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On the Generalized Hydrodynamics of Integrable Quantum Field Theories with Irrelevant Deformations

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“Mi parve tutto ad un tratto che la mia umile vita e i reami del vero non fossero così separati come avevo creduto, che coincidessero persino in alcuni punti.”

Marcel Proust, *Alla ricerca del tempo perduto*

“L’amore istruisce dei e uomini, perchè nessuno impara senza desiderio di imparare. La verità è ricercata non in quanto verità ma in quanto bene. L’attenzione è legata al desiderio. Non alla volontà ma al desiderio.”

Simone Weil, *L’ombra e la grazia*

Abstract

The purpose of this Master Thesis is to study the Generalized Hydrodynamics (GHD) of Integrable Quantum Field Theories perturbed by the famous $T\bar{T}$ deformation and its generalizations. These deformations are irrelevant, and therefore alter dramatically the UV structure of the theory and are not renormalizable, but are still considered consistent and interesting theories since they preserve the integrable structure of the underlying model. The specific focus of the work is to find the average densities and currents related to conserved charges of generic spin, which describe the flow of energy, momentum, and higher charges, between two semi-infinite slabs prepared in two different states and placed in contact in the origin, in the setup known as partitioning protocol. From a theoretical point of view this is the most important protocol in out of equilibrium physics, since it is simple enough to have an analytical description and in some situations to have an analytical solution, but at the same time it is complex enough to gain practical knowledge on the effect of inhomogeneities on integrable systems. In particular, through a convenient way of rewriting the Thermodynamic Bethe Ansatz (TBA) equations, I found exact expressions for the currents and densities in the conformal limit, generalizing previously known results in several directions, with excellent numerical validation. I have also performed an in-depth study of possible extensions of the results out of the conformal point in the simplest possible theory, the free fermion, obtaining interesting expressions for the lowest order corrections in terms of special functions. Finally, I have performed numerical simulations which give a perfect confirmation of the analytical results, and allow to gain insight on some interesting aspects of the models which are not accessible analytically.

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Introduction

The theoretical study of non-equilibrium properties of quantum many-body and statistical systems has undergone a dramatic development in the recent years following new experimental advances [1–5] which now allow to create practically isolated atomic systems, and study their unitary evolution. In particular, the manipulation of one-dimensional systems has drawn attention to understanding the out-of-equilibrium behaviour of `integrable` systems, in which an infinite number of conserved quantities are present. It was early observed [1], that the presence of these conserved charges leads to a lack of thermalization, which was then theoretically studied in the context of quantum quenches [6, 7]. Further, the fact that these systems were found to be non-ergodic posed a challenge to the usual statistical mechanical view of closed systems with many degrees of freedom. The study of quantum quenches led to the realization that the standard Gibbs ensemble, used in the standard formulations of classical and quantum Thermodynamics, is unable to describe the long-time behaviour of the observables in such systems, pointing to the necessity of introducing the Generalized Gibbs Ensemble (GGE) [8–11], in which all the conserved charges characterizing the integrable model are taken into account. Later, in order to describe inhomogeneous situations, Generalized Hydrodynamics (GHD) was proposed [12–15]. This theory is constructed from the postulate of local maximization of entropy, as the standard theory of hydrodynamics, but under the constraint of having infinitely many conserved charges, and it therefore describes the dynamics of integrable systems at a mesoscopic scale, as usual hydrodynamcis does for classical fluids. In the context of Integrable Quantum Field Theories (IQFT), with which this work is mostly concerned, the power of the GHD approach is the possibility of using a quasiparticle description, which is the main feature of the Thermodynamic Bethe Ansatz (TBA) [16], a well known framework to study the thermodynamics of integrable models, of which GHD is essentially the inhomogeneous generalization. The main theoretical application of GHD was since the beginning that of studying the non-equilibrium steady states (NESS) [12, 15, 17, 18], namely the formation of steady currents of energy and momentum between two thermalized reservoirs placed in contact, the simplest and theoretically most significant out-of-equilibrium theory.

This work follows precisely this line of research, in the attempt to extend the GHD description of Integrable Quantum Field Theories to theories perturbed by the famous $T\bar{T}$ deformation and its generalizations [19, 20], irrelevant deformations which have the property of preserving the integrability of the perturbed theory.

As well known, the non-renormalizability of theories with irrelevant deformations makes them particularly subtle to deal with, with many open questions regarding the behaviour in the ultraviolet (UV) limit. Therefore, much space is dedicated to understanding how this deformation affects the basic structure of the theory, such as the TBA equations and the famous Y-systems [21], with particular focus on the integrable theories of ADE type [22]. Subsequently, making use of TBA techniques, and exploiting the peculiar form of the $T\bar{T}$ deformation we find exact expressions in the conformal limit for the higher spin NESS currents in a generic theory with a $T\bar{T}$ deformation. This original accomplishment generalizes previous results by Medenjak, Policastro and Yoshimura [23, 24], and provides expressions of great generality which relate the quantities of the perturbed theory to the ones in the unperturbed theory. This could open the way to a new approach in the use of TBA techniques in the context of $T\bar{T}$ deformed theories. The structure of the Thesis is as follows:

- In chapter 1 the theory of Integrable Quantum Field Theories in (1+1) dimensions is presented, in the framework of S-matrix theory. Since Conformal Field Theories (CFTs) characterize the fixed points of the Renormalization Group flows, the origin of integrable models as perturbations of CFTs is emphazised. Moreover, the $T\bar{T}$ deformation is presented and its main features discussed.

- Chapter 2 is dedicated to the study of the thermodynamics of IQFTs through the Thermodynamic Bethe Ansatz. After a presentation of the main concepts of this approach, namely of how the finite temperature properties of a (1+1)-dimensional IQFT can be derived from its S-matrix, these are applied specifically to $T\bar{T}$ deformed theories.

- In chapter 3 we present the GHD approach to the study of integrable systems out of equilibrium, focusing on the quasiparticle description which characterizes the TBA approach and setting the ground for the study of the partitioning protocol and the NESS currents.

- Chapters 4 and 5 contain the main original achievements of this work. In chapter 4 the general solution to the partitioning protocol of $T\bar{T}$ deformed theories is presented, showing perfect accordance with previously known results, which are then generalized. Although the solution is only valid in the conformal limit, in the second part of the chapter a thorough perturbative analysis shows how one can exit the conformal point, in the simplest case of the free fermion. Finally, chapter 5 is dedicated to numerical analysis of the obtained results.

Chapter 1

S-Matrix Theory and Integrability

Integrability is an extremely wide subject, applying to classical and quantum systems, spin chains, and field theories, both classical and quantum. It is not obvious to provide a unique definition, although the key feature that all integrable systems of the various kinds share is the fact that they are characterized by a certain number of conserved quantities in involution. The presence of these conserved charges makes integrable theories in some sense the simplest theories possible, in which the time evolution is the easiest it could be: in some sense, they are the opposite of chaotic systems [25]. It is however highly nontrivial to find such conserved charges, and therefore to construct generic integrable models, and even to determine whether a given system is integrable or not. The main relevance of integrability in the various possible systems is that integrable models turn out to be exactly solvable, where the notion of "solvability" differs depending on the context: in classical systems, integrability allows to use angle-action variables, which in turn allow to find the equations of motion by quadrature, as originally stated in a theorem by Liouville. In QFTs, solvability is instead related to a constraint of elasticity of the theory, and this leads to the possibility of determining the S-matrix of the theory exactly [26, 27], together with the mass spectrum, and this gives the possibility to find correlation functions, the thermodynamics of the theory through the TBA approach, the out-of-equilibrium dynamics through GHD, and much more. Therefore we see that the natural framework in which to study integrable QFTs is that of S-matrix theory, which is the field-theoretic generalization of the familiar scattering theory of quantum mechanics. In this chapter, we give the theoretical foundations to the theory of Integrable Quantum Field theories, showing how they can be constructed as relevant perturbations of conformal field theories, as was firstly done in [28]. The main features of integrable QFTs, namely the elasticity and factorization of the scattering, are proven following the original arguments by Parke [29]. Subsequently, we discuss the $T\bar{T}$ deformation of such theories, which will be the main focus of the following work.

1.1 Integrable Field Theories

In two dimensional quantum field theories, there exists a subclass of theories which exhibit integrability in the sense mentioned above, namely containing an infinite set $\{Q_i\}$ of conserved charges (of which one has to be the Hamiltonian of the system, otherwise the charges would not be conserved) such that:

$$[Q_i, Q_j] = 0 \qquad \forall i, j \tag{1.1}$$

The presence of an infinite set of conserved charges provides extremely strong constraints on the possible structure of the S-matrix: in fact, these constraints are so strong that they lead to non-trivial theories only in two dimensions (or in (1+1)-dimension, if one takes a Minkowski perspective instead of an Euclidean one), as stated by the Coleman-Mandula theorem [30], if one does not invoke supersymmetry.[1]

We define a quantum field theory to be integrable if it is characterized by an infinite set of conserved charges in involution, namely all commuting with each other [26]. The most natural example of integrable field theories are conformal field theories (CFTs), which as is well known constitute the fixed points of the Renormalization Group flows [31]. In these theories it is possible to construct an infinite set of conserved charges from the descendants of the stress-energy tensor and its conjugate[2]. Indicating with T_{s+1} the descendant of spin s+1 (we recall that in conformal field theory the spin is given by the difference between the left and right conformal dimensions) we have the conservation laws:

$$\partial_{\bar{z}} T_{s+1} = 0$$
$$\partial_z \bar{T}_{s+1} = 0$$

and therefore we can build conserved charges from these currents, as proposed in [28]:

$$q_s(z) = \oint_z d\xi T_s(\xi)(\xi - z)^{s+n-1}, \qquad n = 0, \pm 1, \pm 2, ... \tag{1.2}$$

The modern view of quantum field theories is to consider them as trajectories in coupling constant space, the Renormalization Group (RG) flows [32, 33]. Therefore it is natural to consider integrable field theories which arise by perturbing CFTs by relevant operators, which drive the system away from criticality [28]. The perturbation breaks the factorization into analytic and anti-analytic components, which is a key feature of CFT, hence in general the integrability of the model will be lost.

[1] The intuition behind Coleman and Mandula theorem is essentially the same as Parke's argument, which will be described below to prove the factorization of the scattering any scattering process into two-particle scattering, which is the key to the simplicity of integrable theories.

[2] In a conformal field theory, given a primary field Φ, defined to be an eigenstate of the generator L_0 of the Virasoro algebra, one can construct descendant fields by acting with the generators L_{-n} of negative value, and these form the various levels of the highest weight representation related to the original primary field. The stress energy tensor is a descendant of the identity opeartor, which is a primary field.

However, if the system still has a set of currents if components (T_{s+1}, Θ_{s-1}) which satisfy the new conservation law:

$$\partial_{\bar{z}} T_{s+1} = \partial_z \Theta_{s-1} \tag{1.3}$$

Then one can still build conserved charges as

$$\mathcal{Q}_s = \oint (T_{s+1} dz + \Theta_{s-1} d\bar{z}) \tag{1.4}$$

and it can be shown that these charges are all in involution. We will not focus on how these charges can be constructed, for this we refer to [26] and [28]. We focus mostly on the consequences that integrability has on the properties of the S-matrix, since they are what allows to introduce the Thermodynamic Bethe Ansatz technology and hence to construct GHD. We start by giving a small example of the power of integrability in lagrangian models.

1.1.1 A Lagrangian example

As will be developed in depth in the following sections, the main feature of integrable models is that they give rise to purely elastic scattering with no particle production, and this is what really makes the theories special (and solvable) compared to generic QFTs. To see how this constrains the structure of a theory, following [27] we consider the Z_2 invariant lagrangian:

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \varphi)^2 - m^2 \varphi^2\right] - \frac{g_4}{4!}\varphi^4 \tag{1.5}$$

If we consider the $2 \to 4$ particle production amplitude, this is given by the sum of three possible diagrams admitted by the Feynman rules, and is easily found to be:

$$A = \frac{i}{48 m^2} g_4^2 \tag{1.6}$$

which implies that if we add to the lagrangian a term $-\frac{g_6}{6!}\varphi^6$, where $g_6 = g_4^2$, the amplitude vanishes completely, and the production process is dynamically suppressed. The same can be done to suppress all the (tree-level) processes which involve the production of particles: the potential we obtain, which suppresses the production for any particle number, is then given by:

$$V(\varphi) = m^2 \left[\frac{\varphi^2}{2} \pm \frac{g^2}{4!}\varphi^4 + \frac{g^4}{6!}\varphi^6 \pm ...\right] \tag{1.7}$$

depending on the chosen sign, this gives rise to the Sinh-Gordon and the Sine-Gordon model potentials, which are some of the most important integrable models, and will be used again in chapter 5.

Also, if we do not restrict ourselves to lagrangians with a Z_2 symmetry but we consider the most general Landau-Ginzburg model, the same discussion can be repeated to find the potential:

$$V(\varphi) = m^2 \left[\frac{1}{2}\varphi^2 - \frac{g}{6}\varphi^3 + \frac{g^2}{8}\varphi^4 - ...\right] \tag{1.8}$$

which is precisely the potential of the Bullogh-Dodd model, which is relevant in the following because of its relation with the Yang-Lee singularity model, which will be studied in chapter 5. Therefore we see that at the lagrangian level we can build scalar theories in which particle production is dynamically suppressed, and hence there is no inelastic particle production. This feature greatly simplifies the structure of the theory compared to a generic QFT, in which the infinite amount of possible processes is what makes an exact solution impossible to obtain. However, the lagrangian formulation is not the most transparent and useful way to deal with integrable models, since for example conformal field theories are not defined starting from a lagrangian formulation but rather from an algebra of local fields, and the same extends to their relevant perturbations. The natural framework is that of S-matrix theory, which we now introduce.

1.2 S-matrix theory

In the theory of scattering, the S-matrix is a field theoretic generalization of the quantum mechanical scattering matrix. It is based upon fundamental principles which encode the most essential features of quantum mechanics and special relativity: from quantum mechanics we require the superposition principle, the conservation of probability, which is equivalent to unitarity of the evolution, and the analiticity principle, while from special relativity we require the fundamental aspects of Lorentz invariance, short rangedness of interactions which implements locality, and the causality principle.

The requirement of the interactions being short-ranged allows to have well-defined asymptotic states: in the initial and final states, the ingoing and outgoing multiparticle states are free and hence can be characterized by their momentum (and possibly by other quantum numbers), and therefore they can be expressed as

$$|i\rangle = |p_{1,i}, ...p_{n,i}\rangle \, , |f\rangle = |p_{1,f}, ...p_{n,f}\rangle \tag{1.9}$$

The S-matrix is precisely defined starting from the superposition principle, which implies that any final state can be expressed as $|f\rangle = \mathcal{S}\,|i\rangle$, and therefore the amplitude of a transition $|i\rangle \to |f\rangle$ is given by

$$\mathcal{S}_{fi} = \langle f|\, \mathcal{S}\, |i\rangle \tag{1.10}$$

Therefore the S-matrix encodes all the information about all the possible scattering processes of the theory, connecting the asymptotic initial and final states. In a generic QFT this is an extremely complicated object, since it has to take into account scatterings with arbitrary number of particles in both sides, and it is related to correlation functions by the LSZ reduction formula. In IQFT, however, we will show that the structure is greatly simplified since any scattering process is factorized in two-particle scatterings, and hence the entire S-matrix can be specified by only a relatively small number of matrix elements. The requirement of conservation of probability in scattering processes implies that the S-matrix is unitary, i.e. it satisfies $\mathcal{S}\mathcal{S}^\dagger = 1$, which in turn implies the fundamental optical theorem. Lorentz invariance ensures the independence of the amplitudes from the reference frame which is used to measure the asymptotic momenta. In general we can express the S-matrix in a way which highlights the non-trivial part of the scattering: considering that in absence of any interaction the S-matrix is the identity, we have

$$\mathcal{S} = 1 + i(2\pi)^d \delta^d \left(\sum p_f - \sum p_i \right) \mathcal{T} \tag{1.11}$$

where the modulus squared of the T-matrix determines the probabilities of generic nondiagonal processes $i \to f$, as

$$P_{i\to f} = (2\pi)^d \delta^d \left(\sum p_f - \sum p_i \right) |T_{fi}|^2 \tag{1.12}$$

Finally, the postulate of analyticity is of fundamental importance [34], as it allows to extend the S-matrix elements to the entire complex plane. This in turn allows to study its singularity structure over values of rapidities in $\mathbb{C}$, which is related to the determination of the bound states; then to obtain the bootstrap equations, which essentially allow to solve the system, as will be explained below.

The most important kind of scattering which we are interested in is the two-body scattering process, which takes the form $i_1 + i_2 \to f_1 + f_2$. In this situation $\mathcal{T}$ is dependent on the three independent relativistic invariants of the system, the three Mandelstam variables $s = (p_1 + p_2)^2, t = (p_1 - p_3)^2, u = (p_1 - p_4)^2$, where s is the energy in the centre of mass, t and u are the energies in the centre of mass of the two scattering processes obtained by crossing symmetry. These are not independent, since their sum is equal to the sum of the masses of the four particles, therefore we can focus on s and t alone. We can study $\mathcal{T}$ as function of these variables, but with the constraint of considering the physical strip to consist only of the $s \geq m_1^2 + m_2^2$. At each threshold of production of a new particle, the Optical Theorem requires $\mathcal{T}$ to have a branch point, which then induce branch cuts. in the case of a theory containing a single particle of mass m, this implies the presence of a pole for each value of $(nm)^2$, where $n > 2$. In this way we identify the physical sheet of the theory on the complex plane as the sheet obtained without crossing any of the cuts. Considering $\mathcal{T}$ as an analytic function over all the complex plane allows to introduce the concept of crossing symmetry. First of all it is important to note that the physical region with respect to s and t do not coincide. The "real" process is the one in which s is the total energy in the center of mass frame, namely $i_1 + i_2 \to f_1 + f_2$, but we could also consider the scattering $i_1 + \bar{f}_3 \to \bar{i}_2 + f_4$, where overhead bars are used to indicate the antiparticles. The requirement of crossing symmetry imposes that the amplitude of the second process can be obtained by analytic continuation of $\mathcal{T}$ to the region in which t is physical and s is not. As will be discussed in the following section, the unitarity and crossing invariance conditions will prove essential to determine exactly the S-matrix in those situations in which the system is integrable.

1.3 (1+1) dimensional Scattering Matrices

As discussed previously, (1+1)-dimensional theories may have the feature of being integrable, namely to have infinitely many conserved charges. This poses strong constraints on the structure of these theories, which turn out to be purely elastic and with factorizable scattering. These two features allow then to evaluate exactly the S-matrix element using the bootstrap approach, a general procedure which allows to evaluate some quantities of a given theory by the requirement of internal consistency of the theory itself. In this planar situation it is convenient to parametrize the particles using their rapidities, such that energy and momentum are given by $E_a = m_a \cosh\theta_a$, and $p_a = m_a \sinh\theta_a$. Following [27], we can write a generic asymptotic state as:

$$|A_1(\theta_1), A_2(\theta_2), ...A_n(\theta_n)\rangle \tag{1.13}$$

which is conveniently rewritten defining the noncommuting operators $A_i(\theta)$, which create an asymptotic state by acting on the vacuum $|\Omega\rangle$:

$$|A_1(\theta_1), A_2(\theta_2), ...A_n(\theta_n)\rangle = A_1(\theta_1)A_2(\theta_2)...A_n(\theta_n)\,|\Omega\rangle \tag{1.14}$$

Since asymptotic states at $t \to -\infty$ need to have no interactions, we must order the rapidities in the state as $\theta_1 > \theta_2 > ... > \theta_n$. For the outgoing state, on the other hand, the ordering has to be the opposite: $\theta_1 < \theta_2 < ... < \theta_n$. Using this formalism, the action of the S-matrix can be seen as providing the commutation relations between the noncommuting operators A_n, thus giving rise to the famous Zamolodchikov-Fadeev algebra[3] [35]:

$$A_{a_1}A_{a_2}...A_{a_n} = S^{b_1 b_2 ... b_n}_{a_1 a_2 ... a_n} A_{b_1}A_{b_2}...A_{b_n} \tag{1.15}$$

Now suppose that the system is integrable: we have a set of infinitely many charges $\{Q_s\}$, satisfying $[Q_s, Q_{s'}] = 0$. The requirement of commutativity of all the charges is crucial, because it guarantees that we can diagonalize the charges simultaneously and build the theory on their common eigenstates:

$$Q_s\,|A_a(\theta)\rangle := \; q_s^{(a)} e^{s\theta}\,|A_a(\theta)\rangle \tag{1.16}$$

where the functional dependence on the rapidity guarantees that the operators transform tensorially under Lorentz transformations, since they transform as s

[3] It is important to realize that this notation makes sense only because we are in one spatial dimension, and this matches the "dimensionality" of a string of operators that we can build with the symbols A_n. Extending this reasoning to higher dimensions would be impossible.

copies of the two light-cone components of the momentum. This is then extended to multiparticle states, for which we define the eigenvalue of each conserved charge to be:

$$Q_s \left| A_{a_1}(\theta_1)...A_{a_n}(\theta_n) \right\rangle = \left(q_s^{(a_1)} e^{s\theta_1} + ... + q_s^{(a_n)} e^{s\theta_n} \right) \left| A_{a_1}(\theta_1)...A_{a_n}(\theta_n) \right\rangle \quad (1.17)$$

These few definitions are already sufficient to prove the two fundamental properties of integrable S-matrix theories, following Parke's arguments [29]: the elasticity and factorizability of the scattering.

1.3.1 Elasticity of Scattering

The proof of the elasticity of the scattering simply follows from the action of the conserved charges on multiparticle states given by equation (1.17). In fact, considering that by definition a conserved charge satisfies $\frac{dQ_s}{dt} = 0$, then the value of the total charge eigenvalue on a given multiparticle state has to be a constant of motion. Therefore, considering an initial state represented by the string $A_{a_1}A_{a_2}...A_{a_n}$ and final state $A_{b_1}A_{b_2}...A_{b_m}$, equating the charge at $t = \pm\infty$ leads to:

$$q_s^{(a_1)} e^{s\theta_1} + ... + q_s^{(a_n)} e^{s\theta_n} = q_s^{(b_1)} e^{s\theta_1'} + ... + q_s^{(b_n)} e^{s\theta_m'} \quad (1.18)$$

Since the system is characterized by infinitely many conserved charges, these will give rise to an infinite number of constraints of this kind, which have to hold for every possible configuration of the momenta/rapidities of the particles. The only possible solution to these equation is then to have an equal number of particles on both sides, n=m, and $\theta_i = \theta_i'$ and all the charges to be equal, up to permutations of particles with the same quantum numbers. Therefore the scattering is purely elastic, in the sense that the identity of the particles is preserved, up to a reshuffling of the internal quantum numbers and of the rapidities of the particle, and importantly there cannot be any particle production, as was discussed at the lagrangian level in section 1.1.1 (an exception to this rule is related to the formation of stable bound states, which as will be explained in the following are related to the poles of the S-matrix).

1.3.2 Factorization and Yang-Baxter equations

As mentioned above, an intuitive explanation of the factorization of scattering processes can be given by working on the same lines as in the case of the Coleman and Mandula theorem, by considering the action of the conserved charges on localized wavepackets. From the expression presented above we write in a generic way the action of the charges on the particle states:

$$e^{iQ_s} \left| A_a(p) \right\rangle = e^{i\phi(p)} \left| A_a(p) \right\rangle \quad (1.19)$$

Considering the state in a (strongly peaked) gaussian wavepacket given by the expression:

$$\psi(x) \propto \int_{-\infty}^{+\infty} dp e^{-a(p-p_0)^2} e^{ip(x-x_0)} \quad (1.20)$$

and acting on it with e^{iQ_s} using (1.19) we see that we obtain the transformed wavefunction:

$$\tilde{\psi}(x) \propto \int_{-\infty}^{+\infty} dpe^{-a(p-p_0)^2} e^{ip(x-x_0)} e^{i\phi(p)} \tag{1.21}$$

Expanding $\phi(p)$ in powers of $(p-p_0)$ we see that the transformed wavefunction is simply a new gaussian centered in $\tilde{x}_0 = x_0 - i\phi(p)$ with same momentum as the original one. This allows to devise a trick to deal with multiparticle scattering, as is visualized in figure 1.1, as was first proposed in [29]. Although the two processes

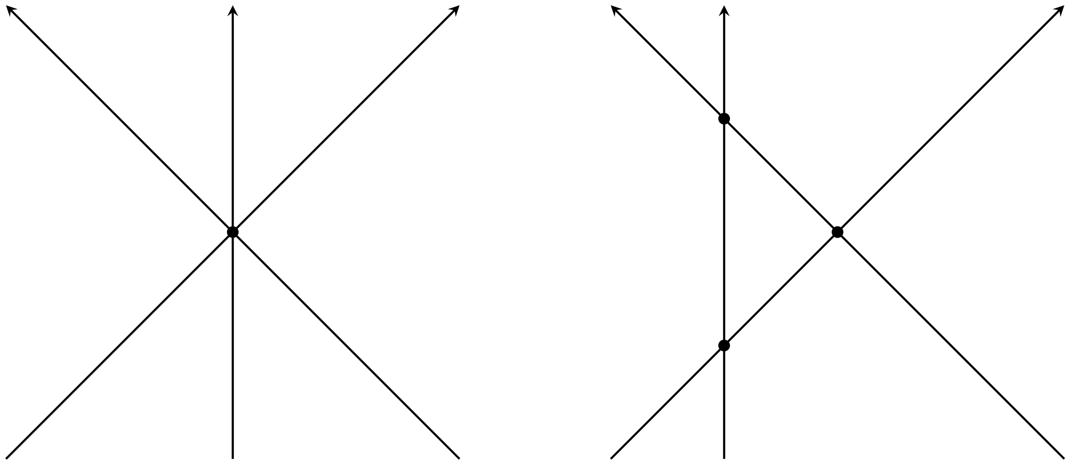


Figure 1.1: A scattering in which all particles collide simultaneously, and a factorized scattering. The two are related to each other by acting with conserved charges on the wavepackets, and hence they lead to the same amplitudes.

are different in a generic QFT, because of integrability and equation (1.21) the two scattering processes are identical: since the two situations can be obtained from one another by the action of a charge which commutes with the hamiltonian, they are the same process (by this we mean that the scattering amplitude is identical). We see that the consequence of the equality of these two scatterings is that `any` scattering process with an arbitrary number of incoming particles can be expressed as a succession of two particle scatterings, by modifying the trajectory of each particle by the necessary amount acting with one of the higher spin charges. This also implies that the total S-matrix will be a product of the two-particle S-matrices. Moreover, following the same reasoning the scattering is also forced to satisfy the famous Yang-Baxter (YB) equations [36–38]. In this situation the YB equation represent a consistency condition of the factorized scattering above, which is graphically represented in figure 1.2. This can be expressed as a overdetermined set of constraints on the 2-particle S-matrix, which can be written as:

$$S_{ij}^{ab}(\theta_{12}) S_{bk}^{cl}(\theta_{13}) S_{ac}^{nm}(\theta_{23}) = S_{jk}^{ab}(\theta_{23}) S_{ia}^{nc}(\theta_{13}) S_{cb}^{ml}(\theta_{12}) \tag{1.22}$$

where we have defined naturally $\theta_{ij} = \theta_i - \theta_j$. Since this set of constraints is overdetermined, namely there are more constraints than amplitudes to be determined,

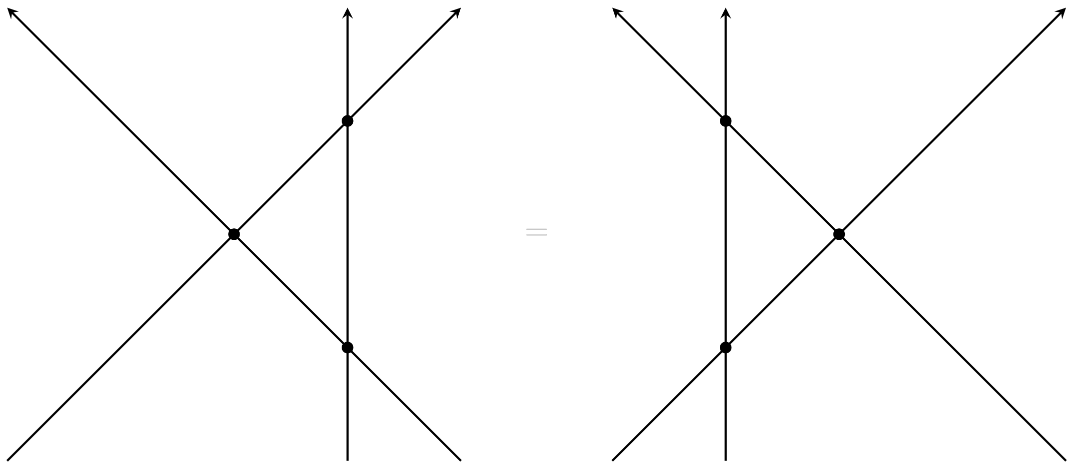


Figure 1.2: Visual representation of the Yang-Baxter equation

only particular choices of the functional form of the S-matrix elements will satisfy the equations. In the simplest case, that of a diagonal S-matrix (in which the only nonvanishing terms have the form S_{ab}^{ab}) the YB equations are automatically satisfied.

1.3.3 The Bootstrap programme

The Bootstrap Programme has the objective of determining all the elements of the scattering matrix between all the particles of the system starting from a very simple dynamical principle, namely that in integrable systems the bound states are placed on the same footing as the asymptotic states, which is a consequence of the fact that the elasticity of the scattering implies that the particles have a macroscopically long lifetime (which is equivalent as saying that there is no particle production). The bound states of a theory are associated to simple poles of the S-matrix, which can be represented as

$$S_{ij}^{kl} = \frac{iR}{\theta - iu_{ij}^{n}} \tag{1.23}$$

where the position of the pole, identified by u_{ij}^{n}, determines the mass of the bound state m_n through the relation:

$$m_n^2 = m_i^2 + m_j^2 + 2m_i m_j \cos(u_{ij}^{n}) \tag{1.24}$$

Since this is Carnot theorem, it implies that the masses of the asymptotic and bound state particles form a "mass triangle". We see that this allows to find all the masses of the bound states starting from the asymptotic ones, and this is the essence of the bootstrap approach. Focusing on diagonal theories (for nondiagonal theories the exact determination of the S-matrix elements is particularly intricated, and was studied in [39]) the S-matrix has only two indices, $S_{ab} = S_{ab}^{ab}$. The unitarity and crossing symmetry which we considered above are able to fix completely the structure of the S-matrix:

$$S_{ab} = \prod_{x \in \mathcal{A}_{ab}} \frac{\sinh(\frac{i\pi x+\theta}{2})}{\sinh(\frac{-i\pi x+\theta}{2})} \tag{1.25}$$

This expression is valid if the particles are charged, i.e. if particles differ from their antiparticle. If the particles are neutral, then the hyperbolic sine is simply substituted by a hyperbolic tangent. Therefore we see that the S-matrix is almost completely determined, except for the position of the poles, which are contained in the set $\mathcal{A}_{ab}$. To determine this set the bootstrap approach comes into play. The strategy is analogous to what was used to prove the factorization of the scattering, namely that it is possible to change the trajectory of the particles by acting with the higher spin conserved charges. This leads to the equivalence shown in 1.3, in which the dashed line represents a bound state of the two full lines. The equality

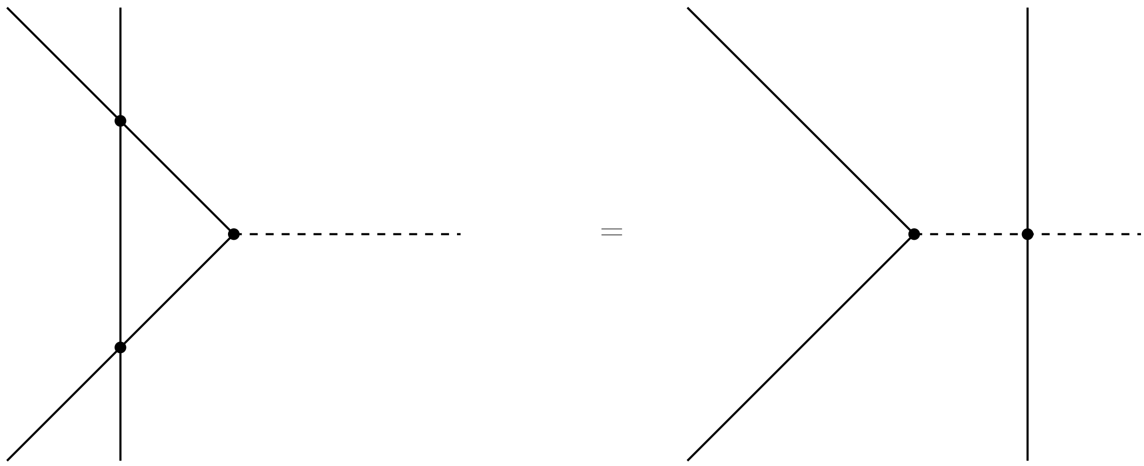


Figure 1.3: The bootstrap approach is based on the equivalence of the two scatterings depicted in figure: here, the dashed line is a bound state of the two continuous lines.

of the two scattering processes above is expressed in terms of S-matrix elements as:

$$S_{i\bar{l}}(\theta) = S_{ij}(\theta + i\bar{u}^k_{jl})S_{ik}(\theta - i\bar{u}^j_{lk}) \tag{1.26}$$

where the overhead bar represents an antiparticle (we assume that the bound state is an antiparticle state $\bar{l}$). Hence the strategy is clear: one needs to determine the pole structure u^k_{ij}, by imposing the bootstrap condition (1.26), and the mass triangle equation (1.24). This is in practice performed iteratively, by starting with the lightest particle, building the bound states, and iterating until the system closes. In this way, it is possible to describe the entire physical content of the theory in terms of a finite number of two-particle scattering amplitudes. This strategy has proven extremely effective, and allows to find the particles, identified by $\{m_1, m_2, ...m_n\}$ which constitute the theory, and their interactions, in all of the known statistical systems which can be expressed as relevant perturbations of some CFT. This allows to implement the TBA technology, as will be developed in Chapter 2, which relies deeply on the possibility of expressing the physical content of the theory in terms of quasiparticles.

1.4 $T\bar{T}$ deformations of IQFT

Many[4] integrable models of interests, as studied in [28,35], are obtained as relevant deformations of a Conformal Field Theory. The main feature of relevant deformations is that they are super-renormalizable in the UV, thus they don't change the structure of the theory in this limit, but still they drastically influence the theory in the IR. Given a massive theory obtained in this way, it is immediately possible to obtain the UV limit by taking the conformal limit, namely sending the mass to zero. The behaviour of these theories is well studied and understood [41, 42], with virtually infinitely many integrable theories which can be obtained by perturbing known conformal models in this way, all of which then can be solved exactly (by finding the S-matrix) via the bootstrap approach presented above, as thoroughly discussed in [26].

In recent years, however, there has been an increasing interest in the study of `irrelevant` deformations of CFTs. These perturbations are non-renormalizable, and hence alter drastically the theory in the UV, but leave it unaltered in the infrared; moreover, in general they break the UV completeness, in the sense that the UV limit of such theories is not in general a consistent local QFT. This signals in general the fact that we are dealing with effective theories, although also interpretations involving a string-like behaviour have being studied. This means that the RG flow of the theory is of the type shown in figure 1.4.

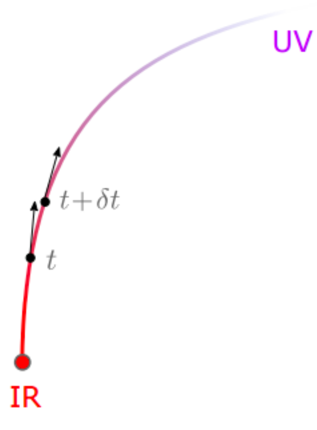


Figure 1.4: Renormalization group flow of a $T\bar{T}$ deformated theory, where the parameter t corresponds to the α in the following, figure taken from [43].

Compared to other irrelevant perturbations, as the one studied in [44], the $T\bar{T}$

[4]Although, as noted in [40], QFTs which are connected to a UV fixed point by a RG flow form only a set of measure zero in the set of all possible (effective) QFTs

deformation and its generalizations have the very special feature, as shown in [19,45] that it is an integrable deformation, i.e. it preserves the conserved charges of the integrable model which is being perturbed[5]. Also, it has been shown that the effect of this perturbation at the S-matrix level consists in a simple introduction of a CDD factor, as first proposed in [20], namely a term in front of the S-matrix of the unperturbed theory with the property of not introducing any new poles. As explained later, this implies that the TBA equations are modified in a very simple way, by a simple change of the scattering kernel.

1.4.1 Definition of the deformation

Following [19], we claim that starting from a generic IQFT (massive or massless) it is possible to construct a series of scalar fields X_s which give rise to integrable perturbations. A perturbation to some field theory of action $\mathcal{A}_g$ identified by a set of parameters $\{g_i\}$ can be defined in terms of variations of such parameters as:

$$\delta\mathcal{A} = \sum_i \int d^2z \delta g_i O^i(z) \tag{1.27}$$

where the $O^i(z)$ are some operators of the algebra of local fields, which in principle can give rise to both relevant and irrelevant perturbations. In particular, integrable perturbations can be built starting from the currents (T_{s+1}, Θ_{s-1}) and $(\bar{T}_{s+1}, \bar{\Theta}_{s-1})$ which characterize the (unperturbed) integrable model; it was shown in [45] that we can construct the composite scalar field:

$$\lim_{z\to z'} \left(T_{s+1}(z)\bar{T}_{s+1}(z') - \Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')\right) = X_s + \text{total derivatives} \tag{1.28}$$

where here we use z as a collective label for the analytic and anti-analytic coordinates. Hence these fields can be explicitly constructed for any IQFT, and they can be shown to be irrelevant since their mass dimension is $[\text{mass}]^{2s+2}$. Placing the theory on a cylinder, where the spectrum becomes discrete and given by some eigenstates $|n\rangle$, the scalars X_s satisfy the important relations:

$$\langle n| X_s |n\rangle = \langle n| T_{s+1} |n\rangle \langle n| \bar{T}_{s+1} |n\rangle - \langle n| \Theta_{s+1} |n\rangle \langle n| \bar{\Theta}_{s+1} |n\rangle \tag{1.29}$$

This is of fundamental importance in the determination of the energy spectrum discussed in the following section. The modification of the action generated by such operators can be easily found following from (1.27):

$$\frac{d}{d\alpha}\mathcal{A}_\alpha \propto \int X_s d^2z \tag{1.30}$$

In fact, this type of perturbations essentially generate the entire set of transformations which preserve the integrability of a theory, except for some situation a

[5]The "solvability" of this perturbation is actually stronger than this, in that also in non-integrable theories it is always possible to find the spectrum of the perturbed theory as a function of the spectrum of the unperturbed one.

`finite` set of extra perturbations exist and have to be added. In some sense, given the space Σ_{int} of all integrable quantum field theory, the integrable perturbations constitute its tangent space $\mathrm{T}\Sigma_{int}$, as discussed in [19]. The fundamental operator is the first of the series, $X_1 = T\bar{T}$, which coincides with the product of the analytic and anti-analytic components of the stress-energy tensor. We observe that the X_1 deformation can be defined direcly at the lagrangian level (if the theory has a lagrangian description) in terms of a parameter α, as:

$$\mathcal{L}^{(\alpha+\delta\alpha)} = \mathcal{L}^{(\alpha)} + \delta\alpha \det T^{(\alpha)}_{\mu\nu} \tag{1.31}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. This approach is particularly useful when one is interested in a holographic or string description (it has been shown that the deformation is equivalent to coupling the theory to 2D topological gravity, in which the cosmological constant Λ is a function of the parameter α of the $T\bar{T}$). However, we will only put the focus on the S-matrix, since it is what we need to study Generalized Hydrodynamics.

1.4.2 Consequences on the theory

We focus on the case of X_1, which is the $T\bar{T}$ deformation, which can be obtained solely in terms of the components of the energy-momentum tensor. The arising of terms of this type in statistical mechanics has been studied for example in flows between minimal conformal models [46–48], the most famous example being the spontaneous supersymmetry breaking flow from the tricritical Ising model to the Ising model [49]. The property of being integrability preserving simply means that the operators X_s commute with the local integrals of motion of the original integrable model, up to total derivatives which vanish in the integrals. This implies a preservation of the conserved charges in the perturbed model. Besides this fundamental property, this particular perturbation has another key feature, namely that the energy levels of the perturbed theory can be found exactly once the spectrum of the unperturbed theory is known. In particular, from (1.29) it is possible to prove that each energy level satisfies the differential equation:

$$\frac{\partial}{\partial\alpha}E_\alpha + E_\alpha \frac{\partial}{\partial R}E_\alpha + \frac{P^2}{R} = 0 \tag{1.32}$$

where R is the radius of the cylinder on which the theory is defined (i.e. the inverse temperature). This is the well known inviscid Burger equation in one dimension, which assuming as initial condition the conformal energy and momentum

$$E(R) = \frac{2\pi}{R}\left(\Delta + \bar{\Delta} - \frac{c}{12}\right), \; P(R) = \frac{2\pi}{R}\left(\Delta - \bar{\Delta}\right) \tag{1.33}$$

can be exactly solved to give

$$E_\alpha = \frac{R}{2\alpha}\left(-1 + \sqrt{1 + \frac{4\alpha E(R)}{R} + \frac{4\alpha^2 P(R)^2}{R^2}}\right) \tag{1.34}$$

This signals the presence of a strange effect which can happen if $\alpha < 0$, namely that the square root becomes negative. This transition, called `Hagedorn Transition`, is actually very general and not present only in the CFT situation. [50, 51]. It was first observed in the TBA context in [52, 53], as a natural consequence of the introduction of a CDD factor in the S-matrix, and is related to the theory not being complete in the UV, but being only an effective theory.

The second consequence on the structure of the theory is the very simple way in which the S-matrix is modified. We can actually define the X_s deformation by multiplying the S-matrix by a CDD (Castillejo, Dalitz, Dyson) factor [20]. These factors arise because the bootstrap approach outlined above is only able to fix the S-matrix up to a scalar meromorphic function $\Phi(\theta)$, bounded in the physical strip, satisfying

$$\Phi(\theta)\Phi(-\theta) = 1 \qquad\qquad \Phi(i\pi + \theta)\Phi(i\pi - \theta) = 1$$

These properties guarantee that the resulting S-matrix still satisfies unitarity, crossing and Yang-Baxter equations, but it still alters the theory dramatically. In the present case, the CDD factor takes the form:

$$S_{ij}^{kl}(\theta) \longrightarrow e^{i\delta_{ij}^{(t)}(\theta)} S_{ij}^{kl}(\theta) \tag{1.35}$$

Where $\delta_{ij}^{(t)}(\theta) = tm_i m_j \sinh(s\theta)$. We observe that this factor is diagonal, and this will come useful in the discussion of the perturbed TBA equations to be analyzed in the following.

The presence of the Hagedorn transition is of great importance since it signals that something strange takes place in the ultraviolet limit, which is interpreted as the presence of a shortest distance in the theory (this simply means that the theory is effective, and some cutoff has to be introduced at some high energy scale). In particular, in general the perturbed theory is `not` UV complete, where by this we mean that it is not always the case that the UV limit of these theories is a consistent local QFT. However, as was shown in [54], this problem can be generally circumvented by adding higher perturbations of the generalized $T\bar{T}$ type, and this allows to find the UV completion, which in general is not unique. Let us now be more precise. The formulation of the problem is the following: since our theory is constructed starting from an S-matrix, on which the $T\bar{T}$ deformation has a very simple effect, is it in general possible to find a local quantum field theory which can give this S-matrix (and hence this scattering theory) using the familiar QFT reduction formulae? The study of [54] shows that this is true only if one is free to add arbitrary number of higher perturbations. This is not necessarily a dramatic problem, since there exist consistent scattering matrices which do not have a underlying consistent QFT. As noted for example in [50], this signals the fact that the theory underlying these S-matrices are string theories and not QFTs. In fact, it was conjectured that except for the situations in which there is a local field theory underlying the S-matrix, the Hagedorn transition is `always` present, being the rule rather than the exception (as mentioned earlier the set of QFTs which are connected to a UV fixed point has measure zero out of the set of all effective QFTs).

In this work, we will not focus particularly on these delicate aspects, and will not explore the aspects related to string theory. We will consider regimes in which the Hagedorn transition does not take place, namely the positive α case, but also the negative α seen as an effective field theory, with α small enough.

Chapter 2

Thermodynamic Bethe Ansatz

The Thermodynamic Bethe Ansatz (TBA) provides an extremely powerful tool which allows to extract the Thermodynamics of a Quantum Field Theory from its S-matrix.[1] First proposed in [16] (as a generalization of [37]) for two simple diagonal theories, it has later been applied to a vast range of diagonal theories [21, 22, 55, 56] and even extended to non-diagonal theories, [47, 57–62]. In this chapter, we will present the fundamental derivation of the TBA for diagonal theories and state its extension to non-diagonal ones. We will show that in many relevant situations the TBA equations can be recast in the extremely general form of the Y-systems.

In this discussion we follow [16], and consider relativistic field theories in (1+1) dimensions defined on a torus, namely a cylinder with periodic boundary conditions. Making use of the mirror symmetry of the theory, namely the invariance of the theory under double Wick rotation, we can choose as time axis both of the directions along the torus, and hence we can quantize the theory in two different ways. Denoting by x and y the two directions of the torus, of lengths L and R respectively, we can either quantize using y as "time direction", with Hamiltonian which can be defined from the corresponding components of the stress-energy tensor:

$$H_L = \frac{1}{2\pi} \int T_{yy} dx \tag{2.1}$$

Equivalently we can consider the x axis as time, to obtain the mirror hamiltonian:

$$H_R = \frac{1}{2\pi} \int T_{xx} dy \tag{2.2}$$

The key idea of the TBA framework is the presence of a mirror symmetry, which implies the equivalence of the statistical mechanics obtained in the two quantization approaches. Considering the situation $L \gg R$, i.e. the limit in which the torus

[1] Actually, in light of the remarks made in the previous chapter, this method only requires a scattering theory and not a QFT. The only thing which is necessary to apply this method is the knowledge of the elements of the S-matrix which are found via the bootstrap approach, irrespective that the underlying theory be a local QFT, a string theory, or anything else.

becomes an infinite cylinder, the statistical physics of the two systems becomes greatly simplified: in the first case, the partition function is dominated by the ground state energy of the theory E(R)[2]:

$$Z_L(R) = e^{-LE(R)} \tag{2.3}$$

while in the second case we need to evaluate the partition function in the thermodynamic limit (since the spatial direction goes to infinity, and we assume that the number of particles which constitutes the system grows correspondingly), and this is immediately found in terms of the free energy density:

$$Z_R(L) = e^{-LRf(R)} \tag{2.4}$$

The mirror symmetry implies the fundamental relation:

$$E(R) = Rf(R) \tag{2.5}$$

and by dimensional considerations, this can be written conveniently as:

$$E(R) = -\frac{\pi c(r_1, r_2, ...r_n)}{6R} \tag{2.6}$$

Where c is the so called scaling function of the theory, and the $\{r_i\}$ are all the dimensionless variables which can be constructed from the various length scales of the system. In the `standard` case, studied for example in [55, 56], the only two relevant scales are m and R, and therefore there is only one parameter $r = mR$. If the $T\bar{T}$ perturbation is introduced, also the additional combination $r' = m^2\alpha$ will have to be considered. The TBA approach is quite transparent: by considering the second of the two quantization schemes, with time direction on the R axis, we will find $f(R)$ from entropy maximization. This will allow us to evaluate the partition function, and immediately also the ground state energy, and the scaling function, thus solving the thermodynamics of the theory. In the following, since R has the statistical interpretation of the inverse temperature of the system, we will use R and β interchangeably, where $\beta = T^{-1}$.

2.1 Derivation of the thermodynamics

2.1.1 The Bethe Wavefunction

Here we follow closely the original proof proposed by Zamolodchikov in [16]. Considering a diagonal IQFT in the torus geometry, we can obtain its spectrum as a set of n particles A_a, a=1,... n, as explained in the previous sections. If the scattering is diagonal we can express the S-matrix as $S_{ab}(\theta) = e^{i\delta_{ab}(\theta)}$, where the δ_{ab}

[2]Recall the usual approach to deal with finite temperature quantum statistical mechanics, in which the inverse temperature β becomes the length of the time direction in the Euclidean formulation. This is easily seen by Wick rotating the generating functional of some QFT.

are the scattering shifts. In general, in relativistic field theories the wavefunction formalism is not appropriate to describe the systems because of particle creation and annihilation. However, the integrability of the theory ensures elasticity of the scattering, and this implies that the evolution of the system conserves the identity of the quasiparticles and their momenta. Moreover, in configuration space we can consider regions in which all the particles are well separated, and since we assume short range interactions they are free (in the sense that the spatial separation of the particles is much larger than the correlation lenght of the system, given by $\xi = 1/m_1$, the mass of the lightest particle). Therefore in each of these regions we can associate to the system a wavefunction of the form [55]

$$\phi(x_1, x_2, ...x_n) = \exp\left(i\sum_j p_j x_j\right) \sum_{Q\in S_n} A(Q)\Theta(x_Q) \tag{2.7}$$

where the sum is over all the permutations of the particles, A(Q) are coefficients, and

$$\Theta(x_Q) = \begin{cases} 1, & \text{if } x_{Q_1} < x_{Q_2} < ... < x_{Q_n} \\ 0, & \text{otherwise} \end{cases}$$

Since our system is 1-dimensional, we can view any exchange of two neighbouring particles as a multiplication of the wavefunction above by the corresponding S-matrix element. We note that this immediately highlights a striking feature of such systems arising because of the presence of a single spatial dimension: the statistics of the particles is inseparable from their interactions. That is, it is impossible to distinguish between, for example, interacting bosons and free fermions (as far as the exchange properties are concerned) and viceversa. This leads to the introduction of the definition of the particle **type**. First of all we recall that the unitarity of the S-matrix implies that $\mathcal{S}^2_{aa}(0) = 1$, and hence it can only be $\mathcal{S}_{aa}(0) = \pm 1$. Then, indicating the particle `statistics` as $(-1)^{F_a} = \pm 1$ if the particle is a boson or a fermion, we define the type as

$$t_a = -(-1)^{F_a} S_{aa}(0) \tag{2.8}$$

Hence we refer to particles with $t_a = \pm 1$ as being of fermionic or bosonic `type` respectively. Having made this premise, we can impose periodic/antiperiodic boundary conditions to the wavefunction on the torus, which by the comment made above imply that:

$$\exp(ip_i L) \prod_{j\neq i} S(\theta_i - \theta_j) = \pm 1 \tag{2.9}$$

which, focusing on the fermionic case [3] can be rewritten as:

$$m_i L \sinh\theta_i + \sum_{j\neq i} \delta_{ij}(\theta_i - \theta_j) = 2\pi n_i \tag{2.10}$$

[3] Although there is no rigorous proof, there is strong evidence [55, 56] of the fact that there are no consistent interacting theory with particles of bosonic type: the only possible theory in this scenario is the trivial one, with S-matrix equal to the identity.

We can consider the values of n_i as quantum numbers which specify completely the Bethe state together with the rapidities of the various particles:

$$|\psi\rangle = |n_1, \theta_1; n_2, \theta_2, ...n_N, \theta_N\rangle \tag{2.11}$$

2.1.2 Thermodynamics

In the thermodynamic limit, where $N \to \infty$ and $L \to \infty$, the spectrum of rapidities becomes continuous, and therefore we can introduce a rapidity density (per unit length) $\rho_a^r(\theta)$, where the r label will be clarified below. Recalling that the energy of a single particle is simply $m \cosh\theta$, we see that the total energy of the system and the quantization condition can be expressed as:

$$E\left[\rho_a^r(\theta)\right] = \sum_a \int_{-\infty}^{+\infty} m_a \cosh\theta \rho_a^r(\theta) d\theta \tag{2.12}$$

$$\frac{m_a}{2\pi} \sinh\theta_i^a + \sum_b (\delta_{ab} * \rho_b^r)(\theta_i) = \frac{n_i^a}{L} \tag{2.13}$$

Where the $*$ in the second equation represents a convolution. If the second equation is satisfied, i.e. there exist some n_i which are admissible quantum numbers corresponding to some θ_i^a which solve the equations, the n_i^a are called `roots` and they contribute to the $\rho_a^r(\theta)$. However, to compute the total density of states we must also consider the `holes` of the theory, which arise for example as a consequence of the selection rules related to the particle type discussed above. The holes give rise to a density $\rho_a^h(\theta)$, and we can compute the total density of states as

$$\rho_a(\theta) = \rho_a^h(\theta) + \rho_a^r(\theta) \tag{2.14}$$

It is clear that, defining $\mathcal{J}_a(\theta) = \frac{m_a}{2\pi} \sinh\theta_i^a + \sum_b (\delta_{ab} * \rho_b^r)(\theta_i)$ we can express the total density as:

$$\rho_a(\theta) = \frac{d}{d\theta}\mathcal{J}_a(\theta) = \frac{m_a}{2\pi} \cosh\theta_i^a + \sum_b (\varphi_{ab} * \rho_b^r)(\theta_i) \tag{2.15}$$

The kernels φ_{ab} are the derivatives of the scattering shifts, $\varphi_{ab}(\theta) = \frac{d}{d\theta}\delta_{ab}(\theta)$, and hence the logarithmic derivatives of the S-matrix elements. The strategy we now use to derive the thermodynamic of the system is to obtain the entropy from the number of ways to distribute particles in the accessible energy levels given by the root density. Since we have that the number of particles in a given rapidity interval is $n_a = L\rho_a^r(\theta)\Delta\theta$ and the total number of accessible states is $N_a = L\rho_a(\theta)\Delta\theta$, we have that the ways of distributing the particles in the levels is:

$$\Omega_a = \frac{(L\rho_a(\theta)\Delta\theta)!}{(L\rho_a^r(\theta)\Delta\theta)!(L(\rho_a(\theta) - \rho_a^r(\theta))\Delta\theta)!} \tag{2.16}$$

This allows to compute the entropy in the usual way, obtaining:

$$S[\rho_a, \rho_a^r] = \sum_a \int_{-\infty}^{+\infty} \left(\rho_a \ln\rho_a - \rho_a^r \ln\rho_a^r - (\rho_a - \rho_a^r)\ln(\rho_a - \rho_a^r)\right) \tag{2.17}$$

Now, we can obtain the free energy as $f = E - TS$, and this is a constrained maximization problem which can be solved using Lagrange multipliers, leading to the final expression for the extremum condition:

$$m_a R \cosh\theta = \epsilon_a(\theta) + \sum_b (\varphi_{ab} * L_b)(\theta) \tag{2.18}$$

where ϵ_a is defined by the equation $\frac{\rho_a^r}{\rho_a} = \frac{1}{1+\exp\epsilon_a}$, while $L_a = \ln(1+e^{-\epsilon_a})$. Finally, by the constrained maximization we obtain the free energy and the partition function, expressed in terms of the ϵ_a which solve equation (2.18):

$$f(R) = -\frac{1}{2\pi R}\sum_a \int_{-\infty}^{+\infty} m_a \cosh\theta \ln(1+e^{-\epsilon_a})d\theta \tag{2.19}$$

$$Z(L,R) = \exp\left(\frac{L}{2\pi}\sum_a \int_{-\infty}^{+\infty} m_a \cosh\theta \ln(1+e^{-\epsilon_a})d\theta\right) \tag{2.20}$$

We clearly see that the whole problem of obtaining the thermodynamics of the theory reduces to finding the solution of the set of coupled integral equations (2.18), the so called TBA equations. The term $\nu_a = m_a R\cosh\theta$ is the driving term of the theory, and as it will become clear later will be modified depending on the situation we are investigating: in the Generalized Gibbs Ensemble, used to describe the long time dynamics of inhomogeneous integrable systems, the driving term will contain terms associated to all the infinite conserved charges. Note that the essence of this discussion is that we can build maximal entropy states for IQFTs. This is crucially important for the Hydrodynamical description of these theories, since the main idea of Hydrodynamics is to consider `local` maximization of entropy instead of global. Therefore this framework almost automatically leads to the formulation of GHD, as will be presented in chapter 3.

2.1.3 The scaling function

As mentioned above the scaling function is related to the definition of the gound state energy and is given by $c(r) = -6R\frac{E_0}{\pi}$. Focusing without loss of generality on a theory with a single particle of mass m=1 in the spectrum, we thus have:

$$c(r) = \frac{3}{\pi^2} r \int_{-\infty}^{+\infty} \cosh\theta \ln(1+e^{-\varepsilon(\theta)})d\theta \tag{2.21}$$

In general this integral cannot be computed exactly (the only exceptions being the free boson and the free fermion, as will be discussed in chapter 4) except in the UV and IR limits. While in the infrared the solution is that of free particles where $\epsilon_a \approx r\cosh\theta$, and the c function is solved in terms of Bessel functions, for $r \to 0$ it can be studied observing that the L-functions and n-functions form pleateaus of constant value in the region $-\ln\frac{2}{r} \ll \theta \ll \ln\frac{2}{r}$. The constant value is solution of the constant TBA equation:

$$\varepsilon_a = \sum_b N_{ab}\ln(1+e^{-\varepsilon_b}), \qquad N_{ab} = -\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\varphi_{ab}(\theta) \tag{2.22}$$

This structure is clearly visible from figure 2.1. Observing that in this limit we can

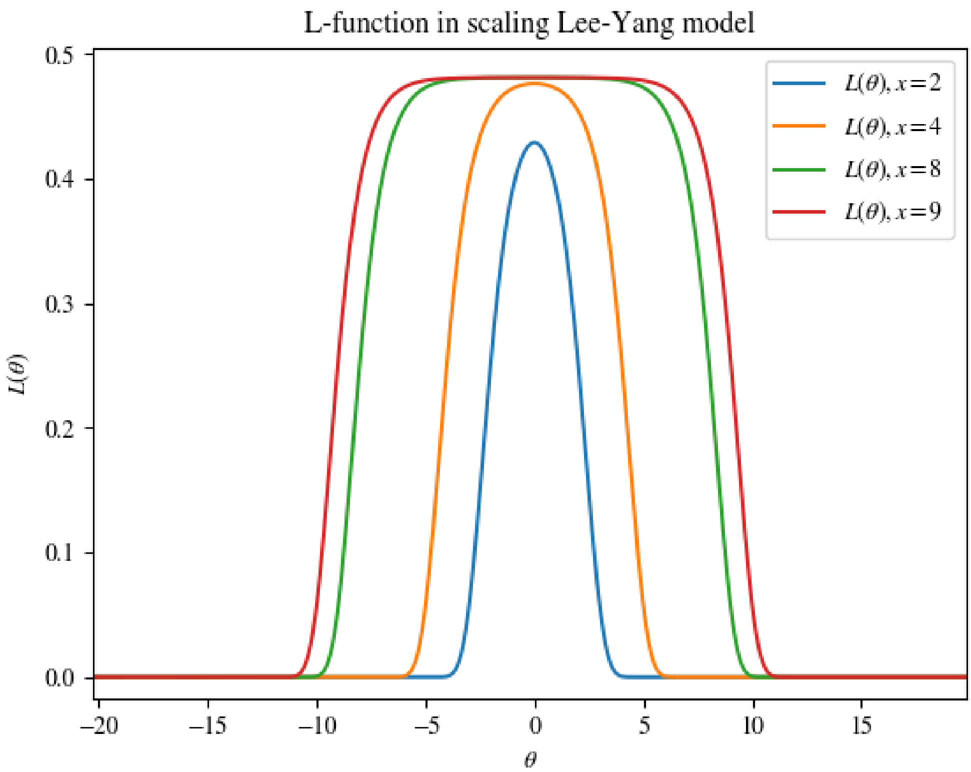


Figure 2.1: Behaviour of the L-functions in the scaling Lee-Yang model, for different values of $x = \ln(2/r)$. For $r \to 0$, which corresponds to $x \to \infty$, the solutions clearly form plateaus of constant height, which is determined by solution of (2.22).

approximate $r \cosh\theta \sim \hat{e}^{\theta - \ln(2/r)}$, one can introduce the equation which describes the system at the edges of the plateau:

$$e^{\theta} = \tilde{\varepsilon} + \varphi * \tilde{L} \tag{2.23}$$

and this allows to find the UV scaling function, as:

$$c(0) = \frac{6}{\pi^2} \int_0^{\infty} d\theta \tilde{L} e^{\theta} \tag{2.24}$$

This can be computed by making use of (2.23) to substitute e^{θ} with the derivative of the right hand side, and standard computations lead to the final expression:

$$c(0) = \frac{6}{\pi^2} L\left(\frac{1}{1 + e^{\epsilon_0}}\right) \tag{2.25}$$

where L is Roger's dilogarithmic function, which is essentially a different normalization of the more common dilogarithm $Li_2(z) = \sum_k \frac{z^k}{k^2}$, defined as:

$$L(x) = Li_2(x) + \frac{1}{2} \ln(x) \ln(1 - x) \tag{2.26}$$

and ϵ_0 appearing in the expression of the scaling function is the one computed via the constant TBA equations. A generalization of this expression, which for now has only led to partial results, will be discussed in the following sections and especially in appendix A.

2.1.4 Diagonal Theories Encoded in Dynkin Diagrams

The authors of [55, 56] studied the S-matrix and thermodynamics of a vast range of diagonal theories which can be associated to Lie algebras. What they discovered was a series of striking features of these theories which show the strong relation with the underlying Lie structure, which are shown in the table below [63]. Moreover,

Lie Algebra	**Scattering Theory**
# nodes in the Dynkin diagram	#particles of the spectrum
Coxeter number g	S-matrix poles at multiples of $\theta = \frac{i\pi}{g}$
Incidence matrix	Universal S-matrix structure
Symmetry of the Dynkin diagram	Charge Conjugation

in these theories the masses and all the conserved charges of higher spins can be arranged into eigenvectors of the incidence matrix G_{ab} of the Dynkin diagram of the corresponding Lie algebra, such that:

$$\sum_b G_{ab} q_b^{(s)} = 2\cos\left(\frac{\pi s}{g}\right) q_a^{(s)} \tag{2.27}$$

This last equation will be crucially important in the following. It is possible to make use of this relation also to find the universal structure of the TBA equations, as was first proposed in [21] and rigorously proven in [22]. To do so, we make use of the fundamental relation, proven in [22], for the S-matrix of such theories:

$$\mathcal{S}_{ab}\left(\theta + \frac{i\pi}{g}\right) \mathcal{S}_{ab}\left(\theta - \frac{i\pi}{g}\right) = \prod_c \mathcal{S}_{ac}(\theta)^{G_{bc}} e^{-2\pi i G_{ab}\Theta(\theta)} \tag{2.28}$$

We can use this to rewrite equation (2.18) in a universal form. To do so we consider equation (2.28) and take the logarithmic derivative to obtain a relation between the kernels, which we then Fourier transform:

$$\varphi_{ab}\left(\theta + \frac{i\pi}{g}\right) + \varphi_{ab}\left(\theta - \frac{i\pi}{g}\right) = \sum_b G_{bc}\varphi_{ac}(\theta) - 2\pi\delta(\theta) G_{ab} \tag{2.29}$$

$$2\cosh\left(\frac{k\pi}{g}\right) \tilde{\varphi}_{ab}(k) = \sum_b G_{bc}\tilde{\varphi}_{ac}(k) - 2\pi G_{ab} \tag{2.30}$$

Using the matrix identity $(1+A)^{-1} = 1-(1+A)^{-1}A$, we can obtain:

$$\left(1-\frac{\tilde{\varphi}}{2\pi}\right)^{-1} = 1-\left(1-\frac{\tilde{\varphi}}{2\pi}\right)^{-1}\left(-\frac{\tilde{\varphi}}{2\pi}\right) = \tag{2.31}$$

$$1+(2\pi\tilde{\varphi}-1)^{-1} = \tag{2.32}$$

$$1+\left(-G^{-1}\left(2\cosh\frac{k\pi}{g}-G\right)-1\right)^{-1} = \tag{2.33}$$

$$= 1-\frac{G}{2\cosh(\frac{k\pi}{g})} \tag{2.34}$$

which in components reads $(\delta_{ab}-\frac{\tilde{\varphi}_{ab}}{2\pi})^{-1} = \delta_{ab}-\frac{G_{ab}}{2\cosh(\frac{k\pi}{g})}$. Now we are ready to find the expression of the TBA equations which highlights the underlying Lie structure. Considering equation (2.18), we can Fourier transform it, multiply by $\delta_{ab}-\frac{G_{ab}}{2\cosh(\frac{k\pi}{g})}$, and sum over the repeated index. Using the relation just obtained, it is easy to see that we obtain the relation:

$$\tilde{\nu}_a(k) = \tilde{\epsilon}_a(k)+\frac{1}{2\pi}\sum_b \tilde{L}_b(k)\tilde{\varphi}_{ab}(k)$$

$$\sum_a(\delta_{ac}-\tilde{R}(k)G_{ac})\tilde{\nu}_a(k) = \sum_a(\delta_{ac}-\tilde{R}(k)G_{ac})(\tilde{\epsilon}_a(k)+\frac{1}{2\pi}\sum_b \tilde{L}_b(k)\tilde{\varphi}_{ab}(k))$$

$$\tilde{\nu}_a(k) = \tilde{\epsilon}_a(k)+\frac{1}{2\pi}\sum_b \tilde{R}(k)G_{ab}(\tilde{\nu}_b(k)-\tilde{\epsilon}_b(k)-\tilde{L}_b(k))$$

Where we have called $\tilde{R}(k) = \frac{1}{2\cosh(\frac{k\pi}{g})}$. Fourier transforming back to rapidity space we obtain the final expression:

$$\nu_a(\theta) = \epsilon_a(\theta)+\frac{1}{2\pi}\varphi * \sum_b G_{ab}(\nu_b-\epsilon_b-L_b)(\theta) \tag{2.35}$$

Where we have introduced the `universal` kernel $\varphi = \frac{g}{2\cosh\frac{g\theta}{2}}$. This equation clearly shows how the entire TBA equation structure can be encoded in the Dynkin diagram via G.

2.2 Non-Diagonal Case

The discussion made above is valid only for diagonal theories, in which the S-matrix simply reduces to the exponential of the phase shift. In non-diagonal theories, however, the solution becomes more complicated, and it involves the difficult problem of diagonalizing the transfer matrix, as was studied in a particular case in [39, 57]. In most situations, however, the TBA equations for nondiagonal theories were proposed without a complete formal derivation by following physical intuition, as in [22, 47, 59, 61]. As discussed above, the elasticity of the S-matrix implies that the only major change that can affect a multiparticle state after a scattering is a reshuffling of the quantum numbers of the particles involved in the scattering. Hence the main feature of non-diagonal theories is that there are two different kind of particles arising in the TBA description: the massive quasiparticles which appear also in the diagonal case and a set of massless particles, the magnons, which are those responsible for the modification of the quantum numbers of the scattered particles. Following [58], and extending the discussion of the previous section, we focus on the theories which can be encoded in a product of Dynkin diagrams, which we write as $G \ltimes H$, where G is the Dynkin diagram related to the massive particles, which works analogously as above, while H is a new Dynkin diagram related to the magnonic excitations of the system. The product implies that to each massive node on the "horizontal direction" of the product of graphs there are several associated massless nodes on the "vertical direction", hence we will refer to the nodes using two indices: a,b,c,... referring to the horizontal position, and i, j, k,... referring to the vertical position. In this notation, we write the general TBA equations as:

$$\nu_a^i = \varepsilon_a^i + \frac{1}{2\pi} * \left(\sum_b \mathcal{G}_{ab} \left(\nu_a^i - \Lambda_a^i \right) + \sum_j \mathcal{H}_{ij} L_a^j \right) \tag{2.36}$$

These theories describe minimal models perturbed by the ϕ_{13} deformation (which is the least relevant operator) in the case G is trivial, while for G nontrivial they describe the coset model $G_k \times G_L / G_{k+l}$ perturbed by the least relevant operator $\phi_{adj}^{id,id}$. Relevant physical examples include for instance the tricritical Ising model perturbed by the vacancy operator t, which generates the flow to the ising model through the supersymmetry preserving direction.[4] Diagonal and non-diagonal theories which can be encoded in Dynkin diagrams in this fashion admit an extraordinarily general formulation in terms of Y-systems.

2.3 Y-systems

As first noted in [21], in some situations the TBA equations can be recast in an extremely general form, that of the Y-system. The main feature of these theories is that their structure is strongly related to some Lie algebra in the diagonal case or to the aforementioned more sophisticated structures which can be represented as products of Dynkin diagrams in the non-diagonal case. The Y-system equation is far more general than the original TBA equation, in the sense that the same Y-system can describe several different theories: for example, in [63] it was shown that systems with different driving terms (in the GGE setup, to be described in the next section) admit the same Y-systems; in [64], moreover, it was shown that also introducing $T\bar{T}$ perturbations does not alter the Y-system. The specific model can be reobtained from the Y-system once the asymptotic behaviour is fixed. The generality of this formulation allows for example even to use the Y-system pole

[4]Interestingly, this flows to the Ising model precisely along the $T\bar{T}$ direction.

structure to find the excited states and energy, something which cannot be done from the sole TBA equations [65, 66].

2.3.1 Diagonal Y-systems

We proceed from equation (2.35) in order to find the generalized expression of the Y-system. Summing equation (2.18) evaluated in $\theta^{\pm} = \theta \pm \frac{i\pi}{g}$, we obtain

$$\nu_a(\theta^+) + \nu_a(\theta^-) = \epsilon_a(\theta^+) + \epsilon_a(\theta^-) + \frac{1}{2\pi} \sum_b \left[\varphi_{ab}(\theta^+) + \varphi_{ab}(\theta^-)\right] * L_b \quad (2.37)$$

$$\nu_a(\theta^+) + \nu_a(\theta^-) = \epsilon_a(\theta^+) + \epsilon_a(\theta^-) + \frac{1}{2\pi} \sum_{b,c} G_{bc}\varphi_{ac} * L_b(\theta) - \sum G_{ab} L_b(\theta) \quad (2.38)$$

where we have used equation (2.29). Thus we see that if we subtract again equation (2.18) evaluated in θ and multiplied by $\sum_b G_{ab}$, we obtain:

$$\nu_a(\theta^+) + \nu_a(\theta^-) - \sum_b G_{ab}\nu_b(\theta) = \epsilon_a(\theta^+) + \epsilon_a(\theta^-) - \sum_b G_{ab}(\epsilon_b(\theta) + L_b(\theta)) \quad (2.39)$$

where the left hand side of this equation vanishes because of (2.27) and of the periodicity properties of the hyperbolic cosine in the imaginary direction. Therefore, defining the Y function to be $Y_a(\theta) = \exp(\epsilon_a(\theta)$, we obtain:

$$\epsilon_a(\theta^+) + \epsilon_a(\theta^-) = \sum_b G_{ab}(\ln(1 + e^{\epsilon_b})) \quad (2.40)$$

$$Y_a(\theta^+)Y_a(\theta^-) = \prod_b (1 + Y_b(\theta))^{G_{ab}} \quad (2.41)$$

which is the famous Y-system. This extremely general equation contains all the information of the TBA equations, and in fact much more. Its stationary solutions (θ independent) can be used to find the UV central charge of the theory, thus they lead immediately to the conformal limit. Moreover, the periodicity of the Y-system can be related to the conformal dimension of the perturbing field: it was shown in [21] that $Y_a(\theta + i\pi P) = Y_{\bar{a}}(\theta)$, where $P = \frac{g+2}{g}$. This is in relation with the dimension of the perturbing field, which can be expressed as $\Delta = 1 - \frac{1}{P}$.

2.4 Non-Diagonal theories

As was investigated in [58, 61], the discussion made in the previous section can be generalized also to non-diagonal theories by encoding them in the product of Dynkin diagrams. Here we just state the final expression, although the proof can be performed following directly from what was done in the previous section, having some care in the treatment of the second Dynkin diagram H. The Y-system appears as a direct generalization of the one for diagonal theories, with an additional term coming from the magnonic theory:

$$Y_a^i(\theta^+)Y_a^i(\theta^-) = \prod_b (1 + Y_b^i(\theta))^{G_{ab}} \prod_j (1 + (Y_a^j(\theta))^{-1})^{-H_{ij}} \tag{2.42}$$

For a theory with a single massive node and several massless nodes, namely a conformal minimal model perturbed by ϕ_{13}, this can be expressed in exctly the same form as the diagonal theory, except for defining $\tilde{Y} = \exp(-\varepsilon)$, and in this case the equations become:

$$\tilde{Y}^i(\theta^+)\tilde{Y}^i(\theta^-) = \prod_j (1 + \tilde{Y}^j(\theta))^{H_{ij}} \tag{2.43}$$

which have the same structure as the previous ones.

2.5 TBA in the presence of $T\bar{T}$ deformations

The particularly simple way in which the $T\bar{T}$ deformation alters the S-matrix of the underlying theories, which was discussed in section 1.4.2, leads immediately to a simple modification of the TBA equations through the introduction of an additional term in the scattering kernel. In particular, since in general we have $\varphi = -i\frac{d}{d\theta}\ln S$, the addition of the CDD factor leads to:

$$\varphi_{ab}^{\alpha} = \varphi_{ab}^{0} - \alpha m_a m_b \cosh\theta \tag{2.44}$$

where α is a parameter of dimension $[M]^{-2}$ which characterizes the strenght of the coupling. Focusing on a theory containing a single particle in which the mass is normalized to 1, postponing a more general discussion to later, the TBA equations contain a term which can be expressed as:

$$\varphi_\alpha \star L(\theta) = \varphi_0 \star L(\theta) - \alpha \cosh \star L(\theta) \tag{2.45}$$

which can be simplified by observing that:

$$\begin{aligned} \cosh \star L(\theta) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cosh(\theta - \theta')L(\theta')d\theta' \\ &= \frac{1}{2\pi}\cosh\theta \int_{-\infty}^{\infty} \cosh\theta' L(\theta')d\theta' - \frac{1}{2\pi}\sinh\theta\int_{-\infty}^{\infty}\sinh\theta' L(\theta')d\theta' \,. \end{aligned}$$

Now (at least when the system is at equilibrium), it is easy to infer from the TBA equations that the symmetry of the L-function is preserved by the addition of the deformation, if the original kernel is symmetric[5] (and this is the case in the situations

[5] This observation simply follows from the fact that in the convolution it is possible to move all the θ dependence on the kernel part, without any change the L-function itself. Therefore, the symmetry follows from the symmetry of the cosh

of interest) and therefore the second integral in the above expression vanishes. On the other hand, the first integral is simply the TBA free energy:

$$E_0^\alpha = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \cosh\theta L(\theta) d\theta\,, \tag{2.46}$$

and therefore we can simplify conveniently the equations above as:

$$\varphi_\alpha \star L(\theta) = \varphi_0 \star L(\theta) + \alpha E_0^\alpha \cosh\theta \tag{2.47}$$

This finally leads to the final TBA equation:

$$\varepsilon_\alpha(\theta) = (\beta - \alpha E_0^\alpha)\cosh\theta - \varphi_0 \star L_\alpha(\theta) \tag{2.48}$$

which shows that in general the effect of the perturbation can be represented as a redefinition of the inverse temperature, $\beta \to \beta - \alpha E_0^\alpha$.[6] In the presence of several different particles of different masses, the discussion is essentially the same, and one easily sees that the structure of the equations is:

$$\begin{aligned} \varepsilon_{\alpha,i} &= \nu_i - \alpha m_i \sum_j E_{0,j}^\alpha \cosh\theta - \sum_j \varphi_0^{ij} \star L_{\alpha,j}(\theta) && (2.49)\\ &= \nu_i - \alpha m_i E_0^\alpha \cosh\theta - \sum_j \varphi_0^{ij} \star L_{\alpha,j}(\theta) && (2.50)\end{aligned}$$

As discussed above, of particular interest in the TBA setting are those theories which can be encoded in a Dynkin diagram or in a product of a massive and a magnonic diagram. In this case there is still the possibility of recasting the equations in a sort of "universal" form as was in the unperturbed theories. This expression, by carrying on the observation that we can simply redefine the inverse temperature and hence the driving term: one obtains the equation, which will be used in the following:

$$\nu_i^a = \varepsilon_i^a + \varphi_g * \{\sum_b G_{ab}(\nu_i^b - \Lambda_i^b) - \sum_j H_{ij} L_j^a\} + \tag{2.51}$$

$$+\sum_b \alpha m_a m_b (\cosh * L_i^b) - \sum_b \alpha G_{ab} m_a m_b (\varphi_g * \cosh * L_i^b)(\theta) \tag{2.52}$$

This is simply obtained by the substitution

$$\nu \to \nu - \sum_b \alpha m_a m_b \cosh\theta \tag{2.53}$$

And can also be immediately proven rigorously using the usual procedure to find the universal form of the TBA equations, [22,61]. This expression shows clearly that

[6] In [19], this effect was reabsorbed as a modification of the bare mass, rather of the inverse temperature, by a factor containing $F_\alpha = E_\alpha/\beta$; the perspective we use here was rather that which was used in [24].

in magnonic theories the perturbation only affects the massive part of the diagram, while the magnonic nodes are left untouched. As will be shown in the following, the possibility of writing the equations in this particularly simple way will have drastic consequences on the properties of the out of equilibrium dynamics of the system, and will allow us to greatly simplify the GHD equations of the theory, which will be exactly solvable in terms of the solutions of the unperturbed theory. The expression found above can be used to show the important fact that the Y-systems in $T\bar{T}$ deformed theories are not altered, a fact which was already implicitly known in [64].

Chapter 3

Integrable Systems Far From Equilibrium

One of the greatest challenges of modern theoretical physics is the study of quantum systems out of equilibrium. The theoretical research in this field has flourished in the latest years because of the recent developments in the experimental control of quantum systems, especially in the field of ultracold trapped atoms. These experimental setups provide an ideal playground to experiment on virtually isolated quantum systems, in and out of equilibrium, thanks to the possibility to simulate any many-body hamiltonian with extreme effectiveness in the tuning of the parameters. In particular, it has been clear since [1] that integrable systems exhibit a rather peculiar behaviour compared to any other model. This is related to the lack of thermalization in such systems, which implies the non-ergodicity of their time evolution, hence the impossibility of describing the long-time relaxation using the standard statistical ensembles. Therefore, a novel approach is needed to study the out of equilibrium dynamics of integrable systems. The study of quantum quenches of integrable models has then led to the formulation of the Generalized Gibbs Ensemble, from the constructive principle of maximization of entropy, first introduced in [8], and studied in a number of contexts including spin chains [67–69] and field theories [10, 70, 71] then even observed in [72]. Generalized Hydrodynamics then arose as the extension of usual hydrodynamics to integrable systems which are described by a GGE [12, 15], and allows to study out of equilibrium and inhomogeneous integrable systems at a mesoscopic level using the usual tools of hydrodynamics. Far from being a purely theoretical and abstract theory, also GHD has been experimentally verified, as in [73, 74].

3.1 Quantum Quenches

The absence of thermalization in integrable systems has been studied in depth by studying quantum quenches [6, 7, 75]. A quantum quench is a very simple protocol which can be set up in the following way: a quantum system is prepared in an eigenstate (for simplicity, the ground state) of some initial hamiltionian H_0. At time t=0, the hamiltonian is abruptly modified (for example by changing a parameter, or by introducing a further term V_I to the hamiltonian) and the system then evolves according to the final hamiltonian. This simply means that the system is found in a superposition of the eigenstates of the hamiltonian H_I which then determines the time evolution. Considering a subsystem of finite number of particles N, what one would expect to observe is that this should thermalize and be described by a standard thermodynamic ensemble in the long time limit: however, this is not what is observed if the system is integrable [13]. An important observation in this regard has been the fact that the same physical system which showed no thermalization when confined in one spatial dimension, instead behaved "normally" when in two or three dimensions. This is a clear signal of the involvement of integrability (or at least dimensionality-related effects), which as well known is absent in in higher dimensions (as explained in the first chapter, in the case of QFT this is due to Coleman and Mandula theorem).

Consider a many-body quantum system characterized by some time-independent Hamiltonian H. The evolution of the system prepared in an some initial state $|\psi(0)\rangle$ is then given by the unitary time evolution $|\psi(t)\rangle = \exp(-iHt)\,|\psi(0)\rangle$. Although the evolution is purely unitary, the system is expected to thermalize for large t [76]. If the initial state is not an eigenstate of the hamiltonian which controls the time evolution, as in the quench situation, it will in general have nontrivial superposition with several eigenstates $|n\rangle$, namely we will write $|\psi(0)\rangle = \sum c_n\,|n\rangle$. It is clear that if we consider the entire system, assuming that it has been prepared in a pure state, then it would be impossible for it to "relax" to a stationary state, because that would mean that averages of observables should become time-independent: considering that the average of any observable on the time-evolved state can be expressed as

$$\langle\psi(t)|\,O\,|\psi(t)\rangle = \sum_{m,n} \langle\psi(0)\rangle\, n\,\langle m\rangle\, \psi(0) e^{-i(E_n - E_m)t} \tag{3.1}$$

To reach a stationary state I should have

$$\lim_{t\to\infty} \lim_{L\to\infty} \langle O\rangle = Tr[\rho_s O] \tag{3.2}$$

where L is the size of the system and ρ_s is a some statistical ensemble which is supposed to describe the system. This is clearly impossible due to the oscillatory behaviour present in equation (3.1). However, a way out is to consider only subsystems of the total system of lenght L [14, 68]. This is equivalent to considering only local operators O, which act on a finite portion of the system. In this way we choose a specific subsystem A, and consider all the rest as a bath. In this case, the density matrix of the reduced system is obtained by the usual procedure of tracing out the degrees of freedom related to the bath, namely $\rho_A = Tr_{\bar{A}}[\rho]$ Supposing that the only conserved quantity of the system is the energy, we make the following assumption on the time evolution [6]: under time evolution, the reduced density matrix of a finite subsystem will retain only the minimum possible amount of information on the initial state. This means that, if energy is the only conserved quantity, the minimal information retained by the final density matrix will precisely depend only on the energy of the system. Therefore we can introduce a Gibbs density matrix $\rho_G = \frac{e^{-\beta_G H}}{Z}$, where β_G is fixed by the initial condition. In the case of integrable systems the application of the same postulate leads to the need of introducing a new ensemble: requirement of minimum information now has to take into account the conservation of an infinite set of conserved charges which were described chapter 1, and therefore the ensemble which has to be used has to be of the form [8]:

$$\rho_{GGE} = \frac{1}{Z_{GGE}} e^{-\sum_n \lambda_n Q_n} \tag{3.3}$$

where the λ_n are generalized chemical potentials and the Q_n are all the conserved charges of higher spin which appear in the theory. This is the Generalized Gibbs Ensemble (GGE), which we will derive on more general grounds in the following section.

3.2 Generalized Gibbs Ensemble

We found in the previous section that in order to describe the long time behaviour of subsystems of integrable models we need to introduce a density matrix which takes into account the presence of all the higher conserved charges. This can be formally and rigorously justified by an argument based on the maximization of entropy. The problem of finding the correct quantum statistical ensemble which describes some quantum system is essentially the problem of maximizing the entropy $S = -Tr[\rho \ln \rho]$, subject to the constraints given by the conservation laws of a system[1]. This constrained maximization problem can be solved introducing Lagrange multipliers β_i and α, such that the maximum of entropy corresponds to:

$$\delta Tr \left[\rho \left(\ln \rho + \sum_i \beta_i Q_i + \alpha \right) \right] = 0 \tag{3.4}$$

$$Tr \left[\delta\rho \left(\ln \rho + \sum_i \beta_i Q_i + \alpha \right) \right] = 0 \tag{3.5}$$

which actually implies a precise expression for the density matrix:

$$\rho \propto e^{-\sum_i \beta_i Q_i} \tag{3.6}$$

[1]Although the idea of entropy maximization in thermodynamic processes was known since the beginning of statistical mechanics, the idea to reverse the process and build statistical mechanics from the maximization of entropy istelf was proposed in [77]. In fact, this work provided already the first hint towards the generalized Gibbs ensemble, in a procedure which is essentially the same as the one which is discussed here.

For an infinite number of charges, this is precisely the Generalized Gibbs Ensemble. The generalized inverse temperatures β_n are to be fixed by solving the self-consistency problem for the initial conditions:

$$\langle Q_s \rangle = \frac{Tr\{Q_s \rho_{GGE}\}}{\mathcal{Z}_{GGE}} \tag{3.7}$$

Although this seems like a rather intuitive and not particularly formal proof, it has been rigorously proven that for particular choices of initial state in quadratic theories the system does indeed relax to a steady state described by the GGE [78,79]. It is worth stressing that in general, except for free theories, it is an insormountable problem to actually evaluate averages as (3.7), and it is also a highly nontrivial question to understand how to evaluate the infinite sum over the charges.

3.2.1 The problem of Quasilocality

The questions of exacly which charges need to included into the sum, and how to deal with the convergence of the infinite sum in the exponential, are highly nontrivial. It was early observed in spin chains [80–82] that including only the usual integer spin charges is in general not sufficient to properly describe the system and can lead to incorrect predictions. This has led to the introduction of a new set of charges [83] satisfying a novel form of locality, called quasilocality. In spin chains, while local charges act on a finite number of sites, quasilocal ones act on an infinite number of sites but with an exponential decay which make the integrals convergent. These new charges were then soon used in [84], to find the definitive expression for the GGE in spin chains. Extensions to IQFT were firstly proposed in [85], where new semilocal charges where defined by taking the scaling limit in such a way to keep the product of the lattice spacing and the number of sites in a given interval fixed (sending $n \to 0$ and $a \to 0$, where a is the lattice spacing. Also, it became clear that in field theories with non-diagonal scattering (as the magnonic theories discussed in section 2.4) a GGE constructed with the standard integer spin charges was not sufficient to fix the magnonic densities, since as discussed above all the charges of integer spins vanish on the magnonic configurations. In [86] an extensive study was carried out on the sine-Gordon model, exploiting the fact that such model can be seen as the continuum limit of the inhomogeneous XXZ chain, which was studied in [15]. Hence, taking the scaling limit of the previously found lattice quasilocal charges, they could find that these give rise in the continuum limit to a set of fractionary spin charges, which however appeared to be local in the above sense. These had exactly the same spin values as the famously nonlocal charges found in [87, 88], and appeared to be a generalization of [89]. However, the exact relation between these different sets of fractionary spin charges is still an open question, and no progress has been made since the publication of [86].

3.2.2 TBA and Y-systems in GGE

It is easy to observe that the TBA equations can be immediately generalized in the situation in which the system is described by a Generalized Gibbs Ensemble. We recall that the main step in the derivation of the thermodynamics of a IQFT was the maximization of the free energy, under the constraint of conserving the total energy. Having an infinite set of conserved charges, the problem can simply be extended by maximizing a generalized free energy [90] instead of the usual free energy. This very simple procedure can be applied both to the diagonal and nondiagonal case, and it shows that the only modification to the theory is a modification of the driving term. In the diagonal case, we can follow the derivation presented in 2.1.2 up to the expression for the entropy 2.17. At this point, we simply need to substitute:

$$\beta E(\theta) \longrightarrow \sum_n \beta_n Q_n \tag{3.8}$$

This then has to be substituted in the free energy to give rise to a generalized free energy which has to be maximized instead of the standard one. Taking the eigenvalues of the higher spin conserved charges to be of the form $\cosh(s\theta)$ (although the same is valid also if one considers hyperbolic sines or exponentials) one immediately obtains:

$$\nu_a(\theta) = \varepsilon_a(\theta) + \sum_b \varphi_{ab} * L_b(\theta) \tag{3.9}$$

where this time the driving term has been upgraded to a generalized driving term:

$$\nu_a(\theta) = \sum_s \beta_s q_s \tag{3.10}$$

where q_s are the higher spin conserved charges. The generalization to the nondiagonal case presents no further difficulty, as one can see following the derivation of [57], where as here one simply has to promote the free energy to a generalized free energy.

If the discussion of the standard TBA equation can be easily generalized to the GGE both in the diagonal and nondiagonal case, this is not true for the Y-system structure. As it was first discovered in [63], in the diagonal theories discussed in section 2.1.4 the Y-system is exactly preserved in the GGE. This observation is based on the previously stated property of the eigenvalues of the conserved charges, which satisfy the fundamental relation:

$$\sum_b G_{ab} q_b^s = 2 q_a^s \cos\left(\frac{\pi s}{g}\right) \tag{3.11}$$

In the derivation of the Y-system in the standard Gibbs case, this relation was exploited only in its mass version $\sum_b G_{ab} m_b = 2 m_a \cos(\frac{\pi s}{h})$, but the fact that it is so general allows to extend the reasoning also to the GGE. The derivation is trivial: we observe that, in the derivation of section 2.3.1, the explicit form of the driving term was only used to show that the left hand side of equation (2.39) vanished. Hence, we just need to show that the driving term still satisfies the condition $\nu_a(\theta^+) + \nu_a(\theta^-) = \sum_b G_{ab}\nu_b(\theta)$. If we take the driving term to be of the form $\nu_a = \sum_s \beta_s q_s \cosh s\theta$, we can easily see that the equation is indeed true:

$$\begin{aligned}
\nu_a(\theta^+) + \nu_a(\theta^-) &= \sum_s \beta_s q_a^s \cosh s\theta^+ + \sum_s \beta_s q_a^s \cosh s\theta^- = \\
&= \sum_s \beta_s q_a^s \left(\cosh s\theta^+ + \cosh s\theta^-\right) \\
&= \sum_s \beta_s q_a^s \left(\cosh\left(s\theta + \frac{i\pi s}{g}\right) + \cosh\left(s\theta - \frac{i\pi s}{g}\right)\right) = \\
&= \sum_s \beta_s q_a^s 2\cos\frac{\pi s}{g}\cosh s\theta = \sum_s \beta_s \sum_b G_{ab} q_b^s \cosh s\theta = \sum_b G_{ab}\nu_b
\end{aligned} \tag{3.12}$$

And therefore this immediately implies that the Y system can be obtained also in this situation and has exactly the same expression as in the standard Gibbs case. In the case of non-diagonal scattering however the discussion is different, and I have shown that the Y-system will be modified by an exponential factor due to the presence of the fractionary spin charges discussed in the previous section. This small result, which I have shown in the context of quantum field theory, is essentially equivalent to what was found in [11] in the case of spin chains. The core of the argument is the fact that there are charges of spin given by [86]:

$$s = \frac{2k}{p}, \qquad k = 1, 2, ... \tag{3.13}$$

These charges, found for the sine-Gordon model, are then immediately moved to the purely magnonic theories described above, which are obtained from the restricted Sine-Gordon model. In the expression p indicates the minimal model of interest, which we refer to as $\mathcal{M}_p$. Performing the discussion as above we see that we have to consider an additional term. Focusing on theories in which G is trivial, we see from (3.12) that if only integer spin charges are present we should have $\nu_a(\theta^+) + \nu_a(\theta^-) = \nu_a(\theta)$. However, in the presence of fractionary spin charges the cosine term does not vanish, and we have

$$\nu_a(\theta^+) + \nu_a(\theta^-) = \sum_s q_a^s \cosh(s\theta)\cos(\frac{k\pi}{p}) \tag{3.14}$$

where s is given by (3.13). Therefore in the final expression this term will not vanish, so we will have a Y-system containing an extra term, namely

$$Y_a^i(\theta^+)Y_a^i(\theta^-) = e^{\lambda}\prod_b (1 + Y_b^i(\theta))^{G_{ab}} \tag{3.15}$$

where $\lambda = \sum_{s\notin\mathbb{N}} q_a^s \cosh(s\theta)\cos(\frac{k\pi}{p})$ contains all the information on the fractionary spin charges. Although for more general quantum field theories the exact expression for higher spin charges are not known, it is natural to assume that the same expression could be extended with no particular modifications to a non-diagonal theory with several massive nodes (of the form $\mathcal{G} \ltimes \mathcal{H}$)

It is interesting to observe that the effect of the $T\bar{T}$ deformation and its generalizations is at the TBA level equivalent to moving from the standard Gibbs Ensemble to a GGE, where the driving term is modified by some hyperbolic cosines and hyperbolic sines (these last ones only if the system is out of equilibrium). Hence the two results we have discussed, that the Y-system in the presence of a $T\bar{T}$ deformation and in a GGE are the same as in the standard theory, are essentially the same result: in fact, considering the $T\bar{T}$ TBA equations with a generic term ν which might in principle contain hyperbolic cosines and sines of any spin,

$$\varepsilon_\alpha(\theta) = \nu(\theta) - \varphi_\alpha \star L_\alpha(\theta) \tag{3.16}$$

This can be expressed in the most general situation as

$$\varepsilon_i = \nu_i - \alpha m_i E_0^\alpha \cosh\theta + \alpha m_i P_0^\alpha \sinh\theta + \sum_j \varphi_0^{ij} \star L_{\alpha,j}(\theta) \tag{3.17}$$

where we have introduced the sinh term which is nonzero if the system is not at equilibrium, and the term

$$P_0^\alpha = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \sinh(\theta) L(\theta) d\theta \tag{3.18}$$

which is analogous to E_0^α. Clearly this is equivalent to a modification of the driving term, by modifying the coefficients $\beta_{\pm 1}$ of $\cosh\theta$ and $\sinh\theta$ respectively. Therefore a simple application of the arguments of [63] leads to the conclusion that the Y-system are preserved by the introduction of the perturbation. The same discussion is valid for the deformations of generalized type, in which the TBA is modified by a term $\cosh(s\theta)$.

3.3 Generalized Hydrodynamics

Generalized hydrodynamics is to the GGE what ordinary Hydrodynamics is to the standard Gibbs Ensemble. As is well known, Hydrodynamics is introduced to describe many-particle systems (both classical and quantum) which are found in non-equilibrium or inhomogeneous situations. The regime of interest of Hydrodynamics is a mesoscopic one, in which the variations happen on a scale which is much larger than the atomic scale, but still small enough compared to a macroscopic scale to consider infinitesimal elements which still contain a (thermodynamically) significant number of constituents. As in the classical case, also in the quantum case the dynamics at large enough scales is expected to admit a hydrodynamical description. The theory can therefore be obtained by `local` maximization of entropy; the main effect of integrability will be the same as what led to the GGE, namely the fact that the entropy will be constrained by the infinite number of conservation laws.

The procedure is the following: one can divide the fluid of interest in mesoscopic[2] fluid cells, in which the entropy is locally maximized in the way described above. Local entropy maximization means that averages of local quantities $\langle O(x,t)\rangle$ tend to averages evaluated in a `generalized` Gibbs ensembles $\langle O\rangle_{\beta(x,t)}$ where the dependence on spacetime appears only in the generalized inverse temperatures of the GGE. Hence, in the context of the quasiparticle TBA description, we just need to introduce a spacetime dependence in the density ρ, the n-functions, eccetera, which in equilibrium situations are functions of rapidity alone[3]: this spacetime dependence indicates the fact that the various quantities are now related to the specific fluid cells. We now consider for simplicity a theory of a single particle, but the extension to more general theories is straightforward [12]. In this context the conserved charges in equation (3.6) can be expressed as integrals of local densities q_i which satisfy the conservation law:

$$\partial_t q_i(x,t) + \partial_x j_i(x,t) = 0 \tag{3.19}$$

This is the Euler equation, which simply expresses the conservation laws of the system. The functions appearing in the Euler equations are local functions. In the local entropy maximization context, however, these can be taken as the averages over the fluid cells, with the derivative meaning the variation with respect to the neighbouring cells. These quantities are $q_i = \langle q_i\rangle_{\beta(x,t)}$, and $j_i = \langle j_i\rangle_{\beta(x,t)}$. Since we will mostly consider these ones from now on, by a slight abuse of notation we will use the same symbol for the densities and their average in the GGE. Considering that in general j and q are related by the equation of state of the system, which can be simply expressed as $j = j(q)$, the conservation equation becomes

$$\partial_t q_i(x,t) + \nabla_q j(x,t)\partial_x q_i(x,t) = 0 \Rightarrow \partial_t q_i(x,t) + J(q)\partial_x q_i(x,t) = 0 \tag{3.20}$$

where clearly $J(q) := \nabla_q j(x,t)$. Considering a driving term of the form $\nu(\theta) = \sum \beta^s h_s(\theta)$, where the h_s are one particle eigenvalues of the conserved charges of the system, as defined in chapter 2, the associated densities are naturally defined in terms of the density of quasiparticles $\rho_p(\theta)$:

$$q_s = \int d\theta \rho_p(\theta) h_s(\theta) \tag{3.21}$$

Differentiating the TBA equation with respect to β_s it is possible to obtain:

$$q_s = \int \frac{d\theta}{2\pi} E(\theta) n(\theta) h_s^{dr}(\theta) \tag{3.22}$$

[2]Namely cells which are large enough to contain a statistically relevant number of particles, but still small enough compared to the macroscopic world, such that we can consider the variations of the thermodynamical quantities as smooth functions in space and time.

[3]The essence of the TBA technology is precisely that we can build maximal entropy states for IQFTs, through the quasiparticle description. In fact, the TBA equations are simply a consequence of the constrained maximization of entropy. Therefore the step to a hydrodynamic description (GHD) appears straightforward.

where we have introduced the dressing operation, which is defined implicitly as:

$$h_s^{dr} \doteq \frac{\partial \varepsilon}{\partial \beta_s} = h_s + \varphi \star (n h_s^{dr}) \tag{3.23}$$

The dressing operation may be interpreted as the modification that the one-particle eigenvalues (such as energy and momentum) undergo because of the (elastic) interactions between the quasiparticles. Similarly, crossing arguments imply that the currents are given by

$$j_s = \int \frac{d\theta}{2\pi} p(\theta) n(\theta) h_s^{dr}(\theta) \tag{3.24}$$

It is important to observe that these expressions are valid both in a homogeneous and equilibrium situation, and in inhomogeneous and out of equilibrium situations. The only difference will be in the fact that in one case entropy will be globally maximized, and all the quantities appearing in the integrals will be dependent on the rapidity alone. In the second case, the quantities will have to be considered as (slowly) varying in space and time. The initial focus of GHD [12] was precisely that of studying the transport due to these conserved currents, which can only be found exactly in some very special situations: most of the next chapters will be dedicated to their evaluation in $T\bar{T}$ perturbed theories. It is possible to introduce the effective velocity of the particles as

$$v^{eff}(\theta) = \frac{(E')^{dr}(\theta)}{(p')^{dr}(\theta)} = \frac{(p)^{dr}(\theta)}{(E)^{dr}(\theta)} \tag{3.25}$$

This represents a dressing of the free velocity, which would be simply $\tanh\theta$ as in usual relativistic contexts, due to the effects of the interactions. The interpretation is simple because of the elasticity of the scatterings, which allows to use the momentum of single particles as a conserved quantity[4]. Therefore the effective velocity is a sort of group velocity which takes into account the scatterings which a quasiparticle undergoes between the asymptotic initial and final states. The effective velocity allows for a further integral representation of the currents:

$$j_s = \int d\theta v^{eff}(\theta) \rho_p(\theta) h_s(\theta) \tag{3.26}$$

An important observation at this stage is the following: although we have used the q_i to describe and fix the state, we can equivalently use the n-function or the density. The simple reason for this is that these quantities fix each other via relations arising from the TBA technology and the hydrodynamical equations. Therefore we can view them as different bases we can choose to proceed with the hydrodynamical

[4]Although in general the momentum of a particle might be changed in an elastic scattering, we can identify quasiparticles by keeping trace of the momentum, instead of the original particles themselves. Therefore the quasiparticles are objects which always carry the same momentum between one scattering process and the next.

description. If one takes the Euler equations, and substitutes the integral expressions of the currents and densities, it is possible to move the attention on a different basis: the conservation laws become

$$\partial_t \rho_p(\theta, x, t) + \partial_x (v^{eff}(\theta, x, t)\rho_p(\theta, x, t)) = 0 \tag{3.27}$$

where the (x,t) dependence of the density represents the fact that this function is the density of the fluid cell placed at position (x,t). It is moreover possible to show that the n-functions give the normal modes of GHD, in the sense that they diagonalize the above expression:

$$\partial_t n_i(\theta, x, t) + v^{eff} \partial_x n_i(x, t) = 0 \tag{3.28}$$

This is the fundamental expression of GHD. It allows, for example, to deal with the partitioning protocol problem, which will be the main interest of the next chapter.

3.3.1 The partitioning protocol

One of the most popular problems in hydrodynamics, dating back to Riemann, is the Partitioning Protocol. Its theoretical importance is related to its relative simplicity, compared to other more general out of equilibrium protocols, which allows for an analytical description and in some situations (most notably in pure CFTs, as in [91]) even an exact solution. It consists in solving the Euler equation (3.28) with piecewise constant initial conditions, separated in the origin of the coordinates. For example, we can consider the infinite (x,t) plane as divided in two halves, which are separately thermalized at two different temperatures T_L and T_R, and then put in contact, as shown in figure 3.1. If we consider a generic GGE, we could also consider the two

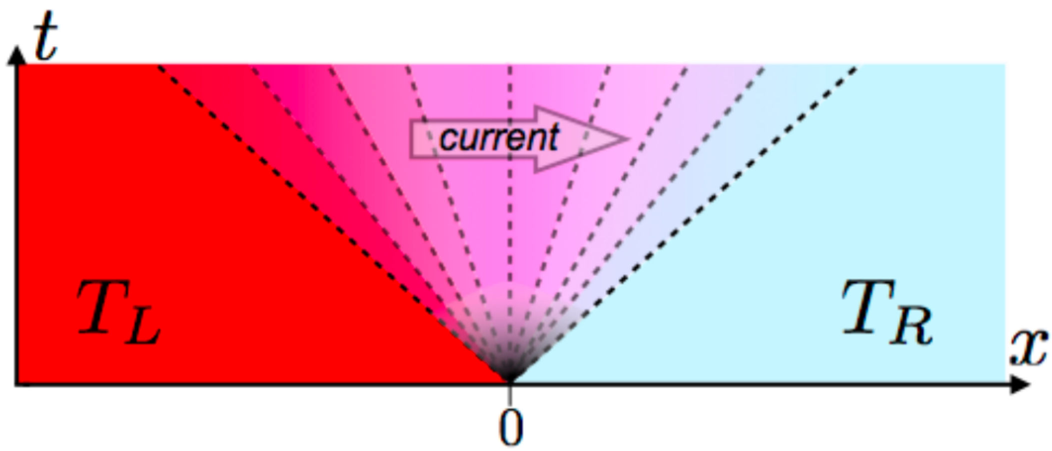


Figure 3.1: Visualization of the Riemann problem for two thermalized halves. Image taken from [12].

halves to be characterized by all the different inverse temperatures and not only by temperature, such that we will have two sets $\{\beta_L^s\}$ and $\{\beta_R^s\}$ for any value of s which is chosen by the initial condition. The time evolution of the system will give rise to matter and energy currents between the two halves, and the theoretical effort is precisely in the direction of evaluating the steady state currents (NESS: non equilibrium steady states) through the origin x=0. We can express the initial condition in n as

$$n(\theta, x, 0) = \begin{cases} n_L(\theta) = n(\theta)\Big|_{\{\beta_L^s\}} & \text{for } x < 0 \\ n_R(\theta) = n(\theta)\Big|_{\{\beta_R^s\}} & \text{for } x > 0 \end{cases} \tag{3.29}$$

We can make use of the manifest symmetry under dilatation of the system, and consider the variable $\xi = x/t$, to which we refer as `ray`, to rewrite the conservation equations as:

$$\left(v^{eff} - \xi\right) \frac{\partial n}{\partial \xi} = 0 \tag{3.30}$$

which has to be solved with the above initial conditions. The solution to this differential equations can be either shocks, rarefaction waves, or contact discontinuities. However, the linear degeneracy of GHD, namely the fact that the effective velocity does not depend on the corresponding mode n, implies that the solution has to be made of contact discontinuities. These are characterized by $v^{eff} = \xi$, and $\frac{\partial n}{\partial \xi} \to \infty$ which gives rise to the discontinuity. Hence for each value of the rapidity we can solve the equation by finding $\xi^\star(\theta)$ such that the implicit equation is solved (since as stated above we must have $v^{eff} = \xi$):

$$\xi^\star(\theta) = v^{eff}(\theta, \xi^\star(\theta)) \tag{3.31}$$

At this point there will be a jump. Requiring that the effective velocity is monotonic we require that there is a single value of $\xi^\star$ for each value of theta which satisfies the above equation, and therefore the solution will be

$$n(\theta, \xi) = n_L(\theta)\Theta(\xi^*(\theta) - \xi) + n_R(\theta)\Theta(\xi - \xi^*(\theta)) \tag{3.32}$$

If the effective velocity is monotonic (as will be in all the situation considered in the following chapter) this solution is also expressed in the form

$$n(\theta, \xi) = n_L(\theta)\Theta(\theta - \theta^*(\xi)) + n_R(\theta)\Theta(\theta^*(\xi) - \theta) \tag{3.33}$$

The NESS refers to the ray $\xi = 0$, and so we can remove it from the above expression and consider a generic $\theta^\star = \theta^\star(0)$

$$n(\theta) = n_L(\theta)\Theta(\theta - \theta^*) + n_R(\theta)\Theta(\theta^* - \theta) \tag{3.34}$$

This is the solution to the partitioning protocol in GHD. Finding this value of the n-function can then be used to find the currents and densities presented above, usually numerically using a recoursive algorithm which will be presented in chapter 5, but in some special case analytical solutions can be extracted.

GHD has been extended and generalized in several directions: it can take in account the presence of external forces which break integrability, if these vary slowly in spacetime as is usually necessary in hydrodynamics, it has been studied by going further than the Euler scale including diffusive higher order contributions. In fact, as explained in [13] it is reasonable to expect that, in contrast to standard hydrodynamics, in the context of integrable systems the derivative expansion is meaningful at all orders, and describes the system more and more precisely. In practice, what has to be done is to consider higher order in the expansion of the generalized chemical potentials: what we have assumed above is that around some point x_0 they can be expanded $\beta(x) = \sum \partial_x^n \beta|_{x_0}(x - x_0)^n$. To obtain the Euler scale, which is what we have used, one takes the expansion to the lower order. The first correction is the diffusive correction, which in the classical hydrodynamic case gives the viscosity term of the Navier-Stokes equations. GHD has also been "quantized" [92] to introduce quantum fluctuations, which become relevant in low entropy states, and lead to correlations between the mesoscopic fluid cells at different positions. However, the basic ingredients presented above are all we need for the discussion in the next chapters, which will be devoted mainly to the study of the average currents and densities of the perturbed theories.

Chapter 4

Generalized Hydrodynamics of $T\bar{T}$ deformed theories: exact results

In this chapter we present analytical results for the generalized hydrodynamics of integrable field theories with $T\bar{T}$ deformations. The main theoretical achievement of this work is presented, namely the proof that the values of the GHD average conserved densities and currents of the perturbed theories can be expressed entirely in terms of non-perturbed quantities. This in particular allows to find expressions for the average densities and currents of higher spin of $T\bar{T}$ perturbed CFTs, a result which generalizes prior known results in which only the energy and momentum currents were found, [24]. In the second part of the chapter a thorough perturbative analysis for the free fermion is presented, together with a study of the scaling function of the perturbed theory, which allows to understand qualitatively how the structure of the solution is changed out of the conformal point. Since in Generalized Hydrodynamics the systems which admit analytical solutions are very few, any exact result is clearly interesting per se, as numerical solutions often fail to give a clear physical understanding of the phenomena at play. In this case, in particular, the exact solution shows clearly the effect that the perturbation has on the theory, since it is exactly decoupled. In the next chapter, the analytical results will be tested numerically, providing an extremely precise validation in the conformal limit.

4.1 TBA and dressing of $T\bar{T}$ deformed theories

For simplicity, we start by considering a theory of a single particle. Generally, in situations where only one particle is present the mass is normalized to one, but here we will keep it explicit for reasons which will become clear in the following. As discussed in section 2.5, the TBA equations are:

$$\varepsilon(\theta) = m\beta \cosh\theta - \varphi_\alpha \star L(\theta)\,, \tag{4.1}$$

where the kernel is modified by the perturbation as

$$\varphi_\alpha = \varphi_0 - \alpha m^2 \cosh\theta$$

If we consider an equilibrium situation, namely the standard TBA context in which there is global maximization of entropy, making use of the properties of the hyperbolic cosine this can be rewritten introducing the TBA ground state energy

$$E_0^\alpha = -\frac{m}{2\pi}\int_{-\infty}^{\infty} \cosh\theta L(\theta) d\theta\,, \tag{4.2}$$

and consequently equation (4.1) becomes:

$$\begin{aligned} \varepsilon(\theta) &= m\beta\cosh\theta - m\alpha E_0^\alpha \cosh\theta - \varphi_0 \star L(\theta) & (4.3)\\ &= m\hat{\beta}\cosh\theta - \varphi_0 \star L(\theta) & (4.4) \end{aligned}$$

where we have reabsorbed the perturbation term in a redefinition of temperature as $\hat{\beta} = \beta - \alpha E_0^\alpha$. The TBA equations for $T\bar{T}$-perturbed theories were famously studied in [20]. One feature that is easy to prove in a similar way is the similar modification of the TBA equations if we consider a more general perturbation, namely where the factor $\Phi_\alpha(\theta)$ is replaced by

$$\Phi(\theta) = e^{-i\sum\limits_{s\in\mathcal{S}} \alpha_s \sinh(s\theta)} \tag{4.5}$$

With the introduction of such term, the TBA equations of the perturbed theory look like the TBA equations of the unperturbed theory in a GGE with couplings which are now related to the perturbation parameters α_s, with driving terms of the form $\cosh(s\theta)$ (and also $\sinh(s\theta)$ if the system is not at equilibrium). In this case, (4.3) becomes instead

$$\varepsilon(\theta) = \sum_s m\hat{\beta}_s \cosh(s\theta) - \varphi_0 \star L(\theta) \tag{4.6}$$

where the modified generalized temperatures are modified analogously (but not in exactly the same way, since in their definition the energy does not appear). In [64] it was shown that the Y-system of these new TBA equations and of the TBA equations of the underformed theory is the same, a result which is essentially equivalent to that of [63], and which constitutes an alternative proof to the one presented in chapter 2. Although this work is mostly concerned with the study of the standard perturbation with s=1, a discussion on the extension of the theory to generalized deformations is presented at the end of the chapter. Except for a few complications, the main discussion is analogous, and exact expressions for the currents can be obtained also in that situation. Observe that out of equilibrium the situation is much more complex, and it is not possible to reabsorb entirely the perturbation as a redefinition of temperature. Therefore in this situation it the solution of the perturbed theory is not always obvious, but as will be shown in this chapter it is only possible to obtain it in the conformal limit.

The particular factorization of the $T\bar{T}$ term into a hyperbolic cosine and a hyperbolic sine part becomes crucial when one evaluates the dressing of the physical quantities which is necessary to study the conserved currents and densities of the theory (both in and out of equilibrium). Starting from (4.1), and recalling that the dressing equation can be obtained by the definition $h_s^{dr} = \frac{\partial \varepsilon(\theta)}{\partial \beta_s}$, it is easy to see that the dressing equation becomes:

$$h_s^{\mathrm{dr}}(\theta) = h_s(\theta) + \varphi_\alpha \star (n h_s^{\mathrm{dr}})(\theta)\,, \tag{4.7}$$

so in the deformed theory we simply have to include an additional term:

$$\varphi_\alpha \star (n h_s^{\mathrm{dr}})(\theta) = \varphi_0 \star (n h_s^{\mathrm{dr}})(\theta) - m^2 \alpha \cosh \star (n h_s^{\mathrm{dr}})(\theta)\,, \tag{4.8}$$

Performing the same considerations as above, the hyperbolic cosine can be rewritten in an extremely interesting way, which will turn out to be particularly useful in the following:

$$\begin{aligned} m^2 \cosh \star (n h_s^{\mathrm{dr}})(\theta) &= \frac{m^2}{2\pi}\left(\cosh\theta \int_{-\infty}^{\infty} \cosh\theta' n(\theta') h_s^{\mathrm{dr}}(\theta') d\theta' - \sinh\theta \int_{-\infty}^{\infty} \sinh\theta' n(\theta') h_s^{\mathrm{dr}}(\theta') d\theta'\right) \\ &= E(\theta) q_s^\alpha - p(\theta) j_s^\alpha\,, \end{aligned} \tag{4.9}$$

Here q_s^α and j_s^α are the average density and the average current associated to the conserved quantity whose one-particle eigenvalue is $h_s(\theta)$ in the deformed theory, and E, p are the (bare) energy and momentum of a single quasiparticle, simply given by $E(\theta) = m\cosh(\theta)$ and $p(\theta) = m\sinh(\theta)$. Therefore the final expression we obtain is:

$$h_s^{\mathrm{dr}}(\theta) = h_s(\theta) - \alpha E(\theta) q_s^\alpha + \alpha p(\theta) j_s^\alpha + \varphi_0 \star (n h_s^{\mathrm{dr}})(\theta)\,. \tag{4.10}$$

The extension to more complicated systems, such as many particle systems, is only slightly more involved. We consider the system to be described by the kernels φ_{ab}, and therefore the $T\bar{T}$ deformed TBA equation is

$$\varepsilon_a = \nu_a - \sum_b \varphi_{ab} * L_b(\theta) + \sum_b \alpha m_a m_b (\cosh * L_b)(\theta) \tag{4.11}$$

Proceding analogously to what was done for a single particle we find the dressed quantities and then use the properties of the hyperbolic cosine to simplify the second convolution:

$$\begin{aligned} h_{a,s}^{dr}(\theta) &= h_{a,s}(\theta) + \sum_b \varphi_{ab} * n_b h_{b,s}^{dr}(\theta) - \alpha \sum_b m_a m_b (\cosh * n_b h_b^{dr})(\theta) \\ &= h_{a,s}(\theta) + \sum_b \varphi_{ab} * n_b h_{b,s}^{dr}(\theta) + \alpha m_a \sum_b \left(j_b^s \sinh(\theta) - q_b^s \cosh(\theta)\right) \end{aligned}$$

The second sum can be simply evaluated to give the `total` spin-s densities and current, q_s and j_s respectively [1]:

$$\begin{aligned} h^{dr}_{a,s}(\theta) &= h_{a,s}(\theta) + \sum_b \varphi_{ab} * n_b h^{dr}_{b,s}(\theta) + \alpha m_a (j_s \sinh\theta - q_s \cosh\theta) \\ &= h_{a,s} + \sum_b \varphi_{ab} * n_b h^{dr}_{b,s}(\theta) + \alpha p_a(\theta) j_s - \alpha E_a(\theta) q_s \end{aligned} \tag{4.12}$$

which is essentially the same as the situation above, except that we see that each particle is coupled to the total currents and densities. As shown explicitly in [64], it is always possible to reabsorb the $T\bar{T}$ term into the TBA equations through a redefinition of the inverse temperature (in general, if we are dealing with a generalized $T\bar{T}$ deformation it will be reabsorbed in some generalized inverse temperature β_s appearing in the GGE). Consider equation (4.12) at equilibrium, with thermal driving term $\nu_a = m_a \beta \cosh\theta$: we will have $j_s = 0$. Then simply I can reabsorb the perturbing term by sending $\beta \to \hat{\beta}$. Therefore it is expected that all physical results of the perturbed theory can be obtained by simply making this substitution. If the system is out of equilibrium, however, the situation is more complicated if one starts with thermal driving terms (as in the partitioning protocol), and other subtleties need to be considered, as will be studied in the following.

4.2 The free fermion

In order to investigate the structure of the TBA equations it is convenient to start with the simplest case, namely the free fermion. In this situation, $\varphi_0(\theta) = 0$ and the TBA equations can be solved exactly, even for the $T\bar{T}$-perturbed theory, at least in the conformal limit. We have

$$\varepsilon(\theta) = m\beta \cosh\theta - m\alpha E_0^\alpha \cosh\theta \,, \tag{4.13}$$

with

$$E_0^\alpha = -\frac{m}{2\pi} \int_{-\infty}^{\infty} \cosh\theta \log(1 + e^{-(\beta - \alpha E_0^\alpha) m \cosh\theta}) d\theta \,, \tag{4.14}$$

This is a non-linear equation for E_0, which can be solved exactly by using Bessel functions, essentially generalizing the standard way of approaching the TBA of a free fermion proposed initially in [55]. Postponing a more general discussion to the end of the chapter, we observe that for $m \ll 1$ we can expand the logarithm, and introduce the modified Bessel function of second kind:

$$K_a(z) = \int_0^\infty e^{-z\cosh t} \cosh at dt \tag{4.15}$$

[1] This is only valid if all the particles are massive. In the presence of magnonic nodes there is a subtlety arising from the fact that the perturbation does not affect directly all the nodes of the diagram: this is however only a small complication.

So to obtain:

$$\begin{aligned}
E_0^\alpha &= \frac{m}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_{-\infty}^{\infty}\cosh\theta e^{-n(\beta-\alpha E_0^\alpha)m\cosh\theta}d\theta \\
&= \frac{m}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}K_1(n(\beta-\alpha E_0^\alpha)m) \approx \frac{m}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2(\beta-\alpha E_0^\alpha)m} \\
&= \frac{1}{\pi(\beta-\alpha E_0^\alpha)}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi}{12(\beta-\alpha E_0^\alpha)} \\
&= -\frac{\pi c}{6(\beta-\alpha E_0^\alpha)}\,,
\end{aligned} \tag{4.16}$$

where we have used the expansion of the Bessel function for small argument, $K_1(z) \sim \frac{1}{z}$, and we have introduced the central charge of the free fermion $c = 1/2$ to relate with more general theories. We observe that for $\alpha = 0$ we recover the known result $E_0^0 := -\frac{\pi c}{6\beta}$. From (4.16) we obtain a quadratic equation in E_0^α

$$\alpha(E_0^\alpha)^2 - \beta E_0^\alpha - \frac{\pi c}{6} \tag{4.17}$$

which can be immediately solved to find the expression:

$$E_0^\alpha = \frac{\beta}{2\alpha}\left(1 \pm \sqrt{1+\frac{2\alpha\pi c}{3\beta^2}}\right)\,, \tag{4.18}$$

We see that, for $\alpha < 0$ the energy can become complex and has a square root branch point. This is related to the famous Hagedorn transition [51] that is often discussed in the context of $T\bar{T}$ perturbations. In order to avoid this complication, we limit ourselves to the $\alpha > 0$ part. Moreover, of the two possible signs we will take only the - sign, as it is the one for which the energy remains finite as $\beta \to \infty$, namely as $T \to 0$. Introducing the scaling function in an obvious way we see that we obtain precisely the same result which was obtained in [23, 24], (up to a redefinition of the perturbing constant $\alpha = -\sigma/2$ to match the notation of these works):

$$c_\alpha = -\frac{3\beta^2}{\pi\alpha}\left(1 \pm \sqrt{1+\frac{2\alpha\pi c}{3\beta^2}}\right) \tag{4.19}$$

Interestingly, the scaling function is β dependent also in the conformal limit. This fact has extremely important consequences: for example, this clearly implies that the scaling function outside the critical point cannot be written as c(r), solely as function of the parameter $r = m\beta$, because otherwise it would become constant in the conformal limit. As discussed in the following, this is a consequence of the addition of a new length scale in the system through α. As mentioned before, the effect of the perturbation can be seen as a redefinition of the temperature. We therefore can define a notion of deformed inverse temperature:

$$\hat{\beta} = \beta - \alpha E_0^\alpha = \frac{\beta}{2}\left(1 \mp \sqrt{1+\frac{2\pi c\alpha}{3\beta^2}}\right)\,. \tag{4.20}$$

This equation coincides exactly to what was found in [24]. Observe that the above expression can be obtained for a generic massive theory through a simple observation: the perturbed energy at temperature β is equivalent to the unperturbed one at temperature $\hat{\beta}$.

$$E_0^\alpha(\beta) = -\frac{m}{2\pi}\int_{-\infty}^{\infty} \cosh\theta \log(1 + e^{-\hat{\beta}\cosh\theta + \varphi * L(\theta)})d\theta \ = E_0^0(\hat{\beta}) \tag{4.21}$$

Using then the fact that $E_0^0(\beta) := -\frac{\pi c}{6\beta}$ in the conformal limit, we can obtain again expression (4.16):

$$E_0^\alpha = -\frac{\pi c}{6(\beta - \alpha E_0^\alpha)} \tag{4.22}$$

which then lead to the same expressions as above for E_0^α and $\hat{\beta}$, this time valid for any kernel. This observation is essentially the same as in both [19,20], namely that the solution of the Burgers equation is of the form

$$E_\alpha(\beta) = E_0(\beta - \alpha E_\alpha(\beta)) \tag{4.23}$$

where the difference in sign is just due to the convention on the sign of the parameter α of the perturbation.

The large mass limit

It is interesting to consider what happens to the theory in the opposite limit, namely when $m \to \infty$. In this situation we can try to expand the logarithm in the integral of the energy, since the exponential is infinitesimal:

$$E_0^\alpha \approx -\frac{m}{2\pi}\int_{-\infty}^{\infty} \cosh(\theta) e^{-(\beta - \alpha E_0^\alpha)m\cosh(\theta)} d\theta \tag{4.24}$$

which is again the modified Bessel function K_1:

$$E_0^\alpha \approx -\frac{m}{2\pi} K_1((\beta - \alpha E_0^\alpha)m) \tag{4.25}$$

This shows an interesting duality compared to the CFT limit. However, in this case we cannot expand the Bessel function in the same way as we did before, but we have to use the large argument expansion $K_1(z) \sim \sqrt{\frac{2\pi}{z}}e^{-z}$ and therefore:

$$E_0^\alpha \approx -\sqrt{\frac{m}{2\pi(\beta - \alpha E_0^\alpha)}} e^{-(\beta - \alpha E_0^\alpha)m} \tag{4.26}$$

Which clearly gives zero in the IR limit (for $\beta \neq 0$, which is obviously the case). This is consistent with the fact that the $T\bar{T}$ perturbation is irrelevant, and therefore does not alter the infrared structure of the theory, and the energy continues to behave the same as in the unperturbed theory(in the unperturbed case, since the theory flows to the trivial fixed point at infinity, we have c=0 for $m \to \infty$).

4.2.1 NESS in $T\bar{T}$ perturbed free fermion

Before giving the complete derivation of the charges and current densities in a generic interacting theory, we start by studying them in the context of the free fermion to test the ground. The results obtained in the following sections will then be shown to reconcile perfectly with the simplest ones considered here. As discussed above, in the free fermion the effect of the perturbation can be reabsorbed in a redefinition of temperature, $\beta \to \hat{\beta}$, where in the conformal limit this deformed temperature is given by (4.20). Using the dressing equation (4.10), we can immediately compute the effective velocity of the theory as (recall that in general $j_E = q_P$):

$$v^{eff}(\theta) = \frac{\sinh\theta - \alpha j_E^\alpha \cosh\theta + \alpha j_P^\alpha \sinh\theta}{\cosh\theta - \alpha q_E^\alpha \cosh\theta + \alpha j_E^\alpha \sinh\theta} \tag{4.27}$$

The crucial quantity in the partitioning protocol is the value θ^* in equation (3.34), which is found by setting the effective velocity to zero. This can be easily found:

$$v^{eff}(\theta^*) = 0 \Leftrightarrow \tanh(\theta^*) = \frac{\alpha j_E}{1 + \alpha j_p} \tag{4.28}$$

which means that

$$\theta^*_\alpha = \text{arctanh} \frac{\alpha j_E^\alpha}{1 + \alpha j_p^\alpha} \tag{4.29}$$

Numerically, this can be used to find the currents and densities in a self-consistent fashion. If we are interested however in an analytical result, this is only possible in the conformal limit, $m \to 0$. In the following general discussion we will show that in the conformal limit:

$$\lim_{m\to 0} \theta^*_\alpha = 0 \tag{4.30}$$

Therefore, in the conformal limit the value of the θ^*_α is precisely the same as in the free fermion without the perturbation. This makes the study of the partitioning protocol much easier. Consider the expression for the currents and densities:

$$j_s^\alpha = \int d\theta \sinh(\theta) n(\theta) \left(h_s(\theta) - \alpha \cosh\theta q_s^\alpha + \alpha \sinh\theta j_s^\alpha\right) \tag{4.31}$$

$$q_s^\alpha = \int d\theta \cosh(\theta) n(\theta) \left(h_s(\theta) - \alpha \cosh\theta q_s^\alpha + \alpha \sinh\theta j_s^\alpha\right) \tag{4.32}$$

Since $\theta^*_\alpha = \theta^*_0 = 0$, it is clear that the n-function is exactly the same as the one for the nonperturbed theory except for the above discussed redefinition of temperature. Therefore the integrals which appear are precisely the densities and currents of the partitioning protocol of a free fermion at temperatures $\hat{\beta}_R$ and $\hat{\beta}_L$:

$$\int d\theta \sinh(\theta) n(\theta) h_s(\theta) = j_s^0(\hat{\beta}) \tag{4.33}$$

$$\int d\theta \cosh(\theta) n(\theta) h_s(\theta) = q_s^0(\hat{\beta}) \tag{4.34}$$

And hence in particular $\int d\theta \sinh(\theta) n(\theta) \cosh(\theta) = j_E^0$, $\int d\theta \cosh(\theta) n(\theta) \cosh(\theta) = q_E^0$, and the same for the momentum. We thus obtain the system of equations in the perturbed current and density:

$$\begin{cases} j_s^\alpha = j_s^0 - \alpha q_s^\alpha j_E^0 + \alpha j_s^\alpha j_p^0 \\ q_s^\alpha = q_s^0 - \alpha q_s^\alpha q_E^0 + \alpha j_s^\alpha j_E^0 \end{cases} \tag{4.35}$$

Assuming that the charges and currents of the unperturbed theory are known (and they are exactly, see [93]) this is a system of equations for two unknowns j_s^α and q_s^α. It can be solved immediately to give:

$$\begin{cases} q_s^\alpha(\beta) = \left. \frac{q_s^0 + \alpha q_p^0 j_s^0 - \alpha j_p^0 q_s^0}{1 - \alpha j_p^0 + \alpha q_E^0 - \alpha^2 j_p^0 q_E^0 + \alpha^2 j_E^0 q_E^0} \right|_{\hat{\beta}} \\ j_s^\alpha(\beta) = \left. \frac{j_s^0 + \alpha j_s^0 q_e^0 - \alpha j_E^0 q_s^0}{1 - \alpha j_p^0 + \alpha q_E^0 - \alpha^2 j_p^0 q_E^0 + \alpha^2 j_E^0 q_E^0} \right|_{\hat{\beta}} \end{cases} \tag{4.36}$$

The fact that the quantities on the right hand side are to be evaluated at the modified temperature will be from now on given as understood, since it will always be the case. Therefore we use the compact notation $q_s^\alpha = q_s^\alpha(\beta)$ and $q_s^0 = q_s^0(\hat{\beta})$, $j_s^\alpha = j_s^\alpha(\beta)$ and $j_s^0 = j_s^0(\hat{\beta})$

Postponing the consistency checks to the general discussion to be done in the next section, we observe that, while these results for the partitioning protocol are only valid in the conformal limit, if we consider an equilibrium situation they are `exact` at all temperatures. In fact, it is easy to see that the n-function at equilibrium is always the same up to a redefinition of temperature, and not only in the conformal limit. Therefore we get the exact equilibrium expression for the conserved charges at all temperatures[2]:

$$q_s^\alpha = \frac{q_s^0 - \alpha j_p^0 q_s^0}{1 - \alpha j_p^0 + \alpha q_E^0 - \alpha^2 j_p^0 q_E^0} \tag{4.37}$$

4.2.2 A comment on the conformal limit of $T\bar{T}$ deformed theories

The conformal limit of a massive theory is defined as the limit in which the correlation lenght diverges, or equivalently the mass tends to zero. In the usual TBA context, since the only two length scales of the theory are identified by β and $1/m$, the process of sending the correlation length to infinity is equivalent to sending $\beta \to 0$. In this limit one obtains expressions at the dominant orders in β which coincide with the conformal values (which, if one starts with a conformal limit or approach the conformal limit sending $m \to 0$ are valid for all values of β). The presence of only two lenght scales is also at the root of the possibility of writing the scaling function c in terms of just r=mβ, as clarified in [55]. However, in the

[2]While the currents associated to cosh(sθ) vanish at equilibrium, the odd ones (like the momentum current) do not. In this section we have for simplicity restricted the attention to the computation of currents and densities associated to even charges, but there is no additional complication if one wants to extend the discussion also to the odd charges.

present context we have an additional lenght scale, namely $\sqrt{\alpha}$. (Interestingly, this lenght scale can be related to a peculiarity which was discussed in [94]: the effect of the $T\bar{T}$ deformation can be understood as an aquisition of a width of the fundamental particles, which thus are no longer pointlike, and this in turn is equivalent to a change of the metric.). Therefore in this situation the conformal limit is `not` simply obtained by sending the inverse temperature to zero. In the perturbed situation, if one wanted to do the usual trick, and approach the conformal limit without modifying the mass, one also would have to send $\alpha \to 0$ together with the inverse temperature. The two procedures lead to exactly the same analytical results in the conformal limit, but perhaps the $m \to 0$ procedure is more transparent, especially when one attemps to to find numerical solutions, as will be shown in the following. Note that α is not renormalized as the mass is varied, as would happen for a relevant coupling. This happens because, considering a massive theory as a relevant deformation of a CFT, the mass is determined by the properties of the corresponding relevant field, as (see [26])

$$m \propto \lambda^{\frac{1}{2-2\Delta}} \tag{4.38}$$

where λ is the coupling and Δ the conformal dimension of the corresponding field. Therefore a modification of the mass corresponds to a modification of the parameter λ. However, in the $T\bar{T}$ deformed case the α couples to the physical mass m just defined, and it is a completely free parameter. Although this peculiarity has not been yet investigated in depth, it is possible that this is related to the fact that in such theories the conformal (massless) limit does not coincide with the UV limit, as is instead the case in the more standard theories which are usually studied in the TBA context, which are obtained as relevant perturbations of CFTs. The presence of three different lenght scales, and two different dimensionless parameters,$r = m\beta$ and $r' = m^2\alpha$, implies that the scaling function will depend on any combination of the two, except in the conformal limit, in which only the combination which eliminates the mass (α/β^2) will survive. In particular, to obtain the conformal limit in practice we can follow the original procedure by Zamolodchikov. Consider a free theory with no $T\bar{T}$ deformation and thermal driving term, such that $\varepsilon(\theta) = m\beta\cosh\theta$. As we take the conformal limit, $m \to 0$, this would appear to vanish identically. However, we have to consider that as the mass goes to zero the particles become relativistic, id est their rapidity would diverge. So we can imagine to match the divergence of the hyperbolic cosine with the vanishing of the mass. In particular, considering $\theta = \theta_0 + \tilde{\theta}$, where $\theta_0 \to \infty$, we can impose that $M = me^{\theta_0}$ is constant along the RG flow, and

$$m\beta\cosh\theta \sim \frac{m\beta}{2}e^{\theta} = \frac{m\beta}{2}e^{\theta_0}e^{\tilde{\theta}} = \frac{M}{2}\beta e^{\tilde{\theta}} \tag{4.39}$$

(sometimes the factor 2 is absorbed in the new mass scale M). This becomes the driving term of TBA equation for the right movers of the CFT in the massless limit, and we will have an analogous expression with opposite sign in the exponential for the left movers. In the case of theories perturbed by $T\bar{T}$ a similar discussion can be performed. Considering (4.1), we see that it might seem that the mass term cannot be reabsorbed since it is squared. Here comes at hand the expression in terms of the energy, such as (4.13). The fact that we can write:

$$\varepsilon(\theta) = m\beta\cosh(\theta) - m\alpha E_0^\alpha \cosh\theta \tag{4.40}$$

implies that we can use exactly the same trick as above on the two terms separately, so we will have in the conformal limit:

$$\varepsilon(\theta) = (\beta - \alpha E_0^\alpha)\frac{M}{2}e^{\tilde{\theta}} \tag{4.41}$$

Hence we see immediately that the notion of $\hat{\beta}$ remains consistently defined upon varying the mass of the particles and flowing to the CFT limit [3]. This will allow in the following sections to apply the obtained results to the pure CFT case, which we will assume to be defined as the limit we just discussed. As before, this particularly simple expression is only valid at equilibrium: in a partitioning protocol, for example, the deformation would also introduce an interaction term between the right and left movers of the theory, as in [23].

4.3 The general solution to the $T\bar{T}$ deformation in interacting theories

Having tested the ground with the free fermion, we can move to study generic theories (for now, with a single particle type) specified by a generic scattering kernel φ. Since all the masses would eventually cancel anyway, we can directly not write them from the beginning.[4] In this situation, the TBA equation is of the form (4.1). To be as general as possible, we will not specify until the end which particular context we are working with, but the two situations we have in mind are the equilibrium case and the partitioning protocol. We start making an observation on the dressing equation in the unperturbed theory. Considering the convolution as an integral operator $\mathbf{T}$ acting on h^{dr}, we can write 4.10 as

$$h_0^{dr}(\theta) = h(\theta) + \mathbf{T}n_0(\theta)h_0^{dr}(\theta) \tag{4.42}$$

$$h_0^{dr}(\theta) = (1 - \mathbf{T}n_0(\theta))^{-1}h(\theta) \tag{4.43}$$

where, following [13], we have treated formally the integral operator: the above equation should be understood as a formal power series in $\mathbf{T}$:

$$f(\mathbf{T}) = \sum_{n=0}^{\infty}\frac{f'(0)}{n!}\mathbf{T}^n \tag{4.44}$$

[3] Clearly, the value of E_0^α varies along the flow. However, the exact result for $\hat{\beta}$ in the conformal limit guarantees that there are no pathologies arising, and this value remains finite.

[4] This is only true in the standard case. In the generalized deformations, studied at the end of the chapter it will be necessary to keep them explicit. We will come back to this problem in section 4.6.

where the powers of the integral operator are interpreted as multiple convolutions. Therefore we are identifying the dressing operation in the free theory with the action of the integral operator $(1-\mathbf{T}n_0(\theta))^{-1}$ on the bare charge eigenvalues. The addition of the $T\bar{T}$ deformation leads to the addition of two extra terms in the dressing equations, and the above manipulation leads to:

$$h_\alpha^{dr}(\theta) = (1-\mathbf{T}n_\alpha(\theta))^{-1}(h(\theta) - \alpha q_s^\alpha \cosh\theta + \alpha j_s^\alpha \sinh\theta) \tag{4.45}$$

which we rewrite conveniently as:

$$h_\alpha^{dr}(\theta) = \tilde{h}_\alpha - \alpha q_s^\alpha \tilde{E}_\alpha + \alpha j_s^\alpha \tilde{P}_\alpha \tag{4.46}$$

where we can take αq_s^α and αj_s^α outside of the action of the convolution because of its linearity. The charges with the tilde are defined as:

$$\tilde{h}(\theta) = (1-\mathbf{T}n_\alpha(\theta))^{-1}h(\theta) \tag{4.47}$$

This is almost identical to the dressing found above for the non-perturbed theory, since the integral operator $\mathbf{T}$ is the same in the two situations, except for the difference in the n-function, which in this situation will contain the information of the perturbation through the modification of the temperature, through a modification of θ^*, and possibly through other effects. Therefore, only in those situations in which the two n-functions are in some way comparable, as it was in the case of the free fermion, they will have the same effect. The situations in which this indeed happens will be analyzed further on. For now, we can use the equation above substituting them in the equations for the currents and densities, as done above:

$$j_s^\alpha = \int d\theta \sinh(\theta) n(\theta) \left(\tilde{h}_s^\alpha(\theta) - \alpha\tilde{E}_\alpha(\theta) q_s^\alpha + \alpha\tilde{P}_\alpha(\theta) j_s^\alpha\right) \tag{4.48}$$

$$q_s^\alpha = \int d\theta \cosh(\theta) n(\theta) \left(\tilde{h}_s^\alpha(\theta) - \alpha\tilde{E}_\alpha(\theta) q_s^\alpha + \alpha\tilde{P}_\alpha(\theta) j_s^\alpha\right) \tag{4.49}$$

Introducing the tilded charges and currents $\tilde{q}_s^\alpha, \tilde{j}_s^\alpha$ defined in an obvious way from the corresponding tilded dressed quantities, we obtain again a system of two equations in the two unknowns j_s^α, q_s^α:

$$\begin{cases} j_s^\alpha = \tilde{j}_s^\alpha - \alpha q_s^\alpha \tilde{j}_E^\alpha + \alpha j_s^\alpha \tilde{j}_P^\alpha \\ q_s^\alpha = \tilde{q}_s^\alpha - \alpha q_s^\alpha \tilde{q}_E^\alpha + \alpha j_s^\alpha \tilde{q}_P^\alpha \end{cases} \tag{4.50}$$

This can be solved easily to give the final expressions:

$$\begin{cases} q_s^\alpha = \frac{\tilde{q}_s^\alpha + \alpha\tilde{q}_p^\alpha \tilde{j}_s^\alpha - \alpha\tilde{j}_p^\alpha \tilde{q}_s^\alpha}{1-\alpha\tilde{j}_p^\alpha + \alpha\tilde{q}_E^\alpha - \alpha^2\tilde{j}_p^\alpha\tilde{q}_E^\alpha + \alpha^2\tilde{j}_E^\alpha\tilde{q}_p^\alpha} \\ j_s^\alpha = \frac{\tilde{j}_s^\alpha + \alpha\tilde{q}_E^\alpha \tilde{j}_s^\alpha - \alpha\tilde{j}_E^\alpha \tilde{q}_s^\alpha}{1-\alpha\tilde{j}_p^\alpha + \alpha\tilde{q}_E^\alpha - \alpha^2\tilde{j}_p^\alpha\tilde{q}_E^\alpha + \alpha^2\tilde{j}_E^\alpha\tilde{q}_p^\alpha} \end{cases} \tag{4.51}$$

For the moment it seems that these equations add nothing to our understanding of the problem, since we are not able to find the tilded quantities exactly. However, we now analyze two fundamental situations in which the two expressions above can be written in terms only of (potentially) known quantities, namely the equilibrium situation and the partitioning protocol. In these situations, the relationship between the tilded and unperturbed quantities can be clarified, and this allows to solve the system exactly in terms of potentially known quantities.

4.3.1 System at equilibrium

We stated above that the system greatly simplifies if n_α and n_0 are the same functions. Considering a system at equilibrium, described by a standard Gibbs ensemble, with driving term $\beta \cosh\theta$, it is clear that this is the case, because of the arguments made at the end of section 4.1: the currents and densities calculated using n_α can be exactly calculated from n_0 simply by adding a prescription of modifying the temperature, $\beta \to \hat{\beta}$. It is then clear that the operation (4.47) is exactly equal to the dressing operation in the free theory at the modified temperature. This means that we can make the identifications:

$$\begin{cases} \tilde{q}_s^\alpha \to q_s^0 \\ \tilde{j}_s^\alpha \to j_s^0 \end{cases} \tag{4.52}$$

where the implicit convention that the quantities with apex 0 are evaluated in $\hat{\beta}$ and those with apex α are evaluated in β is being used. Hence the above system becomes (assuming that in the unperturbed theory we are only considering even charges, such that $j_s^0 = 0$ it is immediate to see from (4.50) that necessarily this implies $j_s^\alpha = 0$[5]):

$$q_s^\alpha = \frac{q_s^0 - \alpha j_p^0 q_s^0}{1 - \alpha j_p^0 + \alpha q_E^0 - \alpha^2 j_p^0 q_E^0} \tag{4.53}$$

This can be further simplified by collecting some terms:

$$q_s^\alpha = \frac{q_s^0(1 - \alpha j_p^0)}{(1 + \alpha q_E^0)(1 - \alpha j_p^0)} = \frac{q_s^0}{1 + \alpha q_E^0} \tag{4.54}$$

Hence we have obtained the complete set of conserved charges at equilibrium for a $T\bar{T}$ deformed theory in terms only of the non-perturbed quantities evaluated at the temperature $\hat{\beta}$, as we found previously for the free theory. Note that this equation is valid at any temperature, and it is exact. However, for generic non-conformal theories $\hat{\beta}$ is difficult to compute exactly, the only exception being the free fermion, in which progress towards a perturbative expression can be made thanks to the properties of the Bessel functions, as discussed in section 4.5. Therefore in general this equation will lead to formal expression in terms of a $\hat{\beta}$ which can be evaluated precisely only for small α or small β. However, this expression is still extremely useful since approximate expression for $\hat{\beta}$ can usually be found in various regimes: for example, for $\beta \gg \alpha$ we have $\hat{\beta} \approx \beta$, so we can evaluate (4.53) at the real temperature. Another problem is then that the values of the unperturbed currents and densities are unknown out of the critical point for all theories of interest (even for the free fermion, the exact expression which can be found is so intricated, since expressed as a series of Bessel functions, that it is scarcely useful).

A consistency check for this expression can be done by analyzing the free fermion in the conformal limit. In this situation, the conserved charges can be found in a

[5] Actually, this is only true if αj_p is different from one.

different way as follows: observing that the dressing becomes $h_s^{\rm dr}(\theta) = h_s(\theta) - \alpha\cosh(\theta)q_s^\alpha$, the charges can be easily computed:

$$\begin{aligned}
q_s^\alpha(\beta) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\cosh\theta' n(\theta')h_s^{\rm dr}(\theta')d\theta' && (4.55)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\cosh\theta' n(\theta')h_s(\theta')d\theta' - \alpha q_s^\alpha\frac{1}{2\pi}\int_{-\infty}^{\infty}\cosh^2\theta' n(\theta')d\theta' \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\cosh\theta'\cosh(s\theta')}{1+e^{\hat\beta\cosh\theta'}}d\theta' - \alpha q_s^\alpha\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\cosh^2\theta'}{1+e^{\hat\beta\cosh\theta'}}d\theta'\,. && (4.56)
\end{aligned}$$

Here we have taken $h_s(\theta) = \cosh(s\theta)$ but we could have taken a combination of $\cosh(s\theta)$ and $\sinh(s\theta)$. The integrals can be computed in the limit of $\beta \ll 1$. We have that

$$\begin{aligned}
\frac{1}{1+e^{\hat\beta\cosh\theta'}} &= e^{-\hat\beta\cosh\theta'}\sum_{n=0}^{\infty}(-1)^n e^{-n\hat\beta\cosh\theta'} \\
&= \sum_{n=1}^{\infty}(-1)^{n+1}e^{-n\hat\beta\cosh\theta'}\,, && (4.57)
\end{aligned}$$

and thus we can again make use modified Bessel functions, this time of higher order, to rewrite the integrals:

$$\begin{aligned}
\int_0^{\infty}\cosh\theta\cosh(s\theta)e^{-A\cosh\theta}d\theta &= \int_0^{\infty}\cosh(s+1)\theta\, e^{-A\cosh\theta}d\theta \\
&- \int_0^{\infty}\sinh\theta\sinh(s\theta)e^{-A\cosh\theta}d\theta \\
&= K_{s+1}(A) - \frac{s}{A}K_s(A)\,, \quad \text{for} \quad A \neq 0
\end{aligned}$$

Using now the asymptotic expansion for the generic modified Bessel function, i.e. $K_s(x) \sim \frac{s!2^{s-1}}{x^s}$ for $x \sim 0$, we can rewrite it to obtain a sum which can be solved exactly in terms of the Riemann zeta function:

$$\begin{aligned}
q_s^\alpha &= \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(K_{s+1}(n\hat\beta) - \frac{s}{n\hat\beta}K_s(n\hat\beta)\right) - \frac{\alpha q_s^\alpha}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(K_2(n\hat\beta) - \frac{1}{n\hat\beta}K_1(n\hat\beta)\right) \\
&\approx \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{s!2^s}{(n\hat\beta)^{s+1}} - \frac{s!2^{s-1}}{(n\hat\beta)^{s+1}}\right) - \frac{\alpha q_s^\alpha}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{2}{(n\hat\beta)^2} - \frac{1}{(n\hat\beta)^2}\right) \\
&= \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{s!2^{s-1}}{(n\hat\beta)^{s+1}} - \frac{\alpha q_s^\alpha}{(n\hat\beta)^2}\right)\,. && (4.58)
\end{aligned}$$

Using the known sum:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{s+1}} = \zeta(s+1)(1-2^{-s}) \qquad (4.59)$$

we obtain the final expression:

$$q_s^\alpha = \frac{s!2^{s-1}\zeta(s+1)(1-2^{-s})}{\pi\hat{\beta}^{s+1}} - \frac{\alpha q_s^\alpha \pi}{12\hat{\beta}^2}\,, \tag{4.60}$$

which can finally be rewritten as

$$q_s^\alpha = \frac{s!2^{s-1}\zeta(s+1)(1-2^{-s})}{\pi\hat{\beta}^{s+1}\left(1+\frac{\alpha\pi}{12\hat{\beta}^2}\right)}\,. \tag{4.61}$$

This expression can be used to test the validity of the fundamental expression (4.53), using the result found in [93], in which the non-perturbed conserved charges are found to be given by (here they are directly evaluated at the modified temperature, since it is what we need):

$$q_s^0 = \frac{s!}{2\pi\hat{\beta}^{s+1}}(2^s-1)\zeta(s+1) = \frac{s!2^{s-1}}{\pi\hat{\beta}^{s+1}}(1-2^{-s})\zeta(s+1) \tag{4.62}$$

and in particular $q_E = \frac{\pi}{12\beta^2} = j_P$ [6]. Therefore, substituting these free results into (4.53), with the modification of the temperature prescription, we get:

$$\begin{aligned} q_s^\alpha(\beta) &= q_s^0 \frac{1-\alpha q_E^0}{1-\alpha^2 q_E^2}\Big|_{\hat{\beta}} = \frac{q_s^0}{1+\alpha q_E^0}\Big|_{\hat{\beta}} \\ &= \frac{s!}{2\pi\hat{\beta}^{s+1}}(2^s-1)\zeta(s+1) = \frac{s!2^{s-1}(1-2^{-s})\zeta(s+1)}{\pi\hat{\beta}^{s+1}\left(1+\frac{\alpha\pi}{12\hat{\beta}^2}\right)} \end{aligned} \tag{4.63}$$

This is precisely the same as (4.61), and hence expression (4.53) leads to consistent results which are in agreement with the previous results known for unperturbed theories. Although these results are only valid in the conformal limit, expression (4.67) is valid for all values of m. Therefore, we expect that going to higher orders in the expansion of the Bessel functions would still lead to consistent results.

4.3.2 Partitioning Protocol

We can now investigate how to deal with the system (4.51) in the most typical of out of equilibriums situations, the partitioning protocol. In this case, the situation is more complicated since it is not enough to consider a modification of the temperature. In fact, in the context of the partitioning protocol the n-function of the system is not only determined by β, but also by the angle θ^*, and can be expressed, assuming the effective velocity to be monotonic, as (since we are interested in the NESS we can set $\xi = 0$):

$$\begin{cases} n_\alpha(\theta) = n_\alpha^L(\theta)\Theta(\theta-\theta_\alpha^*) + n_\alpha^r(\theta)\Theta(\theta_\alpha^*-\theta) \\ v^{eff}(\theta_\alpha^*) = 0 \end{cases} \tag{4.64}$$

[6]The equality $j_p = q_E$ not strictly always valid. However, it is valid up to terms which are always negligible in the conformal limit, since they diverge slower than $1/\beta^{s+1}$.

Since it is reasonable to assume that the effective velocity is monotonic, a claim which will be proven true below, we are not losing any generality with these constraint. The main difference with the equilibrium case is that now it is not only enough to modify the inverse temperature, since we also have $\theta^*_\alpha \neq \theta^*_0$. In the most general case the solution to the partitioning protocol has to be evaluated numerically as it is usually done in GHD, using a recursive approach. However, proceeding in line to what was done for the free fermion, we can show that in the conformal limit one can take $\theta^*_\alpha \approx \theta^*_0$ and obtain consistent results. In fact, the identity is not exact, but as far as the integrals are concerned it makes no difference. To see this, we observe that since $v^{eff}_\alpha = p^{dr}_\alpha / E^{dr}_\alpha$, we can find θ^* by finding the zeros of the dressed momentum:

$$p^{dr}_\alpha(\theta^*_\alpha) = \tilde{p}_\alpha(\theta^*_\alpha) - \alpha q^\alpha_p \tilde{E}_\alpha(\theta^*_\alpha) + \alpha j^\alpha_p \tilde{p}_\alpha(\theta^*_\alpha) = 0 \tag{4.65}$$

Using the fact that $q_p = j_E$ we see that the zero of the dressed momentum corresponds to:

$$\frac{\tilde{p}_\alpha}{\tilde{E}_\alpha} := \tilde{v}^{eff} = \frac{\alpha j^\alpha_E}{1 + \alpha j^\alpha_p} \tag{4.66}$$

In general, the solution of this equation will lead to a θ^*_α different from that of the unperturbed theory. However in the conformal limit we can use make use of the fact that the solutions for the n-functions give a plateau around $\theta = \theta^*$. This happens for both the perturbed and unperturbed case, the only difference being that it is necessary to go to lower values of m in order to see the plateau as α is increased. The plateau structure is shown in figure 4.1.

The key observation is that in the presence of the plateau the exact position of the zero of the effective velocity is totally irrelevant. The value of $n(\theta)$ in the plateau is independent on temperature, because it can be found in terms of the solutions to the constant TBA equations [16]. In a partitioning protocol, the value of θ^* is important because it allows to divide the integrals in the right and left part, but this makes no difference in this situation, since the n-function is continuous in the separation point and constant over a large portion of space (going to infinity as $m \to 0$). Therefore in this limit we can set $\theta^*_\alpha = \tilde{\theta}^*$, and the results of the integrals will be exactly identical. But from the definition of the tilded quantities we see that to find $\tilde{\theta}^*$ one simply has to solve a partitioning protocol which is identical to the one of the unperturbed theory, up to the shift of temperature. So finally we see that in the conformal limit, also in the case of a partitioning protocol, the $T\bar{T}$ can be obtained as functions of the unperturbed quantities with the now familiar prescription $\beta \to \hat{\beta}$. We observe that, although this is a (extremely precise) approximation at small but finite m, it is exact in the conformal limit. Therefore the expressions that will be obtained in the following are expected to be `exact` in the conformal limit, and this will be strongly confirmed by the numerical simulations in chapter 5.

Therefore, since the tilded quantities simply correspond to the unperturbed dressed quantities, system (4.51) can be rewritten as:

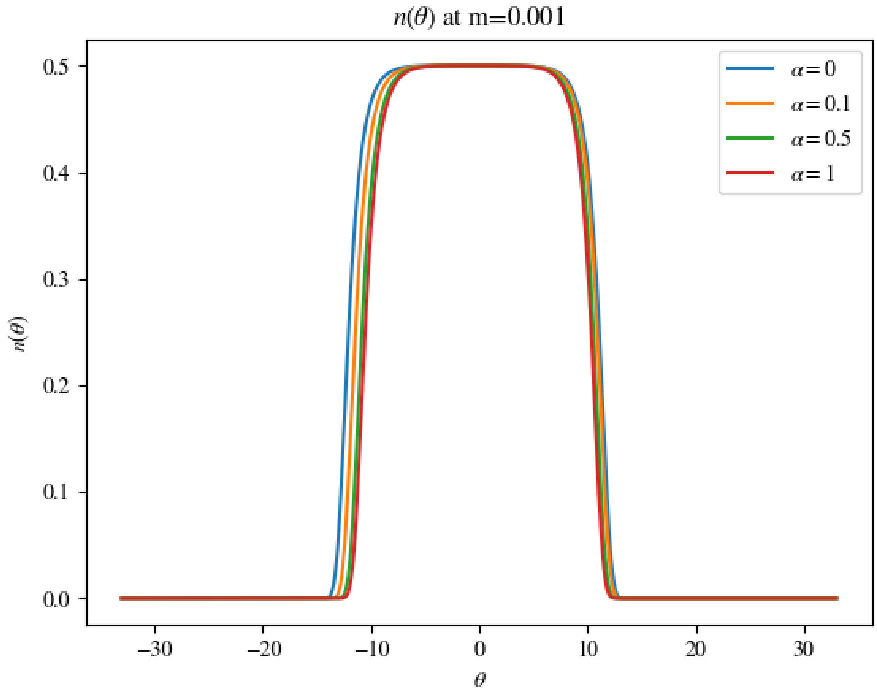


Figure 4.1: Since in the two halves of the partitioning protocol the effect of the perturbation is a redefinition of temperature, then clearly the usual plateau structure will still be preserved. In particular, in the conformal limit the n-function tends to the same value on both sides, although the two temperatures are different. Note however that the effect of the perturbation remains still visible by the presence of an asymmetry.

$$\begin{cases} q_s^\alpha = \frac{q_s^0 + \alpha q_p^0 j_s^0 - \alpha j_p^0 q_s^0}{1 - \alpha j_p^0 + \alpha q_E^0 - \alpha^2 j_p^0 q_E^0 + \alpha^2 j_E^0 q_p^0} \\ j_s^\alpha = \frac{j_s^0 + \alpha q_E^0 j_s^0 - \alpha j_E^0 q_s^0}{1 - \alpha j_p^0 + \alpha q_E^0 - \alpha^2 j_p^0 q_E^0 + \alpha^2 j_E^0 q_p^0} \end{cases} \tag{4.67}$$

These expressions are one of the main theoretical achievements of this work. They allow to find the conserved densities and currents at any spin, once the unperturbed ones are known, and as shown in the next section can be used in particular to generalize the result of [23, 24] to higher spin charges and currents. Moreover, they can be easily generalized to theories in which there are several massive particles, and with some modifications also to magnonic theories, in which there are both massive and massless excitations.

Relationhip with previously known charges

As a test of the fundamental expression just found, we now show that it can be used to recover correctly the results of [24]. Since their study regarded $T\bar{T}$ perturbations of pure CFTs, the unperturbed charges we need to substitute in (4.67) are the famous energy and momentum currents first studied in [91], which were shown to be:

$$j_E^0 = \frac{c\pi}{12}\left(\frac{1}{\beta_L^2} - \frac{1}{\beta_R^2}\right) \tag{4.68}$$

$$q_E^0 = \frac{c\pi}{12}\left(\frac{1}{\beta_L^2} + \frac{1}{\beta_R^2}\right) \tag{4.69}$$

A feature which famously characterizes the pure CFT situation, as clear from the two expressions above, is the separation into a contribution coming from the left reservoir and one coming from the right reservoir. This is essentially related to the separation of the TBA equations into two parts related to the right and left movers of the theory, which in pure CFT are completely free. However, substituting them in equation (4.67), we obtain:

$$\begin{aligned} j_E^\alpha &= \frac{j_E^0}{1-\alpha^2(q_E^0)^2+\alpha^2(j_E^0)^2} \\ &= \frac{j_E^0}{1-\alpha^2(\frac{c\pi}{12})^2\left(\frac{1}{\hat{\beta}_L^2}+\frac{1}{\hat{\beta}_R^2}\right)^2+\alpha^2(\frac{c\pi}{12})^2\left(\frac{1}{\hat{\beta}_L^2}-\frac{1}{\hat{\beta}_R^2}\right)^2} \\ &= \frac{j_E^0}{1-(\frac{\alpha\pi c}{6})^2\frac{1}{\hat{\beta}_L^2\hat{\beta}_R^2}} = \frac{\frac{c\pi}{12}}{1-(\frac{\alpha\pi c}{6})^2\frac{1}{\hat{\beta}_L^2\hat{\beta}_R^2}}\left(\frac{1}{\hat{\beta}_L^2}-\frac{1}{\hat{\beta}_R^2}\right) \end{aligned} \tag{4.70}$$

and similarly, observing that the denominator is exaclty the same,

$$\begin{aligned} q_E^\alpha &= \frac{q_s^0+\alpha\left((j_E^0)^2-(q_E^0)^2\right)}{1-\alpha^2(q_E^0)^2+\alpha^2(j_E^0)^2} \\ &= \frac{\frac{c\pi}{12}}{1-(\frac{\alpha\pi c}{6})^2\frac{1}{\hat{\beta}_L^2\hat{\beta}_R^2}}\left(\frac{1}{\hat{\beta}_L^2}+\frac{1}{\hat{\beta}_R^2}+\frac{\pi c\alpha}{3\hat{\beta}_R^2\hat{\beta}_L^2}\right) \end{aligned} \tag{4.71}$$

These two results are exactly what was obtained through a different approach in [24], and therefore this provides a substantial confirmation of the validity of expression (4.67) in the conformal limit. Following the notation of [24], we now define $c_{LR} := \frac{1}{1-(\frac{\alpha\pi c}{6})^2\frac{1}{\hat{\beta}_L^2\hat{\beta}_R^2}}$, to write the compact results:

$$\begin{aligned} j_E^\alpha &= \frac{c\pi}{12}c_{RL}\left(\frac{1}{\hat{\beta}_L^2}-\frac{1}{\hat{\beta}_R^2}\right) \\ q_E^\alpha &= \frac{c\pi}{12}c_{RL}\left(\frac{1}{\hat{\beta}_L^2}+\frac{1}{\hat{\beta}_R^2}+\frac{\pi c\alpha}{3\hat{\beta}_R^2\hat{\beta}_L^2}\right) \end{aligned}$$

The main feature of this result, compared to the CFT situation, is the absence of factorization into right and left. The perturbation can be interpreted precisely as the addition of an interaction term between the right and left movers, which arises because the S-matrix is no longer trivial.

Generalization of previous results

While in [23, 24] only the energy and momentum currents were found, the method presented in this work allows to generalize the result to charges and currents of any spin. We start by noting that for a generic unperturbed theory these are not known analytically; however expression (4.67) is still useful in that it greatly reduces the computational effort of solving the partitioning protocol numerically. Moreover, there are situations in which such charges are known, for example the free fermion and conformal field theories. The situation for the free fermion was studied above, while for CFTs it is possible to use recently found, and still unpublished, values of the unperturbed conserved charges at any spin [95] (although a particular example of these formulae can be found in [93]):

$$j_s^0 \propto \left(T_L^{s+1} - T_R^{s+1}\right) \tag{4.72}$$

$$q_s^0 \propto \left(T_L^{s+1} + T_R^{s+1}\right) \tag{4.73}$$

The complete proof of this behaviour is provided in appendix A. The proportionality constant is equal for the two quantities, and we shall call it G(s)[7]. Substituting in the fundamental equation (4.67), and observing that again the denominator is still the same and gives the c_{LR} factor, we get:

$$\begin{aligned} j_s^\alpha &= c_{LR}(j_s^0 + \alpha j_s^0 q_E^0 - \alpha q_s^0 j_e^0) \\ &= c_{LR}\left(j_s^0 + \frac{\alpha\pi c}{6} G(s)(\hat{T}_L^{s+1}\hat{T}_R^2 - \hat{T}_R^{s+1}\hat{T}_L^2)\right) \end{aligned}$$

From which we obtain the final expression for the higher spin currents:

$$j_s^\alpha = G(s)c_{LR}\left(\left(\hat{T}_L^{s+1} - \hat{T}_R^{s+1}\right) + \frac{\alpha\pi c}{6}(\hat{T}_L^{s+1}\hat{T}_R^2 - \hat{T}_R^{s+1}\hat{T}_L^2)\right) \tag{4.74}$$

For the charge densitites the calculations are essentially identical, and therefore:

$$\begin{aligned} q_s^\alpha &= c_{LR}(q_s^0 + \alpha j_s^0 j_E^0 - \alpha q_s^0 q_e^0) \\ &= c_{LR}\left(q_s^0 - \frac{\alpha\pi c}{6} G(s)(\hat{T}_L^{s+1}\hat{T}_R^2 + \hat{T}_R^{s+1}\hat{T}_L^2)\right) \end{aligned}$$

Which lead to the final expression:

$$q_s^\alpha = G(s)c_{LR}\left(\left(\hat{T}_L^{s+1} + \hat{T}_R^{s+1}\right) - \frac{\alpha\pi c}{6}(\hat{T}_L^{s+1}\hat{T}_R^2 + \hat{T}_R^{s+1}\hat{T}_L^2)\right) \tag{4.75}$$

These two expressions for the densities and currents in the perturbed theory are the main theoretical finding of this work. The results shows clearly that the presence of the $T\bar{T}$ deformation breaks the left-right factorization also in the higher currents, as

[7] This quantity is an integral which in general cannot be evaluated exactly, except for the free fermion. It consists of integrals which are generalizations of those which are performed to find the UV central charge in the TBA context, which are solved by dilogarithmic functions. Therefore, it appears natural that also these integral might be solved by higher order polylogarithms, although for now we have shown this only for the free theory.

we expected. The two expressions provide a generalization of equations (4.70) and (4.71), to which they reduce exactly when s=1. It is worth making a comment on the relationship between the results of [24] and the results of this work. In [24], the model considered is a pure CFT perturbed by the irrelevant deformation. Therefore, the theory they investigate is defined on the critical surface of the starting CFT. In the present context, we start by considering a massive theory, which itself is a relevant perturbation of a CFT, and consider the irrelevant deformation of this theory. The theories we consider are therefore on some sort of "critical surface" (although it is not critical, since there is no critical point) of the massive theory, namely the submanifold in renormalization group space spanned by the irrelevant deformations of this massive theory. What we do taking the conformal limit is therefore to move back along the direction of the `relevant` perturbation, dragging all the massive irrelevant surface in the flow. The fact that the results coincide means simply that the "critical" massive surface becomes the real critical surface when the mass goes to zero. Since in the TBA context massive theories are usually more transparent than CFTs, the technique shown in this work could be seen as a method of studying irrelevant perturbations of CFT more generally, by studying first the perturbation of the massive theory and then sending $m \to 0$, without having to deal with the more complicated technicalities which usually have to be introduced in the TBA formulation of pure CFT. In the next chapter we will compare these results with numerical simulations of a partitioning protocol and this will confirm that (4.74) and (4.75) are exact results in the conformal limit.

Monotonicity of the effective velocity

To conclude we can make a comment on the monotonicity of the effective velocity. Using expression (4.46), we see that the effective velocity can be expressed as:

$$v^{eff} = \frac{\tilde{p}_\alpha - \alpha q_s^\alpha \tilde{E}_\alpha + \alpha j_s^\alpha \tilde{p}_\alpha}{\tilde{E}_\alpha - \alpha q_s^\alpha \tilde{E}_\alpha + \alpha j_s^\alpha \tilde{p}_\alpha} \tag{4.76}$$

And its derivative, after some algebraic manipulation, and suppressing the α label which is now assumed implicitly, becomes:

$$\frac{dv^{eff}}{d\theta} = \frac{(\tilde{p}'\tilde{E} - \tilde{E}'\tilde{p})(1 - \alpha q_s + \alpha j_s)}{(\tilde{E} - \alpha q_s \tilde{E} + \alpha j_s \tilde{p})^2} \tag{4.77}$$

Where the prime indicates a rapidity derivative. Therefore we see that the sign of the derivative of the effective velocity is entirely given by the behaviour of the term $\tilde{p}'\tilde{E} - \tilde{E}'\tilde{p}$, since the denominator is always positive and the other term in the numerator does not depend on θ (this only means that the sign is fixed. It is not obvious from here that it is always positive, although the simulations show that this should be the case. Anyhow, this does not technically affect the monotonicity property which is needed in the discussion above). This can be conveniently rewritten as:

$$\tilde{p}'\tilde{E} - \tilde{E}'\tilde{p} = \tilde{E}^2 \frac{d\tilde{v}^{eff}}{d\theta} \tag{4.78}$$

Since in the conformal limit the quantities with the tilde become the quantities in the unperturbed theory, we see that if the effective velocity is monotonic in the unperturbed situation it remains so in the perturbed case, and therefore the discussion made above can be applied. On the other hand, if the original theory has a non-monotonic velocity then clearly this will remain the same also in the presence of the perturbation, and the theory will be significantly more complicated. A situation of this sort has been studied for example in [93]. Although the analytical proof shows the monotonicity in the conformal limit, since it is sufficient in the discussion above, numerical simulations show that the monotonicity appears to be preserved for any value of the mass. In particular, the effect of the presence of $T\bar{T}$ is mostly that of introducing an asymmetric deformation of the effective velocity, as shown in figure 4.2

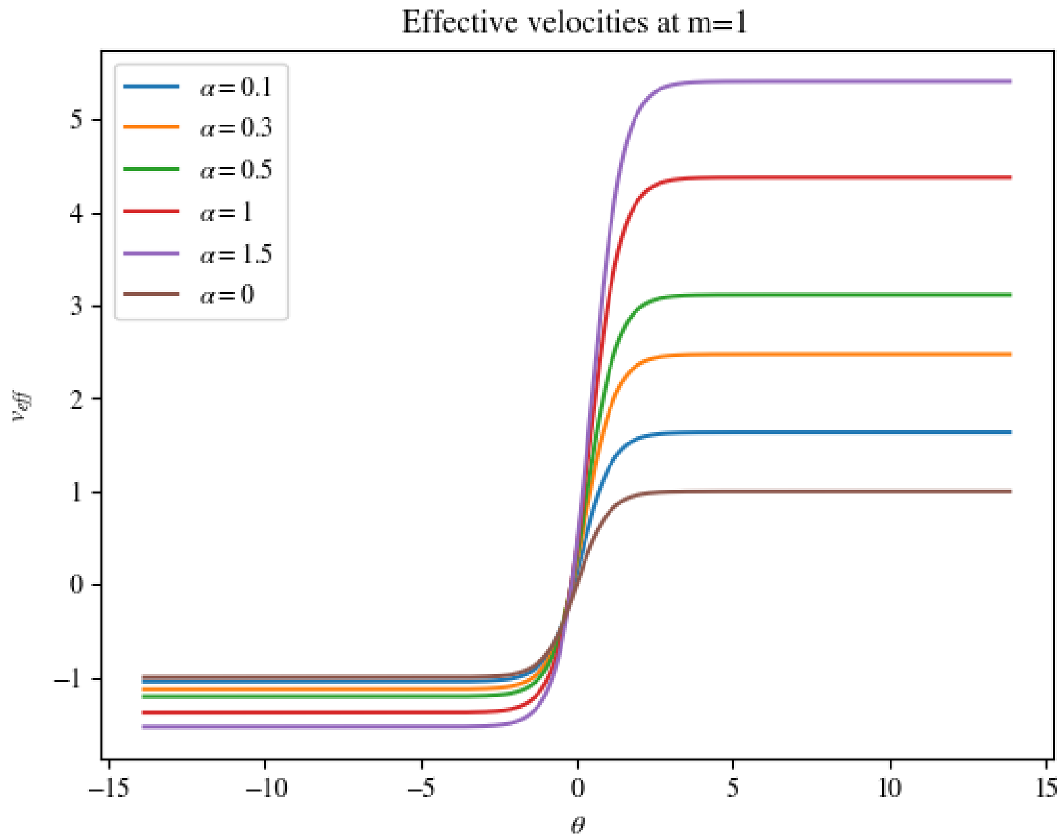


Figure 4.2: Numerical study of the monotonicity of the effective velocity shows that the claimed result about its monotonicity should be valid also out of the conformal limit.

Indeed, the numerical analysis highlights another peculiar feature: it is clear that the zero of the effective velocity deviates only sligthly from the unperturbed value even out of the conformal limit (clearly, this is valid when α is not too large).

Therefore this justifies the attempt to look for perturbative corrections to the above solution as $\theta^*_\alpha \approx \theta^*_0$, as will be investigated in the end of this chapter. We note that the structure of the effective velocity appears a bit odd. In fact, if we consider the effective velocity to be the velocity at which the dressed quasiparticles move, by taking into account all the interactions which might take place, this should be between -1 and 1, since we work with natural units and we are dealing with relativistic systems. Although this might seem a proof of the inconsitency of the discussion, this is not the case, and the reason is simply that the effective velocity in the perturbed theories can be shown not to have the interpretation as a group velocity. Let us now make a comment on this point. In general, the effective velocity is defined by equation (3.25). The interpretation as a "dressed" group velocity arises because it is usually possible to write:

$$v_{eff} = \frac{dE^{dr}}{dp^{dr}} \tag{4.79}$$

which coincides with a dressed version of the textbook group velocity $v_g = \frac{d\omega}{dk}$. This is not always the case, obviously, because in the original definition the derivative is `inside` the dressing and not outside. In fact, in the $T\bar{T}$ deformed theories it is possible to show that equation (4.79) is not actually valid! To see this we use the free fermion for simplicity. The effective velocity in this simple case is given by equation (4.27). Hence to check the validity of (4.79) we need to evaluate the derivative:

$$\frac{dE^{dr}}{dp^{dr}} = \frac{d\theta}{dp^{dr}} \frac{dE^{dr}}{d\theta} \tag{4.80}$$

where we use the generic dressing equations (4.10) to obtain:

$$\frac{dE^{dr}}{dp^{dr}} = \frac{\sinh\theta - \alpha q^\alpha_E \sinh\theta + \alpha j^\alpha_E \cosh\theta}{\cosh\theta - \alpha q^\alpha_P \sinh\theta + \alpha j^\alpha_p \cosh\theta} \tag{4.81}$$

which is different from the effective velocity. Therefore, we see that in the perturbed situation we have a behaviour which is subtly different from what happens normally. This might imply that also the usual notion of dressing has to be taken with additional care, as will be also clear when studying the generalized deformations, in which an even more strange behaviour of the effective velocity will be observed. In any case, this discussion clarifies the reason why the behaviour of the effective velocity is not inconsistent and does not imply some sort of superluminal propagation.

4.4 Systems with several particle types

The discussion can be generalized to the case of multiparticle systems. As proven above, for systems with more than one particle type the dressing equation can be written as

$$h^{dr}_a = h_a + \sum_b \varphi_{ab} * n_b h^{dr}_b + \alpha m_a (j_s \sinh\theta - q_s \cosh\theta) \tag{4.82}$$

The presence of a nested sum over all particle types makes this system extremely complicated to deal with. However, we can apply a similar reasoning to what was done previously, to obtain formally the dressing in the perturbed context as function of the free one. Since the discussion is extremely formal, in the two following sections we will show in two particular situations how it can be applied in practice. Considering (4.82), we can rewrite it in matrix form introducing the matrix of integral operators $\hat{\varphi}$ of components $\mathbf{T}_{ab}n_b$ (we recall that T is just the expression of the convolution between φ and the argument as a single integral operator), and obtain:

$$h^{dr} = h + \hat{\varphi}h^{dr} + \alpha(j_s p(\theta) - q_s E(\theta)) \tag{4.83}$$

where $h^{dr} = (h_1^{dr}, h_2^{dr}, ...)$, and similarly for the other terms. This can be formally inverted as:

$$h^{dr} = (1-\hat{\varphi})^{-1}(h + \alpha(j_s p(\theta) - q_s E(\theta))) \tag{4.84}$$

The inversion of a matrix of integral operators is delicate and has to be dealt with carefully. In this context, as done above for a single particle, we define it as its Taylor expansion, namely

$$(1-\hat{\varphi})^{-1} = \sum_n \hat{\varphi}^n \tag{4.85}$$

which must converge for physical reasons, otherwise the dressing operation would be ill-defined. To make the contact with the discussion made above, we now see that (if the above considerations on the n-functions are still applicable) then we can again see the operation $(1-\hat{\varphi})^{-1}h := \tilde{h}$ where $\tilde{h} \to h_0^{dr}$ in the conformal limit, and therefore also in this case we can express the dressing in the perturbed theory as:

$$h^{dr} = h_0^{dr} - \alpha q_s E_0^{dr} + \alpha j_s p_0^{dr} \tag{4.86}$$

Therefore we see that the possibility of writing the equations in this way is purely given by the way in which the $T\bar{T}$ contribution factorizes in the dressing equation. In this context we are interested in the total currents and densities, so we sum over the various components (multiplying each by the corresponding mass):

$$\sum_a m_a h_a^{dr} = \sum_a m_a h_{a,0}^{dr} + \alpha \sum_a m_a (q_s E_{a,0}^{dr} + j_s p_{a,0}^{dr}) \tag{4.87}$$

and so we can compute the total charges:

$$q_s = \sum_a m_a \int h_{a,0}^{dr} + \alpha(q_s E_{a,0}^{dr} + j_s p_{a,0}^{dr}) n_a(\theta) \cosh\theta \tag{4.88}$$

$$= q_s^0 - \alpha q_s q_E^0 + \alpha j_s q_p^0 \tag{4.89}$$

which is precisely the same as above. Similarly, we can find for the currents the same expressions which se found previously:

$$j_s = j_s^0 - \alpha q_s j_E^0 + \alpha j_s j_p^0 \tag{4.90}$$

Hence the system of equations which allow to find the currents and densitites is exactly the same as in the single-particle case, and the solution will be exactly the same as that given in (4.67). In the next sections, we will show explicitly with two examples that the inversion of the matrix of the integral operators is a sound procedure which lead to the claimed result.

4.4.1 Magnonic case

We start by considering the simplest case, namely a magnonic theory with a single massive excitation, since in this situation the $T\bar{T}$ deformation will only act on a single term (since it is mass-dependent). The Dynkin representation of this theory is given by a single massive node with several magnons forming the adjacency matrix of the Dynkin diagram of A_n. The structure of this theory is particularly simple, since the $T\bar{T}$ perturbation acts only on the single massive node of the theory. To obtain this equation we can start from equation (2.52):

$$\nu_i^a = \varepsilon_i^a + \varphi_g * \{\sum_b G_{ab}(\nu_i^b - \Lambda_i^b) - \sum_j H_{ij} L_j^a\}$$
$$+ \sum_b \alpha m_a m_b(\cosh * L_i^b) - \sum_b \alpha G_{ab} m_a m_b (\varphi_g * \cosh * L_i^b)(\theta)$$

which in the present case gets simplified drastically since we take $G = 1$, and therefore the perturbed TBA equations becomes:

$$\nu_i = \varepsilon_i - \varphi * \sum_j H_{ij} L_j + \delta_{i1} \alpha m_i^2 (\cosh * L_i) \tag{4.91}$$

where now $\varphi = \frac{1}{\cosh\theta}$. Since we have a single mass we can normalize it to one, so the final TBA equation in this situation becomes:

$$\nu_i = \varepsilon_i - \varphi * \sum_j H_{ij} L_j + \delta_{i1} \alpha (\cosh * L_i) \tag{4.92}$$

In order to deal with the system we consider the simplest situation, with one massive particle and one particle (which corresponds to the Tricritical ising model perturbed by ϕ_{13}). In this case the equation above give rise to two dressing equations, where 1 and 2 indicize the particle and the magnon respectively::

$$\begin{cases} h_1^{dr} = h_1 - \mathbf{T} n_2 h_2^{dr} - \alpha E_1(\theta) q_1^s + \alpha p_1(\theta) j_1^s \\ h_1^{dr} = h_2 - \mathbf{T} n_1 h_1^{dr} \end{cases} \tag{4.93}$$

It is easy to see that in the `absence` of the perturbation the dressing operation can be formally inverted to give:

$$\begin{cases} h_1^{dr} = (1 - \mathbf{T} n_2 \mathbf{T} n_1)^{-1} (h_1 - \mathbf{T} n_2 h_2) \\ h_2^{dr} = (1 - \mathbf{T} n_1 \mathbf{T} n_2)^{-1} (h_2 - \mathbf{T} n_1 h_1) \end{cases} \tag{4.94}$$

So, proceeding by analogy with what was done in the case of a single particle, we want to write the perturbed charge densities and currents in terms of the tilded quantities, defined analogously as above (recall that the tilded quantities are obtained by the dressing of the unoerturbed theory where just the n-function is modified). Then, we will show that in relevant physical situations the tilded quantities correspond to the quantities from the nonperturbed theory. Therefore we can take system (4.93) and invert it:

$$\begin{cases} h_1^{dr} = (1 - \mathbf{T}n_2\mathbf{T}n_1)^{-1}(h_1 - \mathbf{T}n_2 h_2 - \alpha E_1 q_1^s + \alpha p_1 j_1^s) \\ h_2^{dr} = (1 - \mathbf{T}n_1\mathbf{T}n_2)^{-1}(h_2 - \mathbf{T}n_1 h_1 - \alpha q_1^s \mathbf{T}n_1 E_1 + \alpha j_1^s \mathbf{T}n_1 p_1) \end{cases} \tag{4.95}$$

To transform this in the tilded quantities we now observe that $p_2 = E_2 = 0$, hence from equation 4.94 we see that $\tilde{E}_1 = (1 - \mathbf{T}n_2\mathbf{T}n_1)^{-1}(E_1)$, $\tilde{E}_2 = (1 - \mathbf{T}n_1\mathbf{T}n_2)^{-1}(-Tn_1 E_1)$, and similarly for the momenta. Hence:

$$\begin{cases} h_1^{dr} = \tilde{h}_1 - \alpha q_1^s \tilde{E}_1 + \alpha j_1^s \tilde{p}_1 \\ h_2^{dr} = \tilde{h}_2 - \alpha q_1^s \tilde{E}_2 + \alpha j_1^s \tilde{p}_2 \end{cases} \tag{4.96}$$

And this leads for the usual system of equations for q_s and j_s: in this case, we will have four equations because of the presence of two particles

$$\begin{cases} q_1^s = \tilde{q}_1^s - \alpha q_1^s \tilde{q}_1^E + \alpha j_1^s \tilde{q}_1^p \\ j_1^s = \tilde{j}_1^s - \alpha q_1^s \tilde{j}_1^E + \alpha j_1^s \tilde{j}_1^p \\ q_2^s = \tilde{q}_2^s - \alpha q_1^s \tilde{q}_2^E + \alpha j_1^s \tilde{q}_2^p \\ j_2^s = \tilde{j}_2^s - \alpha q_1^s \tilde{j}_2^E + \alpha j_1^s \tilde{j}_2^p \end{cases} \tag{4.97}$$

Now, the discussion is identical as before. At equilibrium, the n-function is only changed by the shift in the temperature and therefore the tilded quantities correspond to the dressing in the unperturbed theory with $\beta \to \hat{\beta}$. The same is valid for the partitioning protocol, where now one has to take in consideration two different θ^* for the two particles, but it is immediate to see that the same considerations as above apply. Therefore we get the final system:

$$\begin{cases} q_1^s = q_{10}^s - \alpha q_1^s q_{10}^E + \alpha j_1^s q_{10}^p \\ j_1^s = j_{10}^s - \alpha q_1^s j_{10}^E + \alpha j_1^s j_{10}^p \\ q_2^s = q_{20}^s - \alpha q_1^s q_{20}^E + \alpha j_1^s q_{20}^p \\ j_2^s = j_{20}^s - \alpha q_1^s j_{20}^E + \alpha j_1^s j_{20}^p \end{cases} \tag{4.98}$$

which can be easily solved as before: the first two equations have exactly the same solution as before, while the second ones can be fund by substituting the results of the first two. We are mainly interested in the total densitites and currents, which are obtained by summing the two to obtain:

$$\begin{cases} q^s = q_0^s - \alpha q_1^s q_0^E + \alpha j_1^s q_0^p \\ j^s = j_0^s - \alpha q_1^s j_0^E + \alpha j_1^s j_0^p \end{cases} \tag{4.99}$$

where q^s_1 and j^s_1 can be found solving (4.67). It is immediate to see that it is only terms which depend on the massive particle which make the densities and currents deviate from their unperturbed value, as expected.

In the case of theories with a larger number of magnons the discussion is exactly the same, with the sole difference of having to deal with much more complicated systems of equations. However, one can convince himself that the solution for the total charges is always of the form given by (4.99), which therefore provide the general solution for magnonic problems of this kind. We notice that this is slightly different from the discussion above, and this is related to the fact that in this situation I only have the perturbation on the first node, and not on all the nodes as would happen in a massive theory.

4.4.2 ADE theories

For theories of ADE type the discussion follows directly from what was done in section 4.4. However, in this context we can provide an additional argument in support of the solidity of the discussion made previously, which does not involve the inversion of a matrix of integral operators. We consider the TBA equations, which can be obtained from (2.52) by neglecting the terms related to the magnonic Dynkin diagram H:

$$h_a^{dr} - \sum_b G_{ab}\varphi * (1+n_b)h_b^{dr} = h_a - \sum_b G_{ab}\varphi * [h_b - \alpha(q_s E_b - j_s p_b))] - \alpha q_s E_a + \alpha j^s p_a \tag{4.100}$$

Since we are interested in the total currents and densities, we sum over the particle type a, and observe that we can write $\sum_a G_{ab} = \tilde{c}_b$, where the constant $\tilde{c}_b$ is simply the number of links that start from node b:

$$\sum_a h_a^{dr} - \sum_b \tilde{c}_b\varphi * (1+n_b)h_b^{dr} = \sum_a h_a - \sum_b \tilde{c}_b\varphi * [h_b - \alpha(q_s E_b - j_s p_b))] + \sum_a (-\alpha q_s E_a + \alpha j^s p_a) \tag{4.101}$$

Relabeling the index b to a and rearranging the equations we obtain:

$$\sum_a (1 - \tilde{c}_a \mathbf{T}(1+n_a))h_a^{dr} = \sum_a (1 - \tilde{c}_a \mathbf{T})(h_a - \alpha q^s E_a + \alpha j^s p_a) \tag{4.102}$$

This can be compared to the free case in which by the same argument we would have $\sum_a (1 - \tilde{c}_a \mathbf{T}(1+n_a))h_a^{dr} = \sum_a (1 - \tilde{c}_a \mathbf{T})h_a$, we see that (4.102) is essentially equivalent to the unperturbed theory in which the object which has to be dressed in the equations to find q_s^α and j_s^α is precisely $(h_a - \alpha q^s E_a + \alpha j^s p_a)$, which leads again to the usual result (under the assumptions made above for the general theory). Note that since we are equating two sums, this is not necessarily the unique solutions, but this subtlety is of no interest. As long as this is a solution, since we are only interested in the `total` densities and currents, in any case the final expressions will be the same. Therefore, for this class of theories we do not even need to make use of subtle procedures such as the inversion of integral operators to show the validity of expression (4.67), giving further corroboration to the previously obtained results.

4.5 Going out of the conformal point

All the results obtained thus far are applicable only in the conformal limit. This is also true at equilibrium: even if the expression we found is valid at all temperatures, the expression of $\hat{\beta}$ is only known for $\beta \to 0$. It is however instructive to try to study, at least perturbatively, how the system can be treated outside the critical point. To do so, we focus on the case of a free fermion, for which we can find some analytical expressions thanks to the absence of the scattering kernel. The discussion will highlight the peculiarity of the scaling function in the presence of $T\bar{T}$ deformations.

4.5.1 The scaling function

Focusing on the scaling function, we start by studying it in the absence of the perturbation, as done in [56]. Proceeding analogously as above, and using the properties of the modified Bessel functions, we can obtain the value of c for every value of r:

$$\begin{aligned} c(r) &= \frac{6r}{\pi^2}\int d\theta \cosh\theta \ln(1+e^{-r\cosh\theta}) \\ &= \frac{6r}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}K_1(nr) \\ &= \frac{1}{2}-\frac{3r^2}{2\pi^2}\left[\ln r-\frac{1}{2}-\ln\pi+\gamma_E\right]+ \\ &- \frac{6}{\pi}\sum_{n=1}^{\infty}\left(\sqrt{(2n-1)^2\pi^2+r^2}-(2n-1)\pi-\frac{r^2}{2(2n-1)\pi}\right) \end{aligned} \tag{4.103}$$

where γ_E is Euler-Mascheroni constant. This property of the free theory becomes useful in the present context thanks to the observation made in (4.21), that $E_0^\alpha(\beta) = E_0^0(\hat{\beta})$: out of the conformal limit, we will have $E_0^0(\hat{\beta}) = -\frac{\pi c(\hat{r})}{6\hat{\beta}}$, where c is the unperturbed scaling function which we just found exactly and $\hat{r} := m\hat{\beta}$. Since $\hat{\beta}$ is a function of $E_0^\alpha(\beta)$, the evaluation of the expression above allows to find the modified temperature exactly. It is possible to compute exactly the lowest order corrections to E_0^α and $\hat{\beta}$. To do so, we need to rewrite the sum by expanding the square root in series:

$$\sqrt{(2n-1)^2\pi^2+r^2} = (2n-1)\pi\sum_{k=0}^{\infty}\binom{1/2}{k}\left(\frac{r^2}{(2n-1)^2\pi^2}\right)^k \tag{4.104}$$

$$= (2n-1)\pi+(2n-1)\pi\frac{r^2}{2(2n-1)^2\pi^2}+\sum_{k=2}^{\infty}\binom{1/2}{k}\left(\frac{r^2}{(2n-1)^2\pi^2}\right)^k \tag{4.105}$$

where we see that the first two terms cancel out . Therefore the scaling function becomes:

$$c(r) = \frac{1}{2} - \frac{3r^2}{2\pi^2}\left[\ln r - \frac{1}{2} - \ln\pi + \gamma_E\right] + 6\sum_{k=2}^{\infty}\binom{1/2}{k}\frac{r^{2k}}{\pi^{2k}}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2k-1}} \tag{4.106}$$

The sum in n can be exactly solved using the Riemann zeta function, thanks to the expression

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^p} = (1-2^{-p})\zeta(p) \tag{4.107}$$

and therefore we reach the final expression:

$$c(r) = \frac{1}{2} - \frac{3r^2}{2\pi^2}\left[\ln r - \frac{1}{2} - \ln\pi + \gamma_E\right] + 6\sum_{k=2}^{\infty}\binom{1/2}{k}\frac{r^{2k}}{\pi^{2k}}(1-2^{1-2k})\zeta(2k-1) \tag{4.108}$$

This formulation shows clearly that the sum only contains terms of $O(r^4)$, and can then be truncated at the desired order to find corrections to the expression for E_0^α discussed above. We can approximately simplify the sum even more by observing that

$$(1-2^{1-2k})\zeta(2k-1) = \eta(2k)\frac{\zeta(2k-1)}{\zeta(2k)} \approx \eta(2k) \text{ for } k > 2$$

where η is the Dirichlet function; also, $\eta(x) \approx 1$ for x large, and the approach to this value is very fast even for the lowest term in which the argument is 4. Therefore we can set to 1 all these terms, and obtain a sum which is exactly solved:

$$\begin{aligned} c(r) &\sim \frac{1}{2} - \frac{3r^2}{2\pi^2}\left[\ln r - \frac{1}{2} - \ln\pi + \gamma_E\right] + 6\sum_{k=2}^{\infty}\binom{1/2}{k}\frac{r^{2k}}{\pi^{2k}} \\ &= \frac{1}{2} - \frac{3r^2}{2\pi^2}\left[\ln r - \frac{1}{2} - \ln\pi + \gamma_E\right] + 6\left(-1 - \frac{r^2}{2\pi^2} + \frac{\sqrt{\pi^2+r^2}}{\pi}\right) \end{aligned}$$

In any case, we will only consider the first term in the square bracket, since the second term is subleading, and the presence of the logarithm already makes it difficult to solve. Considering the above mentioned fact that $E_0^\alpha = -\frac{\pi c_0(\hat{r})}{6\hat{\beta}}$, and further using that $E_0^\alpha = \frac{\beta-\hat{\beta}}{\alpha}$ from the definition of $\hat{\beta}$, we obtain an expression relating the inverse temperature to the modified inverse temperature, which holds away from criticality:

$$\frac{\beta}{\alpha} = \frac{\hat{\beta}}{\alpha} - \frac{\pi}{12\hat{\beta}} + \frac{\hat{\beta}m^2}{4\pi}\left[\ln\hat{\beta}m + \chi\right] - \sum_{k=2}^{\infty}\binom{1/2}{k}\frac{\hat{\beta}^{2k-1}m^{2k}}{\pi^{2k-1}}(1-2^{1-2k})\zeta(2k-1) \tag{4.109}$$

where $\chi = -\frac{1}{2} - \ln\pi + \gamma_E$, and this can be solved (at least numerically) to find the value of $\hat{\beta}$ at all orders.[8] The presence of the logarithm makes it anyway

[8]Note that, however, the radius of convergence of the expansion of the square root we have used is only 1, by Cauchy-Hadamard theorem, according to which the radius of convergence of a

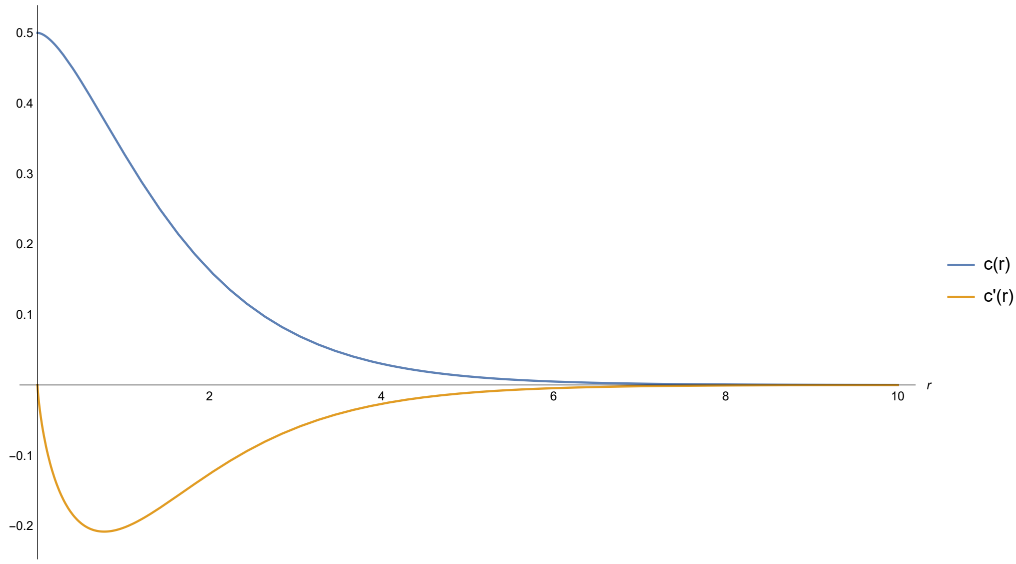


Figure 4.3: Behaviour of the exact scaling function and its derivative as a function of the argument, as given in equation (4.103). We see that the correct conformal limit c=1/2 is obtained as the argument tends to zero.

extremely challenging to invert the expression to find $\hat{\beta}$ as function of β, even at the lowest orders. However, an approximate solution can be found using Lambert functions. We are interested in the lowest order corrections in m, which represents the behaviour of the theory just outside the critical point. For $m \approx 0$, it is justified to neglect entirely the sum, which contains terms $O(m^4)$), and therefore we are left with:

$$\frac{\beta}{\alpha} = \frac{\hat{\beta}}{\alpha} - \frac{\pi}{12\hat{\beta}} + \frac{\hat{\beta}m^2}{4\pi}\left[\ln \hat{\beta}m\right] \tag{4.110}$$

We have also neglected the χ term, since it is a next order correction, and in any case in is easy to reintroduce it at the end of the calculations. We can start by finding the modified temperature at $\beta = 0$. Exponentiating the truncated equation (4.110) at $\beta = 0$ we obtain:

$$\begin{aligned}\hat{\beta} &= \frac{\pi\alpha}{12\hat{\beta}} - \frac{\alpha\hat{\beta}m^2}{4\pi}\left(\ln \hat{\beta}m\right)\\ \Rightarrow \frac{4\pi}{\alpha m^2} &= \frac{\pi^2}{3\hat{\beta}^2 m^2} - \ln \hat{\beta}m\\ \Rightarrow \hat{\beta}m \exp\left(-\frac{\pi^2}{3\hat{\beta}^2 m^2}\right) &= \exp\left(-\frac{4\pi}{\alpha m^2}\right)\end{aligned}$$

power series can be evaluated as $\frac{1}{R} = \limsup_{n\to\infty}\left(|c_n|\right)^{1/n}$, where c_n are the coefficients of the series. In the present case, we have $\limsup_{n\to\infty}\left(|\binom{1/2}{k}|\right)^{1/n} = 1$. Hence the expression above can only be used to study the deviation from the conformal point for small values of β.

This equation can be solved exactly using the Lambert function $W(x)$ [96], which is defined by the equation

$$W(x)e^{W(x)} = x \tag{4.111}$$

To reduce to a form solvable by the Lambert function, we introduce $t = \frac{1}{m^2\hat{\beta}^2}$, then take square of both sides:

$$\begin{aligned}
\sqrt{t}\exp\left(\frac{\pi^2}{3}t\right) &= e^{\frac{4\pi}{\alpha m^2}} \\
t\exp\left(\frac{2\pi^2}{3}t\right) &= e^{\frac{8\pi}{\alpha m^2}} \\
\frac{2\pi^2}{3}t\exp\left(\frac{2\pi^2}{3}t\right) &= \frac{2\pi^2}{3}e^{\frac{8\pi}{\alpha m^2}}
\end{aligned}$$

This can immediately be solved using the defining equation of the Lambert function (4.111), and substituting $t = m^{-2}\hat{\beta}^{-2}$ one obtains:

$$\hat{\beta}(\beta = 0) = \frac{\sqrt{\frac{2\pi^2}{3}}}{m\sqrt{W\left(\frac{2\pi^2}{3}e^{\frac{8\pi}{\alpha m^2}}\right)}} \equiv \hat{\beta}_0 \tag{4.112}$$

Note that using the defining relation $e^{W(x)} = \frac{x}{W(x)}$, this can be rewritten as:

$$\hat{\beta}_0 = \frac{\exp\frac{1}{2}W(\eta)}{m\exp(\frac{4\pi}{\alpha m^2})} = \sqrt{\frac{2\pi^2}{3\eta m^2}}\exp\frac{1}{2}W(\eta) \tag{4.113}$$

where we have introduced the parameter $\eta = \frac{2\pi^2}{3}\exp(\frac{8\pi}{\alpha m^2})$. This expression will be useful in the following. For $m \to 0$ $\hat{\beta}_0$ is finite, since the Lambert function behaves asymptotically as the logarithm of the argument. In particular, the limit can be evaluated:

$$\lim_{m\to 0}\hat{\beta}(\beta = 0) = \sqrt{\frac{\pi\alpha}{6}} \tag{4.114}$$

which corresponds precisely to the result which can be obtained in the $\beta \to 0$ limit of equation (4.20), with c=1/2. If $\beta \neq 0$, the solution in terms of Lambert funtions is not exact, but if we require the temperature to still be small (with respect to $\sqrt{\alpha}$, so that $\hat{\beta}(\beta = 0) \approx \sqrt{\frac{\pi\alpha}{6}} \gg \beta$. The procedure is analogous to what was done before, with the introduction of an extra term:

$$\begin{aligned}
\frac{4\pi\beta}{\alpha\hat{\beta}m^2} &= \frac{4\pi}{\alpha m^2} - \frac{\pi^2}{3\hat{\beta}^2m^2} + \ln\hat{\beta}m \\
\hat{\beta}\exp\left(-\frac{\pi^2}{3\hat{\beta}^2m^2} - \frac{4\pi\beta}{\alpha\hat{\beta}m^2}\right) &= \frac{1}{m}e^{-\frac{4\pi}{\alpha m^2}} \\
m\hat{\beta}\exp\left(-\frac{\pi^2}{3m^2\hat{\beta}^2}\left(1 + \frac{12\beta\hat{\beta}}{\pi\alpha}\right)\right) &= e^{-\frac{4\pi}{\alpha m^2}}
\end{aligned}$$

Unfortunately, despite the great similarity with the previous one, this equation is not exactly solvable in terms of Lambert functions, because of the extra term in the exponential in the left hand side. Although generalizations of the Lambert functions exist, there appear to be no generalization yet which allows to deal with this type of equation[9]. However, assuming that β is small as mentioned above, we perform an approximation which restores the possibility of using Lambert functions: we can approximate the $\hat{\beta}$ appearing in the term $\frac{12\beta\hat{\beta}}{\pi\alpha}$ with its value at $\beta = 0$, since adding extra terms would only lead to higher order corrections in β, which we neglect. Doing so we obtain:

$$m\hat{\beta}\exp\left(-\frac{\pi^2}{3m^2\hat{\beta}^2}\left(1+\frac{12}{\pi\alpha}\beta\hat{\beta}_0\right)\right) = e^{-\frac{4\pi}{\alpha m^2}} \tag{4.115}$$

which can be solved exactly as before, with the addition of an extra β-dependent term:

$$\hat{\beta} = \frac{\sqrt{\frac{2\pi^2}{3}}\sqrt{1+\frac{12}{\pi\alpha}\beta\hat{\beta}_0}}{m\sqrt{W\left(\frac{2\pi^2}{3}\left(1+\frac{12}{\pi\alpha}\beta\hat{\beta}_0\right)e^{\frac{8\pi}{\alpha m^2}}\right)}} \tag{4.116}$$

Notice that we have neglected for simplicity the χ term, which can anyway be reinserted trivially, by adding a term $e^{-2\chi}$ into the Lambert function. In any case, this correction is neglible for small values of m. Again, we can test if expression (4.116) lead to the correct conformal limit, when compared to the conformal definition (4.20). Indeed, we see that

$$\lim_{m\to 0}\hat{\beta} = \sqrt{\frac{\pi\alpha}{12}\left(1+\beta\sqrt{\frac{12}{\pi\alpha}}\right)} \approx \sqrt{\frac{\pi\alpha}{12}} + \frac{\beta}{2} \tag{4.117}$$

which is precisely the correct result, as can be seen by expanding the conformal expression to first order in β. By using the same relation with the exponential of the function, we can obtain an expression similar to (4.113),

$$\hat{\beta} = \sqrt{\frac{2\pi^2}{3\eta m^2}}\exp\left(\frac{1}{2}W(\eta(1+\mathcal{K}\exp(\frac{1}{2}W(\eta))))\right) \tag{4.118}$$

where the new parameter is $\mathcal{K} = \frac{12\beta}{\pi\alpha}\sqrt{\frac{2\pi^2}{3\eta m^2}}$. Therefore we see that this suggests that the complete solution will be given by infinitely many "nested" Lambert functions: this can be seen by iterating the calculation to reach higher values of β, by substituting the i-th obtained value of $\hat{\beta}$(in this notation we have computed above

[9]While the usual Lambert function allows to evaluate $xe^x = a$, generalizations have been studied to deal with equations as $(x-a)(x-b)...(x-c)e^x = z$. However, in this situation we would need a solution to the problem $xe^{(x-a)(x-b)...(x-c)} = z$, which have never been studied. The two lead to different functions which cannot be reduced to one another, since intricated square roots appear in this attempt for any possible reparametrization.

$\hat{\beta}_0$ and $\hat{\beta}_1$) into the expression for the i+1-th component. This will lead to the equation:

$$\hat{\beta}_{i+1} = \frac{\sqrt{\frac{2\pi^2}{3}}\sqrt{1+\frac{12}{\pi\alpha}\beta\hat{\beta}_i}}{m\sqrt{W\left(\frac{3\pi^2}{2}\left(1+\frac{12}{\pi\alpha}\beta\hat{\beta}_i\right)e^{\frac{8\pi}{\alpha m^2}}\right)}} \tag{4.119}$$

which is finally expressed as the nested exponential formula:

$$\hat{\beta} = \sqrt{\frac{2\pi^2}{3\eta m^2}}\exp\left(\frac{1}{2}(W(\eta(1+\mathcal{K}\exp(\frac{1}{2}W(\eta(1+\mathcal{K}\exp(\frac{1}{2}W(...))))\right) \tag{4.120}$$

This is the exact expression of $\hat{\beta}$ for any value of temperature, with mass at second order. We see that this becomes extremely complicated; going higher order in the mass, introducing m^4 terms, would make the solution even more intricated. Therefore we see that an exact solution at any mass appears impossible to obtain, although the nice structure of equation (4.120) gives some hope. Perhaps some special property of the Lambert functions may allow us to go further in the description to simplify the higher orders of the expansion (this would be a necessary step if the attempt to generalize the function in this direction is undertaken); in any case the results which is obtained by the iteration for $i \to \infty$ is exact at any β, and corresponds to the second order in the mass. In light of the considerations made above, we could even venture to define a novel generalized function, with will need to have the properties just found: this extension could even in principle allow to consider evaluations at higher orders in m. We define the modified function as the inverse of the expression:

$$x = ze^{(z-z_1)(z-z_2)...(z-z_n)} \Rightarrow z = \tilde{W}_{z_1,...z_n}(x) \tag{4.121}$$

Although an in-depth study of these functions is well beyond the scope of this thesis, we see that in terms of these functions the solution would simply be:

$$\hat{\beta} = \frac{3/\pi^2}{m\tilde{W}_{0,\frac{12\beta}{\pi\alpha m}}\left(\frac{\pi^2}{3}\exp(\frac{4\pi}{\alpha m})\right)} \tag{4.122}$$

Comparison with the result found above shows the relation between these generalized functions to the standard Lambert function. We leave a more thorough investigation of the properties of such functions for following research: this would be a study of interest because generalizing the function even more would allow to solve also for the higher orders in the mass.

It is then easy to study the small temperature (large β) asymptotics. It is immediate to obtain that the modified temperature satisfies a formal quadratic expression:

$$\hat{\beta}^2 - \beta\hat{\beta} - \frac{\alpha\pi c(\hat{\beta})}{6} = 0 \tag{4.123}$$

Which leads to the expression:

$$\hat{\beta} = \frac{\beta + \sqrt{\beta^2 + \frac{2\alpha\pi c(\hat{\beta})}{3}}}{2} \tag{4.124}$$

This is identical to the one obained above, except for the fact that we have the full complicated scaling function, which is plotted with its derivative in figure 4.3. We see however clearly that for $\beta \to \infty$ the modified temperature tends to the unperturbed temperature, because the scaling function vanishes, and $\hat{\beta} \geq \beta$ for any values of m, β, and α (in the range $\alpha > 0$ considered in this work).

$$\hat{\beta} \approx \beta \text{ for } \beta \to \infty \tag{4.125}$$

The behaviour of the modified temperature is shown in figure 4.4, for m=1 and different values of α. It is obvious from the plot and from the consideration we just made that the $T\bar{T}$ perturbation only affects the low-β region, by increasing the value of $\hat{\beta}$. Similarly, it is mostly effective if the mass is small, while its effects are negligible for $m \to \infty$, and this is related to the perturbation being irrelevant.

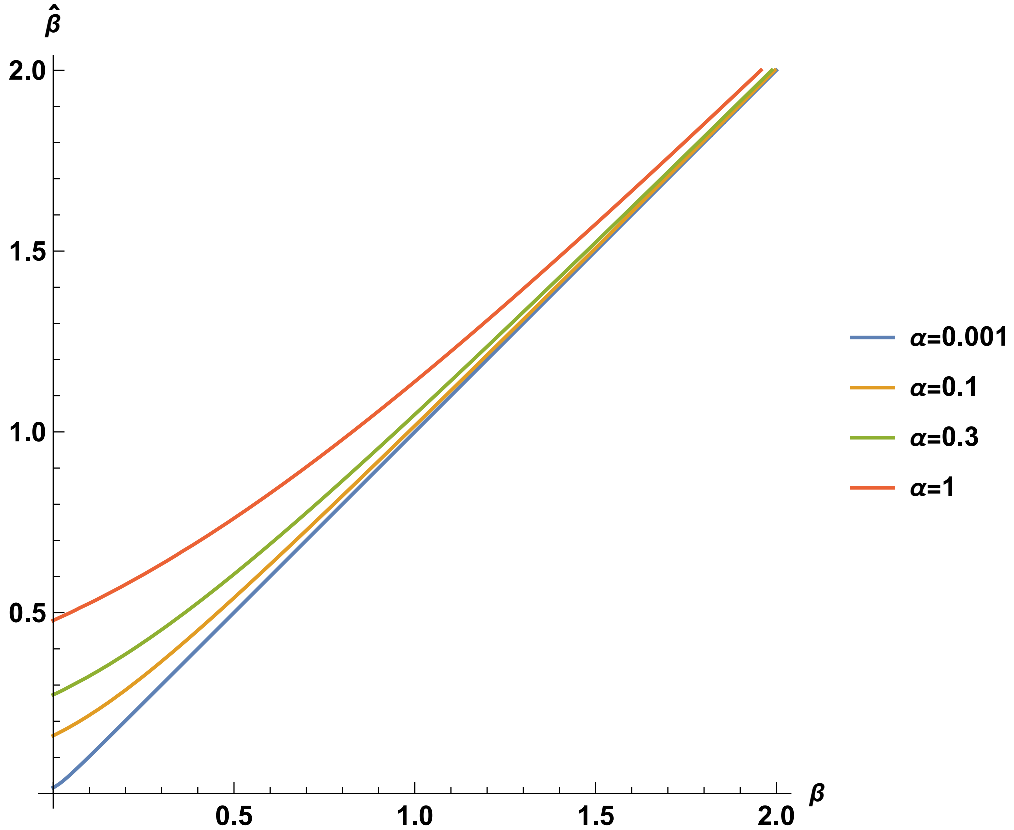


Figure 4.4: Value of $\hat{\beta}$ at different values of temperature. As expected, for large β the two coincide, while for small β the modified temperature tends to a value which is solution to the equation $\hat{\beta}^2 = \frac{\alpha\pi c(\hat{\beta})}{6}$

4.5.2 Monotonicity of the scaling function and c-theorem

This discussion shows clearly that the possibility of obtaining exact results, even at the lowest orders in perturbation theory, is extremely specific to the free fermion, and already extremely complicated. We can however check in general the properties of the scaling function of the perturbed theory c_α, which we define in the most natural way in the TBA context:

$$E_0^\alpha = -\frac{\pi c_0(\hat{r})}{6\hat{\beta}} = -\frac{\pi c_\alpha(r)}{6\beta} \tag{4.126}$$

where the c-function c_0 is the one evaluated in (4.103). From this we see again that we can relate the scaling function of the perturbed theory, c_α, to the one in the free theory evaluated at $\hat{\beta}$:

$$c_\alpha(r) = \frac{\beta c(\hat{r})}{\hat{\beta}} \tag{4.127}$$

which can be expressed solely in terms of the modified temperature as:

$$c_\alpha(r) = c_0(\hat{r}) \left(1 - \frac{\alpha\pi}{6} \frac{c_0(\hat{r})}{\hat{\beta}^2}\right) \tag{4.128}$$

In the unperturbed theories, the scaling function is a monotonically decreasing function, as stated in the famous c-theorem [42][10]. The c-theorem however is constructed only to follow the RG flow of integrable theories constructed as relevant deformations of CFTs, therefore it is not obvious that it should apply to theories with irrelevant deformations, especially because the UV structure of these theories is rather obscure. Hence it is interesting to analyse the structure of the scaling function in this theory, to check wether it is or not decreasing along the flow, and therefore if some sort of formulation of the c-theorem might still apply, as was already investigated in [97–99].

Before doing this we stress a very simple yet fundamental fact: since in this context the theory depends on two dimensionless quantities, which we define as $r = m\beta$ and $r' = m^2\alpha$, then the c-function of the perturbed theory will not in general be a pure function of the quantity r, but will be a function of $r, r', r/r'$, and in principle any combination of these quantities. This is quite obvious when compared to the results obtained in [23] regarding the scaling function, which is not a constant (neither in β nor in α) even in the conformal limit. Specifically, in that work they obtain a functional dependence on r^2/r', which is the same result shown in expression (4.19). Therefore, as was remarked also above, it is not so obvious which is the correct way to approach the conformal limit, and if this limit corresponds to the UV limit of the theory. Hence we will refer to the conformal limit as equivalent to the `massless` limit of the theory, which leads to the results of [24]. The flow is therefore a flow with respect to the variation of the mass parameter,

[10] Although the c-function of Zamolodchikov's theorem and the scaling function of the TBA context are different functions, they are generally believed to carry the same information.

from zero in the conformal limit, and increasing to obtain a generic massive theory. Starting from equation (4.127), we obtain:

$$\frac{\partial c_\alpha}{\partial m} = \beta\left(\frac{\frac{\partial c_0(\hat{r})}{\partial m}\hat{\beta} - c_0(\hat{r})\frac{\partial\hat{\beta}}{\partial m}}{\hat{\beta}^2}\right) \tag{4.129}$$

$$= \frac{\beta}{\hat{\beta}^2}\left(\frac{\partial c_0(\hat{r})}{\partial \hat{r}}\frac{\partial \hat{r}}{\partial m}\hat{\beta} - c_0(\hat{r})\frac{\partial\hat{\beta}}{\partial m}\right) \tag{4.130}$$

$$= \frac{\beta}{\hat{\beta}^2}\left(\frac{\partial c_0(\hat{r})}{\partial \hat{r}}\hat{\beta}^2 - c_0(\hat{\beta})\frac{\partial\hat{\beta}}{\partial m}\right) \tag{4.131}$$

To find the mass derivative of the modified temperature we can use $-\frac{\pi c_0(\hat{r})}{6\hat{\beta}} = \frac{\beta-\hat{\beta}}{\alpha}$, from which we obtain, rearranging and differentiating with respect to m:

$$\beta = \hat{\beta} - \frac{\alpha\pi c_0(\hat{r})}{6\hat{\beta}} \tag{4.132}$$

$$0 = \frac{\partial\hat{\beta}}{\partial m} - \frac{\alpha\pi}{6\hat{\beta}^2}\left(\hat{\beta}^2\frac{\partial c_0(\hat{r})}{\partial \hat{r}} - c_0(\hat{r})\frac{\partial\hat{\beta}}{\partial m}\right) \tag{4.133}$$

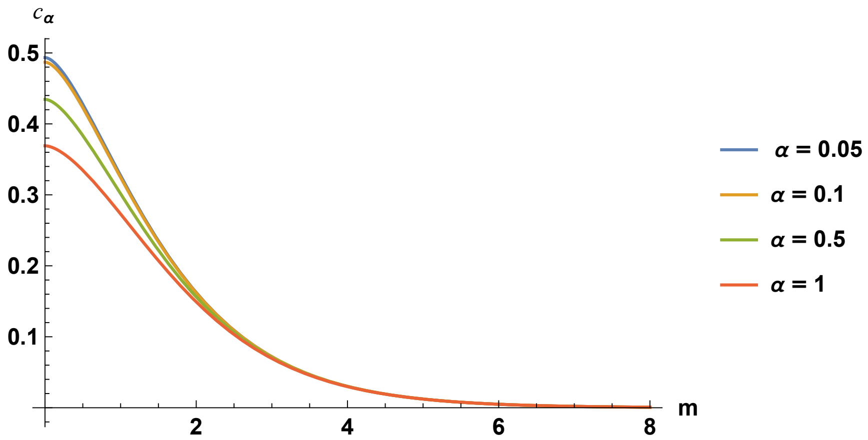
$$\frac{\partial\hat{\beta}}{\partial m} = \frac{\frac{\alpha\pi}{6}\frac{\partial c_0(\hat{r})}{\partial \hat{r}}}{1 + \frac{\alpha\pi c_0(\hat{r})}{6\hat{\beta}^2}} \tag{4.134}$$

which substituted in (4.131) leads to:

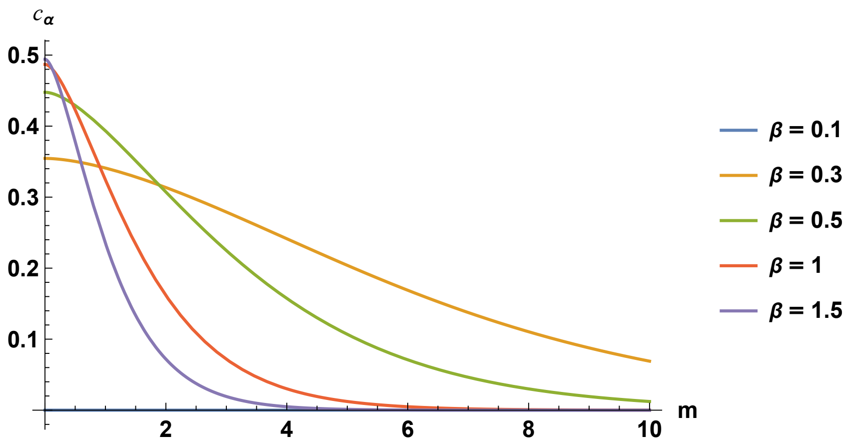
$$\frac{\partial c_\alpha}{\partial m} = \frac{\beta}{\hat{\beta}^2}\frac{\partial c_0(\hat{r})}{\partial \hat{r}}\left(\frac{\hat{\beta}^2}{1 + \frac{\alpha\pi}{6}\frac{c_0(\hat{r})}{\hat{\beta}^2}}\right) \tag{4.135}$$

which is always decreasing with m, since the term in the parenthesis is positive while the free c-function c_0 is always decreasing with its argument, as shown in figure 4.3 for the free fermion. The monotonic behaviour might suggest that some c-theorem might still hold in theories with irrelevant deformations. A similar discussion can be found in [97]. Figures 4.5a and 4.5b show the monotonic behaviour for different values of temperature and α.

An important observation is that the monotonic behaviour is not observed when considering the functional dependence of c_α on β, and this is a further indication of the fact that something completely new is happening, related to the presence of two length scales in the system. This is shown in figure 4.6. The difference of monotonicity of c_α with respect to m and β is a necessary and sufficient condition to conclude that the scaling function does not depend on r only. The interplay of the two different adimensional factors r and r' will give rise to an interesting range of phenomena which would be interesting to study more thoroughly in the future.



(a) The scaling function in terms of the mass, at fixed $\beta = 1$ and at different values of α. As expected, the scaling function is monotonic, and the effect of the perturbation is to decrease its value in the conformal limit.



(b) The scaling function in terms of the mass, at fixed $\alpha = 0.1$ and at different values of β. We see that, when the inverse temperature becomes comparable to the value of α, the behaviour of the scaling function changes drastically. In particular, for $\beta \approx \alpha$ we see that it drops to extremely small values.

Figure 4.5: Behaviour of c_α as function of the mass, varying β and α.

4.5.3 First Corrections in the Partitioning Protocol

The main assumption which was used to study the partitioning protocol was that $\theta^*_\alpha = \theta^*_0$ in the conformal limit, and this allowed to obtain the exact result for higher currents and densities. As in the previous sections, we only consider the case of the free fermion, since it is the only one in which it is plausible to obtain some analytical results, and we study the first order correction to the results found above. The correction can be found by expanding the integrals as function of the parameter $\theta^* \approx \frac{\alpha j_e^\alpha}{1+\alpha j_p^\alpha} \approx 0$ (where we have already expanded the arctangent which we obtained previously). This leads to corrections first to the energy and momentum currents, which propagate to corrections to all the higher currents. It is clear that

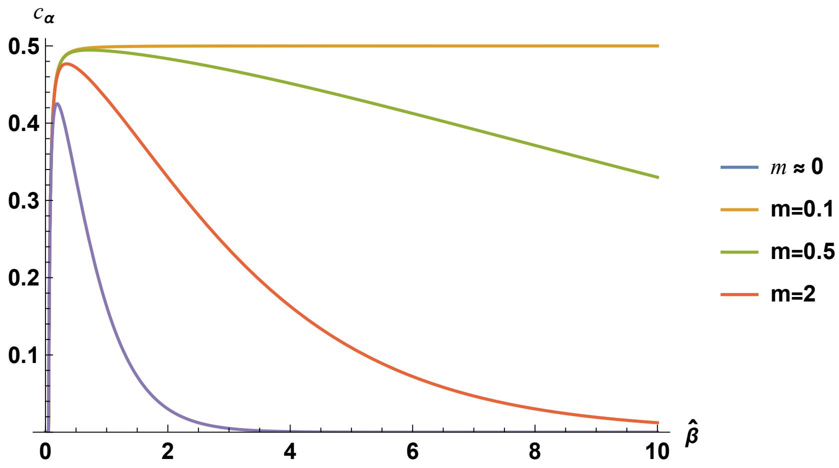


Figure 4.6: The non-monotonic behaviour of the scaling function with respect to the inverse temperature is shown here for different values of the mass, for $\alpha = 0.1$. We note that, for $m \to 0$, the plot is exactly what was found in [24], namely a β-dependent scaling function which tends to c=1/2 for large β. We observe that the plot is a function of $\hat{\beta}$ and not of β, but since the modified temperature is monotonic in β the change of variables would not alter the monotonicity structure of c_α.

the addition of such a term will lead to significant complication of the expressions already at the level of the free theory. Following the discussion of section 4.2.1, we need to solve the system (4.32), this time with θ^* non-zero, but still small. For simplicity, we only consider the corrections at first order. We write conveniently expression (4.32) as:

$$j_s^\alpha = I_{ps} - \alpha I_{pe} q_s^\alpha + \alpha I_{pp} j_s^\alpha \tag{4.136}$$
$$q_s^\alpha = I_{es} - \alpha I_{ee} q_s^\alpha + \alpha I_{ep} j_s^\alpha \tag{4.137}$$

where the constant factors are defined as

$$\begin{aligned} I_{ij} &= \int_{-\infty}^{\infty} h_i(\theta) h_j(\theta) n(\theta) \\ &= \int_{-\infty}^{\theta^*} h_i(\theta) h_j(\theta) \frac{1}{1+e^{\hat{\beta}_R \cosh(\theta)}} + \int_{\theta^*}^{\infty} h_i(\theta) h_j(\theta) \frac{1}{1+e^{\hat{\beta}_L \cosh(\theta)}} \end{aligned}$$

Expanding the integrals at fist order in θ^*, and focusing only on energy and momentum for simplicity, it is possible to obtain:

$$\begin{aligned} I_{ee} &\approx q_e^0 + \left(\frac{1}{1+e^{\hat{\beta}_R}} - \frac{1}{1+e^{\hat{\beta}_L}} \right) \theta^* + O(\theta^*)^2 \\ I_{pe} &\approx j_e^0 + O(\theta^*)^2 \\ I_{pp} &\approx j_p^0 + O(\theta^*)^3 \end{aligned}$$

hence only one term gets a correction. Note that here the quantities q_e^0, j_e^0, j_p^0 are not only the quantities of the unperturbed theory, but they are the quantities evaluated in the conformal limit, namely the well known result by Bernard-Doyon. This leads to the following system, defining $k = \frac{1}{1+e^{\hat{\beta}_R}} - \frac{1}{1+e^{\hat{\beta}_L}}$:

$$\begin{cases} j_e^\alpha = j_e^0 - \alpha j_e^0 q_e^\alpha + \alpha j_p^0 j_e^\alpha \\ q_e^\alpha = q_e^0 + k\theta^* - \alpha(q_e^0 + k\theta^*)q_e^\alpha + \alpha j_e^0 j_e^\alpha \\ j_p^\alpha = j_p^0 + \alpha j_e^0 j_e^\alpha + \alpha j_p^0 j_p^\alpha \end{cases} \tag{4.138}$$

Substituting $\theta^* \approx \frac{\alpha j_e^\alpha}{1+\alpha j_p^\alpha}$, it can be solved exactly. As anticipated, the expressions which one obtains are rather involved, and we report here the correction to the energy current:

$$j_e^\alpha = \frac{1 + \alpha^2(2(j_e^0)^2 - j_p^0 q_e^0) + (\alpha j_p^0 - 1)\sqrt{1 + 2\alpha q_e^0 + 4\alpha^2 k j_e^0 + \alpha^2 (q_e^0)^2}}{2(\alpha^2(j_e^0 - k) + 2\alpha^3 k j_p^0 + \alpha^4((j_e^0)^3 - k(j_p^0)^2 - j_e^0 j_p^0 q_e^0))} \tag{4.139}$$

These calculations show explicitly that obtaining even the lowest order corrections becomes quickly extremely complicated even for the simplest possible theory. At higher orders in θ^*, we would have to consider even terms as $\left(\frac{\alpha j_E}{1+\alpha j_p}\right)^2$ and this would make the system virtually impossible to solve, even just at second order. Therefore, a perturbative approach as this one does not lead to particularly interesting results, and we can say that the power of the methods developed in this work to obtain exact solutions is mostly limited to the conformal case. As stressed previously, this should not be seen as a particular weakness of the theory, since it is the rule rather than the exception exact solutions in GHD are extremely hard to find, even in unperturbed theories.

4.6 Generalized Deformations

It is possible, with some caveats, to extend the discussions of this chapter to the case of a generalized $T\bar{T}$ deformation, in which the kernel gets modified by a factor $\cosh s\theta$. Since the calculations in this situation become more complicated, and this section serves more as an illustration, we will focus on the free fermion theory, and assume boldly that the results will apply with eventual small modifications also in the general interacting case. As stated before, the generalized deformation affects the TBA equations in a simple way:

$$\varepsilon(\theta) = \nu(\theta) + \alpha_s \cosh s\theta * L(\theta) \tag{4.140}$$

where α_s is a dimensionless parameter, of which we do not know the mass dependence exactly. We see that, in order to absorb the perturbation into a redefinition of the generalized inverse temperatures we need to have in the driving term the same $\cosh s\theta$. So for simplicity we consider the case $\nu(\theta) = \beta_s \cosh s\theta$, namely a situation in which the system is described by a GGE with only one nonzero potential. This is the only situation in which some of the integrals can be solved exactly, or reduced to the integrals discussed in the previous sections, and therefore we focus solely on this case since we are interested in finding analytical results [11]. We will consider a TBA with driving term containing potential $\beta_s = m^s\beta^s$ at spin s: this is the situation in which Zamolodchikov's argument explained in section 4.2.2 can be still applied. At equilibrium, we can repeat the calculations made above analogously, as:

$$\begin{aligned}\varepsilon(\theta) &= m^s\beta^s\cosh(s\theta) + \alpha_s\cosh(s\theta)\star L(\theta)\\ &= m^s\beta^s\cosh(s\theta) + \alpha_s\cosh(s\theta)\int\frac{d\theta'}{2\pi}\cosh(s\theta')L(\theta')\end{aligned}$$

Now, we can define a generalization of the free TBA energy that we considered before:

$$\tilde{E}_0^\alpha = -\frac{m^s}{2\pi}\int d\theta\cosh(s\theta)L(\theta) \tag{4.141}$$

where the tilde used here should not be confused with the tilded quantities of section 4.3. This object is interpreted as the analogous of E_0^α for higher spin charges, and we will refer to it as a generalized energy. Introducing this quantity the TBA equation becomes:

$$\varepsilon(\theta) = m^s\beta^s\cosh(s\theta) - \frac{\alpha_s}{m^s}\tilde{E}_0^\alpha\cosh(s\theta) \tag{4.142}$$

hence this allows to find a self consistent equation for the generalized energy:

$$\tilde{E}_0^\alpha = -\frac{m^s}{2\pi}\int\cosh(s\theta)\log(1+e^{-(\beta^s-\frac{\alpha_s}{m^{2s}}\tilde{E}_0^\alpha)m^s\cosh(s\theta)}) \tag{4.143}$$

In order to avoid divergencies we impose that α/m^s goes to zero as m goes to zero. Solving the integral in terms of Bessel functions as above, and performing a change of variables $s\theta\to\theta$ to remove the spin in the exponent, leads to an analogous expression:

$$\begin{aligned}\tilde{E}_0^\alpha &= \frac{m^s}{2\pi s}\sum_n\frac{(-1)^n}{n}\int\cosh(\theta)\exp\left(-nm^s(\beta^s-\frac{\alpha_s}{m^{2s}}\tilde{E}_0^\alpha)\cosh(\theta)\right)\\ &= \frac{m^s}{2\pi s}\sum_n\frac{(-1)^n}{n}K_1(-nm^s(\beta^s-\frac{\alpha_s}{m^{2s}}\tilde{E}_0^\alpha))\\ &= \frac{-\pi c}{6s(\beta^s-\frac{\alpha_s}{m^{2s}}\tilde{E}_0^\alpha)}\end{aligned}$$

Where we still have a mass term, contrarily to the standard case in which all the mass dependence gets canceled. But now we see clearly that in the conformal limit

[11] It is known that the TBA equations become problematic when there are two hyperbolic cosine with different arguments, as there are convergence issues, so we directly avoid also this problem.

this leads to a vanishing of the energy, making the theory unphysical. Therefore we have several reasons to assume that the value of the parameter α_s has the form:

$$\alpha_s = \alpha m^{2s} \tag{4.144}$$

Where α has the dimensions of $[M]^{-2s}$.[12] This is a direct analogy with the standard $T\bar{T}$ case, where s=1. In this situation, the energy remains finite, and in particular the difference with the usual case is the addition of a factor s and a modification of the inverse temperature dependence as $\beta \to \beta^s$. This leads to the expression:

$$\tilde{E}_0^\alpha = \frac{\beta^s}{2}\left(1 - \sqrt{1 + \frac{2\pi c\alpha}{3s\beta^{2s}}}\right) \tag{4.145}$$

Therefore, defining the modified inverse temperature by analogy as above, as

$$\hat{\beta}^s = \beta^s - \alpha \tilde{E}_0^\alpha \tag{4.146}$$

this leads to the expression:

$$\hat{\beta}^s = \frac{\beta^s}{2}\left(1 + \sqrt{1 + \frac{2\pi c\alpha}{3s\beta^{2s}}}\right) \tag{4.147}$$

The final expression for the modified inverse temperature is therefore

$$\hat{\beta} = \frac{\beta}{2^{1/s}}\left(1 + \sqrt{1 + \frac{2\pi c\alpha}{3s\beta^{2s}}}\right)^{1/s} \tag{4.148}$$

In terms of these objects the (equilibrium) TBA equation becomes simply:

$$\varepsilon(\theta) = \hat{\beta}^s m^s \cosh(s\theta) \tag{4.149}$$

where eventually one can add a interacting term through the addition of a kernel. Therefore it should be possible to carry a similar analysis as for the standard case. This clearly requires (4.144) to hold. In a many particle theory, this should look something like:

$$\alpha_s = \alpha m_i^s m_j^s \tag{4.150}$$

Now, we can also proceed by analogy with 4.2.2, since the modified inverse temperature has no mass dependence, to find the CFT limit of (4.149): for right movers and left movers respectively, we will have:

$$\varepsilon_R(\theta) = \frac{M^s \hat{\beta}^s}{2} e^{s\tilde{\theta}} \ , \ \varepsilon_L(\theta) = \frac{M^s \hat{\beta}^s}{2} e^{-s\tilde{\theta}} \tag{4.151}$$

where as usual $M = me^{\theta_0}$, with θ_0 the divergent part of the rapidity. In the following two sections we discuss how the previous results can be extended to include these generalized perturbations.

[12] Note that this form of α_s is the only one which allows to use Zamolodchikov argument presented in 4.2.2 also on the perturbing term, in order to reabsorb the infinity of the hyperbolic cosine into the vanishing of the masses. Therefore we have a double reason to take this mass dependence.

4.6.1 Free fermion charges

Setting now m=1, we move to the evaluation of the charges: the discussion is slightly more involved than before because both the spin of the perturbation and the spin of the charge of interest will have to be considered separately, but we can essentially generalize the results of 4.3.1. We start by observing that, in the presence of a spin s in the deformation, even at equilibrium we cannot use the trick of writing the dressing at some spin s in terms of the charge density of the same spin. This is because we have:

$$h_{\tilde{s}}^{dr} = h_{\tilde{s}} - \alpha \cosh(s\theta) \int_{-\infty}^{\infty} \cosh(s\theta') n(\theta') h_{\tilde{s}}^{dr}(\theta') \tag{4.152}$$

where s is the spin appearing in the modified kernel, while $\tilde{s}$ is the spin of the charge we are performing the dressing operation on. We can avoid this problem in the case of the energy, since the dressing operation can be exchanged between the terms of the integral, such that one obtains (assuming equilibrium from the start):

$$E^{dr} = \cosh\theta - \alpha q_s \cosh(s\theta) \tag{4.153}$$

From this it is possible in a first instance to find the charge density of the same spin s corresponding to the spin of the perturbation:

$$\begin{aligned} q_s &= \frac{1}{2\pi} \int d\theta \cosh(\theta) n(\theta) h_s^{dr}(\theta) = \frac{1}{2\pi} \int d\theta E^{dr}(\theta) n(\theta) h_s(\theta) \\ &= \frac{1}{2\pi} \int d\theta (\cosh\theta - \alpha q_s \cosh s\theta) n(\theta) \cosh(s\theta) \\ &= \frac{1}{2\pi} \int d\theta (\cosh\theta - \alpha q_s \cosh s\theta) \sum_n (-1)^{n+1} e^{-n\hat{\beta}\cosh(s\theta)} \cosh(s\theta) \\ &= \frac{1}{2\pi s} \int d\theta (\cosh\frac{\theta}{s} - \alpha q_s \cosh\theta) \sum_n (-1)^{n+1} e^{-n\hat{\beta}\cosh(\theta)} \cosh(\theta) \end{aligned}$$

where in the last step we have changed the integration variable. Now we can use the modified Bessel function exactly as done for the standard perturbation:

$$q_s = \sum_n (-1)^{n+1} \frac{1}{\pi s} \left[\left(K_{1+1/s}(n\hat{\beta}) - \frac{1}{sn\hat{\beta}} K_{1/s}(n\hat{\beta}) \right) - \alpha q_s \left(K_2(n\hat{\beta}) - \frac{1}{n\hat{\beta}} K_1(n\hat{\beta}) \right) \right] \tag{4.154}$$

which is clearly identical to (4.58) except for the substitution $s \to \frac{1}{s}$, and hence the final solution for the charges is identical up to this substitution: defining $s' = 1/s$, we simply rewrite [13]

$$q_s^{\alpha} = \frac{(s')^2 \Gamma(s') 2^{s'-1} \zeta(s'+1)(1-2^{-s'})}{\pi \hat{\beta}^{s'+1} \left(1 + \frac{\alpha\pi}{12\hat{\beta}^2}\right)} . \tag{4.155}$$

[13] Note that since s' is not an integer we cannot use the factorial as we did above, but we need to use instead $s\Gamma(s)$: the two expressions coincide for integer values.

Now, the knowledge of this fundamental charge allows to find the charges of all the others, with spin $\tilde{s}$ different from that of the perturbation:

$$q_{\tilde{s}} = \frac{1}{2\pi}\int d\theta E^{dr}(\theta)n(\theta)h_{\tilde{s}}(\theta) \tag{4.156}$$

$$= \frac{1}{2\pi}\int d\theta(\cosh\theta - \alpha q_s \cosh s\theta)n(\theta)h_{\tilde{s}}(\theta) \tag{4.157}$$

which can then simply be solved by analogy and substituting (4.155), obtaining the complicated expression for the generic-spin charge densities:

$$q_{\tilde{s}}^{\alpha} = q_s^0 - \frac{\alpha}{8\pi}q_s^{\alpha}\left[(2^{|s-\tilde{s}|}-2)\cdot\frac{(|s-\tilde{s}|!)}{\tilde{\beta}^{|s-\tilde{s}|}}\zeta(|s-\tilde{s}|) + (2^{s+\tilde{s}}-2)\cdot\frac{(s+\tilde{s})!}{\tilde{\beta}^{s+\tilde{s}}}\zeta(s+\tilde{s})\right] \tag{4.158}$$

where s is the spin of the perturbing operator and $\tilde{s}$ is the spin of the generic charge that we are taking into consideration, q_s^0 is the charge in the unperturbed theory, while q_s^{α} is (4.155). Therefore the equilibrium situation can be dealt with in a rather straightforward way, althouth the exact expressions become much more intricated.

4.6.2 Non-Equilibrium

On the other hand, the main discussion which was presented in sections 4.3 onwards, which allowed to solve the partitioning protocol with great generality in the conformal limit, relied essentially on the possibility of writing the dressing of a spin s quantity in terms of the density and current at that precise spin. In the present case this is in general not true, except for the energy and momentum which satisfy:

$$E^{dr} = m\cosh(\theta) - \alpha m^s\cosh(s\theta)q_s + \alpha m^s\sinh(s\theta)q_{-s} \tag{4.159}$$

$$p^{dr} = m\sinh(\theta) - \alpha m^s\cosh(s\theta)j_s + \alpha m^s\sinh(s\theta)j_{-s} \tag{4.160}$$

where the -s is used to indicate the odd spin densities and currents, which in general will be different from zero if the system is not at equilibrium (in this notation, we would have $q_1 = q_E$, and $q_{-1} = q_P$, and similarly for the currents). At least in the conformal limit, we can use the equality $q_s = j_{-s}$ and viceversa (which is the same as $q_E = j_p$ and $j_E = q_p$), to obtain:

$$\begin{aligned} E^{dr} &= m\cosh(\theta) - \alpha m^s\cosh(s\theta)q_s + \alpha m^s\sinh(s\theta)j_s \\ p^{dr} &= m\sinh(\theta) - \alpha m^s\cosh(s\theta)j_s + \alpha m^s\sinh(s\theta)q_s \end{aligned}$$

Therefore we see that the discussion presented for the standard s=1 $T\bar{T}$ cannot be applied directly, because it is not possible to obtain a close solvable system containing the quantities at a fixed spin s. The main difficulty arising when the generalized deformations are introduced regards the effective velocity. As already clear from (4.159), contrarily to what happened for the standard deformation in this case the dressed energy vanishes for two quasi-symmetric values of the rapidity, as shown in figure (4.7). This behaviour is quite problematic, and rather unique,

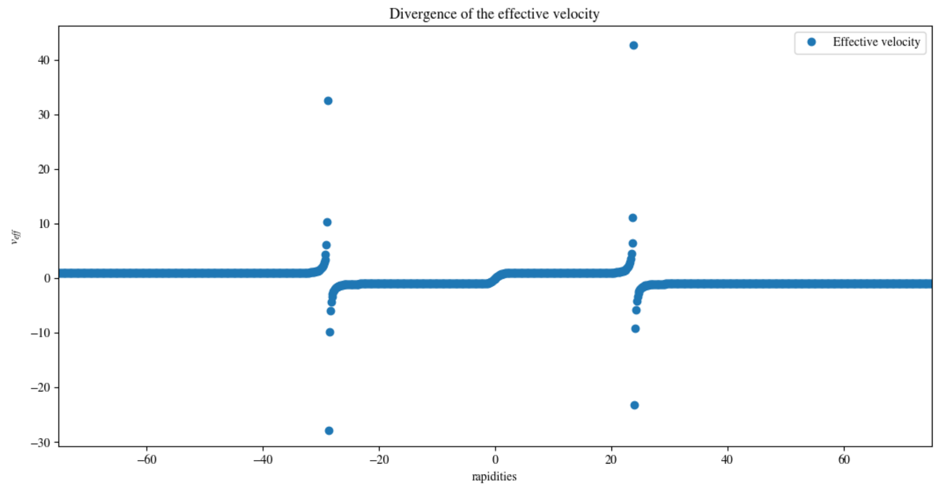


Figure 4.7: In the presence of generalized deformations the effective velocity exhibits a peculiar behaviour, namely it diverges at two values of rapidity which corresponds to the zeros of the dressed energy. This makes the solution to the partitioning protocol extremely interesting in the non-conformal case. More work should be done to give an interpretation to these singularities.

although it had been observed already for the hard rods in GHD. First of all, the solution to the partitioning protocol described in chapter 3 requires to have a smooth effective velocity, which is clearly not the case, and therefore the general solution will not be given by a stacking of contact discontinuities as usual. Another strange feature is the asymptotic behaviour: usually, the hyperbolic tangent-like shape of the effective velocity implies that it is negative for negative values of the rapidity, and positive for positive values, but this is not the case in the present situation. This inverted behaviour might suggest unphysical and puzzling phenomena such as the flow of energy from the coldest to the hottest side of the partitioning protocol, unless the solution for the n-function is significantly different from the one we have in the case of continuous effective velocities, namely (3.34). These features signal an interesting kind of transition taking place: the fact that the dressed energy can become negative implies that, for example, the TBA quasiparticle density also becomes negative, since:

$$\rho_p(\theta) = \frac{1}{2\pi} n(\theta) E^{dr}(\theta) \tag{4.161}$$

These are all unusual properties whose physical meaning is not clear at this stage and is left for future investigation. Fortunately, however, the situation greatly simplifies in the conformal limit, in light of the following observation: as $m \to 0$, the two poles are moved to $\pm\infty$, disappearing completely as $m = 0$. Moreover, it is also easy to see that in the conformal limit the `central` part of the effective velocity, is exactly the arctangent (this is exact, since we are dealing with the free fermion).

This can be easily shown by calculating the effective velocity from (4.159):

$$v_{eff} = \frac{m\sinh(\theta) - \alpha m^s \cosh(s\theta) j_s + \alpha m^s \sinh(s\theta) q_s}{m\cosh(\theta) - \alpha m^s \cosh(s\theta) q_s + \alpha m^s \sinh(s\theta) j_s} \tag{4.162}$$

To take the limit we make the assumption that the currents and densities are well behaved functions in the conformal limit, as it happens in the standard $T\bar{T}$ case, and this immediately implies that:

$$\lim_{m\to 0} v_{eff} = \tanh(\theta) \tag{4.163}$$

This behiour is shown in figure 4.8, and it is completely different from the standard

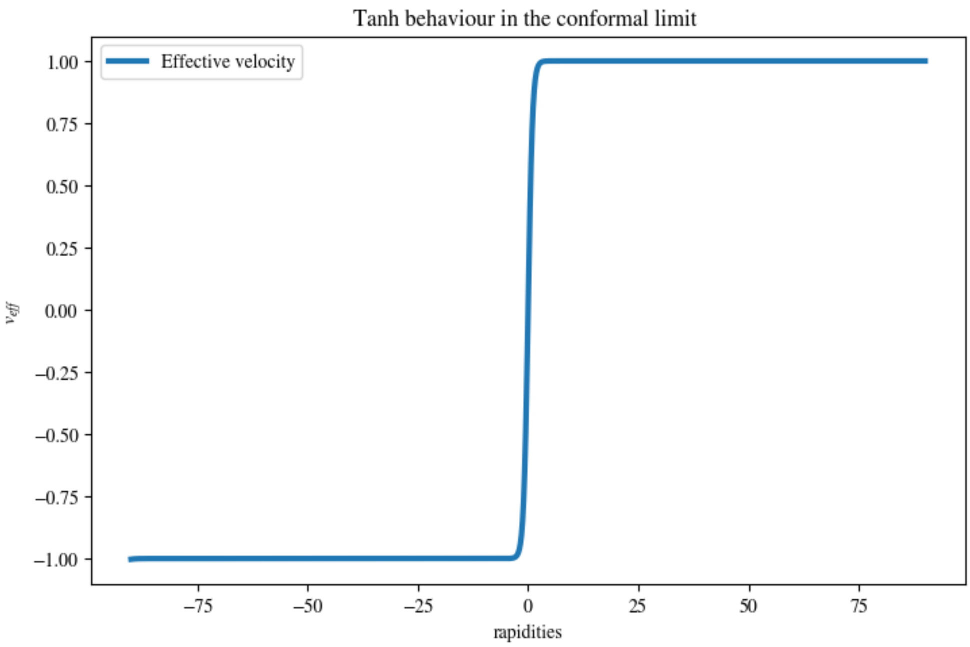


Figure 4.8: In the conformal limit, the effective velocity becomes a perfect hyperbolic tangent, exactly as in a free theory. In this sense the effect of the perturbation modifies the structure of the theory less than in the s=1 $T\bar{T}$ case studied above. To favour the convergence of the numerical simulation, which is greatly spoiled by the higher value of s (equal to 2 in this case) which tends to overflow the hyperbolic cosines in the TBA equations, also the temperatures have been taken extremely small in this case, $T = O(10^{-5})$

$T\bar{T}$ context, in which it is easy to show that:

$$\lim_{m\to 0} v_{eff}^{s=1} = \frac{\sinh(\theta) - \alpha j^\alpha \cosh(\theta) + \alpha q^\alpha \sinh(\theta)}{\cosh(\theta) - \alpha q^\alpha \cosh(\theta) + \alpha j^\alpha \sinh(\theta)} \tag{4.164}$$

because all the masses cancel in the fraction. In this sense the conformal limit of the generalized theories is even easier, since we have a perfect arctangent so it is not even necessary to repeat the considerations on the flatness of the n-functions which were instead necessary above, precisely because the effective velocity is of the form (4.164). These considerations justify the idea of using the same strategy used above also in this situation. The main problem resides in the aforementioned fact that here it is not possible to obtain immediately a closed system for the currents and densities, since the spin of the perturbation will always appear. A way out is to define modified densities and currents as:

$$
\begin{aligned}
q_{\tilde{s}}^{s} &:= \int_{-\infty}^{\infty} \cosh(s\theta')n(\theta')h_{\tilde{s}}^{dr}(\theta') \\
j_{\tilde{s}}^{s} &:= \int_{-\infty}^{\infty} \sinh(s\theta')n(\theta')h_{\tilde{s}}^{dr}(\theta')
\end{aligned}
$$

where again s is the spin of the perturbing term, and $\tilde{s}$ is the spin of the charge of which the average is being calculated. Observe that

$$
n(\theta) \equiv n_s(\theta) = \frac{1}{1+e^{\hat{\beta}^s \cosh(s\theta)}} \tag{4.165}
$$

So what we are doing is taking the $\cosh s\theta$ which performs the averages with the same spin as the n-function. These objects do not have the interpretation of conserved currents and densities, but appear to be their natural generalizations. It is thus possible to perform the same discussion as above in terms of these new quantities, and this would allow to find the values of these charges in the perturbed theory in terms of the ones of the unperturbed theory. The unperturbed versions of these object are easy to compute in the case of the free fermion, and in the general theory a similar discussion to that which will be done in appendix A can be performed: their meaning can be understood as the density associated to a spin $\tilde{s}$ charge with a driving term containing a $\cosh(s\theta)$ potential. It is easy to convince oneself that these quantities are easy to compute from the quantities we already studied in chapter 4. Starting with the case $\tilde{s} = s$ we see immediately:

$$
(j^0)_s^s = \int \sinh(s\theta) \frac{\cosh(s\theta)}{1+e^{\beta^s \cosh(s\theta)}} = \tag{4.166}
$$

$$
\frac{1}{s}\int \sinh(\theta) \frac{\cosh(\theta)}{1+e^{\beta^s \cosh(\theta)}} = \frac{1}{s} j_1^1(\beta^s) = \mathcal{G}(s,s)\left(T_L^{2s} - T_R^{2s}\right) \tag{4.167}
$$

where j_1^1 is the standard CFT energy current. The same discussion can be applied for the densities, hence we obtain:

$$
\begin{aligned}
(q^0)_s^s &= \mathcal{G}(s,s)\left(T_L^{2s} + T_R^{2s}\right) \\
(j^0)_s^s &= \mathcal{G}(s,s)\left(T_L^{2s} - T_R^{2s}\right)
\end{aligned}
$$

where $\mathcal{G}(s,s) = \frac{\pi c}{12 s}$. Similarly, always considering a driving term of the form $\beta^s \cosh(s\theta)$, the energy currents and densities are easily computed:

$$
j_1^s = \int \sinh(s\theta) \frac{\cosh(\theta)}{1+e^{\beta^s \cosh(s\theta)}} = \frac{1}{s}\int \sinh(\theta)\frac{\cosh(\theta/s)}{1+e^{\beta^s \cosh(\theta)}} = \frac{1}{s} j_{1/s}^1(\beta^s)
$$

where $j^1_{1/s}(\beta^s)$ are the current found above, for example in (4.74), only evaluated at a value of spin 1/s, and at temperature β^s (although our result was technically valid for integer values of spin, there is no difficulty in extending it to contain also 1/s), hence the final result is

$$j^s_1 = \frac{1}{s}\mathcal{G}(1/s)(T_L^{1+s} - T_R^{1+s}) \tag{4.168}$$

and the same for the density:

$$q^s_1 = \frac{1}{s}\mathcal{G}(1/s)(T_L^{1+s} + T_R^{1+s}) \tag{4.169}$$

while all the other generic currents will contain terms as $\tilde{s}/s$:

$$j^s_{\tilde{s}} = \int \sinh(s\theta)\frac{\cosh(\tilde{s}\theta)}{1+e^{\beta^s\cosh(s\theta)}} = \frac{1}{s}\int \sinh(\theta)\frac{\cosh(\frac{\tilde{s}}{s}\theta)}{1+e^{\beta^s\cosh(\theta)}} = \frac{1}{s}j^1_{\tilde{s}/s}(\beta^s) \tag{4.170}$$

which then leads to the last interesting current in the unperturbed theory:

$$j^s_{\tilde{s}} = \frac{1}{s}\mathcal{G}(\tilde{s}/s)(T_L^{\tilde{s}+s} - T_R^{\tilde{s}+s}) \tag{4.171}$$

Now we can use the same approach as in the standard case, although the calculations are slightly more intricate. We show how to obtain the energy current as an illustration. First of all we consider the general dressing operation of the charge at spin s, which coincides with the spin appearing in the $T\bar{T}$ deformation:

$$h^{dr}_s = h_s - \alpha q^s_s \cosh(s\theta) + \alpha j^s_s \sinh(s\theta) \tag{4.172}$$

We see immediately that we obtain exactly the same system as the one we had in the main discussion, for q^s_s and j^s_s, by simply multiplying by $\cosh(s\theta)n(\theta)$ or $\sinh(s\theta)n(\theta)$ and integrating over the rapidities, and the solutions are immediately[14]:

$$\begin{aligned}(q^s_s)^\alpha &= \frac{(q^0)^s_s - \alpha((q^0)^s_s)^2 + \alpha((j_0)^s_s)^2}{1-\alpha^2((q^0)^s_s)^2 + \alpha^2((j_0)^s_s)^2}\\ (j^s_s)^\alpha &= \frac{(j^0)^s_s}{1-\alpha^2((q^0)^s_s)^2 + \alpha^2((j_0)^s_s)^2}\end{aligned}$$

Then it is possible to obtain the charges and currents of the form q^s_1, j^s_1, again starting from (4.172), and this time multiplying by $\sinh(\theta)n(\theta)$ and integrating over rapidities; the result depends on the two quantities just found:

$$\begin{aligned}(q^s_1)^\alpha &= (q^0)^s_1 - \alpha(q^0)^s_1(q^\alpha)^s_s + \alpha(j^0)^s_1(j^\alpha)^s_s\\ (j^s_1)^\alpha &= (j^0)^s_1 - \alpha(j^0)^s_1(q^\alpha)^s_s + \alpha(q^0)^s_1(j^\alpha)^s_s\end{aligned}$$

[14]The only difference with (4.67) is that here we have introduced immediately the fact that the odd currents are equal to the even charges, and viceversa, in the conformal limit.

These are already solved since $(q^\alpha)^s_s$ and $(j^\alpha)^s_s$ are known. Substituting the expressions for the charges in the unperturbed theory, (4.168), and the results for $(q^\alpha)^s_s$ and $(j^\alpha)^s_s$, one has:

$$(j_1^s)^\alpha = \mathcal{G}(1/s)\frac{\left(\hat{T}_L^{s+1} - \hat{T}_R^{s+1} + 2\alpha\mathcal{G}(s,s)\left(\hat{T}_L^{2s}\hat{T}_R^{s+1} - \hat{T}_R^{2s}\hat{T}_L^{s+1}\right)\right)}{(1-4\alpha^2\mathcal{G}(s,s)^2\hat{T}_L^{2s}\hat{T}_R^{2s})} \tag{4.173}$$

Observe that this is not the usual energy current. The energy current would be:

$$j_E = \int d\theta \sinh(\theta) n_s(\theta) E^{dr}(\theta) \tag{4.174}$$

while this current we have found is a generalized version,

$$j_1^s = \int d\theta \sinh(s\theta) n_s(\theta) E^{dr}(\theta) \tag{4.175}$$

So it is not clear what interpretation to give to this object. The "real" energy current can be computed from (4.159), which implies that:

$$j_E = \int \sinh(\theta) n_s(\theta)\cosh(\theta) - \alpha(j_1^s)^0(q_1^s)^\alpha + \alpha(q_1^s)^0(j_1^s)^\alpha \tag{4.176}$$

The first integral cannot be easily reduced to known quantities, but from the discussion of appendix A it is easy to see that the temperature dependence will be again of the form $(T_L^2 - T_R^2)$ as in the case of the thermal GGE. We will refer to the prefactor generically as $\mathcal{G}_{1s}$, which for the s=1 case will have to reduce to known quantities to have the correct limit, while for s>1 it is possible to evaluate it only for the free fermion. Considering all the pieces together we obtain the final expression for the energy current (from now on we don't write the hat on the temperatures for legibility):

$$j_E = \mathcal{G}_{1s}(T_L^2 - T_R^2) + \frac{4\alpha^2\mathcal{G}(1/s)^2\mathcal{G}(s,s)T_L^{1+s}T_R^{1+s}(T_L^{2s}-T_R^{2s})}{s^2(1-4\alpha^2\mathcal{G}(s,s)^2T_L^{2s}T_R^{2s})}$$

where the s^2 in the denominator comes from the 1/s appearing in the definition of the current (4.168). Since for s=1 this has to reduce to the result of [24], we consider:

$$j_E(s=1) = \frac{(T_L^2 - T_R^2)\left[\mathcal{G} - 4\alpha^2(\mathcal{G}\mathcal{G}(1,1) - \mathcal{G}(1)^2)T_L^2T_R^2\right]}{1-4\alpha^2\mathcal{G}(s,s)^2T_L^{2s}T_R^{2s}} \tag{4.177}$$

and therefore we see that in order to have the correct value for s=1 we need to have $\mathcal{G} = \mathcal{G}(1)^2\mathcal{G}(1,1)^{-1}$, such that the second term in the square bracket vanishes. But this is simply $\frac{\pi}{24} = \mathcal{G}(1)$. The other values $\mathcal{G}_{1s}$ are computed for the free fermion in the appendix. The final equation for the energy current is conveniently expressed as:

$$j_E = G_{1s}(T_L^2 - T_R^2) + \frac{4\alpha^2\mathcal{G}(1/s)^2\mathcal{G}(s,s)T_{LR}(3s+1,s+1)}{s^2(1-4\alpha^2\mathcal{G}(s,s)^2T_L^{2s}T_R^{2s})} \tag{4.178}$$

where for compactness we have introduced the composite temperature-dependent function:

$$T_{LR}(a,b) = T_L^a T_R^b - T_R^a T_L^b \tag{4.179}$$

which satisfies $T_{LR}(a,b) = -T_{LR}(b,a) = -T_{RL}(a,b)$ and $T_{LR}(a,a) = 0$, properties which immediately show that the correct limit is obtained when s=1. Also, the first property implies the (obvious) fundamental fact that the currents vanish when the two temperatures are equal. By analogy with the previous situation, we can define the prefactor $c_{LR} = (1 - 4\alpha^2 \mathcal{G}(s,s)^2 T_L^{2s} T_R^{2s})^{-1}$. Note that the values of $\mathcal{G}_{ss} = \frac{\pi c}{12s}$ are the same for every theory, upon changing the central charge c, since they are associated to the evaluation of the integral $\mathcal{G}(1)$ of appendix A. So they are easily known quantities. This is not the case instead for $\mathcal{G}(1/s)$ and $\mathcal{G}_{1s}$, which in the free fermion can be computed using expression (5.5), while it is unknown for generic theories. For example, the s=2 case can be expressed as:

$$j_E(s=2) = \frac{\ln 2}{8\pi}(T_L^2 - T_R^2) - \alpha^2 \frac{\pi c}{24} \mathcal{G}(1/2)^2 c_{LR} T_{LR}(7,3) \tag{4.180}$$

where we have used the exact expression for $\mathcal{G}_{12}$. Repeating essentially the same derivation one is lead to a similar expression for the conserved densities:

$$q_E = \frac{\pi}{24}(T_L^2 + T_R^2) + \alpha \mathcal{G}_{1/s}^2 c_{LR} \left(\alpha \mathcal{G}_{ss} T_{(LR)}(3s+1, s+1) - \frac{1}{2} T_{(LR)}(s+1, s+1) \right) \tag{4.181}$$

where the $T_{(LR)}$ are the symmetrized versions of the functions defined above,

$$T_{(LR)}(a,b) = T_L^a T_R^b + T_R^a T_L^b \tag{4.182}$$

Again it is easy to see that in the case s=1 expression (4.181) reduces to the expression of [24]. The procedure can also be generalized to arbitrary spins, leading to analogous expressions. As highlighted by the complicated expression for the equilibrium charges (4.158), and the one just found for the energy current, however, the situation becomes rapidly more involved if one attempts to compute the currents for higher spins, when the spin of the charge is different from the spin of the perturbation. It is important to realize however that the reasoning is always in principle applicable, if enough information on the unperturbed quantities is accessible. Although the calculations in this section have been performed only for the free fermion, the main discussion made for the standard $T\bar{T}$ deformation indicates clearly that this result for the partitioning protocol should be valid for a generic theory, simply by modifying the central charge which appears in the expressions. In fact, the formal discussion in which we considered the inversion of the integral operators should be still applicable in this context. Therefore we see that the equilibrium discussion can be extended with no particular problem to the general case. However in this situation it is the behaviour out of the conformal limit which is particularly puzzling, since the density of quasiparticles can become negative, and this has strange implications on the energy flows. This situation is clearly extremely difficult to treat analytically, and therefore numerical studies will be necessary to study these interesting phenomena.

Chapter 5

Generalized Hydrodynamics of $T\bar{T}$ deformed theories: numerics

Although the results obtained in the previous chapter appear solid because of the several discussed comparisons with previously known results, numerical testing is the final analysis needed to check the overall validity of the theoretical framework presented, in particular of the two expressions (4.74) and (4.75), and their generalizations proposed in section 4.6. These expressions show the behaviour of densities and currents in the $m \to 0$ limit, and to test them is rather straightforward. First of all, one can simulate a partitioning protocol, in the standard way which will be explained below, to obtain the numerical values of the currents j_{simul} for different values of m. Since we are interested in the NESS currents, we evaluate them for $\xi = 0$. Then, one can normalize these currents by the factor which appears in the analytical expression (such as (4.74) for the currents) in order to obtain a quantity which should not depend neither on the choice of α nor of the temperatures:

$$j_{norm} = \frac{j_{simul}}{c_{LR}\left(\left(\hat{T}_L^{s+1} - \hat{T}_R^{s+1}\right) + \frac{\alpha\pi c}{6}(\hat{T}_L^{s+1}\hat{T}_R^2 - \hat{T}_R^{s+1}\hat{T}_L^2)\right)} \tag{5.1}$$

In the conformal limit this is expected to tend to the value of $\mathcal{G}$(s), which in the case of the free fermion can be computed exactly, whereas in other situation only numerically, but still quite easily solving the integrals shown in appendix A. Clearly, to check that the overall dependence on the temperature is correctly given by (4.74) it is enough to check that j_{norm} is independend from T_L and T_R, since it means that through (5.1) we are removing all the temperature dependence. Several other minor checks, such as regarding the monotonicity of the effective velocity which we have shown in the conformal limit, can be also performed, and some interesting quantities relevant for the TBA can be studied. In this chapter we show the perfect agreement of the analytical results with the simulations. In the final part we also address the intuition presented in [94], namely the interpretation of the effect of the $T\bar{T}$ deformation as the introduction of a finite lenght of the particles.

5.1 Simulating GHD protocols

The approach which is usually followed to simulate the different GHD protocols is to solve recursively the integral equations which characterize the theory, namely the TBA equations and the dressing equations, and use the results to compute all other relevant quantities appearing in the system, such as n-functions, densities, ground state energy, and currents. To find the TBA pseudoenergies, for example, it is sufficient to start the iterative process by setting as first step the energy equal to the driving term, $\varepsilon_0(\theta) = \nu(\theta)$[1]. Then, after i iterations the approximated pseudoenergy will be:

$$\varepsilon_i(\theta) = \nu(\theta) + \varphi * L(\varepsilon_{i-1})(\theta) \tag{5.2}$$

If nothing patological happens with the algorithm, it is natural to expect that the "true" pseudoenergy will be the limit of the sequence:

$$\varepsilon(\theta) = \lim_{i\to\infty} \varepsilon_i(\theta) \tag{5.3}$$

Numerically, the algorithm can be set to stop once the difference between two the pseudoenergies at the i-th and (i+1)-th iterations is below a certain value. Since in general this approach is known not to have particular convergence issues, we do not bother with the eventuality of pathologies in the approach to the solution: the presence of a term $\exp(-\cosh(\theta))$ inside the L-function should in any case make everything converge eventually, even if the kernel contains hyperbolic cosine terms coming from the $T\bar{T}$.

A similar method can be applied immediately also to the dressing operation, although with some more complications: taking as 0-th order approximation the bare quantity, $h_0^{dr}(\theta) = h(\theta)$, the iteration becomes

$$h_i^{dr}(\theta) = h(\theta) - \varphi * (n h_{i-1}^{dr}) \tag{5.4}$$

While the TBA equation could be iterated directly, here it is necessary to find the n-function first. In the partitioning protocol, one can perform a circular argument, to find both the quantities at the same time. First, note that to find n it is sufficient to find the numerical value of θ^*, and then use (4.64). The iterative algorithm is defined as follows: one starts with $\theta_0^* = 0$, then this value can be used to find n_0 and hence the dressed energy and momentum using (5.4). These quantities can then be used to find the corresponding approximated effective velocity using (3.25), which can be set to zero to finding the new value for θ_1^*. Iterating this process, we can perform the algorithm until the error in θ_i^* reaches a certain threshold. Therefore in this way it is possible to obtain the dressed quantities and $n(\theta)$ simultaneously. Once these quantities are known, any quantity relating to the partitioning protocol can be immediately computed, since all that is needed are integrals containing the bare quantities, the dressed quantities, and the n-function. Hence it is possible

[1]Note that here the 0 subscript does not refer to the unperturbed theory as above, but just as the 0-th approximation of the pseudoenergy

for instance to obtain the values of j_{simul} and hence of j_{norm} at different values of the parameters and of the masses, which can then be used to compare with the analytical expressions.

The comparison between the analytical results and the simulations is straightforward: j_{norm} has to tend to $\mathcal{G}(s)$ for $m \to 0$ in order for our result (4.74) to be true. This clearly has to be valid for all values of α, T_L and T_R. In the case of generalized deformations, the result will depend on the various different parameters which appear in the equation, but the procedure is identical. In the following sections, we show that the analytical results of this work agree very well with simulations performed on three fundamental examples of relativistic theories, namely the free theory, the sinh-Gordon model, and the Lee-Yang model. Extensions to theories with several particle types requires a slightly more intricate procedure, to take into account the coupling between the various dressing equations, but the main procedure is exactly the same.

5.2 Free fermion

The case of the free fermion is the simplest, since as shown in Appendix A we can even use the exact values of $\mathcal{G}(s)$ for the comparison, where:

$$\mathcal{G}(s) = \frac{s2^s}{4\pi}(1-2^{-s})\Gamma(s)\zeta(1+s) \tag{5.5}$$

Figure 5.1 shows the normalized currents compared to the values of $\mathcal{G}(s)$. The convergence to the conformal value is exact and also extremely fast, with slower convergence as the spin is increased. In the plots, the dots represent the simulated values of the currents at the different values of the mass, normalized as in (5.1), and the dashed lines the analytical expected value of (5.5). The variable on the x axis is one which is usually used when studying the approach to conformal limits, namely

$$x = \ln(2/r), \quad m = \frac{2}{\beta}e^{-x} \tag{5.6}$$

This is used to present more clearly the conformal limit as $x \to \infty$. In the simulation, since it is the mass which is varied to reach the conformal limit, the two temperatures and α enter as parameters. However, repeated simulations show clearly that the result in the conformal limit is completely independent by the choice of these three parameters, as expected (since the normalization is precisely performed to remove the temperature and α dependence), although clearly the behaviour for small values of x (and hence large masses) will strongly depend on this choice, since the analytical expressions have no predictive power in this regime [2]

[2] even if the expressions (4.67) were at least approximately valid out of the conformal limit, using arguments such as the fact that the zero of the effective velocity appears not to change only slightly from the correct value, it is the unperturbed results which are strictly dependent on the conformal limit, and hence make the simulated results deviate significantly from the analytical

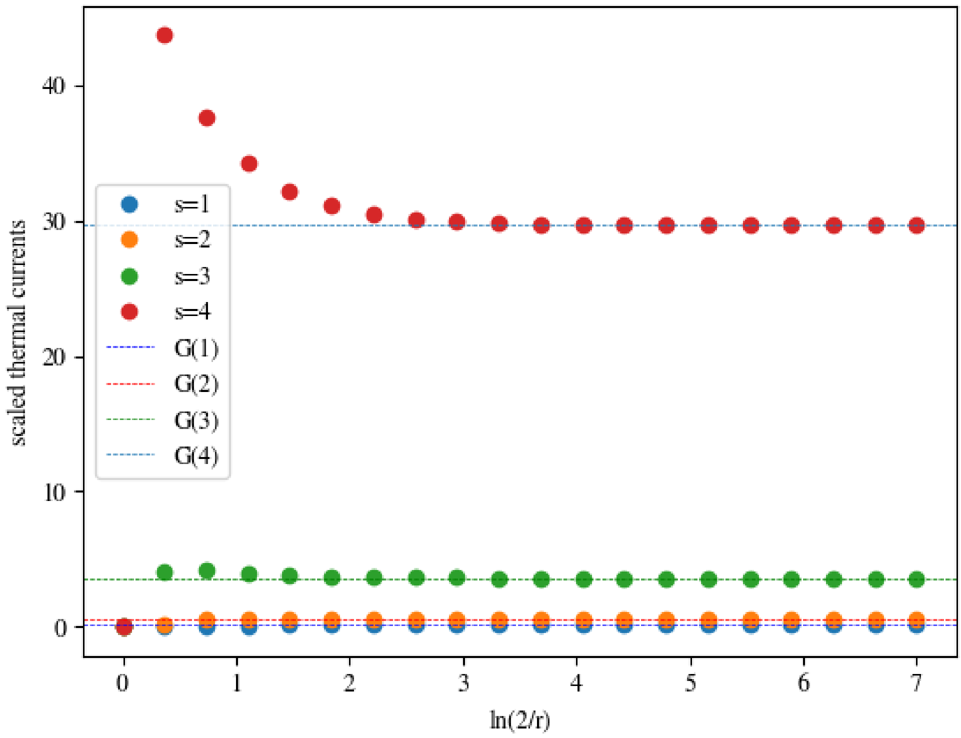


Figure 5.1: The bullet points are the numerical values of the normalized currents, evaluated at four different values of spins. The dashed lines are the values of $\mathcal{G}(s)$ at which the normalized currents are expected to tend in the conformal limit, calculated by (5.5). The plot is done against the variable $x = \ln(2/r)$, and therefore the conformal limit is reached as $x \to \infty$.

5.2.1 Generalized $T\bar{T}$

Similar results are also found for the generalized deformations discussed at the end of the previous chapter. The three discussed currents, namely j_s^s, j_1^s, and j_E are found to be in good agreement with the analytical expressions, in which we have used the values of the constants $\mathcal{G}$ exactly calculated in appendix A. As the spin of the perturbation is increased, the convergence of the algorithms becomes incredibly slow, because of the hyperbolic cosine in the kernel which gives a strongly divergent term. This signals a clear strength of the expressions obtained, since together with giving exact expressions they also allow to greatly simplify the computational effort of numerically studying such models by relating their solution to the solutions of the unperturbed theory. Since the non-conformal situation is problematic because of two divergences of the effective velocity, it is necessary to take extremely low value

prediction. Therefore only expressions (4.75) and (4.74) are problematic, while (4.67) is still approximately true. If we had access to unperturbed expressions out of the conformal limit, we could extend the result with high precision.

of the masses to see the correct results. To obtain this without spoiling convergence the simulations have been performed at small temperatures, $T_L, T_R \approx 10^{-8}$, and hence $\beta \approx 10^8$, such that the mass can be extremely small even if x is of order unity.

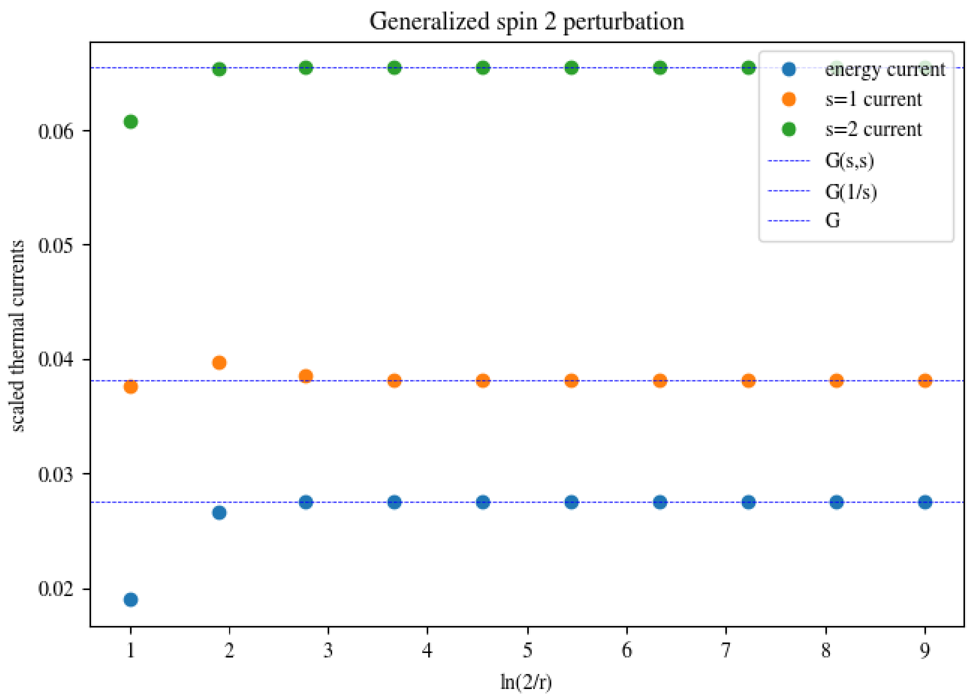


Figure 5.2: In the deep conformal limit, also the results obtained in section 4.6 appear to be verified. The parameters of the simulation are $T_L = 5 \cdot 10^{-8}$, $T_R = 10^{-8}$, $\alpha = 0.01$.

5.3 Interacting theories

5.3.1 Scaling Lee-Yang model

The scalng Lee-Yang model, first studied in [16] as the first example of application of the TBA technology, is built from the Lee-Yang minimal model, a minimal non-unitary conformal model with central charge $c = -22/5$ and effective central charge $c_{eff} = 2/5$, perturbed by the only relevant operator of the theory, of dimensions $(-1/5, -1/5)$. The spectrum of the theory consists of a single particle B, and therefore the S-matrix has only one element, which can be found via the bootstrap approach presented above, and is shown to be [100]:

$$S_{BB}(\theta) = \frac{\sinh\theta + i \sinh \pi/3}{\sinh\theta - i \sinh \pi/3} \tag{5.7}$$

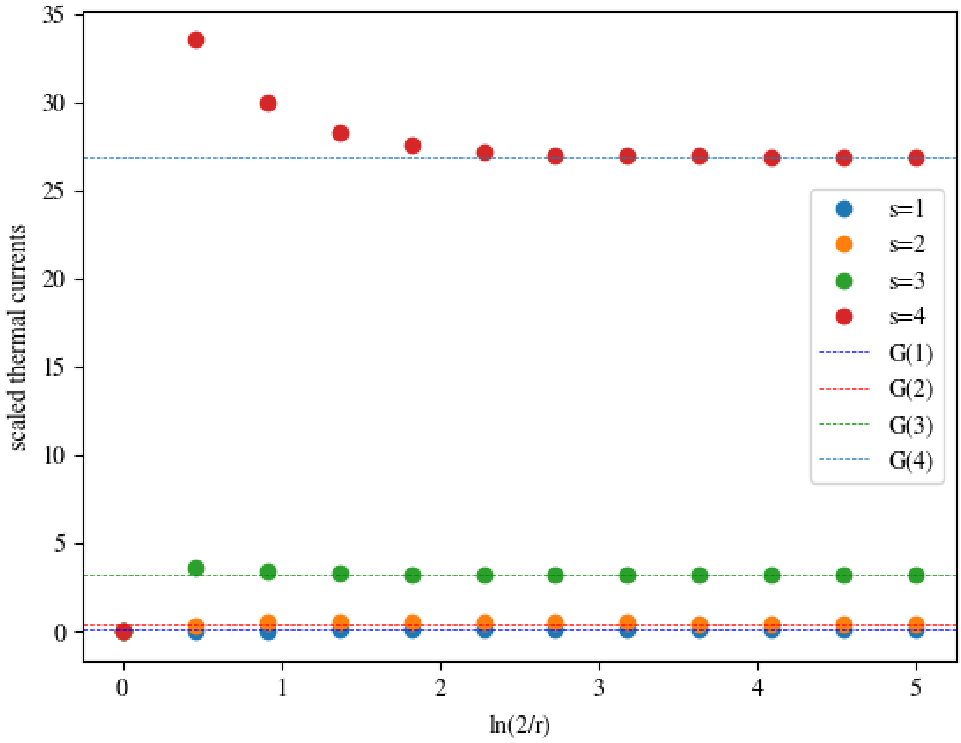


Figure 5.3: Also the comparison with the scaling Lee-Yang theory shows an extremely rapid convergence to the conformal expected values. Therefore the theoretical results appear to be confirmed also in the case of interacting theories. The values of $\mathcal{G}(s)$ are clearly smaller than the free fermion values, because the (effective) central charge is smaller.

Recalling that the kernel appearing in the TBA equations is the logarithmic derivative of the S-matrix, we obtain immediately:

$$\varphi = -\frac{4\sqrt{3}\cosh(\theta)}{1+2\cosh(2\theta)} \tag{5.8}$$

In this situation, the factors $\mathcal{G}(s)$ cannot be computed exactly, so they have to be evaluated by solving numerically the integral in (A.9), which in the conformal limit is equivalent to solve

$$\frac{s2^s}{4\pi}\left(\frac{\beta m}{2}\right)^s \int_{-\infty}^{\infty} \cosh(s\theta)L(\theta) \tag{5.9}$$

The numerical result of this integral can be compared to the simulations[3]. The results are shown in figure 5.3, which shows again an extremely fast convergence

[3]Note that in this situation it is the effective central charge that has to be used in equations (4.20) and (4.74), as is standard in the TBA context: when quantizing a theory on the cylinder, the relevant quantity is $c_{eff} = c - 24\Delta_{min}$, see for example [26].

to the expected values. As should be expected, the values of $\mathcal{G}(s)$ are smaller than in the case of the free fermion, as they depend on the plateau solution of the TBA equations and this in turn determines the central charge, which is smaller for this model than in the free case. This gives a small confirmation of the idea that these objects, as discussed in appendix A, are some sort of generalization of the central charge of the theory, and perhaps could be obtained as functions of the central charge itself.

5.3.2 Sinh-Gordon Model

The Sinh-Gordon model is an integrable model defined by the lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{g^2}(\cosh(g\phi) - 1) \tag{5.10}$$

This can be seen as a CFT of central charge $c = 1$ (the free boson) perturbed by the hyperbolic cosine term, or equivalently as the perturbation of a Liouville CFT by the relevant vertex operator $e^{-\sqrt{2}g\phi}$. In any case, the application of the c-theorem shows that $c_{uv} = 1$, so this is the value we need to use to evaluate $\hat{\beta}$ and the other quantities in the conformal limit. The exact S-matrix for the model is

$$S_{BB} = \frac{\tanh\frac{1}{2}(\theta - i\pi B)}{\tanh\frac{1}{2}(\theta + i\pi B)} \tag{5.11}$$

where the factor B is expressed in terms of the coupling g as $B = \frac{g^2}{4\pi}\frac{1}{1+\frac{g^2}{8\pi}}$. The kernel of the theory is given by

$$\varphi = \frac{2\cosh\theta\sin(\frac{g^2\pi}{g^2+8\pi})}{\cosh^2\theta\sin^2(\frac{g^2\pi}{g^2+8\pi}) + \sinh^2(\theta)\cos^2(\frac{g^2\pi}{g^2+8\pi})} \tag{5.12}$$

We focus on the reflectionless point $g = \sqrt{8\pi}$, for which the kernel greatly simplifies:

$$\varphi = \frac{2}{\cosh\theta} \tag{5.13}$$

The simulation with this model shows a much slower convergence compared to the other two, but still leads to excellent results, as shown in figure 5.4. This is not a suprise, since it is well known that the convergence of GHD algorithms involving the Sinh-Gordon model is particularly slow compared to other models (The convergence issues are actually related to the TBA structure in general, since for example the L-functions do not form plateaus in this situation). Again, we see also that the values of $\mathcal{G}(s)$ in this case are larger than in the two previous situations, and this could be related to the fact that the UV central charge of the model is c=1, and this provides a further hint to the fact that the $\mathcal{G}(s)$ could be functionally dependent on the UV central charge in some way.[4]

[4]However, simply giving a look at the expression of the higher spin charges for the free case shows that this dependence cannot be particularly easy. It is reasonable to imagine that if the $\mathcal{G}(s)$ can be expressed in terms of c this will be a consequence of some special property of polylogarithms of Zeta functions, as it happened in the calculation of [16] of the central charge.

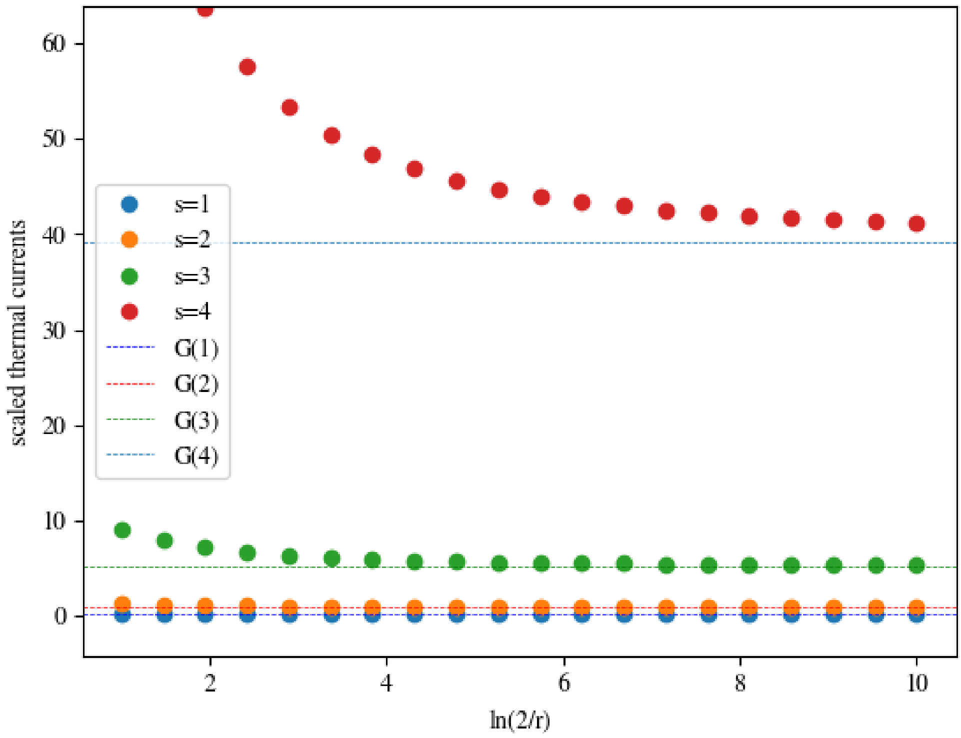


Figure 5.4: In the Sinh-Gordon model the convergence to the conformal value is much slower, but still in the conformal limit all the currents tend to the correct results.

5.4 Out of the Conformal limit

As clear from all the figures of the last section, the results which have been obtained are only valid in the CFT limit. An interesting question to consider is wether this is related to the fact that the discussion performed in this work is strictly valid in that limit, or if it is a fault of the unperturbed currents and densities (4.72). That is, we can ask if, although equations (4.75) and (4.74) clearly appear to be wrong in the conformal limit, the fundamental expression (4.67) is at least approximately valid in a massive theory. This is suggested by a numerical analysis of the effective velocity, which shows that the θ^* of the perturbed theory is still very close to that of the unperturbed theory even for massive theories, and therefore all the discussion of chapter 4 should apply. Since we do not know the analytical expression of the unperturbed quantities for massive theories, we need to simulate them. The approach is quite simple: it is necessary to simulate the unperturbed currents and densities (at inverse temperature $\hat{\beta}$ which is found numerically using the definition $\hat{\beta} = \beta - \alpha E_0^\alpha$ since the analytical expression is not valid out of the conformal limit), substitute the simulated values in (4.67), and compare the result with the direct

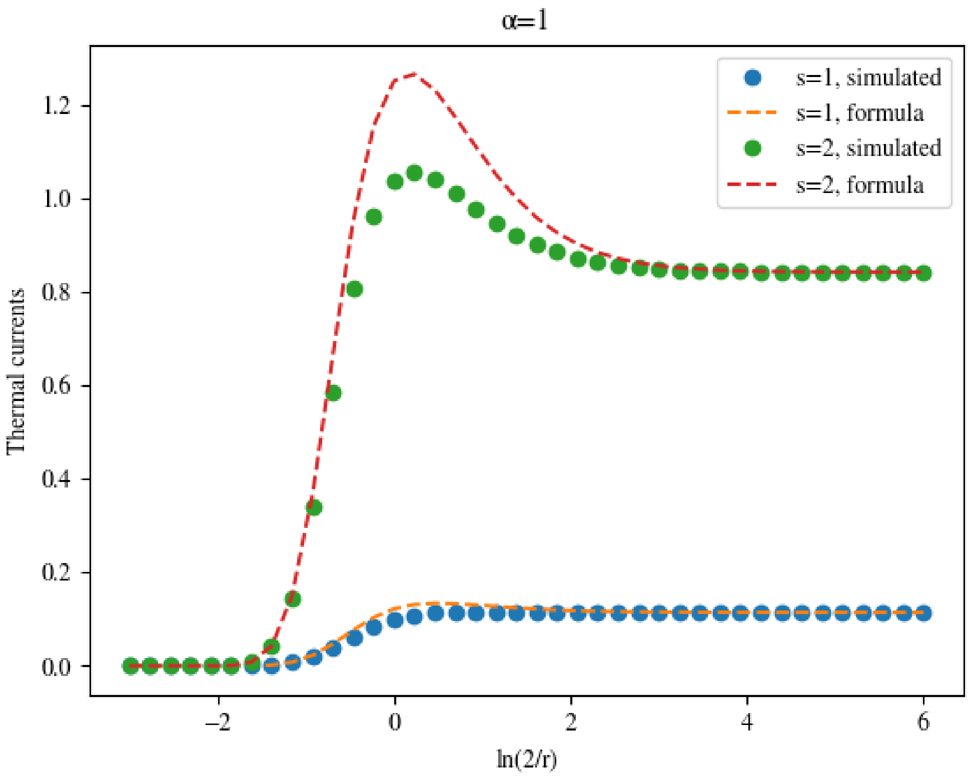


Figure 5.5: The numerical values compared to the dashed lines calculated from (4.67) using the values of the unperturbed theory. It is clear that the results coincide not only in the conformal limit, as it should be, but also in the large mass limit. The plot uses generic values of temperatures of the two halves, and its shape is largely independent on them. (the value of r in the plot is actually r_L, since it is different for the two sides). As in the simulations of the previous section, also in this situation the error becomes increasingly large as the spin is increased

simulation of the perturbed theory, evaluated at the real temperature β. In this way we can test the validity of (4.67) without having to take in consideration the validity of expressions like (4.72). The simulations show clearly that the range of validity of the expression is not only the conformal limit, but also in the large mass limit, as can be seen from figure 5.5. This was expected by considerations on the modification of the effective velocity, and is valid for all three of the models studied in the previous section. The deviation between the formula and the real value of the current is mostly visibe in the intermediate region $m \approx \alpha$, namely outside of the two interesting limits. By increasing the value of α while keeping all other parameters fixed, the discrepance keeps growing in this intermediate region of the mass values, as clear by the comparison of figures 5.5 and 5.6. However the results show clearly that expression (4.67) should be at least perturbatively valid when α is small: by introducing higher corrections as in 4.5.3 one could have results of higher precision.

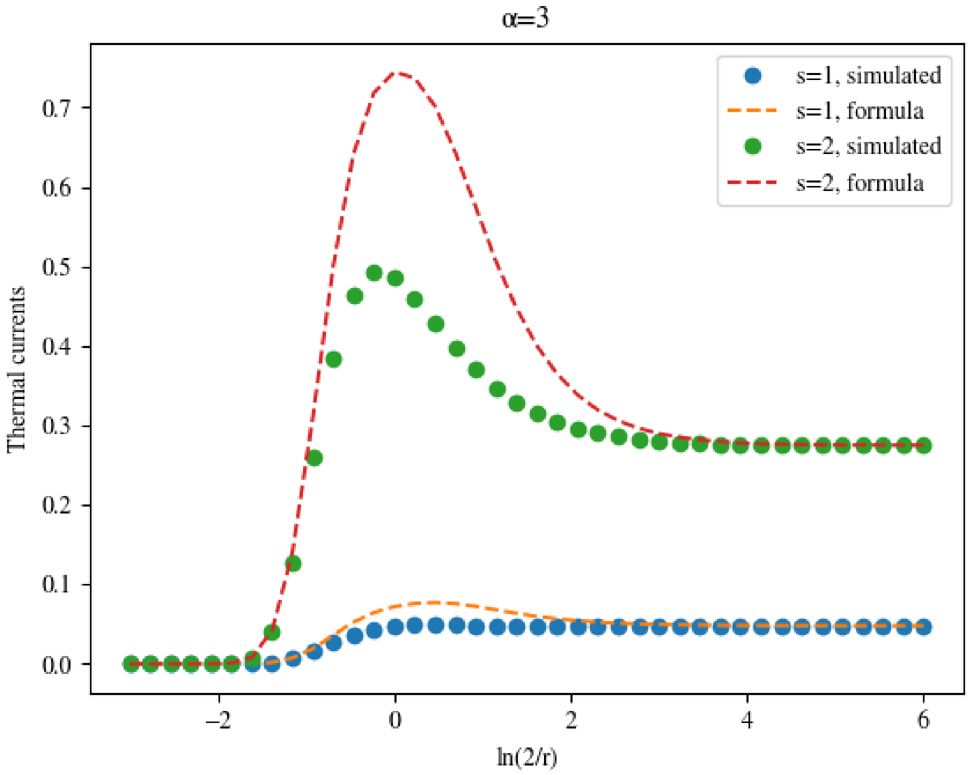


Figure 5.6: By increasing the value of alpha, while keeping the temperatures fixed, the disagreement between the predicted and simulated value grows significantly larger. However the validity of the results in the $m \to \infty$ and $m \to 0$ is clear for any value of α.

The most interesting check that can be performed out of the conformal pointis a test of the equilibrium expression (4.53), which we claimed is valid for any value of the temperature. Indeed, at equilibrium the effect of the perturbation is (as already found by other authors) exactly identical to the modification $\beta \to \hat{\beta}$, with no further change. This claim is perfectly confirmed by figure 5.7, which shows the first three values of spin for the free fermion. Simulations for the Lee Yang scaling model and for Sinh-Gordon model lead to analogously perfect result. Therefore we see that the approach of this work allows to obtain exact results out of the conformal point in the case of equilibrium charges (and clearly also of equilibrium odd-spin currents, which do not vanish at equilibrium).

5.5 Width of fundamental particles?

Once the algorithm discussed above to evaluate the basic TBA quantities is set up, it is then easy to study how the statistical quantities of the system vary as a function of α. For example one can study the partition function, the free energy, the

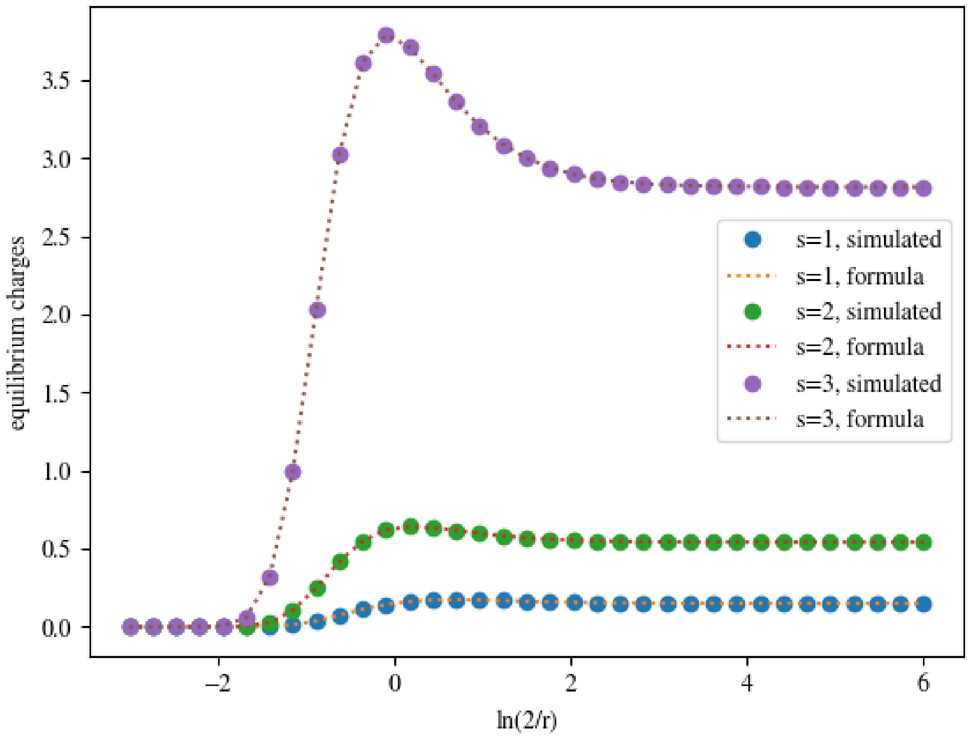


Figure 5.7: As expected, the charges at equilibrium given by formula (4.53) are valid for every value of the mass, as exactly confirmed by the simulations. Here the values of α and β are chosen freely, but no essential difference is observed varying them.

number of quasiparticles, and all of the relevant quantities which were introduced in chapter 2. In this section, to conclude the thesis, we focus on an interesting aspect related to the TBA density of quasiparticles, expressed as ρ_p in the discussion above, and the total number of these quasiparticles (per unit length) which is simply the integral of the density over all rapidity space. This will allow us to give a numerical test of an idea first appeared in [94], namely that the effect of the $T\bar{T}$ perturbation is to give a finite width to the fundamental particle which constitute the system. The considerations of these sections are just meant to give an intuition, a visual explanation of the effect that the perturbation has on the theory.

As shown in figures 5.8 and 5.9, except for a slight antisymmetric deformation, the main visible effect of the perturbation on the particle density is to reduce it drastically as α is increased. The double peak structure of the solution is characteristic of the small r situation: increasing the mass (or the inverse temperature) the two peaks merge and form a maximum in the middle. This is simply related to the fact that the peaks are located in $\theta = \pm \ln(2/r)$, by considerations which mimic those used to discuss the constant plateau TBA. Hence for $r \gtrsim 2$ this would become

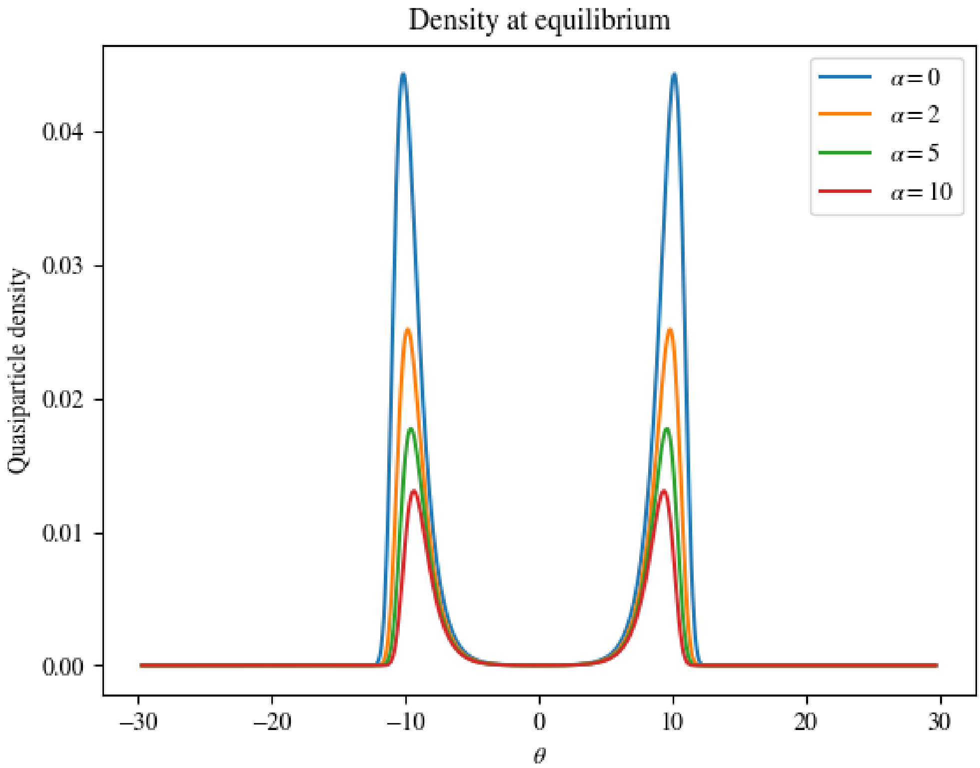


Figure 5.8: Values of ρ_p at equilibrium for different values of α.The symmetry of the system is preserved by the deformation, which only appears to have the effect of reducing the overall density. For different values of the simulation parameters another effect might show, namely a merging of the peaks as a consequence of the increase of the effective inverse temperature. This second effect dominates if the peaks are close enough in the beginning, namely if $m \sim \alpha$.

negative, meaning that the peaks merge in $\theta = 0$ [5]. Since the net effect of the $T\bar{T}$ perturbation is a net increase of the inverse temperature as β is substituted by $\hat{\beta}$, this could lead to an effect of this type, namely a merging of the peaks. However, in general this effect appears clearly much less visible than the drastic reduction of the height of the peaks. This is indeed what happens in figure 5.8: the peaks slightly move towards the origin, but this effect is overshadowed by the reduction of the density of states. The behaviour of the density also implies that the total number of quasiparticles of the system will have a strong α dependence. There is a clear physical intuition behind this fact: following [94], let us suppose that each fundamental particle has a width equal to some parameter l_0, which we assume to have a functional dependence on α, monotonically increasing with the parameter.

[5]Note that this behaviour is the opposite of that of the n-function, in which the peaks merge in the conformal limit and are well separated when the mass is increased, as was used to solve the partitioning protocol.

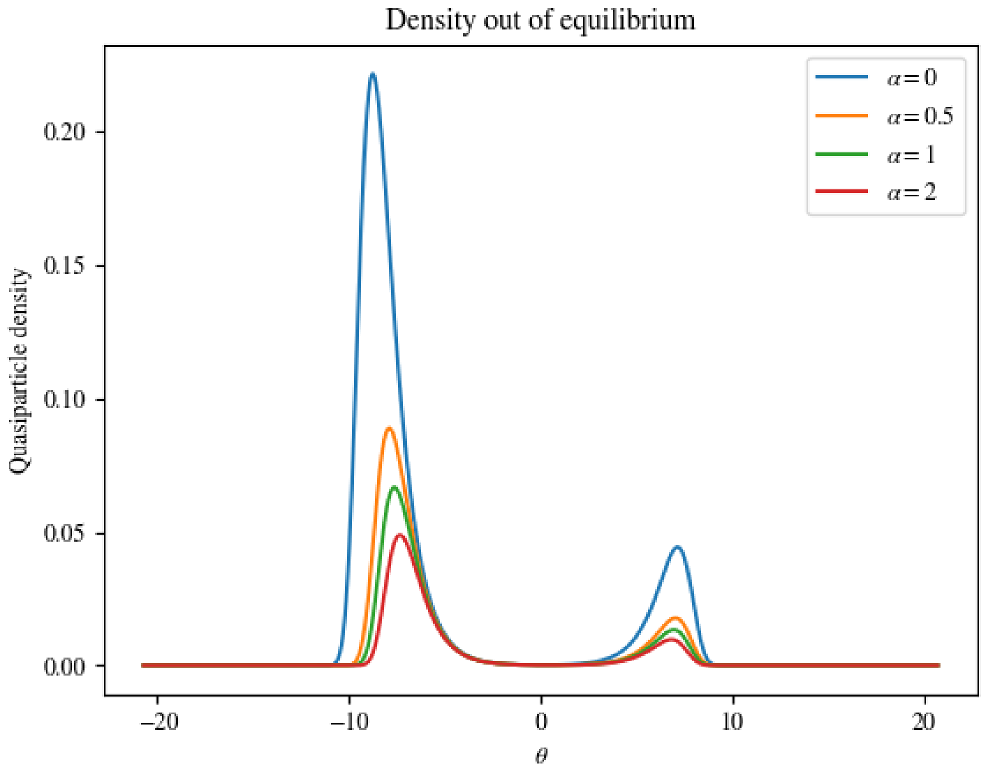


Figure 5.9: In the non-equilibrium (partitioning protocol) situation, the densities appear not only to be reduced, but also to undergo a slight antisymmetric deformation, which is induced by the current factor in expression (4.10). Also in this case an interesting and rich phenomenology can be observed by studying various configurations of the parameters.

Then, except for an eventual transient region when α is small (where the quantification of this depends on the situation), we will have a saturation of the number of particles per unit length, due to the fact that the particles themselves occupy an increasing amount of space:

$$\frac{N}{L} \propto l_0(\alpha) \tag{5.14}$$

Therefore a study of $\frac{N}{L}$ as a function of α will allow to test this prediction of the model.We can use dimensional considerations to make educated guesses on the expression for $l_0(\alpha)$. Since the only dimensional quantities of the system are m, α and β, a combination with dimension $[M]^{-1}$ will have to be constructed through their combinations. Since α has the units of $[M]^{-2}$, the most natural possibility is something like $l_0(\alpha) = m\alpha$, and in fact this is the one which was found in [94] for relativistic massive particles. This is indeed confirmed by the plot of figure 5.10, which shows a saturation of $N\alpha$ for values of α large enough. Clearly, increasing the value of m this effect becomes visible for lower values of α, while for m of order unity the simulation overflows before this effect becomes relevant.

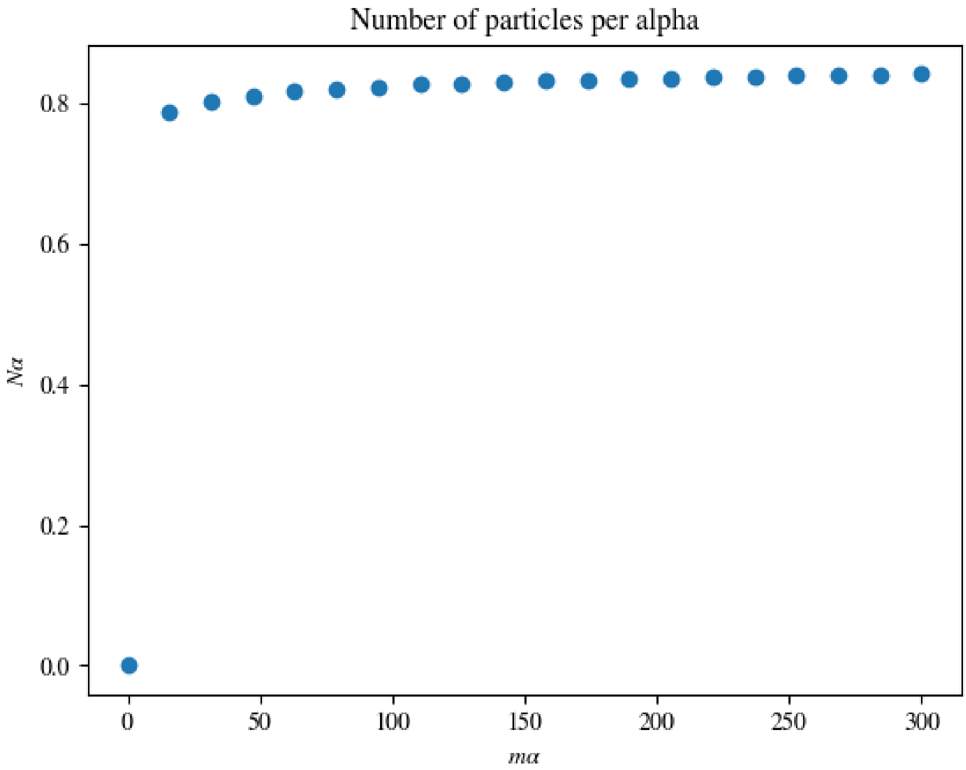


Figure 5.10: In the massive case the result proposed in [94] is conformed by the simulations: since the value of $N\alpha$ saturates for α large enough, we infer that the particles acquire a width $l_0 = m\alpha$. In order to amplify the visibility of the saturation, the simulation is performed with m=40.

In the conformal limit this is clearly not the case (since m=0), so we might have two possibilities: either the particles don't acquire a length or the length will not involve the mass, something as $l_0(\alpha) = \sqrt{\alpha}$, which dimensionally is still acceptable. This is indeed what the simulations predict when computing the total number of particles, as shown in 5.11. This plot, in which the mass parameter was taken as $m = 10^{-8} \approx 0$ (not zero because the algorithms would fail) show that after a small transient until $\alpha \approx 10$ the value of $N\sqrt{\alpha}$ saturates to a fixed value k, showing that we can interpret the length of the particles as $l_0 = k\sqrt{\alpha}$, as suggested by the dimensional considerations. This hints at the possibility of extending the theoretical description of [94] in order to extend their result to the massive case, in order to explain this difference in the behaviour.

In conclusion, numerical simulations provide very robust confirmation of the predictions of this work. To explore the behaviour out of the conformal point, this approach is the only possible way to follow, since the analytical solutions are inadequate and a perturbative solution is unfeasable in practice. An interesting line

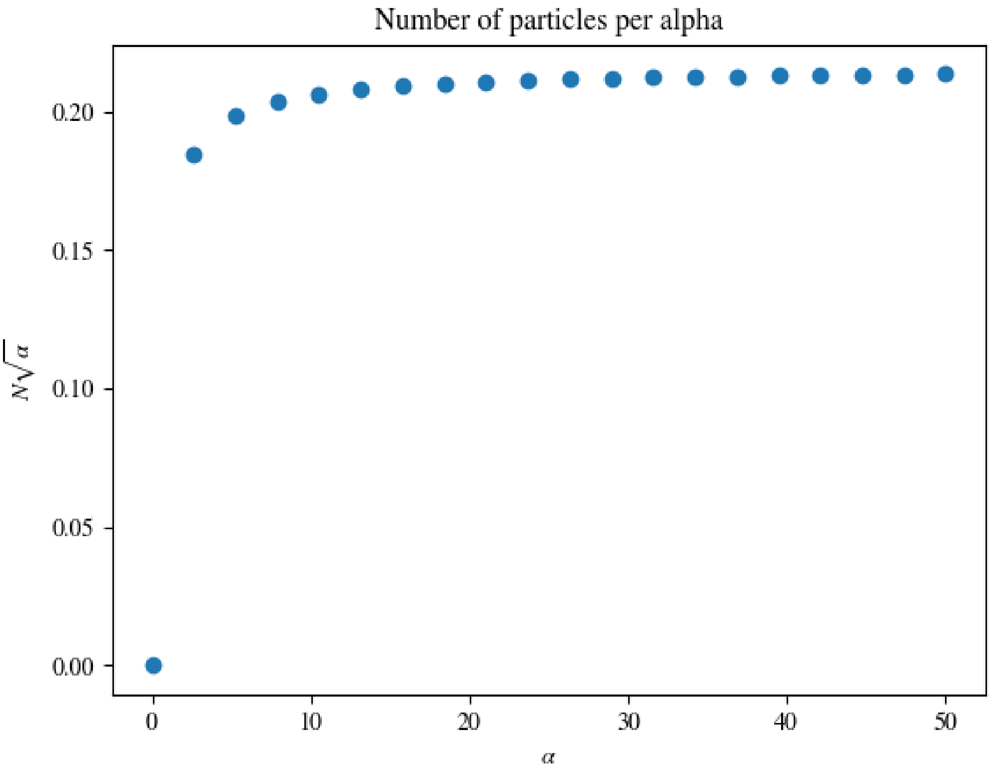


Figure 5.11: Number of particles per unit lenght as a function of the parameter α, multiplied by $\sqrt{\alpha}$, in the case $m \approx 0$. The plot clearly shows that N$\propto \frac{1}{\sqrt{\alpha}}$, suggesting a lenght of the particles given by $l_0 \propto \sqrt{\alpha}$

of research which we will pursue in the future is precisely the numerical study of the generalized $T\bar{T}$ deformations, which out of the conformal limit exhibits novel and strange behaviours, such as the divergence of the effective velocity, which completely spoils the possibility of finding the n-function in the standard manner.

Conclusion and Outlook

In this work we have discussed the out of equilibrium dynamics of Quantum Field Theories with $T\bar{T}$ deformations in the framework of Generalized Hydrodynamics. We have done this by building a general theoretical approach which, by considering a perturbed massive theory and successively taking the massless limit, allows us to obtain exact solutions to the Riemann problem of Hydrodynamics in the conformal limit. Thanks to the particular factorization of the TBA equations, exact analytical expressions for the average higher spin currents and densities have been obtained, with excellent numerical validation. These expressions both generalize and reproduce the results of [24]. The achieved results show that it is possible to encode the effect of the perturbation into expressions of unperturbed quantities, with a dependence on the perturbing parameter α which is purely algebric. This leads to a great simplification of the problem, both from the conceptual and numerical viewpoint, since simulations involving perturbed theories show significantly slower convergence rates, and expressions such as (4.67) can be used to reduce drastically the duration of numerical simulations.

Although it is a common feature that the out of equilibrium dynamics of Quantum Field Theories is only exactly solvable in the CFT limit, a thorough perturbative analysis has been performed in order to go beyond the massless case for the simplest possible theory on which the $T\bar{T}$ deformation can act, namely the free theory. With the help of special functions, we found interesting expressions for the lowest order corrections at $m \sim 0$. We have conjectured that extensions to higher orders (and eventually some exact results) could be obtained by introducing a new class of generalized Lambert functions, of which we outlined the main features, although a precise mathematical definition and characterization of such functions constitutes a mathematical challenge which is far beyond the scope of this thesis.

All the results obtained have also been generalized to find similar expressions which are valid in higher spin $T\bar{T}_s$ deformed theories, in which the term appearing in the TBA equations is a generic $\cosh(s\theta)$. In particular, this configuration exhibits an even nicer behaviour compared to the standard deformation, since the effective velocity of the perturbed theory tends exactly to the unperturbed effective velocity in the conformal limit. Out of the conformal limit, however, the study of the generalized deformations leaves a number of essential questions open. In particular, the behaviour of the effective velocity exhibits some completely novel and unexpected features: for example, it presents divergences at several points where the dressed energy is vanishing, and this implies a failure of the standard GHD solution of the partitioning protocol in terms of contact discontinuities; also, it implies that the dressed energy becomes negative, and this could lead to a negative energy density, currents flowing in the "wrong" direction (from the cold slab to the hot one), and other striking and unexpected phenomena. It is yet unclear whether these solutions have to be discarded as unphysical, or have some deep physical meaning, or if they are actually even solutions, since the non-analyticity of the effective velocity could imply the need for a completely different solution of the Riemann Problem. This study will mainly have to be performed numerically, using the techniques described in the last chapter, because of the difficulties inherent in the purely theoretical approach.

Therefore it is clear that together with answering many interesting questions regarding the structure of IQFTs with irrelevant deformations, this thesis also leaves several important problems open. In particular, the main prospected lines of further research regard the study of the strange behaviour of the effective velocity which has been observed and discussed, and the related peculiar behaviours arising in the $T\bar{T}_s$ deformations, which would imply a rethinking of the approach to solve the partitioning protocol, and could shed light on the serious issues mentioned above. Moreover, a complete study of the scaling function and hence of $\hat{\beta}$ for the free fermion can be in principle performed,along the lines of the famous discussion of [56], if generalizations of the Lambert functions can be defined. Although this is work for mathematicians, it should be possible to at least define more precisely what the properties of these new special functions should be. Finally, another direction which has been left completely untouched is the study of more complicated hydrodynamical quantities, such as correlation functions, or the hydrodynamic matrices, to test if also in this case it would be possible to obtain them as functions of the unperturbed quantities as was found above. This would imply a complete decoupling of the perturbation in any Hydrodynamical problem, and could have profound implications on a complete characterization of the effect of the $T\bar{T}$ deformation on Integrable Quantum Field Theories.

Appendices

Appendix A

Higher currents in CFT

We now prove the aforementioned expression for the NESS currents and densities in pure CFT, following the discussion of [95], with some original additions. We will assume that the effective velocity is a monotonic function of the rapidity, which is the case in all the situations of interest for this work. We consider TBA systems in GGE which take the form:

$$\varepsilon_s(\theta) = \beta^s \cosh(s\theta) - \varphi * L_s(\theta) \tag{A.1}$$

We normalize the mass to 1, and look for expressions for the densities and currents q_s and j_s related to the one-particle eigenvalues $h_s = \cosh s\theta$. The idea of the calculation follows Zamolodchikov's approach to the study of the CFT limit of the TBA equations [16]. The densities are evaluated as:

$$\begin{aligned}
q_s &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} n(\theta) h_s^{dr}(\theta) \\
&= \int_{-\infty}^{\theta^*} \frac{d\theta}{2\pi} \cosh(\theta) n_R(\theta) h_s^{dr}(\theta) + \int_{\theta^*}^{\infty} \frac{d\theta}{2\pi} \cosh(\theta) n_L(\theta) h_s^{dr}(\theta) \\
&= \int_{-\infty}^{\theta^*+x_R} \frac{d\theta}{2\pi} \cosh(\theta - x_R) n_R^-(\theta) h_s^{dr}(\theta - x_R) \\
&+ \int_{\theta^*}^{\infty} \frac{d\theta}{2\pi} \cosh(\theta + x_L) n_L^+(\theta) h_s^{dr}(\theta + x_L)
\end{aligned}$$

where $x = \ln(2/\beta)$. Now, in the conformal limit, as well known, the solutions to the TBA equations form plateaus centered in $\theta = 0$. For the inverse temperature going to zero we can approximate the hyperbolic cosines with exponentials, with $\pm\theta$ for the right and left movers respectively. We obtain two TBA equations for the right and left movers, and therefore two dressing equations, namely:

$$\varepsilon_s^\pm(\theta) = 2^{s-1} e^{\pm s\theta} - \varphi * L_s^\pm(\theta) \tag{A.2}$$

From which we get to the following equations:

$$\begin{aligned}
\varepsilon_s'^{\pm}(\theta) &= \pm s 2^{s-1} e^{\pm s\theta} + \varphi * n_s^{\pm}\varepsilon_s'^{\pm}(\theta) &\text{(A.3)}\\
h_s^{dr\pm} &= h_s^{\pm} + \varphi * n_s^{\pm}\varepsilon_s'^{\pm}(\theta) &\text{(A.4)}\\
h_s^{dr\pm} &= \pm\frac{1}{s\beta^s}\varepsilon_s'^{\pm} &\text{(A.5)}
\end{aligned}$$

and so we obtain:

$$q_s = \frac{2^{s-1}}{2\pi\beta_R^{s+1}}\int_{-\infty}^{\theta^*+x_R} e^{-\theta}n_R^-(\theta)(e^{-s\theta})^{dr} + \frac{2^{s-1}}{2\pi\beta_L^{s+1}}\int_{\theta^*-x_L}^{\infty} e^{\theta}n_L^+(\theta)(e^{s\theta})^{dr} \tag{A.6}$$

We now assume that $e^{k\theta}$ and $e^{-k\theta}$ are dressed using $n_R(\theta)$ and $n_L(\theta)$ respectively, in the conformal limit. Under this assumption the above interals can be solved thanks to (A.5), and the final result after integration by parts is:

$$q_s = \mathcal{G}(s)\left(\frac{1}{\beta_L^{s+1}} + \frac{1}{\beta_R^{s+1}}\right) \tag{A.7}$$

and similarly for the currents:

$$j_s = \mathcal{G}(s)\left(\frac{1}{\beta_L^{s+1}} - \frac{1}{\beta_R^{s+1}}\right) \tag{A.8}$$

where the unknown prefactor in front is defined by:

$$\mathcal{G}(s) = \frac{s2^s}{4\pi}\int_{-\infty}^{x} L^-(\theta)e^{-s\theta} \tag{A.9}$$

where x has to be taken in the limit going to infinity. Clearly, for s=1 this corresponds to a well known integral which has to be solved in the TBA context to find the central charge in the UV limit, expression (2.21).

However, for higher spins there appears to be no way to solve the integral exactly, except for the free theory. It appears natural that, since for s=1 the result is expressed in terms of the dilogarithmic function, the result for higher spin could be in a similar way be expressed in terms of higher order polylogarithms. This is indeed the case for the free theory, as can be easily computed, using the fact that, since $e^{\theta} = \varepsilon$ in the absence of the kernel, then $e^{k\theta} = \varepsilon^k$, but also $e^{\theta} = \varepsilon'$. Hence, neglecting the $\pm$ superscript for simplicity:

$$\begin{aligned}
\mathcal{G}(s) &\propto \int_0^{\infty} L(\theta)e^{s\theta}d\theta = \int_0^{\infty} L(\theta)\varepsilon^{s-1}(\theta)\varepsilon'(\theta)d\theta &\text{(A.10)}\\
&= \int_{\epsilon_0}^{\infty} L(\varepsilon)\varepsilon^{s-1}d\varepsilon = \int_{\epsilon_0}^{\infty} \ln(1+e^{-\varepsilon})\varepsilon^{s-1}d\varepsilon &\text{(A.11)}
\end{aligned}$$

This integral can be solved exactly, through this original expression:

$$\int \ln(1+e^{-x})x^{s-1}dx = \sum_{p=2}^{s+1}\frac{(s-1)!}{(s+1-p)!}Li_p(-e^{-x})x^{s+1-p} \tag{A.12}$$

where $Li_p(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^p}$ is the p-th order polylogarithm. This however is just a formal calculation, since in the free case we have $\epsilon_0 = 0$, and therefore the final expression that one obtains is:

$$\mathcal{G}(s) = \frac{s2^s}{4\pi}(1-2^{-s})\Gamma(s)\zeta(1+s) \tag{A.13}$$

which indeed corresponds to the result obtained in (4.62), as can be seen after substituting $s! \to s\Gamma(s)$. Therefore, the expression in terms of polylogarithms is purely formal, since essentially all terms are canceled by the fact that $\varepsilon_0 = 0$. However, this shows the plausibility of the idea that the prefactor can be expressed in terms of polylogarithms as a direct generalization of the standard s=1 case, even in interacting theories. A definitive proof of this, however, is still beyond grasp.

A similar discussion can be applied to the "generalized" currents, defined by an expression of the form:

$$\tilde{q}_s^k = \int \frac{d\theta}{2\pi} \cosh(\theta) n_s(\theta) \cosh^{dr}(k\theta) \tag{A.14}$$

where n_s is built using a driving term which contains only the $\cosh(s\theta)$ term. These currents arise naturally when studying the $T\bar{T}_s$ deformed theories with s>1. By performing a shift of variables $\theta \to \theta/s$ it is easy to see that in light of the previous discussion the temperature dependence will turn out to be independent on s:

$$\tilde{q}_s^k = \frac{1}{2\pi s} \int d\theta \cosh\left(\frac{\theta}{s}\right) n_1(\theta) \cosh^{dr}\left(\frac{k}{s}\theta\right) \tag{A.15}$$

where the only influence of the spin s remaining in the n-function is in the temperature, which is substituted by β^s. Now lets consider the case of a free fermion in a partitioning protocol: this integral immediately becomes:

$$\tilde{q}_s^k = \frac{1}{2\pi s} \int_0^{\infty} d\theta \cosh\left(\frac{\theta}{s}\right) n_L(\theta) \cosh\left(\frac{k}{s}\theta\right) + \frac{1}{2\pi s} \int_{-\infty}^{0} d\theta \cosh\left(\frac{\theta}{s}\right) n_R(\theta) \cosh\left(\frac{k}{s}\theta\right)$$

which can be immediately evaluated in terms of the familiar modified Bessel functions, observing that the second integral can be moved from 0 to ∞ by parity. Focusing on the left component:

$$\begin{aligned}
(\tilde{q}_s^k)_L &= \frac{1}{2\pi s} \sum_n (-1)^{n+1} \int_0^{\infty} \cosh(\theta/s) \cosh(k\theta/s) \exp(-n\beta^s \cosh(\theta)) \\
&= \frac{1}{4\pi s} \sum_n (-1)^n \left[K_{\frac{1+k}{s}}(\beta^s(n+1)) + K_{\frac{1-k}{s}}(\beta^s(n+1)) \right]
\end{aligned} \tag{A.16}$$

In the conformal limit, using the usual small argument expansion of k, and reintroducing the right part which is calculated analogously, we obtain:

$$\tilde{q}_s^k = \frac{2^{\frac{1+k}{s}}}{8\pi s} \Gamma\left(\frac{1+k}{s}\right) \eta\left(\frac{1+k}{s}\right) \left(T_L^{1+k} + T_R^{1+k}\right) \tag{A.17}$$

where Γ and η are the Euler and Dirichlet functions respectively. The main feature of this solution is the interesting temperature dependence, which has no information of the value of the perturbing spin s. For k=1, we get the values which were necessary in the study of the energy current in the generalized context:

$$\tilde{q}_s^1 = \frac{2^{\frac{2}{s}}}{8\pi s}\Gamma\left(\frac{2}{s}\right)\eta\left(\frac{2}{s}\right)\left(T_L^2 + T_R^2\right) \tag{A.18}$$

which confirms the T^2 dependence used in the main text. For the currents the discussion is similar and leads to the same result except for the difference between the temperatures:

$$\tilde{j}_s^1 = \frac{2^{\frac{2}{s}}}{8\pi s}\Gamma\left(\frac{2}{s}\right)\eta\left(\frac{2}{s}\right)\left(T_L^2 - T_R^2\right) \tag{A.19}$$

and this confirms the claimed value for $\mathcal{G}_{12}$ used above, since $\eta(1) = \frac{\ln 2}{2\pi}$ and $\Gamma(1) = 1$. Note that for s=1 this reduces to the expression found above (A.13), where the s used above corresponds to the k in this context.

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