix:

$$J = \begin{pmatrix} 0 & a^2(t) \cdot \mathbb{1}_d \\ -a^{-2}(t) \cdot \mathbb{1}_d & 0 \end{pmatrix} \quad \rightarrow \quad \dot{J} = 2H \begin{pmatrix} 0 & a^2(t) \cdot \mathbb{1}_d \\ a^{-2}(t) \cdot \mathbb{1}_d & 0 \end{pmatrix} = 2HS. \tag{69}$$

We note that $J^2 = -\mathbb{1}_{2d}$. Then for the even powers of $\dot{S}$ we have:

$$\dot{S}^2 = -4H^2 \mathbb{1}_{2d} \quad \rightarrow \quad \dot{S}^{2k} = -(-1)^{k-1}(2H)^{2k}\mathbb{1}_{2d}, \tag{70}$$

while for the odd ones:

$$\dot{S}^{2k-1} = \dot{S}^{2(k-1)}\dot{S} = (-1)^{k-1}(2H)^{2k-1}J, \tag{71}$$

implying:

$$\frac{d}{dt}\dot{S}^{2k-1} = (-1)^{k-1}(2k-1)2\dot{H}(2H)^{2k-2}J + (-1)^{k-1}(2H)^{2k}S. \tag{72}$$

The last term that we need to substitute is given by:

$$S\dot{S}^{2k} = -(-1)^{k-1}(2H)^{2k}S. \tag{73}$$

A key point for our future development is that we can introduce exclusively single-trace terms in the action, neglecting multitrace ones. This works for a FLRW isotropic ansatz as stated previously. By computing the single-trace terms that appear in the action (57) using Eq. (70) we get:

$$c_k tr(\dot{S}^{2k}) = (-1)^k 2^{2k+1} c_k d H^{2k}, \tag{74}$$

whereas for example, for a double-trace term of same order we would have:

$$c_{k,l} tr(\dot{S}^{2(k-l)}) tr(\dot{S}^{2l}) = (-1)^k 2^{2k+1} c_{k,l} 2d^2 H^{2k}. \tag{75}$$

The only effect at order $(\alpha')^{k-1}$ would be a redefinition of the c_k coefficients $c_k \to c_k + 2dc_{k,l}$. As multitrace terms enter at third order (52) and these coefficients $c_{k\geq 3}$, are not fixed, we can neglect multitrace terms.

By evaluating Eqs. (62) with Eqs. (70)-(73), we get the EOM for our FLRW ansatz. For $E_S = 0$:

$$\begin{aligned} 0 &= -2\sum_{k=1}^{\infty}(-\alpha')^{k-1} k\, c_k \Big((2k-1)2\dot{H}(2H)^{2k-2}J + (2H)^{2k}S - \dot{\Phi}(2H)^{2k-1}J - (2H)^{2k}S\Big) \\ &= -\sum_{k=1}^{\infty}(-\alpha')^{k-1} 2k\, c_k \Big((2k-1)2\dot{H}(2H)^{2k-2} - \dot{\Phi}(2H)^{2k-1}\Big) J. \end{aligned} \tag{76}$$

We note that the dependence in the S matrix vanished, obtaining a unique equation instead of a matrix one. For future convenience we multiply Eq. (76) by $4d$. For $E_\Phi = 0$ we have:

$$0 = 2\ddot{\Phi} - \dot{\Phi}^2 + 2d\sum_{k=1}^{\infty}(-\alpha')^{k-1} c_k (2H)^{2k}, \tag{77}$$

where the factor $2d$ comes from the trace of $\mathbb{1}_{2d}$. Finally for $E_n = 0$:

$$0 = -\dot{\Phi}^2 - 2d\sum_{k=1}^{\infty}(-\alpha')^{k-1}(2k-1) c_k (2H)^{2k}. \tag{78}$$

The key point in this development is to understand that we can express all three EOM by means of defining a unique function $F(H)$:

$$F(H) = 4d\sum_{k=1}^{\infty}(-\alpha')^{k-1} 2^{2k-1} c_k H^{2k}. \tag{79}$$

We get the EOM for the isotropic FLRW metric in vacuum:

$$\dot{H}\, F''(H) - \dot{\Phi}\, F'(H) = 0, \tag{80a}$$

$$\ddot{\Phi} + \frac{1}{2} H\, F'(H) = 0, \tag{80b}$$

$$\dot{\Phi}^2 + H\, F'(H) - F(H) = 0, \tag{80c}$$

where we have $E_S = 0$, $E_\Phi + E_n = 0$ and $E_n = 0$ respectively, and primes denote derivatives with respect to H. It is useful to note the expression for E_Φ for the next section in which we introduce matter:

$$E_\Phi = 2\ddot{\Phi} - \dot{\Phi}^2 + F(H). \tag{81}$$

We can note that the action can also be rewritten in terms of the $F(H)$ function as:

$$I = \int dt e^{-\Phi} \left(-\dot{\Phi}^2 - F(H)\right). \tag{82}$$

The $F(H)$ function encodes the α' corrections at all orders. It has been used to look for non-perturbative de Sitter solutions [8], [4], [3]. The functional forms found for $F(H)$ in these works are not at all guaranteed to be stringy solutions, apart from the fact that is not clear that after our perturbative development of the theory we could just choose a non-perturbative function to play the role of $F(H)$.
We can observe that, as expected, the O(d,d) invariance turned into the duality invariance of the scale factor, since under a transformation of the scale factor, we get:

$$a \to a^{-1} \quad , \quad H \to -H \quad , \quad \Phi \to \Phi \quad , \quad F(H) \to F(H) \quad , \quad F'(H) \to -F'(H). \tag{83}$$

3.2 Matter

To have a more realistic cosmological model, we need to add a matter action to (57). In order to make it as general as possible, we can suppose that the matter action depends also on the shifted dilaton. For this part we will add the dimensionful constant $2\kappa^2 = 16\pi G$ where G is the Newton constant in D dimensions and has mass dimensions $2-D$. We take advantage of this comment on dimensional analysis to point out that the action in Eq. (84) does not have zero mass dimension because we have integrated out the $D-1$ spatial dimensions. The action then reads:

$$I = \frac{1}{2\kappa^2} \int dt\, n e^{-\Phi} \left(-(\mathcal{D}\Phi)^2 + \sum_{k=1}^{\infty} \alpha'^{k-1} c_k\, tr(\mathcal{D}S)^{2k}\right) + S_m[\Phi, n, S, \chi], \tag{84}$$

where for consistence with the symmetries of the background we suppose the matter fields χ to be exclusively time dependent. We follow [2], where the matter action at first order in α' in an O(d,d) covariant way is first introduced. Similarly to what is done in [8], by means of field redefinitions they argue that by introducing a matter action in this way, we are not neglecting α' corrections to the matter sector, the EOM at first order are used to rearrange the total action into the form in Eq. (84). Since the extra matter factors that contribute to the corrected action at order k (since they enter in the EOM and then in the redefinitions of the fields) can be eliminated with the new degree of freedom χ. To get the EOM, the gravitational part is derived in the same way as in the previous section. Now for the matter part, by varying the action with respect to the shifted dilaton we obtain:

$$\delta_\Phi I = \int dt n e^{-\Phi} \delta\Phi E^T_\Phi, \tag{85}$$

$$E^T_\Phi = \frac{1}{2\kappa^2} \left(2\mathcal{D}^2\Phi - (\mathcal{D}\Phi)^2 - \sum_{k=1}^{\infty} \alpha'^{k-1} c_k tr((\mathcal{D}S)^{2k})\right) + \frac{e^\Phi}{n} \frac{\delta S_m}{\delta\Phi}. \tag{86}$$

So that for E_Φ on-shell, the EOM reads:

$$2\mathcal{D}^2\Phi - (\mathcal{D}\Phi)^2 - \sum_{k=1}^{\infty} \alpha'^{k-1} c_k tr((\mathcal{D}S)^{2k}) = -2\kappa^2 \frac{e^\Phi}{n}\frac{\delta S_m}{\delta\Phi} = \kappa^2 e^\Phi \bar{\theta}, \tag{87}$$

where bared variables are multiplied by $\sqrt{g}$, and we define θ as the dilatonic charge that can be seen as the measure of how strongly the dilaton couples to matter:

$$\theta = -\frac{2}{\sqrt{g}}\frac{\delta S_m}{\delta\Phi}. \tag{88}$$

Now, by varying the action with respect to the lapse function we obtain:

$$\delta_n I = \int dt n e^{-\Phi} \frac{\delta n}{n} E_n^T, \tag{89}$$

$$E_n^T = \frac{1}{2\kappa^2}\left((\mathcal{D}\Phi)^2 - \sum_{k=1}^{\infty} \alpha'^{k-1}(2k-1) c_k tr((\mathcal{D}S)^{2k})\right) + e^\Phi \frac{\delta S_m}{\delta n}, \tag{90}$$

and the EOM then reads:

$$(\mathcal{D}\Phi)^2 - \sum_{k=1}^{\infty} \alpha'^{k-1}(2k-1) c_k tr((\mathcal{D}S)^{2k}) = -2\kappa^2 e^\Phi \frac{\delta S_m}{\delta n} = 2\kappa^2 e^\Phi \bar{\rho}, \tag{91}$$

with ρ the energy density in the energy-momentum tensor. We can get this expression from the definition of the energy-momentum tensor :

$$T_{\mu\nu} = -\frac{2}{\sqrt{-G}}\frac{\delta S_m}{\delta g^{\mu\nu}} \to T_{00} = -\frac{n^2}{\sqrt{g}}\frac{\delta S_m}{\delta n} \to \frac{\delta S_m}{\delta n} = -\bar{\rho}. \tag{92}$$

For the S field, the variation reads:

$$\delta_S I = \int dt n e^{-\Phi} tr(\delta S F_S^T) \tag{93}$$

$$F_S^T = \frac{1}{2\kappa^2}\left(\sum_{k=1}^{\infty} \alpha'^{k-1} c_k (2k)\left[\mathcal{D}\Phi(\mathcal{D}S)^{2k-1} - \mathcal{D}(\mathcal{D}S)^{2k-1}\right]\right) + \frac{e^\Phi}{2n}\frac{\delta S_m}{\delta S}. \tag{94}$$

We must recall that S is a constrained field, so that the EOM is given by the vanishing of $E_S^T = (F_S^T - S F_S^T S)/2$. We obtain:

$$E_S^T = -\frac{1}{\kappa^2}\sum_{k=1}^{\infty} \alpha'^{k-1} k\, c_k \left(\mathcal{D}(\mathcal{D}S)^{2k-1} - \mathcal{D}\Phi(\mathcal{D}S)^{2k-1} + S(\mathcal{D}S)^{2k}\right) - \frac{e^\Phi}{4} S\eta\mathcal{T}, \tag{95}$$

where we define $\mathcal{T}$ as the O(d,d) covariant energy-momentum tensor:

$$\mathcal{T} = \frac{1}{n}\left(\eta\frac{\delta S_m}{\delta S}S - \eta S\frac{\delta S_m}{\delta S}\right). \tag{96}$$

We assume the variation of the matter action is unconstrained, so that:

$$\frac{\delta S_m}{\delta S} = -S\frac{\delta S_m}{\delta S}S. \tag{97}$$

We can verify that an unconstrained variation satisfies this property since by taking $F_S = \frac{\delta S_m}{\delta S}$ and introducing it into the expression of E_S to obtain the unconstrained equation of motion using Eq. (97), we get $E_S = F_S = \frac{\delta S_m}{\delta S}$. Later on, in section 3.2.1, we will see the reason for this assumption. Then the O(d,d) covariant energy-momentum tensor takes the simpler form:

$$\mathcal{T} = -\frac{2}{n}\eta S\frac{\delta S_m}{\delta S}. \tag{98}$$

In order to get the continuity equation, we derive Equation (91) getting:

$$2\kappa^2 e^{\Phi}\bar{\rho}\mathcal{D}\Phi + 2\kappa^2 e^{\Phi}\mathcal{D}\rho = 2\mathcal{D}^2\Phi\mathcal{D}\Phi - \sum_{k=1}^{\infty}\alpha'^{k-1}(2k-1)2kc_k tr(\mathcal{D}^2 S(\mathcal{D}S)^{2k-1}). \tag{99}$$

The derivative inside the trace can be simplified in this way because of the ciclicity property of the trace. We introduce the expression for $\mathcal{D}^2\Phi$ from (91). Moreover, for the expression inside the trace we take the trace of Eq. (95) and then multiply by $\mathcal{D}S$ obtaining:

$$\mathcal{D}\bar{\rho} = -\frac{1}{4}tr(S(\mathcal{D}S)\eta\mathcal{T}) + \frac{1}{2}\mathcal{D}\Phi\bar{\sigma}. \tag{100}$$

The continuity equation can be obtained in the same way by using the equations at first order, since α' corrections should not violate diffeomorphism invariance and then the Bianchi identities.

3.2.1 FLRW

We now only have to substitute the expressions obtained, as we did for the gravitational part. The only new part that requires a bit of caution is the computation of the E_S equation. From our ansatz for the S matrix (66) we get the variation:

$$\frac{\delta S_m}{\delta S} = \begin{pmatrix} 0 & \frac{\delta S_m}{\delta g^{-1}} \\ \frac{\delta S_m}{\delta g} & 0 \end{pmatrix}. \tag{101}$$

Then, from the usual definition of the energy momentum tensor in terms of the variation of the matter action we have:

$$T_i^k g_{kj} = (Tg)_{ij} = -\frac{2}{\sqrt{-G}}\frac{\delta S_m}{\delta g^{ij}} \to Tg = -\frac{2}{\sqrt{-G}}\frac{\delta S_m}{\delta g^{-1}}, \tag{102}$$

$$g^{ik} T_k^j = (g^{-1}T)^{ij} = \frac{2}{\sqrt{-G}}\frac{\delta S_m}{\delta g_{ij}} \to g^{-1}T = \frac{2}{\sqrt{-G}}\frac{\delta S_m}{\delta g}. \tag{103}$$

We can easily check the assumption of Eq. (97). It is satisfied for whatever metric, with the condition of vanishing b-field, as we can see:

$$S\frac{\delta S_m}{\delta S}S = \frac{\sqrt{-G}}{2}\begin{pmatrix} 0 & g \\ g^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & -Tg \\ g^{-1}T & 0 \end{pmatrix}\begin{pmatrix} 0 & g \\ g^{-1} & 0 \end{pmatrix} = \frac{\sqrt{-G}}{2}\begin{pmatrix} 0 & Tg \\ -g^{-1}T & 0 \end{pmatrix} = -\frac{\delta S_m}{\delta S}. \tag{104}$$

For the expression of the covariant energy momentum tensor, using Eq. (98), we obtain:

$$\mathcal{T} = \frac{2}{n}\frac{\sqrt{-G}}{2}\begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} 0 & Tg \\ -g^{-1}T & 0 \end{pmatrix} = \sqrt{g}\begin{pmatrix} 0 & g^{-1}Tg \\ -T & 0 \end{pmatrix} = \sqrt{g}\begin{pmatrix} 0 & p\mathbb{1}_d \\ -p\mathbb{1}_d & 0 \end{pmatrix}, \tag{105}$$

where we supposed an isotropic energy momentum tensor with spatial coordinates $T_i^j = p\delta_i^j$. The gravitational part is exactly the same as the computed for the vacuum case, so by fixing $n = 1$ and using Eq. (95), together with Eq. (76), and (105) we find:

$$\frac{e^{\Phi}}{2}\kappa^2 S\eta\mathcal{T} = -\sum_{k=1}^{\infty}(-\alpha')^{k-1}2k\,c_k\left((2k-1)2\dot{H}(2H)^{2k-2} - \dot{\Phi}(2H)^{2k-1}\right)J. \tag{106}$$

Next, we have to multiply Eq. (106) by the matrix J, and recalling that $J^2 = -\mathbb{1}_{2d}$, we compute:

$$S\eta\mathcal{T}J = \sqrt{g}\begin{pmatrix} 0 & a^2\mathbb{1}_d \\ a^{-2}\mathbb{1}_d & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1}_d \\ \mathbb{1}_d & 0 \end{pmatrix}\begin{pmatrix} 0 & p\mathbb{1}_d \\ -p\mathbb{1}_d & 0 \end{pmatrix}\begin{pmatrix} 0 & a^2\mathbb{1}_d \\ -a^{-2}\mathbb{1}_d & 0 \end{pmatrix} = -\sqrt{g}p\mathbb{1}_{2d}. \tag{107}$$

Note that, as in the vacuum case, we obtain a unique equation. Multiplying both sides of Eq. (106) by $4d$, we finally obtain the equations [2]:

$$\dot{H}\,F''(H) - \dot{\Phi}\,F'(H) = -2d\kappa^2 e^{\Phi}\bar{p}, \tag{108a}$$

$$2\ddot{\Phi} - \dot{\Phi}^2 + F(H) = \kappa^2 e^{\Phi}\bar{\theta}, \tag{108b}$$

$$\dot{\Phi}^2 + H\,F'(H) - F(H) = 2\kappa^2 e^{\Phi}\bar{\rho}, \tag{108c}$$

which are the equations for the S matrix, for the dilaton Φ and for the lapse function n.

For the continuity equation from Eq. (100), as $S\dot{S} = -\dot{S}S$ by fixing the lapse function $n = 1$, we compute:

$$tr(S\dot{S}\eta\mathcal{T}) = -tr(S\eta\mathcal{T}\dot{S}) = -2Htr(S\eta\mathcal{T}J) = 4dH\sqrt{g}p. \tag{109}$$

We then get to the final expression of the continuity equation for a FLRW like metric, which reads:

$$\dot{\bar{\rho}} + dH\bar{p} - \frac{1}{2}\dot{\Phi}\bar{\theta} = 0. \tag{110}$$

3.3 Einstein frame

We must not forget that we are working in the string frame, so that our Hubble parameter is also in the string frame. In order to get it in the D-dimensional Einstein frame we must perform a Weyl transformation of our metric, so that:

$$g_{\mu\nu} = \Omega^{-2}\tilde{g}_{\mu\nu} \quad \rightarrow \quad \sqrt{-g} = \Omega^{-D}\sqrt{-\tilde{g}}, \tag{111}$$

where $\tilde{g}_{\mu\nu}$ is the metric in the Einstein frame. The conformal factor Ω is fixed by requiring the gravitational part of the action to be exclusively: $\int dt\sqrt{-\tilde{g}}\tilde{R}$, with $\tilde{R}$ the Ricci scalar in the new frame. The Ricci scalar varies with Ω^2, so that the Weyl transformation is:

$$\Omega = e^{-\frac{2\phi}{d-1}}. \tag{112}$$

Moreover, as explained in [8] and [11], we imposed $g_{00} = -n^2 = -1$, so in order to have $\tilde{g}_{00} = -1$ we will have to reparametrise the time coordinate $dte^{\frac{-2\phi}{d-1}} = dt'$. This procedure will be repeated in a different way in section 6, where we will not fix the lapse function so that we do not need this last time reparametrisation. By expressing $H_E = \dot{a}_E/a_E$ in terms of the string frame parameters, the Hubble parameter in the Einstein frame H_E parametrised with the cosmic time in the Einstein frame t' reads:

$$H_E(t') = -\frac{e^{\frac{2\phi}{d-1}}}{d-1}(\dot{\Phi} + H) = -\frac{a^{\frac{d}{d-1}}e^{\frac{\Phi}{d-1}}}{d-1}(\dot{\Phi} + H), \tag{113}$$

where H, a, Φ, ϕ, and the derivatives are in the string frame. In section 4.3 we will see how to derive this Einstein frame Hubble parameter step by step. In order to look for the different solutions for which the Hubble parameter is constant in the Einstein frame, we can derive with respect to t':

$$\begin{aligned} \frac{dH_E}{dt'} = \frac{dt}{dt'}\frac{dH_E}{dt} &= -\frac{e^{\frac{4\phi}{d-1}}}{d-1}\left(\frac{2\dot{\phi}}{d-1}(\dot{\Phi} + H) + (\ddot{\Phi} + \dot{H})\right), \\ &= -\frac{e^{\frac{4\phi}{d-1}}}{d-1}\left(\ddot{\Phi} + \dot{H} + \frac{1}{d-1}(dH + \dot{\Phi})(\dot{\Phi} + H)\right), \end{aligned} \tag{114}$$

where in the last line we used the identity (67). We obtain an interesting result, since in order to obtain a constant Hubble parameter in both frames, one of the possibilities is to have the velocity of the shifted dilaton and the Hubble parameter in the string frame constants. So that, with $\dot{\Phi} = -H$ or $\dot{\Phi} = -dH$ we get respectively Minkowski and de Sitter vacuum in the Einstein frame, as commented in [4]. We recall that until now we have not assumed that we are either in the vacuum or matter case.

By putting Eq. (114) to 0, and substituing the dilaton components with the EOM (80) of the vacuum case, we can get a differential equation for $F(H)$ and H. For the case in which $F'(H) \neq 0$:

$$\begin{aligned} 0 &= (d-1)\ddot{\Phi} + (d-1)\dot{H} + dH^2 + \dot{\Phi}H(d+1) + \dot{\Phi}^2, \\ &= (d-1)(\dot{H} - \frac{1}{2}HF'(H)) + dH^2 + (d+1)\frac{F''(H)}{F'(H)}H\dot{H} + F(H) - HF'(H) \end{aligned} \quad (115)$$

We can look for some special cases, for example a constant Hubble parameter, from Eq. (80a), imposes a vanishing $\dot{\Phi}$ or a vanishing $F'(H)$. For the first option, we get a vanishing shifted dilaton acceleration and then $F'(H) = 0$, or $H = 0$. We impose $H = 0$, introducing this into Eq. (80c) we also get a vanishing $F(H)$ function. Substituting these values into Eq. (114) we get an expression that vanishes identically. This solution of $H = 0$ (in the string frame) gives a constant Hubble parameter in the Einstein frame. Moreover, looking at Eq. (113), we can see that it is a Minkowski solution.
Looking at the neglected case $F'(H) = 0$ we get a constant but non zero shifted dilaton velocity given by the value of the F function $\dot{\Phi}^2 = F(H)$, which introduced into Eq. (114) allows $\dot{\Phi} = -H$ and $\dot{\Phi} = -dH$, giving a Minkowski and a de Sitter solution in the Einstein frame respectively. A systematic analysis of the fixed points of the system will be done in section 5.

We can also do different substitutions so that we do not get $\dot{H}$ terms in the differential equation. From Eqs. (80) we now carry out the substitutions:

$$\ddot{\Phi} = -\frac{1}{2}HF' \quad , \quad \dot{\Phi}^2 = F - HF' \quad , \quad \dot{H} = \dot{\Phi}\frac{F'}{F''}. \quad (116)$$

This relation gives a different differential equation. The positive point is that now we have a differential equation to get the F function in terms of derivatives with respect to H exclusively, but with the cost of introducing square roots. The differential equation now reads:

$$-\frac{1}{2}HF'F'' + F's\sqrt{F - HF'} + \frac{dH^2}{d-1}F'' + \frac{d+1}{d-1}sHF''\sqrt{F - HF'} + \frac{1}{d-1}(F - HF')F'' = 0, \quad (117)$$

where $s = \pm 1$ by the duality in the sign of $\dot{\Phi}$ by its definition in Eq. (116) from the EOM. This equation coincides with the work done in [12]. But as we can suppose, this differential equation cannot be solved analitycally. We finish this section by commenting that adding matter further complicates the equation by making the time dependent factors of the dilatonic charge $\theta(t)$, the pressure $p(t)$ and the energy density $\rho(t)$ appear.

4 Anisotropic case

4.1 Vacuum

With the objective of compactifying the theory, and seeing what aspect it should have in 4 dimensions, one of the first ideas is to use an anisotropic metric, with a 3-dimensional part with the scale factor $a(t)$, and a D-4-dimensional part with the scale factor $b(t)$ for the extra dimensions. The metric and S matrix are:

$$g_{ij} = \begin{pmatrix} a^2 \cdot \mathbb{1}_3 & 0 \\ 0 & b^2 \cdot g_{mn} \end{pmatrix} \, , \; S = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & a^2 \cdot \mathbb{1}_3 & 0 \\ & & 0 & b^2 \cdot g_{mn} \\ \hline a^{-2} \cdot \mathbb{1}_3 & 0 & \multicolumn{2}{c}{0} \\ 0 & b^{-2} \cdot g^{mn} & & \end{array} \right) = S_3 + S_{D-4}, \tag{118}$$

where we assume a FLRW metric for the 3-dimensional part and a metric $b^2 g_{mn}$ for the extra dimensions. Then, we require the time dependence of the metric of the extra dimensional part to be in the scale factor $b(t)$. Thus the time derivative reads:

$$\dot{S} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & 2Ha^2 \cdot \mathbb{1}_3 & 0 \\ & & 0 & 2\sigma b^2 \cdot g_{mn} \\ \hline -2Ha^{-2} \cdot \mathbb{1}_3 & 0 & \multicolumn{2}{c}{0} \\ 0 & -2\sigma b^{-2} \cdot g^{mn} & & \end{array} \right) = \dot{S}_3 + \dot{S}_{D-4}, \tag{119}$$

where $\sigma = \dot{b}/b$ is the Hubble factor for the extra dimensions. Let us first compute the first order Lagrangian in this case. An important feature is the fact that the $\dot{S}_3$ and $\dot{S}_{D-4}$ matrices will not have cross-terms in the single-trace terms:

$$\dot{S}^2 = \dot{S}_3\dot{S}_3 + \dot{S}_{D-4}\dot{S}_{D-4} + \dot{S}_3\dot{S}_{D-4} + \dot{S}_{D-4}\dot{S}_3 = \dot{S}_3^2 + \dot{S}_{D-4}^2. \tag{120}$$

Taking (13) and (120) we obtain the form of the zero-order action:

$$I_0 = \int dt\, ne^{-\Phi} \frac{1}{n^2} \left(-\dot{\Phi}^2 - \frac{1}{8} tr(\dot{S}_3^2) - \frac{1}{8} tr(\dot{S}_{D-4}^2) \right), \tag{121}$$

where the shifted dilaton is defined as usual:

$$e^{-\Phi} = e^{-2\phi}\sqrt{g}, \tag{122}$$

with $\sqrt{g}$ being the determinant of the spatial part of the full metric in Eq. (118).

For the α' corrections part, by simply taking Eq.(57) and substituting we would obtain an action where the 2 scale factors do not talk to each other directly. However, taking into account that one of the simplifications that was done for the isotropic FLRW case no longer holds. One can no longer neglect the multi-trace factors. Let us see an example, for a multi-trace term of order k, such that $2l = k$, we have:

$$(tr(\dot{S}^{2l}))^2 = (tr(\dot{S}_3^{2l}))^2 + (tr(\dot{S}_{D-4}^{2l}))^2 + 2\,tr(\dot{S}_3^{2l})tr(\dot{S}_{D-4}^{2l}). \tag{123}$$

Now multitrace terms provide new factors to our action. We can introduce the coefficients that will arise from the multitrace terms in the $c_{k,l}$ coefficients at each order. The complete action can be written as:

$$I = \int dt\, ne^{-\Phi}\left(-(\mathcal{D}\Phi)^2 + \sum_{k=1}^{\infty}\alpha'^{k-1}\sum_{l=0}^{k} c_{k,l}\, tr(\mathcal{D}S_3)^{2(k-l)}tr(\mathcal{D}S_{D-4})^{2l}\right). \tag{124}$$

We do not have to demonstrate the 5 properties of [8] since we start from the action found in there, but with the difference that we are forced to also include multitrace terms. An interesting property is given by the redefinitions of the time lapse function, which allowed us to make all the $tr(\mathcal{D}S)^2$ disappear for $k > 1$. Having a look at Eqs. (51) and (52) we see that it is not until third order where the mixed terms will appear in the action.
We have the total action for the anisotropic case in Eq. (124), we can redefine the S matrices, since this redefinition does not have any impact on the equations. The principal reason is to get an S matrix similar to the one in [8], and in such a way that they are constrained to $S_3^2 = \mathbb{1}_6$ and $S_{D-4}^2 = \mathbb{1}_{2(D-4)}$. From now on we define:

$$S_3 = \begin{pmatrix} 0 & a^2\cdot\mathbb{1}_3 \\ a^{-2}\cdot\mathbb{1}_3 & 0\end{pmatrix}\ ,\ S_{D-4} = \begin{pmatrix} 0 & b^2\cdot g_{mn} \\ b^{-2}\cdot g^{mn} & 0\end{pmatrix}. \tag{125}$$

To get the EOM we have to vary our action. For S_3 we obtain:

$$\begin{aligned}\delta_{S_3}I =& \int dte^{-\Phi}\sum_{k=1}^{\infty}\alpha'^{k-1}\sum_{l=0}^{k}c_{k,l}2(k-l)\Big(\dot{\Phi}tr(\delta S_3(\dot{S}_3)^{2(k-l)-1})tr(\dot{S}_{D-4})^{2l}+\\ &+tr(\delta S_3\frac{d}{dt}(\dot{S}_3)^{2(k-l)-1})tr(\dot{S}_{D-4})^{2l} + tr(\delta S_3(\dot{S}_3)^{2(k-l)-1})tr(\frac{d}{dt}(\dot{S}_{D-4})^{2l})\Big)\\ =& \int dte^{-\Phi}tr\left[\delta S_3\sum_{k=1}^{\infty}\alpha'^{k-1}\sum_{l=0}^{k}c_{k,l}2(k-l)\Big(\dot{\Phi}(\dot{S}_3)^{2(k-l)-1}tr(\dot{S}_{D-4})^{2l}+\right.\\ &\left.+(\frac{d}{dt}(\dot{S}_3)^{2(k-l)-1})tr(\dot{S}_{D-4})^{2l} + (\dot{S}_3)^{2(k-l)-1}tr(\frac{d}{dt}(\dot{S}_{D-4})^{2l})\Big)\right].\end{aligned} \tag{126}$$

We already have the form $\int dte^{-\Phi}tr(\delta S_3 F_{S_3})$, but δS_3 is a constrained variation, which means that the EOM will be given by $E_{S_3} = \frac{1}{2}(S_3F_{S_3}S_3 - F_{S_3}) = 0$ [8]. We can recover the relationship demonstrated in Eq. (61):

$$S_3\frac{d}{dt}\dot{S}_3^{2(k-l)-1}S_3 = -\frac{d}{dt}\dot{S}_3^{2(k-l)-1} - 2S_3S_3^{2(k-l)-1}, \tag{127}$$

and obtain:

$$E_{S_3}=\sum_{k=1}^{\infty}\alpha'^{k-1}\sum_{l=0}^{k}c_{k,l}2(k-l)\Bigg(\dot{\Phi}(\dot{S}_3)^{2(k-l)-1}tr(\dot{S}_{D-4})^{2l}+(\frac{d}{dt}(\dot{S}_3)^{2(k-l)-1})tr(\dot{S}_{D-4})^{2l}$$
$$-(\dot{S}_3)^{2(k-l)-1}tr(\frac{d}{dt}(\dot{S}_{D-4})^{2l})-S_3(\dot{S}_3)^{2(k-l)}tr(\dot{S}_{D-4})^{2l}\Bigg). \tag{128}$$

We proceed in the same way for S_{D-4} and obtain:

$$E_{S_{D-4}}=\sum_{k=1}^{\infty}\alpha'^{k-1}\sum_{l=0}^{k}c_{k,l}2l\Bigg(\dot{\Phi}tr((\dot{S}_3)^{2(k-l)})\dot{S}_{D-4}^{2l-1}+tr((\dot{S}_3)^{2(k-l)})\frac{d}{dt}(\dot{S}_{D-4})^{2l-1}$$
$$-tr((\dot{S}_3)^{2(k-l)})\frac{d}{dt}(\dot{S}_{D-4})^{2l-1}-tr((\dot{S}_3)^{2(k-l)})S_{D-4}\dot{S}_{D-4}^{2l}\Bigg). \tag{129}$$

By varying the action with respect to the shifted dilaton we get:

$$E_{\Phi}=2\ddot{\Phi}-\Phi^2-\sum_{k=1}^{\infty}\alpha'^{k-1}\sum_{l=0}^{k}c_{k,l}tr(\dot{S}_3^{2(k-l)})tr(\dot{S}_{D-4}^{2l}). \tag{130}$$

Finally, we vary with respect to the lapse function and then fix n=1, we obtain E_n:

$$E_n=\dot{\Phi}-\sum_{k=1}^{\infty}\alpha'^{k-1}(2k-1)\sum_{l=0}^{k}c_{k,l}tr(\dot{S}_3^{2(k-l)})tr(\dot{S}_{D-4}^{2l}). \tag{131}$$

4.1.1 FLRW

Let us now compute the EOM for the FLRW like metric. First we compute some of the expressions. Using (125) we can get the derivatives and the expressions needed as:

$$\dot{S}_3=2H\begin{pmatrix}0 & a^2\cdot\mathbb{1}_3\\ -a^{-2}\cdot\mathbb{1}_3 & 0\end{pmatrix}=2HJ_3\quad\rightarrow\quad\dot{S}_3^{2k}=(-1)^k(2H)^{(2k)}\mathbb{1}_6, \tag{132}$$

$$\dot{S}_{D-4}=2\sigma\begin{pmatrix}0 & b^2\cdot g_{mn}\\ -b^{-2}\cdot g^{mn} & 0\end{pmatrix}=2\sigma J_{D-4}\quad\rightarrow\quad\dot{S}_{D-4}^{2k}=(-1)^k(2\sigma)^{(2k)}\mathbb{1}_{2(D-4)}. \tag{133}$$

We proceed with the computation of the odd powers of the S matrices:

$$\dot{S}_3^{2(k-l)-1}=\dot{S}_3^{2(k-l-1)}\dot{S}_3=(-1)^{(k-l-1)}(2H)^{2(k-l)-1}J_3, \tag{134}$$
$$\dot{S}_{D-4}^{2l-1}=\dot{S}_{D-4}^{2(l-1)}\dot{S}_{D-4}=(-1)^{(l-1)}(2\sigma)^{(2l-1)}J_{2(D-4)}. \tag{135}$$

The time derivatives for the powers of $\dot{S}_3$ matrices, using the fact that $\dot{J}_3 = 2HS_3$ read:

$$\frac{d}{dt}\dot{S}_3^{2(k-l)} = (-1)^{k-l}2(k-l)2^{2(k-l)}H^{2(k-l)-1}\dot{H}\mathbb{1}_6, \tag{136}$$

$$\frac{d}{dt}\dot{S}_3^{2(k-l)-1} = (-1)^{k-l-1}(2(k-l)-1)2^{2(k-l)-1}H^{2(k-l-1)}\dot{H}J_3 + (-1)^{k-l-1}(2H)^{2(k-l)}S_3. \tag{137}$$

In a symmetric way, for S_{D-4}, we use $\dot{J}_{D-4} = 2\sigma S_{D-4}$, and get:

$$\frac{d}{dt}\dot{S}_{D-4}^{2l} = (-1)^{l}(2l)2^{2l}\sigma^{2l-1}\dot{\sigma}\mathbb{1}_{2(D-4)}, \tag{138}$$

$$\frac{d}{dt}\dot{S}_{D-4}^{2l-1} = (-1)^{l-1}(2l-1)2^{2l-1}\sigma^{2(l-1)}\dot{\sigma}J_{D-4} + (-1)^{l-1}(2\sigma)^{2l}S_{D-4}. \tag{139}$$

All that remains is to introduce in Eqs. (128)-(131) the expressions (132)-(139) in order to get the EOM for our ansatz (118). For E_{S_3} we have:

$$\begin{aligned} E_{S_3} = \sum_{k=1}^{\infty}\alpha'^{k-1}\sum_{l=0}^{k}c_{k,l}2(k-l)(-1)^k 2^{2k}(D-4)\Big(\dot{\Phi}H^{2(k-l)-1}\sigma^{2l} \\ +(2(k-l)-1)\dot{H}H^{2(k-l-1)}\sigma^{2l} + 2l\dot{\sigma}\sigma^{2l-1}H^{2(k-l)-1}\Big)J_3, \end{aligned} \tag{140}$$

which after a few simplifications reads:

$$E_{S_3} = -\sum_{k=1}^{\infty}(-\alpha')^{k-1}\sum_{l=0}^{k}c_{k,l}2(k-l)2^{2k-1}2(D-4)\left(\left[\frac{d}{dt}-\dot{\Phi}\right]H^{2(k-l)-1}\sigma^{2l}\right)J_3. \tag{141}$$

In the same way we can get the equation for S_{D-4}:

$$E_{S_{D-4}} = -\sum_{k=1}^{\infty}(-\alpha')^{k-1}\sum_{l=0}^{k}c_{k,l}(2l)2^{2k-1}6\left(\left[\frac{d}{dt}-\dot{\Phi}\right]H^{2(k-l)}\sigma^{2l-1}\right)J_{D-4}. \tag{142}$$

We see that as in [8] a huge simplification comes out, since we have a unique EOM for each case and not a matrix equation, since all the terms with the S matrices vanish. We complete the set of equations with E_n and E_Φ:

$$E_n = \Phi^2 + \sum_{k=1}^{\infty}(-\alpha')^{k-1}(2k-1)12(D-4)\sum_{l=0}^{k}c_{k,l}2^{2k}H^{2(k-l)}\sigma^{2l}, \tag{143}$$

$$E_\Phi = 2\ddot{\Phi} - \Phi^2 + \sum_{k=1}^{\infty}(-\alpha')^{k-1}12(D-4)\sum_{l=0}^{k}c_{k,l}2^{2k}H^{2(k-l)}\sigma^{2l}. \tag{144}$$

Looking at the resulting EOM, we can define 3 functions $f(H,\sigma)$, coming from (141), $g(H,\sigma)$, from (142), and $h(H,\sigma)$, from (143).

$$f(H,\sigma) = 12(D-4)\sum_{k=1}^{\infty}(-\alpha')^{k-1}\sum_{l=0}^{k} c_{k,l}2^{2k}2(k-l)H^{2(k-l)-1}\sigma^{2l}, \tag{145}$$

$$g(H,\sigma) = 12(D-4)\sum_{k=1}^{\infty}(-\alpha')^{k-1}\sum_{l=0}^{k} c_{k,l}2^{2k}2lH^{2(k-l)}\sigma^{2l-1}, \tag{146}$$

$$h(H,\sigma) = 12(D-4)\sum_{k=1}^{\infty}(-\alpha')^{k-1}\sum_{l=0}^{k} c_{k,l}2^{2k}(2k-1)H^{2(k-l)}\sigma^{2l}, \tag{147}$$

where for convenience we multiplied the $f(H,\sigma)$ function, and then Eq. (141) by 12, and $g(H,\sigma)$, i.e. Eq. (142) by $4(D-4)$. These factors are consistent with the results of the isotropic case [8], where we multiply the S EOM by 4d, i.e. 4 times the dimensionality of the metric with respect to which we are varying. Moreover, we can look for a unique function, which is linked to those three via their derivatives. We define:

$$F(H,\sigma) = 12(D-4)\sum_{k=1}^{\infty}(-\alpha')^{k-1}\sum_{l=0}^{k} c_{k,l}2^{2k}H^{2(k-l)}\sigma^{2l}. \tag{148}$$

Via a trivial calculation we find:

$$\begin{gathered} f(H,\sigma) = \partial_H F(H,\sigma) \quad , \quad g(H,\sigma) = \partial_\sigma F(H,\sigma), \\ h(H,\sigma) = H\partial_H F(H,\sigma) + \sigma\partial_\sigma F(H,\sigma) - F(H,\sigma), \end{gathered} \tag{149}$$

with $\partial_H F = \frac{\partial F}{\partial H}$, and $\partial_\sigma F = \frac{\partial F}{\partial \sigma}$. We point out that, for a more general case in which we study an anisotropic D-dimensional space with d dimensions expanding with scale factor a, and D-d-1 dimensions with scale factor b, we just have to replace the factors of 3 by d, and the factors D-4 by D-d-1. Using Eq. (148), we get to the final EOM for the anisotropic case, as a function of a unique $F(H,\sigma)$ and its derivatives:

$$\dot{H}\partial_H^2 F(H,\sigma) + \dot{\sigma}\partial_{H\sigma}F(H,\sigma) - \dot{\Phi}\partial_H F(H,\sigma) = 0, \tag{150a}$$

$$\dot{\sigma}\partial_\sigma^2 F(H,\sigma) + \dot{H}\partial_{\sigma H}F(H,\sigma) - \dot{\Phi}\partial_\sigma F(H,\sigma) = 0, \tag{150b}$$

$$\ddot{\Phi} + \frac{1}{2}\left(H\partial_H F(H,\sigma) + \sigma\partial_\sigma F(H,\sigma)\right) = 0, \tag{150c}$$

$$\dot{\Phi}^2 + H\partial_H F(H,\sigma) + \sigma\partial_\sigma F(H,\sigma) - F(H,\sigma) = 0, \tag{150d}$$

where (150c) is the sum of (143) and (144). A positive point of these equations is that having a constant shifted dilaton does no longer mean a constant Hubble parameter and vanishing F and F' as in the isotropic case (Eqs. (80)). As expected, the EOM are totally symmetric in H and σ. We can see that with the use of $F(H,\sigma)$ we can rewrite the total action for $n=1$ as:

$$I = \int dte^{-\Phi}\left(-\dot{\Phi}^2 - F(H,\sigma)\right). \tag{151}$$

4.1.2 First order

In order to check the results, we can compare Eqs. (150) with the first order anisotropic stringy Friedman equations in vaccum [7]. Evaluating F and its derivatives at first order:

$$\begin{gathered} F = 2^4 3(D-4)(c_{1,0}H^2 + c_{1,1}\sigma^2), \\ \partial_H F = 2^5 3(D-4)c_{1,0}H \quad , \quad \partial_\sigma F = 2^5 3(D-4)c_{1,1}\sigma, \\ \partial_H^2 F = 2^5 3(D-4)c_{1,0} \quad , \quad \partial_\sigma F = 2^5 3(D-4)c_{1,1} \quad , \quad \partial_{H\sigma}F = 0. \end{gathered} \tag{152}$$

By introducing them into Eqs. (150) we obtain the stringy equations at first order:

$$\dot{H} - \dot{\Phi}H = 0, \tag{153a}$$

$$\dot{\sigma} - \dot{\Phi}\sigma = 0, \tag{153b}$$

$$\ddot{\Phi} - 3H^2 - (D-4)\sigma^2 = 0, \tag{153c}$$

$$\dot{\Phi}^2 - 3H^2 - (D-4)\sigma^2 = 0, \tag{153d}$$

imposing the coefficients at first order to be:

$$c_{1,0} = -\frac{1}{(D-4)\cdot 2^4} \quad , \quad c_{1,1} = -\frac{1}{3\cdot 2^4}. \tag{154}$$

Those coefficients are consistent, since comparing them with the usual $c_1 = -1/8$, we have a difference of wether $1/(2(D-4))$, or $1/(2\cdot 3)$. These factors are due to the presence of the factor with the trace of $\dot{S}_{D-4}^{2l}$ in E_{S_3}, and viceversa, as can be seen in Eqs. (128) and (129). Moreover, the function $F(H,\,\sigma)$ at leading order will have the expected form [12]:

$$F(H,\,\sigma) = -3H^2 - (D-4)\sigma^2 + \mathcal{O}(\alpha'). \tag{155}$$

4.2 Matter

The vacuum case shows a total symmetric appearence of H and σ in the EOM. To make it more interesting, and break the degeneracy of H and σ we can introduce a matter action:

$$I = \frac{1}{2\kappa^2}\int dtne^{-\Phi}\left(-(\mathcal{D}\Phi)^2 + \sum_{k=1}^{\infty}\alpha'^{k-1}\sum_{l=0}^{k}c_{k,l}\,tr(\mathcal{D}S_3)^{2(k-l)}tr(\mathcal{D}S_{D-4})^{2l}\right) + S_m[\Phi, n, S_3, S_{D-4}, \chi]. \tag{156}$$

The EOM are obtained in the exact same way as in the isotropic case. We show here the results obtained, which have been compared with the works [2], [12], [7]:

$$\dot{H}\partial_H^2 F(H,\,\sigma) + \dot{\sigma}\partial_{H\sigma}F(H,\,\sigma) - \dot{\Phi}\partial_H F(H,\,\sigma) = -2d\kappa^2 e^{\Phi}\bar{p}, \tag{157a}$$

$$\dot{\sigma}\partial_\sigma^2 F(H,\,\sigma) + \dot{H}\partial_{\sigma H}F(H,\,\sigma) - \dot{\Phi}\partial_\sigma F(H,\,\sigma) = -2(D-d-1)\kappa^2 e^{\Phi}\bar{q}, \tag{157b}$$

$$2\ddot{\Phi} - \dot{\Phi}^2 + F(H,\,\sigma) = \kappa^2 e^{\Phi}\bar{\theta}, \tag{157c}$$

$$\dot{\Phi}^2 + H\partial_H F(H,\,\sigma) + \sigma\partial_\sigma F(H,\,\sigma) - F(H,\,\sigma) = 2\kappa^2 e^{\Phi}\bar{\rho}, \tag{157d}$$

where p now denotes the pressure in the d-dimensional space with Hubble parameter $H(t)$ and q the pressure in the $D-d-1$ extra dimensions, which come from the spatial part of the energy momentum tensor $T = diag(\rho, p, p, p, q, q...)$. The continuity equation reads:

$$\dot{\bar{\rho}} + dH\bar{p} + (D-d-1)\sigma\bar{q} - \frac{1}{2}\dot{\Phi}\bar{\theta} = 0. \tag{158}$$

In order to compare with the isotropic case, we leave the more general case where we make the factor d appear, which in this section is $d=3$. It is easy to see that now the degeneracy between H and σ can be broken by means of their pressure. We will mostly suppose the extra dimensions to be pressureless.

4.3 Einstein frame

In this section we want to compute the Hubble parameter in the 4-dimensional Einstein frame in terms of the string variables and derivatives. To have an idea of the Hubble parameter in the Einstein frame, we have to look at the action in the string frame, which is of the form:

$$\int d^D x\sqrt{-G^{(S)}}e^{-2\phi}(R^{(S)} + ...), \tag{159}$$

with $G^{(S)}$ the determinant of our D-dimensional metric in the string frame, and $R^{(S)}$ the associated Ricci scalar. In order to go to the Einstein frame we need to perform a conformal transformation so that we only have our 4-dimensional metric factor in front of the Ricci scalar, this is normally done in 3 steps. First we go to the Einstein D-dimensional frame, then we compactify the extra dimensions so that we get to the 4-dimensional Jordan frame, and finally we do a second Weyl rescaling to the 4-dimensional Einstein frame, as can be seen in [14]. The Weyl rescaling is defined by:

$$g_{MN} = \Omega^{-2}\hat{g}_{\mu\nu} \quad , \quad \sqrt{-g} = \Omega^{-D}\sqrt{-\hat{g}}, \tag{160}$$

where $\hat{g}$ is the metric in the new frame. As it is constructed from the metric, the Ricci scalar will also transform. Focusing only in the part of the Ricci scalar in the new frame we have:

$$R = \Omega^2(\hat{R} + ...). \tag{161}$$

In this section we use the index s for the variables in the D-dimensional string frame, the index ed for the variables in the D-dimensional Einstein frame, which coincides with the 4-dimensional Jordan frame after splitting the metric and Ricci in 4 and D-4-dimensional parts, and finally the index E for the 4-dimensional Einstein frame.
Let us focus on the first Weyl transformation to the D-dimensional Einstein frame. First, we have to conformally transform the complete D-dimensional metric in order to eliminate the $e^{-2\phi}$ factor. With the transformation of the determinant of the metric in Eq. (160), and the one of the Ricci scalar we can see from Eq. (159) that we need a Weyl parameter such as:

$$\Omega_1 = e^{-\frac{2\phi}{D-2}}, \tag{162}$$

with the metrics in the string and Einstein D-dimensional frame respectively given by:

$$G^{(S)}_{MN} = \begin{pmatrix} -n_s^2(t) & 0 & 0 \\ 0 & a_s^2(t)\cdot \mathbb{1}_3 & 0 \\ 0 & 0 & b_s^2(t)\cdot g_{mn} \end{pmatrix}, G^{(ED)}_{MN} = \begin{pmatrix} -n_{ed}^2(t) & 0 & 0 \\ 0 & a_{ed}^2(t)\cdot \mathbb{1}_3 & 0 \\ 0 & 0 & b_{ed}^2(t)\cdot g_{mn} \end{pmatrix}. \tag{163}$$

Then, to go to the 4-dimensional Jordan frame, we decompose the Ricci into a 4 and a D-4-dimensional part, we do not obtain any extra factor in front of the new 4-dimensional Ricci scalar. The process makes only appear the b_{ed}^{D-4} that comes from the metric: $\sqrt{-G^{(ED)}} = \sqrt{-g^{(ED)}}b_{ed}^{D-4}$, where $g^{(ED)}$ is the determinant of the 4-dimensional part of $G^{(ED)}$. In such a way that we will need a second Weyl transformation, from the determinant of the metric we will have a factor Ω_2^{-4}, and from the Ricci scalar we have Ω_2^2. A very important point to remark is that the factor b_{ed}^{D-4} that comes from splitting the metric is in the Einstein D-dimensional frame, we should then express it in terms of the b_s scale factor in the string frame. Thus, the second conformal transformation, now only on the 4-dimensional metric, will be given by:

$$\Omega_2 = b_{ed}^{\frac{D-4}{2}} = (b_s e^{-\frac{2\phi}{D-2}})^{\frac{D-4}{2}}, \tag{164}$$

where in the last equality we expressed everything in terms of string variables. Moreover, b_s now is not interpreted as the scale factor of the extra dimensions, but as a scalar field. The two metrics involved in the transformation are:

$$g^{(ED)}_{\mu\nu} = \begin{pmatrix} -n_{ed}^2(t) & 0 \\ 0 & a_{ed}^2(t)\cdot \mathbb{1}_3 \end{pmatrix}, g^{(E)}_{\mu\nu} = \begin{pmatrix} -n_E^2(t) & 0 \\ 0 & a_E^2(t)\cdot \mathbb{1}_3 \end{pmatrix}. \tag{165}$$

The total transformation on the 4-dimensional part of the string metric will be:

$$\Omega = e^{-\frac{2\phi}{D-2}}(b_s e^{-\frac{2\phi}{D-2}})^{\frac{D-4}{2}} = e^{-\phi} b_s^{\frac{D-4}{2}} \quad \to \quad g^{(E)}_{\mu\nu} = e^{-2\phi} b_s^{D-4} g^{(S)}_{\mu\nu}, \tag{166}$$

where $g^{(E)}_{\mu\nu}$ is the 4-dimensional metric in the Einstein frame, and $g^{(S)}_{\mu\nu}$ the 4-dimensional part of the metric in the string frame. We can use the freedom to reparametrise the time in order to have the cosmic time T in the Einstein frame also, as done in [4], and [11]. The time reparametrisation is such that in the new time T, we have $g_{00}'^{(E)} = -1$. In this way, the scale factor in the Einstein frame and the differential equation for the time reparametrisation will be given by:

$$a_E(T) = a_s(t)e^{-\phi(t)}b_s^{\frac{D-4}{2}}(t) \quad , \quad \frac{dT}{dt} = \frac{n_E}{n_s} = e^{-\phi(t)}b_s^{\frac{D-4}{2}}(t). \tag{167}$$

Instead of the time reparametrisation technique we recover the covariant derivatives formalism. We define them for the string frame and the 4-dimensional Einstein frame respectively by the following relation:

$$\mathcal{D}_s = \frac{1}{n_s}\frac{d}{dt} \quad , \quad \mathcal{D}_E = \frac{1}{n_E}\frac{d}{dt}, \tag{168}$$

they are related by:

$$\mathcal{D}_E = e^{\frac{2\phi}{D-2}}b^{-\frac{D-4}{2}}\mathcal{D}_s. \tag{169}$$

We recall the definition of the shifted dilaton, and its derivatives for the anisotropic case:

$$e^{2\phi} = e^{\Phi}a_s^3 b_s^{D-4} \quad \rightarrow \quad 2\mathcal{D}_s\phi = \mathcal{D}_s\Phi + 3H + (D-4)\sigma, \tag{170}$$

where $H = \frac{1}{a_s}\mathcal{D}_s a_s$, and $\sigma = \frac{1}{b_s}\mathcal{D}_s b_s$. We compute the Hubble parameter in the Einstein frame:

$$H_E(T) = \frac{1}{a_E}\mathcal{D}_E a_E = e^{\phi}b_s^{-\frac{D-4}{2}}\left(-\mathcal{D}_s\phi + \frac{D-4}{2}\sigma + H\right). \tag{171}$$

Using relation (170), H_E can also be written as:

$$H_E(T) = -\frac{1}{2}e^{\frac{\Phi}{2}}a_s^{\frac{3}{2}}(\mathcal{D}_s\Phi + H). \tag{172}$$

We can now compare with the expression for the isotropic case in Eq. (113). It is the same Hubble factor taking $d = 3$. Let us now compute the derivative with respect to T:

$$\frac{dH_E(T)}{dT} = -\frac{1}{4}e^{\Phi}a^3\left((\mathcal{D}_s\Phi + 3H)(\mathcal{D}_s\Phi + H) + 2(\mathcal{D}_s^2\Phi + \mathcal{D}_s H)\right). \tag{173}$$

Imposing then constant Hubble parameter and shifted dilaton velocity, we find two equations giving a constant Hubble parameter in the Einstein frame:

$$\mathcal{D}_s\Phi = -H \quad , \quad \mathcal{D}_s\Phi = -3H. \tag{174}$$

We can observe that unlike what we could expect, the parameters of the extra dimensions do not appear in these expressions, neither in that of H_E, nor in that of its derivative when we write the expressions in terms of the shifted dilaton Φ. But at this point it is important to note that we express everything in terms of Φ for the dynamical system analysis in section 5. In Eq. (171) which is in terms of the dilaton ϕ, the real physical degree of freedom, we can see that in fact the Hubble parameter does depend on σ. Moreover, we obtain the same results as for the isotropic case, but here we do not have d as a free parameter of the theory, but it is fixed at $d = 3$.

4.3.1 Differential equation for the vacuum case

With the objective of getting a de Sitter solution in the Einstein 4-dimensional frame, we now impose Eq.(173) to vanish, and using Eqs. (150) we try to establish a differential equation for $F(H,\,\sigma)$ in terms of H and σ exclusively. As in [8], we suppose that the second derivatives: $\partial_\sigma^2 F(H,\,\sigma)$ and $\partial_H^2 F(H,\,\sigma)$ are non vanishing, so with the expressions:

$$\mathcal{D}_s H = \mathcal{D}_s\Phi \frac{\partial_\sigma^2 F \partial_H F - \partial_{\sigma H} F \partial_\sigma F}{\partial_\sigma^2 F \partial_H^2 F - (\partial_{\sigma H} F)^2} \quad , \quad \mathcal{D}_s \sigma = \mathcal{D}_s\Phi \frac{\partial_H^2 F \partial_\sigma F - \partial_{\sigma H} F \partial_H F}{\partial_\sigma^2 F \partial_H^2 F - (\partial_{\sigma H} F)^2},$$
$$\mathcal{D}_s^2 \Phi = -\frac{1}{2}(H\partial_H F + \sigma \partial_\sigma F) \quad , \quad \mathcal{D}_s \Phi^2 = F - (H\partial_H F + \sigma\partial_\sigma F), \tag{175}$$

we get an extra condition on the derivatives of the F function, the determinant of its Hessian has to be different from zero. By imposing Eq. (173) to vanish, we get:

$$0 = \mathcal{D}_s\Phi^2 + 4H\mathcal{D}_s\Phi + 3H^2 + 2\mathcal{D}_s^2\Phi + 2\mathcal{D}_s H. \tag{176}$$

Then introducing expressions (175) we obtain:

$$\begin{aligned} 0 = F - 3(H\partial_H F + \sigma\partial_\sigma F) + 4Hs\sqrt{F - (H\partial_H F + \sigma\partial_\sigma F)} + 3H^2 \\ +2s\sqrt{F - (H\partial_H F + \sigma\partial_\sigma F)}\frac{\partial_\sigma^2 F \partial_H F - \partial_{\sigma H} F \partial_\sigma F}{\partial_\sigma^2 F \partial_H^2 F - (\partial_{\sigma H} F)^2}, \end{aligned} \tag{177}$$

where, as in the isotropic case $s = \pm 1$, due to the square root of $\mathcal{D}_s\Phi^2$. This expression cannot be compared with the work of Padilla et. al. [12] since they take a general anisotropic ansatz, no 3 and D-4-dimensional parts, so when looking for the Hubble parameters in the Einstein frame, they only perform the first Weyl rescaling as we did in the isotropic part. But as in our work we are looking forward to a compactification we decided to proceed with the complete scheme. We can remark that finally we see appear the extra dimensional terms in our expression, they appear through the EOM.

5 Fixed point analysis

As we have seen, the equations for constant Hubble parameter in the Einstein frame for time evolving H, y, and σ are difficult to solve analytically. Let us now look for the fixed points in the EOM of all four dynamical systems we have seen until now. For this section we fix the time lapse function in the string frame to be 1. We will look for solutions with constant H, σ and $\dot{\Phi}$, comparing our results with [4] and [3] for the isotropic case. These fixed points are especially interesting, since what we are looking for are constant stable Hubble parameters, even if they are in the string frame. Within this framework we will also analyse the linear stability of these points. Finally, we will look at the expressions of these constant Hubble parameters solutions in the Einstein frame. Let us first introduce how we will obtain the fixed points and their stability.

We follow the approach in [18]. We suppose a vector $\mathbf{x}$ with the dynamical variables of our system, i.e. (H, y) for the isotropic case, or (H, σ, y) for the anisotropic one, the dynamical system is the following:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \tag{178}$$

where $\mathbf{f}(\mathbf{x})$ is a vector-valued function of the dimensionality of $\mathbf{x}$. The linear analysis lies on Taylor expanding our vector valued function up to first order, and according to the Hartman-Grobman theorem, the linearized system represents the full nonlinear system near the fixed point. For this we expand the variables as $\mathbf{x} = \mathbf{x}_0 + \delta\mathbf{x}$, where $\mathbf{x}_0$ is the fixed point. The $\mathbf{f}(\mathbf{x})$ are expanded as:

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + \mathcal{J}(\mathbf{f}(\mathbf{x}_0))\delta\mathbf{x} + \mathcal{O}(\delta\mathbf{x}^2), \tag{179}$$

where $\mathcal{J}(\mathbf{f}(\mathbf{x}_0))$ is the Jacobi matrix of the system evaluated in the fixed point. Thus we find:

$$\frac{d\delta\mathbf{x}}{dt} = \mathcal{J}(\mathbf{f}(\mathbf{x}_0))\delta\mathbf{x} + \mathcal{O}(\delta\mathbf{x}^2). \tag{180}$$

The solution for each of the variables is then $\delta x_i \approx e^{m_i t}$ where m_i are the eigenvalues of the Jacobian matrix. Therefore, if the real part of the eigenvalues are negative, this means that the linear perturbations will decrease with time, giving a linearly stable fixed point. If one of them is positive, one of the perturbation mode will grow exponentially with time, making the fixed point unstable. Instead, if we have one of the eigenvalues vanishing and the rest negative, the fixed point is called a bifurcation. In this case, one of the perturbation modes remains constant at linear order and the others decrease exponentially.

5.1 Isotropic in vacuum

We recall the EOM for the isotropic case in vacuum (80), for the search of fixed points we will rearrange them as:

$$\dot{H} = y\frac{F'(H)}{F''(H)}, \tag{181a}$$

$$\dot{y} = \frac{1}{2}(y^2 - F(H)), \tag{181b}$$

$$y^2 + HF'(H) - F(H) = 0, \tag{181c}$$

where we define $y = \dot{\Phi}$, and impose $F'' \neq 0$. To compute the fixed points, we establish $\dot{H} = \dot{y} = 0$. So Eq. (181b) fixes $y_0^2 = F_0$, we have then:

$$y_0 F_0' = 0 \quad , \quad H_0 F_0' = 0 \quad , \quad y_0^2 = F_0. \tag{182}$$

Recalling the form of $F(H)$ in Eq. (79), we can see that for vanishing H we have vanishing $F(H)$ and $F'(H)$, but this relation is not an equivalence. We can find a function which has a zero value for $F(H_0)$ or its derivative $F'(H_0)$ and still have a non-vanishing Hubble parameter $H_0 \neq 0$. As the solutions y_0 and H_0 will be two constants, a part from the special cases in which H_0 is zero, we can suppose that they are proportional $y_0 = cH_0$. This idea is also motivated by Eq. (114), where we will have constant Hubble parameter in both string and Einstein frames for $\dot{\Phi} = -H_0$ or $\dot{\Phi} = -dH_0$. We present the fixed points of this system in table 1.

y	$F'(H)$	Stability
0	0	Bifurcation
cH_0	0	Stable for $c < 0$

Table 1: Fixed points for the isotropic case in vacuum.

As explained previously, and done in [4] and [6] in order to determine the linear stability, we calculate the derivatives of (181a) and (181b) with respect to H and y and we evaluate the eigenvalues in the fixed points. We get:

$$\begin{pmatrix} \frac{\partial \dot{H}}{\partial H} & \frac{\partial \dot{H}}{\partial y} \\ \frac{\partial \dot{y}}{\partial H} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} y_0 & \frac{F_0'}{F_0''} \\ -\frac{1}{2}F_0' & y_0 \end{pmatrix} \quad \rightarrow \quad m_\pm = y_0 \pm \sqrt{-\frac{1}{2}\frac{F_0'^2}{F_0''}}, \tag{183}$$

where $m_\pm$ are the two eigenvalues of the Jacobian matrix. For it to be a linearly stable point we need to have both eigenvalues negative. First we see that the square root cancels in both cases since $F_0' = 0$. For the first fixed point, which has a vanishing y_0, we see that both eigenvalues vanish, we then have to look at non-linear stability [3]. The linearly stable point is then for $y_0 = cH_0$, and the condition of stability is $c < 0$, which is in accord with the solutions of constant Hubble parameter in the Einstein frame. So by introducing the set of infinite α' corrections codified in the function $F(H)$ we can find de Sitter solutions with the condition: $F'(H) = 0$. For the fixed points in vacuum to be stable, we need to ask a non vanishing shifted dilaton velocity, which has the opposite sign to H_0.

5.1.1 Hubble parameter in the Einstein frame

Let us examine the Hubble parameter in the Einstein frame for the stable fixed points. For a non vanishing $\dot{\Phi}$, we will have $2\dot{\phi} = \dot{\Phi}_0 + dH_0 = (c+d)H_0$. As the derivative of the dilaton ϕ is a constant, we can integrate the expression in order to obtain the variation of the dilaton as a function of the time t:

$$\phi(t) = (c+d)H_0(t - t_0)/2 + \phi_0, \tag{184}$$

that following Eq. (113) gives a Hubble parameter in the Einstein frame such as:

$$H_E(t') = -\frac{e^{\frac{(c+d)H_0(t-t_0)+2\phi_0}{d-1}}}{d-1}(c+1)H_0. \tag{185}$$

It is easy to see in this expression the 2 special cases mentioned above. For $c = -1$ we obtain a zero H_E, and for $c = -d$ the velocity of the dilaton vanishes. then the H_E is constant.

If we look for a positive H_E, it is important to note that the linear stability condition is to have $c < 0$, and H_0 is in the string frame, so its sign is not really relevant for us. We have two possibilities: one in which $H_0 < 0$, then $-1 < c < 0$, and another situation for $H_0 > 0$, and $c < -d$. For both of them we start with a constant positive Hubble parameter and in the limit $t \to \infty$ it vanishes.

We must now remember that t is the cosmic time in the string frame, with the expression for ϕ and recall the relation:

$$dte^{\frac{-2\phi}{d-1}} = dt', \tag{186}$$

where t' is the reparametrised cosmic time in the Einstein frame. By integrating this equation we obtain a relation between the two time parametrisations:

$$t' - t'_0 = -\frac{2(d-1)}{(c+d)H_0}e^{\frac{-(c+d)H_0(t-t_0)-2\phi_0}{d-1}}, \tag{187}$$

with t'_0 and t_0 two integration constants. By introducing it into Eq. (185) we obtain a final expression for $H_E(t')$:

$$H_E(t') = \left(\frac{c+1}{c+d}\right)\frac{1}{t' - t'_0}. \tag{188}$$

Thus, for $t' > t'_0$ we will need the prefactor to be positive, so that we start with a positive Hubble parameter, and for $t' \to \infty$ we have $H_E \to 0$. For it to be positive, we need to impose either $c < -d$, or $-1 < c < 0$. We can note from Eq. (187) that in the case of $c < -d$ for a positive H_0; and in the case $-1 < c < 0$ for a negative H_0, we have both times parametrised in the same direction. These two cases can be interpreted as a large and negative shifted dilaton velocity case; and a small but positive shifted dilaton velocity respectively.

So, as already advanced in section 3.3, for $\dot{\Phi} = -H_0$ we have a Minkowski solution, and for $\dot{\Phi} = -dH_0$ we have a de Sitter universe in the Einstein frame. For a different proportionality factor between these two, we can get stable solutions where the Hubble parameter in the Einstein frame is always positive and decreases asymptotically to a vanishing value.

5.2 Isotropic with matter

Using the EOM for the isotropic case with matter (108), assuming a barotropic equation of state for the fluid $p = \omega\rho$ and a dilatonic charge such that $\theta = \lambda\rho$, we can rearrange them for the study of the fixed points as:

$$\dot{H} = \frac{1}{F''}(yF' - d\omega(y^2 + HF' - F)), \tag{189a}$$

$$\dot{y} = \frac{1}{2}\left(y^2 - F + \frac{\lambda}{2}(y^2 + HF' - F)\right), \tag{189b}$$

$$y^2 + H\,F' - F = 2\kappa^2 e^{\Phi}\bar{\rho}. \tag{189c}$$

Let us compute the fixed points for a general case. Motivated by Eq. (114), we choose $y_0 = cH_0$. By introducing this expression we get:

$$-(c^2H_0^2 - F_0^2)d\omega - (d\omega - c)H_0F'_0 = 0, \tag{190a}$$

$$(c^2H_0^2 - F_0)(1 + \frac{\lambda}{2}) + \frac{\lambda}{2}H_0F'_0 = 0, \tag{190b}$$

$$c^2H_0^2 + H_0F'_0 - F_0 = 2\kappa^2 e^{\Phi}\bar{\rho}. \tag{190c}$$

By substituting (190a) into (190b) for $\omega \neq 0$ we have:

$$H_0F'_0\left(\omega d - c\left(1 + \frac{\lambda}{2}\right)\right) = 0, \tag{191}$$

where clearly we can have $F'_0 = 0$ which looking at Eq. (190a) will give us $H_0^2c^2 = F_0$, which coincides with the general case of the vacuum equations. We can also obtain an interesting alternative option which reads: $\omega d = c(1 + \lambda/2)$, giving a relation: $(F_0 -$

$c^2H_0^2) = \lambda\kappa^2 e^{\Phi}\bar{\rho}$. The Eq. (191) relates the parameter of the equation of state of our fluid, with the Hubble factor, the shifted dilaton velocity and the dilatonic charge. By reintroducing the Hubble factor in the equation we have $\omega d H_0 = \dot{\Phi}_0(1+\lambda/2)$.
We observe two special cases: the case for which we have a high coupling between the dilaton and the matter action $\lambda = -2$ which fixes the fluid to be pressureless, ordinary matter $\omega = 0$ and $F_0' = 0$, and the case where the dilaton is not coupled to matter $\lambda = 0$ which gives a negative pressure for stable fixed points $\omega = c/d$ and a value of the Hubble factor given by $H_0^2 c^2 = F_0$.

We now have to take a look at the solutions for $\omega = 0$, the Eqs. (189) read now:

$$0 = y_0 F_0', \tag{192a}$$

$$0 = y_0^2 - F_0 + \frac{\lambda}{2}(y_0^2 + H_0 F_0' - F_0), \tag{192b}$$

$$y_0^2 + H_0\, F_0' - F_0 = 2\kappa^2 e^{\Phi}\bar{\rho}. \tag{192c}$$

We consider first the special case in which we have vanishing $F_0'(H)$, then the case of vanishing y_0. For the case in which $F'(H) = 0$ we get:

$$(y_0^2 - F_0)\left(1 + \frac{\lambda}{2}\right) = 0, \tag{193a}$$

$$y_0^2 - F_0 = 2\kappa^2 e^{\Phi}\bar{\rho}, \tag{193b}$$

for which we have again two options: either $\lambda = -2$ for a general y_0 giving $y_0^2 - F_0 = 2\kappa^2 e^{\Phi}\bar{\rho}$, either $y_0^2 - F_0 = 0$ in which we recover the vacuum case. For the $y_0 = 0$ case the equations read:

$$-F_0 + \frac{\lambda}{2}(H_0 F_0' - F_0) = 0, \tag{194a}$$

$$H_0 F_0' - F_0 = 2\kappa^2 e^{\Phi}\bar{\rho}. \tag{194b}$$

We obtain a vacuum case for which $F_0 = F_0' = 0$, already commented in the previous section, then one for which from (194a) we get $H_0 F_0' = (2+\lambda)\kappa^2 e^{\Phi}\bar{\rho}$. We summarise the fixed points in table 2 neglecting the vacuum cases. In the table we have added a last line for a fixed point from the general case $y_0 = cH_0$ with $\lambda = 0$, since as we will see later, it has a special stability condition.

In order to evaluate the stability of these fixed points we proceed in the same way; we compute the Jacobian of the differential equations system:

$$\begin{pmatrix} \frac{\partial \dot{H}}{\partial H} & \frac{\partial \dot{H}}{\partial y} \\ \frac{\partial \dot{y}}{\partial H} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} y_0 - d\omega H_0 & \frac{F_0' - 2d\omega y_0}{F_0''} \\ -\frac{1}{2}(\frac{\lambda}{2}H_0 F_0'' - F_0') & y_0(1+\frac{\lambda}{2}) \end{pmatrix}. \tag{195}$$

After some simplifications we obtain the two eigenvalues $m_\pm$:

$$m_\pm = -\frac{d\omega H_0}{2} + y_0(1+\frac{\lambda}{4}) \pm \frac{1}{2}\sqrt{(d\omega H_0 - \frac{\lambda}{2}y_0)^2 + \lambda F_0' H_0 - \frac{2F_0'}{F_0''}(F_0' - 2d\omega y_0)}. \tag{196}$$

y	F	F'	ω	λ	Stability
0	$\lambda\kappa^2 e^{\Phi}\bar{\rho}$	$H_0 F_0' = (2+\lambda)\kappa^2 e^{\Phi}\bar{\rho}$	0	All	Unstable
cH_0	$c^2 H_0^2 - F_0 = -\lambda\kappa^2 e^{\Phi}\bar{\rho}$	$H_0 F_0' = (2+\lambda)\kappa^2 e^{\Phi}\bar{\rho}$	$\omega d = c(1+\frac{\lambda}{2})$		$c<0$ $-2<\lambda<0$
cH_0	$c^2 H_0^2 - F_0 = 2\kappa^2 e^{\Phi}\bar{\rho}$	0	0	-2	Bifurcation for $c<0$
cH_0	$H_0^2 c^2$	$H_0 F_0' = 2\kappa^2 e^{\Phi}\bar{\rho}$	c/d	0	Bifurcation for $c<0$

Table 2: Fixed points for the isotropic case with matter

So if both eigenvalues are negative, we will have a stable point, and extra conditions on our parameters. For the case $\omega d = c(1+\frac{\lambda}{2})$ we need to simplify the eigenvalues from Eq. (196) and obtain [4]:

$$m_{\pm} = \frac{H_0}{2}|c|(sgn(c) \pm \sqrt{1+2\lambda+\lambda^2}). \tag{197}$$

Thus fixing the interval for λ for which our solution is stable, being $(-2;0)$ and also giving negative c as the only class of solutions. Now for $\lambda = 0$ the two eigenvalues are:

$$m_+ = 0 \quad , \quad m_- = cH_0. \tag{198}$$

From [4], in order for this fixed point to be linearly stable we need a negative c, arguing that this is exactly what happens when perturbing around dS vacuum in GR, but in this case we have the constant shift of the Hubble parameter, and of the shifted dilaton.
For the case with vanishing ω and y_0 we will have a positive and a negative eigenvalue, meaning that this fixed point is not stable.

These solutions are quite interesting, since for the case $c = -1$ we get Minkowski, and for $c = -d$ we get de Sitter in both, the string and the Einstein frame.
A very important verification is confirming under which conditions $\bar{\rho}e^{\Phi}$ is constant. For this we can compute its derivative and use the continuity equation (110), we get:

$$\begin{aligned} \frac{d}{dt}\bar{\rho}e^{\Phi} &= e^{\Phi}(\dot{\bar{\rho}} + \dot{\Phi}\bar{\rho}) \\ &= e^{\Phi}\left(\dot{\Phi}\left(\frac{1}{2}\bar{\theta} + \bar{\rho}\right) - dH\bar{p}\right). \end{aligned} \tag{199}$$

By supposing $p = \omega\rho$ and $\theta = \lambda\rho$ as usual, we obtain that the condition for vanishing Eq. (199) with $\bar{\rho}e^{\Phi} \neq 0$ is given by:

$$y\left(1+\frac{\lambda}{2}\right) = d\omega H. \tag{200}$$

We remark that by imposing $\bar{\rho}e^{\Phi}$ to be constant, we obtain an equivalent equation as for the study of the fixed points of the dynamical system.

5.2.1 Hubble parameter in the Einstein frame

Let us study as in the vacuum case, the Hubble parameter in the Einstein frame for the stable solutions. For the general case we have $\omega d = c(1 + \frac{\lambda}{2})$ and the two conditions are $c < 0$ and $-2 < \lambda \leq 0$, which means that $\omega < 0$. For the solution to be stable we need a fluid with negative pressure. The expression of the Hubble parameter is the same as for the vacuum case:

$$\begin{aligned} H_E(t') &= -\frac{e^{\frac{(c+d)H_0(t-t_0)+2\phi_0}{d-1}}}{d-1}(c+1)H_0 \\ &= \left(\frac{c+1}{c+d}\right)\frac{1}{t'-t'_0} \\ &= \left(\frac{\omega d + 1 + \frac{\lambda}{2}}{d(\omega + 1 + \frac{\lambda}{2})}\right)\frac{1}{t'-t'_0}, \end{aligned} \tag{201}$$

where we used the stability equation: $c(1+\lambda/2) = \omega d$. As in the vacuum case, for this last equation, we suppose $c \neq -d$, since for $c = d$ we get a de Sitter solution in the Einstein frame: $H_E = e^{2\phi_0}H_0$, and an equation of state parameter given by: $\omega = -(1 + \frac{\lambda}{2})$. The case where the dilaton is not coupled to matter $\lambda = 0$ will give us vacuum energy.

In general for $\omega \neq -(1+\frac{\lambda}{2})$ we can see from Eq. (201), that for the Hubble parameter to be positive for $t' > t'_0$, we need to impose $\omega < -(1 + \frac{\lambda}{2})$. As the stability condition on the dilatonic charge parameter is $-2 < \lambda \leq 0$, the upper bound for ω in the case of a very weak coupling between the dilaton and matter is $\omega < -1$, on the contrary for a strong coupling we obtain $\omega < 0$. Which means that the presence of the dilatonic charge allows a smaller pressure in absolute value. A more profound analysis will be performed in the anisotropic section where the Hubble parameter is in the 4-dimensional Einstein frame and not in the D-dimensional Einstein frame.

5.3 Anisotropic in vacuum

In vacuum the EOM for our anisotropic ansatz are Eqs. (150); as usual we rearrange them in order to study the fixed points of the system, they read:

$$\dot{H} = y\frac{\partial_\sigma^2 F \partial_H F - \partial_{\sigma H} F \partial_\sigma F}{\partial_\sigma^2 F \partial_H^2 F - (\partial_{\sigma H} F)^2}, \tag{202a}$$

$$\dot{\sigma} = y\frac{\partial_H^2 F \partial_\sigma F - \partial_{\sigma H} F \partial_H F}{\partial_\sigma^2 F \partial_H^2 F - (\partial_{\sigma H} F)^2}, \tag{202b}$$

$$\dot{y} = \frac{1}{2}(y^2 - F), \tag{202c}$$

$$y^2 + H\partial_H F + \sigma\partial_\sigma F - F = 0, \tag{202d}$$

where the denominator of Eqs. (202a) and (202b) is the determinant of the hessian of $F(H,\sigma)$, and we ask it to be non-vanishing. We must now remember that in contrast to the previous case, having one of the Hubble factors H or σ vanishing does not mean that F or any of its derivatives is zero, we need both of them to be 0.

Looking for the fixed points, the Eqs. (150a) and (150b) tell us that for constant H and σ we have either $y_0 = 0$, or $\partial_H F_0 = 0$ and $\partial_\sigma F_0 = 0$ respectively. For a zero velocity of the shifted dilaton, Eq. (202d) imposes $H_0 \partial_H F_0 + \sigma_0 \partial_\sigma F_0 = 0$. For the case of vanishing first derivatives we see that the equations are satisfied for whatever y having $y_0^2 = F_0$. The fixed points are shown in table 3.

H	σ	y	$\partial_H F$	$\partial_\sigma F$	Stability
H_0	σ_0	0	$H_0 \partial_H F_0 + \sigma_0 \partial_\sigma F_0 = 0$		Unstable
H_0	σ_0	y_0	0	0	$y_0 < 0$

Table 3: Fixed points for the anisotropic case in vacuum

Let us now look at the stability of these fixed points in order to look for extra constraints. As usual, we look at the Jacobian of the differential system:

$$\begin{pmatrix} \frac{\partial \dot{H}}{\partial H} & \frac{\partial \dot{H}}{\partial \sigma} & \frac{\partial \dot{H}}{\partial y} \\ \frac{\partial \dot{\sigma}}{\partial H} & \frac{\partial \dot{\sigma}}{\partial \sigma} & \frac{\partial \dot{\sigma}}{\partial y} \\ \frac{\partial \dot{y}}{\partial H} & \frac{\partial \dot{y}}{\partial \sigma} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} y_0 & 0 & \alpha_0 \\ 0 & y_0 & \beta_0 \\ -\frac{1}{2}\partial_H F_0 & -\frac{1}{2}\partial_\sigma F_0 & y_0 \end{pmatrix} , \tag{203}$$

where for example $\partial_\sigma F_0 = \partial_\sigma F\,|_0$, is the derivative evaluated in the fixed point, and we called the factors $\frac{\partial_H^2 F_0 \partial_\sigma F_0 - \partial_{\sigma H} F_0 \partial_H F_0}{\partial_\sigma^2 F_0 \partial_H^2 F_0 - (\partial_{\sigma H} F_0)^2} = \alpha_0$ and $\frac{\partial_\sigma^2 F_0 \partial_H F_0 - \partial_{\sigma H} F_0 \partial_\sigma F_0}{\partial_\sigma^2 F_0 \partial_H^2 F_0 - (\partial_{\sigma H} F_0)^2} = \beta_0$ for simplicity. The three eigenvalues are given by:

$$m_1 = y_0 \quad , \quad m_{2,3} = y_0 \pm \sqrt{-\frac{1}{2}(\partial_H F_0\, \alpha_0 + \partial_\sigma F_0\, \beta_0)}. \tag{204}$$

For the fixed point with non vanishing y_0, the square root vanishes and the three eigenvalues are given by the value of y_0, thus the stability of the solution requires $y_0 < 0$. For the other family of fixed points we get a vanishing eigenvalue, a positive and a negative one, so these fixed points are unstable for non vanishing first derivatives.

5.3.1 Hubble parameter in the Einstein frame

We proceed now with the computation of the Hubble parameter. The derivative of the dilaton is also constant and is given by $2\dot{\phi} = \dot{\Phi}_0 + 3H_0 + (D-4)\sigma_0$. By integrating to obtain the expression for ϕ we obtain:

$$\phi = \frac{1}{2}(\dot{\Phi}_0 + 3H_0 + (D-4)\sigma_0)(t - t_1), \tag{205}$$

with t_1 an integration constant, introducing it into Eq. (171), where we neglect s indices since everything is now in the string frame, we get:

$$H_E(T) = -\frac{1}{2} e^{\frac{1}{2}(\Phi_0 + 3H_0 + (D-4)\sigma_0)(t-t_1)} b^{-\frac{D-4}{2}} (\dot{\Phi}_0 + H_0). \tag{206}$$

In order to express Eq. (206) in terms of T, we integrate the relation between T the cosmic time in the Einstein frame and t the string frame time in Eq. (167). But here we encounter a difficulty, since we also have the scale factor in the extra dimensions $b(t)$ in the integral. But as σ_0 is a fixed point, we have whether $b = e^{\sigma_0 t}$, whether $b = b_0$ a constant. Let us first integrate Eq. (167) for a constant $b = b_0$ and vanishing σ_0:

$$\begin{aligned} \frac{dT}{dt} &= e^{-\frac{1}{2}(\Phi_0 + 3H_0)(t-t_1)} b_0^{\frac{D-4}{2}} \\ T - T_1 &= -\frac{2b_0^{\frac{D-4}{2}}}{(\Phi_0 + 3H_0)} e^{-\frac{1}{2}(\Phi_0 + 3H_0)(t-t_1)}, \end{aligned} \tag{207}$$

with T_1 is an integration constant. The Hubble parameter then reads:

$$\begin{aligned} H_E(T) = & -\frac{1}{2} e^{\frac{1}{2}(\dot{\Phi}_0 + 3H_0)(t-t_1)} b_0^{-\frac{D-4}{2}} (\dot{\Phi}_0 + H_0) \\ = & \frac{\dot{\Phi}_0 + H_0}{\dot{\Phi}_0 + 3H_0} \frac{1}{(T - T_1)}. \end{aligned} \tag{208}$$

Again, in this expression we are neglecting the special cases of constant Hubble parameter, $\dot{\Phi}_0 = -H_0$ and $\dot{\Phi}_0 = -3H_0$ that give Minkowski and de Sitter solutions respectively in the Einstein frame. We emphasise on the fact that we obtain a de Sitter solution in the Einstein 4-dimensional frame, which is moreover a stable fixed point solution in the string frame, this Hubble factor is given by:

$$H_E = H_0 b_0^{-\frac{D-4}{2}}, \tag{209}$$

where we see that H_E is given by the string frame H_0 and the inverse of the square root of the volume of the extra dimensions which is constant.
An important thing to remark for the general case is that, by computing the H_E for the spatial 3-dimensional parts in terms of the shifted dilaton Φ, all traces of the total dimensions parameter D disappear.

Now, let us look at the solutions in order to have a positive Hubble parameter for $T > T_1$. We recall that the stability condition imposes $c_1 < 0$, i.e. opposite signs for H_0 and Φ_0. It is important to note that H_0 represents the Hubble parameter of the 3-dimensional space in the D-dimensional string frame, so we can take it to be negative.

For a negative H_0, then $\dot{\Phi}_0$ needs to be positive in order to obtain stable solutions. We need in the last expression of Eq. (208) both the numerator and the denominator to have the same sign. That imposes two intervals for $\dot{\Phi}_0$, given by:

$$0 < \dot{\Phi}_0 < |H_0| \quad , \quad 3|H_0| < \dot{\Phi}_0, \tag{210}$$

For a positive H_0 we get a negative $\dot{\Phi}_0$, and the allowed intervals to have a positive H_E are:

$$\dot{\Phi}_0 < -3H_0 \quad , \quad -H_0 < \dot{\Phi}_0 < 0. \tag{211}$$

For both cases we have two intervals for $\dot{\Phi}_0$, which give a high and a slowly varying shifted dilaton with respect to the value of the Hubble parameter in the string frame. Thus, when we consider the extra dimensions to have a constant volume, we get a Hubble parameter which is positive and decreasing for:

$$3|H_0| < |\dot{\Phi}_0| \quad , \quad |\dot{\Phi}_0| < |H_0|. \tag{212}$$

Moreover, the only constraint that we have on the Hubble parameter in the string frame for the general case is to satisfy $\partial_H F(H_0, \sigma_0) = 0$.

We now look for the case of non vanishing σ_0, with a scale factor in the extra dimensions expanding or contracting exponentially $b = e^{\sigma_0 t}$; from Eq. (167) the differential equation between the two times now reads:

$$\begin{aligned}\frac{dT}{dt} =& e^{-\frac{1}{2}(\dot{\Phi}_0+3H_0+(D-4)\sigma_0)(t-t_1)} e^{\frac{D-4}{2}\sigma_0 t}\\ =& e^{-\frac{1}{2}(\dot{\Phi}_0+3H_0)(t-t_1')},\end{aligned} \tag{213}$$

where we redefined t_1'. The expression relating the two time parametrisations reads:

$$T - T_1' = -\frac{2}{\dot{\Phi}_0 + 3H_0} e^{-\frac{1}{2}(\dot{\Phi}_0+3H_0)(t-t_1')}, \tag{214}$$

with T_1' a new integration constant. By introducing it into Eq. (171), we obtain a Hubble parameter given by:

$$\begin{aligned}H_E(T) =& -\frac{1}{2} e^{\frac{1}{2}(\dot{\Phi}_0+3H_0+(D-4)\sigma_0)(t-t_1)} e^{-\frac{D-4}{2}\sigma_0 t}(\dot{\Phi}_0 + H_0)\\ =& \frac{\dot{\Phi}_0 + H_0}{\dot{\Phi}_0 + 3H_0} \frac{1}{T - T_1'},\end{aligned} \tag{215}$$

where again the case $\dot{\Phi}_0 = -3H_0$ corresponds to a de Sitter solution in the Einstein 4-dimensional frame. Its value for an exponential b scale factor, by recovering the original t_1 integration constant reads:

$$H_E = e^{-\frac{1}{2}(D-4)\sigma_0 t_1} H_0, \tag{216}$$

where again the extra dimensions factors appear in the expression of H_E. It is also important that both expressions are only dependent of the scale factors of both the 3 and D-4-dimensional spaces.
We observe that the H_E for both Eqs. (208) and (215) is the same, with different initial times T_1 and T_1' which are just integration constants. So we can do a totally analogous interpretation of these results. Meaning that, no matter the volume of the extra dimensions being exponentially expanding or contracting, or constant, the Hubble parameter in the 4-dimensional Einstein frame does not feel any difference. Thus, we can have stable fixed points solutions in the string frame, which, when translated to the 4-dimensional Einstein frame give a positive H_E. This H_E has a decreasing behaviour as time increases and tends asymptotically to zero.

Unlike in section 5.1.1, this Hubble parameter is in the 4-dimensional Einstein frame. We can compute the deceleration parameter q which is given by:

$$\begin{aligned}1+q=&-\frac{dH_E}{dT}\frac{1}{H_E^2}\\=&\frac{\dot{\Phi}_0+3H_0}{\dot{\Phi}_0+H_0}.\end{aligned} \tag{217}$$

The deceleration parameter then reads:

$$q=\frac{2H_0}{\dot{\Phi}_0+H_0}. \tag{218}$$

The expansion of the universe is accelerated for $q<0$. Looking at Eq. (218) we need the numerator and the denominator to have opposite signs. Knowing that H_0 and $\dot{\Phi}_0$ have also opposite signs for the fixed points to be stable, the condition in absolute value to obtain an accelerating expanding universe is:

$$|\dot{\Phi}_0|>|H_0|. \tag{219}$$

Now using the condition to obtain a positive H_E, in Eq. (212) we get that in order to obtain an accelerating expanding universe:

$$\dot{\Phi}_0<-3H_0. \tag{220}$$

5.3.2 Hubble parameter in terms of the dilaton

It is now crucial to recall that the shifted dilaton is not the physical degree of freedom. So in this section we will look at the stability condition translated in terms of the dilaton ϕ, after we will look what expression holds for the the dilaton in the special cases where we obtain exact Minkowski and de Sitter in the Einstein frame. Finally we look at the

Hubble parameter H_E for the general case in terms of ϕ.

The derivative of the dilaton is given by: $2\dot{\phi}_0 = \dot{\Phi}_0 + 3H_0 + (D-4)\sigma_0$. So that for the special case in which we obtain a Minkowski solution, $H_E = 0$ we have:

$$2\dot{\phi}_0 = 2H_0 + (D-4)\sigma_0. \tag{221}$$

Then a solution in which we have a constant dilaton is allowed for $H_0 = -\frac{D-4}{2}\sigma_0$. For the exact de Sitter solution, the derivative of the dilaton is given by:

$$2\dot{\phi}_0 = (D-4)\sigma_0, \tag{222}$$

it is then proportional to the Hubble factor of the extra dimensions and vanishes only for $\sigma_0 = 0$, a constant volume for the extra dimensions.
Let us now take a look to the general stable fixed points case, where the condition is to have $\dot{\Phi}_0$ and H_0 with opposite sign. We must then split the two cases for positive and negative H_0. If $H_0 > 0$ the condition for the fixed points to be stable is:

$$2\dot{\phi}_0 - 3H_0 - (D-4)\sigma_0 < 0. \tag{223}$$

For a negative H_0 we need:

$$2\dot{\phi}_0 + 3|H_0| - (D-4)\sigma_0 > 0. \tag{224}$$

We now recall from the previous section that H_E has the same expression for an exponential ($\sigma = \sigma_0$) or a constant scale factor ($\sigma_0 = 0$). In terms of the dilaton H_E reads:

$$H_E(T) = \frac{\dot{\phi}_0 - H_0 - \frac{D-4}{2}\sigma_0}{\dot{\phi}_0 - \frac{D-4}{2}\sigma_0}\frac{1}{T-T_1}. \tag{225}$$

When expressing the Hubble parameter in terms of the physical degree of freedom we obtain that in fact, H_E depends on the Hubble factor of the extra dimensions, and the total dimensionality of the theory D. Again we want this parameter to be positive and decreasing, for it we need the fraction to be positive. We can split the condition in two possibilities depending on the sign of H_0. If we have a positive Hubble parameter in the extra dimensions, to have $H_E > 0$ we have two intervals:

$$\dot{\phi}_0 < \frac{D-4}{2}\sigma_0 \quad , \quad H_0 + \frac{D-4}{2}\sigma_0 < \dot{\phi}_0. \tag{226}$$

But at the same time the stability condition of Eq. (223) must be satisfied then the two intervals are constrained to:

$$\dot{\phi}_0 < \frac{D-4}{2}\sigma_0 \quad , \quad H_0 + \frac{D-4}{2}\sigma_0 < \dot{\phi}_0 < \frac{3}{2}H_0 + \frac{D-4}{2}\sigma_0. \tag{227}$$

Then for a negative H_0 the two intervals together with the stability condition in Eq. (224) read:

$$\frac{D-4}{2}\sigma_0 - \frac{3}{2}|H_0| < \dot{\phi}_0 < \frac{D-4}{2}\sigma_0 - |H_0| \quad , \quad \frac{D-4}{2}\sigma_0 < \dot{\phi}_0. \tag{228}$$

We can now look at the deceleration parameter q which in terms of the dilaton reads:

$$q = \frac{H_0}{\dot{\phi}_0 - H_0 - \frac{D-4}{2}\sigma_0}. \tag{229}$$

If we have a positive H_0, then $\dot{\phi}_0 < H_0 + \frac{D-4}{2}\sigma_0$, so putting it together with the condition of a positive H_E we get:

$$\dot{\phi}_0 < \frac{D-4}{2}\sigma_0, \tag{230}$$

and for the case with $H_0 < 0$ we obtain:

$$\dot{\phi}_0 > \frac{D-4}{2}\sigma_0. \tag{231}$$

These results are consistent with the intervals of the previous section. Their physical meaning is clearer expressed in this form. In conclusion, the stable fixed points in the string frame provide, apart than a Minkowski and a de Sitter solution, a positive Einstein frame Hubble parameter that gives an accelerated universe. The conditions depend on the sign of H_0, for both cases the condition rests on the dilaton velocity and the Hubble parameter of the extra dimensions, given by Eq. (230) for $H_0 > 0$ and Eq. (231) for $H_0 < 0$.

5.4 Anisotropic with matter

From the EOM for the case of anisotropic space with matter, Eqs. (157), we assume a barotropic equation of state $p = \omega\rho$, a dilatonic charge $\theta = \lambda\rho$, and pressureless matter in the extra dimensions. We can rearrange this system of equations in order to leave the usual form for the fixed points study:

$$\dot{H} = \frac{y(\partial_\sigma^2 F \partial_H F - \partial_{\sigma H} F \partial_\sigma F) - \partial_\sigma^2 F d\omega(y^2 + H\partial_H F + \sigma\partial_\sigma F - F)}{\partial_\sigma^2 F \partial_H^2 F - (\partial_{\sigma H} F)^2}, \tag{232a}$$

$$\dot{\sigma} = \frac{y(\partial_H^2 F \partial_\sigma F - \partial_{\sigma H} F \partial_H F) + \partial_{H\sigma} F d\omega(y^2 + H\partial_H F + \sigma\partial_\sigma F - F)}{\partial_\sigma^2 F \partial_H^2 F - (\partial_{\sigma H} F)^2}, \tag{232b}$$

$$\dot{y} = \frac{1}{2}\left(y^2 - F + \frac{\lambda}{2}(y^2 + H\partial_H F + \sigma\partial_\sigma F - F)\right), \tag{232c}$$

$$y^2 + H\partial_H F + \sigma\partial_\sigma F - F = 2\kappa^2 e^\Phi \bar{\rho}, \tag{232d}$$

where we need to impose $\partial_H^2 F$, $\partial_\sigma^2 F$, and $\partial_\sigma^2 F \partial_H^2 F - (\partial_{\sigma H} F)^2$ to be non-vanishing. It is important to look at the original EOM, from Eq. (157b), we see that we still need whether $y_0 = 0$, whether $\partial_\sigma F_0 = 0$.

To obtain the fixed points, we can follow a method which is motivated by Eq. (174) and based on the fact that with this way of expressing the solutions, it is easier to compare with the fixed points of the isotropic section. For it, we parametrise both y_0 and σ_0 as $y_0 = c_1 H_0$ and $\sigma_0 = k_1 H_0$. Next, we sum up Eq. (232a) and Eq. (232b) in order to get an expression for $y_0^2 - F_0$:

$$y_0^2 - F_0 = H_0 \left[\partial_H F_0 \left(\frac{c_1}{d\omega} - 1 \right) + \partial_\sigma F_0 \left(\frac{c_1}{d\omega} \frac{(\partial_H^2 F_0 - \partial_{H\sigma} F_0)}{(\partial_\sigma^2 F_0 - \partial_{H\sigma} F_0)} - k_1 \right) \right], \tag{233}$$

and by introducing it into Eq. (232c) we obtain:

$$H_0 \left[\partial_H F_0 \left(\frac{c_1}{d\omega} \left(1 + \frac{\lambda}{2} \right) - 1 \right) + \partial_\sigma F_0 \left(\frac{c_1}{d\omega} \frac{(\partial_H^2 F_0 - \partial_{H\sigma} F_0)}{(\partial_\sigma^2 F_0 - \partial_{H\sigma} F_0)} \left(1 + \frac{\lambda}{2} \right) - k_1 \right) \right] = 0, \tag{234}$$

where we obtain a structure quite similar to the isotropic case (191), with the difference that we got an extra term with an overall factor $\partial_\sigma F_0$.
It is now important to recall that in contrast to the isotropic case [4], we got Eq. (233) by summing up two equations Eq. (232a) and Eq. (232b); then, we have to impose an extra condition. As we are considering a pressureless fluid in the extra dimensions, we recall Eq. (157b), which states that whether $\partial_\sigma F_0$ whether y_0 then c_1 vanishes. For the vanishing c_1 case we get the vacuum solution since it establishes $H_0 \partial_H F_0 + \sigma_0 \partial \sigma F_0 = 0$, which also gives $F_0 = 0$. Then, for a vanishing $\partial_\sigma F_0$ we get two cases: vanishing $\partial_H F_0$, which for $\omega \neq 0$ goes back to the vacuum case, and a second option which establishes:

$$c_1 \left(1 + \frac{\lambda}{2} \right) = \omega d \iff \dot{\Phi}_0 \left(1 + \frac{\lambda}{2} \right) = \omega d H_0, \tag{235}$$

already studied in the isotropic case.
To finish with the discussion of anisotropic fixed points with matter we have to look for the general case with pressureless 3-dimensional fluid, i.e. $\omega = 0$. We get the following equations for the fixed points:

$$0 = y_0 (\partial_\sigma^2 F_0 \partial_H F_0 - \partial_{\sigma H} F_0 \partial_\sigma F_0), \tag{236a}$$
$$0 = y_0 (\partial_H^2 F_0 \partial_\sigma F_0 - \partial_{\sigma H} F_0 \partial_H F_0), \tag{236b}$$
$$0 = (y_0^2 - F_0) \left(1 + \frac{\lambda}{2} \right) + \frac{\lambda}{2} (H_0 \partial_H F_0 + \sigma_0 \partial_\sigma F_0), \tag{236c}$$
$$y_0^2 + H_0 \partial_H F_0 + \sigma_0 \partial_\sigma F_0 - F_0 = 2\kappa^2 e^\Phi \bar{\rho}. \tag{236d}$$

Again, from the complete EOM, either $y_0 = 0$ or $\partial_\sigma F_0 = 0$. For $y_0 = 0$:

$$-F_0 + \frac{\lambda}{2}(H_0 \partial_H F_0 + \sigma_0 \partial_\sigma F_0 - F_0) = 0, \tag{237a}$$
$$H_0 \partial_H F_0 + \sigma_0 \partial_\sigma F_0 - F_0 = 2\kappa^2 e^\Phi \bar{\rho}. \tag{237b}$$

We come back to the vacuum case for $H_0 \partial_H F_0 + \sigma_0 \partial_\sigma F_0 = 0$ and $F_0 = 0$. By introducing Eq. (237b) into Eq. (237a) we get the relations $F_0 = \lambda\kappa^2 e^\Phi \bar{\rho}$, together with $H_0 \partial_H F_0 + \sigma_0 \partial_\sigma F_0 = (2 + \lambda)\kappa^2 e^\Phi \bar{\rho}$, which is the analogous of the $\omega = y_0 = 0$ fixed point in table 2 of the isotropic case.
Now for $\partial_\sigma F_0 = 0$, we get:

$$y_0(\partial^2_\sigma F_0 \partial_H F_0) = 0, \tag{238a}$$
$$y_0(\partial_{\sigma H} F_0 \partial_H F_0) = 0, \tag{238b}$$
$$y_0^2 - F_0 + \frac{\lambda}{2}(y_0^2 + H_0 \partial_H F_0 - F_0) = 0, \tag{238c}$$
$$y_0^2 + H_0 \partial_H F_0 - F_0 = 2\kappa^2 e^\Phi \bar{\rho}. \tag{238d}$$

As we imposed $\partial^2_\sigma F_0 \neq 0$ the first two equations are equivalent. For $\partial_H F_0 = 0$ we get then two kinds of fixed points, the vacuum case in which $y_0^2 = F_0$, and the case in which $\lambda = -2$ where we get a relation between F_0, y_0 and the energy density from Eq. (238d) . Not taking into account the vacuum cases, we show the fixed points in table 4.

y	$\partial_H F$	$\partial_\sigma F$	ω	λ	Stability
0	$H_0 \partial_H F_0 + \sigma_0 \partial_\sigma F_0 = (2 + \lambda)\kappa^2 e^\Phi \bar{\rho}$		0	All	Unstable
$c_1^2 H_0^2 - F_0 = -\lambda\kappa^2 e^\Phi \bar{\rho}$	$H_0 \partial_H F_0 = (2 + \lambda)\kappa^2 e^\Phi \bar{\rho}$	0	$\omega d = c_1 \left(1 + \frac{\lambda}{2}\right)$		$c < 0$ $-2 < \lambda < 0$
$c_1^2 H_0^2 - F_0 = 2\kappa^2 e^\Phi \bar{\rho}$	0	0	0	-2	Bifurcation for $c < 0$
$c_1^2 H_0^2 = F_0$	$H_0 \partial_H F_0 = 2\kappa^2 e^\Phi \bar{\rho}$	0	c_1/d	0	Bifurcation for $c < 0$

Table 4: Fixed points for the anisotropic case with matter

The values of H_0 and σ_0 are determined by the conditions imposed on $F(H, \sigma)$ and its derivatives. It is very interesting that we obtain the exact same fixed points as in the isotropic case. The advantage is in the second fixed point of table 4, which gives an extra degree of freedom in order to fix the value of H_0 while considering the stability, even if we must not forget that the Hubble parameter is in the string frame. As usual, we compute the stability of these fixed points by computing the eigenvalues of the Jacobian of the system in the fixed points:

$$\begin{pmatrix} \frac{\partial \dot{H}}{\partial H} & \frac{\partial \dot{H}}{\partial \sigma} & \frac{\partial \dot{H}}{\partial y} \\ \frac{\partial \dot{\sigma}}{\partial H} & \frac{\partial \dot{\sigma}}{\partial \sigma} & \frac{\partial \dot{\sigma}}{\partial y} \\ \frac{\partial \dot{y}}{\partial H} & \frac{\partial \dot{y}}{\partial \sigma} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} =$$

$$\begin{pmatrix} y - d\omega\partial_\sigma^2 F H\alpha_1 & -d\omega\partial_\sigma^2 F H\beta_1 & \frac{\partial_\sigma^2 F\partial_H F - \partial_{\sigma H}F\partial_\sigma F - 2d\omega y\partial_\sigma^2 F}{\partial_\sigma^2 F\partial_H^2 F - (\partial_{\sigma H}F)^2} \\ d\omega\partial_{\sigma H}FH\alpha_1 & y + d\omega\partial_{\sigma H}FH\beta_1 & \frac{\partial_H^2 F\partial_\sigma F - \partial_{\sigma H}F\partial_H F + 2d\omega y\partial_{\sigma H}F}{\partial_\sigma^2 F\partial_H^2 F - (\partial_{\sigma H}F)^2} \\ -\frac{1}{2}\partial_H F + \frac{\lambda}{4}(H\partial_H^2 F + \sigma\partial_{\sigma H}F) & -\frac{1}{2}\partial_\sigma F_0 + \frac{\lambda}{4}(H\partial_{\sigma H}F + \sigma\partial_\sigma^2 F) & y\left(1+\frac{\lambda}{2}\right) \end{pmatrix}, \tag{239}$$

with $\alpha_1 = \frac{\partial_H^2 F + k_1\partial_{\sigma H}F}{\partial_\sigma^2 F\partial_H^2 F - (\partial_{\sigma H}F)^2}$ and $\beta_1 = \frac{\partial_{\sigma H}F + k\partial_\sigma^2 F}{\partial_\sigma^2 F\partial_H^2 F - (\partial_{\sigma H}F)^2}$. As the expression for the eigenvalues is too laborious, we can first simplify it for each of the solutions.

For $y_0 = \omega = 0$ we get a vanishing eigenvalue $m_1 = 0$ and then:

$$m_{2,3} = \pm\frac{1}{\sqrt{2}}\left[\left(-\partial_H F + \frac{\lambda}{2}(H\partial_H^2 F + \sigma\partial_{\sigma H}F)\right)\left(\frac{\partial_\sigma^2 F\partial_H F - \partial_{\sigma H}F\partial_\sigma F}{\partial_\sigma^2 F\partial_H^2 F - (\partial_{\sigma H}F)^2}\right) + \left(-\partial_\sigma F_0 + \frac{\lambda}{2}(H\partial_{\sigma H}F + \sigma\partial_\sigma^2 F)\right)\left(\frac{\partial_H^2 F\partial_\sigma F - \partial_{\sigma H}F\partial_H F}{\partial_\sigma^2 F\partial_H^2 F - (\partial_{\sigma H}F)^2}\right)\right]^{1/2}, \tag{240}$$

being then an unstable fixed point.

For the general case of non vanishing ω and $\partial_\sigma F = 0$, we use the result of [4], which for $\lambda \neq -2$ gives $\partial_H F = \frac{2\omega^2 d^2 H}{1+\lambda/2}$. Note that from our definition of the coefficients α_1 and β_1 we obtain a relation $\partial_\sigma^2 F\alpha_1 - \partial_{\sigma H}F\beta_1 = 1$. This relation helps us in the simplification of the fixed points but erases all trace of the extra dimensions parameters, i.e. k_1 or σ derivatives of the F function. With all that we get 3 eigenvalues:

$$m_1 = cH_0\left(1+\frac{\lambda}{2}\right) \quad , \quad m_{2,3} = \frac{cH_0}{2}\left(1-\frac{\lambda}{2} \pm \sqrt{1+\lambda+\lambda^2/4}\right). \tag{241}$$

By imposing the condition of all three being negatives, we observe that we need $c_1 < 0$ together with $-2 < \lambda < 0$, so that the presence of the anisotropy does not affect the stability of these fixed points at all.

Let us now look at the special cases for the values of λ. First, for $\lambda = -2$ and $\omega = 0$ we get a quite a simple Jacobian matrix with eigenvalues:

$$m_1 = 0 \quad , \quad m_{2,3} = cH_0, \tag{242}$$

which establishes then that for a negative shifted dilaton velocity y_0 we have to look for the non linear stability of the mode with constant perturbation. For the case of $\lambda = 0$

we will have the same results. This means that one of the perturbation modes will be constant, but the other two will vanish exponentially, giving a bifurcation.

5.4.1 Hubble parameter in the Einstein frame

The derivation of $H_E(T)$ is the same as in the vacuum case, so we will have the two cases, the one for vanishing σ_0, in Eq. (208), and the one for exponential expansion or contraction of the extra dimension, in Eq. (215). For consistency with the derivation of H_E we recover $d = 3$ in this section.
As we have seen both cases have the same time dependency and prefactor, only the initial time which is an integration constant changes. We then treat both cases equally as we have been doing without distinguishing $\sigma = 0$ from $\sigma = \sigma_0$, the Hubble parameter is given by:

$$H_E(T) = \frac{\dot{\Phi}_0 + H_0}{\dot{\Phi}_0 + 3H_0} \frac{1}{(T - T_1)}. \tag{243}$$

Again, the two special values that give a Minkowski solution $\dot{\Phi}_0 = -H_0$, and de Sitter solution $\dot{\Phi}_0 = -3H_0$, behave in the same way as in the vacuum case. Although, in here they relate the matter parameters λ and ω. For the Minkowski case we obtain from Eq. (235) which holds for $\lambda \neq -2$, $\omega \neq 0$, and $\partial_H F_0 \neq 0$:

$$\left(1 + \frac{\lambda}{2}\right) = -3\omega, \tag{244}$$

and for the exact de Sitter solution:

$$\left(1 + \frac{\lambda}{2}\right) = -\omega. \tag{245}$$

Let us now look at the more general case. We introduce the stability condition Eq. (235) into Eq. (243). We obtain the following Hubble parameter in the 4-dimensional Einstein frame:

$$H_E(T) = \frac{3\omega + 1 + \frac{\lambda}{2}}{3\left(\omega + 1 + \frac{\lambda}{2}\right)} \frac{1}{(T - T_1)}. \tag{246}$$

Where now we cannot make appear the extra dimensions parameter but it actually appears in the stability conditions. First in Eq. (235), which in terms of the dilaton reads:

$$\begin{aligned}\dot{\phi} =& 3H_0 \left(\frac{\omega}{2 + \lambda} + \frac{1}{2}\right) + \frac{D - 4}{2}\sigma_0 \\ =& \frac{3H_0}{2}\left(\frac{2\omega + 2 + \lambda}{2 + \lambda}\right) + \frac{D - 4}{2}\sigma_0.\end{aligned} \tag{247}$$

As in the vacuum case having $c_1 < 0$ means that for $H_0 > 0$:

$$2\dot{\phi}_0 - 3H_0 - (D-4)\sigma_0 < 0, \tag{248}$$

and for $H_0 > 0$ we have:

$$2\dot{\phi}_0 + 3|H_0| - (D-4)\sigma_0 > 0. \tag{249}$$

We can again look for more constraints on our parameters by imposing a positive H_E and an accelerating solution with $q < 0$. First we recall that as in the isotropic case, the stability conditions, Eq. (235) along with $c_1 < 0$ and $-2 < \lambda < 0$ imposes $\omega < 0$. Then for H_E we have two options, given by the numerator and the denominator having the same sign, they read:

$$\omega < -\left(1 + \frac{\lambda}{2}\right) \quad , \quad -\frac{1}{3}\left(1 + \frac{\lambda}{2}\right) < \omega < 0. \tag{250}$$

We now look at the deceleration parameter, in this case we have:

$$1 + q = -\frac{\dot{H}_E}{H_E^2} = \frac{3\left(\omega + 1 + \frac{\lambda}{2}\right)}{3\omega + 1 + \frac{\lambda}{2}}, \tag{251}$$

which gives a deceleration parameter:

$$q = \frac{2-\lambda}{3\omega + 1 + \frac{\lambda}{2}}. \tag{252}$$

The numerator is always positive for linearly stable fixed points, so by imposing the denominator to be negative, we get:

$$\omega < -\frac{1}{3}\left(1 + \frac{\lambda}{2}\right) \tag{253}$$

In order to satisfy the latter, and obtain a positive H_E, we then need:

$$\omega < -\left(1 + \frac{\lambda}{2}\right), \tag{254}$$

which means that the upper bound on the ω parameter depends on the strength of the coupling between matter and the dilaton. The lower the coupling (nearer to zero), the nearer the upper bound of the equation of state parameter ω is from the one of vacuum energy asymptotically. For a dilaton coupling non vanishing, the upper bound increases allowing solutions with $\omega = -1$. We obtain the same results as in the isotropic case for $d = 3$.
Again for a general fixed point in the string frame, we manage to obtain for the Hubble parameter in the 4-dimensional Einstein frame a solution which has $H_E > 0$, and is in

accelerating expansion. The H_E decreases asymptotically to zero as time increases.
We can now look at the meaning of the bound on ω for the dilaton and the extra dimensions Hubble parameter. If we look at Eq. (247) we see that the parameter that goes with $3H_0$ is going to be always negative, and in absolute magnitude, the bigger is the pressure in the 3-dimensional space, the more important will be the contribution from H_0 to $\dot{\phi}_0$.
Moreover, we can obtain a solution for a vanishing dilaton velocity given a relation between the Hubble parameters such as:

$$\frac{D-4}{2}\sigma_0 = -3H_0\left(\frac{\omega}{2+\lambda}+\frac{1}{2}\right), \tag{255}$$

as mentioned the term multiplying H_0 is negative, so that means that the two Hubble parameters in the string frame must have the same sign if we want $\dot{\phi} = 0$. Thus the introduction of matter gives a relation between the Hubble parameters in order to obtain a non dynamical dilaton.

5.4.2 Non vanishing pressure in the extra dimensions

We can sketch the fixed points for the case of non vanishing pressure for the fluid in the extra dimensions. For this we can use the conservation of $e^{\Phi}\bar{\rho}$ and barotropic equations of states $p_3 = \omega\rho$ and $p_{D-4} = \gamma\rho$, together with $\theta = \lambda\rho$. We introduce p_3, the pressure of the fluid in the 3-dimensional space, and p_{D-4} its analogous in the D-4-dimensional space, with all three parameters ω, γ, and λ constants. The continuity equation now reads:

$$\dot{\bar{\rho}} + 3H\bar{p}_3 + (D-4)\sigma\bar{p}_{D-4} - \frac{1}{2}\dot{\Phi}\bar{\theta} = 0. \tag{256}$$

After imposing the conservation of $e^{\Phi}\bar{\rho}$ it reads:

$$\dot{\Phi}_0(1+\frac{\lambda}{2}) = 3\omega H_0 + (D-4)\gamma\sigma_0. \tag{257}$$

As expected, by introducing pressure in the extra dimensions, we come back to a symmetric system in which we replace $d\omega$ by $3\omega + (D-4)\gamma\sigma_0$. But this replacement is giving us an extra degree of freedom to fix our Hubble parameter.

By introducing pressure for the D-4-dimensional part, we are not changing the Hubble parameter in the Einstein frame, it still reads:

$$H_E(T) = \frac{\dot{\Phi}_0 + H_0}{\dot{\Phi}_0 + 3H_0}\frac{1}{(T-T_1)}, \tag{258}$$

which introducing Eq. (257), for $\lambda \neq -2$ reads:

$$H_E(T) = \frac{H_0\left(3\omega + 1 + \frac{\lambda}{2}\right) + (D-4)\gamma\sigma_0}{3H_0\left(\omega + 1 + \frac{\lambda}{2}\right) + (D-4)\gamma\sigma_0}\frac{1}{(T-T_1)}, \tag{259}$$

where now H_E depends explicitly on the Hubble parameters of the string frame.

6 On the F(H) function formulation

The $F(H)$ function is the novelty of the work of Hohm and Zwiebach. Given the way in which it appears in the action, we could think of it as a potential of the string frame Hubble factor. In this section we will try to see some interpretations of this F function in the string frame. Then we will perform the compactification of the action to the 4-dimensional Einstein frame, in order to see the form of this function in 4 dimensions.

6.1 String frame

Let us first see how Meissner and Veneziano [13] got to the two derivative theory in Eq. (13). Let us work on Eq. (11), for a vanishing b-field and a FLRW metric, we have a Ricci scalar given by [13]:

$$\begin{aligned} R =& 2\partial_t^2 \ln\sqrt{\det g_{ij}} + (\partial_t \ln\sqrt{\det g_{ij}})^2 - \frac{1}{4}\mathrm{Tr}((\partial_t g_{ij})(\partial_t g^{jk})) \\ =& d(d+1)H^2 + 2d\dot{H}. \end{aligned} \tag{260}$$

Then the zero order action (11) can be rewritten as:

$$I_0 = \int dt \sqrt{g} e^{-2\phi}(-4\dot{\phi}^2 + d(d+1)H^2 + 2d\dot{H}). \tag{261}$$

For the formulation in the O(d,d) covariant way, Eq. (13) is just Eq. (82) at first order, which in terms of the usual dilaton ϕ reads:

$$\begin{aligned} I_0 &= \int dt e^{-\Phi}(-\dot{\Phi}^2 + dH^2) \\ &= \int dt \sqrt{g} e^{-2\phi}(-4\dot{\phi}^2 - d^2H^2 + 4dH\dot{\phi} + dH^2) \\ &= \int dt \sqrt{g} e^{-2\phi}(-4\dot{\phi}^2 - d(d-1)^2H^2) - \int dt \sqrt{g}\left(\frac{d}{dt}e^{-2\phi}\right) 2dH \\ &= \int dt \sqrt{g} e^{-2\phi}(-4\dot{\phi}^2 + d(d+1)H^2 + 2d\dot{H}), \end{aligned} \tag{262}$$

where in the last line we used integration by parts, recalling that $\sqrt{g} = a^d$. We could then think that we can compare the action that we obtained at all orders:

$$I = \int dt e^{-\Phi}(-\dot{\Phi}^2 - F(H)), \tag{263}$$

with some kind of perturbative $f(R)$ theory in the string frame, with an action:

$$S = \int dt \sqrt{g} e^{-2\phi}(-4\dot{\phi}^2 + f(R)), \tag{264}$$

where we should impose a form for the $f(R)$ function in such a way that $f(R) = R + b_1\alpha' R^2 + \mathcal{O}(\alpha'^2)$, so that at first order those two actions are exactly equivalent.
For higher order the Ricci scalar will make appear terms with derivatives of H, which do not appear when developing the F(H) function. For example, for the second order:

$$b_1\alpha' \int dt\sqrt{g}e^{-2\phi}R^2 = b_1\alpha' \int dt\sqrt{g}e^{-2\phi}[d^2(d+1)^2H^4 + 4d^2(d+1)H^2\dot{H} + 4d^2\dot{H}^2], \quad (265)$$

while $F(H)$ gives us a term proportional to H^4. The problem is that by integrating by parts we will also get factors of derivatives of the dilaton. But we must now recall the way in which we derived the form of Eq. (263) in section 3.1.2. We have constrained the most general O(d,d) covariant action enormously. In these constraints we eliminate all the higher-derivatives of the S matrix, and the shifted dilaton Φ. That is why we do not get derivative terms of hte Hubble parameter H in Eq. (263). So we can expect that by computing the EOM at first order for Eq. (264), we can then redefine the fields with corrections of order k, so that order by order we eliminate all the $\dot{H}$ terms, obtaining only H^{2k} terms. Then we should only match the prefactors at each order with the use of the free coefficients.

However, we must now recall that the $F(H)$ function encodes all the higher-derivative corrections of the string action such as $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$. Therefore, it is clear that if we wanted to compare the function $F(H)$ with a function $f(R)$ we would need to impose several extra conditions on our function $f(R)$. First, we would need a perturbative function in the parameter α'. Second, certainly this function at order α' cannot consist only of a square term of the Ricci scalar.

In order to look for some other interpretation of $F(H)$, we can separate the first order (without α') from the higher terms. After that we can take a look at the total action in terms of the dilaton ϕ:

$$I = \int dt\sqrt{g}e^{-2\phi}(-4\dot{\phi}^2 + R_d - F_{k>1}(H)), \quad (266)$$

where R_d is the Ricci scalar in d dimensions, and $F_{k>1}(H)$ encodes all the α' corrections, and its leading order is $\mathcal{O}(\alpha')$. The action obtained is formed by a kinetic term for the dilaton with the wrong sign, although this is not important since we are in the string frame, the Ricci scalar in D dimensions, and finally the so-called $F_{k>1}(H)$. Thus, the function $F(H)$ can be seen as a first "kinetic" term (which contributes to the Ricci scalar) plus a perturbative potential for the Hubble factor in the string frame. In order to have a clearer view of it, we can take a look at its final form in the 4-dimensional Einstein frame.

6.2 Compactification

As previously announced, the idea of splitting the D-dimensional theory in the part of our three spatial dimensions, and other D-4 extra dimensions, was to bring this theory closer to reality, so that we could see what form the $F(H)$ function has, and hence, see what form all the higher derivative corrections of the O(d,d) covariant theories in 4 dimensions take. For this, we will need to extract the dependencies of the parameters in the string frame in their corresponding parameters in the Einstein frame. With this purpose in mind we could follow the [8] approach, although the time reparametrisations explained in the previous sections make it all a bit more cumbersome. So we will instead recover the covariant derivative, as done in [12] to obtain the dependencies between the parameters in the two frames.

We want then to recover time reparametrisations invariance with the use of the covariant derivative:

$$\frac{d}{dt} \to \frac{1}{n_s(t)}\frac{d}{dt} = \mathcal{D}_s. \tag{267}$$

From now on we indicate the quantities in the D-dimensional string frame with the sub-index s. As the second step of the compactification is to separate the 4-dimensional terms from the D-4-dimensional, we use our development of the anisotropic action:

$$I = \int dtne^{-\Phi}(-(\mathcal{D}_s\Phi)^2 - F(H_s,\sigma_s)), \tag{268}$$

where now $H_s = (\mathcal{D}_s a_s)/a_s$ and $\sigma_s = (\mathcal{D}_s b_s)/b_s$. To simplify the notation, we introduce all the constant terms of $F(H,\sigma)$ defined in Eq. (148), by redefining the coefficients $c_{k,l}$, in such a way that:

$$F(H_s,\sigma_s) = \sum_{k=1}^{\infty}(-\alpha')^{k-1}\sum_{l=0}^{k} c_{k,l}H_s^{2(k-l)}\sigma_s^{2l}. \tag{269}$$

We rescale them as $12(D-4)2^{2k}c_{k,l} \to c_{k,l}$, so that now we have the two coefficients at first order for string theory being: $c_{1,0} = -3$, and $c_{1,1} = -(D-4)$. Let us first rewrite the action in terms of the dilaton:

$$I = \int dtn_s\sqrt{g_s}e^{-2\phi}(-4(\mathcal{D}_s\phi)^2 - (3H_s+(D-4)\sigma_s)^2 + 4(3H_s+(D-4)\sigma_s)\mathcal{D}_s\phi - F(H,\sigma)), \tag{270}$$

where we used the definition of the shifted dilaton, and its first derivative in Eq. (170). As the covariant derivative satisfies the usual integration by parts rule [8]:

$$\int dtnB\mathcal{D}A = \int dt\frac{d(BA)}{dt} - \int dtn(\mathcal{D}B)A, \tag{271}$$

we can integrate by parts the term with $\mathcal{D}_s\phi$ in order to obtain:

$$I = \int dt n_s \sqrt{g_s} e^{-2\phi}(-4(\mathcal{D}_s\phi)^2 + (3H_s + (D-4)\sigma_s)^2 + 2(3\mathcal{D}_s H_s + (D-4)\mathcal{D}_s\sigma_s) - F(H,\sigma)). \tag{272}$$

Note that with this last formulation of the action we can already recognise the term that will be the Ricci scalar in the 4-dimensional action.

6.2.1 To the D-dimensional Einstein frame

As explained in [14], we must first use a Weyl transformation in order to go to the D-dimensional Einstein frame. Such a Weyl transformation reads:

$$G^s_{MN} = \Omega^{-2} G^{ed}_{MN} \quad \rightarrow \quad \sqrt{-G_s} = n_s\sqrt{g_s} = \Omega^{-D} n_{ed}\sqrt{g_{ed}} = \Omega^{-D}\sqrt{-G_{ed}}. \tag{273}$$

We analogously introduce the *ed* index for the D-dimensional Einstein frame quantities. $\tilde{G}$ is the metric in the D-dimensional Einstein frame; the two metrics have the following form:

$$G^s_{MN} = \begin{pmatrix} -n_s^2(t) & 0 & 0 \\ 0 & a_s^2(t)\cdot \mathbb{1}_3 & 0 \\ 0 & 0 & b_s^2(t)\cdot g_{mn} \end{pmatrix}, \ \tilde{G}^{ed}_{MN} = \begin{pmatrix} -n_{ed}^2(t) & 0 & 0 \\ 0 & a_{ed}^2(t)\cdot \mathbb{1}_3 & 0 \\ 0 & 0 & b_{ed}^2(t)\cdot g_{mn} \end{pmatrix}. \tag{274}$$

We can see, that the transformations of individual parameters are:

$$n_s^2 = \Omega^{-2} n_{ed}^2 \quad , \quad a_s^2 = \Omega^{-2} a_{ed}^2 \quad , \quad b_s^2 = \Omega^{-2} b_{ed}^2. \tag{275}$$

We now want to find the expressions of the terms in the action as a function of Ω and the Einstein frame variables. For this we note that the covariant derivative now reads:

$$\mathcal{D}_s = \frac{1}{n_s}\frac{d}{dt} = \frac{\Omega}{n_{ed}}\frac{d}{dt} = \Omega\mathcal{D}_{ed}. \tag{276}$$

The covariant derivative of the dilaton in the Einstein D-dimensional frame simply reads:

$$\mathcal{D}_s\phi = \Omega\mathcal{D}_{ed}\phi. \tag{277}$$

Now, for the computation of the scale factors we must be more careful, for H_s we find:

$$\begin{aligned} H_s =& \frac{\Omega}{a_{ed}}\Omega\mathcal{D}_{ed}(\Omega^{-1}a_{ed}) \\ =& \Omega\left(-\frac{\mathcal{D}_{ed}\Omega}{\Omega} + H_{ed}\right), \end{aligned} \tag{278}$$

where we defined $H_{ed} \equiv (\mathcal{D}_{ed} a_{ed})/a_{ed}$. The same happens for σ_s:

$$\sigma_s = \Omega \left(-\frac{\mathcal{D}_{ed}\Omega}{\Omega} + \sigma_{ed} \right), \tag{279}$$

with σ_{ed} defined analogously as $\sigma_{ed} \equiv (\mathcal{D}_{ed}\sigma_{ed})/\sigma_{ed}$. Now we look at the covariant derivatives of the Hubble parameters:

$$\begin{aligned} \mathcal{D}_s H_s =& \Omega \mathcal{D}_{ed} \left(\Omega \left(-\frac{\mathcal{D}_{ed}\Omega}{\Omega} + H_{ed} \right) \right) \\ =& \Omega^2 \left(\mathcal{D}_{ed} H_{ed} + \frac{\mathcal{D}_{ed}\Omega}{\Omega} H_{ed} - \frac{\mathcal{D}^2_{ed}\Omega}{\Omega} \right). \end{aligned} \tag{280}$$

Performing the same computation we obtain $\mathcal{D}_s \sigma_s$:

$$\mathcal{D}_s \sigma_s = \Omega^2 \left(\mathcal{D}_{ed}\sigma_{ed} + \frac{\mathcal{D}_{ed}\Omega}{\Omega}\sigma_{ed} - \frac{\mathcal{D}^2_{ed}\Omega}{\Omega} \right). \tag{281}$$

We note that by expressing all the terms inside the brackets in the action of Eq. (272) as Einstein frame variables, we get an overall Ω^2 parameter. We then impose the same for the $F(H_s, \sigma_s)$ function, in such a way that $F(H_s, \sigma_s) = \Omega^2 F_{ed}(H_{ed}, \sigma_{ed})$. Let us see what form the F function takes in the Einstein D-dimensional frame:

$$F_{ed}(H_{ed}, \sigma_{ed}) = \sum_{k=1}^{\infty} (-\alpha')^{k-1} \sum_{l=0}^{k} c_{k,l} \Omega^{2k-2} \left[-\frac{\mathcal{D}_{ed}\Omega}{\Omega} + H_{ed} \right]^{2(k-l)} \left[-\frac{\mathcal{D}_{ed}\Omega}{\Omega} + \sigma_{ed} \right]^{2l}. \tag{282}$$

We have now obtained all the replacements to be done, so in order to go to the Einstein frame we need to remove the $e^{-2\phi}$ factor with the product of Ω factors. We have an Ω^2 from the terms in brackets, and an Ω^{-D} from $\sqrt{-G_{ed}}$. Then, we get:

$$\Omega = e^{-\frac{2\phi}{D-2}}. \tag{283}$$

By computing its derivatives we have:

$$\frac{\mathcal{D}_{ed}\Omega}{\Omega} = -\frac{2\mathcal{D}_{ed}\phi}{D-2} \quad , \quad \frac{\mathcal{D}^2_{ed}\Omega}{\Omega} = \frac{4(\mathcal{D}_{ed}\phi)^2}{(D-2)^2} - \frac{2\mathcal{D}^2_{ed}\phi}{D-2}. \tag{284}$$

The only thing to do is to introduce everything into the action in Eq. (272). In order to simplify the action we integrate by parts the second derivative term in Eq. (284), and by noting that the order $k = 1$ inside the F function does not contain any α' correction, we can develop the function up to first order in α':

$$F_{ed} = -3H^2_{ed} - (D-4)^2\sigma_{ed} - 4\frac{(D-1)}{(D-2)^2}(\mathcal{D}_{ed}\phi)^2 - \frac{4\mathcal{D}_{ed}\phi}{(D-2)^2}(3H_{ed} + (D-4)\sigma_{ed}) + F^{k>1}_{ed}, \tag{285}$$

where we define $F_{ed}^{k>1}$ which encodes all the α' corrections. Finally, the expression for the action in the D-dimensional Einstein frame is:

$$I = \int dt n_{ed}\sqrt{g_{ed}} \left(\frac{4}{D-2}(\mathcal{D}_{ed}\phi)^2 + (3H_{ed} + (D-4)\sigma_{ed})^2 + 6\mathcal{D}_{ed}H_{ed} \right.$$
$$\left. +2(D-4)\mathcal{D}_{ed}\sigma_{ed} + 3H_{ed}^2 + (D-4)\sigma_{ed}^2 - F_{ed}^{k>1} \right). \quad (286)$$

We can note from this action that the dilaton has now the correct sign in the kinetic term, and we can still recognise the Ricci scalar now in the D-dimensional Einstein frame.

6.2.2 To the 4-dimensional Jordan frame

We now want to separate the 4 and D-4 dimensions. Half of the work is already done, since we have developed the action in terms of H, and σ, already establishing a difference between these 2 spaces. We now only have to develop the determinant of the metric:

$$n_{ed}\sqrt{g_{ed}} = n_{ed}\sqrt{\tilde{g}_3}\sqrt{\tilde{g}_{D-4}} = n_{ed}\sqrt{\tilde{g}_3} \cdot b_{ed}^{D-4}\sqrt{det(g_{mn})}, \quad (287)$$

where $\tilde{g}_3$ represents the determinant of the spatial part of the 4-dimensional metric in the string frame. As we assumed the metric in the extra dimensions to be $b(t)$ times a constant metric g_{mn}, the $\sqrt{det(g_{mn})}$ factor can go out the integral with the $1/2\kappa^2$ mentioned in the sections 3.2 and 4.2, so we will also neglect this proportionality factor in our development. The difference between our compactification and a general one, is that as we are assuming a cosmological ansatz, we have already been able to integrate out the spatial dimensions $\int d^{D-1}x$.
From now on, for our 4-dimensional theory, the $b_{ed}(t)$no longer represents a scale factor but a time-dependent scalar field. We write it: $b(t)$. So that this time from the metric, we see a new scalar field term appear in front of our Ricci scalar. We then have to do a second Weyl transformation in order to remove it.

6.2.3 To the 4-dimensional Einstein frame

We then go on with the second Weyl rescaling, to have clear the notation we rewrite the transformation:

$$G_{\mu\nu}^{ed} = \Omega^{-2}g_{\mu\nu} \quad \rightarrow \quad \sqrt{-G_{ed}} = n_{ed}\sqrt{\tilde{g}_3} = \Omega^{-4}n\sqrt{g_3} = \Omega^{-4}\sqrt{-g}. \quad (288)$$

The Weyl transformation does not affect the $b(t)$ field anymore. Now for simplicity we define the quantities without extra index, as the ones in the 4-dimensional Einstein frame. Again, the metrics read:

$$G_{\mu\nu}^{ed} = \begin{pmatrix} -n_{ed}^2(t) & 0 \\ 0 & a_{ed}^2(t)\cdot \mathbb{1}_3 \end{pmatrix}, \ \tilde{g}_{\mu\nu} = \begin{pmatrix} -n^2(t) & 0 \\ 0 & a^2(t)\cdot \mathbb{1}_3 \end{pmatrix}. \quad (289)$$

The individual parameters as new frame variables read:

$$n_{ed}^2 = \Omega^{-2} n^2 \quad , \quad a_{ed}^2 = \Omega^{-2} a^2, \tag{290}$$

where Ω is the new Weyl factor to be determined in order to cancel the $b(t)$ contribution in front of the metric. The covariant derivative will now be replaced by:

$$\mathcal{D}_{ed} = \Omega \frac{1}{n} \frac{d}{dt} = \Omega \mathcal{D}, \tag{291}$$

with $\mathcal{D}$ being the covariant derivative in the Einstein 4-dimensional frame. As in the previous case, the derivative of the dilaton and the Hubble parameter H_{ed} and its covariant derivative are replaced by:

$$\mathcal{D}_{ed}\phi = \Omega \mathcal{D}\phi, \tag{292}$$

$$H_{ed} = \Omega \left(-\frac{\mathcal{D}\Omega}{\Omega} + H \right), \tag{293}$$

$$\mathcal{D}_{ed} H_{ed} = \Omega^2 \left(\mathcal{D}H + \frac{\mathcal{D}\Omega}{\Omega} H - \frac{\mathcal{D}^2\Omega}{\Omega} \right), \tag{294}$$

defining the Hubble parameter with the proper time covariant derivative $H = (\mathcal{D}a)/a$. For the sigma parameter we now get a different substitution since it does not get transformed by the Weyl rescaling:

$$\sigma_{ed} = \frac{1}{b} \frac{\Omega}{n} \frac{d}{dt} b = \Omega \frac{\mathcal{D}b}{b}, \tag{295}$$

and its covariant derivative is replaced by:

$$\begin{aligned} \mathcal{D}_{ed}\sigma_{ed} =& \Omega \mathcal{D} \left(\Omega \frac{\mathcal{D}b}{b} \right) \\ =& \Omega^2 \left(\frac{\mathcal{D}\Omega}{\Omega} \frac{\mathcal{D}b}{b} + \frac{\mathcal{D}^2 b}{b} - \left(\frac{\mathcal{D}b}{b} \right)^2 \right). \end{aligned} \tag{296}$$

Again we collect an overall Ω^2 factor from the terms in brackets; we can now see how the transformation translates in terms of the $F_{ed}^{k>1}$. We factorise again an overall factor Ω^2 to define the function in the Einstein frame $F_{ed}^{k>1} = \Omega^2 F_{k>1}$, with $F_{k>1}$ given by:

$$\begin{aligned} F_{k>1}(H, b, \phi) = \sum_{k=2}^{\infty} (-\alpha')^{k-1} \sum_{l=0}^{k} c_{k,l} e^{-\frac{(2k-2)2\phi}{D-2}} \Omega^{2k-2} \left[2\frac{\mathcal{D}\phi}{D-2} - \frac{\mathcal{D}\Omega}{\Omega} + H \right]^{2(k-l)} \\ \cdot \left[2\frac{\mathcal{D}\phi}{D-2} + \frac{\mathcal{D}b}{b} \right]^{2l}. \end{aligned} \tag{297}$$

We must now deduce the Ω factor, and compute its derivatives to introduce them in the previous relations. We get an Ω^{-4} from the metric contribution, and an Ω^2 from the terms in brackets, so that in order to cancel with the b^{D-4} we impose:

$$\Omega = b^{\frac{D-4}{2}}. \tag{298}$$

Then, the derivatives of Ω are given by:

$$\frac{\mathcal{D}\Omega}{\Omega} = \frac{D-4}{2}\frac{\mathcal{D}b}{b}, \tag{299}$$

$$\frac{\mathcal{D}^2\Omega}{\Omega} = \frac{D-4}{2}\left(\frac{D-6}{2}\left(\frac{\mathcal{D}b}{b}\right)^2 + \frac{\mathcal{D}^2 b}{b}\right). \tag{300}$$

If we now introduce all these expressions into the action, by integrating by parts the resulting $\mathcal{D}^2 b/b$ we get to this final action:

$$I = \int dtn\sqrt{g_3}\left(\frac{4(\mathcal{D}\phi)^2}{D-2} + 6(2H^2 + \mathcal{D}H) + \frac{(D-4)(D-2)}{2}\left(\frac{\mathcal{D}b}{b}\right)^2 - F_{k>1}(H,b,\phi)\right), \tag{301}$$

where the gravitational part is consistent with the work of Pedro et. al. [14], although we have done it without the Ricci scalar appearing explicitly and then without using its transformation laws. The $F_{k>1}$ which is of order α' and encodes all the higher derivative corrections reads:

$$F_{k>1}(H,b,\phi) = \sum_{k=2}^{\infty}(-\alpha')^{k-1}\sum_{l=0}^{k} c_{k,l}\left(e^{-\frac{2\phi}{D-2}}\cdot b^{\frac{D-4}{2}}\right)^{2k-2}\left[2\frac{\mathcal{D}\phi}{D-2} - \frac{D-4}{2}\frac{\mathcal{D}b}{b} + H\right]^{2(k-l)} \cdot \left[2\frac{\mathcal{D}\phi}{D-2} + \frac{\mathcal{D}b}{b}\right]^{2l}. \tag{302}$$

Therefore, by compactifying the O(d,d) covariant action, we obtain a 4-dimensional action with two scalars and the Ricci scalar in the 4-dimensional Einstein frame for a FLRW metric. Note that from the kinetic term of the b field we see that it has the correct sign for $D > 4$ what we have supposed all along the anisotropic development. Moreover, the α' corrections give us a kind of potential with all kinds of mixed terms. To obtain the canonical kinetic term for the $b(t)$ field we can define:

$$\mathcal{D}ln(b(t)) \equiv \mathcal{D}B(t) \quad \rightarrow B(t) = ln(b(t)) \quad , \quad b(t) = e^{B(t)}. \tag{303}$$

As b was initially a scale factor we assume it is always positive. By also introducing the Ricci scalar $R = 6(2H^2 + \mathcal{D}H)$, and recovering the original coefficients $c_{k,l}$ the action and the perturbative potential take the form:

$$I = \int dtn\sqrt{g_3}\left(R + \frac{4(\mathcal{D}\phi)^2}{D-2} + \frac{(D-4)(D-2)}{2}(\mathcal{D}B)^2 - F_{k>1}(H,B,\phi)\right), \tag{304}$$

$$\begin{aligned} F_{k>1}(H,B,\phi) = 12(D-4)\sum_{k=2}^{\infty}(-\alpha')^{k-1}2^{2k}\sum_{l=0}^{k}c_{k,l}\left(e^{\frac{D-4}{2}B-\frac{2\phi}{D-2}}\right)^{2k-2} \\ \cdot\left[2\frac{\mathcal{D}\phi}{D-2} - \frac{D-4}{2}\mathcal{D}B + H\right]^{2(k-l)}\left[2\frac{\mathcal{D}\phi}{D-2} + \mathcal{D}B\right]^{2l}. \end{aligned} \tag{305}$$

We obtain some quite reasonable action; the function which encodes all the α' corrections has a much more complicated form. It contains an overall exponential factor of the dilaton and the B scalar, which multiply powers of kinetic terms of the scalars and the Hubble parameter.

7 Discussion

In this work we have analysed the O(d,d) covariant theory of Hohm and Zwiebach in the string frame and in the Einstein frame. We will conclude with a discussion on the results obtained.

After introducing the formalism and deriving the equations of motion, we have performed a systematic search for the fixed points of the dynamical system. This study was performed both in isotropic and anisotropic backgrounds. The results are summarised in tables 1, 2, 3 and 4. By comparing fixed points in isotropic and anisotropic backgrounds we conclude that no new points or stability conditions are added to the simple isotropic case by allowing for more general backgrounds. The fixed points extend naturally between the two geometries, as commented in [12]. Quantitatively this means that in the anisotropic case ($\partial_\sigma F \neq 0$), the conditions where we had $H_0 \partial_H F_0$ for the isotropic case will be replaced by $H_0 \partial_H F_0 + \sigma_0 \partial_\sigma F_0$, and the $d\omega H_0$, will come substituted by $3\omega H_0 + (D-4)\gamma\sigma_0$. For the anisotropic fixed points analysis, by assuming a pressureless fluid in the extra dimensions in Eq. (157b) we are fixing either the velocity of the shifted dilaton or $\partial_\sigma F$ to vanish.
If $y_0 = \dot{\Phi}_0 = 0$, by looking at Eq. (157a) we impose the fluid in the three-dimensional space to be pressureless matter. In this case we still have the presence of the Hubble parameter in the extra dimensions, σ, in the equations since it enters with $\partial_\sigma F$, although it enters in the same way as H does. The σ_0 parameter would give us an extra degree of freedom to fix H_0, but those solutions are not stable, like de Sitter solutions with vanishing pressure in the isotropic space.
If $\partial_\sigma F_0 = 0$ the system goes back to the isotropic solutions. We could expect the σ parameter to be relevant in the stability condition of such fixed points, and give rise to a condition on the value of σ_0, but that turns out not to be the case. The condition of vanishing equation of state for the fluid in the extra dimension could be relaxed. We would again obtain symmetric equations in H and σ, and in the two parameters of the equations of state.
Moreover, for some stable fixed points in the string frame, we obtain Minkowski ($\dot{\Phi}_0 = -H_0$) and exact de Sitter ($\dot{\Phi}_0 = -3H_0$) solution in the 4-dimensional Einstein frame. Furthermore, the remaining stable fixed points give rise to solutions with a decreasing Hubble parameter in the Einstein frame. If we compute the latter as a function of the physical degree of freedom ϕ, we manage to get a dependence of the Hubble parameter in the 4-dimensional Einstein frame H_E on the extra dimension Hubble parameter in the string frame σ. These solutions are very interesting since, by imposing from a certain time T_0, H_E to be positive, and the deceleration parameter q to be negative (corresponding to accelerating expansion), we get to constraint the dilaton ϕ and the Hubble of the extra dimensions, and in the matter case we can also constraint the parameters of the equation of state ω and of the dilatonic charge λ. Furthermore, for these solutions, as

the cosmic time in the Einstein frame (t', T) increases, H_E decreases asymptotically to zero.
In future research one could analyse the solutions for a slowly varying dilaton velocity, i.e. $c(t) = c + \epsilon(t)$. It would be interesting to see if there is any solution in which the Hubble parameter in the Einstein frame starts positive and arbitrarily large, and then decreases exponentially. If such solutions have an ϵ which gets to zero in a finite time T_1, we would obtain a late constant and tiny Hubble parameter $H_E(T_1)$ for $T > T_1$.

In the final section of this thesis we have compactified the total O(d,d) covariant action to all orders in α'. In the four-dimensional Einstein frame we get an expected extra scalar field describing the volume of the extra dimensions, the dilaton and the 4-dimensional Ricci scalar for our FLRW metric. The final form of the F function tells us that it acts as a perturbative potential mixing up all the dynamical fields of the system. However, these interaction terms enter along with the corrections, since at first order the action only displays the kinetic terms of the dilaton and the B scalar field minimally coupled to gravity.

To conclude, the α' corrections allow for stable de Sitter solutions in the string frame. Some of the stable points of this family of solutions are also compatible with a de Sitter solution in the Einstein frame. It should also be highlighted that some stable de Sitter points in the string frame give us decreasing Hubble parameters in the Einstein frame that tend assymptotically to zero. In this manner, it would be interesting to study the hybrid solution that goes from a decreasing H_E to a constant and small one.
The development of the anisotropic action does not come with an increase in the variety of the fixed points, nor in their stability conditions, but the extra dimensions have a role to play in the Hubble parameter in the 4-dimensional Einstein frame. Nevertheless, this separation was needed in order to obtain a correct $F(H, \sigma)$ so that the compactification was performed in an easier way, without forgetting multi-trace terms in the original action. The theory obtained in the Einstein frame has a perturbative potential with infinite corrections in α', which carries all the interaction terms.
An interesting future direction could be to analyse this perturbative potential in the Einstein frame. We could also study the fixed points of the system, and compare with the ones issued from the first order equations, i.e. two scalars minimally coupled to gravity, in order to see what are the new solutions that this F function provides.

8 Bibliography

References

[1] Oren Bergman and Barton Zwiebach. "The Dilaton theorem and closed string backgrounds". In: *Nucl. Phys. B* 441 (1995), pp. 76–118. DOI: 10.1016/0550-3213(95)00022-K. arXiv: hep-th/9411047.

[2] Heliudson Bernardo, Robert Brandenberger, and Guilherme Franzmann. "O(d, d) covariant string cosmology to all orders in α'". In: *JHEP* 02 (2020), p. 178. DOI: 10.1007/JHEP02(2020)178. arXiv: 1911.00088 [hep-th].

[3] Heliudson Bernardo, Jan Chojnacki, and Vincent Comeau. "Non-linear stability of α'-corrected Friedmann equations". In: *JHEP* 03 (2023), p. 119. DOI: 10.1007/JHEP03(2023)119. arXiv: 2212.11392 [hep-th].

[4] Heliudson Bernardo and Guilherme Franzmann. "α'-Cosmology: solutions and stability analysis". In: *JHEP* 05 (2020), p. 073. DOI: 10.1007/JHEP05(2020)073. arXiv: 2002.09856 [hep-th].

[5] Raphael Bousso. "TASI Lectures on the Cosmological Constant". In: *Gen. Rel. Grav.* 40 (2008), pp. 607–637. DOI: 10.1007/s10714-007-0557-5. arXiv: 0708.4231 [hep-th].

[6] Edmund J. Copeland, Andrew R Liddle, and David Wands. "Exponential potentials and cosmological scaling solutions". In: *Phys. Rev. D* 57 (1998), pp. 4686–4690. DOI: 10.1103/PhysRevD.57.4686. arXiv: gr-qc/9711068.

[7] M. Gasperini and G. Veneziano. "The Pre - big bang scenario in string cosmology". In: *Phys. Rept.* 373 (2003), pp. 1–212. DOI: 10.1016/S0370-1573(02)00389-7. arXiv: hep-th/0207130.

[8] Olaf Hohm and Barton Zwiebach. "Duality invariant cosmology to all orders in α'". In: *Phys. Rev. D* 100.12 (2019), p. 126011. DOI: 10.1103/PhysRevD.100.126011. arXiv: 1905.06963 [hep-th].

[9] Olaf Hohm and Barton Zwiebach. "Non-perturbative de Sitter vacua via α' corrections". In: *Int. J. Mod. Phys. D* 28.14 (2019), p. 1943002. DOI: 10.1142/S0218271819430028. arXiv: 1905.06583 [hep-th].

[10] Olaf Hohm and Barton Zwiebach. "T-duality Constraints on Higher Derivatives Revisited". In: *JHEP* 04 (2016), p. 101. DOI: 10.1007/JHEP04(2016)101. arXiv: 1510.00005 [hep-th].

[11] Chethan Krishnan. "de Sitter, α'-Corrections & Duality Invariant Cosmology". In: *JCAP* 10 (2019), p. 009. DOI: 10.1088/1475-7516/2019/10/009. arXiv: 1906.09257 [hep-th].

[12] Yang Liu et al. "On de Sitter vacua in O(d,d) invariant cosmology". In: (Apr. 2024). arXiv: 2404.15401 [hep-th].

[13] K. A. Meissner and G. Veneziano. "Symmetries of cosmological superstring vacua". In: *Phys. Lett. B* 267 (1991), pp. 33–36. DOI: 10.1016/0370-2693(91)90520-Z.

[14] Santiago Pajón Otero, Francisco G. Pedro, and Clemens Wieck. "$R+\alpha R^n$ Inflation in higher-dimensional Space-times". In: *JHEP* 05 (2017), p. 058. DOI: 10.1007/JHEP05(2017)058. arXiv: 1702.08311 [hep-th].

[15] Antonio Padilla. "Lectures on the Cosmological Constant Problem". In: (Feb. 2015). arXiv: 1502.05296 [hep-th].

[16] P. J. E. Peebles and Bharat Ratra. "The Cosmological Constant and Dark Energy". In: *Rev. Mod. Phys.* 75 (2003). Ed. by Jong-Ping Hsu and D. Fine, pp. 559–606. DOI: 10.1103/RevModPhys.75.559. arXiv: astro-ph/0207347.

[17] Ashoke Sen. "O(d) x O(d) symmetry of the space of cosmological solutions in string theory, scale factor duality and two-dimensional black holes". In: *Phys. Lett. B* 271 (1991), pp. 295–300. DOI: 10.1016/0370-2693(91)90090-D.

[18] Steven H. Strogatz. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry and Engineering.* Westview Press, 2000.

[19] Steven Weinberg. "The Cosmological Constant Problem". In: *Rev. Mod. Phys.* 61 (1989). Ed. by Jong-Ping Hsu and D. Fine, pp. 1–23. DOI: 10.1103/RevModPhys.61.1.