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# Flattened 5-Brane Axion Monodromy in the Large Volume Scenario

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**Abstract**

After a quick review of the fundamentals of $\mathcal{N} = 1$ supergravity theories and inflationary cosmology, we focus on a class of promising inflation models built within the framework of warped compactifications of type IIB string theory on Calabi-Yau orientifolds. In these models the inflaton is an orientifold-odd closed string axion whose potential is induced by an NS5-brane wrapped on a 2-cycle of the compactification space. This system enjoys a monodromy property that breaks the periodic nature of the axion potential, generating a linear axion potential that can drive inflation with the prediction of large primordial tensor modes. We show that this class of models can be combined with successful moduli stabilization in the Large Volume Scenario improving both their theoretical robustness and the accordance of their predictions with current bounds from CMB observations. A crucial effect which we exploit is the flattening of the axion potential due to the moduli backreaction.

# Contents

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We can assert that this is the most challenging part of the thesis, as, at the moment, no one has found a mathematical formula to express gratitude and affection to other people. Since I have never experienced being on a closed time-like curve, I cannot write the people I want to thank all at once.

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# Introduction

Current observations of the universe imply that we live in a spatially almost homogeneous, isotropic and almost spatially flat universe at large scales. Furthermore, measurements of the Cosmic Microwave Background (CMB) provide us, with strong evidence, that regions in our universe which we now see to be causally disconnected have been correlated in the past.
One of the most successful models addressing these issues with predictions confirmed by the experiments is the scenario of cosmological inflation, which describes a period of the early universe when space underwent extremely rapid exponentially accelerated expansion.
One of the characteristics of this model is that such an epoch of accelerated expansion is determined by the dynamics of a scalar field and, in particular, of its scalar potential. This field is called *inflaton*, and, by now, there have been many proposed candidates. More precisely, for successful inflation to happen, the inflaton dynamics has to satisfy certain constraints. These can be rather easily accommodated in the case of inflationary models where the inflaton traverses a large field range between beginning and the end of inflation. This class of models constitutes one of the least fine-tuned mechanisms of inflation and assumes the inflaton traversing a trans-Planckian field space.
Furthermore, the inflationary dynamic is intrinsically sensitive to the assumptions about the physics at energies far above those probed by particle colliders at the moment. That means that even if we have the best of the effective field theories, which has been proven with success at low energy, if we want to describe inflation, we need to make some assumptions on the UV completion of our theory. Therefore inflation provides a good framework in which we can test any UV complete theory of gravity. We might say that one of the most promising quantum gravity theory that we have at the moment is string theory which can reproduce, in the low energy limit, Standard Model-like constructions and Einstein gravity.

In this thesis we will therefore explore the possibility to realize inflation in the framework of the effective field theories coming from the compactification of ten-dimensional supergravity theories which correspond to the massless spectrum of certain types of string theories.

To be more precise, we will work with ten-dimensional type IIB supergravity theories, compactified on six-dimensional Calabi-Yau orientifolds, in order to obtain a four-dimensional effective field theory description.

We will develop a model of large-field-inflation in which the inflaton is an axion field with a linear potential.
We recall that the axion is in general a scalar field which enjoys a shift symmetry, that in supergravity theory arises after dimensional reduction from the integration of a $p$-form over a $p$-cycle in the compactification space. In our case we are going to consider a 2-form axion coming from the $C_2$ 2-form.
In order to construct the above mentioned linear potential, we will consider a monodromy introduced by an NS5-brane wrapped on the same 2-cycle which is the domain of the 2-form field in the compactification space. This monodromy will explicitly break the periodic nature of the $C_2$ axion and give a viable potential to drive inflation.

In order to write the inflaton potential, we will have to stabilize all the moduli fields arising from the compactification, which are scalar fields associated to the geometry of the compactification space. In more detail, we will assume that the axio-dilaton and the complex structure moduli are stabilized supersymmetrically, while we will explicitly stabilize the Kähler moduli in the large volume scenario where the compactification space volume turns out to be large in Planck units.

The thesis is organized in four chapter: in the first we will discuss the general features of supersymmetry and supergravity theories; in the second chapter we will explain all the background concerning inflationary cosmology, specifying the relevant experimental constraints that our model has to satisfy; in the third chapter we will discuss the compactification of type IIB theories on Calabi-Yau orientifolds; and in the fourth chapter we will present our model discussing moduli stabilization and comparing its predictions with the measured values of the main cosmological observables.

# Chapter 1

# 4D $\mathcal{N} = 1$ Supersymmetry and Supergravity Effective Actions

In the writing of this chapter I followed the following main references [WB83; FP12; QKS10], which summarize the work done in the field of supersymmetric field theories.

## 1.1 Supersymmetry in 4D

### 1.1.1 Poincaré Algebra and Representations

The Poincaré group is the symmetry group of special relativity, which any kind of QFT has to satisfy and it acts on spacetime coordinates $x^\mu$ as follows:

$$x^\mu \mapsto x'^\mu = \underbrace{\Lambda^\mu{}_\nu}_{\text{Lorentz}} x^\nu + \underbrace{a^\mu}_{\text{translation}} \tag{1.1}$$

The generators of the Poincaré group are $M^{\mu\nu}$ and $P^\sigma$ with the following algebra:

$$\begin{cases} [P^\mu, P^\nu] &= 0 \\ [M^{\mu\nu}, P^\sigma] &= i\left(P^\mu \eta^{\nu\sigma} - P^\nu \eta^{\mu\sigma}\right) \\ [M^{\mu\nu}, M^{\rho\sigma}] &= i\left(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho}\right) \end{cases} \tag{1.2}$$

Where a 4-dimensional matrix representation for the $M^{\mu\nu}$ is given by:

$$(M^{\rho\sigma})^\mu_\nu = i\left(\eta^{\mu\nu}\delta^\rho_\nu - \eta^{\rho\mu}\delta^\sigma{}_\nu\right) \tag{1.3}$$

Let us remind us some properties of the Lorentz group, i.e. $SO(3,1)$. First we might say that:

$$SO(3,1) \cong SU(2) \oplus SU(2) \quad \text{(locally)} \tag{1.4}$$

In fact we can express the generators of rotations $J_i$ and the ones of the boosts $K_i$ as:

$$\begin{cases} J_i & = \frac{1}{2}\epsilon_{ijk} M_{jk} \\ K_i & = M_{0i} \end{cases} \tag{1.5}$$

Then we might consider the following linear combinations of $A_i$ and $B_i$:

$$\begin{cases} A_i & = \frac{1}{2}\left(J_i + iK_i\right) \\ B_i & = \frac{1}{2}\left(J_i - iK_i\right) \end{cases} \tag{1.6}$$

Then from the Poincaré algebra we may find:

$$\begin{cases} [J_i, J_j] & = i\epsilon_{ijk} J_k \\ [J_i, K_j] & = i\epsilon_{ijk} K_k \\ [K_i, K_j] & = -i\epsilon_{ijk} K_k \end{cases} \quad \Rightarrow \quad \begin{cases} [A_i, A_j] & = i\epsilon_{ijk} A_k \\ [B_i, B_j] & = i\epsilon_{ijk} B_k \\ [A_i, B_j] & = 0 \end{cases} \tag{1.7}$$

Which is nothing but the $SU(2)$ algebra. Furthermore under parity transformations we have that:

$$P = \begin{cases} x^0 & \mapsto x^0 \\ \mathbf{x} & \mapsto -\mathbf{x} \end{cases} \quad \Rightarrow \quad \begin{cases} J_i & \mapsto J_i \\ K_i & \mapsto -K_i \end{cases} \quad \Rightarrow \quad A_i \quad \leftrightarrow \quad B_i \tag{1.8}$$

Thus we can interpret $\mathbf{J} = \mathbf{A} + \mathbf{B}$ as the physical spin.

Let us recall the following homeomorphism between $SO(3,1)$ and $SL(2,\mathbb{C})$ :

$$SO(3,1) \cong SL(2,\mathbb{C}) \tag{1.9}$$

In order to see such homeomorphism, let us take a 4-vector $X$ and a corresponding $2 \times 2$-matrix $\tilde{x}$:

$$X = x_\mu e^\mu = (x_0, x_1, x_2, x_3) \qquad \tilde{x} = x_\mu \sigma^\mu = \begin{bmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{bmatrix} \tag{1.10}$$

Where $\sigma^\mu$ is the 4-vector of Pauli matrices, that we recall to be:

$$\sigma^\mu = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{1.11}$$

Under $SO(3,1)$ transformations, i.e. $X \mapsto \Lambda X$ where: $\Lambda \in SO(3,1)$, $|X|^2$ is an invariant:

$$|X|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \tag{1.12}$$

Under $SL(2,\mathbb{C})$ transformations, i.e. $\tilde{x} \mapsto N\tilde{x}N^\dagger$ where: $N \in SL(2,\mathbb{C})$, $\det\tilde{x}$ is an invariant:

$$\det\tilde{x} = x_0^2 - x_1^2 - x_2^2 - x_3^2 \tag{1.13}$$

The map between $SL(2,\mathbb{C})$ is not an isomorphism, since $N = \pm\mathbb{1}$ both correspond to $\Lambda = \mathbb{1}$. That means that it is a $2 \mapsto 1$ map. Since $SL(2,\mathbb{C})$ is also simply connected, then $SL(2,\mathbb{C})$ is said to be the **universal covering group**.

The relevant representations of $SL(2,\mathbb{C})$ for our discussion are:

- The **fundamental** representation

$$\psi_\alpha \mapsto \psi'_\alpha = N_\alpha{}^\beta\psi_\beta \quad \text{where: } \alpha,\beta = 1,2 \qquad (\textbf{left-handed Weyl Spinor}) \tag{1.14}$$

- The **conjugate** representation

$$\bar{\chi}_{\dot{\alpha}} \mapsto \bar{\chi}'_{\dot{\alpha}} = N^{*\dot{\beta}}_{\dot{\alpha}}\bar{\chi}_{\dot{\beta}} \quad \text{where: } \dot{\alpha},\dot{\beta} = 1,2 \qquad (\textbf{right-handed Weyl Spinor}) \tag{1.15}$$

- The **contravariant** representation

$$\begin{cases} \psi^\alpha & \mapsto \psi'^\alpha = \psi^\beta \left(N^{-1}\right)_\beta{}^\alpha \\ \bar{\chi}^{\dot{\alpha}} & \mapsto \bar{\chi}'^{\dot{\alpha}} = \bar{\chi}^{\dot{\beta}} \left(N^{*-1}\right)_{\dot{\beta}}{}^{\dot{\alpha}} \end{cases} \tag{1.16}$$

The fundamental and conjugate are irreducible representations of $SL(2,\mathbb{C})$, while the contravariant is not.
In order to raise and lower indices, we have no more the metric tensor $\eta^{\mu\nu} = (\eta_{\mu\nu})^{-1}$, which is invariant under $SO(3,1)$, but in $SL(2,\mathbb{C})$ we must consider this other invariant:

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot{\alpha}\dot{\beta}} \tag{1.17}$$

Which we might prove to be explicitly invariant as follow:

$$\epsilon'^{\alpha\beta} = \epsilon^{\rho\sigma}N_\rho{}^\alpha N_\sigma{}^\beta = \epsilon^{\alpha\beta}\cdot\det N = \epsilon^{\alpha\beta} \tag{1.18}$$

Therefore we can use $\epsilon$ to raise and lower spinor indices:

$$\begin{cases} \psi^{\alpha} & = \epsilon^{\alpha\beta}\psi_{\beta} \\ \bar{\chi}^{\dot{\alpha}} & = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{\dot{\beta}} \end{cases} \tag{1.19}$$

Hence we have explicitly showed that contravariant representations are not independent.
In order to handle mixed $SO(3,1)$ and $SL(2,\mathbb{C})$ indices, recall that the transformed components $x_{\mu}$ should look the same, whether we transform the vector $X$ via $SO(3,1)$ or the matrix $\tilde{x} = x_{\mu}\sigma^{\mu}$, then:

$$\left(x_{\mu}\sigma^{\mu}\right)_{\alpha\dot{\alpha}} \mapsto N_{\alpha}^{\beta}\left(x_{\nu}\sigma^{\nu}\right)_{\beta\dot{\gamma}} N_{\dot{\alpha}}^{*\dot{\gamma}} = \Lambda_{\mu}{}^{\nu}x_{\nu}\sigma^{\mu} \quad \Rightarrow \quad \begin{cases} \left(\sigma^{\mu}\right)_{\alpha\dot{\alpha}} & = N_{\alpha}^{\beta}\left(\sigma^{\nu}\right)_{\beta\dot{\gamma}}\left(\Lambda^{-1}\right)^{\mu}{}_{\nu}N_{\dot{\alpha}}^{*\dot{\gamma}} \\ \left(\bar{\sigma}^{\mu}\right)^{\dot{\alpha}\alpha} & = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\left(\sigma^{\mu}\right)_{\beta\dot{\beta}} = \left(\mathbb{1}, -\vec{\sigma}\right) \end{cases} \tag{1.20}$$

Let us define the generators of $SL(2,\mathbb{C})$ by the tensors $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$, which are antisymmetrized products of $\sigma$ matrices:

$$\begin{cases} \left(\sigma^{\mu\nu}\right)_{\alpha}{}^{\beta} & = \frac{i}{4}\left(\sigma^{\mu}\bar{\sigma}^{\nu} - \sigma^{\nu}\bar{\sigma}^{\mu}\right)_{\alpha}{}^{\beta} \\ \left(\bar{\sigma}^{\mu\nu}\right)^{\dot{\alpha}}{}_{\dot{\beta}} & = \frac{i}{4}\left(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu}\right)^{\dot{\alpha}}{}_{\dot{\beta}} \end{cases} \tag{1.21}$$

Which satisfy the Lorentz algebra:

$$\left[\sigma^{\mu\nu}, \sigma^{\lambda\rho}\right] = i\left(\eta^{\mu\rho}\sigma^{\nu\lambda} + \eta^{\nu\lambda}\sigma^{\mu\rho} - \eta^{\mu\lambda}\sigma^{\nu\rho} - \eta^{\nu\rho}\sigma^{\mu\lambda}\right) \tag{1.22}$$

Then, under a finite Lorentz transformation parametrized by $\omega_{\mu\nu}$, Weyl spinors transform as follows:

$$\begin{cases} \psi_{\alpha} & \mapsto \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_{\alpha}{}^{\beta}\psi_{\beta} & \text{(left-handed)} \\ \bar{\chi}^{\dot{\alpha}} & \mapsto \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\chi}^{\dot{\beta}} & \text{(right-handed)} \end{cases} \tag{1.23}$$

Now, for completeness, let us relate these spinors to the $SU(2)$ representation spanned by the $A_i$ and $B_i$:

$$\begin{cases} \psi_{\alpha}: & (A,B) = \left(\frac{1}{2}, 0\right) & \Longrightarrow & J_i = \frac{1}{2}\sigma_i \quad K_i = -\frac{i}{2}\sigma_i \\ \bar{\chi}^{\dot{\alpha}}: & (A,B) = \left(0, \frac{1}{2}\right) & \Longrightarrow & J_i = \frac{1}{2}\sigma_i \quad K_i = +\frac{i}{2}\sigma_i \end{cases} \tag{1.24}$$

Let us now mention some useful relations starting from the so called **self-duality** and **anti-self-duality** relations:

$$\begin{cases} \sigma^{\mu\nu} & = \frac{1}{2i}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} \\ \bar{\sigma}^{\mu\nu} & = -\frac{1}{2i}\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma} \end{cases} \tag{1.25}$$

At this point we are able to define the product of two Weyl spinors as:

$$\begin{cases} \chi\psi & = \chi^\alpha \psi_\alpha = -\chi_\alpha \psi^\alpha \\ \bar{\chi}\bar{\psi} & = \bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} = -\bar{\chi}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}} \end{cases} \tag{1.26}$$

Furthermore, if we chose $\psi_\alpha$ to be anti-commuting Grassmann variables we gain:

$$\psi\psi = \psi^\alpha \psi_\alpha = \epsilon^{\alpha\beta} \psi_\beta \psi_\alpha = \psi_2 \psi_1 - \psi_1 \psi_2 = 2\psi_2 \psi_1 = -2\psi_1 \psi_2 \tag{1.27}$$

Now we can define the adjoint and the complex conjugate as follows:

$$\begin{cases} \psi_\alpha^\dagger & = \bar{\psi}_{\dot{\alpha}} \\ \bar{\psi}^{\dot{\alpha}} & = \psi_\beta^* (\sigma^0)^{\beta\alpha} \end{cases} \quad \Rightarrow \quad \begin{cases} (\chi\psi)^\dagger & = \bar{\chi}\bar{\psi} \\ (\chi\sigma^\mu\bar{\psi})^\dagger & = \chi\sigma^\mu\bar{\psi} \end{cases} \tag{1.28}$$

### 1.1.2 Supersymmetry Algebra and Representations

Let us introduce the concept of **graded algebra**. Let us consider $O_a$ to be an operator of a Lie algebra, then we can define a graded algebra as follows:

$$O_a O_b - (-1)^{\eta_a \eta_b} O_a O_b = i C^e{}_{ab} O_e \qquad \text{where: } \eta_a = \begin{cases} 0 & \text{if } O_a \text{ is bosonic} \\ i & \text{if } O_a \text{ is fermionic} \end{cases} \tag{1.29}$$

In the case of supersymmetry, the generators of the algebra are: the Poincaré generators $P^\mu$, $M^{\mu\nu}$ and the spinor generators $Q_\alpha^A$, $\bar{Q}_{\dot{\alpha}}^A$, where: $A = 1, \dots, \mathcal{N}$. In case $\mathcal{N} = 1$ we speak of a **simple** SUSY, in case $\mathcal{N} > 1$ of an **extended** SUSY.
For the sake of simplicity, let us work in the $\mathcal{N} = 1$ case, i.e. we can forget about the index $A$, an focus only on the spinorial nature of $Q_\alpha$ and $\bar{Q}_{\dot{\alpha}}$. Considering $Q_\alpha$ as a Weyl spinor implies that it transform according to $SL(2, \mathbb{C})$:

$$Q_\alpha \mapsto Q'_\alpha = \exp\left\{ -\frac{i}{2} \omega_{\mu\nu} \sigma^{\mu\nu} \right\}_\alpha{}^\beta Q_\beta \sim \left( \mathbb{1} - \frac{i}{2} \omega_{\mu\nu} \sigma^{\mu\nu} \right)_\alpha{}^\beta Q_\beta \tag{1.30}$$

Furthermore, since it is a spinor it has also to transform under Lorentz transformations of the type: $U = \exp\{ -\frac{i}{2} \omega_{\mu\nu} M^{\mu\nu} \}$, then:

$$Q_\alpha \mapsto Q'_\alpha = U^\dagger Q_\alpha U \sim \left( \mathbb{1} - \frac{i}{2} \omega_{\mu\nu} M^{\mu\nu} \right) Q_\alpha \left( \mathbb{1} - \frac{i}{2} \omega_{\mu\nu} M^{\mu\nu} \right) \tag{1.31}$$

Then, since we have fermionic operators the SUSY algebra is a graded algebra, i.e. we have both commutators and anti-commutators depending on the nature of the operator. Since knowing the algebra of a theory means knowing its commutation and anti-commutation relations, now we have all the tools to calculate all these quantities:

$$
\begin{cases}
[Q_\alpha, M^{\mu\nu}] & = (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta \\
[Q_\alpha, P^\mu] & = [\bar{Q}_{\dot{\alpha}}, P^\mu] = 0 \\
\{Q_\alpha, Q_\beta\} & = 0 \\
\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} & = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu
\end{cases}
\tag{1.32}
$$

And this, added to the Poincaré algebra, is the $\mathcal{N} = 1$ SUSY algebra, or to be more precise the $\mathcal{N} = 1$ super-Poincaré algebra [1]. Let us comment the last relation in which the factor 2 is arbitrary, but we take it to be 2 for future convenience. That means that acting on a state with $Q\bar{Q}$ only produce a translation in the state, more explicitly we can say that if we take into the account a bosonic state $|B\rangle$ and a fermionic state $|F\rangle$:

$$
\begin{cases}
Q_\alpha |F\rangle & \sim |B\rangle \\
\bar{Q}_{\dot{\beta}} |B\rangle & \sim |F\rangle
\end{cases}
\qquad \Rightarrow \qquad Q\bar{Q}|B\rangle = |B\rangle_{\text{translated}}
\tag{1.33}
$$

And this relation implies that in every **supermultiplet** the number of fermions $n_F$ is equal to the number of bosons $n_B$:

$$
n_F = n_B \tag{1.34}
$$

As we are used to do in QFT, after having obtained the algebra of the group under which our theory has to be invariant, we want to find the operators which commute with all the generators of the group, in order to be able to define the states.
In the case of the Poincaré group we obtain the two Casimir operators which we remind us to be:

$$
C_1 = P^\mu P_\mu \qquad\qquad C_2 = W^\mu W_\mu \tag{1.35}
$$

Where $W_\mu$ is the Pauli-Ljubanski vector which we recall to be:

$$
W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma} \tag{1.36}
$$

However we can imagine that due to the new commutation relations introduced just above those Casimir operator have to be a bit modified. More in the detail, since $P_\mu$ commutes with the fermion generators, the first Casimir will not change, but the second will. Then we can write the two Casimir operators of the super-Poincaré group to be the following:

$$
C_1 = P^\mu P_\mu \qquad\qquad \tilde{C}_2 = C_{\mu\nu} C^{\mu\nu} \tag{1.37}
$$

[1] We have not mentioned the commutators between generic internal symmetry generators $T_i$, which are not always vanishing, e.g. in the case of $\mathcal{N} = 1$ we can consider the $U(1)$ automorphisms of the SUSY algebra, known as **R-symmetry**, but we do not need them explicitly in our present work

Where $C_{\mu\nu}$ is defined as follows:

$$B_\mu = W_\mu - \frac{1}{4}\bar{Q}_{\dot{\alpha}}(\bar{\sigma}_\mu)^{\dot{\alpha}\beta}Q_\beta \qquad\qquad C_{\mu\nu} = B_\mu P_\nu - B_\nu P_\mu \tag{1.38}$$

Now we are ready to define a massless super-multiplet for an $\mathcal{N}=1$ SUSY. Then we know from special relativity that without loss in generality we can think about its momentum as $p_\mu = (E,0,0,E)$, hence both the Casimirs are vanishing. But to construct the multiplet let us explicitly consider the anti-commutator between the Q's:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu = 2E(\sigma^0+\sigma^3)_{\alpha\dot{\beta}} = 4E\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{\alpha\dot{\beta}} \tag{1.39}$$

which means that $Q_2$ is zero in the representation, since:

$$\left\{Q_2, \bar{Q}_{\dot{2}}\right\} = 0 \quad \Rightarrow \quad \left\langle p^\mu, \lambda \left| \bar{Q}_{\dot{2}} Q_2 \right| \tilde{p}^\mu, \tilde{\lambda} \right\rangle = 0 \quad \Rightarrow \quad Q_2 = 0 \tag{1.40}$$

On the other hand the $Q_1$ satisfies instead: $\left\{Q_1, \bar{Q}_{\dot{1}}\right\} = 4E$. Then we can define the ladder operators to be $a$ and $a^\dagger$:

$$a = \frac{Q_1}{2\sqrt{E}} \quad a^\dagger = \frac{\bar{Q}_{\dot{1}}}{2\sqrt{E}} \qquad \text{such that: } \begin{cases} \{a, a^\dagger\} & = 1 \\ \{a, a\} & = 0 = \left\{a^\dagger a^\dagger\right\} \end{cases} \tag{1.41}$$

Furthermore, since $[a, J^3] = \frac{1}{2}(\sigma^3)_{11}\, a = \frac{1}{2}a$, then we can calculate:

$$J^3\left(a\left|p^\mu,\lambda\right\rangle\right) = \left(aJ^3 - \left[a, J^3\right]\right)\left|p^\mu,\lambda\right\rangle = \left(aJ^3 - \frac{a}{2}\right)\left|p^\mu,\lambda\right\rangle = \left(\lambda - \frac{1}{2}\right)a\left|p^\mu,\lambda\right\rangle \tag{1.42}$$

From which we might conclude that: $a\left|p^\mu,\lambda\right\rangle$ has helicity $\lambda - \frac{1}{2}$, and by similar reasoning, we can find that the helicity of $a^\dagger\left|p^\mu,\lambda\right\rangle$ is $\lambda + \frac{1}{2}$.
Hence we have understood that such ladder operators change the helicity of the state of 1/2, thus in order to build the representation we have to start from a vacuum state of minimum helicity $\lambda$, which we define as $|\Omega\rangle$.
Since $a|\Omega\rangle = 0$ and $a^\dagger a^\dagger|\Omega\rangle = 0|\Omega\rangle = 0$, then the whole multiplet consists of two states:

$$|\Omega\rangle = |p^\mu, \pm\lambda_0\rangle \quad a^\dagger|\Omega\rangle = |p^\mu, \pm(\lambda_0 + 1/2)\rangle \tag{1.43}$$

In which we have not specified the sign to include also the CPT the conjugate states. Then depending on $\lambda$ we can define:

- **chiral multiplet**, $\lambda_0 = 0$, $(\lambda = 0, \lambda = 1/2)$, (e.g. squark/quark)
- **vector multiplet**, $\lambda_0 = 1/2$, $(\lambda = 1/2, \lambda = 1)$, (e.g. photino/photon)

- **gravity multiplet**, $\lambda_0 = 3/2$, $(\lambda = 3/2, \lambda = 2)$, (e.g. gravitino/graviton)

Let us see what happens when we enlarge the SUSY, i.e. we consider $\mathcal{N} > 1$. Then let us take back into the account the indices $A, B = 1, \dots, \mathcal{N}$. The anti-commutator between the $Q$s will change accordingly and furthermore we have also to introduce the concept of **central charges** as follows:

$$\begin{cases} \{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} & = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \delta^A{}_B \\ \{Q^A_\alpha, Q^B_\beta\} & = \epsilon_{\alpha\beta} Z^{AB} \end{cases} \tag{1.44}$$

Where $Z^{AB}$ are the central charges, such that: they are anti-symmetric in the indices (i.e. $Z^{AB} = -Z^{BA}$) and commute with all the other generators.

Now we can discuss the massless multiplets also when $\mathcal{N} > 1$. As we did in the case of $\mathcal{N} = 1$, we will start from $p_\mu = (E, 0, 0, E)$, from which (similar to the $\mathcal{N} = 1$), we can derive the following relation:

$$\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} \delta^A_B \quad \Rightarrow Q^A_2 = 0 \quad \Rightarrow \{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} Z^{AB} = 0 \Rightarrow Z^{AB} = 0 \tag{1.45}$$

In analogy with what we have done before, in order to obtain the full representation, we can define $\mathcal{N}$ creation- and annihilation-operators, since $Q^A_1 \neq 0$ and $A = 1, \dots, \mathcal{N}$:

$$\begin{cases} a^A & = \frac{Q^A_1}{2\sqrt{E}} \\ a^{A\dagger} & = \frac{\bar{Q}^A_{\dot{1}}}{2\sqrt{E}} \end{cases} \qquad \text{such that:} \quad \left\{a^A, a^\dagger_B\right\} = \delta^A_B \tag{1.46}$$

Then i order to get all the states we have to start to a vacuum state $|\Omega\rangle$, which is annihilated by all the $a^A$, and start to increase by 1/2 its helicity, by the use of the ladder operators:

| states | helicity | number of states |
|---|---|---|
| $\vert\Omega\rangle$ | $\lambda_0$ | $1 = \binom{\mathcal{N}}{0}$ |
| $a^{A\dagger}\vert\Omega\rangle$ | $\lambda_0 + \frac{1}{2}$ | $\mathcal{N} = \binom{\mathcal{N}}{1}$ |
| $a^{A\dagger}a^{B\dagger}\vert\Omega\rangle$ | $\lambda_0 + 1$ | $\frac{1}{2!}\mathcal{N}(\mathcal{N}-1) = \binom{\mathcal{N}}{2}$ |
| $a^{A\dagger}a^{B\dagger}a^{C\dagger}\vert\Omega\rangle$ | $\lambda_0 + \frac{3}{2}$ | $\frac{1}{3!}\mathcal{N}(\mathcal{N}-1)(\mathcal{N}-2) = \binom{\mathcal{N}}{3}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $a^{\mathcal{N}\dagger}a^{(\mathcal{N}-1)\dagger}\dots a^{1\dagger}\vert\Omega\rangle$ | $\lambda_0 + \frac{\mathcal{N}}{2}$ | $1 = \binom{\mathcal{N}}{\mathcal{N}}$ |

Thus it is easy to see that the total number of states is given by:

$$\sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} = 2^{\mathcal{N}} \tag{1.47}$$

Also in this case we can name the multiplets and we can give some examples for the $\mathcal{N} = 2$ multiplets:

- **vector multiplet**, $(\lambda_0 = 0)$

$$\begin{array}{ccc} & \lambda = 0 & \\ \lambda = \frac{1}{2} & & \lambda = \frac{1}{2} \\ & \lambda = 1 & \end{array} \tag{1.48}$$

  It is easy to see that this $\mathcal{N} = 2$ multiplet can be decomposed in terms of $\mathcal{N} = 1$ multiplets: one $\mathcal{N} = 1$ vector and one $\mathcal{N} = 1$ chiral multiplet.

- **hyper multiplet**, $\left(\lambda_0 = -\frac{1}{2}\right)$

$$\begin{array}{ccc} & \lambda = -\frac{1}{2} & \\ \lambda = 0 & & \lambda = 0 \\ & \lambda = \frac{1}{2} & \end{array} \tag{1.49}$$

  Also in this can be decomposed in terms of two $\mathcal{N} = 1$ chiral multiplets, and we will see it explicitly in the developing of the work

We will not treat the SUSY algebra in the context of the massive states and the **BPS condition**, since even if we are going to use branes (which are BPS objects), we will never explicitly use the BPS constraint between their mass and charge.

## 1.2 Superspace and Superfields

### 1.2.1 Superspace

By now we have always assume to describe one-particle states, but since we want to describe interactions, that is not enough.
In QFT what we call "particles" are the quantum fields, e.g. $\varphi(x)$, where: $x^\mu$ belongs to the Minkowski space, and the fields $\varphi(x)$ transform according to the Lorentz group as tensors. Since we are taking into the account Lie groups, then we can say that the elements of the groups span a smooth Manifold by the so called **exponential map** $\Lambda$:

$$\Lambda : G \mapsto \mathcal{M}_G \qquad \Longleftrightarrow \qquad \Lambda(g \in G) : \left\{g = e^{i\alpha_a T^a}\right\} \mapsto \{\alpha_a\} \tag{1.50}$$

In this sense we can define the Minkowski spacetime as the quotient between the Poincaré group and the Lorentz group:

$$\mathbb{R}^{1,3} = \frac{\{\omega^{\mu\nu}, a^\mu\}}{\{\omega^{\mu\nu}\}} = \{a^\mu\} \quad \text{(Minkowski-generators)} \tag{1.51}$$

Then we may think that also the **superspace**, e.g. for an $\mathcal{N} = 1$ theory might be defined in the same way by quotienting the super-Poincaré group with the Lorentz's one. And indeed we can define it as such co-set:

$$\frac{\left\{\omega^{\mu\nu}, a^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}}\right\}}{\{\omega^{\mu\nu}\}} = \left\{a^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}}\right\} \quad \text{(superspace-generators)} \tag{1.52}$$

Where the most generic element of the super-Poincaré group might be written as:

$$g = \exp\left\{i\left(\omega^{\mu\nu} M_{\mu\nu} + a^\mu P_\mu + \theta^\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}\right)\right\} \tag{1.53}$$

Where $\theta^\alpha$ and $\bar{\theta}_{\dot{\alpha}}$ are Grassmann variables. Then the anti-commutation relation, between the $Q$s becomes a commutation relation when we add such parameters:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \quad \Rightarrow \quad \left[\theta^\alpha Q_\alpha, \bar{Q}_{\dot{\beta}} \bar{\theta}^{\dot{\beta}}\right] = 2\theta^\alpha Q_\alpha (\sigma^\mu)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} P_\mu \tag{1.54}$$

### 1.2.2 Superfields

From now on we will work in the superspace and hence we will have to see how the fields transform in such space.
Let us start to evaluate the translations which are presents also in the Minkowski space. Let us recall that a scalar fields $\varphi\left(x^\mu\right)$ is a function of the spacetime coordinates $x^\mu$. Then treating $\varphi$ as an operator, let us see how it transforms under translation of a parameter $a_\mu$:

$$\varphi \mapsto \exp\{-ia_\mu P^\mu\}\varphi \exp\{ia_\mu P^\mu\} \tag{1.55}$$

Since $\varphi(x^\mu)$ is also an Hilbert vector in some function space $\mathcal{F}$, then:

$$\varphi(x^\mu) \mapsto \exp\{-ia_\mu \mathcal{P}^\mu\}\varphi(x^\mu) = \varphi(x^\mu - a^\mu) \quad \Rightarrow \quad \mathcal{P}_\mu = -i\partial_\mu \tag{1.56}$$

In other words $\mathcal{P}$ is a representation of the abstract operator $P^\mu$ acting on $\mathcal{F}$. Comparing the two transformation rules to first order in $a_\mu$ we can find:

$$(1 - ia_\mu P^\mu)\varphi(1 + ia_\mu P^\mu) = (1 - ia_\mu \mathcal{P}^\mu)\varphi \quad \Rightarrow \quad i[\varphi, a_\mu P^\mu] = -ia^\mu \mathcal{P}_\mu \varphi = -a^\mu \partial_\mu \varphi \tag{1.57}$$

And this is true also in the SUSY case since we also have the translations in the superspace. Then we now have to evaluate the same type of transformations in the case of the $\theta$s.
The most the general form to write a scalar superfield $S\left(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\right)$ can be obtained by expanding in powers of $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$ with a finite number of non-vanishing terms:

$$\begin{aligned} S\left(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\right) =& \varphi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta M(x) + \bar{\theta}\bar{\theta}N(x) + \left(\theta\sigma^\mu\bar{\theta}\right)V_\mu(x) \\ &+ (\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\rho(x) + (\theta\theta)(\bar{\theta}\bar{\theta})D(x) \end{aligned} \tag{1.58}$$

Then, we have to evaluate the transformation of $S\left(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\right)$ under the $\theta$s, firstly as a field operator, taking $\epsilon$ and $\bar{\epsilon}$ as parameter of the transformation:

$$S\left(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\right) \mapsto \exp\left\{-i(\epsilon Q + \bar{\epsilon}\bar{Q})\right\} S \exp\left\{i(\epsilon Q + \bar{\epsilon}\bar{Q})\right\} \tag{1.59}$$

And secondly as an Hilbert space vector:

$$S\left(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\right) \mapsto \exp\left\{i(\epsilon \mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}})\right\} S\left(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\right) = S\left(x^\mu - ic\left(\epsilon\sigma^\mu\bar{\theta}\right) + ic^*\left(\theta\sigma^\mu\bar{\epsilon}\right), \theta + \epsilon, \bar{\theta} + \bar{\epsilon}\right) \tag{1.60}$$

Where: $\mathcal{Q}$ a representation of the spinorial generators $Q_\alpha$ acting on functions of $\theta, \bar{\theta}$, and $c$ is a constant, which is involved in the translations:

$$x^\mu \mapsto x^\mu - ic\left(\epsilon\sigma^\mu\bar{\theta}\right) + ic^*\left(\theta\sigma^\mu\bar{\epsilon}\right) \tag{1.61}$$

Hence we find in analogy with the previous case that:

$$\begin{cases} \mathcal{Q}_\alpha &= -i\frac{\partial}{\partial\theta^\alpha} - c\left(\sigma^\mu\right)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\frac{\partial}{\partial x^\mu} \\ \bar{\mathcal{Q}}_{\dot{\alpha}} &= +i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + c^*\theta^\beta\left(\sigma^\mu\right)_{\beta\dot{\alpha}}\frac{\partial}{\partial x^\mu} \\ \mathcal{P}_\mu &= -i\partial_\mu \end{cases} \tag{1.62}$$

We can determine Re$\{c\}$ from the commutation relation which has to hold in any representation:

$$\left\{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot{\alpha}}\right\} = 2\left(\sigma^\mu\right)_{\alpha\dot{\alpha}} \mathcal{P}_\mu \quad \Rightarrow \quad \mathrm{Re}\{c\} = 1 \tag{1.63}$$

Then for convenience we set $c = 1$.
Following the same logic by comparing the two transformations at first order in $\epsilon$ we can get the commutation relation of $S$ with $Q_\alpha$, and hence its variation $\delta S$:

$$i[S, \epsilon Q + \bar{\epsilon}\bar{Q}] = i(\epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}})S = \delta S \tag{1.64}$$

Furthermore, since we explicitly know $\mathcal{Q}$, $\bar{\mathcal{Q}}$ and $S$, we can get how the different parts of $S$ change under the $\mathcal{Q}$s action:

$$\begin{cases} \delta\varphi & = \epsilon\psi + \bar{\epsilon}\bar{\chi} \\ \delta\psi & = 2\epsilon M + \sigma^\mu\bar{\epsilon}\left(i\partial_\mu\varphi + V_\mu\right) \\ \delta\bar{\chi} & = 2\bar{\epsilon}N - \epsilon\sigma^\mu\left(i\partial_\mu\varphi - V_\mu\right) \\ \delta M & = \bar{\epsilon}\bar{\lambda} - \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\epsilon} \\ \delta N & = \epsilon\rho + \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\bar{\chi} \\ \delta V_\mu & = \epsilon\sigma_\mu\bar{\lambda} + \rho\sigma_\mu\bar{\epsilon} + \frac{i}{2}\left(\partial^\nu\psi\sigma_\mu\bar{\sigma}_\nu\epsilon - \bar{\epsilon}\bar{\sigma}_\nu\sigma_\mu\partial^\nu\bar{\chi}\right) \\ \delta\bar{\lambda} & = 2\bar{\epsilon}D + \frac{i}{2}\left(\bar{\sigma}^\nu\sigma^\mu\bar{\epsilon}\right)\partial_\mu V_\nu + i\bar{\sigma}^\mu\epsilon\partial_\mu M \\ \delta\rho & = 2\epsilon D - \frac{i}{2}\left(\sigma^\nu\bar{\sigma}^\mu\epsilon\right)\partial_\mu V_\nu + i\sigma^\mu\bar{\epsilon}\partial_\mu N \\ \delta D & = \frac{i}{2}\partial_\mu\left(\epsilon\sigma^\mu\bar{\lambda} - \rho\sigma^\mu\bar{\epsilon}\right) \end{cases} \tag{1.65}$$

However we may prove that the scalar superfield $S$ is not an irreducible representation of the $\mathcal{N} = 1$ SUSY algebra, so we can eliminate some of its components maintaining it still as a superfield. In general we can impose consistent constraints on S, leading to smaller superfields that can be irreducible representations of the supersymmetry algebra. In order to introduce such reductions on the general scalar field, let us introduce the following covariant derivative:

$$\begin{cases} \mathcal{D}_\alpha & = \partial_\alpha + i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \\ \bar{\mathcal{D}}_{\dot{\alpha}} & = -\bar{\partial}_{\dot{\alpha}} - i\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu \end{cases} \tag{1.66}$$

Since we have constructed such derivative in a way such that it anti-commutes with all the $\mathcal{Q}$, $\bar{\mathcal{Q}}$, we gain that:

$$\left[\mathcal{D}_\alpha, \epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{Q}}\right] = 0 \quad \Rightarrow \quad \mathcal{D}_\alpha S \quad \text{is a superfield} \tag{1.67}$$

Hence we can define a **chiral-superfield**, as a field $\Phi$, such that $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi$, which takes the general form:

$$
\begin{aligned}
\Phi\left(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\right) = & \varphi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) + i\left(\theta\sigma^\mu\bar{\theta}\right)\partial_\mu\varphi(x) \\
& - \frac{i}{\sqrt{2}}(\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu\varphi(x)
\end{aligned}
\tag{1.68}
$$

Off shell, there are 4 bosonic (complex $\varphi$, $F$) and 4 fermionic (complex $\psi_\alpha$) components, which belong to a chiral multiplet, and that is why this superfield is named in this way.

## 1.3 4D $\mathcal{N} = 1$ Supergravity

### 1.3.1 Chiral Superfield Lagrangian

In order to write a Lagrangian for a chiral field $\mathcal{L}(\Phi)$ such that $\delta\mathcal{L}$ is a total derivative under supersymmetry transformation, we recall that:

- For a general scalar superfield $S = \ldots + (\theta\theta)(\bar{\theta}\bar{\theta})D(x)$, the $D$ term transforms as:

$$
\delta D = \frac{i}{2}\partial_\mu\left(\epsilon\sigma^\mu\bar{\lambda} - \rho\sigma^\mu\bar{\epsilon}\right) \tag{1.69}
$$

- For a chiral superfield $\Phi = \ldots + (\theta\theta)F(x)$, the $F$ term transforms as:

$$
\delta F = i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi \tag{1.70}
$$

From these facts, the most general Lagrangian for a chiral superfield $\Phi$ can be written as:

$$
\mathcal{L} = K\left(\Phi, \Phi^\dagger\right)\Big|_D + [W(\Phi) + \text{ h.c. }]\Big|_F \tag{1.71}
$$

Where $|_D$ refers to the $D$ term of the corresponding superfield, while $|_F$ is stands for the $F$ term.

- The function $K$ is known as the **Kähler potential**, it is a real function of $\Phi$ and $\Phi^\dagger$. Considering to have more than one superfield $\Phi$, that we can denote as $\Phi^i$ we can expand $K(\Phi^i, \Phi^{j\dagger})$ around $\Phi^i = \varphi^i$ as:

$$
\left(\frac{\partial^2 K}{\partial\varphi^i\partial\varphi^{\bar{\jmath}}}\right)\partial_\mu\varphi^i\partial^\mu\varphi^{\bar{\jmath}} = K_{i\bar{\jmath}}\partial_\mu\varphi^i\partial^\mu\varphi^{\bar{\jmath}} \tag{1.72}
$$

Where $K_{i\bar{j}}$ is called **Kähler metric**, and in the next sections we will describe the space in which this metric is defined, which is called **Kähler manifold**. Furthermore by the **no-go theorem** we know that it can get corrections at the perturbative and non-perturbative level (in particular we will consider the perturbative corrections).

- $W(\Phi)$ is known as the **superpotential**, it is a holomorphic function of the chiral superfield $\Phi$ (and therefore is a chiral superfield itself). In this case in order to obtain the $F$ term we Taylor expand $W(\Phi)$ around $\Phi = \varphi$:

$$W(\Phi) = W(\varphi) + \underbrace{(\Phi - \varphi)}_{\cdots+\theta\theta F+\ldots} \frac{\partial W}{\partial \varphi} + \underbrace{\frac{1}{2}(\Phi - \varphi)^2}_{\cdots+(\theta\psi)(\theta\psi)+\ldots} \frac{\partial^2 W}{\partial \varphi^2} \tag{1.73}$$

Also in this case the no-go theorem tell us that the super potential may get only non-perturbative corrections, and we will see in the developing of the work what kind of non-perturbative corrections we will consider.

- The part of the Lagrangian depending on the auxiliary field $F(x)$ takes the form:

$$\mathcal{L}_{(F)} = FF^* + \frac{\partial W}{\partial \varphi} F + \frac{\partial W^*}{\partial \varphi^*} F^* \tag{1.74}$$

Since the action is quadratic and without any derivatives, then the field $F(x)$ does not propagate. We can it explicitly by eliminating $F$ using its field equations:

$$\begin{cases} \frac{\delta \mathcal{S}_{(F)}}{\delta F} & = 0 \Longrightarrow F^* + \frac{\partial W}{\partial \varphi} = 0 \\ \frac{\delta \mathcal{S}_{(F)}}{\delta F^*} & = 0 \Longrightarrow F + \frac{\partial W^*}{\partial \varphi^*} = 0 \end{cases} \tag{1.75}$$

That we can directly plug into the Lagrangian, to find:

$$\mathcal{L}_{(F)} \mapsto - \left| \frac{\partial W}{\partial \varphi} \right|^2 = -V_F(\varphi) \tag{1.76}$$

This defines a positive definite scalar potential $V_F(\varphi)$.

### 1.3.2 $\mathcal{N} = 1$ Supergravity in Superspace

When we refer to supergravity (SUGRA) we means a supersymmetric theory which includes gravity. As Minkowski space is only a local description of the curved spacetime, we can say the same for the superspace that we have defined in the previous sections. In other words we can say that supergravity is a supersymmetric theory in which the supersymmetry is not global, but local, and in this sense it is a gauge theory. That means that we can formulate supergravity in terms of superfields, generalising the superfield formulation of global supersymmetry.

Let us start by consider that under diffeomorphisms, the superspace coordinates $z^M = \left\{x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\right\}$, in general transform as: $z^M = z^M + \zeta^M$.
In order to achieve the invariance of the action under such transformations, we have to include the supergravity multiplet into the Lagrangian as a superfield with components: $\{e^\mu_a, \psi^\mu_\alpha, M, b_a\}$. $e^\mu_a$ is the **vierbein** describing the metric $g_{\mu\nu} = e^a_\mu e_{a\nu}$, $\psi^\mu_\alpha$ is the **gravitino**, $M$ is a complex scalar auxiliary field and $b_a$ a real vector auxiliary field.

We can generalize the vierbein in the superspace by $E^M_A$. Then we have to define a superspace tensor density, which generalize $\sqrt{-g} = e = \det e^\mu_a$ to the superspace, i.e. $\det E^M_A \equiv \mathbf{E}$. Then, we can write the supergravity action (in Planck units $M_P^2 = 1$ ) as:

$$\begin{aligned}\mathcal{S}_{\mathrm{SG}} = -3\int \mathrm{d}^8 z\mathbf{E} \;&=\; -\frac{1}{2}\int \mathrm{d}^4 xe\left\{R - \frac{1}{3}\bar{M}M + \frac{1}{3}b^a b_a\right.\\ &\left.+\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left(\bar{\psi}_\mu\bar{\sigma}_\nu\mathcal{D}_\rho\psi_\sigma - \psi_\mu\sigma_\nu\mathcal{D}_\rho\bar{\psi}_\sigma\right)\right\} \qquad (1.77)\end{aligned}$$

Where we have written: $d^8 z = \mathrm{d}^4 x d^4\theta$ and $\mathcal{D}$ is the covariant derivative. The non-propagating auxiliary fields (which in the SUSY Lagrangian seen before are the analogous of the $F(x)$ auxiliary field) complete the supergravity multiplet providing an off-shell invariant action. As before, integrating them out by their field equations give rise to the Einstein-Hilbert plus the Rarita-Schwinger actions, which describe respectively the graviton ($\lambda = 2$) and the gravitino ($\lambda = 3/2$).

Let us now see some relevant properties of $\mathcal{N} = 1$ supergravity actions coupled to matter. then we can write the total Lagrangian as the sum of supergravity contribution $\mathcal{L}_{\mathrm{SG}}$ and the SUSY Lagrangian discussed before:

$$\mathcal{L} = \mathcal{L}_{\mathrm{SG}} + \mathcal{L}(K, W) \qquad (1.78)$$

Where the second term is understood to be covariantized under general coordinate transformations. Then we can write the full action in analogy of the previous SUSY Lagrangian as:

$$\mathcal{S} = -\frac{3}{\kappa^2}\int \mathrm{d}^4 x\ \mathrm{d}^4\theta\mathbf{E}e^{-\frac{\kappa^2}{3}K} + \left(\int \mathrm{d}^4 x\ \mathrm{d}^4\theta\mathcal{E}W + \text{ h.c. }\right) \qquad (1.79)$$

Where we have restored $M_P$ and defined $\kappa^2 = 8\pi G_N^{(4)} = 1/M^2_{\mathrm{pl}}$. As for $\mathcal{S}_{\mathrm{SG}}$, $\mathbf{E}$ is the determinant of the super-vierbein, while on the other hand $\mathcal{E}$ is defined by $2\mathcal{R}\mathcal{E} = \mathbf{E}$, where $\mathcal{R}$ is the curvature superfield (having components $R, \psi^\mu, M, b_a$ ), which we can see as a supersymmetric generalization of the Ricci scalar. Let us notice that the first term of this action, when expanded in powers of $\kappa^2$ includes the pure supergravity action plus the standard kinetic term for matter fields:

$$e^{-\frac{\kappa^2}{3}K} = 1 - \frac{\kappa^2}{3}K + \mathcal{O}\left(\kappa^4\right) \tag{1.80}$$

Now is easy to see that the flat space-time limit corresponds to $\kappa \to 0, \int d^2\bar{\theta}\mathcal{E} \to 1$, and $\mathbf{E} \to 1$ and the flat space global supersymmetric action from in terms of $K$ and $W$ is reproduced.

Since for any finite value of $\kappa$ $K$ appears explicitly in the pure supergravity part of the action, then the coefficient of the Einstein-Hilbert term, which is the effective Planck mass, depends on the chiral matter fields as in Brans-Dicke-like theories.
In order to go to the Einstein frame (i.e. $M_P = const$) we need to rescale the metric. If we want to rescale the metric, we then have to rescale the fermionic fields accordingly. Since we have a local SUSY, such a rescaling will complicate substantially the derivation of the action in components. In order to overcame these complications we can add an extra auxiliary superfield $\varphi$, known as **Weyl compensator field**. The introduction of such field makes the action invariant under conformal transformations.Then the action after having added the compensator field looks like:

$$\mathcal{S} = -3\int \mathrm{d}^4x\ \mathrm{d}^4\theta\mathbf{E}\varphi\bar{\varphi}e^{-K/3} + \left(\int \mathrm{d}^4x\ \mathrm{d}^4\theta\mathcal{E}\varphi^3W + \text{ h.c.}\right) \tag{1.81}$$

This action is invariant under rescalings of the metric such that: $\mathbf{E} \to \mathbf{e^{2(\tau+\bar{\tau})}}$ and $\mathcal{E} \to e^{6\tau}\mathcal{E} + \cdots$ with $\tau$ a chiral superfield. Furthermore all the matter fields are invariant under $\varphi \to e^{-2\tau}\varphi$.

After having computed the action in components, we can fix $\varphi$ by imposing that the Einstein-Hilbert term is canonically normalized, then we brake the fictitious conformal invariance that we have before after the addiction of the compensator. More precisely $\varphi$ has to be fixed to $\varphi\bar{\varphi}e^{-K/3} = M_{\text{pl}}^2$, breaking explicitly the (artificial) conformal invariance and leaving the physical fields properly normalised with standard kinetic terms.

In the following section we will derive the full component action in the case of a type IIB SUGRA, but at this level we are interested in obtaining the scalar potential which will play a very important role in our discussion. In order to see it we can consider the case of a flat spacetime, i.e. $\mathbf{E} = 1, \int d^2\bar{\theta}\mathcal{E} = 1$, and the covariant derivatives reduce to the global covariant derivatives, then the above action becomes simpler:

$$\mathcal{S} = -3\int \mathrm{d}^4x\ \mathrm{d}^4\theta\varphi\bar{\varphi}e^{-K/3} + \left(\int \mathrm{d}^4x\ \mathrm{d}^2\theta\varphi^3W + \text{ h.c.}\right) \tag{1.82}$$

From which we can derive, in a similar way of the global SUSY case, the scalar potential in supergravity:

$$V_F = e^{\frac{K}{M_P^2}} \left[ \left(K^{-1}\right)^{i\bar{j}} D_i W D_{\bar{j}} \bar{W} - 3\frac{|W|^2}{M_P^2} \right] \tag{1.83}$$

Where the covariant derivative $D_i$ is defined as:

$$D_i W = \partial_i W + \frac{1}{M_P^2} (\partial_i K) W \tag{1.84}$$

Let us notice that in the limit in which $M_P \to \infty$ gravity is decoupled and the global supersymmetric scalar potential $V_F$ is restored. Let us notice that for finite values of the Planck mass, the scalar potential is no longer positive definite, where the extra negative piece $\propto 3\frac{|W|^2}{M_P^2}$ comes from the Weyl compensator. However from now on we will use only Planck units, i.e. $M_P = 1$.

# Chapter 2

# Basics of Inflationary Cosmology

In the writing of this chapter I followed the following main references [Bau22; BM15], which are very complete and exhaustive reviews of cosmology.

## 2.1 Friedmann-Lemaitre-Robertson-Walker Metric

Our current interpretation of Gravity is as a manifestation of the spacetime geometry. Since the purpose of cosmology is the explanation of the origin and evolution of the universe, we have to take into the account its geometry. In other words we need a metric which describes the universe and allow us to do some physical considerations, within the validity regime of general relativity.
As always when we are dealing with geometry, the symmetries guide our research for a metric. More in the detail from the cosmological observations we can infer that our universe is homogeneous and isotropic at large scale. This fact implies that we can foliate our spacetime with space-like slices which are homogeneous and isotropic. Then we can write the most general metric of such spacetime in the following form:

$$ds^2 = -dt^2 + a^2(t)dl^2 \qquad \text{where: } dl^2 = \gamma_{ij}(x)dx^i dx^j \tag{2.1}$$

where $a(t)$ is called **scale factor**, and we have elevated it to the square just for further computational convenience. Let us now discuss the different possibilities for the spatial metric: $\gamma_{ij}(x)$. In particular the isotropy and homogeneity conditions tell us that we should have a constant curvature for our $3D$ space, which can be negative, zero or positive, which lead to hyperbolic ($H^3$), flat ($E^3$), or spherical ($S^3$) spaces respectively. Hence we can write such a metric in the spherical coordinates as:

$$dl^2 = \frac{dr^2}{1+k\left(\frac{r}{R_0}\right)^2} + r^2 d\Omega^2 \quad \text{where: } d\Omega^2 = d\theta^2 + sin(\theta)d\phi^2 \text{ and } \begin{cases} k=-1 & \text{if } H^3 \\ k=0 & \text{if } E^3 \\ k=+1 & \text{if } S^3 \end{cases} \tag{2.2}$$

Then we can now write the complete Friedmann-Lemaitre-Robertson-Walker (FLRW) metric as:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1+k\left(\frac{r}{R_0}\right)^2} + r^2 d\Omega^2\right] \tag{2.3}$$

Let us make a couple of comments on such a metric. First of all we can immediately see that it is invariant with respect to the following rescaling:

$$\begin{cases} a(t) & \longrightarrow \lambda a(t) \\ r & \longrightarrow \frac{r}{\lambda} \\ R_0 & \longrightarrow \frac{R_0}{\lambda} \end{cases} \tag{2.4}$$

Usually we refer to $r$ as the **comoving coordinate**, and $a(t)r$ as the **physical coordinate**, since we cannot measure directly $a(t)$, in fact we can e.g. measure the speed of an object to be:

$$v_{physical} = \frac{dr_{physical}}{dt} = \dot{a}r + a\dot{r} = \frac{\dot{a}}{a}r_{physical} + a\dot{r} = Hr_{physical} + a\dot{r} \tag{2.5}$$

where $H$ is usually called the **Hubble parameter**. Furthermore we can define $Hr_{physical}$ as the **Hubble flow** and $v_{peculiar} = a\dot{r}$ is the so called **peculiar velocity**, which is the velocity measured by an observer moving in the Hubble flow.

Let us now rewrite the metric in a more compact form, using the following substitution:

$$d\chi = \frac{dr}{\sqrt{1+k\left(\frac{r}{R_0}\right)^2}} \tag{2.6}$$

Then after this substitution we can write the metric in the following form:

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + S_k^2(\chi)d\Omega^2\right] \quad \text{where: } S_k(\chi) = R_0 \begin{cases} sinh\left(\frac{\chi}{R_0}\right) & \text{if } k=-1 \\ \frac{\chi}{R_0} & \text{if } k=0 \\ sin\left(\frac{\chi}{R_0}\right) & \text{if } k=+1 \end{cases} \tag{2.7}$$

In order to write our metric in a conformal way, let us define the conformal time to be:

$$d\tau = \frac{dt}{a(t)} \tag{2.8}$$

with this substitution we end up with the following metric:

$$ds^2 = a^2(\tau) \left[-d\tau^2 + d\chi^2 + S_k^2(\chi) d\Omega^2\right] \tag{2.9}$$

## 2.2 Friedmann Equations

The Friedman equations, are nothing else but the Einstein equations for a perfect fluid in the FLRW metric background. Thus let us recall the Einstein equations to be in the most general form the following:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.10}$$

Sometimes we may prefer to put the **cosmological constant** term $\Lambda g_{\mu\nu}$ into the energy-momentum tensor defining a kind of vacuum energy:

$$T^{\Lambda}_{\mu\nu} = \frac{\Lambda}{8\pi G} g_{\mu\nu} = \rho_\Lambda g_{\mu\nu} \tag{2.11}$$

which is often called **dark energy**[1], since it is measured to be a lot smaller than the expected vacuum energy of the quantum fields of the Standard Model. This opens the cosmological constant problem, which is beyond the scope of the present work.

After this initial remark let us proceed with the derivations of the Friedmann equations. Let us start by calculating the Einstein tensor, which we recall to be:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} \tag{2.12}$$

Let us assume to work with the FLRW metric in the form: $g_{\mu\nu} = -dt^2 + a^2(t)\gamma_{ij} dx^i dx^j$. Since we can define the Levi-Civita connections as functions of the metric, we recall the famous relation between the metric and the Christoffel symbols:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{g^{\alpha\beta}}{2} \left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right) \tag{2.13}$$

from which we can calculate:

[1]We know that in general, when people talk about dark energy, they are referring to a more general fluid than the one which appear in the same form of a vacuum energy, but, for the purpose of our work, we will not make any distinction

$$\begin{cases} \Gamma^0_{ij} &= a\dot{a}\gamma_{ij} \\ \Gamma^i_{0j} &= \frac{\dot{a}}{a}\delta^i_j \\ \Gamma^i_{jk} &= \frac{\gamma^{il}}{2}\left(\partial_k\gamma_{lj} + \partial_j\gamma_{lk} - \partial_l\gamma_{jk}\right) \end{cases} \tag{2.14}$$

Hence we now have all the tools to calculate the Riemann tensor and therefore the Ricci tensor:

$$R_{\mu\nu} = \partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\lambda\mu} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\rho\nu} \tag{2.15}$$

Hence in our case the only non vanishing components of such tensor are:

$$\begin{cases} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{ij} &= \left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{(aR_0)^2}\right] g_{ij} \end{cases} \tag{2.16}$$

while the components $R_{0i} = 0$, since the metric is isotropic, therefore all the 3-vectors must vanish. Now we can calculate the Ricci scalar as:

$$R = g^{\mu\nu}R_{\mu\nu} = -R_{00} + \frac{R_{ij}}{a^2} = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{(aR_0)^2}\right] \tag{2.17}$$

Hence we can write the non-vanishing Einstein tensor components in the form: $g^{\mu\alpha}G_{\alpha\nu} = G^\mu_\nu$ as follows:

$$\begin{cases} G^0_0 &= -3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{(aR_0)^2}\right] \\ G^i_j &= -\left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{(aR_0)^2}\right]\delta^i_j \end{cases} \tag{2.18}$$

At this point let us evaluate the RHS of the Einstein equations. Hence we have to find the description of matter which is homogeneous and isotropic, which in the most general case corresponds to a perfect fluid. Let us recall its energy-momentum tensor to be:

$$T_{\mu\nu} = (\rho + P)\,U_\mu U_\nu + P g_{\mu\nu} \tag{2.19}$$

where $P$ is the pressure, $\rho$ is the density and $U_\mu$ is the 4-speed. This implies that in the comoving frame of the cosmological flow $U_\mu = (1; \mathbf{0})$. Hence we assume $T^\nu_\mu$ to be in this frame: $T^0_0 = -\rho$, while: $T^i_j = Pg^i_j$.

Then finally we can write the Friedmann equations to be:

$$\begin{cases} G^0_0 = 8\pi G T^0_0 \longrightarrow & \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{(aR_0)^2} \quad \textbf{(Friedmann equation)} \\ G^i_j = 8\pi G T^i_j \longrightarrow & \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \quad \textbf{(Raychaudhuri equation)} \end{cases} \tag{2.20}$$

where $\rho$ is the density matter and energy of the universe, e.g. matter, radiation, dark matter and dark energy.
We can easily see that the Raychaudhuri equation is nothing but the time derivative of the Friedmann equation, if we take the expression for $\dot{\rho}$ coming from the continuity equation of the fluid which we recall to be: $\nabla_\mu T^\mu_\nu = 0 \longrightarrow \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$.

## 2.3 Slow-roll Inflation

In order to formulate a theory of Big-Bang cosmology which ends up with an homogeneous isotropic and almost flat universe, we need to tune in a very special way the initial conditions. However if we take into account inflation, so a very early epoch of extremely fast expansion of the space, we can start with a more generic setup to obtain the same result.
More in technical terms we can say that inflation is a simple solution to 3 fundamental problems coming from cosmological observations:

- **Horizon problem**
- **Flatness problem**
- **Super-horizon correlations**

Let us now analyze qualitatively those problems and see how inflation solves them all in an elegant way.

### 2.3.1 Horizon problem

Let us define the concept of the **particle horizon**, which is the comoving distance at which light can reach an observer at a certain time $t$. Hence the particle horizon tells us about the causal horizon between the different patches of spacetime, and give us the maximal distance from which a particle can be influenced by all the past events.
Let us suppose that the Big Bang starts with its singularity at $t_i = 0$ on a space-like hyper-surface, then we can write the particle horizon of a point in spacetime at time $t$ as:

$$d_h(\tau) = \tau - \tau_i = \int_{t_i}^{t} \frac{dt}{a(t)} = \int_{a_i}^{a} \frac{da}{a\dot{a}} = \int_{\ln(a_i)}^{\ln(a)} (aH)^{-1} d\left[\ln(a)\right] \tag{2.21}$$

Conventionally $\tau_i$ is set to zero in order to identify the conformal time and the particle horizon, and the size of the horizon could be found simply intersecting the past light-cone of the observer at time $\tau$ with the singularity hyper-surface. This formula shows furthermore how to relate the particle horizon to the **comoving Hubble radius**, which is defined by: $(aH)^{-1}$, in which: $a_i = 0$ corresponds to the Big-Bang singularity.

Before addressing the horizon problem let us introduce the concept of the **last scattering surface**, which is the space-like hyper-surface at which the universe becomes transparent, i.e. photons from this moment on can propagate without scattering, and this is possible only if the density of our cosmological fluid decreases to a certain valour. The photons coming from that surface generate thermal radiation, called the **Cosmic Microwave Background** (**CMB**) with a temperature today measured to be about $3K$. The problem is that, since the time from the Big Bang singularity to the last scattering surface is very short compared to the current age of the universe, in principle exists different areas of the universe which are not in causal contact. More in the detail if we look at the CMB from the earth only regions with less angular separation than $2°$ are in causal contact, so why is the temperature so uniform in all the directions?

### 2.3.2 Flatness Problem

Next, we are considering the spatial curvature. Let us consider the the Friedman equation written in terms of the Hubble parameter:

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho - \frac{k}{(aR_0)^2} \tag{2.22}$$

Let us denote the quantities at our time with the 0 subscript, e.g our time $t = t_0$. If the universe would be completely flat $k = 0$, then we can define a **critical density**, by measuring the current Hubble constant:

$$\rho_{critical,0} = \frac{3H_0^2}{8\pi G} \tag{2.23}$$

Then we can define for each type of substance a dimension-less density parameter as:

$$\Omega_{i,0} = \frac{\rho_{i,0}}{\rho_{critical,0}} \qquad \text{where: } i = r, m, \Lambda \tag{2.24}$$

Let us now recall that all the fluids that appear in the energy-momentum tensor of the Einstein equations have to satisfy a continuity equation $\nabla_\mu T^{\mu\nu} = 0$. We look at 3 types of fluid which are relevant in cosmology and are characterized by the following equation of state:

$$P_i = \omega_i \rho_i \quad \text{where: } \begin{cases} \omega_m & \sim 0 \quad \textbf{matter} \\ \omega_r & = 1/3 \quad \textbf{radiation} \\ \omega_\Lambda & = -1 \quad \textbf{dark energy} \end{cases} \tag{2.25}$$

Hence the continuity equation turns out to be the following:

$$\frac{\dot{\rho}}{\rho} = -3(1+\omega_i)\frac{\dot{a}}{a} \Rightarrow \rho \propto a^{-3(1+\omega_i)} \tag{2.26}$$

Then, defining $\Omega_{k,0} = -\frac{k}{(a_0 R_0 H_0)^2}$, we can rewrite the Friedmann equation as functions of these density parameters which we can measure now:

$$\left(\frac{H}{H_0}\right)^2 = \Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{k,0}a^{-2} + \Omega_{\Lambda,0}a^0 \tag{2.27}$$

Hence if we evaluate this expression at the present time, assuming $a(t_0) = a_0 = 1$, we obtain:

$$1 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{k,0} + \Omega_{\Lambda,0} = \Omega_{0,0} + \Omega_{k,0} \quad \Rightarrow \quad \Omega_{k,0} = -\frac{k}{(R_0 H_0)^2} = 1 - \Omega_{0,0} \tag{2.28}$$

where we have defined: $\Omega_{0,0} = \Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0}$. Hence the Friedmann equation tell us that we can measure now $\Omega_{0,0}$ to obtain information about $\Omega_{k,0}$, which is bounded by CMB observations to be: $|\Omega_{k,0}| < 0.005$. The flatness problem consists in the fact that this value of $\Omega_{k,0} \sim 0$. In order to understand, why that is a problem let us derive an expression for $\dot{\Omega}_k(t)$. Hence:

$$\dot{\Omega}_k(t) = \left(-\frac{k}{R_0^2}\right)\frac{d}{dt}\left(\frac{1}{a^2H^2}\right) = \frac{2k}{(R_0 aH)^2}\left(\frac{\dot{H}}{H} + \frac{\dot{a}}{a}\right) = -2H\Omega_k(t)\left(\frac{\dot{H}}{H^2} + 1\right) \tag{2.29}$$

Let us calculate $\frac{\dot{H}}{H^2}$, recalling the Raychaudhuri equation:

$$\frac{\dot{H}}{H^2} = \frac{1}{H^2}\left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right] = -\frac{4\pi G}{3H^2}(\rho + 3P) - 1 \tag{2.30}$$

Then, summing over fluids:

$$\frac{\dot{H}}{H^2} + 1 = -\frac{4\pi G}{3H^2}\sum_i (1+3\omega_i)\rho_i = -\frac{1}{2}\sum_i (1+3\omega_i)\Omega_i(t) \tag{2.31}$$

Thus we have found:

$$\dot{\Omega}_k(t) = H\Omega_k(t)\sum_i (1+3\omega_i)\Omega_i(t) \sim H\Omega_k(t)\left[\Omega_m(t) + 2\Omega_r(t)\right] \tag{2.32}$$

where in the last section we have neglected the dark energy contribution which dominates only in a brief time of the history of the universe. Then from this equation we can understand the nature of the problem, since all the terms on the RHS that multiply $\Omega_k(t)$ are positive, therefore if $\Omega_k(t) \neq 0$ its derivative becomes always more positive or negative depending on the sign of $\Omega_k(t)$, i.e. this will led to runaways in both cases.

### 2.3.3 Super-Horizon Correlations

The previous 2 problems which we have depicted might be bypassed arguing that the flat, homogeneous and isotropic spacetime is the most symmetric solution which has to be the starting point for the Big Bang. However in the CMB we can see that in that very uniform radiation background there are some fluctuations which are of the order of 0.01% of the temperature (dipole fluctuations). The analysis of such fluctuations shows that they are not random, but are instead are correlated. The problem consists in the fact that these correlations should exist only between causally connected points of the universe, but that is not the case, in fact these correlations exist also between causally disconnected regions.
To give a qualitative idea of what is happening let us suppose to have a fluctuation with a wave-length $\lambda$. Let us consider to be in the comoving frame, then $\lambda$ remains fixed in time, while the Hubble radius $(aH)^{-1}$ increases as the particle horizon does. Observations show the existence of correlations with wave-length $\lambda$ greater than the particle horizon radius at the surface of last scattering.

In a certain sense this is analogous to the previous horizon problem, but in this case, since correlations exist between causally disconnected regions, we cannot avoid the problem by assuming that we do not need causal contact between the different patches of spacetime in order to have the same temperature.

### 2.3.4 Inflationary Solution

Let us define the inflation as a period, before the hot Big Bang, of accelerated expansion of the universe. i.e. $\ddot{a} > 0$.
We can see the consequence of this fact on the comoving Hubble sphere, in particular let us consider:

$$\frac{d}{dt}\left[(aH)^{-1}\right] = \frac{d}{dt}\left(\frac{1}{\dot{a}}\right) = -\frac{\ddot{a}}{\dot{a}^2} < 0 \tag{2.33}$$

Hence we can say that if the expansion accelerates, then the comoving Hubble sphere is shrinking.

Let us now clarify in physical terms the distinction between the Hubble and the particle horizon. Let us suppose to have two observers, namely A and B. Let us further suppose that they are separated by the Hubble radius, then, if A send a light ray to B now, the light will never reach B, since all the points on the Hubble sphere are receding at light-speed, however if A would have sent the light ray just before finding himself at the Hubble radius the light would have reached B, hence the observer B would have concluded that at the time he has received the ray, B would have passed the Hubble radius.

By this interpretation of the horizon we can immediately see why accelerated expansion will lead to shrinking of the Hubble sphere.
From the cosmological perspective: the Hubble radius tells us if two observers at a given time could ever be in causal contact in the future, while the particle horizon tells us if two observers have ever been in causal contact, in the past. Then returning to our example the fact that the observer A finds himself at a given time outside the Hubble radius of the observer B does not imply that in an early time the two observers were not in causal contact. However if A is at a given time is outside the particle horizon of B this implies that they have never been in causal contact, and they will never be in causal contact in the future. In this sense we can say that the Hubble radius is always smaller than the particle horizon distance.

Let us now see how by assuming inflation we can solve the previous problems. Let us start from the Horizon problem: let us recall the definition by which the particle horizon is defined:

$$d_h(\tau) = \int_{\ln(a_i)}^{\ln(a)} (aH)^{-1} d\left[\ln(a)\right] \tag{2.34}$$

Then, if $(aH)^{-1}$, is an increasing quantity, since the integral is an infinite weighted sum, we can argue that the integral would almost be of the same order of the integrand function evaluated at late time: $d_h(\tau) \sim (aH)^{-1}$. In the case $(aH)^{-1}$ is decreasing on the other hand the situation is the opposite and the integral will take the value of the integrand function at early time: $d_h(\tau) \sim (a_i H_i)^{-1} \gg (aH)^{-1}$. I.e. the particle horizon is much bigger than the Hubble horizon at a given time.
Let us now give a more quantitative explanation of this phenomenon. Let us assume to have a perfect cosmological fluid in a flat spacetime background. Hence under these assumption the Friedmann equations become:

$$\begin{cases} \left(\frac{\dot{a}}{a}\right)^2 & = \frac{8\pi G}{3}\rho \\ \left(\frac{\ddot{a}}{a}\right) & = -\frac{4\pi G}{3}\rho\,(1+3\omega) \end{cases} \Rightarrow \dot{a} = (\mathrm{const})a^{-\frac{1+3\omega}{2}} \Rightarrow \frac{(aH)^{-1}}{(a_0H_0)^{-1}} = a^{\frac{1+3\omega}{2}} \tag{2.35}$$

Hence, if we plug this expression into the integral to calculate the particle horizon distance, we find:

$$d_h(\tau) = \tau - \tau_i = \frac{2H_0^{-1}}{1+3\omega}\left(a^{\frac{1+3\omega}{2}} - a_i^{\frac{1+3\omega}{2}}\right) \tag{2.36}$$

Since before we have said that the Big Bang starts at $\tau_i = 0$, if we want to keep this convention also with $a_i = 0$, then we have to impose a kind of "inverse" **strong energy condition** (**SEC**) for our fluid but with a different sign: $1 + 3\omega > 0 \rightarrow 1 + 3\omega < 0$.

$$\tau_i = \frac{2H_0^{-1}}{1+3\omega} a_i^{\frac{1+3\omega}{2}} \longrightarrow -\infty \quad (\text{when: } a_i \to 0) \tag{2.37}$$

In other words, inflation puts the original Big Bang singularity to $-\infty$ in conformal time. In terms of the ordinary time coordinate it remains at $t = 0$, but the light-cones are extremely stretched. Hence, it is simpler to reason in terms of conformal time where the light-cones are at 45° angles always.
In this way we can explain the homogeneity and isotropy of the CMB, since all the patches of the spacetime were initially in causal contact.

Let us come to the solution of the flatness problem provided by inflation. Let us consider a cosmological fluid, and let us recall the relation between the density parameters:

$$\Omega_k(t) = \left(\frac{a_i H_i}{aH}\right)^2 \Omega_k(t_i) \tag{2.38}$$

Next, let us consider the Friedman equations in the full form(without setting $k = 0$):

$$\begin{cases} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{(aR_0)^2} \\ \left(\frac{\ddot{a}}{a}\right) &= -\frac{4\pi G}{3}\rho\,(1+3\omega) \end{cases} \Rightarrow \left(\frac{a_i H_i}{aH}\right)^2 = \left[(1-\Omega_{k,i})\left(\frac{a_i}{a}\right)^{1+3\omega} + \Omega_{k,i}\right]^{-1} \tag{2.39}$$

Then we can rewrite the previous equation as:

$$\Omega_k(t) = \frac{\Omega_{k,i}}{(1-\Omega_{k,i})\left(\frac{a_i}{a}\right)^{1+3\omega} + \Omega_{k,i}} \tag{2.40}$$

If we define the number of **e-folds** as: $N = \ln\left(\frac{a}{a_i}\right)$, then we end up with a more useful expression, which is:

$$\Omega_k(t) = \frac{\Omega_{k,i} e^{(1+3\omega)N}}{(1-\Omega_{k,i}) + \Omega_{k,i} e^{(1+3\omega)N}} \tag{2.41}$$

Since in this period the cosmological fluid has to follow the inverse SEC, then for sufficiently large $N$ we find $\Omega_k = 0$ is a reasonable (stable) solution, since we have exponential suppression.

For what does it concern the super-horizon correlation we have already seen that the shrinking of the Hubble sphere enlarge the particle horizon and allows to have such type of correlation. A pictorial way of seeing that is by imagine that the fluctuations which were inside the Hubble radius during inflation go out of the sphere at a certain point (since the radius is shrinking), but when the Hubble radius (after inflation) restart to grow they will appear again to our sight, and in fact were detected.

### 2.3.5 Slow-roll Parameters

The key feature of the physical description of inflation is that all the quantities involved are slowly varying except from the space expansion, parameterized by $a(t)$.
Before to describe how inflation works, let us fix by an estimation, some bounds for on the time that inflation has to last in order to solve the previous described problems. A reasonable assumption to set all the universe in causal contact before inflation starts, is that the Hubble radius has to be bigger than the universe's size when inflation starts, i.e let us set:

$$(a_0 H_0)^{-1} < (a_i H_i)^{-1} \tag{2.42}$$

Let us assume that the Hubble parameter does not varies much during inflation and let us set to $H_i \approx H_e$, where the subscript $e$ stand for end (of inflation). the parameter which tell us how the Hubble sphere shrinks during inflation could be the number of e-folds: $N = \ln\left(\frac{a_i}{a_e}\right)$. It can be showed, but we wouldn't, that the amount by which the Hubble radius has grown during the hot Big Bang evolution depends on the maximal temperature of the thermal plasma at the beginning of the hot Big Bang: the **reheating temperature**, which we will denote as $T_R$. Since we are giving an estimation, let us assume to be only in the radiation dominated era, then: $H \propto a^{-2}$, thus:

$$\frac{a_0 H_0}{a_R H_R} = \frac{a_0}{a_R}\left(\frac{a_R}{a_0}\right)^2 = \frac{a_R}{a_0} \sim \frac{T_0}{T_R} \sim 10^{-28}\left(\frac{10^{15}\,\text{GeV}}{T_R}\right) \tag{2.43}$$

where $10^{15}$ GeV is a typical scale for the reheating temperature, then we will also assume that the energy density at the end of inflation is converted quickly into the particles of the thermal plasma, so that the Hubble radius do not grows significantly between the end of inflation and the beginning of the hot Big Bang, i.e. $(a_e H_e)^{-1} \sim (a_R H_R)^{-1}$. Then we can rewrite the previous inequality as:

$$a_i H_i > 10^{-28}\left(\frac{10^{15}\,\text{GeV}}{T_R}\right)(a_e H_e)^{-1} \sim 10^{-28}\left(\frac{10^{15}\,\text{GeV}}{T_R}\right)(a_e H_i)^{-1} \tag{2.44}$$

Hence we can set the bound on the e-fold to be:

$$N = \ln\left(\frac{a_e}{a_i}\right) > 64 + \ln\left(\frac{T_R}{10^{15}\,\text{GeV}}\right) \tag{2.45}$$

Which from a more careful analysis might be proven to be 60 e-folds, which is the value which we will use during the whole discussion, even if we know that there are other models in which this number is about 40, which depend on which reheating temperature we are considering, as shown in the formula qualitative formula above.

By now we have presented a qualitatively very promising solution, but let us now develop a physical model which describes inflation.

Let us start to rewrite the time derivative of the Hubble radius in terms of a **slow-roll parameter** $\epsilon$, defined by:

$$\epsilon = -\frac{\dot{H}}{H^2} = -\frac{d}{dN}(\ln H) \quad \text{since: } dN = d(\ln a) = Hdt \tag{2.46}$$

Thus:

$$\frac{d}{dt}\left[(aH)^{-1}\right] = -\frac{a\dot{H} + \dot{a}H}{(aH)^2} = -\frac{1-\epsilon}{a} \tag{2.47}$$

Since the inflation corresponds to keep this quantity negative, then we have to impose that $\epsilon < 1$, and we will see more in the details that $\epsilon \ll 1$, because of the near scale invariance observed in the CMB fluctuations, Furthermore we can say that we recover **De Sitter** spacetime in the limit of $\epsilon \to 0$, i.e $H$ is constant and the metric is transnational invariant:

$$ds^2 = -dt^2 + e^{2Ht}d\mathbf{x}^2 \tag{2.48}$$

This parameter will remain different from zero and small for the whole period of inflation, that is why the inflationary time is often called **quasi-De-Sitter period**.

However $\epsilon$ is not enough to describe inflation since we need an other slow roll parameter which tells us that inflation lasts for a sufficient amount of time. We will call this parameter $\eta$, and we will see that a good definition of it is the following:

$$\eta = \frac{d}{dN}(\ln \epsilon) = \frac{\dot{\epsilon}}{\epsilon H} \tag{2.49}$$

Hence for $|\eta| < 1$ we ensure that the the change of $\frac{\dot{\epsilon}}{\epsilon}$ is sufficiently small to keep $\epsilon$ small and non vanishing for the whole duration of inflation.

### 2.3.6 Slow-rolling Inflaton Field

The most simple model to describe inflation is by the use of a scalar field called the **inflaton**, then let us recall the action form of a generic scalar field:

$$S[\phi] = \int dt d\mathbf{x}\left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right] \tag{2.50}$$

From which, following the minimum action principle we can find the **Klein-Gordon** equation:

$$\ddot{\phi} - (\nabla\phi)^2 = -\frac{\partial V}{\partial \phi} \tag{2.51}$$

However in our case we have not Minkowski spacetime, but the FLRW metric, hence the action becomes:

$$S[\phi] = \int dtd\mathbf{x}a^3 \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2a^2}(\nabla\phi)^2 - V(\phi)\right] \tag{2.52}$$

from which always following the least action principle we end up with thee following equation:

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial \phi} \tag{2.53}$$

The second term of the equation is called the **Hubble friction** term, since it appears with a dependence in the first derivative of the field.

Let us now assume that in this first period of the universe this scalar field dominates over all the other fields, then its energy density might be read from the action to be the sum of the kinetic and potential energy:

$$\rho_\phi = \frac{1}{2}\dot{\phi} + V(\phi) \tag{2.54}$$

Having in mind the continuity equation which every cosmological fluid has to satisfy: $\dot{\rho} = -3H(\rho + P)$, then we can calculate:

$$\dot{\rho} = \left(\ddot{\phi} + \frac{\partial V}{\partial \phi}\right)\dot{\phi} = -3H\dot{\phi}^2 \quad \Rightarrow \quad P_\phi = \frac{1}{2}\dot{\phi} - V(\phi) \tag{2.55}$$

Then we obtain that, if the kinetic energy of the field is negligible with respect to its potential energy (slow-roll approximation), then: $P_\phi \sim -\rho_\phi$. In this sense we might interpret it as a temporary cosmological constant.

Let us now explain in what it consist the slow-rolling of the inflaton field, by the use of the already defined slow-rolling parameters. Let us recall the two relevant equation in the description of the process, which are the first Friedmann equation and the Klein-Gordon equation:

$$\begin{cases} H^2 & = \frac{1}{3M_P^2}\left(\frac{1}{2}\dot{\phi} + V\right) \\ -3H\dot{\phi} & = \ddot{\phi} + \frac{\partial V}{\partial \phi} \end{cases} \quad \Rightarrow \quad \dot{H} = -\frac{1}{2}\left(\frac{\dot{\phi}}{M_P}\right)^2 \tag{2.56}$$

Where, in order to simplify the notation we have defined the 4D Plank's mass as: $M_P = \sqrt{\frac{1}{8\pi G}}$. The the slow-roll parameter $\epsilon$ becomes:

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{\frac{1}{2}\dot{\phi}^2}{(M_P H)^2} = \frac{\frac{1}{2}\frac{\dot{\phi}^2}{M_P^2}}{\frac{1}{3M_P^2}\left(\frac{1}{2}\dot{\phi}^2 + V\right)} = \frac{\frac{3}{2}\dot{\phi}^2}{\frac{1}{2}\dot{\phi}^2 + V} = \frac{\frac{3}{2}\dot{\phi}^2}{\rho_\phi} \tag{2.57}$$

Hence now is clear why having a slow-rolling scalar field in which $\frac{1}{2}\dot{\phi}^2 \ll V$, imply: $\epsilon \ll 1$.

Since we want that the inflation, to last for a sufficient amount of time, then, having in mind the Klein-Gordon equation, we can define a dimensionless acceleration per Hubble time to be:

$$\delta = -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{2.58}$$

Hence we can say that when $\delta$ is small in the Klein-Gordon equation the friction term becomes dominant and the velocity of inflation is purely determined by the slope of the potential.
At this point we are able to calculate the other slow-roll parameter $\eta$, previous introduced, hence:

$$\eta = \frac{\dot{\epsilon}}{\epsilon H} = (\epsilon H)^{-1}\left[\frac{\dot{\phi}\ddot{\phi}}{(M_P H)^2} - \frac{\dot{\phi}^2 \dot{H}}{H(M_P H)^2}\right] = 2\left(\frac{\ddot{\phi}}{H\dot{\phi}} - \frac{\dot{H}}{H^2}\right) = 2(\epsilon - \delta) \tag{2.59}$$

This implies that: if $\{\epsilon, |\delta|\} \ll 1$, then $\{\epsilon, |\eta|\} \ll 1$. Hence, in other words if the velocity and acceleration of the inflaton field are small, inflation will last for a sufficient long period.

Let us now apply the slow-roll approximation to simplify the equation of motion for the inflaton field:

$$\{\epsilon, |\delta|\} \ll 1 \quad \Rightarrow \quad \begin{cases} H^2 & \sim \frac{V}{3M_P^2} \\ 3H\dot{\phi} & \sim -V_{,\phi} \end{cases} \quad \Rightarrow \quad \epsilon = \frac{\frac{1}{2}\dot{\phi}^2}{(M_P H)^2} \sim \frac{M_P^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \tag{2.60}$$

Then, taking the derivative with respect to the field of the approximate version of the Klein-Gordon equation, we can find:

$$3\left(\dot{H} + \frac{H\ddot{\phi}}{\dot{\phi}}\right) \sim -V_{,\phi\phi} \quad \Rightarrow \quad -3H^2\left(\frac{\dot{H}}{H^2} + \frac{\ddot{\phi}}{H\dot{\phi}}\right) \sim V_{,\phi\phi} \quad \Rightarrow \quad \delta + \epsilon = M_P^2 \frac{V_{,\phi\phi}}{V} \tag{2.61}$$

Then we can define new slow-roll parameters for the potential in a more useful way to be:

$$\epsilon_V = \frac{M_P^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2, \qquad \eta_V = M_P^2 \frac{V_{,\phi\phi}}{V} \tag{2.62}$$

Then, now we are able to calculate the number of e-folding as a function of the parameter $\epsilon_V$:

$$N = \int_{a_i}^{a_e} d(\ln a) = \int_{t_i}^{t_e} H(t)dt = \int_{\phi_i}^{\phi_e} \frac{H}{\dot{\phi}} d\phi \sim \int_{\phi_i}^{\phi_e} \frac{|d\phi|}{M_P\sqrt{2\epsilon_V}} \tag{2.63}$$

Then this expression give us certain bounds that the potential should satisfy, e.g. supposing to have a potential, which is linear in the field: $V(\phi) = \xi\phi$, we obtain:

$$\epsilon_V = \frac{M_P^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 = \frac{M_P^2}{2}\phi^{-2} \quad \Rightarrow \quad N \sim \int_{\phi_i}^{\phi_e} \frac{|d\phi|}{M_P\sqrt{2\epsilon_V}} = \frac{\phi_e^2 - \phi_i^2}{2M_P^2} \sim 60 \tag{2.64}$$

For completeness, since in our current time we do not see any inflaton field, we must say that the inflaton decay into the Standard model particle. This is what is called **reheating**, which we can see in an approximate version as a dumped oscillation of the inflaton potential around its minimum, reached when the inflation ends, hence when the inflaton's kinetic energy, overcomes its potential one and the fields stabilize in a certain amount of time to its VEV. This process set up the initial conditions for the hot Big-Bang thermal bath, but in the present work we are not interested in the treatment of this phenomenon.

### 2.3.7 Effective Field Theory perspective

In absence of a UV complete theory of gravity, we might try to tackle the problem of inflation by an **effective field theory** (**EFT**) approach, based on what we claim to be a viable theory of quantum gravity (in our discussion we assume string theory to be such kind of theory). In general when we are dealing with an EFT, we can always write the Lagrangian of our system in the following form:

$$\mathcal{L}_{EFT}[\phi] = \mathcal{L}_0[\phi] + \sum_n c_n \frac{O_n[\phi]}{\Lambda^{\delta_n - 4}} \tag{2.65}$$

Declined to our case: $\mathcal{L}_0[\phi]$ is the Lagrangian of the chosen inflationary model, $O_n[\phi]$ are operators which parameterize corrections coming from the couplings to additional high-energy degrees of freedom, while $c_n$ are $O(1)$ constants and $\Lambda$ is the cut-off scale of the EFT. By now we have seen that, in order to have a slow rolling scalar field, we have to require a flat potential (in Planck's scale units). However when we treat inflation in terms of an effective field theory we have to be careful since it is intrinsically sensitive even to effect smaller than the Planck scale, in fact:

$$\mathcal{L}_{EFT}[\phi] = -\frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) - \sum_n c_n V(\phi)\frac{\phi^{2n}}{\Lambda^{2n}} - \sum_n d_n \frac{(\partial_\mu\phi)^{2n}}{\Lambda^{4n}} + ... \tag{2.66}$$

Assuming $|\phi| \ll \Lambda$, i.e. the value of the inflaton field is much smaller than the cut-off scale, we can argue that the leading contribution comes from the term:

$$\Delta V = c_1 V(\phi)\frac{\phi^2}{\Lambda^2} \ll V(\phi) \quad \Rightarrow \quad \Delta\eta_V = \frac{M_P}{V}\Delta V_{,\phi\phi} \sim 2c_1\left(\frac{M_P}{\Lambda}\right)^2 > 1 \tag{2.67}$$

Then we have encountered what is called the **$\eta$-problem**, from which all the EFT model of inflation are affected. The possible solutions to this problem might come from an enlargement of the symmetries of the inflaton field. Naively one can expect that imposing a **supersymmetry** (**SUSY**) can help us to cancel the corrections of the EFT, however it turns out that, even if it will mitigate the problem, it does not solve the problem, since we know that SUSY must be broken during inflation. Thus we will end up with a massive inflaton, with a mass of the scale of the Hubble parameter $H$, which will still give us a first order correction $\Delta\eta_V \sim 1$. Hence also in this case we cannot avoid the fine-tuning on the mass of $\phi$

A more **natural** (without fine-tuning) solution comes from imposing a **shift-symmetry** (also called **Peccei-Quinn** symmetry), which consist in imposing that the inflaton field $\phi$ is invariant under the following transformation:

$$\phi \longrightarrow \phi + c \quad \forall c \in \mathbb{R} \tag{2.68}$$

In other words we are imposing that the inflaton field is an **axion**.

### 2.3.8 Problems of Inflation

Even if we believe that a fine tuning of the starting conditions of the universe can occur in nature, we might agree that the explanation for the a-causal correlations measured can only come from a shrinking of the Hubble sphere (always assuming an interpretation of the spacetime in the general relativity framework). We can then reasonably move some critics about the dynamic of inflation: why we consider a scalar field? Are reasonable all the assumption on its potential? These questions have not an answer, but we might say that this is the simplest model found in agreement with the current observations, however we can ask more precise questions even assuming this model to be valid.

Assuming that the dynamic of inflation is given by the action of a scalar field, we can in fact ask: why it have to start to the top of a certain potential? Why it has to slowly roll?

In order to answer the first question let us e.g. assume that the inflaton field has only a minimum, and in different regions of the space this scalar field takes different values, then we can say that the regions in which the value of the field finds on the top of this potential will experience inflation and, weighting them by their volume, we can say that

globally we obtain an inflationary process, since most of the volume is made up by the regions in which inflation occurs. Let us now assume that the inflaton potential have more than a minimum, then let us suppose for simplicity that it has 2 minima, then there will be one at higher energy which we define to be the false vacuum, and one at lower energy, which we will call the true vacuum. Then, since we are working in the quantum mechanic framework, tunnelling effects might occur, then also in this case if we give different values of the inflaton field to different regions of the space, the places in which inflation occur will become dominant on the regions in which it does not.
To answer the second question, the slow-rolling assumption implies that the kinetic energy is much smaller than the potential one, then, we might try to overcome such problem assuming the inflaton field to be large, in such a way that the Hubble friction term can effectively slow down the speed of the field, i.e. the slow-rolling solution enjoys an attractor behaviour.
We might furthermore say that the **large-field** inflationary model can mitigate also the problems of perturbations to the field, which are under control under such an assumption.

A more subtle problem is the so called **eternal inflation problem**. Let us come back to the case in which we have at least 2 vacua: inflation can still occur even if we are approaching the false vacua. The problem occurs when the inflation rate is larger than the the tunnelling rate from the false to the true vacuum, since inflation continues to enlarge the space and forbid the tunnelling to the real vacuum, hence it becomes eternal. An other case in which we end up with the eternal inflation, even only with a minimum, is when the quantum fluctuations dominates over the classical slow-rolling dynamics, since in that case the inflaton field could always be placed to the top of the potential to roll, then inflation does not end.

Despite all these problems inflation is by now the best known and tested model that we have for cosmology, and we will assume to be true for the rest of the present work in particular working with an axionic inflaton in the large-field approximation, since both the conditions give us a better control on the inflationary dynamics.

## 2.4 Gravitational Perturbation Theory

In this section we will treat how the inflationary mechanism described in term of a single slow-rolling scalar field provides a natural source to primordial density fluctuations. Such fluctuations are built-in the theory, since we define a scalar field in the context of QFT, which is nothing but a relativistic generalization of Quantum Mechanics. In very simple terms we can imagine the inflaton field as a clock which tell us in which part of the inflationary period we are, but it is a quantum clock, hence we have to take into the account Heisenberg's principle.

In the GR formulation we have two tensors from which we can reconstruct the dynamic of the system: the metric $g_{\mu\nu}$ and the stress-energy tensor $T_{\mu\nu}$, then let us express them as derived by a combination of a perturbative and non-perturbative contribution:

$$\begin{cases} g_{\mu\nu}(t,\mathbf{x}) & = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}(t,\mathbf{x}) \\ T_{\mu\nu}(t,\mathbf{x}) & = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}(t,\mathbf{x}) \end{cases} \tag{2.69}$$

What we now will derive is what comes from the expansion of the continuity and Einstein equations keeping the terms only up to the linear order, hence we will derive the linear perturbations of such tensors, which will be useful in the analysis of the primordial perturbations.

### 2.4.1 Metric Perturbations

In order to simplify the analysis which can be in principle very general let us assume the FLRW to be our background metric, then we can perturb the metric at the linear order in the following way:

$$ds^2 = a^2(\tau)\left[-(1+2A)d\tau^2 + 2B_i dx^i d\tau + (\delta_{ij} + 2E_{ij})dx^i dx^j\right] \tag{2.70}$$

Where $A$, $B_i$ and $E_{ij}$ are functions of the space and conformal time, $\delta_{ij}$ is the 3D Euclidean metric and the 2 factors will simplify the calculations.
Let us perform what is called: **scalar-vector-tensor** (**SVT**) decomposition of the perturbations for later convenience, which consists in the following decomposition:

$$B_i = \partial_i B + \hat{B}_i \qquad \text{where: } \partial^i \hat{B}_i = 0 \tag{2.71}$$

And:

$$E_{ij} = C\delta_{ij} + \partial_{(i}\partial_{j)}E + \partial_{\{i}\hat{E}_{j\}} + \hat{E}_{ij} \quad \text{where: } \begin{cases} \partial^i \hat{E}_i = \partial^i \hat{E}_{ij} = \hat{E}^i_i = 0 \\ \partial_{(i}\partial_{j)}E = \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\nabla^2\right)E \\ \partial_{\{i}\hat{E}_{j\}} = \frac{1}{2}\left(\partial_i\hat{E}_j + \partial_j\hat{E}_i\right) \end{cases} \tag{2.72}$$

Then from such a decomposition we have rearranged the 10 metrics degrees of freedom as:

- 4 scalars d.o.f. : $A$, $B$, $C$, $E$
- 4 vectors d.o.f. : $\hat{B}_i$, $\hat{E}_i$
- 2 tensors d.o.f. : $\hat{E}_{ij}$

The beauty of such SVT decomposition lies in the fact that at the linear order in perturbations, for the FLRW metric, we do not have any mixing between the scalar, vector and tensor modes in the Einstein equation, then we can treat them separately. According to the inflationary mechanism depicted in the previous pages we might show that the vectors perturbations are not produced, and even if they were produced by other types of inflationary mechanisms they would decay quickly during the universe's expansion. On the other hand scalar and tensor perturbations are produced even if our focus will be for obvious reasons on the scalar modes.

However even if such a decomposition helps us in solving the perturbation analysis we have a more difficult problem given by the nature of GR. If we want to give a description of the perturbations, we have to find a description which is invariant under diffeomorphisms, which is the GR gauge group. Implicitly in the previous presentation we have assumed a particular time-slice, however we can imagine that a change of coordinates will lead to a change in the perturbations, hence we can have 2 possible situations: by a coordinate transformation we add some new fictitious perturbations, or we lost some of the relevant ones. In other words we want to find a way to distinguish a perturbation from a slight change of coordinate, thus let us start to consider a generic coordinate transformation in the form:

$$x^\mu(q) \mapsto \tilde{x}^\mu(q) \equiv x^\mu(q) + \xi^\mu(q) \qquad \text{where:} \qquad \begin{cases} \xi^0 & = T \\ \xi^i & = L^i = \partial^i L + \hat{L}^i \end{cases} \tag{2.73}$$

Where $\xi^\mu$ is small and can therefore be treated as a perturbation. The function $T(\tau, \mathbf{x})$ defines the hyper-surfaces of constant time in the new coordinates, while $L^i(\tau, \mathbf{x})$ determines the spatial coordinates on these hyper-surfaces. We also in this case we have split the spatial shift $L^i$ into a scalar, $L$, and a divergenceless vector, $\hat{L}^i$.
In order to determine the transformation of the metric let us remind that the spacetime interval is an invariant:

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = \tilde{g}_{\alpha\beta}(\tilde{x}) d\tilde{x}^\alpha d\tilde{x}^\beta \quad \Rightarrow \quad g_{\mu\nu}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \tilde{g}_{\alpha\beta}(\tilde{x}) \tag{2.74}$$

Then we can explicitly workout the transformation matrix to be:

$$\frac{\partial \tilde{x}^\alpha}{\partial x^\mu} = \begin{pmatrix} \partial\tilde{\tau}/\partial\tau & \partial\tilde{\tau}/\partial x^i \\ \partial\tilde{x}^i/\partial\tau & \partial\tilde{x}^i/\partial x^j \end{pmatrix} = \begin{pmatrix} 1 + \partial_0 T & \partial_i T \\ \partial_0 L^i & \delta^i_j + \partial_j L^i \end{pmatrix} \tag{2.75}$$

We can show, that, after applying such transformation we obtain in terms of the SVT decomposition the following transformations of the perturbations' d.o.f.:

$$
\begin{cases}
A \mapsto A - \partial_0 T - \mathcal{H}T \\
B \mapsto B + T - \partial_0 L & \hat{B}_i \mapsto \hat{B}_i - \partial_0 \hat{L}_i \\
C \mapsto C - \mathcal{H}T - \frac{1}{3}\nabla^2 L \\
E \mapsto E - L, & \hat{E}_i \mapsto \hat{E}_i - \hat{L}_i \qquad \hat{E}_{ij} \mapsto \hat{E}_{ij}
\end{cases}
\tag{2.76}
$$

where: $\mathcal{H}$ is the Hubble parameter in conformal time, i.e. $\mathcal{H} = \frac{a'}{a}$.
However we might see that some combinations of the perturbations' variables are invariant under change of coordinates. Such invariant quantities are defined to be the **Bardeen coordinates**:

$$
\begin{cases}
\hat{E}_{ij} \\
\hat{\Phi}_i = \hat{B}_i - \partial_0 \hat{E}_i \\
\Phi = -C + \frac{1}{3}\nabla^2 E - \mathcal{H}(B - \partial_0 E) \\
\Psi = A + \mathcal{H}(B - \partial_0 E) + \partial_0 (B - \partial_0 E)
\end{cases}
\tag{2.77}
$$

Then now we have a good base to write all the perturbations, and then we can chose the Gauge which we most prefer in order to perform the calculations. We can as example make the following gauge choices:

$$
\begin{cases}
A = E = 0 & \textbf{Synchronous gauge} \\
B = E = 0 & \textbf{Newtonian gauge} \\
C = E = 0 & \textbf{Spatially flat gauge}
\end{cases}
\tag{2.78}
$$

We will see that in order to compute the inflaton field's perturbations, the most convenient gauge is the spatially flat gauge.

### 2.4.2 Matter Perturbations

Let us write the relativistic perfect fluid energy momentum tensor at the linear order in pressure and density perturbations as follows:

$$
\begin{cases}
\rho & = \bar{\rho} + \delta\rho \\
P & = \bar{P} + \delta P
\end{cases}
\Rightarrow
\begin{cases}
T^0_0 = -(\bar{\rho} + \delta\rho) \\
T^0_i = (\bar{\rho} + \bar{P})v_i = -T^i_0 \quad (T_{i0} = T_{0i}) \\
T^i_j = (\bar{P} + \delta P)\delta^i_j + \Pi^i_j \quad \Pi^i_i = 0
\end{cases}
\tag{2.79}
$$

Where $v_i$ is called: **bulk velocity**, while $\Pi^i_j$ is the **anisotropic stress**. Let us also introduce the **momentum density** to be $q_i = (\bar{\rho} + \bar{P})v_i$. Let us now decompose SVT components also the energy momentum tensor, then:

$$\begin{cases} v_i & = \partial_i v + \hat{v}_i \\ q_i & = \partial_i q + \hat{q}_i \\ \Pi_{ij} & = \partial_{(i}\partial_{j)}\Pi + \partial_{\{i}\hat{\Pi}_{j\}} + \hat{\Pi}_{ij} \end{cases} \tag{2.80}$$

Where we define the "parentheses in the indices" as we have done in the case of the metrics perturbations. Let us furthermore introduce the **velocity divergence**: $\theta = \partial_i v^i = \nabla^2 v$, and the **density contrast** to be: $\delta = \frac{\delta\rho}{\rho}$. In particular this last parameter, will tell us when is possible to do a perturbative expansion or not.

Also for the stress energy tensor the gauge fixing will play a crucial role in the discussion of the perturbations then let us remind that such tensor transforms in the following way:

$$T^{\mu}_{\nu}(x) = \frac{\partial x^{\mu}}{\partial \tilde{x}^{\alpha}} \frac{\partial \tilde{x}^{\beta}}{\partial x^{\nu}} \tilde{T}^{\alpha}_{\beta}(\tilde{x}) \tag{2.81}$$

We have already obtained the direct transformations' matrix, and since we are supposing infinitesimal coordinates' transformations, we can consider the direct matrix as $\mathbf{1} + \epsilon$ and its inverse as $\mathbf{1} - \epsilon$ , i.e. :

$$\frac{\partial x^{\mu}}{\partial \tilde{x}^{\alpha}} = \begin{pmatrix} 1 - \partial_0 T & -\partial_i T \\ -\partial_0 L^i & \delta^i_j - \partial_j L^i \end{pmatrix} \tag{2.82}$$

then at the linear order in the perturbation theory, we might explicitly found the following transformation limit:

$$\begin{cases} \delta\rho & \mapsto \delta\rho - T\partial_0\bar{\rho} \\ \delta P & \mapsto \delta P - \bar{P}\partial_0 T \\ q_i & \mapsto q_i + \frac{q_i}{v_i}\partial_0 L_i \\ v_i & \mapsto v_i + \partial_0 L_i \\ \Pi_{ij} & \mapsto \Pi_{ij} \end{cases} \tag{2.83}$$

Also in this case we can find the analogous of the Bardeen coordinates, i.e. some gauge invariant variables, which will help us in the discussion:

$$\begin{aligned} &\textbf{comoving density contrast}: \ \bar{\rho}\Delta = \delta\rho + (v + B)\partial_0\bar{\rho} \\ &\textbf{curvature perturbations}: \begin{cases} \zeta = -C + \frac{1}{3}\nabla^2 E + \mathcal{H}\frac{\delta\rho}{\partial_0\bar{\rho}} \\ \mathcal{R} = -C + \frac{1}{3}\nabla^2 E + \mathcal{H}(v + B) \end{cases} \end{aligned} \tag{2.84}$$

And we can easily check that these 3 perturbations are related by the following equation:

$$\zeta = \mathcal{R} - \frac{\mathcal{H}}{\partial_0 \bar{\rho}} \bar{\rho} \Delta \tag{2.85}$$

Also in this case we can chose the gauge that we prefer which will makes the computations easier, i.e. :

$$\begin{cases} \delta\rho & = 0 \longrightarrow \delta g_{ij} = a^2(1-2\zeta)\delta_{ij} \quad \textbf{uniform density gauge} \\ q & = 0 \longrightarrow \delta g_{ij} = a^2(1-2\mathcal{R})\delta_{ij} \quad \textbf{comoving gauge} \end{cases} \tag{2.86}$$

However there are more version of such gauges depending on which of the metric fluctuations is set to zero, but for future convenience we will chose: $E = 0$, in order to have the metric spatial part isotropic.

### 2.4.3 Primordial Perturbations

As we have said before the slow-rolling inflation is characterized by the dynamic of a scalar field which is supposed to dominates over the other fields during the inflationary period, then let us start from the inflaton action which appears in the following form:

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \tag{2.87}$$

Immediately we can see that if we suppose inflaton perturbations $\delta\phi$ we will have to couple them to the metric perturbations $\delta g_{\mu\nu}$, furthermore, as we might expect this mixing is gauge dependent, then for our purposes is useful to work in the spatially flat gauge in which the line element is defined as:

$$ds^2 = a^2(\tau) \left[ -(1+2A)d\tau^2 + 2\partial_i B dx^i dx^\tau + \delta_{ij} dx^i dx^j \right] \tag{2.88}$$

In order to derive the equations of motions for the inflaton perturbations let us remind the Klein-Gordon equation in a generic spacetime (derived by imposing the vanishing of the variations of the previous equation) to be:

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) = V_{,\phi} \tag{2.89}$$

Then expanding $\phi = \bar{\phi} + \delta\phi$, and hence expanding $V(\phi)_{,\phi}$, we can find from the field equation of motion also the equations of motion for the perturbations at the linear order:

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2 \delta\phi = (A' + \nabla^2 B)\bar{\phi}' - 2a^2 V_{,\phi} A - a^2 V_{,\phi\phi} \delta\phi \tag{2.90}$$

Where we have substituted the partial derivatives with respect to the conformal time with the prime, in order to simplify the notation for further manipulations. In fact, since we want to obtain the equation of motion for the field perturbation we had better to express the metric perturbations as function of the inflaton one. in other words we have to use the linearized version of the Einstein equation in the spatially flat gauge. Let us start from the definition of the the 2 slow-roll parameters:

$$\begin{cases} \epsilon &= -\frac{\dot{H}}{H^2} = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} = 4\pi G \frac{(\bar{\phi}')^2}{\mathcal{H}^2} \\ \delta &= -\frac{\ddot{\bar{\phi}}}{H\dot{\bar{\phi}}} = 1 - \frac{\bar{\phi}''}{\mathcal{H}\bar{\phi}'} \end{cases} \tag{2.91}$$

In order to use the Einstein's equation, let us remind the stress-energy tensor of a scalar field to be:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right] \tag{2.92}$$

Then, since we just need to calculate $A$ and $\nabla^2 B$, we do not need to solve all the equations. Let us tart to compute $A$, then let us consider:

$$\delta G^0_i = -\frac{2\mathcal{H}}{a^2}\partial_i A = 8\pi G\delta T^0_i = -8\pi G\frac{\bar{\phi}'}{a^2}\partial_i\delta\phi \quad \Rightarrow \quad A = 4\pi G\frac{\bar{\phi}'}{\mathcal{H}}\delta\phi = \epsilon\frac{\mathcal{H}}{\bar{\phi}'}\delta\phi \tag{2.93}$$

While for calculating $\nabla^2 B$ we need to consider:

$$\delta G^0_0 = \frac{2\mathcal{H}}{a^2}\left(3\mathcal{H}A + \nabla^2 B\right) = 8\pi G\delta T^0_0 = -8\pi G\left[\frac{\bar{\phi}'\delta\phi' - (\bar{\phi}')^2 A}{a^2} + V_{,\phi}\delta\phi\right] \tag{2.94}$$

In order to solve the equation, let us plug in the expression for $A$, and also use the Klein-Gordon equation for the background field, which we remind to be:

$$-\frac{\bar{\phi}'' + 2\mathcal{H}\bar{\phi}'}{a^2} = V_{,\phi} \tag{2.95}$$

Thus, we end up with:

$$\nabla^2 B = -\epsilon\frac{\mathcal{H}}{\bar{\phi}'}\left[\delta\phi' + (\delta - \epsilon)\mathcal{H}\delta\phi\right] \tag{2.96}$$

Then by a substitution of the results for $A$ and $\nabla^2 B$ into the original equation for the field perturbations we can arrive to the following equation, which is only function of the scalar field:

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi = \left[2\epsilon(3 + \epsilon - 2\delta) - \frac{a^2 V_{,\phi\phi}}{\mathcal{H}^2}\right]\mathcal{H}^2\delta\phi \tag{2.97}$$

Furthermore by deriving an other time with respect to the field the scalar the Klein-Gordon equation and writing it in terms of the slow roll parameters, we can bring the previous equation in the following form:

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi = \left[2\epsilon(3+\epsilon-2\delta) - \frac{\delta'}{\mathcal{H}}\right]\mathcal{H}^2\delta\phi \tag{2.98}$$

We can go further and write this equation even in a nicer form, making the following substitutions:

$$\begin{cases} f &= a\delta\phi \\ z &= a\frac{\bar{\phi}'}{\mathcal{H}} \end{cases} \quad \Rightarrow \quad f'' + \left(k^2 - \frac{z''}{z}\right) f = 0 \quad \textbf{Mukhanov-Sasaki equation} \tag{2.99}$$

In which we have hidden a more complicated quantity in $k^2$, which however is a square. We have written it as $k$ since we have in mind a Fourier expansion of f, hence $k$ has to be seen as a wave number. In order to make such consideration more explicit we could have started directly from the scalar field action:

$$S = \int d\tau d\mathbf{x}\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \quad \text{where: } \phi(\tau,\mathbf{x}) = \bar{\phi} + \frac{f(\tau,\mathbf{x})}{a(\tau)} \tag{2.100}$$

Obtaining an action for the field $f$, which represent the inflaton's variations. And if we perform all the previous substitutions of variables we will end up with the following action:

$$S = \int d\tau d\mathbf{x}\left[(f')^2 - (\nabla f)^2 + \frac{z''}{z}f^2\right] \quad \text{where: } f(\tau,\mathbf{k}) = \int \frac{d\mathbf{x}}{(2\pi)^{3/2}} f(\tau,\mathbf{x})e^{-i\mathbf{x}\mathbf{k}} \tag{2.101}$$

Hence, by imposing the vanishing of the variations, we will recover the same form of the Mukhanov-Sasaki equation.
Before to proceed in the quantization of the theory, let us make a few comment on such equation. By now we have not made any assumption about the parameters which regulate the dynamics of the fluctuations, but now let us assume to be in the slow-roll approximation, i.e. let us consider $H$, $\partial_t\bar{\phi}$, approximately constant, thus:

$$\frac{z''}{z} \sim \frac{a''}{a} \sim 2\mathcal{H}^2 \tag{2.102}$$

Let us furthermore define:

$$\omega^2(\tau,k) = k^2 - \frac{z''}{z} \tag{2.103}$$

Then we can say that at early times $\mathcal{H}^{-1}$ will be large and all the modes will be inside the horizon, hence $\omega^2(\tau,k) \sim k^2$, then MS equation reduces to the equation of an harmonic oscillator:

$$f'' + k^2 f = 0 \quad \Rightarrow \quad f \propto e^{\pm ik\tau} \quad \text{(sub-horizon)} \tag{2.104}$$

At a later time, when the comoving horizon shrinks, then it can happen that we can pass from the regime in which $k^2 \gg |\frac{z''}{z}|$ to one in which: $k^2 \ll |\frac{z''}{z}|$, then the MS equation becomes:

$$f'' - \frac{z''}{z} f = 0 \quad \Rightarrow \quad f \propto \begin{cases} z & \text{growing mode} \\ z^{-2} & \text{decaying mode} \end{cases} \quad \text{(super-horizon)} \tag{2.105}$$

Both those situations are extremal, but let us consider a more refined version of the sub-horizon limit of the MS equation in the slow-roll approximation, then we will have to treat $H \sim$ constant , and consequently $a \sim -(H\tau)^{-1}$, hence we will consider the MS equation in the following form:

$$f'' + \left(k^2 - \frac{2}{\tau^2}\right) f = 0 \quad \Rightarrow \quad f(\tau, k) = f_k(\tau) = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) \quad \textbf{Bunch-Davies mode} \tag{2.106}$$

Upon the quantization of $f$ we might define the following operator:

$$\hat{f}(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \left[ f_k(\tau) \hat{a}_{\mathbf{k}} + f_k^*(\tau) \hat{a}^\dagger_{\mathbf{-k}} \right] e^{i\mathbf{kx}} \tag{2.107}$$

In which $\{\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{-k}}\}$ are the usual ladder operators which satisfy the canonical commutation relations. Then we can define a vacuum state $|0\rangle$ as the state such that: $\hat{a}_{\mathbf{k}}|0\rangle = 0$ $\forall \mathbf{k}$. Hence it is not difficult to see that: $\langle \hat{f} \rangle = \langle 0|\hat{f}|0\rangle = 0$, however its square will be non vanishing:

$$\begin{aligned}
\langle |\hat{f}|^2 \rangle =& \langle 0|\hat{f}(\tau, \mathbf{0})\hat{f}(\tau, \mathbf{0})|0\rangle = \\
=& \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \langle 0| \left[ f_k(\tau)\hat{a}_{\mathbf{k}} + f_k^*(\tau)\hat{a}^\dagger_{\mathbf{-k}} \right] \left[ f_{k'}(\tau)\hat{a}_{\mathbf{k'}} + f_{k'}^*(\tau)\hat{a}^\dagger_{\mathbf{-k'}} \right] |0\rangle = \\
=& \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} f_k(\tau) f_{k'}^*(\tau) \langle 0|[\hat{a}_{\mathbf{-k'}}, \hat{a}^\dagger_{\mathbf{-k'}}]|0\rangle = \int \frac{d^3k}{(2\pi)^3} |f_k(\tau)|^2 = \\
=& \int d[\ln k] \frac{k^3}{2\pi^2} |f_k(\tau)|^2 = \int d[\ln k] \Delta_f^2(\tau, k)
\end{aligned} \tag{2.108}$$

Where $\Delta_f^2(\tau, k)$ is called the **power spectrum**, and is a dimensionless quantity. Then rephrasing such quantity in terms of the inflaton's fluctuations ($f = a\delta\phi$) we can write the inflaton's zero point power spectrum as:

$$\Delta^2_{\delta\phi}(\tau,k) = \frac{\Delta^2_f(\tau,k)}{a^2(\tau)} = \left(\frac{H}{2\pi}\right)^2 \left[1+(k\tau)^2\right] \quad \Rightarrow \quad \lim_{k\tau\to 0}\Delta^2_{\delta\phi}(\tau,k) = \left(\frac{H}{2\pi}\right)^2 \tag{2.109}$$

Let us notice that in the super-horizon limit we lose the momentum dependence, hence we obtain a scale-invariant power spectrum. In order to remain in the above limit, let us evaluate such power spectrum at horizon crossing, i.e. $k = aH(t)$, then:

$$\Delta^2_{\delta\phi}(k) \sim \left(\frac{H(t)}{2\pi}\right)\bigg|_{k=aH(t)} \tag{2.110}$$

From this evaluation we might see that we have lost the scale invariance, since $H(t)$ decreases during inflation. Thus long-wave-length fluctuations, which exit the horizon at the beginning of inflation, will be slightly larger.

At this point let us find out the explicit relation between the curvature and the field perturbation in the spatially flat gauge ($C = E = 0$). Let us recall that in such gauge we have the curvature perturbation are:

$$\mathcal{R} = \frac{\mathcal{H}}{\bar{\rho}+\bar{P}}\delta q \tag{2.111}$$

Furthermore let us remind from our previous calculations that:

$$\delta T^0_i = \partial_i \delta q = g^{0\mu}\partial_\mu\phi\partial_i\delta\phi = g^{00}\partial_0\bar{\phi}\partial_j\delta\phi = -\frac{\bar{\phi}'}{a^2}\partial_i\delta\phi \tag{2.112}$$

Since: $a^2(\bar{\rho}+\bar{P}) = (\bar{\phi}')^2$, then:

$$\mathcal{R} = -\frac{\mathcal{H}}{\bar{\phi}'}\delta\phi \quad \Rightarrow \quad \lim_{k\tau\to 0}\mathcal{R} = constant \tag{2.113}$$

In other words we can say that $\mathcal{R} = H\delta t$, which tell us that the curvature perturbations are induced by the time delay at the end of inflation. This relation is also important because allow us to relate the curvature and field power spectrum in the following way:

$$\Delta^2_{\mathcal{R}}(k) = \left(\frac{H}{\dot{\bar{\phi}}}\right)^2 \Delta^2_{\delta\phi} \sim \left(\frac{H^2}{2\pi\dot{\bar{\phi}}^2}\right)^2\bigg|_{k=aH} = \frac{1}{8\pi^2\epsilon}\frac{H^2}{M_P^2}\bigg|_{k=aH} \tag{2.114}$$

Moreover we could have arrived to a more general conclusion, by starting to solve explicitly the MS equation arriving to express:

$$\Delta^2_{\mathcal{R}}(k) = A_s\left(\frac{k}{k^*}\right)^{n_s-1} \quad \text{where} \begin{cases} A_s & = \frac{1}{8\pi^2\epsilon_*}\frac{H_*^2}{M_P^2} \quad \textbf{scalar amplitudes} \\ n_s & = 1-2\epsilon_* - \eta_* \quad \textbf{spectral index} \end{cases} \tag{2.115}$$

Where the quantities with the star are supposed to be fixed. The current experimental bounds are: $A_s = (2.098 \pm 0.023) \times 10^{-9}$, $n_s = 0.9603 \pm 0.0073$, when $k_* = 0.05 Mpc^{-1}$. And this tell us that, as we expect, the power spectrum is not scale-invariant (i.e. $n_s \neq 1$), because of the time dependence which we have already mentioned.

### 2.4.4 Primordial Gravitational Waves

An other relevant prediction of inflation is a spectrum of primordial gravitational waves. In the language of gravitational perturbation theory these are just tensor perturbations to the spatial metric, hence we can write:

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right] \qquad \text{where: } \partial_i h_{ij} = 0 \quad h_{ii} = 0 \tag{2.116}$$

We might show that the tensor fluctuations satisfy, in the FLRW background the following wave equation:

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = 0 \quad . \tag{2.117}$$

We can derive this from the Einstein-Hilbert action expanded to 2nd order in the tensor perturbation $h_{ij}$ and derivatives $\partial_\mu$ as an action of the following form:

$$S = \frac{M_P^2}{8}\int d\tau dx^3 a^2(\tau)\left[(h'_{ij})^2 - (\nabla h_{ij})^2\right] \tag{2.118}$$

In order to perform the calculations is useful to use rotational symmetry to align the z-axis of the coordinate system with the momentum of the mode, i.e. $\mathbf{k} = (0, 0, k)$ (where $\mathbf{k}$ is the wave-vector), and write the perturbations in function of the 2 polarizations $f_+$ and $f_\times$ :

$$\frac{M_P}{2}a(\tau)h_{ij} = \frac{1}{\sqrt{2}}\begin{bmatrix} f_+ & f_\times & 0 \\ f_\times & f_+ & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.119}$$

thus we can now rewrite the above action as:

$$S = \frac{1}{2}\sum_{\lambda=+,\times}\int d\tau dx^3 a^2(\tau)\left[(f'_\lambda)^2 - (\nabla f_\lambda)^2 + \frac{a''}{a}f_\lambda^2\right] \tag{2.120}$$

Which is simply the double of the Mukhanov-Sasaki equation for the massless scalar field. Then since we have already done the calculation for power spectrum of the scalar perturbations, we can re-use such results to calculate the tensor fluctuations, rescaling the previous derivation according to the action normalization, hence:

$$\Delta_h^2(k) = 2 \times \frac{4}{M_P^2} \times \Delta_{\delta\phi}^2(k,\tau)\Big|_{k=aH} = \frac{2}{\pi^2}\left(\frac{H}{M_P}\right)\Big|_{k=aH} \tag{2.121}$$

And as before we can show that we can write such fluctuations as function of an amplitude $A_t$ and a spectral index $n_t$ (in the slow-roll approximation) in the following way:

$$\Delta_h^2(k) = A_t\left(\frac{k}{k_*}\right)^{n_t} \qquad \text{where} \begin{cases} A_t & = \frac{2}{\pi^2}\frac{H_*^2}{M_P^2} \\ n_t & = -2\epsilon_* \end{cases} \tag{2.122}$$

However observational constraints on the tensor amplitude are usually expressed in terms of the **tensor-to-scalar ratio**, which is simply the ratio between the tensor and scalar amplitude.

$$r = \frac{A_t}{A_s} = 16\epsilon_* \tag{2.123}$$

Since the amplitude of scalar fluctuations has been measured, while a primordial tensor signal so far has not been seen, we express its amplitude in terms of the tensor-to-scalar ratio $r$. A detection of this quantity might be seen as a direct measurement of the slow roll parameter $\epsilon$. From current observations the bound on this ratio is $r < 0.038\,(95\%)$.

# Chapter 3

# Type IIB Fluxed Orientifold Compactifications

## 3.1 Towards Calabi-Yau Manifolds

### 3.1.1 Definition of Calabi-Yau Manifolds

Let us recall that the Calabi-Yau manifolds are a very special type of **complex differential manifolds**. Let us recall the definition of a $n$-dimensional complex differential manifold $\mathcal{M}$ as a generalization of a $2n$-dimensional real differential manifold. Hence we can define an Atlas on it in the form:

$$\bigcup_i (\phi_i, U_i) \quad \text{where:} \qquad \phi_i : \mathcal{M} \supset U_i \to \phi_i(U_i) \subset \mathbb{C}^n \tag{3.1}$$

Hence as a real $n$-dimensional manifold looks locally as $\mathbb{R}^n$, we expect that an $n$-dimensional complex manifold looks locally as $\mathbb{C}^n$, i.e. it should be invariant under the group of complex diffeomorphisms, which we can explicitly write as:

$$z'^i = z'^i(z^1, ..., z^n) \tag{3.2}$$

Hence having in mind the analogy between a $2n$-dimensional real and an $n$-dimensional complex differential manifold, we can define a basis for the tangent and co-tangent space for all points of the manifold $\mathcal{M}$ respectively as:

$$\left\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^{\bar{\imath}}}\right\}_{i=1,\dots,n} \qquad ; \quad \left\{dz^i, d\bar{z}^{\bar{\imath}}\right\}_{i=1,\dots,n} \tag{3.3}$$

However, already from what we know from the relation between $\mathbb{R}^2$ and $\mathbb{C}$ we need to implement and generalize the notion of the multiplication by $i$. The proper way of doing this operation is by the introduction of a 2-form, called **almost complex structure** $J$, which might be written in the previous defined basis to be:

$$J = idz^i \otimes \frac{\partial}{\partial z^i} - id\bar{z}^{\bar{\imath}} \otimes \frac{\partial}{\partial \bar{z}^{\bar{\imath}}} = \begin{bmatrix} i\mathbb{1} & \mathbb{0} \\ \mathbb{0} & -i\mathbb{1} \end{bmatrix} \Rightarrow J^2 = -1 \tag{3.4}$$

Such structure define an **almost complex manifold**, and if, after a coordinate transformation, we can always bring the tensor $J$ $\forall U_p \in \mathcal{M}$ (where: $U_p$ is a neighbourhood of a point $p \in \mathcal{M}$) in the canonical form just described, then we can define many sets of **holomorphic coordinates**. Since the coordinates are defined in every point's neighbourhood, we can hence define an **holomorphic atlas** and this gives to the manifold a **complex structure**, which induces the almost complex structure. We can furthermore prove under such assumptions that the complex structure is unique and coincide with the almost complex structure, which guaranties **integrability**. Furthermore, the Newlander–Nirenberg theorem, tell us that if the Nijenhuis tensor $N_J$[1] of an almost complex structure is vanishing, then it is integrable, hence the almost complex structure is promoted to a complex structure.
Then we can define an $n$-**dimensional complex manifold** as a $2n$-dimensional real manifold with a complex structure, which e.g. is what we are implicitly doing when we consider the Argan-Gauss plane and not $\mathbb{R}^2$.

Now that we have introduced the notion of a complex manifold, we can be even more restrictive in our choice of the complex manifold, more in the detail we can ask to this complex manifold to have a **Riemann structure**, i.e. a metric, and that makes our complex manifold an **Hermitian manifold**. Furthermore if we require also that our manifold have a **symplectic structure**, i.e. a closed, non degenerate and smooth 2-form, then our Hermitian manifold become a **Kähler manifold**. Since the complex structure, the metric and the symplectic structure are all 2-forms and they have to be compatible one to an other, that means that they have to be related. In fact in a Kähler manifolds the symplectic form is named **Kähler form** defined as follows:

$$\omega_{\text{Kähler}} = g_{i\bar{\jmath}}(Jdz^i) \wedge d\bar{z}^{\bar{\jmath}} = ig_{i\bar{\jmath}}dz^i \wedge d\bar{z}^{\bar{\jmath}} \tag{3.5}$$

Those property imply that we can express the metric of a Kähler manifold locally as:

$$g_{i\bar{\jmath}}(p) = \frac{\partial^2 K(p)}{\partial z^i \partial \bar{z}^{\bar{\jmath}}} \quad \text{where: } p \in \mathcal{M} \tag{3.6}$$

Since we have a metric we know that the Levi-Civita connection exists and is unique, hence we can parallel transport the tangent vectors to the manifold. Hence we can define

[1]Let us recall the definition of: $N_A(X,Y) = -A^2[X,Y] + A([AX,Y] + [X,AY]) - [AX,AY]$, hence: $N_J(X,Y) = [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY]$, where $X,Y$ are smooth vector fields and $A$ a generic rank 2 tensor.

an **holonomy group** of the manifold. Let us consider a point $p \in \mathcal{M}$, and any closed curve $\gamma_i \in \mathcal{M}$, such that $p \in \gamma_i$

$$Hol(\mathcal{M}) = \bigcup_i R(\gamma_i) \quad \text{where: } R(\gamma_i) : T_p \to T_p \tag{3.7}$$

If we furthermore assume that $\mathcal{M}$ is connected and orientable (but orientability is guaranteed by the existence of the complex structure), then we can argue that $Hol(\mathcal{M}) \subset SO(2n)$. Let us notice that $Hol(\mathcal{M})$ does not depend op the choice of the point $p$ but only on the geometrical properties of $\mathcal{M}$. It can be shown that require that $Hol(\mathcal{M}) = U(n) \subset SO(2n)$ is equivalent to require that $\mathcal{M}$ is Kähler.

Now we can define a **Calabi-Yau manifold**(**n-fold**) as a Kähler manifold such that $Hol(\mathcal{M}) = SU(n)$. Let us recall that:

$$U(n) = SU(n) \times U(1) \tag{3.8}$$

Thus we can think the connection to be the direct sum of a $SU(n)$ and $U(1)$ part, hence we can imagine a Calabi-Yau manifold as Kähler manifold with a vanishing "field strength" $F_{i\bar{\jmath}}$ associated to the $U(1)$ part of the connection. More explicitly we can construct such field strength starting from the Riemann tensor, and the complex structure:

$$F_{i\bar{\jmath}} = (R_{i\bar{\jmath}})^{\alpha}{}_{\beta} J^{\beta}{}_{\alpha} = i(R_{i\bar{\jmath}})^{k}{}_{k} - i(R_{i\bar{\jmath}})^{\bar{k}}{}_{\bar{k}} = 2i(R_{i\bar{\jmath}})^{k}{}_{k} = -2iR_{i\bar{\jmath}} \tag{3.9}$$

At this point it might be clear that $Hol(\mathcal{M}) = SU(n) \iff R_{i\bar{\jmath}} = 0$, hence asking that the holonomy group of a Kähler manifold to be $SU(n)$ is equivalent to impose the Ricci flatness condition.

Let us now give an alternative definition of Calabi-Yau manifold by the use of the so called **Chern classes**. Hence let us consider the tangent bundle of a generic Kähler manifold, it can naturally be viewed as a complex vector bundle in which the curvature is determined by the Riemann tensor $R_{i\bar{\jmath}}{}^{k}{}_{l}$. Then let us define the following 2-form:

$$R(T_{\mathcal{M}}) = dz^i \wedge d\bar{z}^{\bar{\jmath}} R_{i\bar{\jmath}}{}^{k}{}_{l} \tag{3.10}$$

Then let us write the following multiform:

$$c(\mathcal{M}) = \det\left[\mathbb{1} + R(T_{\mathcal{M}})\right] \tag{3.11}$$

Let us expand it:

$$\begin{aligned} c(\mathcal{M}) &= 1 + Tr\{R(T_{\mathcal{M}})\} + Tr\Big\{R(T_{\mathcal{M}}) \wedge R(T_{\mathcal{M}}) - 2Tr\left[R(T_{\mathcal{M}})^2\right]\Big\} + ... \\ &= 1 + c_1(\mathcal{M}) + c_2(\mathcal{M}) + ... \end{aligned} \tag{3.12}$$

Let us define the $k^{\text{th}}$ Chern class as the $2k$-form $c_k(\mathcal{M})$. As we might guess from our previous considerations we might be interested to the vanishing of the first Chern Class in order to restrict the homology group of the Kähler manifold to $SU(n)$. More precisely we can require the first Chern class to be an **exact** 2-form, i.e. it is zero in **cohomology**. Furthermore, since we have defined the Chern classes starting from the metric, they will be invariant (up to exact forms) under smooth metric's variations. In other words they represent topological invariant. This naive guess is well formulated in the **Yau's theorem** [Nak03]:

**Theorem 1** *Let $\mathcal{M}$ be a Kähler manifold and $\omega_K$ its Kähler form. If the $1^{st}$ Chern class vanishes, then it is possible to define a Ricci flat metric, with correspondent Kähler form $\omega'_K$ in the same cohomology class. This metric is the Calabi-Yau metric and is unique.*

### 3.1.2 Hodge Theory

Let us first recap some concept coming from real differential geometry in order to extend them to the complex manifolds.

Let us recall the definition of $p$**-chain** belonging to a compact manifold $\mathcal{M}$

$$c_p = \sum_i \alpha_i S_{p,i} \quad \text{where: } \alpha_i \in \mathbb{R}, \quad S_{p,i} \ p\text{-dimensional subsets (\textbf{divisors})} \in \mathcal{M} \tag{3.13}$$

Let us define the **boundary operator** $\partial$ as the nihilpotent operator such that:

$$\partial S_{p,i} = S_{(p-1),i} \qquad \partial^2 S_{p,i} = 0 \tag{3.14}$$

Then we can define the $p$**-cycles** as the $p$-chains without boundaries, i.e. $\partial c_p = 0$. Furthermore we can define an **homology group** as follows:

$$H_p = \frac{Ker\{\partial c_p\}}{Im\{\partial c_{p+1}\}} = \frac{p\text{-cycles}}{\text{boundaries of } (p+1)\text{-chains}} \quad . \tag{3.15}$$

The elements of $H_p$ are classes of cycles, called **homology classes**.

Let us introduce the $p$**-form** ad dual objects to the $p$-chains, where the duality relation is given by:

$$\omega_p(c_p) = \int_{c_p} \omega_p = \sum_i \alpha_i \int_{S_{p,i}} \omega_p \tag{3.16}$$

Let us now introduce the **exterior derivative** $d$ as the nilpotent dual operator of $\partial$ for the forms:

$$d : \omega_p \to d\omega_p = \omega_{p+1} \quad \text{such that: } d^2\omega_p = 0 \tag{3.17}$$

Let us remind that an form is said to be **closed** when $d\omega_p = 0$, and **exact** when $\omega_p = d\omega_{p-1}$ (hence all the exact forms are closed). Thus in a similar way as before we can define the so called **de Rham co-homology** as:

$$H^p = \frac{Ker\{d\omega_p\}}{Im\{d\omega_{p-1}\}} = \frac{Ker\{d_p\}}{Im\{d_{p-1}\}} = \frac{\text{closed p-forms}}{\text{exact p-forms}} \tag{3.18}$$

In order to see that $H_p(\mathcal{M}) = H^p(\mathcal{M})^*$, let us consider that the pairing between the forms and chains:

$$\omega_p(c_p) = \int_{c_p} \omega_p \tag{3.19}$$

And we can show explicitly that: if we take 2 representatives of the homology and co-homology classes respectively $[c_p]$ and $[\omega_p]$, such pairing does not depend on the representative:

$$\int_{c_p} \omega_p + d\omega_{p-1} = \int_{c_p} \omega_p + \int_{\partial c_p} \omega_{p-1} = \int_{c_p} \omega_p = \int_{c_p} \omega_p + \int_{c_{p+1}} d\omega_p = \int_{c_p + \partial c_{p+1}} \omega_p \tag{3.20}$$

Since this pairing between the classes is non degenerate, i.e. we have a one to one correspondence between closed non exact forms and cycles, we might prove that $H_p(\mathcal{M})$ and $H^p(\mathcal{M})$ are dual vector spaces (de Rham's theorems). This duality implies that they will have the same dimension, which is expressed by the so called **Betti numbers** as follows:

$$b_p(\mathcal{M}) = dim\{H_p(\mathcal{M})\} = dim\{H^p(\mathcal{M})\} \tag{3.21}$$

Let us now define an other duality in the forms' realm, namely the **Poincaré duality**. Let us suppose to have a real compact $n$-dimensional manifold $\mathcal{M}$, and let us define the following pairing operation between representative of the co-homology classes:

$$[\omega_p][\omega_{n-p}] = \int_{\mathcal{M}} \omega_p \wedge \omega_{n-p} \tag{3.22}$$

We might prove that such pairing is non degenerate, hence there is a duality between $H^p(\mathcal{M})$ and $H^{n-p}(\mathcal{M})$. Thus, this implies that $H^p(\mathcal{M})$ is dual to $H_{n-p}(\mathcal{M})$, i.e we have found a so called **canonical isomorphism**, and this one is defines the Poincaré duality. More explicitly we can say that a $p$-form $\omega_p$ is Poincaré dual to an $(n-p)$-cycle $c_{n-p}$ if:

$$\int_{c_{n-p}} \omega_{n-p} = \int_{c_{n-p}} \omega_p \wedge \omega_{n-p} \qquad \forall \omega_{n-p} \tag{3.23}$$

Furthermore if we can provide to our manifold even a metric, we can define the famous **Hodge star operator** as follows:

$$* : \omega_p \to (*\omega)_{n-p} \quad \text{where:} \quad (*\omega)_{\mu_{p+1}\dots\mu_n} = \frac{\sqrt{g}}{p!}\epsilon_{\mu_1\dots\mu_n}\omega^{\mu_1\dots\mu_p} \tag{3.24}$$

Hence we now are able to define a scalar product between forms as:

$$(\omega_p, \alpha_p) = \int_X \omega_p \wedge *\alpha_p \tag{3.25}$$

Furthermore we can define the **co-differential** $d^\dagger$, which is the adjoint of $d$:

$$d^\dagger = (-1)^p(*)^{-1}d(*) \tag{3.26}$$

Then we are able to properly define the **Laplace operator** as:

$$\Delta = d^\dagger d + dd^\dagger \tag{3.27}$$

In our case this operator is fundamental since it allow us to define the **harmonic forms**, which are the forms $\omega$ such that: $\Delta\omega = 0$. At this point we can introduce the **Hodge decomposition theorem**:

**Theorem 2** *Let $\mathcal{M}$, be a real compact manifold, then any form defined on $\mathcal{M}$ has a unique decomposition in an exact, co-exact and harmonic piece:*

$$\omega = d\alpha + d^\dagger\beta + \gamma \quad \textit{where: } \Delta\gamma = 0 \tag{3.28}$$

As a corollary we can show that if $\omega$ is closed, then $\beta = 0$. This implies that every representative of a cohomology class has a unique decomposition in terms of an exact and harmonic form.

Let us now extend all this machinery that we have developed for real manifolds to the complex manifolds. Let us start by reminding that we can always decompose a one form in the following way:

$$\omega(z,\bar{z}) = \omega(z,\bar{z})_i dz^i + \omega(z,\bar{z})_{\bar{\imath}} d\bar{z}^{\bar{\imath}} = \omega_{(1,0)} + \omega_{(0,1)} \tag{3.29}$$

And following a similar reasoning we have can expand a 3-form as:

$$\omega_3 = \omega_{(3,0)} + \omega_{(2,1)} + \omega_{(1,2)} + \omega_{(0,3)} \tag{3.30}$$

Let us extend to the complex field the exterior derivative, in the following way:

$$d = dz^i\frac{\partial}{\partial z^i} + d\bar{z}^{\bar{\imath}}\frac{\partial}{\partial \bar{z}^{\bar{\imath}}} = \partial + \bar{\partial} \quad \text{where: } i = 1, ..., n \tag{3.31}$$

Since both: the **holomorphic** $\partial$ and **anti-holomorphic** $\bar{\partial}$ are nihilpotent (i.e $\partial^2 = \bar{\partial}^2 = 0$), we can define a cohomology. Conventionally we use to define such cohomology which goes under the name of **Dolbeault cohomology** by the operator $\bar{\partial}$ in the following way:

$$H^{p,q} = \frac{Ker\{\bar{\partial}\omega_{(p,q)}\}}{Im\{\bar{\partial}\omega_{(p,q-1)}\}} = \frac{Ker\{\bar{\partial}_{p,q}\}}{Im\{\bar{\partial}_{p,q-1}\}} \tag{3.32}$$

Which we might understand as a more refined version of the de Rham cohomology, since it gives us also an information about the relation between the non trivial cycles and the complex structure of the manifold. more explicitly we can represent as follows the relation between the 2 cohomologies:

$$H^k = \bigoplus_{p+q=k} H^{p+q} \tag{3.33}$$

At this point we can define the **Hodge numbers** to be the dimensions of the Dolbeault cohomology groups (similarly to the Betti numbers for the relal manifolds):

$$h^{(p,q)}(\mathcal{M}) = dim\{H^{p,q}(\mathcal{M})\} \tag{3.34}$$

Usually such Hodge numbers are arranged in a beautiful way named the **Hodge diamond**. We present here the case of a 3-dimensional complex manifold, since is the type of manifold in which we will work:

$$\begin{array}{ccccccc}
 & & & h^{(0,0)} & & & \\
 & & h^{(1,0)} & & h^{(0,1)} & & \\
 & h^{(2,0)} & & h^{(1,1)} & & h^{(0,2)} & \\
h^{(3,0)} & & h^{(2,1)} & & h^{(1,2)} & & h^{(0,3)} \\
 & h^{(3,1)} & & h^{(2,2)} & & h^{(1,3)} & \\
 & & h^{(3,2)} & & h^{(2,3)} & & \\
 & & & h^{(3,3)} & & &
\end{array} \tag{3.35}$$

### 3.1.3 Calabi-Yau 3-folds and their Moduli Space

The compactification spaces that we are going to consider are the so called **Calabi-Yau 3-folds**, in 3 complex dimensions, which under the previous considerations we can see as a particular kind of 6D real manifold [Heb20]. The reason why we are choosing such space is because we want to not break a certain amount of SUSY during the compactification. More precisely since we are considering the compactification of a type IIB theory, we want to start with a space in which are preserved the original $\mathcal{N} = 2$ SUGRA corresponding to the SUSY in which are organized the multiplets of the mass-less spectrum of the type IIB string theory.

We know that every SUSY correspond to a conserved supercharge, then in our case we want to conserve locally 2 10D supercharges. We Know that such supercharges operators transform as spinors in 10D ($SO(1,9) = SO(6) \times SO(1,3)$), and we know by the Einstein equivalence principle that locally the 4D manifold is flat. That implies that, in order to conserve these supercharges, we have to impose that the 6D part of a spinor is conserved in the compactification space, which in general is not flat. In other words we are requiring the existence of covariantly constant spinors, i.e. spinors which does not rotate under $Hol(\mathcal{M})$, when they are parallely transported along a closed curve. More explicitly we can define a covariant constant spinor $\xi$ as:

$$\nabla_a \xi = 0 \tag{3.36}$$

Since: $SO(6) \cong Spin(6)/\mathbb{Z}_2 \cong SU(4)/\mathbb{Z}_2$, then we might say that the presence of a covariantly constant spinor in the compactification space correspond to require that the holonomy group of the compactification space is smaller than $SU(4)$, thus $SU(3)$ is the greatest holonomy group that we can have (even that in other situation to enhance SUSY is taken a smaller group, e.g. $SU(2)$).

The fact that $Hol(\mathcal{M}) = SU(3)$, lead to some simplifications to the previous discussed Hodge diamond of a generic 3-dimensional complex manifold, which will take the form:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{(1,1)} & & 0 & \\ 1 & & h^{(2,1)} & & h^{(2,1)} & & 1 \\ & 0 & & h^{(1,1)} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{3.37}$$

Let us notice that such diamond is symmetric along the 2 diagonal, and that is a symmetry enjoyed by all the Calabi-Yau $n$-folds. The fact that $h^{(0,0)} = h^{(3,3)} = 1$, comes from the connectedness of the space, while $h^{(3,0)} = h^{(0,3)} = 1$ is peculiar of the 3-fold. This last property implies, by the Hodge decomposition theorem, the existence of a unique holomorphic 3-form, which usually is called $\Omega$. The presence of such form might be seen as a defining property of the Calabi-Yau 3-fold, since it is related to the $SU(3)$ holonomy group of the manifold.
Hence the Hodge diamond of a Calabi-Yau 3-fold is characterized only by two numbers: $h^{(1,1)}$ and $h^{(2,1)}$.

Let us now discuss the importance of these 2 numbers in the geometrical description of the 3-fold, i.e. let us introduce the concept of **moduli space**.
In order to define the moduli space let us start from the metric of the Calabi-Yau space

and ask if is possible to deform it keeping the space Ricci flat. By the Yau theorem we know that that might be possible only if we change accordingly also the Kähler class or the complex structure of the manifold. The possibility of existence of such deformations imply the existence of the manifold's moduli space spanned by these metric deformations. But let us see what are these possible metric's deformations, which are also called **breathing modes** for obvious reasons:

$$g_{i\bar{j}}dz^i d\bar{z}^{\bar{j}} \longrightarrow g_{i\bar{j}}dz^i d\bar{z}^{\bar{j}} + \delta g_{i\bar{j}}dz^i d\bar{z}^{\bar{j}} + \delta g_{ij}dz^i dz^j + h.c. \tag{3.38}$$

From what is our previous definition of the Kähler form, we can suppose that $\delta g_{i\bar{j}}$ will be accompanied with a change of the harmonic representative of the Kähler class, and therefore are called **Kähler deformations**. It is straightforward to guess that the number of such deformations correspond to the number of possible representative of the $H^{1,1}$ cohomology ($h^{(1,1)}$), since the Kähler form is a $(1,1)$-form. Hence the the Kähler moduli space will be a $h^{(1,1)}$ space and its dimension will always be greater than one, since it is always possible to rescale the metric.
On the other hand the other type of deformations $\delta g_{ij}$ break explicitly the Hermitian structure of the manifold, and therefore will be related to a change in the complex structure of the manifold. Those deformations are therefore called **complex structure deformations**.
In order to count this this type of deformation it is useful to define the following $(2,1)$-form:

$$\delta\chi = \Omega_{ij}{}^{\bar{k}}\delta g_{\bar{k}\bar{l}}dz^i \wedge dz^j \wedge d\bar{z}^{\bar{l}} \tag{3.39}$$

In other words what we have done is relate the $H^{2,1}$ cohomology group to the complex structure deformations, and we can show that this correspondence is in fact a one-to-one map, by the uniqueness of the $\Omega$. Hence the possible complex deformations will be counted by $h^{(2,1)}$
More in the details, by our previous consideration, we can parameterize both the deformations in the following way:

$$\begin{cases} \delta g_{i\bar{j}} = iv^a(\omega_a)_{i\bar{j}} & \text{where: } a = 1, ..., h^{(1,1)} \\ \delta g_{ij} = \frac{i}{\|\Omega\|^2}\bar{z}^k(\bar{\chi}_k)_{i\bar{i}\bar{j}}\Omega^{\bar{i}\bar{j}}{}_j & \text{where: } k = 1, ..., h^{(2,1)} \end{cases} \tag{3.40}$$

Where: $\omega_a$ is a basis of $H^{1,1}$, while $\chi_k$ is a basis of $H^{2,1}$ (hence $\bar{\chi}_k$ is a basis of $H^{2,1}$). On the other hand $v^a$ and $z^k$ are respectively real and complex scalar fields which span the moduli space of the Calabi-Yau 3-fold, which therefore might be seen as a Kähler manifold $\mathcal{M}$, such that:

$$\mathcal{M} = \mathcal{M}_{cs}^{h^{(1,2)}} \times \mathcal{M}_K^{h^{(1,1)}} \tag{3.41}$$

After this brief discussion we might have understood that our compactification space has to be treated as a dynamical geometrical object in which its geometry varies according to its moduli space and the main focus of the present work is on the stabilization of this moduli space, since we want to have a certain control on the geometry an hence on the dynamics of the fields.
This idea is very similar to one of General Relativity in which the massive objects moves according to the spacetime geometry, but also tell to the spacetime how to curve modifying its geometry. In this sense we might say that Sting Theory is in some sense an extension of GR in which we want to include in our geometry not only massive object, but also charged objects under some gauge group like e.g. the Standard Model fields.
We might say that this is the same idea that guided Theodor Kaluza and Oskar Klein in the proposal of a unifying theory of Gravity and Electromagnetism, which is therefore called **Kaluza–Klein theory**.

We will see in the next section how String Theory Compactifications inherit the ideas of these two mathematicians, who were the first to add one space compact dimension ($S^1$) in order to obtain a $U(1)$ gauge field coming directly from an Einstein–Hilbert-like action in 5D. In fact e.g. in order to connect $\mathcal{M}$-theory, who lives in 11D, to the other coherent 10D String Theories is explicitly used a $S^1$ compactification. In other contexts, e.g. $F$-theory are employed toroidal compactifications, but in our present discussion we will focus only on compactifications in Calabi-Yau 3-folds, which from now on we will call as $\mathcal{Y}$.

## 3.2 Type IIB Compactifications

### 3.2.1 Scalar Field in $\mathbb{R}^{1,3} \times S^1$

Before to tackle the problem of the compactification for the mass-less sector of the type IIB String Theory, let us treat the simplest example that we have in which we can see all the key features of the compactification procedure.
The example consists in a scalar field in 5D, where the extra dimension is represented by a circle $S^1$ of radius $R$., i.e. $\mathcal{M} = \mathbb{R}^{1,3} \times S^1$. Let us start by writing its action:

$$S_{5D} = \int_{\mathcal{M}} d^5x \partial^i \phi(x) \partial_i \phi(x) \qquad \text{where: } i = 0,1,2,3,4 \tag{3.42}$$

Without loss in generality let us impose the condition that $x^4$ correspond to a compact circle of radius $R$:

$$x^4 = y \qquad \text{where: } y = y + 2\pi R \tag{3.43}$$

The periodicity in the $y$ direction allow us to expand in Fourier series the field as follows:

$$\phi(x) = \phi(x^\mu, y) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu) e^{\frac{iny}{R}} \qquad \text{where: } \mu = 0, 1, 2, 3 \tag{3.44}$$

Let us reason on the equations of motions, obtained by imposing the vanishing of the action's variation:

$$\partial^i \partial_i \phi(x) = 0 \quad \Rightarrow \quad \sum_{n=-\infty}^{\infty} \left( \partial^\mu \partial_\mu - \frac{n^2}{R^2} \right) \phi_n(x^\mu) e^{\frac{iny}{R}} = 0 \quad \Rightarrow \quad \partial^\mu \partial_\mu \phi_n(x^\mu) - \frac{n^2}{R^2} \phi_n(x^\mu) = 0 \tag{3.45}$$

In other words we have obtained an infinite series of massive scalar fields, in which the mass is given by $m_n^2 = \left(\frac{n}{R}\right)^2$. This series of massive states goes under the name of **Kaluza–Klein tower** and we can notice that the only non-massive state correspond to $n = 0$.
Let us now substitute the Fourier expansion also in the 5D action:

$$\begin{aligned} S_{5D} &= \int d^4x \int dy \sum_{n=-\infty}^{\infty} \left( \partial^\mu \phi_n(x^\mu) \partial_\mu \phi_n^*(x^\mu) - \frac{n^2}{R^2} |\phi_n(x^\mu)|^2 \right) = \\ &= 2\pi R \int d^4x \left[ \partial^\mu \phi_0(x^\mu) \partial_\mu \phi_0^*(x^\mu) + ... \right] = S_{4D} + ... \end{aligned} \tag{3.46}$$

Thus we see also from the action that in 4D we and up with the action for a massless scalar field and a tower of massive states. Hence if we consider states with an energy smaller then $1/R$, i.e. we impose this energy to be the cut-off of out theory, then we can truncate all the massive fields and we have **dimensional reduced** the system.

This will be the leitmotiv of our analysis and in our case, but instead of using circle we will use cycles and harmonic forms belonging to a certain cohomology of the compactification space.

### 3.2.2 Type IIB Action

Let us now recall that the low energy massless spectrum of a type IIB ST, is a 10D $\mathcal{N} = 2$ SUGRA, since we will develop out inflationary model in this theory, let us remind its bosonic content. Since we will use a boson to drive inflation and we have SUSY that allow us to relate bosons and fermions uniquely, we will just focus on the bosonic content of the theory, neglecting the fermionic part.

Let us recall that in string theory, in the **target space**, we need 2 coordinates to parameterize the **world–sheet** (in which the string moves), which are called $\tau$ and $\sigma$. When we add world-sheet fermions to the theory, i.e. we add a fermionic action to the **Polyakov action**, thus formulating a **superstring** theory, we also need to specify their boundary conditions:

$$\psi^M(\tau,\sigma{+}2\pi) = \begin{cases} +\psi^M(\tau,\sigma) & \in \textbf{Ramond sector } (R) \\ -\psi^M(\tau,\sigma) & \in \textbf{Neveu–Schwarz sector } (NS) \end{cases} \qquad \text{where: } M = 0,\ldots,9 \tag{3.47}$$

However, since we have both left-moving and right-moving fermions on the world-sheet, we can have 4 possible combinations, which in type IIB give raise to:

$$\text{spacetime bosons: } \begin{cases} NS-NS \\ R-R \end{cases} \qquad \text{spacetime fermions: } \begin{cases} NS-R \\ R-NS \end{cases} \tag{3.48}$$

After this clarification, let us recall that in the massless bosonic spectrum of type IIB theory we find in the NS-NS sector: the dilaton $\hat{\phi}$, the metric $\hat{g}$ and a two-form $\hat{B}_2$, while in the R-R sector we have: axion $\hat{l}$, a two-form $\hat{C}_2$ and a four-form $\hat{C}_4$. Then, by the use of the form notation, the type IIB low energy effective action in the $D = 10$ Einstein frame is given by:

$$\begin{aligned} S^{(10)}_{IIB} = &- \int \left( \frac{1}{2}\hat{R} * \mathbf{1} + \frac{1}{4} d\hat{\phi} \wedge * d\hat{\phi} + \frac{1}{4} e^{-\hat{\phi}} \hat{H}_3 \wedge * \hat{H}_3 \right) \\ &- \frac{1}{4} \int \left( e^{2\hat{\phi}} d\hat{l} \wedge * d\hat{l} + e^{\hat{\phi}} \hat{F}_3 \wedge * \hat{F}_3 + \frac{1}{2} \hat{F}_5 \wedge * \hat{F}_5 \right) - \frac{1}{4} \int \hat{C}_4 \wedge \hat{H}_3 \wedge \hat{F}_3 \end{aligned} \tag{3.49}$$

In which the field strengths are defined as:

$$\begin{aligned} \hat{H}_3 = d\hat{B}_2 \qquad \hat{F}_3 = d\hat{C}_2 - \hat{l} d\hat{B}_2 \\ \hat{F}_5 = d\hat{C}_4 - \frac{1}{2} d\hat{B}_2 \wedge \hat{C}_2 + \frac{1}{2} \hat{B}_2 \wedge d\hat{C}_2 \end{aligned} \tag{3.50}$$

Where we have put a hat on all the fields to remind that they are in 10D. Let us furthermore remind that the self-duality condition $\hat{F}_5 = *\hat{F}_5$ must be imposed at the level of the equations of motion, in order not to spoilt the dimensional reduction.

The assumption that we do in Calabi-Yau compactifications is that the 10-dimensional background metric is block diagonal or in other words the line element to take the form:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + g_{i\bar{\jmath}} dy^i d\bar{y}^{\bar{\jmath}} \qquad \text{where: } \mu,\nu = 0,\ldots,3 \quad i,\bar{\jmath} = 1,\ldots,3 \tag{3.51}$$

Where $g_{\mu\nu}$ is the Minkowski metric and $g_{i\bar{j}}$ is the metric on the Calabi-Yau manifold $\mathcal{Y}$. Let us remind from the previous section that deformations of this metric which respect the Calabi-Yau condition correspond to scalar fields in 4D. The deformations of the Kähler form $J = ig_{i\bar{j}}dy^i \wedge d\bar{y}^{\bar{j}}$ give rise to $h^{(1,1)}$ real scalar fields $v^A(x)$, hence we can expand the Kähler form, which is usually called $J$ as:

$$J = v^A(x)\omega_A, \quad A = 1, \dots, h^{(1,1)} \tag{3.52}$$

where as before $\omega_A$ are harmonic $(1,1)$-forms on $Y$ which form a basis of the cohomology group $H^{(1,1)}(\mathcal{Y})$. Let us also recall that the complex structure deformations are parameterized by complex scalar fields $z^K(x)$ and are in one-to-one correspondence with harmonic $(1,2)$-forms:

$$\delta g_{ij} = \frac{i}{\|\Omega\|^2} \bar{z}^K \left(\bar{\chi}_K\right)_{i\bar{\imath}\bar{j}} \Omega^{\bar{\imath}}_{\ jj}, \quad K = 1, \dots, h^{(1,2)} \tag{3.53}$$

Using the our metric Ansatz for the gauge potentials appearing in the Lagrangian, we can expanded them in terms of harmonic forms on $\mathcal{Y}$ as:

$$\begin{aligned} &\hat{B}_2 = B_2(x) + b^A(x)\omega_A, \quad \hat{C}_2 = C_2(x) + c^A(x)\omega_A, \quad A = 1, \dots, h^{(1,1)}, \\ &\hat{C}_4 = D_2^A(x) \wedge \omega_A + V^{\hat{K}}(x) \wedge \alpha_{\hat{K}} - U_{\hat{K}}(x) \wedge \beta^{\hat{K}} + \rho_A(x)\tilde{\omega}^A, \quad \hat{K} = 0, \dots, h^{(1,2)}. \end{aligned} \tag{3.54}$$

Where the $\tilde{\omega}^A$ are harmonic $(2,2)$-forms which form a basis of $H^{2,2}(\mathcal{Y})$ dual to the $(1,1)$-forms $\omega_A$. While $\left(\alpha_{\hat{K}}, \beta^{\hat{L}}\right)$ are harmonic three-forms and form a real, symplectic basis on $H^3(\mathcal{Y})$ i.e. they satisfy the following relation:

$$\int_{\mathcal{Y}} \alpha_{\hat{K}} \wedge \beta^{\hat{L}} = \delta^{\hat{L}}_{\hat{K}} \qquad \int_{\mathcal{Y}} \alpha_{\hat{K}} \wedge \alpha_{\hat{L}} = \int_{\mathcal{Y}} \beta^{\hat{K}} \wedge \beta^{\hat{L}} = 0 \tag{3.55}$$

In the following tabular we resume the relevant cohomologies to perform the dimensional reduction, that we can also read from the previous Hodge diamond, that we have written in the previous section:

| cohomology group | dimension | basis |
|---|---|---|
| $H^{1,1}$ | $h^{(1,1)}$ | $\omega_A$ |
| $H^{2,2}$ | $h^{(1,1)}$ | $\tilde{\omega}^A$ |
| $H^3$ | $2h^{(2,1)} + 2$ | $\left(\alpha_{\hat{K}}, \beta^{\hat{L}}\right)$ |
| $H^{2,1}$ | $h^{(2,1)}$ | $\chi_K$ |

The 4D fields appearing in the expansions are: the scalars $b^A(x)$, $c^A(x)$ and $\rho_A(x)$, the one-forms: $V^{\hat{K}}(x)$ and $U_{\hat{K}}(x)$ and the two-forms: $B_2(x), C_2(x)$ and $D_2^A(x)$. The self-duality condition of $\hat{F}_5$ eliminates half of the degrees of freedom in $\hat{C}_4$ and we choose to eliminate $D_2^A$ and $U_{\hat{K}}$ in favor of $\rho_A$ and $V^{\hat{K}}$. There are also the two type IIB scalars $\hat{\phi}, \hat{l}$ which also appear as scalars in $D = 4$ and therefore we drop the hats henceforth and denote them by $\phi, l$.

Altogether the massless $D = 4$ spectrum consists of the gravity multiplet with bosonic components $(g_{\mu\nu}, V^0)$, $h^{(2,1)}$ vector multiplets with bosonic components $\left(V^K, z^K\right)$, $h^{(1,1)}$ hypermultiplets with bosonic components $\left(v^A, b^A, c^A, \rho_A\right)$ and one double-tensor multiplet 44 with bosonic components $(B_2, C_2, \phi, l)$ which can be dualized to an additional (universal) hypermultiplet. We summarize the bosonic 4D massless spectrum in the following table:

| | | |
|---|---|---|
| gravity multiplet | 1 | $(g_{\mu\nu}, V^0)$ |
| vector multiplets | $h^{(2,1)}$ | $\left(V^K, z^K\right)$ |
| hypermultiplets | $h^{(1,1)}$ | $\left(v^A, b^A, c^A, \rho_A\right)$ |
| double-tensor multiplet | 1 | $(B_2, C_2, \phi, l)$ |

If we careful plug all our decomposition into the full 10D Lagrangian and we integrate the compact part expanded in harmonic forms we can show that we will end up with the following 4D Lagrangian:

$$
\begin{aligned}
S_{IIB}^{(4)} = \int & -\frac{1}{2} R * \mathbf{1} + \frac{1}{4} \operatorname{Re} \mathcal{M}_{\hat{K}\hat{L}} F^{\hat{K}} \wedge F^{\hat{L}} + \frac{1}{4} \operatorname{Im} \mathcal{M}_{\hat{K}\hat{L}} F^{\hat{K}} \wedge * F^{\hat{L}} \\
& - G_{KL} dz^K \wedge * d\bar{z}^L - G_{AB} dv^A \wedge * dv^B - \frac{1}{4} d\ln\mathcal{K} \wedge * d\ln\mathcal{K} - \frac{1}{4} d\phi \wedge * d\phi \\
& - \frac{1}{4} e^{2\phi} dl \wedge * dl - e^{-\phi} G_{AB} db^A \wedge * db^B - e^{\phi} G_{AB} \left(dc^A - l db^A\right) \wedge * \left(dc^B - l db^B\right) \\
& - \frac{9 G^{AD}}{4\mathcal{K}^2} \left(d\rho_A - \frac{1}{2}\mathcal{K}_{ABC} \left(c^B db^C - b^B dc^C\right)\right) \wedge * \left(d\rho_D - \frac{1}{2}\mathcal{K}_{DEF} \left(c^E db^F - b^E dc^F\right)\right) \\
& - \frac{\mathcal{K}^2}{144} e^{-\phi} dB_2 \wedge * dB_2 - \frac{\mathcal{K}^2}{144} e^{\phi} \left(dC_2 - l dB_2\right) \wedge * \left(dC_2 - l dB_2\right) \\
& + \frac{1}{2} \left(db^A \wedge C_2 + c^A dB_2\right) \wedge \left(d\rho_A - \mathcal{K}_{ABC} c^B db^C\right) + \frac{1}{4} \mathcal{K}_{ABC} c^A c^B dB_2 \wedge db^C ,
\end{aligned}
\tag{3.56}
$$

This action might seems complicated, but is already a simplified version of the action in which we have rearranged the terms in order to define some metrics and matrices.

First of all let us say that we have defined the field strength $F^{\hat{K}}$ as:

$$
F^{\hat{K}} = dV^{\hat{K}} \tag{3.57}
$$

Then we have introduced the **gauge kinetic matrix** $\mathcal{M}_{\hat{K}\hat{L}}$, which is related to the metric on $H^3(Y)$ as follows:

$$
\begin{aligned}
&\int \alpha_{\hat{K}} \wedge * \alpha_{\hat{L}} = -\left(\operatorname{Im}\mathcal{M} + (\operatorname{Re}\mathcal{M})(\operatorname{Im}\mathcal{M})^{-1}(\operatorname{Re}\mathcal{M})\right)_{\hat{K}\hat{L}} \\
&\int \beta^{\hat{K}} \wedge * \beta^{\hat{L}} = -(\operatorname{Im}\mathcal{M})^{-1\hat{K}\hat{L}} \\
&\int \alpha_{\hat{K}} \wedge * \beta^{\hat{L}} = -\left((\operatorname{Re}\mathcal{M})(\operatorname{Im}\mathcal{M})^{-1}\right)_{\hat{K}}^{\hat{L}}
\end{aligned}
\tag{3.58}
$$

Let us recall that we can expand the holomorphic three-form $\Omega(z)$ in terms of the so called **periods** $X^{\hat{K}}$ and $\mathcal{F}_{\hat{K}}$ :

$$
\begin{cases} X^{\hat{K}} & = \int_{\mathcal{Y}} \Omega \wedge \beta^{\hat{K}} \\ \mathcal{F}_{\hat{K}} & = \int_{\mathcal{Y}} \Omega \wedge \alpha_{\hat{K}} \end{cases} \Rightarrow \Omega(z) = X^{\hat{K}}(z)\alpha_{\hat{K}} - \mathcal{F}_{\hat{K}}(z)\beta^{\hat{K}} \tag{3.59}
$$

Where both $X^{\hat{K}}(z)$ and $\mathcal{F}_{\hat{K}}(z)$ depend holomorphically on the complex structure deformations $z^K$ and it has been shown that $\mathcal{F}_{\hat{K}}$ is the derivative of a holomorphic prepotential $\mathcal{F}$, i.e. $\mathcal{F}_{\hat{K}} = \frac{\partial \mathcal{F}}{\partial X^{\hat{K}}}$.
Hence which we can define the **period matrix** to be:

$$
\mathcal{F}_{\hat{K}\hat{L}} = \frac{\partial \mathcal{F}_{\hat{K}}}{\partial X^{\hat{L}}} = \frac{\partial^2 \mathcal{F}}{\partial X^{\hat{K}} \partial X^{\hat{L}}} \tag{3.60}
$$

Then we are able to define $\mathcal{M}_{\hat{K}\hat{L}}$ in terms of the period matrix as:

$$
\mathcal{M}_{\hat{K}\hat{L}} = \bar{\mathcal{F}}_{\hat{K}\hat{L}} + 2i \frac{(\operatorname{Im}\mathcal{F})_{\hat{K}\hat{M}} X^{\hat{M}} (\operatorname{Im}\mathcal{F})_{\hat{L}\hat{N}} X^{\hat{N}}}{X^{\hat{N}} (\operatorname{Im}\mathcal{F})_{\hat{N}\hat{M}} X^{\hat{M}}} \tag{3.61}
$$

Furthermore we can define a set of special coordinates in which: $X^{\hat{K}} = \left(1, z^K\right)$.

This simplification allow us to write the metric $G_{KL}(z, \bar{z})$ defined in the complex structure moduli space as:

$$
G_{KL} = \frac{\partial^2 K_{cs}}{\partial z^K \partial \bar{z}^L} \quad \text{where:} \quad K_{cs} = -\ln\left[-i \int_{\mathcal{Y}} \Omega \wedge \bar{\Omega}\right] = -\ln\left[i\left(\bar{X}^{\hat{K}} \mathcal{F}_{\hat{K}} - X^{\hat{K}} \bar{\mathcal{F}}_{\hat{K}}\right)\right] \tag{3.62}
$$

Then we are left over with the other metric $G_{AB}$, and the $\mathcal{K}$s, and we might notice that they both belong to the Kähler moduli space and are in fact related. Let us define the $\mathcal{K}$'s to be:

$$
\begin{cases}
\mathcal{K}_{ABC} & = \int_{\mathcal{Y}} \omega_A \wedge \omega_B \wedge \omega_C \\
\mathcal{K}_{AB} & = \int_{\mathcal{Y}} \omega_A \wedge \omega_B \wedge J = \mathcal{K}_{ABC} v^C \\
\mathcal{K}_{A} & = \int_{\mathcal{Y}} \omega_A \wedge J \wedge J = \mathcal{K}_{ABC} v^B v^C \\
\mathcal{K} & = \int_{\mathcal{Y}} J \wedge J \wedge J = \mathcal{K}_{ABC} v^A v^B v^C
\end{cases} \tag{3.63}
$$

And therefore we can write the Kähler metric $G_{AB}$ to be:

$$G_{AB} = \frac{3}{2\mathcal{K}} \int_{\mathcal{Y}} \omega_A \wedge *\omega_B = -\frac{3}{2} \left( \frac{\mathcal{K}_{AB}}{\mathcal{K}} - \frac{3}{2} \frac{\mathcal{K}_A \mathcal{K}_B}{\mathcal{K}^2} \right) \tag{3.64}$$

Then according to our conventions in the $\mathcal{K}$s definition, we will express the volume of the Calabi-Yau as:

$$\mathcal{V}(\mathcal{Y}) = \frac{\mathcal{K}}{6} = \frac{1}{6} \int_{\mathcal{Y}} J \wedge J \wedge J = \frac{1}{6} \mathcal{K}_{ABC} v^A v^B v^C \tag{3.65}$$

For completeness we mention that is possible to work out a more beautiful action by dualizing the 2-forms $B_2$ and $C_2$ to scalar fields, so that the tensor multiplet becomes an hypermultiplet, and we can express the the action in a more compact way by the use of a metric defined on a quaternionic manifold, but since we do not need it for our discussion, we will not comment forward.

## 3.3 Type IIB Orientifold Action

### 3.3.1 Orientifold Projection in type IIB Theories

Before to work out the the type IIB orientifold action let us define what we meant by the **orientifold projection**.
Let us remind that until now we have considered oriented strings, i.e. we are parameterizing in the world-sheet a string which goes from one point to an other. Let us call the world-sheet coordinates as $\{\tau, \sigma\}$, where $\tau$ represent the time coordinate, while $\sigma$ the space coordinate [Zwi04]. In order to in vert the string orientation we might think to a transformation which keeps $\tau$ invariant and change the sign of $\sigma$, and hence we define the so called **world-sheet parity** transformation to be $\Omega_p$:

$$\Omega_p = \begin{cases} \tau & \longrightarrow \tau \\ \sigma & \longrightarrow -\sigma \end{cases} \tag{3.66}$$

Furthermore on a Calabi-Yau threefold $\mathcal{Y}$ one can define an isometric holomorphic involution $\sigma$, such that it leaves unchanged the 4D Minkowskian manifold as the Kähler form $J$ unchanged, but might act non-trivially on the holomorphic 3-form $\Omega$. More in the detail when we want to consider the action of $\sigma$ on $\Omega$, we have to consider its pull-back, that acts on forms which we will denote as $\sigma^*$. In this context the pull-back of $\sigma$ acting on a generic $k$-form $\alpha_{M_1 \ldots M_K}(x)$ as follows:

$$\sigma^* \alpha_{M_1 \ldots M_k}(x) = \partial_{M_1} \sigma^{N_1} \ldots \partial_{M_k} \sigma^{N_k} \alpha_{N_1 \ldots N_k}(x) \tag{3.67}$$

Requiring that $\sigma$ is an involution[2] means that the square of its action correspond to the identity, while requiring it to be an isometry means that it keeps the metric and the complex structure invariant (thus also the Kähler form will be invariant to), but the action of $\sigma^*$ on the holomorphic 3-form is not completely fixed [Tom22]:

$$(\sigma^*)^2\Omega = \mathbb{1}\Omega \quad \Rightarrow \quad \begin{cases} \sigma^*\Omega = +\Omega \\ \sigma^*\Omega = -\Omega \end{cases} \tag{3.68}$$

At this point we might think to combine both the transformations, $\Omega_p$ and $\sigma$, to define in a consistent way the orientifold projection as an holomorphic isometric involution. By the previous argument we can have 2 possibilities:

$$\begin{cases} \mathcal{O}_1 = (-1)^{F_L}\Omega_p\sigma^* & \text{if } \sigma^*\Omega = -\Omega \\ \\ \mathcal{O}_2 = \Omega_p\sigma^* & \text{if } \sigma^*\Omega = +\Omega \end{cases} \tag{3.69}$$

The reason why we have putted the factor $(-1)^{F_L}$ in front of the first projection is because of consistency arguments. $F_L$ stands for the number of left moving fermions on the world-sheet, hence each time that we encounter an odd number of world-sheet fermions it will give us a minus sign which is needed to make $\mathcal{O}_1$ an involution.

At this point we might think to use these 2 isometries to quotient our compactification space in order to reduce the amount of SUSY from $\mathcal{N} = 2$ to $\mathcal{N} = 1$, without changing the compactification's space geometry, and this is what we are going to do.
Furthermore we can say that there will be some fixed points in such construction, which more in the detail correspond to planes called **O-planes**. Since $\sigma$ act holomorphically on the coordinate we might argue that the possible dimension of the O-planes will only be even. Since the only time direction is unaffected by such transformations, then we indicate, including the time direction, the O$n$-planes in which $n$ tell us only about the spatial direction which is preserved. Since the Minkowski 4D space is left invariant, then the O-planes will be always spacetime-filling. Hence the smallest O-plane is an O3-plane, and since we can have only even dimensional O-planes, the only possible O-planes are: O3-, O5-, O7- ,O9-planes.

Let us see how we can obtain this type of reflection planes, by studying the action of $\sigma^*$ on the holomorphic 3-form $\Omega$.
Let us define $\{y_i\}_{i=1,2,3}$ to be the coordinates of $\mathcal{Y}$, without loss in generality we might think $\Omega \propto dy^1 \wedge dy^2 \wedge dy^3$. Let us act with $\sigma^*$ on $dy^1 \wedge dy^2 \wedge dy^3$, then we might obtain:

[2] In general we define an involution as a function $f(x)$, such that $f(x)^2 = \mathbb{1}$, hence it does not only consists in a change of sign, e.g. also $f(x) = \frac{1}{x}$ is an involution, but it has to be an isometry it will just correspond at most to a change in sign

$$\begin{cases} -(dy^1 \wedge dy^2 \wedge dy^3) & \text{if } \sigma^*(dy^1) = -dy^1 & \text{or } \sigma^*(dy^i) = -dy^i \ \forall i = 1,2,3 \\ +(dy^1 \wedge dy^2 \wedge dy^3) & \text{if } \sigma^*(dy^i) = -dy^i \ \forall i = 1,2 & \text{or } \sigma^*(dy^i) = +dy^i \ \forall i = 1,2,3 \end{cases} \tag{3.70}$$

Then quotienting with respect to $\mathcal{O}_1$ we obtain, as suggested by the previous formula, respectively O7-, O3-planes, while for $\mathcal{O}_2$ we will have respectively O5-, O9-planes.

In the present work we will only consider the projection $\mathcal{O}_1$ which we will simply denote as $\mathcal{O}$, from now on. After having defined $\mathcal{O}$, we can see what changes it brings to the cohomologies classes, modding out all the forms which are not invariant under its action.

Let us start by analyzing the behaviour of the type IIB fields under the action of: $(-1)^{F_L}\Omega_p$, then:

$$\begin{array}{rclrcl} (-1)^{F_L}\Omega_p\hat{\phi} & = & \hat{\phi} & (-1)^{F_L}\Omega_p\hat{l} & = & \hat{l} \\ (-1)^{F_L}\Omega_p\hat{g} & = & \hat{g} & (-1)^{F_L}\Omega_p\hat{C}_2 & = & -\hat{C}_2 \\ (-1)^{F_L}\Omega_p\hat{B}_2 & = & -\hat{B}_2 & (-1)^{F_L}\Omega_p\hat{C}_4 & = & \hat{C}_4 \end{array} \tag{3.71}$$

Then, since we want to keep only the fields which are invariant under $\mathcal{O}$, we will have to keep only the field which transform under $\sigma^*$ in the following way:

$$\begin{array}{rclrcl} \sigma^*\hat{\phi} & = & \hat{\phi} & \sigma^*\hat{l} & = & \hat{l} \\ \sigma^*\hat{g} & = & \hat{g} & \sigma^*\hat{C}_2 & = & -\hat{C}_2 \\ \sigma^*\hat{B}_2 & = & -\hat{B}_2 & \sigma^*\hat{C}_4 & = & \hat{C}_4 \end{array} \tag{3.72}$$

Furthermore, since the holomorphic involution $\sigma^*$ is such that $\sigma^*\Omega = -\Omega$, then the cohomology groups $H^{(p,q)}$, and thus the harmonic $(p,q)$-forms, will split into two eigenspaces under the action of $\sigma^*$, which we denote as follow:

$$H^{p,q} = H^{p,q}_+ \oplus H^{p,q}_- \quad \Rightarrow \quad h^{(p,q)} = h^{(p,q)}_+ + h^{(p,q)}_- \tag{3.73}$$

Where the $+$ denotes the even eigenspace of $\sigma^*$, while $-$ the odd one.
Since $\sigma$ preserves the orientation and the metric of $\mathcal{Y}$, then the $*$-operator commutes with $\sigma^*$, and thus we can argue that $h^{(1,1)}_\pm = h^{(2,2)}_\pm$. The holomorphicity of $\sigma$ implies that $h^{(2,1)}_\pm = h^{(1,2)}_\pm$. The fact that $\sigma^*\Omega = -\Omega$ leads to $h^{(3,0)}_+ = h^{(0,3)}_+ = 0$ and $h^{(3,0)}_- = h^{(0,3)}_- = 1$, which in a certain sense we might see as the survival of just one SUSY.
Since the volume-form which is proportional to $\Omega \wedge \bar{\Omega}$, then it is invariant under $\sigma^*$, hence: $h^{(0,0)}_+ = h^{(3,3)}_+ = 1$, while $h^{(0,0)}_- = h^{(3,3)}_- = 0$.

In order to be clear, let us express all the relevant cohomologies in the following table:

| cohomology group | | dimension | | basis | |
|---|---|---|---|---|---|
| $H^{1,1}_+$ | $H^{1,1}_-$ | $h^{(1,1)}_+$ | $h^{(1,1)}_-$ | $\omega_\alpha$ | $\omega_a$ |
| $H^{2,2}_+$ | $H^{2,2}_-$ | $h^{(1,1)}_+$ | $h^{(1,1)}_-$ | $\tilde{\omega}^\alpha$ | $\tilde{\omega}^a$ |
| $H^{2,1}_+$ | $H^{2,1}_-$ | $h^{(2,1)}_+$ | $h^{(2,1)}_-$ | $\chi_\kappa$ | $\chi_k$ |
| $H^3_+$ | $H^3_-$ | $2h^{(2,1)}_+$ | $2h^{(2,1)}_- + 2$ | $\alpha_\kappa, \beta^\lambda$ | $\alpha_{\hat{k}}, \beta^{\hat{l}}$ |

Now we can find the 4D invariant spectrum by using the Kaluza-Klein expansion according to the modification of the cohomologies. We see immediately that both the 4D scalar fields arising from $\hat{\phi}$ and $\hat{l}$ remain in the spectrum and as before we denote them by $\phi$ and $l$.
Since $\sigma^*$ leaves the Kähler form $J$ invariant, then, only the $h^{(1,1)}_+$ even Kähler deformations $v^\alpha$ remain in the spectrum, hence we can expand:

$$J = v^\alpha(x)\omega_\alpha \quad \text{where: } \alpha = 1, \dots, h^{(1,1)}_+ \tag{3.74}$$

On the other hand the transformation rule for $\Omega$ implies that the complex structure deformations kept in the spectrum correspond to elements belonging to $H^{1,2}_-$, thus we might write:

$$\delta g_{ij} = \frac{i}{\|\Omega\|^2} \bar{z}^k \, (\bar{\chi}_k)_{i\bar{\imath}\bar{\jmath}} \, \Omega^{\bar{\imath}\bar{\jmath}}{}_j \quad \text{where: } k = 1, \dots, h^{(1,2)}_- \tag{3.75}$$

From what does it concern the other fields, we can say, according to our previous discussion, that in the expansion of $\hat{B}_2$ and $\hat{C}_2$ only odd elements survive, while for $\hat{C}_4$ only even elements are kept, then:

$$\begin{aligned} &\hat{B}_2 = b^a(x)\omega_a \qquad \hat{C}_2 = c^a(x)\omega_a \qquad \text{where: } a = 1, \dots, h^{(1,1)}_- \\ &\hat{C}_4 = D^\alpha_2(x) \wedge \omega_\alpha + V^\kappa(x) \wedge \alpha_\kappa + U_\kappa(x) \wedge \beta^\kappa + \rho_\alpha(x)\tilde{\omega}^\alpha \quad \text{where: } \kappa = 1, \dots, h^{(1,2)}_+ \end{aligned} \tag{3.76}$$

As before by imposing the self-duality at the level of the equation of motion on $\hat{F}_5$, we eliminate half of the degrees of freedom in the expansion of $\hat{C}_4$. For the one-forms $V^\kappa, U_\kappa$ this corresponds to the choice of electric instead of magnetic gauge potentials. An other freedom is in the choice of the two forms $D^\alpha_2$ or the scalars $\rho_\alpha$, which determines the structure of the $N = 1$ multiplets to be respectively either a linear or a chiral multiplet, but we will treat the chiral multiplet case.

According to the explained choices, the resulting $N = 1$ spectrum assembles into a gravitational multiplet, $h^{(2,1)}_+$ vector multiplets, $h^{(2,1)}_- + h^{(1,1)} + 1$ chiral multiplets, and $h^{(1,1)}_+$ chiral or linear multiplets (but we will chose the chiral multiplet representation).
For the sake of clarity we present the spectrum of our $\mathcal{N} = 1$ type IIB SUGRA in the following table:

| | | |
|---|---|---|
| gravity multiplet | 1 | $g_{\mu\nu}$ |
| vector multiplets | $h_+^{(2,1)}$ | $V^\kappa$ |
| chiral multiplets | $h_-^{(2,1)}$ | $z^k$ |
| | 1 | $(\phi, l)$ |
| | $h_-^{(1,1)}$ | $(b^a, c^a)$ |
| chiral/linear multiplets | $h_+^{(1,1)}$ | $(v^\alpha, \rho_\alpha)$ |

We can now compare the previous $\mathcal{N} = 2$ spectrum of the Calabi-Yau compactification: we see that the graviphoton is projected out of the gravitational multiplet, the $h^{(2,1)}$ $\mathcal{N} = 2$ vector multiplets are now decomposed into $h_+^{(2,1)}$ $\mathcal{N} = 1$ vector multiplets plus $h_-^{(2,1)}$ chiral multiplets. Furthermore, the $h^{(1,1)} + 1$ hypermultiplets lost half of their physical degrees of freedom since they are reduced to $h^{(1,1)} + 1$ chiral multiplets. We might also notice that the two 4D two-forms $B_2$ and $C_2$ present in the $\mathcal{N} = 2$ compactification have been projected out leaving in the expansion of $\hat{B}_2$ and $\hat{C}_2$ only the scalar fields $c^a, b^a$.
We can see the non-vanishing of $c^a, b^a$ and $V^\kappa$ as related to the presence of O3- and O7-planes. More in the detail let us recall that the presence of O3-planes implies that the fixed points locus of $\mathcal{Y}$ is zero dimensional, i.e all the tangent vectors to $\mathcal{Y}$ are odd under the action of $\sigma$, while the 2-form are even under the action of $\sigma^*$. The appearance of O7-planes, i.e. having a 2 complex dimensional fixed locus under the orientifold projection gives the support to the harmonic forms belonging to $H_-^{1,1}$ and $H_+^{2,1}$, and hence ensures the non vanishing of $c^a, b^a$ and $V^\kappa$.

### 3.3.2 Type IIB Orientifold Action with O3/O7-planes

After having discussed the new cohomologies groups which survive to the $\mathcal{O}$ projection, let us do once again the Kaluza-Klein procedure, hence let us expand the fields in 4D fields and harmonic forms, which belong to the orientifolded $\mathcal{Y}$ [GL04]:

$$\begin{aligned}
&\hat{H}_3 = db^a \wedge \omega_a + H_3 \qquad \hat{F}_3 = dc^a \wedge \omega_a - l db^a \wedge \omega_a + F_3 - l H_3 \\
&\hat{F}_5 = dD_2^\alpha \wedge \omega_\alpha + dV^\kappa \wedge \alpha_\kappa - dU_\kappa \wedge \beta^\kappa + d\rho_\alpha \tilde{\omega}^\alpha - \frac{1}{2}(c^a db^b - b^a dc^b) \wedge \omega_a \wedge \omega_b
\end{aligned} \tag{3.77}$$

where we notice that we allowed for the presence of background fluxes $H_3$ and $F_3$, which we see as background fluxes of the compactification space which are terms that might be present since they are not ruled out by the orientifold projection. In principle we could expect that these 3-form fluxes influence also the 5-form field-strength, however the only possibility to obtain such contribution is to couple to them with the 2-forms $B_2$ or $C_2$, which however are projected out. Hence such 3-form fluxes do not affect the 5-form, and furthermore the self duality constraint prevent us to insert a background flux for the 5-form.

At this point we just have to insert the new expansion of the field-strengths in the action and integrate over $\mathcal{Y}$. In order to do so we first need to reconsider the complex structure and Kähler metrics, and the intersection numbers which have been modified by the orientifold projection.

Let us start with the complex structure deformations. Let us recall that:

$$H^{(3)} = H_+^{(3)} \oplus H_-^{(3)} \quad \Rightarrow \quad \left\{\alpha_{\hat{K}}, \beta^{\hat{L}}\right\} = \left\{\alpha_\kappa, \alpha_{\hat{k}}, \beta^\lambda, \beta^{\hat{l}}\right\} \tag{3.78}$$

Where, as before we have used symplectic basis, such that the only non vanishing pairing are:

$$\int \alpha_\kappa \wedge \beta^\lambda = \delta_\kappa^\lambda \qquad \int \alpha_{\hat{k}} \wedge \beta^{\hat{l}} = \delta_{\hat{k}}^{\hat{l}} \tag{3.79}$$

Furthermore, since from the previous $h^{(2,1)}$ complex structure deformation $z^K$ only $h_-^{(2,1)}$ (denoted by $z^k$ ) survive, then the three-form $\Omega$ will be an element of $H_-^{(3)}$, and thus might be expanded according to:

$$\Omega\left(z^k\right) = X^{\hat{k}}\alpha_{\hat{k}} - \mathcal{F}_{\hat{k}}\beta^{\hat{k}} \qquad \text{where: } \hat{k} = 0\dots, h_-^{(1,2)} \tag{3.80}$$

Where the only non vanishing periods are: $X^{\hat{k}}$ and $\mathcal{F}_{\hat{k}}$, which we remind to be:

$$X^{\hat{k}} = \int_{\mathcal{Y}} \Omega \wedge \beta^{\hat{k}} \qquad \mathcal{F}_{\hat{k}} = \int_{\mathcal{Y}} \Omega \wedge \alpha_{\hat{k}} \tag{3.81}$$

Hence the metric on the space of complex structure deformations reduces to:

$$G_{kl} = \frac{\partial^2 K_{cs}}{\partial z^k \partial \bar{z}^l} \quad \text{where: } K_{cs} = -\ln\left[-i\int_Y \Omega \wedge \bar{\Omega}\right] = -\ln\left[i\left(\bar{X}^{\hat{k}}\mathcal{F}_{\hat{k}} - X^{\hat{k}}\bar{\mathcal{F}}_{\hat{k}}\right)\right] \tag{3.82}$$

Let us now analyze the Kähler deformations. let us remind that:

$$H^{1,1} = H_+^{1,1} \oplus H_-^{1,1} \quad \Rightarrow \quad \{\omega_A\} = \{\omega_\alpha, \omega_a\} \tag{3.83}$$

Thus also the decomposition of the intersection numbers $\mathcal{K}_{ABC}$ will be different. In fact under the orientifold projection only $\mathcal{K}_{\alpha\beta\gamma}$ and $\mathcal{K}_{\alpha bc}$ can be non-zero, while $\mathcal{K}_{\alpha\beta c}$ and $\mathcal{K}_{abc}$ vanish. Furthermore, since the Kähler-form $J$ is invariant under $\mathcal{O}$, then $\mathcal{K}_{\alpha b} = \mathcal{K}_a = 0$. To sum up, the vanishing intersection numbers are:

$$\mathcal{K}_{\alpha\beta c} = \mathcal{K}_{abc} = \mathcal{K}_{\alpha b} = \mathcal{K}_a = 0 \tag{3.84}$$

While the non-vanishing intersection numbers are:

$$\mathcal{K}_{\alpha\beta} = \mathcal{K}_{\alpha\beta\gamma}v^\gamma \qquad \mathcal{K}_{ab} = \mathcal{K}_{ab\gamma}v^\gamma \qquad \mathcal{K}_\alpha = \mathcal{K}_{\alpha\beta\gamma}v^\beta v^\gamma \qquad \mathcal{K} = \mathcal{K}_{\alpha\beta\gamma}v^\alpha v^\beta v^\gamma \tag{3.85}$$

Which allow us to write the Kähler metric as follow:

$$G_{\alpha\beta} = -\frac{3}{2}\left(\frac{\mathcal{K}_{\alpha\beta}}{\mathcal{K}} - \frac{3}{2}\frac{\mathcal{K}_\alpha\mathcal{K}_\beta}{\mathcal{K}^2}\right) \qquad G_{ab} = -\frac{3}{2}\frac{\mathcal{K}_{ab}}{\mathcal{K}} \qquad G_{\alpha b} = G_{a\beta} = 0 \tag{3.86}$$

At this point we are ready to calculate the 4D action by plugging in all the derived expansions into the original 10D type IIB action. Furthermore in order to impose the self-duality condition $\hat{F}_5 = *\hat{F}_5$ we can add the following total derivative to the action:

$$\delta S^{(4)}_{O3/O7} = \frac{1}{4}dV^\kappa \wedge dU_\kappa + \frac{1}{4}dD_2^\alpha \wedge d\rho_\alpha \tag{3.87}$$

In other words we are saying that the equation of motions for $D_2^\alpha$ and $U_\kappa$ (or equivalently for $\rho_\alpha, V^\kappa$ ) coincide with the self-duality condition, therefore we can consistently eliminate $D_2^\alpha$ and $U_\kappa$ (or $\rho_\alpha, V^\kappa$) by inserting their equations of motions into the action. As we said before keeping $V^\kappa$ corresponds to the choice of expressing the action in terms of an electric instead of a magnetic gauge potential $U_\kappa$. Choosing to eliminate $D_2^\alpha$ or $\rho_\alpha$ corresponds to the choice of expressing the action in terms of linear or chiral multiplets. Since the standard $\mathcal{N} = 1$ supergravity formulation uses the chiral multiplets it is more convenient to eliminate $D_2^\alpha$ in favor of $\rho_\alpha$ and express everything in terms of chiral multiplets.

Then, after having eliminated $D_2^\alpha$ and $U_\kappa$ by its equations of motion and having performed a Weyl rescaling of the four-dimensional metric $g_{\mu\nu} \to \frac{\mathcal{K}}{6}g_{\mu\nu}$ to obtain the canonically normalized Einstein-Hilbert term, we arrive to write the following 4D action:

$$\begin{aligned}
S^{(4)}_{O3/O7} = \int & -\frac{1}{2}R * \mathbf{1} - G_{kl}dz^k \wedge * d\bar{z}^l - G_{\alpha\beta}dv^\alpha \wedge * dv^\beta - \frac{1}{4}d\ln\mathcal{K} \wedge * d\ln\mathcal{K} \\
& - \frac{1}{4}d\phi \wedge * d\phi - \frac{1}{4}e^{2\phi}dl \wedge * dl - e^{-\phi}G_{ab}db^a \wedge * db^b \\
& - e^\phi G_{ab}\left(dc^a - l db^a\right) \wedge * \left(dc^b - l db^b\right) \\
& - \frac{9G^{\alpha\beta}}{4\mathcal{K}^2}\left[d\rho_\alpha - \frac{1}{2}\mathcal{K}_{\alpha ab}\left(c^a db^b - b^a dc^b\right)\right] \wedge * \left[d\rho_\beta - \frac{1}{2}\mathcal{K}_{\beta cd}\left(c^c db^d - b^c dc^d\right)\right] \\
& + \frac{1}{4}\operatorname{Im}\mathcal{M}_{\kappa\lambda}F^\kappa \wedge * F^\lambda + \frac{1}{4}\operatorname{Re}\mathcal{M}_{\kappa\lambda}F^\kappa \wedge F^\lambda - V * \mathbf{1},
\end{aligned} \tag{3.88}$$

Where, as before: $F^\kappa = dV^\kappa$ and $\mathcal{M}_{\kappa\lambda}$ is the $\mathcal{N} = 2$ gauge kinetic matrix already defined, evaluated at $z^\kappa = \bar{z}^\kappa = 0$.

In the last term we have defined the potential $V$, which is manifestly positive semi-definite and in a more explicit form is given by:

$$V = \frac{18ie^{\phi}}{\mathcal{K}^2 \int \Omega \wedge \bar{\Omega}} \left( \int \Omega \wedge \bar{G}_3 \int \bar{\Omega} \wedge G_3 + G^{kl} \int \chi_k \wedge G_3 \int \bar{\chi}_l \wedge \bar{G}_3 \right) \tag{3.89}$$

In which we see the 3-form $G_3$, which we define to be:

$$G_3 = F_3 - SH_3 \qquad \text{where: } S = e^{-\phi} + il \tag{3.90}$$

Now we have to understand by which change of coordinates we can bring the present 4D action into the standard $\mathcal{N} = 1$ SUGRA form, where it is expressed in terms of a Kähler potential $K$, a holomorphic superpotential $W$ and the holomorphic gauge-kinetic coupling functions $f$ as follows:

$$S^{(4)} = -\int \frac{1}{2} R{*}\mathbf{1} + K_{I\bar{J}} DM^I \wedge {*}D\bar{M}^{\bar{J}} + \frac{1}{2} \operatorname{Re} f_{\kappa\lambda} F^{\kappa} \wedge {*}F^{\lambda} + \frac{1}{2} \operatorname{Im} f_{\kappa\lambda} F^{\kappa} \wedge F^{\lambda} + V{*}\mathbf{1} \tag{3.91}$$

Where we define the coordinates $\{M^I\}$ to denote all complex scalars in the theory and $K_{I\bar{J}}$ as the Kähler metric such that: $K_{I\bar{J}} = \partial_I \bar{\partial}_{\bar{J}} K(M, \bar{M})$. Furthermore the scalar potential $V$ is expressed in terms of the Kähler-covariant derivative $D_I W = \partial_I W + (\partial_I K) W$ as:

$$V = e^K \left( K^{I\bar{J}} D_I W D_{\bar{J}} \bar{W} - 3|W|^2 \right) + \frac{1}{2} \left[ (\operatorname{Re} f)^{-1} \right]^{\kappa\lambda} D_\kappa D_\lambda \tag{3.92}$$

However we will never consider the gauge kinetic functions in our discussion, since we will never treat gauge interactions, hence we can immediately set them to zero obtaining:

$$V = e^K \left( K^{I\bar{J}} D_I W D_{\bar{J}} \bar{W} - 3|W|^2 \right) \tag{3.93}$$

In order to define a metric which is manifestly Kähler, we first need to find a complex structure on the space of scalar fields. As we saw by the definition of the complex structure deformations' metric, the $z^k$ are already good Kähler coordinates. For the remaining fields the definition of the Kähler coordinates is more involved, and is suggested by a the action expressed not by the use of the linear multiplets instead of the chiral ones, but we will not discuss it. We can verify that a good definition for the remaining Kähler coordinates is the following:

$$\begin{cases} S &= e^{-\phi} + il \\ G^a &= \bar{S} b^a + \mathrm{i} c^a \\ T_\alpha &= i\rho_\alpha + \frac{1}{2}\mathcal{K}_\alpha(v) - \frac{1}{2}\zeta_\alpha(S, \bar{S}, G, \bar{G}) \\ &= i\rho_\alpha + \frac{1}{2}\mathcal{K}_{\alpha\beta\gamma} v^\beta v^\gamma + \frac{1}{2(S+\bar{S})}\mathcal{K}_{\alpha bc} G^b (G + \bar{G})^c \end{cases} \tag{3.94}$$

Then we can explicitly write the Kähler potential as the sum of the sum of the complex structure moduli's Kähler potential $K_{cs}$ and the Kähler moduli's one as follow:

$$K = K_{cs}(z, \bar{z}) + K_k(S, T, G) \tag{3.95}$$

Where:

$$\begin{cases} K_{cs} &= -\ln\left[-i\int \Omega(z)\wedge\bar{\Omega}(\bar{z})\right] \\ K_k &= -\ln\bigl(S+\bar{S}\bigr) - 2\ln\left[\frac{1}{6}\mathcal{K}(S,T,G)\right] = -\ln\bigl(S+\bar{S}\bigr) - 2\ln\mathcal{V} \end{cases} \tag{3.96}$$

Furthermore, we may show that the superpotential $W$, from which we obtain $V$ might be written in the following form:

$$W(S, z^k) = \int_{\mathcal{Y}} \Omega \wedge G_3 \tag{3.97}$$

Hence we can explicitly calculate the covariant derivatives of such potential with respect to all the Kähler coordinates, having in mind that $D_{z^k}\Omega = i\chi_k$:

$$\begin{cases} D_S W &= \frac{ie^{\phi}}{2}\int_{\mathcal{Y}} \Omega\wedge\bar{G}_3 + iG_{ab}b^a b^b W \\ D_{z^k}W &= i\int_{\mathcal{Y}} \chi_k \wedge G_3 \\ D_{G^a}W &= K_{G^a}W = 2iG_{ab}b^b W \\ D_{T_\alpha}W &= K_{T_\alpha}W = -\frac{2v^\alpha}{\mathcal{K}}W \end{cases} \tag{3.98}$$

From which we can find the previous potential $V$. Let us furthermore stress that $W$ depend only on the axio-dilaton and the complex structure moduli, and on the background fluxes, hence having fixed those quantities allow us to have a Kähler potential which is only function of the Kähler moduli.

## 3.4 Fluxes and Warped compactifications in type IIB SUGRA

### 3.4.1 Flux Quantization

In the previous section we have mentioned the possibility to have the 3-form fluxes $H_3$ and $F_3$, which can be combined into the imaginary self dual $G_3$, such that $dH_3 = dF_3 = 0$. However, as we can imagine, such fields cannot take continuous real values, but they are quantized [IU12].

Let us now show how a q-form flux $F_q$ might be quantized by the **Dirac quantization procedure**, which refers to the way in which Dirac quantized the magnetic charge

as consequence of the quantization of the electric charge, but which in a wider sense we can think as a Path Integral quantization.
The considered flux $F_q$, will have support on an integer cohomology q-cycle an is such that $dF_q = 0$. Let us consider a d-dimensional compact manifold $\mathcal{M}$ in which exist $\Sigma_q$ as support of $F_q$. Let us consider a trivial $(q-1)$-cycle $\Pi_{q-1} \subset \Sigma_q$. In general such cycle split $\Sigma_q$ in 2 parts $\Sigma_+$ and $\Sigma_-$, such that $\Sigma_+ - \Sigma_- = \Sigma_q$ (we have putted a minus sign since we are considering the surface's orientation). Hence, that means that both the boundaries of the 2 regions will be such that $\partial\Sigma_+ = \partial\Sigma_- = \Pi_{q-1}$.
Let us consider the $(q-1)$-form field $C_{q-1}$ associated to the $F_q$ field strength, i.e. such that: $F_q = dC_{q-1}$. Let us furthermore suppose that exist an object that is charged under such gauge field, which wraps $\Pi_{q-1}$ and has a charge of $Q_e$. In string theory these states are the **branes** and in our example we have to consider a $(q-2)$-**Euclidean D-brane** (an ED$(q-2)$-brane), since we want a $(q-1)$-dimensional object. In fact in the branes' description we indicate only the space-dimension $(q-2)$, and implicitly one dimension refers to the time, but for Euclidean-branes the time dimension is converted into a space dimension (through Wick's rotation).

Let us now take into the account path integral quantization: we can describe amplitudes in 2 ways, by the use of the gauge potential $C_{q-1}$ or the field strength by the Gauss theorem in the following way:

$$\Gamma \propto \exp\left\{ iQ_e \int_{\Pi_{q-1}} C_{q-1} \right\} \qquad \text{or} \qquad \Gamma' \propto \exp\left\{ iQ_e \int_{\Sigma_\pm} F_q \right\} \tag{3.99}$$

Where we can chose either $\Sigma_+$ or $\Sigma_-$ since we are in a compact space. However this choice differs by a phase, since from the above decomposition of $\Sigma_q$ we have that:

$$Q_e \int_{\Sigma_q} F_q = Q_e \left( \int_{\Sigma_+} F_q - \int_{\Sigma_-} F_q \right) \tag{3.100}$$

Since we want that both the amplitudes describe the same process, i.e. $\Gamma = \Gamma'$, this difference in phase is not arbitrary but has to be a multiple of $2\pi$. As analogy we can think about the double slit experiment: in which the interference pattern is given by light rays that arrive in the same point, coming from one slit or the other, with an integer number of wave-length difference.
Thus in our case we can state that:

$$Q_e \int_{\Sigma_q} F_q = 2\pi\mathbb{Z} \tag{3.101}$$

This is the quantization condition which all the fluxes that we can consider has to satisfy, and it can be shown that that implies the **BPS** quantization of the branes' "electric" and "magnetic" charge.

### 3.4.2 Fluxes and Warping

In order to stabilize the moduli space of $\mathcal{Y}$ and find a viable potential we will consider some hierarchies between the different terms in the potential and this construction can be realized in the context of flux compactifications [GKP02; CQS05]. To be more precise the to in pose such hierarchies we are considering warped metrics in which the warping factor will make some contributions higher or smaller. For warped metrics we mean metrics of following form:

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A(y)}g_{i\bar{j}}dy^i dy^{\bar{j}} \tag{3.102}$$

Calabi-Yau manifolds are generically non-singular, but at special values of the parameters they can develop singularities. The most generic singular space is a **conifold**. The basic idea is that locally in the vicinity of a conifold point, we can find solutions with fluxes that generate smooth supergravity solutions with large relative warpings. Locally a conifold singularity can be described as the sub-manifold of $\mathbb{C}^4$ defined by:

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0 \tag{3.103}$$

It is easy to see that such sub-manifold is singular at $(w_1, w_2, w_3, w_4) = \mathbf{0}$. This singularity can be regarded as a cone, whose base has the topology of $S^3 \times S^2$. At the singular point, both the $S^3$ and the $S^2$ shrink to zero size. This decomposition tell us that the conifold can be smoothed into a non-singular point in two ways: we can blow-up $S^2$ or $S^3$ to finite size. Let us discus the second possibility that will be relevant for us. The deformed conifold might be described by the following sub-manifold:

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = z \qquad \text{where: } z \in \mathbb{R} \tag{3.104}$$

Where $z$ is the modulus which controls the size of the $S^3$.
Dirac quantization implies that these fluxes, integrated over all of the three-cycles of $\mathcal{Y}$, have to be multiple of integers numbers. In the neighbourhood of the conifold, there are two relevant cycles which we will denote as $A$ and $B$, such that they intersect each other only once. Let us call $A$ the cycle which vanishes as $z \to 0$, that can be constructed taking all the $\{w_i\} \in \mathbb{R}$. On the other hand we can construct the $B$ cycle in order to intersect $A$ only once to be simply the cycle in which all the $\{w_i\}$ are purely imaginary except for one, that we take to be $w_4$ without loss in generality.

According to the Klebanov-Strassler construction in the context of type IIB $\mathcal{N} = 1$ supergravity theories (that we know is done for non-compact manifolds, but it can be embedded in compact ones) we can put $M$ units of the flux $F_3$ on the $A$ cycle and $K$ units of the flux $H_3$ on the dual cycle $B$, then:

$$\frac{1}{2\pi\alpha'}\int_A F_3 = 2\pi M \qquad \frac{1}{2\pi\alpha'}\int_B F_3 = -2\pi K \tag{3.105}$$

Where the minus sign arises from the Poincaré duality. Furthermore, taking into the account the equation of motion of $F_5$ in the Einstein frame (i.e. its Bianchi identity) we obtain:

$$dF_5 = H_3 \wedge F_3 + \frac{(\alpha')^2}{(2\pi)^4}\rho_3^{loc} = 0 \quad \Rightarrow \quad \frac{(2\pi)^4}{(\alpha')^2}\int_{\mathcal{Y}} H_3 \wedge F_3 + Q_3^{loc} = 0 \tag{3.106}$$

Which is nothing but the conservation of the D3-brane condition. Then we can write in our case $Q_3^{loc}$ to be entirely given by the fluxes, i.e:

$$\frac{(2\pi)^4}{(\alpha')^2}\int_{\mathcal{Y}} H_3 \wedge F_3 = MK \tag{3.107}$$

Thus by this condition we can use Poincaré duality to write:

$$F_3 = (2\pi)^2\alpha' M[B] \qquad H_3 = (2\pi)^2\alpha' K[A] \tag{3.108}$$

Now we have the element to fix the superpotential of type IIB theory to be given as:

$$W = \int_{\mathcal{Y}} G_3 \wedge \Omega = (2\pi)^2\alpha' \left( M\int_B \Omega - SK\int_A \Omega \right) \quad . \tag{3.109}$$

Here $\int_A \Omega$ and $\int_B \Omega$ are the periods defined above, which in this case describe the conifold complex structure. It has been shown that we can write the $A$ period as $z$, where $z$ is a complex structure modulus (an we just take one for the sake of simplicity), and that implies:

$$z = \int_A \Omega \quad \Rightarrow \quad \int_B \Omega = \mathcal{G}(z) = \frac{z}{2\pi}\ln z + f_{\text{holomorphic}}(z) \tag{3.110}$$

This implies that we are able to write $W$ as follows:

$$W = (2\pi)^2\alpha' \left[M\mathcal{G}(z) - SKz\right] \tag{3.111}$$

We have already discussed that the complex structure moduli and the axio-dilaton are said to be stabilized supersymmetrically, then let us consider theirs covariant derivatives. Without entering in the details of the axio-dilaton stabilization let us set it to its expectation value, i.e. $g_s$, and let us just consider the derivative with respect to the only complex structure modulus $z$:

$$D_z W \propto M\partial_z\mathcal{G}(z) - SK + \partial_z K_{cs}\left[M\mathcal{G}(z) - SKz\right] \tag{3.112}$$

In order to obtain a large hierarchy, we might think to take $K/g_s \gg 1$ to have z exponentially small, then we can write the dominant terms of the derivative to be:

$$D_zW \propto \frac{M}{2\pi}\ln z - \frac{K}{g_s} + O(1) \tag{3.113}$$

Hence, if $K/(Mg_s) \gg 1$, we obtain the desired exponential relation:

$$z \sim \exp\left\{-\frac{2\pi K}{Mg_s}\right\} \tag{3.114}$$

In order to relate this modulus to the warped metric, one in principle should solve the 10-dimensional Einstein equation, but we will not do it.
Instead, we can recall a relation which was derived in order to give an estimation of the value of the warped factor in the neighbourhood of a stack of $N$ coincident D3-branes:

$$e^{-4A(y)} \sim \frac{4\pi g_s N}{r^4} \tag{3.115}$$

Where $r$ is the distance in terms of the $g_{i\bar{\jmath}}$ metric of $\mathcal{Y}$ from the D3-branes. Is easy to guess that in our case $r \propto w_i^{2/3} \propto z^{1/3}$, and then to conclude that:

$$e^a \propto z^{\frac{1}{3}} \sim e^{-\frac{2\pi K}{3Mg_s}} \tag{3.116}$$

An then in this sense we have understood by a simple example that the fluxes can fix the axio-dilaton and the complex structure, which are related to the warping factor, which will turn out to be crucial in our discussion.

# Chapter 4

# Moduli Stabilization: LVS and Axion Monodromy

## 4.1 Kähler Metrics of Type IIB Orientifolds

In order to give a concrete example of the realization of inflation due to the $C_2$ axion, let us consider a Calabi-Yau threefold $\mathcal{Y}$, orientifolded with a particular choice in the cohomologies and its splitting.
Since we will consider that the complex structure moduli and the dilaton are stabilized supersymmetrically, we are interested only in $H^{1,1}(\mathcal{Y})$ and its splitting under the action of the previously described orientifold action $\mathcal{O}$.
More in the detail, for the sake of simplicity, let us assume the following cohomology structure:

$$H^{1,1}(\mathcal{Y}) = H^{1,1}_+(\mathcal{Y}) \oplus H^{1,1}_-(\mathcal{Y}) \qquad \text{such that: } dim\left\{H^{1,1}_+(\mathcal{Y})\right\} = 2 \quad dim\left\{H^{1,1}_-(\mathcal{Y})\right\} = 1 \tag{4.1}$$

In other words, we are supposing that $dim\left\{H^{1,1}(\mathcal{Y})\right\} = h^{1,1} = 3$ and that, after orientifold projection, we obtain $h^{1,1}_+ = 2$ and $h^{1,1}_- = 1$, having in mind the explicit example of $\mathbb{P}^4_{[1,1,1,6,9]}$ [CCQ08].
Let us start by expressing the 4-cycles volumes as a function of the $T_\alpha$ and $G^a$ fields, following the definition of such Kähler coordinates given in the previous chapter, which we rewrite here:

$$\begin{cases} S &= \frac{1}{g_s} + il \\ G^a &= \bar{S}b^a + \mathrm{i}c^a = \frac{b^a}{g_s} + \mathrm{i}\left(c^a - lb^a\right) \\ T_\alpha &= i\rho_\alpha + \frac{1}{2}\mathcal{K}_\alpha(v) - \frac{1}{2}\zeta_\alpha(S,\bar{S},G,\bar{G}) \\ &= i\rho_\alpha + \frac{1}{2}\mathcal{K}_{\alpha\beta\gamma}v^\beta v^\gamma + \frac{g_s}{4}\mathcal{K}_{\alpha bc}G^b(G+\bar{G})^c \end{cases} \tag{4.2}$$

From which we can find an analytical expression for the 4-cycles volumes $\tau_\alpha$ to be:

$$\tau_\alpha = \frac{1}{2}\mathcal{K}_{\alpha\beta\gamma}v^\beta v^\gamma = T_\alpha + \bar{T}_\alpha + \frac{g_s}{4}\mathcal{K}_{\alpha ab}(G+\bar{G})^a(G+\bar{G})^b\,. \tag{4.3}$$

In our case we know the topology of the manifold $\mathcal{Y}$, i.e. we are supposing to know all the intersection numbers $\mathcal{K}_{\alpha ab}$. Since $h^{1,1}_+ = 2$, we define the 2 possible Greek indices as $(L,+)$, while, since we can have only one Latin index (since $h^{1,1}_- = 1$), we will denote it as $-$. The choice of the name is due to the fact that in the LVS procedure we always suppose to have at least a big and a small 2-cycle which give rise to the so called **Swiss-cheese Calabi-Yau**. In our case $L$ will denote the big cycle and $+$ the small one.
Hence we are interested only in 2 intersection numbers with mixed indices, namely $\mathcal{K}_{+--}$ and $\mathcal{K}_{L--}$.

In order to know the metric of the compactification manifold, and hence perform the stabilization of the Kähler moduli, we have to start from considering the Kähler potential which we have already defined in the previous chapter and we rewrite here:

$$K = K_{cs}(z,\bar{z}) + K_k(S,T,G) = K'_{cs} - 2\ln(\mathcal{V})\,, \tag{4.4}$$

where we have defined $K'_{cs}$ to include all the already fixed contributions, i.e. $K'_{cs} = K_{cs} - \ln\big(S+\bar{S}\big)$. In other words, the unfixed part of the Kähler potential depend only on the volume of $\mathcal{Y}$. We may then consider different ways to write the volume, according to a certain choice of the intersection numbers. In the present work we will consider 3 cases:

- $G_-$ is coupled to the small cycle, i.e. $\mathcal{K}_{+--} = 1$ and $\mathcal{K}_{L--} = 0$
- $G_-$ is coupled to the big cycle, i.e. $\mathcal{K}_{+--} = 0$ and $\mathcal{K}_{L--} = -1$
- $G_-$ is coupled to both cycles, i.e. $\mathcal{K}_{+--} \neq 0$ and $\mathcal{K}_{L--} \neq 0$

We are aware that the intersection numbers are not always 1, but to simplify the calculations we will set them to 1, because we are mainly interested in the possibility of developing a viable model which can be refined and applied also to more complex topologies in the future. However the sign of the intersection numbers is fixed by the fact that the kinetic energy has to be positive definite, i.e. $K_0^{G\bar{G}} > 0$.

At this point we are ready to calculate in the 3 mentioned cases the resulting Kähler metrics corresponding to a different definition of the volume form. We will define them by the notation $(K_0)_{AB}$, since we will see in the developing of the work that the Kähler potential will acquire perturbative corrections.

Let us start by analysing the first case in which $\mathcal{K}_{+--} = 1$ and $\mathcal{K}_{L--} = 0$. Let us write the volume of $\mathcal{Y}$ as:

$$\mathcal{V} = (T_L + \bar{T}_L)^{\frac{3}{2}} - [(T_+ + \bar{T}_+) + g_+(G^- + \bar{G}^-)^2]^{\frac{3}{2}} , \tag{4.5}$$

where we have implicitly defined: $g_+ \equiv \frac{g_s}{4}\mathcal{K}_{+--}$. To proceed in the calculations with a more compact notation, let us make the following substitutions:

$$\begin{cases} T_L + \bar{T}_L = \tilde{T}_L \\ T_+ + \bar{T}_+ = \tilde{T}_+ \\ G^- + \bar{G}^- = \tilde{G}^- \end{cases} \tag{4.6}$$

so that the volume reads:

$$\mathcal{V} = (\tilde{T}_L)^{\frac{3}{2}} - [\tilde{T}_+ + g_+(\tilde{G}^-)^2]^{\frac{3}{2}} \tag{4.7}$$

Furthermore, let us also define $\tilde{T}_+ + g_+(\tilde{G}^-)^2 \equiv \tilde{T}_S$ in order to obtain the familiar form of the volume written as $\mathcal{V} = (\tilde{T}_L)^{\frac{3}{2}} - (\tilde{T}_S)^{\frac{3}{2}}$.
From this volume form we can now work out the Kähler metric starting from the Kähler potential in the following form:

$$(K_0)_{A\bar{B}} = \partial_A \partial_{\bar{B}} K_0 = \partial_A \partial_B \left(-2\ln\left\{(\tilde{T}_L)^{\frac{3}{2}} - [\tilde{T}_+ + g_+(\tilde{G}^-)^2]^{\frac{3}{2}}\right\}\right) \tag{4.8}$$

After an explicit calculation we find the following metric:

$$(K_0)_{A\bar{B}} = \frac{3}{\mathcal{V}^2\sqrt{\tilde{T}_S}} \begin{bmatrix} \sqrt{\frac{\tilde{T}_s}{\tilde{T}_L}}\left(\tilde{T}_L^{3/2} + \frac{\tilde{T}_S^{3/2}}{2}\right) & -\frac{3}{2}\sqrt{\tilde{T}_L}\tilde{T}_S & -3g_+\tilde{G}^-\sqrt{\tilde{T}_L}\tilde{T}_S \\ -\frac{3}{2}\sqrt{\tilde{T}_L}\tilde{T}_S & \frac{\tilde{T}_L^{3/2}}{2} + \tilde{T}_S^{3/2} & 2g_+\tilde{G}^-\left(\frac{\tilde{T}_L^{3/2}}{2} + \tilde{T}_S^{3/2}\right) \\ -3g_+\tilde{G}^-\sqrt{\tilde{T}_L}\tilde{T}_S & 2g_+\tilde{G}^-\left(\frac{\tilde{T}_L^{3/2}}{2} + \tilde{T}_S^{3/2}\right) & 2g_+\left[2g_+\tilde{G}^-\left(\tilde{T}_L^{3/2} + \frac{\tilde{T}_S^{3/2}}{2}\right) + \mathcal{V}\tilde{T}_+\right] \end{bmatrix} \tag{4.9}$$

The inverse of this metric takes instead the form:

$$(K_0)^{A\bar{B}} = \begin{bmatrix} \frac{2\sqrt{\tilde{T}_L}}{3}\left(\tilde{T}_S^{3/2} + \frac{\tilde{T}_L^{3/2}}{2}\right) & \tilde{T}_L\tilde{T}_S & 0 \\ \tilde{T}_L\tilde{T}_S & \frac{2\left[g_+(\tilde{G}^-)^2\mathcal{V} + \tilde{T}_S\left(\tilde{T}_L^{3/2} + \frac{\tilde{T}_S^{3/2}}{2}\right)\right]}{3\sqrt{\tilde{T}_S}} & -\frac{\tilde{G}^-\mathcal{V}}{3\sqrt{\tilde{T}_S}} \\ 0 & -\frac{\tilde{G}^-\mathcal{V}}{3\sqrt{\tilde{T}_S}} & \frac{\mathcal{V}}{6g_+\sqrt{\tilde{T}_S}} \end{bmatrix} \tag{4.10}$$

Let us now calculate the Kähler metric in the second case where $\mathcal{K}_{+--} = 0$ and $\mathcal{K}_{L--} = 1$. Thus we will have the following volume form:

$$\mathcal{V} = [\tilde{T}_L + g_L(\tilde{G}^-)^2]^{\frac{3}{2}} - (\tilde{T}_+)^{\frac{3}{2}}\,, \tag{4.11}$$

where in this case $g_L \equiv \frac{g_s}{4}\mathcal{K}_{L--}$. After defining $\tilde{T}_{gL} \equiv \tilde{T}_L + g_L(\tilde{G}^-)^2$ in the same way as before, we can work out the explicit form of the Kähler metric to be:

$$(K_0)_{A\bar{B}} = \frac{3}{\mathcal{V}^2\sqrt{\tilde{T}_{gL}}}\begin{bmatrix} \tilde{T}_{gL}^{3/2} + \frac{\tilde{T}_+^{3/2}}{2} & -\frac{3}{2}\sqrt{\tilde{T}_+\tilde{T}_{gL}} & 2g_L\tilde{G}^-\left(\tilde{T}_{gL}^{3/2} + \frac{\tilde{T}_+^{3/2}}{2}\right) \\ -\frac{3}{2}\sqrt{\tilde{T}_+\tilde{T}_{gL}} & \sqrt{\frac{\tilde{T}_{gL}}{\tilde{T}_+}}\left(\frac{\tilde{T}_{gL}^{3/2}}{2} + \tilde{T}_+^{3/2}\right) & -3g_L\tilde{G}^-\sqrt{\tilde{T}_+\tilde{T}_{gL}} \\ 2g_L\tilde{G}^-\left(\tilde{T}_{gL}^{3/2} + \frac{\tilde{T}_+^{3/2}}{2}\right) & -3g_L\tilde{G}^-\sqrt{\tilde{T}_+\tilde{T}_{gL}} & 2g_L\left[2g_L\tilde{G}^-\left(\frac{\tilde{T}_{gL}^{3/2}}{2} + \tilde{T}_+^{3/2}\right) - \mathcal{V}\tilde{T}_L\right] \end{bmatrix} \tag{4.12}$$

and its inverse is the following:

$$(K_0)^{A\bar{B}} = \begin{bmatrix} \frac{2\left[g_L(\tilde{G}^-)^2\mathcal{V} + \tilde{T}_{gL}\left(\frac{\tilde{T}_{gL}^{3/2}}{2} + \tilde{T}_+^{3/2}\right)\right]}{3\sqrt{\tilde{T}_{gL}}} & \tilde{T}_{gL}\tilde{T}_+ & \frac{\tilde{G}^-\mathcal{V}}{3\sqrt{\tilde{T}_{gL}}} \\ \tilde{T}_{gL}\tilde{T}_+ & \frac{2\sqrt{\tilde{T}_+}}{3}\left(\tilde{T}_{gL}^{3/2} + \frac{\tilde{T}_+^{3/2}}{2}\right) & 0 \\ \frac{\tilde{G}^-\mathcal{V}}{3\sqrt{\tilde{T}_{gL}}} & 0 & -\frac{\mathcal{V}}{6g_L\sqrt{\tilde{T}_{gL}}} \end{bmatrix} \tag{4.13}$$

where we can see a clear similarity with the previous functional form.

Now we can move to the more general case where both $\mathcal{K}_{+--} \neq 0$ and $\mathcal{K}_{L--} \neq 0$. Before showing the result, let us recall the previous conventions, in order to present a cleaner form of the metric:

$$\begin{cases} g_+ = \dfrac{g_s}{4}\mathcal{K}_{+--} \\ g_L = \dfrac{g_s}{4}\mathcal{K}_{L--} \\ \tilde{T}_S = \tilde{T}_+ + g_+(\tilde{G}^-)^2 \\ \tilde{T}_{gL} = \tilde{T}_L + g_L(\tilde{G}^-)^2 \\ \mathcal{V} = (\tilde{T}_{gL})^{\frac{3}{2}} - (\tilde{T}_S)^{\frac{3}{2}} \end{cases} \tag{4.14}$$

Focusing on the tree-level Kähler potential $K_0 = -2\ln(\mathcal{V})$, we can write the Kähler metric $(K_0)_{A\bar{B}}$ as:

$$\begin{cases}
(K_0)_{L\bar{L}} = \dfrac{3}{\mathcal{V}^2}\left[\dfrac{1}{\sqrt{\tilde{T}_{gL}}}\left(\tilde{T}_{gL}^{3/2} + \dfrac{\tilde{T}_S^{3/2}}{2}\right)\right] \\
(K_0)_{L\bar{+}} = \dfrac{3}{\mathcal{V}^2}\left[-\dfrac{3}{2}\sqrt{\tilde{T}_S\tilde{T}_{gL}}\right] = (K_0)_{+\bar{L}} \\
(K_0)_{L\bar{-}} = \dfrac{3}{\mathcal{V}^2}\left[\dfrac{\tilde{G}^-}{\sqrt{\tilde{T}_{gL}}}\left[g_L\left(2\tilde{T}_{gL}^{3/2} + \tilde{T}_S^{3/2}\right) - 3g_+\tilde{T}_{gL}\sqrt{\tilde{T}_S}\right]\right] = (K_0)_{-\bar{L}} \\
(K_0)_{+\bar{+}} = \dfrac{3}{\mathcal{V}^2}\left[\dfrac{1}{\sqrt{\tilde{T}_S}}\left(\tilde{T}_S^{3/2} + \dfrac{\tilde{T}_{gL}^{3/2}}{2}\right)\right] \\
(K_0)_{+\bar{-}} = \dfrac{3}{\mathcal{V}^2}\left[\sqrt{\dfrac{\tilde{T}_{gL}}{\tilde{T}_S}}\left(g_+\tilde{T}_{gL} - 3g_L\tilde{T}_S\right) + 2g_+\tilde{T}_S\right]\tilde{G}^- = (K_0)_{-\bar{+}} \\
(K_0)_{-\bar{-}} = \dfrac{6}{\mathcal{V}^2}\left[(\tilde{G}^-)^2\left(g_L\sqrt{\tilde{T}_{gL}} - g_L\sqrt{\tilde{T}_{gL}}\right)^2 + \dfrac{g_L\left(\tilde{T}_{gL} + g_L(\tilde{G}^-)^2\right)}{\sqrt{\tilde{T}_{gL}}} + \dfrac{g_+\left(\tilde{T}_S + g_+(\tilde{G}^-)^2\right)}{\sqrt{\tilde{T}_S}}\right]
\end{cases} \tag{4.15}$$

We can see that the analytic expression starts to be a bit involved, and so we expect its inverse to be even more complicated. Therefore, before performing a brute force calculation, we systematically approximate the metric in order to simplify its form and then calculate its inverse. However, during the following discussion, we will see that, due to the stabilization of $G^-$, the two previous results will converge to a unique solution, and we expect also the third case to converge to the same result in the large volume limit.

## 4.2 Supergravity and Large Volume Scenario

### 4.2.1 Kähler Potential and Superpotential

Let us recall that the $\mathcal{N} = 1$ F-term supergravity 4D scalar potential is given by:

$$V = e^K\left(\sum_{i=T,S,z} K^{i\bar{j}} D_i W D_{\bar{j}}\bar{W} - 3|W|^2\right). \tag{4.16}$$

Here we denote by $T$ the Kähler moduli, $S$ is the axio-dilaton, and $z$ are the complex structure moduli. We furthermore recall the definition of the covariant derivative appearing in the above expression:

$$\begin{cases} D_i W = \partial_i W + W \partial_i K, \\ D_{\bar{j}} \bar{W} = \partial_{\bar{j}} \bar{W} + \bar{W} \partial_{\bar{j}} K \end{cases} \tag{4.17}$$

Since classically the superpotential does not depend on the Kähler moduli, the sum over the Kähler moduli gives:

$$\sum_{i,j} \left( \frac{\partial^2 K_0}{\partial T_i \partial \bar{T}_j} \right)^{-1} \left( W \frac{\partial K_0}{\partial T_i} \right) \left( \bar{W} \frac{\partial K_0}{\partial \bar{T}_j} \right) = 3|W|^2 \tag{4.18}$$

This term cancels the second term in the potential. This process is called **no-scale cancellation** and gives rise to the following tree-level **no-scale potential**:

$$V_{\text{no-scale}} = e^{K_0} \left( \sum_{i=S,z} K_0^{i\bar{j}} D_i W D_{\bar{j}} \bar{W} \right) \tag{4.19}$$

This implies that at the semi-classical level we can only stabilize supersymmetrically the dilaton and complex structure moduli, imposing vanishing covariant derivatives: $D_S W = D_z W = 0$.
Hence at semi-classical level the Kähler moduli are flat directions in the potential which can be lifted however by including quantum corrections.

Let us first discuss the non-perturbative corrections to the superpotential, since by the **non-renormalization theorem** we cannot have perturbative ones.
Let us now call the minimum of the superpotential that stabilizes the complex moduli and the axio-dilaton at tree-level as:

$$W_0 = \langle W_{\text{tree}} \rangle = \left\langle \int_{\mathcal{Y}} G_3 \wedge \Omega \right\rangle \tag{4.20}$$

The exact value of such superpotential depends on the flux choice that we make to stabilize the complex structure moduli and the axio-dilaton, but in our discussion we will always consider it as a parameter of $O(1)$. Since the choice of fluxes can be very broad, we will fix this value in the developing of our discussion according to the possible values founded in literature.

Having in mind instanton effects and gaugino condensation, these non-perturbative corrections to $W$ appear as exponentials in the Kähler moduli. Therefore we can write the superpotential as:

$$W = W_0 + \sum_i A_i \, e^{-a_i T_i} \tag{4.21}$$

where the $A_i$ are coefficients which, in general, depend on the stabilized value of the axio-dilaton and complex structure moduli determined by the flux choice, while $a_i = 2\pi/N_i$ (with $N_i = 1$ for stringy instantons while $N_i$ is the rank of the condensing gauge group for gaugino condensation). In our work we will treat them as $O(1)$ coefficients, and we will fix them when we will consider a concrete example. There may additionally be higher instanton effects, but these can be neglected as long as each $\mathrm{Re}(T_i)$ is stabilised such that: $a_i\mathrm{Re}(T_i) \gg 1$.

Let us now see what type of corrections to introduce for the Kähler potential in order to stabilize the Kähler moduli. In this case we do not have any non-renormalization theorem which forbids perturbative corrections. The first leading order correction is in fact given by $\alpha'^3$ effects [Bec+02]. Other subleading order corrections appear at one-loop order in the $g_s$ expansion, but for our purposes we assume to work in a regime where we can neglect them, i.e. the volume of the small cycle is sufficiently large and $g_s$ is sufficiently small.
Assuming to have stabilized all complex structure moduli and the axio-dilaton allow us to write the full Kähler potential, including the first order $\alpha'^3$ corrections as:

$$\begin{aligned}
K &= \left\langle -\ln\left(-i\int_Y \Omega\wedge\bar{\Omega}\right)\right\rangle - \ln\left(\frac{2}{g_s}\right) - 2\ln\left(\mathcal{V} + \frac{\xi}{2g_s^{3/2}}\right) \\
&= K_{cs} - \ln\left(\frac{2}{g_s}\right) - 2\ln\left(\mathcal{V} + \frac{\xi}{2g_s^{3/2}}\right) \\
&= K'_{cs} - 2\ln\left(\mathcal{V} + \frac{\xi}{2g_s^{3/2}}\right). \qquad (4.22)
\end{aligned}$$

Here we recall that $\xi = \frac{\zeta(3)\chi(\mathcal{Y})}{2(2\pi)^3}$ depends on the topology of the compactification manifold $\mathcal{Y}$. Furthermore, in the last equality we have redefined the complex structure term in the same way as above. In the rest of the discussion we will refer to $K'_{cs}$ as $K_{cs}$.

To sum up the results, defining $\hat{\xi} \equiv \xi/g_s^{3/2}$, by now we have found the Kähler and superpotential in the following form:

$$\begin{cases} K = K_{cs} - 2\ln\left(\mathcal{V} + \frac{\hat{\xi}}{2}\right) \\ W = W_0 + \sum_i A_i\, e^{-a_i T_i} \end{cases} \qquad (4.23)$$

At this point we can plug in these expressions into the original equation for the scalar potential, assuming as always supersymmetric stabilization of the complex structure

moduli and the axio-dilaton by fluxes: $D_S W = D_z W = 0$.
Then we see that the previous summation runs only over the Kähler moduli. The no-scale cancellation will still occur, but we get 3 more terms, which give us a non-vanishing potential. Since we have added $\alpha'^3$ corrections to the Kähler potential, one of these terms will be [BB04]:

$$V_{\alpha'} = 3e^K \hat{\xi}\, \frac{\left(\hat{\xi}^2 + 7\hat{\xi}\mathcal{V} + \mathcal{V}^2\right)}{(\mathcal{V} - \hat{\xi})(2\mathcal{V} + \hat{\xi})^2} |W|^2\,. \tag{4.24}$$

Meanwhile, from the non perturbative corrections to the superpotential, we obtain 2 additional terms:

$$\begin{cases} V_{np1} = e^K K^{jk} a_j A_j a_k \bar{A}_k\, e^{-\left(a_j T_j + a_k \bar{T}_k\right)} \\ V_{np2} = e^K K^{jk} \left(a_j A_j e^{-a_j T_j} \bar{W} \partial_{\bar{T}_k} K + a_k \bar{A}_k e^{-a_k \bar{T}_k} W \partial_{T_j} K\right) \end{cases} \tag{4.25}$$

Gathering all the terms, we end up with the following potential:

$$\begin{aligned} V_{\mathrm{LVS}} &= V_{np1} + V_{np2} + V_{\alpha'} = \\ &= e^K \Bigg\{ K^{jk} \left[a_j A_j a_k \bar{A}_k e^{-\left(a_j T_j + a_k \bar{T}_k\right)} - \left(a_j A_j e^{-a_j T_j} \bar{W} \partial_{\bar{T}_k} K + a_k \bar{A}_k e^{-a_k \bar{T}_k} W \partial_{T_j} K\right)\right] + \\ &+ 3\hat{\xi} \frac{\left(\hat{\xi}^2 + 7\hat{\xi}\mathcal{V} + \mathcal{V}^2\right)}{(\mathcal{V} - \hat{\xi})(2\mathcal{V} + \hat{\xi})^2} |W|^2 \Bigg\} \end{aligned} \tag{4.26}$$

This is the starting point to discuss the large volume limit, and that is why we have denoted such potential as $V_{\mathrm{LVS}}$. So far we have considered the full expression of the scalar potential, but now we start to focus on the **large volume limit**.

### 4.2.2 Large Volume Limit

Let us see explicitly what we mean by taking the large volume limit of the scalar potential, and let us discuss if it is reasonable.
After defining as $\tau_i$ the size of the 4-cycles of a generic Calabi-Yau $\mathcal{Y}$, we can state the large volume limit ansatz in the following way [CCQ08]:

$$\exists N_{small} \in \mathbb{N} < h^{1,1}(\mathcal{Y}) : \begin{cases} \tau_i \text{ remains small} & \forall i = 1, ..., N_{small} \\ \tau_i \to \infty & \forall i = N_{small} + 1, ..., h^{1,1}(\mathcal{Y}) \\ \mathcal{V}(\mathcal{Y}) \to \infty & \end{cases} \tag{4.27}$$

The small $\tau_i$ have to be thought off as **blow-up cycles** resolving point-like singularities of $\mathcal{Y}$, without altering the geometry of the bulk of the Calabi-Yau which we are considering to be much bigger. In this limit, our starting forms of the Kähler potential and superpotential become:

$$\begin{cases} K = K_{cs} - 2\ln\left(\mathcal{V} + \frac{\hat{\xi}}{2}\right) \\ W = W_0 + \sum_i^{N_{small}} A_i\, e^{-a_i T_i} \end{cases} \tag{4.28}$$

Since the corrections coming from the cycles whose volume goes to infinity will be exponentially damped in this limit, we just have to perform the sum over the small cycles. For instance, we can immediately see a simplification in the Kähler potential $K$, because in our limit $\mathcal{V} \gg \hat{\xi}/2$:

$$K = K_{cs} - 2\ln\left(\mathcal{V} + \frac{\hat{\xi}}{2}\right) \simeq K_{cs} - 2\ln(\mathcal{V}) = K_{cs} + K_0 \tag{4.29}$$

Then we can immediately say that the exponential of the Kähler potential in this limit becomes:

$$e^K \simeq e^{K_{cs}+K_0} = \frac{e^{K_{cs}}}{\mathcal{V}^2} \tag{4.30}$$

In fact, due to this argument, in the following calculations we will use the tree-level inverse metrics which we have already computed for our case of interest. Then we can rewrite the full potential as [Bal+05]:

$$\begin{aligned} V_{\mathrm{LVS}} =& \frac{e^{K_{cs}}}{\mathcal{V}^2}\Bigg\{K_0^{jk}\left[a_j A_j a_k \bar{A}_k e^{-\left(a_j T_j + a_k \bar{T}_k\right)} - \left(a_j A_j e^{-a_j T_j}\bar{W}\partial_{\bar{T}_k}K_0 + a_k \bar{A}_k e^{-a_k \bar{T}_k} W \partial_{T_j} K_0\right)\right] \\ &+ 3\hat{\xi}\frac{\left(\hat{\xi}^2 + 7\hat{\xi}\mathcal{V} + \mathcal{V}^2\right)}{(\mathcal{V}-\hat{\xi})(2\mathcal{V}+\hat{\xi})^2}|W|^2\Bigg\} \end{aligned} \tag{4.31}$$

Let us start to analyze the different contributions, beginning from $V_{\alpha'}$:

$$V_{\alpha'} = \frac{e^{K_{cs}}}{\mathcal{V}^2}\frac{3\hat{\xi}\left(\hat{\xi}^2 + 7\hat{\xi}\mathcal{V} + \mathcal{V}^2\right)}{(\mathcal{V}-\hat{\xi})(2\mathcal{V}+\hat{\xi})^2}|W|^2 \sim \frac{e^{K_{cs}}}{\mathcal{V}^2}\frac{3\hat{\xi}}{4\mathcal{V}}|W|^2 + O\left(\frac{1}{\mathcal{V}^4}\right) = \frac{3\hat{\xi}|W|^2 e^{K_{cs}}}{4\mathcal{V}^3} + O\left(\frac{1}{\mathcal{V}^4}\right) \tag{4.32}$$

Let us now consider the first non-perturbative correction. More in the detail, let us restrict to our model in order to simplify consistently the calculations. In fact, in our case we have only one small cycle which size is given by $T_+$, and the only Kähler coordinates are $\{T_L, T_+, G^-\}$. We can write our model in the large volume limit as:

$$\begin{cases} K(T_L, T_+, G^-) = K_{cs} - 2\ln\left[\mathcal{V}(T_L, T_+, G^-)\right] = K_{cs} + K_0(T_L, T_+, G^-) \\ W(T_+) = W_0 + A_+\, e^{-a_+ T_+} \end{cases} \tag{4.33}$$

Let us evaluate directly $V_{np1}+V_{np2}$ computing first the explicit expressions of the covariant derivatives involved. As first thing we can immediately say that $\partial_{T_L}W = \partial_{\bar{T}_L}\bar{W} = \partial_{G^-}W = \partial_{\bar{G}^-}\bar{W} = 0$ since $W = W(T_+)$. Then we can directly calculate:

$$\begin{cases} D_{T_L}W &= -\frac{2W}{\mathcal{V}}\partial_{T_L}\mathcal{V} \\ D_{T_+}W &= -a_+A_+e^{-a_+T_+} - \frac{2W}{\mathcal{V}}\partial_{T_+}\mathcal{V} \\ D_{G^-}W &= -\frac{2W}{\mathcal{V}}\partial_{G^-}\mathcal{V} \end{cases} \qquad \begin{cases} D_{\bar{T}_L}\bar{W} &= -\frac{2\bar{W}}{\mathcal{V}}\partial_{\bar{T}_L}\mathcal{V} \\ D_{\bar{T}_+}\bar{W} &= -a_+\bar{A}_+e^{-a_+\bar{T}_+} - \frac{2\bar{W}}{\mathcal{V}}\partial_{\bar{T}_+}\mathcal{V} \\ D_{\bar{G}^-}\bar{W} &= -\frac{2\bar{W}}{\mathcal{V}}\partial_{\bar{G}^-}\mathcal{V} \end{cases} \tag{4.34}$$

Since in the end we are only taking derivatives of the volume form, we can say that all the derivatives with respect to the bar-fields will be equal to the unbar-fields, since they always appear in the form $T_i + \bar{T}_i$, and this leads to some simplifications of $V_{np1}+V_{np2}$ which takes the form:

$$\begin{aligned} V_{np1}+V_{np2} =& \frac{e^{K_{cs}}}{\mathcal{V}^2}\left(K_0^{i\bar{j}}D_iWD_{\bar{j}}\bar{W} - 3|W|^2\right) = \\ =& \frac{e^{K_{cs}}}{\mathcal{V}^2}\Bigg\{K_0^{T_+\bar{T}_+}|a_+A_+|^2e^{-a_+(T_++\bar{T}_+)} + \\ &+ \frac{4\,\mathrm{Re}\left\{\bar{W}\left(a_+A_+e^{-a_+T_+}\right)\right\}}{\mathcal{V}}\left(K_0^{T_L\bar{T}_+}\partial_{T_L}\mathcal{V} + K_0^{T_+\bar{T}_+}\partial_{T_+}\mathcal{V} + K_0^{G^-\bar{T}_+}\partial_{G^-}\mathcal{V}\right) + \\ &+ \frac{4|W|^2}{\mathcal{V}^2}\left[K_0^{T_L\bar{T}_L}\left(\partial_{T_L}\mathcal{V}\right)^2 + K_0^{T_+\bar{T}_+}\left(\partial_{T_+}\mathcal{V}\right)^2 + K_0^{G^-\bar{G}^-}\left(\partial_{G^-}\mathcal{V}\right)^2\right] + \\ &+ \frac{8|W|^2}{\mathcal{V}^2}\left[K_0^{T_L\bar{T}_+}\partial_{T_L}\mathcal{V}\partial_{T_+}\mathcal{V} + K_0^{T_L\bar{G}^-}\partial_{G^-}\mathcal{V}\partial_{T_L}\mathcal{V} + K_0^{T_+\bar{G}^-}\partial_{T_+}\mathcal{V}\partial_{G^-}\mathcal{V}\right] - 3|W|^2\Bigg\} \end{aligned} \tag{4.35}$$

Due to the no-scale cancellation, the last 2 lines sum to zero:

$$\begin{aligned} V_{np1}+V_{np2} =& \frac{e^{K_{cs}}}{\mathcal{V}^2}\left(K_0^{i\bar{j}}D_iWD_{\bar{j}}\bar{W} - 3|W|^2\right) = \\ =& \frac{e^{K_{cs}}}{\mathcal{V}^2}\Bigg\{K_0^{T_+\bar{T}_+}|a_+A_+|^2e^{-a_+(T_++\bar{T}_+)} + \\ &+ \frac{4\,\mathrm{Re}\left\{\bar{W}\left(a_+A_+e^{-a_+T_+}\right)\right\}}{\mathcal{V}}\left(K_0^{T_L\bar{T}_+}\partial_{T_L}\mathcal{V} + K_0^{T_+\bar{T}_+}\partial_{T_+}\mathcal{V} + K_0^{G^-\bar{T}_+}\partial_{G^-}\mathcal{V}\right)\Bigg\} \end{aligned} \tag{4.36}$$

Now let us assume that when the big cycle and the volume go to infinity, in order to preserve the volumes hierarchy, the small cycle goes like $a_+T_+ \sim \ln(\mathcal{V})$. The validity of this assumption will be checked a-posteriori using the main features of the LVS stabilization procedure. We now use it to have an idea of the magnitude of the different terms. This implies that in this regime we can consider:

$$W = W_0 + A_+e^{-a_+T_+} \sim W = W_0 + \frac{A_+}{\mathcal{V}} \sim W_0 + O\left(\frac{1}{\mathcal{V}}\right) \tag{4.37}$$

Another remark that is necessary to make is on the sign of the term:

$$\frac{4\,\mathrm{Re}\left\{\bar{W}\left(a_+ A_+ e^{-a_+ T_+}\right)\right\}}{\mathcal{V}} \tag{4.38}$$

Let us recall that $\mathrm{Im}(T_+) = \rho_+$. Then, since we have to minimize the potential, we can argue that this phase leads to a minus sign, fixing the value of $\rho_+$.
After these remarks, we are able to write the non-perturbative corrections to the scalar potential as:

$$\begin{aligned} V_{np1} + V_{np2} =& \frac{e^{K_{cs}}}{\mathcal{V}^2}\left(K_0^{i\bar{j}} D_i W D_{\bar{j}} \bar{W} - 3|W_0|^2\right) = \\ =& \frac{e^{K_{cs}}}{\mathcal{V}^2}\left[\frac{K_0^{T_+\bar{T}_+}|a_+ A_+|^2}{\mathcal{V}^2} + \right. \\ & \left. - \frac{4\,\mathrm{Re}\left\{a_+ A_+ \bar{W}_0\right\}}{\mathcal{V}^2}\left(K_0^{T_L \bar{T}_+}\partial_{T_L}\mathcal{V} + K_0^{T_+\bar{T}_+}\partial_{T_+}\mathcal{V} + K_0^{G^-\bar{T}_+}\partial_{G^-}\mathcal{V}\right)\right] \end{aligned} \tag{4.39}$$

The full LVS potential, including $\alpha'^3$ effects, turns out to be:

$$\begin{aligned} V_{\mathrm{LVS}} =& V_{np1} + V_{np2} + V_{\alpha'} = \\ =& \frac{e^{K_{cs}}}{\mathcal{V}^2}\left[\frac{K_0^{T_+\bar{T}_+}|a_+ A_+|^2}{\mathcal{V}^2} - \frac{4\,\mathrm{Re}\left\{a_+ A_+ \bar{W}_0\right\}}{\mathcal{V}^2}\left(K_0^{T_L \bar{T}_+}\partial_{T_L}\mathcal{V} + K_0^{T_+\bar{T}_+}\partial_{T_+}\mathcal{V} + K_0^{G^-\bar{T}_+}\partial_{G^-}\mathcal{V}\right)\right] + \\ & + \frac{3\hat{\xi}|W_0|^2 e^{K_{cs}}}{4\mathcal{V}^3} \end{aligned} \tag{4.40}$$

## 4.3 Axion-Monodromy Inflation

### 4.3.1 Axion Field Normalization

Before introducing the concept of **monodromy**, let us consider the correct normalization for the axion field which comes from the axion kinetic terms in the Lagrangian. Let us take into the account the axion $c$ which enters in the 2-form decomposition:

$$C_2 = c_a(x)\omega^a \tag{4.41}$$

Its kinetic terms are given by the square of its associated field-strength in the form:

$$\begin{aligned} S_{\text{kin},c} &= -\int d^{10}X \frac{g_s\sqrt{-\det(G_{MN})}}{2(2\pi)^7\alpha'^4}|dC_2|^2 \\ &= -\int d^{10}X \frac{g_s\sqrt{-\det(G_{MN})}}{2(2\pi)^9(6!)\alpha'^4} G^{\mu\nu}\partial_\mu c_a \partial_\nu c_b \omega^a_{ij}\omega^b_{\bar{i}\bar{j}} G^{i\bar{i}}G^{j\bar{j}} \\ &\supset -\frac{1}{2}\int d^4x\sqrt{\det(g_{\mu\nu})}\gamma^{ab}G^{\mu\nu}\partial_\mu c_a\partial_\nu c_b \qquad (\text{since: } a=b=-) \\ &= -\frac{1}{2}\int d^4x\sqrt{g}f^2(\partial_\mu c)^2 = -\frac{1}{2}\int d^4x\sqrt{g}(\partial_\mu\phi)^2 \end{aligned} \tag{4.42}$$

Since in the Einstein frame we have the following relation:

$$\alpha' M_P^2 = \frac{\mathcal{V}}{\pi} \tag{4.43}$$

we can write:

$$\frac{f^2}{M_P^2} = \frac{g_s}{48\pi^2\mathcal{V}}\left[\frac{\int \omega\wedge *\omega}{(2\pi)^6\alpha'^3}\right] = \frac{g_s}{8\pi^2}\frac{\sqrt{\tilde{T}_S}}{\mathcal{V}} \tag{4.44}$$

There the canonically normalized $C_2$-axion field becomes:

$$c^2 = \frac{\phi^2}{f^2} = \frac{8\pi^2}{g_s}\frac{\mathcal{V}}{\sqrt{\tilde{T}_S}}\left(\frac{\phi}{M_P}\right)^2 \tag{4.45}$$

From now on, we will just express the field $\phi$ in Planck units [Fla+10].

### 4.3.2 5-brane Potential and the Axion-monodromy

At this point let us remind that the main goal of our discussion is to provide a mechanism to drive inflation by the use of axions which couple to branes that are objects with NS-NS or R-R charge (or both). In our model, since we have already fixed the axio-dilaton and the $C_4$ axion, we are left with the R-R axion $c$ and the NS-NS axion $b$, which are both 2-form axions.

Let us recall that the action of a brane consists of the so-called **Dirac-Born-Infeld action** $S_{\text{DBI}}$ and the **Chern-Simons action** $S_{\text{CS}}$. Since the Chern-Simons action is topological, we are interested in the DBI-piece of the brane action. Since we have 2-form axions, if we want to couple them to a brane, this brane has to wrap a 2-cycle in the compactification space $\mathcal{Y}$. This is the case of spacetime-filling 5-branes which are 6D objects with 2 dimensions in $\mathcal{Y}$ since they wrap a 2-cycle.

For our purposes, we can have an NS5-brane in the case of $c$, and a D5-brane in the case of $b$. In this sense we can argue that these are **S-dual** descriptions, due to the $SL(2,Z)$

symmetry which allows us to exchange the fields (up to a sign) but we will not comment further on this duality.
From a historical point of view, the DBI action was first applied to Maxwell's electromagnetism and in fact we can see the DBI action for branes as a generalization of this idea to more than 4D where we can see the R-R and NS-NS charges as the electric and magnetic charges, which are nothing but dual manifestations of the electromagnetic interaction.

In our specific model, however, we have lost such duality since in the Kähler potential we can see the explicit appearance of $b$ in the $G^-$ fields. Therefore, during the stabilization this will set such field to a certain value, breaking its shift symmetry. Furthermore, this will set $\eta \sim 1$ obstructing inflationary dynamics (we refer to such issue as the $\eta$-problem). On the other hand, we have the $c$ field whose shift-symmetry is not spoiled in our potential. Hence this axion can be used as a good candidate for inflation [MSW10; McA+14].[1]

Let us remind that the DBI action for a generic NSp-brane in the Einstein-frame coupled to the 2-form $C_2$ for an unwarped metric is [Pol98]:

$$S_{\text{DBI}}^{NSp} = -\int \frac{(\alpha')^{-\frac{p-1}{2}}}{(2\pi)^p} e^{-2\Phi} d^{p+1}\xi \sqrt{\det\left\{(G_{MN} + C_{MN})\partial_\alpha X^M \partial_\beta X^N\right\}} \tag{4.46}$$

where $G_{MN}$ is the 10D metric and $C_{MN}$ is the 2-form related to the $c$ field. $\xi$ are the coordinates on the brane denoted by $\alpha, \beta$, while $X^I = X^I(\xi)$ (where I= M,N) are the coordinates of 10D spacetime. In other words, under the square root we have the determinant of the pullback of the sum of the metric and the other 2-form on the brane, that we have indicated in a more explicit way.
Let us now decline such action in the context of the NS5-brane, with a warped metric of the form:

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A(y)}g_{i\bar{j}}dy^i dy^{\bar{j}} \tag{4.47}$$

With respect to the previous DBI action, we have to include a warp factor:

$$S_{\text{DBI}}^{NSp} = -e^{4A} \int \frac{(\alpha')^{-2}}{(2\pi)^5} g_s^{-2} d^6\xi \sqrt{det(G_{MN} + C_{MN})\partial_\alpha X^M \partial_\beta X^N} \tag{4.48}$$

We can then derive the following contribution to the potential for the NS5-brane:

$$V_{NS5}(c) = \frac{\epsilon}{g_s^2(2\pi)^5(\alpha')^2}\sqrt{l^4 + (g_s c)^2} \tag{4.49}$$

[1]Axion monodromy inflation can also be described in 4D effective field theory in terms of 4-form field strength coupled to the axion [KS09]. This is very useful as all known string models of axion monodromy reduce to this 4D EFT description.

where the warp factor is hidden in the small prefactor $\epsilon$. In our case we have constructed our compactification space, therefore we can explicitly calculate the size of the 2-cycle by the 4-cycles volume. Furthermore, since $\alpha' M_P^2 = \mathcal{V}/\pi$, we can rewrite the potential as dependent on the inverse of the compactification volume $\mathcal{V}$ as follows:

$$V_{NS5}(c) = \frac{\epsilon}{\mathcal{V}^2}\sqrt{\tilde{T}_S + g_s^2 c^2} \tag{4.50}$$

where we have redefined $\epsilon$ in order to include all the other constants involved, and $\tilde{T}_S$, depending on the choice of the volume form, can be equal to $\tilde{T}_S$ or $\tilde{T}_+$, since we will suppose that our 5-brane wraps always the small cycle which size changes according to the value of the $G$ field.

Then including the canonical normalization of the field, we can express the NS5-brane potential as:

$$V_{NS5}(\phi) = \frac{\epsilon}{\mathcal{V}^2}\sqrt{\tilde{T}_S + 8\pi^2 g_s \frac{\mathcal{V}}{\sqrt{\tilde{T}_S}}\,\phi^2} \tag{4.51}$$

### 4.3.3 Axion Monodromy

Without the NS5-brane the axion potential would have been periodic with a period of $2\pi f$ due to instanton effects which produce a cosine dependence, and a discrete shift symmetry would have been preserved. However, the introduction of the NS5-brane potential explicitly breaks the shift symmetry and the periodicity of the potential.
As in the case of complex analysis with a spiral staircase when the configuration space is a circle but the system changes upon transport by $2\pi$, and therefore we say that a function is monodromic (which in Greek means in fact "one value"), something very similar occurs in the scalar potential for axions. In fact, the coupling with the NS5-brane realizes a potential which always take a different value $\forall$ values of $\phi$. **Axion monodromy** consists exactly in this phenomenon.

It is easy to see that in the large volume limit this potential becomes linear in $\phi$:

$$\lim_{\mathcal{V}\to\infty} V_{NS5}(\phi) = \mu_\epsilon\,\phi\,, \tag{4.52}$$

where $\mu_\epsilon$ is a constant defined by the value of the volume which we obtain after the compactification. Hence this might be a good model if we want to implement linear inflation driven by an axion field.

At this point we have to specify a specific setup of branes defined in a warped throat, since we want to have a warped metric.
To be more precise, since we have to wrap an NS5-brane on a 2-cycle, by charge conservation we must have another object which cancels its charge, i.e. an anti-NS5-brane, and

we have to keep them at distance, because otherwise, as 2 electric charges of opposite sign would do, they would annihilate. Then we might think that in our orientifold $\mathcal{Y}$ we do not realize a typical **Klebanov-Strassler warped throat** [KS00], but a double warped throat where the warping keeps the 2 5-branes distant from each other. In fact, each brane will wrap a 2-cycle in one of the 2 maximally warped regions.
In order to have charge cancellation, we must have that both 2-cycles wrapped by the 5-branes belong to a family of homologous 2-cycles, and by the Gauss theorem we have the flux cancellation outside the throat, which is what we need not to spoil the other regions of the Calabi-Yau.

What we have in mind is to realize a mechanism in which our field starts at $\sim 11M_P$ (a little bit less since the potential is not completely linear), and then decreases its value until it vanishes and the reheating starts to give rise to the hot Big-Bang.

## 4.4 LVS and Axion Monodromy

Now we have finally all the elements to write and discuss the full potential of our model gathering the different contributions, i.e. $V_{\text{LVS}}$ and $V_{NS5}$:

$$
\begin{aligned}
V &= V_{\text{LVS}} + V_{NS5} = \\
&= \frac{e^{K_{cs}}}{\mathcal{V}^2}\left[\frac{K_0^{T_+\bar{T}_+}|a_+A_+|^2}{\mathcal{V}^2} - \frac{4\,\mathrm{Re}\left\{a_+A_+\bar{W}_0\right\}}{\mathcal{V}^2}\left(K_0^{T_L\bar{T}_+}\partial_{T_L}\mathcal{V} + K_0^{T_+\bar{T}_+}\partial_{T_+}\mathcal{V} + K_0^{G^-\bar{T}_+}\partial_{G^-}\mathcal{V}\right)\right] + \\
&\quad + \frac{3\hat{\xi}|W_0|^2 e^{K_{cs}}}{4\mathcal{V}^3} + \frac{\epsilon}{\mathcal{V}^2}\sqrt{\tilde{T}_S + 8\pi^2 g_s \frac{\mathcal{V}}{\sqrt{\tilde{T}_S}}\phi^2}
\end{aligned}
\tag{4.53}
$$

Now what we are left to do is just to check how this potential behaves calculating each volume derivative depending on the chosen volume form.

Let us begin from the first case where $\mathcal{V} = (\tilde{T}_L)^{\frac{3}{2}} - (\tilde{T}_S)^{\frac{3}{2}}$. Let us recall the relevant derivatives and calculate the partial derivatives involved:

$$
\begin{cases}
K^{T_L\bar{T}_+} &= \tilde{T}_L\tilde{T}_S \\
K^{T_+\bar{T}_+} &= \frac{2}{3\sqrt{\tilde{T}_S}}\left[g_+(\tilde{G}^-)^2\mathcal{V} + \tilde{T}_S\left(\tilde{T}_L^{3/2} + \frac{\tilde{T}_S^{3/2}}{2}\right)\right] \\
K^{G^-\bar{T}_+} &= -\frac{\tilde{G}^-\mathcal{V}}{3\sqrt{\tilde{T}_S}}
\end{cases}
\qquad
\begin{cases}
\partial_{T_L}\mathcal{V} &= \frac{3}{2}\sqrt{\tilde{T}_L} \\
\partial_{T_+}\mathcal{V} &= -\frac{3}{2}\sqrt{\tilde{T}_S} \\
\partial_{G^-}\mathcal{V} &= -3g_+\tilde{G}^-\sqrt{\tilde{T}_S}
\end{cases}
\tag{4.54}
$$

If we carefully plug in such expressions and the one for the inverse metric calculated at the beginning of this chapter, we find the following expression:

$$
\begin{aligned}
V_I = & \frac{2e^{K_{cs}}}{\mathcal{V}^4} \left\{ \frac{|a_+ A_+|^2}{3\sqrt{\tilde{T}_S}} \left[ g_+ (\tilde{G}^-)^2 \mathcal{V} + \tilde{T}_S \left( \tilde{T}_L^{\frac{3}{2}} + \frac{\tilde{T}_S^{\frac{3}{2}}}{2} \right) \right] - \tilde{T}_S \mathcal{V} \operatorname{Re}\{a_+ A_+ \bar{W}_0\} \right\} \\
& + \frac{3\hat{\xi}|W_0|^2 e^{K_{cs}}}{4\mathcal{V}^3} + \frac{\epsilon}{\mathcal{V}^2} \sqrt{\tilde{T}_S + 8\pi^2 g_s \frac{\mathcal{V}}{\sqrt{\tilde{T}_S}} \phi^2}
\end{aligned}
\tag{4.55}
$$

Substituting $(\tilde{T}_L)^{\frac{3}{2}} \simeq \mathcal{V}$, the potential becomes:

$$
\begin{aligned}
V_I = & \frac{2e^{K_{cs}}}{\mathcal{V}^3} \left\{ \frac{1}{3} |a_+ A_+|^2 \left[ \sqrt{\tilde{T}_S} + g_+ \frac{(\tilde{G}^-)^2}{\sqrt{\tilde{T}_S}} \right] - \tilde{T}_S \operatorname{Re}\{a_+ A_+ \bar{W}_0\} + \frac{3}{8} \hat{\xi} |W_0|^2 \right\} \\
& + \frac{\epsilon}{\mathcal{V}^2} \sqrt{\tilde{T}_S + 8\pi^2 g_s \frac{\mathcal{V}}{\sqrt{\tilde{T}_S}} \phi^2}
\end{aligned}
\tag{4.56}
$$

Let us perform the same calculation on the second volume form, $\mathcal{V} = \tilde{T}_{gL}^{\frac{3}{2}} - \tilde{T}_+^{\frac{3}{2}}$. The relevant metric components and the derivatives of the volume form are:

$$
\begin{cases}
K^{T_L \bar{T}_+} & = \tilde{T}_{gL} \tilde{T}_+ \\
K^{T_+ \bar{T}_+} & = \frac{2\sqrt{\tilde{T}_+}}{3} \left( \tilde{T}_{gL}^{3/2} + \frac{\tilde{T}_+^{3/2}}{2} \right) \\
K^{G^- \bar{T}_+} & = 0
\end{cases}
\qquad
\begin{cases}
\partial_{T_L} \mathcal{V} & = \frac{3}{2} \sqrt{\tilde{T}_{gL}} \\
\partial_{T_+} \mathcal{V} & = -\frac{3}{2} \sqrt{\tilde{T}_+} \\
\partial_{G^-} \mathcal{V} & = 3 g_L \tilde{G}^- \sqrt{\tilde{T}_{gL}}
\end{cases}
\tag{4.57}
$$

from which we can find the following potential:

$$
\begin{aligned}
V_{II} = & \frac{2e^{K_{cs}}}{\mathcal{V}^4} \left\{ \frac{\sqrt{\tilde{T}_+}}{3} \left( \tilde{T}_{gL}^{\frac{3}{2}} + \frac{\tilde{T}_+^{\frac{3}{2}}}{2} \right) |a_+ A_+|^2 - \operatorname{Re}\{a_+ A_+ \bar{W}_0\} \tilde{T}_+ \mathcal{V} \right\} + \frac{3\hat{\xi}|W_0|^2 e^{K_{cs}}}{4\mathcal{V}^3} \\
& + \frac{\epsilon}{\mathcal{V}^2} \sqrt{\tilde{T}_+ + 8\pi^2 g_s \frac{\mathcal{V}}{\sqrt{\tilde{T}_+}} \phi^2}
\end{aligned}
\tag{4.58}
$$

Substituting $(\tilde{T}_{gL})^{\frac{3}{2}} \simeq \mathcal{V}$, the potential simplifies to:

$$
\begin{aligned}
V_{II} = & \frac{2e^{K_{cs}}}{\mathcal{V}^3} \left\{ \frac{1}{3} |a_+ A_+|^2 \sqrt{\tilde{T}_+} - \operatorname{Re}\{a_+ A_+ \bar{W}_0\} \tilde{T}_+ + \frac{3\hat{\xi}|W_0|^2}{8} \right\} \\
& + \frac{\epsilon}{\mathcal{V}^2} \sqrt{\tilde{T}_+ + 8\pi^2 g_s \frac{\mathcal{V}}{\sqrt{\tilde{T}_+}} \phi^2}
\end{aligned}
\tag{4.59}
$$

Let us now write the 2 versions of the potential obtained by the different volume forms and discuss the stabilization of the $G^-$ field:

$$\begin{cases} V_I &= \frac{2e^{K_{cs}}}{\mathcal{V}^3}\left\{\frac{|a_+A_+|^2}{3\sqrt{\tilde{T}_S}}\left[g_+(\tilde{G}^-)^2+\tilde{T}_S\right]-\tilde{T}_S\,\mathrm{Re}\{a_+A_+\bar{W}_0\}+\frac{3\hat{\xi}|W_0|^2}{8}\right\}+ \\ & +\frac{\epsilon}{\mathcal{V}^2}\sqrt{\tilde{T}_S+8\pi^2 g_s\frac{\mathcal{V}}{\sqrt{\tilde{T}_S}}\phi^2} \\ V_{II} &= \frac{2e^{K_{cs}}}{\mathcal{V}^3}\left\{\frac{\sqrt{\tilde{T}_+}}{3}|a_+A_+|^2-\mathrm{Re}\{a_+A_+\bar{W}_0\}\tilde{T}_+ +\frac{3\hat{\xi}|W_0|^2}{8}\right\} \\ & +\frac{\epsilon}{\mathcal{V}^2}\sqrt{\tilde{T}_+ +8\pi^2 g_s\frac{\mathcal{V}}{\sqrt{\tilde{T}_+}}\phi^2} \end{cases} \tag{4.60}$$

The first thing that we have to check is that when $\tilde{G}^-=0$ we should recover the same potential, otherwise we have made some errors in the calculations, and we see that this statement is verified. At this point we can discuss the stabilization of the $G^-$ field, or the $b^-$ axion since $\tilde{G}^-=2b^-/g_s$.
In the case of $V_I$ we have an explicit quadratic dependence in the first term which fixes $b=0$. For $V_{II}$ we seem to have lost the dependence in $\tilde{G}$. However, if we had been more careful, we would have obtained also a quadratic term which fixes again $b=0$, as shown in [CSS22].

Thus we end up with the same form of the potential for both cases, which we denote simply by $V$:

$$\begin{aligned} V = \frac{e^{K_{cs}}}{\mathcal{V}^3}&\left(\frac{2}{3}\,|a_+A_+|^2-2\,\mathrm{Re}\{a_+A_+\bar{W}_0\}\tilde{T}_+ +\frac{3}{4}\hat{\xi}|W_0|^2\right)+ \\ &+\frac{\epsilon}{\mathcal{V}^2}\sqrt{\tilde{T}_+ +8\pi^2 g_s\frac{\mathcal{V}}{\sqrt{\tilde{T}_+}}\,\phi^2} \end{aligned} \tag{4.61}$$

Until now we have assumed a logarithmic relation between the total volume and the small cycle size without justifying this assumption. Hence, let us now make a step back and understand better the approximation in which we are working. More in the detail, let us re-express the potential treating the volume $\mathcal{V}$ and the size of the 4-cycle $\tilde{T}_+$ as 2 independent quantities, and see if we can work out such a logarithmic relation.
In order to simplify the notation let us introduce $2\tau_+=\tilde{T}_+$, obtaining:

$$\begin{aligned} V = e^{K_{cs}}&\left(\frac{2\sqrt{2}}{3\mathcal{V}}|a_+A_+|^2\sqrt{\tau_+}\,e^{-2a_+\tau_+}-\frac{4\,\mathrm{Re}\{a_+A_+\bar{W}_0\}}{\mathcal{V}^2}\tau_+ +\frac{3\hat{\xi}|W_0|^2}{4\mathcal{V}^3}\right) \\ &+\frac{\epsilon\sqrt{2\tau_+}}{\mathcal{V}^2}\sqrt{1+\frac{4\pi^2 g_s\mathcal{V}}{\tau_+^{3/2}}\,\phi^2} \end{aligned} \tag{4.62}$$

In order to simplify the calculations, let us give some typical numerical values to the constants involved which in principle should come from a complete stabilization also of the axio-dilaton and complex structure moduli, which we are not going to discuss in the present work:

$$\begin{cases} e^{K_{cs}} & \sim 1 \\ \frac{2\sqrt{2}}{3}|a_+A_+|^2 & = \frac{8\pi^2\sqrt{2}}{3}A_+^2 \qquad\qquad (a_+ = 2\pi, \quad A_+ \in \mathbb{R}) \\ 4a_+\,\mathrm{Re}\{A_+\bar{W}_0\} & = 8\pi W_0 A_+ \qquad\qquad (W_0 \in \mathbb{R}) \\ \frac{3}{4}|W_0|^2\hat{\xi} & = \frac{3}{4}\hat{\xi}W_0^2 \end{cases} \tag{4.63}$$

We obtain:

$$V = \frac{8\pi\sqrt{2}A_+^2}{3\mathcal{V}}\sqrt{\tau_+}e^{-4\pi\tau_+} - \frac{8\pi W_0 A_+}{\mathcal{V}^2}\tau_+ e^{-2\pi\tau_+} + \frac{3\hat{\xi}W_0^2}{4\mathcal{V}^3} + \frac{\epsilon\sqrt{2\tau_+}}{\mathcal{V}^2}\sqrt{1 + \frac{4\pi^2 g_s\mathcal{V}}{\tau_+^{3/2}}\phi^2} \tag{4.64}$$

At this point we might be tempted to stabilize the volume and $\tau_+$ by imposing the vanishing of their derivatives, since we do not want that our Calabi-Yau decompactifies:

$$\frac{\partial V}{\partial \mathcal{V}} = \frac{\partial V}{\partial \tau_+} = 0 \tag{4.65}$$

This might be a good idea. However we cannot solve these equation analytically, and so we tackle the problem in a different way.
Let us notice that in the $\epsilon \to 0$ limit, we recover the LVS potential. Then we can use the LVS stabilization which gives us a relation between the volume and $\tau_+$ which we know to be exponential. In other words, for consistency reasons, since we have always assumed an exponential dependence, we have to consider a negligible contribution coming from the NS5-brane potential. In order to realize this approximation, we might think that, because of the warping in the throat, $\epsilon$ is set to a small value. This is plausible since we have set the NS5-brane at the bottom of the throat.
Then let us stabilize the LVS part of the potential. Let us simplify the notation by introducing the following variables:

$$\begin{cases} \lambda & \equiv \frac{8\pi\sqrt{2}A_+^2}{3} \\ \mu & \equiv 8\pi W_0 A_+ \\ \nu & \equiv \frac{3\hat{\xi}W_0^2}{4} \end{cases} \tag{4.66}$$

The LVS contribution to the potential looks like:

$$V_{\mathrm{LVS}}(\mathcal{V}, \tau_+) = \frac{\lambda}{\mathcal{V}}\sqrt{\tau_+}e^{-4\pi\tau_+} - \frac{\mu}{\mathcal{V}^2}\tau_+ e^{-2\pi\tau_+} + \frac{\nu}{\mathcal{V}^3} \tag{4.67}$$

We can find the minimum of such potential by imposing the vanishing of the partial derivatives:

$$\frac{\partial V_{\mathrm{LVS}}}{\partial \mathcal{V}} = \frac{\partial V_{\mathrm{LVS}}}{\partial \tau_+} = 0 \tag{4.68}$$

From the derivative with respect to the volume we find:

$$\mathcal{V} = \frac{\mu}{\lambda}\sqrt{\tau_+}e^{2\pi\tau_+}\left(1 \pm \sqrt{1 - \frac{3\nu\lambda}{\mu^2\tau_+^{\frac{3}{2}}}}\right) \tag{4.69}$$

while from the derivative with respect to $\tau_+$ we obtain:

$$\frac{\lambda\mathcal{V}e^{-2\pi\tau_+}}{\sqrt{\tau_+}}\left(\frac{1}{2} - 4\pi\tau_+\right) - \mu\left(1 - 2\pi\tau_+\right) = 0 \tag{4.70}$$

Combining the 2 equations we find:

$$\left(1 \pm \sqrt{1 - \frac{3\nu\lambda}{\mu\tau_+^{\frac{3}{2}}}}\right)\left(\frac{1}{2} - 4\pi\tau_+\right) = (1 - 2\pi\tau_+) \tag{4.71}$$

In order to solve it in a simpler way, we require $a_+\tau_+ \gg 1$, which is the assumption that we have done until now to be able to ignore higher instanton corrections. By solving these equations we end up with:

$$\begin{cases} \tau_+ & = \left(\frac{4\nu\lambda}{\mu^2}\right)^{\frac{2}{3}} \\ \mathcal{V} & = \frac{\mu}{2\lambda}\left(\frac{4\nu\lambda}{\mu^2}\right)^{\frac{1}{3}} e^{2\pi\left(\frac{4\nu\lambda}{\mu^2}\right)^{\frac{2}{3}}} = \frac{\mu}{2\lambda}\sqrt{\tau_+}e^{2\pi\tau_+} \sim \left(\frac{1-2\pi\tau_+}{\frac{1}{2}-4\pi\tau_+}\right)\frac{\mu}{\lambda}\sqrt{\tau_+\, e^{2\pi\tau_+}} \end{cases} \tag{4.72}$$

In this setup the volume and the 4-cycle size $\tau_+$ are fixed, but these values do not take into the account the presence of the NS5-brane potential. However, this analysis gives us a relation between the two variables which we can use to find the minimum of the previous potential. Then, let us plug such expression for the volume into the full potential:

$$V(\tau_+, \phi) = \left(\frac{2\lambda}{\mu}e^{-3\pi\tau_+}\right)^2\left(\frac{2\lambda\nu}{\mu}\tau_+^{-\frac{3}{2}} - \frac{\mu}{2} + \frac{\epsilon\sqrt{2}e^{2\pi\tau_+}}{\tau_+}\sqrt{\tau_+ + \frac{2\pi^2\mu g_s}{\lambda}e^{2\pi\tau_+}\phi^2}\right) \tag{4.73}$$

In order to check if we have obtained a viable inflationary potential, we can set $A_+ = W_0 = 1$ and $g_s = 0.2$, which are generic values that such variables can take, and we can begin to require some properties to this potential. We want that the volume of the 4-cycle does not decompactify, i.e. it has to minimize the potential. Furthermore we want that the minimum of our potential is zero at $\phi = 0$. We also want enough field range to inflate, that we have estimated before to be almost 11 Planck units to have

60-efoldings (even if we expect a smaller field range since our potential is linear only asymptotically).
Hence we can start to impose $V(\tau_+^{min}, \phi) = 0$ in order to fix $\epsilon$, finding numerically $\epsilon = 0.00295589$. Substituting this value in the potential, we can calculate the maximum field value, i.e. the value where the minimum in $\tau$ is lost, that in this case is $\phi = 0.0068763$, as shown in Fig. 4.1. We then conclude that inflation cannot occur.

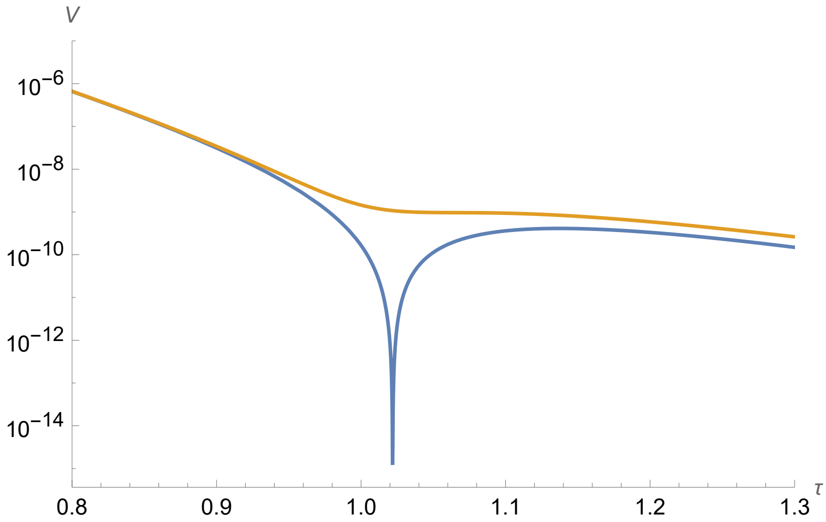


Figure 4.1: Logarithmic plot of $V(\tau_+, \phi)$ for $\phi = 0$ (the blue line), and for $\phi = 0.0068763$ (the orange line)

Furthermore, having a value for $\epsilon \sim 10^{-3}$ means that we cannot neglect consistently the contribution of the NS5-brane when we compute the relation between the volume and $\tau_+$. In fact, we have a back-reacted value of $\tau_+ = 1.02184$ at $\phi = 0$ against the LVS value $\tau_+ = \left(\frac{4\nu\lambda}{\mu^2}\right)^{\frac{2}{3}} = 0.879024$, i.e. an almost 15% of discrepancy.

In order to enlarge the field range, we have to introduce a contribution which substitutes the uplifting given by the NS5-brane, and make the NS5-part of the potential negligible for the volume stabilization.

## 4.5 Potential Uplifting

The most simple solution which we can consider in order to solve the criticality of the previously proposed potential, is to consider an uplifting term of the following form:

$$V_{up} \sim \frac{\tilde{\epsilon}}{\mathcal{V}^2} \tag{4.74}$$

where $\tilde{\epsilon}$ is a small and tunable parameter. We know from the literature that such an uplifting term can be provided by the effect of a $\bar{\text{D}}$3-brane in a different Klebanov-Strassler throat [KPV02], a T-brane [CQV16] or non-zero F-terms of the complex structure moduli [Cic+22]. Note that we need $\epsilon \ll \tilde{\epsilon} \ll 1$. Adding this contribution, our potential becomes:

$$V(\mathcal{V},\tau_+,\phi) = \frac{\lambda}{\mathcal{V}}\sqrt{\tau_+}e^{-4\pi\tau_+} - \frac{\mu}{\mathcal{V}^2}\tau_+ e^{-2\pi\tau_+} + \frac{\nu}{\mathcal{V}^3} + \frac{\epsilon\sqrt{2\tau_+}}{\mathcal{V}^2}\sqrt{1+\frac{4\pi^2 g_s\mathcal{V}}{\tau_+^{3/2}}\phi^2} + \frac{\tilde{\epsilon}}{\mathcal{V}^2} \quad (4.75)$$

As we have seen before, with such a strong uplifting we cannot trust anymore the LVS stabilization of the volume. However, since we are imposing that the highest contribution is given by the uplifting term, we can consistently neglect the NS5-brane contribution in the volume stabilization. Thus, we consider the following potential:

$$V(\mathcal{V},\tau_+) = V_{\text{LVS}} + V_{\text{up}} = \frac{\lambda}{\mathcal{V}}\sqrt{\tau_+}e^{-4\pi\tau_+} - \frac{\mu}{\mathcal{V}^2}\tau_+ e^{-2\pi\tau_+} + \frac{\nu}{\mathcal{V}^3} + \frac{\tilde{\epsilon}}{\mathcal{V}^2} \quad (4.76)$$

Let us minimize $V$ with respect to the volume and $\tau_+$ to find a relation between them:

$$\begin{cases} \frac{\partial V}{\partial \mathcal{V}} = 0 \quad \Rightarrow \mathcal{V} = \left(\frac{\mu\sqrt{\tau_+}}{\lambda}e^{2\pi\tau_+}\right)\left[\left(1-\frac{\tilde{\epsilon}\,e^{2\pi\tau_+}}{\mu\tau_+}\right) \pm \sqrt{\left(1-\frac{\tilde{\epsilon}\,e^{2\pi\tau_+}}{\mu\tau_+}\right)^2 - \frac{3\nu\lambda}{\mu^2\tau_+^{\frac{3}{2}}}}\right] \\ \frac{\partial V}{\partial \tau_+} = 0 \quad \Rightarrow \left(\frac{\lambda}{\mu\sqrt{\tau_+}}e^{-2\pi\tau_+}\right)\mathcal{V}\left(\frac{1}{2} - 4\pi\tau_+\right) = (1-2\pi\tau_+) \end{cases} \quad (4.77)$$

Gathering the equations we end up with the following expression in $\tau_+$:

$$\left[\left(1-\frac{\tilde{\epsilon}\,e^{2\pi\tau_+}}{\mu\tau_+}\right) \pm \sqrt{\left(1-\frac{\tilde{\epsilon}\,e^{2\pi\tau_+}}{\mu\tau_+}\right)^2 - \frac{3\nu\lambda}{\mu^2\tau_+^{\frac{3}{2}}}}\right]\left(\frac{1}{2} - 4\pi\tau_+\right) = (1-2\pi\tau_+) \quad (4.78)$$

Since we are assuming $a_+\tau_+ = 2\pi\tau_+ \gg 1$, this equation, similarly to the previous case without uplifting, reduces to:

$$\pm\sqrt{\left(1-\frac{\tilde{\epsilon}\,e^{2\pi\tau_+}}{\mu\tau_+}\right)^2 - \frac{3\nu\lambda}{\mu^2\tau_+^{\frac{3}{2}}}} = \left(\frac{\tilde{\epsilon}\,e^{2\pi\tau_+}}{\mu\tau_+} - \frac{1}{2}\right) \quad (4.79)$$

However this time, because of the presence of the $\tilde{\epsilon}$ term, taking the square of both sides does not help in the calculation which still has to be done numerically. In any case this expression is still useful since we can evaluate the sign of the solution for the volume form which we have to consider.
Since by previous experience we expect $\tilde{\epsilon} \sim 0.001$ and $\tau_+ \sim 1$, we can consider the first term on the RHS of (4.79) to be:

$$\frac{\tilde{\epsilon}\,e^{2\pi\tau_+}}{\mu\tau_+} \sim \frac{0.001 \times e^{2\pi}}{8\pi} \sim 0.0213 \ll \frac{1}{2} \quad (4.80)$$

This result tells us that we have to consider the solution with the minus sign. Thus we can rewrite the previous system of equations, which fixes the value of the volume and $\tau_+$ uniquely, as:

$$\begin{cases} \mathcal{V}(\tau_+) = \left(\frac{\mu\sqrt{\tau_+}}{\lambda}\, e^{2\pi\tau_+}\right)\left[\left(1-\frac{\tilde{\epsilon}\, e^{2\pi\tau_+}}{\mu\tau_+}\right) - \sqrt{\left(1-\frac{\tilde{\epsilon}\, e^{2\pi\tau_+}}{\mu\tau_+}\right)^2 - \frac{3\nu\lambda}{\mu^2\tau_+^{\frac{3}{2}}}}\,\right] \\ \mathcal{V}(\tau_+) = \left(\frac{1-2\pi\tau_+}{\frac{1}{2}-4\pi\tau_+}\right)\frac{\mu\sqrt{\tau_+}}{\lambda}\, e^{2\pi\tau_+} \end{cases} \tag{4.81}$$

As before, such equations do not take into account the NS5-brane contribution, but they give us a reasonable relation between the total volume and the size of the 4-cycle $\tau_+$. In fact, the first relation gives us a more precise estimate of the value of the volume, while the second is the same as before where we have not approximated $2\pi\tau_+ \gg 1$, and that is straightforward since the uplifting term does not depend explicitly on $\tau_+$.

At this point we have to fix the values for $\epsilon$ and $\tilde{\epsilon}$. Hence, let us retake into account the full potential, assuming the previous relation between $\mathcal{V}$ and $\tau_+$:

$$\begin{aligned} V(\tau_+,\phi) &= \frac{\lambda}{\mathcal{V}}\sqrt{\tau_+}e^{-4\pi\tau_+} - \frac{\mu}{\mathcal{V}^2}\tau_+e^{-2\pi\tau_+} + \frac{\nu}{\mathcal{V}^3} + \frac{\epsilon\sqrt{2\tau_+}}{\mathcal{V}^2}\sqrt{1+\frac{4\pi^2 g_s\mathcal{V}}{\tau_+^{3/2}}\phi^2} + \frac{\tilde{\epsilon}}{\mathcal{V}^2} \\ \text{where } \mathcal{V}(\tau_+) &= \left(\frac{1-2\pi\tau_+}{\frac{1}{2}-4\pi\tau_+}\right)\frac{\mu\sqrt{\tau_+}}{\lambda}e^{2\pi\tau_+} \end{aligned} \tag{4.82}$$

In order to fix these quantities, we can impose the conditions that $V(\tau_+^{min}, \phi = 0) = 0$ and that the potential has a minimum in $\tau_+$ up to $V(\tau_+^{inf}, \phi = 11)$ where the minimum turns into an inflection point.

The numerical values that we have found are $\epsilon = 3.1 \times 10^{-7}$ and $\tilde{\epsilon} = 0.00449811$. If we want our model also to match the observed properties of the spectrum of primordial perturbations, we have to require also that $V(\tau_+^*, \phi^*) \sim 2\times 10^{-9}$ where $*$ denotes the field values at CMB horizon exit. However, in our case we arrive just to $V = 6.38825 \times 10^{-10}$. In order to satisfy this phenomenological requirement, we can set $A_+ = W_0 = \sqrt{2}$. Given that we have a quadratic dependence in the potential, we will increase its value almost by a factor of 4, readjusting also the warping factors. In fact, we can find that for $\epsilon = 6.5 \times 10^{-6}$ and $\tilde{\epsilon} = 0.0179924$ we obtain a potential at horizon exit of order $V \sim 1.4 \times 10^{-9}$. We will not proceed further in the refinement of this result since one should consider the fluxes in order to give values to $g_s$, $A$ and $W$, and we are supposing the most simple example that we can consider. However Fig. 4.2 shows how, by the use of the uplifting term, we have increased the field range up to the desired value to realize a working inflationary model.

After moduli stabilization, all the flat directions have become a valley in which the inflaton field $\phi$ can slow-roll and drive inflation as shown in Fig. 4.3.

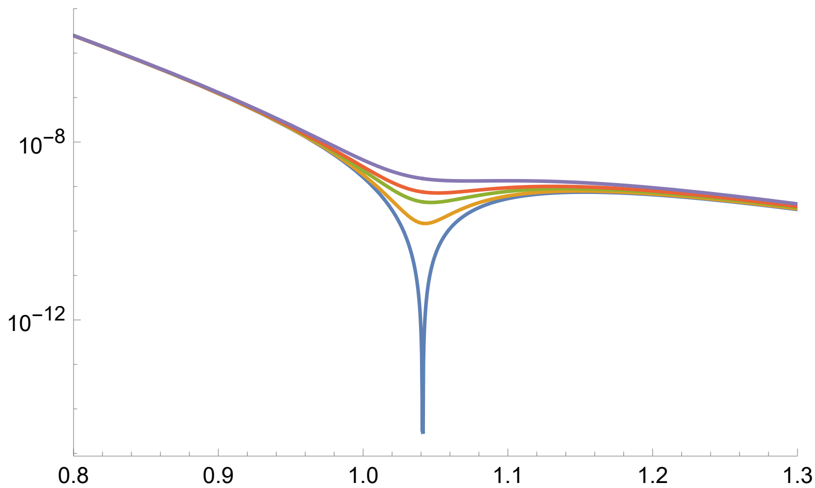


Figure 4.2: Logarithmic plot of $V(\tau_+, \phi)$ between $\phi = 0$ and $\phi = 11$ (which are the upper and lower blue lines)

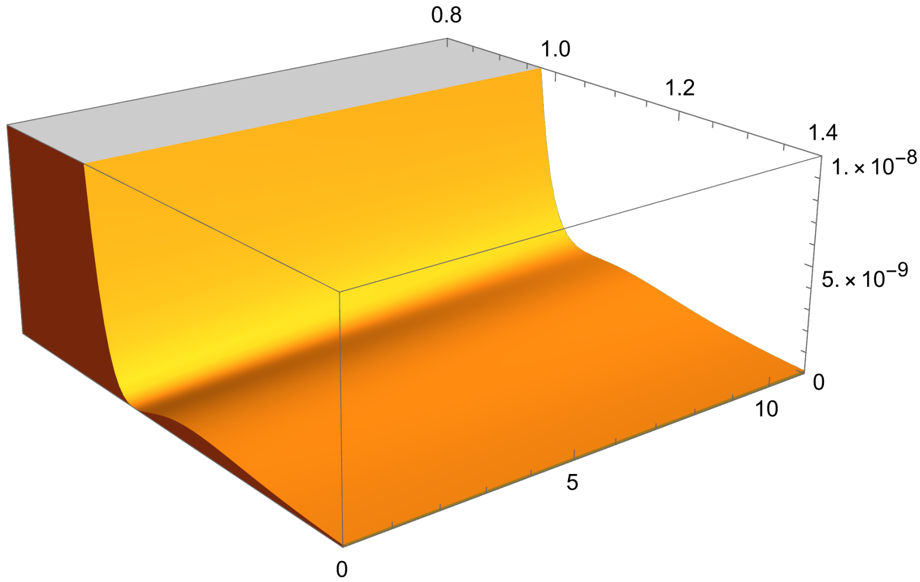


Figure 4.3: 3D plot of the inflationary potential $V(\tau_+, \phi)$ where $\tau_+$ is on the $x$-axis and $\phi$ on the $y$-axis.

## 4.6 Flattening

When we are referring to **flattening** [Don+11; Lan+17] in the context of axion monodromy, we refer to the fact that the "inflation valley" that we have seen before is not a straight line, as we expect from an asymptotically linear potential, but it is slightly

curved. The reason behind this phenomenon is the fact that the potential for $\tau_+$ tends to increase while the value of $\phi$ increases, until it decompactifies. That means that the values of the negative exponentials in the LVS part of the potential, which we have considered to be the dominant part of the potential, decrease a bit the asymptotically linear behaviour of the axion potential, and this effect increases until the cycle is destabilized. In other words, this effect is given by the back-reaction of the NS5-brane which varies a bit the size of the 4-cycle.
To be more clear, indicating with $\tau_+^{min}$ the value of $\tau_+$ which minimizes the potential, we have:

$$\tau_+^{min} = \tau_+^{min}(\phi) \qquad \text{and} \qquad V(\tau_+, \phi) = V(\tau_+^{min}(\phi), \phi) = V(\phi) \tag{4.83}$$

In Fig. 4.4 we see clearly this effect, by comparing $V(\phi)$ to the black line, which is $V(\tau_+^{min}(0), \phi)$. To be more precise the red line comes from an interpolation of points obtained by setting a value for $\phi$, calculating the value of $\tau_+$ which minimizes the potential, and then evaluating the potential for such values of $\phi$ and $\tau_+$.

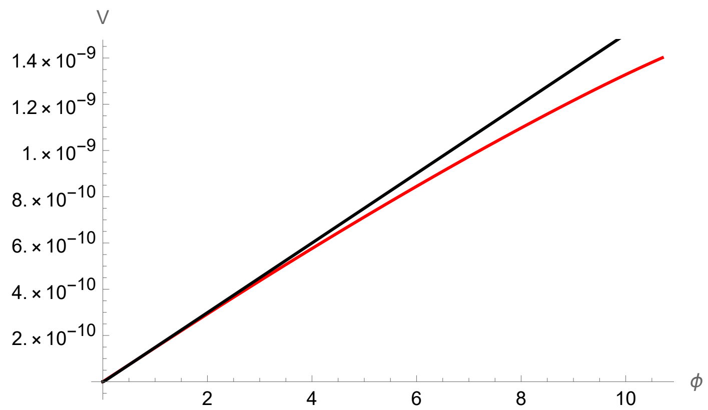


Figure 4.4: The plot in red corresponds to $V(\tau_+(\phi), \phi)$, that is the value of the potential obtained by setting $\tau_+$ to its minimal value which depends on the choice of $\phi$; the plot in black is instead the value of the potential with the volume set to its value at the minimum where $\phi = 0$.

Having the back-reacted potential $V = V(\phi)$ allow us to compute numerically all the parameters relevant for inflation. Let us start from the slow roll parameter $\epsilon_V$ in Planck units, from which we can calculate the number of e-foldings:

$$\epsilon_V(\phi) = \frac{1}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \qquad \Rightarrow \qquad N = \int_{\phi_i}^{\phi^*} \frac{d\phi}{\sqrt{2\epsilon_V(\phi)}} \tag{4.84}$$

where $\phi_*$ is the point where we reach 60-efoldings. From a numerical calculation we find that in our model $\phi_* \simeq 10.366$ which, as expected, is smaller than the 11 because of the flattening effect.

Now that we have calculated $\phi_*$ we can furthermore calculate the amplitude of the scalar perturbations recalling the relation between the Hubble constant $H$ and $V(\phi)$ in Planck units in the slow-roll approximation:

$$H^2(\phi) \sim \frac{V(\phi)}{3} \qquad \Rightarrow \qquad A_s = \frac{H^2(\phi_*)}{8\pi^2\epsilon(\phi_*)} \sim \frac{V(\phi_*)}{24\pi^2\epsilon_V(\phi_*)} \simeq 2.0044 \times 10^{-9} \ . \tag{4.85}$$

This is consistent with the observed value up to 2 error-bars; we recall that $A_s = (2.098 \pm 0.023) \times 10^{-9}$. Moreover, we compute the other slow-roll parameter $\eta_V$ and the scalar spectral index as:

$$\eta_V(\phi) = \frac{V_{,\phi\phi}}{V} \quad \Rightarrow \quad n_s = 1 - 2\epsilon_* - \eta_* = 1 - 6\epsilon_V(\phi_*) + 2\eta_V(\phi_*) \simeq 0.966 \tag{4.86}$$

As we could have expected from the previous results, this is in agreement with the measured value of $n_s = 0.9603 \pm 0.0073$.

We also calculate the tensor-to-scalar ratio, finding:

$$r = 16\epsilon_V(\phi_*) \simeq 0.046\,. \tag{4.87}$$

This value is comparable to the size of the current experimental upper bound $r < 0.038\,(95\%)$, but still not acceptable. We may at first try to argue that this problem comes from the fact that we have not chosen the right parameters, e.g. in the choice of the values of $A_+$ or $W_0$. However, by varying parameter choices we find that this problem persists even in more refined models. A possible way to address this issue discussed in the literature [DKL18; DKW21; DKW22] is to consider higher derivative kinetic terms which additionally suppress the value of $r$. The application of this possibility to our present model we leave for future work.
However is remarkable that neglecting the flattening, we would have obtained a value of $\simeq 0.067$, which cannot be fixed by taking into account the already mentioned corrections.

# Conclusions

In this thesis we have shown, supporting our ideas with precise calculations, that the class of models of large field inflation coming from string compactifications provide us a viable solution for inflation. In particular, we have shown that if we take into account axion monodromy, we can reconstruct the inflationary dynamics with almost no fine-tuning of the parameters involved in the description of the model.
Furthermore, we have shown that is possible to stabilize the Kähler moduli coming from a type IIB orientifold compactification not only in the KKLT-framework [Kac+03], in which the fluxes have to stabilize the value of the superpotential $W_0 \ll 1$, but also in the large volume scenario in which we also include perturbative corrections to the Kähler potential.

We claim that we have obtained a viable class of models since, with a very simple and general model, we have found a good agreement with the current experimental data, without strong fine-tuning. We recall that we have obtained a value of $A_s = 2.0044 \times 10^{-9}$ for the amplitude of the scalar primordial perturbations, with a spectral index $n_s = 0.966$, and a tensor-to-scalar ratio $r = 0.046$.
From such values, we may conclude that $A_s$ and $n_s$ can be compatible with the observations (after a more refined analysis of background fluxes), while the value of $r$ is larger than present cosmological bounds from observations.

In fact, we may argue that we can be satisfied by the results obtained by our simple model, but at the same time we have understood that the limitation of this study stays also in its simplicity. To be more explicit, we have neglected higher instantons corrections or higher derivative kinetic terms contributions, e.g. coming from one-loop calculations. The introduction of these corrections could improve the agreement with observations, especially regarding the amplitudes of the tensor modes.

Our work represents however a very important step forward in this direction, since it provides the first explicit embedding of axion monodromy inflation in a model with full moduli stabilization following the LVS procedure and realizing a flattening effect.

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