Alma Mater Studiorum · University of Bologna

School of Science
Department of Physics and Astronomy
Master Degree in Physics

DIAGRAMMATIC ENCODING OF SINE-GORDON MODEL TBA EQUATIONS

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Academic Year 2022/2023

Abstract

In this work it is presented a study on the sine-Gordon model thermodynamic Bethe Ansatz (TBA) equations, with particular emphasis on its underlying mathematical structures and diagrammatic encoding.

After an outline of sine-Gordon model foundations and main features, it is proposed an in-depth review of the TBA system derivation up to its 'raw' formulation. Some Fourier-space identities are introduced in order to simplify the equations and bring them in their 'universal' form. This is done for all the rational values of the sine-Gordon parameter. The 'universal' TBA equations enjoy a diagrammatic representation that casts light on the inherent mathematical structures of the theory. The Y-system formulation is also considered at reflectionless points: possible generalizations are discussed.

Contents

Introduction

Mathematical structures are at the very root of physics. Math does not only represent the optimal language in which physics is written, but allows to unveil deeper meanings and further significance. Arguably, all modern physics is testimony to that and, in particular, statistical field theory. In this theoretical framework, statistical mechanics and quantum field theory are perceived as indissolubly bounded one to another in a deep mathematical way.

An instance of this is represented by the thermodynamic Bethe Ansatz (TBA): a method able to gift a thermodynamic description to quantum field theory. From its first formulation in [1], through its early developments and up to very recent days it has yet not ceased to find interesting applications and to stimulate new discussions. The topic, in fact, is far from being closed.

Consider, as an example, the sine-Gordon model. This is one of the more widely researched models for both its applications and theoretical features. Many of its aspects are known: this is also due to its property of being an integrable theory, showcasing an infinite set of mutually commuting charges. Regarding its thermodynamic description, the TBA method can be applied to relate its finite-temperature thermodynamics to its finite-volume ground state energy. This relies on a set of highly non-trivial equations known simply as TBA equations. A system of non-linear integral equations (NLIE) [2] stemming from light-cone lattice approaches can also be formulated, confirming the results arising in the TBA method. The crucial point is that the TBA system is known only for some special values of the model's coupling constant. In [3] it is conjectured a general form (Y-system) for all the values of the sine-Gordon parameter, but no formal proof is provided. A step forward is represented by the recent works [4][5], where the 'universal' form of the TBA system is discussed and a further generalized hydrodynamic description is provided. However, the question of whether the two lastly-mentioned formulations are compatible is still open.

The key to this problem might reside in the rich and fascinating mathematical structure of these equations. In fact, their derivation relies on a set of identities inherently satisfied by the scattering amplitudes of the theory. These are able to link the sine-Gordon model to a pure mathematical description in terms of much abstract algebras. When brought to the surface, it allows to test the intimate structures that lie within the model, possibly

allowing for further insights.
Also, similar structures seem to appear in other theories. For instance, very recently [6], a generalization to higher-spin models is proposed where the familiar mathematical architecture of the sine-Gordon model appears. (On reverse, the sine-Gordon model may be deemed as a particular example of this wider class of theories).

The aim of this thesis work is that of providing a solid background to this so present discussion. The main features of TBA systems are presented, with a particular attention to the sine-Gordon model (at the center of recent developments). However, the description is kept as general as possible, allowing for wider applications. Also some contributions are added to the up-to-date discussion, in order to complete the general framework.

The presentation is organized as follows:

- Chapter §1 opens on a presentation of the main features of the 2-dimensional sine-Gordon model. Starting from its classical formulation, the famous soliton, antisoliton and breather solutions are seen to arise from the classical field equations of motion. When undergoing quantization, these build up the massive particle content of the theory. Crucial properties may be understood by means of the equivalence with the massive Thirring model: among others, the attractive/repulsive regimes of the theory and the $O(2)$ symmetry of the model. Capitalizing on the results obtained, a review of scattering theory is subsequently presented, up to the derivation of the sine-Gordon S-matrix.

- Chapter §2 revolves around the general formulation of the TBA system. The celebrated Zamolodchikov mirror argument is presented, followed by an in-depth analysis of Bethe-Yang equations, both for the coordinate and algebraic Bethe Ansatz. The thermodynamic limit of the equations is then discussed, up to the so-called 'raw' TBA equations.

- Chapter §3 presents the 'universal' form of TBA equations, where their mathematical structure is brought to the surface. They are obtained for all the values of the coupling through sets of pivotal identities, adapted here to the sine-Gordon model.

- Chapter §4 closes the discussion presenting the Y-system formulation for the sine-Gordon model at 'reflectionless points'. Some general features of the system are swiftly named and the derivation is subsequently generalized to 'integer points'. Further outlooks are quickly mentioned.

Chapter 1

Foundations of the sine-Gordon model

This opening chapter is dedicated to presenting an essential framework for the 2-dimensional sine-Gordon model, on which this master thesis mainly focuses.

Widely acknowledged as one of the most studied and researched models, it finds applications ranging from partial differential equation theory to condensed matter and particle physics [7][8][9][10][11][12]. Attempting to summarize its successes in few words proves challenging, therefore only a basic outline of its fundamental features is provided here.

The aim is to sketch the foundations of this model's theoretical framework: subsequent sections delve into further developments, as this entire work showcases the 2-dimensional sine-Gordon model as a paradigmatic example.

1.1 Classical sine-Gordon model

Before delving into the rich quantum formulation of the theory, a concise analysis of its classical features may offer some valuable insights. While a thorough development is beyond the scope of this section, the few characteristics discussed here aim to provide a solid foundation for the subsequent presentation.

1.1.1 Lagrangian formulation

The 2-dimensional sine-Gordon model is a theory that involves a single real scalar field $\phi(x,t)$ in $(1+1)$-dimensions, described by the minkowskian lagrangian density

$$\mathscr{L}^{cl}_{sG}[\phi,\partial_\nu\phi] = \frac{1}{2}(\partial^\nu\phi)(\partial_\nu\phi) - \mathscr{U}[\phi] = \frac{1}{2}(\partial^\nu\phi)(\partial_\nu\phi) + \frac{\mu^2}{\beta^2}(\cos(\beta\phi)-1)\,, \tag{1.1}$$

where the parameters μ and β are both real.

Then, the lagrangian field equation of motion takes the form

$$\partial^\nu \partial_\nu \phi + \frac{\mu^2}{\beta} \sin(\beta\phi) = 0\,, \tag{1.2}$$

whence the name of the model. Key observation is that this field equation is non-linear, thus new solutions can not be constructed simply by linear combination.

Both the above enjoy the discrete symmetries

$$\phi \to -\phi \tag{1.3}$$

and

$$\phi \to \phi + \frac{2\pi k}{\beta}, \quad k \in \mathbb{Z}\,. \tag{1.4}$$

Consistently with Eq.(1.4), the potential $\mathscr{U}[\phi]$ shows an infinite series of degenerate minima $\phi(x,t) = \frac{2\pi k}{\beta}$, $k \in \mathbb{Z}$.

An expansion about the configuration of minimum energy $\phi(x,t) = 0$ can give a rough physical interpretation of the parameters' meaning. Since $\mathscr{U}[\phi] = \frac{1}{2}\mu^2\phi^2 - \frac{\mu^2\beta^2}{4!}\phi^4 + \ldots$, μ may be seen as the inverse wavelength (mass) associated with the spectrum of small oscillations about the minimum (ϕ particle excitations), while β regulates the interactions between them.

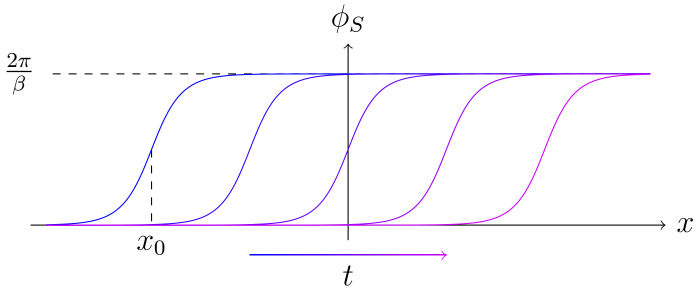
More significantly, given the structure of the potential, all finite-energy field configurations $\phi(x,t)$ can be divided into an infinite number of topological sectors [13, §2.5][14, §16.3]. Each of them is specified by a conserved pair of integer numbers (k_1, k_2), labeling the asymptotic values of the field at spatial infinity $\phi(-\infty, t) \overset{def}{=} \frac{2\pi k_1}{\beta}$, $\phi(+\infty, t) \overset{def}{=} \frac{2\pi k_2}{\beta}$. Then, a topological charge can be defined:

$$Q = k_2 - k_1 = \frac{\beta}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}x\, \frac{\partial \phi}{\partial x}\,. \tag{1.5}$$

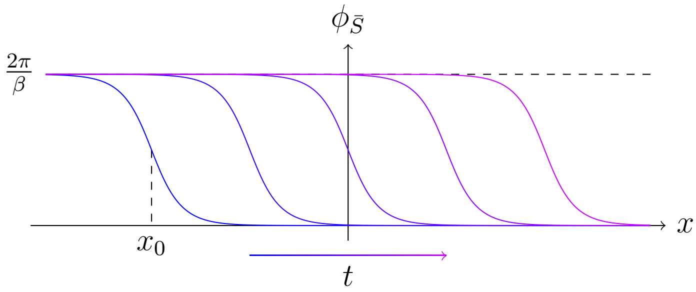
Q is left invariant by transformations of the kind in Eq.(1.4) and can be used to characterize these topological excitations $\phi(x,t)$.

1.1.2 Classical solitons

An expression for the most fundamental topological excitations, those carrying charge $Q = \pm 1$ (thus interpolating between two adjacent minima), can be derived directly from the field equation, Eq.(1.2). By initially looking for static solutions, a mechanical analogy can be constructed [13, §2.3], so that the desired expression can be obtained integrating



(a)



(b)

Figure 1.1: Sketch of the sine-Gordon (a) soliton (Eq.(1.6a)) and (b) antisoliton (Eq.(1.6b)) solutions. The field periodicity of $\frac{2\pi}{\beta}$ is here subtended. The velocity u is chosen to be positive, so that the curves drawn for increasing time values appear to be 'right-moving'.

by quadrature. Then, it is sufficient to apply a Lorentz boost to get the traveling (with velocity u and without dispersion) solutions

$$\phi_S(x,t) = \frac{4}{\beta}\arctan\left(\exp\left(\mu\,\frac{x-x_0-ut}{\sqrt{1-u^2}}\right)\right) \tag{1.6a}$$

$$\phi_{\bar{S}}(x,t) = -\frac{4}{\beta}\arctan\left(\exp\left(\mu\,\frac{x-x_0-ut}{\sqrt{1-u^2}}\right)\right), \tag{1.6b}$$

known respectively as soliton (Eq.(1.6a), $Q=+1$) and antisoliton (Eq.(1.6b), $Q=-1$). Their mathematical behaviors are represented in Fig.(1.1). These finite-energy field configurations show two pivotal features:

- at any time instant, their energy density is localised, allowing for a particle interpretation with rest mass (the same for the soliton and the antisoliton since the two solutions are related by the symmetry in Eq.(1.3))

$$M^{cl} = \frac{8\mu}{\beta^2}\,. \tag{1.7}$$

- in a scattering process, these solutions behave in a 'transparent' way. More precisely, there exist exact solutions of Eq.(1.2) like $\phi_{S\bar{S}}(x,t)$ (and similarly for $\phi_{SS}(x,t)$ and $\phi_{\bar{S}\bar{S}}(x,t)$) that behave as

$$\phi_{S\bar{S}}(x,t) \underset{t\to-\infty}{\sim} \phi_S\left(\mu\,\frac{x-x_0+u(t+\frac{\Delta}{2})}{\sqrt{1-u^2}}\right)+\phi_{\bar{S}}\left(\mu\,\frac{x-x_0-u(t+\frac{\Delta}{2})}{\sqrt{1-u^2}}\right) \tag{1.8a}$$

$$\phi_{S\bar{S}}(x,t) \underset{t\to+\infty}{\sim} \phi_S\left(\mu\,\frac{x-x_0+u(t-\frac{\Delta}{2})}{\sqrt{1-u^2}}\right)+\phi_{\bar{S}}\left(\mu\,\frac{x-x_0-u(t-\frac{\Delta}{2})}{\sqrt{1-u^2}}\right), \tag{1.8b}$$

  thus corresponding to a soliton-antisoliton pair with same shape and velocities in initial and final configuration: the only residual effect of the collision is a time delay Δ.

Other exact solutions of Eq.(1.2) exist. A fundamental one is the so-called doublet or breather, carrying $Q=0$ and of the form

$$\phi_{B(v)}(x,t)=\frac{4}{\beta}\arctan\left(\frac{\sin\left(\mu\frac{vt}{\sqrt{1+v^2}}\right)}{v\cosh\left(\mu\frac{x}{\sqrt{1+v^2}}\right)}\right). \tag{1.9}$$

The breather solution is periodic in time: its mathematical behavior is represented for half a period in Fig.(1.2). By a transformation of the parameter $v\to -iu$, the breather is mapped into the aforementioned $\phi_{S\bar{S}}(x,t)$ solution. This indicates that the breather may be interpreted as a bound solution of a soliton-antisoliton pair, oscillating with respect to one another periodically in time.

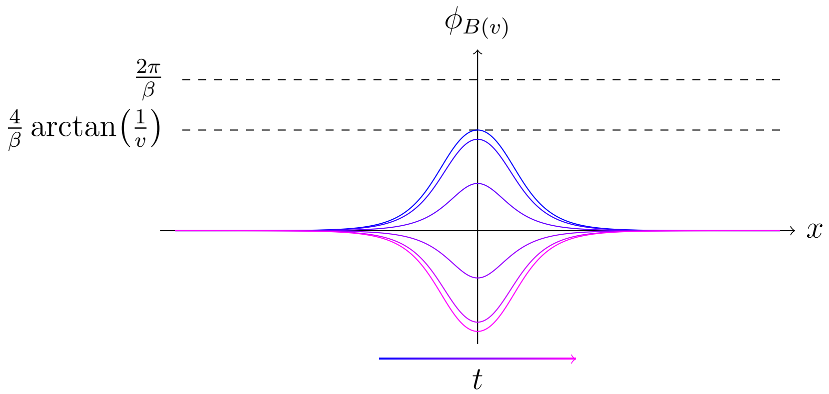


Figure 1.2: Sketch of the sine-Gordon breather solution: different curves are drawn for increasing time values for half a period. For a better legibility, the field periodicity of $\frac{2\pi}{\beta}$ is here subtended again. Already at a graphical level, comparing with Fig(1.1), it may be possible to guess the composite nature of this solution.

Similarly, more complicated multi-solitons solutions may be generated out of simpler ones applying Bäcklund transformations [15]. Through the inverse scattering method, it is possible to prove that the solutions both show a localised energy density and behave transparently in scattering processes.

All the solutions of the classical sine-Gordon equation are, thus, known. They carry an infinite number of conserved quantities: the sine-Gordon model is classically integrable [16][17][18].

1.2 Quantum sine-Gordon model

The classical relativistic field theory discussed earlier (§1.1) can undergo quantization, albeit in a highly nontrivial way. Both semiclassical and exact results exist, that are able to bridge the classical and quantum formulation of the sine-Gordon model. Of great significance is understanding how the classical features persist through quantization. This section will emphasize the most noteworthy accomplishments.

1.2.1 Lagrangian formulation

One way of defining a quantum field theory (QFT) is through local fields (the other way being through scattering theory of asymptotic particles (§1.2.4)), i.e. by explicitly writing a lagrangian. This is feasible for the 2-dimensional sine-Gordon model, whose lagrangian density takes the form

$$\mathscr{L}_{sG} = \frac{1}{2}(\partial^\nu \Phi)(\partial_\nu \Phi) + \frac{\mu^2}{\beta^2} :\cos(\beta\Phi): \ . \tag{1.10}$$

The classical lagrangian formulation (§1.1.1) aids in clarifying the notations adopted (the colons denoting normal ordering, equivalent to renormalization of the theory).

By studying the theory ground state through the Rayleigh-Ritz variational method, it has been shown [19] that its energy becomes unbounded from below when $\beta^2 \geq 8\pi$. The sine-Gordon model is, thus, sensible only for $0 < \beta^2 < 8\pi$.

For later convenience, a new parameter can be introduced

$$p = \frac{\beta^2}{8\pi - \beta^2} \, , \tag{1.11}$$

which, consistently with previous observations, is defined in the range $0 < p < +\infty$.

The lagrangian in Eq.(1.10) still enjoys the discrete symmetries in Eqs.(1.3)(1.4) and shows infinitely many degenerate minima, giving rise to spontaneous symmetry breaking: a vacuum sector of states can be defined around any of these minima.

The topological charge in Eq.(1.5) turns into a (superselection) quantum number for the particle states of the theory

$$Q = \frac{\beta}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{\partial \Phi}{\partial x} \, . \tag{1.12}$$

States with different topological quantum numbers are orthogonal and can not evolve into one another, for Q is a conserved hermitian operator.

This model may be seen as a perturbation of a 2-dimensional free massless scalar boson, $c = 1$ conformal field theory (CFT). When compactified on a cylindrical spacetime (the reasons behind this compactification become evident in later sections (§§2.1,2.2))

$$(x, t) \in [0, L] \times \mathbb{R} \, , \tag{1.13}$$

the lagrangian of the latter can be written as

$$\mathcal{L}_{CFT} = \frac{1}{8\pi} \int_0^L \mathrm{d}x \, (\partial^\nu \Phi)(\partial_\nu \Phi) \, . \tag{1.14}$$

Quasiperiodic boundary conditions can be imposed in the form $\Phi(x + L, t) = \Phi(x, t) + 2\pi m \mathsf{R}$, where the field winds m times while circling once around the cylinder [20, §6.3.5]. This also induces a quantization of the field conjugate momentum in integer (n) multiples of $\frac{1}{\mathsf{R}}$. It is then possible to define vertex operators $\mathcal{V}_{(n,m)}$, which are Kac-Moody primary fields of left and right conformal dimensions $\Delta = \frac{1}{2}\big(\frac{n}{\mathsf{R}} + \frac{1}{2} m \mathsf{R}\big)^2$ and $\bar{\Delta} = \frac{1}{2}\big(\frac{n}{\mathsf{R}} - \frac{1}{2} m \mathsf{R}\big)^2$, so that the perturbed lagrangian

$$\mathcal{L} = \mathcal{L}_{CFT} + \lambda \int_0^L \mathrm{d}x \, (\mathcal{V}_{(1,0)} + \mathcal{V}_{(-1,0)}), \tag{1.15}$$

through the mapping

$$\lambda \to \frac{\mu^2}{2\beta^2} \qquad \mathsf{R} \to \frac{\sqrt{4\pi}}{\beta} \qquad \Phi \to \sqrt{4\pi} \Phi \, , \tag{1.16}$$

coincides with the sine-Gordon lagrangian

$$\mathcal{L}_{sG} = \int_0^L \mathrm{d}x \left(\frac{1}{2} (\partial^\nu \Phi)(\partial_\nu \Phi) + \frac{\mu^2}{\beta^2} :\!\cos(\beta \Phi)\!: \right) \tag{1.17}$$

[21, §1.3].

Of paramount importance is the fact that this CFT deformation proves to be integrable, i.e. it admits an infinite number of mutually commuting local conserved charges. Consequently, the classical integrability of the sine-Gordon model (§1.1.2) persists as quantum integrability after quantization [22].

1.2.2 Quantum solitons

The mass spectrum of the 2-dimensional sine-Gordon model has been found by semi-classical methods [23][24].

Starting from the classical (anti)soliton solution (Eq.(1.6)), it may be observed that, in its rest frame, it is a static solution, thus a minimum of the potential functional. The essential idea [13, §§5,7.1.1] is to realize quantization through perturbation theory in a weak-coupling (i.e. small quantum fluctuations) approximation: to leading order, this amounts to building a tower of harmonic oscillator states around the potential functional local minimum. An (anti)soliton sector of topological quantum number $Q = (-)1$ is constructed in this way and the quantum corrections to the classical particle mass (Eq.(1.7)) read

$$M = M^{cl} - \frac{\mu}{\pi} + \mathcal{O}(\beta^2)\,. \tag{1.18}$$

The quantization of the classical breather solutions (Eq.(1.9)) requires a more refined treatment due to their periodicity in time. The core idea is to repeatedly apply a stationary phase approximation to the propagator path integral around the family of classical periodic orbits: this is at the basis of the WKB method [13, §§6,7.1.2]. The a^{th} quantum breather is, thus, found to be a particle carrying topological quantum number $Q = 0$ of mass

$$M_a = 2M \sin\left(\frac{\pi}{2} pa\right) \qquad a = 1, \ldots, N_B\,, \tag{1.19}$$

where M denotes the (anti)soliton mass (Eq.(1.18)), p the sine-Gordon parameter defined in Eq.(1.11) and $\lfloor x \rfloor$ the integer part of x. For a given value of p, there can be at most $N_B = \lfloor \frac{1}{p} \rfloor$ breather species ($N_B = \frac{1}{p} - 1$ when $p \in \mathbb{N}$): this condition stems from existence requirements of the semiclassical period. Eq.(1.19) proves to be exact.

Perturbation theory in a weak-coupling approximation reveals that the 'elementary' Φ boson of mass μ (also of topological quantum number $Q = 0$) can be identified with the lowest breather. More in general, a loosely bound state of a such particles corresponds to the a^{th} breather [13, §7.2].

Still, breathers can be considered as bound states of a soliton and an antisoliton, similarly to what seen in the classical formulation of the theory. The subsequent sections (§§1.2.3,1.2.4) will provide a more solid basis for this interpretation.

1.2.3 Massive Thirring model equivalence

The 2-dimensional massive Thirring model is a theory describing a current-current self-interaction of a massive Dirac fermion in $(1 + 1)$ dimensions. Its dynamics is determined by the lagrangian density

$$\mathscr{L}_{mT} = \bar{\Psi}(i\gamma^\nu\partial_\nu - \mu_f)\Psi - \frac{1}{2}g(\bar{\Psi}\gamma^\nu\Psi)(\bar{\Psi}\gamma_\nu\Psi)\,, \tag{1.20}$$

where γ^ν denotes the Dirac matrices, m_f the (renormalized) fermionic mass parameter, $j^\nu = \bar{\Psi}\gamma^\nu\Psi$ the fermionic currents (obeying Ward identities) and g the interaction parameter.

The massless ($\mu_f = 0$) Thirring model is well known to be exactly solvable [25], thus allowing Eq.(1.20) to be treated perturbatively in μ_f. The same can be done for Eq.(1.10) w.r.t. μ^2 (as the sine-Gordon model can be seen as a deformation of an integrable $c = 1$ CFT (§1.2.1)) and the two perturbative series can be compared. The correlation functions of the perturbing operator $\bar{\Psi}\Psi$ remarkably prove to coincide with those of $:\cos(\beta\Phi):$ at any order, once the following identifications are made between the two theories

$$\frac{4\pi}{\beta^2} = 1 + \frac{g}{\pi}\,, \quad \text{i.e.}\ \frac{\pi}{2}\left(\frac{1}{p} - 1\right) = g \tag{1.21a}$$

$$-\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\Phi = j^\mu \tag{1.21b}$$

$$\frac{\mu^2}{\beta^2}\,:\cos(\beta\Phi): = -\mu_f\bar{\Psi}\Psi \tag{1.21c}$$

[19]. Thus, the sine-Gordon model is found to be equivalent to the massive Thirring model, more precisely to its charge-zero sector, since $\bar{\Psi}\Psi$ has fermionic charge zero.

The fermionic charge is defined in general as

$$Q = \int_{-\infty}^{+\infty} \mathrm{d}x\, j^0 = \int_{-\infty}^{+\infty} \mathrm{d}x\, \bar{\Psi}\gamma^0\Psi\,. \tag{1.22}$$

Through Eq.(1.21b), it can be seen to coincide with the quantum topological charge of Eq.(1.12), further sustaining an identification between the massive Thirring (anti)fermion ($Q = (-)1$) and the quantum sine-Gordon (anti)soliton ($Q = (-)1$). Indeed, without the use of perturbation theory, it is possible to explicitly define sine-Gordon soliton creation and annihilation operators, which are proven to satisfy the anticommutation relations and the field equations of the massive Thirring model [26].

This remarkable equivalence may be exploited to deepen the understanding of the sine-Gordon model.
By extracting the sign of the static limit force from the lagrangian in Eq.(1.20), it can be tested the attractive/repulsive nature of the fermion-antifermion interaction. Correspondingly, by Eq.(1.21a), it is found that (Fig.(1.3))

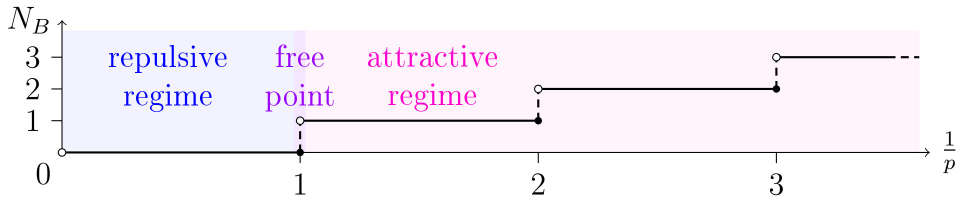


Figure 1.3: Sine-Gordon regimes and number of breather species N_B as a function of $\frac{1}{p}$.

- if $0 < p < 1$ ($g > 0$), then the soliton-antisoliton interaction is attractive, i.e. bound states can be formed. This is in accordance to the semiclassical result in Eq.(1.19), for which there exist $N_B = \lfloor \frac{1}{p} \rfloor$ breather species ($N_B = \frac{1}{p} - 1$ when $p \in \frac{1}{\mathbb{N}}$). As $p \to 0^+$ ($g \to +\infty$), the attraction intensifies resulting in a high number of bound states, the values $p = \frac{1}{n}$, $n \in \mathbb{N}$ being the thresholds where a new bound states appears; as $p \to 1^-$ ($g \to 0^+$), the force intensity decreases, along with the number of breather species, until eventually only one is left.

- if $p = 1$ ($g = 0$), then no interaction is present, i.e. it's a free point (the massive Thirring lagrangian (Eq.1.20) becomes a free fermion lagrangian). Correspondingly, the breather that last remains in the limit $p \to 1^-$ ($g \to 0^+$), at this value of the sine-Gordon parameter, shows a mass $M_1 = 2M$ (Eq.(1.19)): this can be interpreted as a free soliton-antisoliton pair, thus not as a breather bound state.

- if $1 < p < +\infty$ ($g < 0$), then the soliton-antisoliton interaction is repulsive, i.e. there can be no bound states. As a consequence, in this repulsive regime the sine-Gordon spectrum is composed only by the soliton and the antisoliton.

As last notice on the sine-Gordon $\equiv$ massive Thirring equivalence, it may be argued that, through the latter, a hidden $O(2)$ invariance of the former is revealed. Though it can be explicitly shown by recasting Eq.(1.10) by making use of the disorder operator [27], some insights may already be obtained exploiting this remarkable equivalence. Starting from the lagrangian in Eq.(1.20), it enjoys the $U(1)$ symmetry

$$\Psi \to \exp(i\alpha)\Psi\,. \tag{1.23}$$

This is basically $O(2)$ symmetry, where the combinations $\frac{1}{2}(\Psi + \Psi^*)$ and $\frac{1}{2i}(\Psi - \Psi^*)$ act as an $O(2)$ doublet. By means of the sine-Gordon $\equiv$ massive Thirring equivalence, if A and $\bar{A}$ stand for the soliton and the antisoliton respectively, then $A_1 \overset{def}{=} \frac{1}{2}(A + \bar{A})$ and $A_2 \overset{def}{=} \frac{1}{2i}(A - \bar{A})$ form a doublet under the sine-Gordon $O(2)$ hidden symmetry

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \to \mathcal{R}(\alpha) \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad \text{with}\, \mathcal{R}(\alpha) \in O(2)\,. \tag{1.24}$$

This observation is crucial for defining a S-matrix theory for the sine-Gordon model (§1.2.4).

1.2.4 Scattering theory

In principle, a QFT can be completely defined through its S-matrix, i.e. through the description of the scattering of its particle content: this approach is indeed followed for QFTs lacking a lagrangian formulation. Even when the latter is known, the connection between the two formulations is far from trivial, with integrability playing a pivotal role.

The 2-dimensional sine-Gordon model exhibits both the features of allowing a lagrangian formulation and enjoying quantum integrability (§1.2.1). Thus, the aim is to outline the derivation of the S-matrix for this theory, building upon the previously presented properties.

S-matrix, an introduction

For any given massive theory with a short range interaction it is possible to define asymptotic states, where the particles are considered distant enough to neglect their interaction. As a consequence, the particles appearing in such states can not be virtual particles and are considered on-shell, i.e. with momentum satisfying

$$p_i^{\mu} p_{i\,\mu} = m_i^2 \,, \tag{1.25}$$

where m_i denotes the asymptotic mass of the i^{th} particle. In spacetime dimension 2, this allows for the convenient parametrization in terms of the rapidity ϑ

$$p_i^0 = m_i \cosh\vartheta \quad p_i^1 = m_i \sinh\vartheta \,. \tag{1.26}$$

The most fundamental asymptotic state is the vacuum $|0\rangle$.

Then, single-particle states $|A(\vartheta)\rangle$, specified by both ϑ and a set containing the mass and all the internal quantum numbers A, will be constructed out of the vacuum by the action of some vertex operators $Z_A(\vartheta)$ (generalizations of a creation/destruction operators) known as Faddeev-Zamolodchikov operators

$$|A(\vartheta)\rangle = Z_A^{\dagger}(\vartheta)\,|0\rangle \ . \tag{1.27}$$

By Eq.(1.26), these states form an irreducible representation of the $(1+1)$-dimensional Poincaré group with casimir m^2.

Similarly, multi-particle asymptotic states can be defined in the form

$$|A_1(\vartheta_{A_1})\dots A_N(\vartheta_{A_N})\rangle = Z_{A_1}^{\dagger}(\vartheta_{A_1})\dots Z_{A_N}^{\dagger}(\vartheta_{A_N})\,|0\rangle \ . \tag{1.28}$$

Let's suppose to have two asymptotic states: one ($|i\rangle$) at $t = -\infty$ describing the initial state of the system and one ($|f\rangle$) at $t = +\infty$ describing the final state of the

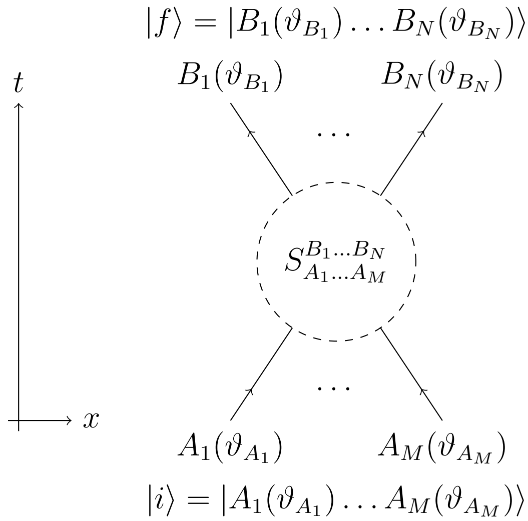


Figure 1.4: Depiction of a $M \to N$ scattering process in 2 spacetime dimensions. Each oriented line may be thought of as a particle world line, even if the aim of the figure is that of presenting a schematic representation, rather than an accurate one.

system, after the interaction has taken place. Then, by the superposition principle of quantum mechanics, it exists a linear operator (time evolution operator) that maps the initial state into the final state:

$$|f\rangle = S^{-1} |i\rangle \ . \tag{1.29}$$

If the initial state describes M particles ($|i\rangle = |A_1(\vartheta_{A_1}) \dots A_M(\vartheta_{A_M})\rangle$) and the final state describes N particles ($|f\rangle = |B_1(\vartheta_{B_1}) \dots B_N(\vartheta_{B_N})\rangle$), then the S-matrix is defined with entries

$$|A_1(\vartheta_{A_1}) \dots A_M(\vartheta_{A_M})\rangle = S_{A_1 \dots A_M}^{B_1 \dots B_N} |B_1(\vartheta_{B_1}) \dots B_N(\vartheta_{B_N})\rangle \ , \tag{1.30}$$

which are related to the probability of the scattering event. A pictorial representation is shown in Fig.(1.4). For the scattering to be nontrivial, the rapidity orders $\vartheta_{A_1} \geq \vartheta_{A_2} \geq \dots \geq \vartheta_{A_M}$ and $\vartheta_{B_1} \leq \vartheta_{B_2} \leq \dots \leq \vartheta_{B_N}$ are subtended.

In order to obtain a sensible theory, the S-matrix must be constrained by some general requirements.

Essential principles such as probability conservation and causality turn into unitarity, analyticity and crossing invariance conditions on the scattering operator [28][29].

Also, for the theory to be relativistic, Lorentz invariance should be imposed, potentially alongside other internal symmetries. The way in which these symmetries act on any massive QFT is regulated by the Coleman-Mandula theorem [30], which imposes severe constraints on the S-matrix. As a particular case, it is found that, in $(1 + d)$ dimensions with $d > 1$, the existence of one conserved charge of (Lorentz) tensor rank larger than 2 implies a trivial S-matrix, i.e. $S = \mathbb{I}$[1]. Very roughly, the reason is that in such spacetimes the symmetries allow to translate differently particles of different momenta, establishing the equivalence between a scattering and an event where the particles move freely without crossing their trajectories. However, this does not prevent theories in $(1+1)$ dimensions to present a more interesting dynamics, which is the reason why the 2-dimensional formulation of the sine-Gordon model is considered in this work.

Nevertheless, the integrability of the theory can further condition the form of the S-matrix in a drastic way. One of the most noteworthy results in this regard is the Parke theorem [32], which may be summarized as follows:

Theorem 1.1 (Parke). *Given a massive local QFT (i.e. a quantum relativistic scattering theory of massive particles) in $(1+1)$ dimensions, if it admits two conserved charges Q_+ and Q_- such that*

1. *do not transform under the Lorentz group as scalars nor vectors, but as higher-rank tensors: $Q_\pm \to Q'_\pm = \Lambda^{\pm q_\pm} Q_\pm$, with $\Lambda \in O(1+3)$, q_+ and q_- odd and $q_+ \geq q_- > 1$ (in parity invariant theories for any Q_+ it exists the parity conjugated Q_- with $q_+ = q_- > 1$, so that only one charge is sufficient)*

2. *can be written as integrals of the time component of local conserved current densities, i.e. $Q_\pm = \int_{-\infty}^{+\infty} \mathrm{d}x \, j^0_\pm$, with $\partial_\mu j^\mu_\pm = 0$*

3. *mutually commute: $[Q_+, Q_-] = 0$*

4. *do not annihilate any nontrivial linear combination of particles in a multiplet*

then, in any scattering event,

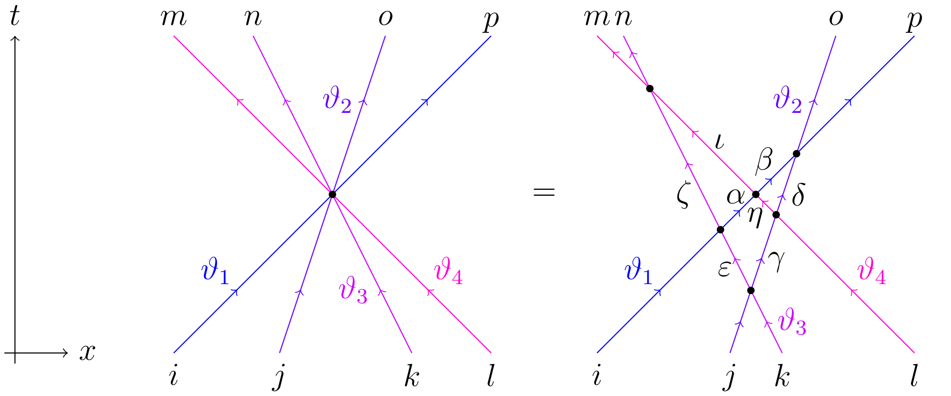
1. *there can be no particle production, i.e. the number of particles of given mass m is conserved, being the same in the initial and the final asymptotic state: particles can at most exchange quantum numbers inside a multiplet*

2. *the set of final momenta of the particles coincides with that of initial momenta, namely the scattering process is elastic*

3. *the S-matrix factorizes, i.e., if it describes a $N \to N$ scattering process, then it can be obtained out of products of $\frac{1}{2}N(N-1)$ S-matrices describing $2 \to 2$ scattering processes*

---

[1] Still, there exist cases for which the hypotheses of the Coleman-Mandula theorem do not apply, thus allowing for nontrivial dynamics in spatial dimensions higher than 1. One notable example is represented by supersymmetric theories, for which a superalgebras generalization, the Haag-Łopuszański-Sohnius theorem [31], holds.



$$S^{mnop}_{ijkl}(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4) =$$

$$= \sum_{\alpha\beta\gamma\delta\varepsilon\zeta\eta\iota} S^{mn}_{\zeta\iota}(\vartheta_3,\vartheta_4) S^{op}_{\beta\delta}(\vartheta_1,\vartheta_2) S^{\iota\beta}_{\alpha\eta}(\vartheta_1,\vartheta_4) S^{\eta\delta}_{\gamma l}(\vartheta_2,\vartheta_4) S^{\zeta\alpha}_{i\varepsilon}(\vartheta_1,\vartheta_3) S^{\varepsilon\gamma}_{jk}(\vartheta_2,\vartheta_3)$$

Figure 1.5: Pictorial representation of an elastic $4 \to 4$ S-matrix possible factorization. As per results *1.* and *2.* of Th.(1.1), there is the same number of particles in the initial and final state and the set of rapidities $\{\vartheta_i\}_{i=1}^4$ is conserved in the scattering (a color scheme is used to highlight this point). The letters in black are indices that take values in the set of labels identifying the particles of the theory: indeed the scattering is in general nontrivial since particles of the same mass can exchange rapidities or be replaced by other particles of the same mass (i.e. the indices may be different along the same line). Finally, it is explicitly written the decomposition of the $4 \to 4$ S-matrix as a sum of products of six $2 \to 2$ S-matrices, as allowed by the Parke theorem. It may be noticed that this is not the sole possibility. For instance, without altering the asymptotic states rapidity order, the line ϑ_2 in the diagram on the right may be shifted left until it crosses the $S^{\eta\beta}_{\alpha\iota}$ scattering point. Then the right-hand side of the decomposition would look like $S^{mn}_{\zeta\iota} S^{\iota o}_{\alpha\eta} S^{\eta p}_{\delta l} S^{\delta\beta}_{\alpha\gamma} S^{\zeta\alpha}_{i\varepsilon} S^{\varepsilon\gamma}_{jk}$ (summing on the repeated greek indices and omitting rapidity dependency for simplicity).

For an integrable theory like the 2-dimensional sine-Gordon model, that enjoys both the elasticity (reminiscent of the 'transparency' of scattering at classical level (§1.1.2)) and the factorizability (for an example, Fig.(1.5)) granted by the Parke theorem, it is sufficient to study its $2 \to 2$ S-matrices in order to reconstruct the full scattering theory.

It may be noticed that the factorizability result of Th.(1.1) does not come with any prescription for ordering the products. Indeed, the final scattering matrix must be independent of this choice, yielding as consistency condition the celebrated cubic identity known as Yang-Baxter equation [33]. This equation appears also in the description of other systems, such as integrable 2-dimensional lattice models [34], allowing for powerful insights. Plus, it is known that its solutions are proportional to the R-matrix of some quantum group [35][36][37][38].

Still another proportionality factor is left ambiguous by the conditions just briefly summarized. It is known as Castillejo-Dalitz-Dyson (CDD) factor [39] and it can be fixed only through dynamical requirements. In the sine-Gordon model case, it is trivial (the S-matrix is 'minimal'), but that's not true in general. They influence the dynamics of the theory (i.e. change its lagrangian formulation), without modifying its particle content. No simple poles of the $2 \to 2$ S-matrix are added in the 'physical strip' $0 < \mathfrak{Im}(\vartheta) < \pi$, which would correspond to bound states when on the imaginary ϑ-axis $\mathfrak{Re}(\vartheta) = 0$.

However, such bound states can be already present in the minimal part of the $2 \to 2$ S-matrix. In this case, the scattering amplitudes involving them should also be computed. This poses bound states (BS) on the same footing of asymptotic states (AS), generating further relations. Known as bootstrap equations, they allow to obtain the AS-BS and BS-BS amplitudes in terms of the AS-AS amplitudes (or viceversa).

It is convenient to encapsulate all the information about the general structure of the scattering theory in a special algebraic construction: the Faddeev-Zamolodchikov algebra [41, §2].

Thanks to elasticity and factorization, it may be seen that the $2 \to 2$ S-matrices enter the commutation relations of the Faddeev-Zamolodchikov operators, firstly introduced in Eq.(1.27). In fact, exchanging two of them would be equivalent to realize a scattering between the two corresponding particles.
To make this more intuitive, it is possible to introduce some slight changes of notation. Let the particles (i.e. the Faddeev-Zamolodchikov operators) be represented by special noncommutative symbols $A(\vartheta)$, $B(\vartheta)$, $C(\vartheta)$,..., with each uppercase letter standing for a different particle species and ϑ denoting the particle rapidity. The multi-particle asymptotic states of Eq.(1.28) can then be identified with products of such symbols, arranged in the order of their appearance along the spatial direction (i.e. of increasing/decreasing rapidities as per previous conventions). Rearranging the products by means of a number of subsequent commutations of neighboring particles corresponds to introducing $2 \to 2$ collisions, leading to commutation rules for these symbols. Let the scattering operator

be defined as in Eq.(1.29). Then, for instance, if particles A and B have different masses, exchanging them would yield

$$A(\vartheta_1)B(\vartheta_2) = S_T[AB](\vartheta_1,\vartheta_2)B(\vartheta_2)A(\vartheta_1)\,, \tag{1.31}$$

where $S_T[AB]$ denotes the transmission amplitude of $AB \to AB$. By Parke theorem (Th.1.1), the reflection amplitude $S_R[AB]$ would be zero in this case, thus it does not appear on the right-hand side of the commutation relation. For identical particles there's no distinction between transmission and reflection amplitudes, so it can be simply written

$$A(\vartheta_1)A(\vartheta_2) = S_0[AA](\vartheta_1,\vartheta_2)A(\vartheta_2)A(\vartheta_1)\,. \tag{1.32}$$

However, if particles A and B have the same mass, then

$$A(\vartheta_1)B(\vartheta_2) = S_T[AB](\vartheta_1,\vartheta_2)B(\vartheta_2)A(\vartheta_1) + S_R[AB](\vartheta_1,\vartheta_2)A(\vartheta_2)B(\vartheta_1)\,, \tag{1.33}$$

where, if there existed other particles of the same mass, corresponding terms should have been added.

Consistency relations for this algebra can be seen to yield conditions of the S-matrix, such as unitarity and the Yang-Baxter equation.

To conclude this introductory and far from exhaustive paragraph on S-matrices (for a full development in $(1+1)$ dimensions, [40]), it is important to notice that the scattering theory obtained through the imposition of the constraints above will be compatible with a possible lagrangian formulation of the theory (at least once the CDD ambiguities are fixed). However, in principle the equivalence may be not fully established: different lagrangians might show the same compatibility. The two descriptions have to be linked to the conformal theory valid in the vicinity of the critical point (UV limit) in order to assert their complete correspondence (CFT perturbation $\leftrightarrow$ factorized scattering theory). This is where TBA becomes essential (§2).

S-matrix, constraints

Prior to presenting the final form of the sine-Gordon S-matrix (as obtained in [41, §3]), it may be worthwhile to explicitly write the constraints that it must satisfy, also adding some visual aids (Fig.(1.6)) to the previous discussion.

Starting by Lorentz invariance, as Lorentz boosts shift rapidities by a constant, the $2 \to 2$ (elastic) S-matrix only depends on rapidities differences

$$S_{ij}^{kl}(\vartheta_1,\vartheta_2) = S_{ij}^{kl}(\vartheta_1-\vartheta_2)\,, \tag{1.34}$$

where the difference $\vartheta_1-\vartheta_2$ will often be denoted as ϑ_{12} or simply as ϑ when no other particular specification is required. The indices take values in the set of labels identifying

the particles of the theory.
The general requirements of C (charge conjugation), P (parity) and T (time reversal) invariance read respectively

$$S_{ij}^{kl}(\vartheta) = S_{\bar{i}\bar{j}}^{\bar{k}\bar{l}}(\vartheta) \quad S_{ij}^{kl}(\vartheta) = S_{ji}^{lk}(\vartheta) \quad S_{ij}^{kl}(\vartheta) = S_{lk}^{ji}(\vartheta)\,, \tag{1.35}$$

with $\bar{i}$ denoting the antiparticle state of i.

For what concerns the analytic structure, the $2 \to 2$ S-matrix is a meromorphic function of ϑ (polynomially bounded), the 'physical strip' being defined by the interval $0 \leq \mathfrak{Im}(\vartheta) \leq \pi$ (Fig.(1.6a)). It can possibly present isolated poles only on the imaginary ϑ-axis $\mathfrak{Re}(\vartheta) = 0$, where the real analyticity condition assures that the S-matrix takes real values

$$S_{ij}^{kl}(i\,\mathfrak{Im}(\vartheta)) \in \mathbb{R}\,. \tag{1.36}$$

Then, the unitarity of the scattering operator implies that

$$S_{ij}^{\alpha\beta}(\vartheta) S_{\alpha\beta}^{kl}(-\vartheta) = \delta_i^k \delta_j^l\,, \tag{1.37}$$

where a sum on the greek indices is subtended.

The crossing symmetry, instead, is obtained by requiring that amplitudes in different 'channels' are described by the same S-matrix. This yields

$$S_{ij}^{kl}(\vartheta) = S_{\bar{k}i}^{l\bar{j}}(i\pi - \vartheta)\,. \tag{1.38}$$

Of paramount importance is the Yang-Baxter equation (Fig.(1.6b)) that explicitly reads

$$S_{\beta\gamma}^{lm}(\vartheta_{23}) S_{\alpha k}^{\gamma n}(\vartheta_{13}) S_{ij}^{\beta\alpha}(\vartheta_{12}) = S_{\alpha\beta}^{mn}(\vartheta_{12}) S_{i\gamma}^{l\alpha}(\vartheta_{13}) S_{jk}^{\gamma\beta}(\vartheta_{23})\,. \tag{1.39}$$

As anticipated, this implies that the S-matrix is proportional to the R-matrix of some quantum group

$$S(\vartheta) = f(\vartheta) R(\vartheta)\,. \tag{1.40}$$

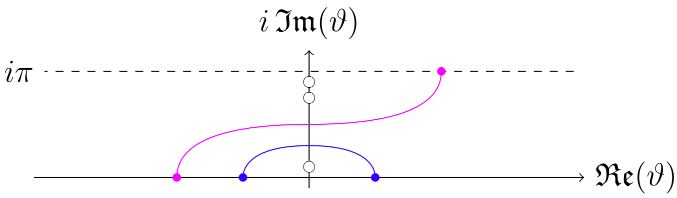
Some few words on R-matrices will be spent in subsequent sections (§2.3, App.A)

Delving more into the details of the sine-Gordon model, it has been noticed that this theory admits an $O(2)$ doublet, made of particles A_1 and A_2 which are combinations of the soliton and the antisoliton (Eq.(1.24)). Their S-matrix is then constrained by the (hidden) $O(2)$ symmetry, taking the form

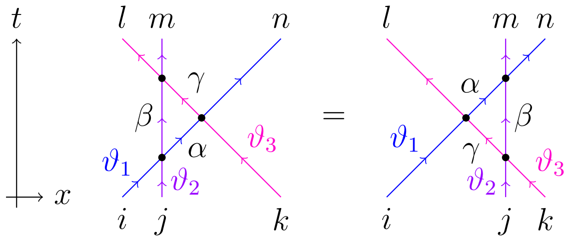
$$S_{ij}^{kl}(\vartheta) = \delta_{ij}\delta^{kl} S_1(\vartheta) + \delta_i^l \delta_j^k S_2(\vartheta) + \delta_i^k \delta_j^l S_3(\vartheta)\,, \tag{1.41}$$

where the functions $S_1(\vartheta)$, $S_2(\vartheta)$ and $S_3(\vartheta)$ represent respectively the processes of annihilation, transmission and reflection during the scattering of A_1 and A_2 (Fig.(1.6c)).

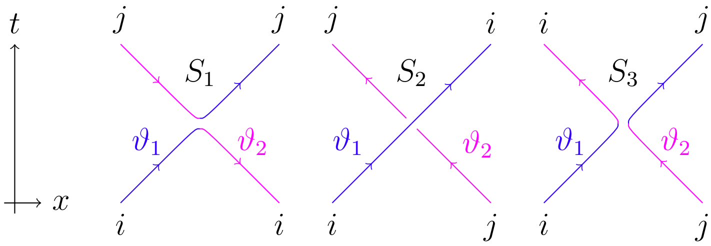
Collecting together all these constraints, the $2 \to 2$ S-matrix for the particles in the doublet can be written exactly (up to a multiplicative factor), thus yielding also



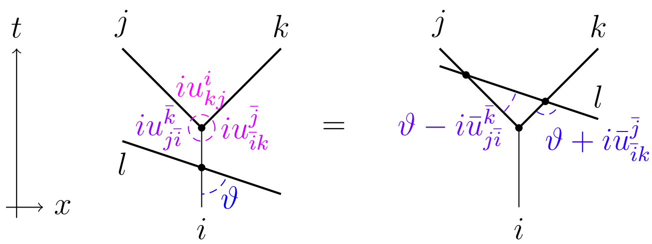
(a) Analyticity, unitarity, crossing symmetry. It is represented here the physical strip $0 \leq \mathfrak{Im}(\vartheta) \leq \pi$, where the S-matrix is meromorphic. Some bound states are depicted as poles on the imaginary axis. The points connected by the darker line are related by unitarity, while the ones connected by the lighter line are related by crossing symmetry.



(b) Yang-Baxter equation.



(c) O(2) symmetry. Scattering processes allowed by the (hidden) $O(2)$ symmetry of the sine-Gordon model.



(d) Bootstrap equation. The thicker lines represent asymptotic states particles, while the thinner line a bound state particle of theirs.

Figure 1.6: Pictorial representation of the sine-Gordon S-matrix constraints.

the scattering information regarding the soliton $A = A_1 + iA_2$ and the antisoliton $\bar{A} = A_1 - iA_2$.

To obtain $2 \to 2$ S-matrices involving breathers (when present) it is, then, sufficient to follow the bootstrap program by subsequent applications of the bootstrap equation (Fig.(1.6d))

$$S_{il}^{li}(\vartheta) = S_{kl}^{lk}(\vartheta - i\bar{u}_{j\bar{i}}^{\bar{k}}) S_{jl}^{lj}(\vartheta + i\bar{u}_{\bar{i}k}^{\bar{j}}) \,, \tag{1.42}$$

with $\bar{u} = \pi - u$, u_{ij}^{k} denoting the (imaginary part) of the rapidity corresponding to the k bound state pole in the scattering $ij \to ij$.

In this way, all the $2 \to 2$ S-matrices can be obtained, granting by factorizability the knowledge of the full scattering theory.

S-matrix, *sine-Gordon model*

The Faddeev-Zamolodchikov algebra for the soliton (A) and the antisoliton ($\bar{A}$) symbols reads

$$A(\vartheta_1)A(\vartheta_2) = S_0(\vartheta_{12})A(\vartheta_2)A(\vartheta_1) \tag{1.43a}$$
$$A(\vartheta_1)\bar{A}(\vartheta_2) = S_T(\vartheta_{12})\bar{A}(\vartheta_2)A(\vartheta_1) + S_R(\vartheta_{12})A(\vartheta_2)\bar{A}(\vartheta_1) \tag{1.43b}$$
$$\bar{A}(\vartheta_1)\bar{A}(\vartheta_2) = S_0(\vartheta_{12})\bar{A}(\vartheta_2)\bar{A}(\vartheta_1) \,, \tag{1.43c}$$

where the same amplitude S_0 appears on both the soliton-soliton and antisoliton-antisoliton scattering because of the charge conjugation symmetry of the model. These amplitudes can be collected into one matrix of the form

$$\boldsymbol{S}(\vartheta) = \begin{bmatrix} S_0(\vartheta) & & & \\ & S_T(\vartheta) & S_R(\vartheta) & \\ & S_R(\vartheta) & S_T(\vartheta) & \\ & & & S_0(\vartheta) \end{bmatrix} \tag{1.44}$$

They are found to be [41, §3, ($\gamma \equiv 8\pi p$)]

$$\begin{aligned} S_0(\vartheta) &= \prod_{n=0}^{+\infty} \frac{\Gamma(\frac{1}{\pi p}((2n+1)\pi - i\vartheta))}{\Gamma(\frac{1}{\pi p}((2n+1)\pi + i\vartheta))} \frac{\Gamma(\frac{1}{\pi p}(\pi p + (2n+1)\pi - i\vartheta))}{\Gamma(\frac{1}{\pi p}(\pi p + (2n+1)\pi + i\vartheta))} \cdot \\ &\quad \cdot \frac{\Gamma(\frac{1}{\pi p}((2n+2)\pi + i\vartheta))}{\Gamma(\frac{1}{\pi p}((2n+2)\pi - i\vartheta))} \frac{\Gamma(\frac{1}{\pi p}(\pi p + (2n)\pi + i\vartheta))}{\Gamma(\frac{1}{\pi p}(\pi p + (2n)\pi - i\vartheta))} \\ &= -\,\mathrm{e}^{\chi(\vartheta)} \,, \end{aligned} \tag{1.45}$$

with

$$\chi(\vartheta) = \int_{-\infty}^{+\infty} \mathrm{d}k \, \frac{\mathrm{e}^{i\frac{\vartheta}{\pi}k}}{k} \frac{\sinh \frac{p-1}{2}k}{2\sinh \frac{p}{2}k \, \cosh \frac{1}{2}k} \,, \tag{1.46}$$

and

$$S_T(\vartheta) = \frac{\sinh\frac{\vartheta}{p}}{\sinh\frac{i\pi-\vartheta}{p}} S_0(\vartheta) \tag{1.47}$$

$$S_R(\vartheta) = \frac{\sinh\frac{i\pi}{p}}{\sinh\frac{i\pi-\vartheta}{p}} S_0(\vartheta)\,. \tag{1.48}$$

When defining [42]

$$a(\vartheta) = \sinh\frac{i\pi-\vartheta}{p}\,, \quad b(\vartheta) = \sinh\frac{\vartheta}{p}\,, \quad c(\vartheta) = \sinh\frac{i\pi}{p}\,, \tag{1.49}$$

the matrix in Eq.(1.44) can be written as

$$\boldsymbol{S}(\vartheta) = \frac{S_0(\vartheta)}{a(\vartheta)}\boldsymbol{R}(\vartheta)\,, \tag{1.50}$$

with $\boldsymbol{R}(\vartheta)$ denoting the $\mathcal{U}_q(\mathfrak{su}_2)$ spin $1/2$ R-matrix in the principal gradation with $q = \mathrm{e}^{i\frac{\pi}{\alpha}}$ (where α is defined in Eq.(2.41)) and spectral parameter $\frac{\vartheta}{p}$

$$\boldsymbol{R}(\vartheta) = \begin{bmatrix} a(\vartheta) & & & \\ & b(\vartheta) & c(\vartheta) & \\ & c(\vartheta) & b(\vartheta) & \\ & & & a(\vartheta) \end{bmatrix}. \tag{1.51}$$

This is the same form of the R-matrix of the 6-vertex lattice model, profoundly related with the $XXZ_{\frac{1}{2}}$ spin chain model (App.A).

Then, the amplitudes involving the breathers B_a can also be obtained. Indeed, when $0 < p < 1$, the amplitudes in Eqs.(1.45)(1.47)(1.48) show a set of $N_B = \lfloor \frac{1}{p} \rfloor$ ($N_B = \frac{1}{p} - 1$ when $p \in \frac{1}{\mathbb{N}}$) simple poles in the physical strip, which is compatible with the presence of breather bound states of masses given in Eq.(1.19)[2]. The Faddeev-Zamolodchikov algebra should be completed with

$$A(\vartheta_1)B_a(\vartheta_2) \;= S_a(\vartheta_{12})B_a(\vartheta_2)A(\vartheta_1) \tag{1.52a}$$

$$\bar{A}(\vartheta_1)B_a(\vartheta_2) \;= S_a(\vartheta_{12})B_a(\vartheta_2)\bar{A}(\vartheta_1) \tag{1.52b}$$

$$B_a(\vartheta_1)B_b(\vartheta_2) = S_{ab}(\vartheta_{12})B_b(\vartheta_2)B_a(\vartheta_1)\,, \tag{1.52c}$$

[2]When looking at the term $\Gamma(\frac{1}{\pi p}(\pi p + (2n)\pi + i\vartheta))$ in Eq.(1.45), it may be clearly see that it shows simple poles at $\vartheta_n^{\text{pole}} = in\pi p$, $n = 0, 1, \ldots$, which fall in the physical strip $0 < \mathfrak{Im}(\vartheta) < \pi$ for $0 < p < \frac{1}{n}$, $n \geq 1$, so that there are at most $N_B = \lfloor \frac{1}{p} \rfloor$ ($N_B = \frac{1}{p} - 1$ when $\frac{1}{p} \in \mathbb{N}$) bound states. The pre-factors in Eqs.(1.47)(1.48) give the crossing-symmetric poles $\vartheta_n^{\text{pole}} = i\pi - in\pi p$ and additional poles fall out of the physical strip.

The masses of these bound states may be computed by evaluating the residues of the conveniently-defined amplitudes $S_\pm(\vartheta) = S_R(\vartheta) \pm S_T(\vartheta)$.

where $a, b = 1, \ldots, N_B$ and the amplitudes are found to be [41, §3, ($\gamma \equiv 8\pi p$)]

$$S_a(\vartheta) = \frac{\sinh\vartheta\ + i\cos\frac{1}{2}a\pi p}{\sinh\vartheta\ - i\cos\frac{1}{2}a\pi p} \prod_{l=1}^{a-1} \frac{\sin^2\left(\frac{1}{4}(a-2l)\pi p - \frac{\pi}{4} + i\frac{\vartheta}{2}\right)}{\sin^2\left(\frac{1}{4}(a-2l)\pi p - \frac{\pi}{4} - i\frac{\vartheta}{2}\right)} \tag{1.53}$$

$$\begin{aligned} S_{ab}(\vartheta) =& \frac{\sinh\vartheta\ + i\sin\frac{1}{2}(a+b)\pi p}{\sinh\vartheta\ - i\sin\frac{1}{2}(a+b)\pi p}\,\frac{\sinh\vartheta\ + i\sin\frac{1}{2}(a-b)\pi p}{\sinh\vartheta\ - i\sin\frac{1}{2}(a-b)\pi p}\cdot \\ &\cdot \prod_{l=1}^{\min\{a,b\}-1} \frac{\sin^2\left(\frac{1}{4}(-|a-b|-2l)\pi p - \frac{\pi}{4} + i\frac{\vartheta}{2}\right)\ \cos^2\left(\frac{1}{4}(a+b-2l)\pi p - \frac{\pi}{4} + i\frac{\vartheta}{2}\right)}{\sin^2\left(\frac{1}{4}(-|a-b|-2l)\pi p - \frac{\pi}{4} - i\frac{\vartheta}{2}\right)\ \cos^2\left(\frac{1}{4}(a+b-2l)\pi p - \frac{\pi}{4} - i\frac{\vartheta}{2}\right)}\,. \end{aligned} \tag{1.54}$$

Some special values of the sine-Gordon parameter (Eq.(1.11)) may be considered inside this general description.

Of particular interest are the points for which $p \in \frac{1}{\mathbb{N}} - \{1\}$, where the theory entails $N_B = \frac{1}{p} - 1$ breather species along with the soliton and the antisoliton. It may be noticed that, for these special values, the reflection amplitude of Eq.(1.48) vanishes: they are named reflectionless points. Then, the matrix in Eq.(1.44) becomes diagonal. This means that, in any scattering event, the particles (the soliton and the antisoliton) can not exchange their quantum numbers even if in the same ($O(2)$) multiplet: not just the global set of momenta (Th.(1.1)), but also the momentum and the quantum numbers of each individual particle are left unchanged by the collision.
For these features, the scattering at the reflectionless points is said to be diagonal or purely elastic.

Subsequent sections delve deeper into the structures behind the sine-Gordon scattering theory for the reflectionless points and other particular values of p (§3.1).

Chapter 2

Thermodynamic Bethe Ansatz

The thermodynamic Bethe Ansatz (TBA) is a powerful method that essentially provides an integrable model with a thermodynamic description, finding numerous instances of implementation [43].

Firstly introduced in the study of the Lieb-Liniger model (Bose gas with δ-function interactions) [1], the TBA was quickly adapted to lattice integrable models [44] and generalized to relativistic theories [45][46][47][48].
The core idea is that of studying in the thermodynamic limit the momenta and energy distributions of particles, as provided by the Bethe Ansatz description. This results in a set of nonlinear integral equations for the (pseudo)particles' roots (and holes) distributions at thermodynamic equilibrium: the TBA equations. If solved, they allow to compute the main thermodynamic functions for the theory. Simply starting by the S-matrix formulation, the TBA provides an expression for the model's free energy at finite temperature, which turns out to be related to the ground state energy of the associated integrable field theory at finite volume. Regarding the latter as an integrable deformation of a conformal field theory (CFT) [49], its high-energy (UV) limit can be studied, firmly connecting the starting scattering theory to CFT perturbation.

Here it is provided a hopefully accessible overview of the derivation of this equations' system, up to its most fundamental formulation: the 'raw' TBA, as it is often called. Strong of the knowledge previously presented (§1), this chapter keeps the sine-Gordon model under particular consideration, even if the TBA method can be applied to many other models. Then, rather than explicitly or numerically solve the TBA equations, subsequent sections delve deeper into their fascinating structure.

2.1 Zamolodchikov mirror argument

Let's open the discussion by clarifying the program behind the TBA method. The procedure's roots reside in the connection between free energy at finite temperature and ground state energy at finite volume: the two can be bridged thanks to the celebrated Zamolodchikov mirror argument [45, ($R \equiv L'$)].

Consider a 2-dimensional (euclidean) theory defined on a flat torus, i.e. with periodic boundary conditions imposed for both dimensions. This specifies two orthogonal geodesic circumferences, $\mathcal{C}_{L'}$ along the x-direction and $\mathcal{C}_L$ along the y-direction, of lengths L' and L respectively: they act as generators for the toroidal geometry (Fig.(2.1)).
The relativistic invariance of the theory allows, then, two possible quantization schemes or 'channels', depending on the choice of space and (euclidean) time directions. Regarding the y-direction as the time direction, the states on $\mathcal{C}_{L'}$, belonging to the Hilbert space $\mathcal{H}_{L'}$, evolve by the hamiltonian

$$H_{L'} = \frac{1}{2\pi} \int_{\mathcal{C}_{L'}} \mathrm{d}x \, T_{yy} \,, \tag{2.1}$$

where $T_{\mu\nu}$ denotes the stress-energy tensor of the theory. Alternatively, the time direction can be chosen along the $(-x)$-direction (the minus sign to preserve the frame orientation), so that the evolution of the states on $\mathcal{C}_L$, living in the Hilbert space $\mathcal{H}_L$, is described by the hamiltonian

$$H_L = \frac{1}{2\pi} \int_{\mathcal{C}_L} \mathrm{d}y \, T_{xx} \,. \tag{2.2}$$

When considering the partition function of the theory $\mathscr{Z}(L, L')$, it can be equivalently expressed in the two channels as

$$\mathscr{Z}(L, L') = \operatorname*{Tr}_{\mathcal{H}_{L'}} \mathrm{e}^{-L H_{L'}} = \operatorname*{Tr}_{\mathcal{H}_L} \mathrm{e}^{-L' H_L} \,. \tag{2.3}$$

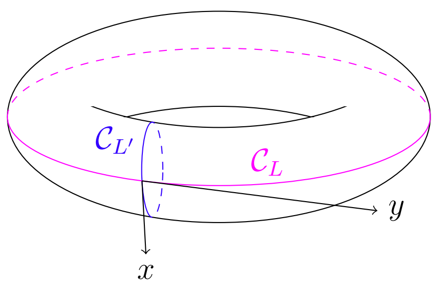


Figure 2.1: Sketch of the toroidal geometry at the basis of Zamolodchikov mirror argument.

The limit $L \to +\infty$ can be studied, showing different physical interpretations for the two quantization schemes. In the first channel this corresponds to relax the time periodicity hypothesis: the partition function is dominated by the ground state energy $E_0(L')$ contribution

$$\mathscr{Z}(L, L') \underset{L\to+\infty}{\sim} \mathrm{e}^{-LE_0(L')} \,. \tag{2.4}$$

In the second channel it coincides, instead, with the thermodynamic limit: the partition function reads

$$\mathscr{Z}(L, L') \underset{L\to+\infty}{\sim} \mathrm{e}^{-L'Lf(L')} \,, \tag{2.5}$$

where $f(L')$ denotes the free energy density at inverse temperature $L' = \frac{1}{T}$ (and at vanishing chemical potentials). Comparing the last two equations, it can be obtained the fundamental relation

$$E_0(L') = L'f(L') \,. \tag{2.6}$$

This result connects the finite-temperature free energy density to the finite-volume ground state energy, i.e. the thermodynamics and the vacuum energy of the theory.

In this context, the TBA arises as a method to investigate the thermodynamics of the states on $\mathcal{C}_L$. Through Eq.(2.6), the ground state energy of $H_{L'}$ can be deduced.

It is of interest, then, to study the scaling behavior of the latter [46][47].
Once possible bulk terms are subtracted, the vacuum energy reads

$$E_0(L') = -\frac{\pi\tilde{c}(\ell)}{6L'} \,. \tag{2.7}$$

$\tilde{c}(\ell)$ is the finite-size scaling function of the theory, depending on the dimensionless scaling parameter ℓ[1]. Besides the length L', the other independent length scale of the massive theory is provided by the correlation length $L_c = \frac{1}{m}$, related to the mass m of the lightest particle. So, it is possible to define $\ell = \frac{L'}{L_c}$: the conformal limit (i.e. the limit in which the correlation length diverges, the massless UV limit of the theory) corresponds to $\ell \to 0$. In this limit,

$$E_0(L') \xrightarrow[\ell\to 0]{} -\frac{\pi c_{\mathrm{eff}}}{6L'} = \frac{2\pi}{L'}\left(\Delta_{\mathrm{min}} + \bar{\Delta}_{\mathrm{min}} - \frac{c}{12}\right), \tag{2.8}$$

where c is the central charge of the CFT (c_{eff} being the effective central charge), while Δ_{min} and $\bar{\Delta}_{\mathrm{min}}$ denote the left and right dimensions of the lowest operator. This means that, from the knowledge of the ground state energy, it can be extracted information regarding the CFT reached in the UV limit of the theory along the renormalization group flow.

[1]Here considered real, it is worth mentioning that analytic continuations for imaginary values of the scaling parameter have been shown to reproduce the excited state energies for some models [50].

Further developments are also possible. For instance, a perturbation of the CFT obtained can be realized in order to obtain the massive QFT corresponding to the starting theory. When the latter is a scattering theory, this allows to fully establish an equivalence with a quantum field formulation.

For instance, in the case of the sine-Gordon model at reflectionless points, it is found that, following the program just outlined, the central charge in the UV limit of the scattering theory of §1.2.4 is $c = 1$ [47], corroborating the correspondence to the lagrangian formulation of §§1.2.1,1.2.3.

It may be noticed that these latter results rely heavily on Zamolodchikov mirror argument (Eq.(2.6)), so on the possibility of obtaining a thermodynamic description (i.e. the finite-temperature free energy density) of the (scattering) theory on $\mathcal{C}_L$. Crucially, that's what the TBA aims to provide. The subsequent sections (§§2.2-2.5) are dedicated to its derivation, showcasing the sine-Gordon model as a paradigmatic example.

2.2 Periodic boundary conditions

Consider, more in detail, a massive elastic factorizable scattering theory (at finite temperature) on a space of length L with periodic boundary conditions ($\mathcal{C}_L$ in Fig.(2.1)). Given a system of N particles (possibly of different species), the space periodicity induces a quantization of their momenta: the equations obtained, known as Bethe-Yang equations, result in the momenta and energy distributions of the particles, which can be subsequently studied in the thermodynamic limit $N, L \to +\infty$, with the ratio N/L kept fixed (§2.4).

The Bethe-Yang equations are, thus, at the core of the TBA. To gain a clearer insight into these equations, let's start by considering the special case of diagonal scattering theory. The derivation of the diagonal Bethe-Yang equations goes under the name of coordinate Bethe Ansatz.

2.2.1 Diagonal Bethe-Yang equations

In a purely elastic scattering theory, all particle momenta are asymptotically conserved: any reflection scattering amplitude vanishes, forbidding the exchange of quantum numbers (even in the same multiplet) during a collision event. Intuitively, in such instances the asymptotic particle momentum may be considered as part of the quantum numbers characterizing it, so that identical particles are meant to have identical momenta. Of course, this allows for a simpler description than in the non-diagonal case.

Suppose that the theory entails different particle species and, as previously introduced, consider on $\mathcal{C}_L$ a system of N particles, N_s of which are of species s (mass m_s), $N = \sum_s N_s$. Being prepared to take the limit $L \to +\infty$, in the space of all possible configurations of these particles there can be found regions in which their positions $\{x_i\}_{i=1}^N$ are widely separated, so that $|x_i - x_j| \gg L_c$. The particle interactions become negligible and these can be regarded as effectively free regions. Off-mass-shell effects can also be neglected, thus allowing for an asymptotic wave function description of the system. Such wave function, called Bethe wave function [51], results to be proportional to that of free particles (i.e. plane waves of well-defined momenta $\{p_i\}_{i=1}^N$)

$$\psi(x_1, \ldots, x_N) = \mathrm{e}^{i \sum_{j=1}^N x_j p_j} \sum_{\sigma \in P_N} \mathrm{B}_\sigma \Theta_\sigma \,. \tag{2.9}$$

Here the second sum runs over the $N!$ permutations σ in the permutation group of N elements P_N, corresponding to a decomposition of the configuration space of the particles based on all the possible orderings. The latter are specified by

$$\Theta_\sigma = \begin{cases} 1\,, & \text{if } x_{\sigma(1)} < \cdots < x_{\sigma(N)} \\ 0\,, & \text{otherwise} \end{cases} \,, \tag{2.10}$$

while the coefficients B_σ are wave amplitudes depending on particle momenta (i.e. rapidities $\{\vartheta_i\}_{i=1}^N$, by Eq.(1.26)).
For the periodicity of the space dimension, the Bethe wave function enjoys the symmetry

$$\psi(x_1, \ldots, x_i + L, \ldots, x_N) = \pm \psi(x_1, \ldots, x_i, \ldots, x_N)\,, \tag{2.11}$$

where the sign depends, as usual, on the statistics obeyed by the i^{th} particle. This may be interpreted as obtained by taking the i^{th} particle and repeatedly exchanging its position order with the neighboring particle along the circumference $\mathcal{C}_L$ until the starting configuration is recovered (Fig.(2.2)).

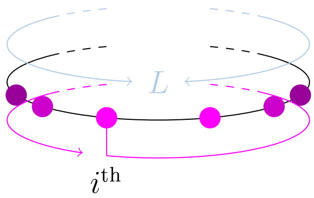


Figure 2.2: Depiction of the Bethe wave function (anti)symmetrization: the i^{th} particle is moved around the circumference until it returns to its starting position. In the process, it scatters with the other particles, but the same Bethe wave function (up to a phase) must be recovered as soon as the system reaches its initial asymptotic configuration.

When considering a transition between two adjacent configuration space free regions the scattering theory provides conditions to connect the two asymptotic wave functions. If the permutations σ and σ' differ only by the exchange of the indices i and j with x_i and x_j adjacent in the space ordering, then

$$\mathrm{B}_{\sigma'} = S_{ij}(\vartheta_{ij})\mathrm{B}_{\sigma}\,, \tag{2.12}$$

where just two indices are enough to denote the entries of the S-matrix, the latter being diagonal. This implies that

$$\psi(x_1,\ldots,x_i+L,\ldots,x_N) = \mathrm{e}^{im_iL\sinh\vartheta_i}\prod_{\substack{j=0\\ j\neq i}}^{N} S_{ij}(\vartheta_{ij})\psi(x_1,\ldots,x_i,\ldots,x_N)\,. \tag{2.13}$$

Inserting it in the left-hand side of Eq.(2.11), the Bethe-Yang equations (i.e. a set of quantization conditions for the asymptotic particle real rapidities ϑ_i) are obtained in the form

$$\mathrm{e}^{im_iL\sinh\vartheta_i}\prod_{\substack{j=1\\ j\neq i}}^{N} S_{ij}(\vartheta_{ij}) = \pm 1\,, \tag{2.14}$$

or (taking the logarithm on both sides)

$$m_iL\sinh\vartheta_i - i\sum_{\substack{j=1\\ j\neq i}}^{N}\log S_{ij}(\vartheta_{ij}) = 2\pi z_i\,, \tag{2.15}$$

with N integer/half-odd integer numbers z_i, $i = 1,\ldots,N$. The latter may be thought of as specifying the admissible rapidities, since the set of rapidities solution of the Bethe-Yang equations is in correspondence with them (the left-hand side of Eq.(2.15) is assumed to define a monotonic function). Thus, they are interpreted as quantum occupation numbers.

Further selection rules have to be taken into account when identical particles are present: depending on their statistics, the wave function describing the system should be either symmetrized or antisymmetryzed in the exchange of their rapidities. Recalling that identical particles also share identical rapidities, and that the unitarity condition for the S-matrix (Eq.(1.37)) requires that $S_{ii}^2(0) = 1$, this leaves only two possible cases. One option is represented by

$$S_{ii}(0) = -1\,, \tag{2.16}$$

meaning that the Bethe wave function is antisymmetric in the identical particles' coordinates. This is compatible with Fermi-Dirac statistics, meaning that fermions can occupy each rapidity value in any number. In a sense, they behave as bosons, so they

are referred to as of 'bosonic type'. On the other hand, Bose-Einstein statistics would result incompatible, meaning that bosons can not share the same rapidity in the first place. This reminds of a sort of exclusion principle, so they are regarded as of 'fermionic type'.
The other possibility is given by

$$S_{ii}(0) = +1\,, \tag{2.17}$$

forcing the Bethe wave function to be symmetric in the identical particles' coordinates. It is easy to understand that in this case the situation is inverted w.r.t. the previous one. Particles obeying Fermi-Dirac statistics are to be intended as of 'fermionic type', while if they obey Bose-Einstein statistics then they are of 'bosonic type'.
Notice, in particular, that the integer/half-odd integer numbers appearing in the Bethe-Yang equations can assume any value for particles of bosonic type, but have to be all different for particles of fermionic type. This is a distinction that influences the form of the entropy of the system, once the thermodynamic limit is considered.

Bethe-Yang equations, *sine-Gordon model at reflectionless points*

As noticed at the end of §1.2.4, the sine-Gordon model at reflectionless points $p \in \frac{1}{\mathbb{N}}$, $p \neq 1$, describes a diagonal scattering theory, for which the coordinate Bethe Ansatz method can be applied.

The particle content of the theory at these points is composed by the soliton, the antisoliton and by a number $N_B = \frac{1}{p} - 1$ of different breather species.
By looking at the form of their identical-particles scattering matrices (Eqs.(1.45)(1.54) specialized at $\vartheta = 0$), it can be immediately noticed that, even if (anti)solitons are fermions and breathers are bosons (§1.2.3), they all are of fermionic type by the definitions above (this holds in general at any value of the sine-Gordon parameter p).
Then, following the coordinate Bethe Ansatz, consider a system of N such particles, N_S of which are solitons (mass M), $N_{\bar{S}}$ antisolitons (mass M), while N_{B_a}, $a = 1, \ldots, N_B$, of them are of the same species as the a^{th} breather (mass M_a), $N = N_S + N_{\bar{S}} + \sum_{a=1}^{N_B} N_{B_a}$.
The diagonal Bethe-Yang equations obtained are different, depending on whether the i^{th} particle is a (anti)soliton or a breather: in the first case the scattering amplitudes of Eqs.(1.45)(1.47)(1.53) should be considered when exchanging particles along $\mathcal{C}_L$; in the second case the ones in Eqs.(1.53)(1.54) instead. The resulting equations read

$$\begin{aligned}
1 = \mathrm{e}^{iM_a L \sinh \vartheta_i^{(B_a)}} \prod_{b=1}^{N_B} \prod_{\substack{j=1 \\ a=b \Rightarrow j \neq i}}^{N_{B_b}} S_{ab}(\vartheta_i^{(B_a)} - \vartheta_j^{(B_b)}) \prod_{k=1}^{N_{\bar{S}}} S_a(\vartheta_i^{(B_a)} - \vartheta_k^{(\bar{S})}) \cdot \\
\cdot \prod_{l=1}^{N_S} S_a(\vartheta_i^{(B_a)} - \vartheta_l^{(S)})
\end{aligned} \tag{2.18a}$$

$$-1 = \mathrm{e}^{iML\sinh\vartheta_i^{(S)}} \prod_{a=1}^{N_B}\prod_{j=1}^{N_{B_a}} S_a(\vartheta_i^{(S)} - \vartheta_j^{(B_a)}) \prod_{k=1}^{N_{\bar S}} S_T(\vartheta_i^{(S)} - \vartheta_k^{(\bar S)}) \prod_{\substack{l=1\\ l\neq i}}^{N_S} S_0(\vartheta_i^{(S)} - \vartheta_l^{(S)}) \tag{2.18b}$$

$$-1 = \mathrm{e}^{iML\sinh\vartheta_i^{(\bar S)}} \prod_{a=1}^{N_B}\prod_{j=1}^{N_{B_a}} S_a(\vartheta_i^{(\bar S)} - \vartheta_j^{(B_a)}) \prod_{\substack{k=1\\ k\neq i}}^{N_{\bar S}} S_0(\vartheta_i^{(\bar S)} - \vartheta_k^{(\bar S)}) \prod_{l=1}^{N_S} S_T(\vartheta_i^{(\bar S)} - \vartheta_l^{(S)})\,, \tag{2.18c}$$

for $a = 1,\ldots,N_B$, $i = 1,\ldots,N_{B_a}$, in Eq.(2.18a), for $i = 1,\ldots,N_S$ in Eq.(2.18b) and for $i = 1,\ldots,N_{\bar S}$ in Eq.(2.18c).
It may be noticed that at reflectionless points the amplitudes in Eqs.(1.45)(1.47) coincide up to a sign. Supposing that N_S and $N_{\bar S}$ are either both even (upper sign in Eq.(2.19b)) or both odd (lower sign in Eq.(2.19b))[2], define $N_{\tilde S} = N_S + N_{\bar S}$, so that $(\tilde S)$ can be used to denote both a soliton or an antisoliton. For simplicity, the ' ˜ ' symbol will often be subtended in what follows. Then, after few passages,

$$-\,1 = \mathrm{e}^{iM_aL\sinh\vartheta_i^{(B_a)}} \prod_{b=1}^{N_B}\prod_{j=1}^{N_{B_b}} S_{ab}(\vartheta_i^{(B_a)} - \vartheta_j^{(B_b)}) \prod_{k=1}^{N_S} S_a(\vartheta_i^{(B_a)} - \vartheta_k^{(S)}) \tag{2.19a}$$

$$\pm\,1 = \mathrm{e}^{iML\sinh\vartheta_i^{(S)}} \prod_{a=1}^{N_B}\prod_{j=1}^{N_{B_a}} S_a(\vartheta_i^{(S)} - \vartheta_j^{(B_a)}) \prod_{k=1}^{N_S} S_0(\vartheta_i^{(S)} - \vartheta_k^{(S)})\,, \tag{2.19b}$$

or (recalling the definition of $\chi(\vartheta)$, Eqs.(1.45)(1.46))

$$2\pi z_i^{(B_a)} = M_aL\sinh\vartheta_i^{(B_a)} - i\sum_{b=1}^{N_B}\sum_{j=1}^{N_{B_b}} \log S_{ab}(\vartheta_i^{(B_a)} - \vartheta_j^{(B_b)}) - i\sum_{k=1}^{N_S} \log S_a(\vartheta_i^{(B_a)} - \vartheta_k^{(S)}) \tag{2.20a}$$

$$2\pi z_i^{(S)} = ML\sinh\vartheta_i^{(S)} - i\sum_{a=1}^{N_B}\sum_{j=1}^{N_{B_a}} \log S_a(\vartheta_i^{(S)} - \vartheta_j^{(B_a)}) - i\sum_{k=1}^{N_S} \chi(\vartheta_i^{(S)} - \vartheta_k^{(S)})\,, \tag{2.20b}$$

for $a = 1,\ldots,N_B$, $i = 1,\ldots,N_{B_a}$ in Eqs.(2.19a)(2.20a) and for $i = 1,\ldots,N_S$ in Eqs.(2.19b)(2.20b).
These are the Yang-Baxter equations for the sine-Gordon model at reflectionless points.

[2] Since the thermodynamic limit is soon to be studied, the number of (anti)solitons on $\mathcal{C}_L$ may be selected as stated above, in such a way that the total topological charge of the system is even (equivalent to an even-sites finite spin chain description (§2.3.1, Eq.(2.42))). Still possible different choices lead to other sectors of the theory [21][52].

As already mentioned, the derivation of this set of equations is greatly simplified by the pure elasticity of the scattering theory. At reflectionless points, the sine-Gordon soliton and antisoliton show a vanishing reflection amplitude (Eq.(1.48)). This means that the two particle species decouple, even if part of the same $O(2)$ multiplet. In other words, when taking an (anti)soliton around $\mathcal{C}_L$ to realize the boundary conditions, it never changes to the particle of opposite charge after scattering.
The behavior is profoundly different outside of the reflectionless points, hence the necessity of a way to 'disentangle' the soliton and the antisoliton, to diagonalize their S-matrix on new eigenstates: this is the aim of the algebraic Bethe Ansatz method.

2.3 Algebraic Bethe Ansatz

The algebraic Bethe Ansatz is a powerful technique that originates within the context of quantum integrability (for a review, [53] and references therein). Its natural language would be that of spin chain models, for which it was initially formulated. Nevertheless, it has already been noticed the deep connection that can be established between such models and the sine-Gordon model (Eqs.(1.50)(1.51)). Here, it will be provided a formulation of the algebraic Bethe Ansatz method more finalized at the issues raised in previous sections.

2.3.1 Non-diagonal Bethe-Yang equations

The aim is still that of obtaining Bethe-Yang equations, as stemming from momentum quantization induced by periodic boundary conditions for the space $\mathcal{C}_L$. So, retracing the steps followed for the coordinate Bethe Ansatz, a similar description can be obtained. The crucial difference is that the scattering theory is now considered elastic and factorizable, but not purely elastic: in presence of particle multiplets, the resulting equation (Eq.(2.14)) does not contain isolated amplitudes (as S_0 for Eq.(2.19b)), but traces of products of entire non-diagonal S-matrices. More explicitly, if the i^{th} particle is inside a multiplet and the system contains $N = N_1 + N_2$ particles, with N_1 outside and N_2 inside that same multiplet, the Bethe-Yang equation for the i^{th} particle reads

$$\mathrm{e}^{im_i L \sinh\vartheta_i} \prod_{j=1}^{N_1} S_{ij}(\vartheta_i - \vartheta_j) \operatorname*{Tr}_{\mathcal{V}_i} \prod_{\substack{k=1\\k\neq i}}^{N_2} \boldsymbol{S}[ik](\vartheta_i - \vartheta_k) = \pm 1\,, \tag{2.21}$$

where $\boldsymbol{S}[ik]$ denotes the S-matrix describing the scattering between particles of the multiplet in which the i^{th} and k^{th} particles are contained, while $\mathcal{V}_i$ is the Hilbert space in which the i^{th} particle lives: $\mathcal{H}_L \overset{def}{=} \mathcal{H}_1 \otimes \mathcal{H}_2 \overset{def}{=} \mathcal{H}_1 \otimes \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_i \otimes \cdots \otimes \mathcal{V}_{N_2}$. The algebraic Bethe Ansatz aids in the computation of these traces.

Very roughly, the essential idea is that of enlarging the Hilbert space of the circumference multiplet particles $\mathcal{H}_2$ with an auxiliary space $\mathcal{V}_a$, where a labels possible internal degrees of the space added. This allows to decouple the interactions in the physical space (i^{th}-k^{th}), considering the ones with the auxiliary space only (a^{th}-k^{th}). Intuitively, this may be thought as being equivalent to consider a new probe particle of rapidity ϑ, which is taken along the circumference, scattering with the others. In this way,

$$\mathrm{e}^{im_i L \sinh\vartheta} \prod_{j=1}^{N_1} S_{ij}(\vartheta - \vartheta_j) \operatorname*{Tr}_{\mathcal{V}_a} \prod_{k=1}^{N_2} \boldsymbol{S}[ak](\vartheta - \vartheta_k) = \pm 1 \,, \tag{2.22}$$

or, exploiting the R-matrix proportionality of Eq.(1.40),

$$\mathrm{e}^{im_i L \sinh\vartheta} \prod_{j=1}^{N_1} S_{ij}(\vartheta - \vartheta_j) \prod_{k=1}^{N_2} f(\vartheta - \vartheta_k) \operatorname*{Tr}_{\mathcal{V}_a} \prod_{k=1}^{N_2} \boldsymbol{R}[ak](\vartheta - \vartheta_k) = \pm 1 \,. \tag{2.23}$$

These R-matrices may be regarded simply as operators $\boldsymbol{R}[ak](\vartheta - \vartheta_k) \colon \mathcal{V}_a \otimes \mathcal{V}_k \to \mathcal{V}_a \otimes \mathcal{V}_k$. Their product may be used to define a new operator $\boldsymbol{T}[a](\vartheta|\vec{\vartheta}) \colon \mathcal{V}_a \otimes \mathcal{H}_2 \to \mathcal{V}_a \otimes \mathcal{H}_2$ (with $\vec{\vartheta}$ collecting all the rapidities $\{\vartheta_k\}_{k=1}^{N_2}$), called monodromy matrix[3], so that

$$\mathrm{e}^{im_i L \sinh\vartheta} \prod_{j=1}^{N_1} S_{ij}(\vartheta - \vartheta_j) \prod_{k=1}^{N_2} f(\vartheta - \vartheta_k) \operatorname*{Tr}_{\mathcal{V}_a} \boldsymbol{T}[a](\vartheta|\vec{\vartheta}) = \pm 1 \,. \tag{2.24}$$

Due to its form, it may be thought as a matrix in $\mathrm{End}(\mathcal{V}_a)$ with entries given by operators acting on $\mathcal{H}_2$. What remains is, then, to evaluate its trace (whose presence, let's recall it, descends directly from the periodic boundary conditions), named transfer matrix.
This can be done by formulating an Ansatz for eigenstates of the monodromy matrix diagonal operators

$$|\Psi(\vec{u})\rangle = \hat{\mathrm{B}}(u_1) \dots \hat{\mathrm{B}}(u_{\mathcal{M}}) \, |\Theta\rangle \tag{2.25}$$

(the use of a similar notation to Eq.(2.9) is to underline the common Ansatz nature of the two equations, still they are well distinct mathematical entities). Here $|\Theta\rangle$ denotes a reference state, a sort of pseudo-vacuum for $\mathcal{H}_2$. The operators $\hat{\mathrm{B}}(u_l)$ act on it just as creation operators, adding $\mathcal{M} \le N_2$ massless particle excitations of (possibly complex) rapidity $\{u_l\}_{l=1}^{\mathcal{M}}$ (collected in a vector $\vec{u}$): they are called elementary magnons. These describe the different possible internal states of the N_2 particles in the same multiplet and can be exploited to diagonalize the scattering of the latter.
It is sufficient to work out the commutation relations between the elementary magnon

[3]More precisely, the monodromy matrix is usually defined as a product of Lax operators $\boldsymbol{T}[a](\vartheta) = \prod_{k=1}^{N_2} \boldsymbol{L}[ka](\vartheta)$, being related with R-matrices by $\boldsymbol{L}[ka](\vartheta) = \boldsymbol{R}[ak](\vartheta - \vartheta_k)$ for some constants $\{\vartheta_k\}_{k=1}^{N_2}$.

operators and the entries of the monodromy matrix and impose that the Ansatz of Eq.(2.25) is actually an eigenstate. The former descend directly from the Yang-Baxter equation and, once applied, the latter condition completes the Yang-Baxter equations with relations for the newly introduced rapidities $\vec{u}$.
In short, expressing again the rapidity of the probe particle ϑ as a physical rapidity ϑ_i,

$$\mathrm{e}^{im_i L\sinh\vartheta_i}\prod_{j=1}^{N_1} S_{ij}(\vartheta_i-\vartheta_j)\,\Sigma(\vartheta_i|\vec{\vartheta},\vec{u})=\pm 1 \tag{2.26a}$$

$$\varsigma'(u_m|\vec{\vartheta},\vec{u})\varsigma''(u_m|\vec{u})=\pm 1\,, \tag{2.26b}$$

or

$$m_i L\sinh\vartheta_i - i\sum_{j=1}^{N_1}\log S_{ij}(\vartheta_{ij}) - i\log\Sigma(\vartheta_i|\vec{\vartheta},\vec{u}) = 2\pi z_i \tag{2.27a}$$

$$-i\log\varsigma'(u_m|\vec{\vartheta},\vec{u}) - i\log\varsigma''(u_m|\vec{u}) = 2\pi z_m\,, \tag{2.27b}$$

where $\Sigma(\vartheta_i|\vec{\vartheta},\vec{u})$ is a short-hand notation for the eigenvalue of the diagonalized scattering, while the second lines denote a general form of the equations for the rapidities $\{u_m\}_{m=1}^{\mathcal{M}}$. As last notice, it can be observed that the N_1 particles outside the multiplet are left untouched by the algebraic Bethe Ansatz method: by Parke theorem (Th.(1.1)) their scattering with the particles inside the multiplet is already diagonal, thus their Bethe-Yang equations are in the form of Eq.(2.14).

Bethe-Yang equations, *sine-Gordon model*

Outside of reflectionless points, the sine-Gordon scattering theory is non-diagonal for the soliton and the antisoliton: the algebraic Bethe Ansatz method can be followed to decouple the two particle species.

Let's start by the usual setup of N particles on $\mathcal{C}_L$, N_S of which of soliton and antisoliton species (here and in what follows (S) is to be intended as $(\tilde{S})$ defined in previous sections, denoting both solitons and antisolitons), N_{B_a} of which of a^{th} breather species, $N = N_S + \sum_{a=1}^{N_B} N_{B_a}$.
Momentum quantization by periodic boundary conditions results in

$$1 = \mathrm{e}^{iM_a L\sinh\vartheta_i^{(B_a)}}\prod_{b=1}^{N_B}\prod_{\substack{j=1\\ a=b\Rightarrow j\neq i}}^{N_{B_b}} S_{ab}(\vartheta_i^{(B_a)}-\vartheta_j^{(B_b)})\prod_{k=1}^{N_S} S_a(\vartheta_i^{(B_a)}-\vartheta_k^{(S)}) \tag{2.28a}$$

$$-1 = \mathrm{e}^{iML\sinh\vartheta_i^{(S)}}\prod_{a=1}^{N_B}\prod_{j=1}^{N_{B_a}} S_a(\vartheta_i^{(S)}-\vartheta_j^{(B_a)})\,\mathop{\mathrm{Tr}}_{\mathcal{V}_i}\prod_{\substack{k=1\\ k\neq i}}^{N_S}\boldsymbol{S}(\vartheta_i^{(S)}-\vartheta_k^{(S)})\,, \tag{2.28b}$$

for $a = 1, \dots, N_B$, $i = 1, \dots, N_{B_a}$ in Eq.(2.28a) and for $i = 1, \dots, N_S$ in Eq.(2.28b). It may be already noticed that Eq.(2.28a) coincides with Eq.(2.18a): when present, the breathers do not enter any multiplet and their scattering is diagonal. For what concerns Eq.(2.28b), the trace of the soliton-antisolition S-matrix (Eq.(1.44)) should be computed.

Following the steps of the algebraic Bethe Ansatz, an auxiliary space $\mathcal{V}_a = \mathbb{C}^2$ is introduced, so that (subtending the labels (S) for simplicity and making use of Eq.(1.50))

$$\operatorname*{Tr}_{\mathbb{C}^2} \prod_{\substack{k=1 \\ k \neq i}}^{N_S} \boldsymbol{S}(\vartheta - \vartheta_k) = \prod_{k=1}^{N_S} \frac{S_0(\vartheta - \vartheta_k)}{a(\vartheta - \vartheta_k)} \operatorname*{Tr}_{\mathbb{C}^2} \prod_{k=1}^{N_S} \boldsymbol{R}(\vartheta - \vartheta_k) = \prod_{k=1}^{N_S} \frac{S_0(\vartheta - \vartheta_k)}{a(\vartheta - \vartheta_k)} \operatorname*{Tr}_{\mathbb{C}^2} \boldsymbol{T}(\vartheta|\vec{\vartheta}), \quad (2.29)$$

where the R- matrix is the one of Eq.(1.51) and the monodromy matrix may be written as

$$\boldsymbol{T}(\vartheta|\vec{\vartheta}) = \begin{bmatrix} \hat{\mathrm{A}}(\vartheta|\vec{\vartheta}) & \hat{\mathrm{B}}(\vartheta|\vec{\vartheta}) \\ \hat{\mathrm{C}}(\vartheta|\vec{\vartheta}) & \hat{\mathrm{D}}(\vartheta|\vec{\vartheta}) \end{bmatrix} . \quad (2.30)$$

The eigenstate Ansatz may be formulated with the help of the operators appearing as monodromy matrix entries. It reads

$$\left| \Psi(\vec{\vartheta}, \vec{u}) \right\rangle = \prod_{l=1}^{\mathcal{M}} \hat{\mathrm{B}}(u_l|\vec{\vartheta}) \left|\Theta\right\rangle , \quad (2.31)$$

with

$$\hat{\mathrm{C}}(\vartheta|\vec{\vartheta}) \left|\Theta\right\rangle = 0 \quad (2.32a)$$

$$\hat{\mathrm{A}}(\vartheta|\vec{\vartheta}) \left|\Theta\right\rangle = \prod_{k=1}^{N_S} a(\vartheta - \vartheta_k) \left|\Theta\right\rangle \quad (2.32b)$$

$$\hat{\mathrm{D}}(\vartheta|\vec{\vartheta}) \left|\Theta\right\rangle = \prod_{k=1}^{N_S} b(\vartheta - \vartheta_k) \left|\Theta\right\rangle . \quad (2.32c)$$

The commutation relations for $\hat{\mathrm{A}}(\vartheta|\vec{\vartheta})$, $\hat{\mathrm{B}}(\vartheta|\vec{\vartheta})$, $\hat{\mathrm{C}}(\vartheta|\vec{\vartheta})$, $\hat{\mathrm{D}}(\vartheta|\vec{\vartheta})$ may be worked out directly from the R-matrix Yang-Baxter equation. Denoting with $\boldsymbol{P}$ the (transposition) matrix

$$\boldsymbol{P} = \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix} \quad (2.33)$$

and defining $\check{\boldsymbol{R}}(\vartheta) \overset{def}{=} \boldsymbol{P}\boldsymbol{R}(\vartheta)$, it may be formulated in general as (App.A)

$$\check{\boldsymbol{R}}(\vartheta_1 - \vartheta_2)(\boldsymbol{T}(\vartheta_1|\vec{\vartheta}) \otimes \boldsymbol{T}(\vartheta_2|\vec{\vartheta})) = (\boldsymbol{T}(\vartheta_2|\vec{\vartheta}) \otimes \boldsymbol{T}(\vartheta_1|\vec{\vartheta}))\check{\boldsymbol{R}}(\vartheta_1 - \vartheta_2) , \quad (2.34)$$

whence, introducing the ratios

$$s(\vartheta) = \frac{a(\vartheta)}{b(\vartheta)}\,, \quad \tilde{s}(\vartheta) = -\frac{c(\vartheta)}{b(\vartheta)}\,, \tag{2.35}$$

the more useful (for this derivation) commutation relations read

$$\hat{\mathrm{B}}(\vartheta_1|\vec{\vartheta})\hat{\mathrm{B}}(\vartheta_2|\vec{\vartheta}) = \hat{\mathrm{B}}(\vartheta_2|\vec{\vartheta})\hat{\mathrm{B}}(\vartheta_1|\vec{\vartheta}) \tag{2.36a}$$

$$\hat{\mathrm{A}}(\vartheta_1|\vec{\vartheta})\hat{\mathrm{B}}(\vartheta_2|\vec{\vartheta}) = s(\vartheta_2-\vartheta_1)\hat{\mathrm{B}}(\vartheta_2|\vec{\vartheta})\hat{\mathrm{A}}(\vartheta_1|\vec{\vartheta}) + \tilde{s}(\vartheta_2-\vartheta_1)\hat{\mathrm{B}}(\vartheta_1|\vec{\vartheta})\hat{\mathrm{A}}(\vartheta_2|\vec{\vartheta}) \tag{2.36b}$$

$$\hat{\mathrm{D}}(\vartheta_1|\vec{\vartheta})\hat{\mathrm{B}}(\vartheta_2|\vec{\vartheta}) = s(\vartheta_1-\vartheta_2)\hat{\mathrm{B}}(\vartheta_2|\vec{\vartheta})\hat{\mathrm{D}}(\vartheta_1|\vec{\vartheta}) + \tilde{s}(\vartheta_1-\vartheta_2)\hat{\mathrm{B}}(\vartheta_1|\vec{\vartheta})\hat{\mathrm{D}}(\vartheta_2|\vec{\vartheta})\,. \tag{2.36c}$$

Applying these, it may be seen that the transfer matrix

$$\operatorname*{Tr}_{\mathbb{C}^2} \boldsymbol{T}(\vartheta|\vec{\vartheta}) = \hat{\mathrm{A}}(\vartheta|\vec{\vartheta}) + \hat{\mathrm{D}}(\vartheta|\vec{\vartheta}) \tag{2.37}$$

acts on the eigenstate Ansatz of Eq.(2.31) as

$$\boldsymbol{t}(\vartheta|\vec{\vartheta},\vec{u}) = (\hat{\mathrm{A}}(\vartheta|\vec{\vartheta}) + \hat{\mathrm{D}}(\vartheta|\vec{\vartheta}))\left|\Psi(\vec{\vartheta},\vec{u})\right\rangle = \textit{(w.t.)} + \textit{(u.t.)}\,. \tag{2.38}$$

The 'wanted terms' *(w.t.)* are those for which the state $\left|\Psi(\vec{\vartheta},\vec{u})\right\rangle$ actually behaves as a transfer matrix eigenstate

$$\textit{(w.t.)} = \left(\prod_{k=1}^{N_S} a(\vartheta-\vartheta_k)\prod_{l=1}^{\mathscr{M}} s(u_l-\vartheta) + \prod_{k=1}^{N_S} b(\vartheta-\vartheta_k)\prod_{l=1}^{\mathscr{M}} s(\vartheta-u_l)\right)\left|\Psi(\vec{\vartheta},\vec{u})\right\rangle\,, \tag{2.39}$$

while the 'unwanted terms' show a different form

$$\begin{aligned}\textit{(u.t.)} = -\sum_{m=1}^{\mathscr{M}} &\tilde{s}(\vartheta-u_m)\,\cdot\\ &\cdot\left(\prod_{k=1}^{N_S} a(u_m-\vartheta_k)\prod_{\substack{l=1\\ l\neq m}}^{\mathscr{M}} s(u_l-u_m) - \prod_{k=1}^{N_S} b(u_m-\vartheta_k)\prod_{\substack{l=1\\ l\neq m}}^{\mathscr{M}} s(u_m-u_l)\right)\cdot\\ &\cdot\,\hat{\mathrm{B}}(\vartheta|\vec{\vartheta})\prod_{\substack{n=1\\ n\neq m}}^{\mathscr{M}}\hat{\mathrm{B}}(u_n|\vec{\vartheta})\left|\Theta\right\rangle\,.\end{aligned} \tag{2.40}$$

The latter disappear when the term in round brackets on the right-hand side of Eq.(2.40) vanishes. This happens when the elementary magnon rapidities, after the shift

$$u_m \to u_m + i\beta^{-1}\,, \quad \beta = \frac{2\alpha}{\pi p}\,, \quad \alpha = \frac{1}{\frac{1}{p} - N_B}\,, \tag{2.41}$$

satisfy (possibly up to a sign, chosen here accordingly to fn.(2, p.33))

$$\prod_{k=1}^{N_S} \frac{\sinh \frac{\pi}{2\alpha}(\beta u_m - \beta\vartheta_k + i)}{\sinh \frac{\pi}{2\alpha}(\beta u_m - \beta\vartheta_k - i)} = -\prod_{l=1}^{\mathcal{M}} \frac{\sinh \frac{\pi}{2\alpha}(\beta u_m - \beta u_l + 2i)}{\sinh \frac{\pi}{2\alpha}(\beta u_m - \beta u_l - 2i)}, \tag{2.42}$$

for $m = 1, \ldots, \mathcal{M}$.

Of the upmost importance is that these equations are in correspondence with the Bethe-Yang equations for the $XXZ_{\frac{1}{2}}$ spin chain model [44, Eq.(2.8), $(N \equiv N_S)$, $(\theta \equiv \frac{\pi}{\alpha})$, $(x \equiv \beta u)$], once again highlighting the strong connection between the two models (and allowing for the description of §2.4.2).

Then, the transfer matrix eigenvalue, under the same elementary magnon rapidity shift of Eq.(2.41) and specified for a physical (S) rapidity, may be written as

$$t\left(\vartheta_i \middle| \vec{\vartheta}, \vec{u} + i\beta^{-1}\right) = (-1)^{1+(N_B+1)\mathcal{M}} \prod_{k=1}^{N_S} a(\vartheta_i - \vartheta_k) \prod_{l=1}^{\mathcal{M}} \frac{\sinh \frac{\pi}{2\alpha}(\beta u_l - \beta\vartheta_i - i)}{\sinh \frac{\pi}{2\alpha}(\beta u_l - \beta\vartheta_i + i)}, \tag{2.43}$$

so that, provided that Eq.(2.42) holds, the (anti)soliton scattering is diagonalized as

$$\begin{aligned} \Sigma\left(\vartheta_i \middle| \vec{\vartheta}, \vec{u} + i\beta^{-1}\right) &= \prod_{k=1}^{N_S} \frac{S_0(\vartheta_i - \vartheta_k)}{a(\vartheta_i - \vartheta_k)}\, t\left(\vartheta_i \middle| \vec{\vartheta}, \vec{u} + i\beta^{-1}\right) = \\ &= (-1)^{1+(N_B+1)\mathcal{M}} \prod_{k=1}^{N_S} S_0(\vartheta_i - \vartheta_k) \prod_{l=1}^{\mathcal{M}} \frac{\sinh \frac{\pi}{2\alpha}(\beta u_l - \beta\vartheta_i - i)}{\sinh \frac{\pi}{2\alpha}(\beta u_l - \beta\vartheta_i + i)}. \end{aligned} \tag{2.44}$$

Finally, introducing the notation

$$\varsigma_n(\vartheta) = \frac{\sinh \frac{\pi}{2\alpha}(\vartheta - in)}{\sinh \frac{\pi}{2\alpha}(\vartheta + in)}, \tag{2.45}$$

the sine-Gordon model Bethe-Yang equations read

$$-1 = \mathrm{e}^{iM_a L \sinh \vartheta_i^{(B_a)}} \prod_{b=1}^{N_B} \prod_{j=1}^{N_{B_b}} S_{ab}(\vartheta_i^{(B_a)} - \vartheta_j^{(B_b)}) \prod_{k=1}^{N_S} S_a(\vartheta_i^{(B_a)} - \vartheta_k^{(S)}) \tag{2.46a}$$

$$\begin{aligned} (-1)^{(N_B+1)\mathcal{M}} &= \mathrm{e}^{iML \sinh \vartheta_i^{(S)}} \prod_{a=1}^{N_B} \prod_{j=1}^{N_{B_a}} S_a(\vartheta_i^{(S)} - \vartheta_j^{(B_a)}) \cdot \\ &\cdot \prod_{k=1}^{N_S} S_0(\vartheta_i^{(S)} - \vartheta_k^{(S)}) \prod_{l=1}^{\mathcal{M}} \varsigma_{+1}(\beta(u_l^{(M)} - \vartheta^{(S)})) \end{aligned} \tag{2.46b}$$

$$-1 = \prod_{k=1}^{N_S} \varsigma_{+1}(\beta(u_m^{(M)} - \vartheta_k^{(S)})) \prod_{l=1}^{\mathcal{M}} \varsigma_{+2}^{-1}(\beta(u_m^{(M)} - u_l^{(M)})) \tag{2.46c}$$

or

$$2\pi z_i^{(B_a)} = M_a L \sinh \vartheta_i^{(B_a)} - i \sum_{b=1}^{N_B} \sum_{j=1}^{N_{B_b}} \log S_{ab}(\vartheta_i^{(B_a)} - \vartheta_j^{(B_b)}) - i \sum_{k=1}^{N_S} \log S_a(\vartheta_i^{(B_a)} - \vartheta_k^{(S)}) \tag{2.47a}$$

$$2\pi z_i^{(S)} = ML \sinh \vartheta_i^{(S)} - i \sum_{a=1}^{N_B} \sum_{j=1}^{N_{B_a}} \log S_a(\vartheta_i^{(S)} - \vartheta_j^{(B_a)}) - i \sum_{k=1}^{N_S} \chi(\vartheta_i^{(S)} - \vartheta_k^{(S)}) + \\ - i \sum_{l=1}^{\mathcal{M}} \log \varsigma_{+1}(\beta(u_l^{(M)} - \vartheta_i^{(S)})) \tag{2.47b}$$

$$2\pi z_i^{(M)} = - i \sum_{k=1}^{N_S} \log \varsigma_{+1}(\beta(u_m^{(M)} - \vartheta_k^{(S)})) + i \sum_{l=1}^{\mathcal{M}} \log \varsigma_{+2}(\beta(u_m^{(M)} - u_l^{(M)}))\,, \tag{2.47c}$$

for $a = 1, \dots, N_B$, $i = 1, \dots, N_{B_a}$ in Eqs.(2.46a)(2.47a), for $i = 1, \dots, N_S$ in Eqs.(2.46b)(2.47b) and for $m = 1, \dots, \mathcal{M}$ in Eqs.(2.46c)(2.47c).
This is a rather general formulation for the sine-Gordon Bethe-Yang equations, since it holds for any value of the sine-Gordon parameter. In fact, some particular cases can be read directly from Eqs.(2.46)(2.47): the equations for reflectionless points $p \in \frac{1}{\mathbb{N}} - \{1\}$ (where the coordinate Bethe Ansatz is sufficient (§2.2.1, Eq.(2.20))) are recovered disregarding all the elementary magnon terms and, similarly, those for $p > 1$ (considered in later sections (§3.2.2)) are obtained omitting the breather terms. For this reason, the following sections focus on this general formulation.

2.4 Thermodynamic limit

Once the Bethe-Yang equations have been obtained, they should be studied in the thermodynamic limit (TL). As anticipated, in the setup of §2.2 this correspond to consider the limit $N, L \to +\infty$, where both N and L show the same asymptotic growth: $N \underset{TL}{\sim} L$. When applying it, a new set of equations (the TBA equations) for the thermodynamic distributions of rapidities is obtained, whence all the thermodynamics may be deduced.

This section is centered on the analysis of the rapidities behavior in the TL, closely following [45][46][47][43].

2.4.1 Rapidity densities

Let's start by considering real rapidities, i.e. the ones belonging to the physical particles on $\mathcal{C}_L$, previously denoted as $\{\vartheta_i\}_{i=1}^N$. Even if the TL treatment of complex rapidities shows affinities, still it requires a more delicate approach, presented in the following section (§2.4.2).

As the number of particles increases, the Bethe-Yang rapidities solutions become denser and denser ($|\vartheta_{i+1} - \vartheta_i| \underset{TL}{\sim} \frac{1}{mL}$). It is, thus, convenient to introduce rapidity densities to study the TL and use them to define the state of the system.
Divide the real rapidity space in intervals $\Delta\vartheta$, small enough to let such densities vary on orders of many rapidity intervals, but large enough to possibly accommodate a high number of rapidity solutions. The density of states for the particles of species s can be defined as

$$\rho^{(s)}(\vartheta) = \frac{\textit{(\# of possible } s \textit{ states with } \vartheta < \textit{rapidity} < \vartheta + \Delta\vartheta\textit{)}}{L\,\Delta\vartheta}\,. \tag{2.48}$$

This can be further specialized as a sum of a density for occupied states and a density for unoccupied states

$$\rho^{(s)}(\vartheta) = \rho_r^{(s)}(\vartheta) + \rho_h^{(s)}(\vartheta)\,. \tag{2.49}$$

As stated above (§2.2.1), the integer/half-odd integer numbers z_i appearing in the Bethe-Yang equations can be interpreted as quantum occupation numbers for the rapidities ϑ_i and are in correspondence with them: the rapidity-dependent side of the Bethe-Yang equations defines a (decreasing or increasing) monotonic function. Once considered the statistics of the species s particles and their type, the set of all the allowed quantum numbers can exceed the set of rapidities for the particles on $\mathcal{C}_L$. The term 'root' is used to denote the Bethe-Yang rapidity solutions for which the corresponding state is actually occupied. When the opposite happens, they are called 'holes'. It is starting from the number of roots (holes) in a given rapidity interval that the roots (holes) density appearing in Eq.(2.49) can be defined in a similar form to Eq.(2.48):

$$\rho_r^{(s)}(\vartheta) = \frac{\textit{(\# of } \vartheta < \textit{roots} < \vartheta + \Delta\vartheta\textit{)}}{L\,\Delta\vartheta} \tag{2.50}$$

$$\rho_h^{(s)}(\vartheta) = \frac{\textit{(\# of } \vartheta < \textit{holes} < \vartheta + \Delta\vartheta\textit{)}}{L\,\Delta\vartheta}\,. \tag{2.51}$$

Further useful quantities in the TL description are the so-called pseudoenergies, naturally arising in the thermodynamic functions' expressions (§2.4.3). Distinguishing between particles of bosonic (upper sign) and fermionic (lower sign) type, they are defined as

$$\varepsilon^{(s)}(\vartheta) = \log\frac{\rho^{(s)}(\vartheta) \pm \rho_r^{(s)}(\vartheta)}{\rho_r^{(s)}(\vartheta)}\,, \quad \text{i.e.}\ \frac{\rho_r^{(s)}(\vartheta)}{\rho^{(s)}(\vartheta)} = \frac{\exp(-\varepsilon^{(s)}(\vartheta))}{1 \mp \exp(-\varepsilon^{(s)}(\vartheta))}\,, \tag{2.52}$$

whence their names.
It is worthwhile to introduce a notation for their combination

$$L^{(s)}(\vartheta) = \mp\log(1 \mp \exp(-\varepsilon^{(s)}(\vartheta)))\,. \tag{2.53}$$

2.4.2 Bethe strings hypothesis

Some more attention should be paid when dealing with possibly complex rapidities $\{u_m\}_{m=1}^{\mathcal{M}}$. Since elementary magnons are introduced to decouple the internal structure of particle multiplets, they are but pseudoparticles, whose rapidity can indeed be complex.

Consider more in detail the second line of Eq.(2.26), describing the Bethe-Yang equations for the magnon rapidities. As the number of physical particles is taken to $+\infty$, the first term $\varsigma'(u_m|\vec{\vartheta},\vec{u})$, which depends on the set of physical rapidities, diverges. In order to still get a finite right-hand side in the equation, this divergence should be compensated by the second term $\varsigma''(u_m|\vec{u})$, which depends on the set of pseudoparticles rapidities only. Thus, the latter organize themselves into particular configurations in the complex ϑ plane: the Bethe strings. They can be thought as bound states of elementary magnons, thus magnon strings or simply magnons.
The form of such configurations will become more apparent once considering an instance of application. For the moment, it is enough to state that each Bethe string can be identified by a real rapidity value, the string center: intuitively, it corresponds to the magnon rapidity. Then, rapidity densities can be introduced for string centers, just like done in the previous section.

Notice that the reasoning followed is not quite rigorous. As the thermodynamic limit is considered, the number of elementary magnons also increases, possibly compensating the divergence of the first term. There exist solutions that do not approach Bethe string configurations in the thermodynamic limit, even if rather atypical. Nevertheless, the measurable contributions to the thermodynamic functions are mainly due to string complexes. It is, thus, reasonable to formulate the 'string hypothesis', i.e. that all the thermodynamically relevant Bethe-Yang solutions are in the form of Bethe string configurations.

Bethe strings, *sine-Gordon model*

In the case of the sine-Gordon model, the string hypothesis can be borrowed directly from the $XXZ_{\frac{1}{2}}$ spin chain model. As observed above, the elementary magnon Bethe-Yang Eq.(2.42) is in perfect correspondence with the one for the spin chain elementary magnons (with the addition of inhomogeneities). It is, thus, sufficient to parallel the derivation of [44, ($p_0 \equiv \alpha$)].

Looking at Eq.(2.42), it may be immediately noticed that it enjoys a periodicity in the elementary magnon rapidities

$$u_m \to u_m + i2\alpha\beta^{-1} = u_m + i\pi p\,, \tag{2.54}$$

so that solutions shifted by the above period can be identified. Also, if u_m is a solution, its complex conjugate $\bar{u}_m$ is a solution too. This means that they are symmetric either

w.r.t the real axis or w.r.t the $\mathfrak{Im}(u) = i\frac{\pi p}{2}$ axis (identified with the $\mathfrak{Im}(u) = -i\frac{\pi p}{2}$ axis by periodicity).
Following the Bethe string argument, it may be seen that, in the limit $N_S \to +\infty$, the right-hand side of Eq.(2.42) should develop poles. These are obtained when the elementary magnon rapidities organize themselves in groups sharing a common real abscissa and distanced in the imaginary ordinates by $\Delta u = i2\beta^{-1} = i\pi p\alpha^{-1}$: the magnon strings. Each string entails n (string length) elementary magnon rapidities and can be of two types $v = \pm$ (string parity), the latter depending on the symmetry described previously. In short, in the thermodynamic limit, strings consist of sets

$$u_{c,l}^{(n)^+} = u_c^{(n)^+} + i\frac{\pi p}{2\alpha}(n+1-2l) \qquad \bmod i\pi p \tag{2.55a}$$

$$u_{c,l}^{(n)^-} = u_c^{(n)^-} + i\frac{\pi p}{2\alpha}(n+1-2l) + i\frac{\pi p}{2} \quad \bmod i\pi p \tag{2.55b}$$

i.e.

$$u_{c,l}^{(n)^v} = u_c^{(n)^v} + i\frac{\pi p}{2\alpha}(n+1-2l) + \delta_{v,-}\, i\frac{\pi p}{2} \quad \bmod i\pi p\,, \tag{2.56}$$

where $l = 1, \ldots, n$, while $u_c^{(n)^v} \in \mathbb{R}$ denotes the string (of length n and parity v) center ($\delta_{i,j}$ standing for the Kronecker δ-symbol as usual). They are shown in Fig.(2.3).
The study of magnon strings reveals much more. The parameter α can always be uniquely expressed as a simple continued fraction (finite for $\alpha \in \mathbb{Q}$, infinite otherwise). For the

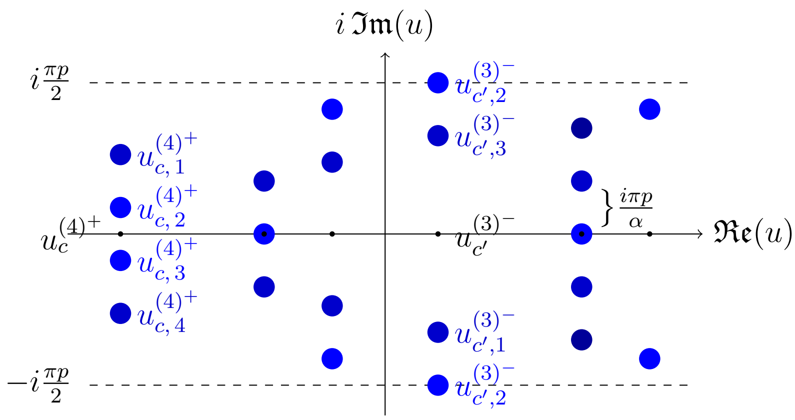


Figure 2.3: Sketch of Bethe strings in the rapidity complex plane. From left to right, are represented (for one period): a +string of length 4; a +string of length 3; a −string of length 4; a −string of length 3; a +string of length 5; a −string of length 2. In some cases, the notation of Eq.(2.56) is explicitly illustrated. Notice that the string center (in black) can both belong to the magnon string or not: in either case, it is real by definition.

sine-Gordon parameter this means that (recalling the definition of α (Eq.(2.41)))

$$p = \frac{1}{N_B + \frac{1}{\alpha}} = \frac{1}{N_B + \frac{1}{\nu_1 + \frac{1}{\nu_2 + \dots}}} = c.f.\{N_B, \nu_1, \nu_2, \dots\} \tag{2.57}$$

where it is considered $1 \leq \alpha \notin \frac{1}{\mathbb{N}}$ and where $\nu_i \in \mathbb{N}$ are positive integers, with the label i identifying the 'level' of the continued fraction. ν_i represents the number of magnon species at level i, so that the sum of such integers results in the total number of magnon species N_M for a particular value of the sine-Gordon parameter: the magnon species can, thus, be denoted as M_k, $k = 1, \dots, N_M$. Some further quantities can be defined as

$$\textit{(auxiliary real numbers):} \quad \mathfrak{p}_0 = \alpha, \quad \mathfrak{p}_1 = 1, \quad \mathfrak{p}_i = \mathfrak{p}_{i-2} - \mathfrak{p}_{i-1}\nu_{i-1} \tag{2.58a}$$

$$\textit{(magnon species up to level i):} \quad \mathfrak{m}_0 = 0, \quad \mathfrak{m}_i = \sum_{j=1}^{i} \nu_j \tag{2.58b}$$

$$\textit{(auxiliary integer numbers):} \quad \mathfrak{y}_{-1} = 0, \quad \mathfrak{y}_0 = 1, \quad \mathfrak{y}_i = \mathfrak{y}_{i-2} + \mathfrak{y}_{i-1}\nu_i \tag{2.58c}$$

$$\textit{(auxiliary integer numbers):} \quad r(i)\colon \mathfrak{m}_{r(i)} \leq i < \mathfrak{m}_{r(i)+1}\,. \tag{2.58d}$$

It is clear that $\nu_i = \left\lfloor \frac{\mathfrak{p}_{i-1}}{\mathfrak{p}_i} \right\rfloor$. More importantly, the length $\mathfrak{n}_k$ and parity $\mathfrak{v}_k$ of each magnon species M_k can be determined as

$$\mathfrak{n}_k = \mathfrak{y}_{i-1} + (k - \mathfrak{m}_i)\mathfrak{y}_i \colon \mathfrak{m}_i \leq k < \mathfrak{m}_{i+1} \tag{2.59a}$$

$$\mathfrak{v}_1 = +1, \quad \mathfrak{v}_{\mathfrak{m}_1} = -1, \quad \mathfrak{v}_k = \mathrm{e}^{\pi i \left\lfloor \frac{\mathfrak{n}_k - 1}{\mathfrak{p}_0} \right\rfloor}\,. \tag{2.59b}$$

These are exactly the dimensions and parities of the irreducible highest-weight representations of the quantum algebra $\mathcal{U}_q(\mathfrak{su}_2)$, $q = \mathrm{e}^{i\frac{\pi}{\alpha}}$ (App.B) [54][55], deepening even further the connection firstly established in Eq.(1.50).

2.4.3 Thermodynamics

The rapidity densities introduced thus far (for the rapidities of physical particles and of Bethe string centers, both real and denoted simply as ϑ in what follows) allow to define the thermodynamic functions for the system in the TL (where $\Delta\vartheta \underset{TL}{\rightarrow} \mathrm{d}\vartheta$).
Starting from the energy density, it is obtained from the sum of the energies of all the particles on $\mathcal{C}_L$. Recalling the parametrization of Eq.(1.26),

$$\begin{aligned} \mathcal{E}[\rho_r^{(s)}(\vartheta)] &= \frac{1}{L}\sum_i m_i \cosh\vartheta_i = \sum_s m^{(s)} \frac{1}{L}\sum_i \cosh\vartheta_i^{(s)} \underset{TL}{\rightarrow} \\ &\underset{TL}{\rightarrow} \sum_s m^{(s)} \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \rho_r^{(s)}(\vartheta) \cosh\vartheta \end{aligned} \tag{2.60}$$

Notice that magnons do not contribute to the energy.
For what concerns the entropy per unit length, it may be observed that, in every rapidity interval (suitably chosen as per considerations in §2.4.1), the number of roots and holes distributions of species s that correspond to a given state is

$$\omega_{\mathscr{B}}[\rho^{(s)}(\vartheta),\rho_r^{(s)}(\vartheta)] = \frac{(\rho^{(s)}(\vartheta)L\,\Delta\vartheta + \rho_r^{(s)}(\vartheta)L\,\Delta\vartheta - 1)!}{(\rho_r^{(s)}(\vartheta)L\,\Delta\vartheta)!(\rho_h^{(s)}(\vartheta)L\,\Delta\vartheta)!} \tag{2.61}$$

if the species is of bosonic type, while takes the form

$$\omega_{\mathscr{F}}[\rho^{(s)}(\vartheta),\rho_r^{(s)}(\vartheta)] = \frac{(\rho^{(s)}(\vartheta)L\,\Delta\vartheta)!}{(\rho_r^{(s)}(\vartheta)L\,\Delta\vartheta)!(\rho_h^{(s)}(\vartheta)L\,\Delta\vartheta)!} \tag{2.62}$$

if the species is of fermionic type. Then, applying the Stirling approximation, the entropy per unit length reads

$$\begin{aligned}
&\mathcal{S}[\rho^{(s)}(\vartheta),\rho_r^{(s)}(\vartheta)] = \\
&= \sum_{s\in\mathscr{B}} \frac{1}{L} \sum_{\Delta\vartheta} \log \omega_{\mathscr{B}}[\rho^{(s)}(\vartheta),\rho_r^{(s)}(\vartheta)] + \sum_{s\in\mathscr{F}} \frac{1}{L} \sum_{\Delta\vartheta} \log \omega_{\mathscr{F}}[\rho^{(s)}(\vartheta),\rho_r^{(s)}(\vartheta)] \underset{TL}{\rightarrow} \\
&\underset{TL}{\rightarrow} \sum_{s\in\mathscr{B}} \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \big(-\rho^{(s)}(\vartheta)\log\rho^{(s)}(\vartheta) - \rho_r^{(s)}(\vartheta)\log\rho_r^{(s)}(\vartheta) + \\
&\qquad + (\rho^{(s)}(\vartheta) + \rho_r^{(s)}(\vartheta))\log(\rho^{(s)}(\vartheta) + \rho_r^{(s)}(\vartheta))\big) + \\
&\quad + \sum_{s\in\mathscr{F}} \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \big(\rho^{(s)}(\vartheta)\log\rho^{(s)}(\vartheta) - \rho_r^{(s)}(\vartheta)\log\rho_r^{(s)}(\vartheta) + \\
&\qquad - (\rho^{(s)}(\vartheta) - \rho_r^{(s)}(\vartheta))\log(\rho^{(s)}(\vartheta) - \rho_r^{(s)}(\vartheta))\big)
\end{aligned} \tag{2.63}$$

In the entropy evaluation, the contribution of magnons should be considered: indeed they are related to internal states of the multiplet particles, thus influencing the final form for $\mathcal{S}$.
From the two above, the Helmholtz free energy density can be defined as usual as

$$\mathcal{F}[\rho^{(s)}(\vartheta),\rho_r^{(s)}(\vartheta)] = \mathcal{E} - T\mathcal{S}\,, \tag{2.64}$$

with temperature given by $T = \frac{1}{L'}$ (§2.1).

At thermodynamic equilibrium, the latter function reaches an extremum point: the rapidity densities should be varied until such a point is reached, always imposing Bethe-Yang equations as constraints. The resulting extremum conditions are the TBA equations (more on these passages in §2.5). They yield an expression for the free energy density to which each particle species contributes with a term

$$\mathcal{F}^{(s)}(L') = -\frac{1}{2\pi L'} \int_{-\infty}^{+\infty} \mathrm{d}\vartheta\, L^{(s)}(\vartheta) m^{(s)} \cosh\vartheta\,, \tag{2.65}$$

where $L^{(s)}(\vartheta)$ is defined in Eq.(2.53). For the Zamolodchikov mirror argument (§2.1), this is related with the ground state energy at finite volume for the theory seen in the other channel (Eq.(2.6)). Thus, by Eq.(2.7), it can be seen that the finite-size scaling function receives contributions from each particle species in the form

$$\tilde{c}^{(s)}(\ell) = \frac{3\ell}{\pi^2} \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \, L^{(s)}(\vartheta) m^{(s)} \cosh \vartheta \, . \tag{2.66}$$

Of course, this can be further specialized in the limit of high (UV) or low (IR) energies. As discussed, in the first case this is connected to the central charge of the CFT of which the scattering theory represents an integrable perturbation (Eq.(2.8)): in the sine-Gordon model case, it is found that $c = 1$ [47].

It is clear that the results above depend on the form of the rapidity densities satisfying the free energy extremum conditions (through the pseudoenergies and their logarithms). Starting from the Bethe-Yang equations in the TL, the next section is centered on the derivation of such equations in their most direct formulation: the 'raw' TBA equations. Both cases of bosonic and fermionic type particles are taken into exam, in order to present a derivation as general as possible. For the sine-Gordon model, however, the particles are all of fermionic type, so just this case is sufficient.

2.5 Raw TBA equations

Consider the system of Bethe-Yang equations in its most general form (Eqs.(2.15)(2.27)). When regarding elementary magnon terms as generalized scattering amplitudes, these equations may be roughly written as

$$m_i L \sinh \vartheta_i - i \sum_{j=1}^{N'} \log S_{ij}(\vartheta_{ij}) = 2\pi z_i \, , \tag{2.67}$$

for $i = 1, \ldots, N'$, where now N' entails both physical particles and pseudoparticles. A more precise analysis is presented for the sine-Gordon model at the end of the current section. Nevertheless, the one above is indeed a general formulation that, when necessary, may be specialized with due attention.
This set of equations may be divided in subgroups sharing the same particle species and the index i may be reassigned in such a way that the $\vartheta_i^{(s)}$ (representing string center rapidities in case of pseudoparticles) are in increasing order within a given subgroup:

$$m_i^{(s)} L \sinh \vartheta_i^{(s)} - i \sum_{j=1}^{N'} \log S_{ij}(\vartheta_i^{(s)} - \vartheta_j^{(s')}) = 2\pi z_i^{(s)} \, , \tag{2.68}$$

with $\vartheta_i^{(s)} \leq \vartheta_{i+1}^{(s)}$ for $i = 1, \ldots, N_s$, $\forall s$. When taking two neighboring equations, subtracting them and dividing by $2\pi L\,\Delta\vartheta$, with $\Delta\vartheta = \vartheta_{i+1} - \vartheta_i$, it is obtained that

$$\frac{m_i^{(s)}}{2\pi}\frac{\Delta \sinh \vartheta_i^{(s)}}{\Delta\vartheta} + \frac{1}{L}\sum_{j=1}^{N'}\frac{1}{2\pi i}\frac{\Delta \log S_{ij}(\vartheta_i^{(s)} - \vartheta_j^{(s')})}{\Delta\vartheta} = \frac{\Delta z_i^{(s)}}{L\Delta\vartheta}\,, \tag{2.69}$$

with the usual ranges for the labels (subtended in what follows). As stated in previous sections, in the TL the rapidities become denser on the real axis ($\Delta\vartheta \underset{TL}{\rightarrow} \mathrm{d}\vartheta$), allowing for the introduction of rapidity densities. It's easy to see that: difference quotients turn into derivatives; the sum over rapidities may be written as an integral with roots density as integration kernel; the right-hand side of the equation coincides with the definition for the states density (Eq.(2.48)), up to a sign when the monotonic $\vartheta_i \leftrightarrow z_i$ relation is decreasing and not increasing.
Thus, introducing the kernels

$$K^{(ss')}(\vartheta) = \frac{1}{2\pi i}\frac{\mathrm{d}\log S^{(ss')}(\vartheta)}{\mathrm{d}\vartheta} \tag{2.70}$$

and assuming the following convention for function convolution

$$(f * g)(x) = \int_{-\infty}^{+\infty} \mathrm{d}y f(x-y)g(y)\,, \tag{2.71}$$

the TL Bethe-Yang equations are in the form

$$\frac{m^{(s)}}{2\pi}\cosh\vartheta - \sum_{s'}(K^{(ss')} * \rho_r^{(s')})(\vartheta) = \eta^{(s)}\rho^{(s)}(\vartheta)\,, \tag{2.72}$$

where the signs $\eta^{(s)} \in \{+1, -1\}$ are introduced to guarantee the rapidity densities positivity. Clearly, this is a set of integral equations for the rapidity densities, that act as constraints in the free energy density extremization.
Proceeding with the latter, given the expressions for the thermodynamic functions described in the previous section (Eqs.(2.60)(2.63)(2.64)), the extremum conditions read

$$m^{(s)}R\cosh\vartheta = \varepsilon^{(s)}(\vartheta) + \sum_{s'}\frac{\delta\rho^{(s')}(\vartheta)}{\delta\rho_r^{(s)}(\vartheta)}L^{(s')}(\vartheta)\,, \tag{2.73}$$

where R denotes the radius of $\mathcal{C}_L$ (i.e. $L \overset{def}{=} 2\pi R$), while the pseudoenergies $\varepsilon^{(s)}(\vartheta)$ and their logarithms $L^{(s)}(\vartheta)$ are defined in Eqs.(2.52)(2.53). The functional derivatives can be computed regarding Eq.(2.72) as definitions for the states densities (i.e. constraint relations). This finally yields the set of raw TBA equations

$$m^{(s)}R\cosh\vartheta = \varepsilon^{(s)}(\vartheta) + \sum_{s'}(\eta^{(s')}K^{(ss')} * L^{(s')})(\vartheta)\,. \tag{2.74}$$

Raw TBA equations, *sine-Gordon model*

The previous derivation overlooks many details in favor of a general validity. In order to recover some specific features, let's delve deeper into the details of the sine-Gordon model.

Its Bethe-Yang equations are presented in Eqs.(2.46)(2.47). Before proceeding on the line of the general derivation above, it is worthwhile to spend some words on the magnon terms of the equations.
As per the Bethe string hypothesis (§2.4.2), in the TL elementary magnons organize themselves into magnon strings (or simply magnons) of species M_k, $k = 1, \ldots, N_M$, $N_M = \sum_j \nu_j$ (Eq.(2.57)), with length n_k (Eq.(2.59a)) and parity v_k (Eq.(2.59b)): the magnon rapidities are assumed in the form of Eq.(2.56). If for each magnon species M_k there exist N_{M_k} strings, products over elementary magnons may be written in general as

$$\prod_{l=1}^{\mathcal{M}} f(u_l) \underset{TL}{\rightarrow} \prod_{k=1}^{N_M} \prod_{c=1}^{N_{M_k}} \prod_{l=1}^{n_k} f(u_{c,l}^{(n_k)^{v_k}}) \,. \tag{2.75}$$

This can be applied to Eqs.(2.46b)(2.46c). After some calculations, it may be verified that

$$\prod_{l=1}^{n} \varsigma_{+1}(\beta(-\vartheta + u_{c,l}^{(n)^v})) = \varsigma_{+n}(\beta(-\vartheta + u_c^{(n)^v}) + \delta_{v,-}\, i\alpha) \tag{2.76a}$$

$$\prod_{l=1}^{n} \prod_{m=1}^{n'} \varsigma_{+2}(\beta(u_{c,l}^{(n)^v} - u_{c',m}^{(n')^{v'}})) = {}_{(2)}\prod_{\ell=|n-n'|}^{n+n'-2} \varsigma_{+\ell}(\beta(u_c^{(n)^v} - u_{c'}^{(n')^{v'}}) + (\delta_{v,-} - \delta_{v',-}) i\alpha) \cdot$$
$$\cdot\, \varsigma_{+(\ell+2)}(\beta(u_c^{(n)^v} - u_{c'}^{(n')^{v'}}) + (\delta_{v,-} - \delta_{v',-}) i\alpha) \,, \tag{2.76b}$$

where the symbol ${}_{(2)}\prod$ is used here to indicate that the product indices have to be increased by 2 for each term. Recall also that $u_c^{(n)^v}$ denotes the length n parity v string center: being a real rapidity on par with ϑ, it is denoted as $\vartheta_c^{(n)^v}$ in what follows. Then, introducing for simplicity the notation

$$\mathfrak{S}_{(k,c)(k',c')} = \beta\left(\vartheta_c^{(n_k)^{v_k}} - \vartheta_{c'}^{(n_{k'})^{v_{k'}}} + \delta_{v_k\, v_{k'},-}\, i\frac{\pi p}{2}\right) , \tag{2.77}$$

under the TL Bethe string hypothesis, the Bethe-Yang equations for the sine-Gordon model may be written as

$$
\begin{aligned}
2\pi z^{(B_a)}(\vartheta^{(B_a)}) \quad =& M_a L \sinh \vartheta^{(B_a)} + \\
& - i \sum_{b=1}^{N_B} \sum_{i=1}^{N_{B_b}} \log S_{ab}(\vartheta^{(B_a)} - \vartheta_i^{(B_b)}) + \\
& - i \sum_{j=1}^{N_S} \log S_a(\vartheta^{(B_a)} - \vartheta_j^{(S)}) \qquad (2.78a) \\
2\pi z^{(S)}(\vartheta^{(S)}) \quad =& M L \sinh \vartheta^{(S)} + \\
& - i \sum_{a=1}^{N_B} \sum_{i=1}^{N_{B_a}} \log S_a(\vartheta^{(S)} - \vartheta_i^{(B_a)}) + \\
& - i \sum_{j=1}^{N_S} \chi(\vartheta^{(S)} - \vartheta_j^{(S)}) + \\
& - i \sum_{k=1}^{N_M} \sum_{c=1}^{N_{M_k}} \log \varsigma_{+\,n_k} \left(\beta \left(\vartheta_c^{(n_k)^{v_k}} - \vartheta^{(S)} + \delta_{v_k,-}\, i\frac{\pi p}{2} \right) \right) \qquad (2.78b) \\
2\pi z^{(M_k)}(\vartheta_c^{(n_k)^{v_k}}) =& - i \sum_{j=1}^{N_S} \log \varsigma_{+\,n_k} \left(\beta \left(\vartheta_c^{(n_k)^{v_k}} - \vartheta_j^{(S)} + \delta_{v_k,-}\, i\frac{\pi p}{2} \right) \right) + \\
& + i \sum_{k'=1}^{N_M} \sum_{c'=1}^{N_{M_{k'}}} \Big((1 - \delta_{n_k, n_{k'}}) \log \varsigma_{+|n_k - n_{k'}|}(\mathfrak{S}_{(k,c)(k',c')}) + \\
& \quad + 2 \sum_{\ell_{kk'}=1}^{\min\{n_k, n'_k\}-1} \log \varsigma_{+(|n_k - n_{k'}| + 2\,\ell_{kk'})}(\mathfrak{S}_{(k,c)(k',c')}) + \\
& \quad + \log \varsigma_{+(n_k + n_{k'})}(\mathfrak{S}_{(k,c)(k',c')}) \Big) \,, \qquad (2.78c)
\end{aligned}
$$

for $\vartheta^{(B_a)} \in \{\vartheta_i^{(B_a)}\}_{i=1}^{N_{B_a}}$, $a = 1, \ldots, N_B$ in Eq.(2.78a), for $\vartheta^{(S)} \in \{\vartheta_i^{(S)}\}_{i=1}^{N_S}$ in Eq.(2.78b) and for $c = 1, \ldots, N_{M_k}$, $k = 1, \ldots, N_M$ in Eq.(2.78c).

It is apparent, now, that the magnon terms depend only on the (real) string center rapidity. Thus, it is possible to introduce the rapidity densities for each particle species (§2.4.1) and follow the general steps outlined above.

Defining the kernels

$$
K^{(B_a B_b)}(\vartheta) = \frac{1}{2\pi i} \frac{\mathrm{d}}{\mathrm{d}\vartheta} \log S_{ab}(\vartheta) \qquad (2.79a)
$$

$$
K^{(S B_a)}(\vartheta) = \frac{1}{2\pi i} \frac{\mathrm{d}}{\mathrm{d}\vartheta} \log S_a(\vartheta) \qquad (2.79b)
$$

$$K^{(SS)}(\vartheta) = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\chi(\vartheta) \tag{2.79c}$$

$$K^{(SM_k)}(\vartheta) = \kappa(\vartheta; n_k, v_k) \tag{2.79d}$$

$$\begin{aligned} K^{(M_k M_{k'})}(\vartheta) = &- (1 - \delta_{n_k, n_{k'}})\, \kappa(\vartheta; |n_k - n_{k'}|, v_k \cdot v_{k'}) + \\ &- 2 \sum_{\ell_{kk'}=1}^{\min\{n_k, n_{k'}\}-1} \kappa(\vartheta; |n_k - n_{k'}| + 2\,\ell_{kk'}, v_k \cdot v_{k'}) + \\ &- \kappa(\vartheta; n_k + n_{k'}, v_k \cdot v_{k'}) \end{aligned} \tag{2.79e}$$

with

$$\kappa(\vartheta; n_k, v_k) = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\vartheta} \log \varsigma_{-\, n_k}(\beta\vartheta) \tag{2.80}$$

and following the notations presented above, the final set of sine-Gordon raw TBA equations reads

$$\begin{aligned} \mathcal{M}^{(B_a)}(\vartheta) = &\varepsilon^{(B_a)}(\vartheta) + \\ &+ \sum_{b=1}^{N_B} (\eta^{(B_b)} K^{(B_a B_b)} * L^{(B_b)})(\vartheta) + \\ &+ (\eta^{(S)} K^{(SB_a)} * L^{(S)})(\vartheta) \end{aligned} \tag{2.81a}$$

$$\begin{aligned} \mathcal{M}^{(S)}(\vartheta) = &\varepsilon^{(S)}(\vartheta) + \\ &+ \sum_{a=1}^{N_B} (\eta^{(B_a)} K^{(SB_a)} * L^{(B_a)})(\vartheta) + \\ &+ (\eta^{(S)} K^{(SS)} * L^{(S)})(\vartheta) + \\ &+ \sum_{k=1}^{N_M} (\eta^{(M_k)} K^{(SM_k)} * L^{(M_k)})\left(\vartheta + \delta_{v_k,-}\, i\frac{\pi p}{2}\right) \end{aligned} \tag{2.81b}$$

$$\begin{aligned} \mathcal{M}^{(M_k)}(\vartheta) = &\varepsilon^{(M_k)}(\vartheta) + \\ &+ (\eta^{(S)} K^{(SM_k)} * L^{(S)})\left(\vartheta + \delta_{v_k,-}\, i\frac{\pi p}{2}\right) + \\ &+ \sum_{k'=1}^{N_M} (\eta^{(M_{k'})} K^{(M_k M_{k'})} * L^{(M_{k'})})\left(\vartheta + \delta_{v_k\, v_{k'},-}\, i\frac{\pi p}{2}\right) \end{aligned} \tag{2.81c}$$

with (the so-called 'driving terms')

$$\mathcal{M}^{(B_a)}(\vartheta) = M_a R \cosh(\vartheta) \tag{2.82a}$$

$$\mathcal{M}^{(S)}(\vartheta) = M R \cosh(\vartheta) \tag{2.82b}$$

$$\mathcal{M}^{(M_k)}(\vartheta) = 0 \tag{2.82c}$$

for $a = 1, \ldots, N_B$ in Eqs.(2.81a)(2.82a) and for $k = 1, \ldots, N_M$ in Eqs.(2.81c)(2.82c). Recall that the signs η are introduced in order to assure the rapidity densities positivity in Eq.(2.72): it is not difficult to see that they have to be positive for solitons and breathers (thanks to the presence of the mass term), while for magnons they vary depending on the species [44]

$$\eta^{(S)} = +1 \tag{2.83a}$$

$$\eta^{(B_a)} = +1\,, \qquad \text{for } a = 1, \ldots, N_B \tag{2.83b}$$

$$\eta^{(M_k)} = -(-1)^{r(k)}\,, \quad \text{for } k = 1, \ldots, N_M \tag{2.83c}$$

(with $r(k)$ being defined in Eq.(2.58d)).
The system of Eq.(2.81) is written for a general value of the sine-Gordon parameter, allowing for possible specializations.

Focusing on the structure of these equations, some further observations can be made. It may be noticed that, crucially, their derivation relies uniquely on the knowledge of the scattering theory: the only information required concerns the mass spectrum and the scattering amplitudes in the Ks' definitions. Thus, it may be expected that the TBA equations could reflect somehow a similar 'mathematical architecture' of the starting S-matrix theory. This point is at the core of the discussion presented in following sections.

Chapter 3

Universal TBA equations

What's 'raw' in Eq.(2.74) (or in Eq.(2.81) for the sine-Gordon model)? The reasons behind this informal label are actually quite deep.

As discussed in previous sections, these equations are at the basis of integrable models' thermodynamic descriptions, further allowing to bridge a given factorizable scattering theory (§1.2.4) with its field theory formulation (§2.1). They may be read as a set of nonlinear integral equations for the pseudoenergies (defined starting from roots and holes distributions in Eq.(2.52)). The set of raw TBA equations showcases a situation where all the pseudoenergies appear in each equation through logarithmic terms of Eq.(2.53), thus defining a very non-trivially coupled system.
However, it may be easily seen that these equations hinge solely on the scattering theory knowledge. If the latter shows an inherent mathematical structure, the TBA equations should be able to mirror it somehow. Indeed, this is what has been firstly found for reflectionless $\mathcal{ADE}$ scattering theories (i.e. related to simply-laced affine Lie algebras in the $\{\mathcal{ADE}\}$ series) in [48] and further developed in [56][57][58][59][60]. The raw TBA system can be rearranged, so to highlight the underlying mathematical structure. The new form is often referred to as 'universal' TBA equations.

This chapter, key in the current work, presents a formulation of universal TBA equations for the sine-Gordon model. After a quick mention of the scattering theory structures described above, these equations are derived for all the model's regimes. As done previously, the discussion is kept general as much as possible, before specializing it to the case of interest.

3.1 Scattering theory structures

The mathematical structure of integrable theories is quite a deep and rich topic, which proves difficult to be summarized in few words. Therefore, this section aims to provide just a swift overview, essential for subsequent developments. App.C is meant to represent a short handy reference for the more frequently mentioned mathematical objects. For a more detailed discussion on these subjects, [14][20].

Let's start by connecting with the previous discussion on scattering theories. In §1.2.4 are presented the main constitutive features of such theories and their close relation to conformal perturbation theory.
If the (unperturbed) CFTs are based on Kac-Moody algebras[1] [61], then they can enjoy a coset construction [62][63], where the underlying symmetries are highlighted. Of particular interest are the cases where the conformal families of the CFT are classified according to an algebra in the $\{\mathcal{ADE}\}$ series [64]: these are the simply-laced Lie algebras $\mathcal{A}_{\mathtt{n}}$, $\mathcal{D}_{\mathtt{n}}$, $\mathcal{E}_{\mathtt{n}}$[2]. When this happens, it is possible to perturb the theory by a relevant operator still preserving integrability [49]. The Lorentz spins of the perturbed CFT integrals of motion show, then, the remarkable pattern of being the exponents of one such Lie algebra, modulo its Coxeter number. The theory constructed is related to a central extension of the starting algebra: a simply-laced affine Lie algebra. For instance, this is explicit in the Lagrangian of affine Toda field theories[3] (for a review, [65]).
The diagonal scattering theories related to such CFT perturbations [46] also show remarkable patterns, connecting them to affine Lie algebras. These are features of the 'minimal' part of the scattering matrix, thus being independent on CDD factors. Very schematically,

- the number of particles in the $\mathcal{G}$-related scattering theory corresponds to the rank $\mathtt{r}_{\mathcal{G}}$ of the algebra.
- recalling that scattering amplitudes are meromorphic functions of rapidity difference, their poles are all equally spaced by $\Delta\vartheta = i2\pi\mathtt{h}_{\mathcal{G}}^{-1}$ along the imaginary rapidity axis. The symbol $\mathtt{h}_{\mathcal{G}}$ denotes here the Coxeter number of $\mathcal{G}$.
- the mass spectrum can be seen as building the components of the Perron-Frobenius eigenvector of the $\mathcal{G}$-associated Dynkin diagram incidence matrix $\boldsymbol{I}_{\mathcal{G}}$[4].

[1] Stated in a very basic way, this happens when a CFT enjoys additional symmetries besides the conformal one: the Kac-Moody algebra stems from commutation relations of the operators from the currents' mode expansion, in a similar way to what happens for the Virasoro algebra with the stress-energy tensor.

[2] More precisely, of the last type only the Lie algebras $\mathcal{E}_6$, $\mathcal{E}_7$ and $\mathcal{E}_8$ are present, while the case $\mathcal{A}_{2\mathtt{n}}$ requires the special reduction $\mathcal{T}_{\mathtt{n}} \stackrel{def}{=} \mathcal{A}_{2\mathtt{n}}/\mathcal{Z}_2$ [59].

[3] The sine-Gordon model can be viewed as a complex-parameter analytic continuation of one of such theories, the sinh-Gordon model.

[4] In short, that's a positive eigenvector related to the highest eigenvalue. The incidence matrix encodes the root structure of the algebra, featuring 0 or 1 entries depending on whether the roots are orthogonal or not. The Dynkin diagram is a graphical representation of the incidence matrix (App.C).

These patterns clearly highlight how that of simply-laced Lie algebras is the natural language to describe the physics of scattering theories. Let's stress, however, that what stated above refers to diagonal $\mathcal{ADE}$ theories only. Crucial generalizations are found in [59][60], but much more attention has to be paid in case of other theories. Still, this language is so powerful that it is worth to stretch slightly the notations, adapting to new meanings.

Scattering theory structures, *sine-Gordon model*

To get a better grasp on the concepts presented above, let's see how they apply to the sine-Gordon model, focusing on different values of the sine-Gordon parameter (Eq.(1.11)).

Being diagonal, the reflectionless points $0 < p < 1$, $p \in \frac{1}{\mathbb{N}}$ are well suited to display the patterns discussed.

Let's recall that for these couplings there exist $N_B = \frac{1}{p} - 1 \overset{def}{=} \mathtt{n} - 2$ breather bound states (the regime is attractive, as depicted in Fig.(1.3)), while the magnons do not need to be introduced (the scattering is already diagonal (§2.2.1)). With the soliton and the antisoliton this adds up to a total of $\mathtt{n} = \mathtt{r}_{\mathcal{D}_\mathtt{n}}$ different particles.

The simple poles in the physical strip of $S_0(\vartheta)$ are found to be evenly spaced by $\Delta\vartheta = i\pi p = i2\pi(2\mathtt{n}-2)^{-1} = i2\pi \mathtt{h}_{\mathcal{D}_\mathtt{n}}^{-1} \overset{def}{=} i2\pi\mathtt{h}^{-1}$ along the imaginary axis, as explained in fn.(2, p.24).

Also, the simple application of trigonometric identities reveals that the masses of Eqs. (1.18)(1.19) satisfy the relations

$$2\cos\left(\frac{\pi}{\mathtt{h}}\right)M_a \quad = M_{a-1} + M_{a+1}\,, \quad \text{for } 1 \le a \le N_B - 1 \tag{3.1a}$$

$$2\cos\left(\frac{\pi}{\mathtt{h}}\right)M_{N_B} = M_{N_B-1} + 2M \tag{3.1b}$$

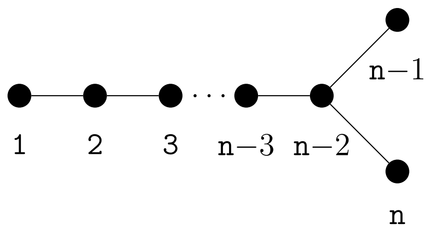
$$2\cos\left(\frac{\pi}{\mathtt{h}}\right)M \quad = M_{N_B}\,. \tag{3.1c}$$

These masses can be organized in a vector with $\mathtt{n}$ components $\{M_\mathtt{i}\}_{\mathtt{i}=1}^{\mathtt{n}} = \{M_1, \ldots, M_{N_B}, M, M\}$ which is evidently the Perron-Frobeniuns eigenvector of a $\mathcal{D}_\mathtt{n}$ Dynkin diagram incidence matrix:

$$\sum_{\mathtt{j}=1}^{\mathtt{n}} [\boldsymbol{I}_{\mathcal{D}_\mathtt{n}}]_{\mathtt{i},\mathtt{j}} M_\mathtt{j} = 2\cos\left(\frac{\pi}{\mathtt{h}}\right) M_\mathtt{i}\,, \tag{3.2}$$

with

$$
\boldsymbol{I}_{\mathcal{D}_{\mathtt{n}}} = \begin{bmatrix}
 & 1 & & \cdots & & & & \\
1 & & 1 & \cdots & & & & \\
 & 1 & & \ddots & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
 & & & \ddots & & 1 & & \\
 & & & \cdots & 1 & & 1 & 1 \\
 & & & \cdots & & 1 & & \\
 & & & \cdots & & 1 & &
\end{bmatrix}
$$



. (3.3)

This depiction shows that each node of the Dynkin diagram may be thought to as corresponding to a different particle (mass) of the theory, with links encoding the relations between them via the common eigenvalue term.
It is, then, clear that the sine-Gordon model at reflectionless points is a $\mathcal{D}_{\frac{1}{p}+1}$ scattering theory.

The structure changes when leaving the reflectionless points. When looking at points in the repulsive regime $p > 1$, breathers vanish from the mass spectrum leaving only the soliton-antisoliton doublet (and magnons). A diagram for the masses would result in just a single node in this case.

Also different is the behavior for the points $0 < p < 1$, $p \in \mathbb{Q} - \frac{1}{\mathbb{N}}$. These are attractive non-diagonal points with a finite number of magnons.
Recalling the continued fraction expression of Eq.(2.57), they can be written in general as $p = \frac{1}{N_B + \frac{1}{\alpha}} \stackrel{def}{=} \frac{1}{(\mathtt{n}-1)+\frac{1}{\alpha}}$, with $\alpha > 1$, $\alpha \in \mathbb{Q} - \frac{1}{\mathbb{N}}$. It is recovered a number $N_B \stackrel{def}{=} \mathtt{n} - 1$ of breather species, whose corresponding simple poles are again evenly spaced by $\Delta\vartheta = i2\pi p$. However, this number can be related in general to no Lie algebra Coxeter number. Furthermore, looking at the mass spectrum, it is composed by $\mathtt{n}$ different masses (the soliton and the antisoliton sharing the same mass), for which Eq.(3.1) generalizes to

$$2\cos\left(\frac{\pi p}{2}\right) M_a = M_{a-1} + M_{a+1}\,, \qquad \text{for } 1 \le a \le N_B - 1 \tag{3.4a}$$

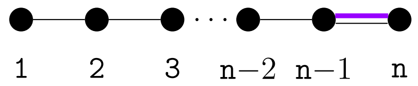
$$2\cos\left(\frac{\pi p}{2}\right) M_{N_B} = M_{N_B-1} + 2\cos\left(\frac{\pi(\alpha-1)p}{2\alpha}\right) \tag{3.4b}$$

$$2\cos\left(\frac{\pi p}{2\alpha}\right) M = M_{N_B}\,. \tag{3.4c}$$

This roughly reminds of the structure of a (not simply-laced) $\mathcal{B}_{\mathtt{n}}$ Dynkin diagram incidence matrix, albeit just for the position of its nonzero entries. In fact, posing $M_{\mathtt{n}} = M$, the vector built from the mass spectrum satisfies

$$\sum_{\mathtt{j}=1}^{\mathtt{n}} [\breve{\boldsymbol{I}}]_{\mathtt{i},\mathtt{j}} M_{\mathtt{j}} = \begin{cases} 2\cos\left(\frac{\pi p}{2}\right) M_{\mathtt{i}}\,, & \text{for } 1 \le \mathtt{i} \le \mathtt{n}-1 \\ 2\cos\left(\frac{\pi p}{2\alpha}\right) M_{\mathtt{n}}\,, & \text{for } \mathtt{i} = \mathtt{n} \end{cases}\,, \tag{3.5}$$

with

$$\breve{\boldsymbol{I}} = \begin{bmatrix} & 1 & & \dots & & & \\ 1 & & 1 & \dots & & & \\ & 1 & & \ddots & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & & 1 & \\ & & & \dots & 1 & & 2\cos\left(\frac{\pi(\alpha-1)p}{2\alpha}\right) \\ & & & \dots & & 1 & \end{bmatrix}$$ 

, (3.6)

but the latter is no incidence matrix in general (let alone in the $\mathcal{D}_{\mathtt{n}}$ form), as well as the former is no (Perron-Frobenius) eigenvector.

So, it is left open the question on whether the $\mathcal{D}_{\mathtt{n}}$-like structure of the sine-Gordon scattering theory is just an accident of the particular reflectionless points or not. Especially when reflected on the TBA equations' structure, this has great physical implications.

3.2 Sine-Gordon model universal TBA equations

As argued above, the TBA equations should present similar structures. The core idea is that of making them emerge rearranging the system, by means of pivotal identities for the kernels in Eq.(2.70) (Eq.(2.79) for the sine-Gordon model).

The latter are made explicit when formulating the TBA equations in Fourier space. Here it is followed the convention

$$(\mathscr{F} f)(\xi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\vartheta\ \mathrm{e}^{-i\xi\vartheta}\, f(\vartheta) \tag{3.7a}$$

$$(\bar{\mathscr{F}} f)(\vartheta) = \int_{-\infty}^{+\infty} \mathrm{d}\vartheta\ \mathrm{e}^{i\vartheta\xi}\, f(\xi)\,, \tag{3.7b}$$

so that function convolutions and products of Fourier-transformed functions are related by

$$(\mathscr{F}(f * g))(\xi) = 2\pi(\mathscr{F} f)(\xi)\cdot(\mathscr{F} g)(\xi) \tag{3.8a}$$

$$(\bar{\mathscr{F}}(\mathscr{F} f \cdot \mathscr{F} g))(\vartheta) = (f * g)(\vartheta)\,. \tag{3.8b}$$

Adopting similar notations to Eq.(2.74) (with the only introduction of the symbol $\mathcal{M}^{(s)}(\vartheta) = m^{(s)} R\cosh\vartheta$ to denote the driving term for particle species s), the raw TBA equations in Fourier space can be written in general as

$$\mathscr{F}\mathcal{M}^{(s)}(\xi) = \mathscr{F}\varepsilon^{(s)}(\xi) + \sum_{s'} \eta^{(s')}\, 2\pi\mathscr{F} K^{(ss')}(\xi)\,\mathscr{F} L^{(s')}(\xi)\,. \tag{3.9}$$

For the sine-Gordon model (Eq.(2.81)), this explicitly reads

$$\begin{aligned}
\mathscr{F}\mathcal{M}^{(B_a)}(\xi) =& \mathscr{F}\varepsilon^{(B_a)}(\xi)+\\
&+\sum_{b=1}^{N_B}\eta^{(B_b)}\,2\pi\mathscr{F}K^{(B_aB_b)}(\xi)\,\mathscr{F}L^{(B_b)}(\xi)+\\
&+\eta^{(S)}\,2\pi\mathscr{F}K^{(SB_a)}(\xi)\,\mathscr{F}L^{(S)}(\xi) \qquad (3.10a)\\
\mathscr{F}\mathcal{M}^{(S)}(\xi) =& \mathscr{F}\varepsilon^{(S)}(\xi)+\\
&+\sum_{a=1}^{N_B}\eta^{(B_a)}\,2\pi\mathscr{F}K^{(SB_a)}(\xi)\,\mathscr{F}L^{(B_a)}(\xi)+\\
&+\eta^{(S)}\,2\pi\mathscr{F}K^{(SS)}(\xi)\,\mathscr{F}L^{(S)}(\xi)+\\
&+\sum_{k=1}^{N_M}\eta^{(M_k)}\,2\pi\mathscr{F}K^{(SM_k)}(\xi)\,\mathscr{F}L^{(M_k)}(\xi) \qquad (3.10b)\\
\mathscr{F}\mathcal{M}^{(M_k)}(\xi) =& \mathscr{F}\varepsilon^{(M_k)}(\xi)+\\
&+\eta^{(S)}\,2\pi\mathscr{F}K^{(SM_k)}(\xi)\,\mathscr{F}L^{(S)}(\xi)+\\
&+\sum_{k'=1}^{N_M}\eta^{(M_{k'})}\,2\pi\mathscr{F}K^{(M_kM_{k'})}(\xi)\,\mathscr{F}L^{(M_{k'})}(\xi)\,, \qquad (3.10c)
\end{aligned}$$

for $a = 1,\ldots,N_B$ in Eq.(3.10a) and for $k = 1,\ldots,N_M$ in Eq.(3.10c). The $\mathcal{M}$-driving terms are defined in Eq.(2.82), while the η-signs in Eq.(2.83). Lengthy but straightforward calculations (some observations may be found in App.D) show that the Fourier transform of the K-kernels may be written as

$$\begin{aligned}
\mathscr{F}K^{(B_aB_b)}(\xi) =& \frac{1}{2\pi}\left(\delta_{a,b}-2\frac{\cosh\frac{\pi}{2}(1-pa)\xi\,\sinh\frac{\pi}{2}pb\xi\,\cosh\frac{\pi}{2}p\xi}{\sinh\frac{\pi}{2}p\xi\,\cosh\frac{\pi}{2}\xi}\right) \qquad (3.11a)\\
\mathscr{F}K^{(SB_a)}(\xi) =& -\frac{1}{2\pi}\frac{\sinh\frac{\pi}{2}pa\xi\,\cosh\frac{\pi}{2}p\xi}{\sinh\frac{\pi}{2}p\xi\,\cosh\frac{\pi}{2}\xi} \qquad (3.11b)\\
\mathscr{F}K^{(SS)}(\xi) =& -\frac{1}{2\pi}\frac{\sinh\frac{\pi}{2}(1-p)\xi}{2\sinh\frac{\pi}{2}p\xi\,\cosh\frac{\pi}{2}\xi} \qquad (3.11c)\\
\mathscr{F}K^{SM_k}(\xi) =& \mathscr{F}\kappa(\xi;\mathfrak{n}_k,v_k) \qquad (3.11d)\\
\mathscr{F}K^{(M_kM_{k'})}(\xi) =& -(1-\delta_{\mathfrak{n}_k,\mathfrak{n}_{k'}})\,\mathscr{F}\kappa(\xi;|\mathfrak{n}_k-\mathfrak{n}_{k'}|,v_k\cdot v_{k'})+\\
&-2\sum_{\ell_{kk'}=1}^{\min\{\mathfrak{n}_k,\mathfrak{n}_{k'}\}-1}\mathscr{F}\kappa(\xi;|\mathfrak{n}_k-\mathfrak{n}_{k'}|+2\,\ell_{kk'},v_k\cdot v_{k'})+\\
&-\mathscr{F}\kappa(\xi;\mathfrak{n}_k+\mathfrak{n}_{k'},v_k\cdot v_{k'})\,, \qquad (3.11e)
\end{aligned}$$

with

$$\mathscr{F}\kappa(\xi; n, +) = \frac{(1-\delta_{0,n \bmod(0,\alpha)})}{2\pi}\frac{\sinh(n \bmod(0,2\alpha)-\alpha)\beta^{-1}\xi}{\sinh\alpha\beta^{-1}\xi} \tag{3.12a}$$

$$\mathscr{F}\kappa(\xi; n, -) = \frac{(1-\delta_{0,n \bmod(0,\alpha)})}{2\pi}\frac{\sinh(n \bmod(-\alpha,\alpha))\beta^{-1}\xi}{\sinh\alpha\beta^{-1}\xi}\,. \tag{3.12b}$$

Of course, the above equations have to be specialized for the different regimes of the sine-Gordon model, but this is the most complete formulation and it may be assumed as starting point for the subsequent derivation. Of interest are, in general, the rational points $p \in \mathbb{Q}$, for which the (pseudo)particle content of the theory is finite.

3.2.1 Reflectionless points

Let's start from the well-known reflectionless points $0 < p < 1$, $p \in \frac{1}{\mathbb{N}}$. These are the values of the sine-Gordon parameter for which, as discussed in §3.1, the theory shows a $\mathcal{D}_{\frac{1}{p}+1}$ structure. Thus, this is just a particular case of the more general universal TBA formulation for diagonal $\mathcal{ADE}$ scattering theories.

Given a theory connected to the affine Lie algebra $\mathcal{G}$ in the $\{\mathcal{ADE}\}$ series (with $\mathtt{n}$ particles, Coxeter number $\mathtt{h}$ and Dynkin diagram incidence matrix $\boldsymbol{I}_{\mathcal{G}}$), it holds a fundamental matrix identity for the Fourier space K-kernels. Let's address it as 'Zamolodchikov identity': it is in the form

$$[\mathbb{I} - 2\pi\mathscr{F}K]^{-1}(\xi) = \mathbb{I} - \frac{1}{2\cosh\frac{\pi}{\mathtt{h}}\xi}\boldsymbol{I}_{\mathcal{G}}\ . \tag{3.13}$$

Here, $\mathbb{I}$ denotes the identity matrix, while $[2\pi\mathscr{F}K](\xi)$ stands for a $\mathtt{n}\times\mathtt{n}$ matrix whose $(\mathtt{i},\mathtt{j})$ entry $[2\pi\mathscr{F}K]_{\mathtt{i},\mathtt{j}}(\xi)$ is given by $2\pi\mathscr{F}K^{(ss')}(\xi)$, s and s' being the species corresponding respectively to the $\mathtt{i}^{\text{th}}$ and $\mathtt{j}^{\text{th}}$ Dynkin diagram nodes. This relation, which clearly resolves the scattering theory in the corresponding simply-laced Lie algebra, appeared first in [48], while in [59] is presented a very elegant way of proving it. The essential idea is that of formulating the scattering amplitudes of the theory so that the underlying algebraic structure is already manifest: this allows to extract some identities that, suitably manipulated, result in Eq.(3.13).

This relation can indeed be applied to the raw TBA system, greatly simplifying it and making the inherent mathematical architecture emerge. The general Eq.(3.9) (changing to Dynkin diagram indices) can be written as

$$(\mathscr{F}(\varepsilon + L - \mathcal{M})_{\mathtt{i}})(\xi) = \sum_{\mathtt{j}=1}^{\mathtt{r}_{\mathcal{G}}}[\mathbb{I} - 2\pi\mathscr{F}K]_{\mathtt{i},\mathtt{j}}(\xi)\,\mathscr{F}L_{\mathtt{j}}(\xi)\,, \tag{3.14}$$

where, being all the particles of the theory massive, the η-signs are all positive. It is sufficient to multiply by the right-hand side of Eq.(3.13) to get

$$\mathscr{F}\mathcal{M}_{\mathtt{i}}(\xi) = \mathscr{F}\varepsilon_{\mathtt{i}}(\xi) + \sum_{\mathtt{j}=1}^{\mathtt{r}_{\mathcal{G}}} [\boldsymbol{I}_{\mathcal{G}}]_{\mathtt{i},\mathtt{j}} \frac{1}{2\cosh\frac{\pi}{\mathtt{h}}\xi} (\mathscr{F}(\mathcal{M}-\varepsilon-L)_{\mathtt{j}})(\xi)\,, \tag{3.15}$$

or, posing $\tilde{\varphi}_0(\xi) \overset{def}{=} \frac{1}{2\cosh\frac{\pi}{\mathtt{h}}\xi}$ with anti-Fourier transform $\varphi_0(\vartheta) \overset{def}{=} (\bar{\mathscr{F}}\tilde{\varphi})(\vartheta) = \frac{\mathtt{h}}{2\cosh\frac{\mathtt{h}}{2}\vartheta}$,

$$\mathcal{M}_{\mathtt{i}}(\vartheta) = \varepsilon_{\mathtt{i}}(\vartheta) + \sum_{\mathtt{j}=1}^{\mathtt{r}_{\mathcal{G}}} [\boldsymbol{I}_{\mathcal{G}}]_{\mathtt{i},\mathtt{j}} (\varphi_0 * (\mathcal{M}-\varepsilon-L)_{\mathtt{j}})(\vartheta)\,. \tag{3.16}$$

This is the universal form of the TBA system for diagonal $\mathcal{ADE}$ scattering theories. From here, it is manifest that the equations for the pseudoenergies are coupled following the structure of the Dynkin diagram incidence matrix: a much more clear situation when compared with the raw TBA system. Even more, it is actually possible to start with a given algebraic structure and proceed the other way around to the formulation of a theory with the above universal TBA.

For later convenience, it may be introduced a generalized notation for universal TBA equations. Let $\{\tilde{\varphi}_i(\xi)\}_{i\in\Lambda}$ and $\{\tilde{\varphi}_i^{self}(\xi)\}_{i\in\Lambda'}$ (with anti-Fourier transform $(\bar{\mathscr{F}}\,\tilde{\varphi}_i)(\vartheta) \overset{def}{=} \varphi_i(\vartheta)$ and $(\bar{\mathscr{F}}\,\tilde{\varphi}_i^{self})(\xi) \overset{def}{=} \varphi_i^{self}(\vartheta)$) denote two sets of kernels, which can be absorbed into the entries of a matrix $[\breve{\boldsymbol{I}}(\tilde{\varphi})](\xi)$. Notice that the latter is not a Dynkin diagram incidence matrix in general, but it may be used nevertheless to encode a ($\mathtt{n}$-nodes) graph. Also, let $\{\sigma_{\mathcal{M}_{\mathtt{i}}}\}_{\mathtt{i}=1}^{\mathtt{n}}$ and $\{\sigma_{\varepsilon_{\mathtt{i}}}\}_{\mathtt{i}=1}^{\mathtt{n}}$ stand for two sets of numbers with values in $\{0,1\}$.
It is, then, clear that Eqs.(3.16)(3.15) adhere to the form

$$\boxed{\mathscr{F}\mathcal{M}_{\mathtt{i}}(\xi) = \mathscr{F}\varepsilon_{\mathtt{i}}(\xi) + \sum_{\mathtt{j}=1}^{\mathtt{n}} [\breve{\boldsymbol{I}}(\tilde{\varphi})]_{\mathtt{i},\mathtt{j}}(\xi)\ \mathscr{F}\mathcal{W}_{\mathtt{j}}(\xi)} \tag{3.17}$$

or

$$\mathcal{M}_{\mathtt{i}}(\vartheta) = \varepsilon_{\mathtt{i}}(\vartheta) + \sum_{\mathtt{j}=1}^{\mathtt{n}} ([\breve{\boldsymbol{I}}(\varphi)]_{\mathtt{i},\mathtt{j}} * \mathcal{W}_{\mathtt{j}})(\vartheta)\,, \tag{3.18}$$

with

$$\mathcal{W}_{\mathtt{i}}(\vartheta) = (\mathcal{M} - \varepsilon - L)_{\mathtt{i}}(\vartheta)\,, \tag{3.19}$$

when identifying $\{\tilde{\varphi}_i(\xi)\}_{i\in\Lambda} = \{\tilde{\varphi}_0\}$ (the second set of kernels being absent), $\{\sigma_{\mathcal{M}_{\mathtt{i}}}\}_{\mathtt{i}=1}^{\mathtt{n}} = \{\sigma_{\varepsilon_{\mathtt{i}}}\}_{\mathtt{i}=1}^{\mathtt{n}} = \{1,1,\ldots,1\}$, $[\breve{\boldsymbol{I}}(\tilde{\varphi})](\xi) = \boldsymbol{I}_{\mathcal{G}} \cdot \tilde{\varphi}_0(\xi)$ and $\mathtt{n} = \mathtt{r}_{\mathcal{G}}$.

In Eqs.(3.17)(3.18) are relaxed the strict requirements of $\mathcal{ADE}$ Dynkin diagram incidence matrices in favor of an increased adaptability. Still, it is retained the powerful encoding of TBA systems in a graphical way.

Returning to the sine-Gordon model at reflectionless points ($p = (N_B+1)^{-1}$, i.e. $\alpha = 1$), as anticipated it consists in an application of this general discussion. Nevertheless, it may be useful to better appreciate the formalism.

In this case, the scattering theory is connected to a $\mathcal{D}_{\mathtt{n}}$ Lie algebra, whose rank and Coxeter number are respectively $\mathtt{r}_{\mathcal{D}_{\mathtt{n}}} = \mathtt{n}$ and $\mathtt{h}_{\mathcal{D}_{\mathtt{n}}} = 2\,\mathtt{n} - 2$, with $\mathtt{n} = \frac{1}{p} + 1$. Following the derivation presented, it is possible to write the Zamolodchikov identity of Eq.(3.13) with $\boldsymbol{I}_{\mathcal{D}_{\mathtt{n}}}$ as Dynkin diagram incidence matrix. With the use of trigonometric identities and some patience, it may even be proven directly, without invoking more general features. Associating particle species to graph node indices (in an ordered way) as

$$\{B_1, B_2, \ldots, B_{N_B}, S, \bar{S}\} \mapsto \{\mathtt{1}, \mathtt{2}, \ldots, \mathtt{n}-2, \mathtt{n}-1, \mathtt{n}\} \tag{3.20}$$

and defining the kernel

$$\tilde{\varphi}_0(\xi) = \frac{1}{2\cosh\frac{\pi}{2}p\xi}\,, \tag{3.21}$$

(i.e. $\{\tilde{\varphi}_i(\xi)\}_{i\in\Lambda} = \{\tilde{\varphi}_0\}$, $\{\tilde{\varphi}_i^{self}(\xi)\}_{i\in\Lambda'} = \emptyset$) the Zamolodchikov identities explicitly read

$$2\pi\mathscr{F}K_{\mathtt{1},\mathtt{j}} - \tilde{\varphi}_0\, 2\pi\mathscr{F}K_{\mathtt{2},\mathtt{j}} = -\,\tilde{\varphi}_0\,\delta_{\mathtt{2},\mathtt{j}} \tag{3.22a}$$

$$2\pi\mathscr{F}K_{\mathtt{i},\mathtt{j}} - \tilde{\varphi}_0\, 2\pi\mathscr{F}K_{\mathtt{i-1},\mathtt{j}} - \tilde{\varphi}_0\, 2\pi\mathscr{F}K_{\mathtt{i+1},\mathtt{j}} = -\,\tilde{\varphi}_0\,\delta_{\mathtt{i-1},\mathtt{j}} - \tilde{\varphi}_0\,\delta_{\mathtt{i+1},\mathtt{j}} \tag{3.22b}$$

$$\begin{aligned} &2\pi\mathscr{F}K_{\mathtt{n-2},\mathtt{j}} - \tilde{\varphi}_0\, 2\pi\mathscr{F}K_{\mathtt{n-3},\mathtt{j}} - \tilde{\varphi}_0\, 2\pi\mathscr{F}K_{\mathtt{n-1},\mathtt{j}} = \\ &\quad = -\,\tilde{\varphi}_0\,\delta_{\mathtt{n-3},\mathtt{j}} - \tilde{\varphi}_0\,\delta_{\mathtt{n-1},\mathtt{j}} - \tilde{\varphi}_0\,\delta_{\mathtt{n},\mathtt{j}} \end{aligned} \tag{3.22c}$$

$$2\pi\mathscr{F}K_{\mathtt{n-1},\mathtt{j}} - \tilde{\varphi}_0\, 2\pi\mathscr{F}K_{\mathtt{n-2},\mathtt{j}} = -\,\tilde{\varphi}_0\,\delta_{\mathtt{n-2},\mathtt{j}} \tag{3.22d}$$

$$2\pi\mathscr{F}K_{\mathtt{n},\mathtt{j}} - \tilde{\varphi}_0\, 2\pi\mathscr{F}K_{\mathtt{n-2},\mathtt{j}} = -\,\tilde{\varphi}_0\,\delta_{\mathtt{n-2},\mathtt{j}}\,, \tag{3.22e}$$

for $\mathtt{2} \le \mathtt{i} < \mathtt{n-2}$ in Eq.(3.22b); the ξ dependency is here subtended for simplicity.

Applying these to the raw TBA system of Eq.(3.10) (where only breathers and the (anti)soliton are present), it yields as expected Eq.(3.15), i.e. Eq.(3.17) with

$$\{\sigma_{\mathcal{M}_{\mathtt{i}}}\}_{\mathtt{i}=1}^{\mathtt{n}} = \{1, 1, \ldots, 1\} \tag{3.23a}$$

$$\{\sigma_{\varepsilon_{\mathtt{i}}}\}_{\mathtt{i}=1}^{\mathtt{n}} = \{1, 1, \ldots, 1\} \tag{3.23b}$$

and

$$\breve{\boldsymbol{I}}(\tilde{\varphi}) = \boldsymbol{I}_{\mathcal{D}_{\mathtt{n}}}\cdot\tilde{\varphi}_0 = \begin{bmatrix} & \tilde{\varphi}_0 & & \cdots & & & & \\ \tilde{\varphi}_0 & & \tilde{\varphi}_0 & \cdots & & & & \\ & \tilde{\varphi}_0 & & \ddots & & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ & & & \ddots & & \tilde{\varphi}_0 & & \\ & & & \cdots & \tilde{\varphi}_0 & & \tilde{\varphi}_0 & \tilde{\varphi}_0 \\ & & & \cdots & & \tilde{\varphi}_0 & & \\ & & & \cdots & & \tilde{\varphi}_0 & & \end{bmatrix}, \tag{3.24}$$

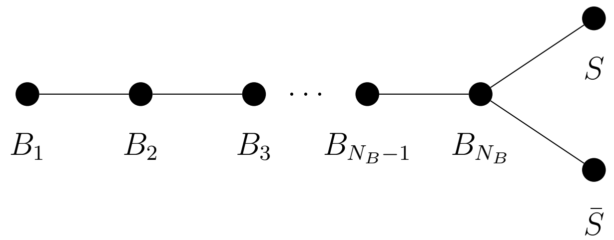
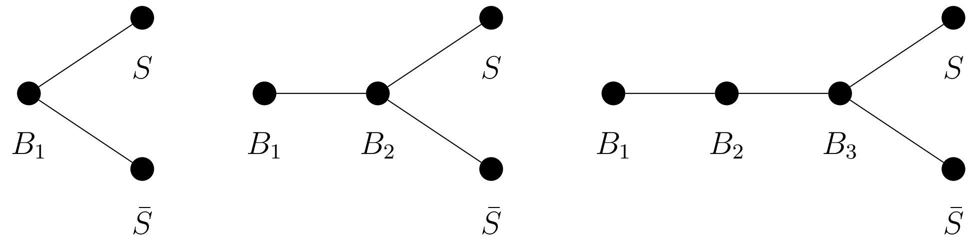


Figure 3.1: Structure of the TBA equations for the sine-Gordon model at reflectionless points. Each massive particle species is represented with a full dot: if the related equations are coupled through the kernel $\tilde{\varphi}_0$, there exist a link connecting them with $0+1$ lines. Since the incidence matrix of Eq.(3.24) is symmetric, the links are not oriented.



(a) $p = \frac{1}{2}$ (b) $p = \frac{1}{3}$ (c) $p = \frac{1}{4}$

Figure 3.2: Structure of the TBA equations for the sine-Gordon model at some reflectionless points. Notice how, as p decreases, the diagram prologues, since new breather species become accessible (Fig.(1.3)).

with diagrammatic representation in Fig.(3.1) (some instances of application may be found in Fig.(3.2)).

Explicitly,

$$\mathcal{M}^{(B_1)} = \varepsilon^{(B_1)} + \varphi_0 * \mathcal{W}^{(B_2)} \tag{3.25a}$$

$$\mathcal{M}^{(B_a)} = \varepsilon^{(B_a)} + \varphi_0 * \mathcal{W}^{(B_{a-1})} + \varphi_0 * \mathcal{W}^{(B_{a+1})} \tag{3.25b}$$

$$\mathcal{M}^{(B_{N_B})} = \varepsilon^{(B_{N_B})} + (\Theta_{N_B>1})\varphi_0 * \mathcal{W}^{(B_{N_B-1})} + \varphi_0 * \mathcal{W}^{(S)} + \varphi_0 * \mathcal{W}^{(\bar{S})} \tag{3.25c}$$

$$\mathcal{M}^{(S)} = \varepsilon^{(S)} + \varphi_0 * \mathcal{W}^{(B_{N_B})} \tag{3.25d}$$

$$\mathcal{M}^{(\bar{S})} = \varepsilon^{(\bar{S})} + \varphi_0 * \mathcal{W}^{(B_{N_B})} \tag{3.25e}$$

for $a = 2, \ldots, N_B - 1$ in Eq.(3.25b), while Eq.(3.25a) is present if $N_B > 1$ (i.e. for the reflectionless $p < \frac{1}{2}$). Θ denotes here the Heaviside-Θ symbol. Again, the ϑ dependence is subtended.

The structure of these equations is well encoded in graphs such as that of Fig.(3.1).

3.2.2 Repulsive points

Another class of sine-Gordon parameter values is characterized by $p > 1$, $p \in \mathbb{Q}$. They are repulsive points, featuring no breather particles and a finite number of magnons. This means that, looking at the continued fraction expression of Eq.(2.57), $p = \alpha$, with a finite continued fraction that extends up to a level λ, i.e. $\frac{1}{\alpha} = c.f.\{\nu_1, \nu_2, \ldots, \nu_\lambda\}$.

The aim is again that of reducing the raw TBA system into a universal form similar to Eq.(3.17), through the application of some pivotal identities. However, new features appear in this case: clearly, the sine-Gordon scattering theory is not diagonal at these values, so differences w.r.t the previous reflectionless points are expected.

To see this better, let's start analyzing the particular sub-case of $p \in \mathbb{N} - \{1\}$. They are points for which the continued fraction extends only up to the first level $\lambda = 1$:

$$p = \cfrac{1}{N_B + \cfrac{1}{\alpha}} = \cfrac{1}{0 + \cfrac{1}{\nu_1}} = \nu_1 \,. \tag{3.26}$$

The particle content of the theory in this case entails the (anti)soliton (which, on the line of §2.3 discussion, is treated as a unique doublet) and $N_M = \alpha = \nu_1 = m_1$ magnons (recall Eq.(2.58)).

Thus, it is expected that the universal TBA equations can be encoded on a graph with $m_1 + 1$ nodes. Let's establish the correspondence to graph nodes indices as

$$\{S, M_1, M_2, \ldots, M_{m_1}\} \mapsto \{\mathtt{1}, \mathtt{2}, \mathtt{3}, \ldots, \mathtt{n}\} \,. \tag{3.27}$$

What remains is to derive a set of identities, which are able to simplify the raw TBA system. To this aim, it is possible to rely on the intimate connection between the sine-Gordon model and the $XXZ_{\frac{1}{2}}$ spin chain model. Concerning the latter, in [44, Eq.(3.8), $(a_j \equiv K_{\mathtt{1},\mathtt{j}+1})$, $(T_{i,j} \equiv K_{\mathtt{i}+1,\mathtt{j}+1})$, $(s_i \equiv \varphi_i)$, $(d_i \equiv \varphi_i^{self})$] appear fundamental relations that are addressed here as 'Takahashi-Suzuki identities'. They are actually formulated for a case that would correspond to a positive real non-rational p, but, through lengthy calculations, they can be verified and adapted to the current case. Introducing the kernel

$$\tilde{\varphi}_1(\xi) = \frac{1}{2\cosh\frac{\pi}{2}\xi} \,, \tag{3.28}$$

(i.e. $\{\tilde{\varphi}_i(\xi)\}_{i\in\Lambda} = \{\tilde{\varphi}_1\}$, $\{\tilde{\varphi}_i^{self}(\xi)\}_{i\in\Lambda'} = \emptyset$) the result reads

$$2\pi\mathscr{F}K_{\mathtt{1},\mathtt{j}} + \tilde{\varphi}_1\, 2\pi\mathscr{F}K_{\mathtt{2},\mathtt{j}} = -\,\tilde{\varphi}_1\,\delta_{\mathtt{2},\mathtt{j}} + (\delta_{0,m_1-2})\,\tilde{\varphi}_1\,\delta_{\mathtt{3},\mathtt{j}} \tag{3.29a}$$

$$\begin{aligned} &2\pi\mathscr{F}K_{\mathtt{2},\mathtt{j}} - (1-\delta_{0,m_1-2})\,\tilde{\varphi}_1\, 2\pi\mathscr{F}K_{\mathtt{3},\mathtt{j}} = \\ &\quad = -\,\tilde{\varphi}_1\,\delta_{\mathtt{1},\mathtt{j}} + (1-\delta_{0,m_1-2})\,\tilde{\varphi}_1\,\delta_{\mathtt{3},\mathtt{j}} - (1-\delta_{0,m_1-3})\,\tilde{\varphi}_1\,\delta_{\mathtt{4},\mathtt{j}} \end{aligned} \tag{3.29b}$$

$$
\begin{aligned}
& 2\pi\mathscr{F}K_{\mathtt{i},\mathtt{j}} - \tilde{\varphi}_1\, 2\pi\mathscr{F}K_{\mathtt{i}-1,\mathtt{j}} - \tilde{\varphi}_1\, 2\pi\mathscr{F}K_{\mathtt{i}+1,\mathtt{j}} = \\
& \quad = \tilde{\varphi}_1\, \delta_{\mathtt{i}-1,\mathtt{j}} + \tilde{\varphi}_1\, \delta_{\mathtt{i}+1,\mathtt{j}} - (\delta_{\mathtt{i},\mathtt{n}-2})\, \tilde{\varphi}_1\, \delta_{\mathtt{n},\mathtt{j}} && \text{(3.29c)} \\
& 2\pi\mathscr{F}K_{\mathtt{n}-1,\mathtt{j}} - \tilde{\varphi}_1\, 2\pi\mathscr{F}K_{\mathtt{n}-2,\mathtt{j}} = \tilde{\varphi}_1\, 2\pi\mathscr{F}K_{\mathtt{n}-2,\mathtt{j}} && \text{(3.29d)} \\
& 2\pi\mathscr{F}K_{\mathtt{n},\mathtt{j}} + \tilde{\varphi}_1\, 2\pi\mathscr{F}K_{\mathtt{n}-2,\mathtt{j}} = -\,\tilde{\varphi}_1\, 2\pi\mathscr{F}K_{\mathtt{n}-2,\mathtt{j}}\ , && \text{(3.29e)}
\end{aligned}
$$

for $2 < \mathtt{i} < \mathtt{n}-2$ in Eq.(3.29c), which holds if $m_1 \neq 2, 3$ (otherwise see Eqs.(3.29a)(3.29b) respectively) while Eq.(3.29d) holds if $m_1 \neq 2$ (otherwise see Eq.(3.29b)).

The raw TBA system equations can, then, be combined to obtain terms in the left-hand side of Eq.(3.29), thus allowing to apply these Takahashi-Suzuki identities. Notice that, differently from the reflectionless case, the η-signs depend on particle species as in Eq.(2.83) and are not all positive ($\eta_{\mathtt{i}} = (-1)^{\delta_{\mathtt{i},1}+\delta_{\mathtt{i},\mathtt{n}}}$): they should be treated carefully. After few passages, this yields a universal TBA system in the form of Eq.(3.17) with

$$
\begin{aligned}
\{\sigma_{\mathcal{M}_{\mathtt{i}}}\}_{\mathtt{i}=1}^{\mathtt{n}} &= \{0, 0, \ldots, 0\} && \text{(3.30a)} \\
\{\sigma_{\varepsilon_{\mathtt{i}}}\}_{\mathtt{i}=1}^{\mathtt{n}} &= \{0, 1, 1, \ldots, 1, 0\} && \text{(3.30b)}
\end{aligned}
$$

and

$$
\breve{\boldsymbol{I}}(\tilde{\varphi}) = \begin{bmatrix}
 & -\tilde{\varphi}_1 & & \cdots & & & & \\
\tilde{\varphi}_1 & & \tilde{\varphi}_1 & \cdots & & & & \\
 & \tilde{\varphi}_1 & & \ddots & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
 & & & \ddots & & \tilde{\varphi}_1 & & \\
 & & & \cdots & \tilde{\varphi}_1 & & \tilde{\varphi}_1 & \tilde{\varphi}_1 \\
 & & & \cdots & & \tilde{\varphi}_1 & & \\
 & & & \cdots & & -\tilde{\varphi}_1 & &
\end{bmatrix} . \qquad (3.31)
$$

The latter is encoded in a diagrammatic form in Fig.(3.3), with some exemplifications in Fig.(3.4)).

Some comments are in order.
It may be observed that the structure obtained vaguely resembles the one at reflectionless points: even if breathers are absent, the magnons seem to help recovering a similar overall behavior. However, albeit for the positions of the non-zero entries, Eq.(3.31) clearly is not in the form of an $\mathcal{ADE}$ Dynkin diagram incidence matrix because of the appearance of negative entries. A clever way to fully recover such a Dynkin structure would be that of absorbing the η-signs in the pseudoenergy definitions: when doing so, $[\breve{\boldsymbol{I}}(\tilde{\varphi}_1)](\xi) \to -\boldsymbol{I}_{\mathcal{D}_{m_1}+1}\,\tilde{\varphi}_1(\xi)$ (a swift elaboration of this point is to be found in §4.1.2). Still, Eqs.(3.30a)(3.30b) show that some driving terms and pseudoenergies disappear from the right-hand side of the universal TBA equations, even if the L-terms (depending on the pseudoenergies, Eq.(2.53)) remain. This marks a difference with the reflectionless case.

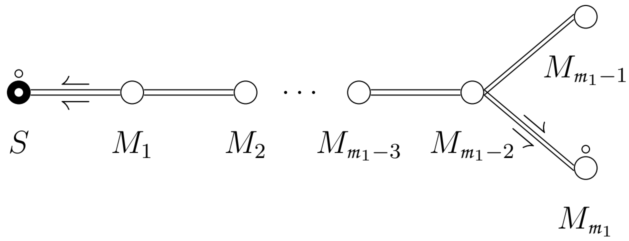


Figure 3.3: Structure of the TBA equations for the sine-Gordon model at integer points. As for the reflectionless points, full dots represent massive particle species, while empty circles stands for the massless ones (i.e.) magnons. However, a white dot is added on those nodes for which the $\sigma_{\mathcal{M}}$-number (Eq.(3.30a)) is zero instead of one (this does not make any difference for the magnons since their driving term is absent anyway, but for the (anti)soliton it is relevant). Similarly, a null σ_ε (Eq.(3.30b)) is represented by a white dot put above the correspondent node. Then, if two equations are coupled through the kernel $\tilde{\varphi}_1$, there exist a link connecting them with $1+1$ lines. Some of these links are oriented since the matrix of Eq.(3.31) features also negative entries: by standard notation, the arrows point towards the node whose corresponding matrix line shows the negative entry.

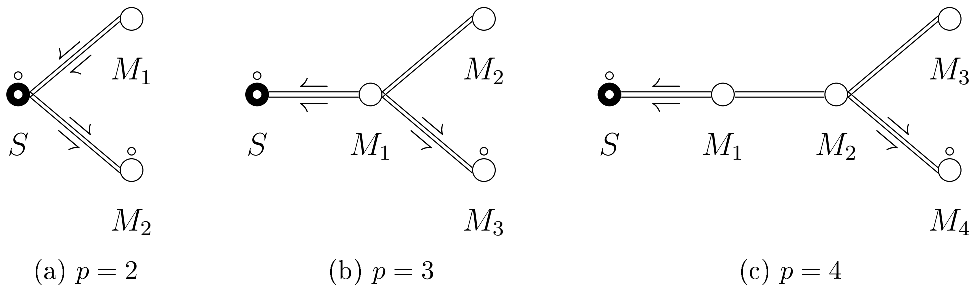


Figure 3.4: Examples of diagrams at some integer values of the sine-Gordon parameter. Thanks to the encoding described in Fig.(3.3), it is possible to read $[\check{\boldsymbol{I}}(\tilde{\varphi})](\xi)$ for these cases simply looking at the diagrams above. Observe that, as more magnon species are introduced, they are added to the diagram, without inducing changes in the overall structure.

Collecting all the above, it is possible to write the universal TBA system for the sine-Gordon model at integer points as follows:

$$\begin{aligned}
\mathcal{M}^{(S)} &= \varepsilon^{(S)} - \varphi_1 * \mathcal{W}^{(M_1)} + (\delta_{0,m_1-2})\varphi_1 * \mathcal{W}^{(M_2)} && (3.32a)\\
\mathcal{M}^{(M_1)} &= \varepsilon^{(M_1)} + \varphi_1 * \mathcal{W}^{(S)} + (1-\delta_{0,m_1-2})\varphi_1 * \mathcal{W}^{(M_2)} + (\delta_{0,m_1-3})\varphi_1 * \mathcal{W}^{(M_3)} && (3.32b)\\
\mathcal{M}^{(M_k)} &= \varepsilon^{(M_k)} + \varphi_1 * \mathcal{W}^{(M_{k-1})} + \varphi_1 * \mathcal{W}^{(M_{k+1})} && (3.32c)\\
\mathcal{M}^{(M_{m_1-2})} &= \varepsilon^{(M_{m_1-2})} + \varphi_1 * \mathcal{W}^{(M_{m_1-3})} + \varphi_1 * \mathcal{W}^{(M_{m_1-1})} + \varphi_1 * \mathcal{W}^{(M_{m_1})} && (3.32d)\\
\mathcal{M}^{(M_{m_1-1})} &= \varepsilon^{(M_{m_1-1})} + \varphi_1 * \mathcal{W}^{(M_{m_1-2})} && (3.32e)\\
\mathcal{M}^{(M_{m_1})} &= \varepsilon^{(M_{m_1})} - \varphi_1 * \mathcal{W}^{(M_{m_1-2})} \, , && (3.32f)
\end{aligned}$$

for $1 < k < m_1 - 2$ in Eq.(3.32c), while Eq.(3.32d) holds if $m_1 \neq 2, 3$ (otherwise see Eqs.(3.32a)(3.32b)) and Eq.(3.32e) holds if $m_1 \neq 2$ (otherwise see Eq.(3.32b)). Let's recall that the $\mathcal{W}$-terms are defined in Eq.(3.19) and in the above equations is subtended the ϑ dependency (as it is often done also in subsequent formulas).

Generalizations

Before proceeding with a wider formulation of the above to include all rational repulsive points, it is worth spending few words on another remarkable result.

Very recently [6], a generalization of Eq.(3.32) has been obtained. This work concerns minimal scattering theories with quantum $\mathcal{U}_q(\mathfrak{su}_2)$ symmetry, such as the sine-Gordon model is (Eq.1.51), but constructed (just out of general principles requirements (§1.2.4)) in higher spin representations. As emerged in the present discussion (§2.3), the sine-Gordon theory is based on a spin $1/2$ representation: when specializing the results of [6] to this value, indeed Eq.(3.32) is obtained.

What's really remarkable is that the conclusions reached in the elegant [6] are spin-independent. Focusing on repulsive regimes (in particular, the analogue of integer points discussed above), a set of Takahashi-Suzuki identities is obtained exactly in the form of Eq.(3.29), where the spin appears as a parameter. As a consequence, the universal TBA system shows the same structure, i.e. it is a common description of different theories: from the here-discussed sine-Gordon model, to the similarly famous 'sausage model' (spin 1) [66][67] and to completely new theories (spin $3/2$ and greater).

In [6] is also present a numerical analysis of the UV behavior, following the program of §2.1. The two theories just explicitly named are found to be UV-completed (i.e. their renormalization group flow reach a well defined CFT at high energies), while other ones are subject to a Hagedorn transition [68]. While this exceeds the scope of the current work, let's just mention that possible explanations to this behavior might reside in the universal TBA diagrammatic structure.

Shifting again the focus on the sine-Gordon model, it is possible to extend the previous description to a more general case $p > 1$, $p \in \mathbb{Q}$. This corresponds to consider sine-Gordon parameter continued fractions with no breather and up to level $\lambda \geq 1$:

$$p = \frac{1}{N_B + \dfrac{1}{\alpha}} = \frac{1}{0 + \dfrac{1}{\nu_1 + \dfrac{1}{\dots + \dfrac{1}{\nu_\lambda}}}} = \alpha = c.f.\{0, \nu_1, \dots, \nu_\lambda\}\,. \tag{3.33}$$

There exist, then, $N_M = m_\lambda$ magnons, besides the (anti)soliton.

Again, the formulation of a universal TBA system relies on the suitable identities that allows to simplify the raw TBA system: these are generalizations of Eq.(3.29). As first step, on the line of Eq.(3.27), it is convenient to define the mapping to diagram nodes as

$$\{S, M_1, M_2, \dots, M_{m_\lambda}\} \mapsto \{\mathtt{1}, \mathtt{2}, \mathtt{3}, \dots, \mathtt{n}\}\,. \tag{3.34}$$

The non-trivial step is to deal with the kernels of Eqs.(3.11c)(3.11d)(3.11e) and to work out the much needed identities. The main inspiration comes from [44] and from the results obtained in the p integer sub-case. Define the universal kernels

$$\tilde{\varphi}_i(\xi) \quad = \frac{1}{2\cosh \beta^{-1}\, p_i\, \xi} \tag{3.35a}$$

$$\tilde{\varphi}_i^{self}(\xi) = \frac{\cosh \beta^{-1}(p_i - p_{i+1})\xi}{2\cosh \beta^{-1}\, p_i\, \xi\ \cosh \beta^{-1}\, p_{i+1}\, \xi}\,, \tag{3.35b}$$

for $i \in \Lambda \overset{def}{=} \{1, \dots, \lambda\}$ in Eq.(3.35a) and for $i \in \Lambda' \overset{def}{=} \{1, \dots, \lambda - 1\}$ in Eq.(3.35b). By Eqs.(2.41)(2.58) it may be seen that the naming of the previously-defined $\tilde{\varphi}_0$ and $\tilde{\varphi}_1$ are consistent with the notation above. The calculations are quite cumbersome, but it is possible to write the Takahashi-Suzuki identities for this case as

$$\begin{aligned}
&2\pi\mathscr{F} K_{\mathtt{1},\mathtt{j}} + (1 - \delta_{0,m_1-1})\,\tilde{\varphi}_1\, 2\pi\mathscr{F} K_{\mathtt{2},\mathtt{j}} + (\delta_{0,m_1-1})\,\tilde{\varphi}_2\, 2\pi\mathscr{F} K_{\mathtt{2},\mathtt{j}} = \\
&\quad = -(1 - \delta_{0,m_1-1})\,\tilde{\varphi}_1\,\delta_{\mathtt{2},\mathtt{j}} + (\delta_{0,m_1-1})\,\tilde{\varphi}_1^{self}\,\delta_{\mathtt{1},\mathtt{j}} - (\delta_{0,m_1-1})\,\tilde{\varphi}_2\,\delta_{\mathtt{2},\mathtt{j}}
\end{aligned} \tag{3.36a}$$

$$\begin{aligned}
&2\pi\mathscr{F} K_{\mathtt{2},\mathtt{j}} - (\delta_{0,m_1-2})\,\tilde{\varphi}_1^{self}\, 2\pi\mathscr{F} K_{\mathtt{2},\mathtt{j}} - (\Theta_{m_1>2})\,\tilde{\varphi}_1\, 2\pi\mathscr{F} K_{\mathtt{3},\mathtt{j}} - (\delta_{0,m_1-1})\,\tilde{\varphi}_2\, 2\pi\mathscr{F} K_{\mathtt{3},\mathtt{j}} = \\
&\quad = -(1 - 2\delta_{0,m_1-1})\,\tilde{\varphi}_1\,\delta_{\mathtt{1},\mathtt{j}} + (\delta_{0,m_1-2})\,\tilde{\varphi}_1^{self}\,\delta_{\mathtt{2},\mathtt{j}} + (\Theta_{m_1>2})\,\tilde{\varphi}_1\,\delta_{\mathtt{3},\mathtt{j}} + \\
&\qquad - (1 - \Theta_{m_1>2})\,\tilde{\varphi}_2\,\delta_{\mathtt{3},\mathtt{j}} - (\delta_{0,m_\lambda-3})\,\tilde{\varphi}_2\,\delta_{\mathtt{4},\mathtt{j}}
\end{aligned} \tag{3.36b}$$

$$\begin{aligned}
&2\pi\mathscr{F} K_{\mathtt{i},\mathtt{j}} - (1 - 2\delta_{\mathtt{i},m_{k-1}})\,\tilde{\varphi}_k\, 2\pi\mathscr{F} K_{\mathtt{i-1},\mathtt{j}} - \tilde{\varphi}_k\, 2\pi\mathscr{F} K_{\mathtt{i+1},\mathtt{j}} = \\
&\quad = (-1)^{k+1}\,\tilde{\varphi}_k\,\delta_{\mathtt{i-1},\mathtt{j}} + (-1)^{k+1}\,\tilde{\varphi}_k\,\delta_{\mathtt{i+1},\mathtt{j}}
\end{aligned} \tag{3.36c}$$

$$\begin{aligned}
&2\pi\mathscr{F} K_{\mathtt{i},\mathtt{j}} - (1 - 2\delta_{\mathtt{i},m_{k-1}})\,\tilde{\varphi}_k\, 2\pi\mathscr{F} K_{\mathtt{i-1},\mathtt{j}} - \tilde{\varphi}_k^{self}\, 2\pi\mathscr{F} K_{\mathtt{i},\mathtt{j}} - \tilde{\varphi}_{k+1}\, 2\pi\mathscr{F} K_{\mathtt{i+1},\mathtt{j}} = \\
&\quad = (-1)^{k+1}\,\tilde{\varphi}_k\,\delta_{\mathtt{i-1},\mathtt{j}} + (-1)^{k+1}\,\tilde{\varphi}_k^{self}\,\delta_{\mathtt{i},\mathtt{j}} - (-1)^{k+1}\,\tilde{\varphi}_{k+1}\,\delta_{\mathtt{i+1},\mathtt{j}}
\end{aligned} \tag{3.36d}$$

$$
\begin{aligned}
&2\pi\mathscr{F}K_{\mathtt{n}-2,\mathtt{j}} - (1-2\delta_{\mathtt{n-2},m_{k-1}})\,\tilde{\varphi}_\lambda\, 2\pi\mathscr{F}K_{\mathtt{n-3,j}} - \tilde{\varphi}_\lambda\, 2\pi\mathscr{F}K_{\mathtt{n-1,j}} = \\
&\quad = (-1)^{\lambda+1}\,\tilde{\varphi}_\lambda\,\delta_{\mathtt{n-3,j}} + (-1)^{\lambda+1}\,\tilde{\varphi}_\lambda\,\delta_{\mathtt{n-1,j}} - (-1)^{\lambda+1}\,\tilde{\varphi}_\lambda\,\delta_{\mathtt{n,j}} && (3.36e)\\
&2\pi\mathscr{F}K_{\mathtt{n-1,j}} - \tilde{\varphi}_\lambda\, 2\pi\mathscr{F}K_{\mathtt{n-2,j}} = (-1)^{\lambda+1}\,\tilde{\varphi}_\lambda\,\delta_{\mathtt{n-2,j}} && (3.36f)\\
&2\pi\mathscr{F}K_{\mathtt{n,j}} + \tilde{\varphi}_\lambda\, 2\pi\mathscr{F}K_{\mathtt{n-2,j}} = -(-1)^{\lambda+1}\,\tilde{\varphi}_\lambda\,\delta_{\mathtt{n-2,j}}\,, && (3.36g)
\end{aligned}
$$

for $1 < m_{k-1} \le \mathtt{i} \le m_k - 2$, $\mathtt{i} \neq \mathtt{n-2}$ in Eq.(3.36c) and for $1 < \mathtt{i} = m_k - 1$, $k \neq \lambda$ in Eq.(3.36d), while Eq.(3.36e) holds if $m_\lambda \neq 3$ (otherwise see Eq.(3.36b)). Also it is implicitly assumed that $m_\lambda \neq 2$, since the set of Takahashi-Suzuki identities has already be presented in Eq.(3.29) for this case.

The universal TBA equations (in Fourier space) are, then, found to be structured as in Eq.(3.17) with

$$
\begin{aligned}
\{\sigma_{\mathcal{M}_\mathtt{i}}\}_{\mathtt{i}=1}^{\mathtt{n}} &= \{0, 0, \ldots, 0\} && (3.37a)\\
\{\sigma_{\varepsilon_\mathtt{i}}\}_{\mathtt{i}=1}^{\mathtt{n}} &= \{0, 1, 1, \ldots, 1, 0\} && (3.37b)
\end{aligned}
$$

and with $[\breve{\boldsymbol{I}}(\tilde{\varphi})](\xi)$ being composed by blocks on the diagonal in the form

$$
\begin{gathered}
\begin{bmatrix}
 & -\tilde{\varphi}_1 & & \cdots \\
\tilde{\varphi}_1 & & \tilde{\varphi}_1 & \cdots \\
 & \tilde{\varphi}_1 & & \ddots \\
\vdots & \vdots & \ddots & \ddots
\end{bmatrix}
\quad
\begin{bmatrix}
-\tilde{\varphi}_1^{self} & -\tilde{\varphi}_2 & & \cdots \\
-\tilde{\varphi}_2 & & \tilde{\varphi}_2 & \cdots \\
 & \tilde{\varphi}_2 & & \ddots \\
\vdots & \vdots & \ddots & \ddots
\end{bmatrix}
\\
\begin{bmatrix}
\ddots & \ddots & \vdots & \vdots & \\
\ddots & & \tilde{\varphi}_i & & \cdots \\
\cdots & \tilde{\varphi}_i & & \tilde{\varphi}_i & \cdots \\
\cdots & & \tilde{\varphi}_i & & \ddots \\
 & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}
\quad
\begin{bmatrix}
\ddots & \ddots & \vdots & \vdots & \\
\ddots & & \tilde{\varphi}_i & & \cdots \\
\cdots & \tilde{\varphi}_i & \tilde{\varphi}_i^{self} & \tilde{\varphi}_{i+1} & \cdots \\
\cdots & & -\tilde{\varphi}_{i+1} & & \ddots \\
 & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}
\\
\begin{bmatrix}
\ddots & \ddots & \vdots & \vdots & \vdots \\
\ddots & & \tilde{\varphi}_\lambda & & \\
\cdots & \tilde{\varphi}_\lambda & & \tilde{\varphi}_\lambda & \tilde{\varphi}_\lambda \\
\cdots & & \tilde{\varphi}_\lambda & & \\
\cdots & & -\tilde{\varphi}_\lambda & &
\end{bmatrix}.
\end{gathered}
\qquad (3.38)
$$

The first line of Eq.(3.38) refers to two possibilities for the entries $[(S), (M_1), (M_2), \ldots]$: on the left is the general block, while the case on the right holds if $m_1 = 1$ (supposing $\lambda > 1$, otherwise see Eq.(3.31)). Next in order are the blocks for $[\ldots, (M_{k-1}), (M_k), (M_{k+1}), \ldots]$ with $1 < m_{i-1} \le k \le m_i - 2$, $k \neq N_M - 2$ (second line, left), $[\ldots, (M_{m_i-2}), (M_{m_i-1}), (M_{m_i}), \ldots]$ with $i \neq \lambda$ (second line right) and $[\ldots, (M_{m_\lambda-2}), (M_{m_\lambda-1}), (M_{m_\lambda})]$

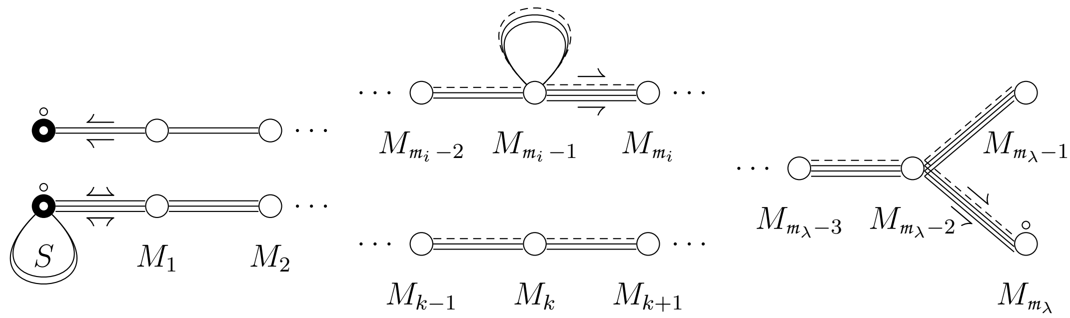
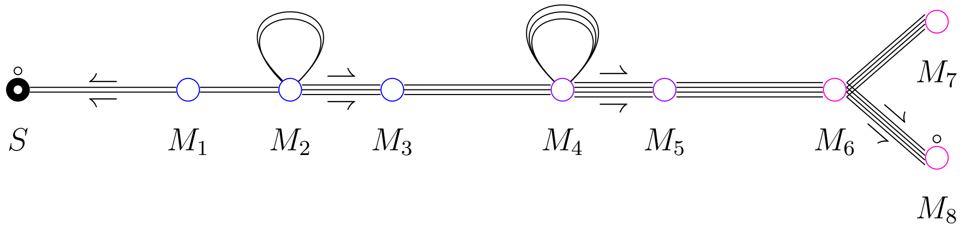
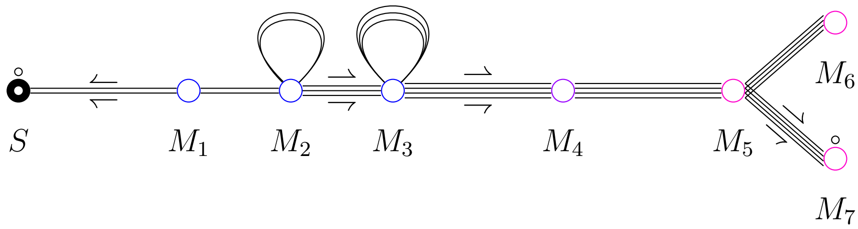


Figure 3.5: Building blocks for sine-Gordon universal TBA diagram at repulsive points: each of them is in correspondence with the matrix blocks of Eq.(3.38) (the assumptions on the labels are the same therein). With respect to Fig.(3.3), similar conventions are followed, but some new features are present here: two nodes are connected with $i+1$ lines to denote the coupling of the equations through $\tilde{\varphi}_i$; similarly, loops with $i+1$ lines stand for $\pm\tilde{\varphi}_i^{self}$, being drawn upwards (downwards) for the positive (negative) sign; the conventions on directed links are maintained, now allowing for doubly negative entries, represented with arrows in both directions. These notations are highly inspired by [69][4][5].

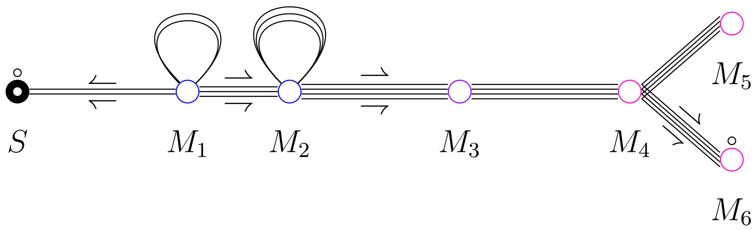
Some examples of application of these blocks to the description of TBA structures are listed in Fig.(3.6).



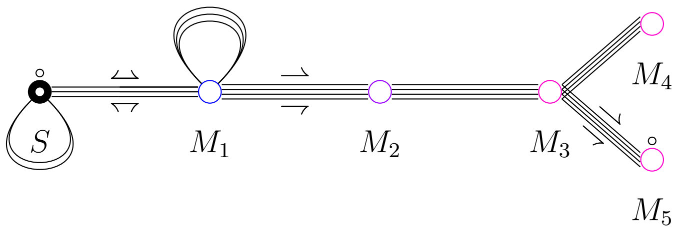
(a) $p = \frac{24}{7}$



(b) $p = \frac{15}{4}$



(c) $p = \frac{11}{4}$



(d) $p = \frac{7}{4}$

Figure 3.6: Universal TBA structure for the sine-Gordon model at $p = \frac{24}{7} = c.f.\{0,3,2,3\}$, $p = \frac{15}{4} = c.f.\{0,3,1,3\}$, $p = \frac{11}{4} = c.f.\{0,2,1,3\}$, $p = \frac{7}{4} = c.f.\{0,1,1,3\}$. The notations explained in Fig.(3.5) are adopted here. Additionally, magnons belonging to the same level are grouped more closely together and share a common color, in order to grant a better legibility. It may be noticed from these examples that, as magnons are subtracted from a level i, the corresponding loop seems to 'move back' until trespassing into the precedent level $i-1$. This behavior is shown here for the second level (ending up on the last first level magnon) and for the first level (ending up on the (anti)soliton).

(last line). The diagrammatic encoding is discussed in Fig.(3.5), while some examples are proposed in Fig.(3.6).

A striking feature of Eq.(3.38) is that it shows diagonal elements, i.e. loops in a diagrammatic language: they describe TBA equations which are self-coupled.

As done in previous cases, the complete formulation of the TBA system is presented here (implicitly supposing $\lambda > 1$, otherwise refer to Eq.(3.32)):

$$\begin{aligned}
\mathcal{M}^{(S)} &= \varepsilon^{(S)} - (\delta_{0,m_1-1})\varphi_1^{self} * \mathcal{W}^{(S)} - (1-\delta_{0,m_1-1})\varphi_1 * \mathcal{W}^{(M_1)} + \\
&\quad - (\delta_{0,m_1-1})\varphi_2 * \mathcal{W}^{(M_1)} && (3.39a)\\
\mathcal{M}^{(M_1)} &= \varepsilon^{(M_1)} + (1-2\delta_{0,m_1-1})\varphi_1 * \mathcal{W}^{(S)} + (\delta_{0,m_1-2})\varphi_1^{self} * \mathcal{W}^{(M_1)} + \\
&\quad + (\Theta_{m_1>2})\varphi_1 * \mathcal{W}^{(M_2)} + (1-\Theta_{m_1>2})\varphi_2 * \mathcal{W}^{(M_2)} + \\
&\quad + (\delta_{0,m_\lambda-3})\varphi_2 * \mathcal{W}^{(M_3)} && (3.39b)\\
\mathcal{M}^{(M_k)} &= \varepsilon^{(M_k)} + (1-2\delta_{k,m_{i-1}})\varphi_i * \mathcal{W}^{(M_{k-1})} + \varphi_i * \mathcal{W}^{(M_{k+1})} && (3.39c)\\
\mathcal{M}^{(M_k)} &= \varepsilon^{(M_k)} + (1-2\delta_{k,m_{i-1}})\varphi_i * \mathcal{W}^{(M_{k-1})} + \varphi_i^{self} * \mathcal{W}^{(M_k)} + \\
&\quad + \varphi_{i+1} * \mathcal{W}^{(M_{k+1})} && (3.39d)\\
\mathcal{M}^{(M_{m_\lambda}-2)} &= \varepsilon^{(M_{m_\lambda}-2)} + \varphi_1 * \mathcal{W}^{(M_{m_\lambda}-3)} + \varphi_1 * \mathcal{W}^{(M_{m_\lambda}-1)} + \varphi_1 * \mathcal{W}^{(M_{m_\lambda})} && (3.39e)\\
\mathcal{M}^{(M_{m_\lambda}-1)} &= \varepsilon^{(M_{m_\lambda}-1)} + \varphi_1 * \mathcal{W}^{(M_{m_\lambda}-2)} && (3.39f)\\
\mathcal{M}^{(M_{m_\lambda})} &= \varepsilon^{(M_{m_\lambda})} - \varphi_1 * \mathcal{W}^{(M_{m_\lambda}-2)}\,, && (3.39g)
\end{aligned}$$

for $1 < m_{i-1} \leq k \leq m_i - 2$, $k \neq m_\lambda - 2$ in Eq.(3.39c) and for $1 < k = m_i - 1$, $i \neq \lambda$ in Eq.(3.39d), while Eq.(3.39e) holds if $m_\lambda \neq 3$ (otherwise see Eq.(3.39b)).
This 'verbose' system of equations is very efficiently encoded into diagrams in the form of Fig.(3.5), whence the power of graphical formalism.

3.2.3 Attractive non-diagonal points

Still some rational values of p (arguably the most interesting ones) have to be considered in the current discussion. They are the attractive points $0 < p < 1$, $p \in \mathbb{Q} - \frac{1}{\mathbb{N}}$, for which the scattering is non-diagonal. In this case, the sine-Gordon parameter is expressed as a continued fraction in the form

$$p = \frac{1}{N_B + \dfrac{1}{\alpha}} = \frac{1}{N_B + \dfrac{1}{\nu_1 + \dfrac{1}{\cdots + \dfrac{1}{\nu_\lambda}}}} = c.f.\{N_B, \nu_1, \ldots, \nu_\lambda\}\,, \tag{3.40}$$

where, differently from the repulsive case, $N_B \neq 0$. Both breathers and magnons are present along with the (anti)soliton: it is expected to obtain a universal TBA description where the two previous cases (§§3.2.1,3.2.2) are 'sewed' together.

As customary, let's introduce the correspondence between (pseudo)particle species and graph nodes. Since the theory entails $N_B + 1 + N_M$ particles, the diagram will feature the same number $\mathtt{n}$ of nodes as per the association

$$
\begin{aligned}
&\{B_1, B_2, \ldots, B_{N_B}, S, M_1, M_2, \ldots, M_{m_\lambda}\} \mapsto \\
&\quad \mapsto \{\mathtt{1}, \mathtt{2}, \ldots, \mathtt{N_B}, \mathtt{N_B}+\mathtt{1}, \mathtt{N_B}+\mathtt{2}, \mathtt{N_B}+\mathtt{3}, \ldots, \mathtt{n}\}\,.
\end{aligned} \tag{3.41}
$$

This allows to formulate 'Zamolodchikov-Takahashi-Suzuki' identities adopting graph indices instead of particle labels. Once introduced the universal kernels in the same form of Eq.(3.35)

$$
\tilde{\varphi}_i(\xi) \quad = \frac{1}{2\cosh \beta^{-1}\, p_i\, \xi} \tag{3.42a}
$$

$$
\tilde{\varphi}_i^{self}(\xi) = \frac{\cosh \beta^{-1}(p_i - p_{i+1})\xi}{2\cosh \beta^{-1}\, p_i\, \xi\, \cosh \beta^{-1}\, p_{i+1}\, \xi}\,, \tag{3.42b}
$$

for $i \in \Lambda \stackrel{def}{=} \{0, \ldots, \lambda\}$ in Eq.(3.42a) and for $i \in \Lambda' \stackrel{def}{=} \{0, \ldots, \lambda-1\}$ in Eq.(3.42b), it may be obtained that

$$
2\pi \mathscr{F} K_{\mathtt{1,j}} - \tilde{\varphi}_0\, 2\pi \mathscr{F} K_{\mathtt{2,j}} = -\,\tilde{\varphi}_0\, \delta_{\mathtt{2,j}} \tag{3.43a}
$$

$$
2\pi \mathscr{F} K_{\mathtt{i,j}} - \tilde{\varphi}_0\, 2\pi \mathscr{F} K_{\mathtt{i-1,j}} - \tilde{\varphi}_0\, 2\pi \mathscr{F} K_{\mathtt{i+1,j}} = -\,\tilde{\varphi}_0\, \delta_{\mathtt{i-1,j}} - \tilde{\varphi}_0\, \delta_{\mathtt{i+1,j}} \tag{3.43b}
$$

$$
\begin{aligned}
&2\pi \mathscr{F} K_{\mathtt{N_B,j}} - (\Theta_{N_B>1})\, \tilde{\varphi}_0\, 2\pi \mathscr{F} K_{\mathtt{N_B-1,j}} - \tilde{\varphi}_0^{self}\, 2\pi \mathscr{F} K_{\mathtt{N_B,j}} = \\
&\quad = -(\Theta_{N_B>1})\, \tilde{\varphi}_{\mathtt{N_B-1,j}} - \tilde{\varphi}_0^{self}\, \delta_{\mathtt{N_B,j}} - \tilde{\varphi}_1\, \delta_{\mathtt{N_B+1,j}}
\end{aligned} \tag{3.43c}
$$

$$
\begin{aligned}
&2\pi \mathscr{F} K_{\mathtt{N_B+1,j}} - (\Theta_{N_B>0})\, \tilde{\varphi}_1\, 2\pi \mathscr{F} K_{\mathtt{N_B,j}} + (1 - \delta_{0,m_1-1})\, \tilde{\varphi}_1\, 2\pi \mathscr{F} K_{\mathtt{N_B+2,j}} + \\
&\qquad + (\delta_{0,m_1-1})\, \tilde{\varphi}_2\, 2\pi \mathscr{F} K_{\mathtt{N_B+2,j}} = \\
&\quad = -(\Theta_{N_B>0})\, \tilde{\varphi}_1\, \delta_{\mathtt{N_B,j}} + (\delta_{0,m_1-1})\, \tilde{\varphi}_1^{self}\, \delta_{\mathtt{N_B+1,j}} + \\
&\qquad - (1 - \delta_{0,m_1-1})\, \tilde{\varphi}_1\, 2\pi \mathscr{F} K_{\mathtt{N_B+2,j}} - (\delta_{0,m_1-1})\, \tilde{\varphi}_2\, 2\pi \mathscr{F} K_{\mathtt{N_B+2,j}} + \\
&\qquad + (1 - 2\delta_{0,N_B})(\delta_{0,m_\lambda-2})\, \tilde{\varphi}_1\, 2\pi \mathscr{F} K_{\mathtt{N_B+3,j}}
\end{aligned} \tag{3.43d}
$$

$$
\begin{aligned}
&2\pi \mathscr{F} K_{\mathtt{N_B+2,j}} - (\delta_{0,m_1-2})\, \tilde{\varphi}_1^{self}\, 2\pi \mathscr{F} K_{\mathtt{N_B+2,j}} - (\Theta_{m_1>2})\, \tilde{\varphi}_1\, 2\pi \mathscr{F} K_{\mathtt{N_B+3,j}} + \\
&\qquad - (\delta_{0,m_1-1})\, \tilde{\varphi}_2\, 2\pi \mathscr{F} K_{\mathtt{N_B+3,j}} = \\
&\quad = -(1 - 2\delta_{0,m_1-1})\, \tilde{\varphi}_1\, \delta_{\mathtt{N_B+1,j}} + (\delta_{0,m_1-2})\, \tilde{\varphi}_1^{self}\, \delta_{\mathtt{N_B+2,j}} + (\Theta_{m_1>2})\, \tilde{\varphi}_1\, \delta_{\mathtt{N_B+3,j}} + \\
&\qquad - (1 - \Theta_{m_1>2})\, \tilde{\varphi}_2\, \delta_{\mathtt{N_B+3,j}} - (\delta_{0,m_\lambda-3})\, \tilde{\varphi}_2\, \delta_{\mathtt{N_B+4,j}}
\end{aligned} \tag{3.43e}
$$

$$
\begin{aligned}
&2\pi \mathscr{F} K_{\mathtt{i,j}} - (1 - 2\delta_{\mathtt{i,N_B+1}+m_{k-1}})\, \tilde{\varphi}_k\, 2\pi \mathscr{F} K_{\mathtt{i-1,j}} - \tilde{\varphi}_k\, 2\pi \mathscr{F} K_{\mathtt{i+1,j}} = \\
&\quad = (-1)^{k+1}\, \tilde{\varphi}_k\, \delta_{\mathtt{i-1,j}} + (-1)^{k+1}\, \tilde{\varphi}_k\, \delta_{\mathtt{i+1,j}}
\end{aligned} \tag{3.43f}
$$

$$
\begin{aligned}
&2\pi \mathscr{F} K_{\mathtt{N_B}+m_k,\mathtt{j}} - (1 - 2\delta_{m_k-1,m_{k-1}})\, \tilde{\varphi}_k\, 2\pi \mathscr{F} K_{\mathtt{N_B-1}+m_k,\mathtt{j}} - \tilde{\varphi}_k^{self}\, 2\pi \mathscr{F} K_{\mathtt{N_B}+m_k,\mathtt{j}} + \\
&\qquad - \tilde{\varphi}_{k+1}\, 2\pi \mathscr{F} K_{\mathtt{N_B+1}+m_k,\mathtt{j}} = \\
&\quad = (-1)^{k+1}\, \tilde{\varphi}_k\, \delta_{\mathtt{N_B-1}+m_k,\mathtt{j}} + (-1)^{k+1}\, \tilde{\varphi}_k^{self}\, \delta_{\mathtt{N_B}+m_k,\mathtt{j}} - (-1)^{k+1}\, \tilde{\varphi}_{k+1}\, \delta_{\mathtt{N_B+1}+m_k,\mathtt{j}}
\end{aligned} \tag{3.43g}
$$

$$2\pi\mathscr{F}K_{\mathtt{n}-2,\mathtt{j}} - (1-2\delta_{\mathtt{n-2},m_{k-1}})\,\tilde{\varphi}_\lambda\, 2\pi\mathscr{F}K_{\mathtt{n-3,j}} - \tilde{\varphi}_\lambda\, 2\pi\mathscr{F}K_{\mathtt{n-1,j}} =$$
$$= (-1)^{\lambda+1}\,\tilde{\varphi}_\lambda\,\delta_{\mathtt{n-3,j}} + (-1)^{\lambda+1}\,\tilde{\varphi}_\lambda\,\delta_{\mathtt{n-1,j}} - (-1)^{\lambda+1}\,\tilde{\varphi}_\lambda\,\delta_{\mathtt{n,j}} \tag{3.43h}$$
$$2\pi\mathscr{F}K_{\mathtt{n-1,j}} - \tilde{\varphi}_\lambda\, 2\pi\mathscr{F}K_{\mathtt{n-2,j}} = (-1)^{\lambda+1}\,\tilde{\varphi}_\lambda\,\delta_{\mathtt{n-2,j}} \tag{3.43i}$$
$$2\pi\mathscr{F}K_{\mathtt{n,j}} + \tilde{\varphi}_\lambda\, 2\pi\mathscr{F}K_{\mathtt{n}-2,\mathtt{j}} = -(-1)^{\lambda+1}\,\tilde{\varphi}_\lambda\,\delta_{\mathtt{n-2,j}}\,, \tag{3.43j}$$

where it has been implicitly assumed $N_M > 0$ (otherwise the reflectionless description holds). It may be noticed that Eqs.(3.43a)(3.43b) are the Zamolodchikov identities of Eqs.(3.22a)(3.22b), while Eqs.(3.43e)-(3.43j) are the Takahashi-Suzuki identities of Eqs.(3.36b)-(3.36g) slightly modified to include integer ps (Eq.(3.29))(the indices in these equations assume values as explained previously). Visible changes come, instead, with Eqs.(3.43c)(3.43d), which now 'sew' together the two sets of identities.

Even if quite extended, Eq.(3.43) represents a quite general and transparent way to obtain the sine-Gordon universal TBA system out of the raw one. As explained, they can be directly applied to yield the universal formulation.

However, in the recent [5] it is proposed an alternative procedure based (quite elegantly) on a matrix formalism, in the fashion of Eq.(3.13). The raw TBA system is put in the form

$$\overrightarrow{(\mathscr{F}(\varepsilon + L - \mathcal{M}))}(\xi) = [\mathbb{I} - \eta\, 2\pi\mathscr{F}K](\xi)\overrightarrow{\mathscr{F}L}(\xi) \tag{3.44}$$

and the universal TBA system is obtained multiplying by the auxiliary matrix

$$\boldsymbol{J} = \left[\begin{array}{c|c|c} [\mathbb{I} - \eta^{(B')}\, 2\pi\mathscr{F}K^{(BB')}]^{-1} & & \\ \hline & 1 & \\ \hline & & [\mathbb{I} - \eta^{(M')}\, 2\pi\mathscr{F}K^{(MM')}]^{-1} \end{array}\right], \tag{3.45}$$

whose entries, besides those explicitly written, are all zero but for $\boldsymbol{J}_{\mathtt{N_B+1,N_B+2}}(\xi) = \tilde{\varphi}_1(\xi)$, $\boldsymbol{J}_{\mathtt{N_B+1,N_B}}(\xi) = -\Theta_{N_B>0}\,\tilde{\varphi}_1(\xi)$ and $\boldsymbol{J}_{\mathtt{N_B+1,N_B+3}}(\xi) = \tilde{\varphi}_1(\xi)$. It is sufficient, then, to further simplify some of the equations obtained to get to the desired result.

It is possible to show that this procedure is equivalent to the application of the identities suggested in this work. Even if more refined, the former is less explicit than Eq.(3.43).

By either method, the form of Eq.(3.17) is reached for the sine-Gordon TBA system. The σ-numbers are specified as

$$\{\sigma_{\mathcal{M}_\mathtt{i}}\}_{\mathtt{i}=1}^{\mathtt{N_B}} = \{1,1,\ldots,1\} \quad \{\sigma_{\mathcal{M}_\mathtt{i}}\}_{\mathtt{i=N_B+1}}^{\mathtt{n}} = \{0,0,\ldots,0\} \tag{3.46a}$$
$$\{\sigma_{\varepsilon_\mathtt{i}}\}_{\mathtt{i}=1}^{\mathtt{N_B}} = \{1,1,\ldots,1\} \quad \{\sigma_{\varepsilon_\mathtt{i}}\}_{\mathtt{i=N_B+1}}^{\mathtt{n}} = \{1,1,\ldots,1\}\,, \tag{3.46b}$$

while $[\breve{\boldsymbol{I}}(\tilde{\varphi})](\xi)$ is made up by the following blocks on the diagonal

$$
\begin{bmatrix}
 & \tilde{\varphi}_0 & & \cdots \\
\tilde{\varphi}_0 & & \tilde{\varphi}_0 & \cdots \\
 & \tilde{\varphi}_0 & & \ddots \\
\vdots & \vdots & \ddots & \ddots
\end{bmatrix}
$$

$$
\begin{bmatrix}
\ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \\
\ddots & & \tilde{\varphi}_0 & & & & \cdots \\
\cdots & \tilde{\varphi}_0 & \tilde{\varphi}_0^{self} & \tilde{\varphi}_1 & & & \cdots \\
\cdots & & \tilde{\varphi}_1 & & -\tilde{\varphi}_1 & & \cdots \\
\cdots & & & \tilde{\varphi}_1 & & \tilde{\varphi}_1 & \cdots \\
\cdots & & & & \tilde{\varphi}_1 & & \ddots \\
 & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}
\begin{bmatrix}
\ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \\
\ddots & & \tilde{\varphi}_0 & & & & \cdots \\
\cdots & \tilde{\varphi}_0 & \tilde{\varphi}_0^{self} & \tilde{\varphi}_1 & & & \cdots \\
\cdots & & \tilde{\varphi}_1 & -\tilde{\varphi}_1^{self} & -\tilde{\varphi}_2 & & \cdots \\
\cdots & & & -\tilde{\varphi}_2 & & \tilde{\varphi}_2 & \cdots \\
\cdots & & & & \tilde{\varphi}_2 & & \ddots \\
 & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}
$$

$$
\begin{bmatrix}
\ddots & \ddots & \vdots & \vdots & \\
\ddots & & \tilde{\varphi}_i & & \cdots \\
\cdots & \tilde{\varphi}_i & & \tilde{\varphi}_i & \cdots \\
\cdots & & \tilde{\varphi}_i & & \ddots \\
 & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}
\begin{bmatrix}
\ddots & \ddots & \vdots & \vdots & \\
\ddots & & \tilde{\varphi}_i & & \cdots \\
\cdots & \tilde{\varphi}_i & \tilde{\varphi}_i^{self} & \tilde{\varphi}_{i+1} & \cdots \\
\cdots & & -\tilde{\varphi}_{i+1} & & \ddots \\
 & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}
$$

$$
\begin{bmatrix}
\ddots & \ddots & \vdots & \vdots & \vdots \\
\ddots & & \tilde{\varphi}_\lambda & & \\
\cdots & \tilde{\varphi}_\lambda & & \tilde{\varphi}_\lambda & \tilde{\varphi}_\lambda \\
\cdots & & \tilde{\varphi}_\lambda & & \\
\cdots & & -\tilde{\varphi}_\lambda & &
\end{bmatrix}
\begin{bmatrix}
\ddots & \ddots & \vdots & \vdots & \vdots \\
\ddots & & \tilde{\varphi}_1 & & \\
\cdots & \tilde{\varphi}_1 & & -\tilde{\varphi}_1 & -(1-2\delta_{0,N_B})\,\tilde{\varphi}_1 \\
\cdots & & \tilde{\varphi}_1 & & \\
\cdots & & -\tilde{\varphi}_\lambda & &
\end{bmatrix}.
\tag{3.47}
$$

The first line of Eq.(3.47) refers to the breather entries $[(B_1), (B_2), (B_3), \ldots]$, when $N_B > 0$. On the second line are shown two possibilities depending on whether $m_1 \neq 1$ (left) or $m_1 = 1$ (right): they correspond to particle species $[\ldots, (B_{N_B-1}), (B_{N_B}), (S), (M_1), (M_2), \ldots]$. The remaining lines are just the same blocks as in Eq.(3.38), but for the last line where a slightly different form (right) is to be used if $m_\lambda = 2$. The diagrammatic encoding conventions are the same as in Figs.(3.1)(3.3)(3.5) (see Fig.(3.7) for a swift summary): the example of Fig.(3.8) may help in visually clarifying the structure outlined above.

An interesting point is that the last breather shows a diagonal element, which in the reflectionless case is absent. This may be regarded as a consequence of magnons' appearance.

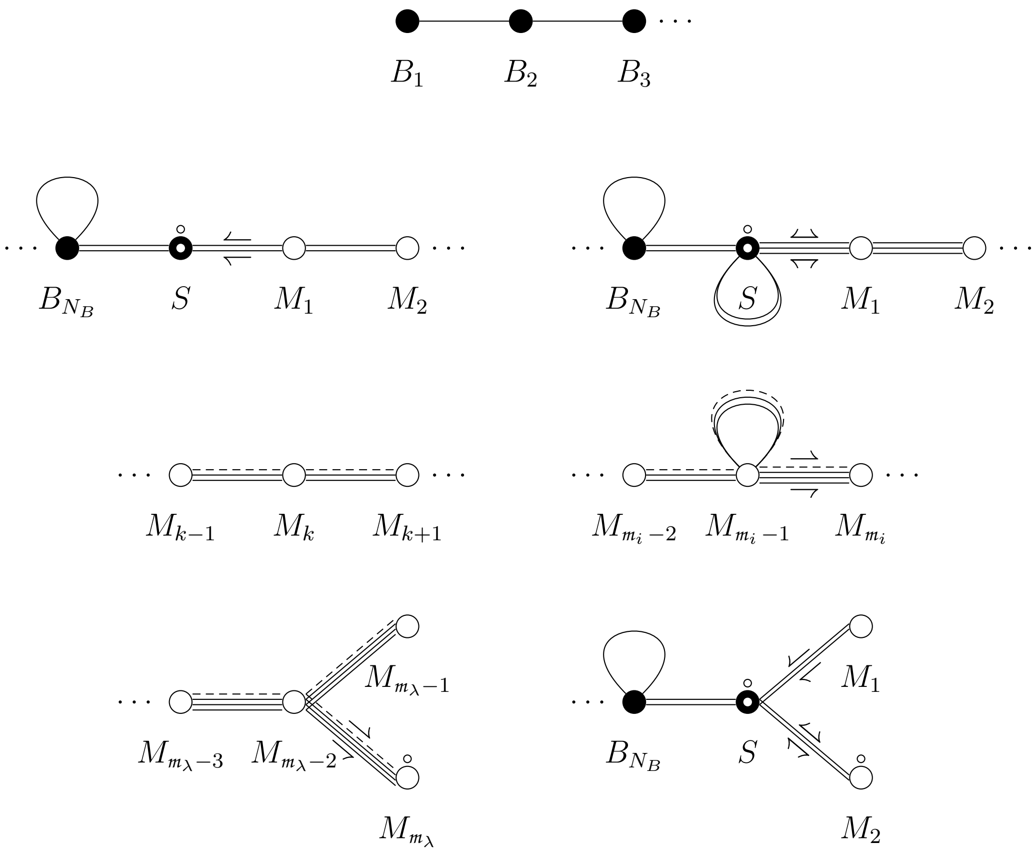


Figure 3.7: Diagrammatic representation of the Eq.(3.47) matrix blocks. The encoding is described in Figs.(3.1)(3.3)(3.5).

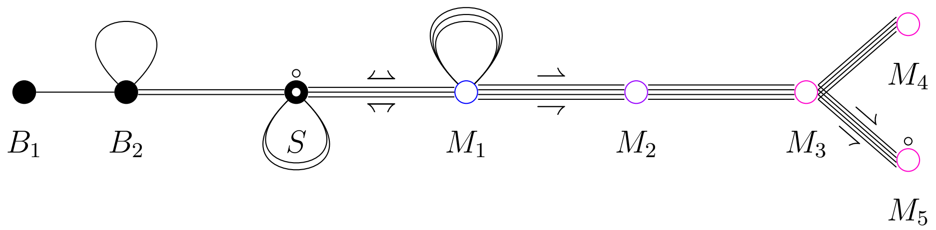


Figure 3.8: Universal TBA structure for the sine-Gordon model at the attractive non-diagonal point $p = \frac{7}{18} = c.f.\{2, 1, 1, 3\}$.

Finally, the universal TBA equations for the sine-Gordon model at attractive non-diagonal points read

$$\mathcal{M}^{(B_1)} = \varepsilon^{(B_1)} + \varphi_0 * \mathcal{W}^{(B_2)} \tag{3.48a}$$

$$\mathcal{M}^{(B_a)} = \varepsilon^{(B_a)} + \varphi_0 * \mathcal{W}^{(B_{a-1})} + \varphi_0 * \mathcal{W}^{(B_{a+1})} \tag{3.48b}$$

$$\mathcal{M}^{(B_{N_B})} = \varepsilon^{(B_{N_B})} + (\Theta_{N_B>1})\varphi_0 * \mathcal{W}^{(B_{N_B-1})} + \varphi_0^{self} * \mathcal{W}^{(B_{N_B})} + \varphi_1 * \mathcal{W}^{(S)} \tag{3.48c}$$

$$\begin{aligned}\mathcal{M}^{(S)} = &\varepsilon^{(S)} + (\Theta_{N_B>0})\varphi_1 * \mathcal{W}^{(B_{N_B})} - (\delta_{0,m_1-1})\varphi_1^{self} * \mathcal{W}^{(S)} + \\ &- (1-\delta_{0,m_1-1})\varphi_1 * \mathcal{W}^{(M_1)} - (\delta_{0,m_1-1})\varphi_2 * \mathcal{W}^{(M_2)} + \\ &+ (1-2\Theta_{N_B>0})(\delta_{0,m_\lambda-2})\varphi_1 * \mathcal{W}^{(M_2)}\end{aligned} \tag{3.48d}$$

$$\begin{aligned}\mathcal{M}^{(M_1)} = &\varepsilon^{(M_1)} + (1-2\delta_{0,m_1-1})\varphi_1 * \mathcal{W}^{(S)} + (\delta_{0,m_1-2})\varphi_1^{self} * \mathcal{W}^{(M_1)} + \\ &+ (\Theta_{m_1>2})\varphi_1 * \mathcal{W}^{(M_2)} + (1-\Theta_{m_1>2})\varphi_2 * \mathcal{W}^{(M_2)} + \\ &+ (\delta_{0,m_\lambda-3})\varphi_2 * \mathcal{W}^{(M_3)}\end{aligned} \tag{3.48e}$$

$$\mathcal{M}^{(M_k)} = \varepsilon^{(M_k)} + (1-2\delta_{k,m_{i-1}})\varphi_i * \mathcal{W}^{(M_{k-1})} + \varphi_i * \mathcal{W}^{(M_{k+1})} \tag{3.48f}$$

$$\begin{aligned}\mathcal{M}^{(M_{m_i}-1)} = &\varepsilon^{(M_{m_i}-1)} + (1-2\delta_{m_i-1,m_{i-1}})\varphi_i * \mathcal{W}^{(M_{m_i}-2)} + \varphi_i^{self} * \mathcal{W}^{(M_{m_i}-1)} + \\ &+ \varphi_{i+1} * \mathcal{W}^{(M_{m_i})}\end{aligned} \tag{3.48g}$$

$$\mathcal{M}^{(M_{m_\lambda}-2)} = \varepsilon^{(M_{m_\lambda}-2)} + \varphi_1 * \mathcal{W}^{(M_{m_\lambda}-3)} + \varphi_1 * \mathcal{W}^{(M_{m_\lambda}-1)} + \varphi_1 * \mathcal{W}^{(M_{m_\lambda})} \tag{3.48h}$$

$$\mathcal{M}^{(M_{m_\lambda}-1)} = \varepsilon^{(M_{m_\lambda}-1)} + \varphi_1 * \mathcal{W}^{(M_{m_\lambda}-2)} \tag{3.48i}$$

$$\mathcal{M}^{(M_{m_\lambda})} = \varepsilon^{(M_{m_\lambda})} - \varphi_1 * \mathcal{W}^{(M_{m_\lambda}-2)}\,. \tag{3.48j}$$

Exactly as in the Zamolodchikov-Takahashi-Suzuki identities, it may be appreciated that Eqs.(3.48a)(3.48b) are just Eqs.(3.25a)(3.25b) as well as Eqs.(3.48e)-(3.48j) coincide with Eqs.(3.39b)-(3.39g) (holding under the conditions discussed therein). This aligns with the results of [4][5].
Once again, it may be appreciated the synthesis gifted by diagrammatic representations.

However, let's remark that the graphs discussed in these sections are not just visual notations. They hold much more meaning at a deeper level. It is not an hyperbole stating that they build the thermodynamics of the scattering theory and are of great importance in the study of the theory's renormalization group flow: subsequent sections elaborate more on this point.

Chapter 4

Y-system and Dynkin diagrams

Strong of the knowledge of previous sections, it is illustrated here a new system of equations, famously known as Y-system. Equivalent to the TBA formulation, that's a set of functional equations where the underlying mathematical structure of a theory truly becomes determinant.

Y-systems are firstly presented in [48] for the general class of $\mathcal{ADE}$ scattering theories: due to their peculiar properties they allow to show the connection with the TBA derivation in a rather manifest way. Important discussions are lead, among others, in [56][57][58][59][60][70][3][50][71][1].

Stated plainly, Y-systems are new sets of functional equations, which can be deduced from the TBA system by means of identities. They are shown to be completely equivalent to the latter formulation, in the sense that solutions of the TBA equations also satisfy the Y-system (they are particular solutions). Even if showing an arguably more elaborated derivation, Y-systems also bring great advantages to the discussion. On par with the universal TBA equations (Eqs.(3.17)(3.18)), they explicitly cast light on the inherent mathematical structure of the theory. Plus, they are intrinsically connected to the renormalization group flow analysis: Y-systems show periodicity properties that reflect the conformal perturbing operator dimensions. Even further, their static solutions (that turn the system into a set of algebraic equations) are directly related to the UV limit central charge (Eqs.(2.7)(2.8)(2.66)). However, a Y-system formulation is not known for all the theories and much there is still to uncover about them.

The subsequent sections are meant to, at least, scratch the surface of this vast topic. With a particular consideration for the sine-Gordon model, it is here proposed a derivation of this system in the reflectionless ($\mathcal{ADE}$) case and some considerations regarding its generalization for other points.

[1]While focusing here on TBA-related aspects, Y-systems are actually related to a huge class of mathematics and physics topics. For a comprehensive collection, see [72].

4.1 Sine-Gordon model Y-system

Similarly to what discussed in the context of universal TBA, the main idea behind the derivation of Y-systems is to simplify the set of raw TBA equations (Eq.(2.74)). This, in some cases, can be done by means of an analytic continuation of the rapidity variable, joint with the application of identities characterizing the mathematical structure of the scattering theory. The goal is that of obtaining a cancellation of the driving terms, so to obtain a set of functional equations in the pseudoenergies (Eq.(2.52)).

As mentioned above, a Y-system formulation is not available for all the scattering theories. Of interest for the current discussion is the particular case of the sine-Gordon model, for which some results can indeed be obtained.

4.1.1 Reflectionless points

The peculiar reflectionless points $0 < p < 1, p \in \frac{1}{\mathbb{N}}$ represent a fortunate example in which a Y-system can be written. As discussed in previous sections (§§3.1,3.2), this consists in an instance of application for the more general class of $\mathcal{ADE}$ scattering theories. As the latter is the context in which Y-systems firstly entered the TBA method, it is worthwhile to study the steps followed in this broader case [59], before specializing to sine-Gordon reflectionless points.

Let's remind that any $\mathcal{ADE}$ theory showcases an intrinsic connection with simply-laced Lie algebra: their features are schematically listed at the beginning of §3.1. Central to this derivation is the property for the masses of the theory's particle content to build up the components of the Perron-Frobenius eigenvector. In fact, Eq.(3.2) holds for any $\mathcal{ADE}$ scattering theory, featuring the Dynkin diagram incidence matrix of the related simple Lie algebra ($\boldsymbol{I}_{\mathcal{G}}$), its Coxeter number ($\mathtt{h}_{\mathcal{G}}$) and the masses of the model ($M_{\mathtt{i}}$), in number equal to the algebra's rank ($\mathtt{r}_{\mathcal{G}}$):

$$\sum_{\mathtt{j}=1}^{\mathtt{r}_{\mathcal{G}}} [\boldsymbol{I}_{\mathcal{G}}]_{\mathtt{i},\mathtt{j}} M_{\mathtt{j}} = 2\cos\left(\frac{\pi}{\mathtt{h}_{\mathcal{G}}}\right) M_{\mathtt{i}}\,. \tag{4.1}$$

Recall also that, for the scattering kernels in Fourier space ($\mathscr{F} K_{\mathtt{i},\mathtt{j}}(\xi)$), it holds the fundamental Zamolodchikov identity of Eq.(3.13). It is convenient to recast it in the form

$$2\pi \mathscr{F} K_{\mathtt{i},\mathtt{j}}(\xi) = -\left[\boldsymbol{I}_{\mathcal{G}}\left(2\cosh\frac{\pi}{\mathtt{h}_{\mathcal{G}}}\xi\cdot\mathbb{I} - \boldsymbol{I}_{\mathcal{G}}\right)^{-1}\right]_{\mathtt{i},\mathtt{j}}\,. \tag{4.2}$$

Now, consider the system of raw TBA equations (Eq.(2.74)): under the notation currently used, it reads

$$\mathcal{M}_{\mathtt{i}}(\vartheta) = \varepsilon_{\mathtt{i}}(\vartheta) + \sum_{\mathtt{j}=1}^{\mathtt{r}_{\mathcal{G}}} (K_{\mathtt{i},\mathtt{j}} * L_{\mathtt{j}})(\vartheta)\,, \tag{4.3}$$

for $\mathtt{i} = 1, \ldots, \mathtt{r}_{\mathcal{G}}$ (notice that the η-signs are all positive if a massive scattering theory is assumed). It is possible to smoothly perform the analytic continuation in the rapidity variable

$$\vartheta \to \pm\, i \frac{\pi}{\mathtt{h}_{\mathcal{G}}}\,. \tag{4.4}$$

Then it can be considered the combination of raw TBA equations (adopting a short-hand notation '*(r.h.s.)*' for the right-hand side of the $\mathtt{i}^{\text{th}}$ equation)

$$\begin{aligned}
&\mathcal{M}_{\mathtt{i}}\left(\vartheta + i\frac{\pi}{\mathtt{h}_{\mathcal{G}}}\right) + \mathcal{M}_{\mathtt{i}}\left(\vartheta - i\frac{\pi}{\mathtt{h}_{\mathcal{G}}}\right) - \sum_{\mathtt{j}=1}^{\mathtt{r}_{\mathcal{G}}} [\boldsymbol{I}_{\mathcal{G}}]_{\mathtt{i},\mathtt{j}}\, \mathcal{M}_{\mathtt{j}}(\vartheta) = \\
&\quad = \textit{(r.h.s.)}_{\mathtt{i}}\left(\vartheta + i\frac{\pi}{\mathtt{h}_{\mathcal{G}}}\right) + \textit{(r.h.s.)}_{\mathtt{i}}\left(\vartheta - i\frac{\pi}{\mathtt{h}_{\mathcal{G}}}\right) - \sum_{\mathtt{j}=1}^{\mathtt{r}_{\mathcal{G}}} [\boldsymbol{I}_{\mathcal{G}}]_{\mathtt{i},\mathtt{j}} \textit{(r.h.s.)}_{\mathtt{j}}(\vartheta)\,.
\end{aligned} \tag{4.5}$$

When formulated in the Fourier space, it's easy to see that, by means of Eq.(4.1), the left-hand side of Eq.(4.5) vanishes, completely eliminating the dependence on driving terms (if not by the same identity Eq.(4.1)). For what concerns the right-hand side, the Zamolodchikov identity of Eq.(4.2) can be used, so that, once anti-Fourier transforms are applied, it is possible to write

$$\varepsilon_{\mathtt{i}}\left(\vartheta + i\frac{\pi}{\mathtt{h}_{\mathcal{G}}}\right) + \varepsilon_{\mathtt{i}}\left(\vartheta - i\frac{\pi}{\mathtt{h}_{\mathcal{G}}}\right) = \sum_{\mathtt{j}=1}^{\mathtt{r}_{\mathcal{G}}} [\boldsymbol{I}_{\mathcal{G}}]_{\mathtt{i},\mathtt{j}} \big(\varepsilon_{\mathtt{j}}(\vartheta) + L_{\mathtt{j}}(\vartheta)\big)\,. \tag{4.6}$$

An important point is that the combination of equations considered yields exactly the incidence matrix $\boldsymbol{I}_{\mathcal{G}}$ after applying the Zamolodchikov identities. Taking the logarithm of both sides, using for the L-terms the expression of Eq.(2.53) and defining the new pseudoenergies functions

$$Y_{\mathtt{i}}(\vartheta) = \mathrm{e}^{\varepsilon_{\mathtt{i}}(\vartheta)}\,, \tag{4.7}$$

it is obtained the Y-system for the $\mathcal{G} \in \{\mathcal{ADE}\}$ scattering theory. It reads

$$Y_{\mathtt{i}}\left(\vartheta + i\frac{\pi}{\mathtt{h}_{\mathcal{G}}}\right) Y_{\mathtt{i}}\left(\vartheta - i\frac{\pi}{\mathtt{h}_{\mathcal{G}}}\right) = \prod_{\mathtt{j}=1}^{\mathtt{r}_{\mathcal{G}}} (1 + Y_{\mathtt{j}}(\vartheta))^{[\boldsymbol{I}_{\mathcal{G}}]_{\mathtt{i},\mathtt{j}}}\,, \tag{4.8}$$

for $\mathtt{i} = 1, \ldots, \mathtt{r}_{\mathcal{G}}$.

It is manifest that Eq.(4.8) has a form completely determined by the underlying Lie algebra structure. Also, as anticipated, only the pseudoenergies enter this system, as the

driving terms are all made vanish in the derivation steps.
Moreover, in [48] it is conjectured the remarkable periodicity

$$Y_{\mathtt{i}}\left(\vartheta + i\pi \frac{\mathtt{h}_{\mathcal{G}} + 2}{\mathtt{h}_{\mathcal{G}}}\right) = Y_{\mathtt{r}_{\mathcal{G}} - \mathtt{i} + 1}(\vartheta) \tag{4.9}$$

for $\mathcal{G} \in \{\mathcal{A}\}$ and

$$Y_{\mathtt{i}}\left(\vartheta + i\pi \frac{\mathtt{h}_{\mathcal{G}} + 2}{\mathtt{h}_{\mathcal{G}}}\right) = Y_{\mathtt{i}}(\vartheta) \tag{4.10}$$

for $\mathcal{G} \in \{\mathcal{DE}\}$[2]. This has been verified numerically with high precision for many instances [59][60]. It is quite meaningful, as this result can be linked with the conformal dimension of the perturbing operator defining the theory's off-critical action[3] .
When, instead, approaching criticality, the Y-system turns out to be of paramount importance for the evaluation of the central charge. It is found that, in the UV limit $\ell \to 0$ (§2.4.3), the pseudoenergies for minimal (i.e. with trivial CDD factors) massive scattering theories can be deemed as constants[4] for the purpose of evaluating the finite size scaling function (Eq.(2.66)) [46]. This allows to write

$$\tilde{c}(0) = \frac{6}{\pi^2} \sum_{\mathtt{i}=1}^{\mathtt{r}_{\mathcal{G}}} \mathcal{L}\left(\frac{1}{1 + y_{\mathtt{i}}}\right), \tag{4.11}$$

where $\mathcal{L}(x)$ stands for the Roger dilogarithm function [73], while $y_{\mathtt{i}}$ denotes a static solution of Eq.(4.8). This means that solving the (algebraic) Y-system in a static regime yields precise information on the conformal central charge.

These observations alone are more than enough to motivate a study of these systems.

Let's, then, discuss the Y-system derivation for the sine-Gordon model. The steps above can be closely followed by specifying $\mathcal{G} \equiv \mathcal{D}_{\frac{1}{p}+1}$, i.e. $\mathtt{r}_{\mathcal{G}} = \frac{1}{p} + 1 \overset{def}{=} \mathtt{n}$, $\boldsymbol{I}_{\mathcal{G}} = \boldsymbol{I}_{\mathcal{D}_{\mathtt{n}}}$, $\mathtt{h}_{\mathcal{G}} = \frac{2}{p}$. With the definition in Eq.(4.7), the Y-system for the sine-Gordon model at reflectionless points reads

$$Y_{\mathtt{i}}\left(\vartheta + i\frac{\pi p}{2}\right) Y_{\mathtt{i}}\left(\vartheta - i\frac{\pi p}{2}\right) = \prod_{\mathtt{j}=1}^{\mathtt{n}} (1 + Y_{\mathtt{j}}(\vartheta))^{[\boldsymbol{I}_{\mathcal{D}_{\mathtt{n}}}]_{\mathtt{i},\mathtt{j}}}, \tag{4.12}$$

for $\mathtt{i} = 1, \ldots, \mathtt{n}$.
Due to its manifest relation to a $\mathcal{D}_{\mathtt{n}}$ Dynkin diagram incidence matrix, Eq.(4.12) also

[2]Symmetry arguments for the case $\mathcal{G} \in \{\mathcal{A}\}$ actually require that $Y_{\mathtt{i}}(\vartheta) = Y_{\mathtt{r}_{\mathcal{G}} - \mathtt{i} + 1}(\vartheta)$, so that also the second periodicity equation is satisfied in this case.

[3]More precisely, it is given by $\Delta = 1 - \frac{1}{P}$ for $\mathcal{G} \in \{\mathcal{ADE}\}$ and by $\Delta = 1 - \frac{2}{P}$ for $\mathcal{G} \equiv \mathcal{T}_{\mathtt{n}}$, with $P = \frac{\mathtt{h}_{\mathcal{G}} + 2}{\mathtt{h}_{\mathcal{G}}}$ [59].

[4]They are indeed constant inside rapidity 'plateau regions': more on this can be found in [14, §20.6].

enjoys a graphical representation, just like the one shown in Fig.(3.1). To each (full) node of the diagram is associated a (massive) particle species, in such a way that the equations' coupling in the above system is completely described by links between them.

4.1.2 Non-diagonal points

Once explained the derivation of $\mathcal{ADE}$ theories' Y-systems, the case of sine-Gordon reflectionless points turns out to be a straightforward application. The same can not be said for non-diagonal points $p \in \mathbb{Q} - \frac{1}{\mathbb{N}}$.

As discussed in previous sections, the mathematical structure inherent to the theory becomes more complicated (§3.1) as well as the sets of identities that enjoy the scattering kernels (Eqs.(3.36)(3.43)). Both these elements are at the root of the Y-system formulation, so it may be expected that many difficulties are met. One class of sine-Gordon parameter values for which the derivation is found to produce some good results is that of integer points $p \in \mathbb{N} - \{1\}$.

As explained in §3.2.2, for these values of the sine-Gordon parameter the theory entails only the (anti)soliton as massive particle, since the m_1 magnons are massless by definition. In view of the Y-system, this means that only the (anti)soliton driving term is different from zero, slightly simplifying the derivation. Even more importantly, the set of identities in Eq.(3.29) features only one universal kernel $\tilde{\varphi}_1(\xi)$ (defined in Eq.(3.28)) and no self-interaction kernels. These are great simplifications.

Recovering the index notation of Eq.(3.27), steps similar to the reflectionless case can be followed, starting from the analytic continuation

$$\vartheta \to \pm i\frac{\pi}{2}\,. \tag{4.13}$$

This choice is only natural, when considered that it reflects the form of $\tilde{\varphi}_1(\xi)$. Next a suitable linear combination of the raw TBA equations has to be considered. Let's take it in a form similar to Eq.(4.5)

$$\begin{aligned}
&\mathcal{M}_{\mathtt{i}}\left(\vartheta + i\frac{\pi}{2}\right) + \mathcal{M}_{\mathtt{i}}\left(\vartheta - i\frac{\pi}{2}\right) - \sum_{\mathtt{j}=1}^{\mathtt{n}} [\breve{\boldsymbol{I}} \leftarrow]_{\mathtt{i},\mathtt{j}}\, \mathcal{M}_{\mathtt{j}}(\vartheta) = \\
&= (r.h.s.)_{\mathtt{i}}\left(\vartheta + i\frac{\pi}{2}\right) + (r.h.s.)_{\mathtt{i}}\left(\vartheta - i\frac{\pi}{2}\right) - \sum_{\mathtt{j}=1}^{\mathtt{n}} [\breve{\boldsymbol{I}} \leftarrow]_{\mathtt{i},\mathtt{j}}\, (r.h.s.)_{\mathtt{j}}(\vartheta)\,.
\end{aligned} \tag{4.14}$$

Notice that in this case not all the η-signs on the right-hand side are positive. When imposing the vanishing of the driving terms, the only condition required is for the first column to be composed of zero entries ($[\breve{\boldsymbol{I}} \leftarrow]_{\mathtt{i},1} = 0,\ \forall\, \mathtt{i}$), which leaves much freedom for the other entries. As the following choice is made

$$
[\breve{\boldsymbol{I}} \leftarrow] = \begin{bmatrix}
 & -1 & & \dots & & & & \\
0 & & 1 & \dots & & & & \\
 & 1 & & \ddots & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
 & & & \ddots & & 1 & & \\
 & & & \dots & 1 & & 1 & 0 \\
 & & & \dots & & 1 & & \\
 & & & \dots & & -1 & &
\end{bmatrix}, \tag{4.15}
$$

the procedure of rewriting Eq.(4.14) in Fourier space yields exactly the left-hand side of the Takahashi-Suzuki identities for the scattering kernels. This implies that, once defined

$$
[\breve{\boldsymbol{I}} \rightarrow] = \begin{bmatrix}
 & -1 & & \dots & & & & \\
1 & & 1 & \dots & & & & \\
 & 1 & & \ddots & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
 & & & \ddots & & 1 & & \\
 & & & \dots & 1 & & 1 & 1 \\
 & & & \dots & & 1 & & \\
 & & & \dots & & -1 & &
\end{bmatrix} \tag{4.16}
$$

(which reflects the right-hand side of the Eq.(3.29) with reabsorbed η-signs, i.e. the matrix of Eq.(3.31)), taking the equations back in the rapidity space it is obtained

$$
\varepsilon_{\mathtt{i}}\left(\vartheta + i\frac{\pi}{2}\right) + \varepsilon_{\mathtt{i}}\left(\vartheta - i\frac{\pi}{2}\right) = \sum_{\mathtt{j}=1}^{\mathtt{n}} \big([\breve{\boldsymbol{I}} \leftarrow]_{\mathtt{i},\mathtt{j}} \varepsilon_{\mathtt{j}}(\vartheta) + [\breve{\boldsymbol{I}} \rightarrow]_{\mathtt{i},\mathtt{j}} L_{\mathtt{j}}(\vartheta)\big)\,. \tag{4.17}
$$

Recalling Eq.(2.83) and defining the Y-functions as

$$
Y_{\mathtt{i}}(\vartheta) = \mathrm{e}^{\eta_{\mathtt{i}} \varepsilon_{\mathtt{i}}(\vartheta)} \tag{4.18}
$$

this yields the Y-system for the sine-Gordon model at integer points

$$
Y_{\mathtt{i}}\left(\vartheta + i\frac{\pi}{2}\right) Y_{\mathtt{i}}\left(\vartheta - i\frac{\pi}{2}\right) = \prod_{\mathtt{j}=1}^{\mathtt{n}} (1 + Y_{\mathtt{j}}^{-1}(\vartheta))^{-[\boldsymbol{I}_{\mathcal{D}_{\mathtt{n}}}]_{\mathtt{i},\mathtt{j}}}\,. \tag{4.19}
$$

The final system obtained here seems to adhere to the form shown in [70] when the latter is specialized at integer points. Notice also the subtle generalization introduced in

Eq.(4.18): this is consistent with the choices proposed in [56][60] and is indeed responsible for the negative exponents appearing on the right-hand side. It may also be clearly observed that a $\mathcal{D}_{\mathtt{n}}$ incidence matrix is fully recovered in this case. The corresponding diagram is of Dynkin type, often depicted in the horizontal (vertical) direction to encode the positive (negative) sign chosen in Eq.(4.18).

The possibility of further generalizations is still an open question.

As mentioned in previous sections (§3.2.2), the case presented above may be seen as a particular instance of a class of higher-spin theories [6]. The hope is for a similar picture to be implemented also in such cases. Indeed, the identities for the scattering kernels are in the same form of Eq.(3.29), suggesting the possibility of a similar derivation.

For what concerns the sine-Gordon model at different values of the parameter, the procedure outlined in this section requires further study.
As for the way of analytically continue the raw TBA equations, shifts of the form $\vartheta \to \pm i\beta^{-1} p_i$ at level i of the continued fraction are to be expected. However, different kernels may appear within the same combination of raw TBA equations: this introduces some difficulties in the derivation. Also self-coupling terms should be properly treated, which is a highly non-trivial point.
This does not prevent some results to be obtained anyways. In [4][5] a study of the UV (and IR) limits of the sine-Gordon Y-system is performed, further elaborating the model in the generalized hydrodynamics context (for a didactic presentation on the topic, [74]). Also in [3] is conjectured a general form for the Y-system at any value of the parameter. The formulation therein is quite promising, but no formal proof has been presented yet.

The mathematical structure revealed by these theories shows a great depth of meaning. The hope is to being able to explore it even deeper.

Conclusion

Throughout this entire work the structure of the TBA equations is studied with a particular focus on the sine-Gordon model. Besides presenting an organized review of the most up-to-date results on the topic, observations are made regarding the formulation of Zamolodchikov-Takahasi-Suzuki identities adapted to generic values of the parameter. Suggestions come also in the form of Y-system description for integer values of p.

Many line of research are left open. The conjecture of [3] is still central in the current debate. The identities presented may shed some light on the topic, but also rise some difficulties yet to be studied. A comforting hope is given by the common structure shown by higher-spin theories, possibly allowing for future developments.

Acknowledgements

A heartfelt thanks goes to Prof.F.Ravanini for the support and the precious discussions, besides for sparking an interest towards this wide and rich branch of physics.

This work is also much indebted to the participants to the 10^{th} *Bologna Workshop on Conformal Field Theory and Integrable Models*. It truly has been an horizon-opening experience and a fertile field for debates.

Appendix A

On R-matrices

R-matrices are mathematical objects that, quite profoundly, rise in the context of (classical and quantum) integrability. The theory concerning them proves hard to be summarized in few words. This appendix, thus, aims at fixing the basic definitions.

Given V complex vector space, let R simply denote an application [33]

$$\begin{aligned} R\colon \mathbb{C} &\to \mathrm{End}(V\otimes V) \\ u &\to R(u) \end{aligned}\,, \tag{A.1}$$

with End(X) representing the space of endomorphisms of X, u being called spectral parameter. Now, consider the space $V^{\otimes 3} = V\otimes V\otimes V$ and define $R_{ij}(u)$ as acting as $R(u)$ on the i^{th} and the j^{th} component, i.e.

$$R_{ij}(u) \in \mathrm{End}(V^{\otimes 3}) : R_{ij}(u)|_{\mathrm{End}(V_i\otimes V_j)} = R(u),\; R_{ij}(u)|_{\mathrm{End}(V_{k\neq i,j})} = \mathbb{I}\,, \tag{A.2}$$

e.g. $R_{12}(u) = R(u)\otimes \mathbb{I}$. The following equation for $R(u)$ is the Yang-Baxter equation

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u)\,, \tag{A.3}$$

which parallels Eq.(1.39).
More in general, let $P\in \mathrm{End}(V\otimes V)$ denote the transposition $P(x\otimes y) = y\otimes x$ and $\check{R}(u)$ be

$$\check{R}(u) = PR(u)\,. \tag{A.4}$$

When considering $V^{\otimes m}$ ($m\geq 2$), the matrices $\check{R}_i(u)$ may be defined as acting as the identity $\mathbb{I}$ on all spaces but the i^{th} and the $(i+1)^{\text{th}}$ ($1\leq i\leq m-1$) where it acts as $\check{R}(u)$

$$\check{R}_i(u) = \mathbb{I}\otimes\cdots\otimes\mathbb{I}\otimes\check{R}(u)\otimes\mathbb{I}\otimes\cdots\otimes\mathbb{I}\,. \tag{A.5}$$

Then the R-matrix satisfies the Yang-Baxter equation in the form

$$\begin{aligned} \check{R}_i(u)\check{R}_j(v) &= \check{R}_j(v)\check{R}_i(u)\,, \quad \text{if } |i-j|>1 \\ \check{R}_{i+1}(u)\check{R}_i(u+v)\check{R}_{i+1}(v) &= \check{R}_i(v)\check{R}_{i+1}(u+v)\check{R}_i(u)\,, \quad \text{otherwise} \end{aligned}\,. \tag{A.6}$$

This admits generalizations. For instance, instead of working with a fixed vector space V, a family of vector spaces $\{V_i\}$ can be considered. The Yang-Baxter equation becomes, then, an equation in $\mathrm{End}(V_1 \otimes V_2 \otimes V_3)$ with $R_{ij}(u) = R_{V_i V_j}(u)$. Supposing $V_1 = V_2 = V$ and regarding $\mathrm{End}(V \otimes V_3) = \mathrm{End}(V) \otimes \mathrm{End}(V_3) = \mathrm{End}(V) \otimes \mathcal{A}$, it is possible to write R_{VV_3} as

$$T(u) = \sum_{ij} t_{ij}(u) E_{ij} \, , \tag{A.7}$$

with E_{ij} in the (canonical) basis of $\mathrm{End}(V)$ and $t_{ij}(u) \in \mathcal{A}$. Then the Yang-Baxter equation becomes

$$\check{R}(u-v)(T(u) \otimes T(v)) = (T(v) \otimes T(u))\check{R}(u-v) \, . \tag{A.8}$$

This can be seen as an equation giving the commutation relations for the generators $t_{ij}(u)$ of the (Hopf) algebra $\mathcal{A}$.

Solutions of the previous equations are known. For instance, in the case $V = \mathbb{C}^2$, the 4×4 matrix in Eq.(1.51) satisfies this requirements (with suitable parametrization of the entries). Let's just mention that it can be seen to rise in the description of the 6-vertex model, collecting the statistical weights assigned to each vertex in a 2-dimensional square lattice with 'ice-type' rules [34]. This model is related to the $XXZ_{\frac{1}{2}}$ spin chain by a remarkable mapping: the 6-vertex model transfer matrix coincides with the exponential of the spin chain quantum hamiltonian.

Appendix B

On $\mathcal{U}_q(\mathfrak{su}_2)$

The quantum algebra $\mathcal{U}_{\mathrm{q}}(\mathfrak{su}_2)$ may be interpreted as a deformation of a $\mathfrak{su}_2$ algebra by means of a (complex) deformation parameter q. The algebra is generated by $S^{\pm}$, $q^{\pm S^z}$ as per the commutation relations

$$[S^z, S^{\pm}] = \pm S^{\pm} \qquad [S^+, S^-] = \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}} \overset{def}{=} [2S^z]_q \,, \tag{B.1}$$

which are manifestly reducing to those of $\mathfrak{su}_2$ in the limit $q \to 1$. The Casimir operator is given in the form

$$C = S^- S^+ + \left(\left[S^z + \frac{1}{2}\right]_q\right)^2 = S^+ S^- + \left(\left[S^z - \frac{1}{2}\right]_q\right)^2 . \tag{B.2}$$

In relation to the discussion presented above, it is interesting to study the definite-parity irreducible highest-weight representations: when q is not a root of unity, they are in a bijective correspondence with the representations of $\mathfrak{su}_2$. Thus, introducing a simultaneous eigenstate of S^z and C and acting on it by means of both the Casimir and the commutation relations, when defining

$$q = \mathrm{e}^{i\theta} \qquad p_0 = \frac{\pi}{\theta} \,, \tag{B.3}$$

it is possible to show [55] that the dimension n and parity v of the representation have to satisfy

$$v = \exp\left(i\pi \left\lfloor \frac{n-1}{p_0} \right\rfloor\right) \qquad \left\lfloor \frac{k}{p_0} \right\rfloor + \left\lfloor \frac{n-k}{p_0} \right\rfloor = \left\lfloor \frac{n-1}{p_0} \right\rfloor . \tag{B.4}$$

While the one on the left coincides exactly with the parity expression of Eq.(2.59b), the one on the right is shown to be verified [44] when the decomposition of Eq.(2.58) holds and n satisfies Eq.(2.59a).

This means that magnon (string) parity and length are in one-to-one correspondence with the parity and length of the irreducible highest-weight representations of $\mathcal{U}_{\mathrm{q}}(\mathfrak{su}_2)$.

Appendix C

On Dynkin diagrams

A Lie algebra $\mathcal{A}$ is defined as a vector space gifted with Lie brackets $[\cdot,\cdot]\colon \mathcal{A}\times\mathcal{A}\to\mathcal{A}$. That's an antisymmetric binary operator which satisfies the Jacobi identity

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0 \qquad \text{for } X,Y,Z\in\mathcal{A}\,. \tag{C.1}$$

Lie algebras are related with Lie groups by the exponentiation map: they may be thought as the tangent space to the connected component of the Lie group containing the unity. Also, they are specified by a set of generators and their commutation relations. If the algebra is of dimension $\mathtt{d}$, then it admits generators $\{J^i\}_{i=1}^{\mathtt{d}}$ such that

$$[J^i,J^j]=\sum_{k=1}^{\mathtt{d}} i f^{ij}_k J^k\,. \tag{C.2}$$

The numbers f^{ij}_k are known as structure constants and are characteristic of the Lie algebra.

The Cartan-Weyl construction allows to select the generators in a particular way. It is possible to extract the maximal set of hermitian generators $H^i{}_{i=1}^{\mathtt{r}}$ commuting one with another, i.e.

$$[H^i,H^j]=0 \qquad \text{for } i,j=1,\ldots,\mathtt{r}\,. \tag{C.3}$$

The number $\mathtt{r}$ is known as rank of the Lie algebra. The choice above constructs the Cartan subalgebra $\mathcal{A}_C$, which is abelian by definition. The $\mathtt{d}-\mathtt{r}$ generators still not defined can be chosen so to satisfy

$$[H^i,E^\alpha]=\alpha^i E^\alpha\,. \tag{C.4}$$

The vector constructed collecting all the αs si called a 'root' for the corresponding ladder operator E^α: it is simply denoted here as $\alpha=(\alpha^1,\ldots,\alpha^{\mathtt{r}})$. It may be seen that roots naturally belong to the dual of the Cartan subalgebra $\alpha\in\mathcal{A}_C^*$, as each of them can define the map $\mathcal{A} \ni H^i \to \alpha(H^i) = \alpha^i \in \mathbb{K}$ (being $\mathbb{K}$ the field on which the algebra is defined). From the above, it is manifest that there are $\mathtt{d} - \mathtt{r}$ roots: in general, they are linearly dependent ($> \mathtt{r}$). It is possible to expand the roots with respect to a given basis. In this way, they are deemed positive (negative) if the first non-zero entry is positive (negative). Simple roots are defined as those roots that can not be obtained as the sum of two positive roots. On the other hand, the highest root is the one for which the sum of the expansion coefficients is maximal (this sum is equal to the Coxeter number minus one). Then, denoting as ρ the set of all roots, the full set of commutation relations for the Cartan-Weyl basis is obtained adding

$$[E^\alpha, E^\beta] = (\delta_{(\alpha+\beta)\in\rho}) C_{\alpha\beta} E^{\alpha+\beta} + (\delta_{\alpha,-\beta}) \frac{2}{|\alpha|^2} \alpha \cdot H \,, \tag{C.5}$$

where $C_{\alpha,\beta}$ are constants and the following definitions have been introduced: $\alpha \cdot H = \sum_{i=1}^{\mathtt{r}} \alpha^i H^i$, $|\alpha|^2 = \sum_{i=1}^{\mathtt{r}} \alpha^i \alpha^i$.

Of interest is the possibility to introduce a Killing form: up to normalization,

$$\mathscr{K}(X, Y) = \mathrm{Tr}[ad(X) ad(Y)] \,, \tag{C.6}$$

where ad stands for the adjoint representation. The latter is the association defined as $\mathcal{A} \ni Y \to ad(X)Y = [X, Y] \in \mathcal{A}$. The Killing form induces a positive definite scalar product in the dual space $\mathcal{A}_C^*$ in the form

$$(\alpha, \beta) = \mathscr{K}(H^\alpha, H^\beta) \,. \tag{C.7}$$

This allows to define the Cartan matrix of the algebra by means of products of simple roots, i.e.

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{\alpha_j^2} \,. \tag{C.8}$$

The study of the Cartan matrix allows to completely reconstruct the whole algebra, as it depends on the relations between roots. The latter are found to show only two possible lengths (often named long and short), while the angles between them (deduced from the scalar product) may assume values only in the set $\{90, 120, 135, 150\}$. This allows to encode the structure of the Lie algebra in a single Dynkin diagram: it entails a node for each simple root (black (white) for short (long) roots) and a number of links connecting them equal to 0, 1, 2, 3 depending on the angle. This can be further encoded into an incidence matrix, featuring the number of links as entries. Algebras whose diagrams show roots all of the same length are called simply laced: they are in the $\{\mathcal{ADE}\}$ series.

Appendix D

On some functions

In order to cast the TBA equations in the universal form, the Fourier transform of the scattering kernels (the Ks of Eqs.(2.79)) has to be evaluated. To this aim, some observations may be useful.

Starting from the easier instance, the kernel $\mathscr{F}K^{(SS)}(\xi)$ may be read directly from the integral form of the soliton-soliton scattering amplitude in Eqs.(1.45)(1.46). The result of Eq.(3.11c) is quickly obtained.

Slightly more attention has to be paid for evaluating the kernels $\mathscr{F}K^{SB_a}$ and $\mathscr{F}K^{B_aB_b}$. On suggestion of [46], it is possible to introduce functions in the form

$$\mathrm{f}_\gamma(\vartheta) = \frac{\sinh\frac{1}{2}(\vartheta + i\pi\gamma)}{\sinh\frac{1}{2}(\vartheta - i\pi\gamma)} \qquad \mathrm{f}_\gamma(i\pi - \vartheta) = \frac{\cosh\frac{1}{2}(\vartheta - i\pi\gamma)}{\sinh\frac{1}{2}(\vartheta + i\pi\gamma)}\,, \tag{D.1}$$

with properties

$$\mathrm{f}_\gamma(\vartheta)\,\mathrm{f}_\gamma(i\pi - \vartheta) = \frac{\sinh\vartheta + i\sin\pi\gamma}{\sinh\vartheta - i\sin\pi\gamma} \tag{D.2a}$$

$$\mathrm{K}_\gamma(\vartheta) = \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}\,\vartheta}\log\mathrm{f}_\gamma(\vartheta) = -\frac{\sin\pi\gamma}{\cosh\vartheta - \cos\pi\gamma} \tag{D.2b}$$

$$\mathrm{K}^\gamma(\vartheta) = \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}\,\vartheta}\log\mathrm{f}_\gamma(i\pi - \vartheta) = -\frac{\sin\pi\gamma}{\cosh\vartheta + \cos\pi\gamma}\,. \tag{D.2c}$$

It may be observed that the scattering amplitudes involving breathers can be completely expressed in terms of the functions of Eq.(D.1). They read

$$S^{(SB_a)} = \mathrm{f}_{\frac{1-pa}{2}}(\vartheta)\,\mathrm{f}_{\frac{1-pa}{2}}(i\pi - \vartheta)\prod_{l=1}^{a-1}\mathrm{f}^2_{\frac{1-(a-2l)p}{2}}(\vartheta) \tag{D.3a}$$

$$\begin{aligned} S^{(B_aB_b)} = {} & \mathrm{f}_{\frac{(a+b)p}{2}}(\vartheta)\,\mathrm{f}_{\frac{(a+b)p}{2}}(i\pi - \vartheta)\,\mathrm{f}_{\frac{(a-b)p}{2}}(\vartheta)\,\mathrm{f}_{\frac{(a-b)p}{2}}(i\pi - \vartheta)\cdot \\ & \cdot \prod_{l=1}^{\min\{a,b\}-1}\mathrm{f}^2_{\frac{(2l+|b-a|)p}{2}}(\vartheta)\,\mathrm{f}^2_{1+\frac{(2l-(b+a))p}{2}}(\vartheta)\,. \end{aligned} \tag{D.3b}$$

It is, then, sufficient to compute the Fourier transform of $\mathrm{K}_\gamma(\vartheta)$ and $\mathrm{K}^\gamma(\vartheta)$ in order to obtain Eqs.(3.11a)(3.11b). It is found that

$$\mathscr{F}\,\mathrm{K}_\gamma(\xi) = -\frac{(1-\delta_{0,|\gamma|\,\mathrm{mod}(0,2)})\,\mathrm{sign}\gamma}{2\pi}\frac{\sinh\pi\xi(1-(|\gamma|\,\mathrm{mod}(0,2)))}{\sinh\pi\xi} \tag{D.4a}$$

$$\mathscr{F}\,\mathrm{K}^\gamma(\xi) = -\frac{(1-\delta_{0,|\gamma|\,\mathrm{mod}(0,2)})\,\mathrm{sign}\gamma}{2\pi}\frac{\sinh\pi\xi(|\gamma|\,\mathrm{mod}(-1,1))}{\sinh\pi\xi} \tag{D.4b}$$

and the (lengthy but straightforward) application of trigonometric identities yields the sought form for the breather scattering kernels.

Only the magnon terms remain (Eq.(2.80)). Let's just mention that, in this case, the Fourier transformation has to be done even more carefully, since imaginary shifts may occur in the argument of the function (depending on the magnon parity). When denoting

$$\phi(\vartheta;\mathfrak{n}) = i\log\varsigma_{-\,\mathfrak{n}}(\beta\theta) = i\log\frac{\sinh\frac{\pi}{2\alpha}(\beta\theta+i\,\mathfrak{n})}{\sinh\frac{\pi}{2\alpha}(\beta\theta-i\,\mathfrak{n})}\,, \tag{D.5}$$

its rapidity derivative shows different forms depending on the presence/absence of the imaginary shifts. It is found that

$$\frac{\partial\phi(\vartheta;\mathfrak{n})}{\partial\vartheta} = \frac{1}{p}\frac{2\sin\frac{\pi}{\alpha}\,\mathfrak{n}}{\cosh\frac{\pi}{\alpha}\beta\vartheta-\cos\frac{\pi}{\alpha}\,\mathfrak{n}} \tag{D.6a}$$

$$\frac{\partial\phi(\vartheta+i\frac{\pi p}{2};\mathfrak{n})}{\partial\vartheta} = -\frac{1}{p}\frac{2\sin\frac{\pi}{\alpha}\,\mathfrak{n}}{\cosh\frac{\pi}{\alpha}\beta\vartheta+\cos\frac{\pi}{\alpha}\,\mathfrak{n}}\,, \tag{D.6b}$$

which are not much diverse w.r.t. Eq.(D.4). Once taken care of the additional factors, the result may be cast as in Eq.(3.12).
However, it is interesting to notice that the derivatives of Eq.(D.6) show different signs depending on the values of $\mathfrak{n}$, i.e. the function $\phi(\vartheta;\mathfrak{n})$ may increase/decrease depending on the magnon length: this line of reasoning is at the basis of the choice of Eq.(2.83) for the magnons.

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