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THE ENERGY-MOMENTUM TENSOR OF PRIMORDIAL GRAVITATIONAL WAVES

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Abstract

In this thesis we have studied the energy momentum tensor (EMT) of cosmological gravitational waves generated in the Early Universe and characterized by a power spectrum described by a power-law in Fourier space. We have computed the EMT by considering only the growing mode of GWs when the scale factor is described as a power of the conformal time, i.e. covering different cosmological stages such as radiation or matter or cosmological constant driven.

We have independently verified previous calculations and found that that the extra term proportional $Hh\dot{h}$ in the energy density is not negligible with respect to the standard terms, i.e. the sum of the kinetic term and laplacian term, in particular on large wavelengths down to the Hubble radius and in the integrated EMT. The full EMT as presented in this thesis should be therefore considered in the comparison with observations.

We have also presented an original comparison of the calculation with the growing mode with the regularized EMT in de Sitter space-time, finding a good agreement. The results of this thesis increase our confidence in the calculation with only the growing mode for the post-inflationary era and encourage us to compute the phenomenological implication of the full EMT also at the time of Big Bang Nucleosynthesis for any n_T.

Contents

Introduction

Gravitational waves carry energy and momentum. The energy momentum tensor of gravitational waves from merging black holes hit the arms of the LIGO detectors in 2016 opening era of gravitational wave astronomy. In this thesis we study the energy momentum tensor (EMT henceforth) for cosmological gravitational waves generated in the Early Universe and characterized by a power spectrum described by a power-law in Fourier space.

This EMT is obtained as a quadratic form in the amplitude of gravitational waves, which are described by transverse and traceless first order linear perturbations in Robertson-Walker space-time, directly from the Einstein equations. Due to non-linearity of the Einstein equations this energy momentum can also distort the space-time which has created them.

This thesis is organized as follows. In the first chapter, we review the basics of cosmology, laying a fundamental foundation through the introduction of GR, the model of our Universe, inflation, and an overview of CMB. In the second we talk about cosmological gravitational waves. Further giving the theoretical overview of the formation of GW. We then talk about Cosmological perturbations which come from quantum fluctuations through primordial effects. In chapter three we set up the theoretical framework to derive the EMT of GWs coming from second-order perturbation, conservation related to it, and more about propagation and back reaction of GW. In chapter four we define constraints on PGWs from CMB and other processes. We will also talk about the Pulsar timing array (PTA) which has claimed the first detection of stochastic GW background and LIGO-Virgo-Kagra which has detected the first observed GW from black hole merger. In the fifth chapter, we present our original results and a connection between the extension of our results to de Sitter space-time to the results obtained for the regularized energy momentum tensor.

Chapter 1

Cosmological Foundation

If our true purpose is understanding the origin of cosmic consciousness, it must be dated back to the newly forged minds of Homo sapiens with the very awakening of their thoughts. With time perceiving towards the civilizations, cosmology was shaped in the form of philosophy, till the 20th century, when we had the first cosmological revolution with three prime milestones:

- Discovery of the true size of our Universe being millions of times larger than our own galaxy.
- Finding the observational results which conflicted with the idea of static universe. Everyone at the face value, including Einstein accepted an expanding universe.
- The acceptance of the idea that our Universe had not been eternal but has a start. Further this idea of beginning was carved with fundamental laws of physics, giving birth to inflation.

As Einstein's theory of General Relativity was too complex to directly counter constraints on the degree of freedom related to the distribution of matter [Coles and Lucchin [2003]] so to define a new science there were few assumptions taken in early 20th century which were:

- Cosmological Principle - states that, on sufficiently large scale our universe is *homogeneous*, defined on an average sense, and *isotropic*, so we do not have any favorite direction or position in the spacetime;
- General Relativity- on the large scales, the dominant force is gravity and Einstein's theory of General Relativity is its best description.

To reach the true foundation of modern cosmology we need to step back to 1917.

1.1 The history of modern cosmology

More than a century ago, 37 year old Albert Einstein presented his paper "Cosmological considerations in the general theory of relativity" laying the foundational pillars of modern cosmology, which conferred a tenet that structural geometry of domain of spacetime is not unconstricted rather is shaped by mass-energy and the idea revels with his field equation.

$$G_{\mu\nu} = -8\pi G \, T_{\mu\nu} \ , \tag{1.1}$$

where both $G_{\mu\nu}$ and $T_{\mu\nu}$ are four-dimensional quantity. $G_{\mu\nu}$ is describing the geometry of the region of spacetime, while $T_{\mu\nu}$ describes flux of mass energy. The quantity $8\pi G$ gives a constant which is also equated as M_{Pl}^{-2}. After completion of his theory, it was natural for him to ask if the theory could formulate the consistent model of universe [O'Raifeartaigh [2017]]. He found that considering a static distribution of matter, it was very hard to reach a clear solution considering both the model of relativity and Mach's principle which says that the inertia is entirely determined by the presence of other masses. Einstein found that a non zero solution to his equation can be obtained by adding a new term called cosmological constant in field equation:

$$G_{\mu\nu} + \lambda g_{\mu\nu} = -8\pi G \, T_{\mu\nu} \ . \tag{1.2}$$

Just few months after the publication, de Sitter found that the modified equation (1.2) allowed for a universe without any matter content. Einstein was perturbed with this model of de Sitter as it merely resembles the reality and it conflicted directly with Mach's principle. In 1922 Alexender Friedmann proposed a non static solution of field equation considering a relativistic universe and derived two Friedmann equation, by assuming a positive spatial curvature of the universe which connects the evolution of time with matter density and cosmological constant.
However Einstein was captivated by the contribution of Friedmann. In 1927 Georges Lemaître derived differential equations for the radius of universe which was an independent work and it resembled with Friedmann equations. Being aware about the observation of recession of spiral nebula, he found it evident to say the universe is expanding but Einstein did not welcome his work as well. Ultimately in 1929 Edwin Hubble published the observational evidence of redshift and change in spiral nebula's radial distance and it was a strong evidence for a non static universe. In 1930 Einstein corrected his opinion, publishing a model for an expanding universe abandoning the idea of static model of universe and also abandoning the cosmological constant from his equation although today there is a dramatic return of cosmological constant due to an observation of an accelerating universe.

1.2 General Relativity

The fundamental idea of every modern theory for description of gravity comes from General Relativity. The theory formulated in 1917 is a natural generalization of Einstein's theory of Special Relativity and ultimately this theory formulates the Einstein's Field Equation. Let's take a general overview of some of its concepts.

The following equation

$$ds^2 = -g_{\mu\nu} dx^\mu dx^\nu \tag{1.3}$$

provides the measure between two infinitesimally distanced events where $g_{\mu\nu}$ is the tensor which describes the space time's geometry. The particle in the tensor geometry moves on the geodesic path following the equation of motion which is written as

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} \ . \tag{1.4}$$

The equation above is very important as it sketches the idea of free particle travelling in background of $g_{\mu\nu}$ which is fixed with the matter distribution. This geodesic has components $\Gamma^\mu_{\alpha\beta}$ which are called Christoffel symbols and are written as

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}\left(\frac{\partial g_{\lambda\alpha}}{\partial x^\beta} + \frac{\partial g_{\lambda\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\lambda}\right) \ . \tag{1.5}$$

We can carve out the generalized expression for the conservation of energy and momentum

$$T_{\mu\nu} = (p + \rho c^2)u_\mu u_\nu - pg_{\mu\nu} \ . \tag{1.6}$$

The above equation is the expression for energy momentum tensor, written for a perfect fluid having pressure P and density ρ . The term u_μ and u_ν are 4- velocity expressed as

$$u_\mu = g_{\mu\nu}\frac{dx^\nu}{ds} \ . \tag{1.7}$$

The fundamental equation that Einstein found to correlate the metric tensor with energy momentum tensor is known as Einstein field equation, which is written as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4} \ .T_{\mu\nu} \tag{1.8}$$

Here $R_{\mu\nu}$ is the Ricci tensor crafted from the Riemann Tensor $R^\alpha_{\mu\alpha\nu}$. The Riemann Tensor is written as

$$R^\alpha_{\beta\gamma\delta} = \frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha_{\kappa\gamma}\Gamma^\kappa_{\beta\delta} + \Gamma^\alpha_{\kappa\delta}\Gamma^\kappa_{\beta\gamma} \ ; \tag{1.9}$$

$R = R^\alpha_\alpha$ gives the Ricci Scalar and $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = G_{\mu\nu}$ is the Einstein Tensor.

1.3 Model of our Universe

The standard cosmological model that we follow now is the ΛCDM Model of Hot Big Bang which agrees with the vast majority of data but still holds some problems. The basic assumptions taken for ΛCDM universe other than General Relativity are

- Cosmological Expansion - as given by Hubble- Lemaître law, we can clearly state that our universe is now expanding in an accelerated way, so we can rely on models of expansion.
- Temperature Evolution - The history of our universe can be stated as a thermal history as expansion makes temperature a decreasing function of time.

To understand more about the model, let's form the metric first.

1.3.1 Friedmann-Lemaître-Robertson-Walker (FLRW) metric

Establishing the Cosmological Principle requires us to hold the geometric property of homogeneity and isotropy. Now the property of this defined geometry can be expressed using a line element as

$$ds^2 = -(cdt)^2 + a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] . \tag{1.10}$$

The equation uses spherical polar coordinate r, θ, ϕ, $a(t)$ is again the expansion parameter with dimension of a length, r is dimensionless and K being the curvature parameter which corresponds to three different solutions: with $K < 0$ there is a negative curvature for the geometrically open universe, $K = 0$ gives solution for a flat Euclidean sphere and $K > 0$ has the property of hypersphere, which has closed geometry.

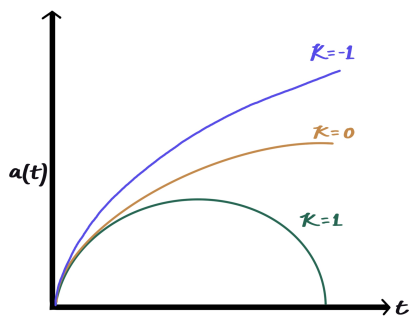

Figure 1.1: Geometrical variation of Universe, Open ($K = -1$), Closed ($K = 1$), flat ($K = 0$).

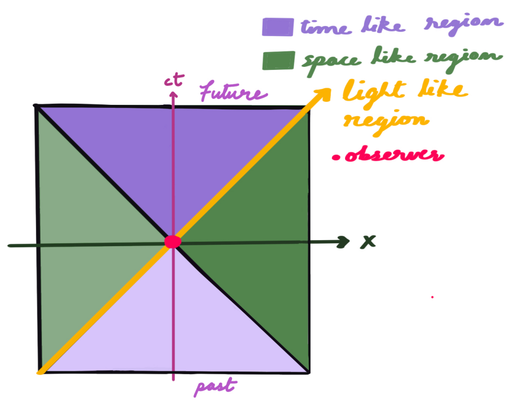

Figure 1.2: Minkowski space - showing time-like, light-like and space-like regions, and the observer is in the hypersurface of present.

The Einstein field equation relates spacetime's geometrical property with the total content of universe that is energy momentum tensor. Specifically it connects perfect fluid with pressure p and energy density of rest mass ρc^2. The two equations of Friedmann are solutions of the field equation and are written as

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) a \tag{1.11}$$

$$\dot{a}^2 + Kc^2 = \frac{8\pi G}{3}\rho a^2 \ . \tag{1.12}$$

Here a represents the expansion factor and K is curvature parameter. The two equations are related with the adiabaticity condition.The relation can be framed as

$$\dot{\rho} = -\frac{3\dot{a}}{a}\left[\frac{p}{c^2} + \rho\right] \ . \tag{1.13}$$

The equations are displaying a time evolution of parameter of expansion a. The space defined by Gaussian curvature can be closed, flat or open depending on density parameter

$$\Omega(t) = \frac{\rho}{\rho_c} \tag{1.14}$$

is greater than or equal to or less than 1. The evolution of Universe can be understood as homogeneous expansion characterized by time evolution of scale factor. This gets structured in Hubble Laîmatre law which gives the Hubble parameter as

$$H = \frac{\dot{a}}{a} \ . \tag{1.15}$$

1.3.2 An overview of the ΛCDM Universe

The contemporary Euclidean universe is characterized by a cosmological constant and by non-baryonic cold dark matter (CDM) [Dodelson and Schmidt [2020]]. These components, along with initial perturbations from early-universe's inflation, constitute the concordance model of cosmology, often referred to as (flat) ΛCDM. It is noteworthy that these elements diverge from the Standard Model of particle physics. Regarding CDM, its "Cold" attribute stems from the requirement for dark matter particles to efficiently clump in the early universe, crucial for structure formation. Contrarily to this, hot dark matter, characterized by large velocities (e.g., neutrinos), fails to create the observed structures. The existence of non-baryonic matter is supported by observations of cosmic structure, along with the smoothness predicted in a baryon-only universe. Measurements of galaxy velocities within clusters and galaxy rotation curves prove the presence of roughly 5 times more dark matter than baryons.

Evidence from distant supernovae suggests that there exists a dark energy component alongside the ordinary matter and radiation. Unlike dark matter, dark energy, presented as a cosmological constant, does not cluster. The concept that empty space itself carries energy, consistent with quantum mechanics and Heisenberg's uncertainty principle, poses challenges in quantifying the cosmological constant, raising questions about its inexplicably large value compared to observations.

We will see further that the inflation, which is added to the hot big bang model, gives mechanisms for generating initial perturbations, leading to observed cosmic structures. Inflation is a brief epoch of exponential expansion in the early universe. Although sharing certain features with the present universe, such as a roughly constant dominant energy form during expansion, the energy scales associated with inflation are vastly different. The challenge lies in probing these high-energy scales experimentally, with potential signatures of inflation offering insights into physics at unprecedented energy scales.

1.3.3 Problems of Standard Model

There are few assumptions upon which the hot Big Bang model is framed upon:

- Physics laws which are verified today should also uphold in the early universe.
- The cosmological principle holds true at all times.
- The initial conditions are bound to be in thermal equilibrium with a temperature larger than $10^{12}K$ and the baryon asymmetry exists also taking into consideration that there is some initial density fluctuations giving rise to formation of structures later.

The model has achieved few outstanding successes like:

- Prediction of cosmological production of Helium in huge quantity during primordial Nucleosynthesis stands correct with observations.
- A framework for understanding the galaxy formation and evolution.
- Cosmic Microwave Background (CMB) is explained as a relic of the hot initial phase.

But there remains many unexplained important features. In fact there are more problems than the solutions provided by standard model:

- The flatness problem
- The Cosmological constant problem
- The baryon asymmetry problem - monopole problem
- The Problems of the evolution of the universe at high energy
- The origin of the primordial spectrum of density fluctuations
- The understanding of the nature of Dark matter.

Many of these problems are solved with the introduction of inflation. we can look into some of the problems and later jump to the solution on how inflation solves these.

The Flatness Problem

The flatness problem corresponds to the problem with the selection of geometry of our universe. The initial conditions for flat geometry were so special that a very very small deviation from this value would have led us to a very high curvature and so it is also called fine tuning problem as it seems that the universe itself chose this value for density of matter and energy. To understand this, let's extrapolate that Einstein equations are valid till Planck era with particle energy of $10^{19} GeV$ .The second Friedmann equation (1.4) can be re-written as curvature equation

$$\Omega - 1 = \frac{K}{H^2 a^2} \ , \tag{1.16}$$

where Ω is the density parameter and is going to be unity for a perfectly flat universe. It is a dimensionless quantity describing the density of matter or energy relative to the critical density ($\Omega = \frac{\rho}{\rho_c}$). K is the curvature parameter which is 0 for the flat universe. H is the Hubble parameter which describes the rate of expansion of the universe and is defined as ratio of velocity of recession to the distance and a is the scale factor, which represents the relative size of universe at given time.
During Radiation dominated era $H^2 \propto a^{-4}$ making $\Omega - 1 \propto a^2$and in matter dominated era $H^2 \propto a^{-3}$ leading to $\Omega - 1 \propto a$. In both cases $\Omega - 1$ decreases going back in time. So when we compare its value at the Planck time to today we find that

$$\frac{|\Omega - 1|_{T_{Pl}}}{|\Omega - 1|_{T_0}} \propto O(10^{-64}) \tag{1.17}$$

and the ratio for the same at the time of primordial nucleosynthesis and today is

$$\frac{|\Omega - 1|_{T_N}}{|\Omega - 1|_{T_0}} \propto O(10^{-16}) \ . \tag{1.18}$$

To get the correct value of ($\Omega_0 - 1 \sim 1$) for today the value must be very fine tuned amazingly near to zero and Ω must had been extremely close to unity in the early epochs. This is the reason of it being called as fine tuning problem. The problem gets solved after we introduce inflation. The requirement to solve the problem is when the density parameter at the start of inflation is closer to unity than today: this condition constrains the duration of inflation and we find that to solve the flatness problem number of e-folding, i.e. the logarithm of the ratio between the scale factor at the end and at the start of inflation should be greater than 60. What it does is if there would had been any deviation or change of curvature from flatness in the early universe gets stretched out during this phase. The accelerated expansion of inflation would dilute the curvature making it looking flat on the large scales. Inflation also takes the universe towards critical density, which flattens the geometry, making also the universe more homogeneous and isotropic on large scales.

The Horizon Problem

The Big Bang theory has the fundamental assumption of the cosmological principle along with the two Friedmann equations which predict an initial singularity. The

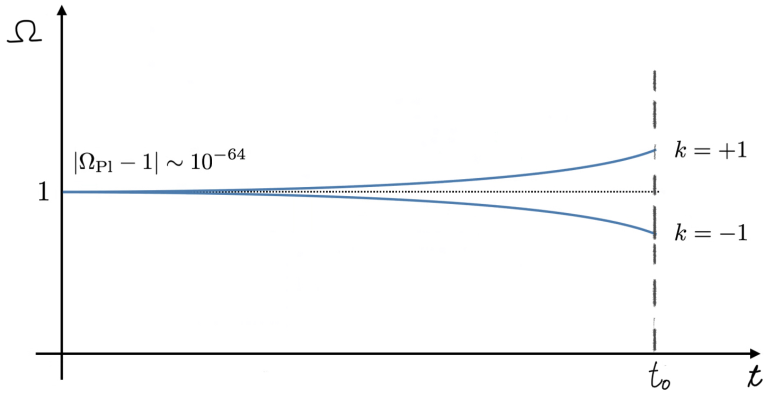

Figure 1.3: Plot showing the flatness problem

Friedmann models with equation of state $P = w\rho c^2$ with $w \geq 0$ (for standard physics), possesses a particle horizon. The particle horizon is given by

$$R_H(t) = a(t)\int_0^t \frac{cdt'}{a(t')} \; . \tag{1.19}$$

The existence of a cosmological horizon makes it very hard to accept the Cosmological Principle. As observed from Cosmic Microwave Background (CMB) radiation, which is the background radiation which came out after the epoch of last scattering (about $z \sim 1100$ corresponding to about 300,000 years after the big bang) the temperature fluctuations are homogeneous and isotropic, which means that they are in causal connection having reached the thermal equilibrium. The radius of last scattering surface at that epoch can be written as

$$r_{ls}(t) \approx \frac{c(t_o - t_{ls})}{1+z_{ls}} \approx \frac{ct_0}{z_{ls}} \; , \tag{1.20}$$

while the radius of the particle horizon at the same epoch can be written as

$$R_H(z_{ls}) \approx (0.1)r_{ls} \ll r_{ls} \; . \tag{1.21}$$

This clearly tells us that CMB has properties related to the causal connection on scales at least ten times larger than that of the particle horizon. This is a big problem in standard model which will be further solved by inflation. The solution to this comes using two conditions: the first one is that we have an overall accelerated expansion in the very beginning and the second one is that the comoving horizon radius at that time must had been larger than that today. In order to have the first condition $\ddot{a} > 0$ we need to have w of equation of state giving $w < -1/3$ as $P = w\rho c^2$. The other condition of a decrease of the comoving horizon does not mean a decrease in the proper horizon since it is defined $R_{H,c} = \int_0^t \frac{cdt}{a(t)}$ and so when there would be an enormous increase in scale factor the cumulative integral would give a comoving horizon bigger radius than at the present time when the scale factor is not growing as fast as in early times. So R_H increases with time while the comoving horizon decreases after the early period of accelerated expansion. In fact if we use

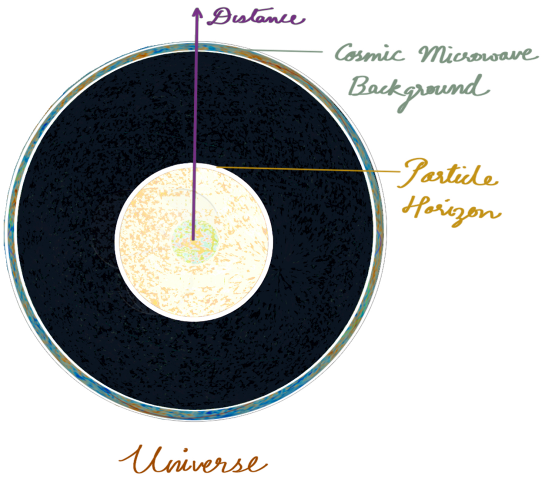

Figure 1.4: An illustration of the Horizon problem showing the difference between the distance from which radiation of CMB is coming today with respect to the particle horizon.

the first condition, we break it in three different case: $-1 < w < -1/3$ called as sub inflation, $w < -1$ as super inflation and $w = -1$ would correspond to the standard exponential inflation. Now, if we do the math to find the condition to solve horizon problem, we find again a constraint on the duration of accelerated expansion which in terms of number of e- foldings should be $N > 60$. This gets satisfied once we introduce inflation to solve the problem of horizon.

The Cosmological Constant problem

Today there's another problem which still is unsolved, which is the problem related to the cosmological constant.

$$\Omega_{0,\Lambda} \approx 0.7 \tag{1.22}$$

which is a major contributor of density parameter taking the energy density to be

$$\rho_{0,\Lambda} = constant = \frac{\Lambda c^2}{8\pi G} \approx \rho_{0,critical} = 10^{-28} g/cm^3 \tag{1.23}$$

leading to $\Lambda \leq 10^{-55} cm^{-2}$ and to a mass of $m_\Lambda = 10^{-32} eV$ which is so small that can't even be compared with particle physics predictions. The energy of the vacuum at Planck time is equal to today's value added with the value lost at phase transition. The problem is this small value is the most dominant 'today', giving another problem which is coincidence problem. The problem remains unsolved and is still a challenge.

1.4 Inflation

Inflation is the most elegant solution to the problems of Standard Model, such as flatness and horizon problems. It embodies a sufficiently long period of Universe's accelerated expansion very near to the creation or Big Bang. Other than solving the above problems it also acts as a solution to the initial quantum fluctuation

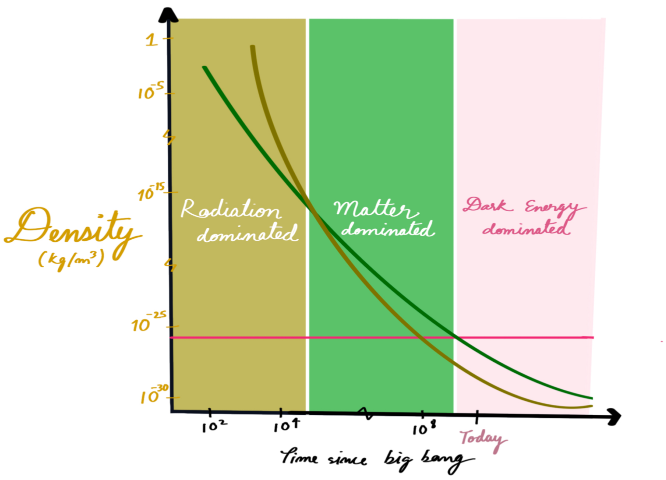

Figure 1.5: Graph showing Dark energy density domination era

which remains responsible for generation of initial seeds of all the physical structure of Universe along with the anisotropies found in CMB radiation, which otherwise would have to be added by hands with no motivation.Once perturbation theory start developing within General Relativity, we find that other than some perturbations integrated with universe's energy density, there is the production of tensorial perturbation due to the fluctuations in metric tensor and this is what composes GWs background.Now we will look at the Physics of classical approach to basic inflatonary model.

1.4.1 The governing principles of Inflation

As we know, the modern cosmology finds its root in cosmological principle described by FLRW metric which get its identification from evolution of a(t) which is the scale factor and K which is the curvature parameter. To understand the evolution of a(t) with Einstein's equation we would require to have cosmic medium defined to be the form of energy. With isotropy and homogeneity for a perfect fluid EMT is (eq. 1.6) Using Friedmann equations if we take into consideration, the negligible spacial curvature due to observation constrain we set K=0 [Ade et al. [2016]].In this case Friedmann Equation answers the type of fluid that can lead the dynamics.
The fundamental necessity for the application of inflatory dynamics consists that that $\ddot{a} > 0$ that means an expansion in Universe which remains accelerated. If we imply this condition in (eq. 1.11) we get that

$$p < -\frac{\rho}{3} \tag{1.24}$$

This makes it self evident that, there cannot be some ordinary matter of radiation that can drive the inflation. An easy way to get such dynamics is when we equate $p \approx -\rho$ which provides the evolution of scale factor as

$$a(t) = a_I e^{H_i(t-t_i)} \tag{1.25}$$

here,at the start of inflation, a_I is the initial scale factor and H_i is the Hubble constant(although it doesn't change being a constant) while t_i is the initial time

. This is the period which is defined for the evolution of scale factor as de Sitter stage. A Hubble radius or Horizon is defined as

$$R_H \equiv \frac{1}{H(t)} \propto ct \tag{1.26}$$

Hubble horizon sets a boundary of casually connected regions at times.In the model by de Sitter, he kept the physical Hubble radius constant in time while letting the physical lengths grow and ultimately they exit from the radius at a time defined as horizon crossing time. It allows fulfilling of requirement of such elongated period of inflation so all relevant scales for cosmological observations were able to overcome the Hubble Radius. The way to implement the condition of $p \equiv -\rho$ on source of energy momentum is by the introduction of Scalar field φ along with its coupling potential energy V(φ).Now minimally coupled scalar field can be introduced and written with Lagrangian as

$$\mathcal{L} = -\frac{1}{2}\partial^{\mu}\varphi\partial_{\mu}\varphi - V(\varphi) \tag{1.27}$$

If we vary the action of φ, we can obtain field's equation of motion know as Klein-Gordon equation and written as

$$\Box\varphi = \frac{\partial V}{\partial\varphi} \quad \text{where} \quad \Box = \partial^{\mu}\partial_{\mu} = \eta^{\mu\nu}\partial_{\mu}\partial_{\nu} \tag{1.28}$$

If we consider these equation, the FRLW background takes the form

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{1}{a^2}\nabla^2\varphi + V'(\varphi) = 0 \tag{1.29}$$

here $V'(\varphi)$ is derivative with respect to φ. If we recollect the original equation of EMT, we can easily find that the scalar field introduced behaves exactly like a perfect fluid. We can write the pressure and energy density background as

$$p_{\varphi} = \frac{\dot{\varphi}^2}{2} - V(\varphi) \quad \text{and} \quad \rho_{\phi} = \frac{\dot{\varphi}^2}{2} + V(\varphi) \tag{1.30}$$

Now we take a look at the quantity which shows the acceleration of universe from the first Friedmann equation (1.21), reads as

$$\rho_{\varphi} + 3p_{\phi} = 2[\dot{\varphi^2} - V(\varphi)] \tag{1.31}$$

In conclusion we can write that the condition to get accelerated expansion as

$$V(\varphi) > \dot{\varphi^2} \tag{1.32}$$

A quasi de Sitter expansion corresponds to a scalar field rolling slowly to its minimum potential.

$$V(\varphi) \gg \dot{\varphi^2} \tag{1.33}$$

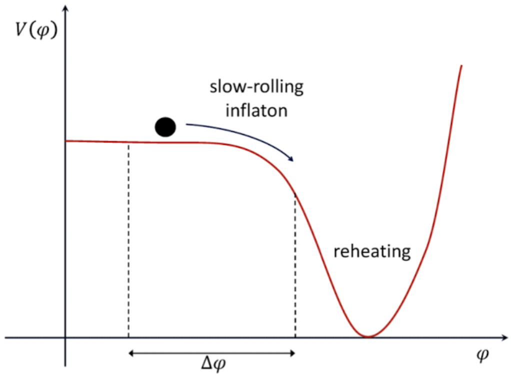

Figure 1.6: The diagram showing example of potential of inflation with flat region.After the slow roll,in reheating phase the scalar field oscillates in the minimum of potential and this happens between the time gap of horizon exit and inflation's end. The picture taken from [Guzzetti et al. [2016]]

1.4.2 Slow-roll conditions

The condition that we introduced in (eq. 1.27), requires the spatial region of field configuration having sufficiently flat potential In these conditions , at late times, friction drives the evolution of scalar field which can be written as $3H\dot{\varphi} \gg \ddot{\varphi}$. With the help of Friedmann equation, condition like these could be summarized with restricted form of potential of inflaton field and its derivative. Using (eq. 1.27) and other previous conditions, the equation becomes

$$H^2 \approx \frac{8\pi G}{3}V(\varphi) \quad \text{and} \quad 3H\dot{\varphi} + V_\varphi \approx 0 \tag{1.34}$$

Assuming inflaton to be homogeneous field, dominating Universe's energy density.Subscipt φ shows derivative w.r.t that field and $\dot{\varphi}$ is a function of V'(φ) and hence the slow roll condition gets satisfied provided

$$\epsilon \equiv \frac{M_{pl}^2}{2}(\frac{V_\varphi}{V})^2 \ll 1 \quad \textit{and} \quad \eta \equiv M_{pl}^2\frac{V_{\varphi\varphi}}{V} \ll 1 \tag{1.35}$$

where $M_{pl} = (8\pi G)^{-1/2}$ and is called reduced plank mass and $\epsilon and \eta$ are the two slow roll parameters and during inflation these can be thought of as time constant at first order. We can also write ϵ w.r.t Hubble parameter $\epsilon = -\dot{H}/H^2$ and similarly $\eta = \dot{\epsilon}/H\epsilon$. As soon as these condition fails, inflaton stops.

1.4.3 The duration of inflation

The duration of inflaton should have to be enough to solve problems like horizon and flatness which means everything we know now had been in casual contact in the past. We need to have a primordial period of such duration with accelerated expansion that a region even smaller than Hubble radius can grow to form the entire observable universe. We can therefore define this feature with number of e-folds

$$N_{tot} \equiv \int_{t_i}^{t_f} H dt \tag{1.36}$$

her $t_i$andt_f are initial and final time when inflation started and end respectively.We could also write the equation from the perspective of scale factor growth with the help of equation (1.24), which reads as $N = \ln \frac{a_f}{a_i}$ so the lower limit required to problem of horizon is

$$N \gtrsim ln10^{26} \sim 60 \tag{1.37}$$

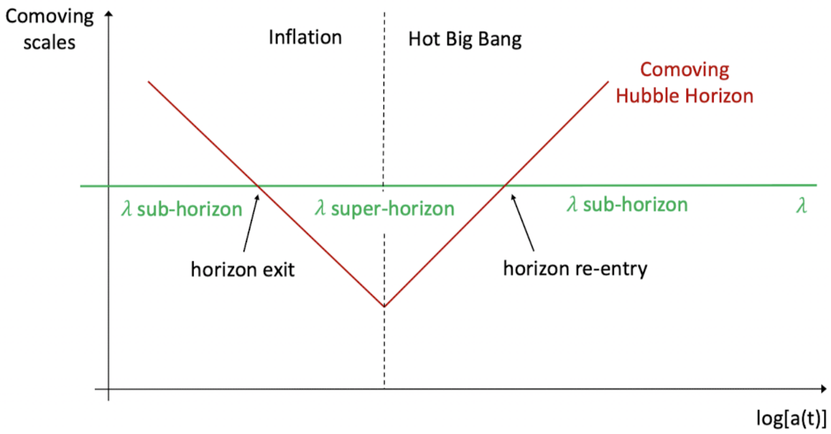

Figure 1.7: This graph shows the time evolution of comoving Hubble horizon,during the epoch of inflation and following ones, being compared with the comoving scale λ.The comoving horizon during the inflationary phase decreses with time while growing back in radiation and matter domination. At certain time in inflation, the comoving scale exits the comoving Hubble radius making a comeback once the inflation gets over. Graph taken from [Guzzetti et al. [2016]]

1.4.4 The phase of reheating

Inflationary phase has to stop to let the standard model take over later.If we consider single field slow roll inflation scenario, the inflaton field should roll out fast to descent to the minimum and then oscillates. It need to take place so to transfer the inflationary mechanism to the radiation domination of hot big bang model and this process of change is called reheating. With the perspective of our work on primordial gravitational waves (PGWs) this phase becomes important as there are many models for this phase which could also allow generation of GWs, even beside the phase of inflation.To move towards the post inflationary period of radiation domination at $T \approx 1MeV$, the amount of energy stored in scalar fields at the end of inflation gets converted to other forms to exhibit toward radiation domination while being in thermal equilibrium at the temperature termed as reheat temperature. There had been models for transition like perturbative decay and non perturbative as well.The condition of small fluctuation may lead to decay to relativistic things when the decay rate of inflaton (Γ) can be comparable to that of Hubble constant but when the process of decay is slow, that may lead to fermionic decay. Afterwards another process takes place converting energy of decay products to radiation.In case the decay is fast,its called preheating at the time it leads to bosonic decay through non perturbative mechanism of parametric resonance making the end of inflaton field with just few oscillation.

1.5 Cosmic Microwave Background

Cosmic Microwave Beckground (CMB) was discovered by [Penzias and Wilson [1979]] and has been a smoking gun for Big Bang Model. The universe of standard model begins from hot and dense initial state and further it expands as well as cools. Gamow and collaborators noted [Gamow [1948]] that the Big Bang model predicts a radiation of cosmic background, a hot early Universe's phase relic as in the early universe photons interacted with charged particles in ionized and dense plasma but as the cooling happens due to expansion, temperature dropped significantly and density reduced as well. At the time of recombination which happened about 380,000 years after Big Bang, the charged particles like protons and electrons combined to form the first neutral hydrogen element which marked the change from charged plasma to a neutral one which allowed photons to move freely without constant scattering which used to happen earlier.Now the universe was no longer opaque but free for movement of photons through space. The boundary of last scattering of photons usually refereed as the last scattering surface and the photons coming from that is CMB radiation that we observe today. These photons of visible and ultraviolet have been travellers of spacetime from that moment along the expansion of universe making the photons loose its energy as well as going through elongation marking itself redshifted. The temperature of CMB estimated by Penzias and Wilson was

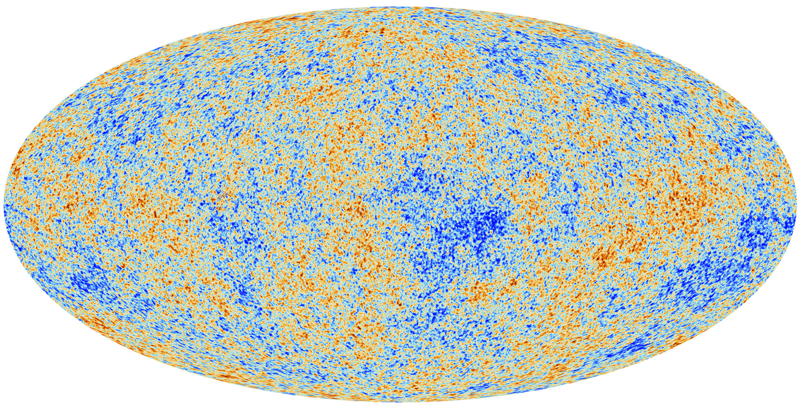

Figure 1.8: Temperature annisotropies of CMB observed by Plank, Picture credit:ESA and the Planck Collaboration

3K. The COBE satellite, launched in 1989, made detailed measurements of the CMB's temperature fluctuations, confirming its isotropy and homogeneity. COBE's findings provided strong support for the Big Bang theory.Further missions such as WMAP and Planck, refined the understanding of the CMB, mapping its temperature variations with unprecedented precision. These missions have helped establish the standard cosmological model, including the prevalence of dark matter and dark energy.

Today the CMB spectrum is well described as function of blackbody and the temperature ($T = 2.72763K \pm^{0.072}_{0.074} K$) [Haug and Tatum [2024]] .The lack of deviation seen in the spectrum of blackbody constrains the process of creation of these radiational phenomenon over cosmic history at the redshift of $z \lesssim 10^7$ [Scott and Smoot [2010]]. The variation in CMB is angular variation of temperature's correlation as well as a growing extent of polarization. This treatment of polarization of CMB brings more difficulty as the description of pattern of polarization in sky is no longer function defined on position of sky like temperature, infact it is a headless vector, which we can deal by assuming flat sky approximation allowing us to trace only small angular portion of sky. Another problem regarding these polarization is keeping an eye on the geometry of compton scattering and this is important to understand why the polarization is crucial for probes of tensor modes.

Polarization dispenses another dimentional study of CMB. The promise of CMB Polarization does not just confine scalar perturbation but GWs from the tensor perturbation carves a particular pattern that are very different from scalar ones. In general we can categorize them as E mode polarization and B mode polarization depending on mathematical property of polarization pattern. E modes as electric mode have no preferred handedness and can be thought of as having a gradient or curl-free nature while B mode showing reminiscent of the behavior of magnetic field lines have a curl free or divergence free character. So this can be used to study and search GWs coming from inflation.

Chapter 2

Gravitational waves

2.1 The origin story

The idea fabricating GWs is said to not have origin with Einstein. In fact Clifford gave his vision much earlier in 1870s, imagining *curvature waves* [Chen et al. [2017]], i.e. being curved or distorted and it gets continuously passed in the portions of space, like a manner of wave [Clifford [1976]]. The term gravitational waves was first coined by Poincaré [Poincaré [1906]],[Katzir [2005]] when he was speculating about relativistic (special) gravity, which would tend to have waves defined by acceleration and travelling at the speed of light.

In 1915 Einstein finally revealed a covariant general theory of relativity. In February 1916, after three months of carving field equations of GR, Einstein's letter to Schwarzchild stated about how he handled the Newtonian case of gravity to the new theory, but he found no waves of gravity analogous to that of waves of light which he said is related to one-sidedness of scalar T. This refers to the dipole's non existence, initializing a lack of belief in expectation of GWs as these waves are commonly associated with the initial interaction fields. Einstein might have extended his calculation from mercury perihelion to higher orders, reaching out his calculations to $(\frac{v}{c})^3$ while the damping of GWs can be seen only after 2.5 of Newtonian order, which means an addition of factor $(\frac{v}{c})^5$ as we know today.

The perspective on gravitational radiation changed in June of 1916 while Einstein was having difficulties when he was integrating his equation. de Sitter suggested to use a different coordinate system ($\sqrt{|g|} = 1$) which would be better suited. In the same year Einstein published his paper "Approximate integration of the field equation of gravity" [Einstein [1916]] where he developed the linearized weak field theory after using the de Sitter suggested gauge condition: in this paper he was able to predict the existence of GWs, which should be travelling at speed of light and would be time dependent on source of quadrupole moment. Although he made few mistakes and he predicted three types of waves

- longitudinal-longitudinal
- longitudinal-transverse
- transverse-transverse

and argued at that time based on incorrect equations that only transverse- transverse carried energy .
In January 1918, after correcting his mistakes in this previous paper, Einstein published "On gravitational waves". The calculation included changes in the quadrupole formula and fixed a huge error on conservation of linearized energy momentum.This work in 1918 was so prominent that laid the foundation of GWs found in textbooks today.Later in 1922 Eddington rechecked the issue with the vision of Clifford about waves of curvature. He confirmed the findings obtained by Einstein in 1918 while suggesting small corrections to the quadrupole amplitude.
In 1936 Einstein and Rosen submitted a paper titled "Do gravity waves exist?" which came as surprise as it says that gravitational plane waves cannot exist. Even after the paper went for detailed referee, Einstein withdrew it. In fact he never published paper in the 'Physical Review' this was because he didn't want any referee to read his submission before publication.Later he submitted his paper to the 'Journal of Franklin Institute' whose referee expressed skepticism about the surprising result. After discussion with Robertson Einstein figured out the problem and revised the conclusion later submitting the paper titled "On Gravitational waves"[Einstein and Rosen [1937]]. This paper claimed the existence of cylindrical waves which is also refereed as Einstein Rosen wave's existence rather than asserting the absence of plane waves. This happened because the singularity that Einstein and Rosen found in the given coordinate system was thought to be a physical singularity, making them argue about presence of such spacetime. Nevertheless, Einstein published paper without the consent of Rosen and he was not happy with that which can be shown with his argument about no transportation of energy by GWs in a conference in 1955 and later by his publication of paper titles "Does Gravitational Radiation exists?"[Rosen [1979]]
1974 marks itself as an important year as it paves the path of first indirect evidence for the existence of GWs, due to the discovery of PSR B1913+16. In fact this discovery by Taylor and Hulse obtained the Nobel Prize in 1993. They discovered that pulsar's orbital parameter was shortening with time which was confirming the quadrupole formula [Taylor et al. [1979]]. It provided the evidence that these GWs carry energy and the formula is valid just for binaries.
After 100 years of recognizing the idea of GWs in February 2016 LIGO collabration finally announced the direct detection of GWs from Laser Interferometer Gravitational Wave Observatory [Abbott et al. [2016]]. LIGO is made of two interferometers which are identical and placed in Louisiana and Washington. The interferometers have two arms, where there is a circulation of laser beams. When the gravitational wave arrive, they distort the space in between them making an interference pattern in the laser beams. The event was of astrophysical origin and was due to the merger of a black hole pair at a distance of 1.3 billion light years. The observed event GW150914 produced a signal with a characteristic waveform matching the theoretical predictions for a signal originated by black hole merger. This detection coincided with several observations in different electromagnetic bands: this helped constraint gravitational wave's propagation speed making implications on scenarios of modified gravity as well as provided a measurement of Hubble rate today giving an important contribution for cosmological studies.[Caprini and Figueroa [2018]].The GWs of cosmological background have not been discovered yet but they are very

interesting for studying the early moments of Universe. Theories agree that these waves would carry lots of information about the epoch of inflation but the detection of these waves remains for the moment a challenge.

2.1.1 Cosmological GWs

The cosmological origin of GWs leads to the primordial background which means it originates near the Universe's start or might have been produced due to quantum processes near the Planck time ($10^{-43}s$). The primordial gravitational waves (PGWs) would imply effect on Universe's dynamics even if their density(Ω_g) remains small, it could provide chaos. The background of GWs can be defined by the common feature of continuity and in some sense being stochastic. The GWs which originate at $z > 1$ are what we define as of cosmological origin or Cosmological GWs [Carr [1980]]. B.J. Carr in his paper says that primordial waves are even defined as acausal as they can be traced back to the Big Bang. They are considered to have wavelength bigger than that of Planck but as they cannot be regarded as waves at Planck time and they are said to be identified by shear fluctuation.

So the stochastic and isotropic nature of GWs which pervade the Universe orig-

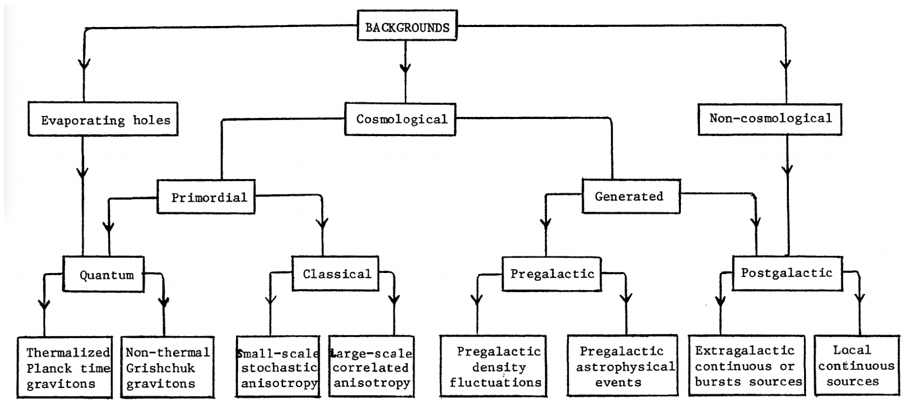

Figure 2.1: Classification of radiation of gravitational background showing different origins of cosmological background from 1979, before Inflation was propsed.Image credit : Carr [1980]

inating from inflationary epoch is the background primordial gravitational waves. This background being produced in the early Universe allows a statistical characterisation of the amplitude of tensor perturbation, which is said to be a random variable.[Caprini and Figueroa [2018]]. The stochastic nature comes from the ergodic hypothesis which is employed at the time when we transit from ensamble average to spacial and temporal averages. This nature of background makes it impossible to resolve individual realizations of signals of GWs.The stochastic GWs background can be summarized as GWs superposition having different wave number both in terms of magnitude and direction.The cosmological backgrounds of these waves have the typical characterization of being isotropic and Gaussian[dos Santos and van Manen [2023]]. Although the model of these stochastic GWs background is very similar to CMB, but it can receive from times, CMB cannot and this is because GWs were allowed to travel through early Universe's hot plasma which was a barrier for the photons of CMB. The signals from different phenomena in the early Universe act as cosmic sources and are said to be reaching us all the time from every direction. However its detection is very hard since the signal is very weak and the the Earth based observatories can measure just noise. Today these stochastic PGWs background would have the frequency range of $10^{-16} - 10^{10}$ Hz [Buonanno and Sathyaprakash [2015]], this covers the frequency band of ground and space detectors existing now or planned for the future. We will come back to observational properties and challenges later, now let's go back to sources of cosmological GWs.

Categorization of cosmological gravitational waves

The period of inflation does not just provide the solution of shortcomings of standard big bang but its major success lies in explaining primordial density fluctuation's physical origin, which is very important as initial conditions for the process of formation of structures in our Universe. Inflation accounts for stretching of quantum fluctuations being stretched to the perturbation of classical density [Polarski and Starobinsky [1996]]. When a perturbation re-enters the Hubble radius when the Universe returns to the deceleration phase after inflation, the formation of structures starts, being confirmed by data from CMB and observational findings from large scale structure.

The categorization by Caprini and Figueroa [2018] in five different forms of back-

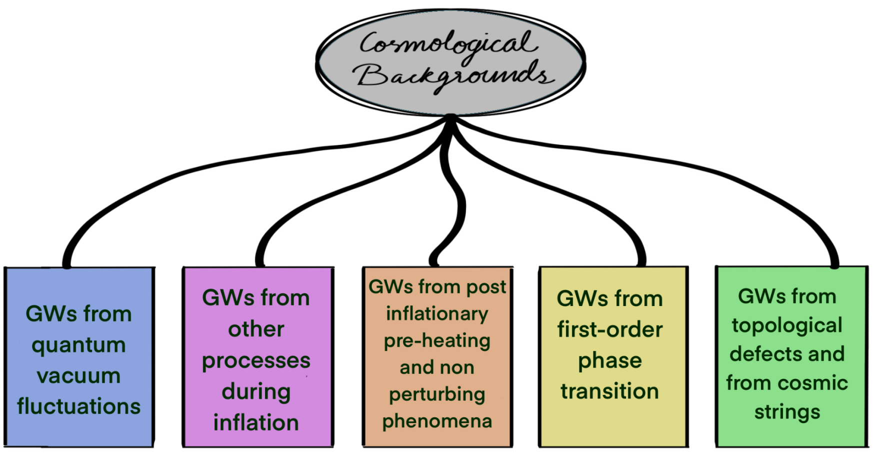

Figure 2.2: Classification of Cosmological background of GWs in the five different categories

ground gives a deep understanding of Cosmological background of GW(see 2.2)

- *GWs from quantum vacuum fluctuations at the time of standard slow roll inflation* - During the inflation perturbations happen in the inflaton field due to quantum vacuum fluctuations which are extended to super Hubble scales . This happens with tensor metric perturbations as they interact with mass less field Fabbri and Pollock [1983]. As the tensor modes come back inside the Hubble radius after inflation they change into classical and stochastic background of GWs having quasi scale invariant spectrum which in turn carves a pattern of polarization of B-mode in the anisotropies of CMB Kamionkowski et al. [1997].

- *GWs from other processes of inflation* - The extensions of standard inflationary model require to have multiple fields or additional sources of quantum fluctuations for GWs. These have different dynamics giving variations to spacetime curvature in return which provides a different spectrum of gravitational wave. In the future these waves can be detectable for detectors like LISA Cai et al. [2016]. This extended model gives an overall background of GWs.

- *GWs from inflaton to the hot classical preheating after inflation* - When reheating happens, it converts the energy of inflaton to particles [Abbott et al. [1982]], defining the change from a quantum cold phase to the standard radiative phase. The phase requires coupling of inflaton field to formation of other particles species.The mechanism of preheating during reheating, specifically non perturbative effects like parametric resonance [Kofman et al. [1997]] can produce GWs contributing to the overall spectrum .

- *GWs from first-order phase transition*-The first-order phase transition which could have happened in the early Universe involves the nucleation of bubbles of a true vacuum to the false vacuum which are filling the space. Near the end of transition, the bubbles collide and form a non zero tensor anisotropic stress which ultimately acts as GWs [Witten [1984]]. These collisions break the spherical symmetry leading to the production of GWs and Gravitational radiation comes in form as entire Universe transits to a symmetry broken phase.

- *GWs due to topological defects like the case of cosmic strings* - Cosmic defects as an aftermath being produced at the end of phase transition like cosmic strings, which have the scaling behaviour, allow them for the continuous creation of GWs. These defects that come out because of symmetry breaking in early Universe carve waves that remain over cosmic timescales. If we talk more specifically about cosmic strings, the detachment of loops from these produces the GWs. Gravitational radiations are produced by collisions and other dynamics of these loops, giving the overall GW background.

We have got the overview of cosmological origin of the gravitational waves, we now need to turn our attention towards the theoretical unpinnings that allow the gravitation waves to originate and to be traced and later we will drift towards the cosmological GWs from a theoretical perspective.

2.2 A theoretical overview of GWs

2.2.1 Linearized Gravity

Linearized theory can be understood as the theory of perturbations of Minkowski spacetime and it is applied when gravitational fields are weak and at the time when there are small perturbations in comparison of the background geometry.The linearized Einstein equation when derived taking the terms attached to just first-order perturbations gives a very simplified idea of gravitational interaction. To apply the linearization, the metric tensor is written as

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu} \tag{2.1}$$

here $h_{\mu\nu} \ll 1$ is the perturbation which is much smaller than η and $\eta_{\mu\nu}$ which represent a flat Minkowski metric. The inverse metric tensor can be written as

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{2.2}$$

From here we can straightforward compute the metric tensor's associated Christoffel symbols

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}) \tag{2.3}$$

which can be written with perturbations as

$$\Gamma^{\mu}_{\alpha\beta} \approx \frac{1}{2} \eta^{\mu\nu}(\partial_\alpha h_{\nu\beta} + \partial_\beta h_{\nu\alpha} - \partial_\nu h_{\alpha\beta}) \tag{2.4}$$

Now we can calculate the Riemann Tensor

$$R^{\mu}_{\alpha\beta\gamma} = \partial_\beta \Gamma^{\mu}_{\alpha\gamma} - \partial_\gamma \Gamma^{\mu}_{\alpha\beta} + \Gamma^{\delta}_{\alpha\gamma}\Gamma^{\mu}_{\delta\beta} - \Gamma^{\delta}_{\alpha\beta}\Gamma^{\mu}_{\delta\gamma} \tag{2.5}$$

which can be computed and written as

$$R^{\mu}_{\alpha\beta\gamma} \approx \frac{1}{2}\eta^{\mu\nu}[(\partial_\alpha\partial_\beta h_{\nu\gamma} + \partial_\beta\partial_\gamma h_{\nu\alpha} - \partial_\beta\partial_\nu h_{\alpha\gamma}) - (\partial_\alpha\partial_\gamma h_{\nu\beta} + \partial_\beta\partial_\gamma h_{\nu\alpha} - \partial_\nu\partial_\gamma h_{\alpha\beta})] \tag{2.6}$$

$$R^{\mu}_{\alpha\beta\gamma} = \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha\partial_\beta h_{\nu\gamma} - \partial_\beta\partial_\nu h_{\alpha\gamma} - \partial_\alpha\partial_\gamma h_{\nu\beta} + \partial_\nu\partial_\gamma h_{\alpha\beta}) \tag{2.7}$$

Similarly we can get the linearized equation for the Ricci tensor from the Riemann tensor as both get same when Riemann tensor has upper and middle index summed together

$$R_{\alpha\beta} = R^{\mu}_{\alpha\beta\mu} \approx \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha\partial_\beta h_{\nu\mu} - \partial_\beta\partial_\nu h_{\alpha\mu} - \partial_\alpha\partial_\mu h_{\nu\beta} + \partial_\nu\partial_\mu h_{\alpha\beta}) \tag{2.8}$$

$$R_{\alpha\beta} = \frac{1}{2}(\partial_\alpha\partial_\beta h^{\mu}_{\mu} - \partial_\beta\partial^\mu h_{\alpha\mu} - \partial_\alpha\partial_\mu h_{\mu\beta} + \partial^\mu\partial_\mu h_{\alpha\beta}) \tag{2.9}$$

$$R_{\alpha\beta} = \frac{1}{2}(\Box^2 h_{\alpha\beta} + \partial_\alpha\partial_\beta h - \partial_\beta\partial^\mu h_{\alpha\mu} - \partial_\alpha\partial^\mu h_{\mu\beta}) \tag{2.10}$$

here we have used h scalar as

$$h = h^{\mu}_{\mu} = \eta^{\mu\nu}h_{\mu\nu} \tag{2.11}$$

$$\Box^2 \equiv \partial^\mu\partial_\mu = \eta^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{c^2}\frac{\partial^2}{\partial t} - \nabla^2 \tag{2.12}$$

The above equation gives the d'Alambertian operator. We can move from here to formulate the Ricci scalar as

$$R = \eta^{\alpha\beta}R_{\alpha\beta} = \Box^2 h - \partial_m u \partial \nu h^{\mu\nu} \tag{2.13}$$

Putting everything back in the Einstein field equation, we get

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R = \frac{1}{2}(\Box^2 h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\mu\partial^\alpha h_{\alpha\nu} - \partial_\nu\partial^\nu h_{\mu\alpha} - \eta_{\mu\nu}\Box^2 h + \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta}) \tag{2.14}$$

So now the linearized gravity imposed Einstein Equation becomes

$$G_{\mu\nu} = \frac{1}{2}(\Box^2 h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\mu\partial^\alpha h_{\alpha\nu} - \partial_\nu\partial^\nu h_{\mu\alpha} - \eta_{\mu\nu}\Box^2 h + \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta}) = -\frac{8\pi G}{c^4}T_{\mu\nu} \tag{2.15}$$

The above equation is redefined by using perturbation variables to make it simpler:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{2.16}$$

These variable are trace -reversed so we can write them such that

$$\bar{h} = \bar{h}^\mu_\mu = h - \frac{1}{2}\eta^{\mu\nu}\eta_{\mu\nu} = h - \frac{1}{2}\delta^\mu_\mu h = -h \tag{2.17}$$

and so the eq. (2.17) can be altered as

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \tag{2.18}$$

Now using eq. (2.17) we can write eq. (2.15) as

$$\Box^2\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box^2\bar{h} - \partial_\mu\partial_\nu\bar{h} - \partial_\mu\partial^\alpha\bar{h}_\alpha\nu + \frac{1}{2}\partial_\mu\partial_\nu\bar{h} - \partial_\nu\partial^\alpha\bar{h}_\mu\alpha + \frac{1}{2}\partial_\nu\partial_\mu\bar{h} + \eta_{\mu\nu}\Box^2\bar{h} + \eta_{\mu\nu}\partial_\alpha\partial_\beta\bar{h}^{\alpha\beta} - \frac{1}{2}\eta_{\mu\nu}\Box^2\bar{h} = -\frac{16\pi G}{c^4}T_{\mu\nu}$$

which can be further simplified as

$$\Box^2\bar{h}_{\mu\nu} - \partial_\mu\partial^\alpha\bar{h}_{\alpha\nu} - \partial_\nu\partial^\alpha\bar{h}_{\mu\alpha} + \eta_{\mu\nu}\partial_\alpha\partial_\beta\bar{h}^{\alpha\beta} = -\frac{16\pi G}{c^4}T_{\mu\nu} \tag{2.19}$$

This is the final Einstein equation written for linearized gravity having trace-reversed variables of perturbations.

2.2.2 Gauge Transformation

We know that there are ambiguities related to the freedom of choice of a system of coordinates and so we can assume that the metric tensor can fluctuate due to the perturbations created in space-time or it can be due to the coordinate system, in fact it could happen because of both. The effect of coordinate system can be understood by allowing a gauge transformation by taking into account considerations of two different systems of coordinates with a small difference of ξ^μ and so we can write $x'^\mu = x^\mu + \xi^\mu \quad and \quad x^\mu = x'^\mu - \xi^\mu$ so now the transformation matrices of coordinates are

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu\xi^\mu \tag{2.20}$$

$$\frac{\partial x^\mu}{\partial x'^\nu} = \delta^\nu_\mu - \partial'_\nu\xi^\mu = \delta^\mu_\nu - \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial\xi^\mu}{\partial x^\alpha} = \delta^\mu_\nu - (\delta^\alpha_\nu - \partial' - \nu\xi^\alpha)\frac{\partial\xi^\mu}{\partial x^\alpha} \approx \delta^\mu_\nu - \partial_\nu\xi^\mu \tag{2.21}$$

If we look at metric tensor, its transformation is like

$$\begin{aligned} g'_{\mu\nu} &= \frac{\partial x^{\alpha}}{\partial x'^{\nu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta} = (\delta^{\alpha}_{\mu} - \partial_\mu \xi^{\alpha})(\delta^{\beta}_{\mu} - \partial_\nu \xi^{\beta}) g_{\alpha\beta} \\ &\approx g_{\mu\nu} - \partial_\nu \xi^{\beta} g_{\mu}\beta - \partial_\mu \xi^{\alpha} g_{\alpha}\nu \\ &= g\mu\nu - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu \end{aligned} \tag{2.22}$$

we find that, for this small transformation, we get $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$ so now if we show in terms of variable of perturbations, eq. (2.22) transforms as

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu \tag{2.23}$$

We can calculate the transformation of the trace- reversed variables of perturbations by first calculating the transformed trace variables

$$h' = \eta^{\mu\nu} h'_{\mu\nu} = \eta^{\mu\nu} h_{\mu\nu} - \eta^{\mu\nu}\partial_\nu \xi_\mu - \eta^{\mu\nu}\partial_\mu \xi_\nu = h - 2\partial_\mu \xi^{\mu} \tag{2.24}$$

and then we can solve for

$$\begin{aligned} \bar{h}' &= h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h' = h_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu - \frac{1}{2}\eta_{\mu\nu}(h - 2\partial_\alpha \xi^{\alpha}) \\ &= \bar{h}_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu + \eta_{\mu\nu}\partial_\alpha \xi^{\alpha} \end{aligned} \tag{2.25}$$

We finally find that these coordinates do not change the Riemann tensor which can be calculated and written as

$$\begin{aligned} R'^{\mu}_{\alpha\beta\gamma} &= \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha \partial_\beta h'_{\nu\gamma} - \partial_\beta \partial_\nu h'_{\alpha\gamma} - \partial_\alpha \partial_\gamma h'_{\nu\beta} + \partial_\nu \partial_\gamma h'_{\alpha\beta}) \\ = \frac{1}{2}\eta^{\mu\nu}[&\partial_\alpha \partial_\beta (h_{\nu\gamma} - \partial_\nu \xi_\gamma - \partial_\gamma \xi_\nu) - \partial_\beta \partial_\nu (h_{\alpha\gamma} - \partial_\alpha \xi_\gamma - \partial_\gamma \xi_\alpha) - \\ &\partial_\alpha \partial_\gamma (h_{\nu\beta} - \partial_\nu \xi_\beta - \partial_\beta \xi_\nu) + \partial_\nu \partial_\gamma (h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha)] \end{aligned} \tag{2.26}$$

$$R'^{\mu}_{\alpha\beta\gamma} = \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha \partial_\beta h_{\mu\gamma} - \partial_\beta \partial_\nu h_{\alpha\gamma} - \partial_\alpha \partial_\gamma h_{\nu\beta} + \partial_\nu \partial_\gamma h_{\alpha\beta}) = R^{\mu}_{\alpha\beta\gamma} \tag{2.27}$$

Now if we go back to eq. (2.19), we find that the coordinate system can be fixed in beneficial way if we use the condition

$$\partial_\nu \bar{h}^{\mu\nu} = 0 \tag{2.28}$$

This condition is what we refer as Lorenz gauge. Once we apply this condition, we can directly crumble eq. (2.19) to

$$\Box^2 \bar{h}_{\mu\nu} = -\frac{16\pi G}{C^4} T_{\mu\nu} \tag{2.29}$$

Checking which coordinate system satisfies this condition, would require a transformation of coordinates like

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_\nu \xi_\nu + \eta_{\mu\nu}\partial_\alpha \xi^{\alpha} \tag{2.30}$$

which satisfies the condition of eq. (2.28).So using the above equation we can write

$$\begin{aligned} \partial^\nu \bar{h}_{\mu\nu} = 0 &= \partial^\nu \bar{h}_{\mu\nu} - \partial^\nu \partial_\nu \xi_\mu - \partial^\nu \partial_\mu \xi_\nu + \eta_{\mu\nu}\partial^\nu \partial_\alpha \xi^{\alpha} \\ &= \partial^\nu \bar{h}_{\mu\nu} - \Box^2 \xi_\mu - \partial_\mu \partial^\nu \xi_\nu + \partial_\mu \partial^\nu \xi_\nu \\ &= \partial^\nu \bar{h}_{\mu\nu} - \Box^2 \xi_\mu \end{aligned} \tag{2.31}$$

We finally get to the result which is a wave equation

$$\partial^\nu \bar{h}_{\mu\nu} = \Box^2 \xi_\mu \tag{2.32}$$

The above wave equation is composed of a source term on the r.h.s. Under some general assumptions we the solution could be found by considering a source term equated to the general solution of the particular homogeneous equation associated with it and allowing us to always find a transformation suitable for holding the Lorenz gauge condition.

2.2.3 Transverse Traceless Gauge

The transverse and traceless is a coordinate system specifically chosen to study the GWs. This system describes transverse oscillations by the metric perturbations of the spacial component and traceless condition allows tidal effects to be applied by GWs without affecting the volume of objects. This coordinate system gives the simplistic solution of polarization of GWs allowing the analysis of cosmic ripples.
In vacuum, eq. (2.29) reduces to $\Box^2 \bar{h}_{\mu\nu} = 0$. We know eq. (2.32) allows gauge transformation to any coordinate system which is bound by the Lorenz condition and there are many such systems of ξ_μ that satisfy the equation. This allows us to further constraint the coordinate system. The wave equation is written as

$$\frac{\partial^2 \bar{h}_{\mu\nu}}{\partial t^2} - c^2 \nabla^2 \bar{h}_{\mu\nu} = 0 \tag{2.33}$$

and its solution is

$$\bar{h}^{\mu\nu} = Re[A^{\mu\nu} e^{ik_\alpha x^\alpha}] \tag{2.34}$$

as true GWs corresponds always to the real part. The solution has to satisfy the constraints of symmetry $A^{\mu\nu} = A^{\nu\mu}$, the constraint from wave equation ($k^\alpha k_\alpha = 0$), which implies the dispersion relation telling that these waves move at speed of light c and it must also satisfy the Lorenz condition $k_\nu A^{\mu\nu} = 0$. If we consider the propagation of wave in z or x^3 direction , its 4-vector can be written as

$$k^\mu = (k, 0, 0, k) \qquad k_\mu = (k, 0, 0, -k) \tag{2.35}$$

as $k^0 = k^3$, it says that this is a null vector having k(wave vector)=ω/c, with this choice of k^μ, we find that the Lorenz condition gives

$$kA^{\mu 0} - kA^{\mu 3} = 0 \tag{2.36}$$

Because of this symmetry and Lorenz condition,the A matrix and reduces to just 6 independent quantities. To further constraint we use another class of Lorenz gauge transformation which is going to be wave dependent.For this we define,

$$\xi^\mu = -Re[i\epsilon^\mu e^(ik_\alpha x^\alpha)] \tag{2.37}$$

The equation satisfies the condition of Lorenz gauge and from eq. (2.30) we find

$$A'^{\mu\nu} = A\mu\nu - k^\nu \epsilon^\mu - \epsilon^\nu k^\mu + \eta^{\mu\nu} k_\alpha \eta^\alpha \tag{2.38}$$

From the above equation we can find the value of all 6 independent components ($A'^{00} = A^{00} - k(\epsilon^0 + \epsilon^3), A'^{01} = A^{01} - K\epsilon^1, A'^{02} = A^{02} - k\epsilon^2, A'^{11} = A^{11} - k(\epsilon^0 - \epsilon^3), A'^{12} = A^{12}, A'^{22} = A^{22} - k(\epsilon^0 - \epsilon^3)$). With these values we add four constraints and so we can find that just A^{12} remains unchanged after gauge transformation, which is having symmetry($A^{12} = A^{21}$). Further we select one more value which is independent by setting the value of ϵ and we finally find the transformed A matrix as

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A^{11} & A^{12} & 0 \\ 0 & A^{12} & -A^{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A^{11} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + A^{12} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.39}$$

which defines the two different types of linear polarization

$$\epsilon_+^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \epsilon_\times^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.40}$$

which can be shown generically as

$$A'^{\mu\nu} = A_+ \epsilon_+^{\mu\nu} + A_\times \epsilon_\times^{\mu\nu} \tag{2.41}$$

Both matrices are drawn to be traceless and three components vanishes for transverse propagation and this choice of gauge is called Transverse-Traceless gauge which leaves us with just two independent components. There is no difference between variable of perturbation and the trace-reverse one as $\bar{h} = 0 \;\; and \;\; h = 0$.With the theory of linear gravity and Transverse-Traceless gauge, we can form the Christoffel connection and find that

$$\Gamma^\mu_{00} = 0 \quad \Gamma^\mu_{0\nu} = \frac{1}{2}\partial_0 h^\mu_\nu \tag{2.42}$$

With this result and taking a rest particle with initial 4-velocity of (c,0,0,0), we can find the geodesic equation as

$$\frac{d^2x^\alpha}{d\eta^2} + \Gamma^\alpha_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0 = \Gamma^\alpha_{00}c^2 = 0 \tag{2.43}$$

This means that the velocity remains constant with the TT gauge, as well as the small spacial vectors remain constant but not the spacial separation d, which is equated as

$$d^2 = -g_{\mu\nu}\xi^\mu\xi^\nu \approx (\xi_i - \frac{1}{2}h_{ik}\xi^k)(\xi^i - \frac{1}{2}h^i_k\xi^k) \tag{2.44}$$

The new variable being defines as $(\xi^i - \frac{1}{2}h^i_k\xi^k)$ gives the correct separation of space.We can also say that in TT gauge, there is no change in propagation direction showing that the wave is transverse. When we take a particle and mark it as a position marker, we find that a ring of N equally spaced masses, changes the separation w.r.t radius R (see 2.3) and is given by

$$r_n^2 = R^2\left(1 - \frac{A_+}{2}\cos\omega t\right)^2 \cos^2\theta_n + R^2\left(1 + \frac{A_+}{2}\cos\omega t\right)^2 sin^2\theta_n \tag{2.45}$$

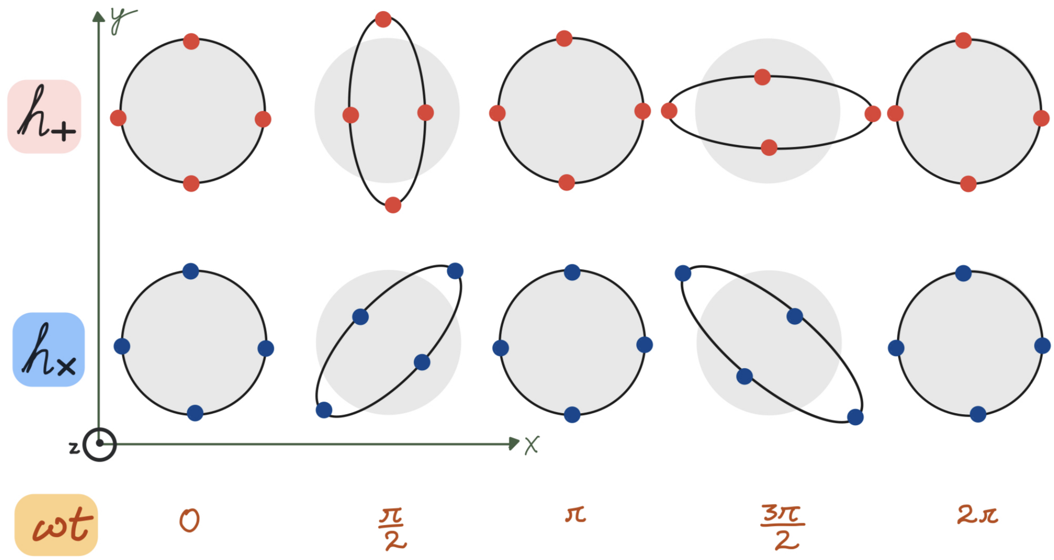

Figure 2.3: The diagram shows the effect of a passing GW propagating in z-direction, deforming the relative distance between test masses. In the upper row h_+ shows the case of plus polarisation while in lower row the $h_\times$ shows the case of cross polarization.The deformation maintains the continuity of wobbling between different states of ωt as the wave passes

where $\theta_n = 2\pi n/N$ is defined as angular position. The equation reduces to

$$r_n \approx R\left(1 - \frac{A_+}{2} cos 2\theta_n \cos(\omega t)\right) \tag{2.46}$$

We find that the variable of perturbation defines a relative deformation between distances of test masses. Till now we saw how in general linearized gravity with gauge transformation and TT gauge theoretically allows the formation of GWs with two polarizations. In the next section we are moving to the cosmological context of perturbations and decomposition of the metric tensor.

2.3 Scalar-Vector-Tensor decomposition

We now turn to Cosmology. In theory, Cosmological perturbations [Mukhanov and Feldman [1992]] can also be understood by decomposing the metric tensor into different components, as the Einstein equation is defined as equality of tensors and thus provides a set of equations which has coupling but we are allowed to decouple to find the different modes. To do so we start by writing the metric tensor of first order as

$$\delta g_{00} = -2a^2\psi \tag{2.47}$$

: the equation shows time-time perturbation which is defined for scalar while

$$\delta g_{0i} = \delta g_{0i} = a^2 B \tag{2.48}$$

is defined scalar as well as vector

$$\delta g_{ij} = 2a^2 C_{ij} \tag{2.49}$$

in space-space perturbation is defined on scalar, vector as well as on tensor perturbation. We can find that 3 vectors can be decomposed in a curl free (not having source or sink due to scalars) and a divergent free part while 3 scalars are decomposed on their trace(scalar) and derivative of 3 vector and also a trace and divergent free tensor which can neither be created by scalar nor by vectors.These tensors are very essential for producing the GWs during the inflation. We can write this decomposition as

$$B_i = B, i - S_i \tag{2.50}$$

$$C_{ij} = -\phi\gamma_{ij} + E_, ij + F_i, j + \frac{1}{2}h_{ij} \tag{2.51}$$

where ',' represents a derivative. So taking the above notation we can say ψ, B, ϕ, E represent scalars F_i, S_i represent vectors which are divergence free and finally h_{ij} represent tensor perturbations [Paci [2009]] which are trace and divergence free. We can write the line elements as

$$\begin{aligned} ds^2 = a^2(\eta)[-1 - 2\psi]d\eta^2 + 2(B_{,i} - S_i)d\eta dx^i \\ +((1-2\phi)\gamma_{ij} + 2E_{,ij]} + 2F_{i,j} + h_{ij}dx^i dx^j \end{aligned} \tag{2.52}$$

As the choice of coordinate system is important , we can look again for the transformation,by taking

$$\tilde{x}^\mu = x^\mu + \xi^\mu \tag{2.53}$$

the ξ part needs to be decomposed and when we apply this to the coordinate transformation $(ds^2 + \tilde{ds}^2)$ so we get set of equations for the perturbation variables. For scalars they are given as

$$\begin{aligned} \tilde{\psi} &= \psi - \mathcal{H}\xi^0 - \xi^{0\prime} \\ \tilde{\phi} &= \phi + \mathcal{H}\xi^0 \\ \tilde{B} &= B + \xi^0 - \xi' \\ \tilde{E} &= E - \xi \end{aligned} \tag{2.54}$$

for vectors they can be written as

$$\begin{aligned} \tilde{F}_i &= F_i - \tilde{\xi} \\ \tilde{S}_i &= S_i + \tilde{\xi}' \end{aligned} \tag{2.55}$$

and finally for tensor it can be written as

$$\tilde{h}_{ij} = h_{ij} \tag{2.56}$$

We can clearly see that the tensor perturbations are invariant of gauge transformation whereas the other scalars and vectors are not fully invariant but just few of their combinations are like for scalars

$$\begin{aligned} \Phi &= \phi - \mathcal{H}(B - E') \\ \Psi &= \psi + \mathcal{H}(B - E') + (B - E')' \end{aligned} \tag{2.57}$$

whereas for vectors, it is

$$V_i = S_i + F_i' \tag{2.58}$$

We just saw the perturbations of metric tensor and so we can now start with how anisotropies affect the energy-momentum tensor

$$T^{\mu}_{\nu} = p\delta^{\mu}_{\nu} + (\rho + p)u^{\mu}u_{\nu} + \pi^{\mu}_{\nu} \tag{2.59}$$

here π^{μ}_{ν} represents the anisotropic stress which is there to allow the matter free streaming which is not described by a perfect fluid. For a background we can write $T^0_0 = -\rho_0, T^0_i = 0$ and $T^i_j = \delta^i_j p_0$. These equations when exposed to the first-order perturbations give

$$\begin{aligned} T^0_0 &= -\delta\rho \\ T^i_0 &= -(\rho_0 + p_0)v^i \\ T^0_i &= (\rho_0 + p_0)(v_i + B_i) \\ T^i_j &= \delta p \delta^i_j + \pi^i_j \end{aligned} \tag{2.60}$$

In the above equations, v^i gives the spatial part coming from u^i and π^i_j which can further be splitted in π which is the scalar part,(π_i) which is the vector part and ($\prod^i_j$) is the tensor contribution. These all can be summarized in as

$$\pi^i_j = \pi'^i_j - \frac{1}{3}\nabla^2\pi\delta^i_j + \frac{1}{2}(\pi^i_{,j} - \pi^{;i}_j) + \prod{}^i_j \tag{2.61}$$

2.4 Quantum fluctuations and primordial power spectrum

With the theories we can move back to the question of how Quantum fluctuations generate the PGWs. The inflaton field's background written as $\bar{\phi}(t)$ is dependent on time and it works like a clock in that regime.Due to the quantum effects the uncertainty principle allows inflaton to fluctuate locally near the value of its background

$$\phi(t, x) = \bar{\phi}(t) + \delta\phi(t, x) \tag{2.62}$$

This denotes different rates of change during inflation at different locations of space-time which produce inhomogeneities in the Universe which leads to the formation of structure. We can start the theoretical approach from quantization of the action of inflaton [Wang [2017]] in an unperturbed metric of FLRW

$$\begin{aligned} S &= \int d\eta d^3x\sqrt{-g}[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)] \\ &= \int d\eta d^3x \frac{1}{2}a^2[\phi'^2 - (\nabla\phi)^2 - 2a^2V(\phi)] \end{aligned} \tag{2.63}$$

here ϕ' is the derivative taken on ϕ w.r.t. η which is the conformal time while $\dot{\phi}$ is w.r.t. cosmic time t. We introduce the field re-definition which is

$$f(\eta, x) = a(\eta)\delta\phi(\eta, x) \tag{2.64}$$

We also neglect the fluctuations in metric of inflationary background.With the different approximations done in slow roll like $H \approx 0 \quad \rho \approx V$ and considering $\eta \ll 1$ from eq. (1.35), the Friedmann equation changes from

$$H^2 = \frac{1}{3M^2_{Pl}}\rho \tag{2.65}$$

to

$$\frac{a''}{a} \approx 2a^2H^2 \approx \frac{1}{3\eta v}a^2V_{,\phi\phi} \tag{2.66}$$

and if we expand the action in eq. (2.63) to 2nd order, with these conditions it crumbles to

$${}^{(2)}S \approx \int d\eta d^3x \frac{1}{2}[f'^2 - (\nabla f)^2 + \frac{a''}{a}f^2] \tag{2.67}$$

By considering the Lagrangian associated with the action (Euler-Lagrangian equation)

$$\mathcal{L} = \frac{1}{2}[f'^2 - (\nabla f)^2 + \frac{a''}{a}f^2] \tag{2.68}$$

we formulate the Mukhanov-Sasaki equation

$$f'' - \nabla^2 f - \frac{a''}{a}f^2 = 0 \tag{2.69}$$

If we consider cannonical quantisation which is a quantum procedure to promote the classical field as well as conjugate field momenta to quantum operator, where we take $f(\eta, x)$ and its momentum conjugate $\pi(\eta, x) \equiv \partial\mathcal{L}/\partial f' = f'$, these operators are now allowed to obey equal time and follow the cannonical commutation relation (CCR), which is written as

$$[\hat{f}(\eta, x), \hat{\pi}(\eta, x')] - i\delta(x - x') \tag{2.70}$$

Using the Fourier space we can expand $\hat{f}(\eta, x)$ and $\hat{\pi}(\eta, x)$ as

$$\begin{aligned}\hat{f}(\eta, x) &= \int \frac{d^3k}{(2\pi)^(3/2)}(f_k^* a_k^\dagger e^{-ik\dot{x}} + f_k a_k e^{ik\dot{x}}) \\ \hat{\pi}(\eta, x) &= \int \frac{d^3k}{(2\pi)^(3/2)}(f_k'^* a_k^\dagger e^{-ik\dot{x}} + f_k^* a_k e^{ik\dot{x}})\end{aligned} \tag{2.71}$$

Here a_k *and* $a_k^\dagger$ are creation operators which undergo annihilation which are time independent and they satisfy the condition $[a_k, a_{k'}^\dagger] = \delta(k - k')$ while f_k obeys the Mukhanov-Sasaki equation which can be written in Fourier space as

$$f_k'' + \omega_k^2(\eta)f_k = 0 \tag{2.72}$$

when $\omega_k^2 = k^2 - \frac{a''}{a}$ and $k \equiv |k|$.To study the existence of a solution we can define the Wronskian using the Canonical commutation relation and by definition of creation operator, we can write

$$\mathcal{W}(f_k^*, f_k) \equiv f_k^*, f_k' - f_k, f_k'^* = -i \tag{2.73}$$

As the expansion at inflation follows the quasi-de Sitter model the expansion factor $a \approx e^{Ht}$and Hubble parameter will be constant. So we can write

$$\eta(t) = -\int_t^\infty \frac{dt'}{a(t')} \approx -\int_t^\infty dt' e^{-Ht'} = \frac{1}{aH} \tag{2.74}$$

Just to remind again, η is conformal time and t is cosmic time. Using the above equation for conformal time, eq. (2.72) becomes

$$f_k'' + (k^2 - \frac{2}{\eta^2})f_k = 0 \tag{2.75}$$

and its exact solution is given by

$$f_k(\eta) = A\frac{e^{-ikr}}{\sqrt{2k}}(1 - \frac{i}{k\eta}) + B\frac{e^{ikr}}{\sqrt{2k}}(1 + \frac{i}{kr}) \tag{2.76}$$

To satisfy wronskian and for the application of ground state of Hamiltonian to be vacuum we choose just the first portion giving positive frequency of the solution. Now we can determine the power spectrum for a physical observable q written as

$$\langle q(k)q^*(k')\rangle \equiv \frac{2\pi^2}{k^3}\mathcal{P}_q(k)\delta(k-k') \tag{2.77}$$

Here $\mathcal{P}$ denotes the power spectrum.For an inflation field $q = \delta\phi = f/a$ and using eq. (2.71) we can find zero-point fluctuations which come as

$$\langle 0|\hat{f}(\eta,0)\hat{f}^\dagger(\eta,0)|0\langle = \int d\ln k \frac{k^3}{2\pi^2}|f_k|^2 \tag{2.78}$$

and the function reads off $P_f = (k^3/2\pi)|f_k(\eta)|^2$ and with first solution of eq. (2.76) we can solve to find

$$\mathcal{P}_\delta\phi(k) = a^{-2}\mathcal{P}_f(k) \to \left(\frac{H}{2\pi}\right)^2 \tag{2.79}$$

and we know at at super-Horizon scale $k \ll aH$.So now we can conclude that as H is changing very slowly, the power spectrum of inflation can be approximated at the horizon crossing ($k = aH$) where $H_k \equiv \frac{k}{a}$, given as

$$\mathcal{P}_\delta\phi(k) \approx \frac{H_k}{2\pi}^2 \tag{2.80}$$

With this we can go back to scalar-vector-tensor decomposition to understand how the power spectrum gets affected.

Scalar Perturbations - The freedom of gauge pushed us to apply scalar perturbations in curvature, so spacial metric uses comoving gauge when $\delta\phi = 0$ and it is written as

$$g_{ij} = a(t)^2 e^{2\tilde{\zeta}}\delta_{ij} \tag{2.81}$$

here $\tilde{\zeta}$ is the comoving curvature perturbation which is also gauge invariant. If we have spatially flat gauge it becomes

$$\zeta = -H\frac{\delta\phi}{\dot{hi}} \tag{2.82}$$

So now if we compare this equation with eq.(2.80), we can get the power spectrum of scalar perturbations written as

$$P_\zeta = \frac{1}{2M_{Pl}^2\epsilon}\left(\frac{H_k}{2\pi}\right)^2 \tag{2.83}$$

The measure of the dependence on scale of this power spectrum is related to scalar spectral index written as

$$n_s = 1 + \frac{d \ln \mathcal{P}_\zeta}{d \ln k} \tag{2.84}$$

We can approximate the power spectrum with a power law and scale of reference of k_*, finally taking the form

$$\mathcal{P}_\zeta(k) = A_s(k_*) \left(\frac{k}{k_*}\right)^{n_s - 1} \tag{2.85}$$

Vector Perturbations- After inflation all the vector perturbations become negligible as they are related to vorticity, which dilutes because of the increase of the scale factor and conservation of angular momentum Hu and White [1997].

Mathematically, PGWs are pure tensor perturbation of spacetime metric. As we have already seen the perturbation in FLRW metric is written as

$$ds^2 = a(\eta)^2[-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j] \tag{2.86}$$

and h_{ij} which is the perturbation in itself is symmetric, traceless and transverse, as we can absorb the scalar and vector perturbations which comes from the true tensor perturbation at linear order finally getting us to just two degrees of freedom which here we can call helicity $p \pm 2$. We can expand h_{ij} in the Fourier modes, which can be written as

$$h_{ij} = \sigma_{p=\pm 2} \int \frac{d^3 k}{(2\pi)^{3/2}} h_{ij}^{(p)}(\eta, k) e^{ik \cdot x} \tag{2.87}$$

we set the basis vector for k along z direction

$$m^{(\pm 2)}(\hat{z}) = \frac{1}{2}(\hat{x} \pm i\hat{y}) \otimes (\hat{x} \pm i\hat{y}) \tag{2.88}$$

which stands on the condition of orthogonality and reality condition [Challinor et al. [2009]]

$$m_{ij}^{p}(\hat{k})[m^{(q)ij}(\hat{k})]^* = \delta^{pq} \tag{2.89}$$

$$[m_{ij}^{p}(\hat{k})]^* = m_{ij}^{(-p)}(\hat{k}) = m_{(ij)}^{(p)}(-\hat{k}) \tag{2.90}$$

here we also have to follow the condition

$$h_{ij}^{(\pm 2)}(\eta, k) = \frac{1}{\sqrt{2}} m_{ij}^{(\pm 2)}(\hat{k}) h^{(\pm 2)}(\eta, k) \tag{2.91}$$

When the anisotropic stress itself is not present in Einstein field equation, the traceless *equation of motion* takes the form

$$\ddot{h}^{(\pm 2)} + 2\mathcal{H}\dot{h}^{(\pm 2)} + k^2 h^{(\pm 2)} = 0 \tag{2.92}$$

where the solution of $h^{(\pm 2)}$ for $k \gg \frac{a''}{a}$ is given as

$$h^{(\pm 2)} \propto \frac{e^{\pm ik\eta}}{a} \tag{2.93}$$

In case of inflation we start from the Einstein-Hilbert action and action of matter as well

$$S = \frac{M_{Pl}^2}{2}\int d^4x\sqrt{-g}R + \int d^4x\sqrt{(-g)}[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)] \tag{2.94}$$

Here R represents Ricci tensor.Also, we use dot instead of superscript to show derivation w.r.t conformal time η here to avoid clustering. We expand the action to second order written as

$$^{(2)}S = \frac{M_{Pl}^2}{8}\int d\eta d^3x a^2(\dot{h}_{ij}\dot{h}^{ij} - \partial_i h_{jk}\partial^i h^{jk}) \tag{2.95}$$

Taking eqs. (2.87), (2.89), (2.90) and (2.91), we reframe the second order action in Fourier space and after calculation we obtain the rewritten action as

$$^{(2)}S = \frac{M_{Pl}^2}{16}\sum_{P=\pm 2}\int d\eta d^3k a^2[(\dot{h}^p)^2 + k^2(h^{(p)})^2] \tag{2.96}$$

If we compare the above equation with the action of eq. (2.63) in Fourier space, we find that they are related by

$$\delta\phi \to \left(\frac{M_{Pl}}{\sqrt{8}}\right)h^{(p)} \tag{2.97}$$

that is for every independently evolving state of helicity. Now we can find the power spectrum as mentioned in the two-point correlator as

$$\langle h^{(p)}(k)[h(p)(k')]^*\rangle \equiv \frac{2\pi^2}{k^3}\mathcal{P}_h(k)\delta(k-k') \tag{2.98}$$

which makes the power spectrum at the horizon crossing to be

$$P_h(k) = \frac{8}{M_{Pl}^2}\left(\frac{H_k}{2\pi}\right)^2 \tag{2.99}$$

So in this case we evaluate the ***tensor spectral index* or *spectral tilt*** as

$$n_T = \frac{d\ln\mathcal{P}_h(k)}{d\ln k} \tag{2.100}$$

so now we can approximate by power law and then the *power spectrum of tensor perturbations* can be written as

$$\mathcal{P}_h(k) = A_T(k_*)\left(\frac{k}{k_*}\right)^{n_T} \tag{2.101}$$

If we compare the tensor power spectra eq. (2.99) to the scalar power spectra eq. (2.83) as well as their power law approximations which are given by eqs. (2.101) and (2.85), we can find the *tensor to scalar ratio* which is formulated as

$$r = \frac{A_t}{A_s} = 16\epsilon \tag{2.102}$$

In CMB polarization we find that this value is crucial for the measurements important information coming from inflation. Further we calculate taking into account eq. (2.99), eq. (2.100) and the slow roll parameters eq. (1.35) that are

$$\epsilon = -\frac{d\ln H}{d\ln a} \equiv \frac{-\dot{H}}{H^2} \tag{2.103}$$

and

$$\eta = \frac{d\ln \epsilon}{d\ln a} \equiv \dot{\epsilon}/H\epsilon \tag{2.104}$$

obtaining

$$\begin{aligned} n_T &= \frac{d\ln \mathcal{P}_h}{d\ln a}\frac{d\ln a}{d\ln k} \\ &= 2\frac{d\ln H}{d\ln a}(\frac{d\ln k}{d\ln a})^{-1}|_{k=aH} \\ &= -2\epsilon(1-\epsilon)^{-1} \end{aligned} \tag{2.105}$$

which comes out as

$$n_T \approx -2\epsilon \tag{2.106}$$

here, at Horizon crossing, we used $\ln k = \ln a + \ln H$ so $\frac{d\ln k}{d\ln a} = 1-\epsilon$ and so we finally get the ***consistency condition*** for canonical single field slow roll inflation, which is

$$r \approx -8n_T \tag{2.107}$$

There can also be different models in which the condition of eq (2.107) is not respected. The power spectrum changes and with that also the consistency condition changes correspondingly, we get the modified equation for generic single field as $r = -8n_T c_s$, where c_s is the speed of sound of inflation, while for multi field we get $r = -8n_T sin^2\nabla$ where ∇ is correlator between curvature and isocurvature perturbations.

Chapter 3

Energy Momentum Tensor of GWs

We can decipher the prime meaning of the Einsterin's field equation, where geometry gives matter direction of movement and matter in return tells geometry how to curve. This exchange would be null until we calculate the amount of mass-energy present in a unit volume and this tool is what we call energy momentum tensor [EMT].
Let's consider a spacetime .Think of the flow in time and space as a river of moving particles along world lines having associated a 4-momentum, so with this thought we would have many particles which are drifting in many world lines in a continuum approximation, producing a continuous flow. To quantify the flow of this flow, energy momentum is the used tool. We understood that GWs must carry energy allowing it to propagate and distort the spacetime along the perpendicular direction of propagation. The energy momentum carried by the GWs is not allowed to be localized within a wavelength so we can understand that there is no implication of energy being confined on crest of wave or on its trough or within the walls [Misner et al. [1973]] but something certain which we can say is, there is certain quantity of energy momentum present in an average area of a few wavelengths which we define as microscopic region. Hence we can associate a tensor for an averaged smeared out effective energy momentum of GWs which can be denoted as $T^{GW}_{\mu\nu}$.

3.1 Energy Momentum Tensor

In the inertial frame associated with linearized theory, this EMT is given by

$$T^{GW}_{\mu\nu} = \frac{1}{32\pi}\langle h^{TT}_{jk,\mu} h^{TT}_{jk,\mu}\rangle \tag{3.1}$$

here $\langle\rangle$ defines smeared average of few wavelengths. h^{TT}_{jk} is the metric perturbation which is gauge invariant as well as transverse and traceless defined by TT gauge conditions.We can also formulate the energy momentum for arbitrary gauge with trace reversed perturbation defined in eq. (2.17) with conditions like $\bar{h} \neq 0$, $\partial_\alpha \bar{h}^\alpha_\mu \neq 0$ and $\bar{h}_{0\mu} \neq 0$ as

$$T^{GW}_{\mu\nu} = \frac{1}{32\pi}\langle \partial_\mu \bar{h}_{\alpha\beta}\partial_\nu \bar{h}^{\alpha\beta} - \frac{1}{2}\partial_\mu \bar{h}\partial_\nu \bar{h} - \partial_\beta \bar{h}_{\alpha\beta}\partial_\nu \bar{h}^{\alpha\mu} - \partial_\beta \bar{h}_{\alpha\beta}\partial_\mu \bar{h}^{\alpha\nu}\rangle \tag{3.2}$$

Any defined EMT along with the above defined ones are divergence free in vacuum which means there is conservation of the EMT of GWs and in vacuum it means to have no source or sink.So we can write

$$\partial_\nu {T^{(GW)}}^{\nu}_{\ \mu} = 0 \tag{3.3}$$

The formalism provides background curvature on large scale which is being ignored in linearized theory similar to what other EMTs do

$$G^{B}_{\mu\nu} = 8\pi(T^{GW}_{\mu\nu} + T^{radiation}_{\mu\nu} + T^{matter}_{\mu\nu} + T^{otherfields}_{\mu\nu}) \tag{3.4}$$

If we define $h_{\mu\nu}$ for plane wave as

$$h_{\mu\nu} = \mathbb{R}(A_{+}\epsilon_{+\mu\nu} + A_{\times}\epsilon_{\times\mu\nu})e^{-i\omega(t-z)} \tag{3.5}$$

the EMT associated with that would be

$$T^{GW}_{tt} = T^{GW}_{zz} = -T^{GW}_{tz} = \frac{1}{32\pi}\omega^2(|A_{+}|^2 + |A_{\times}|^2) \tag{3.6}$$

We can find that there is no background associated to radius of curvature $\mathcal{R}$ because of linearized theory. The condition which is satisfied through mean reduced wavelength $\bar{\lambda} = wavelength/2\pi$ as well by the amplitude A for GWs are

- $\mathcal{R}^{-2} \sim$ gives typical magnitude of Riemann Tensor for background $R^{(B)}_{\alpha\beta\gamma\delta}$
- $\mathcal{R}^{-2} \sim T^{GW}_{\mu\nu} \sim \frac{A^2}{\bar{\lambda}^2}$ when EMT of GWs is the main source of background curvature
- $\mathcal{R}^{-2} \gg T^{GW}_{\mu\nu} \sim \frac{A^2}{\bar{\lambda}^2}$ when it is not the main source

The relation between dimensionless numbers of A and $\frac{\bar{\lambda}}{\mathcal{R}}$ is given as

$$A \lesssim \frac{\bar{\lambda}}{\mathcal{R}} \tag{3.7}$$

So the whole concept of ripples of small scale, travelling in background curvature of large scale, breaks down and all the formalism here losses its meaning when the gravitational wave's dimensionless amplitude goes to unity. We should always have $A \ll 1$ and also $\bar{\lambda} \ll \mathcal{R}$ for the formalism of GWs to hold any validity

With these if we pass from linearized gravity to full general relativity considerations, leading to nonlinear effects to second-order terms , we start finding the effect in curvature of background of spacetime due to the energy present in GWs and other effect as well.
Linearized theory allows considering source of GWs as something with steady oscillation, allowing the formation of very periodic waves but if we take into account the full theory, it insists about decrease of source energy which counters the balance of energy being carried off by dispersal. This imposes restriction to have exact periodic waves. The curvature of our universe, is not just carved out of GWs but also matter and other contents. So with the propagation of these waves there are changes in

wave fronts, redshift (changes in wavelength) and also in few extent it can backreact. If we find wave as pulse there will be consecutive changes in shape as well as polarization due to back scatter which will lead to the production of tails [DeWitt and Brehme [1960]] which would move behind the pulse with a velocity smaller than that of speed of light.These effects are very small until we follow the condition of $A \ll 1$ and also $\bar{\lambda} \ll \mathcal{R}$. It sometimes increases when the size is comparable to the order of $\mathcal{R}$ (radius of curvature). So we can say that linearized theory is accurate just locally . Let's consider to have a model of universe just made of GWs, only the interaction between waves and their produced background curvature creates tails and back scatters. The self gravitational attraction is also evident when we have reduced wavelength $\bar{\lambda}$ and considering m as mass energy focusing on a small area of radius r less than m, a part of wave would get affected by its own attraction and go for gravitational collapse forming singularity (black hole) which happens generally in astrophysical GWs but in general if the wave has less energy, it does not collapse but in undergoes delay before it gets re-expanded [Christodoulou [1970]].

3.2 Evaluating $T_{\mu\nu}$ for GWs

To evaluate the effective EMT $T^{GW}_{\mu\nu}$ we require to have averages of different quantities done for few wavelengths. The rules which need to be followed inside $\langle\rangle$ are

- The covariant derivatives are allowed to commute

$$\langle h \;\; h_{\mu\nu|\alpha\beta}\rangle = \langle h \;\; h_{\mu\nu|\beta\alpha}\rangle \tag{3.8}$$

  and the small errors which are made are $\sim (\bar{\lambda}/\mathcal{R})^2$, much lower than the computational inaccuracy

- The gradients when averaged give 0,i.e.

$$\langle (h_{|\alpha} h_{\mu\nu})_{|\beta}\rangle = 0 \tag{3.9}$$

  so we get fractional errors around $\leq (\bar{\lambda}\mathcal{R})$

- Integration by parts can be done if we flip the derivatives from h of one side to the other side:

$$\langle h \;\; h_{\mu\nu|\alpha\beta}\rangle = \langle -h_{|\beta} h_{\mu\nu|\alpha}\rangle \tag{3.10}$$

Using these conditions when we calculate for Ricci tensor of second rank associated with h $R^{(2)}_{\mu\nu}(h)$, considering the definition of the trace reversed metric perturbation, taking the propagation equation and using another definition of $T^{GW}_{\mu\nu}$ [Misner et al. [1973]] which is

$$T^{GW}_{\mu\nu} \equiv -\frac{1}{8\pi}(\langle R^{(2)}_{\mu\nu}(h) - \frac{1}{2} g^{B}_{\mu\nu}\langle R^{(2)}(h)\rangle) \tag{3.11}$$

we get,

$$\langle R^{(2)}(h)\rangle = 0 \tag{3.12}$$

and

$$T^{GW}_{\mu\nu} = \frac{1}{32\pi}\langle \bar{h}_{\alpha\beta|\mu}\bar{h}^{\alpha\beta}_{|\nu} - \frac{1}{2}\bar{h}_{|\mu}\bar{h}_{|\nu} - 2\bar{h}^{\alpha\beta}_{|\beta}\bar{h}_{\alpha(\mu|\nu)}\rangle \tag{3.13}$$

This is the equation we have written in eq. (3.2). The changes that we encounter can be seen in linearized theory, where there is no difference in covariant derivative and ordinary derivative. If we consider the gauge conditions $\bar{h}^{\alpha}_{\mu|\alpha} = 0$ we can remove the last term from eq (3.13), traceless condition $\bar{h} = 0$ removes the second term as well so the equation becomes

$$T^{GW}_{\mu\nu} = \frac{1}{32\pi}\langle \bar{h}_{\alpha\beta|\mu}\bar{h}^{\alpha\beta}_{|\nu}\rangle \tag{3.14}$$

The accuracy of above expression for effective EMT of GWs is such that it errors just in fractional part of A order which is due to neglection of corrections due to second order of $h_{\mu\nu}$, it also gives fractional error corresponding to $\bar{\lambda}/\mathcal{R}$ because of averaging which does not have any significant impact when the magnitude of $\bar{\lambda}$ tends to $\mathcal{R}$. As the condition $A \lesssim \bar{\lambda}/\mathcal{R}$ gets satisfied , the main error in $T^{GW}_{\mu\nu}$ becomes $\bar{\lambda}/\mathcal{R}$. To summarize the property of effective EMT $T^{GW}_{\mu\nu}$ we can say to this accuracy , it has same footing as any other EMT. It offers the same role in fabricating the curvature background and obeying the same laws of conservation.

3.3 Towards the second order of field equation

To reach to the effective EMT of GWs we need to explore than second order perturbation by adding non linearity to the Field equation. We have to consider the background of spacetime obeying homogeneity and isotropy. We add scalar metric perturbations as well as tensor metric perturbations consisting of small amplitudes to satisfy the linear fluctuation that we have derived before in linearized gravity and performing the decomposition of scalar vector and tensor. When we add non linearities in the Einstein Field equations, the metric fails to provide solutions at second order of the equation so to have the solution we require to annex terms of second order in the metric which requires to have correction in metric of background as well as in the fluctuations.
To add the corrections to metric of background we can use ansatz for metric with just linearized fluctuations in equation which can be further expanded to second order in fluctuation's amplitude and further taking the spatial average of the resultant equation to get to the effects of back reaction over the metric of background. In the back reaction each Fourier mode independently gives its contribution. At this second order of perturbation theory, these spatial averages also cancel the coupling of scalar and tensor fluctuations allowing us to study scalar and tensor perturbation modes separately. To understand the effect on GWs we need to set the fluctuations in scalar modes to zero and continue with just the tensor modes. The second order of Einstein tensor $G_{\mu\nu}$ is written as [Brandenberger and Takahashi [2018]]

$$\begin{aligned}\delta^{(2)}G^0_0 = \frac{1}{a^2}[&\mathcal{H}h^{km}h'_{km} + \frac{1}{8}h'^{km}h'_{km} - \frac{1}{8}h'^k_{\ k}h'^m_{\ m} - \frac{1}{2}h^{km}\nabla^2 h_{km} - \frac{1}{2}h^{km}h^j_{\ j,km}\\ &+ h^{km}h^j_{\ k,mj} - \frac{1}{4}h^m_{\ m}{}^{,k}h^j_{\ j,k}h^{\ ,j}_{kj} + \frac{1}{8}h^m_{\ m}{}^{,k}h^j_{\ j,k} - \frac{3}{8}h^{km,j}h_{km,j}\\ &\frac{1}{4}h^{km}_{\ \ ,k}(2h^{\ ,j}_{mj} - h^j_{\ j,m}) + \frac{1}{4}h^{jk,m}h_{jm,k}],\end{aligned} \tag{3.15}$$

$$\delta^{(2)}G^0_{\ i} = \frac{1}{a^2}[\frac{1}{2}h^{mk}_{\ \ ,m}h'_{\ ki} + h^{mk}h'_{\ k[i,m]} - \frac{1}{4}h'^{mk}h_{mk,i} - \frac{1}{4}h^k_{\ k}{}^{,j}h'_{ij}], \tag{3.16}$$

$$\delta^{(2)}G^i_{\ 0} = \frac{1}{a^2}[h^{ij}h'^{k}_{[k,j]} + \frac{1}{4}h^k_{\ k,j}h'^{ij} - \frac{1}{2}h'^{j}_{jk}h'^{ik} + \frac{1}{4}h'_{jk}h^{jk,i} - h_{jk}h'^{k[i,j]}], \tag{3.17}$$

$$\begin{aligned}\delta^{(2)}G^i_{\ j} = &\frac{1}{a^2}\delta^i_{\ j}[\mathcal{H}h'_{km}h^{km} + \frac{1}{2}h''_{km}h^{km} + \frac{3}{8}h'_{km}h'^{km} - \frac{1}{8}h'^k_{\ k}h'^m_{\ \ m}\\ &- \frac{1}{2}h^{km}(\nabla^2 h_{km} + h^j_{\ j,km} - 2h^j_{\ (k,m)j}) - \frac{3}{8}h^{km,j}h_{km,j}\\ &- \frac{1}{8}h^m_{\ m}{}^{,k}(2h^{,j}_{kj} - h^j_{\ j,k}) + \frac{1}{4}h^k m_{\ ,k}(2h^{,j}_{mj} - h^j_{\ j,m}) + \frac{1}{4}h^{km,i}h_{jm,k}]\\ &+ \frac{1}{a^2}[-\mathcal{H}h^{ik}h'_{kj} - \frac{1}{2}h^{ik}h''_{kj} - \frac{1}{2}h'^{ik}h'_{kj} + \frac{1}{4}h'^k_k h'^i_j + \frac{1}{4}h^{km,i}h_{km,j}\\ &\frac{1}{2}h^{ik}h^m_{\ m,kj} + \frac{1}{2}h^{m,k}_{m}h^{(i}_{\ k,k)} - \frac{1}{4}h^m_{\ m}{}^{,k}h^i_{\ j,k}\\ &+ \frac{1}{2}h^{km}(h^{,i}_{km,j} + h^i_{\ j,km} - 2h^{(i}_{\ m,j)k}) - \frac{1}{2}h^{km}_{\ ,k}(2h_m^{(i}{}_{,j)} - h^i_{\ j,m})\\ &+ h^{i[k,m]}h_{kj,m} - h^{ik}h^m_{\ (k,j)m} + \frac{1}{2}h^{ik}\nabla^2 h_{kj}].\end{aligned} \tag{3.18}$$

We can again use the condition of TT gauge on GW tensor which further simplifies the Einstein tensor as

$$\begin{aligned}\delta^{(2)}G^0_{\ 0} = &\frac{1}{a^2}[\mathcal{H}h^{km}h'_{km} + \frac{1}{8}h'^{km}h'_{km} + \frac{1}{8}h^{km,j}h_{km,j} -\\ &\frac{1}{2}\delta^j(h^{km}h_{km,j}) + \frac{1}{4}\partial_k(h^{jk,m}h_{jm})],\end{aligned} \tag{3.19}$$

$$\delta^{(2)}G^0_{\ i} = \frac{1}{a^2}[h^{mk}h'_{\ k[i,m]} - \frac{1}{4}h'^{mk}h_{mk,i}], \tag{3.20}$$

$$\delta^{(2)}G^i_{\ 0} = \frac{1}{a^2}[\frac{1}{4}h'_{jk}h^{jk,i} - h_{jk}h'^{k[i,j]}], \tag{3.21}$$

$$\begin{aligned}\delta^{(2)}G^i_{\ j} = &\frac{1}{a^2}\delta^i_{\ j}[\mathcal{H}h'_{km}h^{km} + \frac{1}{2}h''_{km}h^{km} + \frac{3}{8}h'_{km}h'^{km} -\\ &\frac{1}{2}\partial_j(h^{km}h_{km}) + \frac{1}{2}h^{km,j}h_{km,j} - \frac{3}{8}h^{km,j}h_{km,j} +\\ &\frac{1}{4}\partial^m(h^{jk}h_{jm,k})] + \frac{1}{a^2}[-\mathcal{H}h^{ik}h'_{kj} - \frac{1}{2}h^{km}_{\ ,j}h^{,i}_{km} + \frac{1}{4}h^{km,i}h_{km,j} +\\ &\frac{1}{2}\partial_j(h^{km}h^{,i}_{km}) - \frac{1}{2}h^{km}_{\ ,j}h^{,i}_{km} + \frac{1}{2}\partial_m(h^{km}h^i_{\ j,k}) -\\ &\partial_k(h^{km}h_m^{(i}{}_{,j)}) - \frac{1}{2}\partial_m(h^{im,k}h_{kj}) + \frac{1}{2}\partial^m(h^{ik}h_{kj,m})].\end{aligned} \tag{3.22}$$

Spatial average of some quantity A can be obtained if we integrate it over the time part and divide it with spatial volume [Abramo et al. [1997]], this can be written as

$$\langle A \rangle \equiv \frac{1}{V}\lim_{v\to\infty}\int AdV \tag{3.23}$$

We should also take into the consideration that matter's presence gives a physical meaning to the hyperspace making it a surface associated to constant matter-energy density. If we use the above equation for spatially averaging (the above equation also corresponds to quantum averaging) the Einstein tensor, it would give

$$\langle\delta^{(2)}G^0{}_0\rangle = \frac{1}{a^2}[\mathcal{H}\langle h^{km}h'_{km}\rangle + \frac{1}{8}\langle h'^{km}h'_{km}\rangle + \langle\frac{1}{8}h^{km,j}h_{km,j}\rangle], \tag{3.24}$$

$$\langle\delta^{(2)}G^0_{\ i}\rangle = \frac{1}{a^2}[\langle h^{mk}h'{}_{k[i,m]}\rangle - \frac{1}{4}\langle h'^{mk}h_{mk,i}\rangle], \tag{3.25}$$

$$\langle\delta^{(2)}G^i{}_0\rangle = \frac{1}{a^2}[\frac{1}{4}\langle h'_{jk}h^{jk,i}\rangle - \langle h_{jk}h'^{k[i,j]}\rangle], \tag{3.26}$$

$$\begin{aligned}\langle\delta^{(2)}G^i{}_j\rangle = \frac{1}{a^2}\delta^i{}_j[\langle\frac{3}{8}h'_{km}h'^{km}\rangle - \langle\frac{3}{8}(h^{km,n}h_{km,n})\rangle] + \\ \frac{1}{a^2}[-\frac{1}{2}\langle h'^{ik}h'_{kj}\rangle - \frac{1}{4}\langle h^{km,i}h_{km,j}\rangle + \frac{1}{2}\langle h^{ik,m}h_{kj,m}\rangle].\end{aligned} \tag{3.27}$$

here the total derivative terms has been dropped off as we are considering just the spatial average, also we have taken equation of motion eq. (2.92) into account. Now if we use the correction terms of Einstein tensor and take it to the matter side of the cosmological equation then we can finally get the gravitational wave's effective energy density as well as its effective pressure.The effective density that we finally get is

$$\bar{\rho}_{GW} = \frac{1}{8\pi G}\delta G^0{}_0 = \frac{1}{8\pi Ga^2}\left(\frac{1}{8}\langle(h'_{ij})^2\rangle + \frac{1}{8}\langle(\nabla h_{ij})^2\rangle + \mathcal{H}\langle h^{ij}h'_{ij}\rangle\right) \tag{3.28}$$

and the effective pressure is

$$\begin{aligned}\bar{p}_{GW} &= \frac{1}{3}\frac{1}{8\pi G}G^i{}_i + \frac{1}{3\mathcal{H}}\langle^{(2)}\Gamma^\alpha{}_{\alpha 0}\rangle(\rho^{(0)} + p^{(0)}) \\ &= \frac{1}{3}\frac{1}{8\pi Ga^2}\left(-\frac{5}{8}\langle(h'_{ij})^2\rangle + \frac{7}{8}\langle(\nabla h_{ij})^2\rangle\right) - \frac{1}{3\mathcal{H}}\langle^{(2)}\Gamma^\alpha{}_{\alpha 0}\rangle(\rho^{(0)} + p^{(0)}) \\ &= \frac{1}{3}\frac{1}{8\pi Ga^2}\left(-\frac{5}{8}\langle(h'_{ij})^2\rangle + \frac{7}{8}\langle(\nabla h_{ij})^2\rangle\right) + \frac{1}{2}\frac{1}{8\pi Ga^2}\mathcal{H}\left(1 + w^{(0)}\right)\langle h^{ij}h'_{ij}\rangle.\end{aligned} \tag{3.29}$$

In the above equation $w^{(0)}$ defined as parameter of equation of state which is defined as

$$w^{(0)} = \frac{p^{(0)}}{\rho^{(0)}} \tag{3.30}$$

To come to the final form of eq 3.29 we have used

$$\langle^{(2)}\Gamma^\alpha{}_{\alpha 0}\rangle = -\frac{1}{2}\langle h^{ij}h'_{ij}\rangle \tag{3.31}$$

We can define $\bar{\rho}^{(k)}_{GW} \propto d^3k\rho_{GW}(k,\eta)$ where $\rho_{GW} \propto (|\dot{h}_k|^2, |h_k|^2\text{and}|\dot{h}_k h_k|)$

3.4 Classical conservation of EMT of GWs

The perturbation in first order does not give the same result of EMT as that of second order. There are mixed terms of h_k which do not come in equation of first order perturbation. The corrections of non linear nature to Einstein tensor is composed of quadratic combination of h_k to the lowest order with the form [Giovannini [2010]]

$$\delta_t^2 G_0^0 = -M_{Pl}^{-2} T_0^0 \tag{3.32}$$

here t in subscript denotes the tensor perturbation and 2 in superscript denotes second order perturbation so for EMT T_{00} we get

$$T_0^0 = \frac{M_{Pl}^2}{a^2}\left[\mathcal{H} h_k' h_k + \frac{1}{8}(k^2|h_k|^2 + |h_k'|^2)\right] \tag{3.33}$$

and

$$T_i^j = \frac{T}{3}\delta_i^j + (T_i^j - \frac{T}{3}\delta_i^j) \tag{3.34}$$

where

$$T = \frac{M_{Pl}^2}{a^2}\left[\frac{5}{8}|h_k'|^2 - \frac{7}{8}(k^2|h_k|^2)\right] \tag{3.35}$$

the expression matches the ones which are obtained in [Abramo et al. [1997]] and so the equation for energy and pressure density we obtain by definition $\rho_{GW}^{(2)} = T_0^0$ and $p_{GW}^{(2)} = -T/3$. But it is important to remark that, the mixed term arises in second order which we can find in the above expression of T_0^0. There lies a huge difference first and second order terms, and in reality this difference makes a huge impact when we are talking about wavelengths larger than Hubble radius. The difference comes because our eq. (3.33) and eq. (3.34) are not covariantly conserved but we can use the Bianchi identity $\nabla_\mu G_\nu^\mu = 0$ which has the validity at all orders. The conservation equation, in generic form written as

$$\rho_{GW}' + 3\mathcal{H}(\rho_{GW} + p_{GW}) = 0 \tag{3.36}$$

becomes

$$\bar{\rho}_{GW}' + 3\mathcal{H}(\bar{\rho}_{GW} + p_{GW}) - \frac{2M_{Pl}^2(\mathcal{H}^2 - \mathcal{H}')}{a^2}\delta_t \Gamma_{k0}^k = 0 \tag{3.37}$$

which can also be framed as

$$\bar{\rho}_{GW} + 3\mathcal{H}(\bar{\rho}_{GW} + \bar{p}_{fGW}) = 0 \tag{3.38}$$

where

$$\bar{p}_{fGW} = p_{GW} + \frac{(\mathcal{H}^2 - \mathcal{H}')}{3\mathcal{H}a^2} h_k' h_k \tag{3.39}$$

We will call the second term of effective pressure as the effective term. So now the final equation of effective energy density and pressure after the conservation of EMT becomes in Fourier space is

$$\bar{\rho}_{GW} = \frac{M_{Pl}^2}{4\pi^2 a^2}\int_0^\infty k^2 dk[|h_k'|^2 + k^2|h_k|^2 + 8\mathcal{H}|h_k' h_k|] \tag{3.40}$$

and the effective pressure can be written as

$$\bar{p}_{GW} = \frac{M_{Pl}^2}{4\pi^2 a^2}\int_0^\infty k^2 dk[-\frac{5}{3}|h_k'|^2 + \frac{7}{3}k^2|h_k|^2 + \left(\frac{1+\beta}{3}\right)|h_k' h_k|] \tag{3.41}$$

where we have used $\mathcal{H} = \beta/\eta$, we will look at this further in the last chapter.We can find that the above equations match (3.28) and eq. (3.29).

3.5 Propagation of Inflationary GWs through spacetime

We have understood till now that inflation provides a mechanism for the amplification of tensor fluctuations at super Hubble scale. As they surpass the Hubble radius, tensor modes become constant in time with nearly scale invariant spectrum. As soon as inflation ends, the standard model particles arise out of the energy density stored in inflaton field and the standard big bang cosmology comes into place [Liddle and Lyth [2000]]. The first domination comes of radiation and there after matter domination until almost now when the dark energy becomes dominant. During the radiation and matter domination epoch, the size of cosmological horizon increases and the tensor fluctuation originated at primordial times whose wavelength during inflation were exponentially elongated slowly comes back inside the horizon scale, which happens as the time passes, fluctuation's physical wavelength decreases in comparison to the inverse of parameter of Hubble. Now at this point we cannot consider the the gradient of tensor fluctuation to be negligible in comparison with the rate of expansion and so these tensor modes are expected to oscillate with their primary conditions led by the horizon crossing value: these tensor modes are what we know as PGWs which are travelling through time from very early universe to now. After they re enter, their propagation remains undisturbed as they very feebly interact with the other components of universe, allowing them to have the properties mostly sensible to the geometry of background curvature of universe. If we have a good approximation then the analysis of how they evolve linearly would be enough for characterizing their features today.

Expansion of tensor modes in Fourier space in first order is done using

$$h_{ij}(\eta, x) = \sum_{\lambda=+,\times} \int \frac{d^3k}{(2\pi)^3} \epsilon_{ij}^{(\lambda)}(k) h_k^{(\lambda)}(\lambda) e^{ik\cdot x} \tag{3.42}$$

where $\epsilon_{ij}^{(\lambda)}$ is the polarization tensors and it follows the TT condition. Tensor fluctuation's Fourier mode comes back inside horizon at a certain time given by $\mathcal{H}(\eta) = k$ which happens in radiation or matter domination epoch. As dark energy becomes dominant just recently,we can ignore its contribution. When the tensor modes come back inside the horizon at radiative era the scale factor follows

$$a(\eta) \propto \eta \tag{3.43}$$

For the equation of motion, the two independent solutions we get are

$$h(\eta) = \frac{(sin\ k\eta)}{k\eta} \quad \text{and} \quad h_k(\eta) = \frac{(cos\ k\eta)}{k\eta} \tag{3.44}$$

Using two initial conditions

$$h_k(\eta_{in}) = h_k^{inf} \quad \text{and} \quad h_k'(\eta_{in}) = 0 \tag{3.45}$$

where η_{in} represents the time of beginning of radiation domination and the conditions are taken by assuming a constant solution $h_{(\lambda)}^{inf}(k)$ of inflation at super Hubble scale($k\eta_{in} \ll 1$). We finally find that tensor modes function during radiative era reads as

$$h_k^{RD}(\eta) = \left(\frac{sin(k\eta)}{k\eta}\right) h_k^{inf} \tag{3.46}$$

In the analogy, we can find the solution in the era of matter domination as

$$h_k^{MD}(\eta) = \left[\frac{3}{(k\eta)^2}\left(\frac{sin(k\eta)}{k\eta} - cos(k\eta)\right)\right] h_k^{inf} \tag{3.47}$$

Equality between radiation and matter domination eras can be first approximation and defines k_{eq} as the characteristic scale which can be written as $k_{eq} = \mathcal{H}(\eta_{eq})$. If we transform k_{eq} into the today's GWs frequency with the relation $f = \frac{k}{(2\pi)}$ taking the scale factor for today as $a_0 = 1$, we get [Maggiore [2018]]

$$f \simeq 1.7 \times 10^{-17} \left(\frac{h_0}{0.7}\right)^2 Hz \tag{3.48}$$

So after the end of inflation, when the GWs re enter, there is the formation of a *stochastic GWs background* due to the inflationary tensor modes. As we know, the production of GWs at the time of inflation is carved on principles of quantum mechanics, in which the occupation number related to them is amplified. This happens so that the distribution related to the GWs of inflationary origin be catered in terms of classical variables and stochastic random variables. As a result the spectrum of GW is stationary, homogeneous- statistically as well as isotropic since they must obey the cosmological principle for the space time background during inflationary scenario. This explains the characteristics related to stochastic GW background. In case of the simplest scenario of inflation, the tensor modes are almost Gaussian, scale invariant (nearly) and not polarized, but we also have exceptions
To characterize the properties of stochastic GW background coming from inflation, we can use energy density and $\Omega_{GW}(\eta, f)$ which is a dimensionless parameter of energy density which is defined as

$$\Omega_{GW}(\eta, f) = \frac{1}{\rho_c}\frac{d\ln\rho_{GW}}{d\ln f} \tag{3.49}$$

the above equation is evaluated post inflation at conformal time, here

$$\rho_c = \frac{3H_0^2}{M_{Pl}^2} \tag{3.50}$$

which provides the critical value related the energy density. From here we can write the EMT associated to GW at time η (given in [Isaacson [1968]]) as

$$\bar{\rho}_{GW}(\eta) = \frac{1}{32\pi G a^2}\langle h'_{ij}h'_{ij}\rangle = \int_0^\infty d\rho_{GW} \tag{3.51}$$

here $\langle h'_{ij}h'_{ij}\rangle$ represents many wavelengths averaging.A transfer function gives a mathematical description of how GWs affect the CMB annisotropies and is dependent on η and frequency. The parametrization of tensor modes for its evolution which happens in sub horizon part during both the eras of radiation and matter domination can be done using this Transfer function.

Transfer Function Numerical integration of the equation of cosmological perturbation gives the result, which can be easily summarized with the help of simple fitting formulas. In the linear theory the gravitational potential pleateau at matter domination is connected to the initial value of Radiation domination. The modes separately evolve and for such linear case we introduce a Transfer function with connects the potential during matter domination to that of radiation domination. This is the case of scalar but since we need to focus on transfer function related to tensor modes, with the use of inflationary mode's initial solution, this can be written as [Tasinato [2022]]

$$h_k = \mathcal{T}(\eta, k) h_k^{inf} \tag{3.52}$$

The transfer function is independent of polarization [Maggiore [2018]]. The tensor modes re enter the horizon depending upon comoving momentum and so can re-enter at any time between radiation or matter domination. By using the transfer function we can derive density parameter Ω_{GW} written for today

$$\Omega_{GW}(\eta_0, f) = \frac{1}{12H_0^2} |\mathcal{T}'(\eta_0, f)|^2 \mathcal{P}_T(f) \tag{3.53}$$

this is associated with primordial tensor spectrum from inflation $\mathcal{P}_T$ as the density parameter is dependent on frequency, which comes from the dependence of transfer function on frequency. It depends on the time of re-entering of tensor modes, if it is during radiation domination or if it is in the matter domination era.

Tensor modes having frequency ($f > f_{eq}$), re-enters during the radiation domination era. There are modes of frequency which come inside horizon at the time when the temperature of universe is very high to keep SM particles in thermal equilibrium. We can write the expression for simplified value of dimensionless energy density for today as

$$h_0^2 \Omega_{GW} = 1.36 \times 10^{-17} \left(\frac{H_\star}{10^{-5} M_{Pl}} \right)^2 \left(\frac{f}{f_\star} \right)^{n_T} \tag{3.54}$$

which is written for frequency $f \geq 10^{-4} Hz$, where $H_\star$ provides the value related to Hubble parameter during inflation,$f_\star$ gives a pivot frequency which is $f_\star = 7.7 \times 10^{-17} Hz$ which is related to the primordial momenta ($k_\star = 0.05 Mpc^{-1}$) which is comparable to the scales of CMB and in the equation n_T defines the spectral tilt related to the tensor spectrum.

If we focus on single field slow roll inflationary scenario, n_T is quite small as well as negative eq. (2.106). From eq. (3.54) the suggestion of the required sensitivity for measuring Ω_{GW} from inflation through experiments is

$$h_0^2 \Omega_{GW} \simeq 10^{-18} - 10^{-17} \tag{3.55}$$

The above mentioned magnitudes are below the sensitivity of present or future proposed observatory. In fact it could be easier to measure primordial Ω_{GW} coming from inflation if we do not consider single-field scenarios. In this case the primordial tensor spectrum has the possibility of rising at rise to large frequency which would also amplify the amplitude of the PGWs making it easier to be detected by interferometer existing today, for example we can look at [Bartolo et al. [2016]]. It also opens the portal of having complex frequency shapes of primordial power spectrum taking it beyond what we have now, a single power law (n_T being constant) also the chirality can be probed. All these features are interesting as they lead a way to certainly find and identify inflationary source, cosmological source, stochastic background of GW. We can now look at some of the alternatives of inflation for which the spectral tilt is non zero.

3.6 Alternatives to slow roll inflation

There are many models of inflation and few alternatives of inflation [Wang and Xue [2014]] as well which has $n_T \neq 0$, below we list a few.

- Inflation: externally sourced tensor modes- Other than production of GWs from amplification of quantum fluctuations, GWs could have also been generated by strings or particles. In this case, inflaton's kinetic energy written as $\dot{\phi}^2 \sim 2\epsilon H^2 M_{Pl}^2$ dumps to particle, which is large enough allowing the production of a visible tensor spectrum.In fact, a bound on ϵ can be taken out of the mechanism of tensor production due to sources from particle production, which comes to $\epsilon > 4 \times 10^{-10}$ if the tensor to scalar ratio is 0.2. Due to the increase in mass we can afford to have $n_T > 0$ (blue tensor spectrum) as the particles acquire mass which are time dependent as $M^2 \sim \dot{\phi}^2 t^2$ which is because of the coupling with the inflaton's rolling mechanism.

- Inflation: Beyond slow roll- It has been recently shown that violation of slow roll inflation can produce blue tensor spectrum. If spectral tilt in second order gets calculated to $n_T \simeq -2\epsilon - 2\epsilon^2 - 0.54\epsilon\eta$. The slow roll gets extrapolated to $\eta \lesssim -3.7$ then we obtain blue tensor spectrum.

- Inflation: modified gravity- We know that the tensor modes are related to gravity and so if the gravity changes it would result in the change of tensor spectrum. If we had massive gravity, then the possibility of achieving this goal increases, if $m^2 > 0$ for graviton at the time of inflation, earlier exit tensor modes get more time to slowly roll back to the origin of their mass potential leading to more suppression allowing the generation of blue tensor tilt. In the late universe the massive gravity does not help make changes in the tensor and this is because gravitons have very less mass at that time.

- Second order effects- The scalar modes are not able to source tensor modes at the linear level but beyond that it is possible. The second order effects remain unobserved because of suppression by inflationary power spectrum. However, contributions from second order tensor modes get boosted with small speed of sound of isocurvature scalar sector.These GWs are mostly non-Gaussian and the tilt of this tensor spectrum is not constrained by conditions of null energy thus allowing a blue tilt.

- Matter bounce- As an alternative to inflation, the matter bounced model predicts that the, universe bounced back and heated up due to contraction. Although in this case tensor modes are scale invariant but amplitudes are quite high.This could allow a potential for blue tensor spectrum.

3.7 Back reaction of GWs

From previous sections we now understand that GWs carry energy as well as momentum, this does not just changes the distance between objects but also opens the portal to the effect on dynamics of background spacetime. This effect comes in place because of the introduction of non linearity in field equation which comes with second order of perturbations. This effect of propagation through spacetime, on spacetime itself by GWs is termed as back reaction of GWs. We know that a range of PGWs are produced in the early epoch of universe like for instance in the inflationary scenario or in the early universe but they are generally characterized by two characteristics which are

- Amplitude of particular sourced GWs A_T
- Spectral tilt of particular sourced GWs n_T

If we take k_* as comoving wavenumber at CMB Scales ($k_* = 0.05/Mpc$), then we can define the power spectrum on scale k associated to GWs written in eq. (2.101) By taking into account data from Planck and BICEP/Keck 2018 (BK18), the consistency condition for the 95% limit must uphold given that we have just an upper bound on r, any joint constraint on r and n_T is prior dependent [Paoletti et al. [2022]]
We can categorize the wavelengths as

- Short wavelengths- which are wavelengths of smaller measure than of the Hubble radius (Sub-Hubble mode or ultraviolet mode)
- Long wavelengths- are wavelengths larger than the Hubble radius (Super-Hubble mode or infrared mode)

In case of short wavelength, the oscillation of GWs happens and they fall in the same category for equation of state as of radiation and so the BBN (Big Bang Nucleosynthesis) imposes an upper bound to the energy associated with GWs of short wavelengths. In case of long wavelengths GWs were less studied and the reason for that was sometimes wrongly said to be due to causality preventing modes like these to have a local effect that could be measured. The truth is that, models that give solution of the horizon problem and also where the causal explanation for structural formation is given, in all those models the cosmological horizon is enormously larger than the Hubble radius at the time of imposition of initial conditions. For example if we take the case of physics at inflation, cosmological horizon at the end of inflation is e^N (N defines number of e-foldings here) larger than that of Hubble radius and so this rules out the problem which could arise from causality and would stop just super Hubble modes from showing back reaction effects. We can say that because of a non vanishing contribution given to the local effective EMT which comes from GWs of super Hubble size, they also imprint geometry of background. In the paper [Finelli et al. [2005]], we find that, due to normalization, on EMT, the effect of PGW on sub-Hubble scale gets cancelled. In the last chapter we will find that, for computing the effective EMT through Bessel function, allows as effect on this scale but as we take just the growing modes of Bessel function that leads to cancel the effect of Short wavelengths ruling out the causality problem.

3.8 Effect of scalar modes

The effect of back reaction on super Hubble cosmological perturbations of scalar origin has been in focus. If we take into account cosmological fluctuations that are linearized then the associated effective $T_{\mu\nu}$ which are accountable for back reaction of these fluctuations is going to have an amplitude of quadratic nature .In fact in the paper [Brandenberger [2002]] and [Abramo et al. [1997]] the formulation of effective EMT was derived and it manifested that at super Hubble modes this effective $T_{\mu\nu}$ gets the form of cosmological constant but with a negative value. This becomes evident as temporal and spatial derivatives in effective $T_{\mu\nu}$ are not taken in account on scales of super Hubble just leaving out the terms which work as potential energy of scalar fields. As on the scales of super Hubble, the negativity of effective energy of GWs submerges the positivity of the energy of matter, giving out the contribution which mimics a negative cosmological constant. This could be explained by the mechanism of dynamic relaxation of cosmological constant.

Many papers by (see for instance [Tsamis and Woodard [1994]] and [Tsamis and Woodard [1997]]) investigated the effects in de Sitter Universe of these super Hubble GWs. They conclude that with loop order of two (quadratic nature of amplitude) the super Hubble mode tends to push cosmological constant to negative side and could provide relaxation to it. On the contrary, in the paper by [Abramo et al. [1997]], it is shown that, in matter universe having scalar perturbations, the equation of state for GWs of longer wavelengths is

$$p = -\frac{1}{3}\rho \tag{3.56}$$

The above equation contributes to the spatial curvature and we can notice that the sign associated with effective energy density ρ is negative and this is because of scalar metric perturbations.

In the paper [Unruh [1998]], the effect of back reaction and local observability of large wavelengths was for the first time investigated as an important affair. In the papers [Geshnizjani and Brandenberger and Abramo and Woodard [2002]], it was demonstrated that the effect of back reaction of large wavelengths is similar to that of second order translation of time for cosmological perturbations when happens adiabatically. On the contrary, in the models which have a different clock field for example radiation field present in matter dominated cosmology, this effects of back reaction on super Horizon are measurable locally [Geshnizjani and Brandenberger [2005]] and ultimately they produce a decrease of observed rate of Hubble expansion[Marozzi et al. [2013]]. In fact locally super Hubble mode's back reaction shows a change in local measure of cosmological constant as well as curvature scalar [Brandenberger and Lam [2004]].

Chapter 4

Constraints on primordial GWs

Primordial GWs are smoking gun for probing the early Universe. The tensor perturbations from the simplest models of inflation give rise to nearly scale invariant PGWs' spectrum which is connected directly to the energy scale at which the accelerated stage occurred. At present there is no detection of GWs in temperature and polarization pattern(TT,TE,EE,BB) in the CMB anisotropies, see for reference [Paoletti et al. [2022]]. However, for different mechanisms of production of PGWs, different bounds are present.
We saw in the second chapter that there are many sources of PGWs background. These exact sources are the ones that are searched during any observational probe. The amplitude of such background constraints lowest frequencies that are observable from 10^{-17} Hz to 10^{-16} Hz which is similar to the wavelength of cosmological horizon at present [Smith et al. [2006]], which comes out of large angular fluctuations in temperature of CMB. The anticipation of investigating CMB with lower amplitude associated with this cosmological GW background comes out of future projects for measuring CMB polarization. Apart from the constraints imposed by the Pulsar Timing Array (PTA) on the CMB around 10^{-9}–10^{-8} Hz, the Big Bang Nucleosynthesis (BBN) is the cause of the strongest constraints for frequencies with amplitudes greater than 10^{-10}. The size of the comoving horizon at the BBN time is the source of the constraints for lower frequencies
PGWs of larger frequency add up for radiation density at that time, which increases the rate of expansion and so there is also an increase in the abundances of light element. Their measurement allow us to confine the extra relativistic species at the time of BBN giving us a limit corresponding to the current energy density of cosmological GW background $\Omega_{GW}h^2 \lesssim 7.8 \times 10^{-6}$. So now the range of frequencies $10^{-16} - 10^{-10}$ Hz remains mostly not constricted. Quasi steller object (QSO) astrometry gives an upper bound on energy density of $\Omega_{GW}h^2 \lesssim 0.1$ corresponding to the frequency [Seto and Cooray [2006]], it was also proposed that in future, anisotropy related to the global rate of change corresponding to the redshift observation would one day come down to $\Omega_{GW}h^2 \sim 10^{-5}$. In case of non relativistic matter, the recent measurements related to the angular power spectrum of CMB constrains matter density($\Omega_m h^2$), which is around 10% . This can be refereed to the first approximation related to constraining energy density of radiation at CMB decoupling, limiting some neutrino extra degree of freedom . From this we can understand that CMB allows a limit to energy density with can rival with that of BBN, but that gets to

the lower frequency of about 10^{-15} Hz, which is about the wavelength corresponding to the size of comoving Hubble horizon at the time of CMB decoupling [Smith et al. [2006]], giving an improved four order magnitude limit for the frequency range $10^{-15} - 10^{-10}$ Hz. In reality the cosmological GW background carves itself as massless particles that serves as free streaming gas similar to massless neutrinos and so this allows them to have an influence on the growth of perturbations due to density in different ways as well as producing an increase in the rate of expansion at the time of decoupling.

4.1 CMB constraints

Although inflation produces GWs from E modes as well as from B modes but primordial density fluctuations does not contribute to B modes. Although we have focused on B mode polarization for the detection of PGWs still there are many processes that plague the observation like gravitational lensing along the line of sight with galactic foregrounds that need to be modelled correctly so that we can derive constraints on cosmological parameters at the time when we find that the scalar as well as tensor power spectra are showing usual parametrizations of power law. As we know the tensor to scalar ratio is amplitude's ratio which is examined at scale k. This is usually assigned to the tensor amplitude but as scalar amplitude is resolved enough, so they are allowed to be interchanged, which is easy as well. In fact today the best constraints on tensor to scalar ratio which is obtained from all different combinations of modes TT,TE, EE and BB is

$$r_{0.002} < 0.035 \text{ at } 95\% C.L. \tag{4.1}$$

The constraint on r can be from linear contribution of GWs

Parameters	*Planck*+BK18 $n_t = -r/8$	*Planck*+BK18 free n_t
$\Omega_b h^2$	0.0224 ± 0.0001	0.0224 ± 0.0001
$\Omega_c h^2$	0.120 ± 0.001	0.120 ± 0.001
$100\ \theta$	1.0409 ± 0.0003	1.0409 ± 0.0003
τ	$0.0546^{+0.0073}_{-0.0072}$	0.0544 ± 0.0073
$\ln(10^{10} A_s)$	3.045 ± 0.014	3.044 ± 0.014
n_s	0.9653 ± 0.0041	0.9656 ± 0.0041
$r_{0.005}$	(< 0.032)	$< \mathbf{0.030}$
$r_{0.02}$	(< 0.034)	$< \mathbf{0.098}$
$r_{0.05}$	$< \mathbf{0.035}$	(< 0.71)

Figure 4.1: Data giving constraints using Planck+BICEP Keck Array (BK15) with 68% CL for parameters of ΛCDM. The paper used two method, first by using n_T from consistency equation and other by taking n_T from observational data as well as from simulation giving the two different sets of values of constraints. The bold letters show 95% CL upper bound for parameters of primary tensor while the parenthesis composed of tensor parameters that are derived. Credit: Paoletti et al. [2022]

$$\Omega_{GW}(k)h^2 \propto \Omega_r h^2 \mathcal{P}_T \tag{4.2}$$

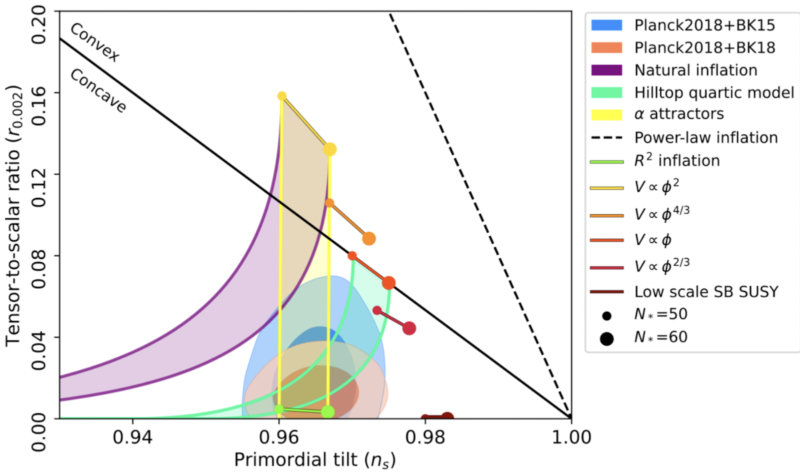

Figure 4.2: The plot shows regions with 68% and 95% confidence level on constraints where of $n_T = -r/8$ from the data by Planck combined with BK18 and BK15 data. The theoretical predictions for different slow roll inflationary models are given with an uncertainity in number of e-foldings to the end of inflation between 50 and 60. Credit: Paoletti et al. [2022]

Here the equation is written in terms of k and power spectrum related to tensor and the density parameter associated to GW that is a function of frequency. Also h has an uncertainity on Hubble parameter which is given as $H_0 = 100h\ h\ km\ s^{-1}\ Mpc^{-1}$ and density related to relativistic species is given by Ω_r, k_{eq} contains the value $\sqrt{2}H_0\Omega_m/\sqrt{\Omega_r}$. Here Ω_m is defined for the matter density, for the modes that enter the Hubble radius at the time of equality of matter and radiation. The tensor power spectrum for single field model of slow roll inflation is given as eq. (2.101) where the value is standarized for the pivot scale $k_* = 0.05\ Mpc^{-1}$ (We know the consistency equation from eq. (2.107), which depicts that prediction of inflation allows a small redshift spectrum having value $-0.007 < n_T < 0$ with 95% C.L.).

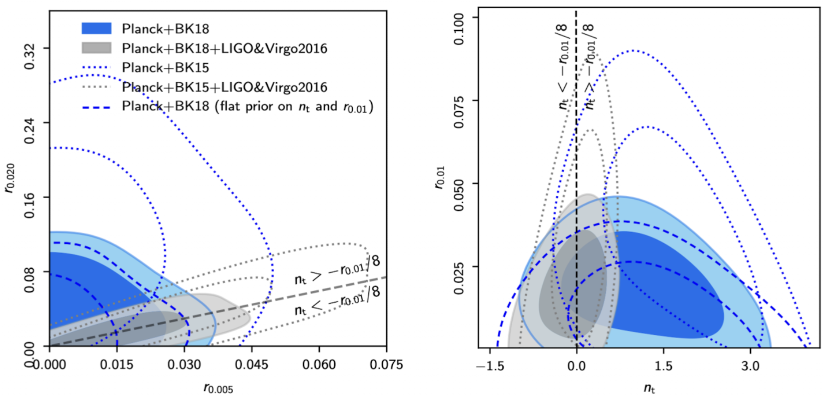

Figure 4.3: The plot shows posterior obtained from the real data of tensor parameters when n_T is allowed to vary. ΛCDM models has been used with MCMC parameters for sampling, likelihood parameter and independent ones for tensor to scalar ration r at $k_1 = 0.005\ \ Mpc^{-1}$ and $k_2 = 0.02 Mpc^{-1}$. The first panel is having contours of 68% and 95% CL for primary tensor parameters having flat priors except of the dashed lines. The second panel has derived n_T as well as $r_{0.02} Mpc^{-1}$ parameters having non flat priors, except for dashed lines. Credit: Paoletti et al. [2022]

4.1.1 Constraints from GWs EMT

In the last section we have shown how GW at linear order affect CMB anisotropies however, their EMT can also affect the background geometry and as a result there are changes in CMB pattern. When we have shortwave approximation, PGWs acts as massless hence contribute to the degree of freedom related to relativistic species' effective number (N_{eff}). If we take adiabatic consideration of initial condition in account, N_{eff} can be used to find the value $\Omega_{GW}h^2$ where

$$\Omega_{GW}h^2 = \int_0^\infty d(log\ f)h^2\Omega_{GW}(f) = 5.6 \times 10^{-6}(N_{eff} - 3.046) \tag{4.3}$$

Assuming value from standard model for $N_{eff} \approx +3.046$ which comes in place when the PGWs are not present.

$$\rho_r = \rho_\gamma[1 + \frac{7}{8}(\frac{4}{11})^{4/3}N_{eff}] \tag{4.4}$$

In the above equation ρ_r is energy density related to relativistic species and ρ_γ is the energy density related to photons. The relation of density parameter with GWs' dof is given by

$$\frac{7}{8}(\frac{4}{11})^{4/3}\Omega_\gamma h^2 = 5.605 \times 10^{-6} \tag{4.5}$$

The constraint on N_{eff} from Planck 2018 data and BAO is

$$N_{eff} = 2.99^{+0.34}_{-0.33}(2\sigma) \tag{4.6}$$

along with the upper limit of $\Omega_{GW}h^2 < 1.6 \times 10^{-6}$.

These approximation allow the perturbations of GWs to be similar to massless neutrinos which gets described by their density, vorticity, sheer etc. The possiblity is of having a non adiabatic initial condition which could come from different PGWs sources. Adiabatic model would work if PGWs would had been thermalized species of particle which would had been produced from inflaton decay, but as we know, most of CGWB generates unperturbed background. The other choice of initial condition related to these GWs of homogeneity does not have initial perturbation density, considering Newtonian gauge. What happens in this case is that the perturbations of GWs evolve differently from that of neutrino's and this allows the breaking of degeneracy between Ω_{GW} and N_{eff}. [Smith et al. [2006]] provides a detail of how CMB anisotropies related to temperature can be taken into consideration for constraining GWs short wavelengths for different initial conditions like adiabaticity and homogeneity. This uses the observation like WMAP(first year), Lyman α forest and SDSS. Infact there are more strict constraint seen for homogeneous GWs than adiabatic ones by factor ($\approx 5 - 10$), which has been updated in data of WMAP (seven year) where we find $\Omega_{GW}h^2 < 8.7 \times 10^{-6}$, this is for adiabatic case and for homogeneous case it is $\Omega_{GW}h^2 < 1.0 \times 10^{-6}$[Sendra and Smith [2012]]. From Planck 2015 data $\Omega_{GW}h < 1.7 \times 10^{-6}$ is what we find but for the case of homogeneous, the results have not been updated recently. In the joint analysis of measurements from CMB and light element abundances from BBN, put constraint on $N_{eff} = 2.862 \pm 0.306$ at 95% CL [Kahniashvili et al. [2022]].

4.2 Pulsar Time Array

Pulsar Time array (PTA) can be refereed as program consisting worldwide distribution of millisecond pulsar(MSP) array for very precise timing observation. Since Pulsars have been promising for observations, after the discovery of millisecond pulsar in 1982, they are considered to be of high accuracy for measurements of pulse arrival time as they show precise regularity in rotation. After discovery of hundreds of such MSP, Foster and Backer proposed that these could be used for detecting GWs and they initiated the formation of MSP program called PTA program. Later three such programs PPTA, NANOgrav and EPTA joined hands for combining expertise and data to form International PTA.[Hobbs and Dai [2017]]

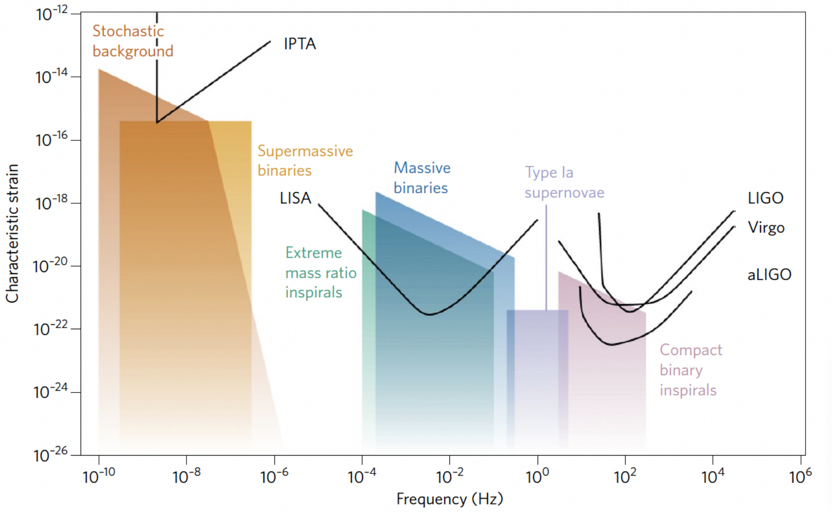

Figure 4.4: The plot shows sensitivity space of different experiments for detection of GWs(in black) for different sources (in different colours). On the left are galaxy based detectors, in middle - space based and on right are ground based detectors, Credit: Lommen [2017]

4.2.1 Effect of GWs on pulsar observation

When a GW pass, it induces a change in observational data related to pulsar frequency

$$\frac{\delta \nu}{\nu} = -H^{ij}[h_{ij}(t_e, x_e^i) - h_{ij}(t_e - \frac{D}{c}, x_p^i)] \tag{4.7}$$

where H^{ij} is dependent on the position of source of GW, it is merely a geometrical term. D is the distance of pulsar from Earth while h_{ij} is a strain of GW in observation, t_e is the time of detection while x_e is position at that time while the time when GW pass the pulsar is t_p, and position is x_p. The measurement on shift of pulse frequency is not determined instead time of arrival of pulse(ToAs) are used which gets further compared with arrival time prediction depending on the model and their difference is celled 'timing residuals'. At initial time of observation

$$R(t) = -\int_0^t \frac{\delta \nu}{\nu} dt \tag{4.8}$$

In terms of polarization, Earth term is written as

$$R_e(t) = \int_0^t \frac{P_+ A_+(t) + P_\times A_\times(t)}{2(1-\gamma)} dt \tag{4.9}$$

where geometrical terms are P_+ and $P_\times$ while γ determines the angle between GW, pulsar and Earth. Although astrophysical GW can be detected with these array but as detection of pulsar happens almost every week and starting from 1982 for MSP there is a huge data set in PTA which are sensitive enough for GW. This corresponds to the wavelengths of weeks to years, for GWs of frequency $10^{-9} - 10^{-8}$Hz opens a possibility for detection of Cosmological GWs.

4.2.2 Results to date

In the first years, research in PTA has been divided into three parts

- Prediction of the signal expected as well as calculation of their detection time
- Searches for such GWs with more sensitivity
- To understand the implication faced by non-detections

The search has started for bounds coming from current PTAs but moreover soon sensitivity of whole IPTA would be required for the detection of such GWs
Most of the research in PTA are related to bound the background of GWs, which is used for detecting spectrum of background, written as

$$h_c(f) = A \left(\frac{f}{f_{1\,yr}} \right)^{\alpha} \tag{4.10}$$

In the above equation, $f_{1\,yr}$ represents frequency per year. α is spectral exponent, which could have different value for example -1 as the value for cosmic strings and -7/6 for relics GWs. The three PTAs have put bound for $A < 10^{-15}$ with 95% confidence.

4.2.3 Evidence of GWB detection

Recently in June 2023, NANOGrav,(EPTA-InPTA), Chinese PTA and PPTA published papers(Reardon et al. [2023],Xu et al. [2023],Antoniadis et al. [2023] and Agazie et al. [2023]) claiming the evidence of GWB. Although the observation points at the source of these background as binary SMBH creating nanohertz frequency GWs but there are also different other processes from early universe which are eligible candidates also for the source of generation, data cannot discriminate source with high confidence level. The collaboration found an unequal intensity for data as they found noise was not 'white' in the data of PTA , but it was 'red' for the presence in lower frequency which tells that GWs signal were produced in decades as a period of oscillation. This was given with Hellings-Downs curve which shows angular correlation with the observed data giving the evidence of such detection

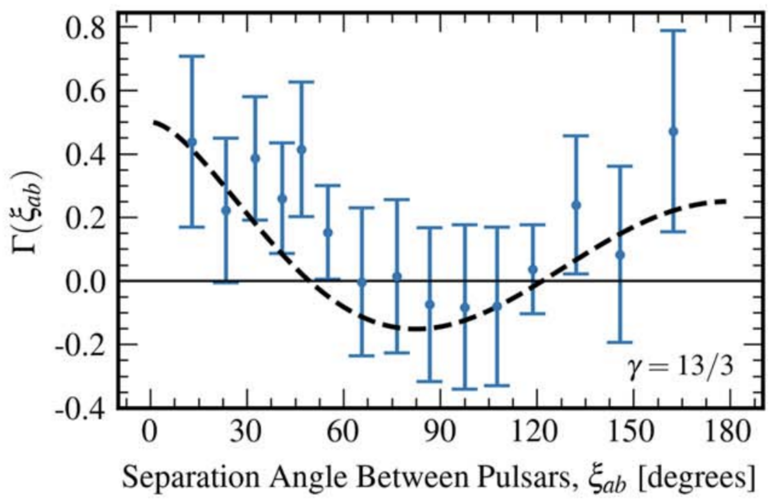

Figure 4.5: The plot shows correlation where it is dependent on the distance of separation between two pulsars in sky which is in degrees. The blue line shows the data with error bars which is compared with Hellings-Down curve, showing an expected correlation.Agazie et al. [2023]

with $\gamma = \frac{13}{3}$ and the spectrum of GW (A) as (eq. 4.10) which is of log uniform distribution from -18 to -14 for power law. [Agazie et al. [2023]] The problem with this is, researchers found the confidence level for the correlation between $3 - 4.6\sigma$ which could be a concern as in this range there are lot of possibility of a false alarm, so for claiming detection with higher confidence of finding GWB, 5σ would be required.

4.3 LIGO-Virgo-Kagra

Another prominent collaboration for GW detection is LIGO-Virgo-Kagra (LVK), which aims at the detection of GWs frequency and cooperate for data combining to get higher confidence in the sensitivity of the detection of weaker sources or distant sources of GWs and also to have more precision in parameters of source . LIGO, the Laser interferometer gravitational wave Observatory, situated in Hanford and Livingston are two L shaped Michelson interferometers of 5 km length. Virgo, a detector of GW having 3 km log interferometer is situated in Pisa, Italy and Kagra, an underground observatory of same 3km length is situated in Kamioka, Japan. The collaboration has been detecting GWs from the merger of black holes, neutron stars.

The bounds on detection on O1 run was taken on flat ω_{GW} from [Abbott et al. [2016]] was $< 1.7 \times 10^{-7}$ which changed for later runs and became $< 2.7 \times 10^{-8}$ for [Abbott et al. [2021]].The improvement that is achieved is due to marginalizing the slope where as earlier $n_T = 0$ was used.

4.3.1 O1:First detection of GW

The first run from Sept 18th 2015 to 12th Jan, 2016 by LIGO was successful with first detection of GW, done on Sept 14, 2015 by the two detectors of LIGO, which observed signal of source (GW150914) [Abbott et al. [2016]]. The frequency increased and was detected from 35 to 250 Hz having GW strain of 1×10^{-21} which matched

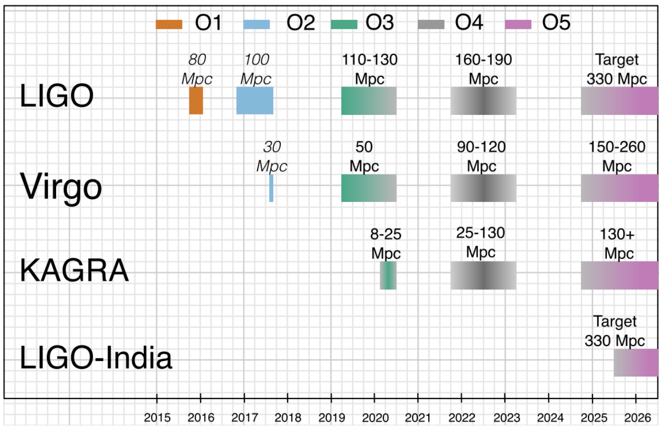

Figure 4.6: The picture shows planned sensitivity for evolution of runs by LIGO-Virgo-Kagra collaboration, Abbott et al. [2020]

with the GR prediction of waveform from merging binary black hole. The SNR of the observation was of 24, for observation occurring for just 10-ms of propagation time. Later GW151012 and GW151226 were also detected with Binary Neutron Star(BNS) range of 80 Mpc, see for reference [Abbott et al. [2020]]

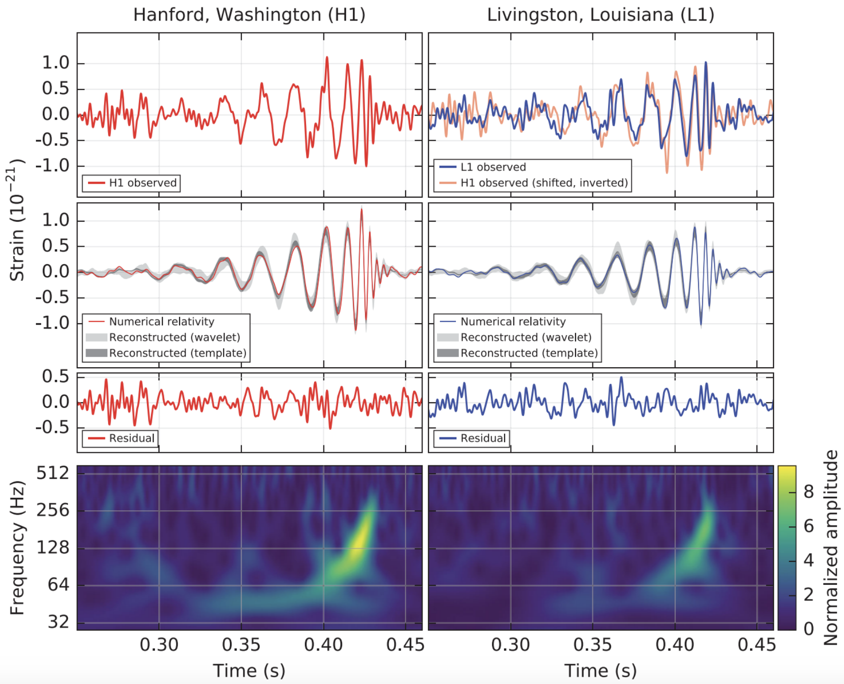

Figure 4.7: The plots show comparison of the detected data from merger at the two LIGO sites. The observed signal overlaps with high precision and graphs show increase in frequency as a result of merging and suddenly a drop in the frequency showing the end of merger, forming a single black hole.[Image credit: Abbott et al. [2016]

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4.3.2 O2: Advanced VIRGO + LIGO

The second run started on 30th November 2016 by LIGO and completed on 25th Aug 2017. The sensitivity which was achieved during this run was of BNS range 80-100 Mpc. On Aug 1 2017 Virgo joined, allowing three detectors to work in the last month. This run detected 8 GWs signal out of with 5 were localized by all three detectors making them precise with 90% credibility in angle with the ability to look for neutrino counterparts as well as electromagnetic counterparts from very high energy to the radio band.

4.3.3 O3:LIGO-Advanced Virgo and Kagra

The third observational run for GW began on April 1 2019 and ended on March 27th 2020 with a gap of a month in between. There was an increase in sensitivity and expectation for 120 Mpc target but detectors at Livington and Hanford ran at BNS rage of 130 and 110 Mpc respectively and Advance Virgo ran at BNS range of 50 Mpc at the start while Kagra gave range of 8-25 Mpc.
Later runs such as O4 and O5 will see a larger sensitivity drawing larger range of BNS for all detectors. Advance Virgo is set to get updated and there are plans to use the observation of LIGO-India later which will be operational from 2025.

Chapter 5

Effective $T_{\mu\nu}$ of primordial gravitational waves

In this chapter, we will compute the EMT related to the fluctuating gravitational waves described by a power law spectrum. The formulation for EMT in terms of Bessel function would be our main aim, deriving out of the energy density and the effective pressure term coming out of the nonlinear effect, which takes place due to stretching of GW beyond the Hubble radius at the time of inflation and re enters at the time between radiation and matter domination epoch (see for instance [**?**]). We will also investigate the equation of state that would follow in the later time of radiation and matter domination and compare with the other results.

It has been thirty years for development of the strong research interest in back reaction of gravitational waves due to the external field. Treating the back reaction from self consistency of GWs is even more difficult which comes out of second order which allows non linear interaction, from the invariance coming through the coordinate transformation.

5.1 Results of EMT with Bessel functions

We can assume that the scale factor evolves as a function of conformal time which is dependent on β which is written as

$$a = a_* \eta^\beta \tag{5.1}$$

where

$$\beta = \frac{2}{1+3w} \tag{5.2}$$

where w is the parameter of state. With this definition of scale factor, the homogeneous solution for the tensor modes (eq. 2.92) can be framed in terms of Bessel functions as

$$h_k = A_k \frac{J_{\beta-\frac{1}{2}(k\eta)}}{(k\eta)^{\beta-\frac{1}{2}}} + B_k \frac{N_{\beta-\frac{1}{2}(k\eta)}}{(k\eta)^{\beta-\frac{1}{2}}} \tag{5.3}$$

The first type of Bessel function is the solution to the growing mode and we keep only this, leaving the second type. We parameterize the tensors related to inflationary spectrum

$$A_k^2 = \tilde{A}^2 k^{-3+n_T} \tag{5.4}$$

Now our motiveis to compute EMT of GWs i.e. energy density and effective pressure, in terms of Gamma function. We can now proceed to the integration part preserving general β and n_T dependence.
In Fourier space the energy density is written as eq. (3.40) and the effective pressure can be written as eq. (3.41), as we want it to keep just the growing mode of solution so now h_k and its derivative can be expressed as

$$\begin{aligned} h_k &= A_k \frac{J_{\beta-\frac{1}{2}}(k\eta)}{(k\eta)^{\beta-\frac{1}{2}}} \\ h_k' &= -A_k \frac{J_{\beta+\frac{1}{2}}(k\eta)}{(k\eta)^{\beta-\frac{1}{2}}} \end{aligned} \tag{5.5}$$

With the assumption of power law expansion in conformal time of scale factor as in eq. 5.23, we can write

$$\mathcal{H} = \frac{\beta}{\eta} \tag{5.6}$$

to get the value of energy density and momentum we need to compute

$$k^2|h_k'|^2 \quad k^4|h_k|^2 \quad k^2|h_k' h_k| \tag{5.7}$$

from eq. (5.2) we can find that for radiation domination ($w = \frac{1}{3}$) $\beta = 1$ and for matter domination($w = 0$) era $\beta = 2$.
Using eq. (5.4), eq. (5.5) and eq. (5.16) we compute for eq. (5.17) individually

$$\begin{aligned} k^2|h_k'|^2 &= \tfrac{k^{2+n_T-2\beta}}{\eta^{2\beta-1}} \tilde{A}^2 J^2_{\beta+\frac{1}{2}}(k\eta) \\ k^4|h_k|^2 &= \tfrac{k^{2+n_T-2\beta}}{\eta^{2\beta-1}} \tilde{A}^2 J^2_{\beta-\frac{1}{2}}(k\eta) \\ k^2|h_k h_k'| &= -\tfrac{k^{n_T-2\beta+1}}{\eta^{2\beta-1}} \tilde{A}^2 (J_{\beta-\frac{1}{2}}(k\eta))(J_{\beta+\frac{1}{2}}(k\eta)) \end{aligned} \tag{5.8}$$

Now we use the integral [**?**,see eq. (2) of 2.12.31]

$$\int_0^\infty x^{\alpha-1} J_\mu(cx) J_\nu(cx) dx = 2^{\alpha-1} c^{-\alpha} \Gamma^{1-\alpha,(\alpha+\nu+\mu)/2}_{1+(\mu-\nu-\alpha)/2, 1+(\nu+\mu-\alpha)/2, 1+(\nu-\mu-\alpha)/2} \tag{5.9}$$

where J is Bessel function of order μ and ν, when

$$Re \propto < 1; \quad Re(\alpha, \nu, \mu) > 0$$

Taking the above equation for solving Bessel function and eq. (5.8), eq. (5.8) and eq. (5.13) into account to solve the integral simultaneously, we get,

$$\int_0^\infty dk k^2 |h_k'|^2 = \frac{\tilde{A}^2 \eta^{-n_T-2}}{2^{2\beta-n_T-2}} \frac{\Gamma(2\beta-n_T-2)\Gamma(2+\frac{n_T}{2})}{\Gamma^2(\frac{2\beta-n_T-1}{2})\Gamma(2\beta-\frac{n_T}{2})} \tag{5.10}$$

$$\int_0^\infty dk k^4 |h_k|^2 = \frac{\tilde{A}^2 \eta^{-n_T-2}}{2^{2\beta-n_T-2}} \frac{\Gamma(2\beta-n_T-2)\Gamma(1+\frac{n_T}{2})}{\Gamma^2(\frac{2\beta-n_T-1}{2})\Gamma(2\beta-\frac{n_T}{2}-1)} \tag{5.11}$$

$$\int_0^\infty dk k^2|h'_k h_k| = \frac{\tilde{A}^2\eta^{-n_T-1}}{2^{2\beta-n_T-1}}\frac{\Gamma(2\beta-n_T-1)\Gamma(1+\frac{n_T}{2})}{\Gamma(\frac{2\beta-n_T-1}{2})\Gamma(\frac{2\beta-n_T+1}{2})\Gamma(2\beta-\frac{n_T}{2})} \tag{5.12}$$

Using eqs. (3.40), (3.41), eq. (5.17), eq. (5.14),eq. (5.11) and eq. (5.12), we calculate the energy density and the effective pressure of the gravitational waves, preserving general dependence on quantity β and n_T with the condition of n_T ranging from -2 to $2\beta-2$ imposed just for high frequency limit. The energy density eq. (3.40) can now be written as

$$\bar{\rho}_{GW} = \frac{M_{Pl}^2}{4\pi^2 a^2}\int_0^\infty k^2dk[|h'_k|^2 + k^2|h_k|^2 + 8\mathcal{H}|h'_k h_k|]$$

$$= \frac{M_{pl}^2\tilde{A}^2\eta^{-2\beta-n_T-2}}{a_*^2\pi^2 2^{2\beta-n_T-1}}[\frac{\Gamma(2\beta-n_T-2)\Gamma(2+\frac{n_T}{2})}{\Gamma^2(\frac{2\beta-n_T-1}{2})\Gamma(2\beta-\frac{n_T}{2})} + \frac{\Gamma(2\beta-n_T-2)\Gamma(1+\frac{n_T}{2})}{\Gamma^2(\frac{2\beta-n_T-1}{2})\Gamma(2\beta-\frac{n_T}{2}-1)} - 4\beta\frac{\Gamma(2\beta-n_T-1)\Gamma(1+\frac{n_T}{2})}{\Gamma(\frac{2\beta-n_T-1}{2})\Gamma(\frac{2\beta-n_T+1}{2})\Gamma(2\beta-\frac{n_T}{2})}] \tag{5.13}$$

and the effective pressure of the EMT eq. (3.41) can be written as

$$\bar{p}_{GW} = \frac{M_{Pl}^2}{4\pi^2 a^2}\int_0^\infty k^2dk[-\frac{5}{3}|h'_k|^2 + \frac{7}{3}k^2|h_k|^2 + \left(\frac{1+\beta}{3}\right)|h'_k h_k|]$$

$$= \frac{M_{pl}^2\tilde{A}^2\eta^{-2\beta-n_T-2}}{a_*^2\pi^2 2^{2\beta-n_T-1}}[-\frac{5\Gamma(2\beta-n_T-2)\Gamma(2+\frac{n_T}{2})}{3\Gamma^2(\frac{2\beta-n_T-1}{2})\Gamma(2\beta-\frac{n_T}{2})} + \frac{7\Gamma(2\beta-n_T-2)\Gamma(1+\frac{n_T}{2})}{3\Gamma^2(\frac{2\beta-n_T-1}{2})\Gamma(2\beta-\frac{n_T}{2}-1)} + \left(\frac{1+\beta}{3}\right)\frac{\Gamma(2\beta-n_T-1)\Gamma(1+\frac{n_T}{2})}{\Gamma(\frac{2\beta-n_T-1}{2})\Gamma(\frac{2\beta-n_T+1}{2})\Gamma(2\beta-\frac{n_T}{2})}] \tag{5.14}$$

Now we can compute the effective equation of state that comes from the ratio of effective pressure and density,

$$w_{eff} = \frac{[-\frac{5\Gamma(2\beta-n_T-2)\Gamma(2+\frac{n_T}{2})}{3\Gamma^2(\frac{2\beta-n_T-1}{2})\Gamma(2\beta-\frac{n_T}{2})} + \frac{7\Gamma(2\beta-n_T-2)\Gamma(1+\frac{n_T}{2})}{3\Gamma^2(\frac{2\beta-n_T-1}{2})\Gamma(2\beta-\frac{n_T}{2}-1)} + \left(\frac{1+\beta}{3}\right)\frac{\Gamma(2\beta-n_T-1)\Gamma(1+\frac{n_T}{2})}{\Gamma(\frac{2\beta-n_T-1}{2})\Gamma(\frac{2\beta-n_T+1}{2})\Gamma(2\beta-\frac{n_T}{2})}]}{[\frac{\Gamma(2\beta-n_T-2)\Gamma(2+\frac{n_T}{2})}{\Gamma^2(\frac{2\beta-n_T-1}{2})\Gamma(2\beta-\frac{n_T}{2})} + \frac{\Gamma(2\beta-n_T-2)\Gamma(1+\frac{n_T}{2})}{\Gamma^2(\frac{2\beta-n_T-1}{2})\Gamma(2\beta-\frac{n_T}{2}-1)} - 4\beta\frac{\Gamma(2\beta-n_T-1)\Gamma(1+\frac{n_T}{2})}{\Gamma(\frac{2\beta-n_T-1}{2})\Gamma(\frac{2\beta-n_T+1}{2})\Gamma(2\beta-\frac{n_T}{2})}]} \tag{5.15}$$

From here we can describe the effective equation of state at the post inflatory epochs. By using the value of β. We get the EoS for radiation domination $\beta = 1$ era as

$$w_{eff}^{RD} = \frac{[-\frac{5\Gamma(-n_T)\Gamma(2+\frac{n_T}{2})}{3\Gamma^2(\frac{1-n_T}{2})\Gamma(2-\frac{n_T}{2})} + \frac{7\Gamma(-n_T)\Gamma(1+\frac{n_T}{2})}{3\Gamma^2(\frac{1-n_T}{2})\Gamma(1-\frac{n_T}{2})} + \frac{2}{3}\frac{\Gamma(1-n_T)\Gamma(1+\frac{n_T}{2})}{\Gamma(\frac{1-n_T}{2})\Gamma(\frac{3-n_T}{2})\Gamma(2-\frac{n_T}{2})}]}{[\frac{\Gamma(-n_T)\Gamma(2+\frac{n_T}{2})}{\Gamma^2(\frac{1-n_T}{2})\Gamma(2-\frac{n_T}{2})} + \frac{\Gamma(-n_T)\Gamma(1+\frac{n_T}{2})}{\Gamma^2(\frac{1-n_T}{2})\Gamma(1-\frac{n_T}{2})} - 4\frac{\Gamma(1-n_T)\Gamma(1+\frac{n_T}{2})}{\Gamma(\frac{1-n_T}{2})\Gamma(\frac{3-n_T}{2})\Gamma(2-\frac{n_T}{2})}]} \tag{5.16}$$

Similarly by using the value of β. We get the EoS for radiation domination era $\beta = 2$ as

$$w_{eff}^{MD} = \frac{[-\frac{5\Gamma(2-n_T)\Gamma(2+\frac{n_T}{2})}{3\Gamma^2(\frac{3-n_T}{2})\Gamma(4-\frac{n_T}{2})} + \frac{7\Gamma(2-n_T)\Gamma(1+\frac{n_T}{2})}{3\Gamma^2(\frac{3-n_T}{2})\Gamma(3-\frac{n_T}{2})} + \frac{\Gamma(3-n_T)\Gamma(1+\frac{n_T}{2})}{\Gamma(\frac{3-n_T}{2})\Gamma(\frac{5-n_T}{2})\Gamma(4-\frac{n_T}{2})}]}{[\frac{\Gamma(2-n_T)\Gamma(2+\frac{n_T}{2})}{\Gamma^2(\frac{3-n_T}{2})\Gamma(4-\frac{n_T}{2})} + \frac{\Gamma(2-n_T)\Gamma(1+\frac{n_T}{2})}{\Gamma^2(\frac{3-n_T}{2})\Gamma(3-\frac{n_T}{2})} - 8\frac{\Gamma(3-n_T)\Gamma(1+\frac{n_T}{2})}{\Gamma(\frac{3-n_T}{2})\Gamma(\frac{5-n_T}{2})\Gamma(4-\frac{n_T}{2})}]} \tag{5.17}$$

5.2 EMT of de Sitter Universe

To solve for EMT of de Sitter, we start by using the Einstein equation, the line element and action which is written as

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}[R - 2\Lambda] \tag{5.18}$$

The gravitational waves are traceless and transverse. We proceed for computing the EMT Finelli et al. [2005]

$$T^{GW}_{\mu\nu} = -M^2_{Pl}\left[R^2_{\mu\nu} - \frac{1}{2}(g_{\mu\nu}g^{\alpha\beta}R_{\alpha\beta})^{(2)}\right] \tag{5.19}$$

As soon as we use the equation of motion 2.92 of the first order on the above equation, it takes the form

$$T^{GW}_{\mu\nu} = -M^2_{pl}(R^{(2)}_{\mu\nu} - \frac{1}{2}g^{(0)}_{\mu\nu}g^{(0)\alpha\beta}R^{(2)}_{\alpha\beta}) \tag{5.20}$$

Here the superscript (2) describes the quadratic in the perturbation of h_{ij}. The Einstein tensor for the second order is described in the eqs. (3.19-3.22) where we have taken the derivative with respect to the conformal time . The action can be described in a background value and a second piece ($S = S^{(0)} + S^{(2)}$) associated with it which is

$$S^{(2)} = \frac{M^2_{Pl}}{8}\int d^4xa^3[\dot{h}^{mn}h_{mn} - \partial_k h_{mn}\partial^k h^{mn}] \tag{5.21}$$

with this we use the h_{ij} given in eq. (3.42). We find that amplitude of h_s, k, though these two equation satisfy the massless minimally coupled continuity equation $\ddot{h}_{s,k} + 3H\dot{h}_{s,k} + \frac{k^2}{a^2}h_{s,k} = 0$ and it gives plane wave as the solution for short wavelengths which is written as

$$h_{s,k} = \frac{1}{a^{3/2}M_{Pl}}\left(\frac{\pi}{2H}\right)^{1/2} H^{(1)}_{3/2}(-k\eta) \tag{5.22}$$

where $H_{\frac{3}{2}}$ is Hankel function of order 3/2, so we can say that the solution is valid for all value but just not k=0 where it becomes pure gauge which is independent of space, which when averaged for vacuum state, given the EMT which is conserved by eq. (3.36) and the solution we obtain for energy density

$$\bar{\rho}_{GW} = \sum_s M^2_{Pl}\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{4}|\dot{h}_{s,k}|^2 + \frac{1}{4}\frac{k^2}{a^2}|h_{s,k}|^2 + H(\dot{h}_{s,k}h_{s,k} + h^*_{s,k}\dot{h}^*_{s,k}h^*_{s,k})\right] \tag{5.23}$$

and pressure becomes

$$p_{GW} = \sum_s M^2_{Pl}\int\frac{d^3k}{(2\pi)^3}\left[-\frac{5}{12}|\dot{h}_{s,k}|^2 + \frac{7}{12}\frac{k^2}{a^2}|h_{s,k}|^2\right] \tag{5.24}$$

where $h_{s,k}$ are solutions of eq. (5.22).In eq. (5.23) the last term corresponds to $Hh\dot{h}$, this formula bears resemblance to scenarios where a scalar field is non-minimally coupled to gravity. Adding a mass term to above equation of energy density and pressure, which allows EMT to be conserved covariantly, given by eq. (3.36) and with renormalization, the final EMT for de Sitter becomes

$$\langle T^{GW}_{\mu\nu}\rangle = -g_{\mu\nu}\frac{361}{960\pi^2}H^4 \tag{5.25}$$

is the principle outcome and from this the earlier assertion is refused. Essentially, the results implies that, contrary to what previous studies had suggested, the energy

density of GWs in the invariant vacuum positively contributes to the cosmological constant in de Sitter space-time. This renormalized EMT contributes to have parameter of state as $w = -1$.
The perturbation amplitude $h_{s,k}$ from the Finelli et al. [2005], which is displayed in eq. (5.21), can now be compared to h_k, which we have used in eq. (5.5), which comes from the growing mode of Bessel function. So, eq. (5.21) uses Hankel function which can be written as

$$H_\alpha = J_\alpha + iY_\alpha \tag{5.26}$$

where J_α is the Bessel function of first kind and Y_α is Bessel function of the second kind. In de Sitter case, we have used the ultraviolet limit ($k >> aH$) where $k\eta \to \infty$ for super Hubble modes the equation turns to

$$h_{s,k} \to \frac{1}{a^{3/2} M_{Pl}} \left(\frac{\pi}{2H}\right)^{1/2} (+i) Y_{3/2}(k\eta) \tag{5.27}$$

which is exactly the eq. (5.21). Furthermore when we move to infrared limit ($-k\eta \to 0$) for which $k << aH$, the Y_α can be written as

$$Y_\alpha \quad - \frac{\Gamma(\alpha)}{\pi} \left(\frac{2}{-k\eta}\right) \tag{5.28}$$

using the above equation we can re write eq. (5.21) for sub Hubble modes

$$\begin{aligned} h_{s,k} &\to \frac{1}{a^{3/2} M_{Pl}} \left(\frac{\pi}{2H}\right)^{1/2} (+i) \left[\frac{-\Gamma(3/2)}{\pi}\right] \left(\frac{2}{-k\eta}\right)^{3/2} \\ &\to \frac{H^{3/2}}{M_{Pl}} \left(\frac{\pi}{2H}\right)^{1/2} (+i) \left[\frac{\Gamma(3/2)}{\pi}\right] \left(\frac{2}{k}\right)^{3/2} \end{aligned} \tag{5.29}$$

For our case of h_k eq. (5.5) explicitly for sub Hubble modes ($k << aH$) becomes

$$h_k(\eta) = A_k \frac{1}{\Gamma(\beta + \frac{1}{2})} \frac{1}{2^{\beta - \frac{1}{2}}} \tag{5.30}$$

We can find from both the eq. (5.29) and eq. (5.30) that for the de Sitter case in the infrared limit the modes become a constant in sub Hubble radius as the dependence of η is eliminated by itself, and by comparison of there two, by putting the value of $\beta = -1$ in eq. (5.30) for de Sitter, we find that

$$A_k \approx \frac{H}{M_{Pl}} \left(\frac{\pi}{2}\right)^{1/2} (+i) \left(-\frac{1}{k^{3/2}}\right) \tag{5.31}$$

The complexity induced through the eq. (5.22) for finding the solution of perturbation h gets must easy and simpler by taking Bessel function (eq. (5.3)) into account, providing the same result. A very minute difference may be seen sometimes but that comes as we took just the growing function into account.
The de Sitter case has been a benchmark of previous studies [**?**,Finelli et al. [2005]]. In this case, the only inhomogeneities are gravitational waves since scalar field fluctuations are only Gauge modes in presence of a cosmological constant. This case can be used to evaluate our final equation coming out of Bessel function to verify if our equation is consistence with the benchmark.

5.3 Comparison with de Sitter results

To check if the eq. (5.15) is correct, we use the de Sitter case, by inserting $\beta = -1$ and $n_T = 0$ to find the parameter of state (w) of de Sitter universe, so the equation becomes

$$w_{eff}^{dS} = \frac{-\frac{5\Gamma(-4)\Gamma(2)}{3\Gamma^2(-\frac{3}{2})\Gamma(-2))} + \frac{7\Gamma(-4)\Gamma(1)}{3\Gamma^2(-\frac{3}{2})\Gamma(-3))}}{\frac{\Gamma(-4)\Gamma(2)}{\Gamma^2(-\frac{3}{2}\Gamma(-2))} + \frac{\Gamma(-4)\Gamma(1)}{\Gamma^2(-\frac{3}{2})\Gamma(-3))} + \frac{4\Gamma(-3)\Gamma(1)}{\Gamma(\frac{-3}{2})\Gamma(\frac{-1}{2})\Gamma(-2)}} = -1 \qquad (5.32)$$

Using the properties of Gamma function the equation provides the parameter of state for de Sitter as -1 which is accurate proving the correctness of the equation.

5.4 Discussion

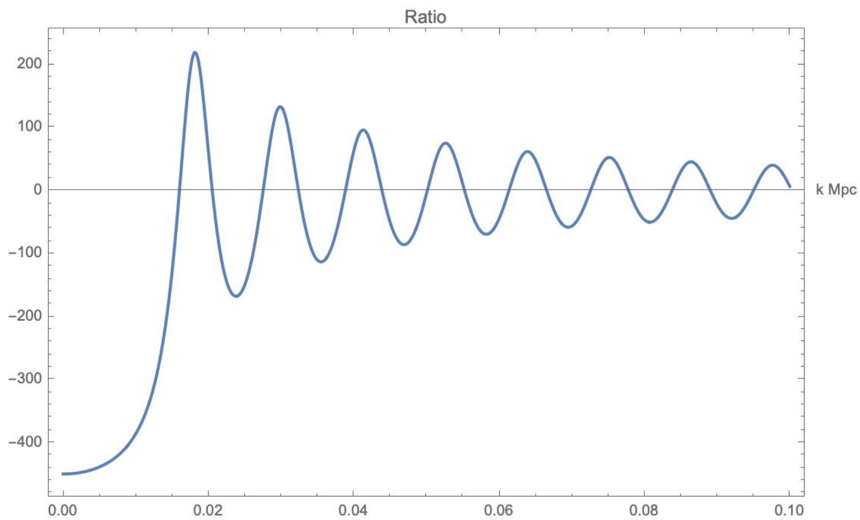

Figure 5.1: The graph shows effective energy density's ratio of mixed term over Laplacian and Kinetic term

The plot shows the ratio of the mixed term $Hh\dot{h}$ of the effective energy density eq. (5.13) over the laplacian and kinetic term of the equation. The ratio has been taken considering the spectral tilt ($n_T = 0$) at the time of CMB recombination which happens at $\frac{\eta}{Mpc} = 280.76$ at a redshift of ($z = 1100$) which falls in matter domination era and so $\beta = 2$ has been taken in account for calculation of Bessel function. We can find thought the graph that the effective term is prominent for long wavelengths and it should not be neglected.

Chapter 6

Conclusions

The discovery of gravitational waves in 2016 [Abbott et al. [2016]] has verified one of the predictions of GR and opened the era of gravitational wave astronomy. In this thesis we have examined the effective energy-momentum tensor of primordial gravitational waves as function of the wavelength - or of the frequency - and its integral. This research work underscores the significance of GWs which exit the Hubble radius during the inflation and later re-enter during radiation or matter domination era. The nonlinear effects due to GR, originating from second order terms, quadratic in the amplitude of primordial gravitational waves induce an EMT of these GWs [Abramo et al. [1997], Abramo [1999]]. An extra term of the form $Hh\dot{h}$ appear, leading to differences with respect to the standard formula for GWs (see [Maggiore [2018]] for instance).

We have computed the EMT by considering only the growing mode of GWs and found that this extra term is not negligible with respect to the standard terms, i.e. the sum of the kinetic term and laplacian term. See (Fig. 5.1) for the ratio of this term of the EMT over the standard terms at the time of CMB recombination. We have then computed the integrated EMT and found that for scale invariant GWs produced during inflation, we have found that this extra term is $-8/3$ of the sum of the kinetic and laplacian terms, for matter dominated era. The effective parameter of state of the integrated EMT is given in eqs. (5.16) and (5.17), based on the dominance era of reentry.

We have also compared our computation of the EMT by using only the growing mode against the regularized EMT in de Sitter space-time [Finelli et al. [2005]], which can be recovered by $\beta = -1$ and $n_T = 0$. We have used the benchmark of de Sitter spacetime, in which the scalar fluctuation are pure gauge modes due to the presence of the sole cosmological constant and GWs are the only propagating physical degrees of freedom. Consequently during our comparison, we find that the effective additional term nullifies following de Sitter parameters, and found a parameter of state $w = -1$, as for the regularized case in [Finelli et al. [2005]]. This increase our confidence in our calculation with only the growing mode for the post-inflationary era and encourage us to compute the phenomenological implication of the full EMT also at the time of Big Bang Nucleosynthesis for any n_T.

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