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Bremsstrahlung emission and astrophysical applications

Bachelor Thesis

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Abstract

Una carica non-relativistica in accelerazione emette su un angolo solido secondo la formula di Larmor:

$$P = -\frac{dE}{dt} = \frac{2q^2\dot{v}^2}{3c^3}$$

Quando la carica in questione è accelerata da un campo di Coulomb, questa radiazione prende il nome di *Bremsstrahlung* (dal Tedesco: *'radiazione di frenamento'*).

In questa tesi vengono riportate le caratteristiche principali dell'emissione di Bremsstrahlung, come varia rispetto alla distribuzione di energia della popolazione di elettroni, in quali contesti astrofisici si osserva e le conclusioni che si possono trarre dal suo studio.

In particolare, il primo capitolo descrive l'emissione generale di una carica in moto, spostando poi l'enfasi ad un moto in un campo Coulombiano ed all'emissione totale proveniente da un plasma che emette per Bremsstrahlung.

Il secondo capitolo descrive due delle tante applicazioni astrofisiche dell'emissione di bremsstrahlung: la stima del gas *intracluster* in un ammasso di galassie, ed il ruolo di questa radiazione nella formazione di cascate elettromagnetiche per raggi gamma.

Chapter 1

Bremsstrahlung

1.1 Emission of a Moving Charged Particle

In order to study bremsstrahlung emission, we first need a general framework to describe the radiation of a non-relativistic accelerating point charge.
Energy emitted over time in a solid angle $d\Omega$ can be obtained multiplying the Poynting vector for the area element $dA = R^2 d\Omega$:

$$\frac{dW}{dt d\Omega} = SR^2 = \frac{c}{4\pi}|\vec{E}_{rad}|^2 R^2 \tag{1.1}$$

We can write the Poynting vector using these radiation fields, obtained by differentiating the Liénard-Wiechert potentials[1]:

$$\vec{E}_{rad}(\vec{r}, t) = \frac{q}{Rc^2}[\hat{n} \times (\hat{n} \times \vec{a})] \tag{1.2}$$

$$\vec{B}_{rad} = \hat{n} \times \vec{E}_{rad} \tag{1.3}$$

Let θ be the angle between the direction of motion $\hat{n}$ and the acceleration $\vec{a}$:

$$|\vec{E}_{rad}| = |\vec{B}_{rad}| = \frac{qa}{Rc^2}\sin\theta \tag{1.4}$$

$$S = \frac{c}{4\pi}(\frac{qa}{Rc^2}\sin\theta)^2 \tag{1.5}$$

$$\frac{dW}{dt d\Omega} = SR^2 = \frac{q^2 a^2}{4\pi c^2}\sin^2\theta \tag{1.6}$$

We can then find the total emitted power by integrating over $d\Omega$:

$$P = \frac{dW}{dt} = \frac{q^2 a^2}{4\pi c^2}\int \sin^2\theta d\Omega = \frac{2q^2 a^2}{3c^3} \tag{1.7}$$

Which is Larmor's formula in a non relativistic regime [1].

[1] The derivation of Liénard-Wiechert potentials will not be treated here. Details in [2] Ch 6.3

1.2 Single Electron in a Coulomb Field

Consider the interaction between an electron e^- and an ion Ze.[2]
The electron experiences 'braking' due to the attractive Coulomb field, of magnitude:

$$\dot{v} = \frac{F}{m} = k\frac{Ze^2}{mx(t)^2} \tag{1.8}$$

The emitted power associated with an accelerating charged particle is given by (1.7):

$$P = -\frac{dE}{dt} = \frac{2q^2\dot{v}^2}{3c^3} \tag{1.9}$$

Inserting here the Coulomb acceleration:

$$P = \frac{2Z^2e^6}{3c^3m^2x^4} \tag{2.0}$$

It is worth noting immediately that $P \sim x^{-4}m^{-2}$, thus it is justified to consider electrons to be the main emitters and for the emission to happen at the minimum distance x_{min} of the interaction, which is called the impact parameter b.
The direct consequence of this dependency is that the radiation is emitted in a very short time, which we can write by assuming that the interaction starts when the electron is at a distance b from the minimum distance point, and ends when it is another b away:

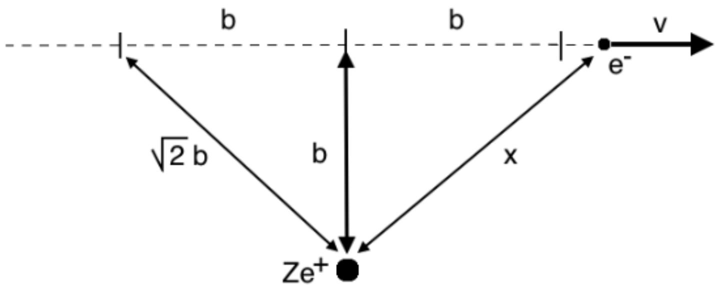

Figure 1.1: Electron-ion interaction. [2]

Such that:

$$\Delta t = \frac{2b}{v} \tag{2.1}$$

The energy emitted in a single interaction is thus:

$$P\Delta t = \frac{2Z^2e^6}{3c^3m^2b^4}\frac{2b}{v} = \frac{4Z^2e^6}{3c^3m^2b^3v} \tag{2.2}$$

[2] Bremsstrahlung due to the interactions of like particles is zero in the dipole approximation.

Notice how the power of a single interaction depends strongly on the impact parameter and the initial velocity.
Furthermore, the two charges involved were free before the interaction, and remain free after it, thus the name "free-free emission" for bremsstrahlung.

1.2.1 Spectral Distribution

As seen above, the radiation is emitted in short pulses of Δt duration, slightly asymmetrical because of the velocity loss after the emission.

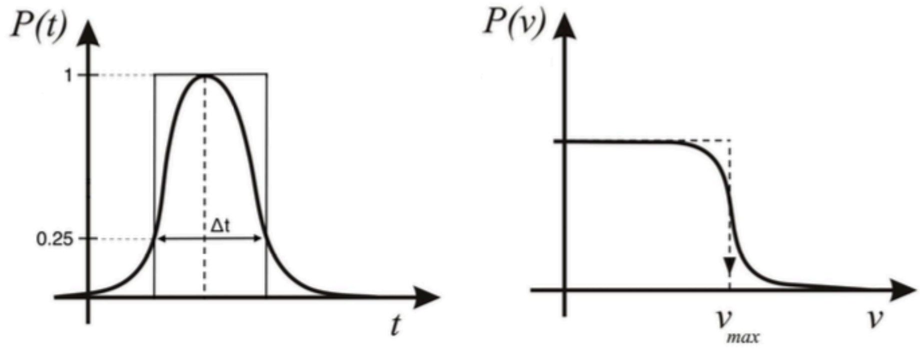

Figure 1.2: Single Event Spectral Distribution [2]

The Fourier analysis of the power over time profile (Figure 1.2, Left) gives the power as a function of frequency (Figure 1.2, Right) $\nu = 2\pi\omega$:

$$P(\omega) = \int_{-\infty}^{\infty} P(t)e^{-i\omega t} \tag{2.3}$$

Which can be written simply by approximating the spectrum to a rectangle of width ν_{max}:

$$\frac{dW}{d\nu} = \frac{P\Delta t}{\nu_{max}} = \frac{16Z^2e^6}{c^3m^2b^2v^2} \tag{2.4}$$

Notice how the spectrum has a cut-off at a certain frequency ν_{max}, followed by an exponential decrease. This cut-off frequency is tied to the electron's kinetic energy, and will be of great importance in estimating the temperature of the emitting region, among other things.

1.2 Total Radiation from a Bremsstrahlung-Emitting Plasma

We will estimate the total emissivity coming from a real plasma by considering the number of collisions happening in a volume element dV.

Let the plasma be made of a e^- component with numerical density n_e, and a ionic component with numerical density n_z. We only want to consider interactions with an impact parameter within b and $b + db$:

$$n_{int} = 2\pi n_z v b db \tag{2.5}$$

In an element of volume dV:

$$N_{coll} = n_e n_{int} dV \tag{2.6}$$

The **total emissivity** from the region will then be:

$$J_{Br}(v,\nu) = \frac{dW}{d\nu dt dV} = 2\pi n_z n_e v \int_{b_{min}}^{b_{max}} \frac{dW}{d\nu} b db \tag{2.7}$$

$$= 2\pi n_z n_e v \int_{b_{min}}^{b_{max}} \frac{16 Z^2 e^6}{3 c^3 m_e^2 b^2 v^2} b db = \tag{2.8}$$

$$\frac{32\pi n_z n_e Z^2 e^6}{3 c^3 m_e^2 v} \int_{b_{min}}^{b_{max}} \frac{db}{b} \tag{2.9}$$

$$J_{Br}(v,\nu) = \frac{32\pi n_z n_e Z^2 e^6}{c^3 m_e^2 v} \ln(\frac{b_{max}}{b_{min}}) \tag{3.0}$$

Where we constrain the interaction between two impact parameters b_{max} and b_{min}: the maximum impact parameter is simply derived from the cut-off frequency:

$$b_{max} \approx \frac{v}{4\nu_{max}} \tag{3.1}$$

While the minimum impact parameter has to satisfy both some classical and quantum requirements.

1.2.1 Quantum limit and Gaunt's Factors

A complete treatment of Bremsstrahlung emission has to be quantistic in nature, but deriving a minimum impact parameter b_{min} will help us understand the limit in which a classical approach is justified, or when quantum corrections must be included.

As aforementioned, b_{min} must satisfy the classical requirement of $\Delta v \leq v$, since electrons can only lose part of their initial kinetic energy.
Consider the acceleration (1.8) and duration (2.1):

$$\Delta v = a\Delta t = \frac{Ze^2}{m_e b^2}\frac{2b}{v} = \frac{2Ze^2}{m_e b v} \leq v \tag{3.4}$$

$$b_{min,class} \approx \frac{2Ze^2}{m_e v^2} \tag{3.5}$$

At the same time, Heisenberg's uncertainty principle must hold true:

$$\Delta x \Delta p \geq \hbar \tag{3.6}$$

Where $\Delta p \approx m_e \Delta v \approx m_e v$ and $\Delta x \approx b$

$$b m_e v \geq \hbar \tag{3.7}$$

$$b_{min,quant} \approx \frac{\hbar}{m_e v} \tag{3.8}$$

We're interested in their ratio, to find a condition to distinguish between the two regimes:

$$\frac{b_{min,quant}}{b_{min,class}} \approx \frac{\hbar}{m_e v}\frac{m_e v^2}{2Ze^2} = \frac{\hbar v}{2Ze^2} \approx \frac{v}{Z\alpha c} \approx \frac{137v}{Zc} \tag{3.9}$$

Where $\alpha = \frac{e^2}{\hbar c}$ is the fine structure constant in CGS units.

We can now approximate a quantum limit:

$$b_{min,quant} > b_{min,class} \leftrightarrow v \geq 0.01c \tag{4.0}$$

That is, we can safely assume $b_{min} = b_{min,class}$ for a colder plasma with a slower electron population ($10^4 - 10^6\,K$), while we have to use $b_{min} = b_{min,quant}$ for calculations regarding hot plasma ($10^8 K$), as Heisenberg's uncertainty principle starts to become more relevant.

This dependence on the impact parameters is collected under a correction factor called *Gaunt Factor* g_{ff} (free-free):

$$g_{ff} = \frac{\sqrt{3}}{\pi} \ln(\frac{b_{max}}{b_{min}}) \tag{4.1}$$

In practical calculations, we use a temperature-averaged gaunt factor $\bar{g}_{ff}$, which lies in the range $[1, 10]$, depending on $u = \frac{h\nu}{kT}$ and ν.

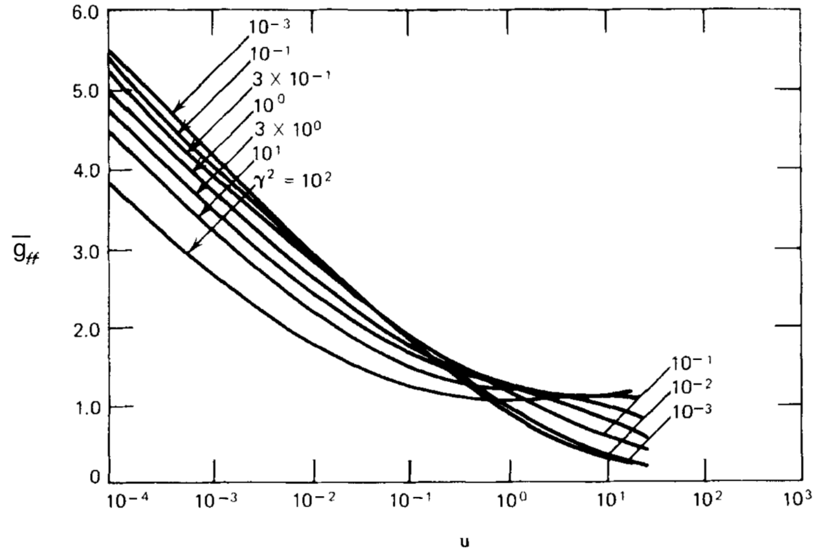

Figure 1.3: Average Gaunt factor as a function of $u = h\nu/kT$ at different γ^2 [3]

1.3 Thermal Bremsstrahlung

Given a real astrophysical plasma made of electrons and ions at thermal equilibrium, the resulting *thermal bremsstrahlung* emission will take into account the statistical distribution of the particle population.

In particular, at thermal equilibrium, the plasma's electrons will have different velocities, according to the Maxwell-Boltzmann distribution:

$$f(v)d^3v = (\frac{m}{2\pi kT})^{3/2}\exp(-\frac{mv^2}{2kT})d^3v \propto v^2\exp(-\frac{mv^2}{2kT})dv \tag{4.2}$$

We will now have a certain electron density $n_e(v) = n_e f(v)$ for electrons with velocities within v and $v + dv$. Once having integrated over all possible velocities allowed by the distribution, we will obtain a formulation for the total emissivity that only depends on the plasma's temperature T, at a fixed frequency ν:

$$J_{Br}(\nu, T) = \int_{v_{min}}^{\infty} J_{Br}(\nu, v)f(v)dv \tag{4.3}$$

We must set a minimum velocity to the integral, due to the same classical limit of the electron having to radiate less than its initial kinetic energy:

$$h\nu \leq \frac{1}{2}m_e v^2 \tag{4.4}$$

$$v_{min} = \sqrt{\frac{2h\nu}{m_e}} \tag{4.5}$$

Solving the integral gives the **emissivity per unit frequency** of Bremsstrahlung $[erg\, s^{-1} cm^{-3} Hz^{-1}]$:

$$J_{Br}(\nu, T) = 6.8 \cdot 10^{-38} T^{-1/2} e^{-hv/kT} n_e n_z Z^2 \bar{g}_{ff}(\nu, T) \tag{4.5}$$

Where the Maxwell-Boltzmann exponential leads to the characteristic high energy cut-off, with the cut-off frequency defined as the frequency of exponential decrease ($\propto 1/e$):

$$\nu_{cut-off} = \frac{kT}{h} \tag{4.6}$$

We see that as temperature increases, so does the cut-off frequency, while the peak emissivity goes down:

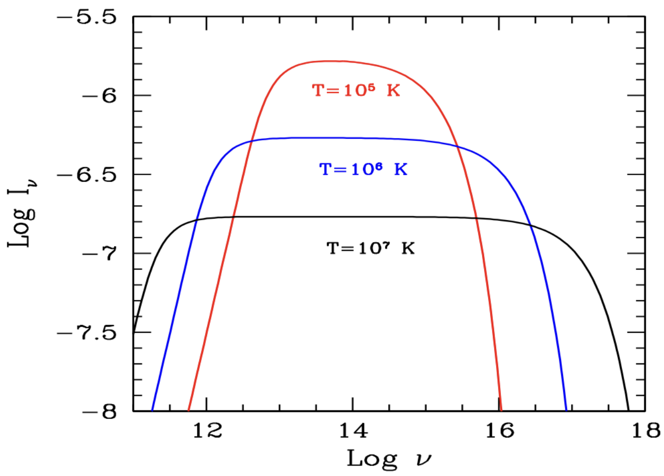

Figure 1.4: Bremsstrahlung intensity from a model radiosource with $R = 10^{15} cm,\ n_e = n_z = 10^{10} cm^{-3},\ \bar{g}_{ff} = 1$ [4]

The overall **emissivity per unit volume** is obtained by integrating over the whole spectrum of frequencies; for simplicity, we can approximate the spectrum as a rectangle of width ν_{min} and height $J_{BR} = n_e^2 T^{-1/2} \bar{g}_{ff}$. Numerically $[erg\ s^{-1}\ cm^{-3}]$:

$$J_{Br}(T) = \frac{dW}{dV dt} = 1.4 \cdot 10^{-27} \sqrt{T} n_e n_z Z^2 \bar{g}_{ff} \tag{4.7}$$

We see an immediate difference between the specific emissivity per unit time and volume and the total emission: the former is proportional to $T^{-1/2}$, while the latter increases as $T^{1/2}$. This is a direct consequence of the high energy cut-off increasing linearly with temperature, leading to more potentially-emitting energetic electrons at high temperatures.

Additionally, once having derived the total emissivity, we can use it to define the **cooling time** as the total thermal energy of a plasma over the energy loss to bremsstrahlung:

$$t_{br} = \frac{E_{th}}{J_{Br}(T)} = \frac{\frac{3}{2}(n_e + n_z)kT}{1.4 \cdot 10^{-27} \sqrt{T} n_e n_z Z^2 \bar{g}_{ff}} \tag{4.8}$$

$$t_{br} = \frac{1.8 \cdot 10^1 1}{n_e \bar{g}_{ff}} \sqrt{T} \texttt{sec} = \frac{6 \cdot 10^3}{n_e \bar{g}_{ff}} \sqrt{T} \texttt{yr} \tag{4.9}$$

This characteristic time can vary greatly, from thousands of years for typical HII regions ($T \approx 10^4 °K$) to tens of billions of years for galaxy clusters ($T \approx 10^7 - 10^8 °K$).

1.3.1 Bremsstrahlung Self-Absorption and Luminosity

In a regime of thermal equilibrium, one must account for emission as well as absorption of energy. The more complete way of modelling a bremsstrahlung plasma is to consider that the emitted photon is absorbed by another free electron in the cloud. This phenomenon is called thermal free-free absorption μ_{br}, and it is related to the emission via Kirchoff's law:

$$J_{Br}(\nu, T) = 4\pi\mu_{br}(\nu, T)B(\nu, T) \tag{5.0}$$

Where $B(\nu, T)$ is Planck's black body law. Numerically $[cm^{-1}]$:

$$\mu_{br}(\nu, T) = \frac{J_{Br}(\nu, T)}{4\pi B(\nu, T)} = 3.7 \cdot 10^8 T^{-1/2} Z^2 n_e n_z \nu^{-3}(1 - e^{-h\nu/kT})\bar{g}_{ff} \tag{5.1}$$

We see a strong dependence on frequency:

- At high frequencies ($h\nu \gg kT$) the exponential term is negligible and so is the absorption:

$$\mu_{br}(\nu, T) \propto \nu^{-3} \ll 1 \tag{5.2}$$

- At low frequencies ($h\nu \ll kT$) we can approximate according to Rayleigh-Jeans:

$$e^{-h\nu/kT} \approx 1 - \frac{h}{kT} \tag{5.3}$$

$$\mu_{br} \propto T^{-1/2} n_e n_z \nu^{-3} \frac{h\nu}{kT} \propto T^{-3/2} n_e n_z \nu^{-2} \tag{5.4}$$

Once having discussed emissivity and absorption, we can write the total bremsstrahlung flux from a plasma as a sum of the two phenomena:

$$B_{Br}(\nu, T) = B_{bb}(\nu, T)(1 - e^{-\tau(\nu, T)}) \tag{5.5}$$

$$B_{Br}(\nu, T) = \frac{\nu^3}{e^{h\nu/kT} - 1}(1 - e^{-\tau(\nu, T)}) \tag{5.6}$$

Where $\tau(\nu,\ T)\ =\ \mu(\nu,\ T)l$ is the optical depth of the plasma in the direction of the observer.

Again, we can take advantage of high and low frequencies approximations:

For $h\nu\ \ll\ kT$ luminosity will depend on the opacity:

$$L \propto T\nu^2(1-e^{-\tau})$$

- For $\tau\ \gg\ 1$, in an optically thick regime (**Figure 1.5** - a):

$$L \propto T\nu^2$$

- For $\tau\ \ll\ 1$, in an optically thin regime (**Figure 1.5** - b):

$$L \propto T\nu^2(1-1+\tau) = \tau T\nu^2$$
$$L \propto T^{-3/2}\nu^0$$

For $h\nu\ \gg\ kT$ we're still in an optically thin regime and there is no relevant absorption, only the exponential high energy cut-off (**Figure 1.5** - c)

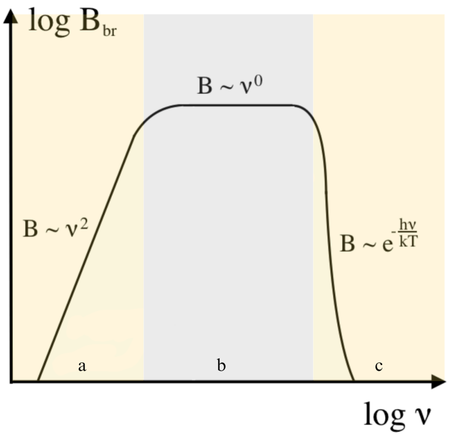

Figure 1.5: Bremsstrahlung Spectrum

1.4 Relativistic Bremsstrahlung

As seen with thermal bremsstrahlung, the emission of an electron population is dependent on its initial energy/velocity distribution. In the case of electrons with relativistic velocity $v \approx c$, the emissivity for *relativistic bremsstrahlung* will follow the relativistic version of Larmor's formula (not treated here):

$$J_{Br,rel}(v,\nu) = \frac{32}{3}\frac{\pi e^6}{c^3 m_e v} n_e n_z Z^2 \ln(\frac{183}{Z^{1/3}}) \tag{5.7}$$

In particular, let's consider a power law energy distribution for the electrons:

$$n_e(E) \approx n_{e,0} E^{-\delta} \tag{5.8}$$

We'll see that integrating over the distribution results in yet another power law for the emission spectrum:

$$J_{Br,rel} \propto \int_{h\nu}^{\infty} E^{-\delta} dE \tag{5.9}$$

$$J_{Br,rel} \approx \frac{E^{-\delta+1}}{1-\delta} \approx \nu^{-\delta+1} \tag{6.0}$$

Numerically, considering an average cosmic abundance of $\bar{Z} = 1.3$, $[erg\ s^{-1} cm^{-3}]$:

$$J_{Br,rel}(v,\nu) \approx 5 \cdot 10^{-42} n_e n_z \tag{6.1}$$

As this is not a thermal process, we see that the emission does not depend on temperature any longer.

Chapter 2

Astrophysical Applications

Bremsstrahlung emission is present in different processes and active regions throughout the universe, from the ionised gas in between stars or galaxies, to the cascading gamma rays in our own atmosphere. In particular, we'll start by going over the bremsstrahlung emission of gas in galaxy clusters, and the fundamental role it played in estimating their mass, hinting at the overwhelming presence of dark matter; finally, we'll briefly discuss the role of Bremsstrahlung in study of gamma rays in our atmosphere.

2.1 Bremsstrahlung in Galaxy Clusters and Dark Matter

Galaxy Clusters (GCs) are collections of hundreds to thousands of gravitationally interacting galaxies. They are among the largest virialized structures in the universe.

Among the galaxies of a cluster there resides a great quantity of hot gas, heated from the gravitational collapse of the cluster formation. Assuming that a given cluster moves at typical velocities $v \approx 10^{2-3} km/s$, and that the composition is hydrogen-dominated, we can estimate the intracluster gas temperature as:

$$kT \approx m_p v^2$$
$$T \approx 10^7 - 10^8 K$$

Which indicates that these are full of hot ionised gas, emitting bremsstrahlung radiation in the X-ray spectrum. In fact, GCs are some of the brightest emitters in the X-ray sky.

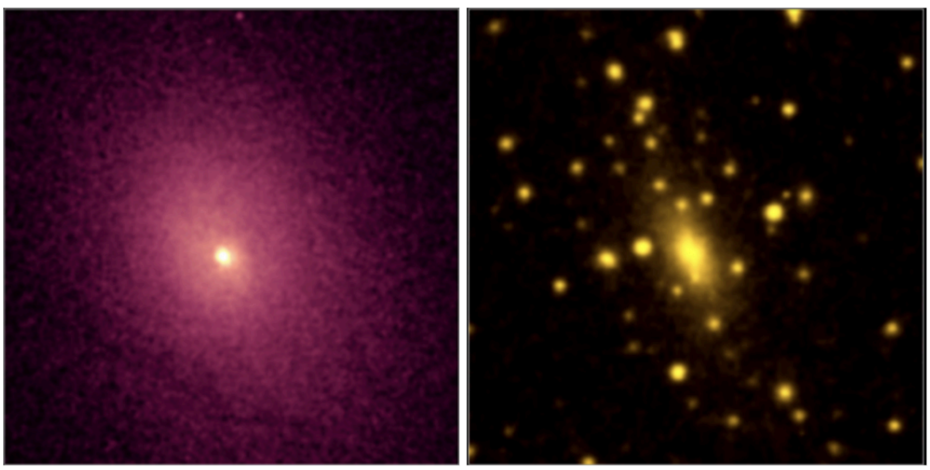

Figure 2.1: GC Abell 2029 in X-ray (left) and in optical wavelengths (right) [5]

Figure 2.1 is a perfect example of just how prominent the gas presence is in GCs.

The overall mass of GCs has been a debated topic since the early '30s, when Swiss astrophysicist F. Zwicky estimated the mass of the *Coma Cluster* from the motion and velocity dispersion of its outer galaxies[3].
His findings were not in accordance with the simple sum of the masses of the individual galaxies, by a non-negligible factor.

A calculation of the intracluster gas mass[4] could perhaps justify such a discrepancy between the mass estimates. Since GCs are in hydrostatic equilibrium, we can write:

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho(r) \tag{6.1}$$

With P gas pressure, ρ gas density and $M(r)$ gas mass within a radius R.
Furthermore, if we approximate the gas as ideal:

$$P = nkT = \frac{\rho kT}{\mu m_p} \tag{6.2}$$

$$\frac{dP}{dr} = \frac{k}{\mu m_p}(\frac{\partial \rho}{\partial r} + \frac{\partial T}{\partial r}) = -\frac{GM(r)}{r^2}\rho(r) \tag{6.3}$$

We need the gas density and temperature to obtain its mass. Thankfully, these are both parameters that can be estimated by looking at the gas' bremsstrahlung emission:

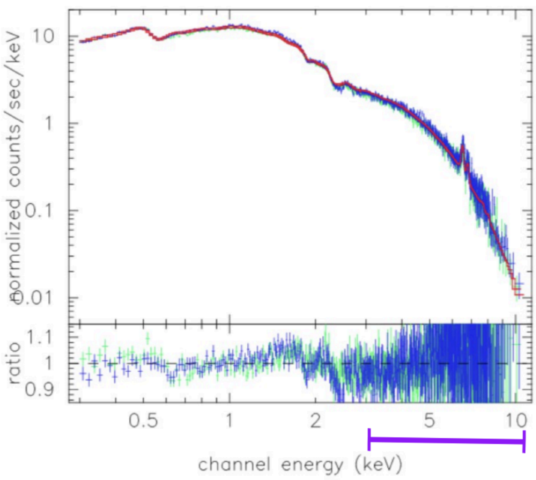

Figure 2.2: Coma Cluster X-Ray best fit flux model, XMM-Newton [8]

- The temperature determines the position of the high energy cut-off as $T = \frac{h\nu}{k}$;
- The total emission is proportional to the square of the electron density, for a hydrogen-dominated gas: $J_{BR} \sim T^{1/2} n_e^{\ 2}$

[3] On the Masses of Nebulae and of Clusters of Nebulae, F. Zwicky, Astrophysical Journal [6]
[4] Bremsstrahlung and Galaxy Clusters, Philip Best, University of Edinburgh [7]

In practice, these observations are not always possible and require great precision, which is why only modern X-ray telescopes like *Chandra*, finally allowed for measurements of nearby and bright clusters.

The results show that the mass of the intracluster is indeed much bigger than the mass of the galaxies, but not quite enough to account for the total mass of the GC.

Studies conducted with X-ray observatories like Chandra and XMM-Newton showed in fact that a GC's mass is made up for 1-5% by galaxies, 10-15% by intracluster gas, while the remaining 80-90% is undetectable dark matter.

2.2 Gamma Rays

Bremsstrahlung radiation also plays a role in the study of cosmic rays and their origin. When extremely energetic photons ($E > 10^{10}\ eV$) interact with the Coulomb fields of particles in our atmosphere they make an ultra-relativistic electron-positron pair, which in turn emit for Bremsstrahlung due to the same fields. This process repeats several times, creating what is known as an *electromagnetic cascade*.

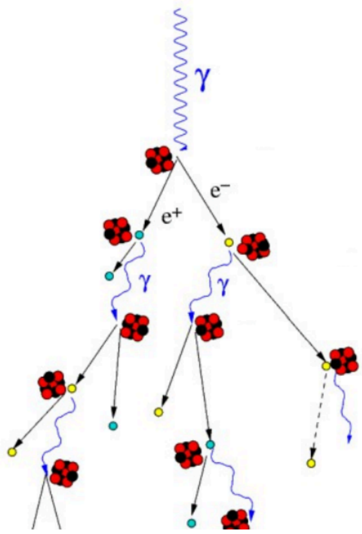

Figure 2.3: Gamma Ray Air Shower [9]

This cascading effect will only last until the energy loss to ionisation is more dominant than the losses for bremsstrahlung, when the energy drops below the critical energy point of $E_c \approx 10^8\ eV$, the *shower max*.

Since these particles are travelling at relativistic speeds not in a vacuum but in our atmosphere, they may reach superluminous velocities $v \geq c_{atm}$, emitting the characteristic Cherenkov radiation[5].

This radiation is not very intense and it causes the cascade to expand in radius and length, but it makes it possible to be observed from the ground with extremely sensitive detectors, called *Imagine Air Cherenkov Telescopes* (IACTs), which can estimate and identify the energy and direction of the original gamma ray.

[5] Radiation emitted by a charge travelling in a dielectric medium at a speed greater than the phase velocity of light in that medium.

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