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Study of the Wheeler-DeWitt equation with the Wigner-Weyl transform

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Abstract

The scope of this thesis is studying a new approach to the Wheeler-DeWitt equation, consisting in a Born-Oppenheimer decomposition of the "Universe wave function" and a generalised description of the gravitational sector through the Wigner function. This new approach is more general than the standard one since it can be applied to mixed state which may arise, in the early Universe, as the consequence of the interaction with some hidden sector (trans-plankian) of the theory.
The gravity wave function gets transformed according to Wigner-Weyl, different solutions to the gravity equation are studied and their association with the Wigner function calculated. The form of the Wigner function which solves the gravity equation then affects the form of the matter (inflaton) equation. In the semi-classical limit the matter equation takes the form of a Schrodinger (or Schwinger-Tomonaga) equation, and the time can be defined in it. In the Born-Oppenheimer approach, non-adiabatic next-to-leading order corrections emerge in the gravitational and in the matter equation. Such corrections have a quantum-gravitational origin in this context. The study of the Wigner function indeed is useful to keep track of the quantum effect of gravity, and to better understand their role in the matter-gravity system.
The above study is applied in particular for two different sets of initial condition for the gravitational wavefunction: the first one is the Hartle-Hawking (HH) initial condition which describe the Universe as a superposition of expanding and contracting solutions, the second one is the Vilenkin initial condition, which describe an expanding Universe. For both cases different approximation methods and procedures are analyzed. The Vilenkin solution, in particular, has been shown to generate quantum-corrections to the definition of time inside the matter equation, which can be described as the presence of an "early-Universe virtual fluid" possibly affecting the slow-roll parameters of and the spectral indices of the primordial spectre.

Contents

Introduction

Quantum Cosmology studies how the quantum effects can affect the evolution and the features of the Universe. Certainly, our knowledge of cosmology is based on the theory of General Relativity, but we expect that Quantum Mechanics/Quantum Field Theory had a fundamental role during the early stages of the Universe. In particular, since the Cosmic Microwave Background (CMB) measurement, it has become evident how the quantum behaviour of matter is relevant for the study of the Universe. Quantum Cosmology is the theory where General Relativity and Quantum Mechanics are intertwined, in particular searching for a quantum origin of our Universe.
The theory of inflation is a bridge between “classical” Cosmology and Quantum Mechanics, and nowadays is the most accepted theory to explain the behaviour and the evolution of the Universe in its early stages, with a period of accelerated expansion, *Inflation*, which may be originated by the presence of a nearly homogeneous scalar field, the *inflaton* and occurred in the early stages of the Universe history at very-high energies, and therefor we expect that, in this regime, new and unknown physics may be taken into account. Inflation can eventually provide insights to the solution of some of the biggest problems of modern physics, such as the formulation of quantum gravity. Indeed, in the early stages of the inflationary expansion, quantum gravity could have affected the spectrum of the fluctuations, leading in principle to tiny observational effects to test candidate quantum theories of gravity.
The study of the quantum effects on the evolution of the Universe had a turn 50 years ago, when John Archibald Wheeler e Bryce DeWitt proposed the canonical quantization of General Relativity, introducing their famous *“Wheeler-DeWitt equation”* [12], thanks to which they were able to introduce the wavefunction of the Universe. The whole Universe was then considered as a quantum object, whose state was described by their equation. This was one of the first and more conservative attempts to describe quantum mechanically the gravitational interaction, and since its formulation, the Wheeler-DeWitt (WdW) equation has been studied by numerous physicists, searching for a most complete description of the quantum effects on the Universe evolution.
A very well-known method to study the WdW equation consists in the Born-Oppheniemer (BO) decomposition of the Universe wavefunction [4] [24] [34] into a gravity wavefunction, which depends on the cosmological factor, and a matter wavefunction, which describes

the modes of the inflaton field or other matter fields. This procedure leads to the separation of the WdW equation into a gravitational equation and a matter equation.
Anyway, in the WdW equation the time variable is not present, so the resulting matter equation will not have a definite time inside itself. This is the consequence of time-reparametrisation invariance of General Relativity. However, time can emerge into the matter equation in the semi-classical limit for gravity, when the cosmological factor follows its classical trajectory. In this way the matter equation becomes equivalent to the time-dependent Schrodinger (Schwinger-Tomonaga) equation, and apart from tiny equation corrections, one recovers the standard results of Quantum Field Theory on curved (classical) space-time.
Let us note that time can be introduced in this semi-classical limit, and since the matter equation depends on the solution of gravitational equation, usually some approximation is needed to obtain the gravity wavefunction. Commonly the WKB approximation is used to solve the gravity equation, since this could be the most efficient way to study both the semi-classical limit and the emergence of quantum corrections in the matter equation.
Recently, an alternative procedure has been proposed to introduce "time" in the matter equation. It consist in transforming the gravity wavefunction according to Wigner-Weyl and then substituting the gravity wavefunction with a Wigner function in the matter equation. This procedure allows to better describe the quantum effects of gravity, and to generalize the BO decomposition to mixture of states of the gravitational wavefunction.
In this thesis we study the matter (inflaton) equation resulting for different choices of the Wigner function for the gravitational sector, corresponding to diverse initial conditions and describing pure and mixed states. Finally, a new way to evaluate the quantum gravitational corrections in the matter equation is proposed.
The appearence of quantum-gravitational corrections in the matter equation are associated with the definition of time. This could lead to the appearance of a "*virtual fluid*" in the early Universe, whose effect becomes negligible for increasing scalar factor, but may still contribute with non-negligible effects on the early stage of Universe/inflation. Studying the continuity equation of this virtual fluid, it is possible to see how its presence could affect the potential of the scalar (inflaton) field.
Further, modifications of the evolution of the scale factor could have consequences on the values of the slow-roll parameter of the inflationary theory, thus leading to small variations of the spectrum of the quantum perturbations described by the Mukhanov-Sasaki equation. This result is an example of how quantum-gravitational effects affect the observables related to inflation, in particular those which describe the anisotropies present in the CMB, and such effects may be relevant to discriminate between different models of the early Universe.
The thesis is organized as follows:

- In Chapter 1 we first recall the basic properties of the Friedmann-Robertson-Walker spacetime and the problems of the Standard cosmological model which motivate the necessity of an inflationary phase. We then briefly discuss a model of inflaton made of a scalar homogeneous field with a nearly constant potential. Lastly we comment the quantum-cosmological perturbation theory and the Mukhanov-Sasaki equation.

- In Chapter 2 we study the Wheeler-DeWitt equation through the Born-Oppenheimer decomposition, obtaining a gravitational equation and a matter equation. We then introduce the Wigner function and study the gravitational equation for the gravitational Wigner function, with Hartle-Hawking initial conditions. Lastly, we overview the quantum theory of measurement and the concept of decoherence in quantum cosmology.

- In Chapter 3 we study the matter equation which is obtained by considering the Wigner function for the Hartle-Hawking case found in Chapter 2. The classical and semi-classical approach are analyzed, showing how the time emerges in both limits, and seeing that the matter equation consistently reduces to the Schrodinger equation in both cases.

- In Chapter 4 we start from the Vilenkin wave function for gravity, find its associated Wigner function and then we study the matter equation in this case. It is shown how the Wigner solution can be coarse-grained and the effects of this procedure are studied. Furthermore, a new approach is illustrated to estimate the matter equation accounting for the quantum gravitational corrections. Lastly we comment on the cosmological effects produced by these quantum corrections.

Chapter 1

The cosmological model

1.1 The FLRW metric

Modern cosmology is built upon two foundational principles that define the intrinsic nature of our Universe. The first principle, known as the Copernican Principle, asserts that:
"*We do not occupy a special or privileged position within the Universe.*"
This implies that the Universe would appear similar to any other observers as it does to us. While this principle has limited practical application, it sets the stage for further exploration. However, it is the Cosmological Principle that carries significant weight:
"*The Universe is homogeneous and isotropic.*"
In this context, isotropy is recognized as an observational fact, while homogeneity is deduced from the assumption that isotropy is observer-independent, as for the Copernican Principle. Essentially, these principles outline the symmetries of our Universe. Let's delve into these seemingly straightforward symmetries.
When observing the night sky, it is evident that it lacks isotropy: our solar system mainly consists of empty space with scattered celestial bodies, forming constellations, clusters, and the Milky Way. Nevertheless, we speculate that if we could detect all matter in the Universe, its average distribution (over a sufficiently large portion of the night sky) would be consistent in all directions. Thus, isotropy is more an assumption than an observation.
It is essential to note that what we perceive in the night sky isn't the instantaneous state of the Universe but rather an image formed by light cones reaching us at the time of observation. Asserting that the Universe is homogeneous and isotropic implies the existence of a time t at which the Universe is uniform and symmetric on each time slice Σ_t. The matter distribution on each Σ_t influences light propagation, highlighting the necessity for experimental validation of the entire framework. Additionally, signals traveling along light cones may have originated at different times Δt in the past from various distances

Δs, requiring separate methods to determine either Δt or Δs to validate specific models. Consequently, the Cosmological Principle serves as a fundamental assumption upon which explicit models of the Universe are constructed and subsequently tested. Specifically, we should anticipate a minimum scale beyond which the Universe exhibits homogeneity. However, for the sake of discussion, let's consider an idealized scenario where galaxies form a uniform (perfect) fluid filling all of space.

1.1.1 The FLRW metric satisfies the Copernican and Cosmological Principles

The configuration of the cosmological metric finds some of its parameters determined by the postulation of Killing vectors, which delineate the directions amenable to the application of metric symmetries. Specifically, we will now encounter three spatial Killing vectors that induce spatial translations, thus characterizing homogeneity, along with three additional spatial Killing vectors responsible for rotations, thus defining isotropy. We must also recall that isotropy with regard to any arbitrary point equates to homogeneity.

Moreover, it is notable that we do not possess a time-like Killing vector in this context, as we seek to describe a dynamically evolving Universe. This insight stems from the groundbreaking observations made by Edwin Hubble and Milton Humason in 1929, revealing that galaxies at greater distances from us exhibit accelerated recession.

The request of having a metric which would describe an homogeneus and isotropic Universe, which at the same time would also fit for the case of an expanding Universe, led Friedmann, Robertson, Walker, and Lemaître to build the FLRW metric [6], representing a cornerstone in the characterization of the cosmological model:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin(\theta)^2 d\phi^2)\right] \tag{1.1}$$

where $\{t, r, \theta, \phi\}$ are the comoving coordinates, $a(t)$ is the cosmic-scale factor, which embeds the effect of the expanding Universe, $R_{curv} \equiv a(t)|k|^{-1/2}$ is the curvature radius, and $k = -1, 0, 1$ is the curvature scalar which, how we will see, will determine, beside the geometric propriety of the space-time, also the evolutionary destiny of the Universe. Depending on the value of the curvature scalar, we can introduce new coordinates such that the topology of the hypersurface Σ_t is apparent from its line element $d\sigma^2$:

- **Flat Universe**: for $k = 0$, the coordinate r is very similar to the usual radial coordinate in $R_{(3)}$:

  $$d\sigma^2 = dr^2 + r^2 d\Omega^2 = dx^2 + dy^2 + dz^2,$$

  and Σ_t is flat (zero spatial curvature).

- **Closed Universe**: for $k = +1$, the proper radius $R_{(3)}$ is bounded from above and

$$r = \sin(X) \rightarrow d\sigma^2 = dX^2 + \sin^2(X)d\Omega^2,$$

and Σ_t is a 3-dimensional sphere.

- **Open Universe**: for $k = -1$, one can write

$$r = \sinh(\psi) \rightarrow d\sigma^2 = d\psi^2 + \sinh^2(\psi)d\Omega^2,$$

and Σ_t is a 3-dimensional hyperboloid.

Finally, the meaning of the coordinate r is very different from that in the Schwarzschild space-time. If we write our metric as:

$$ds^2 = -dt^2 + a^2(t)d\sigma^2$$

we can see that in the FLRW metric the areal radius r_A is given by:

$$r_A = a(t)r$$

and the area of surfaces of constant r therefore depends on time. Likewise, the proper distance between two points is given by:

$$dR = a(t)\frac{dr}{\sqrt{1 - kr^2}} = a(t)dR_{(3)}$$

where $R_{(3)}$ is the proper distance on a surface Σ_t, and this quantity can be bounded. Observations suggest that the distance between galaxies increases in time, whereas their typical size remains the same. We can therefore claim that the Universe is expanding, with the furthest galaxies moving faster away from us, like dots on an inflating balloon. This picture can be mathematically modeled by a modified FLRW metric which locally (around matter sources such as a galaxy) looks like the Schwarzschild metric: local lengths are mostly affected by the localized sources and do not appreciably change in time, whereas the distance between sources increases because of the increasing scale factor. This picture is still debated sometimes, and is the topic of the so-called Einstein-Straus problem in General Relativity.

1.1.2 The Universe as a Perfect Fluid

As we wrote above, we can make a series of simplifying assumptions about the behaviour of the elements that constitute our Universe, such assuming the Universe is filled with a perfect fluid of matter and energy. Its energy-momentum tensor then can be written in the following, diagonal, form:

$$T^{\mu}{}_{\nu} = diag\{-\rho, p, p, p\} \tag{1.2}$$

where ρ is the energy density and p is the pressure, and we can write them as time-dependent quantities $\rho(t)$, $p(t)$ that satisfy the continuity equation to which a perfect fluid is subject:

$$\nabla_\mu T^\mu{}_\nu = 0. \tag{1.3}$$

The 00-component of this equation yields to the condition of energy conservation:

$$\nabla_\mu T^\mu{}_0 = -\partial_0 T^0{}_0 - \sum_{i=1}^{3} \Gamma^i_{i0}(T^0{}_0 - T^i{}_i) = \dot{\rho} + 3H(\rho + p) = 0 \tag{1.4}$$

where the dot stands with the differentiation respect to the cosmological (proper) time, Γ^i_{jk} is the well-known Christoffel symbol and $H \equiv \dot{a}/a$ is the Hubble constant, which measures the rate of expansion of the Universe.
The continuity equation (1.4) can be written in the following form:

$$\frac{d}{dt}(a^3\rho) = -p\frac{d}{dt}(a^3) \tag{1.5}$$

and, assuming an equation of state, ω, for the fluid:

$$p = \omega\rho. \tag{1.6}$$

When ω is constant, then the energy-conservation equation implies:

$$\frac{\dot{\rho}}{\rho} = 3(1+\omega)\frac{\dot{a}}{a} \rightarrow \rho = a^{3(1+\omega)}. \tag{1.7}$$

The simplest components of cosmic fluids are dust (pressure-less matter, or non-relativistic matter almost exactly at rest with the cosmic frame) and radiation (massless matter, or highly-relativistic matter), and we can reproduce their behaviour by a proper choice of the parameter ω:

- **Dust**: in this case no force is present, beside gravity, and we must choose $\omega = 0$ so that the fluid it is pressure-less ($p = 0$). In this case, solving the continuity equation, we find the following expression for ρ from (1.7):

$$\rho_{dust} \propto a^{-3} = \frac{E}{V} \tag{1.8}$$

  which reflects the fact that the energy is proportional to the proper mass of dust particles, $E = n \cdot m_0$, where n is the particle density, and it scales with volume, i.e. like $V \propto a \times a \times a$.

- **Radiation**: in the case of radiation, the mass of the particles is negligible, and that reflects on the energy-momentum tensor, which is traceless:

  $$T = \rho + 3p = 0$$

  so we can conclude that:

  $$p = \frac{1}{3}\rho \rightarrow \omega = \frac{1}{3}, \tag{1.9}$$

  and

  $$\rho_{rad} \propto a^{-4} = \frac{E}{V}. \tag{1.10}$$

  This result can be understood by noting that the number density scales again like $V \propto a^3$, and photon energy also redshifts according to

  $$E \propto a^{-1}. \tag{1.11}$$

For a long time it was thought that we now live in a matter (dust)-dominated Universe, whereas in the early stages, the Universe dynamics was controlled by radiation, since the density of the latter increases faster (going backward in time). We now know that the Universe expansion is presently accelerating ($\ddot{a} > 0$), which is not compatible neither with the effect of dust or radiation. So we must search for a new energy source, which could explain the late inflationary behaviour of the Universe,

- **Vacuum or dark energy**: among possible sources driving cosmic acceleration, we include the fluid with equation of state:

  $$\rho = -p = \frac{\Lambda}{8\pi G_N}, \quad \omega = -1, \quad \rho_\Lambda \propto const. \tag{1.12}$$

  where Λ is the so-called "*Cosmological constant*", and was initially introduced by Einstein on the l.h.s of his equations to search for static solutions of the Universe:

  $$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}.$$

  Einstein himself later describes this attempt as its "*biggest blunder*", but nowadays it seems fundamental to explain the accelerated behaviour we observe for our Universe.

1.1.3 Friedman Equations

Studying the Einstein equation in the context of the FLRW metric, two fundamental equations, known as Friedman equations are found [30]:

$$G_{00} = 8\pi G_N T_{00} \rightarrow 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] = 8\pi G_N \rho \tag{1.13}$$

and

$$G_{ii} = 8\pi G_N T_{ii} \to 3\frac{\ddot{a}}{a} = -4\pi G_N(\rho + 3p). \tag{1.14}$$

The first equation (1.13), is a constraint, which selects the possible combinations of initial conditions $a(t_0) = a_0$ and $\dot{a}(t_0) = \dot{a}_0$ for the truly dynamical (second order) equation (1.14) for the scale factor $a = a(t)$, given a specific matter content. However, the constraint is preserved at all times, as can be seen by deriving Eq.(1.13) with respect to time and using the continuity equation (1.7) to obtain (1.14). So in the end, for a fluid satisfying the continuity equation (1.7), it is easier to just solve for the constraint (1.13) at all times $t \geq t_0$.
We can introduce a new parameter to rewrite the Friedman equations: the *density parameter* Ω:

$$\Omega = \frac{\rho}{\rho_{critical}} = \frac{8\pi G_N}{3H^2}\rho \tag{1.15}$$

where

$$\rho_{critical} \equiv \frac{3H^2}{8\pi G_N} \tag{1.16}$$

it is the so-called critical energy density and represents the energy density of a flat Universe (k=0).
We can rewrite the first Friedman equation (1.13) as:

$$\Omega - 1 = \frac{k}{H^2 a^2} \tag{1.17}$$

and the following observations can be made:

- If $\rho < \rho_{critical}$, then $\Omega < 1$, $k = -1$ which corresponds to a **Open Universe** scenario,
- If $\rho = \rho_{critical}$, then leads to $\Omega = 1$, $k = 0$ which corresponds to a **Flat Universe** scenario,
- If $\rho > \rho_{critical}$, then leads to $\Omega > 1$, $k = 1$ which corresponds to a **Closed Universe** scenario.

The spatial curvature k and the equation of state of the fluid which dominates the energy budget of the Universe then determine the evolution of the scale factor (see Figure 1.1, which displays the cosmic evolution in the matter-domination case, and similar behaviours would also occur for radiation). Observations suggest that our Universe is very close to the critical density. For a flat, matter dominated Universe, one has $\rho \propto a^{-3}$, thanks to which we can wrote the following law for a:

$$\frac{\dot{a}}{a} \sim a^{-3} \to a^{3/2} \sim t. \tag{1.18}$$

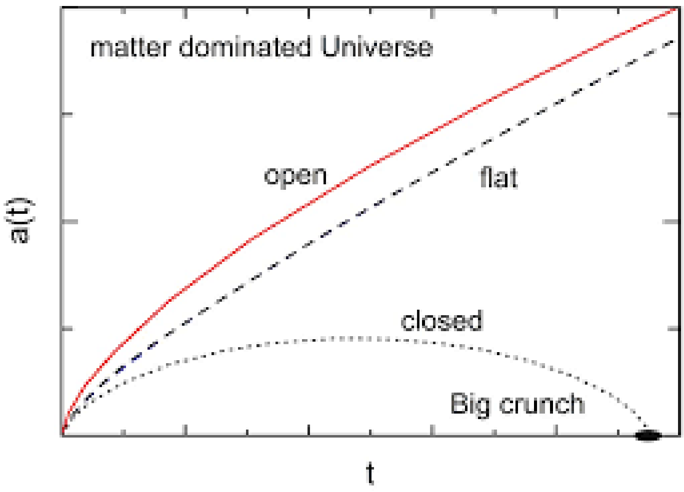

Figure 1.1: The time dependence of the scale factor for open, closed and critical matter-dominated cosmological models.

For a flat, but radiation dominated Universe, one has $\rho_{rad} \propto a^{-4}$, and we can conclude that:

$$\frac{\dot{a}}{a} \sim a^{-4} \rightarrow a^2 \sim t. \tag{1.19}$$

Finally, for a flat and empty Universe, with only a positive vacuum energy present, one can write $\rho_{critical} = \Lambda$ and obtains the exact solution:

$$\frac{\dot{a}}{a} \sim \frac{\Lambda}{3} \rightarrow a \sim e^{H_0 t} \tag{1.20}$$

where H_0 is now a true cosmological constant.

1.1.4 The flatness problem

As we saw in the previous section, the quantity $\dot{a}$ is decreasing in time for the case of a dust/radiation dominated Universe. As we can see from (1.15), for radiation and dust domination, the quantity $|\Omega - 1|$ grows with time:

- radiation domination: $|\Omega - 1| \propto t$,
- dust domination: $|\Omega - 1| \propto t^{2/3}$.

This means that any deviation from the critical density ($\Omega = 1$) present in the early Universe, would have been immensely amplified. Since today the observed value of the Ω parameter is very close to one, one could quantify the maximal deviation from flatness that should be present when our Universe originated at $t = t_i$. We can simply compute this quantity by mean of the relation:

$$\frac{|\Omega_i - 1|}{|\Omega_0 - 1|} = \left(\frac{\dot{a}_0}{\dot{a}_i}\right)^2 \tag{1.21}$$

where the subscription 0 stands for the value measured today.
A rough estimation of the $\dot{a}_0/\dot{a}_i$ can be done considering the initial instant i as the Plank time t_{pl}, the time at which the Universe was at a temperature T_{pl}, and further considering:

$$a \sim T^{-1}, \quad \dot{a} \sim \frac{a}{t} \tag{1.22}$$

and so we obtain:

$$\left(\frac{\dot{a}_0}{\dot{a}_i}\right)^2 = \frac{t_{pl}}{t_0}\frac{T_{pl}}{T_0} \sim \frac{10^{-43}}{10^{17}}10^{32} \tag{1.23}$$

with a final result $|\Omega_i - 1| < 10^{-52}$.
This is a very fine tuned and unnatural initial condition, and this question is addressed as the *flatness problem*. One could say that, for some unknown reasons, our universe emerged from the Planck era with a density very close to the critical one. But, this statement is unsatisfactory because the case $\Omega = \Omega_c$ describes an unstable Universe with respect to the generic case $\Omega \neq \Omega_c$.

1.1.5 CMB and the horizon problem

Different experiments have verified the presence of a nearly uniform radiation pervading the Universe, known as the Cosmic Microwave Background (CMB). It is believed that this radiation emerged in the early stages of the Universe, when the decoupling of matter and radiation occurred at the last scattering surface (LSS), allowing photons to travel freely.
By solving the Friedman equation in presence of matter (dust) and radiation fluids, one easily realises that the Universe was significantly denser and hotter in its early stages. At the onset of matter domination the Universe was opaque: energy density was high and electrons where not bounded by nuclei. Photons interacting with charged particles (protons and electrons) had a very short free path and such a primordial plasma was in thermal equilibrium. When energy dropped below $0.3eV$, neutral hydrogen atoms formed and photons could free stream, constituting the oldest detectable light signals observable today, characterized by a temperature of approximately $3K$. In principle, only gravitational waves we could observe today may be originated in preceding epochs. Considering this picture of the early Universe, the uniformity (or more precisely, the isotropy from our point of observation) of the CMB seems unjustifiable within the standard Big Bang scenario. When observing opposite directions in the celestial sphere, the

CMB reaching us is isotropic, but comes from casually disconnected regions. So the fundamental question arises of how this is possible without assuming very fine tuned or unnatural initial conditions for our Universe. Deriving for the FLRW metric the definition of light cones we have:

$$ds^2 = 0 \rightarrow dr = \frac{dt}{a}. \tag{1.24}$$

Suppose we place ourselves at $r = 0$ and integrate the above expression (along the light-cone) from $t = -t_s$, the time of the last photon-scattering, until now ($t = 0$). We thus find the comoving distance travelled by a CMB photon:

$$r_s = \int_0^{r_s} dr \sim \int_{-t_s}^{0} \frac{dt}{a(t)} = \int_{a(-t_s)}^{a_0} \frac{da}{a^2 H}. \tag{1.25}$$

In a Universe dominated by matter, we have seen that $H \sim H_0(a_0/a)^3$, so

$$r_s \sim \frac{1}{2H_0 a_0^3}(a_0^2 - a_s^2) \sim \frac{1}{2H_0 a_0}$$

where H_0 is the value fo the Hubble constant today.
If then we compute the corrisponding proper distance travelled by that photon from the last-scattering epoch to now, we find

$$R = a_0 r_s = \frac{1}{2H_0} \tag{1.26}$$

This distance is the radius of the LSS today, while r_s is the comoving radius of the LSS. We can also note that H^{-1} is approximately the so-called particle horizon R_H

$$R_H = \frac{1}{H(-t_s)} \sim t_s \tag{1.27}$$

which physically represents the size of a casually connected region in an expanding Universe. In comoving coordinates the size of casually connected region is $r_H = (aH)^{-1}$, and at the time of last scattering $r_H^{ls} = (a_{ls}H_{ls})^{-1}$. If we now compare r_H^{ls} with a region on the LSS subtended by an angle $\Delta\theta$ we find:

$$r_H^{ls} = r_s \Delta\theta \rightarrow \Delta\theta = \frac{r_H^{ls}}{r_s} \approx \frac{a_0 H_0}{a_{ls} H_{ls}} = \left(\frac{a_0}{a_{ls}}\right)^{1/2} \sim 10^{3/2} \sim 1.8^\circ. \tag{1.28}$$

Thus only the distances on the LSS subtended by an angle less than $\sim 1.8^\circ$ are in casual contact. We conclude that photons of CMB are isotropic despite being originated by (apparently) casually disconnected regions.

1.2 Inflaton and Inflation

The Flatness and Horizon problem have a common origin: the fact, for dust or radiation dominated Universe, to have a factor $\dot{a}^{-1} = (aH)^{-1}$ that grows with time. This also mean that they can be simultaneously solved by imposing the condition thanks to which $(aH)^{-1}$ decreases in time. This condition can be written as:

$$\frac{d}{dt}(aH)^{-1} < 0 \rightarrow \ddot{a} > 0.$$

So an accelerating stage during the primordial phases of the evolution of the Universe might be able to solve the flatness and horizon problems. From the second Friedman equation (1.14) we learn that the condition for an accelerating Universe $\ddot{a} > 0$ is satisfied by the request:

$$\rho + 3p < 0 \rightarrow p < -\rho/3 \rightarrow \omega < -\frac{1}{3}. \tag{1.29}$$

An accelerating period is obtainable only if the overall pressure p of the Universe is negative: $p < -\rho/3$, satisfied by the condition $\omega < 1/3$ from the continuity equation (1.4). From the cosmological models we have described, neither a radiation-dominated phase nor a matter-dominated phase (for which $p = \rho/3$ and $p = 0$, respectively) satisfy such a condition. A possible model satisfying the condition (1.29) is that of a vacuum energy-dominated Universe with $p = -\rho = -\Lambda = constant$, and the resulting evolution is known as *de Sitter phase*. Indeed, if one considers a generic value for the parameter ω, (1.25) gives us the following expression of the comoving distance r_s:

$$r_s \sim \int_{a(-t_s)}^{a_0} \frac{da}{a\dot{a}} = \frac{2}{1+3\omega}\left(a_0^{\frac{1}{2}(1+3\omega)} - a_s^{\frac{1}{2}(1+3\omega)}\right) \rightarrow +\infty$$
$$\text{for} \quad a_s \rightarrow 0; \quad 1+3\omega < 0 \tag{1.30}$$

and in the particular case of a vacuum dominated (de Sitter) Universe, we will have:

$$a(t) \propto e^{H_0 t} \rightarrow r_s \sim \frac{e^{H_0 t_s}}{H_0} \rightarrow +\infty \quad \text{for} \quad t_s \rightarrow +\infty. \tag{1.31}$$

Having an increasing particle horizon, every point on the LSS can be in causal contact assuming that the primordial phase of accelerated expansion lasts enough.
This early phase of accelerated expansion is called *inflation*.

1.2.1 Euler-Lagrange analysis for the action of an inflaton field

Previously we have described the various advantages of having a period of accelerated expansion, and showed that the latter required $p < -\rho/3$. Now, we would like to show that this condition can be attained by means of a simple homogeneous scalar field. We

shall call this field the inflaton ϕ.
The action of the inflaton field reads [31]:

$$S_\phi = \int_\Omega d^4x\sqrt{-g}\mathcal{L} = \frac{1}{2}\int_\Omega d^4x\sqrt{-g}[\partial_\mu\phi\partial^\mu\phi - 2V(\phi)] \tag{1.32}$$

where $\sqrt{-g} = a^3$ in the FLRW metric (1.1).
Now considering the Euler-Lagrange equations for the Lagrangian (1.32):

$$\partial^\mu \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta\partial_\mu\phi} - \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta\phi} = 0 \tag{1.33}$$

we can find:

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2\phi}{a^2} + V'(\phi) = 0 \tag{1.34}$$

where $V'(\phi) = (dV(\phi)/d\phi)$ and $\nabla^2\phi = 0$ due to homogeneity. We note in particular the appearance of a term $+3H\dot{\phi}$, known as *friction term*, which originates from the fact that the scalar field rolling down its potential suffers a friction due to the expansion of the Universe.
We can write the energy-momentum tensor for the scalar field:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L} \tag{1.35}$$

and write the corresponding energy density ρ_ϕ and pressure p_ϕ as:

$$T_{00} = \rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi), \tag{1.36}$$

$$T_{ii} = p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi). \tag{1.37}$$

One now could consider a small departure from homogeneity

$$\phi(x,t) = \phi_0(t) + \delta\phi(x,t) \tag{1.38}$$

where $\phi_0(t)$ is the ‘classical’ homogeneous inflaton, while $\delta\phi(x,t)$ represents the quantum fluctuations around $\phi_0(t)$.
If we assume that the potential term is much larger with respect to the kinetic term of the scalar field, $V(\phi) >> \dot{\phi}^2$, we satisfy the following condition:

$$p_\phi \sim -\rho_\phi. \tag{1.39}$$

We then realize that a scalar field whose potential energy dominates over the kinetic term gives inflation. So, inflation is driven by the vacuum energy of the inflaton field.

It is useful to express the inflationary evolution in terms of the so-called "slow-roll parameters", which can be easily related to the conditions necessary for the Universe to undergo inflation. The first one is ϵ_1, which is equivalent to the condition $V(\phi) >> \dot{\phi}^2$:

$$\epsilon_1 = \frac{\dot{\phi}^2}{V(\phi)} \sim -\frac{\dot{H}}{H^2} << 1 \tag{1.40}$$

where in the last relation we exploited the relation coming from (1.34).
In order to solve the horizon and flatness problems, inflation must last for a sufficiently long period of time, so the field acceleration must be small compered to the friction term:

$$|\ddot{\phi}| << |3H\dot{\phi}|$$

and this property can be described by another slow-roll parameter, η:

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}} << 1 \tag{1.41}$$

which can be re-expressed in terms of the second derivative of the inflaton potential $V(\phi)$:

$$\eta_V = -\frac{m_p^2 V''(\phi)}{6V(\phi)} \tag{1.42}$$

such that $\eta = \eta_V - \epsilon_1$.

1.2.2 The case of a constant potential in a mini-superspace

Consider now the simple case of a mini-superspace, containing only two degrees of freedom associated with a minimally-coupled homogeneous scalar field, that we identify with the inflaton ϕ. In this context, we can write an action [23]:

$$\tilde{S} = \frac{1}{2}\int_\Omega d^4x\sqrt{-g}\Big[m_p^2 R - \partial_\mu\phi\partial^\mu\phi - 2V(\phi)\Big] \tag{1.43}$$

where R is the Ricci scalar and $m_p = (8\pi G_N)^{-\frac{1}{2}}$ is the Plank mass.
Now we would write the FLRW metric (1.1) in the ADM decomposition [25] that slices our space-time into homogeneous (space-like) hypersurfaces:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(N^2 - N_i N^i)dt^2 + 2N_i dx^i dt + a^2(t)(dx^2 + dy^2 + dz^2) \tag{1.44}$$

where $N = N(t)$ is the lapse function and $N^i = [N^x(t), N^y(t), N^z(t)]$ are the three shift functions which encodes the displacements on the homogeneous space-like surfaces

$\Sigma = \Sigma(t)$ identified by the fixed time parameter t.
In this contest, we can decompose the Hamiltonian of our system as

$$H = NH_S + N^i H_i$$

where H_S is the *super-Hamiltonian*, which generates diffeomorphism along the synchronous time τ in the direction $\vec{n} = \frac{d}{d\tau}$ orthogonal to the hypersurfaces $\Sigma(t)$, and H_i are the *super-momenta*, which generate spatial diffeomorphism along the hypersurfaces $\Sigma(t)$, such that, for every tensor T:

$$\{T, H_i\} = \mathscr{L}_{\partial_i} T. \tag{1.45}$$

We assume that the basis vector ∂_i are also Killing vectors, so:

$$\{T, H_i\} = \{T, H_i^G + H_i^\phi\} = \mathscr{L}_{\partial_i} T = 0. \tag{1.46}$$

Indeed, for the metric (1.44), the gravitational super-momenta

$$H_i^G \propto G_{0i} = 0 \tag{1.47}$$

and the matter super-momenta, for a scalar field ϕ:

$$H_i^\phi \propto \partial_i \phi = 0 \tag{1.48}$$

vanish identically, so (1.46) becomes an empty identity. In order to proceed we can pick $N^i = 0$ for all the shift functions, such that each point on the surface $\Sigma(t)$ cannot be identified with respect to the others, and so our system becomes essentially 1-dimensional, with only the parameter t which can be used to label different points. This is not a proper gauge, since $\{N^i, H_i\} = 0$ trivially, but the choice $N^i = 0$ is preserved by the Hamiltoninan evolution, so it is compatible with the dynamics.
Also, we will perform a change of variables following the procedure displayed by Louko in [28]

$$\bar{N} \equiv aN, \quad q \equiv a^2$$

and so our metric becomes

$$ds^2 = -\frac{\bar{N}^2}{q} dt^2 + q(t)(dx^2 + dy^2 + dz^2) \tag{1.49}$$

and we must also observe that now our Hamiltonian is decomposed as

$$H = \frac{\bar{N}}{\sqrt{q}} H_S. \tag{1.50}$$

From the above metric redefinition, we can compute the Ricci's scalar as:

$$R = \frac{3}{\bar{N}^2}\left(\ddot{q} - \frac{\dot{q}\dot{\bar{N}}}{\bar{N}} + \frac{\dot{q}^2}{2q}\right) \tag{1.51}$$

where the dot indicates a derivative respect to t. Now, for simplicity, we can limit ourself to the case of constant potential $V(\phi) = \Lambda$, and Λ plays the role of the cosmological constant.
Then we can compute the action, having care of redefining $m_p \to \sqrt{6}m_p$ and adding a vanishing boundary term in order to delete the dependence of the action by terms $\ddot{q}$ and $\dot{\bar{N}}$, obtaining:

$$\begin{aligned} S \equiv \frac{\tilde{S}}{V} &= \frac{1}{2}\int_{t_1}^{t_2} dt\bar{N}q\left[\frac{m_p^2}{2\bar{N}^2}\left(\ddot{q} - \frac{\dot{q}\dot{\bar{N}}}{\bar{N}} + \frac{\dot{q}^2}{2q}\right) + \frac{q\dot{\phi}^2}{\bar{N}^2} - 2\Lambda\right] \\ &= \frac{1}{2}\int_{t_1}^{t_2} dt\bar{N}q\left[\frac{m_p^2}{2\bar{N}^2}\left(\ddot{q} - \frac{\dot{q}\dot{\bar{N}}}{\bar{N}} + \frac{\dot{q}^2}{2q}\right) + \frac{q\dot{\phi}^2}{\bar{N}^2} - 2\Lambda\right] - m_p^2\frac{d}{dt}\left(\frac{\dot{q}q}{2\bar{N}}\right) \\ &= \frac{1}{2}\int_{t_1}^{t_2} dt\left[-m_p^2\frac{\dot{q}^2}{4\bar{N}} + \frac{q^2\dot{\phi}^2}{\bar{N}} - 2\bar{N}q\Lambda\right]. \end{aligned} \tag{1.52}$$

and so, we get the Lagrangian:

$$L = \frac{1}{2}\left[-m_p^2\frac{\dot{q}^2}{4\bar{N}} + \frac{q^2\dot{\phi}^2}{\bar{N}} - 2\bar{N}q\Lambda\right]. \tag{1.53}$$

From here, we can proceed with the Hamiltonian analysis[1].

1.2.3 Hamiltonian analysis

Now, from the Hamiltonian formalism, we can introduce the momenta:

$$\pi_N = \frac{\partial L}{\partial \dot{N}} = 0, \tag{1.54}$$

$$\pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{q^2\dot{\phi}}{N}, \tag{1.55}$$

$$\pi_q = \frac{\partial L}{\partial \dot{q}} = -m_p^2\frac{\dot{q}}{4N}. \tag{1.56}$$

Inverting the above relations, we can find:

$$\dot{\phi} = \frac{N\pi_\phi}{q^2}, \tag{1.57}$$

[1]From now on, we will keep writing $\bar{N}$ as N for simplicity.

$$\dot{q} = -\frac{4N\pi_q}{m_p^2}. \tag{1.58}$$

Now we can insert the relations (1.58) and (1.57) inside the Lagrangian (1.53), and so write the canonical Hamiltonian of our system as:

$$\begin{aligned} H &= \pi_N \dot{N} + \pi_q \dot{q} + \pi_\phi \dot{\phi} - L \\ &= -\frac{4N\pi_q^2}{m_p^2} + \frac{N\pi_\phi^2}{q^2} - \frac{1}{2}\left[-\frac{4N\pi_q^2}{m_p^2} + \frac{N\pi_\phi^2}{q^2} - 2Nq\Lambda \right] \\ &= \frac{N}{\sqrt{q}}\left[-\frac{2\pi_q^2\sqrt{q}}{m_p^2} + \frac{\pi_\phi^2}{2q^{3/2}} + q^{3/2}\Lambda \right] \equiv \frac{N}{\sqrt{q}} H_S \end{aligned} \tag{1.59}$$

where H_S is the super-Hamiltonian, which yields the constraint for the physical points. Indeed, Eq. (1.54) is a primary constraint, so it must be conserved under the Hamiltonian evolution. This yields to the Hamiltonian constraint:

$$0 = \{\pi_N, H\} = \frac{\partial H}{\partial N} = \frac{H_S}{\sqrt{q}} \tag{1.60}$$

this is the full theoretical background we need to quantize the system.

Before proceeding with the quantization, we want to rewrite our Hamiltonian (1.59) in terms of conformal time η. Since (1.54) is a primary constraint, N plays the role of a Lagrangian multiplier, and we can impose the gauge condition $N = q$ that brings us the conformal time $\eta = t$ and the Hamiltonian becomes:

$$H = -\frac{2q\pi_q^2}{m_p^2} + \frac{\pi_\phi^2}{2q} + q^2\Lambda \tag{1.61}$$

where we have

$$\pi_q = -m_p^2 \frac{q'}{4q}, \quad \pi_\phi = q\phi'$$

and we point out that the time is the conformal time η with the choice $N = q$, and differentiation with respect to η will be indicated with the prime ($'$) symbol.

Let us consider the Friedman equation (1.13), with $k = 0$ and $\rho = \Lambda$. Selecting the conformal-time with the gauge-choice $N = q$, we can find the behaviour of q'. From the definition of the energy-momentum tensor of the inflaton field $T_{\mu\nu}$ (1.35) and the current FLRW metric in ADM decomposition (1.44), we have the following form for the component ${T_{00}}$[2]:

$$T_{00} = g_{0\alpha} {T^\alpha}_0 = q\rho = q\Lambda$$

[2]As we said in the previous section, we are considering the condition which leads to inflation in our computations, so we are neglecting the kinetic contribution of ϕ

with the Ricci scalar R given by (1.51) and the component 00 of the Ricci tensor that can be computed to be:

$$R_{00} = -\frac{3}{2q}\left(q'' - \frac{q'^2}{q}\right). \tag{1.62}$$

We then we have that (1.13) becomes[3]:

$$\left(\frac{q'}{q}\right)^2 = \frac{8q\Lambda}{m_p^2} \tag{1.63}$$

that, finally, gives us the following classical relation for q':

$$q' = \pm\left(\frac{8q^3\Lambda}{m_p^2}\right)^{\frac{1}{2}}. \tag{1.64}$$

Comparing this last result with the expression found from the Hamiltonian relations (1.58), considering again the conformal time condition $N = q$, we can find the following expression for the classical value of π_q:

$$\pi_q = \pm\left(\frac{q\Lambda m_p^2}{2}\right)^{\frac{1}{2}}. \tag{1.65}$$

In terms of comoving distance, the particle horizon corresponds to the conformal time measured since the Big Bang:

$$R_H = a(\eta)\int_{\eta_0}^{\eta} d\eta = a(t)\int_0^t \frac{dt'}{a(t')} = a(t)\int_o^r dr' \tag{1.66}$$

where the Big Bang is at $t = 0$. This can be seen also substituting our choices $N = a$, $N_i = 0$ inside the metric (1.44).
We can show that, under the condition to be in a flat, vacuum dominated, de Sitter Universe, we can write the cosmological factor a in the form

$$a(t) \sim e^{H_0 t}$$

and so we can find from (1.66) the relation:

$$R_H \sim -H_0^{-1} \tag{1.67}$$

and so, as a consequence, we can express the conformal time η as:

$$\eta = -\frac{1}{H_0 a(\eta)},$$

or, in terms of Louko variable, as:

$$\eta = -\frac{1}{H_0\sqrt{q(\eta)}}. \tag{1.68}$$

[3]we recall once again that we rescaled the value of the Plank mass $m_p \to \sqrt{6}m_p$

1.3 The theory of cosmological perturbations

So far we have studied the properties of a strictly homogeneous and isotropic cosmology. To obtain a complete description of the Universe we need to take into account also inhomogeneities. From the theory of inflation, it is believed that these primordial inhomogeneities where the cause of the formation of large scale structures, which then started growing because of gravitational instability. This means that inhomogeneities were much smaller in the past and therefore for most of their evolution they can be treated as linear perturbations. But what is their origin? It is believed that the origin of these perturbations is quantum fluctuations originated during the in inflationary period. So another point of interest for the inflationary model, is that inflation can be addressed as the source of primordial perturbations. The perturbations one must consider in the cosmological scenario are of three types (scalar, vectorial, and tensorial), and their dynamics is studied via the SVT decomposition. The vectorial perturbations usually are not important during inflation, since they rapidly decay in an expanding background, so one essentially calculates the evolution of the scalar and tensorial ones. Scalar and tensor perturbations during inflation can be described in terms of a Mukhanov-Sasaki field.

1.3.1 Scalar perturbations

Starting from the Lagrangian density of the minimally coupled (scalar) inflaton field inside the action (1.52), we can generalize it considering the presence of inhomogeneous modes of the scalar field, that will seed the scalar fluctuations. These will be described in terms of a single, Mukhanov-Sasaki (MS) field $v(q,\eta)$, and each Fourier mode of this field will be indicated by $v_k(\vec{x},\eta)$, with $k \neq 0$, as $k = 0$ can be included the homogeneous mode of the inflaton, ϕ. The total action takes the form [21]:

$$\mathcal{L} = \left[-\frac{m_p^2 q'^2}{8q^3} + \frac{\phi'^2}{2q} - \Lambda \right] + \sum_{k\neq 0} \mathcal{L}_k \tag{1.69}$$

where $\mathcal{L}_k$ is the Lagrangian density of the inhomogeneous mode k for the MS field $v(\vec{x},\eta)$, and can be written as:

$$\mathcal{L}_k = \frac{1}{2}\left(v_k'^2 + \omega_k^2 v_k^2 \right) \tag{1.70}$$

with

$$\omega_k^2 = k^2 - \frac{z''}{z}. \tag{1.71}$$

We note that the expression (1.70) describes the action of a time dependent harmonic oscillator. The time dependent frequency term for the scalar, z''/z, is defined in terms of the homogeneous (classical) degrees of freedom by [21], [7]:

$$z \equiv \sqrt{q\epsilon_1} \quad \text{with} \quad \epsilon_1 = -\frac{H'}{\sqrt{q}H^2} = -\frac{d\ln(H^2)}{d\ln(q)} \quad \text{with} \quad H = \frac{q'}{2q^{3/2}} \tag{1.72}$$

so we can estimate z''/z:

$$\frac{z''}{z} = qH^2\left[2 - \epsilon_1 + \epsilon_2\left(\frac{3}{2} + \frac{\epsilon_2}{4} - \frac{\epsilon_1}{2} + \frac{\epsilon_3}{2}\right)\right] \equiv qH^2 f_{MS}(\epsilon_i) \tag{1.73}$$

where $\epsilon_{i+1} = 2q\epsilon_i^{-1}d\epsilon_i/dq$.
The infinite set of ϵ_i form the so-called hierarchy of "Hubble flow functions" of slow-roll parameters. It is important to note that, depending on the model of inflation, other hierarchies are commonly used, and they are associated with the evolution of different (homogeneous) degrees of freedom.
From the Lagrangian (1.69), we can find the following equation of motion in momentum space:

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0 \tag{1.74}$$

known as Mukhanov-Sasaki equation.
The Mukhanov-Sasaki field can be related to the inflationary observables. The scalar perturbations of the metric are described by the "comoving curvature perturbation" $\mathcal{R}$:

$$\mathcal{R} = \frac{v}{z}. \tag{1.75}$$

In the following $\mathcal{R}$ is assumed to be a Guassian random field with Fourier-transform $\mathcal{R}_k$ [13]:

$$\mathcal{R}_k = \int d^3x\mathcal{R}(\vec{x})e^{i\vec{k}\vec{x}}. \tag{1.76}$$

We can compute the two-point correlation function for $\mathcal{R}_k$:

$$\langle\mathcal{R}_k\mathcal{R}_{k'}\rangle = (2\pi)^3\delta(k-k')|\mathcal{R}(k)|^2. \tag{1.77}$$

The power spectrum $\mathcal{P}_\mathcal{R}(k)$ encodes the full information about the curvature perturbation, if they are Gaussian distributed. This can be computed as:

$$\mathcal{P}_\mathcal{R}(k) = \frac{k^3|\mathcal{R}_k|^2}{2\pi^2} \tag{1.78}$$

which, on scales larger than the Hubble radius ($-k\eta << 1$) , takes the form:

$$\mathcal{P}_\mathcal{R}(k) = \frac{3}{4M_p^2\pi^2}\left(\frac{H^2}{\epsilon_1}\right)_{k=q^{1/2}H}. \tag{1.79}$$

Indeed, the spectral index of the power spectrum (1.78), for the perturbation modes outside the horizon (1.79), is a quantity which can be related with CMB observations and is defined as [33]:

$$n_s - 1 = \frac{d\ln(\mathcal{P}_{\mathcal{R}}(k))}{d\ln(k)} = -2\epsilon_1 - \epsilon_2 + \mathcal{O}(\epsilon_1^2) \tag{1.80}$$

Where we exploited the definitin of the slow-roll parameter ϵ_2:

$$\epsilon_2 = 2\frac{d\ln(\epsilon_1)}{d\ln(q)}. \tag{1.81}$$

The spectral index $n_s - 1$ measures the deviation from the scale invariance of the power spectrum.

1.3.2 Tensor perturbations

The case of tensorial perturbations is similar to that of scalar perturbations. We start from the perturbation of the FLRW metric [7]:

$$ds^2 = q(t)[d\eta^2 + (h_{ij} + \sigma_{ij})dx^i dx^j] \tag{1.82}$$

where the quantity $h_{ij}(\eta, \vec{x})$ describes tensorial perturbations, and represents the gravitational waves.
At quadratic order in tensor perturbations, the action takes the form [13]:

$$S^{(2)} = \int d\eta d^3x \left[(h'_{ij})^2 - (\vec{\nabla} h_{ij})^2\right]. \tag{1.83}$$

We can define the tensorial MS field as [9]:

$$v_k^\lambda = \frac{q^{1/2} m_p}{2} h_k^\lambda$$

which obeys a MS-like equation:

$$v_k''^\lambda + \left[k^2 - \left(\frac{q''}{2q} - \frac{q'^2}{4q^{3/2}}\right)\right] v_k^\lambda = 0.$$

Without going in further details, we can also report here the formulas for the power spectrum of the tensorial perturbations for single field inflaton:

$$\mathcal{P}_{\mathcal{T}}(k) = \frac{2}{3\pi^2}\left(\frac{V(\phi)}{m_p^4}\right)_{k=q^{1/2}H} \tag{1.84}$$

and of the spectral index n_t, analogous of $n_s - 1$:

$$n_t = \frac{d\ln(\mathcal{P}_{\mathcal{T}}(k))}{d\ln(k)} = -2\epsilon_1. \tag{1.85}$$

Lastly, we note that the results obtained so far for the scalar and tensor perturbations allow to calculate a consistency relation which holds for the models of inflation driven by one-single field ϕ. On using such a consistency relation one can express the so-called tensor to scalar ratio r, which is proven to be:

$$r = \frac{\mathcal{P}_T}{\mathcal{P}_r} = -8n_t = 16\epsilon_1 \tag{1.86}$$

which is proportional to the slow-roll parameter ϵ_1, and because of the relation between r and n_t, $r = -8n_t$ is also known as *consistency relation*.

Chapter 2

Wheeler-DeWitt equation

2.1 Quantization procedure and Born-Oppenheimer approximation

The constraint (1.60) sets points corresponding to physical configurations in the phase space apart from those which are not. In quantum theory, we need to distinguish between physical states on Hilbert space, for which we can assign a unique probability amplitude, to those not uniquely identifiable. Isometric spaces cannot be identified uniquely by means of their points, therefore their are not physical and is impossible to assign a unique probability amplitude for the states of the Hilbert spaces associated with them. So we assume we removed all the possible isometric configuration and proceed with the quantization of the system. We can quantize the constraint (1.60) by the Dirac prescription, i.e. substituting $\pi_q \to -i\hbar\frac{\partial}{\partial q}$ (and the same for π_ϕ) into it, and applying the *Gupta-Bleuler condition*

$$\hat{H}_s \left|\psi_{phys}\right\rangle = 0.$$

Here, the operator $\hat{H}_s$ arises when we substitute the phase space variables by the corresponding operators and choose some particular operator ordering, while $\left|\psi_{phys}\right\rangle$ is a quantum state of the Universe. We choose a particular simple ordering, such that in the end we obtain[1]:

$$\left[\frac{2q\hbar^2}{m_p^2}\frac{\partial^2}{\partial q^2} - \frac{\hbar^2}{2q}\frac{\partial^2}{\partial \phi^2} + q^2\Lambda\right]\Psi(a,\phi) \equiv \left[\frac{2q\hbar^2}{m_p^2}\frac{\partial^2}{\partial q^2} + \hat{H}_\phi\right]\Psi(a,\phi) = 0. \qquad (2.1)$$

This is the quantum equivalent of the Hamiltonian constraint, which is satisfied only by physical states in Hilbert space, and corresponds to *Wheeler-DeWitt equation* for the wave function of the Universe $\Psi(q,\phi)$. Let us note that different orderings usually affect the small q behaviour of the solutions of (2.1).

[1] we took the freedom to rescale H_s by a factor $\sqrt{q}$

We start our analysis by noting that the (2.1) is not separable because of the term $\frac{\hbar^2}{2q}\frac{\partial^2}{\partial\phi^2}$, and despite it quite-easy form it is not exactly solvable in general. Then we should apply an approximation to this equation in order to find an approximate solution. In particular, the *Born-Oppenheimer factorization* (BO) [4] seems well-suited. The BO approximation consists in separating the total wave function $\Psi(q,\phi)$ into a "slow/heavy" part and a "fast/ light" part as:

$$\Psi(q,\phi) = \psi(q)\chi(q,\phi), \tag{2.2}$$

where the light part $\chi(q,\phi)$ is not further separable. The resulting BO approximation was introduced in order to find a solution for the Schrodinger equation in a system with "heavy/slow" atoms/molecules and "fast/light" electrons. For such systems, the evolution of the fast degrees of freedom depends (almost) adiabatically on the slow ones. The BO approximation has been applied to the study of inflaton-gravity system by many years now [34] [24] [14] [22]. Indeed here the "heavy/slow" part is associated with the gravitational part, and the "light/fast" is associated with the matter part. This can be done since the Plank's mass is really much heavier than any other matter mass [27].
We can therefore rescale the wave-functions by the so-called geometric phase, namely:

$$\chi \to \tilde{\chi}(q,\phi) = \chi e^{-i\int^q \mathcal{A}dq'}; \quad \psi \to \tilde{\psi}(q) = \psi e^{i\int^q \mathcal{A}dq'} \quad with \quad i\mathcal{A} = \langle\chi|\partial_q\chi\rangle$$

such that $\tilde{\chi}\tilde{\psi} = \chi\psi$ and

$$\langle\tilde{\chi}|\frac{\partial\tilde{\chi}}{\partial q}\rangle = -i\mathcal{A}\langle\chi|\chi\rangle + i\mathcal{A} = 0. \tag{2.3}$$

Contracting (2.1) by $\langle\tilde{\chi}|$, and after some manipulations, we obtain coupled equations for the gravitational and the matter wave functions. Indeed, from:

$$\langle\tilde{\chi}|\left(\frac{2q\hbar^2}{m_p^2}\frac{\partial^2}{\partial q^2} + \hat{H}_\phi\right)\tilde{\psi}(q)|\tilde{\chi}\rangle = 0,$$

we get the *gravitational equation*

$$\left[\frac{2q\hbar^2}{m_p^2}\frac{\partial^2}{\partial q^2} + \langle\hat{H}_\phi\rangle\right]\tilde{\psi} = -\frac{2q\hbar^2}{m_p^2}\langle\partial_q^2\rangle\,\tilde{\psi} \tag{2.4}$$

while considering

$$(\mathbb{1} - |\tilde{\chi}\rangle\langle\tilde{\chi}|)\left(\frac{2q\hbar^2}{m_p^2}\frac{\partial^2}{\partial q^2} + \hat{H}_\phi\right)\tilde{\psi}(q)\,|\tilde{\chi}\rangle = 0$$

we get the *matter equation*

$$\frac{4q\hbar^2}{m_p^2}\partial_q\tilde{\psi}\partial_q\tilde{\chi} + \tilde{\psi}(\hat{H}_\phi - \langle\hat{H}_\phi\rangle)\tilde{\chi} = -\frac{2q\hbar^2}{m_p^2}\tilde{\psi}\left[\partial_q^2 - \langle\partial_q^2\rangle\right]\tilde{\chi}, \tag{2.5}$$

where $\langle\hat{O}\rangle = \langle\tilde{\chi}|\hat{O}|\tilde{\chi}\rangle$ and $\langle\tilde{\chi}|\tilde{\chi}\rangle = 1$.
The equations (2.4) and (2.5) are equivalent to the Wheeler-DeWitt equation (2.1).

Notice that on the l.h.s. we have adiabatic contributions, while in the r.h.s. we have non-adiabatic contributions which are associated with quantum gravitational effects. Furthermore, we note that both the equations (2.4) and (2.5) are still not separable, so in general are not exactly solvable, and we need further mathematical treatment for these equations. As first approach, we will ignore the non-adiabatic effects on the r.h.s. of (2.4) and (2.5), which means essentially performing the BO approximation. Since we are considering a constant potential Λ in order to reproduce the slow-roll condition during inflation, we will treat the inflaton kinetic contribution in the gravitational equation as negligible[2]. So we can rewrite (2.4) as:

$$\left[\frac{2\hbar^2}{m_p^2}\frac{\partial^2}{\partial q^2} + q\Lambda\right]\tilde{\psi} = 0 \tag{2.6}$$

and (2.5) as:

$$\frac{4q\hbar^2}{m_p^2}\partial_q\tilde{\psi}\partial_q\tilde{\chi} + \tilde{\psi}(\hat{H}_\phi - \langle\hat{H}_\phi\rangle)\tilde{\chi} = 0. \tag{2.7}$$

Let us note that the gravitational equation can be solved by the WKB approach, since gravity is generally assumed to behave nearly classically. On then substituting the gravitational wave function in the inflaton (matter) equation, one can define a time evolution, and the matter equation then becomes a time-dependent Schrodinger equation for the homogeneous inflaton.
Indeed, let's consider a solution for (2.6) in the WKB form [14] :

$$\tilde{\psi} = W_q \exp\left\{\frac{i}{\hbar}\int^q dq' \frac{m_p(q'\Lambda)^{1/2}}{\sqrt{2}}\right\} = W_q \exp\left\{\frac{i}{\hbar}\int^q dq' \pi_{q'}\right\} \tag{2.8}$$

with $W_q \sim (m_p^2 q\Lambda/2)^{-1/4} = \pi_q^{-1/2}$ and where the relation with π_q comes from (1.65). Inserting now the solution (2.8) inside the matter equation (2.7) we find:

$$i\hbar\left(\frac{8q^3\Lambda}{m_p^2}\right)^{1/2}\partial_q\tilde{\chi} + (\hat{H}_\phi - \langle\hat{H}_\phi\rangle)\tilde{\chi} \approx 0. \tag{2.9}$$

We notice that, from (1.64):

$$q' = \pm\left(\frac{8q^3\Lambda}{m_p^2}\right)^{1/2}$$

and from the fact that $q'\partial_q = \partial_\eta$, we can introduce the (conformal) time η in our theory. In particular we can choose the negative solution of (1.64). In this way, (2.9) becomes:

$$-i\hbar\partial_\eta\tilde{\chi} + (\hat{H}_\phi - \langle\hat{H}_\phi\rangle)\tilde{\chi} \approx 0. \tag{2.10}$$

[2] We will neglect the inflaton kinetic term explicitly for the gravitational equation, while we will keep it in the matter equation.

Then, rephasing the wave function $\tilde{\chi}$ by:

$$\tilde{\chi} \to \tilde{\chi} \exp \left\{ \frac{i}{\hbar} \int^{\eta} d\eta' \langle \hat{H}_\phi \rangle \right\}$$

we find that (2.10) is nothing else that the Schwinger-Tomonaga (or Schrodinger) equation:

$$-i\hbar\partial_\eta \tilde{\chi} + \hat{H}_\phi \tilde{\chi} \approx 0. \tag{2.11}$$

Let us note that a tiny $\mathcal{O}(m_p^{-2})$ quantum gravitational correction, which we shall neglect as we did for the terms on the r.h.s., still may appear. In order for "time" to be introduced, the gravitational quantum state must behave quasi-classically, since "time" is a classical parameter. If the gravitational wave function describes an highly non-classical state, time cannot be properly defined.

Henceforth, we shall consider a different approach, which consists of introducing the Wigner function for the gravitational sector in order to study how time then emerges. Using the Wigner function instead of the gravitational wave function is relevant since by the Wigner function one may describe a quantum system which is not in a pure state, and thus this approach is more general.

A gravitational state being not pure (or mixed) may be the consequence of an interaction of gravity with some hidden sector (before or during inflation) which is not observable and can be traced over. In general, however, we expect that the two approaches (the one with the wave function and with the Wigner function) must lead to the same result for pure states.

Moreover, the classical limit for gravity clearly emerges withing the Wigner approach.

2.2 The Wigner function

In quantum mechanics the measurement of some quantity brings out a probabilistic value, so classical measures are different to quantum ones.

Quantum mechanics describes a microscopic system in terms of a state vector $|\psi\rangle$ or a (probability) density operator $\hat{\rho}$. For a given pure state $|\psi\rangle$, indeed, one can construct a density matrix operator $\hat{\rho} = |\psi\rangle\langle\psi|$, and express it in the position representation $\langle x|\hat{\rho}|x'\rangle = \langle x|\psi\rangle\langle\psi|x'\rangle$, or in the momentum representation $\langle p|\hat{\rho}|p'\rangle = \langle p|\psi\rangle\langle\psi|p'\rangle$. One could also define the density matrix for a quantum system in mixed state, as $\hat{\rho} = \sum_j c_j |\psi_j\rangle\langle\psi_j|$, where c_j specifies the fraction of the ensemble in the pure state $|\psi_j\rangle$, or in other words the probability $p = |c_j^2|$ to find the ensemble in the state $|\psi_j\rangle$ after a measurement.

However, the density matrix operator must satisfy three conditions to give a good description of the quantum system to which it refers [2]:

$$Tr(\hat{\rho}) = 1, \tag{2.12}$$

$$\hat{\rho} = \hat{\rho}^\dagger, \tag{2.13}$$

$$\langle u|\hat{\rho}|u\rangle \geq 0 \quad \forall u. \tag{2.14}$$

These properties of $\hat{\rho}$ reflect the properties of a probability distribution, like the addition to 1 (from (2.12)), the real nature of probability (from (2.13)) and the existence only of probabilities ≥ 0 (from (2.14)). Other descriptions of a quantum system should in the same way take into account the properties (2.12), (2.13) and (2.14).
Anyway, $|\psi\rangle$ and $\hat{\rho}$ remain abstract objects and it is difficult to read off their properties. However, there exists a representation of quantum mechanics which expresses the properties of a (quantum) system, in a way more similar to the classical description. This representation lives in the phase space and is based on the definition of the Wigner function. As we will see, the Wigner function is a (quasi-)probability distribution, an object that could be used to describe a quantum system instead of using the density matrix $\hat{\rho}$, and which lives "between" the momenta and the position representation.
Suppose you want to describe the "motion" of a particle from $x' = x - \frac{s}{2}$ to $x'' = x + \frac{s}{2}$, where $s = x'' - x'$ is the "quantum jump". The quantum jump can be described by the matrix element $\langle x + \frac{s}{2}|\hat{\rho}|x - \frac{s}{2}\rangle$ in position representation, which represents the correlation between the points $x - \frac{s}{2}$ and $x + \frac{s}{2}$ given a certain state for the system, described by the density matrix $\hat{\rho}$. Operating a Fourier transform on "s", brings us from a distribution in configuration space to a distribution in the phase space. So we get [32]:

$$W(x,p) = \int_{-\infty}^{+\infty} ds e^{-i\frac{ps}{\hbar}} \langle x + \frac{s}{2}|\hat{\rho}|x - \frac{s}{2}\rangle \tag{2.15}$$

that is our *Wigner function*[3]. Indeed, we went from a matrix element $\langle x''|\hat{\rho}|x'\rangle$ which depends on two positions, to a function which depends on the Fourier variable associated with the "jump", p, and the center of the jump x. So we are now in the phase space. It is also possible to go from a matrix element in the position representation to a Wigner function in the following way [11]:

$$W(x,p) = \int_{-\infty}^{+\infty} dk e^{i\frac{qk}{\hbar}} \langle p + \frac{k}{2}|\hat{\rho}|p - \frac{k}{2}\rangle \,. \tag{2.16}$$

Both x and p are c-numbers, and not operators, so our Wigner function depends on two classical variables, and has properties analogous to any classical probability distribution on the phase space. Indeed, it can be shown that Wigner function (as other phase space distributions) allows to compute quantum mechanical expectation integrating over the classical variables, similarly to classical statistical mechanics. Integrating the Wigner

[3]in other notations, the Wigner function has an additional factor of $(2\pi\hbar)^{-1}$ in front of it, that here is ignored

function over the position variable, one gets the probability distribution for momentum, and vice versa:

$$\begin{aligned}\int_{-\infty}^{+\infty} W(x,p)dp &= \langle x|\hat{\rho}|x\rangle \equiv W(x);\\ \int_{-\infty}^{+\infty} W(x,p)dx &= \langle p|\hat{\rho}|p\rangle \equiv W(p).\end{aligned} \tag{2.17}$$

Lastly, if we have pure states, such that $\hat{\rho} = |\psi\rangle\langle\psi|$, we can write our Wigner function as:

$$W(x,p) = \int_{-\infty}^{+\infty} e^{-i\frac{ps}{\hbar}}\psi^*\left(x - \frac{s}{2}\right)\psi\left(x + \frac{s}{2}\right)ds \tag{2.18}$$

where $\psi(x) = \langle x|\psi\rangle$ is the position representation of $|\psi\rangle$. So we can see that the generalization of Wigner function for pure and mixed states is really straightforward.
Similarly, one can define the Wigner representation of any operator $\hat{R}$, other than $\hat{\rho}$, which is given by:

$$\hat{R}_W(q,p) = \int_{-\infty}^{+\infty} dse^{-i\frac{ps}{\hbar}}\langle x + \frac{s}{2}|\hat{R}|x - \frac{s}{2}\rangle\,. \tag{2.19}$$

The average of a dynamical variable R in the state ρ is given by $\langle R\rangle = Tr(\rho R)$. If we want to express the trace of a product of two operators in terms of the Wigner function we can start from the position representation:

$$Tr(\rho R) = \int\int \langle q|\rho|q'\rangle\langle q'|R|q\rangle\, dqdq'$$

and then we can express the matrix element in the Wigner representation, finding:

$$\langle R\rangle = Tr(\rho R) = \int\int W(q,p)R_W(q,p)dqdp. \tag{2.20}$$

One can see now how the properties (2.12), (2.13) and (2.14) for the density matrix $\hat{\rho}$ are reflected as properties of the Wigner function.
The property (2.12) becomes the following property for the Wigner function:

$$\int\int W(q,p)dqdp = \int \langle q|\hat{\rho}|q\rangle\, dq = 1 \tag{2.21}$$

so it becomes a condition of normalization in the phase space for the Wigner distribution. The second property (2.13), corresponds to the fact that $W(q,p)$ is a real function. The third property (2.14), however, does not imply the non-negativity of the Wigner function. Indeed, as we said before, the Wigner function is a quasi-probability distribution, and it can take also negative values. It is convenient to replace the property (2.14) with a generalization, which states that for any pair of state operators ρ, ρ' that satisfy the properties (2.12), (2.13) and (2.14), the trace of their product obeys the inequality

$$0 \leq Tr(\rho\rho') \leq 1.$$

So, in the case of the Wigner function, we can generalise the result (2.20) where, instead of a generic operator R, we can consider a second Wigner distribution $W'(q,p)$ associated to the density matrix ρ':

$$0 \leq Tr(\rho\rho') = \int\int W(q,p)W'(q,p)dqdp \leq 1 \tag{2.22}$$

and in particular, for $\rho' = \rho$, we have

$$\int\int |W(q,p)|^2 dqdp \leq 1$$

that can be interpreted as the requirement for the Wigner function to be not too sharply peaked[4].

We have shown that it is possible to describe a quantum system via the Wigner probability distribution $W(q,p)$, and that this new description is completely equivalent to that based on the density matrix ρ, but with the advantage that the Wigner function is defined on the phase space, and we can potentially see how the quantum correlation between the variables (p,q) can affect the behaviour of our physical system.

2.3 Wigner function for gravitational equation

Starting from the gravitational wave equation (2.6), and limiting ourself to the homogeneous part of the gravitational wave function, we can write the Wigner function for $\tilde{\psi}$ [24] :

$$W(q,p_q) \equiv \int_{-\infty}^{+\infty} ds e^{\frac{i}{\hbar}ps} \tilde{\psi}^*_- \tilde{\psi}_+ \tag{2.23}$$

where $\tilde{\psi}_\pm = \tilde{\psi}(q_\pm)$, $q_\pm = q \pm s/2$ and p is related to the (classical) conjugate momenta of the spatial parameter q.

Let us note that the Wigner function allows to calculate quantum expectation values using a formalism analogous to that of (classical) statistical mechanics, and it is very useful in search for the classical limit.

The gravitational wave function must have positive argument and therefore for a fixed s only varies in the interval $[-2q; 2q]$. Since $q = a^2 > 0$, our integration range gets

[4]This can be better visualized if we used the definition with $(2\pi\hbar)^{-1}$ in front of the Wigner function. In this case, indeed, the last condition becomes $\int\int |W(q,p)|^2 dqdp \leq (2\pi\hbar)^{-1}$

restricted to $[0;2q]$. However, for the case of interest, we will assume $q >> 1$, and the wave function $\tilde{\psi}$ has negligible support for $q < 0$, and thus extending the interval $[0;2q]$ to $]\infty, +\infty[$ is a reasonable approximation.
A straight method of calculating (2.23) consists in solving the equation for the gravitational wave function and then evaluating the integral (2.23) to obtain the corresponding Wigner function. This is what is performed in [18], with the gravitational wave function obtained with the WKB method. What we will do instead is to obtain the exact equations satisfied by (2.23), then use some approximation scheme to obtain the Wigner function of the system directly from their resolution, and lastly compare the result with the one from [18].
By substituting $q \to q_+$ into (2.6), multiplying by $\exp(\frac{i}{\hbar}ps)\tilde{\psi}^*_-$ and then integrating over s one finds:

$$\int_{-\infty}^{+\infty} ds e^{\frac{i}{\hbar}ps}\tilde{\psi}^*_- \left[\frac{2\hbar^2}{m_p^2}(\partial_+)^2 + \langle\frac{\hat{H}_\phi}{q}\rangle_+\right]\tilde{\psi}_+ = 0 \tag{2.24}$$

where $\langle\hat{O}\rangle_\pm \equiv \langle\tilde{\chi}(q_\pm,\phi)|\hat{O}|\tilde{\chi}(q_\pm,\phi)\rangle$ and $\partial_\pm \equiv \partial_{q_\pm}$.
We can do the same for $\tilde{\psi}^*$, substituting $q \to q_-$ and multiplying by $\exp(\frac{i}{\hbar}ps)\tilde{\psi}_+$ to obtain:

$$\int_{-\infty}^{+\infty} ds e^{\frac{i}{\hbar}ps}\tilde{\psi}_+ \left[\frac{2\hbar^2}{m_p^2}(\partial_-)^2 + \langle\frac{\hat{H}_\phi}{q}\rangle_-\right]\tilde{\psi}^*_- = 0. \tag{2.25}$$

Then, on summing and subtracting (2.24) with (2.25), we obtain:

$$\int_{-\infty}^{+\infty} ds e^{\frac{i}{\hbar}ps}\left\{\frac{2\hbar^2}{m_p^2}(\partial_+^2 \pm \partial_-^2) + \left(\langle\frac{\hat{H}_\phi}{q}\rangle_+ \pm \langle\frac{\hat{H}_\phi}{q}\rangle_-\right)\right\}\tilde{\psi}^*_-\tilde{\psi}_+ = 0 \tag{2.26}$$

where we used the fact that q_+ and q_- are independent variables.
Let us note that, using the chain rule, we can write the previous integrand as a function of a and s, and:

$$\partial_+^2 + \partial_-^2 = \frac{\partial_q^2 + 4\partial_s^2}{2} \quad and \quad \partial_+^2 - \partial_-^2 = 2\partial_q\partial_s.$$

Integrating by parts and remembering that the gravitational wave functions are zero at the boundaries, we can find the following relations:

$$\int_{-\infty}^{+\infty} ds e^{\frac{i}{\hbar}ps}\partial_s^{(n)}\tilde{\psi}^*_-\tilde{\psi}_+ = \left(-\frac{i}{\hbar}p\right)^n W(a,p) \tag{2.27}$$

and

$$\int_{-\infty}^{+\infty} ds e^{\frac{i}{\hbar}ps}s^n\tilde{\psi}^*_-\tilde{\psi}_+ = (-i\hbar)^n\partial_p^{(n)}W(a,p). \tag{2.28}$$

Therefor, on now Taylor expanding the expectation values of the inflaton Hamiltonian in (2.26) for $s \to 0$ (see [32] and Appendix A in the present work) and using (2.27), we obtain the following two equations (for + and − respectively):

$$\frac{\hbar^2}{m_p^2}\partial_q^2 W - \frac{4p^2}{m_p^2}W + \sum_{n=0}^{\infty}\left[((-1)^n+1)\frac{(i\hbar)^n}{2^n n!}\frac{d^n\left\langle\frac{H_\phi}{q}\right\rangle}{dq^n}\partial_p^{(n)}W\right] = 0 \tag{2.29}$$

and

$$-\frac{4i\hbar p}{m_p^2}\partial_q W + \sum_{n=0}^{\infty}\left[((-1)^n-1)\frac{(i\hbar)^n}{2^n n!}\frac{d^n\left\langle\frac{H_\phi}{q}\right\rangle}{dq^n}\partial_p^{(n)}W\right] = 0. \tag{2.30}$$

Notice that the last equation corresponds to a *quantum Liouville equation*. Its classical counterpart asserts that the phase-space distribution function is constant along the trajectories of the system. Its quantum formulation states that the expectation values of a quantum quantity evolve following:

$$\frac{d\hat{\rho}}{dt} = 0 = \frac{\partial\hat{\rho}}{\partial t} + \frac{i}{\hbar}[\hat{H},\hat{\rho}] \rightarrow \frac{d\langle\hat{A}\rangle}{dt} = \langle\frac{\partial\hat{A}}{\partial t}\rangle + \frac{i}{\hbar}\langle[\hat{H},\hat{A}]\rangle \tag{2.31}$$

for a generic operator $\hat{A}$, where $\langle\hat{A}\rangle = Tr(\hat{A}\hat{\rho})$.
If now we substitute

$$\langle\frac{H_\phi}{q}\rangle = q\Lambda$$

we obtain specific forms for (2.29) and (2.30), respectively:

$$\frac{\hbar^2}{m_p^2}\partial_q^2 W - \frac{4p^2}{m_p^2}W + 2q\Lambda W = 0 \tag{2.32}$$

and

$$-\frac{4i\hbar p}{m_p^2}\partial_q W - i\hbar\Lambda(\partial_p W) = 0. \tag{2.33}$$

The Wigner function must satisfy the above equations simultaneously. From (2.33) we can find the relation

$$\partial_q W = -\frac{m_p^2}{2}\Lambda(\partial_{p^2}W) \tag{2.34}$$

which has the simple solution

$$W = W(x) \equiv W\left(q - \frac{2p^2}{m_p^2\Lambda}\right). \tag{2.35}$$

Eq. (2.32) then simply becomes

$$\hbar^2\partial_q^2 W - 4p^2 W + 2m_p^2\Lambda q W = \hbar^2\partial_x^2 W - 4p^2 W - \lambda q W = 0 \tag{2.36}$$

where we have defined $\lambda \equiv 2m_p^2\Lambda$ an x becomes $x = q - 4p^2/\lambda$.
This last equation can be exactly solved by Fourier transforming:

$$W(q,p) = \int_{-\infty}^{+\infty} \widetilde{W}(y,p)e^{iyq}dy \tag{2.37}$$

and one can then find the following first order differential equation for $\widetilde{W}$:

$$-(\hbar^2y^2 + 4p^2)\widetilde{W} + i\lambda\frac{d\widetilde{W}}{dy} = 0 \tag{2.38}$$

which can be solved to obtain:

$$\widetilde{W} = \tilde{W}_0 \exp\left\{-i\left[\frac{\hbar^2}{3\lambda}y^3 + \frac{4p^2}{\lambda}y\right]\right\}. \tag{2.39}$$

We can now transform back this last expression (2.39), by using the relation[5]:

$$\int_{-\infty}^{+\infty} e^{i(\frac{b}{3}t^3+ct^2+dt)} = \frac{2\pi}{b^{1/3}} \exp\left\{i\frac{c}{b}\left(\frac{2c^2}{3b} - d\right)\right\} Ai\left(\frac{d-\frac{c^2}{b}}{b^{1/3}}\right) \tag{2.40}$$

to obtain the Wigner function. By comparing (2.40), (2.39) and (2.37), and simply replacing:

$$b = -\frac{\hbar^2}{\lambda}; \tag{2.41}$$

$$c = 0; \tag{2.42}$$

$$d = (q - \frac{4p^2}{\lambda}) = x \tag{2.43}$$

inside (2.40) we obtain the expression for the Wigner function:

$$W(q,p) = -\frac{2\pi\lambda^{1/3}}{\hbar^{2/3}}Ai\left[-\frac{x\lambda^{1/3}}{\hbar^{2/3}}\right] = -\frac{2\pi\lambda^{1/3}}{\hbar^{2/3}}Ai\left[\frac{4p^2-\lambda q}{(\lambda\hbar)^{2/3}}\right]. \tag{2.44}$$

Let us note that the argument of the Airy is in agreement with (2.35). One finally needs to normalize the Wigner function.
Since the Airy function is normalized as [35]:

$$\int_{-\infty}^{+\infty} Ai(x)dx = 1,$$

[5]For the sake of simplicity, we fixed the initial condition $\tilde{W}_0 = 1$

we obtain the following correct normalization of our solution (2.44)

$$W(q,p) = \frac{1}{(\lambda\hbar)^{2/3}} Ai\left[\frac{4p^2 - \lambda q}{(\lambda\hbar)^{2/3}}\right] \tag{2.45}$$

where we observe that the argument of the Airy function is equal to zero for:

$$p = \pm\left(\frac{q\lambda}{4}\right)^{1/2} = \pm\left(\frac{q m_p^2 \Lambda}{2}\right)^{1/2} \tag{2.46}$$

that is when p is equal to its classical value π_ϕ (found in (1.65) from the Hamiltonian analysis and the Friedman equation).
As we can see comparing the Wigner function (2.45) with the one reported in [18], the same result (2.45) can be obtained by solving the Wheeler-DeWitt equation for the gravitational wave function $\psi(q)$, and choosing the Hartle-Hawking (HH) initial condition, corresponding to a Universe in a state which is a superposition of an expanding and a contracting phase.
Let us note that the gravitational equation for a constant inflaton potential takes the form of an Airy equation and its solutions can he written exactly as superposition of Ai and Bi Airy's functions.
On imposing initial condition of Vilenkin type [18] one obtains a Universe in an expanding state if the wave function $\tilde{\psi}(q)$ contains Bi. However, the Wigner integral cannot be performed exactly, unless, as we shall see in what follows, some approximation is considered.

2.4 The quantum theory of measurement

From observations and different experimental test of General Relativity, it has been probed that our Universe behaves classically. In the context of classical mechanics, a system may be described by a definite state and its evolution is described in a deterministic manner. However, in the last century we have discovered that the macroscopic world we experience as "classic" has a quantum mechanical origin in its microscopic components. In principle, every phenomenon we observe at all scales, including the entire Universe, should be described by quantum mechanics.
In quantum mechanics, states can be defined, but due to the Heisenberg uncertainty principle, it is generally not possible to measure simultaneously the position and momentum of a system. Moreover, quantum mechanics is deterministic, but a probabilistic interpretation is intrinsic in a quantum model. Given the state of some system at a particular time, such a state describes the outcomes of its measurement in a probabilistic way.
It is obviously fundamental to reconcile, in a certain limit, the predictions of quantum

mechanics with our classical experience. This issue is studied in the context of *quantum theory of measurement*. The so-called quantum to classical transition, and how this could occur, has been discussed by numerous authors [19] [15] [37] [5].
In the context of quantum cosmology, one attempts to apply quantum mechanics to the early Universe. This involves the quantization of the gravitational field, and one should predict the conditions under which the gravitational field behaves classically.
There are at least two requirements that must be satisfied for a system to be regarded as classical. The first requirement is that interference effects are negligible. This involves the notion of *decoherence*, that formally consist of cancelling the non-diagonal terms of the density matrix, which describe interference. Note that we can still think of a system where the knowledge of its observable features can not be determinate with absolute precision, as it happens in classical statistical mechanics.
The second requirement is that evolution should, to a very good approximation, be described by classical laws. This means that observables probability distributions should be strongly peaked about their classical configuration. We shall refer to this second requirement as *correlation*.
So we can see that the recovery of some "classical result", or of correlation between position and momenta, is not sufficient to consider our system as classical, since also decoherence has a fundamental role to this scope.
In the next section, we will discuss the notion of decoherence following primarily the approach of Zurek [37]. The fundamental concepts underlying his study are summarized here. When a system interacts with a measuring apparatus, the states of both entities become correlated (entangled). However, the density matrix of the pure state for the apparatus-system ensemble contains non-zero off-diagonal elements, representing interference among various potential measurement outcomes. Only when these off-diagonal elements become negligible, we can conclude that the measuring apparatus has definitively recorded a measurement.
However, this means that the pure-state density matrix must evolve into a mixed-state diagonal density matrix, which cannot be achieved by unitary evolution. The resolution of this difficulty comes from the realization that the apparatus and system must necessarily be in interaction with the rest of the Universe, commonly referred to as "the environment". If one includes the state of the environment in the initial pure-state density matrix, then one finds that the reduced density matrix, obtained by tracing out the environment states, can evolve non-unitarily, taking an initial pure state to a final mixed state.

2.4.1 Decoherence with the environment-induced superselction by Zurek

Consider a physical system $\mathcal{S}$ described by a set of state vectors $|S_n\rangle$ that we are interested in measuring and put it in interaction with a measuring apparatus $\mathcal{A}$, whose state are described by the state vectors $|A_n\rangle$. Let the initial state of the system be a superposition of states with coefficients c_n, and the initial state of the apparatus be $|A_0\rangle$. Then the initial state of the interacting system $\mathcal{SA}$ is:

$$|\Psi_i\rangle = \sum_n c_n |S_n\rangle |A_0\rangle . \tag{2.47}$$

To operate the measurement, the apparatus and the physical system must interact, and this leads to an evolution of the interacting system to a final state:

$$|\Psi_f\rangle = \sum_n c_n |S_n\rangle |A_n\rangle . \tag{2.48}$$

One can interpret this last result as the fact that the apparatus state $|A_n\rangle$ has become correlated with the system state $|S_n\rangle$, that the apparatus has "measured" the system and finds it to be in state $|S_n\rangle$ with probability $|c_n^2|$.
However, the general formalism of quantum mechanics allows for an arbitrary change of basis. In particular, one may introduce a new orthonormal basis for the apparatus, defined by:

$$|A_n\rangle = \sum_m |\tilde{A}_m\rangle \langle \tilde{A}_m|A_n\rangle \tag{2.49}$$

such that the result (2.48) could be written also as:

$$|\Psi_f\rangle = \sum_n c_n |S_n\rangle \sum_m |\tilde{A}_m\rangle \langle \tilde{A}_m|A_n\rangle \equiv \sum_m \tilde{c}_m |\tilde{S}_m\rangle |\tilde{A}_m\rangle \tag{2.50}$$

where $|\tilde{S}_m\rangle$ are the relative states, and are defined by the upper relation.
In this new basis, it appears that the apparatus states $|\tilde{A}_m\rangle$ have become correlated with the system states $|\tilde{S}_m\rangle$, and where the measuring apparatus finds itself in one of the states $|\tilde{A}_m\rangle$ it has measured the system to be in the state $|\tilde{S}_m\rangle$. So what has actually been measured?
Measuring apparatuses are macroscopic objects which are not observed in superpositions. But what determines the choice of a particular basis?
The situation becomes even more problematic if one considers the pure-state density matrix corresponding to the final state (2.48):

$$\rho_{pure} = |\Psi_f\rangle \langle \Psi_f| = \sum_{n,m} c_n c_m^* |S_n\rangle |A_n\rangle \langle S_m| \langle A_m| \tag{2.51}$$

that involves nonzero off-diagonal terms.
We are searching for a solution that will set a "biunivocal correspondence" between the system and the apparatus states, where the combined system's final state is a definite state in which each system state $|S_n\rangle$ is correlated with the apparatus state $|A_n\rangle$, with probability $|c_n^2|$ of finding the system in state $|S_n\rangle$. Such a situation can only be described by a diagonal mixed-state density matrix of the form:

$$\rho_{mixed} = \sum_n |c_n^2| \, |S_n\rangle \, |A_n\rangle \, \langle S_n| \, \langle A_n| \, . \qquad (2.52)$$

Let's notice that (2.52) and (2.51) differ by the presence of off-diagonal terms in (2.51), which represent interference between the different outcomes of the measurement. It is only when these interference terms can be neglected that the combined system may be said to be decohered.
There is no way that under unitary Schrodinger evolution the pure-state density matrix (2.51) will evolve into the mixed-state density matrix (2.52). It is for this reason that we need an "interpretation" for quantum mechanics measurement, and one is forced to introduce the "second stage" of the measurement process: namely, the "collapse" of the wave function. This is the process thanks which one is able to projects the state vector (2.48), a superposition of states, down onto just one of the states of the superposition.
We are capable to (almost) diagonalize our density matrix. To resolve this issue, we can apply the so-called *"environment-induced superselection"* approach pioneered by Zurek [37] [19].
The key point introduced by Zurek is that no macroscopic system can realistically be considered as closed and isolated from the rest of the Universe (with the possible exception of the entire Universe itself, and we will discuss this later), that we can consider as our "environment".
Then it can be argued that it is the inevitable interaction with the external environment which leads to a continuous "measuring" or "monitoring" of a macroscopic system and it is this that causes the wave function to "collapse". More precisely, the environment causes the off-diagonal terms in ρ_{pure} to become negligible respect the diagonal element, and that it is well-approximated by ρ_{mixed}. This is what we mean for decoherence.
So let's consider again the system which results from the interaction of the measuring appartus $\mathcal{A}$ and the physical system $\mathcal{S}$, but now we consider also the external environment $\mathcal{E}$ described by the state-vectors $|E_n\rangle$. Let the initial state of this total system $\mathcal{SAE}$ be the state $|\Phi_i\rangle$:

$$|\Phi_i\rangle = \sum_n c_n \, |S_n\rangle \, |A_0\rangle \, |E_0\rangle \qquad (2.53)$$

and similarly to what we have previously discussed, let the total system evolve into the final state:

$$|\Phi_f\rangle = \sum_n c_n \, |S_n\rangle \, |A_n\rangle \, |E_n\rangle \, . \qquad (2.54)$$

In this model, not only have the system and apparatus become correlated with each other, but they have also become correlated with the environment.
One is not interested, however, in the state of the environment, and it should be traced out in the calculation of any quantities of interest. The object of relevance, therefore, is the *reduced density matrix* $\tilde{\rho}$, obtained by tracing over the environment states:

$$\tilde{\rho} = Tr_{\mathcal{E}} \left|\Phi_f\right\rangle \left\langle\Phi_f\right| = \sum_{n,m} c_n c_m^* \left\langle E_n | E_m \right\rangle \left|S_n\right\rangle \left|A_n\right\rangle \left\langle S_m\right| \left\langle A_m\right| . \tag{2.55}$$

Let us note that the density matrix $|\Phi_f\rangle \langle\Phi_f|$ still evolves unitarily, but the reduced density matrix $\tilde{\rho}$ does not, so now it is possible to make it evolve from an initial pure state into a final mixed one. However, the environment induced decoherence does not explain why the density matrix of the system becomes diagonal, since the density matrix is always diagonal in some basis, but rather points out some preferred basis, and shows that the density matrix inexorably becomes diagonal in that basis. So what is such a preferred-basis?
To eliminate the non-diagonal matrix element in $\tilde{\rho}$, the product $\langle E_n|E_m\rangle$ must be the smallest possible for $n \neq m$. For Zurek, this depends significantly on the specific nature of the interaction between the system and its surrounding environment. Essentially, one can define a "pointer observable" as any observable that commutes with the Hamiltonian describing the system-environment interaction. A preferred set of states, known as "pointer basis," comprises the eigenstates of this pointer observable. If the system is in one of these eigenstates, its state remains undisturbed by interactions with the environment. Conversely, if the system is not in an eigenstate of the pointer observable, it will evolve in some new state due to interactions with the environment. Let us note that superpositions of states within the pointer basis are not observable because, as previously discussed, interference between such states is eliminated by the environment. This mechanism is called "environment-induced superselection", coined by Zurek. Consequently, we identify these as the "definite states" introduced earlier, representing relatively stable states free from interference.
It is worth noting that the environment plays a dual role in the measurement process: it induces decoherence, and determines the preferred basis. In a typical model of the measurement process, the environment interacts with the system through a specific quantity, such as position (that for us is represented by q, or a). As a result, the system-environment interaction Hamiltonian commutes with position, making the pointer basis equivalent to the position basis. Consequently, the density matrix diagonalizes in the position basis. Indeed, this will generally be true in many situations of interest, not just measurement situations. Fields generally couple to each other through their configuration-space coordinates, and not their momenta.

2.4.2 Decoherence in quantum cosmology

Now we return to the topic in which we are more interested, trying to apply a mechanism of decoherence in the context of quantum cosmology. We will focus on the previously introduced Zurek's method, which anyway requires an external environment to decohere the quantum states, and since in quantum cosmology we are interested in the study the Universe as a unique quantum object, it is problematic to identify some external-environment that could help us in the decoherence. What we can do, is to regard some of the variables describing the Universe as the system, and the rest as environment. The environment should be some kind of large reservoir into which information about correlations can be dissipated. It should, therefore, have a large number of modes. Since minisuperspace usually involves the homogeneous modes of the fields, the inhomogeneous modes, which have so far been ignored, are a natural candidate for the environment. One can use the inhomogeneous modes of either gravitational or matter fields. So we can trace over this unobserved modes to get a decoherence term to our density matrix, and recover a classical limit of our model. In the following, we will apply an effective procedure to our Wigner function to display how this could work.

A side look on the coarse-grained Wigner function

What we will do in the next chapter is to try to find the form of the matter equation coming from the Born-Oppenheimer decomposition of the WdW equation, and using the Wigner function (2.45) associated with the gravitational equation (2.6) inside (2.5), (some approximations are used).
It is important to remind what is the physical meaning of the Wigner function and how we should interpret its form (2.45). The Wigner function is a quasi-probability distribution [20] and we should interpret its peaks as the possible outcomes of measurement of the physical variables, beside the difficulties in the interpretation of negative-valued peaks. So, roughly speaking, the position of the peaks in the (p, q) phase-space are associated with the correlation between the variables, and so we need them to be close to the classical trajectory to recover the classical limit for quantum scale factor. In this context, the dependence of the Wigner function on the Airy function displayed in (2.45), which is a highly-oscillating function, is relevant. We can plot a 2-D graph of it in Figure 2.1. The highly-oscillating behaviour is also originated by the fact that we are considering a solution associated with the Hartle-Hawking (HH) initial condition in (2.45), so a Universe wavefunction which is a superposition of an expanding Universe and a contracting Universe. In this scenario the two solutions produce an "interference" that results in the highly-oscillating behaviour . The fact that the Wigner solution (2.45) is symmetric in p is a consequence of the superposition.

The highly oscillating behaviour of the Wigner function can be alleviated by the effect of the previously cited quantum decoherence, produced by the interaction with the

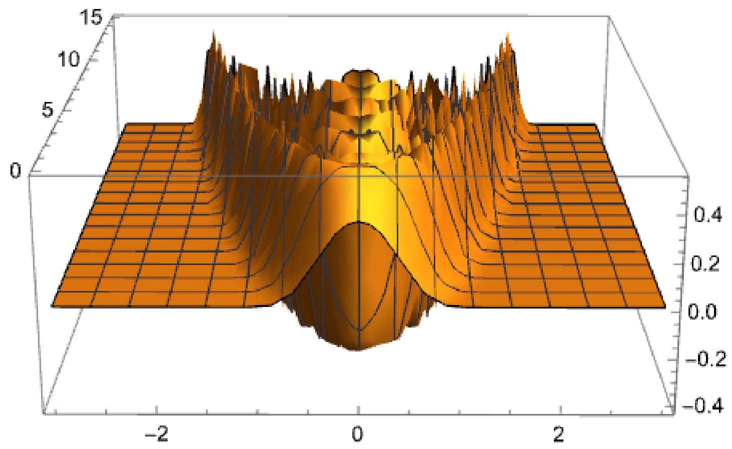

Figure 2.1: 2D graph of the Wigner function (2.45) with $\lambda, \hbar = 1$.

environment. A way to apply this procedure is to "trace over unobservables" (TOU) [17] degrees of freedom, that in our context may be, for example, the inhomogeneous degrees of freedom of the matter fields present in the early Universe. A decoherence factor that multiplies the density matrix ρ, suppressing the off-diagonal terms, may be used to mimic the effect of tracing over unobservable degrees of freedom.
As we already mentioned, an additional effect of this is to effectively operate a coarse-graining over the oscillations of the Wigner function. The average over some "grains" of the domain with another distribution, eventually letting opposite sign oscillations delete each other, pave the way to the classical limit. In this way indeed, we could automatically delete the peaks displaced from the classical trajectory, remaining with just one peak on the classical solution. In principle, we should look only at the coarse-grained Wigner function to describe the quantum behaviour of the gravitational sector (scale factor) in the presence of unobservables degrees of freedom.
Decoherence is necessary to reproduce the classical limit, but at the same time we require our Wigner function to have sharp peaks. The presence of sharp peaks around the classical trajectory indicates a strong correlation between the phase space variables. The amount of coarse-graining will determine different amounts of decoherence and correlation. In general a "gentle" coarse-graining is a good compromise between the two condition we just exposed. This is possible by a wise choice in the values of the parameters that will define our coarse-graining procedure.
As shown in [18], the process of TOU leads to the multiplication of the density matrix $\rho_0(p,q)$ by a decoherence factor $e^{-\alpha(\bar{q})(q-q')^2}$,where $\bar{q} = q + q'$, which leads to the new density matrix $\rho'(p,q)$:

$$\rho'(p,q) = \rho_0(p,q)e^{-\alpha(q)(q-q')^2}. \tag{2.56}$$

The decohered density matrix $\rho'(p,q)$ can be Wigner-transformed to give a new distribution $W'(q,p)$ which replaces the original Wigner $W_0(q,p)$. The effect of the decoherence factor is given by:

$$W'(q,p) = \frac{1}{\sqrt{\alpha(q)\pi}} \int dk W_0(q,k) e^{-\frac{(p-k)^2}{\hbar^2 \alpha(q)}}. \tag{2.57}$$

This shows that the effect of the TOU on the Wigner function is equivalent to averaging the momenta over a scale $\sqrt{\alpha}$. The distribution in (2.57) can be identified with a Husimi distribution[6] on the variable p . The Husimi distribution is indeed equal to a Gaussian smoothing of the Wigner function [2] [10] which, differently from the Wigner distribution itself, is positive definite by construction. Let us note that any strongly pronounced feature of the Husimi distribution will also show up in the Wigner function, although the latter may also contain unphysical structures (peaks).
The necessity to have sharp peaks and at the same time decoherence will constraint the choice of the value for α. Classical correlation require to satisfy the relation (2.46):

$$|p| \sim \left(\frac{q\lambda}{4}\right)^{1/2} \tag{2.58}$$

and, in order to the new distribution $W'(q,p)$ to have sharp peaks, we will require that $p > \sqrt{\alpha}$, so:

$$\sqrt{\alpha} < |p| \sim \left(\frac{q\lambda}{4}\right)^{1/2}. \tag{2.59}$$

On the other hand, the coarse-graining integral requires at least an averaging over few oscillations of the $W_0(q,p)$ distribution, and this implies that:

$$\sqrt{\alpha} > \frac{1}{|p|} \sim \left(\frac{q\lambda}{4}\right)^{-1/2}. \tag{2.60}$$

In Figure 2.2 we plotted the distribution $W'(p,q)$ for different values of α. We can notice the presence of two peaks in correspondence of the classical trajectory $p = \pm p_{cl}$, and the presence of a large negative peak at $p = 0$. Increasing the value of α, the peaks become broader and lower. We can observe that the coarse-graining procedure gives peaks on the classical value of p, reproducing (in a certain limit) the classical behaviour of the system.

The form of α is the consequence of the choice of some form of "cutoff", i.e. how the environment to trace over is defined. As an example, Halliwell [19] introduced a cutoff which depends on the value of the scalar factor, taking in consideration that modes

[6]The Husimi distribution must be on both the variables (q,p), so this is not a proper form of the Husimi distribution, but it is a distribution with the form of the Husimi in the variable p

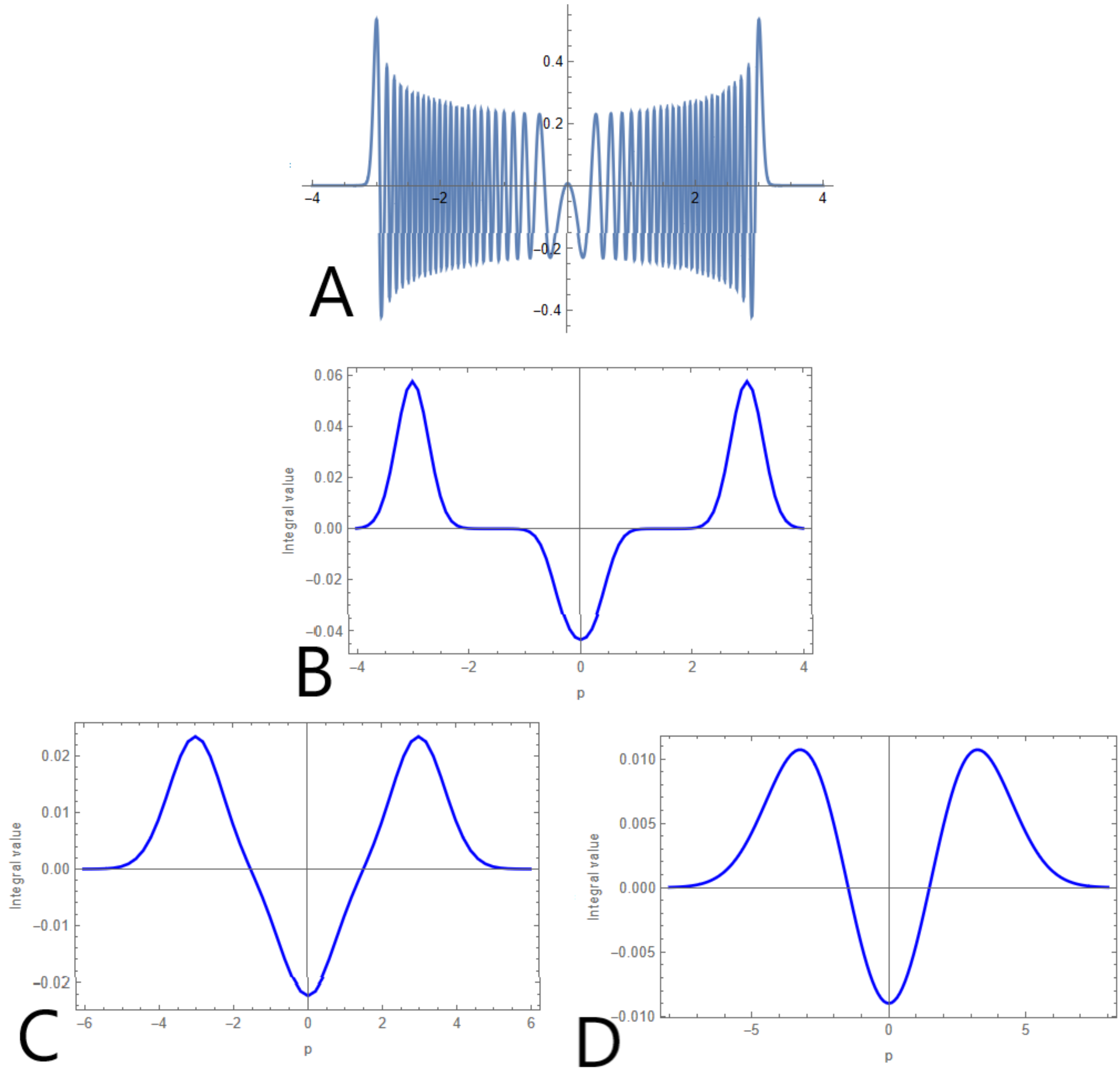

Figure 2.2: Numerical evaluation of the distributions in the variable p, with fixed $\lambda, \hbar = 1$, $q = 36$ and different values of α: A. Wigner function in p without coarse grain; B. Wigner function coarse-grained with $\alpha = 1/6$; C. Wigner function coarse-grained with $\alpha = 1$; D. Wigner function coarse-grained with $\alpha = 4$

outside the horizon are not observable and summing over them in TOU. He estimated:

$$\alpha(q) = \frac{H_0^3}{12q^{1/2}} \tag{2.61}$$

with H_0 being the Hubble constant. Many other choices can be made on the cutoff and many other form for α could be found in the literature, as the ones from Morikawa [29] or Kiefer [26] et. others. In particular, the factors $\alpha(q)$ with a dependence on $q^{-1/2}$, like the Halliwell and the Morikawa ones, respect the conditions (2.59) and (2.60) for a "correct" scale of coarse-graining, while other behaviours for $\alpha(q)$, like the one presented by Kiefer, does not respect those conditions.

Chapter 3

The matter equation for the Hartle-Hawking Wigner

Now that we have found the solution for the gravitational wave function, we can start to approach the matter equation (2.5). It is in this context that we should be able to reach a matter PDE that will consent us to introduce the time in our theory, and we will show that indeed the correct "classical" limit could be found as the matter obeys to a Schrodinger-like (or Schwinger-Tomonaga) equation, apart from tiny quantum-gravitational corrections.
Starting from (2.5), once again we replace $q \to q_+$, multiply by $\tilde{\psi}_-^*$ and integrate over s. In contrast with the case of the gravitational equation, we should also integrate over p and get rid of the p dependence, and since we are interested in studying the solution that describes an expanding Universe, the integration range $\mathcal{I}_+$ can be restricted to the positive values of p:

$$\int_{\mathcal{I}_+} dp \int_{-\infty}^{+\infty} ds e^{\frac{i}{\hbar}ps} \tilde{\psi}_-^* \left[\frac{4q\hbar^2}{m_p^2} (\partial_+ \tilde{\psi}_+)(\partial_+ \tilde{\chi}_+) + \tilde{\psi}_+ (H_{\phi,+} - \langle H_\phi \rangle_+) \tilde{\chi}_+ \right] \tag{3.1}$$

where we have chosen the expanding ($\mathcal{I}_+$) branch.
Taylor expanding the matter wave function in the limit $s \to 0$, we find for its derivative:

$$\partial_+ \tilde{\chi}_+ = \left(\frac{1}{2} \partial_q + \partial s \right) \left(\tilde{\chi} + \partial_q \tilde{\chi} \frac{s}{2} + \partial_q^2 \tilde{\chi} \frac{s^2}{8} + \dots \right) = \partial_q \tilde{\chi} + \partial_q^2 \tilde{\chi} \frac{s}{2} + \dots, \tag{3.2}$$

while the derivative of the product of the gravitational wave functions is:

$$\partial_+ \left(\tilde{\psi}_-^* \tilde{\psi}_+ \right) = \frac{1}{2} \partial_q \left(\tilde{\psi}_-^* \tilde{\psi}_+ \right) + \partial_s \left(\tilde{\psi}_-^* \tilde{\psi}_+ \right). \tag{3.3}$$

Now using the relations (2.27), (2.28) we can find:

$$\int ds e^{\frac{i}{\hbar}ps} \left(\partial_q \tilde{\chi} \right) \partial_s \left(\tilde{\psi}_-^* \tilde{\psi}_+ \right) = -i \frac{p}{\hbar} \left(\partial_q \tilde{\chi} \right) W \sim \hbar^{-1}, \tag{3.4}$$

$$\int dse^{\frac{i}{\hbar}ps}\left(\partial_q\tilde{\chi}\right)\frac{1}{2}\partial_q\left(\tilde{\psi}_-^*\tilde{\psi}_+\right)=\frac{1}{2}\left(\partial_q\tilde{\chi}\right)\left(\partial_q W\right)\sim\hbar^0, \tag{3.5}$$

$$\int dse^{\frac{i}{\hbar}ps}\left(\partial_q^2\tilde{\chi}\right)\frac{1}{2}s\partial_s\left(\tilde{\psi}_-^*\tilde{\psi}_+\right)=-\frac{1}{2}\left(\partial_q^2\tilde{\chi}\right)\left(W+p\partial_p W\right)\sim\hbar^0, \tag{3.6}$$

$$\int dse^{\frac{i}{\hbar}ps}\left(\partial_q^2\tilde{\chi}\right)\frac{1}{4}s\partial_q\left(\tilde{\psi}_-^*\tilde{\psi}_+\right)=\frac{1}{4}\left(\partial_q^2\tilde{\chi}\right)\left(-i\hbar\right)\frac{d^2W}{dpdq}\sim\hbar^1, \tag{3.7}$$

higher powers of s in (3.2) contribute higher powers of $\hbar$ to the first contribution in (3.1) (because of (2.28)) and can be neglected for the moment.
Let us note that, for $\hbar$ small and close to the classical limit, the expression (3.2) seems justified. Let us however note the order of magnitude of the terms (3.4), (3.5), (3.6), (3.7) should be checked a posteriori. The introduction of time is necessarily associated with the classical limit for gravity. Therefore, while quantum corrections could be present, the classical behaviour must dominate over quantum fluctuations. Indeed in the very early Universe, the quantum fluctuations of the metric are large and time cannot be consistently defined.
The remaining term proportional to $(H_{\phi,+}-\langle H_\phi\rangle_+)$ in (3.1) can be Taylor expanded for $s\to 0$ in a similar way of (3.2), and we obtain:

$$\begin{aligned}\int_{-\infty}^{+\infty} & dse^{\frac{i}{\hbar}ps}\left(\tilde{\psi}_-^*\tilde{\psi}_+\right)\left(H_{\phi,+}-\langle H_\phi\rangle_+\right)\tilde{\chi}_+ \\ &=\left[\left(H_\phi-\langle H_\phi\rangle\right)\tilde{\chi}\right]W(q,p)-\frac{1}{2}\partial_q\left[(H_\phi-\langle H_\phi\rangle)\tilde{\chi}\right]i\hbar\partial_p W(q,p)+\dots\end{aligned} \tag{3.8}$$

Now we will see two different approach to the problem, which consist in the "classical" and "semiclassical" approach to the matter equation.

3.1 The classical approach for the matter equation

Following the procedure exposed in [20], we can work with the relation:

$$\lim_{\alpha\to 0}\frac{1}{|\alpha|}Ai\left(\frac{x}{|\alpha|}\right)=\delta(x)+\frac{\alpha^3}{3}\delta^{(3)}(x)+\dots \tag{3.9}$$

in our case $\alpha=(\lambda\hbar)^{2/3}$, which goes to zero in the classical limit with $\hbar\to 0$. The classical limit is realized at the Leading Order (LO) of (3.9), which can be written as:

$$\lim_{(\lambda\hbar)^{2/3}\to 0}\frac{1}{(\lambda\hbar)^{2/3}}Ai\left(\frac{4p^2-\lambda q}{(\lambda\hbar)^{2/3}}\right)=\delta(4p^2-\lambda q)+O(h^2). \tag{3.10}$$

We can notice that we obtain a Dirac delta with support on the classical solution of the Friedman equation (1.13), which correspond to the classical value of the momenta of the inflaton field (2.46) under the condition $V(\phi) >> \phi'^2$. Indeed, we defined λ in the gravitational equation, where we considered the kinetical term of ϕ small enough to be ignored, so in this way $\lambda = \frac{2\langle H_\phi\rangle}{q^2} m_p^2$, and:

$$p = \pm\sqrt{\frac{\lambda q}{4}} = \pm\sqrt{\frac{m_p^2 \langle H_\phi\rangle}{2q}}$$

which is exactly our classical value of the momenta p_{cl}, and we can also rewrite our Dirac delta as $\delta(4p^2 - 2m_p^2 \langle H_\phi\rangle)$.
Now, given the Dirac's delta properties, we can express the classical limit of the Wigner function as

$$W_{cl} = \delta\left(4p^2 - \frac{2m_p^2 \langle H_\phi\rangle}{q}\right) = \frac{\delta\left(p - \sqrt{\frac{m_p^2\langle H_\phi\rangle}{2q}}\right) + \delta\left(p + \sqrt{\frac{m_p^2\langle H_\phi\rangle}{2q}}\right)}{\sqrt{\frac{2m_p^2\langle H_\phi\rangle}{q}}} \tag{3.11}$$

and inserting this last expression inside the relation (3.4), we obtain:

$$-\int_{\mathcal{I}_+} dp\frac{ip}{\hbar}(\partial_q\tilde{\chi})W = -\frac{i}{2\hbar}\partial_q\tilde{\chi}. \tag{3.12}$$

For the considerations made at the beginning of this subsection, we can now rewrite the relation (2.34) in the following way:

$$\partial_q W_{cl} = -\frac{\lambda}{8p}(\partial_p W_{cl}) = -\frac{m_p^2}{4qp}\langle H_\phi\rangle (\partial_p W_{cl})$$

and use it to find the contribution from (3.5):

$$\begin{aligned}
\int_{\mathcal{I}_+} dp\frac{1}{2}(\partial_q\tilde{\chi})(\partial_q W_{cl}) &= -\int_{\mathcal{I}_+} dp\frac{1}{2}(\partial_q\tilde{\chi})\left(\frac{m_p^2}{4qp}\langle H_\phi\rangle \partial_p W_{cl}\right)\\
&= -(\partial_q\tilde{\chi})\langle H_\phi\rangle \frac{m_p^2}{8qp}W_{cl}|_{\mathcal{I}_+} - \int_{\mathcal{I}_+} dp\frac{1}{2}(\partial_q\tilde{\chi})\left(\frac{m_p^2}{4qp^2}\langle H_\phi\rangle W_{cl}\right)\\
&= -(\partial_q\tilde{\chi})\langle H_\phi\rangle \frac{m_p^2}{16qp_{cl}^3} = -\frac{\partial_q\tilde{\chi}}{8qp_{cl}}
\end{aligned} \tag{3.13}$$

where the boundary terms vanish and we have taken the Dirac delta "positive" solution $p_{cl} = \sqrt{\frac{m_p^2\langle H_\phi\rangle}{2}}$.
Moving on, we have for Eq. (3.6)

$$-\int_{\mathcal{I}_+} dp\frac{1}{2}(\partial_q^2\tilde{\chi})(W_{cl} + p\partial_p W_{cl}) = -\frac{1}{4p_{cl}}\partial_q^2\tilde{\chi}\left(1 + pW_{cl}|_{\mathcal{I}_+} - 1\right) = 0 \tag{3.14}$$

where, once again, boundary terms vanish.
Lastly, we find that the expression (3.7) is next to next leading order for $\hbar \to 0$ and so can be neglected compared to other dominant contributions. The remaining contribution in (3.8) can be written as:

$$\int_{\mathcal{I}_+} dp\,[(H_\phi - \langle H_\phi\rangle)\,\tilde{\chi}]\,W(q,p) - \frac{1}{2}\partial_q\,[(H_\phi - \langle H_\phi\rangle)\,\tilde{\chi}]\,i\hbar\partial_p W(q,p) \\ = \frac{(H_\phi - \langle H_\phi\rangle)\,\tilde{\chi}}{2p_{cl}} - \frac{1}{2}\partial_q\,[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}]\,i\hbar W_{cl}|_{\mathcal{I}_+} \tag{3.15}$$

where the last contribution is a boundary term which vanishes.
Summing all the contributions up to order $\hbar^2$, and multiplying by $2p_{cl}$, we finally obtain:

$$-\frac{i\hbar}{m_p^2}4qp_{cl}\partial_q\tilde{\chi} - \frac{\hbar^2}{m_p^2}\partial_q\tilde{\chi} + (H_\phi - \langle H_\phi\rangle)\tilde{\chi} = 0 \tag{3.16}$$

and we observe that the time parameter η is now present. Indeed, the Wigner function is peaked on the classical trajectory for the scale factor and the quantities which depend on q must be calculated on such a trajectory. Morover, we can define the time parameter derivative by the chain rule:

$$\frac{\partial}{\partial t} = q'\frac{\partial}{\partial q} \tag{3.17}$$

where the expression of q' is the classical "velocity" corresponding to (1.58). To identify the associated time variable τ with the conformal time η we must fix the gauge $N = q$, such that (1.58) becomes:

$$q' = -\frac{4pq}{m_p^2} \tag{3.18}$$

and we have:

$$-\frac{i\hbar\lambda}{m_p^2}\partial_q\tilde{\chi} \equiv i\frac{\hbar\lambda}{4pq}\partial_\eta\tilde{\chi}. \tag{3.19}$$

What we find is that (3.16), with the definition of conformal time, ignoring contributions beyond $\hbar$ order, and rescaling the matter wave function $\tilde{\chi}$ [25]:

$$\tilde{\chi} \to \chi = \tilde{\chi}e^{\frac{i}{\hbar}\int^\eta \langle \hat{H}_\phi\rangle d\eta} \tag{3.20}$$

is nothing else that the *Schrodinger (or Schwinger-Tomonaga)* equation for the homogeneous inflaton:

$$i\hbar\frac{\partial\chi}{d\eta} \approx \hat{H}_\phi\chi \tag{3.21}$$

which describes the quantum behaviour of the "fast/light" matter wave function. The contributions at order $\hbar^2$ are quantum gravitational corrections to the matter wave equation. Such corrections include the non-adiabatic effects and the quantum fluctuations arising from the introduction of time.

3.2 The "semiclassical" approach to the matter equation

It is now worth investigating what happens to the matter equation if the limit $\hbar \to 0$ for the Wigner function is not taken. In such a case the Wigner function is essentially different from a Dirac delta and is "spread" around the classical trajectory . One may there expects different results and larger quantum gravitational effects.
If we insert the exact solution of gravitational equation for Wigner function (2.45) inside the relation (3.4), (3.5), (3.6), (3.7) one finds, for Eq.(3.4):

$$\begin{aligned} -\int_{\mathcal{I}_+} dp i\frac{p}{\hbar}(\partial_q\tilde{\chi})\frac{1}{(\lambda\hbar)^{2/3}}Ai(t) &= -\frac{i}{8\hbar}(\partial_q\tilde{\chi})\int_{-\frac{q\lambda^{1/3}}{\hbar^{2/3}}}^{+\infty} Ai(t)dt \\ &= i\frac{\pi}{8\hbar}(\partial_q\tilde{\chi})\left[Ai'\left(-\frac{q\lambda^{1/3}}{\hbar^{2/3}}\right)Gi\left(-\frac{q\lambda^{1/3}}{\hbar^{2/3}}\right) - Ai\left(-\frac{q\lambda^{1/3}}{\hbar^{2/3}}\right)Gi'\left(-\frac{q\lambda^{1/3}}{\hbar^{2/3}}\right)\right]. \end{aligned} \tag{3.22}$$

The integral above and those which follow below are found following the procedure shown in [35]. We indicate with t the argument of the Airy function $Ai(t) = Ai\left(\frac{4p^2-\lambda q}{(\lambda\hbar)^{2/3}}\right)$, such that $Ai'(t) = \frac{dAi(t)}{dt}$. In (3.22), $Gi(t)$ is the inhomogeneous Airy function (or Scorer function), defined as:

$$Gi(t) = \frac{1}{\pi}\int_0^{+\infty} \sin\left(\frac{k^3}{3} + kt\right)dk.$$

Using the result from (2.34), we can solve (3.5) as:

$$\begin{aligned} \int_{\mathcal{I}_+} dp\frac{1}{2}(\partial_q\tilde{\chi})(\partial_q W) &= -\int_{\mathcal{I}_+} dp\frac{1}{2}(\partial_q\tilde{\chi})\left(\frac{\lambda}{8p}\partial_p W\right) \\ &= -\frac{1}{2}(\partial_q\tilde{\chi})\int_{-\frac{q\lambda^{1/3}}{\hbar^{2/3}}}^{+\infty} dt\left(\frac{\lambda}{4\left[(\lambda\hbar)^{2/3}t + \lambda q\right]^{1/2}}\right)\left(\frac{1}{(\lambda\hbar)^{2/3}}Ai'(t)\right) \\ &= -\frac{1}{8\hbar}(\partial_q\tilde{\chi})\int_0^{+\infty} dz\frac{1}{\sqrt{z}}\frac{dAi\left(z - \frac{\lambda q}{(\lambda\hbar)^{2/3}}\right)}{dz} \\ &= -\frac{\pi}{4\hbar}(\partial_q\tilde{\chi})Ai\left[-\frac{q\lambda^{1/3}}{(2\hbar)^{2/3}}\right]Ai'\left[-\frac{q\lambda^{1/3}}{(2\hbar)^{2/3}}\right] \end{aligned} \tag{3.23}$$

where we made the change of variable $z = t + \frac{\lambda q}{(\lambda\hbar)^{2/3}}$.
Moving on, we have for Eq. (3.6)

$$-\int_{\mathcal{I}_+} dp\frac{1}{2}(\partial_q^2\tilde{\chi})(W + p\partial_p W) = -\int_{\mathcal{I}_+} dp\frac{1}{2}(\partial_q^2\tilde{\chi})(W - W) - \frac{1}{2}\left[pW\right]_0^{+\infty} = 0 \tag{3.24}$$

where W is proportional to the Airy function, which goes to zero as its argument goes to infinity and has a finite value for the zero argument. Since the Airy is proportional to $\exp\{-p^2\}$, then $\lim_{p\to+\infty} pW(q,p) = 0$.
Finally, we can integrate in momentum (3.7), to obtain:

$$\begin{aligned} &\frac{-i\hbar}{4}(\partial_q^2\tilde{\chi})\int_0^{+\infty} dp\frac{\partial}{\partial p}\left(\frac{\partial W}{\partial q}\right) = \frac{i\hbar}{4}(\partial_q^2\tilde{\chi})\int_0^{+\infty} dp\frac{\partial}{\partial p}\left(\frac{\lambda}{4}\frac{\partial}{\partial(p^2)}W\right) \\ &= \frac{i\hbar}{4}(\partial_q^2\tilde{\chi})\int_{-\frac{\lambda^{1/3}}{\hbar^{2/3}}q}^{+\infty} dt\frac{d}{dt}\left(\frac{\lambda}{4}\frac{(\lambda\hbar)^{2/3}}{4}\frac{d}{dt}W\right) = i\frac{(\lambda\hbar)^{5/3}}{64}(\partial_q^2\tilde{\chi})\frac{1}{(\lambda\hbar)^{2/3}}\left[Ai'(t)\right]_{-\frac{\lambda^{1/3}}{\hbar^{2/3}}q}^{+\infty} \\ &= -i\lambda\frac{\hbar}{64}(\partial_q^2\tilde{\chi})Ai'\left(-\frac{q\lambda^{1/3}}{\hbar^{2/3}}\right). \end{aligned} \tag{3.25}$$

Since the argument of the Airy function dependens on $\hbar^{-2/3}$, and for other reasons we will clarify in the following, we shall consider contributions due to the expansion in s above the order $O(\hbar)$. We will start from Eq.(3.2) and consider its expansion up to $O(\hbar^3)$:

$$\begin{aligned} \partial_+\tilde{\chi}_+ &= \left(\frac{1}{2}\partial_q + \partial s\right)\left(\tilde{\chi} + \partial_q\tilde{\chi}\frac{s}{2} + \partial_q^2\tilde{\chi}\frac{s^2}{8} + \partial_q^3\tilde{\chi}\frac{s^3}{48} + \partial_q^4\tilde{\chi}\frac{s^4}{384}\dots\right) \\ &= \partial_q\tilde{\chi} + \partial_q^2\tilde{\chi}\frac{s}{2} + \partial_q^3\tilde{\chi}\frac{s^2}{8} + \partial_q^4\tilde{\chi}\frac{s^3}{48} + \dots, \end{aligned} \tag{3.26}$$

and from (3.3), we will obtain the same results in (3.4), (3.5), (3.6), (3.7), plus two new contributions coming from the term $\partial_q^3\tilde{\chi}\frac{s^2}{8}$ inside (3.26), that give:

$$\int dse^{\frac{i}{\hbar}ps}\left(\partial_q^3\tilde{\chi}\right)\frac{s^2}{8}\partial_s\left(\tilde{\psi}_-^*\tilde{\psi}_+\right) = \frac{i\hbar}{4}\left(\partial_q^3\tilde{\chi}\right)\left(\partial_p W + \frac{p}{2}\partial_p^2 W\right) \tag{3.27}$$

and

$$\int dse^{\frac{i}{\hbar}ps}\left(\partial_q^3\tilde{\chi}\right)\frac{s^2}{8}\partial_q\left(\tilde{\psi}_-^*\tilde{\psi}_+\right) = \frac{(-i\hbar)^2}{8}\left(\partial_q^3\tilde{\chi}\right)\frac{d^3W}{dp^2dq}. \tag{3.28}$$

Starting with the evaluation of the integral in dp for the contribution (3.27), we can use integration by parts (IBP) to obtain the following result:

$$\begin{aligned} &\int_0^{+\infty} dp\frac{i\hbar}{4}\left(\partial_q^3\tilde{\chi}\right)\left(\partial_p W + \frac{p}{2}\partial_p^2 W\right) = \int_0^{+\infty} dp\frac{i\hbar}{4}\left(\partial_q^3\tilde{\chi}\right)\left(\partial_p W - \frac{1}{2}\partial_p W\right) + \left[\frac{p}{2}\partial_p W\right]_0^{+\infty} \\ &= \frac{i\hbar^{1/3}}{8\lambda^{2/3}}\left(\partial_q^3\tilde{\chi}\right)Ai\left(-\frac{\lambda^{1/3}}{\hbar^{2/3}}q\right) \end{aligned} \tag{3.29}$$

where, in the last line, we used the same reasoning we used in (3.24).
The integral in dp of the contribution (3.28) can be evaluated by taking the useful relation

(2.34) to obtain:

$$
\begin{aligned}
&-\int_0^{+\infty} dp\frac{(i\hbar)^2}{8}\left(\partial_q^3\tilde{\chi}\right)\frac{d^2}{dp^2}\left(\frac{dW}{dq}\right) = -\int_0^{+\infty} dp\frac{(i\hbar)^2}{8}(\partial_q^3\tilde{\chi})\frac{d^2}{dp^2}\left(-\frac{\lambda}{8p}\partial_p W\right) \\
&= \int_0^{+\infty} dp\frac{(i\hbar)^2}{8}(\partial_q^3\tilde{\chi})\frac{d}{dp}\left[\frac{1}{\lambda^{1/3}\hbar^{4/3}p}Ai'(t) - \frac{\lambda}{8p}\left(\frac{64p^2}{(\lambda\hbar)^2}Ai''(t) + \frac{8}{(\lambda\hbar)^{4/3}}Ai'(t)\right)\right] \\
&= \frac{\left(\partial_q^3\tilde{\chi}\right)}{\lambda}\left[pAi''\left(\frac{4p^2-\lambda q}{(\lambda\hbar)^{2/3}}\right)\right]_0^{+\infty} = 0.
\end{aligned}
\tag{3.30}
$$

Now we need to evaluate the contributions from (3.8). By substituting the solution (2.45) of the gravity equation for the Wigner function one finds:

$$
\begin{aligned}
&\int_0^{+\infty} dp\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]W(q,p) - \frac{1}{2}\partial_q\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]i\hbar\partial_p W(q,p) \\
&= \int_{-\frac{\lambda^{1/3}}{\hbar^{2/3}}q}^{+\infty} dt\left[\frac{\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]}{4\left[(\hbar\lambda)^{2/3}t + \lambda q\right]^{1/2}}Ai(t)\right] - \frac{\partial_q\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]}{2}i\left(\frac{\hbar}{\lambda^2}\right)^{1/3}\partial_t Ai(t) \\
&= \frac{\pi\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right](2)^{2/3}}{4(\lambda\hbar)^{1/3}}Ai^2\left(-\frac{\lambda^{1/3}}{(2\hbar)^{2/3}}q\right) + \frac{\partial_q\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]}{2}i\left(\frac{\hbar}{\lambda^2}\right)^{1/3}Ai\left(-\frac{\lambda^{1/3}}{\hbar^{2/3}}q\right).
\end{aligned}
\tag{3.31}
$$

Since we are keeping contributions beyond $\hbar$, we must consider also terms beyond the ones in (3.31), that is:

$$
\begin{aligned}
&-\int_0^{+\infty} dp\frac{\hbar^2}{8}\partial_q^2\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]\partial_p^2 W(q,p) \\
&= -\int_0^{+\infty} dp\frac{\hbar^{4/3}}{\lambda^{2/3}}\partial_q^2\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]\partial_p\left[Ai'(t)p\right] = 0,
\end{aligned}
\tag{3.32}
$$

and we should also evaluate

$$
\begin{aligned}
&\int_0^{+\infty} dp\frac{i\hbar^3}{48}\partial_q^3\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]\partial_p^3 W(q,p) \\
&= \int_0^{+\infty} dp\frac{i\hbar^{5/3}}{6\lambda^{4/3}}\partial_q^3\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]\partial_p\left[\frac{8p^2}{(\lambda\hbar)^{2/3}}Ai''(t) + Ai'(t)\right] \\
&= -\frac{i\hbar^{5/3}}{6\lambda^{4/3}}\partial_q^3\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]Ai'\left(-\frac{\lambda^{1/3}}{\hbar^{2/3}}q\right).
\end{aligned}
\tag{3.33}
$$

Finally we can sum all the contributions found, properly multiplying the contributions (3.22), (3.23), (3.25), (3.29) by the factor $(4\hbar^2)/(m_p^2)$ present in (3.1) and, at the same time, keeping only the contributions up to order $\hbar^2$. We finally obtain the cumbersome equation:

$$\begin{aligned}
&\frac{\pi i\hbar q}{2m_p^2}\left[Ai'\left(-\frac{q\lambda^{1/3}}{\hbar^{2/3}}\right)Gi\left(-\frac{q\lambda^{1/3}}{\hbar^{2/3}}\right)-Ai\left(-\frac{q\lambda^{1/3}}{\hbar^{2/3}}\right)Gi'\left(-\frac{q\lambda^{1/3}}{\hbar^{2/3}}\right)\right.\\
&\quad\left.+2iAi\left(-\frac{q\lambda^{1/3}}{(2\hbar)^{2/3}}\right)Ai'\left(-\frac{q\lambda^{1/3}}{(2\hbar)^{2/3}}\right)\right](\partial_q\tilde{\chi})+\frac{\pi[(H_\phi-\langle H_\phi\rangle)\tilde{\chi}](2)^{2/3}}{4(\lambda\hbar)^{1/3}}Ai^2\left(-\frac{\lambda^{1/3}}{(2\hbar)^{2/3}}q\right)\\
&\quad+\frac{i\partial_q[(H_\phi-\langle H_\phi\rangle)\tilde{\chi}]\hbar^{1/3}}{2\lambda^{2/3}}Ai\left(-\frac{\lambda^{1/3}}{\hbar^{2/3}}q\right)-\frac{i\hbar^{5/3}}{6\lambda^{4/3}}\partial_q^3[(H_\phi-\langle H_\phi\rangle)\tilde{\chi}]Ai'\left(-\frac{\lambda^{1/3}}{\hbar^{2/3}}q\right)=0.
\end{aligned} \tag{3.34}$$

It is evident that solving exactly this last equation in the present form is very difficult. Further approximations are then necessary. However we can restrict our analysis to the regime where q becomes very large. In such a regime the equation further simplifies and and the Airy functions can be expanded in terms of elementary functions.

3.2.1 Plane wave solution in the asymptotic regime

We can make some considerations about the arguments of the Airy functions inside our matter equation.
Since we are interested in a "semiclassical approximation", with $\hbar$ small in this limit and in a phase of the expanding/inflationary Universe, where the value of the parameter $q >> 1$, it is natural to consider values for the argument of the Airy function

$$|\frac{q\lambda^{1/3}}{\hbar^{2/3}}| >> 1.$$

In such a regime, the asymptotic expression for Airy's, Scorer's function and their derivatives, based on the method of saddle point approximation [3] (displayed in Appendix B) can be used.
In particular we shall use the following expressions [35]:

$$\begin{aligned}
Ai(-x) =&\frac{1}{\pi^{1/2}x^{1/4}}\left[\sin\left(\frac{2}{3}x^{3/2}-\frac{\pi}{4}\right)\left(\frac{5}{48}x^{-3/2}+\dots\right)\right.\\
&\left.+\cos\left(\frac{2}{3}x^{3/2}-\frac{\pi}{4}\right)\left(1-\frac{385}{4608}x^{-3}+\dots\right)\right];
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
Ai'(-x) =&\frac{x^{1/4}}{\pi^{1/2}}\left[\sin\left(\frac{2}{3}x^{3/2}-\frac{\pi}{4}\right)\left(1+\frac{455}{4608}x^{-3}+\dots\right)\right.\\
&\left.+\cos\left(\frac{2}{3}x^{3/2}-\frac{\pi}{4}\right)\left(\frac{7}{48}x^{-3/2}+\dots\right)\right];
\end{aligned} \tag{3.36}$$

$$Gi(-x) = \frac{1}{\pi^{1/2}x^{1/4}}\Bigg[-\sin\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right)\left(1 - \frac{385}{4608}x^{-3} + \dots\right) + \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right)\left(\frac{5}{48}x^{-3/2} + \dots\right)\Bigg] - \frac{1}{\pi x}\left(1 - \frac{2}{x^3} + \dots\right); \tag{3.37}$$

$$Gi'(-x) = \frac{x^{1/4}}{\pi^{1/2}}\Bigg[-\sin\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right)\left(\frac{7}{48}x^{-3/2} + \dots\right) + \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right)\left(1 + \frac{455}{4608}x^{-3} + \dots\right)\Bigg] + \frac{1}{\pi x^2}\left(1 - \frac{8}{x^3} + \dots\right); \tag{3.38}$$

where, in our case, $x = \frac{q\lambda^{1/3}}{\hbar^{2/3}}$, which is strictly positive.
Substituting these last four expressions inside our matter equation (3.34), we can find the following expression, where we just keep the contributions up to $\hbar^2$ order:

$$\begin{aligned}
&- i\frac{q\hbar}{2m_p^2}(\partial_q\tilde{\chi})\Bigg\{1 + \frac{1}{i}\cos\left(\frac{2}{3}\frac{\lambda^{1/2}q^{3/2}}{\hbar}\right) + \frac{\hbar^{1/2}\sin\left(\frac{2}{3}\frac{\lambda^{1/2}q^{3/2}}{\hbar} - \frac{\pi}{4}\right)}{\pi^{1/2}\lambda^{1/4}q^{3/4}} \\
&- \frac{i\hbar}{6\lambda^{1/2}q^{3/2}}\left[\frac{5}{2} + \cos^2\left(\frac{1}{3}\frac{\lambda^{1/2}q^{3/2}}{\hbar} - \frac{\pi}{4}\right)\right]\Bigg\} + \frac{\left[H_\phi - \langle H_\phi\rangle\right]\tilde{\chi}}{4(\lambda q)^{1/2}}\times \\
&\times\left[1 + \sin\left(\frac{2}{3}\frac{\lambda^{1/2}q^{3/2}}{\hbar}\right)\left(1 - \frac{205}{288}\frac{\hbar^2}{\lambda q^3}\right) - \frac{5}{12}\frac{\hbar}{\lambda^{1/2}q^{3/2}}\cos\left(\frac{2}{3}\frac{\lambda^{1/2}q^{3/2}}{\hbar}\right) - \frac{5}{16}\frac{\hbar^2}{\lambda q^3}\right] \\
&+ i\partial_q\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]\left\{\frac{\hbar^{1/2}\cos\left(\frac{2}{3}\frac{\lambda^{1/2}q^{3/2}}{\hbar} - \frac{\pi}{4}\right)}{2\pi^{1/2}q^{1/2}\lambda^{3/4}} + \frac{5\hbar^{3/2}\sin\left(\frac{2}{3}\frac{\lambda^{1/2}q^{3/2}}{\hbar} - \frac{\pi}{4}\right)}{96\pi^{1/2}q^{7/4}\lambda^{5/4}}\right\} \\
&- \frac{i\hbar^{3/2}q^{1/4}}{6\pi^{1/2}\lambda^{5/4}}\partial_q^3\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]\sin\left(\frac{2}{3}\frac{\lambda^{1/2}q^{3/2}}{\hbar} - \frac{\pi}{4}\right) = 0.
\end{aligned} \tag{3.39}$$

We can further simplify this last equation in the regime in which the value of x is very large, and neglect the quantum corrections of order x^{-n}. These corrections become negligible for values of $x \geq 10^3$. We can estimate for which values of q this occurs. If we take the natural units in which $\hbar = 1$, and the value of the nearly constant inflaton potential is [8]:

$$\Lambda = (6 \cdot 10^{-2})^6 m_p^4 r$$

where we considered $r = 10^{-4}$ which is the upper limit for the *"tensor to scalar ratio"* (at 95% CL as measured by Plank 2018 [1]). Thus the value of λ is:

$$\lambda = 2m_p^2\Lambda \approx 10^{-11}m_p^6.$$

Now we can compute the value of q for which the contribution proportional to x^{-n} can be ignored:

$$x = \frac{q\lambda^{1/3}}{\hbar^{2/3}} \geq 10^3 \to q \geq \frac{10^3\hbar^{2/3}}{(10^{-11}m_p^6)^{1/3}} \sim 5 \cdot 10^6 m_p^{-2}. \tag{3.40}$$

Some clarifications are now in order: in (1.52) we have absorbed a squared length L^2 inside $q = a^2$, such that actually they have the dimensions of a squared length $[q] = [a]^2 = [l_p]^2$, and since $[m_p] = [l_p]^{-1}$, this explains the dimensions of the result in (3.40). Therefor the equation (3.39), for values of $q \geq 5 \cdot 10^6 m_p^{-2}$, takes the form [1]:

$$-i\frac{q\hbar}{2m_p^2}(\partial_q\tilde{\chi})\left[1 + \frac{1}{i}\cos\left(\frac{2}{3}\frac{\lambda^{1/2}q^{3/2}}{\hbar}\right)\right] + \frac{[H_\phi - \langle H_\phi\rangle]\,\tilde{\chi}}{4(\lambda q)^{1/2}}\left[1 + \sin\left(\frac{2}{3}\frac{\lambda^{1/2}q^{3/2}}{\hbar}\right)\right] = 0 \tag{3.41}$$

where the only corrections of quantum origin come from the terms $\frac{1}{i}\cos\left(\frac{2}{3}\frac{\lambda^{1/2}q^{3/2}}{\hbar}\right)$, $\sin\left(\frac{2}{3}\frac{\lambda^{1/2}q^{3/2}}{\hbar}\right)$. Now we can change variables to this last equation, and introduce the variable η, that in the classical approximation is the conformal time. By multiplying the equation by a factor $8p = 4(\lambda q)^{1/2}$ and, on using again the relations (3.17) and (3.18), we can rewrite equation (3.41) as:

$$-i\hbar(\partial_\eta\tilde{\chi})\left[1 + \frac{1}{i}\cos\left(\frac{2}{3}\frac{\lambda^{1/2}q^{3/2}}{\hbar}\right)\right] + (H_\phi - \langle H_\phi\rangle)\left[1 + \sin\left(\frac{2}{3}\frac{\lambda^{1/2}q^{3/2}}{\hbar}\right)\right]\tilde{\chi} = 0. \tag{3.42}$$

Now we can try to solve the differential equation:

$$i(\partial_\eta\tilde{\chi}) = \hbar^{-1}\left[\frac{1 + \sin\left(\frac{2}{3}x^{3/2}\right)}{1 + \frac{1}{i}\cos\left(\frac{2}{3}x^{3/2}\right)}\right](H_\phi - \langle H_\phi\rangle)\tilde{\chi}. \tag{3.43}$$

Before proceeding, we point out that the time variable η has the same mathematical definition that the conformal time for the de Sitter Universe. Let us note, however, that the corrections to the matter equation evaluated from the asymptotic expansion of the Airy function, are encoded in the quantum gravitational oscillating contribution

$$\frac{1 + \sin\left(\frac{2}{3}x^{3/2}\right)}{1 - i\cos\left(\frac{2}{3}x^{3/2}\right)},$$

where the oscillations are very large and very fast. The definition of time in the matter equation is not straightforward since the quantum effects are large. We will return on

[1]The reader could notice that we discarded the term $\partial_q^3[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}]$ despite the fact it was multiplied by a function $Ai'(t)$, and in the asymptotic expansion (3.36) this term is proportional to $x^{1/4}$. The reason behind this choice is due to the fact that, after the multiplication by the factor $8p = 4(\lambda q)^{1/2}$, the term becomes proportional to $(q^3\partial_q^3)/(x^{9/4})$, and so it is evident how this contribution becomes negligible in the large x regime.

this point later.
Classically the Louko variable q and the conformal time η are related by:

$$q = \left(-\frac{1}{H_0\eta}\right)^2 \tag{3.44}$$

where H_0 is the Hubble constant during inflation.
Considering the form of our inflaton Hamiltonian H_ϕ, we observe that the contribution coming from the constant inflaton potential Λ disappears, and we remain with:

$$(H_\phi - \langle H_\phi\rangle) = -\frac{\hbar^2}{2q}(\partial_\phi^2 - \langle\partial_\phi^2\rangle)$$

such that our Schrodinger equation (3.43) becomes

$$i(\partial_\eta\tilde{\chi}) = -\frac{\hbar}{2q}\left[\frac{1+\sin\left(\frac{2}{3}x^{3/2}\right)}{1+\frac{1}{i}\cos\left(\frac{2}{3}x^{3/2}\right)}\right](\partial_\phi^2 - \langle\partial_\phi^2\rangle)\chi \equiv -\frac{\hbar}{2q}m(\eta)(\partial_\phi^2 - \langle\partial_\phi^2\rangle)\chi, \tag{3.45}$$

in which we have defined $m(\eta)$ as

$$m(\eta) \equiv \frac{1+\sin\left(-\frac{2}{3}\frac{\lambda^{1/2}}{\hbar H_0^3\eta^3}\right)}{1+\frac{1}{i}\cos\left(-\frac{2}{3}\frac{\lambda^{1/2}}{\hbar H_0^3\eta^3}\right)} \tag{3.46}$$

and where we have substituted $q^{3/2} = -(H_0\eta)^{-3}$ from the relation (3.44).
This is a non-hermitian Schrodinger equation, dependent on the parameter η. We can note that, despite the fact we have a non-hermitian equation (given by the complex factor $m(\eta)$), the following relation holds:

$$\frac{d}{d\eta}\langle\tilde{\chi}|\tilde{\chi}\rangle = i\frac{\hbar}{2q}m(\eta)(\langle\partial_\phi^2\rangle - \langle\partial_\phi^2\rangle) + h.c. = 0 \tag{3.47}$$

and then the norm $\langle\tilde{\chi}|\tilde{\chi}\rangle$ is constant, conserved, and can be arbitrarily set to one. Moreover, we can recover the relation (2.3).
Then, from (3.45), we can observe that the $\langle\partial_\phi^2\rangle$ contribution could be also removed by rescaling $\tilde{\chi}$, in a similar way as we did for the classical limit in (3.20). In this case one can "rephase"[2] $\tilde{\chi}$ function in the following way:

$$\tilde{\chi} = e^{i\zeta}\chi \tag{3.48}$$

with ζ is defined in such a way that:

$$\frac{\partial\zeta}{\partial\eta} = -\frac{\hbar}{2q}\left[\frac{1+\sin\left(\frac{2}{3}x^{3/2}\right)}{1+\frac{1}{i}\cos\left(\frac{2}{3}x^{3/2}\right)}\right]\langle\partial_\phi^2\rangle\,\chi. \tag{3.49}$$

[2]Let us note that this is note a proper rephase, since ψ is a complex function

In this way, Eq. (3.43) becomes:

$$i(\partial_\eta \chi) = -\frac{\hbar}{2q}\left[\frac{1+\sin\left(\frac{2}{3}x^{3/2}\right)}{1+\frac{1}{i}\cos\left(\frac{2}{3}x^{3/2}\right)}\right]\partial^2_\phi \chi. \tag{3.50}$$

This is indeed a non-hermitian Schrodinger equation, with Hamiltonian

$$\bar{H}_\phi \equiv m(\eta)\frac{\hat{\pi_\phi}^2}{2q(\eta)}. \tag{3.51}$$

A set of solutions of the equation (3.45) can be obtained from the ansatz:

$$\chi(\eta,\phi) \sim e^{i\beta\phi + f(\eta)} \tag{3.52}$$

where β is a real free parameter.
Now we can substitute (3.52) in (3.50), to obtain the following differential equation for $f(\eta)$:

$$f'(\eta) = -\frac{i\hbar}{2q}m(\eta)\beta^2 \tag{3.53}$$

where the prime symbol represent a derivative w.r.t η.
The solution of Eq.(3.53) is:

$$f(\eta) = -\frac{i\hbar}{2}\beta^2\int_{\eta_0}^{\eta}\frac{m(\eta')}{q(\eta')}d\eta'. \tag{3.54}$$

In conclusion, we can write our matter wave function in the following form:

$$\chi(\phi,\eta)_\beta = e^{i\beta\phi}exp\left\{-\frac{i\hbar}{2}\beta^2\int_{\eta_0}^{\eta}\frac{m(\eta')}{q(\eta')}d\eta'\right\}. \tag{3.55}$$

Taking back the definition (3.46), we can write $m(\eta)$ as:

$$\begin{aligned} m(\eta) &= \frac{1+\sin\left(\frac{2}{3}x^{3/2}(\eta)\right)}{1-i\cdot\cos\left(\frac{2}{3}x^{3/2}(\eta)\right)} = \frac{1+\sin\left(\frac{2}{3}x^{3/2}(\eta)\right)}{1+\cos^2\left(\frac{2}{3}x^{3/2}(\eta)\right)} + i\frac{\cos\left(\frac{2}{3}x^{3/2}(\eta)\right)+\frac{1}{2}\sin\left(\frac{4}{3}x^{3/2}(\eta)\right)}{1+\cos^2\left(\frac{2}{3}x^{3/2}(\eta)\right)} \\ &\equiv Re(m(\eta)) + i\cdot Im(m(\eta)) \end{aligned} \tag{3.56}$$

and so the exponent in our χ wave function (3.55) has an imaginary part (ϕ-independent), as a consequence of the non-hermitianicity of the Hamiltonian $\bar{H}_\phi$ defined by (3.51). We should also point out that normalised matter wave function is $\tilde{\chi}(\phi,\eta)$ in (3.48). One has:

$$\langle\tilde{\chi}(\eta,\phi)|\tilde{\chi}(\eta,\phi)\rangle = 1 = \langle\chi(\eta,\phi)|\chi(\eta,\phi)\rangle\, e^{-2\cdot Im\{\zeta\}} \rightarrow \langle\chi(\eta,\phi)|\chi(\eta,\phi)\rangle = e^{2\cdot Im\{\zeta\}} \tag{3.57}$$

in which we have decomposed ζ as $\zeta = Re\{\zeta\} + i\cdot Im\{\zeta\}$.

3.2.2 The emergence of a classical time with a coarse-graining of the perturbation

We can see that the matter equation in (3.41) has an highly oscillating behaviour because of the term $m(\eta)$ that multiplies the inflaton Hamiltonian H_ϕ. Because of the large quantum effects, the definition of time is not straightforward. The variable $q(\eta)$, that in the classical case of the de Sitter Universe is given by $(H_0\eta)^{-2}$ as we shown (3.44), could be affected by quantum effects.
As shown in [34], factors like $m(\eta)/q(\eta)$ that give the oscillatory behavior to our Hamiltonian $\bar{H}_\phi$ may be present in the matter equation in quantum cosmology, large oscillations may be dumped with a coarse-graining procedure which consists in averaging the matter equation over a period Δq (which is much shorter than the Plank length). This procedure may be justified since typical oscillations related to matter (inflaton) are much slower that those (trans-plankian) associated with the gravitational wavefunction. Inflaton evolution may be "insensitive" to such trans-plankian oscillations, and we may coarse-grain them.
Indeed if we take the matter equation (3.41) and, just for dimensional convenience, we rewrite it in terms of the scale factor a, and multiply both sides of the equation by a factor a^2, we obtain:

$$-i\hbar\frac{a^4\lambda^{1/2}}{m_p^2}\left[1-i\cos\left(\frac{2\lambda^{1/2}a^3}{3\hbar}\right)\right](\partial_a\tilde{\chi}) = -a^2\left[1+\sin\left(\frac{2\lambda^{1/2}a^3}{3\hbar}\right)\right][H_\phi-\langle H_\phi\rangle]\,\tilde{\chi}=0. \tag{3.58}$$

We can now average the terms $a^2\left[1-i\cos\left(\frac{2\lambda^{1/2}a^3}{3\hbar}\right)\right]$ and $a^2\left[1-i\cos\left[\frac{2\lambda^{1/2}a^3}{3\hbar}\right)\right]$ respectively on the l.h.s. and r.h.s. of the last equation, over a period of oscillation Δa given approximately by:

$$\Delta a \approx \frac{\hbar\pi}{\lambda^{1/2}a^2}. \tag{3.59}$$

Let us note that, for "a" large enough, the period becomes smaller than the Plank length:

$$\Delta a \sim 10^{-1}m_p^{-1} < l_p$$

and we can average the oscillating terms $\mathcal{I}(a) = a^2 f\left(\frac{2\lambda^{1/2}a^3}{3\hbar}\right)$ (where $f(x)$ could be both $\sin(x)$ or $-i\cos(x)$) over a period Δa in the following way:

$$\mathcal{I}(a) = \frac{1}{\Delta a}\int_a^{a+\Delta a} a'^2 f\left(\frac{2\lambda^{1/2}a^3}{3\hbar}\right)da' = \frac{\lambda^{1/2}a^2}{\hbar\pi}\int_a^{a+\Delta a} a'^2 f\left(\frac{2\lambda^{1/2}a^3}{3\hbar}\right)da'.$$

On defining a new integration variable $t=\left(\frac{2\lambda^{1/2}a'^3}{3\hbar}\right)$, one may write

$$\mathcal{I}(a) = \frac{a^2}{2\pi}\int_0^{2\pi} dt f(t). \tag{3.60}$$

This particular form of our integral $\mathcal{I}(a)$ can be now evaluated in the complex plane by using Jordan's Lemma and the Residue Theorem. On setting $z = \exp\{it\}$ one has:

$$\cos(t) = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

$$\sin(t) = \frac{1}{2i}\left(z - \frac{1}{z}\right),$$

$$dt = -i\frac{dz}{z}$$

and the integral (3.60) becomes:

$$\mathcal{I}(\eta) = -\frac{a^2}{4\pi}\oint \frac{z^2 + z \pm 1}{z^2} dz \tag{3.61}$$

where the sign + is for the contribution $i\cos(t)$ on the l.h.s, and viceversa the sign − is for the contribution $\sin(t)$ on the r.h.s.
The function inside the integral has a unique pole of second order in $z = 0$, which lies inside the integration path (a circumference of radius 1 centered in the origin of the complex plane). With the residue theorem, we can compute the integral (3.61) in the following way:

$$\mathcal{I}(\eta) = -\frac{a^2}{4\pi} \cdot (2\pi i)\left[Res\left(\frac{z^2 + z \pm 1}{z^2}\right)_{z=0}\right]. \tag{3.62}$$

The residue of order two can be computed as:

$$\lim_{z\to 0} \frac{d}{dz}\left(\frac{z^2 + z \pm 1}{z^2}(z - 0)^2\right) = 1 \tag{3.63}$$

and the final result is:

$$\mathcal{I}(\eta) = -i\frac{a^2}{2}. \tag{3.64}$$

We can see that the contribution of the oscillating terms is equal on both sides of the matter equation (3.58), so they cancel, and we obtain the final form:

$$i\hbar\frac{a^2\lambda^{1/2}}{m_p^2}(\partial_a\tilde{\chi}) = \left[H_\phi - \langle H_\phi\rangle\right]\tilde{\chi}. \tag{3.65}$$

that is nothing else that the matter equation we found in the classical limit for the Wigner equation in (3.16).
Let us note that in the procedure described above we have neglected the dependence on a for $\tilde{\chi}$. On accounting properly for such a dependence, one expects deviations from the above result of order Δa, obtained by Taylor expanding the integrands close to the first

extremum of the integral.
So we can conclude that a time can still be defined in the matter equation, and that it coincides with the conformal time for the de Sitter Universe.
The Eq.(3.65) has "plane wave" solutions of the form:

$$\chi_{cl}(\eta, \phi) = exp\{i\beta\phi - i\frac{\hbar H_0^2 \beta^2}{6}(\eta^3 - \eta_0^3)\}. \tag{3.66}$$

Chapter 4

The Vilenkin solution

We have seen in the previous chapter that, from the WDW equation with the Born-Oppenheimer decomposition, we obtain a gravitational equation (2.4) and a matter equation (2.5) and that we can find the coupled differential equations for the Wigner-Weyl transform of the gravitational wave function.
By solving these equations for a simplified inflaton Hamiltonian, we where able only to find the Wigner function associated with the HH solution of the gravitational equation. Such a solution describes a Universe in the quantum superposition of the expanding and contracting phase. This is exactly the reason for the large quantum gravitational effects which emerge in the matter equation.
Anyway, the classical Universe is better described by an expanding state. The solution for the gravitational wave function which accounts only for the expanding phase is the Vilenkin solution.
In what follows we calculate an approximate expression of the Vilenkin solution of the gravitational equation (2.4), and then search for the associated Wigner function. As we observed in section 2.3, an analytic form of the Wigner function for the exact Vilenkin solution is unknown, and this is one of the reasons why we studied the exact HH case first. Due to large interference effect, the HH case lead to other problems which could be mitigated by a coarse-graining procedure. In this chapter we search for the Wigner function in the case of the Vilenkin solution, calculated with the asymptotic form of the Airy's functions, like the ones reported in (3.35)-(3.38).

4.1 Vilenkin wave function for gravity and its Wigner function

The gravitational equation for the gravity wave function $\tilde{\psi}$ (2.4) is:

$$\left[\frac{2q\hbar^2}{m_p^2}\frac{\partial^2}{\partial q^2}+\langle\hat{H}_\phi\rangle\right]\tilde{\psi}=-\frac{2q\hbar^2}{m_p^2}\langle\partial_q^2\rangle\,\tilde{\psi}. \tag{4.1}$$

If we neglect the term on the r.h.s and the kinetic term of the inflaton inside $\langle\hat{H}_\phi\rangle$ (since they are sub-leading terms in the context of inflation), we are left with only the contribution of the potential of the scalar field, and (4.1) takes the form:

$$\left[\frac{2q\hbar^2}{m_p^2}\frac{\partial^2}{\partial q^2}+\frac{m_p^2\Lambda}{2\hbar^2}q\right]\tilde{\psi}=0. \tag{4.2}$$

For this equation, we can write exactly the Vilenkin solution $\tilde{\psi}_V$ (see [18] or [34] for more details) as:

$$\begin{aligned}\tilde{\psi}_V&=-iAi\left[-\left(\frac{\lambda}{4\hbar^2}\right)^{\frac{1}{3}}q\right]+Bi\left[-\left(\frac{\lambda}{4\hbar^2}\right)^{\frac{1}{3}}q\right]\\&=-iAi\left[-\left(\frac{\lambda}{4\hbar^2}\right)^{\frac{1}{3}}a^2\right]+Bi\left[-\left(\frac{\lambda}{4\hbar^2}\right)^{\frac{1}{3}}a^2\right].\end{aligned} \tag{4.3}$$

In order to calculate the Wigner-Weyl transform of this last solution, and express $Ai(-x)$ and $Bi(-x)$ in their asymptotic forms, we restrict to the limit of large a, q, as we did in the previous chapter.
We can express the function $Ai(-x)$ in the large x limit with (3.35), while for $Bi(-x)$ in the same limit we can find the following form [35]:

$$Bi(-x)\sim-\frac{1}{\pi^{1/2}x^{1/4}}\sin\left(\frac{2}{3}x^{3/2}-\frac{\pi}{4}\right) \tag{4.4}$$

where we kept just the leading terms in x for both $Ai(-x)$ and $Bi(-x)$.
Substituting $x=\left[\left(\frac{\lambda}{4\hbar^2}\right)^{\frac{1}{3}}a^2\right]$, we can rewrite the solution $\tilde{\psi}_V$ in (4.3) as:

$$\begin{aligned}\tilde{\psi}_V&\approx-i\cdot\frac{\hbar^{1/6}}{\pi^{1/2}\lambda^{1/12}a^{1/2}}\left[\cos\left(\frac{2}{3}\frac{\lambda^{1/2}}{\hbar}a^3-\frac{\pi}{4}\right)-i\cdot\sin\left(\frac{2}{3}\frac{\lambda^{1/2}}{\hbar}a^3-\frac{\pi}{4}\right)\right]\\&=-i\cdot\frac{\hbar^{1/6}}{\pi^{1/2}\lambda^{1/12}a^{1/2}}\exp\left\{-i\left(\frac{2}{3}\frac{\lambda^{1/2}}{\hbar}a^3-\frac{\pi}{4}\right)\right\}.\end{aligned} \tag{4.5}$$

We can use this last approximate solution to find the associated Wigner function, starting from its definition[1]:

$$W(a,p_a) \equiv \int_{-\infty}^{+\infty} dse^{\frac{2ip_a s}{\hbar}} \tilde{\psi}^*_- \tilde{\psi}_+$$

remembering that $\tilde{\psi}_\pm = \tilde{\psi}(a \pm s)$, with s representing the "quantum jump" between different states. More explicitly, the Wigner function for $\tilde{\psi}_V$ results from the following integral:

$$\begin{aligned} W(a,p_a)_V \propto & \int_{-\infty}^{+\infty} dse^{\frac{2ip_a s}{\hbar}} \exp\left\{ i\left(\frac{2}{3}\frac{\lambda^{1/2}}{\hbar}(a^3 - s^3 - 3a^2 s + 3as^2)\right) - i\frac{\pi}{4}\right\} \\ & \times \exp\left\{ -i\left(\frac{2}{3}\frac{\lambda^{1/2}}{\hbar}(a^3 + s^3 + 3a^2 s + 3as^2)\right) + i\frac{\pi}{4}\right\} \\ & = \int_{-\infty}^{+\infty} dse^{\frac{2ip_a s}{\hbar}} \exp\left\{ -\frac{2i\lambda^{1/2}}{\hbar}\left(\frac{s^3}{3} + a^2 s\right)\right\}. \end{aligned} \tag{4.6}$$

Let us note that we omitted the term $a^{-1/2}$ in front of the asymptotic form of the $Ai[x]$ and $Bi[x]$. This is justified by the fact that the factor $a^{-1/2}$ is sub-leading in the large a limit.
We can easily recognize that (4.6) is an Airy function, in its integral form:

$$Ai(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} ds \exp\left\{ i\left(\frac{s^3}{3} + zs\right)\right\}$$

and thus we can obtain the following Airy function:

$$W(a,p_a)_V \propto Ai\left[\frac{2^{2/3}\lambda^{1/2}}{(\lambda\hbar)^{2/3}}(\lambda^{1/2}a^2 - p_a)\right]. \tag{4.7}$$

In terms of the Louko variables q, p_q, the Wigner function takes the following form:

$$W(q,p_q)_V \propto Ai\left[\frac{2^{5/3}(\lambda q)^{1/2}}{(\lambda\hbar)^{2/3}}\left(\frac{\lambda^{1/2}q^{1/2}}{2} - p_q\right)\right] \tag{4.8}$$

where $a^2 = q$ and

$$p_a = \frac{\partial \mathcal{L}}{\partial a'} = \frac{\partial \mathcal{L}}{\partial q'}\frac{\partial q'}{\partial a'} = p_q\frac{\partial q}{\partial a} = 2\sqrt{q}p_q$$

and henceforth, we will use p to indicate p_q for simplicity.
On comparing this last form for the Vilenkin Wigner with the one found previously for the HH solution (2.45), we can see that in (4.8) the argument of the Airy is zero in the

[1] We are still considering coherent states, so we can use the simplified form for the Wigner-Weyl transform

point $p = \frac{\lambda^{1/2}q^{1/2}}{2} = p_{cl}$, while in the HH Airy's argument is null for $p = \pm\frac{\lambda^{1/2}q^{1/2}}{2} = \pm p_{cl}$, therefor the Vilenkin Wigner function is peaked on the positive solution $p = p_{cl}$, as should since it is associated with the solution of an expanding Universe, while the presence of the term p^2 in the argument of (2.45) makes the Wigner peak around $p = \pm p_{cl}$, describing the superposition of an expanding and contracting Universe.

4.2 The matter equation for Vilenkin

Let us now calculate the matter equation from the Wigner function (4.8). We shall repeat the same procedure as the HH case, but due to the fact that we are considering the case of an expanding Universe, we shall integrate on the entire real axis.
The contribution (3.22), in our current case, takes the form:

$$\begin{aligned}
&-\int_{-\infty}^{+\infty} dpi\frac{p}{\hbar}(\partial_q\tilde{\chi})Ai\left[\frac{2^{5/3}(\lambda q)^{1/2}}{(\lambda\hbar)^{2/3}}\left(\frac{\lambda^{1/2}q^{1/2}}{2}-p\right)\right]\\
&=-\frac{i\lambda^{2/3}}{2^{5/3}\hbar^{1/3}(\lambda q)^{1/2}}(\partial_q\tilde{\chi})\int_{-\infty}^{+\infty} dk\left[-\frac{(\lambda\hbar)^{2/3}}{2^{5/3}(\lambda q)^{1/2}}k+\frac{(\lambda q)^{1/2}}{2}\right]Ai(k)\\
&=-\frac{i(\lambda\hbar)^{2/3}}{2^{8/3}\hbar}(\partial_q\tilde{\chi})
\end{aligned} \tag{4.9}$$

where on the second line we changed variable

$$k = \left[\frac{2^{5/3}(\lambda q)^{1/2}}{(\lambda\hbar)^{2/3}}\left(\frac{\lambda^{1/2}q^{1/2}}{2}-p\right)\right]$$

and, for the final result, we exploited the fact that the Airy function is a normalized function and that [35]:

$$\int_{-\infty}^{+\infty} dk \quad k\cdot Ai(k) = 0.$$

This last result, in particular, comes from the limit [16]:

$$\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty} dk \quad k\cdot Ai(k)\exp\{-\epsilon k^2\} = 0.$$

Proceeding with the evaluation of the contribution (3.23), we find:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} dp\frac{1}{2}(\partial_q\tilde{\chi})(\partial_q W)\\
&=\int_{-\infty}^{+\infty} dp\frac{1}{2}(\partial_q\tilde{\chi})Ai'\left[\frac{2^{5/3}(\lambda q)^{1/2}}{(\lambda\hbar)^{2/3}}\left(\frac{\lambda^{1/2}q^{1/2}}{2}-p\right)\right]\left(\frac{2^{2/3}\lambda}{(\lambda\hbar)^{2/3}}-\frac{2^{2/3}\lambda^{1/2}}{q^{1/2}(\lambda\hbar)^{2/3}}p\right)\\
&=\frac{\lambda^{1/6}\hbar^{2/3}}{2^{11/3}q^{3/2}}(\partial_q\tilde{\chi})\int_{-\infty}^{+\infty} dkAi[k]=\frac{\lambda^{1/6}\hbar^{2/3}}{2^{11/3}q^{3/2}}(\partial_q\tilde{\chi})
\end{aligned} \tag{4.10}$$

where we have once again changed variable to k and the term on the second line $Ai'[k]\frac{2^{2/3}\lambda}{(\lambda\hbar)^{2/3}}$ goes to zero because, after the integration, we have a result proportional to $[Ai(z)]_{-\infty}^{+\infty}$ that is null, while the second term multiplying $Ai'(z)$ can be integrated by parts, leading to a null result $[Ai(z)\cdot z]_{-\infty}^{+\infty}$. The integral in the last line can be evaluated by using the normalization condition of the Airy function.
The contribution (3.24) can be evaluated with a similar reasoning and is null as well.
The last contribution we must consider comes from the integral (3.25), and gives us the following result:

$$\begin{aligned}
&\frac{-i\hbar}{4}(\partial_q^2\tilde{\chi})\int_{-\infty}^{+\infty}dp\frac{\partial}{\partial p}\left(\frac{\partial W}{\partial q}\right)=\frac{i\hbar}{4}(\partial_q^2\tilde{\chi})\int_{-\infty}^{+\infty}dp\frac{\partial}{\partial p}\left(\frac{\lambda}{8p}\frac{\partial}{\partial p}W\right)\\
&=-i\hbar\frac{q^{1/2}\lambda^{5/6}}{2^{10/3}\hbar^{2/3}}(\partial_q^2\tilde{\chi})\int_{-\infty}^{+\infty}dp\frac{\partial}{\partial p}\left(\frac{1}{p}Ai'\left[\frac{2^{2/3}\lambda^{1/3}q}{\hbar^{2/3}}-\frac{2^{5/3}q^{1/2}}{\lambda^{1/6}\hbar^{2/3}}\right]\right)\\
&=-i\hbar^{1/3}\frac{q\lambda^{2/3}}{2^{5/3}}(\partial_q^2\tilde{\chi})\int_{-\infty}^{+\infty}dz\frac{\partial}{\partial z}\left(\frac{Ai'[z]}{z\hbar^{2/3}-2^{2/3}\lambda^{1/3}q}\right)\\
&=-i\hbar^{1/3}\frac{q\lambda^{2/3}}{2^{5/3}}\left(\partial_q^2\tilde{\chi}\right)\left[\frac{Ai'[z]}{z\hbar^{2/3}-2^{2/3}\lambda^{1/3}q}\right]_{-\infty}^{+\infty}=0.
\end{aligned}\tag{4.11}$$

where we used the relation in (2.34) $\partial_q W=-\frac{\lambda}{8p}(\partial_p W)$, and we have changed variable

$$z=\frac{2^{2/3}\lambda^{1/3}q}{\hbar^{2/3}}-\frac{2^{5/3}q^{1/2}}{\lambda^{1/6}\hbar^{2/3}}.$$

The contribution coming from $[(H_\phi-\langle H_\phi\rangle)\tilde{\chi}]$, in (3.31) can be evaluated as follows:

$$\int_{-\infty}^{+\infty}dp\,[(H_\phi-\langle H_\phi\rangle)\tilde{\chi}]\,Ai\left[\frac{2^{5/3}(\lambda q)^{1/2}}{(\lambda\hbar)^{2/3}}\left(\frac{\lambda^{1/2}q^{1/2}}{2}-p\right)\right]=\frac{(\lambda\hbar)^{2/3}}{2^{5/3}(\lambda q)^{1/2}}[(H_\phi-\langle H_\phi\rangle)\tilde{\chi}]\tag{4.12}$$

and the continuation:

$$-\frac{i\hbar}{2}\int_{-\infty}^{+\infty}dp\partial_q\,[(H_\phi-\langle H_\phi\rangle)\tilde{\chi}]\,\partial_p\left\{Ai\left[\frac{2^{5/3}(\lambda q)^{1/2}}{(\lambda\hbar)^{2/3}}\left(\frac{\lambda^{1/2}q^{1/2}}{2}-p\right)\right]\right\}=0.\tag{4.13}$$

Now we can properly collect all the contributions computed and reconstruct the matter equation, and find the following form:

$$-i\hbar\frac{2\lambda^{1/2}q^{3/2}}{m_p^2}\partial_q\tilde{\chi}\left[1+\frac{i\hbar}{2\lambda^{1/2}q^{3/2}}\right]+[H_\phi-\langle H_\phi\rangle]\,\tilde{\chi}=0.\tag{4.14}$$

That's exactly the same result we obtained with the classical approach to the matter equation and the HH Wigner function in (3.16). Indeed we note that the factor multiplying $\partial_q\tilde{\chi}$:

$$\frac{2\lambda^{1/2}q^{3/2}}{m_p^2}\partial_q\tilde{\chi}\left[1+\frac{i\hbar}{2\lambda^{1/2}q^{3/2}}\right]$$

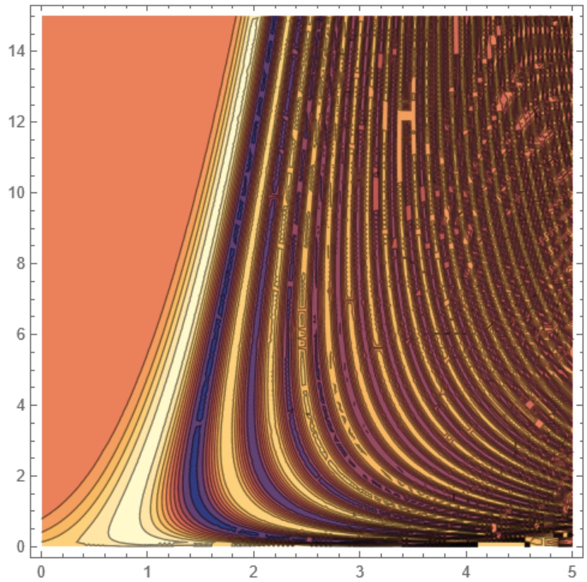

Figure 4.1: Contour Plot for the Vilenkin Wigner in the range $q = [0, 15]$ (vertical axis), $p = [0, 5]$ (horizontal axis), where $\lambda, \hbar$ are fixed to 1.

contains the definition of the classical "velocity" $q' = \frac{2\lambda^{1/2} q^{3/2}}{m_p^2}$ which consistently defines "time" in our theory, and inside the square bracket there is an additional imaginary contribution that becomes negligible in the limit of large q, in particular:

$$\frac{\hbar}{2\lambda^{1/2} q^{3/2}} < 1 \quad \text{for} \quad q > 10^4 m_p^{-2}.$$

4.3 Coarse-graining with the Husimi function

At the end of Chapter 2 we introduced the concept of coarse-graining by the Husimi distribution, and we explained that the presence of the term p^2 inside the argument of the Wigner function for the HH solution prevents from finding an analytic expression for it. We therefore estimated the Husimi distribution numerically. As we discussed earlier, the term p^2 is the consequence of the property of the HH solution to describe a Universe which is a superposition of contracting and expanding phase. Now, by considering the Vilenkin solution, we are describing a Universe which only expands, and this is probably a better description of the Universe we live in. For the Vilenkin case we find a Wigner function proportional to an Airy function with an argument which only depends on p. This allows to calculate the coarse-grained form of the Wigner function.

Let's remind a possible form of the coarse-grained Wigner function (2.57):

$$\mathcal{C}(q,p)=\frac{1}{\sqrt{\alpha(q)\pi}}\int_{-\infty}^{+\infty}dkW_0(q,k)e^{-\frac{(p-k)^2}{\hbar^2\alpha(q)}}.$$

If $W_0(q,k)$ is the Wigner function for the Vilenkin solution (4.8) (with a contour plot displayed in Figure 4.1), we find the following integral:

$$\mathcal{C}(q,p)=\frac{1}{\sqrt{\alpha(q)\pi}}\int_{-\infty}^{+\infty}dkAi\Bigg[\frac{2^{5/3}(\lambda q)^{1/2}}{(\lambda\hbar)^{2/3}}\left(\frac{\lambda^{1/2}q^{1/2}}{2}-k\right)\Bigg]e^{-\frac{(p-k)^2}{\hbar^2\alpha(q)}} \tag{4.15}$$

and with the change of variable $(k-p)/(\hbar\alpha^{1/2})=x$, it becomes:

$$\mathcal{C}(q,p)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}dxAi\Bigg[\frac{2^{5/3}\hbar^{1/3}(\alpha q)^{1/2}}{\lambda^{1/6}}\left(\frac{\lambda^{1/2}q^{1/2}}{2\hbar\alpha^{1/2}}-\frac{p}{\hbar\alpha^{1/2}}-x\right)\Bigg]e^{-x^2}. \tag{4.16}$$

If we define the quantities:

$$m\equiv\frac{\lambda^{1/6}}{2^{5/3}\hbar^{1/3}(\alpha q)^{1/2}} \tag{4.17}$$

and

$$y\equiv\frac{\lambda^{1/2}q^{1/2}}{2\hbar\alpha^{1/2}}-\frac{p}{\hbar\alpha^{1/2}} \tag{4.18}$$

we can rewrite (4.16) as:

$$\mathcal{C}(q,p)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}dxAi\left[\frac{1}{m}(y-x)\right]e^{-x^2}. \tag{4.19}$$

Now we can recognize that (4.19) is a particular *Airy transform* [36], which can be integrated to give [35]:

$$\int_{-\infty}^{+\infty}dxAi\left[\frac{1}{m}(y-x)\right]e^{-x^2}=\sqrt{\pi}\exp\left\{\frac{1}{4m^3}\left(y+\frac{1}{24m^3}\right)\right\}Ai\left(\frac{y}{m}+\frac{1}{16m^4}\right). \tag{4.20}$$

Using this latter result we can write (4.19) as:

$$\begin{aligned}\mathcal{C}(q,p)=\exp&\left\{\frac{8\hbar(q\alpha)^{3/2}}{\lambda^{1/2}}\left(\frac{\lambda^{1/2}q^{1/2}}{2\hbar\alpha^{1/2}}-\frac{p}{\hbar\alpha^{1/2}}+\frac{4\hbar(q\alpha)^{3/2}}{3\lambda^{1/2}}\right)\right\}\\ &\times Ai\Bigg[\left(\frac{(q\lambda)^{1/2}}{2}-p\right)\left(\frac{2^{5/3}q^{1/2}}{\lambda^{1/6}\hbar^{2/3}}\right)+\frac{2^{8/3}\hbar^{4/3}(q\alpha)^2}{\lambda^{2/3}}\Bigg].\end{aligned} \tag{4.21}$$

The plots of the coarse-grained Wigner function for different values for α are shown in Fig.4.2. We can see that increasing the value of α (so increasing the coarse graining effects) we average over more and more oscillations and the principal peak moves closer the classical trajectory, and becomes lower and more spread.
Now we can use the solution (4.21) in the matter equation.

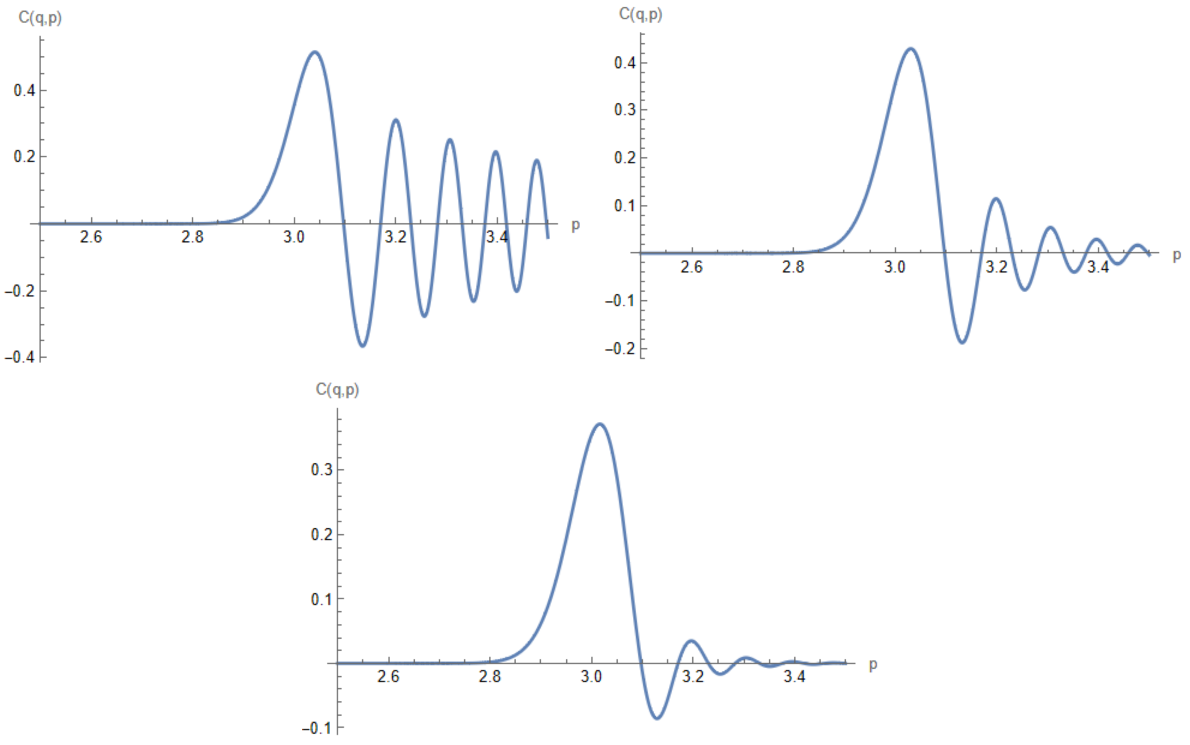

Figure 4.2: Numerical evaluation of the distributions in the variable p, with fixed $\lambda, \hbar = 1$, $q = 36$ and different values of α to fix $\frac{8\hbar(q\alpha)^{3/2}}{\lambda^{1/2}}$. The classical trajectory here is placed at $p = 3$. In the first picture (high-left) we have $\frac{8\hbar(q\alpha)^{3/2}}{\lambda^{1/2}} = 1$, in the second picture (high-right) $\frac{8\hbar(q\alpha)^{3/2}}{\lambda^{1/2}} = 6$, and in the last picture $\frac{8\hbar(q\alpha)^{3/2}}{\lambda^{1/2}} = 36$.

4.3.1 Matter equation with the coarse-grained Wigner

As we explained previously, by coarse-graining one averages on the rapid oscillations of the Wigner function, smoothing the oscillatory behaviour and leaving the principal peak. In order to compute the corresponding matter equation, we must compute the integrals (4.9), (4.10) and (4.12) substituting $W(q,p)$ with its coarse-grained version $\mathcal{C}(q,p)$ on (4.21). We start with (4.9):

$$
\begin{aligned}
&-\frac{i}{\hbar}(\partial_q\tilde{\chi})\int_{-\infty}^{+\infty} p\mathcal{C}(q,p)dp \\
&= -\frac{i}{\hbar}(\partial_q\tilde{\chi})\int_{-\infty}^{+\infty} p\exp\left\{\frac{8\hbar(q\alpha)^{3/2}}{\lambda^{1/2}}\left(\frac{\lambda^{1/2}q^{1/2}}{2\hbar^2\alpha^{1/2}}-\frac{p}{\hbar^2\alpha^{1/2}}+\frac{4\hbar(q\alpha)^{3/2}}{3\lambda^{1/2}}\right)\right\} \\
&\times Ai\left[\left(\frac{(q\lambda)^{1/2}}{2}-p\right)\left(\frac{2^{5/3}q^{1/2}}{\lambda^{1/6}\hbar^{2/3}}\right)+\frac{2^{8/3}\hbar^{4/3}(q\alpha)^2}{\lambda^{2/3}}\right]
\end{aligned} \tag{4.22}
$$

To compute this integral, we recognize the definition of an Airy transform of the function $p\exp\{ikp\}$. The Airy transform, $\Phi_\beta(y)$, of the function $\exp\{ikp\}$ is [36]:

$$
\Phi_\beta(y)=\int_{-\infty}^{+\infty}\exp\{ikp\}Ai\left[\frac{y-p}{\beta}\right]=\beta\exp\left\{iky+i\frac{k^3\beta^3}{3}\right\}. \tag{4.23}
$$

We can apply the Lemma [35]:

Lemma:*If the Airy transform of the function $f(x)$ is $\Phi_\beta(y)$, then the Airy transform of the function $xf(x)$ is:*

$$
\mathcal{A}_\beta[xf(x)]=x\Phi_\beta(y)-\beta^3\Phi''_\beta(y) \tag{4.24}
$$

where the prime symbols $'$ stands for the differentiation with respect to y.
We can compute the Airy transform in (4.22) on using (4.23) and the Lemma (4.24), by identifying:

$$
y=\frac{(q\lambda)^{1/2}}{2},\quad \beta=\frac{\lambda^{1/6}\hbar^{2/3}}{2^{5/3}q^{1/2}},\quad k=i\frac{8q^{3/2}\alpha}{\lambda^{1/2}}.
$$

We then obtain the result:

$$
-\frac{i}{h}(\partial_q\tilde{\chi})\int_{-\infty}^{+\infty}p\mathcal{C}(q,p)dp=-\frac{i\lambda^{2/3}}{2^{8/3}\hbar^{1/3}}(\partial_q\tilde{\chi}) \tag{4.25}
$$

where the exponential that appears in the Airy transform (4.23) is cancelled by the p-independent part of the exponential in $\mathcal{C}(q,p)$. Let us note that we get back the same result we found in (4.9).

Proceeding with the computation of (4.10):

$$\frac{1}{2}(\partial_q\tilde{\chi})\int_{-\infty}^{+\infty}\partial_q\mathcal{C}(q,p)dp$$
$$= \frac{1}{2}(\partial_q\tilde{\chi})\int_{-\infty}^{+\infty}\exp\left\{4q^2\alpha - \frac{8q^{3/2}\alpha p}{\lambda^{1/2}} + \frac{32\hbar^2(q\alpha)^3}{3\lambda^4}\right\} Ai\left[\frac{2^{2/3}q\lambda}{(\lambda\hbar)^{2/3}} - \frac{2^{5/3}(q\lambda)^{1/2}p}{(\lambda\hbar)^{2/3}} + \frac{2^{8/3}\hbar^{4/3}(q\alpha)^2}{\lambda^{2/3}}\right]$$
$$\times\left[8q\alpha + 4q^2\alpha' - \frac{12\alpha q^{1/2}p}{\lambda^{1/2}} - \frac{8\alpha' q^{3/2}p}{\lambda^{1/2}} + \frac{32q^2\hbar^2\alpha^3}{\lambda} + \frac{32q^3\hbar^2\alpha^2\alpha'}{\lambda}\right] dp$$
$$+ \frac{1}{2}(\partial_q\tilde{\chi})\int_{-\infty}^{+\infty}\exp\left\{4q^2\alpha - \frac{8q^{3/2}\alpha p}{\lambda^{1/2}} + \frac{32\hbar^2(q\alpha)^3}{3\lambda^4}\right\} Ai'\left[\frac{2^{2/3}q\lambda}{(\lambda\hbar)^{2/3}} - \frac{2^{5/3}(q\lambda)^{1/2}p}{(\lambda\hbar)^{2/3}} + \frac{2^{8/3}\hbar^{4/3}(q\alpha)^2}{\lambda^{2/3}}\right]$$
$$\times\left[\frac{2^{2/3}\lambda^{1/3}}{\hbar^{2/3}} - \frac{2^{2/3}p}{\hbar^{2/3}\lambda^{1/6}q^{1/2}} + \frac{2^{11/3}q\hbar^{4/3}\alpha^2}{\lambda^{2/3}} + \frac{2^{11/3}q^2\hbar^{4/3}\alpha\alpha'}{\lambda^{2/3}}\right] dp \tag{4.26}$$

where $Ai'[z]$ is the Airy derivative (with respect to its argument) and $\alpha'(q)$ is the derivative with respect to q.
Let's note that we have divided the last integral in two parts, the first part is $\mathcal{I}_1$:

$$\begin{aligned}\mathcal{I}_1 \equiv& \frac{1}{2}(\partial_q\tilde{\chi})\int_{-\infty}^{+\infty}\exp\left\{4q^2\alpha - \frac{8q^{3/2}\alpha p}{\lambda^{1/2}} + \frac{32\hbar^2(q\alpha)^3}{3\lambda^4}\right\}\\ &\times Ai\left[\frac{2^{2/3}q\lambda}{(\lambda\hbar)^{2/3}} - \frac{2^{5/3}(q\lambda)^{1/2}p}{(\lambda\hbar)^{2/3}} + \frac{2^{8/3}\hbar^{4/3}(q\alpha)^2}{\lambda^{2/3}}\right]\\ &\times\left[8q\alpha + 4q^2\alpha' - \frac{12\alpha q^{1/2}p}{\lambda^{1/2}} - \frac{8\alpha' q^{3/2}p}{\lambda^{1/2}} + \frac{32q^2\hbar^2\alpha^3}{\lambda} + \frac{32q^3\hbar^2\alpha^2\alpha'}{\lambda}\right] dp\end{aligned} \tag{4.27}$$

which can be evaluated to obtain:

$$\mathcal{I}_1 = \left[\frac{q^{1/2}\hbar^{2/3}\alpha\lambda^{1/6}}{2^{5/3}} + \frac{2^{7/3}q^{3/2}\hbar^{2/3}\alpha^3}{\lambda^{5/6}} + \frac{2^{7/3}q^{5/2}\hbar^{2/3}\alpha^2\alpha'}{\lambda^{5/6}}\right]\partial_q\tilde{\chi}. \tag{4.28}$$

The second integral is:

$$\begin{aligned}\mathcal{I}_2 \equiv& + \frac{1}{2}(\partial_q\tilde{\chi})\int_{-\infty}^{+\infty}\exp\left\{4q^2\alpha - \frac{8q^{3/2}\alpha p}{\lambda^{1/2}} + \frac{32\hbar^2(q\alpha)^3}{3\lambda^4}\right\}\\ &\times Ai'\left[\frac{2^{2/3}q\lambda}{(\lambda\hbar)^{2/3}} - \frac{2^{5/3}(q\lambda)^{1/2}p}{(\lambda\hbar)^{2/3}} + \frac{2^{8/3}\hbar^{4/3}(q\alpha)^2}{\lambda^{2/3}}\right]\\ &\times\left[\frac{2^{2/3}\lambda^{1/3}}{\hbar^{2/3}} - \frac{2^{2/3}p}{\hbar^{2/3}\lambda^{1/6}q^{1/2}} + \frac{2^{11/3}q\hbar^{4/3}\alpha^2}{\lambda^{2/3}} + \frac{2^{11/3}q^2\hbar^{4/3}\alpha\alpha'}{\lambda^{2/3}}\right] dp\end{aligned} \tag{4.29}$$

where we can compute a change of variable:

$$z = \frac{2^{2/3}q\lambda^{1/3}}{\hbar^{2/3}} - \frac{2^{5/3}q^{1/2}p}{\lambda^{1/6}\hbar^{2/3}} + \frac{2^{8/3}\hbar^{4/3}(q\alpha)^2}{\lambda^{2/3}}$$

and integrate by parts, to get:

$$\begin{aligned}\mathcal{I}_2 = &-\frac{\hbar^{2/3}\lambda^{1/6}}{2^{8/3}q^{1/2}}(\partial_q\tilde{\chi})\exp\left\{-\frac{16q^3\hbar^2\alpha^3}{3\lambda}\right\}\int_{-\infty}^{+\infty}\exp\left\{\frac{2^{4/3}q\hbar^{2/3}\alpha z}{\lambda^{1/3}}\right\}\\ &\times\left[\frac{2^{1/3}z\hbar^{2/3}\alpha}{\lambda^{1/3}}+2q\alpha+\frac{24q^2\hbar^2\alpha^3}{\lambda}+\frac{1}{2q}+\frac{32q^3\hbar^2\alpha^2\alpha'}{\lambda}\right]Ai[z]dz.\end{aligned} \tag{4.30}$$

Using the Airy transform (4.23), we find:

$$\begin{aligned}&-\frac{\hbar^{2/3}\lambda^{1/6}}{2^{8/3}q^{1/2}}(\partial_q\tilde{\chi})\exp\left\{-\frac{16q^3\hbar^2\alpha^3}{3\lambda}\right\}\int_{-\infty}^{+\infty}\exp\left\{\frac{2^{4/3}q\hbar^{2/3}\alpha z}{\lambda^{1/3}}\right\}\frac{32q^3\hbar^2\alpha^2\alpha'}{\lambda}Ai[z]dz\\ &=-\frac{2^{7/3}q^{5/2}\hbar^{2/3}\alpha^2\alpha'}{\lambda^{5/6}}\partial_q\tilde{\chi}.\end{aligned}$$

In the remaining integral, we can change variable:

$$x=-\frac{2^{1/3}z\hbar^{2/3}\alpha}{\lambda^{1/3}}+2q\alpha+\frac{24q^2\hbar^2\alpha^3}{\lambda}+\frac{1}{2q}$$

and obtain the final form for $\mathcal{I}_2$:

$$\begin{aligned}\mathcal{I}_2 = &-\frac{2^{7/3}q^{5/2}\hbar^{2/3}\alpha^2\alpha'}{\lambda^{5/6}}\partial_q\tilde{\chi}+\frac{\lambda^{1/2}}{8q^{1/2}\alpha}(\partial_q\tilde{\chi})\exp\left\{-\frac{2^7q^3\hbar^2\alpha^3}{3\lambda}+4q^2\alpha+1\right\}\int_{-\infty}^{+\infty}\exp\{-2qx\}\\ &\times Ai\left[\frac{\lambda^{1/3}}{2^{1/3}\hbar^{2/3}\alpha}\left(-x+2q\alpha+\frac{24q^2\hbar^2\alpha^3}{\lambda}+\frac{1}{2q}\right)\right]\left[x-4q\alpha-\frac{48q^2\hbar^2\alpha^3}{\lambda}-\frac{1}{q}\right]dx.\end{aligned} \tag{4.31}$$

Now we recognize an integral similar to $\mathcal{I}_1$, containing a combination of an integral of the kind (4.22) and an Airy transform (4.23), where:

$$y=2q\alpha+\frac{24q^2\hbar^2\alpha^3}{\lambda}+\frac{1}{2q},\quad \beta=\frac{2^{1/3}\hbar^{2/3}\alpha}{\lambda^{1/3}},\quad k=2iq.$$

We thus find the final result for $\mathcal{I}_2$:

$$\mathcal{I}_2=-\left[\frac{\lambda^{1/6}q^{1/2}\hbar^{2/3}\alpha}{2^{5/3}}+\frac{2^{7/3}q^{3/2}\hbar^{2/3}\alpha^3}{\lambda^{5/6}}-\frac{\hbar^{2/3}\lambda^{1/6}}{2^{11/3}q^{3/2}}+\frac{2^{7/3}q^{5/2}\hbar^{2/3}\alpha^2\alpha'}{\lambda^{5/6}}\right]\partial_q\tilde{\chi}. \tag{4.32}$$

On now summing the results of $\mathcal{I}_1$ and $\mathcal{I}_2$ in (4.28) and (4.32) we obtain the final result of (4.26):

$$\mathcal{I}_1+\mathcal{I}_2=\frac{\hbar^{2/3}\lambda^{1/6}}{2^{11/3}q^{3/2}}\partial_q\tilde{\chi} \tag{4.33}$$

where is noteworthy the fact that this is the same result we found in (4.10), starting from a very different Wigner function.
Ignoring the contribution (3.24) for the same reasons explained previously, we can now consider (4.11):

$$
\begin{aligned}
&\frac{-i\hbar}{4}(\partial_q^2\tilde{\chi})\int_{-\infty}^{+\infty} dp\frac{\partial}{\partial p}\left(\frac{\partial \mathcal{C}}{\partial q}\right) = \frac{i\hbar}{4}(\partial_q^2\tilde{\chi})\int_{-\infty}^{+\infty} dp\frac{\partial}{\partial p}\left(\frac{\lambda}{8p}\frac{\partial}{\partial p}\mathcal{C}\right)\\
&= -i\hbar^{1/3}\frac{q\lambda^{2/3}}{2^{5/3}}(\partial_q^2\tilde{\chi})\int_{-\infty}^{+\infty} dz\frac{\partial}{\partial z}\left[\frac{1}{z\hbar^{2/3}-2^{2/3}\lambda^{1/3}q}\left(\exp\left\{\frac{8(q\hbar^2\alpha^2)}{\lambda^{1/2}}\left(\frac{\lambda^{1/6}z}{2^{5/3}q^{1/2}\hbar^{1/3}\alpha^{1/2}}+\frac{4q\hbar^2\alpha^2}{3\lambda^{1/2}}\right)\right\}\right.\right.\\
&\left.\left.\times\, Ai'[z] + 2^{4/3}q\lambda^{2/3}\alpha\exp\left\{\frac{8q\hbar^2\alpha^2}{\lambda^{1/2}}\left(\frac{\lambda^{1/6}z}{2^{5/3}q^{1/2}\hbar^{1/3}\alpha^{1/2}}+\frac{4q\hbar^2\alpha^2}{3\lambda^{1/2}}\right)\right\}Ai[z]\right)\right]\\
&= -i\hbar^{1/3}\frac{q\lambda^{2/3}}{2^{5/3}}(\partial_q^2\tilde{\chi})\left[\frac{1}{z\hbar^{2/3}-2^{2/3}\lambda^{1/3}q}\left(\exp\left\{\frac{8(q\hbar^2\alpha^2)}{\lambda^{1/2}}\left(\frac{\lambda^{1/6}z}{2^{5/3}q^{1/2}\hbar^{1/3}\alpha^{1/2}}+\frac{4q\hbar^2\alpha^2}{3\lambda^{1/2}}\right)\right\}Ai'[z]\right.\right.\\
&\left.\left.+\, 2^{4/3}q\lambda^{2/3}\alpha\exp\left\{\frac{8q\hbar^2\alpha^2}{\lambda^{1/2}}\left(\frac{\lambda^{1/6}z}{2^{5/3}q^{1/2}\hbar^{1/3}\alpha^{1/2}}+\frac{4q\hbar^2\alpha^2}{3\lambda^{1/2}}\right)\right\}Ai[z]\right)\right]_{-\infty}^{+\infty}=0
\end{aligned}
\tag{4.34}
$$

where we have followed a procedure similar to that described for (4.11).
Finally, we must compute the contribution proportional to the Hamiltonian, to obtain:

$$\int_{-\infty}^{+\infty} dp\,[(H_\phi-\langle H_\phi\rangle)\tilde{\chi}]\,\mathcal{C}(q,p) = \frac{\lambda^{1/6}\hbar^{2/3}}{2^{5/3}q^{1/2}}(H_\phi-\langle H_\phi\rangle)\tilde{\chi} \tag{4.35}$$

solved recognizing the form of the Airy transform (4.23). The result is identical to that found for (4.12) without coarse-graining. Moreover, this result is also independent from the "weight" over the oscillations α, since it does not appear in any part of the final result.
So in the end we find the same results we got for the matter equation without coarse-graining, and we can rewrite the same matter equation:

$$-i\hbar\frac{2\lambda^{1/2}q^{3/2}}{m_p^2}\partial_q\tilde{\chi}\left[1+\frac{i\hbar}{2\lambda^{1/2}q^{3/2}}\right]+[H_\phi-\langle H_\phi\rangle]\,\tilde{\chi}=0. \tag{4.36}$$

So, once again, we were able to reproduce exactly the classical behaviour we would have if we just consider the Wigner as a Dirac's delta peaked on the classical trajectory.

4.4 The "p^2" matter equation with coarse-grained Wigner

When we introduced the coarse-graining procedure as result of the TOU method, we found the Husimi distribution in p, that is nothing else that a Gaussian-smoothed version

of the Wigner function. Such a coarse-grained Wigner is similar to the Gaussian probability distribution, and by increasing the value of the factor α, the coarse-grained Wigner becomes a normalized, positive definite, distribution, centered on the value $p = p_{cl}$, which spreads as α grows. So we can effectively think to the coarse-grained Wigner as a probability distribution in phase space that, at high values of α, becomes essentially a Gaussian with mean $p = p_{cl}$ and standard deviation dependent on α.
The definition of time in the matter equation is related to the integral $\sim \int_{-\infty}^{+\infty} pW dp$. We calculate such integral in different cases. While for HH the integral, due to large interference effects, led to non-negligible quantum (oscillatory) effects, and the semiclassical limit could be obtained only by averaging over such oscillations, for the other cases the resulting matter equation was the same and consistently reproduced the classical limit. This occurred in the "pure" Vilenkin-case and also for the coarse-grained Wigner function. This result is peculiar since the final matter equation seems to be independent on the shape of the Wigner function.
Let us note that obtaining the classical limit in the pure Vilenkin case was expected. Indeed if gravity (scale factor) is described by a pure state, time can be introduced in the matter equation à la Born-Oppenheimer, and the same result is obtained. The introduction of time employing the Wigner function is necessary if gravity cannot be defined by a pure state (that is the case of a coarse-grained Wigner function). In such cases one would expect slightly different forms for the matter equation.
Let us note that the procedure for obtaining the final form of the matter equation consists in performing an integration over p. While such an integration indeed reproduces the correct result in the classical limit and for the Vilenkin case, one may agree that different integrations would lead to different results, non of them being a priori justifiable. In what follows we investigate the consequences of performing an integration over dp^2 or, similarly, on integral $\sim \int_{-\infty}^{+\infty} pdp$ of both sides of the matter equation.
In the classical limit, when the Wigner is a Dirac delta, the correct matter equation is certainly reproduced and it is worth studying how the matter equation varies in the other cases.
Let's start right with the evaluation of (4.22). By multiplying the integrand by p, we find:

$$
\begin{aligned}
&-\frac{i}{\hbar}(\partial_q\tilde{\chi})\int_{-\infty}^{+\infty} p^2 \mathcal{C}(q,p)dp \\
&\quad= -\frac{i}{\hbar}(\partial_q\tilde{\chi})\int_{-\infty}^{+\infty} p^2 \exp\left\{\frac{8\hbar(q\alpha)^{3/2}}{\lambda^{1/2}}\left(\frac{\lambda^{1/2}q^{1/2}}{2\hbar\alpha^{1/2}} - \frac{p}{\hbar\alpha^{1/2}} + \frac{4\hbar(q\alpha)^{3/2}}{3\lambda^{1/2}}\right)\right\} \\
&\quad\times Ai\left[\left(\frac{(q\lambda)^{1/2}}{2} - p\right)\left(\frac{2^{5/3}q^{1/2}}{\lambda^{1/6}\hbar^{2/3}}\right) + \frac{2^{8/3}\hbar^{4/3}(q\alpha)^2}{\lambda^{2/3}}\right]dp
\end{aligned}
\tag{4.37}
$$

and in in order to evaluate this contribution, we use the lemma of the Airy transform (4.24), finding that:

$$\Phi_\beta(y) = \beta \exp\left\{iky + i\frac{k^3\beta^3}{3}\right\},$$

$$y = \frac{(q\lambda)^{1/2}}{2} + \frac{2q^{3/2}\beta^2}{\hbar^2\lambda^{1/2}}, \quad k = i\frac{8q^{3/2}\beta}{\hbar^2\lambda^{1/2}}, \quad \beta = \frac{\lambda^{1/6}\hbar^{2/3}}{2^{5/3}q^{1/2}}$$

and the integral (4.37) can be evaluated as:

$$\begin{aligned}
&-\frac{i}{\hbar}(\partial_q\tilde{\chi})\left[y\left(y\Phi_\beta(y) - \beta^3\Phi''_\beta(y)\right) - \beta^3\left(y\Phi_\beta(y) - \beta^3\Phi''_\beta(y)\right)''\right] \\
&= -\frac{i}{\hbar}(\partial_q\tilde{\chi})\left(y^2 + 2y\beta^3k^2 - 2ik\beta^3 + \beta^6k^4\right)\Phi_\beta(y) \\
&= -\frac{i}{\hbar^{1/3}}(\partial_q\tilde{\chi})\left(\frac{q^{1/2}\lambda^{7/6}}{2^{11/3}} + \frac{\lambda^{1/6}\hbar^2\alpha}{2^{5/3}q^{1/2}}\right).
\end{aligned} \tag{4.38}$$

Now we can proceed evaluating the new contribution that comes from (4.26), leading to:

$$\begin{aligned}
&\frac{1}{2}(\partial_q\tilde{\chi})\int_{-\infty}^{+\infty} p\partial_q\mathcal{C}(q,p)dp = -\frac{\lambda}{16}(\partial_q\tilde{\chi})\int_{-\infty}^{+\infty} p\frac{\partial_p\mathcal{C}}{p}(q,p)dp \\
&= -\frac{\lambda}{16}(\partial_q\tilde{\chi})\Bigg[\exp\left\{\frac{8\hbar(q\alpha)^{3/2}}{\lambda^{1/2}}\left(\frac{\lambda^{1/2}q^{1/2}}{2\hbar\alpha^{1/2}} - \frac{p}{\hbar\alpha^{1/2}} + \frac{4\hbar(q\alpha)^{3/2}}{3\lambda^{1/2}}\right)\right\} \\
&\times Ai\left[\left(\frac{(q\lambda)^{1/2}}{2} - p\right)\left(\frac{2^{5/3}q^{1/2}}{\lambda^{1/6}\hbar^{2/3}}\right) + \frac{2^{8/3}\hbar^{4/3}(q\alpha)^2}{\lambda^{2/3}}\right]\Bigg]_{p=-\infty}^{p=+\infty} = 0
\end{aligned} \tag{4.39}$$

where we have used the relation $\partial_q W = -[\lambda/(8p)]\partial_p W$ that comes from the gravitational equation (2.29).
For this case we can also compute the contribution coming from (3.24), finding the following result:

$$\begin{aligned}
&-\int_{-\infty}^{+\infty} dp\frac{1}{2}(\partial_q^2\tilde{\chi})(p\mathcal{C} + p^2\partial_p\mathcal{C}) = -\int_{-\infty}^{+\infty} dp\frac{1}{2}(\partial_q^2\tilde{\chi})(p\mathcal{C} - 2p\mathcal{C}) - \frac{1}{2}\left[p^2\mathcal{C}\right]_{-\infty}^{+\infty} \\
&= \int_{-\infty}^{+\infty} dp\frac{1}{2}(\partial_q^2\tilde{\chi})p\mathcal{C} = \frac{(\lambda\hbar)^{2/3}}{2^{11/3}}(\partial_q^2\tilde{\chi}).
\end{aligned} \tag{4.40}$$

The contribution (4.34) now becomes:

$$\begin{aligned}
&\frac{-i\hbar}{4}(\partial_q^2\tilde{\chi})\int_{-\infty}^{+\infty} dpp\frac{\partial}{\partial p}\left(\frac{\partial\mathcal{C}}{\partial q}\right) = \frac{i\hbar}{4}(\partial_q^2\tilde{\chi})\int_{-\infty}^{+\infty} dpp\frac{\partial}{\partial p}\left(\frac{\lambda}{8p}\frac{\partial}{\partial p}\mathcal{C}\right) \\
&= \frac{i\hbar}{4}(\partial_q^2\tilde{\chi})\int_{-\infty}^{+\infty} dp\left(-\frac{\lambda}{8p}\frac{\partial}{\partial p}\mathcal{C} + \frac{\lambda}{8}\frac{\partial^2}{\partial p^2}\mathcal{C}\right) = \frac{i\hbar}{4}(\partial_q^2\tilde{\chi})\int_{-\infty}^{+\infty} dp\left(\frac{\partial}{\partial q}\mathcal{C} + \frac{\lambda}{8}\frac{\partial^2}{\partial p^2}\mathcal{C}\right).
\end{aligned} \tag{4.41}$$

In the first part of the integral, with $\frac{\partial}{\partial q}\mathcal{C}$, we have exploited the gravitational equation (2.4). The resulting contribution is of the same form of (4.26), and can be evaluate it in the same way, leading to:

$$\frac{i\hbar}{4}(\partial_q^2\tilde{\chi})\int_{-\infty}^{+\infty} dp\left(\frac{\partial}{\partial q}\mathcal{C}\right) = i\frac{\hbar^{5/3}\lambda^{1/6}}{2^{14/3}q^{3/2}}(\partial_q^2\tilde{\chi}). \tag{4.42}$$

The remaining part of the integral (4.41), with $\frac{\partial^2}{\partial p^2}\mathcal{C}$, becomes:

$$\frac{i\hbar}{4}(\partial_q^2\tilde{\chi})\int_{-\infty}^{+\infty} dp\left(\frac{\lambda}{8}\frac{\partial^2}{\partial p^2}\mathcal{C}\right) = \frac{i\hbar\lambda}{32}(\partial_q^2\tilde{\chi})\left[\frac{\partial}{\partial p}\mathcal{C}\right]_{-\infty}^{+\infty} = 0. \tag{4.43}$$

So we have that the result of the integral (4.41) is simply given by (4.42).
The result of the contribution from (4.35), is:

$$\int_{-\infty}^{+\infty} dp\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]p\mathcal{C}(q,p) = \frac{\lambda^{2/3}\hbar^{2/3}}{2^{8/3}}\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right] \tag{4.44}$$

where we recognize that the integral is of the same kind of (4.22) and (4.40).
The second contribution from the matter Hamiltonian is:

$$\begin{aligned}&-\frac{i\hbar}{2}\partial_q\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]\int dpp\partial_p\mathcal{C} = \frac{i\hbar}{2}\partial_q\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]\int dp\mathcal{C}\\ &= i\frac{\hbar^{5/3}\lambda^{1/6}}{2^{8/3}q^{1/2}}\partial_q\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]\end{aligned} \tag{4.45}$$

where we integrated by part and evaluated the resulting integral by using the same result found in (4.35).
Now we can put all these contribution together to obtain the new matter equation:

$$\begin{aligned}&-i\hbar\left[\frac{2q^{3/2}\lambda^{1/2}}{m_p^2} + \frac{8q^{1/2}\hbar^2\alpha}{\lambda^{1/2}m_p^2}\right]\partial_q\tilde{\chi} + i\left[\frac{\hbar^3}{m_p^2\lambda^{1/2}q^{1/2}} - 2i\frac{\hbar^2 q}{m_p^2}\right]\partial_q^2\tilde{\chi}\\ &+(H_\phi - \langle H_\phi\rangle)\tilde{\chi} + i\frac{\hbar}{q^{1/2}\lambda^{1/2}}\partial_q\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right] = 0.\end{aligned} \tag{4.46}$$

Let us note that the term multiplying $\partial_q\tilde{\chi}$ contains a first contribution proportional to the classical velocity and necessary to introduce the conformal time in the matter equation, plus a higher-order contribution proportional to α that becomes small in the large q limit. This contribution cancels when $\alpha = 0$ (as it should for a Vilenkin pure state), but for $\alpha \neq 0$ is non-negligible. It is related to the spread of the coarse-grained Wigner function and can be naturally interpreted as quantum gravitational effects. Let us also note that the factor α can be dependent to q and this modifies the time dependence of

such quantum corrections.
Finally, if we can keep the leading, hermitian terms of the last equation, recalling also that the terms $\partial_q^2\tilde{\chi}$ and $\partial_q\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]$ are sub-leading in the context of inflation, we are left with the new version of the matter equation, of the form:

$$-i\hbar\left[\frac{2q^{3/2}\lambda^{1/2}}{m_p^2} + \frac{8q^{1/2}\hbar^2\alpha}{\lambda^{1/2}m_p^2}\right]\partial_q\tilde{\chi} + (H_\phi - \langle H_\phi\rangle)\tilde{\chi} = 0. \tag{4.47}$$

4.4.1 What would happen for a "p^3" matter equation?

Having computed the matter equation in the "p^2" case, it is normal to ask why we have chosen to perform such an integral, and what happens for the "$p^{n>2}$" choice. So, it is worth calculating the "p^3" matter equation, and verify if it will produce the same kind of matter equation, with a NLO term for $\partial_q\tilde{\chi}$ which has the same behaviour of the one found in (4.47), plus NNLO contributions which become rapidly negligible in the high-q limit.
We will consider the contribution from (4.37), with p^3 instead of p^2, which can be solved by the application of the Lemma (4.24) in the following way:

$$-\frac{i}{\hbar}(\partial_q\tilde{\chi})\int_{-\infty}^{+\infty} p^3\mathcal{C}(q,p)dp = -i\left[\frac{q\lambda^{5/3}}{2^{14/3}\hbar^{1/3}} + \frac{3\lambda^{2/3}\alpha\hbar^{5/3}}{2^{11/3}} - \frac{\lambda^{2/3}\hbar^{5/3}}{2^{17/3}q^2}\right](\partial_q\tilde{\chi}). \tag{4.48}$$

Then we can compute the contribution coming from (4.44) with p^2 instead of p inside the integral, and we can solve it recognizing that it is the same kind of integral we computed in (4.37):

$$\int_{-\infty}^{+\infty} dp\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]p^2\mathcal{C}(q,p) = \left[\frac{\hbar^{2/3}q^{1/2}\lambda^{7/6}}{2^{11/3}} + \frac{\hbar^{8/3}\lambda^{1/6}\alpha}{2^{5/3}q^{1/2}}\right]\left[(H_\phi - \langle H_\phi\rangle)\tilde{\chi}\right]. \tag{4.49}$$

We will ignore the other contributions to the matter equation for our actual scope, knowing from previous considerations that they give contributions of higher order to the matter equation.
Considering the contributions (4.48) and (4.49), we can follow the same procedure of the previous case and obtain a matter equation:

$$-i\hbar\frac{2q^{3/2}\lambda^{1/2}}{m_p^2}\left[1 + \frac{2\hbar^2\alpha}{q\lambda} + \mathcal{O}(q^{-2})\right]\partial_q\tilde{\chi} + (H_\phi - \langle H_\phi\rangle)\tilde{\chi} = 0. \tag{4.50}$$

We can see that the matter equation (4.50), at LO, is identical to the ones which gave us the Schrodinger equation in all the other cases, and at NLO ($\mathcal{O}(q^{n<0})$) has a correction for the term $\partial_q\tilde{\chi}$ that, besides coefficients, is identical to the one found in (4.47), and has the same behaviour in the high-q limit. We also obtain a further correction of order

$\mathcal{O}(q^{-3})$, which becomes highly negligible in the regime we are considering.
So we can say that the NLO corrections to the matter equation are already reproduced correctly in the case of the "p^2" matter equation, and considering "$p^{n>2}$" equations will produce basically the same corrections, while considering the standard matter equation (4.36), the NLO (quantum) corrections are completely lost.
Let us finally note that integration over p is necessary in order to obtain the matter equation (which should not depend on p). The "p" dependence is a consequence of the hybrid method used to perform the BO decomposition, using the Wigner function to describe the gravitational sector. A rigorous treatment should consist in describing the entire matter gravity system in terms of a density matrix and its Wigner transform. In our "effective" approach, how the integration over p is performed is left arbitrary, however the results obtained by integrating over p^2 look consistently reproduce the classical limit.

4.4.2 Cosmological outcomes

In this section, we will investigate the potential consequences of the quantum corrections in the matter equation emerging from the "p^2" integration.

Quantum corrections as the effect of a "virtual fluid"

Let us now return to the matter equation (4.47) we found for the "p^2" case:

$$-i\hbar\left[\frac{2q^{3/2}\lambda^{1/2}}{m_p^2}+\frac{8q^{1/2}\hbar^2\alpha}{\lambda^{1/2}m_p^2}\right]\partial_q\tilde{\chi}+(H_\phi-\langle H_\phi\rangle)\tilde{\chi}=0.$$

As we said in different occasions, the term

$$\frac{2q^{3/2}\lambda^{1/2}}{m_p^2}\partial_q\tilde{\chi}=q'\partial_q\tilde{\chi}=\partial_\eta\tilde{\chi}$$

is the one that consent to reintroduce the conformal time inside our matter equation, because of the presence of the term q'. The other contribution multiplying $\partial_q\tilde{\chi}$, $(8q^{1/2}\hbar^2\alpha)/(\lambda^{1/2}m_p^2)$, is a quantum-gravitational corrective term, that inevitably is going to modify the term $\partial_\eta\tilde{\chi}$ inside the matter equation.
One can think that this term modifies the source term of the classical Friedman equation (1.13), and q' is replaced by new quantum-corrected q'_{qc}, such that:

$$q'_{qc}=\left[\frac{2q^{3/2}\lambda^{1/2}}{m_p^2}+\frac{8q^{1/2}\hbar^2\alpha}{\lambda^{1/2}m_p^2}\right]. \tag{4.51}$$

The modification of q' in (4.51) has inevitably cosmological consequences, and we can study them via the Friedman equation, considering a new form of the energy density ρ_{qc}.

We can find the following new form for the first Friedman equation:

$$\left(\frac{q'_{qc}}{q}\right)^2 = \frac{8q\rho_{qc}}{m_p^2}. \tag{4.52}$$

Where ρ_{qc} is the result of a sum $\rho_{qc} = \rho + \rho_{virt}$ between the previous contribution $\rho = \Lambda$ and of a new contribution ρ_{virt} which encodes quantum gravitational corrections. We can then interpret the new contribution ρ_{virt} as the presence of a new "virtual" fluid which fills the Universe, and for the nature of the corrections produced, we can imagine it must be a fluid whose density of energy decreases with the growth of the parameter q, and whose effects were more important in a primordial phase of the Universe, leaving today just a tiny correction to the classical observations.
Solving (4.52) with the form of q'_{qc} given by (4.51), we can find the following expression:

$$\rho_{qc} = \Lambda + \frac{4\hbar^2\alpha}{qm_p^2} + \frac{4\hbar^4\alpha^2}{q^2 m_p^4 \Lambda} \tag{4.53}$$

and, recalling that $\rho_{qc} = \rho + \rho_{virt}$ and $\rho = \Lambda$, we can conclude that:

$$\rho_{virt} = \frac{4\hbar^2\alpha}{qm_p^2} + \mathcal{O}(q^{-2}) \tag{4.54}$$

where we are considering only the first-order correction in the limit of high-value q for ρ_{qc}.
Now, since we are considering this energy contribution as if it was coming from a virtual fluid, we can study its continuity equation, to determine its equation of state. One can simply take back the continuity equation displayed in (1.4) with the result (1.7) coming from the ansatz $p = \omega\rho$, and find:

$$\rho_{virt} = q^{-\frac{3}{2}(1+\omega_{virt})}. \tag{4.55}$$

Now one can compare (4.54) and (4.55), where the coarse-graining parameter α is usually definite as inversely proportional to q [19] [29] [26]($\alpha \propto q^{-n_\alpha}$, with n_α positive). So, the form of ρ_{virt} from (4.54) is $\rho_{virt} \propto q^{-(1+n_\alpha)}$, and we can conclude that:

$$\omega_{virt} = \frac{2}{3}n_\alpha - \frac{1}{3}$$

and we can rewrite (4.54) as:

$$\rho_{virt} = q_0 \cdot q^{-(n_\alpha+1)}. \tag{4.56}$$

Outcomes in the context of cosmological perturbations

In the first chapter we briefly illustrated the theory of of quantum-inflationary perturbations, studied via the Mukhanov-Sasaki (MS) equation (1.74).
From the solutions of this equation, it is possible to obtain the features of the quantum perturbations spectre, which can be then related to the CMB anisotropies. Such features are defined by relations which contains the slow-roll parameters ϵ_1 and ϵ_2. These are $n_s - 1$, n_t and r, that we defined in (1.80), (1.85), (1.86), that we rewrite here for more clarity (in terms of q):

$$\epsilon_1 = -\frac{H'}{\sqrt{q}H^2}, \quad \epsilon_2 = 2\frac{d\ln(\epsilon_1)}{d\ln(q)},$$
$$n_s - 1 = -2\epsilon_1 - \epsilon_2, \quad n_t = -2\epsilon_1, \quad r = 16\epsilon_1.$$

The presence of quantum gravitational corrections alters these parameters since the evolution of the scale factor is modified by the expression $q' = q'_{qc}$ in Eq.(4.51).
One can then compute the values of the slow-roll (SR) parameters ϵ_1 and ϵ_2 for $q' = q'_{qc}$.
The SR parameter ϵ_1, can be first computed for the classical (de Sitter) value of $H = H_0 = const.$, and find:

$$\epsilon_{1cl} = -\frac{H'}{\sqrt{q}H^2} = 0 \tag{4.57}$$

which is the correct parameter for the de Sitter Universe. Since $\epsilon_{1cl} = 0$, ϵ_{2cl} and higher-order SR parameters are identically zero.
If one now considers the evolution modified by the virtual fluid described by the Friedman equation q'_{qc}, a different result for ϵ_1 is found, indeed from (4.51):

$$H^2 \sim \Lambda + q_0 \cdot q^{-n_\alpha - 1}, \tag{4.58}$$

then one finds:

$$\epsilon_1 \cong -(n_\alpha + 1)\frac{q_0}{\Lambda}q^{-n_\alpha - 1}. \tag{4.59}$$

Let us notice that, without virtual fluid ($q_0 \to 0$), on recovers the classical (de Sitter) evolution.
On now calculating ϵ_2 one finds:

$$\epsilon_2 \equiv 2\frac{d\ln(\epsilon_1)}{d\ln(q)} \approx -2(n_\alpha + 1) \tag{4.60}$$

which may be large. This result however is the consequence of the unperturbed de Sitter evolution obtained without the quantum virtual fluid. A large value for ϵ_2 is dynamically related to the evolution driven by the fluid (4.53) which mimics a so called constant roll phase.

For realistic inflationary models one has $\epsilon_1 = \epsilon_{1cl} \neq 0$, and the quantum corrections can be approximately described by:

$$\epsilon_1 \cong \epsilon_{1cl}(q) - (n_\alpha + 1)\frac{q_0}{\Lambda}q^{-1-n_\alpha}, \tag{4.61}$$

and therefore

$$\epsilon_2 \cong \epsilon_{2cl} + 2(n_\alpha + 1)^2\frac{q_0}{\Lambda}q^{-1-n_\alpha}. \tag{4.62}$$

Finally, one can calculate:

$$n_s - 1 = -2\epsilon_1 - \epsilon_2 = (n_s - 1)_{cl} - 2n_\alpha(n_\alpha + 1)\frac{q_0}{\Lambda}q^{-1-n_\alpha}, \tag{4.63}$$

$$n_t = -2\epsilon_1 = -2\epsilon_{1cl} + 2(n_\alpha + 1)\frac{q_0}{\Lambda}q^{-1-n_\alpha}, \tag{4.64}$$

$$r = 16\epsilon_1 = r_{cl} - 16(n_\alpha + 1)\frac{q_0}{\Lambda}q^{-1-n_\alpha} \tag{4.65}$$

Notice, that in the limit $q_0 \to 0$, the virtual fluid disappears, and the variables $n_s - 1$, n_t and r return to their value in classical limit.
As we said, the quantities $n_s - 1$, n_t and r are parameters that describe the inflationary quantum perturbations, and can be implemented in the study of measurement of the CMB anisotropies.
Let us finally note that, the case we have just considered is an approximation of this mechanism, but anyway shows that such effects exists and can be used as a starting point for a more accurate analysis of inflation.

Conclusions

In this thesis we studied a simplified inflationary model with an approximately constant potential $V(\phi) = \Lambda$, and by the Dirac quantization procedure of the Hamiltonian constraints, we were able to obtain the Wheeler-DeWitt (WdW) equation. The WdW equation has been treated by a Born-Oppenheimer decomposition, that in this contest consists of decomposing the total Universe wave function into a gravitational component (the "heavy" degree of freedom), and a (homogeneous) matter component for the inflaton field (the "light" degree of freedom). This allows us to obtain two partial differential equations from the original WdW equation: one for gravity and one for matter.
Subsequently one of the two equations was transformed according to Wigner-Weyl, in particular transforming the gravity wavefunction into a Wigner function, on which also the matter equation will depend. This was done in order to study the quantum behaviour, and possibly the emergence of quantum perturbative effects associated with the introduction of time, and study how that would impact on the propagation of the matter component.
The study of the gravity solution, with Hartle-Hawking (HH) initial conditions, brought to a form of the Wigner function proportional to an Airy function, symmetric respect to the momenta of the scale factor and peaked on the classical trajectory $p = \pm p_{cl}$, which remarks the fact that the HH initial conditions describe a Universe in a quantum superposition of an expanding and a contracting phase. Due to its form, we saw that the coarse-grained version of the HH Wigner function as an Husimi distribution cannot be expressed analytically, despite the fact that a numerical evaluation displays the correct expected behaviour.
The Wigner solution was inserted inside the matter equation, and different approximation methods have been applied. Searching for the classical limit, we showed how the Wigner function can be approximated by a Dirac delta peaked on the classical solutions $p = \pm p_{cl}$, and integrating around the positive (expanding) one, we were able to make the time emerge in our theory, and the matter equation takes the form of a Schrodinger equation.
Considering the contribution of the exact gravitational solution inside the matter equation, and expanding the Wigner (Airy) function for large values of the scale factor, one obtains the appearance of a large, non-hermitian, highly-oscillating contribution to the

matter equation. However, this contribution can be eliminated by coarse-graining the matter equation, which is equivalent to average over fast oscillations (trans-plankian). In this way one is able to recover the conformal time definition inside the matter equation, showing that it can be still written as a Schrodinger equation. This coarse-graining “a posteriori” may mimic the coarse-graining on the HH Wigner function which we could not perform analytically.

Finally, we studied the Vilenkin solution for the gravity equation, Wigner-Weyl transforming the gravity wavefunction and finding an approximate solution for the corresponding Wigner function, which resulted into an Airy function peaked on the positive (expanding) classical solution for the scale factor momentum $p = +p_{cl}$. This was once again inserted into the matter equation, finding the same semi-classical limit as a Schrodinger equation, with the emergence of conformal time. In this case the Wigner function was also coarse-grained into an Husimi distribution, and inserting this function inside the matter equation, we obtained the same (Schrodinger) form of the “pure” (not coarse-grained) Wigner function. In particular, our result was independent from the coarse-graining parameter α.

We then focused on a new approach for obtaining the matter equation with the coarse-grained Wigner function, based in a different integration procedure. The new approach showed the emergence of small quantum corrections in the contribution connected with the emergence of time inside the matter equation (considering just first order approximations). Finally, we probed that considering alternative versions of this method, these would produce substantially the same quantum perturbative effects.

The appearance of the perturbative correction has an effect on the “velocity” q' and, inserted into the Friedman equation, results in a time-dependent correction for the constant inflaton potential Λ. This could be interpreted as the presence, in the early-stages of the Universe, of a “virtual fluid”, which contribution is non-negligible in the early (inflationary) epoch, but becomes almost null in the large cosmological factor regime. The continuity equation and the behaviour of this fluid have been studied.

The presence of quantum-corrections which modify the inflationary evolution determine a redefinition of the value for the slow-roll parameters ϵ_1 and ϵ_2, that in turn has consequences in the estimate of $n_s - 1$, n_t and r. We showed that small corrections emerge from this context, and this last result should be taken as an example of the effects that could emerge when quantum-gravitational corrections are considered, and needs further and deeper studies. Indeed, with increasingly high precision observations coming in the next few years, such features may be relevant to discriminate between different modes of inflation.

Appendix A

Taylor expansion of potential energy terms

Following [32] ,we can make use of position eigenvalue equation of the position eigenstate

$$U(\hat{x})\left|x\right\rangle = U(x)\left|x\right\rangle .$$

Then we have the following Taylor expansion:

$$U\left(x \pm \frac{s}{2}\right) = \sum_{n=0}^{\infty}\left[\frac{1}{n!}\frac{d^nU}{dx^n}\left(\pm\frac{s}{2}\right)^n\right]. \tag{A.1}$$

Now we can multiply by $(i\hbar)^n$ and by $(-\frac{i}{\hbar})^n$ (which is its inverse) and get

$$U\left(x \pm \frac{s}{2}\right) = \sum_{n=0}^{\infty}\left[(\mp 1)^n\frac{(i\hbar)^n}{n!}\frac{d^nU}{dx^n}\left(\frac{is}{2\hbar}\right)^n\right] \tag{A.2}$$

so we get

$$U\left(x + \frac{s}{2}\right) \pm U\left(x - \frac{s}{2}\right) = \sum_{n=0}^{\infty}\left[((-1)^n \pm 1)\frac{(i\hbar)^n}{n!}\frac{d^nU}{dx^n}\left(\frac{is}{2\hbar}\right)^n\right]. \tag{A.3}$$

Now we can substitute this last expression inside a Wigner integral and get

$$\begin{gathered}\int dse^{\frac{ip}{\hbar}}\langle x+\frac{s}{2}|\left(U\left(\hat{x}+\frac{s}{2}\right) \pm U\left(\hat{x}+\frac{s}{2}\right)\right)\hat{\rho}|x-\frac{s}{2}\rangle = \\ \sum_{n=0}^{\infty}\left[((-1)^n \pm 1)\frac{(i\hbar)^n}{n!}\frac{d^nU}{dx^n}\right]\int dse^{\frac{ip}{\hbar}}\left(\frac{is}{2\hbar}\right)^n\langle x+\frac{s}{2}|\hat{\rho}|x-\frac{s}{2}\rangle .\end{gathered} \tag{A.4}$$

In the last line, we can recognize the relation (2.28), so we can rewrite it as

$$\sum_{n=0}^{\infty}\left[((-1)^n \pm 1)\frac{(i\hbar)^n}{2^n n!}\frac{d^nU}{dx^n}\left(\frac{\partial^n W}{\partial p^n}\right)\right] \tag{A.5}$$

that are the expansions that we find in (2.29) and (2.30) for the terms $\langle H_\phi^2\rangle$ and $\langle\partial_a^2\rangle$.

Appendix B

Saddle point approximation for Airy function

The saddle point approximation (SPA) is a method for deriving an asymptotic approximation to integrals of the form [3]:

$$I(x) = \int_{\mathcal{C}} dt f(t) e^{xg(t)}$$

for the limit $x \to +\infty$. We indicate with $\mathcal{C}$ the complex contour which is our integration range and the functions f and g are analytic functions of t. the idea behind this method is to take advantage of the analytic propriety of the $f(t), g(t)$ functions to deform our integration range $\mathcal{C}$ into a new one $\mathcal{C}'$ on which $g(t)$ will have a constant imaginary part. Once this has been done, we can evaluate the new integral with Laplace's method.
Now we take the example of an Airy function $Ai(x)$ with $x > 0$, we can write it as [35]:

$$Ai(x) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dt e^{\frac{t^3}{3} - xt}. \tag{B.1}$$

Now we can choose a new variable u defined as

$$t = \sqrt{x} + iu \quad with \quad -\infty < u < +\infty \tag{B.2}$$

and now we can rewrite the integral (B.1) as:

$$\pi e^{\frac{2}{3}x^{3/2}} Ai(x) = \int_{0}^{+\infty} e^{-u^2\sqrt{x}} \cos\left(\frac{u^3}{3}\right) du = \frac{1}{2x^{1/4}} \int_{-\infty}^{+\infty} e^{-v^2} \cos\left(\frac{v^3}{3x^{3/4}}\right) dv. \tag{B.3}$$

Then we can replace the $\cos\left(\frac{v^3}{3x^{3/4}}\right)$ by its expansion

$$\pi e^{\frac{2}{3}x^{3/2}} Ai(x) = \frac{1}{2x^{1/4}} \int_{-\infty}^{+\infty} e^{-v^2} \left(1 - \frac{v^6}{18x^{3/2}} + \dots\right) \tag{B.4}$$

and integrate it by parts, obtaining:

$$\pi e^{\frac{2}{3}x^{3/2}} Ai(x) = \frac{\pi^{1/2}}{2x^{1/4}} \left(1 - \frac{15}{144x^{3/2}} + \dots \right) \tag{B.5}$$

which gives us the asymptotic behaviour of the Airy function with positive argument. It is worth noting that this method is really useful to evaluate the Airy function in the $x > 0$ range, and it gives an increasingly better approximation for more terms considered in the expansion (B.4) an for larger values of x, but it can't be used as an approximation for the whole range Airy function, since it also diverges in $x = 0$, so its use is only for a limited set of cases [18].

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