Alma Mater Studiorum · University of Bologna

School of Science
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Master Degree in Physics

HAWKING RADIATION AND REGULARITY OF QUANTUM STATES IN TWO-DIMENSIONAL BLACK HOLES SPACETIMES

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Abstract

This Master's Thesis aims to investigate the regularity properties of quantum vacuum states across the horizon in various spacetime geometries. One of the most captivating phenomena in theoretical physics is the Hawking effect, the emission of thermal radiation by a black hole. A straightforward approach to derive this phenomenon involves examining the interplay between distinct quantum states defined in a curved spacetime.

To ensure an accurate description of the field, the quantum state should exhibit regularity throughout the entire spacetime. A valuable method for studying state behavior involves analyzing the expectation value of the energy-momentum tensor. Following a detailed exposition of this analytical procedure within the familiar Schwarzschild spacetime, attention turns to the Reissner-Nordström spacetime, where the construction of a regular quantum state is found to be not possible.

Concluding the investigation, we introduce the concept of regular black holes, characterized by spacetimes without physical singularities while still retaining essential black hole attributes. Our analysis centers on the Simpson-Visser spacetime, where the construction of a vacuum state exhibiting regularity across the entire spacetime is achievable.

Introduction

When we think about a black hole we picture a body so extremely dense that nothing can escape from its gravitational field. This concept of black hole was first introduced in the 18th century by John Michell and Pierre-Simon Laplace.
However, substantial developments have occurred since then. In 1915, Einstein formulated the theory of General Relativity, and just after a couple of month the first solution was discovered. This solution describes the Schwarzschild black hole. The associated metric has two singularities, the physical one at the origin, while the other at the event horizon $r_S = 2M$, where M is the mass of the source generating the gravitational field. Examining the causal structure of this spacetime we find that light within the region enclosed by the event horizon cannot escape. There is no way to know what happens inside the horizon. The region inside the horizon is causally disconnected from the rest of the spacetime, and such behavior is a common feature among various black hole solutions, such as the Reissner-Nordström or Kerr black hole.

The theory of Einstein is a classical theory, and in our pursuit of a unified theory of all physical phenomena, we should obtain a quantum theory of gravity. Such goal has not been achieved yet.
Nevertheless, we can still introduce some quantum aspects in Einstein's theory. This can be done in the framework of *quantum field theory in curved spacetime*, in which we quantize the matter fields in a fixed curved backgroud. This is ofcourse just an approximation, but still allows us to obtain some extremely interesting results, such as the *Hawking radiation*, the emission of thermal radiation by a black hole.
A simple but effective way to derive this result involves studying how different quantum states are related between themselves, which can be done using the Bogoljubov transformations.
In order to correcly describe the state of a physical system, such states should be regular. The aim of this thesis is study this regularity.

This work is structured as follows:

- **Chapter 1**: We provide an overview of the key elements of quantum field theory in flat spacetime and its generalization in curved spacetime. We discuss the absence of a unique vacuum state and introduce the need for the Bogoljubov transformations;

- **Chapter 2**: We use the framework developed previous chapter to derive the Hawking radiation. For mathematical simplicity, we assume a Schwarzschild backgroud and introduce two vacuum states, Boulware and Unruh. After the computation of the Bogoljubov coefficients, we look into the physical meaning of the Hawking radiation, such as black hole thermodynamics;

- **Chapter 3**: We establish the framework for studying the regularity of the quantum states. This is done by studying the behavior of the expectation value of the energy-momentum tensor. We derive the components of the energie-momentum tensor for a conformally flat two-dimensional spacetime and assuming a non-vanishing trace anomaly;

- **Chapter 4**: We apply the formalism of Chapter 3 to the simple and well-known Schwarzschild spacetime. The physical meaning of the Boulware and Unruh is discussed, and we also introduce a new vacuum state, the Israel-Hartle-Hawking vacuum, as a vacuum state regular everywhere;

- **Chapter 5**: The same analysis is done, but in a Reissner-Nordström spacetime. The presence of two horizons with different values of surface gravity will not allow the creation of a regular vacuum state;

- **Chapter 6**: We introduce a new metric, the Simpson-Visser metric, which is shown to be a black hole metric without geometrical singularities. Some of the key properties are exposed;

- **Chapter 7**: We study the regularity of the three vacuum states in the new Simpson-Visser metric and show that it is possible to build a vacuum state regualar everywhere.

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Notation and Conventions

Here are some notations and conventions used in this work.

The metric is taken with signature $(+,-,-,-)$, so for example the Minkowski metric $\eta^{\mu\nu}$ will take values

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

Greek indices take values $0, 1, 2, 3$, while Roman indices $1, 2, 3$.
We will assume a natural set of units in which $c = \hbar = 1$, which might be reinstated if necessary.
Usual differentiation is denoted by

$$\frac{\partial}{\partial x^\mu} = \partial_\mu = {}_{,\mu} \tag{2}$$

while covariant differentiation by

$$\nabla_\mu = {}_{;\mu}. \tag{3}$$

Quantum Field Theory in Curved Spacetime

1 Quantization in Curved Spacetime

1.1 Why Quantum Field Theory in Curved Spacetime?

Before attacking this problem it is important to point out why we are interested in this topic and what we mean by quantization in curved spacetime.
As widely known, we identify in nature four elemental interactions. Three of them, electro-magnetism and strong and weak interactions have a consistent quantum description, in particular are *Quantum Field Theories* (QFT), a theoretical framework that combines special relativity, classical field theory and quantum mechanics.

The fourth interaction, gravity, is often neglected in the context of particle physics, since the characteristic energies of the problem are much smaller than the Planck mass ($M_P \sim 10^{19}\, GeV$).
At the classical level, our best understanding of gravity is given by the theory of general relativity, and extending the quantum framework to this theory is an extremely difficult problem.
By QFT in curved spacetime we mean a theory in which matter and gauge field are quantized, but the metric field is still treated as a classical field. We are studying quantum fields propagating on a fixed classical field describing a curved spacetime.

It is important to remind that in this way we will not obtain a full theoretical picture.

1.2 Quantization in Minkowski Spacetime

Before dealing with the more interesting and involved case of quantization on a curved spacetime, it is useful to review how we can quantize a field in a flat spacetime.
To this purpose consider a massive scalar field φ in Minkowski spacetime, with the line element given by

$$\mathrm{d}s^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

The classical equation of motion is given by the Klein-Gordon equation

$$(\Box + m^2)\varphi(x) = 0 \tag{1.2}$$

where $\Box$ denotes as usual the d'Alembert operator. The Klein-Gordon equation can be derived by varying the Klein-Gordon action

$$\mathcal{S} = \int d^4x \frac{1}{2}\Big(\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - m^2\varphi^2\Big) \tag{1.3}$$

which is invariant under Poincaré transformations

$$x^\alpha \to \Lambda^\alpha{}_\beta x^\beta + a^\beta. \tag{1.4}$$

The solutions to the Klein-Gordon equation can be written in terms of plane waves

$$u_k(t, \vec{x}) \propto e^{-ik_\alpha x^\alpha} \tag{1.5}$$

where $k^\alpha = (\omega, \vec{k})$ and we have the dispersion relation relating wave number to its frequency

$$\omega = \sqrt{k^2 + m^2} > 0. \tag{1.6}$$

Solutions of this form are called *positive frequency solutions* with respect to t. It is easy to see that the solution is the eigenfunction of the operator $\xi^\alpha\partial_\alpha$, where ξ^α is the Killing vector associated with time translation. In particular, we have

$$\xi^\alpha\partial_\alpha u_k(t, \vec{x}) = -i\omega u_k(t, \vec{x}). \tag{1.7}$$

But we also know that the momentum operator is defined as $\hat{p}_\mu = -i\partial_\mu$, so we immediately realize

$$p_0 = \omega > 0. \tag{1.8}$$

Another important concept, which we will use also in the quantization in curved spacetime, is the *scalar product*. Considering two solutions to the Klein-Gordon equation $f_{1,2}$ we can define[1]

$$(f_1, f_2) = -i \int_{t=const} \mathrm{d}^3x \Big[f_1 \partial_t f_2^* - (\partial_t f_1) f_2^* \Big]. \tag{1.9}$$

Scalar product between two solutions is conserved and can be used to normalize solutions, for example

$$(u_p, u_q) = \delta^3(\vec{p} - \vec{q}) \tag{1.10}$$
$$(u_p^*, u_q^*) = -\delta^3(\vec{p} - \vec{q}) \tag{1.11}$$
$$(u_p, u_q^*) = 0. \tag{1.12}$$

Positive (negative) frequency solutions have positive (negative) norm[2], and these solutions form an orthonormal base.

1.2.1 Quantization

Quantization is performed by promoting fields to operators, $\varphi \rightarrow \hat{\varphi}$, imposing the field to satisfy the classical Klein-Gordon equation and imposing equal-time bosonic commutation relations

$$[\hat{\varphi}(t, \vec{x}), \hat{\varphi}(t, \vec{y})] = [\hat{\pi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] = 0 \tag{1.13}$$

and

$$[\hat{\varphi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] = i\delta^3(\vec{x} - \vec{y}) \tag{1.14}$$

where as usual we have the conjugate field to $\hat{\varphi}$

$$\hat{\varphi} = \frac{\partial \mathscr{L}}{\partial(\partial_t \varphi)} = \partial_t \hat{\varphi}. \tag{1.15}$$

The field can be expanded using the orthonormal base of positive and negative frequency solutions and ladder operators

$$\hat{\varphi} = \sum_k \left[\hat{a}_k u_k + \hat{a}_k^\dagger u_k^* \right]. \tag{1.16}$$

Using Eq. 1.13 and Eq. 1.14 we can derive the commutation relations for the ladder operators.

[1] Recall, in Minkowski spacetime, the surface $t = const$ is a Cauchy surface.
[2] This will not be true in general in curved spacetime.

The operators $\hat{a}_k, \hat{a}_k^\dagger$ allow us to construct the Fock space and give a particle interpretation. We define the vacuum state as the state annihilated by $\hat{a}_k$, ie

$$\hat{a}_k|0\rangle = 0 \tag{1.17}$$

while applying $\hat{a}_k^\dagger$ on the vacuum we get excitations of the field.
Particles are associated to unitary irreducible representations of the Poincaré group, so we can identify excitations as particles.

This already shows one of the main problems we have in quantizing a field in curved spacetime. In such a case we do not have anymore invariance under the Poincaré group, so we cannot properly give a particle interpretation of the Fock space.

1.3 Quantization in Curved Spacetime

We are now ready to move to curved spacetime. Many concepts defined before for quantization in flat spacetime will also be present here, such as positive and negative frequency solutions and scalar product.

1.3.1 Classical Field Theory in Curved Spacetime

Given the Klein-Gordon action Eq. 1.3, it is easy to move to a curved spacetime using the general covariance principle. According to this we just need to replace the flat Minkowski metric with a general metric $g^{\mu\nu}$ and replace usual derivatives with covariant derivatives, which, for scalar fields, are just the usual derivative.
The Lagrangian density[3] for a massive scalar in curved spacetime reads

$$\mathscr{L} = \frac{1}{2}\sqrt{g}\Big(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - m^2\varphi^2 - [m^2 + \xi R]\varphi^2\Big). \tag{1.18}$$

In this Lagrangian we also included the coupling between the gravitational field and the scalar with the term $\xi R\varphi^2$, where R is the Ricci scalar curvature and ξ is a numerical factor. We distinguish:

1. $\xi = 0$ is called minimal coupling;

[3]We will often denote this quantity as Lagrangian. To be properly correct, the Lagrangian is the quantity which integrated over time gives the action, while the Lagrangian density has to be integrated over space and time to yield the action.

2. $\xi \neq 0$ is called non-minimal coupling;

3. $\xi = 1/6$ is the so-called *conformal coupling*. In this case for $m = 0$ the theory is invariant under conformal transformations
$$g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) \tag{1.19}$$
where $\Omega(x)$ is a real, non-vanishing, finite function.
Note: in a Schwarzschild spacetime, we have $R = 0$, so the Lagrangian for conformal and non-minimally coupling are the same, but we will get terms depending on ξ in the energy-momentum tensor.

Solving the Euler-Lagrange equation for this Lagrangian yields the generalization to the Klein-Gordon equation in a curved spacetime
$$\left(\hat{\Box} + m^2 + \xi R\right)\varphi = 0, \qquad \hat{\Box} = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g} g^{\mu\nu}\partial_\nu\right) \tag{1.20}$$
which is invariant under general coordinate transformations.
For the scalar product we have instead
$$(\varphi_1, \varphi_2) = -i\int_\Sigma \mathrm{d}\Sigma^\mu \sqrt{g_\Sigma}\left(\varphi_1 \partial_\mu \varphi_2^* - (\partial_\mu \varphi_1)\varphi_2^*\right) \tag{1.21}$$
where Σ is a Cauchy surface, $\sqrt{g_\Sigma}$ the determinant of the induce metric on the surface and $d\Sigma^\mu = d\Sigma n^\mu$, n^μ the normal to the surface. The scalar product does not depend on the Cauchy surface on which we evaluate the integral.

In short, a Cauchy surface is a surface for which for every point in the past and future of the surface, the lightcone of that point intersects the surface. A manifold with at least one Cauchy surface is called *globally hyperbolic*. An example is Schwarzschild spacetime, see Fig. 1.1 for a pictorial description. On the contrary, Reissner-Nordström does not have any Cauchy surface, see Fig. 1.2.
Since we will need to normalize solutions, we will assume to be in a globally hyperbolic manifold.

1.3.2 Quantization

The quantization procedure is similar to the case of flat spacetime: promote fields to operators, impose the fields to satisfy the classical Klein-Gordon equation in curved spacetime and impose commutation relation on a Cauchy surface
$$[\hat{\varphi}(x), \hat{\varphi}(y)]_\Sigma = [n^\mu \partial_\mu \hat{\varphi}(x), n^\mu \partial_\mu \hat{\varphi}(y)]_\Sigma = 0 \tag{1.22}$$
$$[\hat{\varphi}(x), n^\mu \partial_\mu \hat{\varphi}(y)]_\Sigma = \frac{i}{\sqrt{g_\Sigma}}\delta^3(\vec{x} - \vec{y}). \tag{1.23}$$

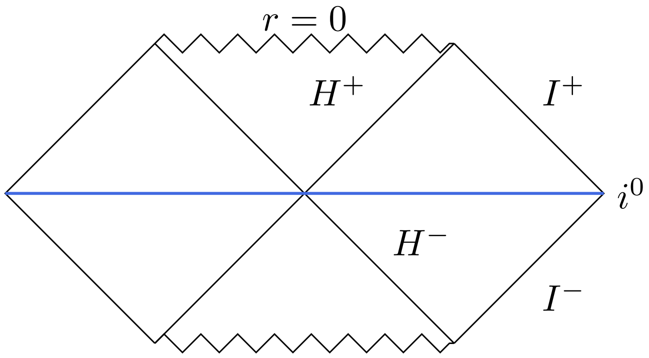


Figure 1.1: The blue line represents a good Cauchy surface for the Schwarzschild spacetime. For every point in the spacetime, its past or future lightcone intersects the Cauchy surface.

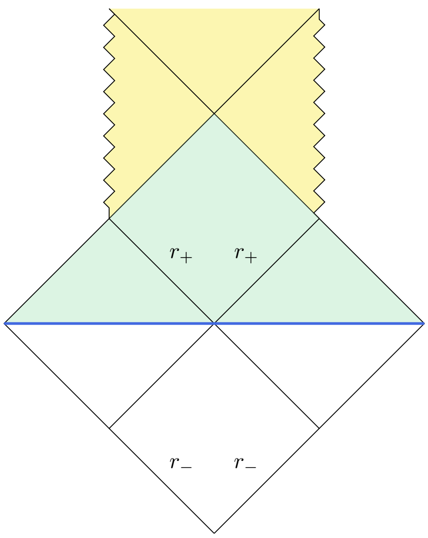


Figure 1.2: The blue line represents a similar surface to the Cauchy surface of Schwarzschild. For the points in the yellow region, the lightcones intersects the singularity. We cannot completely predict what happen in this region by giving initial data on the blue surface.

In full analogy to the previous case, we can expand the field in terms of ladder operator and positive and negative frequency solutions, which are normalized as in the flat spacetime case

$$\hat{\varphi} = \sum_i (\hat{a}_i u_i + \hat{a}_i^\dagger u_i^*) \tag{1.24}$$

$$(u_i, u_j) = -(u_i^*, u_j^*) = \delta_{ij}, \qquad (u_i, u_j^*) = 0. \tag{1.25}$$

One of the big differences with respect to the quantization in flat spacetime concerns particle interpretation. As already mentioned, in Minkowski spacetime, the association between Fock states and particle states is possible thanks to the symmetry group of the theory, the Poincaré group.

In a generic curved spacetime the symmetry group is no longer Poincaré, the theory is now symmetric under general coordinate transformations, and so we cannot give to the Fock states an interpretation in terms of particles. Nonetheless, we will still call the state $|0\rangle$ such that $\hat{a}|0\rangle = 0$ as vacuum, and similar for the excitations, but it is important to remember that this state cannot be considered as the state with no particle.

Another big difference between quantum field theory in flat and curved spacetime is the absence of a unique notion of the vacuum state. In general, there is no privileged coordinate system, no natural set of modes to expand the field and no natural way to define a vacuum state. This is indeed the spirit of general relativity and the principle of general covariance: coordinate systems are physically irrelevant.

A way to overcome this problem is to consider a stationary spacetime. In such spacetime we have a timelike Killing vector ξ^α and the infinitesimal transformation associated with such vector leaves the metric invariant

$$\delta_{\xi^\alpha} g_{\mu\nu} = 0. \tag{1.26}$$

So we can define a *Killing time* t such that $\xi^\alpha \nabla_\alpha t = 1$, and in this case we can define modes which are positive frequency with respect to this Killing time.

In a more physical situation, such as gravitational collapse, the spacetime is not stationary. What we can do is to consider a so-called *sandwich spacetime*, a spacetime which is stationary before and after some time t_1 and t_2, see Fig. 1.3. Asymptotically in the past and in the future we will recover a flat spacetime, and therefore have a particle interpretation.

Having two stationary regions we also have two Killing vectors, which might be different. Since Killing vectors are used to define positive and negative frequency solutions, a positive frequency solution in the past might not be positive in the future.

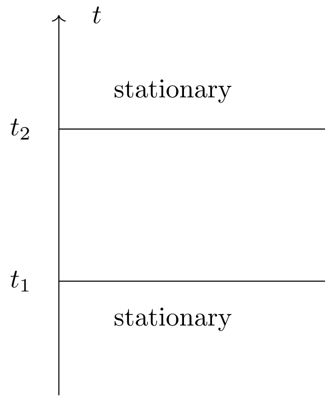


Figure 1.3: Sandwich spacetime.

1.3.3 Bogoljubov Transformations

Let us consider for a moment the *sandwich spacetime* we talked about before.
In each of the two stationary regions we can construct a set of modes, let us call them u_i and $\bar{u}_i$. These modes are both solutions to the Klein-Gordon equation, so they form an orthonormal basis, and they can both be used to expand the field. There is no a-priori criteria to pick a set over the others. So we might ask *are the two sets of modes related*?
To answer this question, we note that, being both u and $\bar{u}$ a full set of modes, we can expand one in terms of the other

$$\bar{u}_i = \sum_j \left(\alpha_{ji} u_i + \beta_{ji} u_i^*\right) . \tag{1.27}$$

Since the basis is complete only if we consider both positive and negative frequency solutions, we find that a positive frequency solution (the "barred" one) is a combination of both positive and negative frequency solutions ("unbarred").
The transformation Eq. 1.27 relates two set of modes, and it is called *Bogoljubov transformation* (Bogoljubov, 1958).
It is easy to see that if $\beta_{ij} = 0$, the Bogoljubov transformation is just a rotation. In this case the transformation is called trivial.

A similar transformation can be obtained to write the "barred" modes in terms of the "unbarred" modes

$$u_i = \sum_i \left(A_{ij} \bar{u}_j + B_{ij} \bar{u}_j^*\right) . \tag{1.28}$$

Using the scalar product and its properties we find a relation between the Bogoljubov coefficients

$$(u_i, \bar{u}_j) = A_{ij} = (\bar{u}_j, u_i)^* = \alpha^*_{ji} \qquad \to A_{ij} = \alpha^*_{ji} \tag{1.29}$$
$$(u_i, \bar{u}^*_j) = -B_{ij} = (\bar{u}^*_j, u_i)^* = \beta_{ji} \qquad \to B_{ij} = -\beta_{ji}. \tag{1.30}$$

Similarly, we can also find a transformation between ladder operators (in the following, the hat symbol to denote an operator will be understood)

$$a_i = \sum_j \left(\alpha_{ji}\bar{a}_j + \beta^*_{ji}\bar{a}^\dagger_j\right) \tag{1.31}$$
$$\bar{a}_j = \sum_i \left(\alpha^*_{ji}\bar{a}_i - \beta_{ji}\bar{a}^\dagger_i\right). \tag{1.32}$$

In analogy to the modes, we find that the creation, or annihilation, operator, is a linear combination of both ladder operators[4].

We finish this subsection on Bogoljubov transformations with two observations:

1. The Bogoljubov coefficients are related. This can be easily seen by writing the "barred" modes in terms of the "unbarred" modes, using the normalization of modes and the property of the scalar product $(af, bg) = ab^*(f, g)$, where f, g are solutions to the Klein-Gordon equation and a, b are $\mathbb{C}$ numbers. We have

$$(\bar{u}_i, \bar{u}_j) = \delta_{ij} = \sum_r \left(\alpha_{ir}\alpha^*_{jr} - \beta_{ir}\beta^*_{jr}\right) \tag{1.33}$$
$$(\bar{u}_i, \bar{u}^*_j) = 0 = \sum_r \left(\alpha_{ir}\beta_{jr} - \beta_{ir}\alpha_{jr}\right) \tag{1.34}$$

   or in matrix form

$$\alpha\alpha^\dagger - \beta\beta^\dagger = \mathbb{1} \tag{1.35}$$
$$\alpha\beta^\dagger - \beta^\dagger\alpha = 0. \tag{1.36}$$

2. The vacuum with respect to a set of modes might not be a vacuum with respect to another set of modes. The basic reason for this is that, as we have already seen, creation, or annihilation, operators can be written as a combination of both ladder operators.

[4]Creation and annihilation operators are ambiguous terms since we do not have a particle interpretation.

This is clear if we evaluate the number operator built with "unbarred" operators between "barred" states, *i.e.*

$$\langle\bar{0}|N_i|\bar{0}\rangle \tag{1.37}$$

where $N_i = a_i^\dagger a_i$. Using Bogoljubov transformations it is easy to see how the operator a_i acts on $|\bar{0}\rangle$

$$a_i|\bar{0}\rangle = \sum_j \left(\alpha_{ji}\bar{a}_i + \beta^*_{ji}\bar{a}^\dagger_j\right)|\bar{0}\rangle = \sum_j \beta^*_{ji}|\bar{1}_j\rangle \tag{1.38}$$

and so

$$\langle\bar{0}|N_i|\bar{0}\rangle = \sum_j |\beta_{ji}|^2 \neq 0. \tag{1.39}$$

An observer performing measurements with respect to the "unbarred" modes will not see $|\bar{0}\rangle$ as the vacuum.

From these two observations is clear why a Bogoljubov transformation with $\beta = 0$ is called trivial: the transformation becomes just a unitary transformation, the creation (annihilation) operator is written only in terms of the creation (annihilation) operator, and $|0\rangle$ and $|\bar{0}\rangle$ are vacuum states with respect to both sets of modes.

Bogoljubov transformations offer a way to derive one of the most fascinating results of theoretical physics: *Hawking radiation*.

2 Hawking Radiation

In this chapter, we will derive the Hawking radiation and Hawking temperature in the maximally analytical extension of the Schwarzschild spacetime.

2.1 Maximally Analytical Extension of Schwarzschild Spacetime

The Schwarzschild metric, that describes the gravitational field outside a spherical charge-less mass, can be expressed as

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2). \tag{2.1}$$

The metric is singular at $r = 0$ and $r = 2M$. While the first singularity is a true geometrical singularity, the one at $r = 2M$ it a *coordinate singularity*, due to the choice of a set of coordinates not regular.
For this reason we can remove this singularity by using a different set of coordinates, built to be regular everywhere except on the physical singularity at $r = 0$. These coordinates are the Kruskal-Szekeres coordinates (U, V). To construct these coordinates we first define the Eddington-Finkelstein coordinates (u, v)

$$u = t - r^*, \qquad v = t + r^* \tag{2.2}$$

with the Regge-Wheeler coordinate r^* given by

$$\mathrm{d}r^* = \int \mathrm{d}r\, C(r)^{-1} \tag{2.3}$$

$$= r + 2M\ln\left|\frac{r}{2M} - 1\right|. \tag{2.4}$$

Now we can define the Kruskal coordinates as

$$U = \mp\frac{1}{\kappa}e^{-u\kappa}, \qquad V = \pm\frac{1}{\kappa}e^{v\kappa} \tag{2.5}$$

where $\kappa = 1/4M$ is the surface gravity. The actual sign depends on the portion of spacetime we are interested in

$$\begin{array}{ll} \text{Region R} : U < 0, V > 0 & \text{Region BH} : U > 0, V > 0 \\ \text{Region L} : U > 0, V < 0 & \text{Region WH} : U < 0, V < 0 \end{array} \tag{2.6}$$

The line element in these coordinates read

$$\mathrm{d}s^2 = \frac{2M}{r} e^{-r/2M} \mathrm{d}U \mathrm{d}V - r^2 \mathrm{d}\Omega^2. \tag{2.7}$$

It is important to mention that now r is not a coordinate, but a function of the Kruskal coordinates $r = r(U, V)$, defined implicitly by

$$e^{r/2M} \left(\frac{r}{2M} - 1 \right) = -\frac{UV}{16M^2}. \tag{2.8}$$

The Penrose diagram associated with this spacetime is shown in Fig. 2.1.

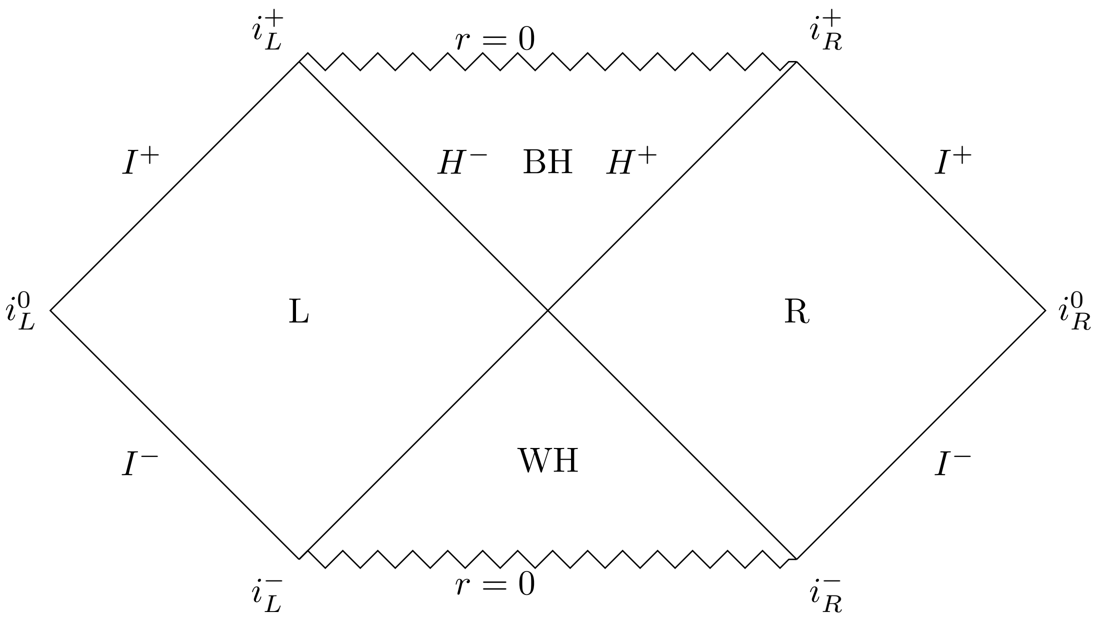


Figure 2.1: Penrose diagram of the maximally analytical extension of the Schwarzschild spacetime.

2.2 Boulware Modes

Consider a massless scalar field f, with the equation of motion given by the "covariantized" Klein-Gordon equation

$$\hat{\Box} f = 0. \tag{2.9}$$

Since Schwarzschild spacetime is spherically symmetric, we can write an ansatz of the solution using spherical harmonics

$$f = \sum_{\ell,m} \frac{f_\ell(r,t)}{r} Y_\ell^m(\theta,\varphi). \tag{2.10}$$

Substituting this in the Klein-Gordon equation we find

$$\left(\partial_t^2 - \partial_{r^*}^2 + V_\ell(r)\right) f_\ell(r,t) = 0 \tag{2.11}$$

$$V_\ell(r) = C(r)\left(\frac{2M}{r^3} + \frac{\ell(\ell+1)}{r^2}\right) \tag{2.12}$$

where

$$r^* = \int \mathrm{d}r C(r)^{-1} \tag{2.13}$$

$$= r + 2M \ln\left|\frac{r}{2M} - 1\right| \tag{2.14}$$

and $C(r) = 1 - 2M/r$. The presence of the effective potential makes it very hard to solve this equation: we should write the potential in terms of the Regge-Wheeler coordinate r^* or conversely, write the derivative with respect to r^* in terms of r. To solve this problem we make two approximations:

- s-wave: we consider only the component $\ell = 0$. This is justified by the fact that this component is the one less affected by the potential, and also we find, from numerical simulations, to be the one giving the biggest contribution to the radiation, see Fig. 2.2;

- Neglect the effective potential. This is a very crude approximation. The effect of the potential is to produce scattering, so neglecting it we basically imply that modes do not mix between themselves.
  This can be motivated by the fact that the potential vanish at the horizon, where the relevant physics is happening, so we neglect it everywhere. As we will see later, this produces a divergent emitted flux, so we will need to reintroduce it.

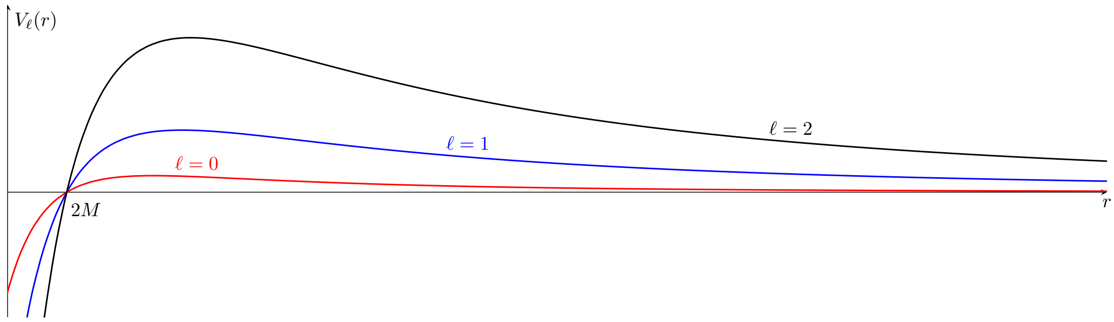


Figure 2.2: The effective potential felt by a massless scalar in Schwarzschild spacetime.
The potential for $\ell = 0$ is very weak, while increases with ℓ. This justify the s - wave approximation.

With these two approximations and using Eddington-Finkelstein coordinates, the wave equation reduces to a simple free wave equation

$$\partial_u \partial_v f(u, v) = 0. \tag{2.15}$$

A general solution can be written as

$$f(u, v) = F(u) + G(v) \tag{2.16}$$

where F, G are arbitrary functions. We can choose them in such a way that at past and future null infinity I^- and I^+, which are asymptotically flat regions, modes approach usual Minkowski modes

$$f_\omega \xrightarrow{r\to\infty} \begin{cases} \exp\{-i\omega(t-r)\}/r & \text{outgoing} \\ \exp\{-i\omega(t+r)\}/r & \text{ingoing} \end{cases} \tag{2.17}$$

This choice gives the *Boulware modes* (Boulware, 1975), which, upon normalization, are expressed as

$$f_\omega = \begin{cases} A\exp\{-i\omega u\}/r & \text{outgoing} \\ B\exp\{-i\omega v\}/r & \text{ingoing} \end{cases} \tag{2.18}$$

Associated with these modes we have a vacuum state, the Boulware vacuum, which at $I^\pm$ approaches the usual Minkowski vacuum.

2.2.1 Normalization

The normalization factors can be derived using the scalar product and imposing

$$(f_1, f_2) = \delta(\omega_1 - \omega_2). \tag{2.19}$$

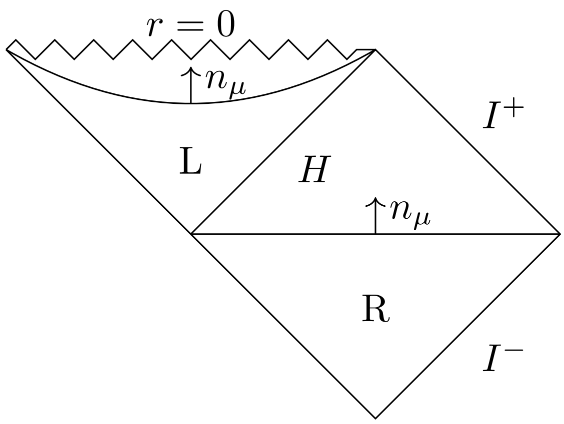


Figure 2.3: Cauchy surface of Boulware modes.

To do so we need Cauchy surfaces, which will be different depending on the sector we are considering of the Schwarzschild spacetime. To our purpose, the relevant sectors are the L and R sectors, which describe respectively the black hole and the asymptotically flat regions.

Asymptotically Flat Region

In this region, the surface $\Sigma : t = const$ is a good Cauchy surface. Pictorially, this surface is the horizontal line in the R region of Fig. 2.3.
Is a spacelike surface, so the norm, which is proportional to the derivative of the surface, is a timelike vector

$$n_\mu \propto \partial_\mu \Sigma = \alpha(x)(1, 0, 0, 0) \tag{2.20}$$

and can be normalized to 1

$$1 = g^{\mu\nu} n_\mu n_\nu \to \alpha = \pm\sqrt{C(r)}. \tag{2.21}$$

Motion is forward in time, so we take the + sign. The normal derivative becomes

$$n_\mu \partial^\mu = \frac{1}{\sqrt{C(r)}} \partial_t. \tag{2.22}$$

The induced metric is

$$\mathrm{d}s^2|_\Sigma = C(r)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 \tag{2.23}$$

which yields a determinant

$$\sqrt{g_\Sigma} = \frac{r^2 \sin\theta}{\sqrt{C(r)}}. \tag{2.24}$$

Thanks to spherical symmetry, nothing depends on the angles, so the integration on θ, φ yields the usual factor 4π.
Putting everything together

$$(f_1, f_2) = -4\pi i \int \mathrm{d}r \frac{r^2}{C(r)} \Big(f_1 \partial_t f_2^* - (\partial_t f_1) f_2^* \Big) \tag{2.25}$$

we immediately recognize the differential of the Regge-Wheeler coordinate

$$\mathrm{d}r^* = \frac{\mathrm{d}r}{C(r)} \tag{2.26}$$

so

$$(f_1, f_2) = -4\pi i \int \mathrm{d}r^* r^2 \Big(f_1 \partial_t f_2^* - (\partial_t f_1) f_2^* \Big). \tag{2.27}$$

We now need to choose an explicit form for the modes, let us first consider both f_1, f_2 to be outgoing modes, so differentiating with respect to Schwarzschild time t yields

$$\partial_t f_1 = -\frac{i\omega_1 A}{r} e^{-i\omega_1 u}. \tag{2.28}$$

The $1/r$ factor from the modes cancels out with the r^2 factor from the determinant of the metric.
In this region, the surface $t = \mathit{const}$ covers the portion $r \in [2M, +\infty[$, so $r^* \in]-\infty, +\infty[$.
It is now trivial to perform the integration

$$\begin{aligned}(f_1, f_2) = 4\pi \int_{-\infty}^{+\infty} & \mathrm{d}r^* |A|^2 (\omega_1 + \omega_2) \qquad (2.29)\\ & \times \exp\Big\{-i(\omega_1 - \omega_2)t\Big\} \exp\Big\{i(\omega_1 - \omega_2)r^*\Big\}\\ = 4\pi |A|^2 & (\omega_1 + \omega_2) \exp\Big\{-i(\omega_1 - \omega_2)t\Big\} \qquad (2.30)\\ & \times \int_{-\infty}^{+\infty} \mathrm{d}r^* \exp\Big\{i(\omega_1 - \omega_2)r^*\Big\}.\end{aligned}$$

The integral is a Dirac delta which imposes $\omega_1 = \omega_2$, and so we get

$$A = \frac{1}{4\pi\sqrt{\omega}}. \tag{2.31}$$

For the ingoing modes, we find in the same way the same normalization. We conclude that the Boulware modes in the asymptotically flat region are

$$\left\{ \frac{1}{4\pi\sqrt{\omega}} \frac{e^{-i\omega u}}{r}, \frac{1}{4\pi\sqrt{\omega}} \frac{e^{-i\omega v}}{r}; \mathrm{c.c.} \right\}. \tag{2.32}$$

Note, as for Minkowski modes, both ingoing and outgoing modes are positive norm and positive frequency.

Black Hole Region

In this region we need to distinguish between ingoing and outgoing modes.
For the ingoing modes, the surface $t = const$ is still a good Cauchy surface, and so ingoing modes are already normalized.
For outgoing modes we take instead the surface $r = const$. Pictorially, this surface is the curved horizontal line in the L region of Fig. 2.3.

It is important to remind that inside the horizon, the role of the coordinates t and r are interchanged. In this region, r play the role of a time coordinate.
The surface $r = const$ is again a spacelike surface, and its norm is timelike. The normalized normal vector is derived in the same way as for the R region, and reads

$$n_\mu = (0, -\sqrt{|C(r)|}, 0, 0). \tag{2.33}$$

When we contract the norm with the derivative, we will get derivation with respect to r. This reflects the fact that, as mentioned above, inside the horizon, t and r exchange their role.
As before we evaluate the determinant of the induced metric to

$$\sqrt{g_\Sigma} = \sqrt{|C(r)|}r^2 \sin\theta \tag{2.34}$$

and when we multiply this by the normal derivative we obtain the derivation with respect to r^*

$$\sqrt{g_\Sigma}n^\mu\partial_\mu = -|C(r)|r^2 \sin\theta\partial_r = C(r)r^2 \sin\theta\partial_r \tag{2.35}$$
$$= r^2 \sin\theta\partial_{r^*} \tag{2.36}$$

where we omitted the minus sign in front of the Schwarzschild factor $C(r)$, as well the absolute value, because inside the horizon $C(r)$ is negative.

The integration can now be performed similarly to the normalization in the asymptotically flat region: the integration over the angles is trivial, so we are left to integrate over t from $-\infty$ to $+\infty$.
The big difference with respect to the previous case is that, when we differentiate the modes with respect to r^*, we do not get anymore the minus sign. Previously, the minus sign from the differentiation canceled out with a minus sign coming from the i factors. In the black hole region this is not the case, so we get an overall minus sign.
Performing the integration we find the same normalization factor

$$A = \frac{1}{4\pi\sqrt{\omega}} \tag{2.37}$$

but now modes are normalized as

$$(f_1, f_2) = -\delta(\omega_1 - \omega_2) \tag{2.38}$$

so the outgoing modes in the black hole region are negative norm.
We can make these modes positive norm by inverting the sign in the exponential

$$f_\omega = \frac{1}{4\pi\sqrt{\omega}}\frac{e^{+i\omega u}}{r} \tag{2.39}$$

but in this way the mode is no longer positive frequency.

So, the normalized modes in the black hole region are

$$\left\{\frac{1}{4\pi\sqrt{\omega}}\frac{e^{+i\omega u}}{r}, \frac{1}{4\pi\sqrt{\omega}}\frac{e^{-i\omega v}}{r}; \text{c.c.}\right\}. \tag{2.40}$$

We stress out that inside the horizon, the Killing energy can be both positive and negative, since these modes are trapped inside.

2.2.2 Quantization

We now have a complete orthonormal base to expand fields. We can define a set of modes in both regions as

$$u_R = \begin{cases} A\exp\{-i\omega u\}/r & r > 2M \\ 0 & r < 2M \end{cases} \tag{2.41}$$

$$u_L = \begin{cases} 0 & r > 2M \\ A\exp\{i\omega u\}/r & r < 2M \end{cases} \tag{2.42}$$

$$u_I = A\exp\{-i\omega v\}/r \tag{2.43}$$

$$A = \frac{1}{4\pi\sqrt{\omega}} \tag{2.44}$$

where:

- u_R: outgoing mode for the asymptotically flat region;
- u_L: outgoing mode for black hole region;
- u_I: ingoing mode for both regions.

The Boulware base is then

$$B : \left\{u_R, u_L, u_I; \text{c.c.}\right\} \tag{2.45}$$

and a field can be expanded as

$$\varphi = \sum_{\omega}\Big(a^R_\omega u_R + a^L_\omega u_L + a^I_\omega u_I + \text{h.c.}\Big). \tag{2.46}$$

The *Boulware vacuum* is defined as the state $|\mathrm{B}\rangle$ such that

$$\begin{cases} a^R|\mathrm{B}\rangle = 0 \\ a^L|\mathrm{B}\rangle = 0 \\ a^I|\mathrm{B}\rangle = 0 \end{cases} \tag{2.47}$$

As already mentioned, Boulware modes reduce to Minkowski modes at infinity, so Boulware vacuum can be interpreted as a state which reproduce Minkowski vacuum at $I^\pm$.

By applying instead the $a^\dagger$ operator on the vacuum we build particle states

$$a^{I\dagger}_\omega|\mathrm{B}\rangle \quad : 1 \text{ particle state on } I^- \tag{2.48}$$

$$a^{R\dagger}_\omega|\mathrm{B}\rangle \quad : 1 \text{ particle state on } I^+ \tag{2.49}$$

$$a^{L\dagger}_\omega|\mathrm{B}\rangle \quad : 1 \textit{ partner} \text{ state with } < 0 \text{ Killing energy} \tag{2.50}$$

2.3 Unruh Modes

As we will see later, Boulware modes yield an expectation value for the energy-momentum tensor which diverges on the horizon, and the ultimate reason for that is a choice of coordinates, the Eddington-Finkelstein coordinates, which are singular on the horizon.
So, to solve this problem, Unruh (Unruh, 1976) proposed a set of modes built using the Kruskal and Eddington-Finkelstein coordinates. In particular, the Unruh modes are built using the ingoing Eddington-Finkelstein coordinate v and the outgoing Kruskal coordinate U. States built with respect to these modes will still approach the Minkowski vacuum at I^-, but not on I^+.

The Unruh outgoing mode is

$$u_k(\omega_k) = A\frac{1}{r}e^{-i\omega_k U} \tag{2.51}$$

where A is a normalization factor and

$$U = \pm\frac{1}{\kappa}e^{-\kappa u}. \tag{2.52}$$

The ingoing mode is the ingoing mode of Boulware modes.
We can immediately make a few observations:

- In the definition of the mode we have a term $1/r$. We are using Kruskal coordinates, so r is not a coordinate but rather a function $r = r(U, V)$;
- Kruskal's coordinate U is regular on the future horizon H^+, so these modes will be regular there;
- The modes u_k are positive frequency with respect to the Kruskal time $T = (V + U)/2$. This means that we have a Killing vector associated to it.

2.3.1 Normalization and Quantization

The procedure to find a normalization for the modes is the same as for the Boulware modes: imposing the scalar product to be a Dirac delta distribution.
The difference is that the Cauchy surfaces we used for the scalar product in the previous case were expressed in terms of cartesian coordinates r, t, but now we use the Kruskal coordinates, so we should express the equation for the surfaces in terms of these coordinates.
This procedure is algebraically lengthy, but we can use the fact that the scalar product is independent on the Cauchy surface, so we can take one which simplify the math, such as the past horizon. The drawback to this choice is that the horizon is a null surface, and scalar product on a null surface requires some more delicacy. The evaluation of the scalar product is not of our interest, so we skip the calculation and find that the normalization factor is the same as the Boulware modes.

So, the Unruh modes are given by

$$U : \Big\{ u_k, u^I_\omega ; \text{ c.c.} \Big\} \tag{2.53}$$

where

$$u_k(\omega_k) = \frac{1}{4\pi\sqrt{\omega_k}} \frac{1}{r} e^{-i\omega_k U} \tag{2.54}$$

$$u^I_\omega(\omega) = \frac{1}{4\pi\sqrt{\omega}} \frac{1}{r} e^{-i\omega v} \tag{2.55}$$

which are both positive norm.

Having a complete normalized set of modes, we can now expand our field

$$\varphi = \sum_{\omega_k} \Big(a_{\omega_k} u_k + \text{h.c.} \Big) + \sum_{\omega} \Big(a^I_\omega u^I_\omega + \text{h.c.} \Big) \tag{2.56}$$

and define the *Unruh vacuum* $|\mathrm{U}\rangle$ by

$$\begin{cases} a_{\omega_k}|\mathrm{U}\rangle = 0 \\ a^I_{\omega}|\mathrm{U}\rangle = 0. \end{cases} \tag{2.57}$$

Similarly, by applying $a^\dagger_{\omega_k}$ and $a^{I\dagger}_{\omega}$ on the Unruh vacuum we build the Fock space. Since the ingoing Unruh mode approaches the ingoing Minkowski mode, we have that $|\mathrm{U}\rangle$ can be interpreted as a vacuum state at I^-.
On the other hand, the outgoing mode do not approaches the Minkowski outgoing mode, therefore the meaning of the state $a^\dagger_{\omega_k}|\mathrm{U}\rangle$ is more involved, and will be discussed later.

2.4 Hawking Radiation

We have now all the essential ingredients to derive the thermal spectrum at the Hawking temperature emitted by a black hole. Before doing so we need to find the physical meaning of the Unruh vacuum, and to do so we need a brief review of the simplest model of gravitational collapse: the *Vaidya spacetime*.

2.4.1 The Vaidya Spacetime

The purpose of this subsection is to present the very basic element of a simple model of gravitational collapse. This is by no means a full and rigorous treatment of the subject.
The Vaidya spacetime is a simplified model of gravitational collapse. It describes a null shell of radiation free-falling at the speed of light toward the origin, see Fig. 2.4. This allows us to split the physically relevant part of the spacetime into two regions. If can consider the null shell to be located at $v = v_0$, we have (see Fig. 2.5):

- For $v < v_0$ we do not have any radiation, the spacetime is flat, which can be described by a Minkowski spacetime. In this region we can define the usual Minkowski vacuum $|\mathrm{in}\rangle$ which can be, thanks to Poincaré invariance, interpreted as a state of no particles. This portion of spacetime is called the “in” region, and we cover it with the null coordinates (u_{in}, v), defined as $u_{in} = t_{in} - r$, $v = t_{in} + r$;

- For $v > v_0$ we are instead in a portion of Schwarzschild spacetime, which describes the final black hole configuration. This portion is covered by another

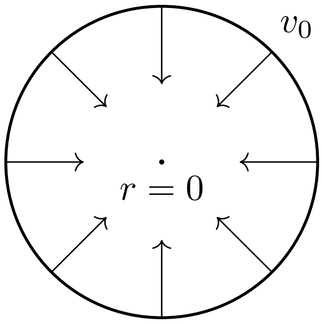


Figure 2.4: Dynamics of the Vaidya collapse. The null shell, located at $v = v_0$, collapse toward the origin.

set of Eddington-Finkelstein coordinates, which we call (u, v), and call this region the "out" region.

While the ingoing coordinate is the same, the outgoing coordinates are different in the two regions, but we can find a relation between them by matching the metrics across the shell

$$u = u_{in} - 4M \ln\left(\frac{v_0 - u_{in}}{4M} - 1\right) \tag{2.58}$$

$$= u_{in} - 4M \ln\left(-\frac{u_{in}}{4M}\right) \tag{2.59}$$

where we have taken, without loss of generality, the null shell to be located at $v_0 = 4M$.
For $u_{in} \to 0^-$ (which also mean at late time $u \to +\infty$) we find that u and u_{in} are related by a Kruskal-like relation

$$u_{in} \sim -4Me^{-u/4M} \tag{2.60}$$

so near the horizon, the outgoing coordinate for the "in" region behaves like a Kruskal coordinate.
This final result does not depend on the specific collapse model.

2.4.2 Physical Meaning of Unruh Modes

Now let's go back to the maximally analytical extension of Schwarzschild spacetime and Unruh modes.
We have seen that, in a gravitational collapse, at late time the outgoing coordinate behaves like a Kruskal coordinate.

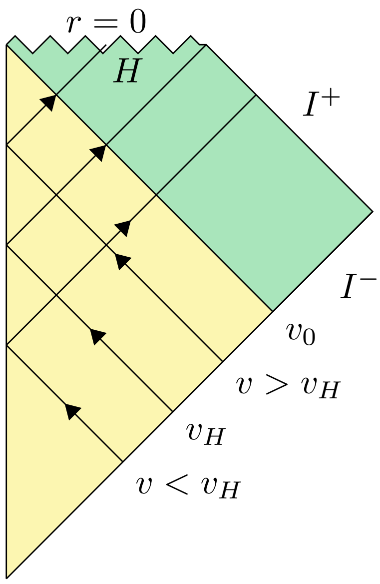


Figure 2.5: Vaidya spacetime. The green region is the Schwarzschild portion of spacetime, ie $v > v_0$, while the yellow region is the Minkowski portion, ie $v < v_0$.

This means that at late time, the maximal analytical extension of Schwarzschild spacetime with Unruh modes mimic the gravitational collapse, and so the effects produced by the collapse on the modes can be replaced by appropriate boundary condition on an empty Schwarzschild spacetime. Unruh modes can be used to describe gravitational collapse at late time $u \to +\infty$

$$\lim_{u\to\infty} \langle \text{in}|\hat{N}_\omega|\text{in}\rangle = \langle \text{U}|\hat{N}_\omega|\text{U}\rangle. \tag{2.61}$$

Besides that, Unruh modes, and in particular how they are related to the Boulware modes, can be used to derive the Hawking spectrum in the full Schwarzschild spacetime.
Indeed, suppose our system is in the initial state at I^-. Here the Unruh modes approach Minkowski modes, and the associated vacuum state $|\text{U}\rangle$ approach the Minkowski vacuum $|0\rangle$, which describes an absence of particles.
If we work in the Heisenberg picture we have that the states do not change over time, but the operators evolve. So if our system is in $|\text{U}\rangle$ at the initial time, it will be in $|\text{U}\rangle$ also at a late time on I^+, but the operators have evolved, so the vacuum at the early time might not be vacuum at late time. An interpretation in terms of particles can be obtained by studying how the Unruh vacuum is perceived by an

observer performing measurements with Boulware operators, *i.e.*

$$\langle \mathrm{U}|N_\omega|\mathrm{U}\rangle = \langle \mathrm{U}|a_\omega^{R\dagger} a_\omega^R|\mathrm{U}\rangle. \tag{2.62}$$

This can be evaluated using Eq. 1.39

$$\langle \mathrm{U}|a_\omega^{R\dagger} a_\omega^R|\mathrm{U}\rangle = \sum_{\omega_k} \left|\beta^R_{\omega_k,\omega}\right|^2 \tag{2.63}$$

so we need an explicit expression for the Bogoljubov coefficients β between Unruh and Boulware modes.

2.4.3 Evaluation of Bogoljubov Coefficients

Formally, an expression for the Bogoljubov coefficients can be obtained using the scalar product and the commutation relation of the ladder operators. In particular, in the asymptotically flat region, we have

$$\alpha^R_{\omega_k,\omega} = \Big(u_k(\omega_k), u^R_\omega\Big) \tag{2.64}$$

$$\beta^R_{\omega_k,\omega} = -\Big(u_k(\omega_k), u^{R*}_\omega\Big) \tag{2.65}$$

and similar relations in the black hole region. In the following we will write the Unruh modes simply as u_k, omitting the dependence on the frequency ω_k.[1]
We are not particularly interested in the Bogoljubov coefficient, but rather its absolute value squared and summed over one index, and to do this it will be useful the relations between Bogoljubov coefficients, Eq. 1.33 and Eq. 1.34.

So, let us start with the α coefficient. To compute the scalar product we need a Cauchy surface, in particular we use the past horizon H^-

$$\alpha^R_{\omega_k,\omega} = -i\int \mathrm{d}U\mathrm{d}\theta\mathrm{d}\varphi\Big[u_k\partial_U u^{R*}_\omega - (\partial_U u_k)\, u^{R*}_\omega\Big] r^2\sin\theta. \tag{2.66}$$

We need to pay attention to the differentiation with respect to U.
For the Unruh mode, the dependence on U is in the exponential and in the term $1/r$, since r is defined implicitly by Kruskal coordinates. Differentiating this term we will eventually get a factor which cancels out in the double product, so in a symbolic way we can write

$$\partial_U u_k = -i\omega_k u_k + \text{"}\frac{1}{r}\text{ factor"}. \tag{2.67}$$

[1] u_k: Unruh mode, u_ω: Boulware mode

For the Boulware mode it is even more involved. Since they are defined using Eddington-Finkelstein coordinates, we need to express them in terms of Kruskal coordinates. To do so we need to invert the relation between U and u

$$U = -\frac{1}{\kappa}e^{-\kappa u} \leftrightarrow u = -\frac{1}{\kappa}\ln(-\kappa U). \tag{2.68}$$

Substituting we get

$$u^{R*}_{\omega} = \frac{1}{4\pi\sqrt{\omega}}\frac{1}{r}(-\kappa U)^{-i\omega/\kappa} \tag{2.69}$$

and differentiating

$$\partial_U u^{R*}_{\omega} = \frac{1}{4\pi\sqrt{\omega}}\frac{-i\omega}{r\,-\kappa U}(-\kappa U)^{i\omega/\kappa} \tag{2.70}$$

where we omitted the contribution coming from the derivative of the $1/r$ term.

Now we can put everything together

$$\alpha^R_{\omega_k,\omega} = \frac{-i}{4\pi\sqrt{\omega}\sqrt{\omega_k}}\int_{-\infty}^{0} \mathrm{d}U \exp\{-i\omega_k U\}(-\kappa U)^{i\omega/\kappa}\left[\frac{-i\omega}{-\kappa U} + i\omega_k\right]. \tag{2.71}$$

This integral can be solved by changing to the variable $x = -\kappa U$ and recognizing the Euler Γ function

$$\begin{aligned}\int_0^{+\infty} \mathrm{d}x x^a e^{-bx} &= b^{-1-a}\Gamma(1+a)\\ &= b^{-1-a}a\Gamma(a).\end{aligned} \tag{2.72}$$

Performing the change of variable we get

$$\begin{aligned}\alpha^R_{\omega_k,\omega} &= \frac{-i}{4\pi\sqrt{\omega\omega_k}\,\kappa}\left[\int_0^{+\infty}\mathrm{d}x \exp\left\{\frac{i\omega_k x}{\kappa}\right\}\right. \qquad (2.73)\\ &\qquad \left.\times x^{-i\omega/\kappa-1}(i\omega)\int_0^{+\infty}\mathrm{d}x\exp\left\{\frac{i\omega_k x}{\kappa}\right\}x^{-i\omega/\kappa}(i\omega_k)\right]\\ &= I_1 + I_2 \qquad (2.74)\end{aligned}$$

where I_1, I_2 just denotes the two integrals.
In our case, we can recognize the Euler function in both I_1 and I_2, with different values for the parameters a, b, in particular

$$I_1: \quad a = -\frac{i\omega}{\kappa} - 1, \quad b = -\frac{i\omega_k}{\kappa} \tag{2.75}$$

$$I_2: \quad a = -\frac{i\omega}{\kappa}, \quad b = -\frac{i\omega_k}{\kappa} \tag{2.76}$$

and we get

$$I_1 = \frac{1}{4\pi\kappa}\sqrt{\frac{\omega}{\omega_k}}\left(-\frac{i\omega_k}{\kappa}\right)^{i\omega/\kappa}\Gamma\left(-\frac{i\omega}{\kappa}\right) \tag{2.77}$$

$$I_2 = \frac{1}{4\pi\kappa}\sqrt{\frac{\omega_k}{\omega}}\left(-\frac{i\omega_k}{\kappa}\right)^{i\omega/\kappa-1}\left(-\frac{i\omega}{\kappa}\right)\Gamma\left(-\frac{i\omega}{\kappa}\right). \tag{2.78}$$

Combining we eventually get

$$\alpha^R_{\omega_k,\omega} = \frac{1}{2\pi\kappa}\sqrt{\frac{\omega}{\omega_k}}\left(-\frac{i\omega_k}{\kappa}\right)^{i\omega/\kappa}\Gamma\left(-\frac{i\omega}{\kappa}\right) \tag{2.79}$$

and similar expressions for the other coefficients

$$\beta^R_{\omega_k,\omega} = \frac{1}{2\pi\kappa}\sqrt{\frac{\omega}{\omega_k}}\left(-\frac{i\omega_k}{\kappa}\right)^{i\omega/\kappa}\Gamma\left(\frac{i\omega}{\kappa}\right) \tag{2.80}$$

$$\alpha^L_{\omega_k,\omega} = \frac{1}{2\pi\kappa}\sqrt{\frac{\omega}{\omega_k}}\left(\frac{i\omega_k}{\kappa}\right)^{i\omega/\kappa}\Gamma\left(\frac{i\omega}{\kappa}\right) \tag{2.81}$$

$$\beta^L_{\omega_k,\omega} = \frac{1}{2\pi\kappa}\sqrt{\frac{\omega}{\omega_k}}\left(\frac{i\omega_k}{\kappa}\right)^{i\omega/\kappa}\Gamma\left(-\frac{i\omega}{\kappa}\right). \tag{2.82}$$

For the moment let us focus only on the R coefficients. The factor $(-i)^{i\omega/\kappa}$ can be written as an exponential, and we get

$$\alpha^R_{\omega_k,\omega} = \frac{1}{2\pi\kappa}\sqrt{\frac{\omega}{\omega_k}}\left(-\frac{\omega_k}{\kappa}\right)^{i\omega/\kappa}\Gamma\left(-\frac{i\omega}{\kappa}\right)\exp\left\{\frac{\pi\omega}{2\kappa}\right\} \tag{2.83}$$

$$\beta^R_{\omega_k,\omega} = \frac{1}{2\pi\kappa}\sqrt{\frac{\omega}{\omega_k}}\left(-\frac{\omega_k}{\kappa}\right)^{i\omega/\kappa}\Gamma\left(\frac{i\omega}{\kappa}\right)\exp\left\{-\frac{\pi\omega}{2\kappa}\right\}. \tag{2.84}$$

If we take the modulo square we find

$$\left|\alpha^R_{\omega_k,\omega}\right|^2 = \exp\left\{\frac{2\pi\omega}{\kappa}\right\}\left|\beta^R_{\omega_k,\omega}\right|^2. \tag{2.85}$$

Now we can use the "pseudo-unitarity" of the Bogoljubov transformation

$$\sum_{\omega_k}\left(\left|\alpha^R_{\omega_k,\omega}\right|^2 - \left|\beta^R_{\omega_k,\omega}\right|^2\right) = 1. \tag{2.86}$$

Using this relation and the expressions Eq. 2.83, Eq. 2.84, after some simple algebra we get the desired result

$$\sum_{\omega_k}\left|\beta^R_{\omega_k,\omega}\right|^2 = \frac{1}{e^{8\pi m\omega}-1} \tag{2.87}$$

and so, according to Eq. 2.63

$$\langle \mathrm{U}|a_\omega^{R\dagger} a_\omega^R|\mathrm{U}\rangle = \frac{1}{e^{\hbar\omega/k_B T_H} - 1} \tag{2.88}$$

where k_B is the Boltzman constant and

$$T_H = \frac{\hbar\kappa}{2\pi k_B} \tag{2.89}$$

is called *Hawking temperature.*

2.4.4 Physical Considerations

What we just showed is one of the most fascinating results of theoretical physics: a black hole is not actually black, but it emits particles, and these particles are emitted with a thermal distribution. This temperature, the Hawking temperature, depends on the surface gravity, which in turn can depend only on mass, electric and magnetic charge and angular momentum.

Hawking derived this result by studying a generic gravitational collapse. As we have seen, in that model the $|\mathrm{in}\rangle$ state approach $|\mathrm{U}\rangle$ only at late time, so in a physical scenario we measure the thermal flux of particles only on I^+.
In the maximally analytical extension instead, modes are always the Unruh modes, so we measure the flux at any time.

From what we have seen we have the creation of particles, so it makes sense to ask where these particles come from. According to general relativity, they cannot come from inside the black hole. Moreover, it seems we are violating energy conservation by creating particles of energy ω. We can answer both questions by writing the relation between Unruh and Boulware vacuum.
Since we are neglecting the effective potential, the ingoing and outgoing modes do not mix, and so we can split these vacuum states into an "in" and "out" component. For our analysis, we are interested in the "out" component. Using the Bogoljubov transformation we can write

$$|\mathrm{U}\rangle_{\mathrm{out}} \sim \exp\left\{\sum_\omega e^{-4\pi M\omega} a_\omega^{L\dagger} a_\omega^{R\dagger}\right\} |\mathrm{B}\rangle_{\mathrm{out}}. \tag{2.90}$$

This relation tells us that Hawking radiation is the creation of a pair particle-partner. The particle is created outside the horizon and it will propagate toward I^+, where is detected as Hawking radiation. The partner instead is created inside the horizon, and it is trapped inside. Since particle and partner have opposite frequencies, globally we have conservation of energy.

Hawking radiation is one of the cornerstones of *black hole thermodynamics* (Davies, 1977b). From general relativity we can derive the four laws of black hole dynamics:

- 0^{th} The surface gravity κ of a static black hole at equilibrium is constant on the horizon;
- 1^{st} The variation of mass, angular momentum and area of the horizon due to a small perturbation is $\delta m = (\kappa/8\pi)\delta A + \Omega_H \delta J$, where A is the area of the horizon, Ω_H is the angular velocity of the black hole and δJ the variation of angular momentum;
- 2^{nd} The area of the horizon never decreases: $\delta A \geq 0$;
- 3^{rd} Surface gravity cannot vanish in a finite advanced time (we cannot destroy a black hole horizon): $\kappa \not\to 0$.

Hawking showed a relation between temperature and surface gravity. Bekenstein (Bekenstein, 1972) derived, a few years before, a relation between entropy and the area of the black hole

$$S = \frac{k_B A}{4G\hbar}. \tag{2.91}$$

Moreover, we have the equality between mass and energy, therefore we have a correspondence between the laws of black holes and laws of thermodynamics

$$T \leftrightarrow \kappa \qquad S \leftrightarrow A \qquad E \leftrightarrow M. \tag{2.92}$$

We finish this section by just mentioning a technical detail regarding black hole radiation in gravitational collapse. We have found that black holes emit particles of frequency ω, or to be more specific, at I^+ we measure a flux of particles. These particles have a definite frequency ω. Definite frequency implies absolute uncertainty in time, so the expectation value of the number operator will represent the number of particles of definite frequency ω emitted at any time, but we are interested in those emitted at late time, once the black hole has settled down to a stationary configuration.
This is related to the fact that the modes we used are all built using plane waves, which are completely delocalized.
This can be solved by replacing these plane waves with wave packets of discrete

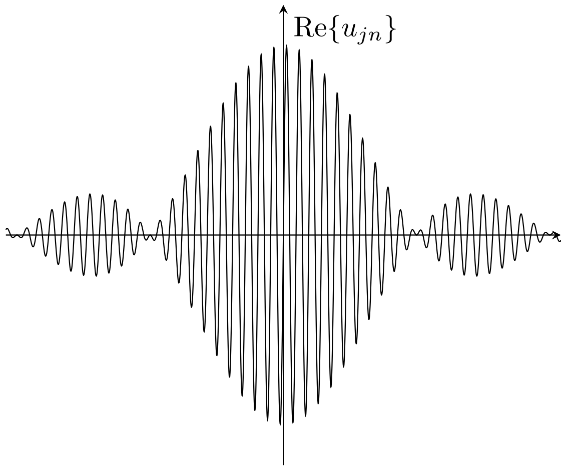


Figure 2.6: Wave packet built from a plane wave, with quantum numbers $\varepsilon = 0.5$, $j = 10$, $n = 0$.

quantum numbers

$$u_{jn} = \frac{1}{\sqrt{\varepsilon}} \int_{j\varepsilon}^{(j+1)\varepsilon} \mathrm{d}\omega e^{2\pi i \omega n/\varepsilon} u_\omega \tag{2.93}$$

where u_ω is the usual mode built with plain waves and j, n are the quantum numbers. We can take ε small to have a wave packet narrowly centered around $\omega = j\varepsilon$, see Fig. 2.6. The expectation value of the number operator built with operators defined with respect to these modes have now a clear physical interpretation.

2.4.5 Backscattering

It is interesting to notice from the number operator

$$\langle \mathrm{U} | a_\omega^{R\dagger} a_\omega^R | \mathrm{U} \rangle = \frac{1}{e^{\hbar\omega/k_B T_H} - 1} \tag{2.94}$$

that for $\omega \to 0$ we have an infinite number of particles emitted. This is clearly wrong.
The reason for this is that we neglected the effective potential

$$V_\ell = \left(1 - \frac{2M}{r}\right)\left[\frac{2M}{r} + \frac{\ell(\ell+1)}{r^2}\right]. \tag{2.95}$$

The potential mixes modes between themselves, some will be reflected, and others will be transmitted, with probability T and R such that $|T|^2 + |R|^2 = 1$. We are interested in studying how the modes which reach I^+ are affected by this scattering, so we need to find an explicit form for the probability T. To do so we will still assume $\ell = 0$.

We have seen that the spectrum diverges for small ω, which corresponds to small values of the potential. This happens in two regions: on the horizon and at infinity.
Since complete analytical solution is not possible, so we work with a technique called *asymptotically matching*. The main idea is to get a solution for $2M < r < \infty$, and then match the solution at $2M$ and ∞.

So, let us consider the wave equation for a scalar field in Schwarzschild spacetime. Thanks to spherical symmetry we can expand the field in spherical harmonics and a function of time and radius

$$f \sim \sum_{\ell,m} \frac{F_\ell(r,t)}{r} Y_\ell^m(\theta,\varphi). \tag{2.96}$$

We can also write F_ℓ as a radial function and a time phase

$$F_\ell \sim e^{-i\omega t}\chi_\ell(r). \tag{2.97}$$

We are interested in the radial part. The wave equation for χ reduces to

$$(\partial^2_{r^*} + \omega^2 + V_\ell)\chi(r) = 0. \tag{2.98}$$

For small frequency we can neglect the factor ω^2 coming from the differentiation with respect to time, so the covariant wave equation reduces to

$$\partial_r \left[r^2 \left(1 - \frac{2M}{r}\right) \partial_r \left(\frac{\chi}{r}\right)\right] = 0 \tag{2.99}$$

which can be integrated up to two constants

$$\chi = ar + br \int \frac{\mathrm{d}r^*}{r^2}. \tag{2.100}$$

To fix these constants we impose how the solution should behave at $2M$ and infinity:

- For $r \to \infty$ we have that $r^* \sim r$, so

  $$\chi = ar - b. \tag{2.101}$$

  But at infinity, modes will be of the form

  $$\chi \sim T e^{i\omega r^*} \xrightarrow{\omega \to 0} T + \omega T r^* \tag{2.102}$$

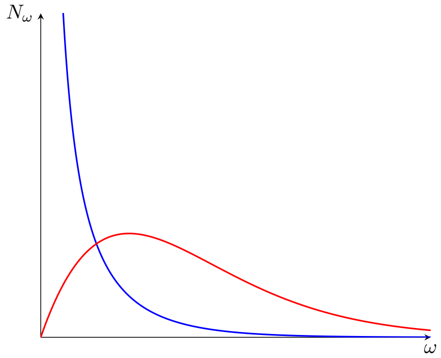


Figure 2.7: Hawking spectrum without grey factor (blue function) and with the grey factor (red function).

from which we get

$$\begin{cases} a = i\omega T \\ b = -T \end{cases} \tag{2.103}$$

- For $r = 2M$

$$\chi = 2Ma + \frac{b}{2M} r^*. \tag{2.104}$$

Here we need to consider both the contributions coming from the past and future horizon

$$\chi = e^{i\omega r^*} + Re^{i\omega r^*} \xrightarrow{\omega \to 0} 1 + R + i\omega(1 - R)r^* \tag{2.105}$$

so

$$\begin{cases} 2Ma = 1 + R \\ b = 2M\omega(1 - R) \end{cases} \tag{2.106}$$

Solving Eq. 2.105 and Eq. 2.106 we find

$$R = -\frac{1 - 4M^2\omega^2}{1 + 4M^2\omega^2} \tag{2.107}$$

and using $|T|^2 + |R|^2 = 1$ we obtain

$$|T|^2 = 16M^2\omega^2. \tag{2.108}$$

Redefining the modes $u^R_\omega \to Tu^R_\omega$ affects the Bogoljubov transformation. For the β coefficient we find

$$|\beta|^2 \to |T^2||\beta|^2 \tag{2.109}$$

and so

$$\langle \mathrm{U}|N_\omega|\mathrm{U}\rangle = \frac{16M^2\omega^2}{e^{\hbar\omega/k_B T_H} - 1} \tag{2.110}$$

which now converge for $\omega \to 0$. In Fig. 2.7 we can see how the spectrum changes with the energy ω, with and without grey factor.

Energy-Momentum Tensor and Regularity of States

3 Energy-Momentum Tensor and Trace Anomaly

In the previous chapter we have seen how, in the presence of curved spacetime, the concept of vacuum and particles do not match with the usual concept in flat spacetime. We can quantize the same spacetime with different sets of modes, which leads to the conclusion that there is no natural, privileged definition of particles. No matter how we define them, the concept of particle is connected to a particular choice of modes, which are defined *globally*, and therefore are sensitive to the large structure of spacetime. However, it is worthwhile to study the behavior of physical quantities defined locally.
An example is the energy-momentum tensor, which in this context is extremely important for two main reasons. First, it allows us to describe the physical structure of the quantum field at a specific spacetime point. Second, it acts as a source of gravity for the Einstein field equation. For this reason, having an explicit expression of the energy-momentum tensor allows us to study the back-reaction problem, determining how Hawking radiation affects spacetime.

We are interested in finding the energy-momentum tensor in order to study the regularity of the different states we introduced before. In the following chapters, we will derive an explicit expression of the energy-momentum tensor starting from the trace anomaly, and then study the regularity of the states in both Schwarzschild and Reissner-Nordström spacetime.

3.1 Trace Anomaly

As mentioned before, our derivation of the Hawking radiation relied on the approximation that the matter field we quantize does not affect the background. But this field carries energy and momentum, so it will affect the energy-momentum tensor, which in turn affects the geometry of the spacetime.
The Einstein field equation should be modified accordingly. For this reason, we

introduce the so-called *semi-classical Einstein equations*

$$G_{\mu\nu}(g_{\mu\nu}) = 8\pi\langle \text{in}|T_{\mu\nu}(g_{\mu\nu})|\text{in}\rangle \tag{3.1}$$

in which the right-hand side represents the expectation value of the energy-momentum tensor for the matter field and also depends on the metric tensor, and the state $|\text{in}\rangle$ is built for a generic spherical symmetric metric. No one managed to solve this problem in full generality, but with some approximations, we can compute the components of the energy-momentum tensor.

From Eq. 3.1 we can infer some properties of $T_{\mu\nu}$. From the Bianchi identity for the Riemann tensor, we find that the Einstein tensor is covariantly conserved

$$\nabla_\mu G^{\mu\nu} = 0. \tag{3.2}$$

Consistency requires also the energy-momentum tensor to be conserved

$$\nabla_\mu \langle T^{\mu\nu}\rangle = 0 \tag{3.3}$$

so quantization of the matter field should be compatible with general covariance.

As mentioned before, an explicit expression of the energy-momentum tensor can be found only when we have enough symmetries on matter fields and spacetime to constrain the system. An example is conformally invariant spacetime with a classical action invariant under conformal transformations

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}. \tag{3.4}$$

Except for $\xi = 1/6$ and $m = 0$, this transformation is not a symmetry.

In a two-dimensional theory of a conformally invariant matter field, we can give an expression of $\langle T_{\mu\nu}\rangle$ for an arbitrary metric. This is because every two-dimensional spacetime is locally conformally flat, meaning we can relate any metric to the flat metric by a conformal transformation

$$g_{\mu\nu}(x) = \Omega^2(x)\eta_{\mu\nu} \tag{3.5}$$

where $\eta_{\mu\nu}$ is the flat metric. Such invariance imposes a big constraint on the form of the energy-momentum tensor.

To see this, consider an infinitesimal conformal transformation $\Omega(x) \simeq 1 + \omega(x)$. The variation of the metric reads

$$\delta g_{\mu\nu} = 2\omega(x) g_{\mu\nu} \tag{3.6}$$

and for the classical action, we find

$$\delta\mathcal{S} = \int \mathrm{d}^n x \sqrt{-g} T_{\mu\nu} g^{\mu\nu} \omega(x) \tag{3.7}$$

where n is the dimension of the spacetime. It is now clear that invariance under this transformation requires the energy-momentum tensor to be traceless

$$T \equiv g^{\mu\nu} T_{\mu\nu} = 0. \tag{3.8}$$

At the quantum level, this is not the case. The quantization procedure produces an energy-momentum tensor with a non-vanishing trace. This is the so-called *trace anomaly*[1] (Capper et al., 1974).
The numerical value of the anomaly is independent of the states on which we compute the expectation value, and depends only on geometrical quantities, such as contractions of Weyl or Ricci tensor, the Ricci scalar or its derivatives. For example, in two dimensions and for massless scalar fields we find

$$\langle T \rangle = \frac{\hbar}{24\pi} R. \tag{3.9}$$

3.2 Components of Energy-Momentum Tensor

An explicit expression of the energy-momentum tensor is not only needed in order to try to solve the semi-classical Einstein equations Eq. 3.1, but also to study the regularity of the different vacuum states we built before. We are interested in this second point.
Straightforward calculation of the expectation value of the energy-momentum tensor yields a divergent quantity, but we can recover a finite result by using the techniques of regularization. This is done, for example, in (Birrell et al., 1982).
However, the same result can be obtained without regularization, but assuming a non-vanishing trace anomaly. This has been done in (Davies, 1977a) and (Fulling, 1986).
In the following, we will omit the $\langle\,\rangle$ for the expectation value of the energy-momentum tensor, and just denote by $T_{\mu\nu}$.
Moreover, the expectation value is computed with respect to some state. For the moment we will not specify the state, but just assume to be a well-defined vacuum state.

[1]By anomaly we mean any transformation which is a symmetry of the classical theory but not of the quantum one.

So, let us consider a two-dimensional spacetime, covered by a couple of null coordinates $x^{\pm}$. Moreover, since in two dimensions any spacetime is conformally flat, we can take it of the form

$$\mathrm{d}s^2 = C(x^+, x^-)\mathrm{d}x^+\mathrm{d}x^-. \tag{3.10}$$

The only non-vanishing component of the Christoffel symbols are

$$\Gamma^+_{++} = C^{-1}\partial_+ C, \qquad \Gamma^-_{--} = C^{-1}\partial_- C \tag{3.11}$$

while the scalar (Ricci) curvature for this metric reads

$$\begin{aligned} R &= 4C^{-1}\partial_\mu\partial^\mu \ln C \\ &= 4C^{-1}\Big(C^{-1}\partial_+\partial_- C - C^{-2}\partial_+ C\partial_- C\Big) \\ &= 4C^{-2}\Big(\partial_+\partial_- C - C^{-1}\partial_+ C\partial_- C\Big). \end{aligned} \tag{3.12}$$

The Bianchi identity for the Riemann tensor implies the covariant conservation of the energy-momentum tensor

$$\nabla_\mu T^{\mu\nu} = 0 \tag{3.13}$$

which for our metric reads

$$\partial_- T_{++} - \frac{1}{4}C\partial_+ T_\mu{}^\mu = 0 \tag{3.14}$$

$$\partial_+ T_{--} - \frac{1}{4}C\partial_- T_\mu{}^\mu = 0. \tag{3.15}$$

For a general Lagrangian in flat spacetime, the energy-momentum tensor is defined as

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_\alpha)}\partial_\nu\varphi_\alpha - \eta_{\mu\nu}\mathcal{L}. \tag{3.16}$$

Taking the usual Lagrangian for a scalar field, and by "covariantize" the last expression, we find

$$T_{\mu\nu}(x) = \nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi. \tag{3.17}$$

Since the covariant differentiation is a local process, the energy-momentum tensor must be a local function of x, so it depends only on C and its derivative evaluated at x.

As said before, classically we expect the trace $T_\mu{}^\mu$ to vanish, but this is not the case. The trace anomaly, also called "conformal anomaly", is expected in quantum field theory.

Here we do not need to assume a particular form for this anomaly, but just that is a non-vanishing local quantity.
In natural units, the trace is a scalar with the dimension of [lenght]$^{-2}$, so it must contain quadratic derivatives of C. Moreover, since in flat spacetime we do not have a conformal anomaly, $T_\mu{}^\mu$ should vanish for certain choices of C. These assumptions and requirements are enough to determine the trace and, using Eq. 3.14, Eq. 3.15 the whole tensor.

The first thing we note is that C cannot be a function of only x^+ or x^-, otherwise the spacetime is flat. Indeed, suppose C to be a function of only x^+, then we could introduce a new coordinate x'^- in such a way that $C(x^+)\mathrm{d}x^+\mathrm{d}x^- \to \mathrm{d}x^+\mathrm{d}x'^-$. So C must be a function of both null coordinates.
Since the trace should contain second derivatives of C, the only possible terms are $\partial_+\partial_- C$ and $\partial_+ C\partial_- C$. Moreover, the trace should be a homogeneous function of degree -1. This fixes $T_\mu{}^\mu$ to be

$$T_\mu{}^\mu = aC^{-2}\partial_+\partial_- C + bC^{-3}\partial_+ C\partial_- C. \tag{3.18}$$

We can fix one of the two coefficients by noticing that the conformal factor of the form $C \sim e^{x^+ + x^-}$ is related to the Milne spacetime, an unconventional parametrization of the usual flat spacetime[2]. A few details about the Milne spacetime are given in Appendix A.
Being the Milne spacetime flat, and since in flat spacetime we do not have any trace anomaly, we can set $T_\mu{}^\mu$ to vanish when $C \sim e^{x^+ + x^-}$, which implies $a + b = 0$. So the trace of the energy-momentum tensor becomes

$$T_\mu{}^\mu = aC^{-2}\Big(\partial_+\partial_- C - C^{-1}\partial_+ C\partial_- C\Big) \tag{3.19}$$

and, using Eq. 3.12 we immediately recognize

$$T_\mu{}^\mu = \frac{1}{4}aR. \tag{3.20}$$

We can now solve Eq. 3.14, Eq. 3.15. We will focus only on T_{++}, but the procedure is the same for T_{--}.

We can rewrite Eq. 3.14 as

$$\partial_- T_{++} = -\frac{1}{2}a\partial_- \left[\sqrt{C}\partial_+^2\sqrt{C^{-1}}\right] \tag{3.21}$$

which can be integrated up to an arbitrary function of x^+

$$T_{++} = -\frac{1}{2}a\sqrt{C}\partial_+^2\sqrt{C^{-1}} + g(x^+). \tag{3.22}$$

[2]Just like the Rindler spacetime.

We now need to fix $g(x^+)$. As already mentioned, T_{++} is a local quantity, so g can depend on the geometry of the spacetime only through the conformal factor C and its derivatives. Moreover, we also saw that in order to have a curved spacetime, C must be a function of both null coordinates (x^+, x^-), but g is a function only of x^+, so it can only be a constant.
To fix this constant we can study the limiting case $C \to 1$. In this limit the spacetime is empty flat Minkowski spacetime, so $T_{++} = 0$, and hence we need $g = 0$.

Now all that is left to do is compute the constant a, which can be again fixed by looking at a special case, the Minkowski spacetime with *moving boundary conditions*.
The emission of thermal particles in curved spacetime can be reproduced in a flat spacetime by introducing a time-dependent boundary condition, also called *mirror*, for the field. In curved spacetime, the emission of particles emerges studying the relation between two different set of modes, let us call them (u_i, v_i) and $(\tilde{u}_i, \tilde{v}_i)$. In this "moving mirror" model, the particular choice of boundary conditions can make the modes (u_i, v_i) evolve as $(\tilde{u}_i, \tilde{v}_i)$.
It is not of our interest to study this model, but some more details are give in (Fulling et al., 1976), (Davies and Fulling, 1977b).
The basic idea is consider a conformal factor of the form

$$C = \frac{\mathrm{d}p(x^+)}{\mathrm{d}x^+} \tag{3.23}$$

where

$$p(x^+) = 2\tau_+ - x^+, \qquad \tau_+ = x^+ + z(\tau_+) \tag{3.24}$$

and $z(t)$ is the mirror trajectory.
A particular trajectory, namely

$$z(t) = -\ln\cosh t \tag{3.25}$$

has the property that the radiation emitted from this mirror is a thermal incoherent radiation of temperature $1/2\pi$.
This means that the energy flux can be computed not only from the energy-momentum tensor, but also by integrating over the incoherent thermal spectrum

$$T_{++} = \frac{1}{2\pi}\int_0^\infty \mathrm{d}\omega \frac{\omega}{e^{2\pi\omega} - 1} = \frac{1}{48\pi}. \tag{3.26}$$

But we can also compute T_{++} by substituting in Eq. 3.22 the conformal factor computed from the trajectory $z(t)$, from which we get

$$T_{++} = \frac{1}{8}a. \tag{3.27}$$

A direct comparison shows that

$$a = \frac{1}{6\pi}. \tag{3.28}$$

So, starting from the basic assumption of a non-vanishing trace anomaly we managed to recover the full explicit form of the energy-momentum tensor and its trace

$$T_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{48\pi} R g_{\mu\nu} \tag{3.29}$$

$$\theta_{++} = -\frac{1}{12\pi}\sqrt{C}\partial_+^2\sqrt{C^{-1}} \tag{3.30}$$

$$\theta_{--} = -\frac{1}{12\pi}\sqrt{C}\partial_-^2\sqrt{C^{-1}} \tag{3.31}$$

$$\theta_{+-} = \theta_{-+} = 0 \tag{3.32}$$

$$T_\mu{}^\mu = \frac{R}{24\pi}. \tag{3.33}$$

These results have been derived without any regularization procedure, but are in complete agreement with those results.

3.3 Transformation of the Energy-Momentum Tensor

In the previous section, we derived the expectation value of the energy-momentum tensor, neglecting the state we use to evaluate the quantity.
These states, which could be for example, the Boulware, the Unruh, or the Israel-Hartle-Hawking vacuums, are not unique, and are built with respect to some modes. Alternatively, we could have chosen a different set of modes, with different coordinates. For this reason, it is interesting to study how the expectation values for different states are related.

So, suppose to cover a two-dimensional spacetime with a metric of the form

$$\mathrm{d}s^2 = C(x^+, x^-)\mathrm{d}x^+\mathrm{d}x^-. \tag{3.34}$$

Quantization in this spacetime is achieved with a set of normalized modes which reads

$$\frac{1}{\sqrt{4\pi\omega}}e^{-i\omega x^+}, \qquad \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega x^-}. \tag{3.35}$$

This set of modes allows us, together with the associated ladder operators, to expand a massless scalar field and to identify a vacuum state, which we denote by $|x^{\pm}\rangle$.
The expectation value of the energy-momentum tensor for the scalar field in the $|x^{\pm}\rangle$ state is (see Eq. 3.29 - Eq. 3.33)

$$\langle x^{\pm}|T_{\pm\pm}|x^{\pm}\rangle = -\frac{1}{12\pi}\sqrt{C}\partial_{\pm}^{2}\sqrt{C^{-1}} \tag{3.36}$$

$$\langle x^{\pm}|T_{+-}|x^{\pm}\rangle = -\frac{1}{48\pi}Rg_{+-} = -\frac{1}{96\pi}RC. \tag{3.37}$$

The coordinates $x^{\pm}$ and the states built with these coordinates are not special. Indeed we could also use a different set of coordinates, let us call them $\tilde{x}^{\pm}$, related to $x^{\pm}$ by

$$\tilde{x}^{\pm} = \tilde{x}^{\pm}(x^{\pm}). \tag{3.38}$$

With these coordinates, we can build a new set of modes

$$\frac{1}{\sqrt{4\pi\tilde{\omega}}}e^{-i\tilde{\omega}\tilde{x}^{+}}, \qquad \frac{1}{\sqrt{4\pi\tilde{\omega}}}e^{-i\tilde{\omega}\tilde{x}^{-}}. \tag{3.39}$$

and an associated vacuum state $|\tilde{x}^{\pm}\rangle$.

Knowing the transformation rule between the two sets of coordinates we can recover a relation between the expectation value of the energy-momentum tensor in the two different vacuum states (Balbinot and Bergamini, 1989)

$$\begin{aligned}\langle \tilde{x}^{\pm}|T_{\pm\pm}|\tilde{x}^{\pm}\rangle &= \langle x^{\pm}|T_{\pm\pm}|x^{\pm}\rangle + \frac{1}{24\pi}F_{\tilde{x}^{\pm}}(x^{\pm})\\ \langle \tilde{x}^{\pm}|T_{+-}|\tilde{x}^{\pm}\rangle &= \langle x^{\pm}|T_{+-}|x^{\pm}\rangle\end{aligned} \tag{3.40}$$

where the functional $F_{\tilde{x}^{\pm}}(x^{\pm})$, proportional to the Schwarzian derivative, relates the two reference frames

$$F_{\tilde{x}^{\pm}}(x^{\pm}) = \left(\frac{\mathrm{d}x^{\pm}}{\mathrm{d}\tilde{x}^{\pm}}\right)^{-2}\left[\frac{\mathrm{d}^{3}x^{\pm}}{\mathrm{d}\tilde{x}^{\pm 3}}\Big/\frac{\mathrm{d}x^{\pm}}{\mathrm{d}\tilde{x}^{\pm}} - \frac{3}{2}\left(\frac{\mathrm{d}^{2}x^{\pm}}{\mathrm{d}\tilde{x}^{\pm 2}}\Big/\frac{\mathrm{d}x^{\pm}}{\mathrm{d}\tilde{x}^{\pm}}\right)^{2}\right]. \tag{3.41}$$

We now see that the only difference in the expectation value of the $T_{\mu\nu}$ with respect to two different sets of coordinates is given by the Schwarzian derivative. Moreover, this difference is non-local, in the sense that does not depend on geometrical quantities, but on the global relation between the coordinates.
We also note that in Eq. 3.40, the components of the energy-momentum tensor are still expressed with respect to the coordinates $x^{\pm}$, but evaluated for the state built with the coordinates $\tilde{x}^{\pm}$. Expressing the components in terms of a specific set of coordinates will be relevant later in discussing the regularity of the states.

4 Energy-Momentum Tensor in Schwarzschild Spacetime

Having obtained an explicit expression for the expectation value of the energy-momentum tensor we can now compute explicitly this value in different vacuum states.

Thanks to the spherical symmetry of the Schwarzschild spacetime we can neglect the angular terms in the metric and focus only on the $t-r$ sector.
We will compute explicitly the expectation value for three states: Boulware, Unruh and Israel-Hartle-Hawking vacuum.

4.1 Boulware Vacuum

Boulware vacuum is defined with respect to modes built with the Eddington-Finkelstein null coordinates. The metric in these coordinates reads

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right) \mathrm{d}u\mathrm{d}v \tag{4.1}$$

and the normalized modes are

$$\frac{1}{4\pi\sqrt{\omega}} e^{-i\omega u}, \qquad \frac{1}{4\pi\sqrt{\omega}} e^{-i\omega v}. \tag{4.2}$$

The key equations are Eq. 3.36, Eq. 3.37, which can be rewritten as

$$\langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle = \langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle = \frac{1}{192\pi}\left(2CC'' - C'^2\right) \tag{4.3}$$

$$\langle \mathrm{B}|T_{uv}|\mathrm{B}\rangle = \frac{1}{96\pi} CC'' \tag{4.4}$$

and a prime denotes differentiation with respect to r.
Now we just need to compute the derivatives. Introducing the conformal factor

for the Schwarzschild spacetime we find

$$\begin{aligned} \langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle = \langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle &= \frac{1}{24\pi}\left[\frac{3}{2}\frac{M^2}{r^4} - \frac{M}{r^3}\right] \\ \langle \mathrm{B}|T_{uv}|\mathrm{B}\rangle &= -\frac{1}{24\pi}\frac{M}{r^3}\left(1-\frac{2M}{r}\right). \end{aligned} \tag{4.5}$$

As discussed previously in Section 2.2, Boulware modes reduce to usual Minkowski modes at infinity, therefore we expect the Boulware vacuum to reduce to the Minkowski vacuum. This is precisely the case. As $r \to \infty$, all components of $T_{\mu\nu}$ vanish, reproducing the familiar concept of vacuum state as a state of no particles.

4.1.1 Regularity and Physical Meaning of Boulware Modes

We can study the behavior of Eq. 4.5 also on the horizon. We know that the Schwarzschild metric is singular at $r = 2M$ when expressed with both null coordinates. But this singularity is not a physical singularity, so the region belongs to the physical spacetime and we expect a finite result for the expectation value. To better investigate this region it is more appropriate to use a coordinate system regular on the horizon, for example, the Kruskal coordinates

$$U = -4Me^{-u/4M}, \qquad V = 4Me^{v/4M}. \tag{4.6}$$

The expectation value of $T_{\mu\nu}$ is regular if its components, in Kruskal coordinates, are finite approaching the horizon

$$\langle \mathrm{B}|T_{UU}|\mathrm{B}\rangle < \infty \tag{4.7}$$

$$\langle \mathrm{B}|T_{UV}|\mathrm{B}\rangle < \infty \tag{4.8}$$

$$\langle \mathrm{B}|T_{VV}|\mathrm{B}\rangle < \infty. \tag{4.9}$$

The components of $T_{\mu\nu}$ transform as

$$\langle \mathrm{B}|T_{UU}|\mathrm{B}\rangle = \left(\frac{\mathrm{d}u}{\mathrm{d}U}\right)^2 \langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle \tag{4.10}$$

$$\langle \mathrm{B}|T_{UV}|\mathrm{B}\rangle = \left(\frac{\mathrm{d}u}{\mathrm{d}U}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}V}\right) \langle \mathrm{B}|T_{uv}|\mathrm{B}\rangle \tag{4.11}$$

$$\langle \mathrm{B}|T_{VV}|\mathrm{B}\rangle = \left(\frac{\mathrm{d}v}{\mathrm{d}V}\right)^2 \langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle. \tag{4.12}$$

The transformation factor can be easily computed by inverting the defining relation of the Kruskal coordinates from Eq. 4.6 and using

$$UV = -8Me^{r/2M}(r - 2M) \tag{4.13}$$

from which we immediately get

$$\langle \mathrm{B}|T_{UU}|\mathrm{B}\rangle \sim V^2(r-2M)^{-2}\langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle \tag{4.14}$$

$$\langle \mathrm{B}|T_{UV}|\mathrm{B}\rangle \sim (r-2M)^{-1}\langle \mathrm{B}|T_{uv}|\mathrm{B}\rangle \tag{4.15}$$

$$\langle \mathrm{B}|T_{VV}|\mathrm{B}\rangle \sim U^2(r-2M)^{-2}\langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle. \tag{4.16}$$

The future horizon is described by $U = 0$, so the regularity conditions take the form

$$(r-2M)^{-2}|\langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle| < \infty \tag{4.17}$$

$$(r-2M)^{-1}|\langle \mathrm{B}|T_{uv}|\mathrm{B}\rangle| < \infty \tag{4.18}$$

$$|\langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle| < \infty \tag{4.19}$$

while for the past horizon, we just exchange u and v.
This shows that the expectation value for $T_{\mu\nu}$ is regular *approaching* the horizon, but not *on* the horizon, since the conditions are not satisfied when $r = 2M$. The reason for that is, as mentioned above, the Schwarzschild metric in double-null form is singular. This is the reason for the introduction of the Unruh modes.

Lastly, we just mention the physical interpretation of these states. Since they reduce to Minkowski modes at infinity and are regular everywhere except on the horizon, Boulware modes can be used to describe the vacuum polarization of spacetime outside a static massive body with a radius bigger than $2M$, a star. In this case, the portion of spacetime that holds physical significance does not include horizons, and so the modes are regular everywhere on the physical spacetime.
A vacuum state with such properties can be built also with other modes, but only the Boulware modes lead to a time-independent energy-momentum tensor.

4.2 Unruh Vacuum

We have seen that the Boulware modes are not regular on the horizon. To solve this issue we could use a set of modes built using coordinates regular on the horizon. An example are the Unruh modes, built with the ingoing Eddington-Finkelstein coordinate v and the outgoing Kruskal U

$$\frac{1}{\sqrt{4\pi\omega}}e^{-i\omega U}, \qquad \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega v}. \tag{4.20}$$

To compute the expectation value of $T_{\mu\nu}$ we can use the relation between the energy-momentum tensor in two different conformally-related frames, Eq. 3.40. We already know the expectation value with respect to the Boulware modes, we just need to compute the functional Eq. 3.41 relating the two sets of coordinates. Before directly computing the value of the functional, we note that this factor will not vanish only for the (uu) component of the energy-momentum tensor. Indeed, Unruh and Boulware states are defined with respect to the same ingoing v coordinate, therefore we do not have an additional term for the (vv) component, $F_v(v) = 0$. The mixed term (uv) instead does not pick up an additional term, no matter the relationship between the coordinates.
Therefore we need to compute only for the (uu) component

$$F_U(u) = \left[\frac{\mathrm{d}^3 u}{\mathrm{d}U^3}/\frac{\mathrm{d}u}{\mathrm{d}U} - \frac{3}{2}\left(\frac{\mathrm{d}^2 u}{\mathrm{d}U^2}/\frac{\mathrm{d}u}{\mathrm{d}U}\right)^2\right]\left(\frac{\mathrm{d}u}{\mathrm{d}U}\right)^{-2}. \tag{4.21}$$

We can invert the relations Eq. 4.6 to write Eddington-Finkelstein u in terms of Kruskal U, and computing the derivative we eventually get

$$F_U(u) = \frac{1}{32\pi M^2}. \tag{4.22}$$

Putting everything together using Eq. 3.40 we find

$$\langle \mathrm{U}|T_{uu}|\mathrm{U}\rangle = \langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle + \frac{1}{24\pi}F_U(u) = \langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle + \frac{1}{768\pi M^2} \tag{4.23}$$

$$= \frac{(1-2M/r)^2}{768\pi M^2}\left[1 + \frac{4M}{r} + \frac{12M^2}{r^2}\right]$$

$$\langle \mathrm{U}|T_{uv}|\mathrm{U}\rangle = \langle \mathrm{B}|T_{uv}|\mathrm{B}\rangle = -\frac{1}{24\pi}\frac{M}{r^3}\left(1 - \frac{2M}{r}\right) \tag{4.24}$$

$$\langle \mathrm{U}|T_{vv}|\mathrm{U}\rangle = \langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle = \frac{1}{24\pi}\left[\frac{3}{2}\frac{M^2}{r^4} - \frac{M}{r^3}\right]. \tag{4.25}$$

Let us look at the regularity on the future horizon. Checking Eq. 4.14 we immediately find that the expectation value of the energy-momentum tensor on Unruh vacuum is regular. This is possible since, recalling what we have seen in Subsection 4.1.1, the condition not satisfied was the one concerning the (uu) component, that is the one which gets the additional term $1/(768\pi M^2)$, introduced by the relation between the two sets of coordinates.
For the (vv) component instead, regularity was assured by the presence of the factor $U^2/(r-2M)^{-2}$, which is regular on the horizon ($r = 2M$ or $U = 0$).
Conversely, on the past horizon $V = 0$, the (uu) component is still regular, but the condition on the (vv) component is now

$$(r-2M)^{-2}|\langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle| < \infty \tag{4.26}$$

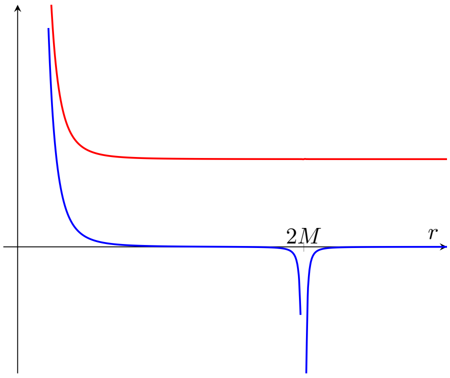


Figure 4.1: $\langle T_{UU} \rangle$ for Boulware and Unruh vacuum in Schwarzschild spacetime. The blue function shows $\langle \mathrm{B}|T_{UU}|\mathrm{B} \rangle$ and its divergence at $r = 2M$. The red function instead shows $\langle \mathrm{U}|T_{UU}|\mathrm{U} \rangle$, with a small vertical offset in order to not overlap with the blue function. As we can see, $\langle \mathrm{U}|T_{UU}|\mathrm{U} \rangle$ is now regular at the horizon.

which is not satisfied.
Therefore we conclude that the Unruh vacuum is regular on the future horizon, but not on the past one. A graphical representation of $\langle T_{UU} \rangle$ is given in Fig. 4.1.

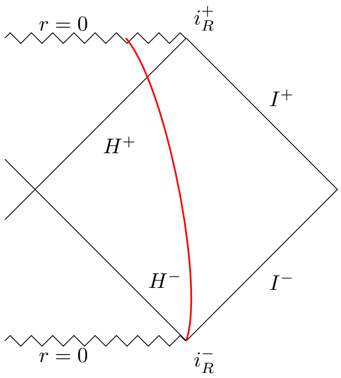


Figure 4.2: Relevant part of Schwarzschild spacetime during gravitational collapse. The red line represents the trajectory of a collapsing surface. The past horizon is "inside" the collapsing body, and therefore physics there should be described according to the interior metric.

As we discussed in Subsection 2.4.2, the Unruh state is the state that correctly describes at late times the quantum state of a field in the spacetime of gravitational collapse (such as the Vaidya model). In this scenario, the past singularity and the past horizon are "covered" by the interior region(see Fig. 4.2), so the divergence of $T_{\mu\nu}$ is not particularly a problem. This is not the case for an *eternal black* *hole*, a black hole whose physical spacetime is the complete maximally analytical extension of the Schwarzschild spacetime. To describe correctly quantum field in this scenario we need different states that do not diverge on the past horizon.

Lastly, we can discuss the behavior at past and future null infinity. The ingoing Unruh mode is the same as the ingoing Boulware mode, which we know approaches the ingoing Minkowski mode at infinity. Therefore the Unruh vacuum describes the familiar Minkowski vacuum at past null infinity.
The scenario is completely different at future null infinity.
Indeed, the outgoing Unruh mode does not approach the outgoing Minkowski mode. The Unruh state describes the state of a field in the spacetime of gravitational collapse, therefore comparing the expectation value for the Unruh vacuum with the expectation value for the Boulware vacuum we find the effects of the gravitational collapse.
The collapse produced an additional term to the "outgoing" (uu) component. This term is relevant at large r and represents a constant flux of energy of magnitude $(768\pi M^2)^{-1}$.

In general, looking at Eq. 4.23, the energy flux is given by two components:

1. At $r \to \infty$ we have a constant outgoing flux of energy given by T_{uu}. The contribution from the Boulware vacuum vanishes and we are left with

$$\langle \mathrm{U}|T_{uu}|\mathrm{U}\rangle \sim \frac{1}{768\pi M^2} \tag{4.27}$$

which represents a constant thermal flux of the Hawking radiation of temperature

$$T_H = \frac{1}{8\pi k_B M}; \tag{4.28}$$

2. Near the horizon we find a negative ingoing flux of energy given by T_{vv}. This negative flux of energy could cause the area of the horizon of the black hole to shrink consistently with the emission measured at infinity (Davies, Fulling, and Unruh, 1976).

It is interesting to notice that this energy flux was obtained without recurring to the backward ray tracing, the argument used by Hawking in (Hawking, 1975) to show the particle creation near a collapsing black hole, or to the Bogoljubov transformations, as we did in the maximally analytical extension of Schwarzschild spacetime in Chapter 2.

4.3 Israel-Hartle-Hawking Vacuum

To solve the problem of having a divergent expectation value energy-momentum tensor on the past horizon we can introduce another vacuum state: the *Israel-Hartle-Hawking state* (Israel, 1976) (Hartle et al., 1976).
The introduction of the outgoing Kruskal coordinate allowed us to solve the divergence on the future horizon of the Boulware state. To remove the divergence on the past horizon we can introduce also the ingoing Kruskal coordinate.
In this way we get ingoing and outgoing modes both built with Kruskal coordinates

$$\frac{1}{\sqrt{4\pi\omega}}e^{-i\omega U}, \qquad \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega V}. \tag{4.29}$$

By the same analysis we performed previously for the Unruh vacuum, we can compute the expectation value of $\langle \mathrm{H}|T_{\mu\nu}|\mathrm{H}\rangle$ summing the vacuum polarization term, coming from the Boulware vacuum, and the term coming the functional $F_U(u)$ relating the two sets of coordinates.
The difference this time is that we get a non-vanishing contribution also to the (vv) component.

Up to a sign (that depends on which sector of the Kruskal sector we are considering), the relation between U and u is of the same form as the relation between V and v, therefore we immediately find that

$$F_U(u) = F_V(v) = \frac{1}{32\pi M^2} \tag{4.30}$$

and

$$\begin{aligned}\langle \mathrm{H}|T_{uu}|\mathrm{H}\rangle &= \langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle + \frac{1}{24\pi}F_U(u) \\ &= \frac{(1-2M/r)^2}{768\pi M^2}\left[1+\frac{4M}{r}+\frac{12M^2}{r^2}\right]\end{aligned} \tag{4.31}$$

$$\langle \mathrm{H}|T_{uv}|\mathrm{H}\rangle = \langle \mathrm{B}|T_{uv}|\mathrm{B}\rangle = -\frac{1}{24\pi}\frac{M}{r^3}\left(1-\frac{2M}{r}\right) \tag{4.32}$$

$$\begin{aligned}\langle \mathrm{H}|T_{vv}|\mathrm{H}\rangle &= \langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle + \frac{1}{24\pi}F_V(v) \\ &= \frac{(1-2M/r)^2}{768\pi M^2}\left[1+\frac{4M}{r}+\frac{12M^2}{r^2}\right].\end{aligned} \tag{4.33}$$

Checking the usual regularity conditions Eq. 4.14 we find that the Israel-Hartle-Hawking vacuum is now regular on both future and past horizons.

However, the Israel-Hartle-Hawking modes do not reduce to Minkowski modes at past and future null infinity. Indeed at infinity

$$\langle \mathrm{H}|T_{uu}|\mathrm{H}\rangle = \langle \mathrm{H}|T_{vv}|\mathrm{H}\rangle \sim \frac{1}{768\pi M^2} \tag{4.34}$$

$$\langle \mathrm{H}|T_{uv}|\mathrm{H}\rangle \sim 0. \tag{4.35}$$

This means that at $I^{\pm}$ we do not recover the usual Minkowski vacuum. Indeed, $|\mathrm{H}\rangle$ is a finite-temperature state representing an ingoing and outgoing thermal flux at the temperature

$$T_H = \frac{1}{8\pi k_B M}. \tag{4.36}$$

The Israel-Hartle-Hawking state describes the so-called *black hole in a box*, a black hole enclosed in a box, in thermal equilibrium with its own radiation.
It is important to note that this state is not the only state which can reproduce these properties. Indeed, we could define another set of coordinates that behave like Kruskal at infinity and on the horizons, and we still would find a non-vanishing contribution to the expectation value of $T_{\mu\nu}$ at infinity. However, only with the Israel-Hartle-Hawking state $\langle \mathrm{H}|T_{uu}|\mathrm{H}\rangle$ and $\langle \mathrm{H}|T_{vv}|\mathrm{H}\rangle$ are time-independent, and therefore correctly describe the thermal equilibrium.

5 Energy-Momentum Tensor in Reissner-Nordström Spacetime

The analysis on the regularity of vacuum states can also be performed on different spacetimes, such as the Reissner-Nordström spacetime, which describes a massive charged black hole. Before studying the regularity we can make a brief review of the key elements of this metric.

5.1 Charged Black Holes

The Reissner-Nordström spacetime is the solution to the Einstein-Maxwell problem, meaning a solution of the Einstein field equation for an energy-momentum tensor solution of the Maxwell equations. The source is supposed to be point-like, of mass M and electric (or magnetic) charge Q. The line element takes the form

$$\mathrm{d}s^2 = C(r)\mathrm{d}t^2 - C(r)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 \tag{5.1}$$

where the conformal factor is

$$C(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right). \tag{5.2}$$

The metric is spherically symmetric, therefore from now on we will neglect the angular terms.
We have a physical singularity at $r = 0$, and geometrical singularities given by the condition

$$C(r) = 0 \to r_\pm = M \pm \sqrt{M^2 - Q^2} \tag{5.3}$$

which, if the argument of the square root is positive, corresponds to horizons. Looking at the numerical values of M and Q we can identify three different scenarios, see also Fig. 5.1:

1. $M < |Q|$: the argument of the square root is negative, therefore we do not have any horizon. The singularity at $r = 0$ is a time-like singularity,

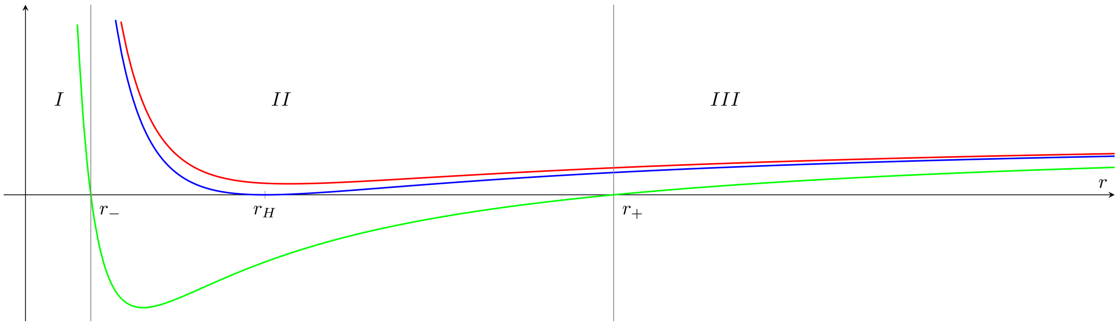


Figure 5.1: Conformal factor for the Reissner-Nordström spacetime.
The red, blue and green functions corresponds respectively to $M < |Q|$, $M = |Q|$ and $M > |Q|$.
For $M = |Q|$ we have a single horizon at r_H.
For $M > |Q|$ we have two horizons located at r_+ and r_-.

not hidden by any horizon. This means that any lightcone in the whole spacetime intersects the singularity and therefore we lose the predictability of the theory. According to the *Penrose cosmic censorship* hypothesis, such singularities are not realized in nature;

2. $M = |Q|$: in this case we have one horizon at $r_H = M$. Such a black hole is called *extremal*, and it has vanishing surface gravity, and therefore vanishing Hawking temperature;

3. $M > |Q|$: in this case we have two different horizons. The outer horizon at r_+ behaves like the horizon of Schwarzschild spacetime. The inner horizon at r_- instead shows an odd behavior. If we consider a free-falling null observer:

   - In the region III, which is for $r > r_+$, we can have ingoing and outgoing trajectories ;

   - In the region II, which is for $r_- < r < r_+$, the only possible trajectories are those of decreasing r, so photons are attracted towards r_-;

   - In the region I, which is for $0 < r < r_-$, we can have both ingoing and outgoing trajectories, but without crossing the inner horizon.

In the following we will focus only on Reissner-Nordström black hole with two horizons, $M > |Q|$.

To remove the geometrical singularities we can adopt a different set of coordinates. In full analogy to the Schwarzschild spacetime, we can introduce the null

Eddington-Finkelstein coordinates

$$u = t - r^*, \qquad v = t + r^* \tag{5.4}$$

where the Regge-Wheeler coordinate is

$$r^* = \int \mathrm{d}r\, C(r)^{-1} \tag{5.5}$$

$$= r + \frac{1}{2\kappa_+} \ln \left| \frac{r - r_+}{r_+} \right| + \frac{1}{2\kappa_-} \ln \left| \frac{r - r_-}{r_-} \right| \tag{5.6}$$

and the surface gravity

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2r_\pm^2}. \tag{5.7}$$

Both extensions built with u and v are not complete, therefore we need to introduce another extension able to cover the whole spacetime: the Kruskal extension. The main difference with respect to the Kruskal extension of Schwarzschild spacetime is that now we have two different values of the surface gravity, and so we have ambiguities in the definition of Kruskal coordinates[1]

$$U_\pm = \pm \frac{1}{\kappa_\pm} e^{-\kappa_\pm u}, \qquad V_\pm = \pm \frac{1}{\kappa_\pm} e^{-\kappa_\pm v}. \tag{5.8}$$

We will face the same ambiguities later when studying the regularity of the states. The Kruskal extension with κ_+ is regular on r_+ but not on r_-, and covers only $r > r_-$. Conversely, the extension with k_- is regular only on r_- and covers $0 < r < r_-$. We can compactify both extensions by a conformal transformation and obtain the Penrose diagrams of the two extensions, which can be combined to obtain the Penrose diagram of the whole Reissner-Nordström spacetime, see Fig. 5.2. It consists of an infinite sequence of asymptotically flat regions connected by wormholes.

The "right-side" horizon at r_-, corresponding to $V_- = 0$, is also called *Cauchy horizon*, since we have predictability only for $r < r_-$. The "left-side" instead is called *inner horizon*, and it is described by $U_- = 0$.

At r_+ instead we have the "right-side" described by $U_+ = 0$ and denoted by *event horizon*, while the "left-side" is called *past horizon*, and it is described by $V_+ = 0$.

Studying what happens to an observer crossing r_-, we can see that the inner horizon is a surface unstable against small perturbations coming from outside. These perturbations affect the spacetime and lead to the formation of a space-like curvature singularity[2] called *mass inflation*. With the presence of this singularity the infinite sequence stops, and the trajectories of free falling observer end on this new curvature singularity.

[1]The overall sign depends on the region we choose to cover.

[2]A singularity similar to $r = 0$ in Schwarzschild spacetime.

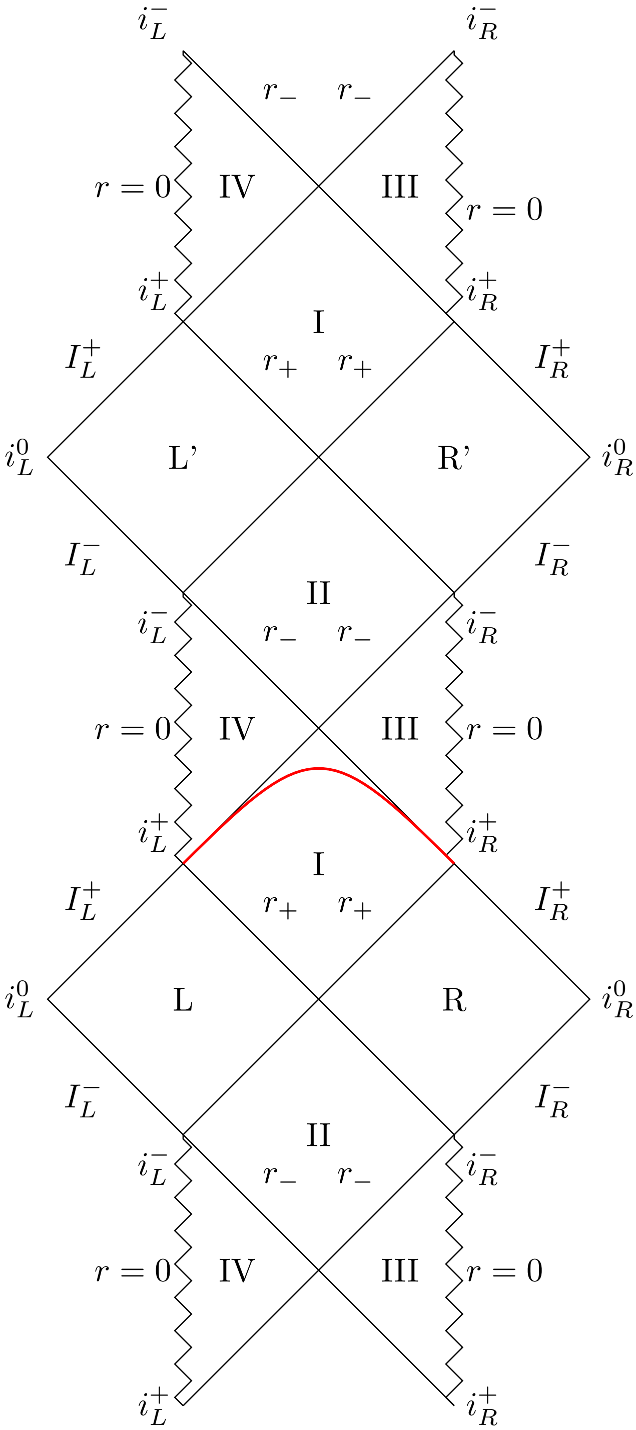


Figure 5.2: Penrose diagram of the whole Reissner-Nordström spacetime. The red surface is the mass inflation singularity.

5.2 Boulware Vacuum

In order to build the Boulware modes we start with the line element in the double null form

$$\mathrm{d}s^2 = \frac{(r-r_+)(r-r_-)}{r^2}\mathrm{d}u\mathrm{d}v. \tag{5.9}$$

Boulware modes for a Reissner-Nordström metric can be built as for the Schwarzschild metric

$$\frac{1}{\sqrt{4\pi\omega}}e^{-i\omega u}, \qquad \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega v}. \tag{5.10}$$

The only difference is that the Regge-Wheeler coordinate, used to build the null coordinates, is defined with respect to the conformal factor of the Reissner-Nordström spacetime.

To compute the expectation value of the energy-momentum tensor we use again Eq. 3.36 and Eq. 3.37 with the conformal factor from the metric in the double null form Eq. 5.9. After computing the derivatives and some algebra we eventually find

$$\begin{aligned}\langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle &= \langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle \\ &= \frac{1}{24\pi}\left[-\frac{M}{r^3}+\frac{3}{2}\frac{M^2}{r^4}+\frac{3}{2}\frac{Q^2}{r^4}-\frac{3MQ^2}{r^5}+\frac{Q^4}{r^6}\right] \end{aligned} \tag{5.11}$$

$$\langle \mathrm{B}|T_{uv}|\mathrm{B}\rangle = \frac{1}{24\pi}\left[-\frac{M}{r^3}+\frac{2M^2}{r^4}+\frac{3}{2}\frac{Q^2}{r^4}-\frac{4MQ^2}{r^5}+\frac{3}{2}\frac{Q^4}{r^6}\right]. \tag{5.12}$$

We immediately see that, as expected, all the contributions vanish at $r \to \infty$, so again the Boulware modes correctly reproduce the usual Minkowski vacuum.

The double null form of the metric is singular on the horizons, so in order to study the regularity of the vacuum, we need a set of coordinates regular in these regions, such as Kruskal coordinates. As for Schwarzschild, we will use again the relations from Eq. 4.10, but now we need to check both horizons.

5.2.1 Regularity of Boulware Vacuum

To check the regularity of the Boulware vacuum on the horizons, we need to study the behavior of the expectation value of the energy-momentum tensor in Kruskal coordinates evaluated on the horizons. As mentioned earlier, the presence of two horizons, and therefore of two different values for the surface gravity, offers two ways to build the Kruskal coordinates.

So, suppose we want to study the regularity at the event horizon, which is located at $r = r_+$, or equivalently, at $U_+ = 0$.
As for the Schwarzschild spacetime, the regularity of the energy-momentum tensor on a point means that the component of the tensor should be regular with respect to a set of coordinates regular on that point. The coordinates regular on the event horizon are (U_+, V_+), so the conditions of regularity can be written as

$$\langle \mathrm{B}|T_{UU}|\mathrm{B}\rangle = \left(\frac{\mathrm{d}u}{\mathrm{d}U_+}\right)^2 \langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle < \infty \tag{5.13}$$

$$\langle \mathrm{B}|T_{UV}|\mathrm{B}\rangle = \left(\frac{\mathrm{d}u}{\mathrm{d}U_+}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}V_+}\right) \langle \mathrm{B}|T_{uv}|\mathrm{B}\rangle < \infty \tag{5.14}$$

$$\langle \mathrm{B}|T_{VV}|\mathrm{B}\rangle = \left(\frac{\mathrm{d}v}{\mathrm{d}V_+}\right)^2 \langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle < \infty. \tag{5.15}$$

Inverting the relation between Kruskal U_+ and Eddington-Finkelstein u, and using the relation

$$U_+V_+ = -e^{2\kappa_+ r}\left(\frac{r-r_+}{r_+}\right)\left(\frac{r-r_-}{r_-}\right)^{\kappa_+/\kappa_-}\frac{1}{\kappa_+^2} \tag{5.16}$$

we find

$$\langle \mathrm{B}|T_{UU}|\mathrm{B}\rangle \sim \left(\frac{r_+}{r-r_+}\right)^2 V_+^2 \langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle \tag{5.17}$$

$$\langle \mathrm{B}|T_{UV}|\mathrm{B}\rangle \sim \left(\frac{r_+}{r-r_+}\right) \langle \mathrm{B}|T_{uv}|\mathrm{B}\rangle \tag{5.18}$$

$$\langle \mathrm{B}|T_{VV}|\mathrm{B}\rangle \sim \left(\frac{r_+}{r-r_+}\right)^2 U_+^2 \langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle. \tag{5.19}$$

On the event horizon $U_+ = 0$ these conditions translate to

$$\left(\frac{r_+}{r-r_+}\right)^2 |\langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle| < \infty \tag{5.20}$$

$$\left(\frac{r_+}{r-r_+}\right) |\langle \mathrm{B}|T_{uv}|\mathrm{B}\rangle| < \infty \tag{5.21}$$

$$|\langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle| < \infty. \tag{5.22}$$

At $r = r_+$, the expectation values are evaluated to

$$\langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle\Big|_{r_+} = \langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle\Big|_{r_+} = -\frac{1}{48}\kappa_+^2 \tag{5.23}$$

$$\langle \mathrm{B}|T_{uv}|\mathrm{B}\rangle\Big|_{r_+} = 0. \tag{5.24}$$

By looking at the conditions above, we see that at the event horizon only the conditions for the (uv) and (vv) components are satisfied, while the one for the (uu) is not.
Therefore, the Boulware vacuum is not regular on the event horizon.
The same arguments holds for the past horizon $V_+ = 0$. In this case the condition not satisfied is the one for the (vv) component.

We can also study the regularity at the inner horizon r_-. The conditions Eq. 5.13 - Eq. 5.15 can be expressed with respect to the coordinates (U_-, V_-). Then we can use

$$U^- V^- = -e^{2\kappa_- r} \left(\frac{r_- - r}{r_-} \right) \left(\frac{r_+ - r}{r_+} \right)^{\kappa_-/\kappa_+} \frac{1}{\kappa_-^2} \tag{5.25}$$

to find the conditions at r_-. The same analysis done before shows that the (uu) and (vv) components evaluated at $r = r_-$ are finite, and in particular proportional to κ_-^2, while again the (uv) component vanishes.
The regularity conditions are similar. At the Cauchy horizon $V_- = 0$ the condition on the (vv) component is not satisfied, while at the inner horizon $U_- = 0$ the problematic condition is the one for the (uu) component.

5.3 Unruh Vacuum

Similarly to the Boulware vacuum for the Schwarzschild spacetime, also in the Reissner-Nordström spacetime the Boulware vacuum is not a regular state.
In full analogy to Schwarzschild spacetime, the ultimate reason for the non-regularity is the choice of a set of coordinates not regular on the horizons. We can try to remove the divergencies by introducing a vacuum state defined with respect to a set of coordinates regular on the horizons: the Kruskal coordinates.
The Unruh vacuum can be built in the same way as we did in the Schwarzschild spacetime. The normalized modes will take the form

$$\frac{1}{\sqrt{4\pi\omega}} e^{-i\omega U_\pm}, \qquad \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega v}. \tag{5.26}$$

The expectation value of the energy-momentum tensor for the Unruh vacuum can be computed using again Eq. 3.40, which relates the expectation value of the energy-momentum tensor in two conformally-related frames. We already know the "vacuum polarization" term, coming from the Boulware vacuum Eq. 5.11 and

Eq. 5.12. We just need to compute the functional

$$F_{U_\pm}(u) = \left(\frac{\mathrm{d}u}{\mathrm{d}U_\pm}\right)^{-2}\left[\frac{\mathrm{d}^3u}{\mathrm{d}U_\pm^3}\Big/\frac{\mathrm{d}u}{\mathrm{d}U_\pm} - \frac{3}{2}\left(\frac{\mathrm{d}^2u}{\mathrm{d}U_\pm^2}\Big/\frac{\mathrm{d}u}{\mathrm{d}U_\pm}\right)^2\right]. \tag{5.27}$$

Computing the derivatives we find

$$F_{U_\pm}(u) = \frac{1}{2}\kappa_\pm^2 \tag{5.28}$$

and finally

$$\langle \mathrm{U}_\pm|T_{uu}|\mathrm{U}_\pm\rangle = \langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle + \frac{1}{48\pi}\kappa_\pm^2 \tag{5.29}$$

$$\langle \mathrm{U}_\pm|T_{uv}|\mathrm{U}_\pm\rangle = \langle \mathrm{B}|T_{uv}|\mathrm{B}\rangle \tag{5.30}$$

$$\langle \mathrm{U}_\pm|T_{vv}|\mathrm{U}_\pm\rangle = \langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle \tag{5.31}$$

where the subscript on the state refer to which values of surface gravity is used to build the associated modes.

5.3.1 Regularity of Unruh Vacuum

Since the (uu) component is the only one that gets an additional term, we will focus only on that one. In order to study the regularity we need to choose one of the two surface gravity.

Suppose we take $F_{U_+}(u) = \kappa_+^2/2$, which means that the Unruh vacuum is built with respect to U_+.
As for the Schwarzschild case, the introduction of this new term makes the energy-momentum tensor regular on the outer horizon. Indeed, at r_+

$$\begin{aligned}\langle \mathrm{U}_+|T_{UU}|\mathrm{U}_+\rangle\Big|_{r_+} &= \langle \mathrm{U}_+|T_{uu}|\mathrm{U}_+\rangle\Big|_{r_+}\left(\frac{r_+}{r-r_+}\right)^2 \\ &= \left(-\frac{1}{48\pi}\kappa_+^2 + \frac{1}{48\pi}\kappa_+^2\right)\left(\frac{r_+}{r-r_+}\right)^2 \qquad (5.32)\\ &< \infty. \qquad (5.33)\end{aligned}$$

The condition on the (uu) component is satisfied thanks to the introduction of the correction term which cancelled the vacuum polarization term. The condition on the (vv) component is also satisfied since at the event horizon we have $U_+ = 0$.
We also see that the Unruh vacuum is not regular on the past horizon $V_+ = 0$.

Indeed, looking at Eq. 5.17 - Eq. 5.19 we see that the regularity condition on (vv) component is not satisfied, since it does not get a correction term.
So, we have found that, just like in the Schwarzschild spacetime, the Unruh vacuum is regular on the event horizon but not on the past horizon.

But what about on r_-?
To answer this question we need to study the regularity of

$$\begin{aligned}\langle \mathrm{U}_+|T_{UU}|\mathrm{U}_+\rangle\Big|_{r_-} &= \langle \mathrm{U}_+|T_{uu}|\mathrm{U}_+\rangle\Big|_{r_-}\left(\frac{r_-}{r_- - r}\right)^2 \\ &= \left(-\frac{1}{48\pi}\kappa_-^2 + \frac{1}{48\pi}\kappa_+^2\right)\left(\frac{r_-}{r_- - r}\right)^2 .\end{aligned} \tag{5.34}$$

At $r = r_-$, the first factor, which is the expectation value for the Unruh vacuum using the Eddington-Finkelstein coordinates, does not vanish and is finite, while the second factor, coming from the transformation between u and U_-, diverges. The Unruh vacuum is not regular on the inner horizon. Moreover, since only the (uu) component received an additional term, we have that the Unruh vacuum is not regular also on the Cauchy horizon, as for the Boulware vacuum.
We conclude that the Unruh vacuum built with the coordinates (U_+, v) is regular on the event horizon, but not on the past, Cauchy and inner horizon.

The same analysis can be performed using the Unruh vacuum built with the coordinates (U_-, v). The expectation value will be

$$\langle \mathrm{U}_-|T_{uu}|\mathrm{U}_-\rangle = \langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle + \frac{1}{48\pi}\kappa_-^2 . \tag{5.35}$$

The additional term will remove the divergence at $U_- = 0$, but not at $U_+ = 0$. Performing the same analysis as before, we see that now the Unruh vacuum will be regular on the inner horizon, but not on the event, past and Cauchy horizon.

Summarizing, the analysis of the regularity of the Unruh vacuum showed that if we construct the vacuum with (U_+, v), the state will be regular on $U_+ = 0$ but not on $V_+ = 0$. Conversely, building the vacuum with the coordinates (U_-, v), we will get a state regular on $U_- = 0$ but not on $V_- = 0$.
It is not possible to define the Unruh vacuum in such a way that it is regular on both horizons. A plot of the different expectation values is given in Fig. 5.3.

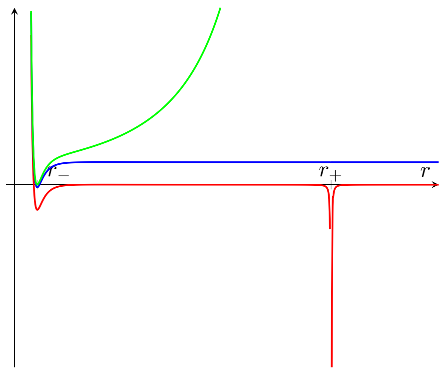


Figure 5.3: $\langle T_{UU}\rangle$ for Boulware and Unruh vacuum in Reissner-Nordström spacetime. The red function shows $\langle \mathrm{B}|T_{UU}|\mathrm{B}\rangle$ and its divergence at $r = r_+$. The blue function instead shows $\langle \mathrm{U}_+|T_{UU}|\mathrm{U}_+\rangle$, with a small vertical offset in order to not overlap with the blue function.
As we can see, $\langle \mathrm{U}_+|T_{UU}|\mathrm{U}_+\rangle$ is now regular at the outer horizon.
Finally, the green function shows $\langle \mathrm{U}_-|T_{UU}|\mathrm{U}_-\rangle$.It diverges at the outer horizon.
Note, for r large enough also the green function approaches zero.
We find the opposite scenario at the inner horizon.

5.4 Israel-Hartle-Hawking Vacuum

The last vacuum state we will study is the Israel-Hartle-Hawking vacuum. It is built with respect to the Kruskal coordinates U, V, therefore, starting from the "vacuum polarization" contribution given by the Boulware vacuum, we will have an additional term for both the (uu) and (vv) components, while the (uv) components remain unchanged as usual.

To derive the explicit expression we just need to follow the same steps as in Section 5.3. Since the relation between V and v is the same as U and u, we immediately find

$$\langle \mathrm{H}|T_{uu}|\mathrm{H}\rangle = \langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle + \frac{1}{48\pi}\kappa_\pm^2 \tag{5.36}$$

$$\langle \mathrm{H}|T_{uv}|\mathrm{H}\rangle = \langle \mathrm{B}|T_{uv}|\mathrm{B}\rangle \tag{5.37}$$

$$\langle \mathrm{H}|T_{vv}|\mathrm{H}\rangle = \langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle + \frac{1}{48\pi}\kappa_\pm^2. \tag{5.38}$$

Before studying the regularity at the horizons, we recall that the Israel-Hartle-Hawking state can be interpreted as the state describing a black hole enclosed in a box, in thermal equilibrium with its own radiation.

Indeed, as $r \to \infty$, the contribution coming from $\langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle$ and $\langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle$ vanish, leaving only the constant term $\kappa_\pm^2/48\pi$, which again can be interpreted as the Hawking flux.

Now let us study the regularity at the horizons. As usual, we need to express the energy-momentum tensor in terms of a set of coordinates regular on the horizons. Since the outgoing Israel-Hartle-Hawking mode is the same as the outgoing Unruh mode, by just looking at the T_{uu} component, we can already say that the also the Israel-Hartle-Hawking vacuum will not be regular on the horizon, or more properly:

- If the state $|\mathrm{H}\rangle$ is built using U_+, the state will be regular on $U_+ = 0$ and $V_+ = 0$;
- If the state $|\mathrm{H}\rangle$ is built using U_-, the state will be regular on $U_- = 0$ and $V_- = 0$.

The same analysis can be performed for the T_{vv}, which also gets an additional term.
In the Schwarzschild spacetime, the additional term coming from the relation between the two different sets of coordinates was enough to remove both the divergences, making the Israel-Hartle-Hawking vacuum a regular state everywhere. In the Reissner-Nordström spacetime instead, this is not the case. We have a similar situation as discussed above for the Unruh vacuum: we can make the state regular on the outer or inner horizon, but not on both at the same time.

So, we have seen that, unlike in the Schwarzschild spacetime, *in the Reissner-Nordström spacetime is not possible to construct a vacuum state that is regular everywhere.* The reason for that is the presence of two different horizons.

Simpon-Visser Black Hole

6 Classical Simpson-Visser Black Hole

In this chapter, we present the basic feature of a class of the so-called *regular black hole*, the *Simpson-Visser Black Hole* (Simpson et al., 2019).
We are interested in this metric since, as we will see in detail later, it presents two horizons but with the surface gravity equal up to a sign. This peculiar property will be then used in Chapter 7, to show the existence of a vacuum state regular everywhere.

6.1 Simpson-Visser Metric

A simple regular metric can be obtained, starting from the usual Schwarzschild metric, by replacing $r \to \sqrt{r^2 + a^2}$, with a some real constant parameter. With this substitution we find

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{\sqrt{r^2 + a^2}}\right)\mathrm{d}t^2 - \left(1 - \frac{2M}{\sqrt{r^2 + a^2}}\right)^{-1}\mathrm{d}r^2 - (r^2 + a^2)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2). \quad (6.1)$$

Usual Schwarzschild metric presents two singularities, linked to the diverges in the g_{tt} and g_{rr} components of the metric. The divergence in the g_{tt} component corresponds to the real physical singularity at $r = 0$, while the one in the g_{rr} component, located at $r = 2M$, is due the choice of a bad set of coordinates and it describes the event horizon.
In the Simpson-Visser metric we have a different scenario. While the divergence in g_{rr} might be present, depending on the values of a, the component g_{tt} never diverges, therefore we do not have any physical singularity.

Depending on the value of the parameter a, the metric Eq. 6.1 describes different physical configurations.

Since with the choice $a = 0$ we get the usual Schwarzschild metric, it makes sense to require $a \neq 0$.
Moreover, since this metric is a minimal modification of ordinary Schwarzschild metric, they share a lot of properties. Indeed, just like the Schwarzschild metric, the Simpson-Visser spacetime is static and spherically symmetric, or equivalently, it admits a non-vanishing timelike Killing vector and there are no off-diagonal components in the matrix representation of the metric.
This allows us to conclude that, like for the Schwarzschild spacetime, fixed r surfaces correspond to spherical surfaces.

A very interesting feature of this spacetime, not shared with Schwarzschild spacetime, is the natural domain of the coordinates, also for the radial one

$$r \in (-\infty, +\infty), \quad t \in (-\infty, +\infty), \quad \theta \in [0, \pi], \quad \varphi \in (-\pi, \pi]. \tag{6.2}$$

As mentioned before, different values of a yield different physical configurations. It is interesting to study the behavior of the radial null curves ($\mathrm{d}s^2 = 0$, $\mathrm{d}\theta = \mathrm{d}\varphi = 0$)

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\left(1 - \frac{2M}{\sqrt{r^2 + a^2}}\right). \tag{6.3}$$

For $a > 2M$, we have that

$$\frac{\mathrm{d}r}{\mathrm{d}t} \neq 0 \quad \forall r \in (-\infty, +\infty). \tag{6.4}$$

It is the geometry of a two-way traversable wormhole. Looking at the behavior of the curvature tensor components and curvature invariants, we find a maximum for both at $r = 0$. This allows us to conclude that the wormhole's throat is located at $r = 0$, and the region of negative r can be interpreted as the universe on the other side of the throat seen by an observer in our own universe. Penrose diagram is showed in Fig. 6.1c.

For $a = 2M$, we have

$$\frac{\mathrm{d}r}{\mathrm{d}t} \xrightarrow{r \to 0} 0. \tag{6.5}$$

Despite the presence of an horizon at $r = 0$, this geometry does not represent a black hole, but rather a one-way traversable wormhole. Also in this case the throat is located at $r = 0$. The associated Penrose diagram is given in Fig. 6.1a.

For $a < 2M$ we can find values of r such that

$$\exists\, r^{\pm} \in \mathbb{R} \text{ s.t. } \frac{\mathrm{d}r}{\mathrm{d}t} = 0. \tag{6.6}$$

This happens for

$$r^{\pm} = \pm\sqrt{4M^2 - a^2}. \tag{6.7}$$

These two locations correspond to a pair of symmetrically placed horizons. Note: one of the horizons is located in the region with $r < 0$.
For regular black holes we are interested in this interval, $a \in (0, 2M)$. We can have different physical interpretations. This spacetime looks like Schwarzschild, but now the hypersurface $r = 0$ does not act as a singularity, but rather as the boundary between our universe and a separate copy of it. The negative values of r describe the copy of the universe once we go through the "bounce" at $r = 0$. The corresponding Penrose diagram is given in Fig. 6.1d. This construction can be repeated *ad infinitum*, moving from universe to universe. In this case, the time coordinate is not constrained and runs from the bottom to the top.

Alternatively, we could set periodic boundary conditions for the time coordinate. Doing so, we identify the "future bounce" with the "past bounce", as we can see in Fig. 6.1b. Crossing the future surface $r = 0$ we get bounced back to the past surface $r = 0$. In this configuration, the time coordinate still runs from the bottom to the top, but it is now cyclical.

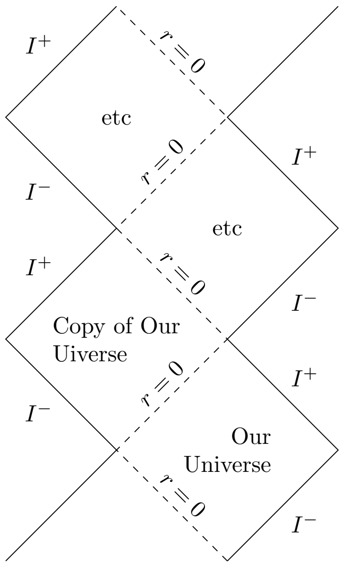
6.2 Curvature Tensor and Invariants

In order to correctly describe a regular geometry, the curvature tensor, scalar and invariant, computed for the Simpson-Visser metric, should be regular everywhere.

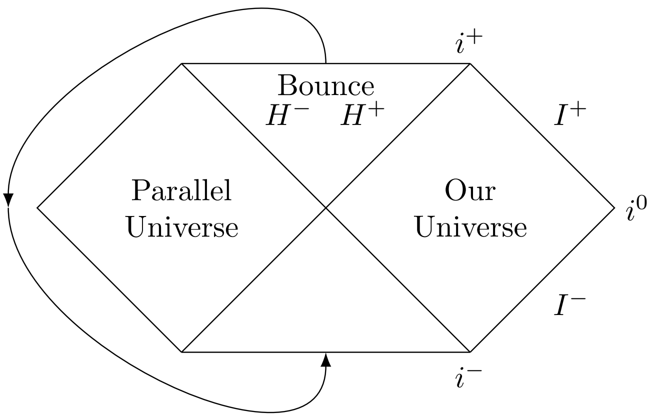
We start with the Weyl tensor. The Weyl tensor is a measure of the curvature of spacetime. Enjoys the same symmetries as the Riemann tensor and can be shown to be the "trace-less part" of the Riemann tensor. Thanks to this property, the Weyl tensor can be used, together with the Ricci tensor, to express the full Riemann tensor. In conformally flat spacetimes[1], the Weyl tensor always vanishes, allowing to compute the Riemann tensor directly from the Ricci tensor. Moreover, the Weyl tensor is invariant under conformal transformations.
Assuming $a \neq 0$, the non-vanishing components are

$$\begin{aligned} C^{t\theta}{}_{t\theta} = C^{t\varphi}{}_{t\varphi} = C^{r\theta}{}_{r\theta} = C^{r\varphi}{}_{r\varphi} = -\frac{1}{2}C^{tr}{}_{tr} = -\frac{1}{2}C^{\theta\varphi}{}_{\theta\varphi} \\ = \frac{6r^2M + a^2\left(2\sqrt{r^2+a^2} - 3M\right)}{6(r^2+a^2)^{5/2}}. \end{aligned} \tag{6.8}$$

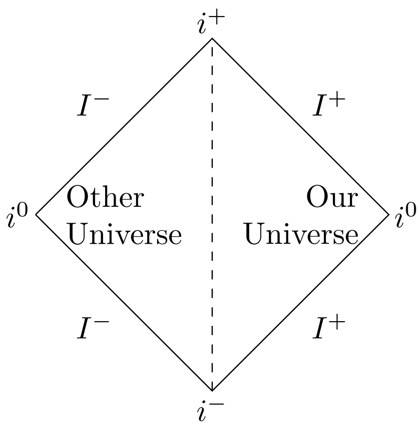
[1]Recall that any 2D spacetime is conformally flat



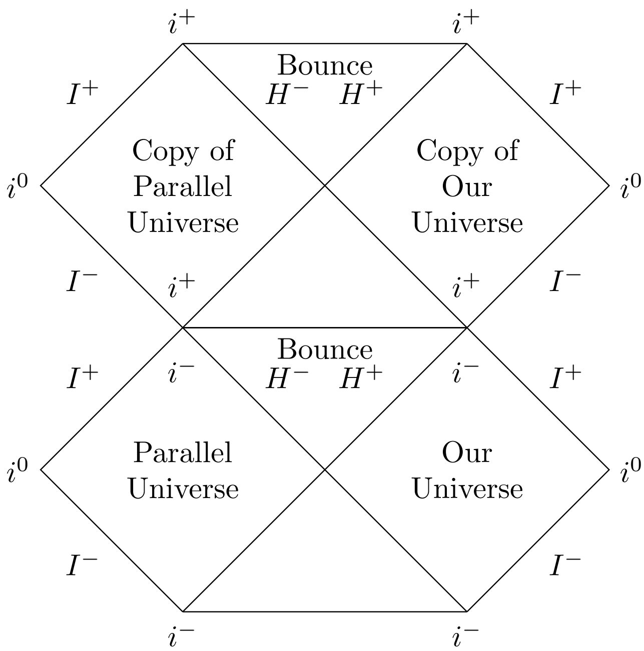
(a) Penrose diagram of Simpson-Visser spacetime for $a = 2M$.



(b) Penrose diagram of Simpson-Visser spacetime for $a \in (0, 2M)$. The time coordinate still runs from bottom to top, but with periodic boundary conditions.



(c) Penrose diagram of Simpson-Visser spacetime for $a > 2M$. The dashed line represents the wormhole throat.



(d) Penrose diagram of Simpson-Visser spacetime for $a \in (0, 2M)$. The time coordinate is not constrained and runs from bottom to top.

which behaves as

$$\begin{aligned} C^{t\theta}{}_{t\theta} = C^{t\varphi}{}_{t\varphi} = C^{r\theta}{}_{r\theta} = C^{r\varphi}{}_{r\varphi} = -\frac{1}{2}C^{tr}{}_{tr} = -\frac{1}{2}C^{\theta\varphi}{}_{\theta\varphi} \\ \xrightarrow{r\to 0} \frac{2a-3M}{6a^3} \end{aligned} \tag{6.9}$$

so not vanishing and regular.

A similar analysis can be done on the Riemann tensor. The non-vanishing components are more complicated, here we just mention the behavior near the origin

$$R^{tr}{}_{tr} \to -\frac{M}{a^3} \tag{6.10}$$

$$R^{t\theta}{}_{t\theta} = R^{t\varphi}{}_{t\varphi} \to 0 \tag{6.11}$$

$$R^{r\theta}{}_{r\theta} = R^{r\varphi}{}_{r\varphi} \to \frac{2M-a}{a^3} \tag{6.12}$$

$$R^{\theta\varphi}{}_{\theta\varphi} \to \frac{1}{a^2}. \tag{6.13}$$

Instead, at infinity $|r| \to +\infty$, all the components approach 0, meaning that at large distances, this geometry models general relativity in the *weak-field* approximation.
We can conclude that in the full domain $r \in (-\infty, +\infty)$, the Riemann tensor is always finite. Moreover, in the interval $a \in (0, 2M]$ there is an horizon, but not a singularity, and the metric describes the geometry of a regular black hole. For the case $a > 2M$ there are also no singularities.

For the Ricci tensor instead, we have

$$-2R^t_t = R^\theta_\theta = R^\varphi_\varphi = \frac{2a^2M}{(r^2+a^2)^{5/2}} \tag{6.14}$$

$$R^r_r = \frac{a^2\left(3M - 2\sqrt{r^2+a^2}\right)}{(r^2+a^2)^{5/2}}. \tag{6.15}$$

At the origin, these components approach a finite limit

$$-2R^t_t = R^\theta_\theta = R^\varphi_\varphi \to \frac{2M}{a^3} \tag{6.16}$$

$$R^r_r \to \frac{3M-2a}{a^3} \tag{6.17}$$

while all the components vanish at infinity.

Lastly, we can consider the Ricci, or curvature, scalar

$$R = \frac{2a^2(3M - \sqrt{r^2 + a^2})}{(r^2 + a^2)^{5/2}} \tag{6.18}$$

which again vanishes at infinity, while is finite at the origin

$$R \to \frac{2(3M - a)}{a^3}. \tag{6.19}$$

Having an explicit expression for the curvature tensor we can also compute their contractions

$$R_{\mu\nu}R^{\mu\nu} = \frac{a^4\left[4\left(\sqrt{r^2 + a^2} - 3M/2\right)^2 + (3M)^2\right]}{(r^2 + a^2)^{5/2}} \tag{6.20}$$

$$C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu} = \frac{4}{3(r^2 + a^2)^{5/2}}\left[3M(2r^2 - a^2) + 2a^2\sqrt{r^2 + a^2}\right]^2 \tag{6.21}$$

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu} + 2R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2 \tag{6.22}$$

which are all finite, well-behaved and vanish at infinity.

6.3 Source of Simpson-Visser Geometry

For $a = 0$, the Simpson-Visser Eq. 6.1 metric is just the Schwarzschild metric and therefore satisfy the vacuum Einstein equations

$$R_{\mu\nu} = 0. \tag{6.23}$$

For $a \neq 0$, the Simpson-Visser metric does not solve the equation above, but rather of the full Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{6.24}$$

This means that, starting from the Ricci tensor and scalar, given in Section 6.2, we can solve the Einstein equations and find the hypothetical source of the gravitational field.

Outside the horizon we can identify

$$\rho = -T_t{}^t, \qquad p_\parallel = T_r{}^r, \qquad p_\perp = T_\theta{}^\theta = T_\varphi{}^\varphi. \tag{6.25}$$

Solving the Einstein equations we find

$$\rho = -\frac{a^2\left(\sqrt{r^2+a^2}-4M\right)}{8\pi G_N\left(r^2+a^2\right)^{5/2}} \tag{6.26}$$

$$p_\parallel = -\frac{a^2}{8\pi G_N\left(r^2+a^2\right)^2} \tag{6.27}$$

$$p_\perp = \frac{a^2\left(\sqrt{r^2+a^2}-M\right)}{8\pi G_N\left(r^2+a^2\right)^{5/2}}. \tag{6.28}$$

Within the framework of classical general relativity, it is reasonable to expect some conditions on the energy-momentum tensor, such as positivity of the energy density and dominance of the energy density over the pressure (see (Poisson, 2004) for more details).
We are interested in the *null energy condition*, which states that the energy density, measured by any null observer, must be non-negative[2]. In mathematical terms, this correspond to the requirment

$$T_{\mu\nu}p^\mu p^\nu \geq 0 \tag{6.29}$$

where p^μ is an arbitrary future-directed null vector.
This condition can be written in a more useful form by decomposing the energy-momentum tensor in its usual diagonal form $T_{\mu\nu} = \mathrm{diag}(\rho, p_1, p_2, p_3)$.
Then, the null energy condition can be expressed as

$$\rho + p_i \geq 0, \quad i = 1, 2, 3. \tag{6.30}$$

Therefore, for the Simpson-Visser metric we need to check $\rho+p_\parallel \geq 0$ and $\rho+p_\perp \geq 0$, which should hold for any r, a, M. From the first condition we find

$$\rho + p_\parallel = \frac{-a^2\left(\sqrt{r^2+a^2}-2M\right)}{4\pi G(r^2+a^2)^{5/2}}. \tag{6.31}$$

Outside the horizon we have $\sqrt{r^2+a^2} > 2M$, and we immediately see that $\rho+p_\parallel < 0$, therefore we conclude that the null energy condition is not satisfied.

Inside the horizon we need to remember that the role of the t and r coordinates are interchanged, therefore we have $\rho = -T_r{}^r$ and $p_\parallel = T_t{}^t$, while nothing changes

[2] We can formulate also other energy condtions, but the violatetion of the null condition imply the violatetion of also the others.

for the $p_\perp$ component. So, inside the horizon we have

$$\rho = \frac{a^2}{8\pi G_N \left(r^2+a^2\right)^2} \tag{6.32}$$

$$p_\parallel = \frac{a^2\left(\sqrt{r^2+a^2}-4M\right)}{8\pi G_N \left(r^2+a^2\right)^{5/2}}. \tag{6.33}$$

For the energy condition we find

$$\rho + p_\parallel = \frac{a^2\left(\sqrt{r^2+a^2}-2M\right)}{4\pi G(r^2+a^2)^{5/2}}, \tag{6.34}$$

but, inside the horizon $\sqrt{r^2+a^2} < 2M$, and so again the null energy condition is not satisfied.
The two results can be combined

$$\rho + p_\parallel = -\frac{a^2\left|\sqrt{r^2+a^2}-2M\right|}{4\pi G(r^2+a^2)^{5/2}} \tag{6.35}$$

from which we see that the null energy condition is violated everywhere, *except on* the horizon(s), when present.

So we have seen that the Simpson-Visser metric describes the geometry of a regular black hole[3], but violates all the energy conditions.

6.4 Radial Null Directions

Studying the curvature tensors and invariants we have seen that indeed the Simpson-Visser geometry does not shows spacetime singularities, for all vales of M and $a \neq 0$.
As already mentioned, the Simpson-Visser metric for $a \in (0, 2M)$ describes a regular black hole. We can see this by studying the causal structure, by looking at the lightcones and the radial null directions, in order to see if it act as black hole. This is done by taking the line element of the metric given by Eq. 6.1, setting $\mathrm{d}\varphi = \mathrm{d}\theta = 0$ since we are interested in radial directions, and $\mathrm{d}s^2 = 0$ since we are considering light, which travels on null directions.
The metric as presented in Eq. 6.1 is singular at the horizons. Since it is, like in Schwarzschild, a coordinate singularity and not a physical one, we can build

[3] Or a wormhole, since a can take any non-vanishing value.

a set of coordinates regular in that region. We can define the advanced(ingoing) Eddington-Finkelstein coordinate

$$v \equiv t + r^*, \quad r^* \equiv \int \mathrm{d}r \frac{1}{C(r)}, \quad C(r) = 1 - \frac{2M}{\sqrt{r^2 + a^2}} \tag{6.36}$$

and write the metric in the advanced Eddington-Finkelstein form (we neglect the angular part)

$$\mathrm{d}s^2 = C(r)\mathrm{d}v^2 - 2\mathrm{d}v\mathrm{d}r. \tag{6.37}$$

Next, we define a timelike coordinate t' as

$$t' = v - r \tag{6.38}$$

and find

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{\sqrt{r^2 + a^2}}\right)\mathrm{d}t'^2 - \left(1 + \frac{2M}{\sqrt{r^2 + a^2}}\right)\mathrm{d}r^2 - \frac{4M}{\sqrt{r^2 + a^2}}\mathrm{d}t'\mathrm{d}r. \tag{6.39}$$

To understand the meaning of this new coordinate, we can study surfaces with $t' = const$. It is easy to see that these surfaces are spacelike, and therefore have a timelike norm. This means that t' is a good "time coordinate", meaning that can be used to foliate spacetime.

Now we can set $\mathrm{d}s^2 = 0$, and dividing everything by $\mathrm{d}r^2$ we obtain

$$\left(1 - \frac{2M}{\sqrt{r^2 + a^2}}\right)\frac{\mathrm{d}t'^2}{\mathrm{d}r^2} - \frac{4M}{\sqrt{r^2 + a^2}}\frac{\mathrm{d}t'}{\mathrm{d}r} - \left(1 + \frac{2M}{\sqrt{r^2 + a^2}}\right) = 0. \tag{6.40}$$

This can be easily solved for $\mathrm{d}t'/\mathrm{d}r$, and we find

$$\frac{\mathrm{d}t'}{\mathrm{d}r} = \frac{2M/\sqrt{r^2 + a^2} \pm 1}{1 - 2M/\sqrt{r^2 + a^2}} = \begin{cases} -1 \\ \dfrac{1 + 2M/\sqrt{r^2 + a^2}}{1 - 2M/\sqrt{r^2 + a^2}} \end{cases} \tag{6.41}$$

Integrating the first solution is immediate, and we get

$$t' = -r + const \quad \leftrightarrow \quad v = const \tag{6.42}$$

which corresponds to the radial *ingoing* null directions.
The radial *outgoing* null directions are obtained by integrating the second solution. The actual analytical solution is rather involved and not of our interest, but can be plotted and we obtain Fig. 6.2.

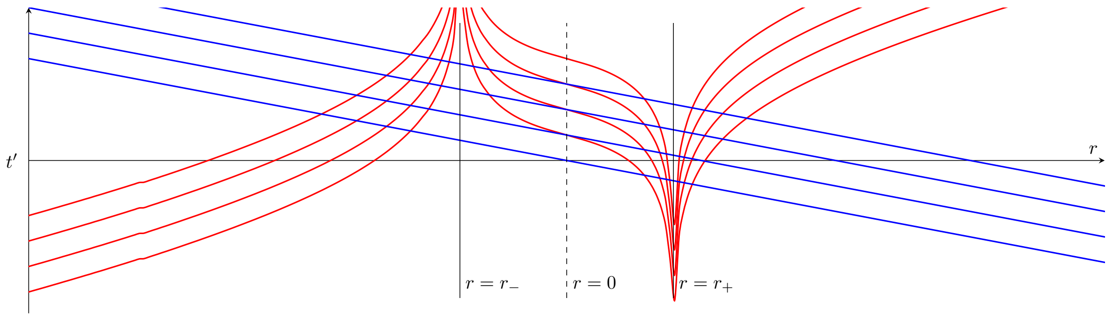


Figure 6.2: Radial null directions for Simpson-Visser spacetime. The red lines are the outgoing directions, while the blue ones are the ingoing. Intersections between the two directions give the lightcones.

As we can see, outside the horizon $r_+ = \sqrt{4M^2 - a^2}$, light trajectories can be both ingoing and outgoing. But once inside the horizon, all trajectories are towards the origin and, since the conformal factor is negative in the region $0 < r < r_+$, they are all characterized by decreasing values of r. There are no physical trajectories with $r = const$.
At negative r we have the mirrored configuration. For $-r_- < r < 0$ all the trajectories are towards the origin.

So, we can conclude that the Simpson-Visser metric, with $a \in (0, 2M)$, correctly describes a spacetime containing a region, enclosed by the horizon, causally disconnected from the rest of the spacetime.

6.5 Surface Gravity

As we mentioned before, the Simpson-Visser metric features a Killing vector associated with invariance under time translation. To this Killing vector, we can associate a quantity called *surface gravity*, which measures the rapidity in r with which vanish the norm of the Killing vector.
Let us consider the Killing vector associated with the time-translation invariance, $\xi_\mu = (1, 0, 0, 0)$. Looking at the metric Eq. 6.1 we can find the norm of this vector to be

$$\xi^\mu \xi_\mu = g_{\mu\nu} \xi^\mu \xi^\nu = C(r) = \left(1 - \frac{2M}{\sqrt{r^2 + a^2}}\right). \tag{6.43}$$

Then we can define the vector

$$\ell_\mu \equiv \partial_\mu \sqrt{|\xi^\mu \xi_\mu|} \tag{6.44}$$

and finally

$$\kappa \equiv \sqrt{|\ell_\mu \ell^\mu|}\Big|_H . \tag{6.45}$$

For the Simpson-Visser metric we have

$$\ell_\mu = \left(0, \frac{Mr}{(r^2+a^2)^{3/2}\sqrt{C(r)}}, 0, 0\right). \tag{6.46}$$

In order to compute the contraction we need the radial component of the inverse metric, which simply is $g^{rr} = C(r)$, and so we find

$$\ell_\mu \ell^\mu = \ell_r \ell_r g^{rr} = \frac{M^2 r^2}{(r^2+a^2)^3}. \tag{6.47}$$

Now we can take the square root and evaluate everything on the horizon

$$\kappa_\pm = \frac{Mr}{(r^2+a^2)^{3/2}}\Big|_{H^\pm} = \frac{\pm\sqrt{4M^2-a^2}}{8M^2}. \tag{6.48}$$

The two values of surface gravity correspond to the two different horizons. An interesting feature of the Simpson-Visser spacetime, not shared with other spacetimes with more than one horizon(such as Reissner-Nordström), is that the two values are equal up to a sign

$$|\kappa_+| = |\kappa_-|, \tag{6.49}$$

and therefore also when squared

$$\kappa_+^2 = \kappa_-^2. \tag{6.50}$$

This will be extremely important in the discussion about the regularity of states in the next chapter.
For consistency, we can easily see that taking $a = 0$ we get the known result for the Schwarzschild metric, $\kappa = 1/4M$.
Knowing the surface gravity we can compute the Hawking temperature

$$T_H = \frac{\hbar\kappa}{2\pi k_B}. \tag{6.51}$$

In the next chapter, we will discuss the regularity and physical meaning of states in a spacetime curved by the presence of a Simpson-Visser regular black hole.

7 Quantization in Simpson-Visser Spacetime

We now have the basic ingredients to study quantum states in a spacetime curved by the presence of a regular black hole.
Both the construction of the states and the analysis of their regularity will be performed in close analogy to the one carried on for Schwarzschild and Reissner-Nordström spacetime in Chapter 4 and Chapter 5.

7.1 Boulware Vacuum

We start with the simplest vacuum state: the Boulware vacuum. As already discussed, the Boulware vacuum is constructed in order to reproduce the Minkowski vacuum at infinity. In order to do so, the associated modes are built using the Eddington-Finkelstein coordinates, which approach the Minkowski null coordinates. The Eddington-Finkelstein coordinates are built using the tortoise radial coordinate r^*, defined as

$$\begin{aligned} r^* &= \int \frac{\mathrm{d}r}{C(r)} \\ &= r + 2M \ln\left(\frac{r}{a} + \sqrt{\left(\frac{r}{a}\right)^2 + 1}\right) + \frac{4M^2}{\sqrt{4M^2 - a^2}} \times \\ &\qquad \times \ln \frac{\left|\sqrt{4M^2 - a^2}\tanh\left(\frac{\sinh^{-1}\left(\frac{r}{a}\right)}{2}\right) - 2M + a\right|}{\left|\sqrt{4M^2 - a^2}\tanh\left(\frac{\sinh^{-1}\left(\frac{r}{a}\right)}{2}\right) + 2M - a\right|}. \end{aligned} \tag{7.1}$$

Then the Eddington-Finkelstein coordinates are built as usual

$$u = t - r^*, \qquad v = t + r^*, \tag{7.2}$$

and the Boulware modes as

$$\frac{\exp\{-i\omega u\}}{\sqrt{4\pi\omega}}, \qquad \frac{\exp\{-i\omega v\}}{\sqrt{4\pi\omega}}. \tag{7.3}$$

The general procedure to build a vacuum state is to start with a matter field, in our case for simplicity a massless scalar, and to expand the field using the modes written above and a set of ladder operators. Then we can define the vacuum state[1] $|\mathrm{B}\rangle$ as the state annihilated by the destruction operator.
Now we can compute the expectation value for the energy-momentum tensor in the Boulware vacuum. We simply use Eq. 4.3 and Eq. 4.4 with the conformal factor of the Simpson-Visser metric. Expressions are slightly more involved, but evaluating the derivatives and after some algebra we find

$$\langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle = \langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle = \frac{1}{192\pi}\left[-\frac{4M^2r^2}{(r^2+a^2)^{3/2}}+ \right. \tag{7.4}$$

$$\left. +\frac{4M\left(\sqrt{r^2+a^2}-2M\right)(a^2-2r^2)}{(r^2+a^2)^5}\right]$$

$$\langle \mathrm{B}|T_{uv}|\mathrm{B}\rangle = \frac{1}{96\pi}\left(1-\frac{2M}{\sqrt{r^2+a^2}}\right)\frac{2M(a^2-2r^2)}{(r^2+a^2)^{5/2}}. \tag{7.5}$$

We are interested in regularity at the horizons, so in order to study how the expectation values behave, we need to express them in a set of coordinates regular there, the Kruskal coordinates. Following the same steps as in Subsection 5.2.1, regularity means regular when expressed in regular coordinates, then we use the relation between Eddington-Finkelstein and Kruskal coordinates to find some regularity conditions. For example, on the outer horizon, corresponding to $U_+ = 0$, we find

$$C(r)^{-2}|\langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle| < \infty \tag{7.6}$$

$$C(r)^{-1}|\langle \mathrm{B}|T_{uv}|\mathrm{B}\rangle| < \infty \tag{7.7}$$

$$|\langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle| < \infty. \tag{7.8}$$

For the (vv) component we do not have any problem. At r_+ it evaluates to

$$\langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle = -\frac{\kappa_+^2}{48\pi} = -\frac{\kappa^2}{48\pi}, \tag{7.9}$$

which is perfectly finite.
It is interesting to notice that, as already mentioned, in this spacetime $\kappa_+^2 = \kappa_-^2 \equiv \kappa^2$. For this reason, we will no longer distinguish between the κ_+^2 and κ_-^2.
Also the (uv) component shows no divergences. Indeed, since it is proportional to the conformal factor, it identically vanishes on the horizons.
Just like the Schwarzschild and Reissner-Nordström spacetime, the problematic

[1] We stress again that this state cannot be interpreted as the state with no particles due to the lack of Poincaré invariance.

condition is the one concerning the (uu) component. At the outer horizon, the term $\langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle$ alone is finite, but not the conformal factor $C(r)^2$, leaving a divergence in the expectation value for the energy-momentum tensor.

We conclude that the Boulware vacuum is not a regular state. Again, since it is not regular on the horizon it can be used to describe vacuum polarization of objects bigger than their Schwarzschild radius.

7.2 Unruh Vacuum

To solve the problem with the (uu) component at the outer horizon we can introduce the Unruh vacuum. It is the state associated with the modes

$$\frac{\exp\{-i\omega U_\pm\}}{\sqrt{4\pi\omega}}, \qquad \frac{\exp\{-i\omega v\}}{\sqrt{4\pi\omega}}. \tag{7.10}$$

It shares the same ingoing mode with the Boulware state, but the outgoing mode is built using the Kruskal coordinate

$$U_\pm = \mp\frac{1}{\kappa_+}e^{-\kappa_\pm u}. \tag{7.11}$$

For the moment let us consider the state built with the Kruskal coordinate for r_+. Since the ingoing mode is the same as the Boulware ingoing mode, the "ingoing" sector of the Unruh vacuum behaves as the Boulware vacuum. Therefore, they have the same expectation value

$$\begin{aligned}\langle \mathrm{U}_+|T_{vv}|\mathrm{U}_+\rangle &= \langle \mathrm{B}|T_{vv}|\mathrm{B}\rangle \\ &= \frac{1}{192\pi}\left[-\frac{4M^2r^2}{(r^2+a^2)^{3/2}}+\right. \\ &\quad\left.+\frac{4M\left(\sqrt{r^2+a^2}-2M\right)(a^2-2r^2)}{(r^2+a^2)^5}\right].\end{aligned} \tag{7.12}$$

Using Eq. 3.40 we find also the other components. The (uv) component does not get an additional term, while for the (uu) component we only need to compute, as we did before, the functional which relates the two sets of coordinates, in this case, the Eddington-Finkelstein u and Kruskal U_+

$$\langle \mathrm{U}_+|T_{uu}|\mathrm{U}_+\rangle = \langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle + \frac{1}{24\pi}F_{U_+}(u). \tag{7.13}$$

From the previous chapters, we already know that the additional term is proportional to the surface gravity squared

$$F_{U_+}(u) = \frac{1}{2}\kappa_+^2 = \frac{1}{2}\kappa^2 \tag{7.14}$$

and we finally have

$$\langle \mathrm{U}_+|T_{uu}|\mathrm{U}_+\rangle = \frac{\kappa^2}{48\pi} + \frac{1}{192\pi}\left[-\frac{4M^2r^2}{(r^2+a^2)^{3/2}} + \right. \tag{7.15}$$
$$\left. + \frac{4M\left(\sqrt{r^2+a^2}-2M\right)(a^2-2r^2)}{(r^2+a^2)^5}\right]$$

$$\langle \mathrm{U}|T_{uv}|\mathrm{U}\rangle = \frac{1}{96\pi}\left(1-\frac{2M}{\sqrt{r^2+a^2}}\right)\frac{2M(a^2-2r^2)}{(r^2+a^2)^{5/2}} \tag{7.16}$$

$$\langle \mathrm{U}_+|T_{vv}|\mathrm{U}_+\rangle = \frac{1}{192\pi}\left[-\frac{4M^2r^2}{(r^2+a^2)^{3/2}} + \right. \tag{7.17}$$
$$\left. + \frac{4M\left(\sqrt{r^2+a^2}-2M\right)(a^2-2r^2)}{(r^2+a^2)^5}\right].$$

To check the regularity of the Unruh vacuum we need the check the usual conditions, namely require the regularity of the components when expressed in terms of regular coordinates. Since the (uv) and (vv) components are the same as the Boulware vacuum, and for these components the regularity conditions are satisfied, we can already assert that also for the Unruh vacuum those components are regular. We only need to check the (uu) component.
In particular, when evaluated at the outer horizon we have

$$\langle \mathrm{U}_+|T_{uu}|\mathrm{U}_+\rangle\Big|_{r_+} = \frac{\kappa_+^2}{48\pi} - \frac{\kappa_+^2}{48\pi} = 0. \tag{7.18}$$

So we have found that, on the outer horizon, the expectation value of the (uu) component of the energy-momentum tensor in Eddington-Finkelstein coordinates vanishes, making the regularity condition satisfied.
The first term comes from the relation between the coordinates, while the second is the vacuum polarization term (expectation value for the Boulware vacuum).

Since we have also a past horizon, we can check regularity there. When evaluated at the past horizon, we find

$$\langle \mathrm{U}_+|T_{uu}|\mathrm{U}_+\rangle\Big|_{r_-} = \frac{\kappa_+^2}{48\pi} - \frac{\kappa_-^2}{48\pi} = 0. \tag{7.19}$$

In the last expression, the first term comes from the relation between Eddington-Finkelstein and Kruskal coordinates, proportional to κ_+^2, while the second one comes from the vacuum polarization evaluated at r_-.
Since the surface gravities for the Simpson-Visser spacetime are equal up to a sign, $\langle \mathrm{U}|T_{uu}|\mathrm{U}\rangle$ identically vanishes, and so the condition on the (uu) component is satisfied.
However, the Unruh vacuum still shows singularities. Indeed, when we consider the past horizon, the regularity conditions are different.
The past horizon is described, in Kruskal coordinates by $V = 0$, so looking at Eq. 4.14, we need

$$\frac{\langle \mathrm{U}_+|T_{vv}|\mathrm{U}_+\rangle}{C^2(r)} < +\infty. \tag{7.20}$$

At r_-, $\langle \mathrm{U}_+|T_{vv}|\mathrm{U}_+\rangle$ is finite, but the conformal factor diverge.

Therefore we conclude that, also in the Simpson-Visser spacetime, the Unruh vacuum is not regular.

7.3 Israel-Hartle-Hawking Vacuum

From the previous analysis is clear that, in order to build a state regular everywhere, we should add a correction term also to the (vv) component. This is what happens when the state used to compute the expectation value is the Israel-Hartle-Hawking state.

The Israel-Hartle-Hawking vacuum state is the state associated with a set of modes built with both Kruskal coordinates

$$\frac{\exp\{-i\omega U_\pm\}}{\sqrt{4\pi\omega}}, \qquad \frac{\exp\{-i\omega V_\pm\}}{\sqrt{4\pi\omega}} \tag{7.21}$$

where as usual

$$V_\pm = \pm\frac{1}{\kappa_+}e^{\kappa_\pm v}. \tag{7.22}$$

Let us assume to build the vacuum with κ_+.
Since we already know the vacuum polarization terms corresponding to the expectation values for the Boulware vacuum, the expectation values for the Israel-

Hartle-Hawking vacuum can be computed using Eq. 3.40

$$\langle \mathrm{H}_+|T_{uu}|\mathrm{H}_+\rangle = \langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle + \frac{1}{24\pi}F_{U_+}(u) \tag{7.23}$$

$$\langle \mathrm{H}_+|T_{uv}|\mathrm{H}_+\rangle = \langle \mathrm{B}|T_{uv}|\mathrm{B}\rangle \tag{7.24}$$

$$\langle \mathrm{H}_+|T_{vv}|\mathrm{H}_+\rangle = \langle \mathrm{B}|T_{uu}|\mathrm{B}\rangle + \frac{1}{24\pi}F_{V_+}(v). \tag{7.25}$$

Computing the additional terms for the (uu) and (vv) components is immediate. We already know the one for the (uu) component, since it is the same as for the Unruh vacuum

$$F_{U_+}(u) = \frac{1}{2}\kappa_+^2 = \frac{1}{2}\kappa^2. \tag{7.26}$$

Given that the relation between U and u is the same as between V and v, we also have

$$F_{V_+}(v) = \frac{1}{2}\kappa_+^2 = \frac{1}{2}\kappa^2. \tag{7.27}$$

Therefore, the expectation value of the energy-momentum tensor in the Israel-Hartle-Hawking state is

$$\langle \mathrm{H}_+|T_{uu}|\mathrm{H}_+\rangle = \frac{\kappa^2}{48\pi} + \frac{1}{192\pi}\left[-\frac{4M^2r^2}{(r^2+a^2)^{3/2}} + \right. \tag{7.28}$$
$$\left. + \frac{4M\left(\sqrt{r^2+a^2}-2M\right)(a^2-2r^2)}{(r^2+a^2)^5}\right]$$

$$\langle \mathrm{H}_+|T_{uv}|\mathrm{H}_+\rangle = \frac{1}{96\pi}\left(1-\frac{2M}{\sqrt{r^2+a^2}}\right)\frac{2M(a^2-2r^2)}{(r^2+a^2)^{5/2}} \tag{7.29}$$

$$\langle \mathrm{H}_+|T_{vv}|\mathrm{H}_+\rangle = \frac{\kappa^2}{48\pi} + \frac{1}{192\pi}\left[-\frac{4M^2r^2}{(r^2+a^2)^{3/2}} + \right. \tag{7.30}$$
$$\left. + \frac{4M\left(\sqrt{r^2+a^2}-2M\right)(a^2-2r^2)}{(r^2+a^2)^5}\right].$$

Let us start the regularity analysis on the outer horizon $U_+ = 0$. Regularity conditions can be expressed as

$$\frac{\langle \mathrm{H}_+|T_{uu}|\mathrm{H}_+\rangle}{C(r)^2} < +\infty \tag{7.31}$$

$$\frac{\langle \mathrm{H}_+|T_{uv}|\mathrm{H}_+\rangle}{C(r)} < +\infty \tag{7.32}$$

$$\langle \mathrm{H}_+|T_{vv}|\mathrm{H}_+\rangle < +\infty. \tag{7.33}$$

At the horizon, the components evaluates to

$$\langle \mathrm{H}_+ | T_{uu} | \mathrm{H}_+ \rangle \Big|_{r_+} = \frac{\kappa_+^2}{48\pi} - \frac{\kappa_+^2}{48\pi} = 0 \tag{7.34}$$

$$\langle \mathrm{H}_+ | T_{uv} | \mathrm{H}_+ \rangle \Big|_{r_+} = 0 \tag{7.35}$$

$$\langle \mathrm{H}_+ | T_{vv} | \mathrm{H}_+ \rangle \Big|_{r_+} = \frac{\kappa_+^2}{48\pi} - \frac{\kappa_+^2}{48\pi} = 0. \tag{7.36}$$

At the outer horizon, all the components of the expectation value of the energy-momentum vanish and therefore satisfy the regularity conditions.

On the past horizon, we have a similar result. The regularity conditions in this case are

$$\langle \mathrm{H}_+ | T_{uu} | \mathrm{H}_+ \rangle < +\infty \tag{7.37}$$

$$\frac{\langle \mathrm{H}_+ | T_{uv} | \mathrm{H}_+ \rangle}{C(r)} < +\infty \tag{7.38}$$

$$\frac{\langle \mathrm{H}_+ | T_{vv} | \mathrm{H}_+ \rangle}{C^2(r)} < +\infty \tag{7.39}$$

while the components evaluated at r_- are

$$\langle \mathrm{H}_+ | T_{uu} | \mathrm{H}_+ \rangle \Big|_{r_-} = \frac{\kappa_+^2}{48\pi} - \frac{\kappa_-^2}{48\pi} = 0 \tag{7.40}$$

$$\langle \mathrm{H}_+ | T_{uv} | \mathrm{H}_+ \rangle \Big|_{r_-} = 0 \tag{7.41}$$

$$\langle \mathrm{H}_+ | T_{vv} | \mathrm{H}_+ \rangle \Big|_{r_-} = \frac{\kappa_+^2}{48\pi} - \frac{\kappa_-^2}{48\pi} = 0 \tag{7.42}$$

For the same reasons as above, the regularity conditions are now satisfied, making the Israel-Hartle-Hawking state regular everywhere in the Simpson-Visser spacetime.
This is a different scenario than the Reissner-Nordström spacetime. In the Reissner-Nordström spacetime we also have two horizons, but the values of the surface gravities are different, so we do not have any cancellation between κ_+^2 and κ_-^2. This is not true in the Simpson-Visser spacetime, and this particular property is the one responsible for the regularity of the Israel-Hartle-Hawking state .

A similar analysis can also performed by constructing the vacuum state with κ_-, obtaining the state $|\mathrm{H}_-\rangle$, and again we find regularity everywhere, thanks to $\kappa_+^2 = \kappa_-^2$.

A Milne Spacetime

The Milne spacetime is a particular case of the much more general Robertson-Walker spacetimes.
The general Robertson-Walker line element can be written as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\sum_{i,j=1}^{3} h_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{A.1}$$

where

$$h_{ij}\mathrm{d}x^i\mathrm{d}x^j = (1-Kr^2)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) \tag{A.2}$$
$$= \mathrm{d}\chi^2 + f^2(\chi)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) \tag{A.3}$$

and

$$f(\chi) = r = \begin{cases} \sin\chi, & 0\le\chi\le 2\pi, \quad K=+1 \\ \chi, & 0\le\chi\le\infty, \quad K=0 \\ \sinh\chi, & 0\le\chi\le\infty, \quad K=-1. \end{cases} \tag{A.4}$$

The spacetime is called hyperbolic, flat or closed depending on the value of K, respectively $K=-1,0,+1$.

The Milne spacetime is the Robertson-Walker spacetime with the choice $a(t)=t$. The resulting four-dimensional spacetime, as well as its two-dimensional counterpart, are unconventional coordinatizations of flat spacetime, in a similar way to the Rindler spacetime.
The line element for the Milne spacetime reads[1]

$$\begin{aligned}\mathrm{d}s^2 &= \mathrm{d}t^2 - b^2t^2\mathrm{d}x^2 \\ &= e^{2b\eta}(\mathrm{d}\eta^2 - \mathrm{d}x^2)\end{aligned} \tag{A.5}$$

where $|t| = b^{-1}e^{b\eta}$, b is a constant and η is defined by

$$\eta = \int^t \mathrm{d}t' a^{-1}(t'). \tag{A.6}$$

[1] The line element for the Rindler spacetime is very similar, but in the Rindler case the exponential contains the space-like coordinate x instead of the time-like η.

We can introduce a new set of coordinates $x^{\pm}$ defined as

$$x^{+} = b^{-1}e^{b\eta}\cosh b\eta, x^{-} = b^{-1}e^{b\eta}\sinh b\eta \tag{A.7}$$

and the line element Eq. A.5 reduces to

$$\mathrm{d}s^2 = (\mathrm{d}x^{+})^2 - (\mathrm{d}x^{-})^2 \tag{A.8}$$

which is the line element of flat Minkowski spacetime.
Moreover, if we introduce a set of null coordinates

$$u = \eta - x, v = \eta + x \tag{A.9}$$

we immediately find

$$\mathrm{d}s^2 = e^{b(u+v)}\mathrm{d}u\mathrm{d}v. \tag{A.10}$$

The conformal factor for the metric in double-null form is the same we used to fix $a + b = 0$ in Section 3.2[2].

[2]Previously we used $x^{\pm}$ as null coordinates, but the form of the conformal factor is the same.

Conclusion

We now have all the ingredients to draw some conclusions.
We started with the simple Schwarzschild spacetime. The analysis showed us that to build a regular quantum state, the modes should be built using regular coordinates. Indeed, the Boulware vacuum, built with both Eddington-Finkelstein null coordinates, is found to be divergent on both past and future horizons. Nonetheless, since this state approaches the Minkowski vacuum, we were able to understand the physical meaning of this state, i.e. the vacuum state which describes the vacuum polarization of a static star.
Introducing one of the Kruskal coordinates we find regularity on one horizon. In particular, the Unruh vacuum is defined with respect to the Kruskal U. This state is found to be regular on the future horizon $U = 0$, but not on the past $V = 0$. This is because the introduction of the Kruskal coordinates adds a "correction" terms only to the (uu) component of the energy-momentum tensor, while the (vv) component, which is the same as for the Boulware vacuum, still diverge.
This suggests that, in order to build a state regular everywhere we should introduce correctional terms also to the (vv) component. This is precisely what happens with the Israel-Hartle-Hawking vacuum. It is built with respect to both Kruskal coordinates, which are regular everywhere, and therefore the state is regular everywhere.

The same cannot be said for the Reissner-Nordström spacetime. While the Boulware vacuum is still singular, the first difference emerges when studying the Unruh vacuum.
Indeed, in this spacetime we have two horizons. For each horizon, we can define a set of Kruskal coordinates that are regular on that horizon but not on the other. And we find the same thing regarding the regularity of the Unruh vacuum.
We can build the Unruh vacuum, for example, with coordinates regular on the event horizon. We will find the state to be regular on the event horizon but not on the other horizons. The opposite happens when we build the state with coordinates regular on the inner horizon.
From a mathematical point of view, the singularity is due to the non-cancellation between the vacuum polarization term and the term coming from the transfor-

mation between the two coordinate systems, and this is because of the different values of the surface gravity.
For this reason, also the Israel-Hartle-Hawking state will not be regular everywhere. We can make the state regular on both $U_+ = 0$ and $V_+ = 0$, but not on $U_- = 0$ and $V_- = 0$, and viceversa.

With this in mind, it is interesting to study the Simpson-Visser spacetime. This regular metric presents some elements of the Reissner-Nordström spacetime, such as two horizons and a somehow similar causal structure, but also of the Schwarzschild spacetime, like the unique value for the surface gravity. In particular, this last property will ensure the existence of a regular state.
Performing the regularity analysis we find that the Boulware vacuum is not regular. The Unruh vacuum can be built to be regular on the outer horizon, but will diverge on the inner horizon. This is because the Unruih vacuum is built with only one Kruskal coordinate. The term coming from the transformation will cancel out the vacuum polarization term.
To solve this we introduced the Israel-Hartle-Hawking vacuum, which is built with both Kruskal coordinates. The transformation to these coordinates will introduce correctional terms to both the (uu) and (vv) components, ensuring regularity everywhere. The regularity is achieved thanks to the same value of the squared surface gravity.

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