Alma Mater Studiorum · University of Bologna

**School of Science**
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**Master Degree in Physics**

# Turnaround radius in Horndeski's theory

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Academic Year 2022/2023

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**Abstract**

The evaluation of the recent concept of turnaround radius could be helpful in discriminating between the various theories of gravity; after a recap of the scalar-tensor theories, with particular relevance to the Horndeski's theory, the concept of mass is exploited in order to give an alternative definition to the turnaround radius. This recent definition has proven more useful than the standard one, mainly because it turned out to be gauge-invariant. Despite its unshakable differences with the standard definition using radial time-like geodesics, the hope is that such differences, as underlined in simpler cases, are small compared to the cosmological quantities involved. Thus, one can compare these simpler (and gauge-independent) predictions of the turnaround radius for the various theories of gravity with future data, finally finding which one will fits better them.

# Contents

# Introduction

The Einstein's General Relativity is one of the most successful physical (or, better, scientific) theories of human being; its results were fundamental in the explanation of some phenomena that previously did not have with the only help of the Newtonian gravity and even more are the applications to our technologies and space research. However, as any other scientific theory, it could be not enough to describe the whole behaviour of our Universe, even though, nowadays, a big effort both in theoretical and experimental setup is made in order to preserve this status. The General Relativity is extremely accurate when we talk about our Solar system (it correctly predicts the Mercury's perihelion precession, the deflection of a far star's light ray by the Sun and, as one of the latest outcomes, it finally has found confirmation in gravitational waves) but its predictability is starting to be weaker, specially when we look at larger scales. The prediction of the rotation velocity of the low surface brightness galaxies (simply LSB galaxies) is only one of the main problems that the theory has encountered on his way: the observed mass is not enough to justify the very large velocity of such galaxies [1]. The attempt to explain such a strange effect still on the light of General Relativity naturally leads to the conclusion that, probably, the actual mass of such systems has to be larger than the observed one: this is how dark matter was born. Many experimental attempts tried and continue to try to find a sign of the existence of this type of matter that exhibits no interaction with the electromagnetic spectrum and even more effort was made only to provide it of a theoretical particle composition, since, by definition, no particles of the Standard Model are useful to the scope [2]. Another way to explain the observational issues found in LSB galaxies' rotation and strong gravitational lensing, is to reject the General Relativity as the fundamental theory of gravity. This would be a strong requirement given the exceptional result that this theory has brought to humanity but, at the same time, opens a new and very vast scenario; indeed, there are pletora of modified gravity theories which nowadays try to explain dark matter effects (and even dark energy) without the need of adding it. The problem is how one could choose between the various theories of gravity, General Relativity included, and why. In this thesis, after a recap of one of the most prominent extended theories of gravity, the scalar-tensor theories, we first give a definition to the mass in the context of a generic gravity theory and then we define the turnaround radius, in the hope that a comparison between its predicted value in the various theories and its observed one may help to understand the way to the "correct" theory of gravity. The turnaround radius, in a certain celestial cluster, is nothing else than that radius value (from an appropriate centre of mass) in which the gravitational attraction due to local sources and the expansion of the Universe cancel each other; therefore, the natural definition of such a quantity is given by considering radial geodesics, evaluating so where the radial acceleration vanishes. However, some discussions about the gauge-invariance of such an approach (since the radial acceleration is calculated in a perturbed FLRW metric,

which is a more realistic metric than the standard FLRW, that does not consider local inhomogeneities), lead to define the turnaround radius in a slighter different way by the use of the mass definition. Despite the fact that the results of such a calculation are mathematically different from the ones made by using the standard definition, one can argue that such difference, specially in astronomical observations and measurements, is not too large to be detected. The advantages of this alternative definition, on the other hand, are that the calculation is simpler but, most of all, the result is certainly gauge-invariant (the mass is a scalar).
The goal of such a thesis therefore is to calculate, whenever possible, the turnaround radius by using the standard definition and compare it with the one computed with the alternative mass definition, with a final application to the Horndeski's theory. The hope, so, is that in the near future the various and different predictions on the turnaround radius made by different gravitational theories, i.e. General Relativity (plus dark matter) and scalar-tensor theory, may be compared with the observations: if one of these will match better, then maybe we could say that we are on the right way to discover the true theory of gravity. However, before becoming too enthusiastic, one should keep his foot on the ground because we have to remind the fact that we do not have a theory of gravity that matches both with data and the realm of Quantum Mechanics; in fact, the theories of gravity are here treated as classical, in the sense that we describe them with a Lagrangian and their field equations are obtained by means of the Hamilton's least action principle. Thus, even if a classical theory of gravity will be found, what will remains is understanding how to quantize it.

# Chapter 1

# Scalar-tensor theories

One of the major benefits that Einstein took with his theory is the geometrical interpretation of the gravity. Until then, Newton's law was the universally accepted theory of gravity and, even in its later formulation, which can be resumed by the Poisson equation, the gravity was thought as a scalar field potential. Slighter modifications of Newtonian's theory began to appear, specially after the discover of Special Relativity, trying to build up a covariant formulation of the gravity and, mainly, solve the problem of the instantaneous change of the scalar potential, forbidden by the emergent theory: a natural guess, made by Nordstrom, was to modify the laplacian of the Poisson equation into a dalambertian. However, still no relationship appears between the gravity and the whole geometry of the space-time.
One of the most discussed statements in Newtonian mechanics was the equivalence of inertial and gravitational masses: when they face up by means of Newton's second law, they simplify each other; however, the only reason to do it was of experimental nature, that is, the mass observed in the kinematical motions has the same value of that undergoing gravitational forces. The starting point of Einstein's General Relativity returns to this discussion, formulating the well known Equivalence Principle. Today, one can state at least two formulations of this principle, a stronger and a weaker one:

> *Strong Equivalence Principle: in any gravitational field, there always exists a local reference frame in which all gravitation effects vanish.*
>
> *Weak Equivalence Principle: the inertial mass, that is the property of any material body to react to motion variations, and the gravitational mass, that is the property of any material body to be a source of or suffer a gravitational field, are numerically equivalent.*

The strong automatically satisfies the weak but the viceversa is not true. In particular, the strong formulation assigns to the gravity a pure geometrical formulation: there is no space for scalar fields, because if one exists, it cannot be gauged away by a simple transformation of coordinates. So, Einstein formulates its famous field equations assuming as the only dynamical field the metric tensor. Nevertheless, the strong principle is in disagreement with another general statement, the Mach's principle:

> *Mach's principle: the inertia of every systems is the result of the interaction of the system itself with the rest of the Universe.*

In a local inertial system, however, by means of Strong Equivalence Principle, it seems that no interaction appears. The solution is to sacrifice one of the two principles (preserving anyway the Weak Equivalence Principle). Neglecting the strong formulation, the natural way to restore the Mach's principle is reintroducing, beyond the metric tensor, the scalar field, as Dicke and his PhD student Brans made for the first time: this was the birth of the scalar-tensor theories of gravity [3].
In the first section, the Einstein field equations, the way how they are obtained and its slighter modifications are briefly discussed; in the second one, the main features of scalar-tensor theories are explored, with particular relevance to the possible origins of such a scalar field; in the final section, some limitations to these theories are considered, leading to the general Horndeski's theory.

## 1.1 Einstein General Relativity and beyond

The Einstein equations were derived in two different ways and almost in the same period by Einstein and Hilbert; the reason why General Relativity is universally attributed to Einstein is accountable to the fact that Hilbert formulation was a pure mathematical construction with no physical consequences derived. The Einstein's derivation concerned conservation law and the recast of the Newtonian theory in weak field limit whereas Hilbert used the variational principle. However, both agree on what the gravity field strenght should be, namely something that at least contains second derivatives of the metric and that is able to distinguish between flat and curved space-times unambiguously, that is the Riemann tensor. Moreover, a well known theorem, sometimes attributed to Vermeil, claims:

> *Vermeil's theorem: the Ricci scalar is the only invariant linear in the second derivatives of the metric tensor.*

This is the starting point of both approaches. Following the Hilbert one will be useful to have a precise procedure to derive field equations also in modified gravity theories. The variational principle make use of the action of the theory to derive its field equation by requiring that the action is stationary; in a 4-dimensional spacetime, the action, provided by its volume invariant is

$$S = \int d^4x \sqrt{-g}\mathcal{L} \tag{1.1}$$

where $\mathcal{L}$ is the Lagrangian density of the theory. One can split it in two components, one concerning the geometry and one the matter contribution: as known, its variation leads naturally to the definition of the energy-momentum tensor. By means of Vermeil's theorem, the choice of the Lagrangian density falls back up Ricci scalar and so one can write

$$\mathcal{L} = \frac{1}{2\chi} R + \mathcal{L}_M \tag{1.2}$$

where $\chi$ represents a coupling constant. The procedure to obtain the field equations is so outlined: let's make use of least action principle

$$\delta S = 0 \tag{1.3}$$

The matter Lagrangian $\mathcal{L}_M$ is a function of the metric field $g_{\mu\nu}$ as well as the Ricci scalar; thus, the variation can be made with respect to this tensor field as

$$\begin{aligned} \delta S &= \int d^4x \frac{\partial \left(\sqrt{-g}\left(\frac{1}{2\chi}R + \mathcal{L}_M\right)\right)}{\partial g^{\mu\nu}} \delta g^{\mu\nu} = \\ &= \int d^4x \sqrt{-g} \frac{1}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g}\left(\frac{1}{2\chi}R + \mathcal{L}_M\right)\right)}{\partial g^{\mu\nu}} \delta g^{\mu\nu} = 0 \end{aligned} \tag{1.4}$$

Then, the field equations can be obtained by the arbitrariness of $\sqrt{-g}\delta g^{\mu\nu}$ as

$$\frac{1}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g}\left(\frac{1}{2\chi}R + \mathcal{L}_M\right)\right)}{\partial g^{\mu\nu}} = 0 \tag{1.5}$$

Applying the Leibniz rule, one gets

$$\frac{\partial R}{\partial g^{\mu\nu}} + \frac{R}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g}\right)}{\partial g^{\mu\nu}} = -2\chi \frac{1}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g}\mathcal{L}_M\right)}{\partial g^{\mu\nu}} \tag{1.6}$$

The variation of right-hand side term with respect to the metric field, as already said, yields the energy-momentum tensor $T_{\mu\nu}$

$$-2\chi \frac{1}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g}\mathcal{L}_M\right)}{\partial g^{\mu\nu}} = \chi T_{\mu\nu} \tag{1.7}$$

The second term on the left-hand side

$$\frac{R}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g}\right)}{\partial g^{\mu\nu}} = \frac{R}{2\left(-g\right)} \frac{\partial \left(-g\right)}{\partial g^{\mu\nu}} = -\frac{1}{2} g_{\mu\nu} R \tag{1.8}$$

since

$$\frac{\partial g}{\partial g^{\mu\nu}} = -g g_{\mu\nu} \tag{1.9}$$

The first term is

$$\frac{\partial R}{\partial g^{\mu\nu}} = \frac{\partial \left(g^{\mu\nu} R_{\mu\nu}\right)}{\partial g^{\mu\nu}} = R_{\mu\nu} + g^{\mu\nu} \frac{\partial R_{\mu\nu}}{\partial g^{\mu\nu}} \tag{1.10}$$

In particular, for the last term we know

$$R_{\mu\nu} = \left(\Gamma^{\rho}_{\mu\nu}\right)_{;\rho} - \left(\Gamma^{\rho}_{\rho\nu}\right)_{;\nu} \tag{1.11}$$

Defining

$$\begin{aligned} V^{\rho} &\equiv g^{\mu\nu} \Gamma^{\rho}_{\mu\nu} \\ W^{\nu} &\equiv g^{\mu\nu} \Gamma^{\rho}_{\rho\mu} \end{aligned} \tag{1.12}$$

then

$$R_{\mu\nu} = \frac{1}{4} g_{\mu\nu} \left(V^{\rho}_{;\rho} - W^{\rho}_{;\rho}\right) \equiv \frac{1}{4} g_{\mu\nu} U^{\rho}_{;\rho} \tag{1.13}$$

So

$$g^{\mu\nu} \frac{\partial R_{\mu\nu}}{\partial g^{\mu\nu}} = g_{\mu\nu} \frac{\partial R_{\mu\nu}}{\partial g_{\mu\nu}} = \frac{1}{4} g_{\mu\nu} U^{\rho}_{;\rho} \tag{1.14}$$

However, a divergence of the vector $U \equiv V - W$ can be neglected by integrating at the action level and assuming that it vanishes at the borders. Thus, putting all together, finally Einstein field equations are obtained

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \chi T_{\mu\nu} \tag{1.15}$$

The trace of this yields

$$R = -\chi T \tag{1.16}$$

The coupling constant can be recovered by requiring that Einstein theory reproduces Newtonian one at weak field limit, easily finding that $\chi = 8\pi G$ in $c = 1$ units. The field equations can be completed by adding by hand a term which could explain the expansion of the Universe: in fact, the equations (1.15) are not enough to describe such an expansion in spatially flat space-times and so they can be modified into

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = \chi T_{\mu\nu} \tag{1.17}$$

Admitted from Einstein himself as the biggest mistake of his life, nowadays the cosmological constant is considered as the dark energy ingredient that allows the General Relativity to describe the expansion [4]. One of the main goals of the Extended theories of gravity, however, is to allow such a situation without adding by hand any term. Thus, the ways that one can proceed are two: keeping the Einstein theory of General Relativity as the fundamental theory of gravity, fine tuning $\Lambda$ and introduce dark matter or trying to go beyond it, analyzing other possible theories.

The "simplest" extensions of Einstein theory are the so-called $f(R)$ theories; indeed, one can consider any function of the Ricci scalar as Lagrangian density, maybe relaxing the linearity of second derivatives of metric tensor. In other words, the new Lagrangian density (apart from volume invariant) will be

$$\mathcal{L} = \frac{1}{2\chi} f(R) + \mathcal{L}_M \tag{1.18}$$

Having already described the variational principle method to get the field equations, one has to repeat the same steps taking care of the last term seen; in order to do not disturb the flow of the argumentation, the whole calculation is performed in the Appendix A and here the field equations are directly shown

$$f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} = \chi T_{\mu\nu} + f'(R)_{;\mu\nu} - g_{\mu\nu} \Box f'(R) \tag{1.19}$$

where a prime denotes a differentiation with respect to the curvature scalar $R$; the following, instead, is the trace equation

$$f'(R) R - 2f(R) + 3\Box f'(R) = \chi T \tag{1.20}$$

Rewriting all in terms of the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$, the field equations can be written as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{f'(R)} \left[ \frac{1}{2} g_{\mu\nu} \left( f(R) - f'(R) R \right) + f'(R)_{;\mu\nu} - g_{\mu\nu} \Box f'(R) \right] + \frac{\chi}{f(R)} T_{\mu\nu} \tag{1.21}$$

Looking at them from this perspective, the field equations so written seem like a sort of "effective" Einstein equations, introducing an extra term which one can call in full generality as

$$T^{curv}_{\mu\nu} \equiv \frac{1}{2} g_{\mu\nu} \left( f(R) - f'(R) R \right) + f'(R)_{;\mu\nu} - g_{\mu\nu} \Box f'(R) \tag{1.22}$$

giving raise to

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{f'(R)} \left( \chi T^M_{\mu\nu} + T^{curv}_{\mu\nu} \right) \tag{1.23}$$

In other words, the extra contribution can act as a source, namely as a sort of fluid which could explain several phenomena, like the expansion of Universe or the possible presence of more matter in our Universe than that observed. Another big difference with the Einstein equations is that the $f(R)$ field equations are of the fourth order (not second) and this comes clear with the following theorem which will be the starting point of the final section

*Lovelock's theorem: the Einstein field equations with cosmological constant are the only possible second-order Euler-Lagrange equations derived from a Lagrangian scalar density in 4 dimensions depending only on the metric tensor.*

Another similar but equivalent approach to obtain the field equations is using the so-called Palatini formalism [5]. The suggestion is to assume the Lagrangian depending separately on the metric tensor and on the connection; the field equations are so computed first varying the action with respect to the first and then varying with respect to the other: the result is the same but this alternative way has the advantage that we do not need to specify the connection, in a generic theory of gravity and, so, can differ from the Levi-Civita.

## 1.2 Reintroducing the scalar field

Hitherto, even in the $f(R)$ theories, the gravity was treated as a pure geometric theory; to make agreement with Mach's principle, the first suggestion is to reintroduce a scalar field, sacrificing the strong formulation of the Equivalence principle. Thus, beyond the metric tensor which represents a tensor field, the idea is to consider a scalar but what kind of field should it represent? To answer this question, Dicke suggested that the possible scalar field could be the inverse of the (effective) gravitational constant [6], attempting the following argument: the first Friedmann equation is

$$3\left(H^2 + \frac{k}{a^2}\right) = 8\pi G\rho \tag{1.24}$$

from which the value of the spatial curvature can be fixed by

$$\frac{k}{H^2 a^2} = \frac{\rho}{\rho_{crit}} - 1 \tag{1.25}$$

with $\rho_{crit} = \frac{3H^2}{8\pi G}$. What it is well known is that the Universe can be considered flat, namely $k = 0$ because the ratio $\frac{\rho}{\rho_{crit}} \sim 1$ by current observations [7]. In other words

$$\frac{\rho}{\rho_{crit}} = \frac{8\pi G\rho}{3H^2} \sim \frac{GM}{R} \sim 1 \tag{1.26}$$

where $\rho = \frac{M}{\frac{4}{3}\pi R^3}$ and $H \sim \frac{1}{R}$ are used and $R$ is the radius of the visible Universe. This would mean that

$$\frac{1}{G} \sim \frac{M}{R} \tag{1.27}$$

and so the gravitational constant is actually dependent of time as the radius of the Universe. This argumentation leads Dicke to consider the scalar field as the inverse of the gravitational constant, which now it is assumed as effective

$$\phi = \frac{1}{G_{eff}} \tag{1.28}$$

From a theoretical point of view, the introduction of a scalar in the theory occurs at the level of the Lagrangian, which now should depend on both metric field and scalar field (and, in principle, on their derivatives)

$$\mathcal{L} = \mathcal{L}\left(g_{\mu\nu}, \phi\right) \tag{1.29}$$

The part of Lagrangian depending on the metric field should be the same of General Relativity (1.2) (in order to obtain field equations which contain second derivatives of metric tensor), but there is a little but relevant difference: the gravitational constant, which naturally enters in $\chi = 8\pi G$, is now effective and represented by the scalar field $\phi$, so at Lagrangian level, it represents the non-minimal coupling between the metric and the scalar field. Thus, the most general form of the Lagrangian of the scalar-tensor theories will be

$$\mathcal{L}_{ST} = \frac{1}{16\pi}\phi\left(R + \mathcal{L}_\phi\right) + \mathcal{L}_M\left(g_{\mu\nu}, \phi, \psi\right) \tag{1.30}$$

where $\psi$ stays for non-gravitational fields. However, in order to restore at least the Weak Equivalence Principle, the matter Lagrangian must not depend on the scalar field, so let us assume

$$\mathcal{L}_M\left(g_{\mu\nu}, \phi, \psi\right) = \mathcal{L}_M\left(g_{\mu\nu}, \psi\right) \tag{1.31}$$

The form of $\mathcal{L}_\phi$ can be chosen as usual as a function of the field and only its first derivative, that is $\mathcal{L}_\phi = \mathcal{L}\left(\phi, \phi_{;\mu}\right)$: this choice is done in order to obtain, when Euler-Lagrange are applied, second order equations. Since the Lagrangian is a scalar, then the most general form of $\mathcal{L}_\phi$ is

$$\mathcal{L}_\phi = -\frac{\omega(\phi)}{\phi^2} g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - V(\phi) \tag{1.32}$$

where $\omega(\phi)$ is a parameter that initially was set constant in the Brans-Dicke theory, $V(\phi)$ is a function of $\phi$ and the scalar field at the denominator is used to restore the right unit dimensionality. Thus, the Lagrangian for scalar-tensor theories can be considered as [8]

$$\mathcal{L}_{ST} = \frac{1}{16\pi}\left(\phi R - \frac{\omega(\phi)}{\phi} g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - V(\phi)\right) + \mathcal{L}_M\left(g_{\mu\nu}, \psi\right) \tag{1.33}$$

The field equations of such a theory can be got, as usual, by applying the principle of least action; this time, however, the variation is with respect to both metric field and scalar field and so there will be a couple of field equations plus a trace equation. Combining them, as shown in the Appendix B, the field equations are

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} =& \frac{8\pi}{\phi} T^M_{\mu\nu} - \frac{\omega(\phi)}{\phi^2} \left( \phi_{;\mu}\phi_{;\nu} - \frac{1}{2} g_{\mu\nu}\phi^{;\sigma}\phi_{;\sigma} \right) + \\
&+ \frac{1}{\phi} \left( \phi_{;\mu\nu} - g_{\mu\nu}\Box\phi \right) - \frac{V}{2\phi} g_{\mu\nu}
\end{aligned}
\tag{1.34}
$$

$$
\Box\phi = \frac{1}{2\omega + 3} \left( 8\pi T^M - \frac{d\omega}{d\phi}\phi^{;\sigma}\phi_{;\sigma} + \phi\frac{dV}{d\phi} - 2V \right) \tag{1.35}
$$

So written, the field equations assume the form of "effective" Einstein equations, since one can identify

$$
T^{curv}_{\mu\nu} = -\frac{\omega(\phi)}{\phi^2} \left( \phi_{;\mu}\phi_{;\nu} - \frac{1}{2} g_{\mu\nu}\phi^{;\sigma}\phi_{;\sigma} \right) + \frac{1}{\phi} \left( \phi_{;\mu\nu} - g_{\mu\nu}\Box\phi \right) - \frac{V}{2\phi} g_{\mu\nu} \tag{1.36}
$$

as the modification of the energy-momentum tensor due to modified gravity. From these equations and the Lagrangian (1.33), it is an easy computation recasting the $f(R)$ theories [9], that are now considered as a subclass of the more general scalar-tensor theories, with the scalar field and the potential that respectively are

$$
\phi_{f(R)} = f'(R) \tag{1.37}
$$

$$
V(\phi)_{f(R)} = \phi R - f(R) \tag{1.38}
$$

It is important to outline that the present equations are obtained in the so-called Jordan frame; this clarification is due to the possibility of perform a conformal transformation such that one obtains, at Lagrangian level, a complete decouple of the Ricci scalar from the scalar field, leading to the so-called Einstein frame. In this frame the field equations appear "easier" and several results are obtained using this frame instead of Jordan one (e.g. the Birkhoff's theorem to the first order in spherically symmetric space-time) [10].
At this point, one can ask from where such a scalar field can come from; the first hint came from the Kaluza-Klein theory [11]: the dimensional reduction of a pure geometrical higher-dimensional theory of gravity results in the appearance of a scalar field, known in the literature as the dilaton. There are many other possible origins for the scalar field, such as supergravity models or string theory; in the current inflationary models, moreover, a scalar field with very low kinematical energy at early times, called inflaton, could be the responsible of the inflation [12].
To end this section, it is important to remark that the choice of the Lagrangian (1.33) was made in order to obtain "simple" equations, namely again the Ricci scalar was considered, by means of Vermeil's theorem, and only first derivatives of scalar field are considered (since the most desired form of a Lagrangian for a scalar field is basically

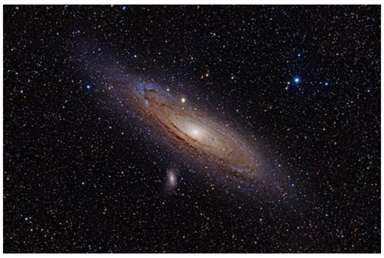
composed by the kinetic energy and the potential); however, one could consider in full generality

$$\mathcal{L} = \mathcal{L}\left(g_{\mu\nu}, g_{\mu\nu;\lambda_1}, g_{\mu\nu;\lambda_1\lambda_2}, ..., g_{\mu\nu;\lambda_1\lambda_2...\lambda_p}, \phi, \phi_{;\lambda_1}, \phi_{;\lambda_1\lambda_2}, ..., \phi_{;\lambda_1\lambda_2...\lambda_q}\right) \tag{1.39}$$

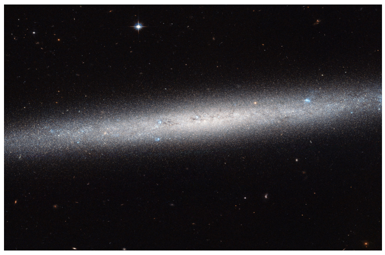
so one can go further than the second derivatives of $g_{\mu\nu}$ (and linear in them) and the first derivatives of $\phi$. Of course, the field equations will be much more involved and the question of whether such theories are somewhat physical is the main argument of the next section. Finally, one could proceed further and add to the theory also other spin-type fields, such as a vector, obtaining a scalar-vector-tensor theory.

### 1.2.1 An application to LSB galaxies

The rotation velocity of some galaxies, called Low Surface Brightness or simply LSB, has highlighted that there are notable differences between the predictions of General Relativity and the observed data [1]. These discrepancies, among others, led the scientific community to consider the possible existence of more matter than that is effectively seen. Considering spiral galaxies, such as the "near" Andromeda or the NGC 5023, we can predict the value of the rotation of the galaxy as follows.



(a) Andromeda



(b) NGC 5023

Figure 1.1: Two examples of spiral galaxies

We first consider a star which is located at the edges of the galaxy, in order to consider the weak field limit and so, one can write

$$g_{00} = 1 + \frac{2\Phi}{c^2} \tag{1.40}$$

where $\Phi$ is the gravitational potential. Considering also the Schwarzschild metric

$$ds^2 = \left(1 - \frac{R_s}{r}\right)c^2dt^2 - \frac{dr^2}{\left(1 - \frac{R_s}{r}\right)} - r^2d\theta^2 - r^2sin^2\theta d\psi^2 \tag{1.41}$$

Comparing the two previous equations, one has that the gravitational potential in the weak field limit is

$$\Phi(r) = -\frac{Gm}{r} \tag{1.42}$$

Since the Birkhoff's theorem is satisfied, the potential does not depend on time. Moreover, given a central potential, the rotational velocity modulus can be calculated as

$$v_c^2(r) = r\frac{d\Phi}{dr} = \frac{Gm}{r} \tag{1.43}$$

This formula is in agreement with Kepler's law and it is obtained considered the star as a point but it is valid also in the case of extended objects, because the Gauss's theorem ensures that the flux of the gravitational field depends solely on the mass of the internal sources and not upon their relative positions.
However, what is found is that the asymptotic behavior of the speed, correctly described by the Kepler's law, is not even measured but it seems that there is a sort of plateau, as if there were additional mass. Thus, this is one of the reasons which led scientists

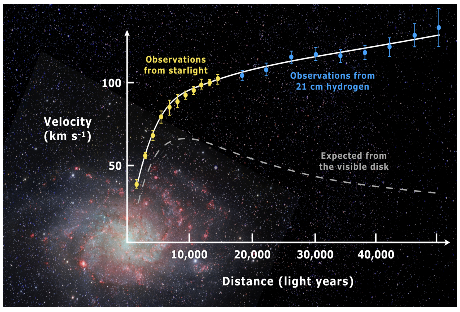



Figure 1.2: Rotation velocity: prediction and observation.

to introduce dark matter; however, another explanation can come from using another theory of gravitation, beyond the General Relativity. For instance, one can consider a $f(R)$ theory; the field equations and their trace are written in (1.21) and (1.20) and the matter energy-momentum is considered vanishing. Again, we make use of the spherical symmetry and, moreover, we assume that Birkhoff's theorem holds in such theories, so we forget about time dependencies

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\Omega^2 \tag{1.44}$$

Moreover, let us assume that, as well as for General Relativity, the following is true

$$A(r) = \frac{1}{B(r)} = 1 + \frac{2\Phi(r)}{c^2} \tag{1.45}$$

in the weak field limit. Considering the 00-component of the (1.21) and the (1.20), one gets

$$f'(R)\left(3\frac{R_{00}}{g_{00}} - R\right) + \frac{1}{2}f(R) - 3\frac{f'(R)_{;00}}{g_{00}} = 0 \tag{1.46}$$

In particular, considering a power law theory

$$f(R) = f_0 R^n \tag{1.47}$$

and substituting in the previous equation, one finds

$$\begin{gathered} R_{00}(r) = \frac{2n-1}{6n}A(r)R(r) - \frac{n-1}{B(r)}\frac{dA(r)}{dr}\frac{d\left(lnR(r)\right)}{dr} \\ \Box R^{n-1}(r) = \frac{2-n}{3n}R^n(r) \end{gathered} \tag{1.48}$$

For $n = 1$, the equation reduces to $R = 0$ and so $R_{00}(r) = 0$, which corresponds to Schwarzschild solution. These differential equations have solutions in $A(r)$ and so, in $\Phi(r)$. In order to determine the solution, one can consider the following Ansatz

$$\Phi(r) = -\frac{Gm}{2r}\left[1 + \left(\frac{r}{r_c}\right)^\beta\right] \tag{1.49}$$

where $\beta$ e $r_c$ are some parameters of the theory. The Newtonian potential is regained when $\beta = 0$; this is a sort of corrective potential so, what is remaining, is to determine $\beta$ from the system (1.48). It is shown in [13] that

$$\beta(n) = \frac{12n^2 - 7n - 1 - \sqrt{36n^4 + 12n^3 - 83n^2 + 50n + 1}}{6n^2 - 4n + 2} \tag{1.50}$$

Moreover, we can neglect those values that make the square root negative, that is the interval $n \simeq [-1.9135, -0.0194]$ and, from a physical point of view, one can require that, even if slowed down, the asymptotic behaviour is recovered, at least at far distances, so $\beta < 1$; moreover, to avoid strong increases of this potential, one can assume also $\beta > 0$ which, together with the previous condition, implies $n > 1$. Thus, the rotational velocity is

$$v_c^2(r) = r\frac{d\Phi}{dr} = \frac{Gm}{2r}\left[1 + (1-\beta)\left(\frac{r}{r_c}\right)^\beta\right] \tag{1.51}$$

where for $\beta = 0$ General Relativity is recovered. This formula says that the rotation velocity is the sum of two contributions, one of which is Newtonian and the second has a corrective nature. Tuning the two parameters, it is possible to recover the trend of the measured curve; the first one, $\beta$, selects the value of $n$ and so choose among all possible (power law) $f(R)$ theories whereas the critical radius, $r_c$, is added to take into account extended objects: in fact, now, given this form of potential, Gauss's theorem is no longer valid and, in principle, the gravitational field can depend on the mass distribution [14].

## 1.3 Horndeski's theory

We have already cited the Lovelock's theorem at the end of first section which states that the only possible second-order field equations derived from a Lagrangian density depending solely on the metric are the Einstein ones. At the light of the scalar-tensor theories, now one can ask what are the equations that are field equations derived from a Lagrangian density depending not only on the metric but also on the scalar field. On the other hand, however, the question is: why we need such restrictions? The answer lies in the so-called Ostrogradsky instability [15]:

> *A non-degenerate higher derivative Lagrangian suffers from ghost-like instabilities.*

In order to understand what are these ghosts or instabilities, let us consider a simple example of a Lagrangian depending on a scalar field, its derivative and second derivative with respect to the time

$$\mathcal{L} = \mathcal{L}\left(\phi, \dot{\phi}, \ddot{\phi}\right) \tag{1.52}$$

where $\phi$ is the scalar field and the dot represents a time derivation. The equations of motion, namely the Euler-Lagrange equations, are so

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{d^2}{dt^2}\frac{\partial \mathcal{L}}{\partial \ddot{\phi}} = \frac{\partial \mathcal{L}}{\partial \phi} \tag{1.53}$$

Considering the following Lagrangian, for instance

$$\mathcal{L} = \frac{a}{2}\ddot{\phi}^2 - V(\phi) \tag{1.54}$$

the equations of motion therefore are

$$a \ddddot{\phi} = \frac{\partial V(\phi)}{\partial \phi} \tag{1.55}$$

Now, in order to reproduce a standard Lagrangian, that is with a canonical kinetic term depending on first derivative (not second) in time, the trick is to consider an auxiliary variable $\psi$ and write the former Lagrangian as

$$\mathcal{L} = a\psi\ddot{\phi} - \frac{a}{2}\psi^2 - V(\phi) \tag{1.56}$$

where our Lagrangian is re-obtained for $\psi = \ddot{\phi}$. Using the Leibniz rule

$$\mathcal{L} = a\frac{d}{dt}\left(\psi\dot{\phi}\right) - a\dot{\psi}\dot{\phi} - \frac{a}{2}\psi^2 - V(\phi) \tag{1.57}$$

The first term, at action level, does not contribute to Euler-Lagrange equations and so can be neglected. Finally, substituting $q = \frac{\phi+\psi}{\sqrt{2}}$ and $Q = \frac{\phi-\psi}{\sqrt{2}}$, one gets

$$\mathcal{L} = -\frac{a}{2}\dot{q}^2 + \frac{a}{2}\dot{Q}^2 - U(q, Q) \tag{1.58}$$

which is the canonical form of a Lagrangian but we immediately notice the minus sign for the kinetic term of variable $q$: this represents something non-physical and it is called ghost term.

This kind of instabilities could affect also theories of gravity; General Relativity is exempt from them because, by construction, it depends at maximum on second derivatives (and linear in them) of the metric tensor and no scalar fields are included. $f(R)$ theories, instead, can include such things and, more generally, scalar-tensor theories are not constrained by anything to satisfy these criteria. The attempt to find a Lagrangian of type (1.39) that fulfills the criterion of having just second-order field equations led Gregory Horndeski to formulate its Lagrangian density in 1973 [16]. However, it went unnoticed until the last decade when it was re-discovered through an apparently very different approach, the generalized Galileon [17], whose Lagrangian was shown to be equivalent to the original Horndeski's [18]. Thus, when talking about the Horndeski's theory, we refer to the following Lagrangian

$$\begin{aligned}\mathcal{L} =& G_2(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi, X)R + \\ &+ G_{4,X}(\phi, X)\left[(\Box\phi)^2 - \phi^{;\mu\nu}\phi_{;\mu\nu}\right] + G_5(\phi, X)G^{\mu\nu}\phi_{;\mu\nu} \\ &- \frac{G_{5,X}(\phi, X)}{6}\left[(\Box\phi)^3 - 3\Box\phi\phi^{;\mu\nu}\phi_{;\mu\nu} + 2\phi_{;\mu\nu}\phi^{;\nu\lambda}\phi^{;\mu}_{;\lambda}\right]\end{aligned} \tag{1.59}$$

where $X = -\frac{1}{2}\phi_{,\mu}\phi^{,\mu}$.

Now, one could be interested in the field equations; instead of fully deriving them, which could be rather long, we try to write them in form of effective Einstein equations, like (1.34), without worrying about the form of the effective energy-momentum tensor. Thus, we are interested in the variation with respect to the metric field only of the action, that is

$$\delta_g S = \int d^4x\sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}\mathcal{L} + \frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}\right)\delta g^{\mu\nu} \tag{1.60}$$

The terms of (1.59) which, by means of the previous variation, can contribute to the Einstein tensor are those with $G_4$ and $G_5$; in particular, for $G_4$ term, the variation of Ricci scalar gives the Ricci tensor and a term that does not contribute to Einstein tensor and then we have to add the contribution of the variation of the volume invariant which gives exactly the Einstein tensor, that is adding them up we get a term $G_4\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)$. For the $G_5$ term, the variation of Einstein tensor should vanish by diffeomorphism invariance, so we are left with the volume invariant variation which gives a term $-\frac{1}{2}G_5\phi_{;\mu\nu}g_{\mu\nu}G^{\mu\nu} =$

$\frac{1}{2}G_5\phi_{,\mu\nu}R$. Thus, in the end, the effective Einstein equations for the Horndeski's theory should be something like

$$G_4R_{\mu\nu} - \frac{1}{2}\left(G_4g_{\mu\nu} - G_5\phi_{,\mu\nu}\right)R = 8\pi T^M_{\mu\nu} + T^{Horn}_{\mu\nu} \tag{1.61}$$

and, for the moment, we are not interested on the form of $T^{Horn}_{\mu\nu}$. The left hand side can be also modified as

$$\begin{aligned} G_4R_{\mu\nu} - \frac{1}{2}\left(G_4g_{\mu\nu} - G_5\phi_{,\mu\nu}\right)R =& G_4R_{\mu\nu} - \frac{1}{2}G_4\left(g_{\mu\nu} - \frac{G_5}{G_4}\phi_{,\mu\nu}\right)R = \\ =& G_4\left(R_{\mu\nu} - \frac{1}{2}g'_{\mu\nu}R\right) \end{aligned} \tag{1.62}$$

where now we have a modified metric $g'_{\mu\nu}\left(g_{\mu\nu},\phi\right) = g_{\mu\nu} - \frac{G_5}{G_4}\phi_{,\mu\nu}$. Thus, the Horndeski's effective Einstein equations are rewritten as

$$G_4\left(\phi, X\right)\left(R_{\mu\nu} - \frac{1}{2}g'_{\mu\nu}\left(g_{\mu\nu},\phi\right)R\right) = 8\pi T^M_{\mu\nu} + T^{Horn}_{\mu\nu} \tag{1.63}$$

Moreover, we can consider also the Ricci tensor in the new metric by including the extra terms again in the unknown $T^{Horn}_{\mu\nu}$

$$G_4\left(\phi, X\right)\left(R'_{\mu\nu} - \frac{1}{2}g'_{\mu\nu}\left(g_{\mu\nu},\phi\right)R\right) = 8\pi T^M_{\mu\nu} + T^{Horn}_{\mu\nu} \tag{1.64}$$

Thus, instead of calculating directly the field equations, one can directly consider the effective Einstein equations which will be much more useful in calculating the mass in Horndeski's theory and then, the turnaround radius.

# Chapter 2

# A new mass definition

The concept of what is a mass in General Relativity is gaining prominence because it provides an alternative way to calculate the turnaround radius [19]. The main problem, however, is that there is no a universal accepted definition of mass in General Relativity (and, of course, in any other extended theories of gravity), neither of what we know as gravitational energy.
In Newtonian Mechanics, there were no doubts on what mass is; considering the Poisson equation

$$\nabla^2 \phi(x) = 4\pi G \rho(x) \tag{2.1}$$

where $\phi(x)$ is the gravitational field potential and $\rho(x)$ is the matter density, a definition of the mass directly occurs through an integration of the matter density over the whole space

$$M \equiv \int_V d\tau \rho(x) \tag{2.2}$$

Moreover, because of the Poisson equation, one can always reduce the integral over the 3-dimensional space to an integration over a surface

$$M = \int_V d\tau \rho(x) = \frac{1}{4\pi G} \int_V d\tau \vec{\nabla} \cdot \left( \vec{\nabla} \phi(x) \right) = \frac{1}{4\pi G} \int_{\partial V} d\Sigma \vec{\nabla} \phi(x) \cdot \vec{n} = \frac{M_{int}}{4\pi G} \tag{2.3}$$

where it is been defined

$$M_{int} \equiv \int_{\partial V} d\Sigma \vec{\nabla} \phi(x) \cdot \vec{n} \tag{2.4}$$

which represents a first definition of "quasi-local" mass since there is no need to measure mass through the entire space but it is enough to know the gravitational flux on the 2-dimensional surface. In particular, given the Newton's gravitational law

$$\vec{\nabla} \phi(x) = G \frac{M}{r^2} \hat{r} \tag{2.5}$$

one can easily check that, being in the hypotheses of Gauss theorem, the quasi-local mass definition is equivalent to that of the mass itself. However, an analogous quasi-local definition of the potential energy cannot be obtained since there is no a Poisson-like equation which links the potential energy to the Laplacian of some function.
In the context of General Relativity, the situation is completely different: there is no Poisson equation but only Einstein field equations (1.15) (or effective ones (1.34), in the case of scalar-tensor theories). A natural definition of mass should, however, belong to the 00-component of the energy-momentum tensor but now there is no general formula which relates this to some potential (that, in this case, translates in something coming from the metric itself). Moreover, any local definition of the gravitational energy would be wrong, because of the Equivalence Principle: a single observer can always put himself in a system where no gravity is felt, not distinguishing between gravitational effect from kinematical ones, so two observers are needed to measure a pure gravitational effect, that is the geodesic deviation

$$\frac{D^2x^\alpha}{Ds} = R^\alpha_{\mu\nu\lambda}u^\mu u^\nu x^\lambda \tag{2.6}$$

In order to solve these issues, one can consider a simple case, like spherical symmetry, and then try to analyse the problem from a full perspective: indeed, the need of at least two observers suggests that studying the evolution of 2-surfaces will surely contain the information of some quasi-local energy, since 1-dimensional ones (lines) are not necessarily measuring it, given a possible geodesical observer. So, the chapter is organised as follows: in the first section, there is an attempt to define mass in presence of spherical symmetry; in the second one, the 2+2 formalism is introduced in order to find a quasi-local definition of the energy; in the third section, a natural generalization of such a definition for scalar-tensor theories is given with some examples and the application to Horndeski's theory is discussed.

## 2.1 The spherically symmetric case

Consider, for instance, the simplest example of spherically symmetric space-time, the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \tag{2.7}$$

where, as usual, $d\Omega^2 = d\theta^2 + sin^2\theta d\varphi^2$; here, there are no ambiguities on the concept of mass, represented by the mass $m$, so one can say that the mass is constant throughout the entire space-time. A different situation appears in the Reissner-Nordström metric

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2 \tag{2.8}$$

which describes a mass $m$ charged by $Q$: in this case one does not have a trivial analogy as in the previous case but what one finds is that the mass is no longer constant everywhere because of the electromagnetic energy-momentum tensor. In order to be more precise, a generic spherically symmetric metric is considered

$$ds^2 = -A(t,r)\,dt^2 + B(t,r)\,dr^2 - 2C(t,r)\,dtdr + D^2(t,r)\,r^2 d\Omega^2 \tag{2.9}$$

Performing a change of coordinates $(t,r,\theta,\varphi) \to (t,R,\theta,\varphi) : R = D(t,r)\,r$, one can express the metric in terms of the areal radius $R$

$$ds^2 = -A'(t,R)\,dt^2 + B'(t,R)\,dR^2 - 2C'(t,R)\,dtdr + R^2 d\Omega^2 \tag{2.10}$$

and, since the unknown functions are in general functions of the time also one can try to find another coordinate transformation such that $(t,R,\theta,\varphi) \to (T,R,\theta,\varphi) : C(T,R) = 0$ so that

$$ds^2 = -A''(T,R)\,dT^2 + B''(T,R)\,dR^2 + R^2 d\Omega^2 \tag{2.11}$$

At this point, without loss of generality, one can re-express the two unknown functions $A'', B'' \to A, B$ in terms of other two unknown functions, $m, \psi$, so built

$$\begin{aligned} A(T,R) &= e^{2\psi(T,R)}\left(1 - \frac{2m(T,R)}{R}\right) \\ B(T,R) &= \left(1 - \frac{2m(T,R)}{R}\right)^{-1} \end{aligned} \tag{2.12}$$

$$ds^2 = -e^{2\psi(T,R)}\left(1 - \frac{2m(T,R)}{R}\right)dT^2 + \left(1 - \frac{2m(T,R)}{R}\right)^{-1} dR^2 + R^2 d\Omega^2 \tag{2.13}$$

The advantage of making so is that now a natural definition of mass can be obtained: indeed, an easy computation of the $\binom{0}{0}$ Einstein equation yields

$$\frac{\partial m(T,R)}{\partial R} = 4\pi G R^2 T^0_0 \tag{2.14}$$

As we know, the $\binom{0}{0}$-component of the energy-momentum tensor represents the energy density and so can lead to a natural description of a mass; in practice, $T^0_0$ has the role that matter density had in Poisson equation, that is

$$M(R) \equiv G \int_V d\tau T^0_0 \tag{2.15}$$

Due to spherical symmetry and inserting (2.14), we get

$$M(R) = 4\pi G \int_0^R dR' T^0_0 = 4\pi G \int_0^R dR' \frac{1}{4\pi G R'^2}\frac{\partial m}{\partial R'} = \int_0^R dR' \frac{\partial m}{\partial R'} = m(R) \tag{2.16}$$

and so the function $m$ will play the role of the effective mass function. This has been possible thanks to the actual form of the metric, without which we would never have achieved the Einstein equation (2.14). So, in the end, the mass function can be determined by analysing the $g^{11}$, obtaining

$$m\left(T,R\right)=\frac{R}{2G}\left(1-g^{RR}\right)=\frac{R}{2G}\left(1-R_{;\mu}R^{;\mu}\right) \tag{2.17}$$

where we want to stress out the fact that the areal radius can be in general a function not only of the radius but also of the time.
In our former examples, the areal radius coincides with the radius itself; in the Schwarzschild case, one consistently finds

$$m\left(t,r\right)=const=m \tag{2.18}$$

whereas in the Reissner-Nordström case

$$m\left(t,r\right)=m\left(r\right)=m-\frac{Q^2}{2r} \tag{2.19}$$

which means that the mass is no longer a constant of the space-time but varies throughout it in a way that an observer, ideally sitting at infinity, would measure a constant value. A similar (but not equivalent) situation can be encountered in the Schwarzschild-de Sitter space-time

$$ds^2=-\left(1-\frac{2m}{r}-\Lambda r^2\right)dt^2+\left(1-\frac{2m}{r}-\Lambda r^2\right)^{-1}dr^2+r^2d\Omega^2 \tag{2.20}$$

where the mass function results to be

$$m\left(t,r\right)=m\left(r\right)=m+\frac{\Lambda}{2}r^3 \tag{2.21}$$

so we found again a change of the value of mass measured but this time an observer at infinity would measure an infinite mass, due to expansion. An example of areal radius dependent on time is given by the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric

$$ds^2=-dt^2+a^2(t)\left(\frac{dr^2}{1-kr^2}+r^2d\Omega^2\right) \tag{2.22}$$

The areal radius is so dependent of time because of the scale factor

$$R(t,r)=a(t)r \tag{2.23}$$

Applying the (2.17), one easily gets

$$m\left(t,r\right)=\frac{R^3}{2G}\left(H^2+\frac{k}{a^2}\right) \tag{2.24}$$

where $H(t) = \frac{a_{,t}}{a}$ is the usual Hubble parameter; so, in this case, the concept of mass is entirely related to the expansion of the Universe. In the flat Universe, one has

$$m\left(t,r\right) = \frac{H^2 R^3}{2G} = \frac{4\pi R^3}{3}\rho \tag{2.25}$$

where in the last line Hamiltonian constraint, i.e. the first Friedmann equation, is applied. This formula is in agreement what one expects: the mass is nothing else than the matter density contained in a spherical volume. A more realistic situation is described by the perturbed FLRW metric; considering the conformal time $t \to \eta : dt = a d\eta$, the (flat) FLRW metric becomes

$$ds^2 = a^2(\eta)\left(-d\eta^2 + dr^2 + r^2 d\Omega^2\right) \tag{2.26}$$

The perturbed FLRW metric is introduced in order to describe inhomogeneities caused by the presence of matter; considering only the scalar perturbations in the Newtonian gauge, the perturbed FLRW metric is

$$ds^2 = a^2(\eta)\left[-\left(1 + 2\psi(t,r)\right)d\eta^2 + \left(1 - 2\phi(t,r)\right)\left(dr^2 + r^2 d\Omega^2\right)\right] \tag{2.27}$$

One further simplification comes from the Birkhoff's theorem, that allows us to write

$$\psi(t,r) = \phi(t,r) = \phi(r) \tag{2.28}$$

A remark here on what we mean for Birkhoff's theorem: of course the metric will depend on time (scale factor) so cannot be meant as static (as well as the theorem states) but what is interesting is that the two scalar perturbations, as result of Einstein equations, are independent of time; thus, the idea is to consider a sort of generalized Bhirkhoff's theorem which is not applying to the entire metric solution but only to its some components. Thus, the mass function, whose calculation is performed in the Appendix C, results

$$m\left(t,r\right) = \frac{1}{G}\left[-\frac{r^3\phi_{,r}^2}{2}\left(1-2\phi\right)^{-\frac{3}{2}} + r^2\phi_{,r}\left(1-2\phi\right)^{-\frac{1}{2}}\right]a + \frac{H^2R^3}{2G\left(1+2\phi\right)} \tag{2.29}$$

We immediately notice that the zeroth order in perturbation coincides with the mass of FLRW metric (2.24) (with $k = 0$); the interpretation of this is simple: one can address to the first term a "local" behaviour, due to clumping and gravitational attraction exercised by matter structures, and to the second term a "cosmological" behaviour, due to the expansion. At the first order in the perturbation

$$m\left(t,r\right) \simeq \frac{r^2\phi_{,r}a}{G} + \frac{H^2R^3}{2G}\left(1-2\phi\right) + O(\phi^2) \tag{2.30}$$

To understand why the first term is responsible of gravitational attraction, one can try to calculate $\nabla^2\phi$ and consider the first order in the perturbation, finding (see Appendix C)

$$\nabla^2\phi \simeq \frac{1}{a^2r^2}\frac{d}{dr}\left(r^2\phi_{,r}\right) \tag{2.31}$$

In other words, the perturbation represents a generalization of the Newtonian gravitational field and, at first order, it can be integrated as made for Poisson equation

$$m_N = \int_V d\tau\nabla^2\phi = 4\pi\int_0^r dr'r'^2\nabla^2\phi = \frac{4\pi}{a^2}\int_0^r dr'\frac{d}{dr'}\left(r'^2\phi_{,r'}\right) = \frac{4\pi}{a^2}r^2\phi_{,r} \tag{2.32}$$

and redefining $m_N \to \frac{a^2}{4\pi G}m_N$, we finally get

$$m\left(t,r\right) \simeq m_N a + \frac{H^2R^3}{2G}\left(1-2\phi\right) \tag{2.33}$$

which clearly shows the different and opposite nature of the two terms.

## 2.2 The 2+2 formalism and the Hawking-Hayward mass

The previous discussion was possible because spherical symmetry allows the mass function to be directly related to the energy density via Einstein equation but one can ask what is a mass in general, that is when such a useful relation is no longer found. The main idea is again trying to find a prescription for the energy and if, from a Lagrangian point of view, the equation of motion are not useful, one may consider the Hamiltonian formalism and try to calculate the Hamiltonian density as a prescription of the so-called "quasi-local" energy. However, in doing so, a 3+1 decomposition of the metric (like ADM) can lead to something that has nothing to do with gravitational energy because, as said in the introduction, at least a 2-dimensional surface is needed: that's why a 2+2 decomposition of the metric could yield a correct gravitational energy without ambiguities.

So, instead having a 3-spatial surface and an evolution vector, in the 2+2 formalism there is a 2-dimensional surface evolving by means of two commuting evolution vectors $u^\alpha = \frac{d}{d\xi}, v^\alpha = \frac{d}{d\eta}$, forming so a basis $\vec{e_\mu} = (\partial_\xi, \partial_\eta, \partial_2, \partial_3)$; since the main goal of decomposition is to write the metric, it is always useful to split the evolution vectors in a component normal to the 2-dimensional surface and a shift vector and choose as basis the normal ones: in other words, calling these normal components $n^\alpha = \frac{d}{d\tau}$ and $l^\alpha = \frac{d}{d\sigma}$, and so choosing a basis $\vec{e_a} = (\partial_\tau, \partial_\sigma, \partial_2, \partial_3)$, the mixing components of the metric vanish

$$g\left(\vec{n}, \partial_{2,3}\right) = g\left(\vec{l}, \partial_{2,3}\right) = 0 \tag{2.34}$$

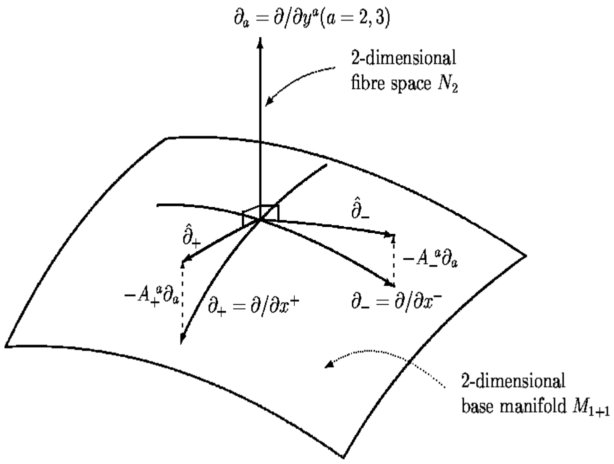



Figure 2.1: A graphical illustration of the 2+2 formalism

Moreover, in general we have

$$\begin{aligned} g(\vec{n}, \vec{n}) &= a \\ g\left(\vec{l}, \vec{l}\right) &= b \\ g\left(\vec{n}, \vec{l}\right) &= -e^m \\ g(\partial_a, \partial_b) &= h_{ab} \end{aligned} \tag{2.35}$$

If the 2-dimensional surface is null, many calculation will be simplified in the context of integrability conditions and $a = b = 0$ can be set [20]. Thus, now it is easy to write the most general line element by means of the following metric tensor

$$g_{ab} = \begin{pmatrix} r_c r^c & r_c s^c - e^{-m} & r_b \\ r_c s^c - e^{-m} & s_c s^c & s_b \\ r_a & s_a & h_{ab} \end{pmatrix} \tag{2.36}$$

where $\vec{r}$ and $\vec{s}$ are the shift vectors, that is $r^a = h^a_b u^b$ and $s^a = h^a_b v^b$. Now, what one has to do is to compute the Lagrangian which now appears at the action level as

$$S = \int_{S \times \mathcal{U} \times \mathcal{V}} \lambda R = \int_{\mathcal{U}} d\xi \int_{\mathcal{V}} d\eta \int_S \mathcal{L} \tag{2.37}$$

where $S$ is the 2-dimensional surface and $\lambda$ is the volume form. Once done, the conjugate momenta have to be calculated in order to develop the Hamiltonian density. After much effort, it can be shown that the Hamiltonian density can be expressed in terms of extrinsic fields as [21]

$$8\pi G\mathcal{H} = -\mu \left( \mathcal{R} + \theta\tilde{\theta} - \frac{1}{2}\sigma_{ab}\tilde{\sigma}^{ab} - 2\omega_a\omega^a \right) \tag{2.38}$$

where $\mu$ is the area 2-form and $\mathcal{R}$ is the Ricci scalar relative to the 2-dimensional surface $S$, namely it is made of $h_{ab}$. Since we are interested in the energy itself and not a density, the previous has to be integrated over; in a 2+2 context, the integration has to be performed over the 2-surface and so, in the end, one has to add a factor of lenght units to get the right energy units

$$E = -\sqrt{\frac{A}{16\pi}} \int_S \mathcal{H} \tag{2.39}$$

where $A = \int_S \mu$ and the factor has been chosen for agreement with the Schwarzschild mass. Thus, taking into account 2.38, the searched expression for the energy could be [22]

$$E = \frac{1}{8\pi G}\sqrt{\frac{A}{16\pi}} \int_S \mu \left(\mathcal{R} + \theta\tilde{\theta} - \frac{1}{2}\sigma_{ab}\tilde{\sigma}^{ab} - 2\omega_a\omega^a\right) \tag{2.40}$$

This is the so-called Hawking-Hayward mass, so called after Hayward revised the previous definition given by Hawking in [23]: indeed, the Hawking definition of mass was similar

$$M_H = \frac{1}{8\pi G}\sqrt{\frac{A}{16\pi}} \int_S \mu \left(\mathcal{R} + \theta\tilde{\theta}\right) \tag{2.41}$$

However, the lack of twist and, mainly, of shear caused this old definition to be incompatible with the simplest case, the flat space-time: in fact, from the contracted Gauss equation

$$\mathcal{R} + \theta\tilde{\theta} - \frac{1}{2}\sigma_{ab}\tilde{\sigma}^{ab} = h^{ac}h^{bd}R_{abcd} \tag{2.42}$$

since flat space-time is such that $R_{abcd} = 0$ and $\omega_a = 0$, the $M_{HH}$ of (2.40) vanishes whereas the $M_H$ of (2.41) does not. Other definitions of mass are also given in literature ([24]) but at present time this is the universally accepted one.
Interesting is the case of spherical symmetry (2.11): the 2-surface is a sphere and so shear and twist naturally disappears; the area is $A = 4\pi R^2$ and the Ricci scalar of the surface is $\mathcal{R} = \frac{2}{R^2}$; so the integration over the surface S yields a factor $4\pi R^2$ since the Ricci scalar and the expansions do not depend on the variables which form the basis of S. Instead of calculating the expansion scalars, one can also use the contracted Gauss equation (2.42) and find that

$$\mathcal{R} + \theta\tilde{\theta} = h^{ac}h^{bd}R_{abcd} = \frac{2}{R^2}\left(1 - R_{;\mu}R^{;\mu}\right) \tag{2.43}$$

So, in the end, the Hawking-Hayward mass in spherical symmetry is

$$M_{HHs} = \frac{1}{8\pi G}\sqrt{\frac{4\pi R^2}{16\pi}} 4\pi R^2 \frac{2}{R^2}\left(1 - R_{;\mu}R^{;\mu}\right) = \frac{R}{2G}\left(1 - R_{;\mu}R^{;\mu}\right) \tag{2.44}$$

in agreement with (2.17). In this special case, the Hawking-Hayward is known as the Misner-Sharp-Hernandez mass [25]. We recall once again that the radial coordinate is represented by the areal radius which in general can depend on time and so, the covariant derivative can lead to something less trivial than expected.

## 2.3 The scalar-tensor generalization

The Hawking-Hayward mass is derived in the context of General Relativity: in fact, in deriving the Hamiltonian (2.38), the action (2.37) was used, where the Lagrangian density is simply the volume form multiplied by the Ricci scalar. If one wants to generalize it to any scalar-tensor theory, he should replace the Lagrangian with the appropriate one (1.33) but, if the actual derivation of the Hamiltonian using the simplest Einstein-Hilbert action is already rather long, the new derivation would be an endless dive in calculations. So, rather than follow such a hopeless construction, an alternative way was suggested in [26]: assuming that the (2.40) is still the correct expression for the mass, the effective Einstein equations (1.34) must be considered. Thus, considering the decomposition of the Riemann tensor into Weyl tensor

$$R_{abcd} = C_{abcd} + g_{a[c}R_{d]b} - g_{b[c}R_{d]a} - \frac{R}{3} g_{a[c}R_{d]b} \tag{2.45}$$

making use of the contracted Gauss equation (2.42) and the effective Einstein equations, one finds

$$\begin{aligned} h^{ac}h^{bd}R_{abcd} =& h^{ac}h^{bd}C_{abcd} + \frac{8\pi}{\phi}h^{ac}h^{bd}\left[g_{a[c}T_{d]b} - g_{b[c}T_{d]a} - \frac{T}{2}\left(g_{a[c}g_{d]b} - g_{b[c}g_{d]a}\right)\right] + \\ &+ \frac{\omega}{\phi^2}h^{ac}h^{bd}\left(g_{a[c}\nabla_{d]}\phi\nabla_b\phi - g_{b[c}\nabla_{d]}\phi\nabla_a\phi\right) + \\ &+ \frac{1}{\phi}h^{ac}h^{bd}\left(g_{a[c}\nabla_{d]}\nabla_b\phi - g_{b[c}\nabla_{d]}\nabla_a\phi\right) + \\ &+ \frac{\Box\phi + V}{2\phi}h^{ac}h^{bd}\left(g_{a[c}g_{d]b} - g_{b[c}g_{d]a}\right) + \\ &+ \left(\frac{8\pi T}{3\phi} - \frac{\omega}{3\phi^2}\nabla^c\phi\nabla_c\phi - \frac{\Box\phi}{\phi} - \frac{2V}{3\phi}\right)h^{ac}h^{bd}g_{a[c}g_{d]b} \end{aligned} \tag{2.46}$$

Since

$$h^{ac}h^{bd}\left(g_{a[c}A_{d]b} - g_{b[c}A_{d]a}\right) = h^{ab}A_{ab} \tag{2.47}$$

where $A_{ab}$ will be at turn the terms $T_{ab}$, $\nabla_a\phi\nabla_b\phi$, $\nabla_a\nabla_b\phi$, $g_{ab}$ that appear in the previous expression and considering by means of the Mach principle the field $\phi$ as the effective gravitational constant, one can consider as the mass prescription for the scalar-tensor theories the following

$$M_{ST} = \frac{1}{8\pi}\sqrt{\frac{A}{16\pi}}\int_S \mu\phi \Bigg[h^{ac}h^{bd}C_{abcd} - 2\omega_a\omega^a + \frac{8\pi}{\phi}h^{ab}T_{ab} - \frac{16\pi T}{3\phi} + \frac{h^{ab}\nabla_a\nabla_b\phi}{\phi} + \\ + \frac{\omega}{\phi^2}\left(h^{ab}\nabla_a\phi\nabla_b\phi - \frac{1}{3}\nabla^c\phi\nabla_c\phi\right) + \frac{V}{3\phi}\Bigg] \tag{2.48}$$

Again, one of the simplest frameworks in which it is comfortable to work is the spherical symmetry; a simple guess [27] can be a natural generalization of the Misner-Sharp-Hernandez mass (2.17) by considering an effective gravitational constant represented by the scalar field $\phi = \frac{1}{G_{eff}}$

$$M_{STs} = \frac{\phi R}{2}\left(1 - R_{;\mu}R^{;\mu}\right) \tag{2.49}$$

However, in order to be careful, let us apply directly the diagonal spherically symmetric metric in the mass prescription above: this means considering a vanishing twist, an area $A = 4\pi R^2$ and an integrand indipendent of integration domain, which returns an extra factor $4\pi R^2$, so the scalar-tensor mass in spherical symmetry should be

$$M_{STs} = \frac{\phi R^3}{4}\Bigg[h^{ac}h^{bd}C_{abcd} + \frac{8\pi}{\phi}h^{ab}T_{ab} - \frac{16\pi T}{3\phi} + \frac{h^{ab}\nabla_a\nabla_b\phi}{\phi} + \\ + \frac{\omega}{\phi^2}\left(h^{ab}\nabla_a\phi\nabla_b\phi - \frac{1}{3}\nabla^c\phi\nabla_c\phi\right) + \frac{V}{3\phi}\Bigg] \tag{2.50}$$

Looking at it, it seems that it has nothing to do with Misner-Sharp-Hernandez mass or even with the (2.49); however, one can show that in several cases, like the FRLW metric, by making use of the Hamiltonian constraint (i.e. field equations), one can obtain the searched relationship. In fact, if one consider the starting point of the previous mass derivation

$$M_{ST} = \frac{1}{8\pi}\sqrt{\frac{A}{16\pi}}\int_S \mu\phi\left(\mathcal{R} + \theta\tilde{\theta} - \frac{1}{2}\sigma_{ab}\tilde{\sigma}^{ab} - 2\omega_a\omega^a\right) \tag{2.51}$$

and the diagonal spherically symmetric metric (2.11), assuming that the scalar field $\phi$ is a function of $t$ and $R$, then it can be extracted out, obtaining [28]

$$M_{ST} = \frac{1}{8\pi}\sqrt{\frac{A}{16\pi}}\phi(t,r)\int_S \mu\left(\mathcal{R} + \theta\tilde{\theta} - \frac{1}{2}\sigma_{ab}\tilde{\sigma}^{ab} - 2\omega_a\omega^a\right) \tag{2.52}$$

At this point, as in the previous section, the metric allows to neglect shear and twist and, again using contracted Gauss equation, one gets the generalization of the Misner-Sharp-Hernandez mass. An alternative way to see the validity of this prescription is by means of the arguments shown in the first section: considering the metric with explicit mass function, one would have

$$\frac{\partial m\left(T,R\right)}{\partial R}=4\pi R^{2}\left(\frac{T_{0}^{0}}{\phi}+T_{0}^{0^{curv}}\right) \tag{2.53}$$

So, considering the energy-momentum tensor as the sum of two contributions, a matter contribution and a curvature one, then we have the same result (2.14) of the section one, with a precise and unambiguous definition of the mass, which is indeed the (2.49).

### 2.3.1 Examples

Applications of the prescription (2.49) to the Schwarzschild, Reissner-Nordström and Schwarzschild-de Sitter metrics are straightforward: indeed, the only modification concerns the effective gravitational constant which naturally appeared in the cited metrics considering $m(t,r)=GM(t,r)$. So, the scalar-tensor mass prescription for those metrics respectively are

$$\begin{aligned} M_{ST}\left(t,r\right)=&\phi(t,r)m\\ M_{ST}\left(t,r\right)=&\phi(t,r)\left(m-\frac{Q^{2}}{2r}\right)\\ M_{ST}\left(t,r\right)=&\phi(t,r)\left(m+\frac{\Lambda}{2}r^{3}\right)\end{aligned} \tag{2.54}$$

Of particular relevance is the fact that, in general, when considering theories which go beyond the Einstein's General Relativity, even the Schwarzschild mass, i.e. the mass of a static, chargeless and asymptotically flat black hole, is no longer a constant through the space-time; this fact can be directly connected to the failure of the Gauss's law: in the General Relativity case, the Newtoninan potential, by means of the diagonal spherically symmetric metric with mass function, was simply $\phi_N=\frac{m}{r}$ and so the hypotheses of the law (i.e. the inverse squared behaviour of the flux) are satisfied, ensuring that, outside the source, an observer will always measure the whole internal mass, no matter where or how it is distributed; in extended theories of gravity, a simple analytical solution of the effective Einstein equations (1.34) is not found and, moreover, there is no proof that Birkhoff's theorem holds even in this simplest case. Thus, unlike the General Relativity, the Newtonian potential cannot be ensured to be of the best form (i.e. an inverse radius behaviour) and so the Gauss's law does not hold. The other cases treated in the first section concern the FLRW metric; in the standard (flat) FLRW metric, the calculation of the mass is straightforward

$$M_{ST}\left(t,r\right)=\frac{H^{2}R^{3}\phi}{2} \tag{2.55}$$

However, the Hamiltonian constraint is quiet different, because, by means of the effective Einstein equations, extra terms appear in the Friedmann equation

$$H^2 = \frac{8\pi\rho}{3\phi} - H\frac{\dot{\phi}}{\phi} + \frac{\omega}{6}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{V}{6\phi} \equiv \frac{8\pi\left(\rho+\rho_\phi\right)}{3\phi} \tag{2.56}$$

These extra terms can be considered as the contribution to the energy density of the curvature modifications induced by modified gravity; thus, the mass results

$$M_{ST}\left(t,r\right) = \frac{H^2R^3\phi}{2} = \frac{4\pi R^3}{3}\left(\rho+\rho_\phi\right) \tag{2.57}$$

So, now the mass is not simply the matter contained in a sphere but one has to take into account the effect of the gravity modification induced by the scalar field: the result would be a prediction of the mass larger than that observed, which could be of main interest in understanding the true nature of the dark matter. The same result can be obtained applying directly the (2.50). In order to proceed further, let us consider the more realistic metric (2.27); however, to not make confusion between the scalar field and the scalar perturbation of the FLRW metric, the metric is rewritten as

$$ds^2 = a^2(\eta)\left[-\left(1+2A(\eta,r)\right)d\eta^2 + \left(1-2B(\eta,r)\right)\left(dr^2+r^2d\Omega^2\right)\right] \tag{2.58}$$

Unlikely, there is no certainty that Birkhoff's theorem holds now and so a simplification like the (2.28) is no longer available. Thus, one has to treat the two scalar perturbations as distinct functions and both to be intended as functions of the time also. What one can say is, by means of [29], the Birkhoff's theorem holds only at zeroth order in the Jordan frame and till the first order in the Einstein frame: however, the prescription of mass given naturally lies in the Jordan frame and so one has to consider the conformal transformation, with a subsequent modification of the mass prescription (2.49); moreover, even at zero order, the maximum that one can say is that the two functions are independent of time but, in general, they are not the same function. Again, one can use either the (2.50) or (2.49); following the easier one, namely the second, it is shown in the Appendix D that

$$\begin{aligned} M_{ST}\left(t,r\right) =& \phi\left[-\frac{r^3B_{,r}^2}{2}\left(1-2B\right)^{-\frac{3}{2}} + r^2B_{,r}\left(1-2B\right)^{-\frac{1}{2}}\right]a+ \\ & -\phi a^2Hr^3B_{,\eta}\frac{\sqrt{1-2B}}{1+2A} + \frac{\phi}{2}ar^3B_{,\eta}^2\frac{1}{\left(1+2A\right)\sqrt{1-2B}}+ \\ & + \frac{\phi H^2R^3}{2\left(1+2A\right)} \end{aligned} \tag{2.59}$$

where all dipendencies of $\phi$, $A$ and $B$ are implicit. Notice that the result is again a first term, composed by two parts, possibly responsible of the gravitational potential and so, exhibiting a "local" behaviour, and three terms that can be addressed as "cosmological" [30]: the last term is already encountered in (2.29) (as well as for the first two terms in the square brackets) whereas the other two terms are new and take into account the variation of the scalar perturbation with respect to the (conformal) time. At first order in the scalar perturbations

$$M_{ST}\,(t,r) \simeq \phi r^2 B_{,r} a - \phi a^2 H r^3 B_{,\eta} + \frac{\phi H^2 R^3}{2}\,(1-2A) \tag{2.60}$$

Calculating $\nabla^2 B$ as done in the first section, one can show that, at first order (check again Appendix D)

$$\nabla^2 B \simeq \frac{1}{a^2}\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2 B_{,r}\right) - \frac{1}{a^2}\frac{d}{d\eta}\left(a^2 B_{,\eta}\right)\right] \tag{2.61}$$

The second term does not allow a similar definition for $m_N$ as in the (2.32); however, at this level of approximation, the Birkoff's theorem hold and so one can neglect that term, allowing for a correct Newtonian mass definition, leading to the following

$$M_{ST}\,(t,r) \simeq \phi m_N a + \frac{\phi H^2 R^3}{2}\,(1-2A) \tag{2.62}$$

where now even $A$ is a function of $r$ only. This expression is comparable to the (2.33). However, here, there is another thing that can be expanded at the first order of approximation, the scalar field

$$\phi(t,r) = \phi^{(0)}(t,r) + \delta\phi(t,r) \tag{2.63}$$

Moreover, one can apply the Hamiltonian constraint and obtain an expression which has to be compared with the one obtained using the general prescription (2.50), whose result is computed in the [31] and shown below

$$\begin{aligned} M_{ST}(t,r) =& \frac{H_{(0)}^2 R_{(0)}^3 \phi_{(0)}}{2}\,(1-3B) + \frac{R_{(0)}^3\phi_{(0)}}{4}\left\{\frac{2}{3a^2}\left(\frac{A_{,r}}{r} + \frac{B_{,r}}{r} - A_{,rr} - B_{,rr}\right) + \right. \\ & + \frac{16\pi}{3}\left[-4\left(\rho_{(0)} + P_{(0)}\right)A + \delta\rho - 2P_{(0)}B\right] + \frac{\omega_{0,r}\delta\phi - 2\omega_0 A}{3a^2\phi_{(0)}}\phi_{(0),\eta}^2 + \\ & \left. + \frac{4a_{,\eta}}{a^3}\phi_{(0),\eta}A + \frac{2\delta\phi_{,r}}{a^2 r} + \frac{V_{0,r}}{3}\delta\phi\right\} \end{aligned} \tag{2.64}$$

### 2.3.2 Horndeski's mass

The way we have calculated the field equations in (1.64) is very useful to understand what is mass function in such a theory, at least in the hypothesis of spherical symmetry.

Indeed, one can repeat the same argumentation of the first section, or even refer to the (2.53), and consider a metric with explicit mass function like the (2.13), finding

$$\frac{\partial m\left(T,R\right)}{\partial R}=4\pi R^{2}\left(\frac{T_{0}^{0}}{G_{4}\left(\phi,X\right)}+T_{0}^{0^{Horn}}\right) \tag{2.65}$$

However, a little clarification has to be made here; the metric that one refers to is, again by means of (1.64), $g'_{\mu\nu}=g_{\mu\nu}-\frac{G_5}{G_4}\phi_{,\mu\nu}$; in practice, the mass function now is

$$m\left(T,R\right)=\frac{G_{4}R}{2}\left(1-g'^{RR}\right)=\frac{G_{4}R}{2}\left(1-g^{RR}+\frac{G_{5}}{G_{4}}\phi^{,RR}\right) \tag{2.66}$$

Thus, the convenience of using the (1.64) without regarding whatever form the energy-momentum acquires is manifest in the simple and analog prescription that one assigns to mass.

# Chapter 3

# Turnaround radius

There is a wide range of applications for the scalar-tensor theories; the final goal is to understand which effectively is the "true" theory of gravity. In particular, what actually the area of research is aiming to is finding, from one hand, confirmations of the existence of dark matter and, in the other, possible tests that could make manifest the authenticity of a theory rather than another. At this point, it could be helpful remarking that we are not considering Quantum Mechanics at all: indeed, General Relativity as well as scalar tensor theories are treated classically and we are far from consider a Quantum theory of gravity. However, until we remain indifferent to Early Universe and singularities formation, we can forget about quantum effects. Having made this clarification, one can proceed to list the possible tests that can reveal us which is the most authentic theory of gravity (at least, for large scales and not early times); there are plethora as, only to cite some, the measurement of the rotational velocity of LSB galaxies and the gravitational lensing. However, another test is gaining prominence in these last years, that is the measurement of the so-called turnaround radius.
The turnaround radius is defined as that point (or better, that 2-surface) of any celestial structure (such as galaxies, clusters) in which the gravitational attraction due to the presence of massive objects is perfectly counterbalanced by the effect of departure due to the expansion of the Universe. Measuring the location of such a radius and comparing it with the predictions made by the various theories of gravity (including the most fundamental one, the General Relativity) could help to identify the most promising theory.

Defining the turnaround radius should be something very familiar: the only thing that one has to do in order to calculate it is considering the radial geodesics and demanding whether and where the radial acceleration vanishes, as it will be illustrated in the first section; however, when the realistic perturbed FLRW metric (2.27) is analysed, the computation is rather involved and the result is highly gauge dependent (in fact, the metric used the Newtonian gauge). This yields to consider another definition of the turnaround radius which makes use of the definition of mass given in the previous chapter:

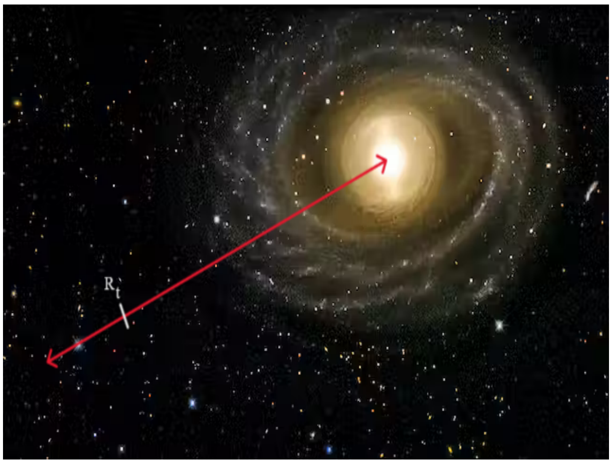



Figure 3.1: A simple visualization of the turnaround radius

even if the two definitions are not perfectly equivalent, their difference is shown to be small [32], specially considering the impossibility to measure such differences in the large cosmological scales. Finally, we try to calculate the turnaround radius in the context of the Horndeski's theory. These will be the topics of the second section. In the final section, there is a new suggestive way to calculate the turnaround radius, which surprisingly make contact with the alternative definition through mass.

## 3.1 A natural definition

As already said, the most naturally definition for the turnaround radius is that value in which the radial acceleration of a geodesic observer becomes vanishing. A geodesic observer satisfies

$$\frac{D^2 r}{Ds^2} = \frac{d^2 r}{ds^2} + \Gamma^1_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \ddot{r} + \Gamma^1_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{3.1}$$

where the dot indicates $\cdot = \frac{d}{ds}$. Therefore, the turnaround radius is that value of $r$ that satisfies the following second-order differential equation

$$\ddot{r} = -\Gamma^1_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \tag{3.2}$$

Let us consider some examples in spherically symmetric space-times; first consider the case in which the metric is independent of time

$$ds^2 = -f(r)^2 dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \tag{3.3}$$

The calculation of (3.2) in this case yields

$$\begin{aligned}\ddot{r} =& -\Gamma^1_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = -\frac{1}{2}g^{1\lambda}\left(g_{\mu\lambda,\nu}+g_{\lambda\nu,\mu}-g_{\mu\nu,\lambda}\right)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \\ &= -\frac{1}{2}g^{11}\left(g_{\mu1,\nu}+g_{1\nu,\mu}-g_{\mu\nu,1}\right)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \\ &= -\frac{1}{2}g^{11}\left(g_{11,1}\frac{dx^1}{ds}\frac{dx^1}{ds}+g_{11,1}\frac{dx^1}{ds}\frac{dx^1}{ds}-g_{00,1}\frac{dx^0}{ds}\frac{dx^0}{ds}-g_{11,1}\frac{dx^1}{ds}\frac{dx^1}{ds}\right) = \\ &= -\frac{1}{2}g^{11}\left(g_{11,1}\dot{r}^2-g_{00,1}\dot{t}^2\right)\end{aligned} \tag{3.4}$$

From normalization condition (time-like geodesic observer)

$$g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = -1 \Rightarrow \dot{r}^2 = f(r)^2\dot{t}^2 - f(r) \tag{3.5}$$

then

$$\begin{aligned}\ddot{r} = &= -\frac{1}{2}g^{11}\left(g_{11,1}\dot{r}^2-g_{00,1}\dot{t}^2\right) = \\ &= -\frac{1}{2}f(r)\left[-\frac{\partial_r f(r)}{f(r)^2}\left(f(r)^2\dot{t}^2-f(r)\right)+\partial_r f(r)\dot{t}^2\right] = \\ &= -\frac{1}{2}\partial_r f(r)\end{aligned} \tag{3.6}$$

So, in this particular case, calculating the turnaround radius is simple since in the previous equation every 4-velocities disappear at the right hand side; thus, the turnaround radius is that value $r_t$ for which the following is satisfied

$$\partial_r f(r)|_{r_t} = 0 \tag{3.7}$$

For instance, one can apply this to the particular cases of Schwarzschild and Reissner-Nordstrom space-times, even if there is no correlation with cosmology. In the first case, $f(r) = 1 - \frac{2m}{r}$ and so turnaround radius satisfies

$$\frac{m}{r_t^2} = 0 \tag{3.8}$$

that is, there is no solution, since, in such simple space-time, the unique effect is that of gravitational attraction. In the Reissner-Nordstrom space-time, $f(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}$, then

$$\frac{2m}{r_t^2} - \frac{2Q^2}{r_t^3} = 0 \tag{3.9}$$

A solution exists, namely the presence of charge causes an effect of repulsion, and the point in which this repulsion and the gravitational attraction are equal is

$$r_t = \frac{Q^2}{m} \tag{3.10}$$

Of more cosmological interest is the Schwarzschild-de Sitter space-time, which belongs to this category of metrics with $f(r) = 1 - \frac{2m}{r} - \Lambda r^2$; the turnaround radius therefore is

$$\frac{2m}{r_t^2} - 2\Lambda r_t = 0 \Leftrightarrow r_t = \sqrt[3]{\frac{m}{\Lambda}} \tag{3.11}$$

Thus, the equilibrium point is reached very far from the centre of gravity of the massive object, since the cosmological constant, sourcing the dark energy and so the expansion, should be "small".

More involved are the cases in which the metric explicitly depends on time coordinate, such as the FLRW standard metric and the perturbed one. In this more general case, the computation of the Christoffel symbols is longer and there will be no a final simplification of first derivatives through the normalization condition. For example, considering the (flat) FLRW metric (2.22), it is convenient to rewrite it in terms of the areal radius $R = a(t)r$, in order to keep track of the evolving 2-surfaces (spheres) but this introduces a mixed term when coordinates $(t, R, \theta, \varphi)$ are considered. Let us then consider the most general spherically symmetric metric (2.9); as said, it is better to express the metric in terms of areal radius $R = D(t, r)r$, so that one has the metric (2.10) provided that

$$\begin{aligned} A' &= A - 2C\frac{RD_{,t}}{D^2} - B\frac{R^2 D_{,t}^2}{D^4} \\ B' &= B\left(\frac{1}{D} - \frac{RD_{,R}}{D^2}\right)^2 \\ C' &= C\left(\frac{1}{D} - \frac{RD_{,R}}{D^2}\right) + B\left(\frac{1}{D} - \frac{RD_{,R}}{D^2}\right)\frac{RD_{,t}}{D^2} \end{aligned} \tag{3.12}$$

Then, in order to compute the turnaround radius by means of (3.2) (with now $R$ as radial coordinate), one has to compute the Christoffel symbols taking into account the fact that there is a mixed term and, in general, the various functions depend on time also. With the help of the software Maple, it is shown that the radial acceleration, provided by the normalization of time-like geodesics, is

$$\ddot{R} = -\frac{1}{2A'B'+2C'^2}\Bigg[\left(A'A'_{,R} - 2A'C'_{,t} + C'A'_{,t} + \frac{A'^2B'_{,R}}{B'} + \frac{A'C'B'_{,t}}{B'} + \frac{2A'C'C'_{,R}}{B'}\right)\dot{t}^2 + \left(2A'B'_{,t} + 2C'A'_{,R} + \frac{2A'C'B'_{,R}}{B'} + \frac{2C'^2B'_{,t}}{B'} + \frac{4C'^2C'_{,R}}{B'}\right)\dot{t}\dot{R} - \left(\frac{A'B'_{,R}}{B'} + \frac{C'B'_{,t}}{B'} + \frac{2C'C'_{,R}}{B'}\right)\Bigg] \tag{3.13}$$

So, the turnaround radius can be calculated requiring this to be zero. The resulting equation is a differential equation which, in general, will depend on the background and the time evolution.

Now, applying to FLRW metric (2.22) one has

$$\begin{aligned} A =&1 \\ B =&a^2(t) \\ C =&0 \\ D =&a(t) \end{aligned} \tag{3.14}$$

such that

$$\begin{aligned} A' =&1 - H^2R^2 \\ B' =&1 \\ C' =&HR \end{aligned} \tag{3.15}$$

Then, applying to these the (3.13), one gets

$$\ddot{R} = H^2R + RH_{,t}\dot{t}^2 = \frac{H^2}{R^2}R^3 + RH_{,t} \tag{3.16}$$

where the reason why there is a rearrangement of the first term becomes clear in the next example. In general, the turnaround radius requires to solve a differential equation and so, even in this case it depends on the particular background, that is, the form of scale factor $a(t)$. In the hypothesis of de Sitter expansion, however, we know that $a(t) = e^{HT}$ and $H$ is a constant, such that the second term in the (3.16) vanishes: in such a scenario no turnaround radius exists.

Another examples, not cited yet, that can account for a generalization of the Schwarzschild-de Sitter metric are the so-called McVittie metrics [33], which include a time dependent Hubble parameter instead of the cosmological constant. The form of this metric is

$$ds^2 = -\left(\frac{1-\frac{m}{2u}}{1+\frac{m}{2u}}\right)dt^2 + a^2\left(1+\frac{m}{2u}\right)^4\left(dr^2 + r^2d\Omega^2\right) \tag{3.17}$$

where $u \equiv ar$. It can be shown that, provided that the areal radius is

$$R = u\left(1 + \frac{m}{2u}\right) \tag{3.18}$$

the transformation of coordinate $r \to R$ modifies the metric into [34]

$$ds^2 = -\left(1 - \frac{2m}{R} - H^2R^2\right)dt^2 + \left(1 - \frac{2m}{R}\right)dR^2 - \frac{2HR}{\sqrt{1-\frac{2m}{R}}}dtdR + R^2d\Omega^2 \tag{3.19}$$

Applying the (3.13), one finds that the radial acceleration is

$$\ddot{R} = \frac{H^2}{R^2}\left(R^3 - \frac{m}{H^2}\right) + RH_{,t}\sqrt{1 - \frac{2m}{R}}\dot{t}^2 \tag{3.20}$$

Comparing to the standard Schwarzschild-de Sitter metric, by considering $\Lambda = H^2$, the previous can rewritten as

$$\ddot{R} = \frac{H^2}{R^2}\left(R^3 - R_t^3\right) + RH_{,t}\sqrt{1 - \frac{2m}{R}}\dot{t}^2 \tag{3.21}$$

where $R_t$ indicates the turnaround radius in the Schwarzschild-de Sitter space-time, as shown in (3.11). It coincides with the value of turnaround radius of this generalized McVittie space-time only if, for instance, the expansion of the Universe is of de Sitter type, such that the Hubble parameter is a constant and the second term vanishes. However, for this generalization of space-time, there is a remaining dependence to the time evolution which can modify the effective value of the turnaround radius.
Now, it is worth to consider the more realistic metric (2.27); then, in general

$$\begin{aligned} A =& 1 + 2\psi(\eta, r) \\ B =& a^2(t)\left(1 - 2\phi(\eta, r)\right) \\ C =& 0 \\ D =& a(t)\sqrt{1 - 2\phi(\eta, r)} \end{aligned} \tag{3.22}$$

By means of Birkhoff's theorem, one can simplify this as done in (2.28) but only if General Relativity is considered as the actual theory of gravity; assuming that this is the case (so $\psi(\eta, r) = \phi(\eta, r) = \phi(r)$), then the previous transforms, considering as areal radius $R = ar\sqrt{1 - 2\phi}$, into

$$\begin{aligned} A' =& a^2\left(1 + 2\phi - \frac{\mathcal{H}^2R^2}{a^2}\right) \\ B' =& \left(1 + \frac{R\phi_{,R}}{1 - 2\phi}\right)^2 \\ C' =& \mathcal{H}R\left(1 + \frac{R\phi_{,R}}{1 - 2\phi}\right) \end{aligned} \tag{3.23}$$

where a $\mathcal{H}$ is introduced to refer to the conformal time $\eta$. The calculation of the radial acceleration is not trivial; by means of (3.13), one expects at least three terms, one of which could identify the turnaround radius in the case of no time evolution or special situations in which extra terms disappear and the other two concerning the evolution of the background. Neglecting so the "evolution" terms in $\dot{t}^2$ and $\dot{t}\dot{R}$, one could show that, at first order in scalar perturbations, and given the definition of Newtonian mass (2.32) that

$$\ddot{R} = H^2 R\,(1 - 2\phi) + \frac{m_N a}{R^2} \tag{3.24}$$

which yields a turnaround radius of

$$R_t = \sqrt[3]{\frac{m_N a}{H^2\,(1 - 2\phi)}} \tag{3.25}$$

Thus, instead of showing the result of a rather long derivation, it is more meaningful to notice the limits of such an approach: nevertheless the assumption of Birkhoff's theorem, the turnaround radius, results, in general, as the solution of a differential equation which depends on the background metric and could be not trivial; moreover, considering the metric (2.27), the choice made was to adopt the Newtonian gauge but, however, with another gauge one could find another result: so, the standard approach, at least for the realistic perturbed FLRW metrics, is highly gauge dependent [31]. Finally, but not less important, the Birkhoff's theorem holds in General Relativity context but not when one is considering other theories of gravity, so, the standard definition of turnaround radius, nevertheless it is certainly correct from a theoretical point of view, could be not enough to give results in some practical contexts.

## 3.2 A definition through the mass

In order to avoid gauge-dependence problems and highly involved calculations, some slightly different definitions of turnaround radius are being proposed in the literature. One of these, outlined in [19], goes through the definition of mass given in the previous chapter, that is the Hawking-Hayward mass (or Misner-Sharp-Hernandez mass in the case of spherical symmetry). As seen in the previous examples where there are both a gravitational effect and a repulsion one (that is, excluding the Schwarzschild case), the mass results in a sum of a least two terms: one is called "local", since it accounts for the effect of attraction due to matter, and the other is called "cosmological", in the sense that is mostly generated by the expansion of the Universe and so, from our actual cosmological model. The suggestion of [19] is to define the turnaround radius as that value for which these two terms, the local and the cosmological one, are equal in modulus. In other words, if the mass prescription yields

$$M_{HH} = M_{local} + M_{cosm} \tag{3.26}$$

then the turnaround radius $r'_t$ can be defined as that value of radius for which

$$|M_{local}| = |M_{cosm}| \tag{3.27}$$

where a prime is used to distinguish between the two different definitions. The reason behind this is rather intuitive: since the turnaround radius identifies the 2-surface in which the net gravitational effect is perfectly counterbalanced by the expansion, then one could expect that on this surface the two contributions to the mass, that are very different from a physical point of view, are equal. However, what one discovers is that the value found with this definition differs from the one obtained using the standard definition; the hope is that such a difference is not so large, in the sense that it can be neglected compared to the accuracy of our observations. From a theoretical point of view, instead, the only reason why this definition is still used, despite its discrepancy with the standard one, is that the calculation is rather "simpler" and, in the realistic situation of the perturbed FLRW metric, it exhibits no gauge dependence.

Thus, moving to our examples, we can start directly with the Schwarzschild-de Sitter case, since the Schwarzschild and the Reissner-Nordstrom ones are related to "local" effects, due to the gravitational field source (even in the second case, the charge, being possibly responsible of a repulsion, is stored in the source and it is not representing any cosmological counterpart); recalling that the mass function in the (2.20) metric is

$$m(r) = m + \frac{\Lambda}{2} r^3 \tag{3.28}$$

the new turnaround radius prescription (3.27) gives

$$r'_t = \sqrt[3]{\frac{2m}{\Lambda}} \tag{3.29}$$

The difference between the two definition is then clear; indeed, comparing this with (3.11), the ratio of the two obtained turnaround radii is

$$\frac{r'_t}{r_t} = \sqrt[3]{2} \tag{3.30}$$

So, the difference is net and cannot be completely canceled (it is a mathematical constant that cannot in any way be removed) but can be neglected since such a value is not appreciated on cosmological scale observations.

In the case of FLRW metric, through this definition, there is no turnaround radius, since only a cosmological term appears in (2.24), as expected and so one can directly analyze the realistic case of perturbed FLRW metric. This can be done either considering General Relativity or any scalar-tensor theory for gravity; in the first case, it is convenient to rewrite the mass function, at first order in scalar perturbation, in $G = 1$ units

$$m\,(t, r) \simeq m_N a + \frac{H^2 R^3}{2}\,(1 - 2\phi) \tag{3.31}$$

Then, the turnaround radius, through the prescription (3.27), is easily found as the solution of the following

$$m_N a = \frac{H^2 R_t'^3}{2}\left(1 - 2\phi\right) \tag{3.32}$$

that is

$$R_t' = \sqrt[3]{\frac{2m_N a}{H^2\left(1 - 2\phi\right)}} \tag{3.33}$$

which is the result found in (3.25) with again an extra factor

$$\frac{R_t'}{R_t} = \sqrt[3]{2} \tag{3.34}$$

Moreover, one can provide the Hamiltonian constraint $3H^2 = 8\pi\rho$ such that

$$R_t' = \sqrt[3]{\frac{3m_N a}{4\pi\rho\left(1 - 2\phi\right)}} \tag{3.35}$$

At this point we can restore the $G$ constant in order to make a future comparison with the modified gravity result

$$R_t' = \sqrt[3]{\frac{3m_N a}{4\pi G\rho\left(1 - 2\phi\right)}} \tag{3.36}$$

This procedure becomes very useful when modified gravity is considered; as said, no Birkhoff's theorem holds and very long calculations are behind the standard definition of turnaround radius; moreover, the result could be gauge-dependent and, in general, one has to consider even the terms in $\dot{t}^2$ and $\dot{t}\dot{R}$ in the game. The advantage of consider the turnaround radius definition through the mass is to be a gauge-invariant result, since mass naturally is. Therefore, one can use the mass prescription (2.62) and the turnaround radius, at this level of approximation, turns to be the solution of the following

$$m_N a = \frac{H^2 R_t'^3}{2}\left(1 - 2A\right) = \frac{4\pi\left(\rho + \rho_\phi\right) R_t'^3}{3\phi(t,r)}\left(1 - 2A\right) \tag{3.37}$$

such that

$$R_t' = \sqrt[3]{\frac{3m_N a\phi(t,r)}{4\pi\left(\rho + \rho_\phi\right)\left(1 - 2A\right)}} \tag{3.38}$$

Notice the similarity with (3.36) (assuming Birkhoff's theorem holds at this level of perturbations even for scalar-tensor theories); the slight difference is that the previous has no gravitational constant but this varies through the space-time and, moreover, the density term at the denominator should be larger due to the presence of the scalar field. The goal of observations should be to find which of these two expressions best fits the data; the previous expression, however, could in principle account for dark matter effects without adding it a priori: in fact, in the context of General Relativity, the existence of dark matter should decrease the value of the turnaround radius (since more gravitational force has to be considered) with respect to the case in which there is no dark matter whereas, adopting scalar-tensor theories, such a reduced value can be obtained directly without adding it. Finally, for completeness, one could calculate the turnaround radius also from (2.64) [31]

$$\begin{aligned} H_{(0)}^2 = & \frac{2}{3a^2}\left(\frac{A_{,r}}{r}+\frac{B_{,r}}{r}-A_{,rr}-B_{,rr}\right)+ \\ & +\frac{16\pi}{3}\left[-4\left(\rho_{(0)}+P_{(0)}\right)A+\delta\rho-2P_{(0)}B\right]+ \\ & +\frac{\omega_{0,r}\delta\phi-2\omega_0 A}{3a^2\phi_{(0)}}\phi_{(0),\eta}^2+ \\ & \left.+\frac{4a_{,\eta}}{a^3}\phi_{(0),\eta}A+\frac{2\delta\phi_{,r}}{a^2 r}+\frac{V_{0,r}}{3}\delta\phi\right\} \end{aligned} \tag{3.39}$$

A final remark is worth to take place: even the last formula is performed in the context of spherical symmetry; indeed, the question whether is possible to calculate the turnaround radius in a more general symmetry goes beyond our scopes. In principle, one has more involved geodesic equations, if first definition is used, or the mass prescription (2.40) should be considered in its full generality, if the second definition is adopted, and so there is no longer Misner-Sharp-Hernandez mass (2.49); however, if one attempts such a calculation, he must specify also the angles because the value of this more involved turnaround radius does not change with respect to the radial coordinate (and time coordinate) only but also with respect to the angular ones.

### 3.2.1 Horndeski's turnaround radius

The time has come to calculate the turnaround radius in Horndeski's theory (in the hypothesis of spherical symmetry), considering the perturbed FLRW metric. As already said, the first definition (3.13) is not useful in practical calculation, so one can directly proceed with the definition through mass. Thus, we can consider the Horndeski's mass prescription given in (2.66): this tells us that there is no difference with the Misner-Sharp-Hernandez mass prescription (2.49) except for the fact that the metric is redefined and instead of a generic scalar field, there is $G_4$ as effective $G_{eff}^{-1}$. Thus, we should have the same result of any other scalar-tensor theory (3.38), that is

$$R_t' = \sqrt[3]{\frac{3m_N a' G_4(\phi, X)}{4\pi\left(\rho+\rho_{Horn}\right)\left(1-2A'\right)}} \tag{3.40}$$

where, instead of the scalar field $\phi$, there is $G_4$, the generic $\rho_\phi$ is substituted with $\rho_{Horn}$ which represent the extra term coming from the Hamiltonian constraint of the effective energy-momentum tensor and a prime indicates that those quantities are part of the redefined metric $g'_{\mu\nu} = g_{\mu\nu} - \frac{G_5}{G_4}\phi_{,\mu\nu}$.

## 3.3 A suggestion: surface gravity

There is a vastity of literature concerning surface gravity and, somewhat like the mass, there is no a true universal definition but only some most accreditated ones. In Newtonian mechanics, there is no ambiguity of what surface gravity is: by means of Newton's gravitational law, a test particle with mass much smaller than that of a massive object suffers an acceleration

$$g = \frac{GM}{R^2_{surf}} \tag{3.41}$$

where the equivalence between gravitational and inertial mass is again assumed and $R_{surf}$ indicates the radius of a 2-surface that feels this acceleration. This is the surface gravity in Newtonian mechanics and it is measured by units of $g \sim 9.8 \quad m/s^2$. In General Relativity, there is no a single scalar field that expresses the gravity through a definite law but the space-time is responsible for it. Let us restrict again ourselves to the spherically symmetric case and let us consider the following definition of surface gravity

$$g = \sqrt{\left| g^{\mu\nu}\left(\sqrt{|\xi_\alpha \xi^\alpha|}\right)_{,\mu}\left(\sqrt{|\xi_\alpha \xi^\alpha|}\right)_{,\nu}\right|}\Bigg|_{R_{surf}} \tag{3.42}$$

where $\xi^\alpha$ is a Killing vector. In order to make some comparison, consider the following metric

$$ds^2 = -f(r)^2 dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \tag{3.43}$$

Due to Birkhoff's theorem, a possible Killing vector is $\xi^\alpha = (1, 0, 0, 0)$; then, through (3.42), the surface gravity results

$$g = \frac{1}{2}\partial_r f(r)|_{r_{surf}} \tag{3.44}$$

which is equivalent to the radial acceleration found in (3.6). In particular, for Schwarzschild space-time $f(r) = 1 - \frac{2m}{r}$, the surface gravity is

$$g = \frac{m}{r^2_{surf}} \tag{3.45}$$

which is equivalent to the Newtonian definition (3.41) (since $m = GM$). Thus, in agreement with the Newtonian definition, the surface gravity even in this case represents the (radial) acceleration felt at a certain radius. The problems arise when there is no time-independence in the metric and the vector used in the previous derivation is no longer a Killing one. But before considering such a case, it is worth to repeat the derivation but using a different Killing vector, since it is not the unique possible one. Indeed, one can likewise consider the following: $\xi^\alpha = (0, 0, 0, 1)$. This yields the following (by using $\theta = \frac{\pi}{2}$)

$$g = \sqrt{f(r)} \tag{3.46}$$

This seems to tell us nothing but, assuming that we can expand $f(r)$, we can write

$$g = \sqrt{f(r)} = \sqrt{1 + (f(r) - 1)} \simeq 1 + \frac{1}{2}(f(r) - 1) \tag{3.47}$$

Recalling the Misner-Sharp-Hernandez definition for the mass, one can surprisingly find that

$$M \simeq -R(g - 1) \tag{3.48}$$

This can be useful in situations in which there is no other choices of Killing vectors. This is the case of the perturbed FLRW metric (2.27); so, using the same Killing vector due to rotational invariance, the surface gravity can be obtained as (Appendix E)

$$g = \sqrt{1 + r^2\phi_{,r}^2 (1 - 2\phi)^{-2} - 2r\phi_{,r}(1 - 2\phi)^{-1} - \mathcal{H}^2 r^2 \frac{1 - 2\phi}{1 + 2\phi}} \tag{3.49}$$

Assuming that expansion is allowed and reintroducing $R = ar\sqrt{1 - 2\phi}$ and $\mathcal{H} = aH$, therefore

$$g \simeq 1 + \frac{1}{2} r^2\phi_{,r}^2 (1 - 2\phi)^{-2} - r\phi_{,r}(1 - 2\phi)^{-1} - \frac{\mathcal{H}^2 R^2}{2(1 + 2\phi)} \tag{3.50}$$

Thus, at this point, it is simple to check that

$$-R(g - 1) \simeq -\frac{1}{2} ar^3\phi_{,r}^2 (1 - 2\phi)^{-\frac{3}{2}} + ar^2\phi_{,r}(1 - 2\phi)^{-\frac{1}{2}} + \frac{\mathcal{H}^2 R^3}{2(1 + 2\phi)} = M \tag{3.51}$$

is equivalent to (2.29), in agreement with (3.48).

# Conclusions

Most probably, we are still far from the true theory of gravity; despite the numerous attempts, there is no a single theory that matches with observations and, specially, with the other fundamental physical theory nowadays, the Quantum Mechanics. General Relativity is still considered the fundamental theory of gravity, maybe because it is the "simplest" geometric theory, in the sense of Lovelock's theorem, and perhaps because its predictions were too strong to think that such a theory could be sat aside. However, such predictions, that were shown to be successful in the context of our solar system (Mercury's perihelion precession, light rays deflected by the Sun), revealed to be inaccurate on larger scales (we have seen, for instance, the problems about the rotation velocity of some galaxies): this led physicists to consider the presence of a type of matter that does not emit light, the dark matter. The introduction of extended theories of gravity, like the scalar-tensor theories, was made in order to avoid such a prescription and to try to explain the observed data within the theory itself. The subleading principle of such theories is that gravity does not merely have to do with the geometry but a scalar field has to be reintroduced; this can be associated to the inverse of the gravitational constant which now can be considered as, at least, time dependent. The rise of such a scalar field can be justified in several ways: it can be associated to the dilaton emerging from the reduction of higher dimensional theories or even with the need to introduce the inflaton, a scalar field that could drive the inflation. The interesting fact, however, is that, despite the fact that the resulting field equations are more difficult and, then, less treatable from an analytical point of view, they can reproduce some predictions that General Relativity does not without adding dark matter. Nevertheless, the new question is how one can choose between the various possible scalar-tensor theories; this led, for example, Horndeski to consider only second-order field equations and one can naturally guess that the solution is finding the theory which best fits with the data (General Relativity plus dark sector included). This is one of the aim of this thesis: introducing a new observable, the turnaround radius, and making predictions of such a value in the context of the various theories of gravity, with the hope that a future research could compare these with observational data. In making such predictions, another concept is exploited, the mass function (and so the quasi-local energy) in a general theory of gravity; motivated by the need of uniquely describing a gravitational energy, the 2+2 formalism is developed in order to unambiguously obtain an Hamiltonian which naturally contains gravity. After showing the connection with the natural concept of mass in Newtonian theory, at least in spherical symmetry, it is shown that, in cosmological context, the mass function is always the sum of two very different contributions, a local one, that is responsible for the attraction of local massive sources, and a cosmological one, which stays for the expansion of Universe (but in general, on the basis of the cosmological model, it can represent whatever the Universe is shown to do, either expanding or contracting). This suggests to define the turnaround radius in an alternative way, that is the point in

which these two contributions are equal. Despite its small difference with the standard definition, the relative simplicity with which it can be calculated, together with the fact that the result is certainly gauge-independent, promotes this alternative definition of the turnaround radius to the most useful one.
At this point, it is important to remark that scalar-tensor theories are only a possible extension of the General Relativity but the truth is that there could be many others, as, for instance, the scalar-vector-tensor theories [35] (in practice, one can think at gravity not only as a spin-2 field). More remarkably, is the fact that all these theories, General Relativity theory included, are classical: the field equations are obtained by means of the least action principle. Indeed, we are very far from a complete theory of Quantum Gravity, despite of numerous attempts nowadays are made [36], [37], [38]. In this thesis, indeed, the realm of Quantum Mechanics is neglected, and this can, in principle, be done if no Early Universe and no singularities are analysed in detail. So, even if a theory could be chosen among all, the next step would be quantizing it; however, are we sure of the correcteness of our quantizing methods? Quantum Field Theories and all its extensions provide a very interesting framework, where miraculous particle experiments confirmed many predictions [39], but it is right treating gravity merely as a spin-2 field? After all, gravity has always had a behaviour of a very different type of force: it seems to be only attractive and it has to do with the structure of the space-time itself; with these premises, is it really possible to connect it with other forces without losing anything? These are the questions that in the future we have to answer; String Theory, among many other promising theories, claims that we already have an answer but, at present time, it is impossible to test even only one of its predictions [40]. Moreover, are we even sure of its theoretical fundations? As it always has been, the scientific research has to proceed step-by-step and there is still much that we can learn from General Relativity, even if one day it will be surpassed: humility is the main ingredient of a physical, or better, scientific theory and there could be something that we did not even think about.

# Appendices

# Appendix A

# f(R) theory's field equations

Let us start with the following Lagrangian in vacuum

$$\mathcal{L}_{vacuum} = f(R) \tag{A.1}$$

As usual, the field equations are obtained varying the metric with respect to the metric field

$$\delta S = \delta \int d^4x\sqrt{-g}f(R) = 0 \tag{A.2}$$

Denoting $f'(R) = \frac{df(R)}{dR}$ and using the (1.9), then

$$\begin{aligned}
\delta S = \delta \int d^4x\sqrt{-g}f(R) &= \int d^4x \left[\delta\left(\sqrt{-g}\right) f(R) + \sqrt{-g}\delta\left(f(R)\right)\right] = \\
&= \int d^4x \left[-\frac{1}{2}\sqrt{-g}g_{\mu\nu}f(R)\delta g^{\mu\nu} + \sqrt{-g}f'(R)\delta R\right] = \\
&= \int d^4x \left[-\frac{1}{2}\sqrt{-g}g_{\mu\nu}f(R)\delta g^{\mu\nu} + \sqrt{-g}f'(R)R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}f'(R)g^{\mu\nu}\delta R_{\mu\nu}\right] = \\
&= \int d^4x\sqrt{-g}\left[f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R)\right]\delta g^{\mu\nu} + \int d^4x\sqrt{-g}f'(R)g^{\mu\nu}\delta R_{\mu\nu}
\end{aligned} \tag{A.3}$$

Recalling the definition given in (1.12) and integrating by parts, the second integral becomes

$$\begin{aligned}
&\int d^4x\sqrt{-g}f'(R)g^{\mu\nu}\delta R_{\mu\nu} = \int d^4x\sqrt{-g}f'(R)W^{\sigma}_{;\sigma} = \\
&= \int d^4x \left[\sqrt{-g}f'(R)W^{\sigma}\right]_{;\sigma} - \int d^4x \left[\sqrt{-g}f'(R)\right]_{;\sigma} W^{\sigma} = \\
&= -\int d^4x \left[\sqrt{-g}f'(R)\right]_{;\sigma} W^{\sigma}
\end{aligned} \tag{A.4}$$

From the definition of Christoffel symbols, one finds that

$$W^\sigma \equiv g^{\mu\nu}\delta\Gamma^\sigma_{\mu\nu} - g^{\mu\sigma}\delta\Gamma^\nu_{\mu\nu} = (g_{\mu\nu}\delta g^{\mu\nu})^{;\sigma} - (g_{\mu\nu}\delta g^{\sigma\nu})^{;\mu} \tag{A.5}$$

so the previous integral becomes

$$-\int d^4x \left[\sqrt{-g}f'(R)\right]_{;\sigma} W^\sigma = \int d^4x \left[\sqrt{-g}f'(R)\right]_{;\sigma} [(g_{\mu\nu}\delta g^{\sigma\nu})^{;\mu} - (g_{\mu\nu}\delta g^{\mu\nu})^{;\sigma}] \tag{A.6}$$

Integrating by parts, canceling the divergence terms that do not contribute and relabelling appropriately the indices, one gets

$$\int d^4x\sqrt{-g}\left[f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + g_{\mu\nu}\left(\sqrt{-g}f'(R)\right)^{;\sigma}_{;\sigma} - g_{\sigma\nu}\left(\sqrt{-g}f'(R)\right)^{;\sigma}_{;\mu}\right]\delta g^{\mu\nu} \tag{A.7}$$

Because of the arbitrariness of $\sqrt{-g}\delta g^{\mu\nu}$, the field equations (1.21) are regained.

# Appendix B

# Scalar-tensor theories' field equations

Let us now consider the following Lagrangian in vacuum

$$\mathcal{L}_{ST} = \frac{1}{16\pi}\left(\phi R - \frac{\omega}{\phi} g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - V(\phi)\right) \tag{B.1}$$

where, for simplicity, as in the Brans-Dicke theory, the parameter $\omega$ is assumed independent from the scalar field. The field equations are now derived first varying with respect to the metric field and then, to the scalar field; proceeding with the first variation, one gets

$$\begin{aligned}\delta_g S = \delta_g \int d^4x\sqrt{-g}\left[\phi R - \frac{\omega}{\phi} g^{\mu\nu}\phi_{;\mu}\phi_{;\nu}\right] = \\ = \int d^4x \phi \delta_g\left(\sqrt{-g}R\right) - \int d^4x \frac{\omega}{\phi}\delta_g\left(\sqrt{-g}g^{\mu\nu}\right)\phi_{;\mu}\phi_{;\nu} = 0\end{aligned} \tag{B.2}$$

The first integral gives exactly the Einstein tensor, as seen in the derivation of the Einstein equations, for arbitrariness of $\sqrt{-g}\delta g^{\mu\nu}$;even the second integral can be expressed in terms of $\sqrt{-g}\delta g^{\mu\nu}$, by means of the (1.9)

$$\begin{aligned}&-\frac{\omega}{\phi}\delta_g\left(\sqrt{-g}g^{\mu\nu}\right)\phi_{;\mu}\phi_{;\nu} = -\frac{\omega}{\phi}\delta_g\left(\sqrt{-g}\right)g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - \frac{\omega}{\phi}\sqrt{-g}\delta_g\left(g^{\mu\nu}\right)\phi_{;\mu}\phi_{;\nu} = \\ &= -\frac{\omega}{\phi}\left(-\frac{1}{2}g_{\mu\nu}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} + \phi_{;\mu}\phi_{;\nu}\right)\sqrt{-g}\delta g^{\mu\nu} = -\frac{\omega}{\phi}\left(\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\alpha}\phi^{;\alpha}\right)\sqrt{-g}\delta g^{\mu\nu}\end{aligned} \tag{B.3}$$

So, for the arbitrariness of $\sqrt{-g}\delta g^{\mu\nu}$, one gets the (1.34) (in the case of constant parameter $\omega$.

In order to find the (1.35), one has to vary with respect to $\phi$, so

$$\begin{aligned}\delta_\phi S &= \delta_\phi \int d^4x\sqrt{-g}\left[\phi R - \frac{\omega}{\phi}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu}\right] = \\ &= \int d^4x\sqrt{-g}R\delta_\phi\phi - \int d^4x\sqrt{-g}\omega g^{\mu\nu}\delta_\phi\left(\frac{1}{\phi}\phi_{;\mu}\phi_{;\nu}\right) = 0\end{aligned} \tag{B.4}$$

Manipulating the second integral as follows

$$\begin{aligned}&-\int d^4x\sqrt{-g}\omega g^{\mu\nu}\delta_\phi\left(\frac{1}{\phi}\phi_{;\mu}\phi_{;\nu}\right) = \\ &= -\int d^4x\sqrt{-g}\omega g^{\mu\nu}\left(-\frac{\phi_{;\mu}\delta_\phi\phi}{\phi^2}\phi_{;\mu}\phi_{;\nu} + \frac{1}{\phi}\delta_\phi\left(\phi_{;\mu}\right)\phi_{;\nu} + \frac{1}{\phi}\phi_{;\mu}\delta_\phi\left(\phi_{;\nu}\right)\right) = \\ &= -\int d^4x\sqrt{-g}\omega g^{\mu\nu}\left(\frac{1}{\phi^2}\phi_{;\mu}\phi_{;\nu} - \frac{2}{\phi}\phi^{;\nu}_{;\mu}\right)\delta_\phi\phi = \\ &= \int d^4x\omega\left(-\frac{1}{\phi^2}\phi_{;\mu}\phi^{;\mu} + \frac{2}{\phi}\phi^{;\mu}_{;\mu}\right)\sqrt{-g}\delta_\phi\phi\end{aligned} \tag{B.5}$$

Substituting, one obtains

$$\int d^4x\left(R - \frac{\omega}{\phi^2}\phi_{;\mu}\phi^{;\mu} + \frac{2\omega}{\phi}\phi^{;\mu}_{;\mu}\right)\sqrt{-g}\delta_\phi\phi = 0 \tag{B.6}$$

For arbitrariness of $\sqrt{-g}\delta_\phi\phi$, the field equations (1.35) are regained (in the hypothesis of constant $\omega$).

# Appendix C

# Mass function in perturbed FLRW metric: the General Relativity case

Let us calculate the mass function in the context of the following perturbed FLRW metric

$$ds^2 = a^2(\eta)\left[-\left(1+2\phi(r)\right)d\eta^2 + \left(1-2\phi(t,r)\right)\left(dr^2 + r^2 d\Omega^2\right)\right] \tag{C.1}$$

through the Misner-Sharp-Hernandez prescription

$$m\left(t,r\right) = \frac{R}{2G}\left(1 - R_{;\mu}R^{;\mu}\right) \tag{C.2}$$

The areal radius, in this case, is

$$R(\eta,r) = a(\eta)r\sqrt{1-2\phi(r)} \tag{C.3}$$

Thus, by direct calculation, one finds

$$\begin{aligned} R_{;\mu}R^{;\mu} &= g^{\eta\eta}R_{,\eta}R_{,\eta} + g^{rr}R_{,r}R_{,r} = \\ &= -\frac{\left(a_{,\eta}r\sqrt{1-2\phi}\right)^2}{a^2\left(1+2\phi\right)} + \frac{\left(a\sqrt{1-2\phi} - \frac{ar\phi_{,r}}{\sqrt{1-2\phi}}\right)^2}{a^2\left(1-2\phi\right)} = \\ &= -\mathcal{H}^2 r^2\frac{1-2\phi}{1+2\phi} + 1 + \frac{r^2\phi_{,r}^2}{\left(1-2\phi\right)^2} - \frac{2r\phi_{,r}}{1-2\phi} \end{aligned} \tag{C.4}$$

where $\mathcal{H}(\eta) = \frac{a_{,\eta}(\eta)}{a(\eta)}$. Knowing that $\mathcal{H}(\eta) = a(t)H(t)$, one obtains

$$m\left(t,r\right) = \frac{1}{G}\left[-\frac{r^3\phi_{,r}^2}{2}\left(1-2\phi\right)^{-\frac{3}{2}} + r^2\phi_{,r}\left(1-2\phi\right)^{-\frac{1}{2}}\right]a + \frac{H^2R^3}{2G\left(1+2\phi\right)} \tag{C.5}$$

which is the (2.29). To the first order in $\phi$

$$m\left(t,r\right) \simeq \frac{r^2\phi_{,r}a}{G} + \frac{H^2R^3}{2G}\left(1-2\phi\right) + O(\phi^2) \tag{C.6}$$

The first term can be written in a slightly different way, using the divergence

$$
\begin{aligned}
\nabla^2\phi &= \frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}g^{ab}\partial_b\phi\right) = \\
&= \frac{1}{a^4\left(1+2\phi\right)^{\frac{1}{2}}\left(1-2\phi\right)^{\frac{3}{2}}r^2 sin\theta}\partial_r\left[a^4\left(1+2\phi\right)^{\frac{1}{2}}\left(1-2\phi\right)^{\frac{3}{2}}r^2 sin\theta\frac{1}{a^2\left(1+2\phi\right)}\phi_{,r}\right] = \\
&= \frac{1}{a^2\left(1+2\phi\right)^{\frac{1}{2}}\left(1-2\phi\right)^{\frac{3}{2}}r^2}\partial_r\left[\left(1+2\phi\right)^{\frac{1}{2}}\left(1-2\phi\right)^{\frac{1}{2}}r^2\phi_{,r}\right] = \\
&= \frac{\phi_{,r}^2}{a^2\left(1+2\phi\right)\left(1-2\phi\right)} - \frac{\phi_{,r}^2}{a^2\left(1-2\phi\right)^2} + \frac{2\phi_{,r}}{a^2\left(1-2\phi\right)r} + \frac{\phi_{,rr}}{a^2\left(1-2\phi\right)}
\end{aligned}
\tag{C.7}
$$

At the first order, one simply has

$$\nabla^2\phi \simeq \frac{2\phi_{,r}}{a^2 r} + \frac{\phi_{,rr}}{a^2} = \frac{1}{a^2 r^2}\frac{d}{dr}\left(r^2\phi_{,r}\right) \tag{C.8}$$

It therefore follows that, by following the (2.32), the mass function can be written, at first order, as

$$m\left(t,r\right) \simeq m_N a + \frac{H^2 R^3}{2G}\left(1-2\phi\right) \tag{C.9}$$

# Appendix D

# Mass function in perturbed FLRW metric: the scalar-tensor theories case

Now, the general perturbed FLRW metric (in Newtonian gauge) is written as

$$ds^2 = a^2(\eta)\left[-\left(1+2A(\eta,r)\right)d\eta^2 + \left(1-2B(\eta,r)\right)\left(dr^2 + r^2 d\Omega^2\right)\right] \tag{D.1}$$

due to the fact that there is no certainty that Birkoff's theorem holds. The areal radius is now

$$R(\eta,r) = a(\eta) r \sqrt{1-2B(\eta,r)} \tag{D.2}$$

In calculating the mass function via the Misner-Sharp-Hernandez prescription (2.49), one needs of

$$\begin{aligned}
R_{;\mu}R^{;\mu} &= g^{\eta\eta}R_{,\eta}R_{,\eta} + g^{rr}R_{,r}R_{,r} = \\
&= -\frac{\left(a_{,\eta} r\sqrt{1-2B} - \frac{arB_{,\eta}}{\sqrt{1-2B}}\right)^2}{a^2\left(1+2A\right)} + \frac{\left(a\sqrt{1-2B} - \frac{arB_{,r}}{\sqrt{1-2B}}\right)^2}{a^2\left(1-2B\right)} = \\
&= -\mathcal{H}^2 r^2 \frac{1-2B}{1+2A} + \frac{2\mathcal{H} r^2 B_{,\eta}}{1+2A} - \frac{r^2 B_{,\eta}^2}{\left(1+2A\right)\left(1-2B\right)} + \\
&\quad + 1 + \frac{r^2 B_{,r}^2}{\left(1-2B\right)^2} - \frac{2rB_{,r}}{1-2B}
\end{aligned} \tag{D.3}$$

Again by knowing that $\mathcal{H}(\eta) = a(t)H(t)$, what one finds is that

$$\begin{aligned} M_{ST}(t,r) =& \phi\left[-\frac{r^3B_{,r}^2}{2}(1-2B)^{-\frac{3}{2}} + r^2B_{,r}(1-2B)^{-\frac{1}{2}}\right]a+ \\ &-\phi a^2Hr^3B_{,\eta}\frac{\sqrt{1-2B}}{1+2A} + \frac{\phi}{2}ar^3B_{,\eta}^2\frac{1}{(1+2A)\sqrt{1-2B}}+ \\ &+\frac{\phi H^2R^3}{2(1+2A)} \end{aligned} \tag{D.4}$$

and, at the first order in scalar perturbations

$$M_{ST}(t,r) \simeq \phi r^2B_{,r}a - \phi a^2Hr^3B_{,\eta} + \frac{\phi H^2R^3}{2}(1-2A) \tag{D.5}$$

The first term are shown to be something related again to the divergence

$$\begin{aligned} \nabla^2 B =& \frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}g^{ab}\partial_b B\right) = \\ =& \frac{1}{\sqrt{-g}}\partial_r\left(\sqrt{-g}g^{rr}\partial_r B\right) + \frac{1}{\sqrt{-g}}\partial_\eta\left(\sqrt{-g}g^{\eta\eta}\partial_\eta B\right) \end{aligned} \tag{D.6}$$

This time the function $B$ depends also on (conformal) time and so the calculation is more involved in the sense that one has to add a piece to the divergence calculated in the previous Appendix, coming from the time dependence. Proceeding in a similar way for the second term, one gets

$$\begin{aligned} &\frac{1}{\sqrt{-g}}\partial_\eta\left(\sqrt{-g}g^{\eta\eta}\partial_\eta B\right) = \\ &-\frac{2a_{,\eta}B_{,\eta}}{a^3(1+2A)} + \frac{A_{,\eta}B_{,\eta}}{a^2(1+2A)^2} + \frac{3B_{,\eta}^2}{a^2(1+2A)(1-2B)} - \frac{B_{,\eta\eta}}{a^2(1+2A)} \end{aligned} \tag{D.7}$$

At the first order

$$\frac{1}{\sqrt{-g}}\partial_\eta\left(\sqrt{-g}g^{\eta\eta}\partial_\eta B\right) \simeq -\frac{2a_{,\eta}B_{,\eta}}{a^3} - \frac{B_{,\eta\eta}}{a^2} = -\frac{1}{a^4}\frac{d}{d\eta}\left(a^2B_\eta\right) \tag{D.8}$$

from which one recovers the (2.61) and then, (2.62).

# Appendix E

# Surface gravity in perturbed FLRW metric

In order to correctly use the definition given in (3.42), let's define the intermediate vector

$$l_\mu = \left(\sqrt{\xi_\alpha \xi^\alpha}\right)_{,\mu} \tag{E.1}$$

Considering the perturbed FLRW metric (in Newtonian gauge)

$$ds^2 = a^2(\eta)\left[-\left(1+2\phi(r)\right)d\eta^2 + \left(1-2\phi(t,r)\right)\left(dr^2 + r^2 d\Omega^2\right)\right] \tag{E.2}$$

one firstly has, considering $\xi^\alpha = (0,0,0,1)$

$$\xi_\alpha \xi^\alpha = g_{\alpha\beta}\xi^\alpha \xi^\beta = g_{33} = a^2\left(1-2\phi\right)r^2 sin^2\theta \tag{E.3}$$

then, selecting the plane identified by $\theta = \frac{\pi}{2}$,

$$\sqrt{\xi_\alpha \xi^\alpha} = ar\sqrt{1-2\phi} \tag{E.4}$$

Thus, one has

$$l_\mu = \left[a_{,\eta} r\sqrt{1-2\phi}, a\left(\sqrt{1-2\phi} - \frac{r\phi_{,r}}{\sqrt{1-2\phi}}\right), 0, 0\right] \tag{E.5}$$

The norm of such a vector therefore turns out to be

$$\begin{aligned} l_\mu l^\mu = g^{\mu\nu} l_\mu l_\nu &= -\frac{a_{,\eta}^2 r^2\left(1-2\phi\right)}{a^2\left(1+2\phi\right)} + \frac{a^2\left(\sqrt{1-2\phi} - \frac{r\phi_{,r}}{\sqrt{1-2\phi}}\right)^2}{a^2\left(1-2\phi\right)} = \\ &= 1 + r^2\phi_{,r}^2\left(1-2\phi\right)^{-2} - 2r\phi_{,r}\left(1-2\phi\right)^{-1} - \mathcal{H}^2 r^2 \frac{1-2\phi}{1+2\phi} \end{aligned} \tag{E.6}$$

Since

$$g = \sqrt{l_\mu l^\mu} \tag{E.7}$$

one finally gets the (3.49).

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