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Moduli Stabilization for the Axiverse

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Abstract

This Master's thesis focuses on the study of phenomenological and cosmological applications of string theory. The low-energy limit of string compactifications has been argued to give rise to a so-called "axiverse," which comprises a plethora of axion-like particles with a mass spectrum that is logarithmically distributed over a wide range of mass scales. However, this scenario is currently unsupported by any explicit computations based on moduli stabilization. In this thesis, we aim to fill in this gap by analyzing the mass spectrum of type IIB Calabi-Yau orientifold compactifications with a large number of moduli and axions. We will focus on the regime where α'-corrections are under control, which implies a very large overall volume. Moreover, we will consider various types of perturbative and higher derivative corrections to the four-dimensional effective action to achieve full moduli stabilization analytically, considering two types of Calabi-Yau manifolds: Swiss-Cheese and Fibred Calabi-Yau threefolds.

The final goal is to determine the axionic mass spectrum while taking into account constraints imposed by consistency with standard cosmology. Consequently, we can explicitly investigate the conditions under which an actual axiverse exists, instead of a plethora of effective massless axions. Furthermore, the existence of the axiverse may give rise to several potential observable effects, which are governed by the mass spectrum. By identifying the spectra, we can establish a concrete connection between cosmological observations and the topology of Calabi-Yau manifolds.

Sommario

Questa tesi di laurea si concentra sullo studio delle implicazioni fenomenologiche e cosmologiche della teoria delle stringhe. Il limite a bassa energia della compattificazione della teoria delle stringhe da origine a un cosiddetto "axiverse", ovvero una moltitudine di particelle dette assioni, con uno spettro di massa distribuito logaritmicamente su una vasta gamma di scale. Tuttavia, attualmente questo modello non è supportato da calcoli espliciti basati sulla stabilizzazione dei moduli. In questa tesi, miriamo a colmare questa lacuna analizzando lo spettro di massa delle compattificazioni della teoria di stringa di tipo IIB con un grande numero di moduli e assioni. Ci concentreremo sul regime in cui le correzioni α' sono sotto controllo, il che implica un volume della varietà di Calabi-Yau considerata molto grande. Inoltre, considereremo vari tipi di correzioni, perturbative e alto-derivative, all'azione efficace quadridimensionale per ottenere una stabilizzazione di tutti i moduli in modo analitico, considerando due tipi di varietà di Calabi-Yau: Swiss-Cheese e Fibred Calabi-Yau threefolds.

L'obiettivo finale è determinare lo spettro di massa degli assioni, imponendo che questi modelli siano coerenti con quello cosmologico standard. Di conseguenza, possiamo verificare esplicitamente in quali condizioni esiste effettivamente un axiverse, invece di una moltitudine assioni a massa nulla. Inoltre, l'esistenza dell'axiverse può dare luogo a diversi effetti osservabili, governati dallo spettro di massa. Identificando gli spettri, possiamo stabilire una connessione concreta tra le osservazioni cosmologiche e la topologia delle varietà di Calabi-Yau.

Ringraziamenti

"[...] Esistono poi i casi davvero rari e speciali: le teorie conformi fortemente interagenti. Anche dopo aver raggiunto l' Estremo Infrarosso loro si portano assieme qualcosa, un settore di particelle con cui continuano a interagire, fortemente, e per sempre. Rimangono eternamente attaccate, fortemente, a qualcosa o a qualcuno. Un settore spesso troppo difficile da spiegare con formule matematiche, o con parole. Sto per scrivere una cosa che forse non sarà capita da tutti, ma non importa. Mi trovavo l'altro giorno a riflettere sul fatto che sarebbe davvero bello riuscire a vivere una vita conforme fortemente interagente."

Grazie a chi rende la mia una vita conforme fortemente interagente.

Contents

Chapter 1

Introduction

String theory is one of the most promising attempts to formulate a quantum theory for gravity and space-time within current theoretical physics. As a physical hypothesis, it needs to be connected with the experiments. Due to the intrinsic difficulties in reaching the scales required to directly observe the key ingredients of string theory (such as strings, branes, and extra dimensions), which are far beyond the current capabilities of particle physics experiments, we must derive constraints from low-energy (relative to the Planck scale) phenomena. Indeed, as unified theory of matter and gravitational physics, string theory must be compatible with both particle physics from our accelerators and from cosmological observations. In particular, we can expect to obtain much new data in the upcoming years from new gravitational interferometers (such as LISA and ET).
Despite all this, obtaining physical information from string theory is subtle. Model building faces several issues:

(*i*) the fundamental theory (M-theory) is far from being fully understood. Only some versions of the theory are well-controlled, in different regimes and approximations (superstring theories);

(*ii*) starting from whatever the fundamental theory may be, several choices (e.g. the version of superstring theory, which Calabi-Yau, the orientifold, etc.) are necessary to recover a low-energy effective theory that reproduces our world (e.g. the SM) or a variation of it compatible with the current data. Each of these choices introduces a certain degree of arbitrariness in the final expectations;

(*iii*) many of the explicit constructions currently known (e.g. KKLT scenario, LVS, uplifting mechanism, etc.) are toy models which are interesting in the perspective of understanding some general features but have not yet accomplished realistic physical model building due to technical challenges.

This work will focus on one general ingredient of string model building, common to every specific realization: the *moduli*. Moduli are complex scalar fields that arise from

the dimensional reduction of the 10-dimensional superstring theory to an effective 4-dimensional effective supergravity theory. They encode geometrical information about the extra-dimensions, such as volume or shape. From this fact, we can immediately understand the origin of the *moduli stabilization* problem: at first glance, the theory predicts that these moduli are massless. This means that we can't determine the size of the extra-dimensions. On the other hand, as scalar particle, they would gravitationally couple with the rest of the matter, mediating unobserved long-range interactions. This, along with the impact that these massless moduli would have on the cosmological evolution of the early universe (e.g. inflation, baryogenesis) constraints the moduli to have masses $\gtrsim$ 30 TeV. Therefore, both theoretical and experimental perspectives highlight the necessity to stabilize the moduli by giving them a mass.
Different kind of moduli come from different sectors of the theory. Consequently, they can play different roles and acquire masses through various mechanisms. For example, the so-called *complex structure moduli* acquire a mass already at the semi-classical level. Things are more involved for the *Kähler moduli*. They play a crucial role because they enter in the evaluation of the volume of the extra-dimensions. Their stabilization can be achieved by considering quantum corrections to the classical theory. Generically, quantum corrections can be perturbative or non-perturbative and affect differently the two building blocks of an $N = 1$ supergravity theory: the *Kähler potential* K and the *superpotential* W.

The second Chapter is dedicated to some preliminaries. Both global and local supersymmetry are necessary as natural endpoints of string compactification. The dimensional reduction is discussed, necessary to deal with the predicted extra-dimensions and how they give rise to scalar fields in the final $4d$ theory.

In the third Chapter, string theory is explored. There is a brief presentation of the bosonic version of the theory to develop familiarity with the quantization of relativistic extended objects. After that, supersymmetry is included, resulting in the superstring theories. A part from discussing some general features (e. g. the critical dimension), the main focus is on the type IIB string theory, because of its connection with phenomenology. Thus, its spacetime $10d$ effective action is presented.

The fourth Chapter provides some bit of standard cosmology, including both evidence and open issues. These problems can be partially solved by inflation, which leads to new challenges, primarily due to the lack of a microscopic understanding of it.

Returning to the dimensional reduction problem, the fifth Chapter takes into account specific requirements of string theory, such as preserving supersymmetry in compactification, resulting in Calabi-Yau manifold for the compact dimensions. This Chapter summarizes recent discoveries that allow us to extract results in physically realistic mod-

els with hundreds of moduli. The aim of realizing viable models necessitates the inclusion of other extended objects like Dp-branes. The final goal is to write the effective $4d$ supergravity theory derived from string theory, in order to start the research of phenomenologically viable vacua.
Next, the embedding of inflation into string theory is discussed. Understanding the interplay between string theory, inflation, and early cosmology is necessary to establish a more direct connection between string theory and observations. This embedding leads to quantitative constraints on moduli masses, as the previously mentioned cosmological moduli problem. Other interesting perspectives include potential observable effects from axion physics, contributing to Dark Matter or causing superradiance.

In the sixth Chapter, explicit moduli stabilization is finally achieved. Stabilization is performed by incorporating both perturbative and non-perturbative corrections, leading to the Large Volume Stabilization scenario. In particular, higher order α'-corrections are included, in order to perform full moduli stabilization. Finally, some classes of Calani-Yau manifolds are considered: the Swiss-Cheese and the fibred. Stabilization is explicitly achieved for several (or even an arbitrary) number of moduli, extracting the axionic mass spectra. They rely directly on topological quantities of the considered manifolds, offering the possibility to verify under what conditions they give rise to an Axiverse, rather then a collection of effectively massless particles and how many axions can have a mass to sustain observable effects.

Chapter 2

Supersymmetry and Extra Dimensions

String theory requires many new ideas for its internal consistency. These ingredients are often interesting independently of strings themselves. For example, supersymmetry unifies bosons with fermions and provides a self-consistent proposal beyond the Standard Model. Extra dimensions enable unified theories where hierarchies are automatically exponentially suppressed. Higher dimensional objects allow for the generalization of field theories.
In this Chapter, we will explore some of these ideas to understand realistic string theories in the upcoming discussions.

2.1 Supersymmetry

Supersymmetry (SUSY) is a symmetry that mixes bosons and fermions. It has many virtues, including solving or alleviating the hierarchy problem of the Higgs mass without (or with much less) fine-tuning. It also provides a dynamical explanation for the Higgs potential and offers different Dark Matter candidates (e. g. weakly interacting massive particles (WIMPs), axions). Additionally, it suggests Grand Unification Theories where all three gauge interactions of the Standard Model (SM) are unified into a single interaction. However, in this work, we are interested into SUSY for its deep connection with string theory, as it seems to be required by the string framework for consistency.
In this section, we will present SUSY from its basic aspects to dynamical systems embedding global or local SUSY.

2.1.1 Supersymmetric algebra

SUSY extends Poincaré symmetry, in a way that is allowed by the Coleman-Mandula no-go theorem. Poincaré transformations act on spacetime coordinates as

$$x^\mu \mapsto x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu, \tag{2.1}$$

where a^μ parameterizes spacetime translations and Λ^μ_ν are the Lorentz transformations, defined by the relation $\Lambda^T \eta \Lambda = \eta$, with the spacetime metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The generators of Poincaré group are P^σ and $M^{\mu\nu}$, which satisfy

$$[P^\mu, P^\nu] = 0, \tag{2.2}$$

$$[M^{\mu\nu}, P^\sigma] = i\left(P^\mu \eta^{\nu\sigma} - P^\nu \eta^{\mu\sigma}\right), \tag{2.3}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho}\right). \tag{2.4}$$

Notice that all these 10 generators are bosonic. In a supersymmetric framework, we allow the presence of fermionic generators $Q^A_\alpha, \overline{Q}^A_{\dot\alpha}$, where $A = 1, \dots, \mathcal{N}$ and $\mathcal{N}$ is the amount of SUSY considered. For $\mathcal{N} = 1$, the algebra is specified by

$$\left\{Q_\alpha, \overline{Q}_{\dot\beta}\right\} = 2\left(\sigma^\mu\right)_{\alpha\dot\beta} P_\mu \tag{2.5}$$

$$[Q_\alpha, M^{\mu\nu}] = \left(\sigma^{\mu\nu}\right)_\alpha^{\ \beta} Q_\beta \tag{2.6}$$

plus the vanishing of all others commutators and anticommutators (with the exception of the $U(1)$ automorphism of SUSY algebra, known as R symmetry). $\mathcal{N} > 1$ is called extended SUSY, but we report some results of just $\mathcal{N} = 1$ SUSY, because it is the only phenomenological viable scenario, as we will shortly see.
Together with the standard Poincaré generators, the new spinor operators define the *super-Poincaré* group, where the generic element can be written as

$$e^{i\left(\omega^{\mu\nu}M_{\mu\nu} + a^\mu P_\mu + \theta^\alpha Q_\alpha + \overline{\theta}_{\dot\alpha}\overline{Q}^{\dot\alpha}\right)}, \tag{2.7}$$

where θ^α and $\overline{\theta}_{\dot\alpha}$ parametrize spinor transformations. This also leads to the introduction of the *superspace*, where the coordinates can be written as

$$x^M = (x^\mu, \theta^\alpha, \overline{\theta}_{\dot\alpha}). \tag{2.8}$$

Superspace can be then thought as the ordinary 4-dimensional spacetime plus extra fermionic dimensions, parametrized by *Grassmann variables*, which capture the anti-commuting nature of fermions. Some properties of Grassmann variables are

$$\theta_1^2 = \theta_2^2 = 0, \theta_1\theta_2 = -\theta_2\theta_1, \int d\theta_\alpha = 0, \int d\theta_\alpha \theta_\alpha = \frac{\partial}{\partial\theta_\alpha}\theta_\alpha = 1 \quad \text{(no sum)}. \tag{2.9}$$

By studying irreducible representations of super-Poincaré group, we introduce dynamical functions of x^M, in analogy with spacetime fields (scalar fields, spinor fields, vector fields, etc). Thus, we call them *superfields*. However, the simplest attempt to define a scalar superfield $\Phi(x^\mu, \theta^\alpha, \overline{\theta}_{\dot{\alpha}})$ without external indices does not yield an irreducible representation. The necessary constraints to find them are given by covariant derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_\mu, \quad \overline{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu. \tag{2.10}$$

They are introduced in order to ensure that, given a superfield S, its derivative with respect to Grassman variables also defines a superfield $D_\alpha S$. These constraints enable us to define the simplest irreducible superfields: the *chiral superfield* Φ such that

$$\overline{D}_{\dot{\alpha}}\Phi = 0 \tag{2.11}$$

or the *anti-chiral* $\overline{\Phi}$ such that $D_\alpha\overline{\Phi} = 0$. The explicit structure of a chiral superfield is

$$\begin{aligned}\Phi(x^\mu, \theta^\alpha, \overline{\theta}^{\dot{\alpha}}) =& \varphi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) + i\theta\sigma^\mu\overline{\theta}\partial_\mu\varphi(x) \\ &- \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi(x)\sigma^\mu\overline{\theta} - \frac{1}{4}(\theta\theta)(\overline{\theta\theta})\partial_\mu\partial^\mu\varphi(x).\end{aligned} \tag{2.12}$$

We see that the chiral superfield is given by several fields, that form a *supersymmetric multiplet* or *supermultiplet*. Supermultiplets are labeled by a half-integer, the *superspin* y. For massless particles, superspin matches the helicity. In the case of massive particles, it coincides with the spin j in the vacuum state. As always in the construction of the Hilbert space of quantum states, we act with creation operators on the vacuum state, defined as the state annihilated by annihilation operator. For $\mathcal{N} = 1$ SUSY, there is just one set of two operators $a^\dagger_{1,2}$ that increases or decreases the spin from $j = y$ of the vacuum to $j = y \pm \frac{1}{2}$ (in the massless case, we end up with just one operator $a^\dagger$). However, being fermionic operators, acting twice simply annihilates the state. Then, in $\mathcal{N} = 1$ SUSY, the supermultiplets determined by $y = 0$ contain just $j = y = 0$ scalar fields and $j = y + \frac{1}{2} = \frac{1}{2}$ fermionic fields. This content matches precisely that of the chiral superfield, where we have two scalars ϕ and F and one Weyl fermion ψ. In this simple example, we also find the realization of a standard SUSY result: in a supermultiplet, the number of bosonic degrees of freedom equals the number fermionic ones. Indeed, in the chiral supermultiplet, we have 4 bosonic components (2 complex scalar fields, ϕ and F) and the 4 fermionic components (2 complex components for the Weyl fermion ψ, thus 4 real components). This counting is performed off-shell, but the equality is also verified on-shell: the Weyl equation cancels 2 fermionic components and F disappears from the final dynamics, resulting in 2 bosonic components and 2 fermionic components on-shell. This reveals that F is an auxiliary field and that ψ is the *supersymmetric partner* (*superpartner*) of the boson ϕ.

Another relevant supermultiplet is the vector one V, with $y = \frac{1}{2}$ and defined by the reality condition $V(x, \theta, \overline{\theta}) = V^\dagger(x, \theta, \overline{\theta})$. With $y = \frac{1}{2}$, apart from the fermions with $j = y$, we have both scalars with $j = y - \frac{1}{2} = 0$ and vectors with $j = y + \frac{1}{2} = 1$. The explicit expression of this multiplet is involved due to the presence of many off-shell components, as is typical of theories with gauge bosons A_μ. Many of them can be gauged away by a proper gauge transformation, that in this more general context looks like

$$V \mapsto V - \frac{i}{2}\left(\Lambda - \Lambda^\dagger\right), \tag{2.13}$$

where Λ is a chiral superfield. A common gauge fixing condition defines the *Wess-Zumino gauge*, where the vector multiplets read as

$$V(x, \theta, \overline{\theta}) = -\theta\sigma^\mu\overline{\theta}A_\mu(x) + i\theta\theta\overline{\theta\lambda}(x) - i\overline{\theta\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\overline{\theta\theta}D(x). \tag{2.14}$$

This gauge breaks SUSY (there are 5 bosonic components and 4 fermionic ones off-shell), but we can identify the gauge fields A_μ (e. g. photons, weak bosons, gluons) and their fermionic superpartners: the *gauginos* (photinos, winos, gluinos). D turns out to be another auxiliary field, as F for the chiral multiplets. Despite not being truly dynamical entities, auxiliary fields have a crucial role in SUSY models, because they govern SUSY breaking, as we will see in a moment.
A last interesting supermultiplet that we need to mention is the *supergravity multiplet* with $y = \frac{3}{2}$. A particle with spin $j = \frac{3}{2}$ is a fermion satisfying the Rarita-Schwinger equation of motion, rather than the Dirac one. It is a fermion with a spacetime index ψ^μ, so we have $4 \times 4 = 16$ fermionic off-shell degrees of freedom. In the multiplet, we have a spin $j = \frac{3}{2} + \frac{1}{2} = 2$ symmetric tensor field $g_{\mu\nu}$, so 10 bosonic degrees of freedom. The remaining 6 bosonic components can be arranged in different ways. In the so-called minimal set, 4 degrees of freedom are used to define a vector and the remaining 2 a complex scalar.
All the fields in a supermultiplet transform in the same representation. In extended SUSY, $\mathcal{N} > 1$ gives a vector in (almost) all supermultiplets. Vectors transform in the adjoint representation, which is real. Then, all fields must transform under real representation, preventing chiral theories. Thus, the only phenomenological viable SUSY theory, in the sense of a chiral theory as Standard Model, is $\mathcal{N} = 1$ SUSY. Extended SUSY is still interesting for several reasons: string theory can result in these extended cases, so we need to understand how to handle them. However, in the following, we will present how to construct dynamical systems with embedded $\mathcal{N} = 1$ SUSY.

2.1.2 Supersymmetric dynamics

The dynamics is encoded in the action of the theory, which we can easily build from Grassmann properties (2.9): there are two possible Grassmann integral measures:

$$d^2\theta = -\frac{1}{4} d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta}, \quad d^2\overline{\theta} = \frac{1}{4} d\overline{\theta}^{\dot{\alpha}} d\overline{\theta}^{\dot{\beta}} \epsilon_{\dot{\alpha}\dot{\beta}}, \tag{2.15}$$

that we can combine in $d^4\theta = d^2\theta d^2\overline{\theta}$. The action is given by an integration over coordinates, now extended to also include the fermionic directions. Thus,

$$\begin{aligned} S = &\int \mathrm{d}^4x \int d^4\theta K(\Phi) + \\ &+ \int \mathrm{d}^4x \int d^2\theta \left(W(\Phi) + h.c.\right). \end{aligned} \tag{2.16}$$

This is the simplest SUSY model, for a single chiral superfield. Notice that, being $d^4\theta$ real, also $K(\Phi)$ must be a real function of Φ. It is the *Kähler potential*. $\mathrm{d}^2\theta W(\Phi)$ instead is not real, then we need to add the hermitian conjugate to preserve unitarity. $W(\Phi)$ must be a holomorphic function of Φ, called *superpotential*. Recall that for Grassmann variables integration equals derivation, then

$$\int d^2\theta(\theta\theta) = 1, \quad \int d^4\theta(\theta\theta)(\bar{\theta}\bar{\theta}) = 1. \tag{2.17}$$

Then, integrations over Grassmann variables in the action simply correspond to prescriptions to extract the $\theta\theta\overline{\theta\theta}$ component of the Kähler potential K_D and the $\theta\theta$ component of the superpotential W_F. The subscripts refer to the auxiliary fields D and F. Finally, the lagrangian of the theory reads as

$$\mathcal{L} = K(\Phi)|_D + \left(W(\Phi)|_F + h.c.\right). \tag{2.18}$$

Furthermore, by dimensional analysis of (2.12), because $[\Phi] = [\phi] = 1$ and $[\psi] = \frac{3}{2}$, $[\theta] = -\frac{1}{2}$. Then, $[d^2\theta] = 1$ and $[d^4\theta] = 2$. Finally, $[K(\Phi)] = 2$ and $[W(\Phi)] = 3$. This means that the most general Kähler potential and superpotential of this system are

$$K = \Phi^\dagger\Phi, \quad W = \alpha + \lambda\Phi + \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3. \tag{2.19}$$

Without loss of generality, α and λ can conveniently set to 0.
The D- and F-components of these potentials can be directly extracted, but we present here a different approach that is more useful when dealing with many chiral superfields Φ_i. We can expand both Kähler potential and superpotential around the scalar components

ϕ_i: starting from W and defining $\frac{\partial W}{\partial \varphi_i} = \frac{\partial W}{\partial \Phi_i}\Big|_{\Phi_i=\varphi_i} \equiv W_i$

$$\begin{aligned} W(\Phi_i) &= W(\varphi_i) + \underbrace{(\Phi_i - \varphi_i)}_{...+\theta\theta F_i+...} \frac{\partial W}{\partial \varphi_i} + \frac{1}{2} \underbrace{(\Phi_i - \varphi_i)}_{...+\theta\psi_i+...} \overbrace{(\Phi_j - \varphi_j)}^{...+\theta\psi_j+...} \frac{\partial^2 W}{\partial \varphi_i \partial \varphi_j} = \\ &= \ldots + \theta\theta \left(F_i \frac{\partial W}{\partial \varphi_i} + \frac{1}{2} \psi_i \psi_j \frac{\partial^2 W}{\partial \varphi_i \partial \varphi_j} \right) + \ldots \end{aligned} \tag{2.20}$$

where the "..." contain just the other components. It is important to stress that the expansion stops at third order because higher terms vanish automatically (too high powers of Grassman variables are involved), so the expansion is *exact*.
We can similarly expand the Kähler potential: for simplicity we define $\Delta_i := \Phi_i - \varphi_i$ and

$$K_i := \frac{\partial K}{\partial \Phi_i}\Big|_{\Phi_i=\varphi_i}, \quad K_{ij} := \frac{\partial^2 K}{\partial \Phi_i \partial \Phi_j}\Big|_{\Phi_i=\varphi_i}, \ldots \tag{2.21}$$

and similarly for higher derivatives. The quantity K_{ij} is also called the *Kähler metric*: it is the metric of the space of field configurations. This space, admitting a metric such as K_{ij} that follows from a Kähler potential, is a complex manifold called *Kähler manifold*. The most general expansion for K is then

$$\begin{aligned} K(\Phi_i) = K(\varphi_i) &+ K_i \Delta_i + K_{\bar{i}} \Delta_{\bar{i}} + \frac{1}{2} K_{ij} \Delta_i \Delta_j + \\ &+ \frac{1}{2} K_{\bar{i}\bar{j}} \Delta_{\bar{i}} \Delta_{\bar{j}} + K_{i\bar{j}} \Delta_i \Delta_{\bar{j}} + \frac{1}{2} K_{ij\bar{k}} \Delta_i \Delta_j \Delta_{\bar{k}} + \\ &+ \frac{1}{2} K_{\bar{i}\bar{j}k} \Delta_{\bar{i}} \Delta_{\bar{j}} \Delta_k + \frac{1}{4} K_{ij\bar{k}\bar{l}} \Delta_i \Delta_j \Delta_{\bar{k}} \Delta_{\bar{l}}. \end{aligned} \tag{2.22}$$

This is the full Kähler potential: we need the D-component, that looks like

$$\begin{aligned} K(\Phi_i)|_D = K_{i\bar{j}} &\left(F_i \overline{F}_{\bar{j}} + \partial_\mu \varphi_i \partial^\mu \overline{\varphi}_{\bar{j}} + i \partial_\mu \psi_i \sigma^\mu \overline{\psi}_{\bar{j}} \right) + \\ &+ \frac{1}{2} K_{ij\bar{k}} \left(i \psi_i \sigma^\mu \overline{\psi}_{\bar{k}} \partial_\mu \varphi_j - \psi_i \psi_j \overline{F}_{\bar{k}} \right) + h.c. + \\ &+ \frac{1}{4} K_{ij\bar{k}\bar{l}} \psi_i \psi_j \overline{\psi}_{\bar{k}} \overline{\psi}_{\bar{l}} \end{aligned} \tag{2.23}$$

In a renormalizable theory, operator mass dimensions must be ≥ 4 or, equivalently, the coupling of that interaction must have a non-negative dimension. This means that in a renormalizable theory, only the first line is allowed. In general, renormalizable theories have $[K_D] \leq 4$ and $[W_F] \leq 4$, so $[K] \leq 2$ and $[W] \leq 3$. However, being non-renormalizable simply means that the theory requires a completion in the UV, necessitating a different approach in that regime. But when working within a SUSY

framework, as a low-energy approximation of string theory, it is quite natural to have non-renormalizable theories since the UV-completed theory has already been integrated out.
A generic renormalizable SUSY lagrangian is then simply

$$\mathcal{L} = K_{i\bar{j}}\left(F_i\overline{F}_{\bar{j}} + \partial_\mu\varphi_i\partial\overline{\varphi}_{\bar{j}} + i\partial_\mu\psi_i\sigma^\mu\overline{\psi}_{\bar{j}}\right) + \left(F_i\frac{\partial W}{\partial\varphi_i} + \frac{1}{2}\psi_i\psi_j\frac{\partial^2 W}{\partial\varphi_i\partial\varphi_j} + h.c.\right) \tag{2.24}$$

This lagrangian can be studied in general, identifying the kinetic terms for the boson and the fermion, their equal masses and the mutual interactions governed by the same coupling. This is at the origin of the "*miraculous cancellation*" that makes SUSY so appealing. We leave this standard calculations to references such as [11]. However, something we are directly interested in are the equations of motion (EOMs) for the auxiliary fields F_i. Their EOMs can be easily deduced:

$$\frac{\partial\mathcal{L}}{\partial\overline{F}_{\bar{j}}} = 0 \quad\Rightarrow\quad K_{i\bar{j}}F_i + \overline{W}_{\bar{j}} = 0 \quad\Rightarrow\quad F_i = -(K^{-1})_{i\bar{j}}\overline{W}_{\bar{j}} \tag{2.25}$$

The fields F_i can now be integrated out by substituting these EOMs directly into the lagrangian. All the F-terms can be summed and, after the substitution, what remains is a potential for the scalar fields:

$$\mathcal{L}_F := K_{i\bar{j}}F_i\overline{F}_{\bar{j}} + F_i\frac{\partial W}{\partial\varphi_i} + \overline{F}_{\bar{i}}\frac{\partial\overline{W}}{\partial\overline{\varphi}_{\bar{i}}} = -(K^{-1})_{i\bar{j}}W_i\overline{W}_{\bar{j}} \equiv -V_F(\varphi) \tag{2.26}$$

2.1.3 Supergravity

So far, we have studied the global version of SUSY, where we apply the same transformation independently of the spacetime point. As is often the case in field theories, many interesting results arise when *gauging* a symmetry, promoting it to a local one. Upon closer inspection of these fermionic transformations, particularly (2.5), we notice that local SUSY involves local spacetime transformations. They correspond to the local diffeomorphisms, thus the symmetry of General Relativity! We are able to introduce gravity in a supersymmetric context thanks to local SUSY, building a supersymmetric extension of GR: it is the *supergravity* SUGRA. A full treatment of SUGRA is beyond the scope of this thesis, nonetheless it defines the natural framework in which to study cosmological effects of string theory: it is a natural class of effective field theories coupled with gravity in a particularly simple way that often results from string compactification. We work in analogy with standard gauging mechanisms, where the parameter α is promoted to a function $\alpha(x)$ and a gauge boson A_μ transform as

$$A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) + \partial_\mu\alpha(x) \Rightarrow \delta A_\mu(x) = \partial_\mu\alpha(x). \tag{2.27}$$

In SUGRA, we have a spinorial gauge parameter η_α and the analog of the gauge boson A_μ is a fermion such that

$$\delta\psi^\mu_\alpha(x) = \partial^\mu \eta_\alpha(x). \tag{2.28}$$

It is a fermion carrying a spacetime index, so a Rarita-Schwinger spin-3/2 fermion. We have already seen an object like this together with the spin-2 metric in the supergravity multiplet. This clarifies the choice of the name. To be more precise, the field ψ^μ_α is called *gravitino*, being the superpartner of the spin-2 graviton, described by the metric field $g_{\mu\nu}$ o, equivalently, by a vierbein field e^μ_a such that $g_{\mu\nu} = e^a_\mu e_{a\nu}$. The other fields in the supermultiplet guarantee SUSY off-shell. All together, $(e^\mu_a, \psi^\mu_\alpha, M, b_a)$ define the SUGRA action

$$S_{\text{SG}} = -\frac{1}{2}\int \mathrm{d}^4 x e \left\{ R - \frac{1}{3}\overline{M}M + \frac{1}{3}b^a b_a + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left(\overline{\psi}_\mu \overline{\sigma}_\nu \mathcal{D}_\rho \psi_\sigma - \psi_\mu \sigma_\nu \mathcal{D}_\rho \overline{\psi}_\sigma\right)\right\}, \tag{2.29}$$

which results, on-shell, in the standard Einstein-Hilbert action in tedrad formalism, plus a Rarita-Schwinger term. The full theory is also specified by the matter content coupled with gravity.

As always, in a locally symmetric theory, we have redundancy. A_μ does not carry 4 degrees of freedom, but only 3 (in the massive case): the 4^{th} can be gauge-fixed by a gauge transformation and expressed in terms of the others. In SUGRA, something similar happens: while in SUSY the dynamics is specified by two building blocks, the Kähler potential and the superpotential, in SUGRA the presence of *Kähler invariance* makes K and W no longer independent. This new symmetry of the action looks like

$$\begin{aligned} K &\mapsto K + h(\Phi) + h^*(\Phi^*), \\ W &\mapsto \exp(-h(\Phi))W, \end{aligned} \tag{2.30}$$

with $h(\Phi)$ an holomorphic function. It forces the only physically relevant combination to be

$$G(\Phi_i) := K(\Phi_i) + \ln|W(\Phi_i)|^2. \tag{2.31}$$

For example, the scalar F-term potential V_F becomes in this framework

$$V_F = e^G \left(G^{i\bar{j}} G_i G_{\bar{j}} - 3\right), \tag{2.32}$$

where the subscripts of G stand for derivatives with respect to Φ_i. However, in the following, we will continue to use K and W explicitly distinct because it is more convenient in practice and makes the effect of gravity more transparent. Consider again V_F: in terms of K and W it reads as

$$V_F = e^K \left\{\left(K^{-1}\right)^{i\bar{j}} D_i W D_{\bar{j}}\overline{W} - 3|W|^2\right\}. \tag{2.33}$$

The *Kähler covariant derivatives* are

$$D_iW := \partial_i W + K_i W. \tag{2.34}$$

We are working in natural units, where the Planck mass is $M_p = 1$. Restoring it in the equations above, we understand how in the gravity decoupling limit ($M_p \to \infty$) global SUSY is restored:

$$\begin{cases} e^{K/M_p^2} \to 1 \\ 3\frac{|W|^2}{M_p^2} \to 0 \\ D_iW \to \partial_i W \end{cases} \Rightarrow V_F \to \left(K^{-1}\right)^{i\bar{j}} \partial_i W \partial_{\bar{j}} \overline{W} \ . \tag{2.35}$$

Finally, we can read directly the gravitino mass $m_{3/2}$

$$m_{3/2}^2 = e^K |W|^2, \tag{2.36}$$

interesting because it is the order parameter of SUGRA breaking. More properly, the order parameters are the vacuum expectation values $\langle F \rangle$: $\langle F \rangle \neq 0$ implies SUGRA breaking and if $\langle V_F \rangle \simeq 0$, $m_{3/2}^2 \simeq \langle |F|^2 \rangle$.

2.2 Dimensional reduction

A highly non-trivial and very general prediction of string theory is the number of spacetime dimensions, denoted as d. For superstring theories, this number is 10, while in M-theory, it is 11. Other numbers have also been found to be intriguing, but in any case, it is unambiguous that within the string framework, $d > 4$. Therefore, the problem arises of how to recover our (3+1)-spacetime. This issue is addressed through a process called "**compactification**", where the extra-dimensions beyond the 4 that we observe are "curled up" or "compactified" in such a way that their effects become hidden or negligible at low energies. More formally, we write the 10-dimensional manifold $\mathcal{M}_{10}$ as

$$\mathcal{M}_{10} = \mathcal{M}_4 \times X_6, \tag{2.37}$$

where $\mathcal{M}_4$ is the effective 4-dimensional one and X_6 are the compact dimensions. The specific shape and size of these compactified dimensions play a crucial role in determining the particle content, symmetries and interactions in the resulting (3+1)-spacetime.

2.2.1 Kaluza-Klein compactification

Let's take a look on some enlightening examples. Consider a simple 5d world, with a scalar matter field in it. The coordinates are $x^M = (x^\mu, y)$, where $y = x^4$. The fifth dimension is not seen, so we imagine that it is curled up and small enough to be ignored at first approximation. A simple way to realize this idea is $\mathbf{M}_4 \times \mathbf{S}^1$, where we have the standard 4d flat Minkowskian spacetime and a circular extra-dimension. This means that the system enjoys the symmetry $\phi(x^\mu, y) = \phi(x^\mu, y + 2\pi R)$, where R is the radius of $\mathbf{S}^1$. Then, we can expand the y-component in Fourier modes simply as

$$\phi(x^\mu, y) = \sum_{k\in\mathbf{Z}} \phi_k(x^\mu)\, e^{iky/R}. \tag{2.38}$$

The 5d action of the theory is

$$S_{5d} = \int_M d^5x \frac{1}{2} (\partial_M \phi)(\partial^M \phi). \tag{2.39}$$

Expanding the field in Fourier modes in the action and collecting them, we rewrite

$$S_{5d} = \int_{M_4} d^4x \int_0^{2\pi R} dy \sum_{k,k'\in\mathbb{Z}} \frac{1}{2}\left(\partial_\mu\phi_k\partial^\mu\phi_{k'} - \frac{kk'}{R^2}\phi_k\phi_{k'}\right) e^{i\frac{(k+k')y}{R}} \tag{2.40}$$

Thanks to the identity

$$\int_0^{2\pi R} dy e^{i\frac{(k+k')y}{R}} = 2\pi R\delta_{k+k',0}, \tag{2.41}$$

the integral in the action can be performed, leading to the 4d action

$$S_{4d} = 2\pi R \int d^4x \left[\frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 + \sum_{k=1}^{\infty}\left(\partial_\mu\phi_k^*\partial^\mu\phi_k + m_k^2\phi_k^*\phi_k\right)\right], \tag{2.42}$$

where $\phi_k^* = \phi_{-k}$. We recognize a massless mode ϕ_0 and *an infinite tower of Kaluza-Klein (KK) massive modes*, with mass given by

$$m_k^2 = \frac{k^2}{R^2}. \tag{2.43}$$

Working at scales $E \ll 1/R$, only the massless mode would be observable. So, we end up with an effective field theory in 4d populated by light modes. This procedure is the so-called *Kaluza-Klein compactification.*
We can also considered a purely 5d generalization of General Relativity, given by an Einstein-Hilbert action as

$$S_{5d} = \frac{M_5^3}{2}\int d^5x\sqrt{-G}R_{5d} \tag{2.44}$$

where M_5 is the 5d gravitational coupling, R_5 is the Ricci scalar built with the 5d metric and G its determinant. Compactifying into $\mathbf{M}_4 \times \mathbf{S}^1$ as before, G_{MN} can be decomposed again in Fourier modes

$$G_{MN}(x^\mu, y) = \sum_{k\in\mathbf{Z}} G_{MN}^k(x^\mu)\, e^{iky/R}. \tag{2.45}$$

The massless modes in this case are

$$\left(G^0_{MN}\right) = e^{\sigma/3}\left(\begin{array}{c|c} g_{\mu\nu}(x) + e^{-\sigma}A_\mu A_\nu & e^{-\sigma}A_\mu(x) \\ \hline e^{-\sigma}A_\mu(x) & e^{-\sigma} \end{array}\right). \tag{2.46}$$

where $g_{\mu\nu}$ is the 4d metric, A_μ a gauge vector enjoying a gauge symmetry inherited by $U(1)$ parametrization invariance of $\mathbf{S}^1$ and σ is called *radion* and it is a *modulus* field. Substituting this decomposition in the action, we can extract a 4d action for the 0 KK modes

$$S^0_{KK} = M_5^3\pi R\int d^4x\sqrt{-g}\left(R_{4d} - \frac{1}{6}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{4e^\sigma}F^2_{\mu\nu}\right). \tag{2.47}$$

We can observe many interesting things: the modulus appears without any potentials, so its vev is not fixed by the theory, but it parametrizes both the compact geometry and the coupling of the gauge interaction $g^2 = e^\sigma$. So, the size of the fifth dimension is arbitrary. This problem (called *moduli problem*) is very general in string theory and must be addressed by introducing a potential for the moduli that forces them to get a vev and to be massive, resulting also in the stabilization of the extra-dimensions.
Furthermore, in the final $4d$ effective theory we have obtained scalar matter coupled with Einstein gravity, but also with an $U(1)$ gauge interaction: this model is the first unification of gravity and electromagnetism. In general, isometries of extra-dimensions determine the gauge symmetry of the resulting effective theory. However, it is not enough to demand extra-dimensions invariant under $SU(3)\times SU(2)\times U(1)$ to reproduce SM.
Finally, there is a connection between the 4d and 5d Planck constants:

$$M_p^2 = 16\pi^2 M_5^3 R. \tag{2.48}$$

Despite $M_p \approx 10^{18}$GeV, the scale of the higher dimensional gravity can be much lower, having large enough R.
The situation of compactification in string theory is more complicated, because of many more ingredients and more extra-dimensions. The compact manifold X_6 is typically taken a complex *Calabi-Yau manifold*. A purely $10d$ (a number that arises in the context of superstring theories) gravitational theory described by the action

$$S^{(10)}_{\text{EH}} = \frac{1}{2\kappa^2}\int d^{10}X\sqrt{-G}R_{10}, \tag{2.49}$$

can be studied by using the general ansatz for the metric

$$ds^2 = G_{MN}dX^M dX^N = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + e^{2u(x)}\hat{g}_{mn}dy^m dy^n, \tag{2.50}$$

where $\hat{g}_{mn}$ is the metric for the compact manifold X_6 and u is the modulus or the *breathing mode*, which represents the variations in size of the internal space X_6 as a function of the four-dimensional coordinates x^μ. By employing this ansatz, we can express the R_{10}

in terms of the standard 4- and the 6-dimensional Ricci scalars R_4 and $\hat{R}_6$, along with the modulus u, as

$$R_{10} = R_4 + e^{-8u}\hat{R}_6 + 12\partial_\mu u \partial^\mu u. \tag{2.51}$$

In string theory, it is often found that the $\hat{R}_6$ vanishes. Then, plugging (2.51) into (2.49), we obtain

$$S_{\rm EH}^{(4)} = \frac{M_{\rm p}^2}{2}\int {\rm d}^4x\sqrt{-g}\left(R_4 + 12\partial_\mu u\partial^\mu u\right) \tag{2.52}$$

where the Plank mass is defined in terms of the volume of compact dimensions $\mathcal{V}$ as

$$M_{\rm p}^2 \equiv \frac{\mathcal{V}}{\kappa^2} \quad , \quad \mathcal{V} := \int_{X_6} d^6y\sqrt{\hat{g}} = e^{6u}, \tag{2.53}$$

where $\hat{g}$ is the determinant of the compact part of the metric (2.50).
A non-trivial fact is that this system can be equivalently rewritten in terms of SUGRA. Promoting the modulus $u(x)$ to a complex field $T(x)$ such that ${\rm Re}^{3/2}(T) = e^{6u} = \mathcal{V}$, the theory specified by the Kähler potential

$$K = -3\ln(T + \overline{T}) = -2\ln(\mathcal{V}), \tag{2.54}$$

generates the same kinetic term for u as in (2.52), following simply the expansion (2.23). The imaginary part of T arises from the dimensional reduction, but we will explore it in greater detail in the analogous string case.[1]
In this analysis, the presence of the constant string coupling g_s has been ignored. This coupling is determined, like many other things, by the vev of a scalar field that is always present in string compactification: the dilaton. We will explore these concepts in greater detail in the part dedicated directly to string theory. Here, we have again obtained the usual General Relativity in 4-dimensions coupled with a real scalar field. Again, the modulus of the theory does not fix the scale of the extra-dimensions (now parametrized by $\mathcal{V}$). Even if we admit $SU(3)\times SU(2)\times U(1)$ as isometries for these extra-dimensions, the model is not able to reproduce SM. However, the relation between scales $M_p^2 \sim M_{10}^8\mathcal{V}$ suggests that the fundamental $10d$ gravitational scale can be much smaller than Planck, it can even be ~ 1 TeV (in order to automatically fix the fine-tuning problem of the electroweak scale), if the extra-dimensions are large enough.

2.2.2 Extra-dimensional scenarios

There are two scenarios for extra-dimensions:

[1] To be more precise, the Kähler potential must be written in terms of some chiral superfield $\Phi = T + \theta\psi + \dots$. Then, we expand around the scalar term T. However, we are interested just into the bosonic sector. Thus, from now on, we will write Kähler potentials as functions of the scalar component of some superfield.

(i) we can achieve large extra-dimensions in the *brane-world scenario*, which takes advantage of the relation $M_p^2 \sim M_d^{d-2}\mathcal{V}$. The volume is regulated by some length scale L such that $\mathcal{V} \sim L^{d-4}$. In order to obtain our $4d$ Plank scale $M_p \approx 10^{18}$GeV from a small fundamental scale as $M_d \sim 1$ TeV, we need large volumes. But in the $5d$ case that we have outlined before, the volume scales too slowly, $\mathcal{V} \sim L$, forcing $L \sim 10^8$ km. However, in $10d$, the case suggested by string theory, $\mathcal{V} \sim L^6$, thus $L \sim 0.1$ mm. We have seen that dimensional reduction results in a tower of massive particles, with mass determined by size of extra-dimensions. Nothing like that has been observed in current accelerators, so we can take $L \lesssim 10^{-16}$ cm. However, this bound is based on particle physics experiments. In the brane-world scenario, we decouple matter and gauge interactions from the dimensional reduction. Particles are forced to live on *branes*, multi-dimensional sub-manifolds of the full spacetime manifold, called *bulk*. We will study branes in more detail in the following, but for now it is sufficient to say that by confining matter to live on them, we work always with fields defined in $4d$, which prevents the appearance of KK modes. On the other hand, gravity propagates in the bulk. Consequently, constraints from particle physics simply do not apply. Experimental bounds can only come from gravitational experiments, but gravity is weak and challenging to measure. A standard experiment involves measuring the gravitational law at small energy scales: the standard Newtonian gravity potential goes as $\frac{1}{r}$, the famous inverse law. In an arbitrary dimensional world, this law would be $\frac{1}{r^{d-3}}$. By investigating gravitational phenomena at small distances, we can determine the number of dimensions in the Universe. Current purely gravitational observations lead to $L \lesssim 0.1$mm, allowing for the existence of large extra-dimensions while remaining compatible with SM.

(ii) A more general ansatz than (2.49) is

$$ds^2 = e^{W(y)}\eta_{\mu\nu}dx^\mu dx^\nu + g_{mn}(y)dy^m dy^n, \tag{2.55}$$

where $W(y)$ is the *warp factor*. Compactification in this case is a bit different. In $5d$, instead of compactifying on $\mathbf{S}^1$, we compactify on $\mathbf{S}^1/\mathbb{Z}_2$ (technically an *orbifold*). So, the extra-dimension acts as an interval rather than a circle. The end-points are specified by the coordinates $y = 0$ and $y = \pi R$ and the surfaces at these points act as branes. The warp factor is $e^{-2|y|/r}$, with r being a constant determined by M_5. Thus, the 4d metric changes between these two branes as

$$g_{\mu\nu}(y = \pi R) = e^{-2\pi R/r} g_{\mu\nu}(y = 0). \tag{2.56}$$

These cause a red-shift of energy scales that naturally solves hierarchy problem. If we let SM physics on the $y = \pi R$ brane and gravity on $y = 0$, the Higgs scale gets naturally suppressed as $m^2_{\text{Higgs}} \simeq e^{-2\pi R/r} M_p^2$. So, we can fix the Higgs hierarchy problem by a small exponential, which also leads $M_p \sim M_5$, because of

$M_p^2 = 4\pi M_5^3 r \left(1 - e^{-2\pi R/r}\right)$. In $5d$, R is slightly bigger than the Planck length, so in principle compatible with current observations. It is a very different situation compared to the large extra-dimensions of the brane-world. Also, phenomenology here is more complicated and KK modes at TeV appear.

To summarize, by starting from a higher dimensional theory we arrive at

(i) an effective $4d$ theory together with some scalar field content;

(ii) gauge interactions from isometries, providing a first simple example of a "*unified theory*";

(iii) a natural solution for the Higgs hierarchy problem.

These results are not reliant on string theory and simply highlight the fact that the existence of extra dimensions allows for the emergence of intriguing scenarios. In this sense, string theory provides an interesting example of a framework in which these extra-dimensions naturally appear. However, the situation in string theory is more complicated than what has been outlined so far. Fermions, which correspond to regular matter, arise from excitations of strings rather than dimensional reduction. The topologies of Calabi-Yau manifolds are non-trivial in order to reproduce particle phenomenology. In the following Chapters, we will study the topology of Calabi-Yau manifolds more directly due to its connection with cosmology.

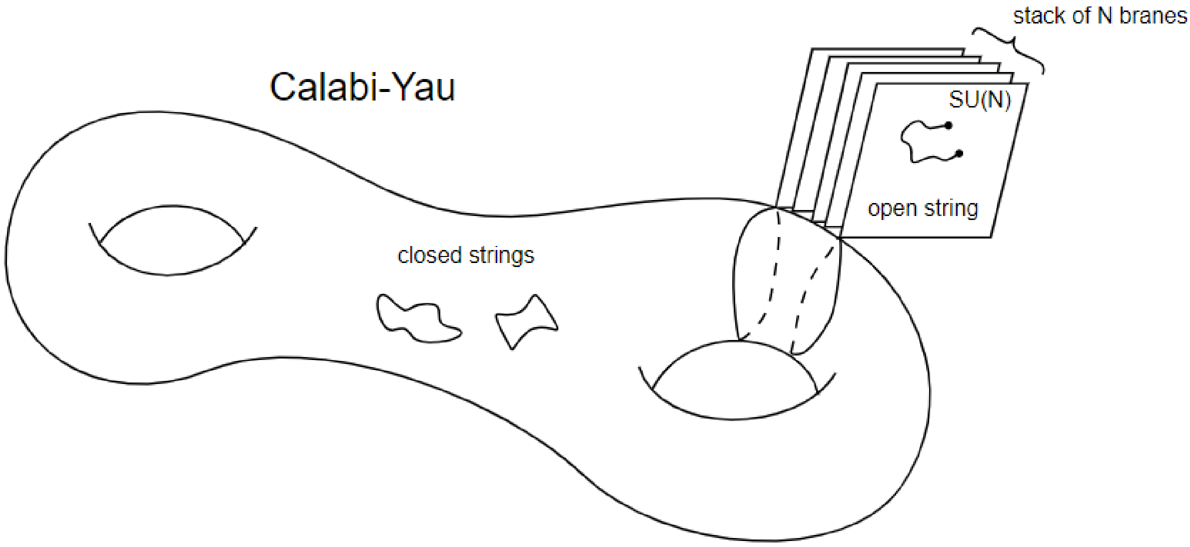

Figure 2.1: The brane-world: matter arises from open strings that exist on a stack of branes. These branes provide the gauge group and the coupling of the resulting effective field theory. Closed strings, on the other hand, are free to oscillate and propagate in the full bulk, and their spectra contain gravity.

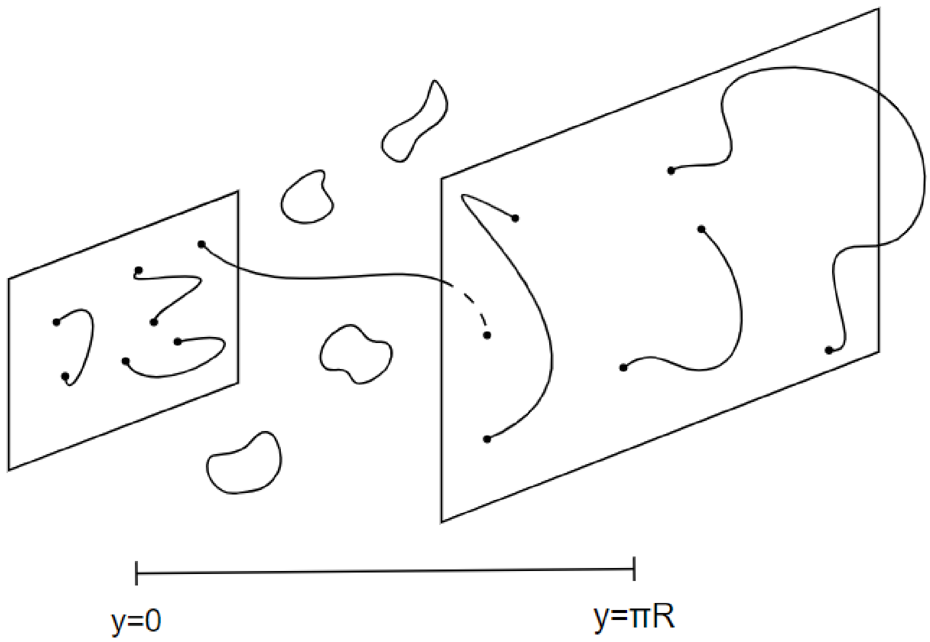

Figure 2.2: Warped extra-dimensions: the Planck brane is localized at $y = 0$, while SM (or at least the Higgs mechanism, connected to electroweak physics) brane is localized at $y = \pi R$. Again, closed strings are free propagate in the bulk.

Chapter 3

String Theory

In this Chapter, some fundamental aspects of string theory are presented. It is a huge and complex topic, and here we provide just a sketch of the relevant arguments for the following discussion on its phenomenological implications.
Its first formulation, known as *bosonic string theory*, was presented in the 1970s. It had the initially appealing feature of automatically containing gravity (in the form of gravitons) by combining quantum relativistic physics of a simple extended object, a string. Apart from its physical inconsistencies, it opened the window to an entire research line that combines many different and independent ideas. By including SUSY, the resulting superstring theories provide an enough well-understood and physically interesting proposal for quantum gravity.

3.1 Bosonic String Theory

This is the first example of a string theory. Although it is not physically viable (tachyonic instabilities, absence of fermions), it captures many relevant aspects of the theory. Starting from a relativistic theory of a $1d$ object, its quantization leads to ordinary particle in spacetime, depending by the topology of the string (i. e. open or closed). The theory fixes the number of spacetime dimensions via internal consistency and also determines the couplings via field vevs, resulting in the absence of free-parameters.

3.1.1 Relativistic strings

The starting point is the well-known action for a relativistic particle

$$S = -m \int d\tau = -m \int dt \sqrt{-\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \frac{\mathrm{d}x^\nu}{\mathrm{d}t}}. \tag{3.1}$$

τ is the proper time, that measures the length of the world-line spanned by the particle. In the right hand side, the action is written in terms of the measured time t. Poincaré

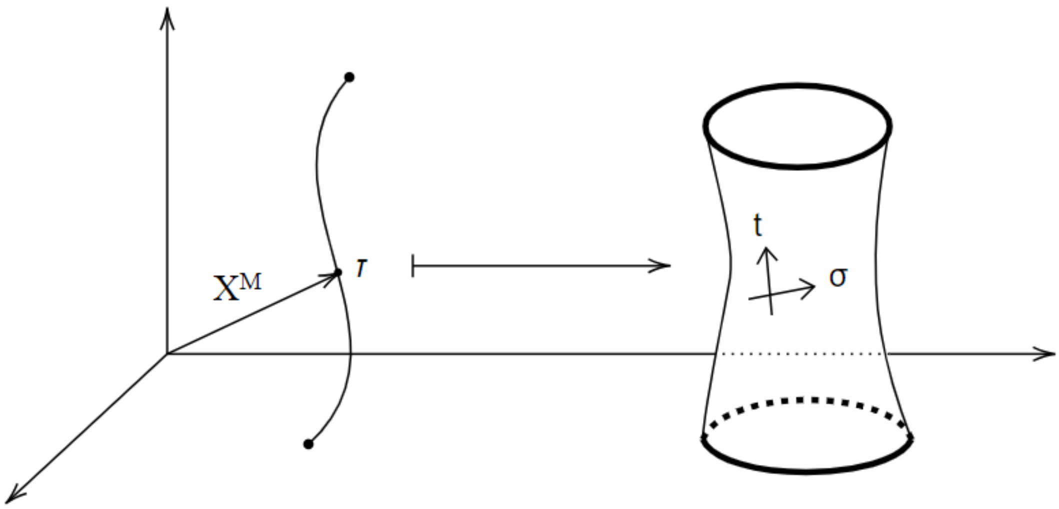

Figure 3.1: on the left, the field view of the particle action, in an arbitrary target spacetime. On the right, the worldsheet surface spanned by a closed string.

symmetry is embedded into S and the right hand side make explicitly the 1-dimensional diffeomorphism invariance, or invariance under coordinate reparametrizations

$$t \mapsto t' = t'(t). \tag{3.2}$$

A different way to look at this action is as a theory of 4 fields x^μ on a $1d$ manifold parametrized by t. This is the starting point towards string theory. We would to generalize (3.1) to a theory of d fields X^M, $M = 0, \dots, d-1$, on a $2d$ manifold parametrized by t and σ, called *worldsheet*. This surface is spanned by a $1d$ object, a *string* with length $L_s = 1/M_s$, where M_s is the string scale. The natural generalization of (3.1) is the Nambu-Goto action

$$S_{\mathrm{NG}} = -\frac{1}{2\pi\alpha'}\int_\Sigma dA = -\frac{1}{2\pi\alpha'}\int_\Sigma dt d\sigma \sqrt{-h}. \tag{3.3}$$

The prefactor $1/(2\pi\alpha') \simeq M_s^2$ is the energy density along the string length, which is the string tension, dA is the unit of worldsheet area and h is the determinant of the 2d worldsheet metric induced by the spacetime metric as

$$h_{tt} = \partial_t X^M \partial_t X_M, \quad h_{\sigma\sigma} = \partial_\sigma X^M \partial_\sigma X_M, \quad h_{t\sigma} = \partial_t X^M \partial_\sigma X_M. \tag{3.4}$$

This action is hard to quantize, so it is replaced by the classically equivalent Polyakov action

$$S_{\mathrm{P}} = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi \sqrt{-\det g}\,\eta_{MN} g^{ab}(t,\sigma)\partial_a X^M \partial_b X^N, \tag{3.5}$$

where $a, b = 0, 1$, and $\xi^0 = t, \xi^1 = \sigma$. They can be used to define a system of two orthonormal coordinates, where t is time-like and σ is space-like. In (3.5), the target spacetime metric η_{MN} is decoupled from the worldsheet metric g^{ab}. However, the two actions are equivalent, so there are the same degrees of freedom. The new degrees of freedom introduced in (3.5) are just gauge redundancy. Indeed, we need to fix the gauge to cancel unphysical states from the final spectrum, corresponding to longitudinal oscillations of the worldsheet. (3.5) has many symmetries:

(*i*) d-dimensional global Poincaré symmetry

$$\begin{aligned} X^M(\xi) &= \Lambda^M_N X^N(\xi) + a^M, \\ g'_{ab}(\xi) &= g_{ab}(\xi). \end{aligned} \tag{3.6}$$

(*ii*) local diffeomorphisms, so invariance under coordinate reparametrizations of the worldsheet coordinates ξ

$$\begin{aligned} \xi'^a &= \xi'^a(\xi), \\ X'^M(\xi') &= X^M(\xi), \\ g'_{ab}(\xi') &= \frac{\partial \xi^c}{\partial \xi'^a}\frac{\partial \xi^d}{\partial \xi'^b} g_{cd}(\xi). \end{aligned} \tag{3.7}$$

(*iii*) local rescalings of the worldsheet metric, or *Weyl symmetry*

$$\begin{aligned} X^M(\xi) &= X^M(\xi), \\ g'_{ab}(\xi) &= \Omega(\xi) g_{ab}(\xi). \end{aligned} \tag{3.8}$$

Another virtue of (3.5) is that it can be easily coupled with gravity by replacing η_{MN} with a general background metric $G_{MN}(X(t, \sigma))$. This is an useful way to recast the interaction between a string and the background, where its quanta (e. g. the gravitons) are given by other strings.

There are only two kinds of fundamental objects in this theory: open strings and closed strings. They result in two distinct topologies for the worldsheet: a $2d$ strip or a cylinder-like topology. Despite being globally distinct, the local dynamics of any string look the same. Therefore, in the following, we will present some details of the closed case and some relevant results also of the open one. More details can be found in standard textbooks, such as [15] and [16].

EOMs of (3.5) can be derived by standard variational approach. The EOMs of the worldsheet metric g_{ab} are

$$\frac{\delta S_P}{\delta g_{ab}} = 0 \quad \Rightarrow \quad -\frac{1}{2} g_{ab} g^{cd} \partial_c X^M \partial_d X_M + \partial_a X^M \partial_b X_M = 0. \tag{3.9}$$

By using diffeomorphisms invariance and Weyl symmetry, we impose the *conformal gauge*, where $g_{ab} = \eta_{ab}$. Then, the EOMs (3.9) are not truly dynamical EOMs, but rather a set of constraints for the physical states, the *Virasoro constraints*

$$\partial_t X^M \partial_t X_M + \partial_\sigma X^M \partial_\sigma X_M = 0, \quad \partial_t X^M \partial_\sigma X_M = 0. \tag{3.10}$$

Considering a flat target manifold, (3.5) becomes

$$S_{\mathrm{P}} = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\xi \eta^{ab} \partial_a X^M \partial_b X^N \eta_{MN} = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\xi \partial^a X^M \partial_a X_M. \tag{3.11}$$

Thus, EOMs are

$$\Box X^M \equiv \left(\partial_t^2 - \partial_\sigma^2\right) X^M = 0. \tag{3.12}$$

It is the standard D'Alambert equation for a set of functions, solved by the standard solution in terms of left and right modes

$$X^M(t,\sigma) = X_L^M(t+\sigma) + X_R^M(t-\sigma). \tag{3.13}$$

For a more explicit form, boundary conditions are necessary. They depend on the topology of the considered string. For closed strings, at $t = 0$, it can be chosen an arbitrary reference line where $\sigma \in [0, \ell]$. ℓ is the length in σ-direction, but it is arbitrary as the parametrization along σ itself. The correct conditions to impose are the periodic conditions

$$X^M(t,\sigma) = X^M(t, \sigma + \ell), \quad \forall \sigma \in [0, \ell] \tag{3.14}$$

3.1.2 Quantization of relativistic strings

(3.14) suggests to expand X^M into Fourier modes. Then, by writing the Hamiltonian of the theory, we can proceed with canonical quantization.
Before doing this, it is simpler to define the light-cone coordinates $X^\pm$

$$X^\pm = \frac{1}{\sqrt{2}} \left(X^0 \pm X^1\right), \tag{3.15}$$

and leave $i = 2, \ldots, d$ for the others. The spacetime metric in these coordinates looks like $\eta_{+-} = \eta_{-+} = -1, \eta_{ij} = \delta_{ij}$, so for a vector V^M we have $V_- = -V^+, V_+ = -V^-$, and $V_i = V^i$.
Thanks to X^+, we can impose the gauge fixing condition

$$X^+(t,\sigma) = t \tag{3.16}$$

that fixes Weyl symmetry. X^- is determined by Virasoro constraints (3.10), that determine X^- as a function of X^i. So, two degrees of freedom are already fixed (the

coordinates of the center of mass are not fixed, but it simply moves linearly and we are interested in internal motion: the oscillations). They break Lorentz invariance explicitly, so the group $SO(d-1,1)$ is broken into $SO(d-2)$.
The classical dynamics is encoded in the lagrangian

$$\mathcal{L} = -\frac{1}{4\pi\alpha'}\int_0^\ell d\sigma\left(2\partial_t X^- - \partial_t X^i \partial_t X^i + \partial_\sigma X^i \partial_\sigma X^i\right) = \int_0^\ell d\sigma L. \tag{3.17}$$

We introduce the conjugate momenta to the coordinates $X^\pm$ and X^i

$$\Pi_- = -\Pi^+ = \frac{\partial L}{\partial\left(\partial_t X^-\right)} = -\frac{1}{2\pi\alpha'}, \tag{3.18}$$

$$\Pi^i = \frac{\partial L}{\partial\left(\partial_t X^i\right)} = \frac{1}{2\pi\alpha'}\partial_t X^i. \tag{3.19}$$

Thus, the hamiltonian is

$$H = \frac{1}{2}\int_0^\ell d\sigma\left(2\pi\alpha'\Pi_i\Pi_i + \frac{1}{2\pi\alpha'}\partial_\sigma X^i\partial_\sigma X^i\right). \tag{3.20}$$

Now, we can expand in Fourier modes as suggested by (3.14)

$$X^i(t,\sigma) = \frac{\Pi^i}{\Pi^+}t + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\left[\frac{\alpha_n^i}{n}e^{-2\pi i n(t+\sigma)\ell} + \frac{\tilde{\alpha}_n^i}{n}e^{-2\pi i n(t-\sigma)\ell}\right]. \tag{3.21}$$

The first term in the square brackets comes from the expansion of left-sector X_L^i which contributes with the amplitudes α_n^i, while the second term from the expansion of right-sector X_R^i which contributes with $\tilde{\alpha}_n^i$.
Finally, we can promote all quantities to operators (specified by small letters: $\Pi^i \to \hat{\Pi}^i \equiv p^i$) and impose the canonical commutators

$$\begin{aligned}&\left[x^-, p^+\right] = -i, \quad \left[x^i, p_j\right] = i\delta^i_j,\\ &\left[\alpha^i_m, \alpha^j_n\right] = \left[\tilde{\alpha}^i_m, \tilde{\alpha}^j_n\right] = m\delta^{ij}\delta_{m,-n}, \quad \left[\alpha^i_m, \tilde{\alpha}^j_n\right] = 0.\end{aligned} \tag{3.22}$$

Hamiltonian operator looks like

$$\begin{aligned}H &= \sum_{i=2}^{D-1}\frac{p_i^2}{2p^+} + \frac{1}{\alpha' p^+}\sum_i\sum_{n>0}\left[\left(\alpha^i_{-n}\alpha^i_n + \frac{1}{2}n\right) + \left(\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + \frac{1}{2}n\right)\right] \equiv\\ &\equiv \frac{1}{2p^+}p_i p^i + \frac{1}{\alpha' p^+}\left[\sum_i\sum_{n>0}\left(\alpha^{i\dagger}_n\alpha^i_n + \tilde{\alpha}^{i\dagger}_n\tilde{\alpha}^i_n\right) + E_0 + \tilde{E}_0\right].\end{aligned} \tag{3.23}$$

where the first term is the kinetic energy associated to the motion of the center of mass, with the constant $p^+ = -\ell/2\pi\alpha'$. The summations in the square brackets give the hamiltonian of two infinite sets of decoupled harmonic oscillators and E_0 and $\tilde{E}_0$ are zero point energies. Clearly, they are equal and involve a divergent sum. They can be regularized by standard techniques, resulting in

$$\begin{gathered}
E_0 = \tilde{E}_0 = \sum_i \frac{1}{2} \sum_{n=1}^{\infty} n \mapsto \lim_{\varepsilon \to 0} \frac{1}{2} \sum_{n=0}^{\infty} n e^{-n\varepsilon} = -\frac{1}{2} \lim_{\varepsilon \to 0} \left(-\frac{1}{\varepsilon^2} + \frac{1}{12} + \dots \right) \Rightarrow \\
\Rightarrow E_0 = \tilde{E}_0 = -\frac{d-2}{24}.
\end{gathered} \tag{3.24}$$

We can now proceed with the construction of the Hilbert space. The operators $\alpha_n^{i\dagger}$ and α_n^i form a pair of creation and annihilation operators for left modes, while the operators with the symbol "$\sim$" are creation and annihilation operators for right modes. The vacuum is defined as always as the state such that

$$\alpha_n^i |0\rangle = \tilde{\alpha}_n^i |0\rangle = 0 \quad \forall n > 0, \quad \forall i = 2, \dots, d. \tag{3.25}$$

The number of left and right modes are

$$N = \sum_i \sum_{n>0} \alpha_n^{i\dagger} \alpha_n^i, \quad \tilde{N} = \sum_i \sum_{n>0} \tilde{\alpha}_n^{i\dagger} \tilde{\alpha}_n^i. \tag{3.26}$$

Before considering excited states, we need to impose invariance under σ reparametrizations, then σ-translations generated by the momentum operator

$$P_\sigma = \int_0^\ell d\sigma \Pi_i \partial_\sigma X^i = \frac{2\pi}{\ell}(N - \tilde{N}), \tag{3.27}$$

via

$$\begin{aligned}
f(\sigma_0) \mapsto f(\sigma_0 + d\sigma) &= f(\sigma_0) + d\sigma \frac{d}{d\sigma}\bigg|_{\sigma_0} f + \frac{1}{2}(d\sigma)^2 \frac{d^2}{d\sigma^2}\bigg|_{\sigma_0} f + \cdots \\
&= \left(\hat{I} + d\sigma \frac{d}{d\sigma}\bigg|_{\sigma} + \frac{1}{2}(d\sigma)^2 \frac{d^2}{d\sigma^2}\bigg|_{\sigma} + \cdots \right) f\,(f_0 = \\
&= e^{d\sigma \frac{d}{d\sigma}}\bigg|_{\sigma_0} f = e^{iP_\sigma d\sigma}\big|_{\sigma_0} f.
\end{aligned} \tag{3.28}$$

Translational invariance corresponds then to $e^{iP_\sigma d\sigma}\big|_{\sigma_0} f = f(\sigma_0)$, ensured by $P_\sigma = 0$. This imposes the *level matching condition*

$$N = \tilde{N}. \tag{3.29}$$

String oscillations correspond to particle in spacetime with mass M given by the on-shell relation

$$M^2 = -p^2 = 2p^+p^- - p_i p_i. \tag{3.30}$$

Because of the gauge fixing condition (3.16), we have

$$p^- = -p_+ = i\frac{\partial}{\partial x^+} = i\frac{\partial}{\partial t} = H. \tag{3.31}$$

So, the on-shell relation becomes $M^2 = 2p^+H - p_ip_i$ and then

$$\frac{\alpha' M^2}{2} = N + \tilde{N} + E_0 + \tilde{E}_0. \tag{3.32}$$

The ground state, with $N = \tilde{N} = 0$, has a mass

$$\frac{\alpha' M^2}{2} = -\frac{(d-2)}{12} \tag{3.33}$$

Finally, we can consider the first excited state, determined by $N = \tilde{N} = 1$

$$\alpha_1^{i\dagger}\tilde{\alpha}_1^{j\dagger}|0\rangle. \tag{3.34}$$

By the level matching condition, both left and right operators act. This means that the state carries two Lorentz indexes: it transforms non-trivially under a representation of $SO(d-2)$. However, a generic momentum p^μ in d dimensions transforms under a representation of the little group $SO(d-1)$, with the only exception being the massless case, where $p^\mu = (E, E, 0, \ldots, 0)$ is invariant under $SO(d-2)$. Therefore, the first excited state must necessarily be massless:

$$\frac{\alpha' M^2}{2} = 2 - \frac{(d-2)}{12} = 0 \Rightarrow d = 26. \tag{3.35}$$

This sets the so-called *critical dimension*: the number of spacetime dimensions required to preserve Lorentz invariance. We need to reduce the state (3.34) into irreducible representations: by doing so, we obtain a two-index symmetric traceless field G_{MN}, representing the *graviton*, a scalar field ϕ called the *dilaton* and a two-index antisymmetric 2-form field B_{MN}. Higher excited states automatically belong to $SO(d-1)$, independently by the value of d ([17]).
Furthermore, using the value for the critical dimension in (3.33), we find a negative squared mass. This corresponds to a *tachyon* T, a particle that moves faster than light. This is clearly problematic: tachyons could potentially indicate that the theory has been expanded around a maximum rather then a minimum. However, the fate of tachyonic instabilities is not clear in closed string theories. We will no longer explore them, because SUSY will automatically cancel them. To summarize, the ground and the first excited state are

Sector	State	$\alpha' M^2$	26d field
$N = \tilde{N} = 0$	$\|0\rangle$	-4	T
$N = \tilde{N} = 1$	$\alpha_1^{i\dagger}\alpha_1^{j\dagger}\|0\rangle$	0	G_{MN}, B_{MN}, ϕ

(3.36)

At energies $E \ll M_s$, the resulting theory is an effective field theory in 26 dimensions. At leading order in α' and by ignoring the tachyon, the effective action is

$$S_{26d}^{\text{E.f.}} = \frac{1}{2\kappa^2} \int d^{26}X (-\tilde{G})^{1/2} \left(\tilde{R} - \frac{1}{12} e^{-\tilde{\phi}/3} H_{MNP} H^{MNP} - \frac{1}{6} \partial_M \tilde{\phi} \partial^M \tilde{\phi} \right) + \mathcal{O}(\alpha') , \tag{3.37}$$

The symbol "$\sim$" refers to the *Einstein frame*, in which there is a more transparent connection with the $4d$ final theory: $\tilde{\phi} = \phi - \phi_0, \quad \tilde{G}_{MN} = e^{(\phi_0 - \phi)/6} G_{MN}$. The strength field H is the analog of F for a Maxwell-like theory, so $H = dB$. $\tilde{G}$ and $\tilde{R}$ are the metric and the Ricci scalar built from $\tilde{G}_{MN}$. Finally, the dilaton vev ϕ_0 governs the gravitational coupling k and the *string coupling constant* $g_s = e^{\phi_0}$. It is the coupling of the interaction between strings. Notice 2 things:

(i) there are no free-parameters in string theory, in the sense that pure couplings are determined by vevs of fields;

(ii) in the EFT there are two expansion parameters, α' and g_s: perturbative string theory works when both are small $\alpha', g_s \ll 1$.

3.1.3 Open strings

Open strings display the same local dynamics of closed strings, so we will very briefly recap some differences. The main point is that open strings have a different boundary conditions with respect to (3.14). Now, we have a strip as worldsheet, where σ varies from 0 to ℓ. The variation of action (3.5) leads to

$$\begin{aligned} 0 = \delta S_P &= -\frac{1}{2\pi\alpha'} \int_\Sigma d^2\xi \eta^{ab} \partial_a X^M \partial_b \delta X_M \\ &= -\frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} dt \left(\delta X^M \partial_\sigma X_M \right) \Big|_{\sigma=0}^{\sigma=\ell} + \frac{1}{2\pi\alpha'} \int_\Sigma d^2\xi \delta X^M \partial_a \partial^a X_M . \end{aligned} \tag{3.38}$$

The second term gives rise again to (3.12) as EOMs. This confirms that local dynamics for open and closed strings are the same. The first term defines the boundary conditions, which can be of two types:

$$\partial_\sigma X^M \big|_{\sigma=0,\ell} = 0 \quad \text{(Neumann)} , \quad \delta X_M \big|_{\sigma=0,\ell} = 0 \quad \text{(Dirichlet)}. \tag{3.39}$$

The Neumann boundary conditions describe endpoints moving freely, conserving the momentum. The Dirichlet boundary conditions correspond to endpoints constrained to move along fixed hyperplanes and break Poincaré invariance. One Dirichlet boundary condition reads as $X^M = \text{const}$ for some $M = 0, \ldots, d$. This defines a hyperplane in the

target spacetime, called a *Dp-brane*, where D stands for Dirichlet and p is the number of the space dimensions filled by the brane, i. e. its spatial dimension. Because they must always fill the time direction, they are $(p+1)$-dimensional objects. Dp-branes are also dynamical objects that play a fundamental role in string theory.
These conditions must be imposed on both string endpoints: both Neumann conditions are called *Neumann-Neumann (NN) boundary conditions*. For the left and right modes, NN boundary conditions read as

$$\partial_\sigma X_L^M + \partial_\sigma X_R^M = 0 \text{ at } \sigma = 0, \ell. \tag{3.40}$$

This means that for open strings, left and right modes can be identified with each other. Physically, this corresponds to the phenomenon where modes are reflected at the endpoints and start moving backward. Expanding in modes analogously to (3.21) and imposing $\partial_\sigma X^i = 0$ at $\sigma = 0$ and $\sigma = \ell$, we obtain

$$\alpha_n^i = \tilde{\alpha}_n^i, \quad n \in \mathbb{Z}. \tag{3.41}$$

Thus, we can consider just left or right modes: in the following, we will focus on left modes. Expanding in these modes the hamiltonian (3.20), we obtain

$$H = \frac{\sum_i p_i^2}{2p^+} + \frac{1}{2\alpha' p^+}\left(\sum_i \sum_{n>0} \alpha_n^{i\dagger}\alpha_n^i - 1\right) \equiv \frac{1}{2p^+}p^i p_i + \frac{1}{2\alpha' p^+}\left(N-1\right). \tag{3.42}$$

The on-shell relation now reads as

$$\alpha' M^2 = N - 1. \tag{3.43}$$

Now, we can build the Hilbert space for these left modes. The first excited state corresponds to $N = 1$

$$\alpha_1^{i\dagger}|0\rangle. \tag{3.44}$$

We see just one Lorentz index, so it corresponds to a massless vector in the target spacetime. The ground state still give rise to a tachyon

Sector	State	$\alpha' M^2$	26d field
$N=0$	$\lvert 0\rangle$	-1	T
$N=1$	$\alpha_1^{i\dagger}\lvert 0\rangle$	0	A_M

(3.45)

3.2 Superstring theories

Bosonic string theory is intriguing: it automatically results in General Relativity through quantization, describes gauge interactions, predicts the number of spacetime dimensions

and prevents free-parameters. However, the absence of fermions excludes this theory from being physically viable. The solution is to include SUSY as *a symmetry on the worldsheet*. This is a subtle point: SUSY in encoded in principle on the worldsheet, not in the target manifold. For example, type 0 string theory has worldsheet SUSY, but it does not admit a SUSY spacetime spectrum, resulting again in the lack of spacetime fermions. Therefore, the presence of spacetime SUSY and spacetime fermions is not requested, but rather a consequence of the theory after applying a GSO projection, a mathematical projection on the spectrum of superstring theories, described in this section.
The process of defining a SUSY version of string theory is not unique, resulting in different theories. However, we are primarily interested in type IIB string theory.

3.2.1 Worldsheet supersymmetry

We can follow a similar procedure to the one outlined in 2.1 to embed SUSY. However, SUSY is defined on the $2d$ worldsheet rather than in the target spacetime. Spacetime SUSY and superpartners will emerge at the end of the process, resulting in a SUSY theory free from tachyonic instabilities.
In practice, we add fermionic directions parametrized θ_α and promote scalar fields X^M to superfields

$$X^M(\xi^a) \mapsto Y^M(\xi^a, \theta^\alpha, \overline{\theta}^{\dot{\alpha}}) = X^M(\xi^a) + \bar{\theta}\psi^M(\xi^a) + \frac{1}{2}\bar{\theta}\theta B^M(\xi^a), \tag{3.46}$$

where fermions $\psi^\mu(\xi^a)$ are fermionic superpartners of X^M and B^M are auxiliary fields. The SUSY generator is

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\left(\gamma^a\theta\right)_\alpha \frac{\partial}{\partial\xi^a} \equiv \partial_\alpha + i\left(\gamma^a\theta\right)_\alpha \partial_a, \tag{3.47}$$

where γ^a are two-dimensional Dirac matrices. Thanks to the supercovariant derivative

$$D_\alpha = \partial_\alpha - i\left(\gamma^a\theta\right)_\alpha \partial_a, \tag{3.48}$$

we can write down the SUSY invariant version of the Polyakov action (3.11)

$$\begin{aligned} S &= \frac{i}{8\pi\alpha'}\int d^2\xi d^2\theta (\overline{D}^{\dot{\alpha}} Y^M)\left(D_\alpha Y_M\right) = \\ &= -\frac{1}{4\pi\alpha'}\int d^2\xi \left(\partial_a X^M \partial^a X_M - i\bar{\psi}^M \partial\!\!\!/ \psi_M - B^M B_M\right). \end{aligned} \tag{3.49}$$

On-shell, auxiliary fields disappear, leading to

$$S = S_{\mathrm{P}} + S_{\mathrm{F}} = -\frac{1}{4\pi\alpha'}\int d^2\xi \left(\partial^a X^M \partial_a X_M - i\bar{\psi}^M \partial\!\!\!/ \psi_M\right). \tag{3.50}$$

This is the new action to be quantized, which now also includes fermions.
As with the bosonic case, it is necessary to specify the topology of the strings being considered. However, the situation is more complex now. Different choices result in distinct theories. For instance, open strings result in *type I string theory*, while closed strings lead to *type II string theories*, particularly interesting for us due to their direct connection with phenomenology.

3.2.2 Neveu-Schwarz and Ramond sectors

When working with closed strings, we have periodic boundary conditions as in (3.14) for both bosonic X^M and fermionic ψ^M fields. This allows us to decouple the left and right modes. They are decoupled, apart from the level matching condition, so we can consider them separately. In light-cone quantization, we are left with only $i = 2, \ldots, d$ degrees of freedom. Fermions always appear quadratically in observables, so we have a further decomposition based on their periodicity or antiperiodicity:

$$\begin{array}{ccc} \text{Neveu-Schwarz} & \text{NS} & \psi_L^i(t+\sigma+\ell) = -\psi_L^i(t+\sigma), \\ \text{Ramond} & \text{R} & \psi_L^i(t+\sigma+\ell) = \psi_L^i(t+\sigma). \end{array} \tag{3.51}$$

This choice is made independently for right modes, defining four coexisting *sectors* of the theory, namely NS-NS, NS-R, R-NS and R-R (the first letter refers to the sector of the left modes).
These boundary conditions result in different mode expansions: the antiperiodic boundary condition requires odd terms in the expansion, as

$$\psi_L^i(t+\sigma) = i\sqrt{\frac{\alpha'}{2}} \sum_{r\in\mathbf{Z}} \psi^i_{r+1/2} e^{-2\pi i(r+1/2)(t+\sigma)/\ell}, \tag{3.52}$$

while R sector has even modes

$$\psi_L^i(t+\sigma) = i\sqrt{\frac{\alpha'}{2}} \sum_{r\in\mathbf{Z}} \psi^i_r e^{-2\pi i r(t+\sigma)/\ell}. \tag{3.53}$$

The amplitudes are promoted to anticommuting operators

$$\left\{\psi_r^i, \psi_s^j\right\} = \delta_{rs}\delta^{ij} \quad \text{with} \quad \left\{ \begin{array}{ll} r, s \in \mathbb{Z} & \text{(R)} \\ r, s \in \mathbb{Z} + \frac{1}{2} & \text{(NS)} \end{array} \right. . \tag{3.54}$$

From the action (3.50), we can extract the hamiltonian of the theory. From the Polyakov part S_P, we obtain a bosonic sector described again by (3.23), now H_B. From the fermionic part S_F, we obtain a hamiltonian for the left modes in NS sector as

$$H_{F_{\text{NS}},L} = \frac{1}{\alpha' p^+} \left[\sum_{r=0}^{\infty} \left(r + \frac{1}{2} \right) \psi^i_{-r-1/2} \psi^i_{r+1/2} + E_0^{F_{NS}} \right], \tag{3.55}$$

where the zero point energy is

$$E_0^{F_{\rm NS}} = -\frac{1}{2}(d-2)\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right) \overset{\rm reg.}{=} -\frac{1}{48}(d-2) \tag{3.56}$$

Analogous equations hold for right modes, replacing with an upper "$\sim$". Then, the full zero energy of the NS sector is

$$E_0^B + \tilde{E}_0^B + E_0^{F_{\rm NS}} + \tilde{E}_0^{F_{\rm NS}} = 2\left(E_0^B + E_0^{F_{\rm NS}}\right) = -\frac{1}{8}(d-2). \tag{3.57}$$

Defining the left and right fermionic number operators

$$N_{F_{\rm NS}} = \sum_i \sum_{r=0}^{\infty}\left(r+\frac{1}{2}\right)\psi^{i\dagger}_{r+1/2}\psi^i_{r+1/2}, \quad \tilde{N}_{F_{\rm NS}} = \sum_i \sum_{r=0}^{\infty}\left(r+\frac{1}{2}\right)\tilde{\psi}^{i\dagger}_{r+1/2}\tilde{\psi}^i_{r+1/2}. \tag{3.58}$$

Finally, the hamiltonian operator is

$$H_{\rm NS} = \frac{\sum_i p_i^2}{2p^+} + \frac{1}{\alpha' p^+}\left(N_{F_{\rm NS}} + \tilde{N}_{F_{\rm NS}} + N_B + \tilde{N}_B - \frac{d-2}{8}\right), \tag{3.59}$$

that must be supplemented by the level matching conditions for both bosons and fermions. The usual on-shell relation (3.30) gives us

$$\frac{\alpha' M_{\rm NS}^2}{4} = N_{F_{\rm NS}} + N_B - \frac{d-2}{8}. \tag{3.60}$$

Now, we can construct the Hilbert space of the NS sector. The vacuum state $|0\rangle_{\rm NS}$ is defined as

$$\begin{aligned} &\psi^i_{r+1/2}|0\rangle_{\rm NS} = \tilde{\psi}^i_{r+1/2}|0\rangle_{\rm NS} = 0 \quad \forall i, \forall r \geq 0; \\ &\alpha^i_n|0\rangle_{\rm NS} = \tilde{\alpha}^i_n|0\rangle_{\rm NS} = 0 \quad \forall i, \forall n > 0. \end{aligned} \tag{3.61}$$

Excited states are given by applying creation operators the ground state. The first excited state corresponds now to $N_{F_{\rm NS}} = 1/2$ and $N_B = 0$. As before, this physical state belongs to representations of $SO(d-2)$, which coincide with those of the little group of massless states. The critical dimension is then

$$0 = \frac{\alpha' M_{\rm NS}^2}{4} = \frac{1}{2} - \frac{d-2}{16} \quad \Rightarrow \quad d = 10, \tag{3.62}$$

common to all sectors and types of superstring theories. With this value, the ground state has a negative mass squared, so again a tachyon.
For R sector, the procedure is the same. The hamiltonian is

$$H_{\rm R} = \frac{\sum_i p_i^2}{2p^+} + \frac{1}{\alpha' p^+}\left(N_{F_{\rm R}} + \tilde{N}_{F_{\rm R}} + N_B + \tilde{N}_B\right), \tag{3.63}$$

where the new number operator is

$$N_{F_{\rm R}} = \sum_{r=1}^{\infty} r \psi_r^{i\dagger} \psi_r^i. \tag{3.64}$$

The absence of a zero-point energy is due to the already fixed critical dimension value:

$$E_0^{F_{\rm R}} = -\frac{1}{2}(d-2)\sum_{r=1}^{\infty} r \overset{\text{reg.}}{=} \frac{1}{24}(d-2) \overset{d=10}{\Longrightarrow} E_0^{F_{\rm R}} = \frac{1}{3}, \tag{3.65}$$

that precisely cancels the bosonic zero-point energy (3.24). Finally, the mass spectrum is now

$$\frac{\alpha' M_{\rm R}^2}{4} = N_{F_{\rm R}} + N_B, \tag{3.66}$$

where any excited state is massive. However, there is an important difference with respect to NS sector: fermionic zero modes ψ_0^i contribute with vanishing energy to the vacuum. Therefore, the vacuum is now more subtle: the full construction is beyond our scope, but it transforms under the 16-dimensional spinorial representation of $SO(8)$. It is reducible in $\mathbf{8}$ and $\mathbf{8}'$, two irreducible representations, with opposite chirality. All of them are ground states. Thus, the vacuum degenerates in 16 ground states, which are the only massless states of the R sector.

3.2.3 Type IIB superstring theory

We have obtained fermions thanks to SUSY. However, we still have a tachyon, so the result is not yet physically allowed. As often in quantum mechanics, the physical Hilbert space of a system composed of many subsystems is not just the tensor product of the single Hilber spaces, but involve also the projection onto some irreducible representation (e. g. the symmetric or the antisymmetric subspaces for a system of bosons or fermions). This suggest defining a projection operator P to cancel the tachyon (and many other states).
The starting point is the operator $(-1)^{N_F}$: it signals the fermionic nature of a state ($(-1)^{N_F} = -1$) rather than the bosonic one ($(-1)^{N_F} = 1$), taking into account that an even number of fermions correspond to bosons. It is a fermionic operator, so it anticommutes with other fermionic operators. Furthermore, we require that it acts on the ground state of NS sector as

$$(-1)^{N_F}|0\rangle_{\rm NS} = -|0\rangle_{\rm NS} \tag{3.67}$$

We write a sector of a given eigenvalue of $(-1)^{N_F}$ as NS-, for instance. By the level matching condition, $N_F = \tilde{N}_F$. So, for NS-, $N_F = 0$ implies that the only possible pairing between left and right modes is (NS-, NS-), corresponding to the tachyon. Others

sectors are free to pair in arbitrary ways, forming sectors as (NS+,NS+), (NS+, R+), (NS+, R-), (R-, NS+), etc. Consequently, there are 10 states: the tachyon (NS-, NS-) and 9 massless states which contain combinations of vectors and spinors in different representations. We need to choose which sectors to use to build a theory, for a total of 2^{10} possible choices. However, we need to exclude the tachyon, and taking into account other consistent criteria, the possible choices reduce to only 2.
Now, we define the projection P as

$$P = \frac{1}{2}\left(1 + (-1)^{N_F}\right). \tag{3.68}$$

It is the *Gliozzi-Scherk-Olive projection* or *GSO projection*. The GSO operator automatically vanishes on $|0\rangle_{\text{NS}}$, because of (3.67), cancelling the tachyon. To move forward, we need to select a possible action for the operator $(-1)^{N_F}$: $(-1)^{N_F} = 1$ for all left and right modes and NS and R boundary conditions. This choice leads to the *type IIB superstring theory*. Its spectrum is

Sector	$SO(8)$ rep.	10 d field
(NS+, NS+)	$8_v \times 8_v = 1 + 28 + 35$	ϕ, B_{MN}, G_{MN}
(NS+, R+)	$8_v \times 8 = 8' + 56$	$\lambda^1_\alpha, \psi^1_{M\alpha}$
(R+, NS+)	$8 \times 8_v = 8' + 56$	$\lambda^2_\alpha, \psi^2_{M\alpha}$
(R+, R+)	$8 \times 8 = 1 + 28 + 35_+$	a, C_{MN}, C_{MNPQ}

(3.69)

We recognize the standard dilaton field ϕ, the 2-form field B_{MN} and the graviton G_{MN}. There are also other bosons, coming from the (R+, R+) sector: a pseudo-scalar axion field a or C_0 (because it is a 0-form), another 2-form C_{MN} or C_2 and a 4-form C_{MNPQ} or C_4. Fermions arise from the mixed sectors (NS+, R+) and (R+, NS+): we have two spinors λ_α known as *dilatinos* and two Rarita-Schwinger gravitinos, which are superpartners of the graviton. Then, there is $\mathcal{N} = 2$ SUSY.
The full bosonic sector of the low-energy theory is described by the action

$$\begin{aligned} S_{\text{IIB}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x (-G)^{\frac{1}{2}} \left[e^{-2\phi} \left(R + 4\partial_M \phi \partial^M \phi - \frac{1}{2} |H_3|^2 \right) - \frac{1}{2} |F_1|^2 - \frac{1}{2} \left|\tilde{F}_3\right| - \frac{1}{4} \left|\tilde{F}_5\right|^2 \right] + \\ - \frac{1}{4\kappa_{10}^2} \int_{10d} C_4 \wedge H_3 \wedge F_3. \end{aligned} \tag{3.70}$$

where $2\kappa_{10}^2 = (2\pi)^7 \alpha'^4$, $H_3 = dB_2$, $F_p = dC_{p-1}$ and the other forms are defined as

$$\tilde{F}_3 = F_3 - C_0 H_3, \quad \tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3. \tag{3.71}$$

They define a higher gauge theory, which is invariant under the gauge transformation

$$C_p \mapsto C_p + d\Lambda_{p-1}. \tag{3.72}$$

Moreover, the self-duality relation

$$\tilde{F}_5 = *\tilde{F}_5, \tag{3.73}$$

halves the number of degrees of freedom. The term in the second line is the *Chern-Simons term* S_{CS}, which is metric independent and then purely topological.
In order to establish a more direct connection with standard fields in $4d$, we can rewrite this action from the string frame to the Einstein frame, defining the *axio-dilaton* τ, the 3-form G_3 and the Einstein metric

$$\begin{aligned} \tau &:= C_0 + ie^{-\phi} \\ G_3 &:= F_3 - \tau H_3 \\ G_{E,MN} &:= e^{-\phi/2} G_{MN} \end{aligned} \tag{3.74}$$

the final action is

$$\begin{aligned} S_{\text{IIB}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G_E} \left[R_E - \frac{|\partial\tau|^2}{2(\text{Im}(\tau))^2} - \frac{|G_3|^2}{2\,\text{Im}(\tau)} - \frac{\left|\tilde{F}_5\right|^2}{4} \right] \\ - \frac{i}{8\kappa_{10}^2} \int \frac{C_4 \wedge G_3 \wedge \bar{G}_3}{\text{Im}(\tau)}. \end{aligned} \tag{3.75}$$

Different choices would give rise to different theories. A different GSO projection, such that

$$\text{left:} \quad (-1)^F = 1 \quad \text{right:} \quad (-1)^{\tilde{F}} = 1(\text{NS})/(-1)^{\tilde{F}} = -1(\text{R}) \tag{3.76}$$

leads to the *type IIA superstring theory*. *Unoriented* strings (i. e. where a symmetry under the parity operator acting on the worldsheet is imposed) result in *type I*. Admitting worldsheet SUSY only on left (or right) modes result in the *heterotic* $E_8 \times E_8$ *theory* and the *heterotic* $SO(32)$ *theory*. All of these theories are connected via relations and dualities, as shown in 3.2 and are different limits of the underlying M-theory.

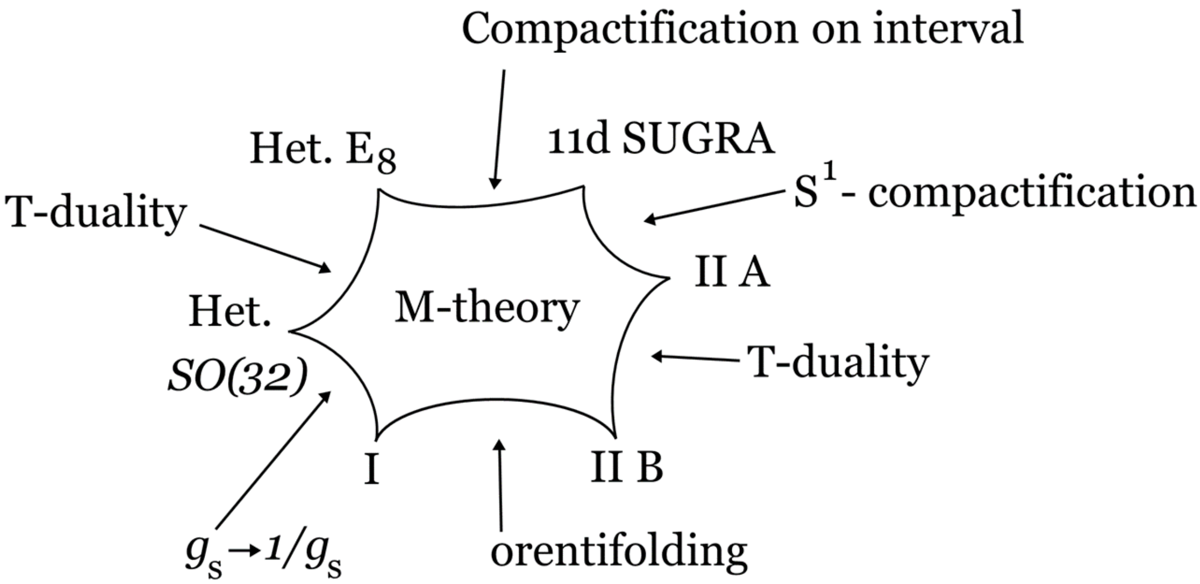

Figure 3.2: all the superstring theories, along with the relations among them. They are approximations of a more fundamental underlying theory, known as M-theory. Picture taken from [4].

Chapter 4

Early Universe Cosmology

Cosmology is currently a significant source of data and observations for fundamental physics. The study of the evolution and history of the Universe as a whole naturally serves as a testing ground for General Relativity. Cosmology also influences the study of elementary particles, by placing constraints on the properties that particles must have in order to reproduce the observed Universe. However, one of the most intriguing perspectives is that, in the coming years, cosmology may provide insights into Quantum Gravity, the theory that aims to unify gravity with other interactions within a single conceptual framework. Among the current attempts to formulate this theory, we will focus on string theory in this work.
To understand its phenomenological implications, it is necessary to review certain concepts of standard cosmology. This includes the theoretical model building based on General Relativity, key observational evidences and unresolved issues.

4.1 Friedman-Lemaître-Robertson-Walker Universe

The dynamics of the Universe is described by the Einstein's theory of General Relativity. Specifically, by assuming homogeneity and isotropy, we can derive the Friedman-Lemaître-Robertson-Walker (FLRW) metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\,\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right], \tag{4.1}$$

written in terms of the comoving coordinates, the *scale factor* $a(t)$ and the curvature k, that after rescaling the coordinates can only takes the values of 0 or ± 1.
By plugging this metric in the Einstein's equations of motion

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N T_{\mu\nu}, \tag{4.2}$$

for an ideal perfect fluid as

$$T^{\mu}_{\nu} = \mathrm{diag}(-\rho, p, p, p), \tag{4.3}$$

we obtain the dynamics of $a(t)$ determined by this matter content, expressed by

$$G_{00} = 8\pi G_{\mathrm{N}} T_{00} \quad \Rightarrow \quad 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = 8\pi G_{\mathrm{N}}\rho, \tag{4.4}$$

$$G_{ii} = 8\pi G_{\mathrm{N}} T_{ii} \quad \Rightarrow \quad 3\frac{\ddot{a}}{a} = -4\pi G_{\mathrm{N}}(\rho + 3p). \tag{4.5}$$

Although (4.5) is the true dynamical equation of motion for $a(t)$ due to the presence of the second-order derivative with respect to the comoving time t, it is more convenient to study (4.4) together with the covariant energy-momentum conservation. This is ensured by the (local) diffeomorphism invariance deeply embedded in General Relativity and expressed as

$$\nabla_{\mu} T^{\mu\nu} = 0. \tag{4.6}$$

For the energy-tensor (4.3), the 00-component of (4.6) reads as

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{4.7}$$

where we have defined the *Hubble parameter*

$$H = \frac{\dot{a}}{a}. \tag{4.8}$$

By this definition, we can rewrite (4.4) as

$$H^2 = \frac{8\pi G_N}{3}\rho - \frac{k}{a^2}, \tag{4.9}$$

Finally, we can rewrite the last equation in a different form that explicitly relates observable quantities and global topological properties of the Universe:

$$\Omega(t) - 1 = \frac{k}{(aH)^2} \tag{4.10}$$

where we have introduced the dimensionless density parameter Ω and the critical density ρ_{critical} defined as

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_{\text{critical}}(t)}, \quad \rho_{\text{critical}}(t) \equiv \frac{3H^2(t)}{8\pi G_N}. \tag{4.11}$$

Thus, the matter distribution in the Universe determines its geometry as

$$\begin{array}{lllllll} \rho < \rho_{\text{critical}} & \Leftrightarrow & \Omega < 1 & \Leftrightarrow & k = -1 & \Leftrightarrow & \text{Open Universe} \\ \rho = \rho_{\text{critical}} & \Leftrightarrow & \Omega = 1 & \Leftrightarrow & k = 0 & \Leftrightarrow & \text{Flat Universe} \\ \rho > \rho_{\text{critical}} & \Leftrightarrow & \Omega > 1 & \Leftrightarrow & k = +1 & \Leftrightarrow & \text{Closed Universe} \end{array} \tag{4.12}$$

Current observations indicate that our Universe is flat, implying that its spatial 3-curvature vanishes. Therefore, we will put $k = 0$ from now on.
As mentioned before, equation (4.5) or the equations (4.4) along with (4.7) determine the dynamics of the scale factor. Explicitly, by assuming the ideal equation of state for the fluid

$$p = \omega\rho, \tag{4.13}$$

with a constant ω, we can extract the scaling of $a(t)$ over time for various cases of interest. These cases are summarized in tab. 4.1.

Stress Energy	ω	Energy Density	Scale Factor $a(t)$
Dust	$\omega = 0$	$\rho_{dust} \sim a^{-3}$	$a(t) \sim t^{2/3}$
Radiation	$\omega = 1/3$	$\rho_{rad} \sim a^{-4}$	$a(t) \sim t^{1/2}$
Vacuum (Λ)	$\omega = -1$	$\rho_\Lambda \sim \frac{\Lambda}{8\pi G_N}$	$a(t) \sim \exp(\sqrt{\Lambda/3}t)$

Table 4.1: Constant ω, scale factor and energy density behaviour for matter (in form of dust), radiation and vacuum dominated universes for $k = 0$.

4.2 Standard Cosmology: the Λ-Cold Dark Matter model

This section presents key evidence supporting the standard model of cosmology, that give rise to the current view of the history of the Universe, which outlines the history of the Universe through various epochs and events. While this model is consistent with current observations, there are still some unresolved issues regarding the initial conditions of the Universe that make our theoretical understanding incomplete.

4.2.1 ΛCDM: main results and evidences

The FLRW equations result in the different scalings shown in tab. 4.1. These scalings demonstrate that different components of the stress-energy tensor dominate during different epochs of the Universe. As a result, the Universe experiences distinct phases, expanding and cooling at varying rates. These phases are summarized in fig. 4.1.
It has been established that our Universe is currently flat, with $k = 0$. Therefore, (4.10) indicates that the density is equal to the critical value. Additionally, we are currently living in an epoch dominated by Dark Energy, so the Hubble parameter is a constant H_0 and

$$\rho_0 = \rho_{\text{critical}} = \frac{3}{8\pi G_N} H_0^2 \sim 10^{-29} \text{ g/cm}^3. \tag{4.14}$$

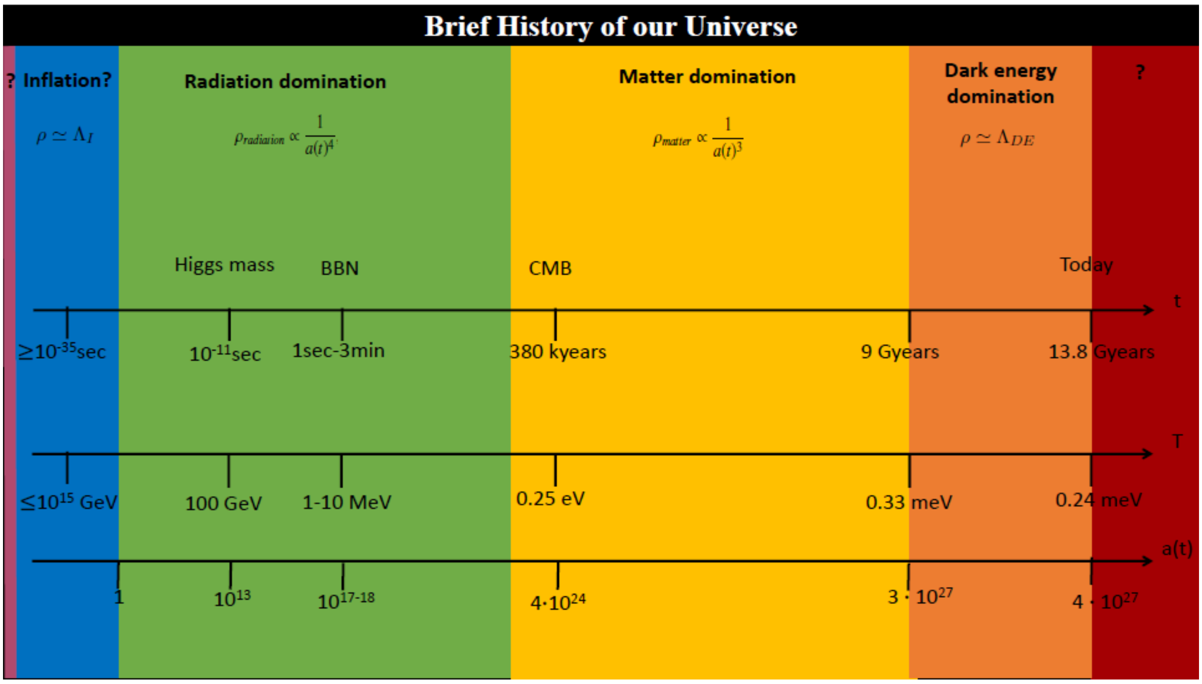

Figure 4.1: A schematic representation of the different epochs and their temperatures within the history of the Universe in the standard ΛCDM cosmological model. Temperature units can be transformed to Kelvin using the conversion factor $1\text{GeV} = 1.16 \times 10^{13}$ K. Picture taken from [13].

An Universe dominated by positive Dark Energy is of *De-Sitter* type. The components of the total energy density ρ_0 of the Universe are

- radiation: ultra-relativistic particles ($v \simeq c$) with $\omega = 1/3$. It is dominated by photons of the *Cosmic Microwave Background* CMB and represents

$$\frac{\rho_{\text{radiation}}}{\rho_0} \simeq 10^{-4}; \tag{4.15}$$

- baryons, which correspond to regular observable matter, with $\omega = 0$ and a fraction

$$\frac{\rho_{\text{matter}}}{\rho_0} \simeq 5\%; \tag{4.16}$$

- Dark Matter, which satisfies the same equation of state than the regular matter with $\omega = 0$, but it remains unobserved directly and its microscopic description is still lacking. However, its fraction of total density is

$$\frac{\rho_{\text{DM}}}{\rho_0} \simeq 25\%; \tag{4.17}$$

- the Dark Energy, with $\omega = -1$ and required by the accelerated expansion of the Universe. Its fraction is
$$\frac{\rho_{\rm DE}}{\rho_0} \simeq 70\%. \tag{4.18}$$

These data describe a dynamic Universe that has undergone various states and significant events. As the Universe expands and grows larger over time, it also cools down. The temperature of the Universe scales as

$$T \propto a^{-1}. \tag{4.19}$$

As a result, different epochs are characterized by distinct temperatures, leading to significant variations in the relevant physics during those times. We would like to highlight some of the more interesting aspects.

- At present, with the Universe being $t \sim 10^{16} - 10^{17}$ s old, galaxy clusters arise in correspondence of small primordial inhomogeneities as a result of gravitational instability. The Universe is homogeneous and isotropic on a large scale and is currently dominated by Dark Energy;

- at $t \sim 10^{12} - 10^{13}$ s, two key events occur. The first is the *recombination*, during which free electrons and protons combine to form neutral Hydrogen nuclei. Simultaneously, photons decouple, making the Universe transparent and allowing light to travel freely, forming the *Cosmic Microwave Background* (CMB) radiation. The CMB spectrum has already been measured and is shown in fig. 4.2. This is one of the strongest evidence supporting the Big Bang model;

- between $t \sim 0.2$ s and $200 - 300$ s, corresponding to $T \sim 1\text{MeV} - 0.05\text{MeV}$, other two significant events happen. Neutrinos decouple from other particles and begin to propagate freely until now, when they are expected to form the *Cosmic Neutrino Background* (CNB). Unfortunately, directly measuring the CNB directly is extremely difficult.
The other important event is the *primordial nucleosynthesis* or *Big Bang Nucleosynthesis* (BBN). During this phase, the Universe is dominated by radiation, so high-energy photons prevent the formation of stable nuclei by protons and neutrons. However, as the Universe cools down, nucleons exhibit a strong tendency to bind together, leading to the formation of nuclei. In particular, light element nuclei such as Hydrogen, Helium-4 and Lithium are synthesized during this phase. The abundance of these elements is fixed at the end of the process and can be calculated: Hydrogen makes up about 75%, Helium-4 25%, Helium-3 and Deuterium about 0.01% and traces of Lithium of order of 10^{-10}. These predictions are in very good agreement with observations, providing another key evidence for the Hot Big Bang model;

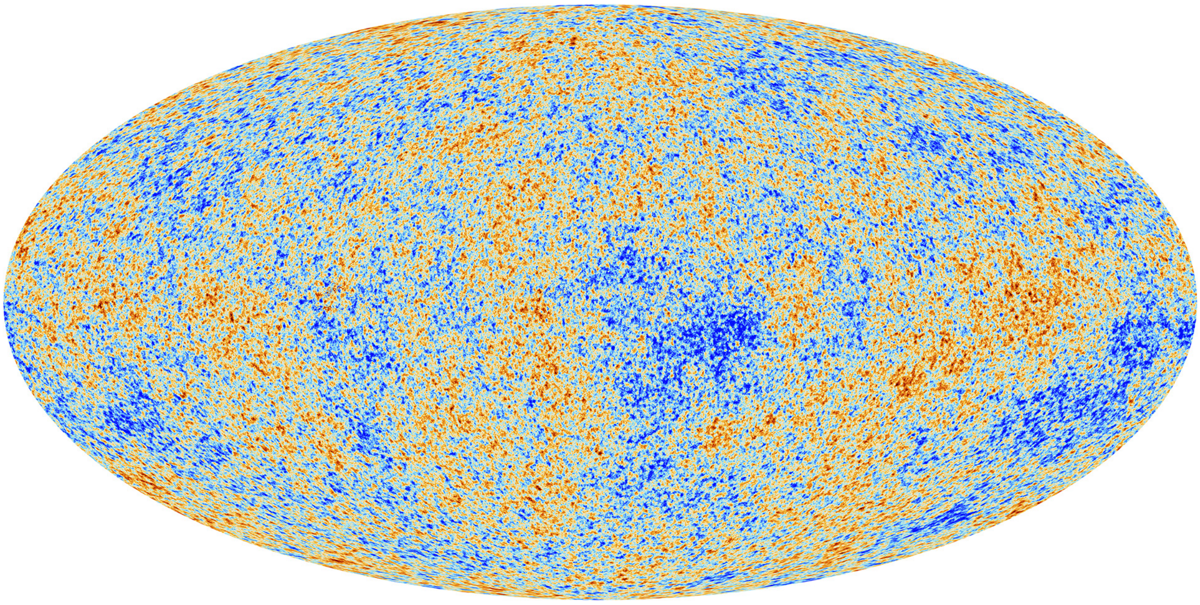

Figure 4.2: Cosmic microwave background radiation seen by ESA's space observatory Planck. Different colors corresponds to very small inhomogeneities in the temperature distribution, less than one part in $10^{-4} - 10^{-5}$. Picture taken from [7].

- between $t \sim 10^{-43} - 10^{-14}$ s, corresponding to $T \sim 10^{19}\text{GeV} - 10^{4}\text{GeV}$, the Universe may be influenced by various proposals that go beyond our current understanding of physics. Hypothesis such as SUSY, string theory and extra-dimensions may play crucial roles in this context. Additionally, a key cosmological mechanism that is expected to occur during this stage is *inflation*, which refers to the rapid accelerated expansion of the early Universe;
- Finally, at $t \sim 10^{-43}$ s ($T \sim 10^{19}\text{GeV}$) we approach the Planck scale, where quantum gravity phenomena are expected to dominate. It is hoped that a theory of quantum gravity will be able to describe this stage and resolve the expected singularity predicted by General Relativity.

4.2.2 ΛCDM: open issues

This picture of the Universe is in very good agreement with many observations. However, there are some discrepancies between predictions and observations, such as the Lithium abundance, known as the *Primordial Lithium Problem* (see [14]), and the lack of a microscopic description of Dark Matter and Dark Energy, which drive the current large-scale structure of the Universe and its accelerated expansion. However, we will consider some problems collectively known as *fine-tuning problems*. These are issues that arise in

many areas of physics and refer to the need to set certain parameters or initial conditions of the model at very unnatural (in some sense) values to reproduce the observed data, without any dynamical (or even physical) explanation. In the cosmological context, we face two such problems: the *Horizon problem* and the *Flatness problem*.
A model of the Universe consists of the laws governing its evolution and the initial values evolved by those laws. We can use the FLRW equations of motion to determine what initial conditions are necessary to reproduce the Universe we observe. These initial conditions are set at the Planck scale, with $t_i = t_P \sim 10^{-43}$ s, assuming that quantum gravity effects are negligible below this scale. The subscript i refers to initial values in this sense, while the subscript 0 stands for current values.
Two fundamental properties that we observe today are the homogeneity and isotropy at the *horizon scale* of $ct_0 \sim 10^{28}$ cm, as confirmed by the CMB in fig. 4.2. Its variations across the sky average less than 0.01%. Initially, this homogeneous region was smaller, with its the relative dimension given by

$$l_i \sim ct_0 \frac{a_i}{a_0}. \tag{4.20}$$

The size of a causal region, i. e. a region in which points can be causally connected, at time t_i is simply $l_c \sim ct_i$. The ratio between these two scales can be estimated as

$$\frac{l_i}{l_c} \sim \frac{t_0}{t_i}\frac{a_i}{a_0} \sim \frac{t_0}{t_i}\frac{T_p}{T_0} \sim 10^{28}. \tag{4.21}$$

This means that, in $3d$ at the Planck time, 10^{84} causally independent regions have the same energy with relative variations smaller than 10^{-4} without any possible physical (in the sense of causal) mechanism allowing for this. The physical dimensions were much larger than the causal scale: this unnatural situation is called horizon problem and is represented in the diagram in fig. 4.3.

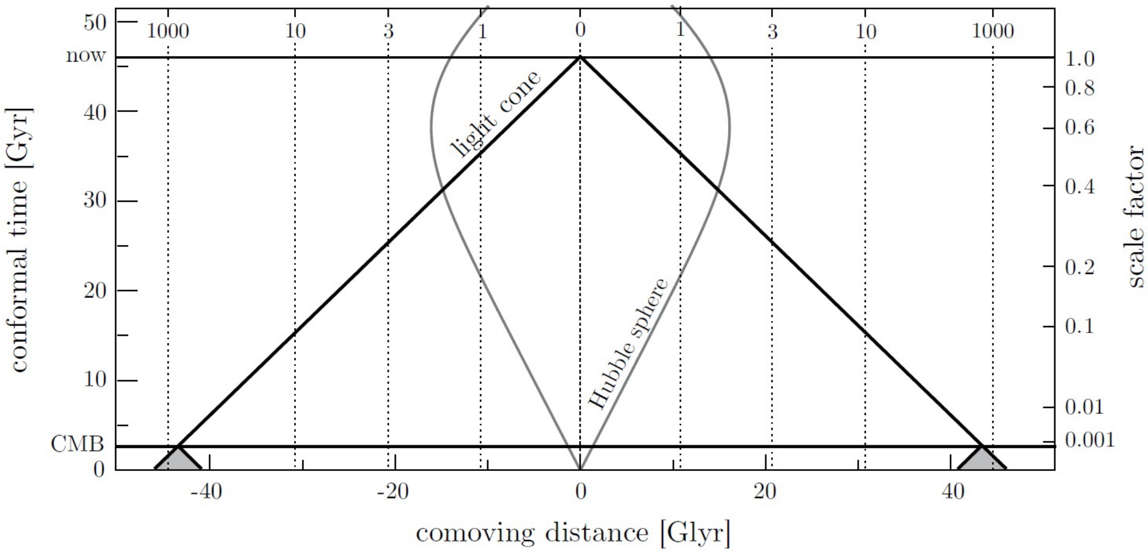

Figure 4.3: Spacetime diagram which illustrates the horizon problem. The vertical axis represents the conformal time, while the horizontal axis represents the comoving distance. The space-like surface called CMB corresponds to the last scattering surface and is the time when the CMB was created. As observers, we are located on the central geodesic. We immediately see that the past light cones of points on the CMB surface do not intersect with each others, meaning that they cannot be in causal contact or thermal equilibrium. Picture taken from [7].

Well-posed initial conditions for a Cauchy problem require assigning values both a quantity and its first derivatives, or in other terms, its velocity. In our context, velocity corresponds to $\dot{a}$, which we can use to estimate the kinetic energy K. The relative value between different instants of time is

$$K_i = K_0 \left(\frac{\dot{a}_i}{\dot{a}_0} \right)^2 . \tag{4.22}$$

The total energy E is the sum of this and the negative gravitational potential energy U and it is conserved. We can then estimate

$$\frac{E}{K_i} = \frac{K_i + U_i}{K_i} = \frac{K_0 + U_0}{K_0} \left(\frac{\dot{a}_0}{\dot{a}_i} \right)^2 \leq 10^{-56} \tag{4.23}$$

because $K_0 \sim U_0$. We see that the total energy is a tiny fraction of the kinetic energy at the Planck scale: this can be explained by *fine-tuning* the potential energy to huge values that precisely cancel out the kinetic energy. A variation in velocity greater than $10^{-54}\%$ would result in either the recollapse of the Universe or an empty one. Because of (4.10), the kinetic energy can be related to the dimensionless parameter Ω and the estimation outlined above leads to an Ω extremely close to unity, resulting in a flat Universe. Thus, this fine-tuning problem is known as flatness problem.

4.3 Cosmological inflation

The two fine-tuning problems mentioned have a common origin: the value of $\dot{a}_i/\dot{a}_0 \gg 1$, due to the attractive nature of gravity. On the other hand, a value $\dot{a}_i/\dot{a}_0 < 1$ would automatically solve these problems. This can be achieved in an accelerated expansion scenario, such as the current one driven by Dark Energy. However, the accelerated phase in the early Universe is called inflation and is usually driven by one or more scalar fields. The simplest realization involves a single field and will be discussed in this section, along with the conditions necessary to preserve good predictions of ΛCDM. Inflation should end at $t_f \sim 10^{-34} - 10^{-36}$ s in a FLRW metric and generate SM matter with the process of reheating.

4.3.1 Inflation and the problems of ΛCDM cosmology

Inflation provides a solution to both the horizon and flatness problems. The former is solved simply by the kinematics of this kind of Universe. The *event horizon* at a given instant t for an event e, denoted by $r_e(t)$, is the boundary separating the events that can influence the future of e from those that cannot. It is given by

$$r_e(t) = a(t) \int_t^{t_{\max}} \frac{dt}{a} = a(t) \int_{a(t)}^{a_{\max}} \frac{da}{\dot{a}a}. \tag{4.24}$$

In an accelerated Universe, this integral is always finite, even for $a_{\max} \to +\infty$. Therefore, events outside a sphere of radius $2r_e(t)$ are too far away to influence e due to simple geometric reasons. This allows us to relax the homogeneity requirement of the initial state of the Universe. In an arbitrarily distributed Universe at the beginning of inflation t_i, we can consider a small, homogeneous domain that will eventually have dimensions determined by

$$r_h\left(t_f\right) = r_e\left(t_i\right) \frac{a_f}{a_i} \tag{4.25}$$

and that preserves homogeneity, because it is protected from any possible inhomogeneities coming from outside the domain. In particular, let's consider energy relative variations that start at order $O(1)$ on the scale $H_i^{-1} = H(t_i)^{-1}$

$$\left(\frac{\delta\varepsilon}{\varepsilon}\right)_t \sim \frac{1}{\varepsilon} \frac{|\nabla\varepsilon|}{a(t)} H(t)^{-1} \sim O(1) \frac{\dot{a}_i}{\dot{a}(t)}. \tag{4.26}$$

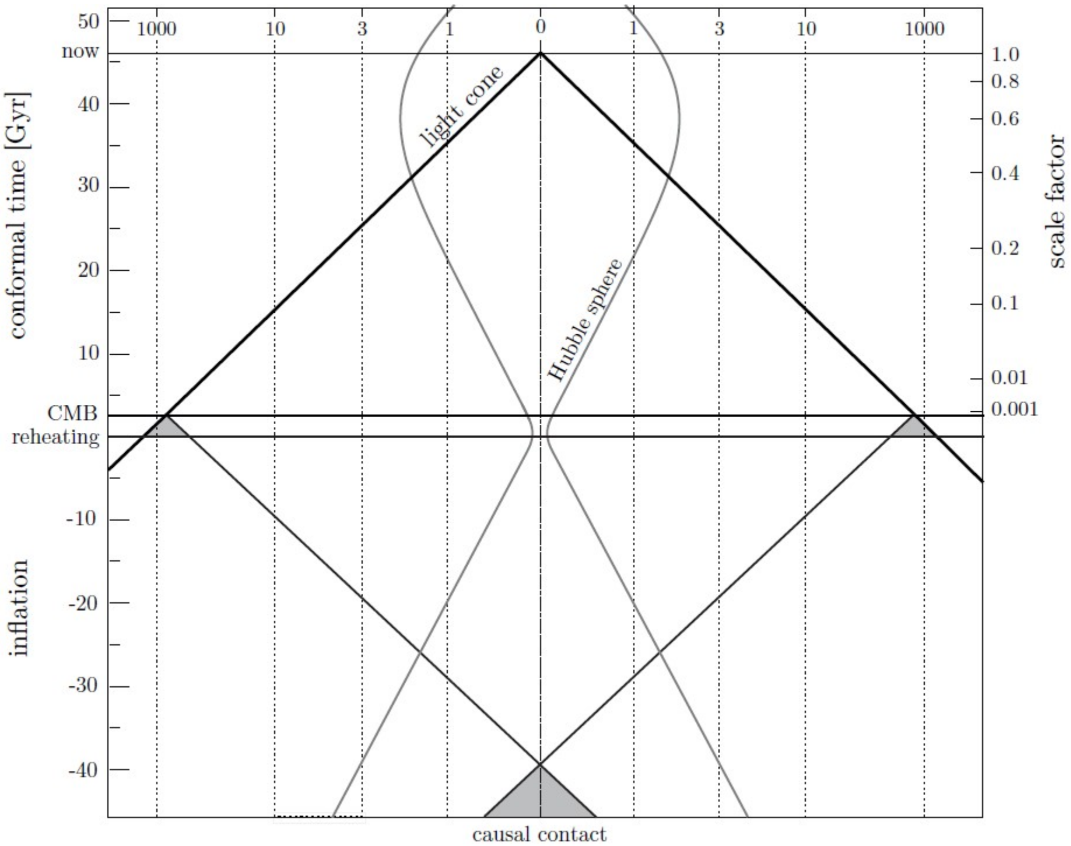

Figure 4.4: Thanks to the new region of negative conformal time η between the singularity (now at $\eta_i \to -\infty$) and the CMB, the past light cones of the events can be projected until they overlap. The events are now in causal contact. Picture taken from [7].

Initial inhomogeneities are "kicked out" after enough time because in an accelerated case $\dot{a} > \dot{a}_i$ for $t > t_i$. A pictorial representation of how inflation solves the horizon problem is given in fig. 4.4.

The flatness problem is also addressed by inflation. By expressing the current parameter Ω_0 (from the Planck mission) in terms of the initial one, we have

$$\Omega_0 - 1 = (\Omega_i - 1)\left(\frac{\dot{a}_i}{\dot{a}_0}\right)^2 < 10^{-4}. \tag{4.27}$$

In a non-inflationary scenario, with $\dot{a}_i/\dot{a}_0 = 10^{28}$ we need to set $\Omega_i = 1 + 10^{-60}$. This is precisely the unnatural fine-tuning. However, thanks to inflation, where $\dot{a}_i/\dot{a}_0 = 10^{-5}$, the simple requirement $\Omega_i \sim O(1)$ *predicts* $\Omega_0 = 1$.

Furthermore, inflation is also capable of introducing the small inhomogeneities present in the CMB. Looking at fig. 4.2, we see regions of different colors, corresponding to temperature differences on the order of $10^{-4} - 10^{-5}$. These inhomogeneities are necessary to explain the large structures that we observe today.

4.3.2 Slow-roll inflation and single field model

Inflation essentially means a phase of accelerated expansion, thus

$$\ddot{a} > 0, \tag{4.28}$$

or, involving the Hubble parameter H,

$$\frac{\ddot{a}}{a} = H^2[1 - \epsilon] > 0, \tag{4.29}$$

where ϵ is the *slow-roll* parameter, which in an accelerated Universe is

$$\epsilon := -\frac{\dot{H}}{H^2} < 1. \tag{4.30}$$

Using (4.7) and (4.9), it can be written as

$$\epsilon \equiv \frac{3}{2}(1 + \omega). \tag{4.31}$$

Note that $\epsilon < 1$ implies $\omega < -1/3$. This confirms the necessity of non-classical matter (e. g. Dark Energy with $\omega = -1$) as the source of acceleration.
A standard realization of an inflationary scenario involves a scalar field ϕ called *inflaton* that drives inflation on a curved space-time. The action of the theory is

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{8\pi G_N}\frac{R}{2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right]. \tag{4.32}$$

This is an example of *single-field* inflation, where the inflaton self-interacts through the potential $V(\phi)$. We will further assume that the kinetic energy is small compared to the potential energy

$$\dot{\phi}^2 \ll V(\phi), \tag{4.33}$$

which is the *first slow-roll* condition and allows us to simplify the equation of motion for ϕ by ignoring the spatial dependence with respect to the time derivatives as

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0. \tag{4.34}$$

This is the equation of motion of a harmonic oscillator with a friction proportional to H. The energy density and the momentum of the inflaton are

$$\begin{aligned} \rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V(\phi), \\ p_\phi &= \frac{1}{2}\dot{\phi}^2 - V(\phi). \end{aligned} \tag{4.35}$$

Accelerated expansion is achieved requiring $\epsilon \ll 1$. In terms of the inflaton, this means

$$\epsilon = \frac{\dot{\phi}^2}{2M_{\mathrm{P}}^2 H^2} \ll 1. \tag{4.36}$$

In order to solve the horizon problem, inflation must act for a sufficient amount of time, so ϵ must remain small for a long enough period, measured by the *second slow-roll* parameter η

$$\eta \equiv \frac{\dot{\epsilon}}{H\epsilon} = \frac{\ddot{H}}{H\dot{H}} + 2\epsilon = 2\frac{\ddot{\phi}}{H\dot{\phi}} + 2\epsilon, \tag{4.37}$$

where the request that $\dot{\phi}^2$ remains small compared to $V(\phi)$ during a sufficiently long time interval is ensured by

$$\delta_\phi \equiv \frac{\ddot{\phi}}{H\dot{\phi}} \ll 1. \tag{4.38}$$

The slow-roll conditions (4.36) and (4.38) depend on the specific model considered. For the single-field inflation, they looks like

$$\epsilon_V \equiv \frac{M_{\mathrm{P}}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \simeq \epsilon \ll 1, \tag{4.39}$$

$$\eta_V \equiv M_{\mathrm{P}}^2 \left|\frac{V_{,\phi\phi}}{V}\right| \ll 1 \tag{4.40}$$

4.3.3 Reheating

We need to consider how inflation ends, particularly how the inflaton decays into Standard Model particles. This phase is not yet clear because it is not known how to go beyond SM and which degrees of freedom are relevant. However, in the simplest picture, we have an inflaton coupled to SM particles Q through the 3-vertex $y\phi\overline{Q}Q$. Thus, ϕ decays at a rate

$$\Gamma_\phi \sim \frac{y^2}{16\pi} m_\phi. \tag{4.41}$$

where m_ϕ is the inflaton mass. After the slow-roll phase, the inflaton begins to oscillate around the minimum of the potential, as shown in fig. 4.5. During this rapidly oscillating phase, the equation of motion for ϕ becomes

$$\ddot{\phi} + (3H + \Gamma_\phi)\dot{\phi} = -m_\phi^2 \phi. \tag{4.42}$$

These oscillations of ϕ looks like a condensate of heavy particles with mass m_ϕ and decay rate Γ_ϕ. We can see that there are 3 different phases here: we start with a Universe dominated by Dark Energy and then in accelerated expansion. After the slow-roll, ϕ begins to oscillate rapidly, forming a condensate of ϕ modes and thus a Universe

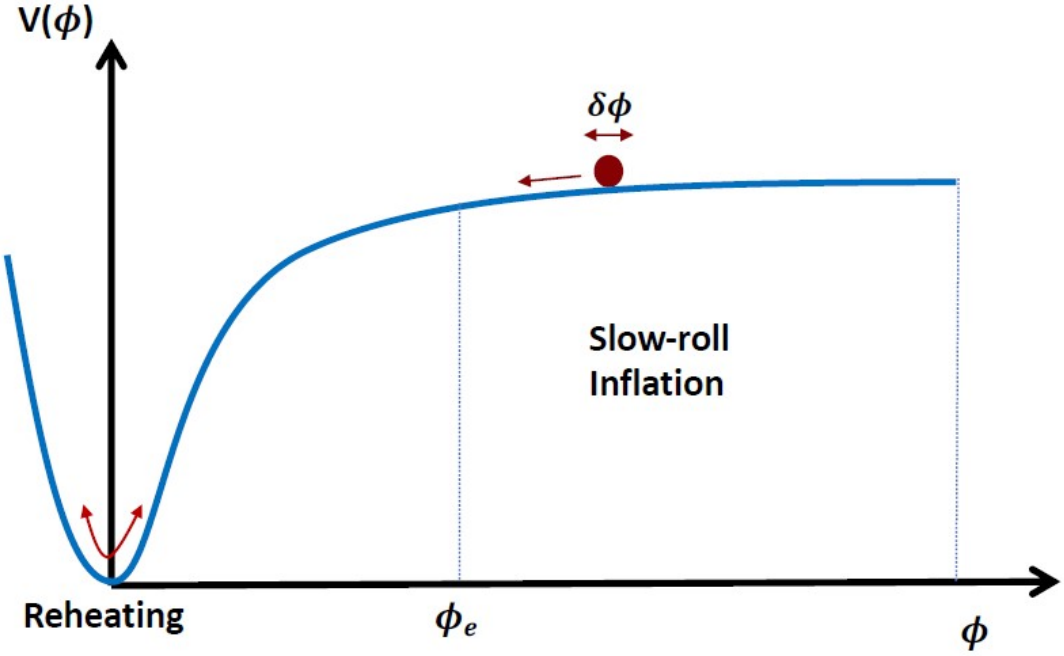

Figure 4.5: An example of a typical inflationary potential: the field ϕ initially slow-rolls driving inflation until it reaches ϕ_e, where inflation ends in a fast roll to a minimum and subsequent reheating of the Universe. Picture taken from [13].

dominated by matter, for $t \ll \tau \equiv \Gamma_\phi^{-1}$. These quanta are unstable and decay into ultra-relativistic SM particles, leading to an Universe dominated by radiation for $t \gg \tau$, when all quanta have undergone decay. This thermal bath has temperature

$$T_{\rm rh} = \left(\frac{90}{g_* \pi^2} H_{\rm dec}^2 M_{\rm p}^2\right)^{1/4} \sim \left(\frac{y^2}{16\pi} m_\phi M_{\rm p}\right)^{\frac{1}{2}}. \tag{4.43}$$

where $H_{\rm dec}$ is the Hubble parameter evaluated at time τ. For $m_\phi \gtrsim 1$ TeV, $T_{\rm rh} \gtrsim 10^8$ TeV. During the accelerated expansion, the temperature drops to lower values, which are maintained until the end of inflation. The process outlined so far allows the temperature to return to the higher pre-inflation values: this is the reason behind the name *reheating*. The reheating temperature is strongly model-dependent, with a range $T_{\rm rh} \sim 1\text{GeV} - 10^{11}\text{GeV}$. The only requirement is that reheating occurs before nucleosynthesis at $t \sim 0.01$ s, $T_{\rm BBN} \sim 1$MeV, in order to preserve the successful predictions of standard cosmology.

Chapter 5

String Compactifications and Axions

Having briefly discussed string theory in Chapter 3 and standard cosmology in Chapter 4, we can now consider their interplay. In this Chapter, we present the compactification of type IIB string theory, in order to derive an effective theory in $4d$. A closer examination of the topology of Calabi-Yau manifolds is necessary, as well as a consideration of how to meet phenomenological requirements, such as SUSY breaking from particle physics. Moreover, studying the implications of string theory on cosmology introduces additional constraints, such as the cosmological moduli problem. This problem establishes a prerequisite for string cosmology (specifically, for string inflation) that must be met in order to align with standard cosmology.

5.1 String compactification

The process of dimensional reduction has already been presented. However, in string theory, the compact submanifold X_6 of the decomposition (2.37) must satisfy certain properties, such as preserves SUSY. Specific properties of Calabi-Yau manifolds are discussed in this section, with a particular focus on recent findings about toric Calabi-Yau at large $h^{1,1}$. These discoveries are of great interest in the view of the fothcoming discussion on moduli stabilization of physically realistic scenarios.
The section concludes by introducing Dp-branes and Op-planes and discussing how to arrange the string degrees of freedom in an effective SUGRA in $4d$.

5.1.1 Calabi-Yau manifolds and string moduli

We are interested into string compactifications that preserve SUSY, because SUSY theories are inherently stable under quantum corrections, tachyons are absent and allow particle model-building, including proposals beyond SM. Compactification preserves some SUSY if there exists a non-trivial 6d spinor ξ that is invariant under parallel transport on X_6

$$\nabla_{X_6}\xi\left(x^m\right)=0. \tag{5.1}$$

This result can be expressed using *holonomy group* of the connection ∇. On a point $p\in\mathcal{M}$ of a N-dimensional manifold $\mathcal{M}$, it is defined as the group of automorphisms of the tangent space $T_p\mathcal{M}$ induced by parallel transport along closed loops

$$\mathrm{Hol}_p(\nabla)=\{G_c:T_p\mathcal{M}\to T_p\mathcal{M}\}\subset GL(N,\mathbb{R}). \tag{5.2}$$

It forms a group with the product of automorphisms, where the composition law is given by following one closed loops after the other and the inverse element by going backwards. (5.1) requires $SU(3)$ holonomy group.
A general construction is based on Calabi's conjecture, which is proven by Yau: a complex N-manifold that is Kähler and Ricci-flat admits an $SU(N)$ holonomy. This means that the X_6 must be considered a complex 3-fold.
More formally, a complex N-fold is a real $2N$-fold $\mathcal{M}$ where the charts $\phi:U\subseteq\mathcal{M}\to\mathbb{C}^N$ and the transition functions are holomorphic. Equivalently, it is a real $2N$-manifold equipped with *complex structure* $\mathcal{J}$: it is a (1,1) tensor such that $\mathcal{J}^2=-1$, admitting the decomposition

$$\mathcal{J}^{\mu}{}_{\nu}=i\delta^{\mu}{}_{\nu}\quad,\quad\mathcal{J}^{\bar{\mu}}{}_{\bar{\nu}}=-i\delta^{\bar{\mu}}{}_{\bar{\nu}} \tag{5.3}$$

in each collection of patches U_α covering the manifold $\mathcal{M}$. (1,1) refers to the two types of coordinates on a complex manifold, the holomorphic z^i and the anti-holomorphic $\bar{z}^{\bar{i}}$, $i=1,\ldots,N$. As a Riemannian manifold, it admits a metric $g:T\mathcal{M}\times T\mathcal{M}\to\mathbb{R}$. If it is compatible with the complex structure $\mathcal{J}$ as

$$g(\mathcal{J}X,\mathcal{J}Y)=g(X,Y) \tag{5.4}$$

the metric is called *Hermitian* and so the manifold. It can be used to express the complex structure as a 2-form:

$$J=ig_{\mu\bar{\nu}}dz^{\mu}\wedge d\bar{z}^{\bar{\nu}}. \tag{5.5}$$

If it is a closed form, meaning $dJ=0$, the manifold is a *Kähler manifold* and J a *Kähler form*. These are interesting objects in their own right, as the moduli space of SUSY theories (i. e. the set of SUSY vacua parametrized by moduli fields) is a Kähler manifold. Ultimately, a Kähler N-fold with holonomy $SU(N)$ is a Calabi-Yau N-fold.
There are different equivalent definitions. For instance, a Calabi-Yau N-fold is a Ricci-flat Kähler N-fold. Another equivalent definition, based on Chern classes, will be presented in the following subsection, along with the largest known class of Calabi-Yau manifolds: the toric Calabi-Yau manifolds in the *Kreuzer-Skarke list*.
Now, we would to determine the number of free parameters in the choice of an $SU(N)$ holonomy. This is equivalent to the number of possible deformations that leave the

Calabi-Yau structure invariant. More concretely, these are the transformations of the metric δg such that

$$R_{i\bar{j}}(g + \delta g) = 0. \tag{5.6}$$

The transformations can be decomposed as

$$\delta g = \delta g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}} + \delta g_{ij} dz^i dz^j + c.c., \tag{5.7}$$

where $\delta g_{i\bar{j}}$ and δg_{ij} define two classes of decoupled deformations. By (5.5), the former leads to deformation of the Kähler form

$$\delta J = i\delta g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}. \tag{5.8}$$

When Ricci-flatness condition is preserved, δJ belongs to the *Dolbeault cohomology* group $H^{1,1}(\mathcal{M}, \mathbb{C})$. This is the complex generalization of de Rham cohomology of real differential geometry, which captures some topological features of the considered manifold. In particular, we need the dimension of these groups: the *Hodge* (or *Betti*, for real case) *numbers* $h^{r,s} := \dim H^{r,s}(\mathcal{M}, \mathbb{C})$.

All the Hodge numbers can be arranged into the so-called *Hodge diamond*, which takes a simple form for a Calabi-Yau 3-fold

$$\begin{matrix} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{matrix} \quad = \quad \begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix} \tag{5.9}$$

Taking a basis of (1,1)-forms ω^I, $I = 1, \ldots, h^{1,1}$, the Kähler form is

$$J = t^I(x)\omega_I. \tag{5.10}$$

Then, there are $h^{1,1}$ deformations of the Kähler form, parametrized by $t^I(x)$ the *Kähler moduli*.

The complex deformations δg_{ij} affect the complex structure $\mathcal{J}$. These can be expanded using the non-vanishing (2,1)-form Ω, whose existence can be taken as another definition for Calabi-Yau manifolds. The deformations are then

$$\delta g_{ij} = \frac{i}{||\Omega||^2} \zeta^A(x) \, (\chi_A)_{i\bar{i}\bar{j}} \, \Omega^{\bar{i}\bar{j}}{}_j, \tag{5.11}$$

where χ_A, $A = 1, \ldots, h^{1,2}$ is a basis of $H^{1,2}$ and ζ^A are the $h^{1,2}$ *complex structure moduli*. Since these originate from complex terms, the number of real moduli is twice this number.

Thus, we end up with $h^{1,1} + 2h^{1,2}$ moduli from compactification.
Additional bosons arise from NSNS and RR sectors of closed string theory. They can be expanded in terms of the Dalbeault cohomology basis as

$$\begin{aligned} B_2 &= B_2(x) + b^I(x)\omega_I, \\ C_2 &= C_2(x) + c^I(x)\omega_I, \\ C_4 &= \theta^I(x)\tilde{\omega}_I, \end{aligned} \tag{5.12}$$

where $\tilde{\omega}_I$ is a basis of $H^{2,2}(\mathcal{M}, \mathbb{R}) \cong H^{1,1}(\mathcal{M}, \mathbb{R})$. All these fields, originating from both compactifications and string spectra, must form some SUSY theory. Up to this point, they can be arranged into the $\mathcal{N} = 2$ SUSY multiplets presented in 5.1.

SUSY multiplet	# of multiplets	Field content
gravity multiplet	1	$(g_{\mu\nu}, C_4^0)$
vector multiplets	$h^{1,2}$	(C_4^A, ζ^A)
hypermultiplets	$h^{1,1}$	$(t^I, b^I, c^I, \theta_I)$
double-tensor multiplet	1	(B_2, C_2, ϕ, C_0)

Table 5.1: $\mathcal{N} = 2$ SUSY multiplets of type IIB compactification.

The fields C_4^0 and C_4^A originate from the full expansion of C_4. Here, there is a truncated expansion for the sake of clarity. The complete expression is discussed in [18]. Type II compactifications result in $4d$ $\mathcal{N} = 2$ SUSY, which is not physically viable due to its lack of chirality. These models require additional features to break a part of SUSY.

5.1.2 Toric Calabi-Yau manifolds

A very useful tool to construct explicit Calabi-Yau threefolds is toric geometry. Some of the geometrical and topological tools are discussed in [3] and [4]. The starting object is

$$X = \left\{ x \in \mathbb{C}^n \left| \sum_{i=1}^{n} Q_i^a \, |x_i|^2 = \xi^a \right. \right\} / U(1)^s \tag{5.13}$$

which is a *toric variety* if $\dim X = n - s$. This is the ground state of a SUSY potential, where ξ^a for $a = 1, ..., s$ are the Fayet–Iliopoulos terms and Q_i^a for $i = 1, ..., n$ are the charges of the X_i chiral superfield charged under the group $U(1)^s$. These objects are of interest to us because they can be taken as an ambient space in which to define the Calabi-Yau manifolds simply and several of their properties are inherited from the toric variety in which they are embedded.
The *divisors* of a variety X are defined as the formal sum

$$D = \sum_A n_A D_A \quad n_A \in \mathbb{Z}, \tag{5.14}$$

where D_A are holomorphic complex codimension one hypersurfaces in X. If all n_A are non-negative, D is called *effective* divisor. If X is a toric variety, there is a simple set of divisors given by

$$D_i : x_i = 0, \; i = 1, ..., s. \tag{5.15}$$

This set is fully characterized by the charges Q_i^a and we can use a linearly independent subset of them as a basis for the divisors.
A key quantity derived from these objects is the *intersection product* or *number* of d divisors

$$D_A \cdots D_B = \int_X \mathrm{PD}(D_A) \wedge \cdots \wedge \mathrm{PD}(D_B) = \#(D_A \cap \cdots \cap D_B) \tag{5.16}$$

where $\mathrm{PD}(D_A)$ denotes the Poincarè dual form of the divisor D_A.
This number number appears in the evaluation of any topological quantity, e.g. the volume of a k-dimensional subvariety $U \subset X$

$$\mathcal{V}_U := \frac{1}{k!} \int_U \wedge^k J \tag{5.17}$$

where $J \in H^{1,1}(X, \mathbb{R})$ is the Kähler form.
Another class of relevant objects is that of holomorphic *curves* or 2-cycles C^a, which are obtained as transversal intersections of $n-1$ of the divisors D_i. The set

$$\mathcal{M}_X \equiv NE(X) = \left\{ \sum c_a [C^a], c_a \geq 0 \right\} \tag{5.18}$$

is called *Mori cone* of X. The mutual intersection between curves and divisors is specified by the charges as

$$D_i \cdot C^a = Q_i^a \tag{5.19}$$

Because $\mathcal{V}_{C^a}$ is the area of C^a, is natural to require $\mathcal{V}_{C^a} = \int_{C^a} J \geq 0$ for any curves in the Mori cone. On the other hand, the subset of $J' \in H^{1,1}(X, \mathbb{R})$ such that $\mathcal{V}_{C^a} = \int_{C^a} J \geq 0$ for all C^a in the Mori cone is the *Kähler cone* of X, $\mathcal{K}_X$.
A final interesting tool that encodes topological information about a (complex) variety X of (complex) dimension r is the *Chern class*

$$c(X) \equiv c(TX) = \det\left(1 + \frac{1}{2\pi} F\right) = 1 + \frac{1}{2\pi} \mathrm{Tr}\, F + \cdots \tag{5.20}$$

where TX is the tangent vector bundle of (complex) rank r of X and F is its the matrix curvature form. For example, we can compute the *Euler characteristic* $\chi(X)$ as

$$\chi(M) = \int_M e(TM) = \int_M c_r(TM) \tag{5.21}$$

with the *Euler class* $e(TM) = c_r(TM) \in H^{2r}(X, \mathbb{R})$ is its top Chern class.
If X is a toric variety, the Chern class is simply

$$c(X) = \prod_{i=1}^{n} (1 + D_i) \tag{5.22}$$

where D_i are actually the Poincarè dual of the divisors. The Chern class of submanifold S of toric variety X can also be easily evaluated: if S is the intersection of a family of hypersurfaces given by polynomials in x_i, we obtain

$$c(\mathcal{S}) = \left.\frac{c(X)}{\prod_\alpha c(S_\alpha)}\right|_{\mathcal{S}} = \left.\frac{\prod_i (1 + D_i)}{\prod_\alpha (1 + S_\alpha)}\right|_{\mathcal{S}} = 1 + \sum_i D_i - \sum_\alpha S_\alpha + \cdots |_{\mathcal{S}} \tag{5.23}$$

We can now finally discuss Calabi-Yau manifolds. The concepts defined so far refer to the ambient space in which the Calabi-Yau manifolds are defined. Specifically, a manifold X is considered to be Calabi-Yau if it is a submanifold of a toric variety and its first Chern class vanishes

$$c_1(X) = \sum_i D_i - \sum_\alpha S_\alpha = 0. \tag{5.24}$$

From now on, we will use the symbol X for CY manifolds and V for the ambient toric variety. We can define a basis of divisors $\{D_i\}$ in X from a collection of divisors $\{\widehat{D}_A\}$ in V simply by intersecting them as

$$\{D_A\} := \left\{\widehat{D}_A \cap X\right\} \quad A = 1, \ldots, h^{1,1}(X) + 4 \tag{5.25}$$

and reordering to extract precisely $\dim H_4(X, \mathbb{Q}) = h^{1,1}$ linearly independent elements. Effective divisors on X are then inherited from effective divisors on V, but the converse is not true: there exist effective divisors on X called *autochthonous*.
Since $J \in H^{1,1}(X, \mathbb{R})$, it is natural to expand it in terms of the Poincaré duals of D_i

$$J = t^i \, [D_i] \tag{5.26}$$

which allow us to write the volumes of curves, divisors and X itself, as defined in (5.17), using intersection numbers and the Kähler moduli t^i

$$\begin{aligned} \mathfrak{t}_a &= M_{ai} t^i, \\ \tau_A &= \frac{1}{2} \kappa_{Ajk} t^j t^k, \\ \mathcal{V} &= \frac{1}{6} \kappa_{ijk} t^i t^j t^k. \end{aligned} \tag{5.27}$$

Because these objects measure areas and volumes, it is natural to consider J inside the Kähler cone $\mathcal{K}_X$. In order to ensure control over both the α'-perturbative and non-perturbative expansions, we need a further requirement: for every holomorphic curve C^a,

$$\mathcal{V}_{C^a} = \mathfrak{t}_a \geq 1 \tag{5.28}$$

We then define the *stretched Kähler cone* $\widetilde{\mathcal{K}_X}$ of X as

$$\widetilde{\mathcal{K}_X} := \left\{ J \in H^{1,1}(X, \mathbb{R}) \mid \mathrm{Vol}_J(C) \geq 1 \quad \forall C \in \mathcal{M}_X \right\} \tag{5.29}$$

Because it is difficult to express $\widetilde{\mathcal{K}_X}$ directly, we prefer to find other sets that contain it and within which $\widetilde{\mathcal{K}_X}$ is contained

$$\widetilde{\mathcal{K}_V} \subseteq \widetilde{\mathcal{K}_X} \subseteq \widetilde{\mathcal{K}_\cap}. \tag{5.30}$$

$\widetilde{\mathcal{K}_V}$ is simply the stretched Kähler cone constructed from the toric variety V, which is the subset of $H^{1,1}(V, \mathbb{R})$ such that any curve $\widehat{C}$ on V has volume greater than 1. The cone $\mathcal{K}_\cap$, on the other hand, is built from a more general set of curves compared to the Mori cone. These are obtained by the transversal intersection of X with surfaces on V given by

$$\left\{ \widehat{S}_{AB} \right\} := \left\{ \widehat{D}_A \cap \widehat{D}_B, A, B = 1, \ldots, h^{1,1} + 4, A \neq B \right\}. \tag{5.31}$$

The curves in X are simply $C_{AB} = D_A \cap D_B$. The *intersection cone* $\mathcal{K}_\cap$ is then

$$\mathcal{K}_\cap := \{ J \mid \mathcal{V}, \tau_A, \mathfrak{t}_{AB} > 0 \}, \tag{5.32}$$

while the *stretched intersection cone* is defined in analogy with the previous cases as the subset where the volumes are greater than 1.

5.1.3 Topology of Calabi-Yau's at large $h^{1,1}$

Currently, it is very difficult to explicitly stabilize all moduli for $h^{1,1} \gg 1$. However, some preliminary and general results have emerged from systematic studies of all possible geometries for the Calabi-Yau manifolds, such as the presence of one effectively massless axion in every mass spectrum.

To go into the details of this discussion, we need to build physical models, so lagrangians that depend on proper fields. Indeed, the Kähler moduli t^i do not define good coordinates for the moduli space of the theory. The complete expression for such fields and the procedure to construct them is presented in [6]. We will focus on the simpler case where the Kähler coordinates T_i are the complexification of the 4-cycle volumes τ_i defined in (5.27)

$$T_i := \tau_i + i\theta_i, \tag{5.33}$$

where θ_i are axions that arise from integration over the corresponding 4-cycle D_i

$$\theta_i := \int_{D_i} C_4. \tag{5.34}$$

Some explicit calculations are presented in 6.1 and 6.2. Here, we continue with a general treatment for arbitrary $h^{1,1}$. The axion Lagrangian, written in terms of canonically normalized fields ϕ_i looks like

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi_i \partial_\mu \phi^i - V(\phi) \tag{5.35}$$

We are interested in stable points, so we need to inspect the Hessian

$$\mathcal{H}_{ij} := \frac{\partial^2}{\partial\phi_i \partial\phi_j} V(\phi), \tag{5.36}$$

which provides information about the stability of the critical points and the masses of the axions, given by the eigenvalues $h_1^2 \leq \ldots \leq h_{h^{1,1}}^2$.
We can evaluate the volumes of the elements of any basis of $H_4(X, \mathbb{Q})$: $\tau_1^{\mathcal{B}} \leq \cdots \leq \tau_{h^{1,1}}^{\mathcal{B}}$.
A *minimal basis* is defined as the basis $\mathcal{B}_{\text{min}}$ that minimizes $\tau_{h^{1,1}}^{\mathcal{B}}$. We then write

$$\tau_{\text{last}}(J) := \tau_{h^{1,1}}^{\mathcal{B}_{\text{min}}}. \tag{5.37}$$

Finally, let's focus on the various cones defined earlier, from which we obtain

$$\tau_{\text{last}}^{\cap} \leq \tau_{\text{last}}^{X} \leq \tau_{\text{last}}^{V}, \tag{5.38}$$

where the upper symbols refer to the corresponding stretched cones. It is then possible to set an upper bound to the superpotential W of the lightest axion in the theory

$$|W| \leq e^{-\tau_{\text{last}}^{X}} \leq e^{-\tau_{\text{last}}^{\cap}} =: |W_\cap|. \tag{5.39}$$

This constitutes an upper bound and at the same time an estimation of the lightest axion mass

$$m_{\text{min}}^2 \lesssim |W_\cap|. \tag{5.40}$$

In conclusion, we can report some of the most interesting results from [2] which focused on the stretched Kähler cone and performed a statistical study of different possible Calabi-Yau manifolds at fixed $h^{1,1}$, covering the range $2 \leq h^{1,1} \leq 491$. This investigation sheds light on the scaling behavior of the quantities introduced so far as a function of $h^{1,1}$.
Furthermore, $\tau_{\text{last}}^{\cap}$ together with (5.40) indicate that at least one axion does not experience superpotential contributions larger than

$$|W_\cap| = \exp\left(-2\pi\tau_{\text{last}}^{\cap}\right) \sim \exp\left(-0.1\left(h^{1,1}\right)^3\right) \tag{5.41}$$

Topological quantity	$h^{1,1}$ dependence
k_{ijk}	$h^{1,1}$
τ^V_{last}	$0.01(h^{1,1})^{4.3}$
$\tau^{\cap}_{\text{last}}$	$0.02(h^{1,1})^{3.2}$
$\mathcal{V}^V$	$0.0004(h^{1,1})^{7.2}$
$\mathcal{V}^{\cap}$	$0.0002(h^{1,1})^{6.2}$

Table 5.2: intersection numbers, volumes of divisors and volumes of stretched cones exhibit a power-law dependence on $h^{1,1}$.

and has a final mass smaller than the cosmological constant. Thus, it is considered to be massless.
This ultra-light axion is a general feature of large $h^{1,1}$ models, which, together with exponentially large volumes, suggest that we consider the class of physical models for moduli stabilization presented in the next Chapter.

5.1.4 Dp-branes and p-form fluxes

As previously mentioned, string theory also incorporates other higher-dimensional objects. We will present some properties of the Dp-branes, which are essential for the partial breaking of SUSY.
Dp-branes are $(p+1)$-dimensional objects, where p represents the spatial dimensions filled by the brane. Their defining property is that they are submanifolds where the endpoints of open strings are constrained to live. In their spacetime motion, they span a multi-dimensional generalization of worldline and worldsheet: the *worldvolume*. Their dynamics is described by a generalization of the Nambu-Goto action, the *Dirac action*

$$S_d = -T_p \int d^{p+1}\xi \sqrt{-\det\left(P[G]_{ab}\right)}, \tag{5.42}$$

where T_p is the brane tension and $P[G]$ is the pullback of the target metric onto the worldvolume. Furthermore, branes can interact via closed strings exchange, thanks to the coupling with RR and NSNS gauge fields F_p and H_3. The interaction can be described by the *Born-Infeld theory*, a non-linear generalization of Maxwell's theory. The full action is a combination of these, known as the *Dirac-Born-Infeld action*

$$S_{\text{DBI}} = -g_s T_p \int d^{p+1}\xi e^{-\phi} \sqrt{-\det\left(P[G+B] - 2\pi\alpha' F\right)}, \tag{5.43}$$

where $P[B]$ is the pullback of the B_2 form of NSNS sector. The combination $B - 2\pi\alpha' F$ is the correct gauge invariant term. Finally, it turns out that the tension is given by

$$T_p = \frac{1}{(2\pi)^p g_s \, (\alpha')^{(p+1)/2}}. \tag{5.44}$$

Thus, branes are heavy for $g_s \ll 1$: the spectra of oscillations of open strings encode the description of fluctuations of the theory around the branes. Type II theories are compatible with the presence of branes, but of different kinds. Open strings satisfy Dirichlet boundary conditions, which flip the sign between left and right GSO projections in the R sector. This implies that type IIB is compatible with Dp-branes with odd p, while type IIA is compatible with even p. The spectrum is given by that of open superstrings, but the strings move inside the (p+1)-dimensional worldvolume, so their states transform under lower dimensional Lorentz group:

Sector	State	$SO(p-1)$	$(p+1)$-dim field
NS	$\psi^{\mu\dagger}_{1/2}\lvert 0\rangle$	Vector	Gauge boson A_μ
	$\psi^{i\dagger}_{1/2}\lvert 0\rangle$	Scalar	$9-p$ real scalars ϕ^i
R	$\lvert \mathbf{8}\rangle$	spinor	fermions λ_α

(5.45)

Finally, branes are naturally *charged* under RR fields, via an electromagnetic-like coupling

$$S_{\text{CS}} = \mu_p \int C_{p+1}, \tag{5.46}$$

which is another Chern-Simons term. μ_p is the charge, given by $\mu_p = g_s T_p$.
The final Dp-brane action is then the sum

$$S_{\text{D}p} = S_{\text{DBI}} + S_{\text{CS}}, \tag{5.47}$$

which must be added to the action (3.75) as a local source of stress-energy and RR charges. The presence of Dp-branes has several consequences. Primarily, they generically turn on the background RR fields C_p. The p-form strengths $F_p = dC_p$ are called *p-form fluxes*, as they are a generalization of Maxwell's 2-form $F = dA$. The condition for consistently quantizing them is

$$\int F_p \in 2\pi\mathbb{Z}, \tag{5.48}$$

which is a generalization of magnetic flux quantization in presence of magnetic monopoles. In the absence of charges, the flux can only be non-zero if a nontrivial p-cycle exists in the geometry. If so, it is determined by a discrete choice that must be made for every such p-cycle. One particularly interesting effect of the presence of these fluxes is a possible solution for the cosmological constant Λ problem. A glimpse of this can be seen in the *Bousso-Polchinski model*, but a complete understanding of it is beyond the scope of this work. Some reviews on this topic can be found in [4] and [19]. Consider h flux quanta. The number h refers to some Hodge number, because the number of quanta counts the number of non-trivial closed cycles of the Calabi-Yau. The classical potential that develops is

$$V_N(\phi) = V_0(\phi) + \int_X ||F||^2 = V_0(\phi) + \sum_{i,j} g_{ij}(\phi) N^i N^j, \tag{5.49}$$

where ϕ collectively are moduli, $i, j = 1, \ldots h$, $N^i \in \mathbb{Z}$ and g_{ij} is a metric on the moduli space. The bare potential $\Lambda = V_N(\phi_0)$ is of the order of some fundamental scale, like M_s. The vacua are characterized by their position in the moduli space and by a flux vector. Because finding these critical points analytically is hard, a more efficient approach is to use statistical methods. Then, searching for cosmological constants Λ less than some fixed scale $\Lambda*$ results is a Λ-distribution that, for $\Lambda = 0$, give a number of solutions of

$$dN_{\text{vac}} \sim 10^{h/2} d\Lambda / |V_0| \,. \tag{5.50}$$

h is naturally of order of few hundred for Calabi-Yau manifolds, so there will be an exponential number of vacua such that $\Lambda \sim 10^{-120} M_p^4$. This, together with *eternal inflation* and the *Weinberg argument*, leads to a genuine string solution to the cosmological constant problem.
This is, as always, a simplification of the full IIB string compactification. Here, the flux quantization

$$\frac{1}{2\pi\alpha'} \int F_3 \in 2\pi\mathbb{Z}, \quad \frac{1}{2\pi\alpha'} \int H_3 \in 2\pi\mathbb{Z}, \tag{5.51}$$

where H_3 is the NSNS form. Together with the axio-dilaton τ, they combine into the complex 3-form flux G_3 defined in (3.74). This form combines with the non-vanishing 3-form Ω_3, developing the (tree-level) superpotential (see [20])

$$W_0 = \frac{1}{(2\pi)^2\alpha'} \int_M G_3 \wedge \Omega. \tag{5.52}$$

This is part of the final $4d$ SUGRA theory, but before we can discuss it further, we need one more ingredient.

5.1.5 Orientifolding

Another effect of the presence of D-branes is related to their positive tension. A fundamental consistency requirement for flux compactifcations with D-branes is cancellation of all tadpole anomalies associated with the charge and tension of the sources. Then, we need objects with negative tension to balance the D-branes. The construction that introduces the proper objects, while also truncating half of SUSY is called *orientifolding*.

Basically, orientifolding involves modding out a transformation (called *orientifold action*) from the Calabi-Yau X_6. The general form of the orientifold action is $\Omega\sigma$, possibly with the left sector operator $(-1)^{N_F}$. Ω is the parity operator on the worldsheet, while σ is a $\mathbb{Z}_2$ symmetry of X_6 acting holomorphically on it. The points left invariant by orientifold are called *orientifold p-planes*, or *Op-planes*, where p is their spatial dimensions. It applies differently in type IIA and IIB string theories. For example, in IIA is $\Omega\sigma(-1)^{N_F}$, where σ acts as

$$J \to -J, \quad \Omega_3 \to \overline{\Omega}_3, \tag{5.53}$$

or in type IIB just Ω, so a trivial σ, that reduces to type I compactification.
More interesting cases in type IIB are

- *(i)* $\Omega\sigma_i(-1)^{F_L}$, where σ_i flips the sign of only z_i coordinate, leaving the other unchanged. This action leads to O7-planes;
- *(ii)* $\Omega\sigma_i\sigma_j$,where just i and j coordinates are flipped, while $k \neq i, j$ is invariant. This action results in O5-planes;
- *(iii)* $\Omega\sigma_1\sigma_2\sigma_3(-1)^{F_L}$ exchanges the sign of all coordinates, providing O3-planes.

It is important to note that both the first and the last involve the operator $(-1)^{N_F}$ and flip the sign of an even number of coordinates, resulting in a more generic action specified by

$$\mathcal{O} = (-1)^{F_L}\Omega\sigma, \quad \sigma\Omega_3 = -\Omega_3, \tag{5.54}$$

resulting in a model with both O3- and O7-planes. This is precisely the action that we will use to truncate type IIB spectrum. Indeed, this σ acts on the various fields as

$$\begin{aligned} &\sigma\phi = \phi, \quad \sigma C_0 = C_0, \\ &\sigma g = g, \quad \sigma C_2 = -C_2, \\ &\sigma B_2 = -B_2, \quad \sigma C_4 = C_4. \end{aligned} \tag{5.55}$$

Thus, the action of this transformation distinguishes between two sectors, specified by the σ-eigenvalue. Then, the cohomology groups break into the positive and negative subspaces as

$$H^{(r,s)} = H_+^{(r,s)} \oplus H_-^{(r,s)}, \tag{5.56}$$

with dimensions $h_+^{r,s}$ and $h_-^{r,s}$ respectively. Clearly, $h^{r,s} = h_+^{r,s} + h_-^{r,s}$. The subspaces admits truncated bases as $\omega_i, i = 1, \ldots, h_+^{1,1}$ for $H_+^{(1,1)}$ or $\chi_\alpha, \alpha = 1, \ldots, h_-^{1,2}$ for $H_-^{(1,2)}$. Finally, because of (5.55), the invariant states under (5.54) are presented in Table 5.3.

SUSY multiplet	# of multiplets	Field content
gravity multiplet	1	$g_{\mu\nu}$
vector multiplets	$h^{1,2}_{+}$	C_4^a
chiral/linear multiplets	$h^{1,2}_{-}$	ζ^α
	1	(ϕ, C_0)
	$h^{1,1}_{-}$	(b^ι, c^ι)
chiral multiplets	$h^{1,1}_{+}$	(t^i, θ_i)

Table 5.3: $\mathcal{N} = 1$ SUSY multiplets of type IIB orientifold with O3/O7-planes.

5.1.6 Effective 4d $\mathcal{N} = 1$ Supergravity

We now have all the degrees of freedom coming from string compactification. These must be cast into an effective SUGRA theory.
The starting point is the 10d action (3.75) together with Dp-branes and Op-planes in order to break SUSY as we have already described. Their effects are collected into a term S_{loc}. The total action is then the sum of these two terms:

$$S = S_{\text{IIB}} + S_{\text{loc}}. \tag{5.57}$$

Due to the presence of local sources, warped compactification must be considered. The proper ansatz for the metric is (2.55). In this case, it looks like

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A(y)}g_{mn}dy^m dy^n \tag{5.58}$$

A well-understood class of solutions is known as *imaginary self dual (ISD) solutions*. The name comes from the imaginary self-duality relation satisfied by the complex 3-form G_3

$$*_6 G_3 = iG_3, \tag{5.59}$$

where $*_6$ is the Hodge dual defined in the compact manifold X_6. The ISD solutions also establish a connection between the trace of the stress-energy tensor from S_{loc} and the local source charge density ρ_3^{loc}

$$\frac{1}{4}\left(\sum_{M=4}^{9} T^M_M - \sum_{M=0}^{3} T^M{}_M\right)_{\text{loc}} = T_3\rho_3^{\text{loc}}\,. \tag{5.60}$$

This equality is satisfied only by D3/D7-branes and O3-planes. Finally, the warp factor is

$$e^{4A} = \alpha, \tag{5.61}$$

where $\alpha(y)$ is a scalar function on X_6 such that

$$\tilde{F}_5 = (1 + *_{10})\, d\alpha(y) \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \tag{5.62}$$

with the $10d$ Hodge dual $*_{10}$. The equations (5.59), (5.60) and (5.61) all together define the ISD solutions.
Let's now focus on S_{IIB} now. Its Chern-Simons term automatically develops a semiclassical potential for all the complex structure moduli ζ^α and the axiodilaton τ:

$$V_{\text{flux}} = \frac{1}{2\kappa_{10}^2}\int d^{10}X\sqrt{-G_E}\left[-\frac{|G_3|^2}{2\,\mathrm{Im}(\tau)}\right]. \tag{5.63}$$

This means that their are automatically massive at leading order in α'. Thus, we will integrate them out from the low-energy theory. However, they are the only degrees of freedom that are already good *Kähler coordinates*, so proper coordinates of the moduli space. Indeed, the Kähler moduli t^i combine with the axions θ_i and the complex 2-forms $G_\iota = c_\iota - \tau b_\iota$ to define the true Kähler coordinate T_i, that is an expansion of the complexified 4-cycle volumes (5.33)

$$T_i \equiv \frac{1}{2}c_{ijk}t^jt^k + i\vartheta_i + \frac{1}{4}e^\phi k_{i\iota\delta}G^\iota(G-\bar{G})^\delta. \tag{5.64}$$

Then, $\mathcal{N}=1$ SUGRA is specified by $h_+^{1,1}$ complexified volumes T_i, $h_-^{1,1}$ 2-forms G_ι, $h_-^{1,2}$ complex structure moduli ζ^α and an axiodilaton, for a total of $h^{1,1}+h_-^{1,2}+1$. Their SUGRA is given by the already mentioned superpotential W_0, which we report here for completeness

$$W_0 = \frac{1}{(2\pi)^2\alpha'}\int G_3\wedge\Omega, \tag{5.65}$$

and an expression similar to (2.54) for the Kähler potential

$$K_0 = -2\ln(\mathcal{V}) - \ln(-i(\tau-\bar{\tau})) - \ln\left(-i\int\Omega\wedge\bar{\Omega}\right). \tag{5.66}$$

The scalar potential V_F is given by the formula (2.33), by plugging in these K_0 and W_0. An important feature of K_0 is the relation

$$\sum_{I,J=T_i} K_0^{I\bar{J}}\partial_I K_0\partial_{\bar{J}}K_0 = 3. \tag{5.67}$$

Such K_0 is of *no-scale type* and, together with the fact that W_0 is independent of Kähler moduli T_i, leads to a simpler scalar potential

$$V_F = e^{K_0}\sum_{i,j\neq T_i} K_0^{i\bar{j}}D_iW_0\overline{D_jW_0}. \tag{5.68}$$

It is positive semi-definite, as in global SUSY, and the minimum $V_F=0$ is realized for $D_iW_0 = 0$ for all moduli except the Kähler ones. However, SUSY can be broken by

them, via $D_{T_i}W_0 = 0$. T_i are flat direction for this potential, so they parametrize a set of different allowed vacua. The problem is that they their vevs govern the overall volume $\mathcal{V}$, so it remains unfixed.
We have already mentioned that ζ^α and τ are automatically massive. By working below their mass scale, we can integrate them out. Since W_0 depends only on them, it turns out to be simply a constant. Also, K_0 reduces to just its first term, $K_0 = -2\ln(\mathcal{V})$.

To summarize, type IIB orientifold results in a $4d$ $\mathcal{N} = 1$ SUGRA theory. It admits a class of ISD solutions where the complex structure moduli ζ^α and the axiodilaton τ are stabilized, while no-scale structure forces the Kähler moduli T_i to be flat directions. The superpotential W_0 is a tunable constant, chosen properly by the quantized flux, and the Kähler potential K_0 is a funtion of Kähler moduli as $K_0 = -2\ln(\mathcal{V})$.

5.2 String cosmology

Before considering how flat directions can be stabilized, it is worthwhile to mention why they are so problematic, even from a phenomenological perspective. To state the problem, it is necessary to understand how string theory influences cosmology, particularly its relation to inflation. By requiring string inflation to be compatible with ΛCDM, this leads to a constraint on the moduli: light moduli would destroy the good predictions of Hot Big Bang model. Apart from consistency with already known physics, string theory suggests new cosmological phenomena. The end of this section is dedicated to the exploration one of them, the axiverse. This is an Universe with hundreds of axions, potentially leading to several observable effects.

5.2.1 String inflation

Inflation is one of the most promising and extensively studied proposals for cosmology beyond ΛCDM model. It has the ability to address certain issues within the ΛCDM model and offer additional explanations for observables that are beyond the reach of ΛCDM. It is quite natural the attempt to explicitly realize inflation in string theory and there are good arguments for doing this:

- the lack of a microscopic description of inflation, which could reasonably be achieved in an UV-complete quantum gravity theory. This is a compelling problem, from both a theoretical perspective and a phenomenological one.
  The former is exemplified by the so called η_V-*problem*. The parameter η_V needs to satisfy (4.40), but quantum corrections tend to affect its value as $\eta_V \simeq O(1)$. This would reintroduce a fine-tuning; however, a more natural solution may arise if the value is protected by underlying symmetries that could be present in the UV

theory.
The latter is provided by the sensitivity to trans-Planckian effects that directly enter into some effectively observable quantities, like the value of inflaton field displacement

$$\Delta\phi \propto M_p, \tag{5.69}$$

where we see the explicit dependence of the Planck scale M_p in a quantity measurable by CMB power spectra. Inflation is a unique arena where the typical scale is not too far from the Planck one, typical of quantum gravity;

- a challenge in inflationary physics is its reliance on the specific model being considered, which leads to a wide range of predictions that are difficult to precisely determine. The scenarios achievable in a complete theory such as string theory are much more constrained, in addition to the requirement of being compatible with current observational data;

- a crucial stage in the complete inflationary paradigm is the process of reheating. Reheating is essential because it establishes the connection with standard physics, including both particle physics and cosmology. Then, it is strongly dependent on which kind of physics beyond Standard Model we choose. String theory proposes which degrees of freedom are relevant at that scale, from the perspective of an UV-embedding of inflation;

- string theory needs to be connected with phenomenology. In the case of cosmology, this means recovering standard cosmology and possibly some extension of it and the inflationary proposal is a natural candidate. Furthermore, the landscape of string theory (i.e. the set of all possible low-energy effective theories that can be obtained in the string framework) is huge. Currently, there is a research line that tends to set constraints to select which physical results can effectively be extracted from the UV-complete theory. Thus, both string theory and inflation constrain each other: naively, their overlap defines the only physical scenarios;

- string theory predicts the existence of extra-dimensions and these influence the $4d$ effective theories that we can formulate. In particular, we end up with a class of fields called moduli T with many interesting properties:

  (i) they are scalar particles neutral under Standard Model but are coupled gravitationally with regular matter. However, these interactions are weak, because they are suppressed by M_p^{-1} factors;

  (ii) their vevs encode key information about the geometry of extra-dimensions (e. g. volume and shape);

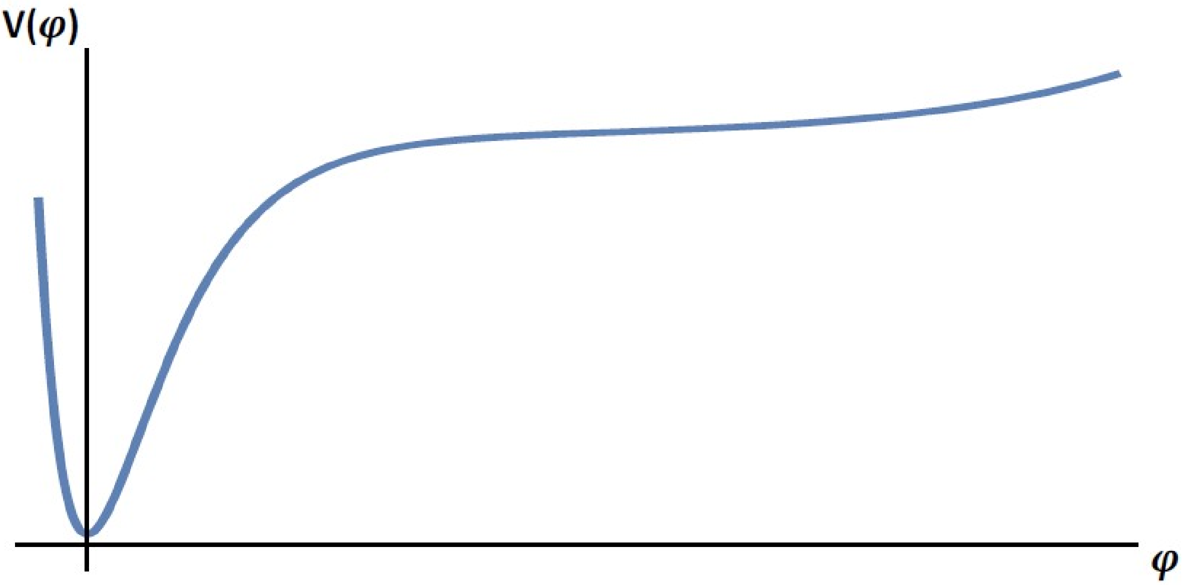

Figure 5.1: the so-called *Fibre Inflation*: an example of an inflationary potential realized in string theory, after moduli stabilization. We see the typical features of inflationary potential: a large plateau which corresponds to slow-roll and a minimum around which the reheating can take place. The field range is here $\Delta\phi = O(5)M_p$. Picture taken from [13].

(iii) they are complex fields that enjoys shift symmetries

$$T \to T' = T + \mathrm{i}\alpha, \quad \alpha \in \mathbb{R} \quad \Rightarrow \quad \theta \to \theta' = \theta + \alpha \tag{5.70}$$

where θ is their imaginary part. These shift symmetries play a fundamental role, for example, in fixing the η_V-problem and making the masses of θ particles vanish at tree-level;

(iv) moduli are the natural string candidates to be the inflaton field. An example of inflation realized in string theory is the *Fibre Inflation*

$$V(\phi) = V_0 \left[3 - 4e^{-\frac{1}{\sqrt{3}}\phi} + e^{-\frac{4}{\sqrt{3}}\phi} + \delta\left(e^{\frac{2}{\sqrt{3}}\phi} - 1\right)\right], \tag{5.71}$$

plotted in fig. 5.1. V_0 and $\delta \propto g_s^4 \ll 1$ are two constants.

5.2.2 Cosmological moduli problem

The importance of ensuring compatibility with current models has already been emphasized. In particular, we need to ensure that the elements introduced by string theory

leave the good predictions of standard Hot Big Bang model invariant. We therefore need to inspect more closely how string theory affects cosmology.
It is already been mentioned that moduli interact gravitationally with SM particles. Thus, their decay times are crucial: a decay that occurs too late would spoil the predictions of Big Bang Nucleosynthesis by diluting everything via entropy injection. In the new picture, we have that inflation is driven by one modulus ϕ and the others σ are trapped until the end of inflation. At this time, the other moduli start to oscillate around their minimum, acting as a classical condensate of heavy particles. Through gravitational coupling, their decay rates are

$$\Gamma_\sigma = \frac{\lambda}{16\pi}\frac{m_\sigma^3}{M_\mathrm{p}^2} \tag{5.72}$$

where $\lambda = O(1)$. A crucial and general prediction in string cosmology is the presence of a new phase in the history of the Universe. By filling the Universe with heavy particles, despite the decay of some of them into relativistic products (i. e. radiation), matter becomes the dominant contribution to the stress-energy tensor: we have a new era of matter domination between cosmic inflation and the radiation-dominated era. It is the *moduli domination* epoch. The reheating process now happens from moduli and the reheating temperature is modified as

$$T_\mathrm{rh} \simeq \left(\frac{\alpha}{4\pi}\right)^{\frac{1}{2}} \left(\frac{m_\sigma}{M_\mathrm{p}}\right)^{\frac{1}{2}} m_\sigma \simeq 1\mathrm{GeV}\left(\frac{m_\sigma}{10^6\mathrm{GeV}}\right)^{3/2}, \tag{5.73}$$

which is generally lower than (4.43), because it is suppressed by M_p, instead to be proportional to it. We need to ensure that these particles decay soon enough, which means, in terms of temperature

$$T_\mathrm{rh} > T_\mathrm{BBN} \sim 1 \text{ MeV} \Rightarrow m_\sigma \gtrsim 30 \text{ TeV}. \tag{5.74}$$

The scenario with light moduli is the "*cosmological moduli problem*". This is a non-trivial bound, also because in some realizations of string theory, this scale is related to the scale of SUSY breaking M_soft. SUSY breaking is a huge topic, but we are interested here in two aspects:

(*i*) the scale of SUSY breaking is the scale of the lightest supersymmetric partner. They have not been observed, so from current particle phenomenology we can set $M_\mathrm{soft} \gtrsim 1$ TeV;

(*ii*) one attractive feature of SUSY is the chance to address the hierarchy problem of the Higgs mass. It provides a good solution if $M_\mathrm{soft} \sim 1$ TeV or slightly larger.

We see that these two requirements, together with (5.74) are incompatible if $m_\sigma \sim M_{\text{soft}}$, unless we accept a little hierarchy problem for the Higgs. So, a class of models is ruled out. A possible solution for the cosmological moduli problem is to decouple these scales, as happens in a scenario of moduli stabilization called *sequestered Large Volume Scenario* (LVS), where

$$m_\sigma \sim 100 \text{ TeV} \gg M_{\text{soft}} \sim 1 \text{ TeV}. \tag{5.75}$$

The cosmological moduli problem is solved and the fine-tuning problem for the Higgs is fixed remaining with a phenomenological viable model. LVS and sequestering will be sketched in the following Chapter.

5.2.3 The Axiverse

The moduli described in the previous section are just the real parts of the moduli T that follows from string theory. The imaginary parts are the ones that enjoy a shift symmetry which is exact at perturbative level, and thus they behave as axions. Axions form a class of particles initially introduced to solve a specific problem in particle physics. In QCD, a parameter that measures the amount of violation of CP symmetry, θ, needs to be fine-tuned to small values to account for the absence of strong CP-violation. A natural solution is to promote θ to a dynamical field that enjoys a $U(1)$ symmetry, which acts on it as $\theta \to \theta + \mathit{const}$. The vev of this field is 0 and this value is protected by the symmetry. This corresponds to the existence of a new particle, historically proposed as the first axion. Many of its properties are governed by the scale at which the symmetry is broken, called *axion decay constant* f_a. For example, the mass is

$$m_a \approx 6 \times 10^{-10}\,\text{eV} \left(\frac{10^{16}\,\text{GeV}}{f_a}\right) \tag{5.76}$$

String theory predicts hundreds of particles that enjoy a perturbative shift symmetry and are therefore massless at perturbative level. These particles acquire a mass only via non-perturbative effects, leading in general to a plethora of very light axions forming the string axiverse.

However, it is not trivial whether one of them realizes the CP-axion since they can turn out to be too heavy. Their mass and axion decay constant could violate relation (5.76). However, requiring that a QCD axion is realized constrains model building of the theory, just like requiring that inflation compatible with actual data is achievable. As we will see, this last condition can constrain the vevs of the moduli, resulting in a plethora of axions with $f_a \simeq 2 \times 10^{16}\,\text{GeV}$, a logarithmically distributed mass spectrum and the hypothesis that the QCD one exists.

Considering several windows for the masses, we have different impacts on phenomenology, observable today or with near-future experiments:

(i) for masses between 10^{-33} eV and 4×10^{-28} eV and a coupling $\vec{E}\cdot\vec{B}$ with Electromagnetism, these axions would cause a rotation of the CMB spectrum. In particular, a linearly polarized photon would experience a rotation of an angle $\Delta\beta$ because of the coupling with an axion such that

$$\Delta\beta \sim \frac{\alpha}{2\sqrt{3}} \approx 10^{-3}. \tag{5.77}$$

Notice that this quantity is independent of scale f_a and that for N axions it gets a factor $\sqrt{N}$. It is constant along the sky and measurable by Planck or CMBPol mission;

(ii) in the range between 10^{-28} eV and 4×10^{-18} eV axions can be a significant fraction of Dark Matter, measurable by impact on the Cold Dark Matter Power Spectrum. The uncertainty principle generates quantum pressure proportional to $1/\sqrt{m}$ and then short modes (high momenta) of these light particles suppress the power spectrum. The suppression exhibits step-like behaviour, where the width is proportional to the comoving momentum and the drop to the axion-fraction of Dark Matter ρ_a/ρ_{DM}. Having a plethora of axions, we expect many steps in CDM power spectrum, which can be observed by the Baryon Oscillation Spectroscopic Survey (BOSS);

(iii) finally, between 10^{-22} eV and 10^{-10} eV they affect the dynamics of rotating black-holes through *superradiance*. This phenomenon is connected to the Penrose process, which extracts energy and spin from rotating black holes. Basically, spinning objects that approach black holes can be scattered off with greater energy and spin. This is true if the object has an initial spin in the so-called superradiance range, bounded by the spin of the black hole horizon. Massive axions can orbit around these black holes along stable orbit, similar to an atomic system where the black hole is the nucleus. However, two differences need to be stressed: we are working with fields and not particles. Thus, there will always be some modes of these fields in the superradiant range. Secondly, these fields are bosons instead of fermions: some energy levels will be highly populated, forming Bose-Einstein condensates. They can move on stable orbits acting as mirrors: they reflect the accelerated emitted particles again towards the black hole and the Penrose process can take place again. So, superradiant modes get amplified, leading to an exponential classical instability. From the superradiance state, an axion can decay into the non-superradiance level emitting a graviton and it could be absorbed by the black hole, restoring part of the energy and spin carried away by Penrose process. Fueled in this way, the black hole acts as a gravitational wave pulsar, emitting signals measurable by upcoming interferometer like LISA.

Chapter 6

Moduli Stabilization for the Axiverse

In this Chapter, we will study how moduli stabilization can be achieved. By including several types of quantum corrections, we arrive at a well-known stabilization scenario, the Large Volume Scenario, which is presented explicitly for an arbitrary number of Kähler moduli. For simplicity, we will consider only Kähler moduli, setting $h^{1,1} = h^{1,1}_{+}$. After the general discussion, we will focus on some special cases: Swiss-Cheese Calabi-Yau manifolds and Fibred ones. Both of these classes will be considered with various $h^{1,1}$ numbers. The final goal is to find the axionic mass spectrum as a function of the topological properties of Calabi-Yau's while constraining the moduli to be massive enough to avoid the cosmological moduli problem.

6.1 Moduli stabilization in LVS

Finally, we can properly consider moduli stabilization. This can be achieved by including different kinds of quantum corrections. Their role is to break the no-scale structure, lift the flat directions and lead to stable points for the scalar potential. The stabilization is realized by a competition between the quantum corrections, which can be both perturbative or non-perturbative. In the former, we have higher-derivative α'^3-corrections from worldsheet loop corrections or loop in spacetime from higher genus worldsheets. Non-perturbative corrections come from non-perturbative effects, i. e. D3-brane instantons or *gaugino condensation*.
By the non-renormalization theorem in SUSY, Kähler potential gets corrections order by order in perturbation theory (+ non-perturbative corrections), while the superpotential is corrected only by non-perturbative effects. Then, we need to choose which quantum corrections to take into account. There can be a competition between different perturbative corrections, while the superpotential is fine-tuned to exponentially suppressed values

by properly choosing the fluxes. This is the so-called *KKLT scenario*. It results in an SUSY AdS vacua in $4d$.
In this work, we will focus on the so-called *Large Volume Scenario (LVS)*. This choice is made because

- (i) it is an understood scenario, where explicit results have been found (even if in very simple cases, see [1]);
- (ii) from the Kreuzer-Skarke database, it is known that the Hodge number $h^{1,1}$ of a typical Calabi-Yau 3-fold is roughly $h^{1,1} \sim 100$ and that in general $h^{1,1} \lesssim 500$. [2] reports how the volume of the Calabi-Yau and other topological quantities grow with $h^{1,1}$ and we find an exponentially large volume $\mathcal{V} \gtrsim 10^8 l_s^6$.

Formally, LVS involves a competition between perturbative corrections with respect to non-perturbative corrections. Generally, non-perturbative corrections are suppressed compared to perturbative ones. To allow for competition between them, it is necessary for some of the Kähler moduli to be exponentially suppressed relative to others. More explicitly, this implies that for some *small* moduli τ_s the limit

$$a_s^r \tau_s^r = \ln \mathcal{V}, \qquad \text{when } \mathcal{V} \to \infty \text{ for some } r = 1, ..., N_{small} \tag{6.1}$$

The remaining modes are then identified as *large* or *big* moduli $N_{large} = h^{1,1} - N_{small}$. In this scenario, it is clearly essential to have $h^{1,1} > 1$ in order to allow for the the hierarchy (6.1).
Now, we report some of the basic results of [1], where the moduli stabilization is done explicitly for $N_{small} = 1$. In natural units ($M_p = 1$) the α'^3-corrections and non-perturbative ones modify the unperturbed expression of $K_0 = -2\ln(\mathcal{V})$ and W_0 as

$$K = -2\ln\left(\mathcal{V} + \frac{\xi}{2g_s^{3/2}}\right) + \ln\left(\frac{g_s}{2}\right), \quad W = \frac{1}{\sqrt{4\pi}}\left(W_0 + A_s e^{-a_s T_s}\right) \tag{6.2}$$

From these, we can extract the F-term SUGRA scalar potential at leading order with a standard calculation, obtaining

$$V_{\text{LVS}} = \left(\frac{g_s}{8\pi}\right)\left[\frac{8}{3}(a_s A_s)^2 \frac{\sqrt{\tau_s} e^{-2a_s\tau_s}}{\mathcal{V}} - 4a_s A_s W_0 \frac{\tau_s e^{-a_s\tau_s}}{\mathcal{V}^2} + \frac{3\xi W_0^2}{4g_s^{3/2}\mathcal{V}^3}\right] \tag{6.3}$$

Now, the analysis can begin. By extremizing with respect to τ_s

$$\begin{aligned} \frac{\partial V_{\text{LVS}}}{\partial \tau_s} = 0 \Leftrightarrow \mathcal{V} = \frac{W_0}{a_s A_s}\frac{1 - a_s\tau_s}{1 - 4a_s\tau_s} 3\sqrt{\tau_s} e^{a_s\tau_s} \equiv \frac{W_0}{a_s A_s} f(\tau_s) e^{a_s\tau_s} \Rightarrow \\ \Rightarrow \frac{\mathcal{V}}{W_0} \simeq \frac{1}{a_s A_s}\frac{3}{4}\sqrt{\tau_s} e^{a_s\tau_s} \end{aligned} \tag{6.4}$$

where $f(\tau_s)$ is expanded in terms of the parameter $\varepsilon_s \equiv 1/(4a_s\tau_s)$

$$f(\tau_s) = \frac{1-1/(4\varepsilon_s)}{1-1/\varepsilon_s}3\sqrt{\tau_s} = \frac{4\varepsilon_s-1}{\varepsilon_s-1}\frac{\varepsilon_s}{4\varepsilon_s}3\sqrt{\tau_s} = \frac{3}{4}\frac{1-4\varepsilon_s}{1-\varepsilon_s}\sqrt{\tau_s} = \\ = \frac{3}{4}\sqrt{\tau_s}(1-3\varepsilon_s) + O(\sqrt{\tau_s}\varepsilon_s^2) \tag{6.5}$$

Plugging this into (6.3) we obtain a function of $\mathcal{V}$

$$\langle V_{\text{LVS}}(\mathcal{V})\rangle = \frac{g_s}{8\pi}\frac{3W_0^2}{2\mathcal{V}^3}\left(\frac{\hat{\xi}}{2} - \tau_s(\mathcal{V})^{3/2}\right) \tag{6.6}$$

where $\hat{\xi} \equiv \xi/g_s^{3/2}$.
The extrema of (6.3) are finally given by

$$0 = \frac{\partial V_{\text{LVS}}}{\partial \mathcal{V}} = \left(\frac{g_s}{8\pi}\right)\left[-\frac{8}{3}(a_sA_s)^2\frac{\sqrt{\tau_s}e^{-2a_s\tau_s}}{\mathcal{V}^2} + 8a_sA_sW_0\frac{\tau_s e^{-a_s\tau_s}}{\mathcal{V}^3} - \frac{9\xi W_0^2}{4g_s^{3/2}\mathcal{V}^4}\right] = \\ = -\frac{g_s}{8\pi}\frac{2W_0^2}{V^4}\left[\frac{4}{3}(a_sA_s)^2\sqrt{\tau_s}e^{-2a_s\tau_s}\left(\frac{\mathcal{V}}{W_0}\right)^2 - 4a_sA_s\tau_s e^{-a_s\tau_s}\frac{\mathcal{V}}{W_0} + \frac{9}{8}\hat{\xi}\right] \tag{6.7}$$

Substituting the relation (6.4) into the last equation, we end up with

$$\frac{4}{3}\sqrt{\tau_s}f^2(\tau_s) - 4\tau_s f(\tau_s) + \frac{9}{8}\frac{\xi}{g_s^{3/2}} = 0 \tag{6.8}$$

and now, expanding as in (6.5)

$$\frac{3}{2}\sqrt{\tau_s}\tau_s(1-6\varepsilon_s) - 6\tau_s\sqrt{\tau_s}(1-3\varepsilon_s) + \frac{9}{4}\hat{\xi} = 0 \\ \Rightarrow \tau_s = \left(\frac{\hat{\xi}}{2}\right)^{2/3}(1+2\varepsilon_s)^{2/3}. \tag{6.9}$$

Substituting this result into (6.6) we find the minimum of the potential V_{LVS}, which is

$$\langle V_{\text{LVS}}\rangle = -\frac{3}{2}\hat{\xi}\frac{m_{3/2}^2}{\mathcal{V}}\varepsilon_s, \tag{6.10}$$

where $m_{3/2}^2$ is the squared gravitino mass that we can easily extract from Kähler potential and superpotential

$$m_{3/2}^2 = e^K|W|^2 \simeq \left(\frac{g_s}{8\pi}\right)\frac{W_0^2}{\mathcal{V}^2}. \tag{6.11}$$

We then end up with the following hierarchy in the mass spectrum

$$m_{\tau_s}^2 \sim m_{c_s}^2 \sim m_{3/2}^2 \gg m_{\mathcal{V}}^2 \sim \frac{m_{3/2}^2}{\mathcal{V}} \gg m_{c\mathcal{V}}^2 \sim M_p^2 e^{-a_b \mathcal{V}^{2/3}} \sim 0 \tag{6.12}$$

where c_s refers to the axion.
According to the hierarchy (6.12), LVS gives rise to stabilized moduli fields and associated axions with masses similar to the gravitino mass $m_{3/2}$ (apart from the large cycles). The cosmological moduli problem states that $m_{3/2} \sim m_{\tau_i} \gtrsim 30$ TeV. The scale of soft SUSY breaking M_{soft} must then be decoupled from them. As already anticipated, this can be achieved in sequestered scenarios. An example of sequestering uses D3-branes, localized at Calabi-Yau singular points. SM degrees of freedom live on them. We have also D7-branes, that wrap Calabi-Yau 4-cycles with volumes τ_i. See figure 6.1. SUSY breaking occurs away from the D3-branes where the MSSM lives, in a *sequestered sector*, corresponding here to the D7-branes. SUSY is then locally preserved on the visible sector, with $M_{\text{soft}} = 0$, but broken globally, resulting in $m_{3/2} \neq 0$. Thus, $M_{\text{soft}} \ll m_{3/2}$ automatically [1].

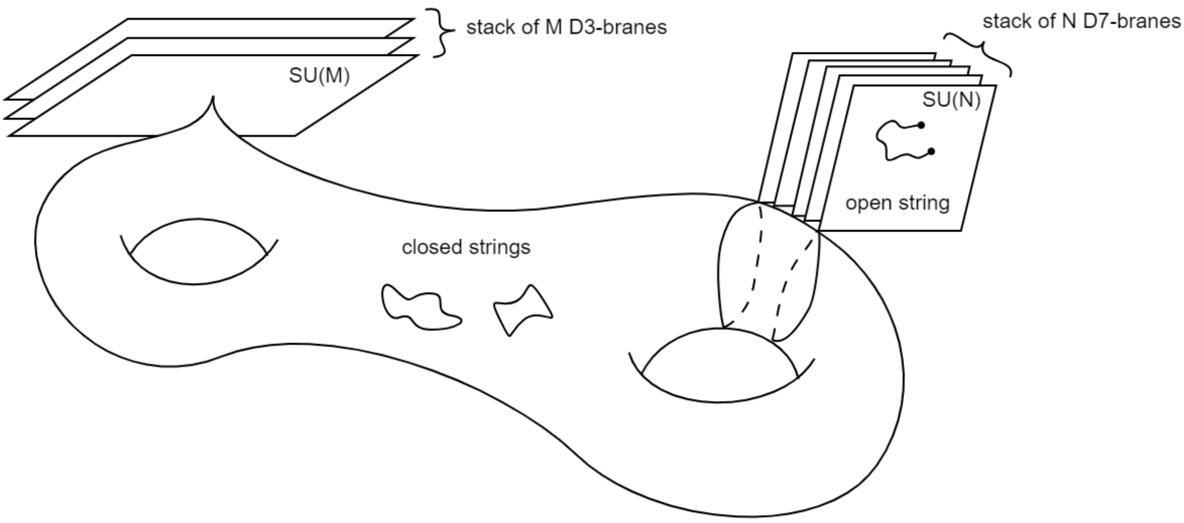

Figure 6.1: sequestered LVS: the stacks of N and M D7/D3-branes define an $SU(N)$ and an $SU(M)$ gauge theory, respectively. SM or its SUSY extensions are constrained to live on the D3-branes.

To summarize, we have a minimum where the small modulus is stabilized and SUSY is broken ($m_{3/2}^2 \neq 0$). However, two issues are still present:

[1] Two technical details: we have two soft terms contributing to SUSY breaking: the gaugino mass $M_{1/2}^{D3}$ and scalar masses (e.g. squarks and sleptons) m_0. Both vanish on D3-branes. However, quantum corrections lift them. In the end, we have the hierarchy $M_{1/2}^{D3} \sim \frac{m_{3/2}}{\mathcal{V}} \ll m_0 \sim \frac{m_{3/2}}{\sqrt{\mathcal{V}}} \ll m_{3/2}$ for $\mathcal{V} \gg 1$, as in LVS. This hierarchy takes into account many aspects: the cosmological moduli problem, Higgs fine-tuning and WIMP Dark Matter compatible with LHC. For more on sequestering, see [21].

(i) $N_{large} = h^{1,1} - 1$ moduli are still massless;

(ii) the solution is anti-de Sitter ($\langle V_{\rm LVS}\rangle < 0$), contrary to the fact that our universe is de Sitter.

(ii) is a very general and one of the most compelling problems in string theory: compactification naturally ends up in an AdS spacetime.
Both (i) and (ii) can be addressed with a similar strategy: we ass appropriate terms to the potential (6.3)

$$V = V_{\rm LVS}\left(\mathcal{V}, t_s\right) + V_{F^4}\left(t_i, t_s\right) + V_{\rm up}(\mathcal{V}) \tag{6.13}$$

where

$$V_{F^4} = -\left(\frac{g_s}{8\pi}\right)^2 \frac{\lambda\left|W_0\right|^4}{g_s^{3/2}\mathcal{V}^4}\Pi_i t_i \tag{6.14}$$

comes from higher-order α'^3-corrections with λ and Π_i constants and t_i are the volumes of 2-cycles that enter directly in both the expression of the volume of the fixed CY

$$\mathcal{V} = \frac{1}{6}\sum_{i,j,k=1}^{N_{\rm large}} k_{ijk}t_i t_j t_k - \frac{1}{6}\sum_{s=1}^{N_{\rm small}} k_{sss}t_s^3 \tag{6.15}$$

and in the moduli τ_i, which are the volumes of divisors of CY

$$\tau_i = \frac{1}{2}k_{ijk}t_j t_k. \tag{6.16}$$

V_{F^4} will stabilize the remaining moduli, while

$$V_{\rm up}(\mathcal{V}) = \frac{\kappa}{\mathcal{V}^\alpha} \quad , \quad 0 < \alpha < 3 \tag{6.17}$$

provides the dS uplift. As before, we proceed in the search for a stable critical point. In particular, we can integrate out the small modulus τ_s ignoring the t_s-dependence of $V_{\rm LVS} \sim \mathcal{O}\left(\mathcal{V}^{-3}\right)$ wrt $V_{F^4} \sim \mathcal{O}\left(\mathcal{V}^{-4}\right)$ for $\mathcal{V} \ll 1$. We can then write the potential as

$$\begin{aligned} V = &\left(\frac{g_s}{8\pi}\right)\frac{3W_0^2}{2\mathcal{V}^3}\left(\frac{\hat{\xi}}{2} - \tau_s(\mathcal{V})^{3/2}\right) - \left(\frac{g_s}{8\pi}\right)^2\frac{\hat{\lambda}W_0^4}{\mathcal{V}^4}\sum_{i=1}^{N_{\rm large}}\Pi_i t_i + \frac{\kappa}{\mathcal{V}^\alpha} \equiv \\ &\equiv \hat{V}(\mathcal{V}) + V_{F^4}\left(t_i, t_s\right). \end{aligned} \tag{6.18}$$

Leaving the details in [1], here we report the fact that by extremising with respect to t_i we are able to express all the now stabilized τ_i in terms of only one of them, namely τ_* and written with the overall volume $\mathcal{V}$

$$0 = \frac{\partial V}{\partial t_i} = \frac{\partial \hat{V}}{\partial \mathcal{V}}\tau_i + \frac{\partial V_{F^4}}{\partial t_i} \Rightarrow \tau_i = \frac{\Pi_i}{\Pi_*}\tau_*(\mathcal{V}), \quad \forall i, \tag{6.19}$$

where its stabilization gives

$$\frac{\hat{\xi}}{2} = \tau_s^{3/2}(\mathcal{V})(1-2\varepsilon_s) - \frac{8\pi}{g_s}\frac{2\alpha\kappa}{9W_0^2}V^{3-\alpha}. \tag{6.20}$$

We notice that this result looks very similar to (6.9). Plugging into (6.18), we obtain

$$\langle V(\mathcal{V})\rangle = -\frac{3cW_0^2}{\mathcal{V}^3}\epsilon_s\tau_s(\mathcal{V})^{3/2} + \frac{2c^2\hat{\lambda}W_0^4\Pi_*}{3h\mathcal{V}^{11/3}} + \frac{\kappa(3-\alpha)}{3\mathcal{V}^\alpha} \tag{6.21}$$

where $c = g_s/8\pi$ and h is a function of Π_i and k_{ijk}.
Finally, by setting (6.21) to 0, we find the value of the constant κ that uplifts the AdS solution obtained so far. Substituting this κ into (6.20) we also fix that $\hat{\xi} > 0$ regardless the negative contribution.
In the end, to ensure that the critical point is indeed a minimum, we need to inspect the Hessian. By requiring $\hat{\lambda} < 0$, the Hessian is positive definite and then we have an uplifted minimum where all moduli are stabilized.

6.2 Axion masses for Swiss-Cheese Calabi-Yau's

6.2.1 Swiss-cheese: $h^{1,1}$=2

The 4-cycles τ_i define complex scalar fields T_i together with an axionic partner θ_i, which comes from the integration of a 4-form over the corresponding 4-cycle. We will now see how axions acquire mass in a model closer to the one discussed in [5].
In order to give mass to the axions associated with the 2 moduli τ_s and τ_b, we need to generalize the superpotential W by including non-perturbative corrections from τ_b. Then, the Kähler potential and the superpotential look like

$$K = -2\ln\left(\frac{1}{9\sqrt{2}}\left(\tau_b^{3/2} - \tau_s^{3/2}\right) + \frac{\xi}{2g_s^{3/2}}\right) \equiv -2\ln\left(\mathcal{V}+\xi'\right) \tag{6.22}$$

$$W = W_0 + A_s e^{-a_s T_s} + A_b e^{-a_b T_b} \tag{6.23}$$

The Kähler potential is the same of [5] and so the Kähler metric, then we can write

$$K_b = \frac{\partial K}{\partial T_b} = -\frac{3\sqrt{\tau_b}}{2\left(\mathcal{V}+\xi'\right)} = K_{\bar{b}} = \frac{\partial K}{\partial \overline{T}_{\bar{b}}} \simeq -\frac{3}{2}\frac{1}{\tau_b} \tag{6.24}$$

$$K_s = \frac{\partial K}{\partial T_s} = \frac{3\sqrt{\tau_s}}{2\left(\mathcal{V}+\xi'\right)} = K_{\bar{s}} = \frac{\partial K}{\partial \overline{T}_{\bar{s}}} \simeq \frac{3}{2}\frac{\tau_s^{1/2}}{\tau_b^{3/2}} \tag{6.25}$$

and finally, the Kähler metric and its inverse are

$$K_{i\bar{j}} = \begin{pmatrix} K_{b\bar{b}} & K_{b\bar{s}} \\ K_{s\bar{b}} & K_{b\bar{s}} \end{pmatrix} = \begin{pmatrix} \frac{3}{4\tau_b^2} & -\frac{9\tau_s^{\frac{1}{2}}}{8\tau_b^{5/2}} \\ -\frac{9\tau_s^{\frac{1}{2}}}{8\tau_b^{5/2}} & \frac{3}{8\tau_s^{\frac{1}{2}}\tau_b^{3/2}} \end{pmatrix} \tag{6.26}$$

$$K_{i\bar{j}}^{-1} = \begin{pmatrix} \frac{4\tau_b^2}{3} & 4\tau_b\tau_s \\ 4\tau_b\tau_s & \frac{8\tau_b^{3/2}\tau_s^{1/2}}{3} \end{pmatrix}. \tag{6.27}$$

These expressions are evaluated using the fact that $\tau_b \gg \tau_s > 1$ by definition of big and small moduli and the decompactification limit

$$\mathcal{V} \to \infty \quad , \quad \tau_b = \mathcal{V}^{2/3} \quad , \quad a_s\tau_s = \ln \mathcal{V} \tag{6.28}$$

The scalar potential is now more complicated

$$\begin{aligned} V_F = e^K \Big[& K^{s\bar{s}}\partial_s W \partial_{\bar{s}} \overline{W} + K^{s\bar{b}}\partial_s W \partial_b \overline{W} + K^{b\bar{s}}\partial_b W \partial_{\bar{s}} \overline{W} + K^{b\bar{b}}\partial_b W \partial_{\bar{b}} \overline{W} + \\ & + K^{s\bar{s}}\left(K_s W \partial_{\bar{s}}\overline{W} + \partial_s W K_{\bar{s}}\overline{W}\right) + K^{s\bar{b}}\left(K_s W \partial_{\bar{b}}\overline{W} + \partial_s W K_{\bar{b}}\overline{W}\right) + \\ & + K^{b\bar{s}}\left(K_b W \partial_{\bar{s}}\overline{W} + \partial_b W K_{\bar{s}}\overline{W}\right) + K^{b\bar{b}}\left(K_b W \partial_b\overline{W} + \partial_b W K_{\bar{b}}\overline{W}\right) + \\ & + \left(K^{s\bar{s}}K_s K_{\bar{s}} + K^{b\bar{s}}K_b K_{\bar{s}} + K^{s\bar{b}}K_s K_{\bar{b}} + K^{b\bar{b}}K_b K_{\bar{b}} - 3\right)|W|^2 \Big] \end{aligned} \tag{6.29}$$

or, collecting the lines

$$V_F = V_{np1} + V_{np2} + V_{\alpha'} \tag{6.30}$$

where V_{np1} is the first line, V_{np2} the second and the third and $V_{\alpha'}$ the last.
In order to evaluate the potential, we need to know also the derivatives of the superpotential W:

$$\begin{array}{ll} \partial_s W = -a_s A_s e^{-a_s T_s}, & \partial_b W = -a_b A_b e^{-a_b T_b} \\ |\partial_s W|^2 = a_s^2 |A_s|^2 e^{-a_s\left(T_s + \bar{T}_s\right)}, & |\partial_b W|^2 = a_b^2 |A_b|^2 e^{-a_b\left(T_b + \bar{T}_b\right)} \\ \partial_s W \partial_b \overline{W} = a_s a_b A_s \overline{A}_b e^{-a_s T_s - a_b \overline{T}_b}, & \partial_{\bar{s}} \overline{W} \partial_b W = a_s a_b \bar{A}_s A_b e^{-a_s \overline{T}_s - a_b T_b} \end{array} \tag{6.31}$$

then V_{np1} becomes

$$\begin{aligned} V_{np1} = e^K \Big[& \frac{8}{3}\tau_b^{3/2}\tau_s^{1/2} a_s^2 |A_s|^2 e^{-2a_s\tau_s} + \frac{4}{3}\tau_b^2 a_b^2 |A_b|^2 e^{-2a_b\tau_b} + \\ & + 4\tau_b\tau_s a_s a_b \left(A_s\overline{A}_b e^{-a_s T_s - a_b\overline{T}_b} + A_b\overline{A}_s e^{-a_b T_b - a_s\overline{T}_s}\right)\Big]. \end{aligned} \tag{6.32}$$

By using (6.31), and (6.24)-(6.27)

$$
\begin{aligned}
K^{s\bar{s}}K_s\partial_{\bar{s}}\overline{W} &= \left(\frac{8}{3}\tau_b^{3/2}\tau_s^{1/2}\right)\left(\frac{3}{2}\frac{\tau_s^{1/2}}{\tau_b^{3/2}}\right)\left(\left(-a_sA_s\right)e^{-a_s\overline{T}_s}\right) = -4\left(a_sA_s\right)\tau_s e^{-a_s\overline{T}_s} \\
K^{s\bar{b}}K_s\partial_{\bar{b}}\overline{W} &= \left(4\tau_b\tau_s\right)\left(\frac{3}{2}\frac{\tau_s^{1/2}}{\tau_b^{3/2}}\right)\left(\left(-a_bA_b\right)e^{-a_b\overline{T}_b}\right) = -6\left(a_bA_b\right)\frac{\tau_s^{3/2}}{\sqrt{\tau_b}}e^{-a_b\overline{T}_b} \\
K^{b\bar{s}}K_b\partial_{\bar{s}}\overline{W} &= \left(4\tau_b\tau_s\right)\left(-\frac{3}{2}\frac{1}{\tau_b}\right)\left(\left(-a_sA_s\right)e^{-a_s\overline{T}_s}\right) = 6\left(a_sA_s\right)\tau_s e^{-a_s\overline{T}_s} \\
K^{b\bar{b}}K_b\partial_{\bar{b}}\overline{W} &= \left(\frac{4}{3}\tau_b^2\right)\left(-\frac{3}{2}\frac{1}{\tau_b}\right)\left(\left(-a_bA_b\right)e^{-a_b\bar{T}_b}\right) = 2\left(a_bA_b\right)\tau_b e^{-a_b\overline{T}_b}
\end{aligned}
\tag{6.33}
$$

from which V_{np2} is

$$
\begin{aligned}
V_{np2} &= e^K\left[W\left(K^{ss}K_s\partial_s\overline{W} + \cdots + K^{b\bar{b}}K_b\partial_b\overline{W}\right) + h.c.\right] = \\
&= e^K\left[W\left(2\left(a_sA_s\right)\tau_s e^{-\alpha_s\overline{T}_s} + \left(a_bA_b\right)\left(2\tau_b - 6\frac{\tau_s^{3/2}}{\sqrt{\tau_b}}\right)e^{-a_l\overline{T}_b}\right) + h.c.\right] \\
&= e^K\left[W_0\left(2\left(a_sA_s\right)\tau_s\left(e^{-a_s\overline{T}_s} + e^{-a_sT_s}\right) + \left(a_bA_b\right)\left(2\tau_b - 6\frac{\tau_s^{3/2}}{\sqrt{\tau_b}}\right)\left(e^{-a_b\overline{T}_b} + e^{-a_bT_b}\right)\right)\right] \\
&= e^K\left[4\left(a_sA_s\right)\tau_sW_0e^{-a_s\tau_s}\cos\left(a_s\theta_s\right) + 2\left(a_bA_b\right)\left(2\tau_b - 6\frac{\tau_s^{3/2}}{\sqrt{\tau_b}}\right)W_0e^{-a_b\tau_b}\cos\left(a_b\theta_b\right)\right]
\end{aligned}
\tag{6.34}
$$

Because the Kähler potential is the same as the one in [5], the term $V_{\alpha'}$ is unchanged

$$
V_{\alpha'} = \frac{3\xi W_0^2}{4g_s^{3/2}\mathcal{V}^3} \tag{6.35}
$$

By using the limit (6.28), the common prefactor becomes

$$
e^K = \frac{1}{\mathcal{V}^2} = \frac{1}{{\tau_b}^3}. \tag{6.36}
$$

Finally, the potential looks like

$$\begin{aligned} V_F = &\frac{8}{3}\left(a_sA_s\right)^2\frac{\tau_s^{1/2}}{\tau_b^{1/2}}e^{-2a_s\tau_s}+\frac{4}{3}\left(a_bA_b\right)^2\frac{1}{\tau_b}e^{-2a_b\tau_b}+\\ &+8\left(a_sA_sa_bA_b\right)\frac{\tau_s}{\tau_b^2}e^{-a_s\tau_s-a_b\tau_b}\cos\left(a_s\theta_s-a_b\theta_b\right)+\\ &+4\left|W_0\right|\left(\left(a_sA_s\right)\frac{\tau_s}{\tau_b^3}e^{-a_s\tau_s}\cos\left(a_s\theta_s\right)+\left(a_bA_b\right)\frac{1}{\tau_b^2}e^{-a_b\tau_b}\cos\left(a_b\theta_b\right)\right)+\\ &+\left|W_0\right|^2\frac{6\xi'}{\tau_b^{9/2}}. \end{aligned} \tag{6.37}$$

Notice that we have reabsorbed the phases of the A_i coefficients into the axions without modifying the physics. Furthermore, by setting $A_b = 0$, (6.37) reduces exactly to the simpler potential in [5].
Before analyzing the critical points of this potential, it is necessary to rewrite the fields in terms of canonically normalized fields. By considering the kinetic term of the Lagrangian

$$\begin{aligned} K_{ij}\partial_\mu T_i\partial^\mu\overline{T}_j \supset\, &K_{s\bar{s}}\left(\partial_\mu\tau_s\partial^\mu\tau_s+\partial_\mu\theta_s\partial^\mu\theta_s\right)+\\ &+K_{b\bar{b}}\left(\partial_\mu\tau_b\partial^\mu\tau_b+\partial_\mu\theta_b\partial^\mu\theta_b\right)=\\ &=\frac{3}{8\tau_s^{1/2}\tau_b^{3/2}}\left(\partial_\mu\tau_s\partial^\mu\tau_s+\partial_\mu\theta_s\partial^\mu\theta_s\right)+\\ &+\frac{3}{4\tau_b^2}\left(\partial_\mu\tau_b\partial^\mu\tau_b+\partial_\mu\theta_b\partial^\mu\theta_b\right), \end{aligned} \tag{6.38}$$

the canonically normalized fields look like

$$\begin{aligned} &\tau_s=\tfrac{2}{\sqrt{3}}\left\langle\tau_s\right\rangle^{1/4}\left\langle\tau_b\right\rangle^{3/4}\tau_s^c, && \theta_s=\tfrac{2}{\sqrt{3}}\left\langle\tau_s\right\rangle^{1/4}\left\langle\tau_b\right\rangle^{3/4}\theta_s^c,\\ &\tau_b=\sqrt{\tfrac{2}{3}}\left\langle\tau_b\right\rangle\tau_b^c, && \theta_b=\sqrt{\tfrac{2}{3}}\left\langle\tau_b\right\rangle\theta_b^c, \end{aligned} \tag{6.39}$$

where $\langle...\rangle$ refers to the location of the minimum and we define the constants $\tau_i = b_i\tau_i^c$. After plugging this into (6.37), the analysis of the axions can begin. By focusing on the part of the potential that depends on θ_i, we define

$$\begin{aligned} V_\theta :=\, &8\left(a_sA_sa_bA_b\right)\frac{\tau_s}{\tau_b^2}e^{-a_s\tau_s-a_b\tau_b}\cos\left(a_s\theta_s-a_b\theta_b\right)+\\ &+4\left|W_0\right|\left(\left(a_sA_s\right)\frac{\tau_s}{\tau_b^3}e^{-a_s\tau_s}\cos\left(a_s\theta_s\right)+\left(a_bA_b\right)\frac{1}{\tau_b^2}e^{-a_b\tau_b}\cos\left(a_b\theta_b\right)\right)\equiv\\ &\equiv a_1\cos\left(a_s\theta_s-a_b\theta_b\right)+a_2\cos\left(a_s\theta_s\right)+a_3\cos\left(a_b\theta_b\right) \end{aligned} \tag{6.40}$$

The coefficients a_1, a_2 and a_3 are introduced for practical convenience. In the regime (6.28), we have the following hierarchy between them

$$a_1 \propto \frac{\tau_s}{\tau_b^2} e^{-a_b\tau_b - a_s\tau_s} \quad , \quad a_2 \propto \frac{\tau_s}{\tau_b^3} e^{-a_s\tau_s} \quad , \quad a_3 \propto \frac{1}{\tau_b^2} e^{-a_b\tau_b} \tag{6.41}$$

$$\begin{cases} \frac{a_1}{a_2} = \tau_b e^{-a_b\tau_b} \ll 1 \Rightarrow a_1 \ll a_2 \\ \frac{a_1}{a_3} = \tau_s e^{-a_s\tau_s} \ll 1 \Rightarrow a_1 \ll a_3 \end{cases} \Rightarrow$$

$$\Rightarrow a_1 \ll a_3 \ll a_2 \tag{6.42}$$

Critical points of (6.37) are given by vanishing the derivatives

$$\begin{cases} \frac{\partial V_\theta}{\partial \theta_s^c} = -a_1 a_s b_s \sin(a_s\theta_s - a_b\theta_b) - a_2 a_s b_s \sin(a_s\theta_s) = 0 \\ \frac{\partial V_\theta}{\partial \theta_b^c} = a_1 a_b b_b \sin(a_s\theta_s - a_b\theta_b) - a_3 a_b b_b \sin(a_b\theta_b) = 0 \end{cases} \Rightarrow \tag{6.43}$$

$$\Rightarrow \begin{cases} a_1 \sin(a_s\theta_s - a_b\theta_b) = -a_2 \sin(a_s\theta_s) \\ a_2 \sin(a_s\theta_s) = a_3 \sin(a_b\theta_b) \end{cases} \tag{6.44}$$

By using some trigonometric identities, the first line becomes

$$\begin{aligned} & a_1 \left(\sin(a_s\theta_s)\cos(a_s\theta_b) - \cos(a_s\theta_s)\sin(a_b\theta_b)\right) = \\ & = \pm a_1 \sin(a_s\theta_s) \sqrt{1 - \frac{a_2^2}{a_3^2}\sin^2(a_s\theta_s)} + \cos(a_s\theta_s)\frac{a_2}{a_3}\sin(a_s\theta_s) = \\ & = -a_2 \sin(a_s\theta_s) \end{aligned} \tag{6.45}$$

We need to distinguish 2 cases:

(i) $\sin(a_s\theta_s) = 0 \Rightarrow \sin(a_b\theta_b) = 0.$

(ii) $\sin(a_s\theta_s) \neq 0 \Rightarrow \pm a_1 \sqrt{1 - \frac{a_2^2}{a_3^2}\sin^2(a_s\theta_s)} + \frac{a_2}{a_3}\sqrt{1 - \sin^2(a_s\theta_s)} = -a_2.$

Notice that because of the hierarchy (6.28), the argument of the first square root in (ii) is negative, so we do not obtain any critical point. Only (i) provides them as

$$(\theta_s, \theta_b) = \left(\frac{n_s\pi}{a_s}, \frac{n_b\pi}{a_b}\right), n_s, n_b \in \mathbb{N}, \tag{6.46}$$

For the sake of simplicity, we have written the minima in terms of the original fields, but by substituting (6.39), we can express them in terms of the canonically normalized ones. It is important to note that the mixed second-order partial derivatives with respect to τ_i and θ_j of the original full potential (6.37) are proportional to $\sin(a_i\theta_i)$. Then, in the critical points given by (i), all these modulus-axion mixed terms vanish. Consequently, the full 4×4 Hessian of (6.37) can be represented by only 2 blocks. Due to this simplification, in order to extract axion masses, we can simply consider the sub-block

$$\left.\frac{\partial^2 V_\theta}{\partial\theta_i^c\partial\theta_j^c}\right|_{(n_s,n_b)} = \begin{pmatrix} (-1)^{n_s-n_b+1}a_1a_s^2b_s^2 + (-1)^{n_s+1}a_2a_s^2b_s^2 & (-1)^{n_s-n_b}a_1a_sb_sa_bb_b \\ (-1)^{n_s-n_b}a_1a_sb_sa_bb_b & (-1)^{n_s-n_b+1}a_1a_b^2b_b^2 + (-1)^{n_b+1}a_3a_b^2b_b^2 \end{pmatrix} \tag{6.47}$$

which, under (6.28), reduces to

$$\left.\frac{\partial^2 V_\theta}{\partial\theta_i^c\partial\theta_j^c}\right|_{(n_s,n_b)} = \begin{pmatrix} (-1)^{n_s+1}a_2a_s^2b_s^2 & (-1)^{n_s-n_b}a_1a_sb_sa_bb_b \\ (-1)^{n_s-n_b}a_1a_sb_sa_bb_b & (-1)^{n_b+1}a_3a_b^2b_b^2 \end{pmatrix} \equiv V_{ij} \tag{6.48}$$

We find minima for $\det V_{ij} > 0$ and positive diagonal terms. These 2 conditions constrain the Hessian to

$$V_{ij} = \begin{pmatrix} a_2a_s^2b_s^2 & a_1a_sb_sa_bb_b \\ a_1a_sb_sa_bb_b & a_3a_b^2b_b^2 \end{pmatrix}. \tag{6.49}$$

Notice that the minima (for odd n_s and n_b) are such that they ensure a negative sign in front of V_{np2}. It plays a key role in the full potential V_F as well. As we can also see from the analysis in the previous section, the minimum of (6.3) is realized through the negative contribution proportional to $|W_0|$. The negative sing is precisely ensured by the minimum of the axions.
Finally, we can express the masses of the θ's. Their squares are simply the eigenvalues of $\frac{1}{2}V_{ij}$ and because $\det(V_{ij}) \ll \mathrm{tr}(V_{ij})$, we can make the following approximations

$$m_{\theta_s}^2 \simeq \mathrm{tr}(\frac{1}{2}V_{ij}) \simeq a_2a_s^2b_s^2 = \frac{16}{3}\left(a_s^3A_s\right)W_0\frac{\tau_s^{\frac{3}{2}}}{\tau_b^{\frac{3}{2}}}e^{-a_s\tau_s} \sim \frac{(\ln\mathcal{V})^{\frac{3}{2}}}{\mathcal{V}^2} \tag{6.50a}$$

$$m_{\theta_b}^2 \simeq \frac{\det\left(\frac{1}{2}V_{ij}\right)}{\mathrm{tr}\left(\frac{1}{2}V_{ij}\right)} \simeq a_3a_b^2b_b^2 = \frac{16}{3}\left(a_b^3A_b\right)W_0e^{-a_b\tau_b} \sim e^{-a_b\mathcal{V}^{2/3}} \tag{6.50b}$$

We end up with 2 light axions, one of which is exponentially suppressed and therefore effectively massless. This is in line with the general analysis reported in [2], which shows the presence of a massless axion as an always-present feature.

6.2.2 Swiss-cheese: arbitrary $h^{1,1}$

In the low-energy regime, realistic string theory vacua exhibit a plethora of axions: the axiverse, which we have already encountered in 5.2.3. We will now consider how they modify the model discussed so far.
Let's start with the volume of the extra-dimensions in a model with $h^{1,1}$ moduli. A

canonical class of LVS vacua arises in the "*Swiss-Cheese*" manifolds, as reported in [7]. In particular, we will focus on the "*strong*" Swiss-Cheese, which looks like

$$\mathcal{V} = \alpha \left(\tau_b^{3/2} - \sum_{j=2}^{h^{1,1}} \lambda_j \tau_j^{3/2} \right) . \tag{6.51}$$

The big modulus τ_b regulates the overall volume (of the cheese), while the small moduli τ_i control the volume of the holes and $\alpha, \lambda_j > 0$. Notice that also the volume in (6.22) is of this kind.

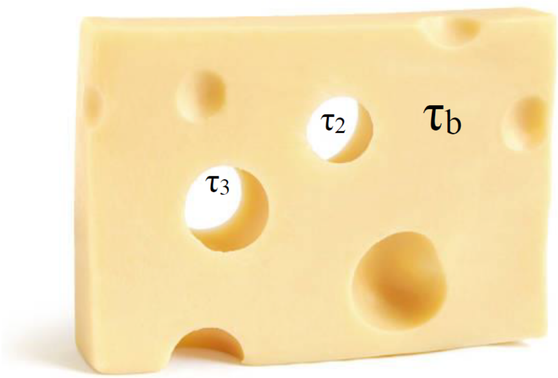

Figure 6.2: Pictorial representation of a Swiss-Cheese manifold: τ_b is the volume of the cheese and τ_2 and τ_3 are the volumes of 2 holes.

First, let's write the new Kähler potential and the superpotential:

$$K = -2\ln(\mathcal{V} + \xi') \tag{6.52}$$

$$W = W_0 + \sum_{j=1}^{h^{1,1}} A_j e^{-a_j T_j} \tag{6.53}$$

where we set $\tau_1 = \tau_b$. Now, the analysis proceeds in a strongly analogous manner to the previous one. In order to express Kähler metric and its inverse, we evaluate

$$K_b = K_{\bar{b}} = -\frac{3\sqrt{\tau_b}}{2(\mathcal{V} + \xi')} \simeq -\frac{3}{2}\frac{1}{\tau_b} \tag{6.54}$$

$$K_i = K_{\bar{i}} = \frac{3\sqrt{\tau_i}}{2(\mathcal{V} + \xi')} \simeq \frac{3}{2}\lambda_i \frac{\tau_i^{1/2}}{\tau_b^{3/2}} \tag{6.55}$$

The second-order derivatives appear in the Kähler metric

$$K_{i\bar{j}} = \begin{pmatrix} \frac{3}{4\tau_b^2} & -\frac{9\lambda_i \tau_i^{1/2}}{8\tau_b^{5/2}} \\ -\frac{9\lambda_i \tau_i^{1/2}}{8\tau_b^{5/2}} & \frac{3\lambda_i \delta_{ij}}{8\tau_i^{1/2}\tau_b^{3/2}} \end{pmatrix} \tag{6.56}$$

It is worthwhile to stress that in the $(2, 2)$-like element of this matrix, there is instead an $(h^{1,1} - 1) \times (h^{1,1} - 1)$ diagonal sub-matrix, where we have ignored $O(1/\mathcal{V}^2)$ corrections. Before expressing its inverse, it is interesting to consider for a moment the determinant

$$\det K = \frac{3}{4\tau_b^2} \prod_{i=2}^{h^{1,1}} \left(\frac{3\lambda_i}{8\tau_i^{1/2}\tau_b^{3/2}} \right) - \sum_{j=2}^{h^{1,1}} \left[\left(\frac{9\lambda_i \tau_i^{1/2}}{8\tau_b^{5/2}} \right)^2 \prod_{\substack{i=2 \\ i \neq j}}^{h^{1,1}} \left(\frac{3\lambda_i}{8\tau_i^{1/2}\tau_b^{3/2}} \right) \right] \tag{6.57}$$

In the decompactification limit, where under (6.1) all the small moduli τ_i behave similarly, the determinant is

$$\det K \simeq \left(\frac{1}{\mathcal{V}} \right)^{h^{1,1}+1/3} \left(\frac{1}{\ln \mathcal{V}} \right)^{\frac{h^{1,1}-1}{2}} - \left(h^{1,1} - 1 \right) \left(\frac{1}{\mathcal{V}} \right)^{h^{1,1}+4/3} \left(\frac{1}{\ln \mathcal{V}} \right)^{\frac{h^{1,1}}{2}} \tag{6.58}$$

In order to understand the dominant term, we need specific information on the scaling of $\mathcal{V}$ with respect to $h^{1,1}$. As reported in 5.1.2, [2] provides a power-law dependence for the volume such that

$$\mathcal{V} \sim \left(h^{1,1} \right)^p, \quad 6 \lesssim p \lesssim 7 \tag{6.59}$$

Thanks to this result, we can compare the 2 terms as

$$\left(h^{1,1} - 1 \right) \frac{(\ln \mathcal{V})^{1/2}}{\mathcal{V}} \simeq h^{1,1} \frac{(\ln h^{1,1})^{1/2}}{(h^{1,1})^p} = \frac{(\ln h^{1,1})^{1/2}}{(h^{1,1})^{p-1}} \sim 0 \tag{6.60}$$

Thus, the determinant can be approximated as

$$\det K \simeq \frac{3}{4\tau_b^2} \prod_{i=2}^{h^{1,1}} \left(\frac{3\lambda_i}{8\tau_i^{1/2}\tau_b^{3/2}} \right) \tag{6.61}$$

This fact is not so general, but specific for (6.59): a different law, such as $\mathcal{V} \sim \log (h^{1,1})$ or an exponent in the range $0 \lesssim p \lesssim 1$ would have led to a completely different result, despite the fact that the volume increases in all cases.
Now, we can write down the inverse of the Kähler metric

$$K^{i\bar{j}} = \begin{pmatrix} \frac{4\tau_b^2}{3} & 4\lambda_i \tau_i \tau_b \\ 4\lambda_i \tau_i \tau_b & \frac{8}{3} \frac{\tau_i^{1/2} \tau_b^{3/2}}{\lambda_i} \delta^{ij} + 12\tau_i \tau_j \left(1 - \delta^{ij} \right) \end{pmatrix} \tag{6.62}$$

The derivatives of the superpotential W are completely analogous to (6.31): the "b" derivative is unchanged, while for the small moduli we simply replace the "s" with the "i's".
We have all the quantities to evaluate the scalar potential V_F. As before, we introduce V_{np1}, V_{np2} and $V_{\alpha'}$ for simplicity. Let us start with V_{np1}

$$V_{np_1} = e^K \left[K^{b\bar{b}} \partial_b W \partial_b \overline{W} + \left(K^{i\bar{b}} \partial_i W \partial_{\bar{b}} \overline{W} + \text{h.c.} \right) + K^{i\bar{j}} \partial_i W \partial_{\bar{j}} \overline{W} \right] \tag{6.63}$$

where the sums from $i,j=2$ to $h^{1,1}$ are left implicit. The first term is the same as in the previous case. The sum inside the round bracket is easily

$$\left(K^{i\bar{b}}\partial_i W\partial_{\bar{b}}\overline{W}+h.c.\right)=8\lambda_i\left(a_iA_ia_bA_b\right)\tau_i\tau_b e^{-(a_i\tau_i+a_b\tau_b)}\cos\left(a_i\theta_i-a_b\theta_b\right) \tag{6.64}$$

The last term is

$$\begin{aligned}K^{i\bar{j}}\partial_i W\partial_{\bar{j}}\overline{W}=\sum_{i=2}^{h^{1,1}}&\left[\frac{8}{3\lambda_i}\left(a_iA_i\right)^2\tau_b^{3/2}\tau_i^{1/2}e^{-2a_i\tau_i}+\right.\\&\left.+\sum_{j\neq i}24\left(a_iA_ia_jA_j\right)\tau_i\tau_j e^{-(a_i\tau_i+a_j\tau_j)}\cos\left(a_i\theta_i-a_b\theta_b\right)\right]\end{aligned} \tag{6.65}$$

In the end, V_{np1} looks like

$$\begin{aligned}V_{np1}=\frac{4}{3}\left(a_bA_b\right)^2\frac{1}{\tau_b}e^{-2a_b\tau_b}+\sum_{i=2}^{h^{1,1}}&\left[\frac{8}{3\lambda_i}\left(a_iA_i\right)^2\frac{\tau_i^{1/2}}{\tau_b^{3/2}}e^{-2a_i\tau_i}+\right.\\&+8\lambda_i\left(a_iA_ia_bA_b\right)\frac{\tau_i}{\tau_b^2}e^{-(a_i\tau_i+a_b\tau_i)}\cos\left(a_i\theta_i-a_b\theta_b\right)+\\&\left.+24\sum_{j\neq i}\left(a_iA_ia_jA_j\right)\frac{\tau_i\tau_j}{\tau_b^3}e^{-(a_i\tau_i+a_j\tau_j)}\cos\left(a_i\theta_i-a_j\theta_j\right)\right]\end{aligned} \tag{6.66}$$

V_{np2} is

$$\begin{aligned}V_{np2}=e^K&\left[W\left(K^{b\bar{b}}K_b\partial_{\bar{b}}\overline{W}+K^{b\bar{j}}K_b\partial_{\bar{j}}\overline{W}+K^{j\bar{b}}K_j\partial_{\bar{b}}\overline{W}+K^{ij}K_i\partial_{\bar{j}}\overline{W}\right)+\right.\\&\left.+h.c.\right]\end{aligned} \tag{6.67}$$

By similar expressions to (6.33), this term is

$$\begin{aligned}V_{np_2}=4\left|W_0\right|&\left\{\left(a_bA_b\right)\frac{1}{\tau_b^2}e^{-a_i\tau_b}\cos\left(a_b\theta_b\right)+\sum_{i=2}^{h^{1,1}}\left[\left(a_iA_i\right)\frac{\tau_i}{\tau_b^3}e^{-e_i\tau_i}\cos\left(a_i\theta_i\right)+\right.\right.\\&\left.\left.+8\sum_{j\neq i}\frac{\tau_i\tau_j}{\tau_b^{9/2}}\left(\lambda_i\left(a_jA_j\right)\tau_i^{1/2}e^{-a_j}\tau_j\cos\left(a_j\theta_j\right)+\left(i\leftrightarrow j\right)\right)\right]\right\}.\end{aligned} \tag{6.68}$$

Finally, $V_{\alpha'}$. It is simpler to evaluate it expanding the Kähler potential as

$$K=-2\ln\mathcal{V}-\frac{2\xi'}{\mathcal{V}}\equiv K_{(0)}+\delta K. \tag{6.69}$$

Because $V_{\alpha'}$ is defined as

$$V_{\alpha'}=e^K\left(K^{i\bar{j}}K_iK_{\bar{j}}-3\right)|W|^2, \tag{6.70}$$

we need the expansion of the Kähler metric and its inverse, given by

$$K_{i\bar{j}} = K^{(0)}_{i\bar{j}} + K^{(1)}_{i\bar{j}}, \tag{6.71}$$

$$K^{i\bar{j}} = \left(K_{i\bar{j}}\right)^{-1} = K^{i\bar{j}}_{(0)} - K^{i\bar{k}}_{(0)} \delta K_{\bar{k}\ell} K^{l\bar{j}}_{(0)}, \tag{6.72}$$

$$\begin{aligned} K^{ij} K_i K_j &= \left(K^{ij}_{(0)} - K^{i\bar{k}}_{(0)} \partial K_{\bar{k}e} K^{(j}_{(0)}\right) \left(K^{(0)}_i + \delta K_i\right) \left(K^{(0)}_j + \delta K_j\right) = \\ &= K^{ij}_{(0)} K^{(0)}_i K^{(0)}_j + 2 K^{i\bar{j}}_{(0)} K^{(0)}_i \delta K_{\bar{j}} - K^{(0)}_i K^{i\bar{k}}_{(0)} \delta K_{\bar{k}l} K^{l\bar{i}}_{(0)} K^{(0)}_{\bar{j}}, \end{aligned} \tag{6.73}$$

where δK_i represents the derivative of δK with respect to T_i and the subscripts (0) and (1) refer to the unperturbed or the first-order in ξ' values, respectively. The first unperturbed term in (6.73) cancels the 3 in (6.70). This is the already discussed no-scale structure, which is correct at the non-perturbative level but broken by perturbative corrections that ultimately give rise to a potential

$$V_{\alpha'} = \frac{3\xi'}{2\tau_b^{3/2}} \left|W_0\right|^2, \tag{6.74}$$

precisely equal to the one obtained in the previous case.
By summing these 3 terms, we are able to express the full V_F potential. However, since we are specifically interested in the axions, we will extract only their potential, which is

$$\begin{aligned} V_\theta = &\sum_{i=2}^{h^{1,1}} \left[X_i \cos\left(a_i\theta_i - a_b\theta_b\right) + \sum_{j\neq i} X_{ij} \cos\left(a_i\theta_i - a_j\theta_j\right)\right] + \\ &+ \left\{\sum_{i=2}^{h^{1,1}} \left[Y_i \cos\left(a_i\theta_i\right) + \sum_{j\neq i} \left(Y_{ij}\cos\left(a_j\theta_j\right) + Y_{ji}\cos\left(a_i\theta_i\right)\right)\right] + \right. \\ &\left. + Z\cos\left(a_b\theta_b\right)\right\}, \end{aligned} \tag{6.75}$$

where the coefficients are

$$X_i = 8\lambda_i a_i A_i a_b A_b \frac{\tau_i}{\tau_b^2} e^{-(a_i\tau_i + a_b\tau_b)} \quad , \quad Y_i = 4|W_0| a_i A_i \frac{\tau_i}{\tau_b^3} e^{-a_i\tau_i} \tag{6.76}$$

$$X_{ij} = 24 a_i A_i a_j A_j \frac{\tau_i\tau_j}{\tau_b{}^3} e^{-(a_i\tau_i + a_j\tau_j)} \quad , \quad Y_{ij} = 36\lambda_i a_j A_j \frac{\tau_i^{3/2}\tau_j}{\tau_b^{9/2}} e^{-a_j\tau_j} \tag{6.77}$$

$$Z = a_b A_b \tfrac{1}{\tau_b^2} e^{-a_b\tau_b}. \tag{6.78}$$

We recognise a generalization of the potential (6.40). The hierarchy between these coefficients in the decompactification limit is

$$X_i \ll Y_{ij} \ll Z \ll X_{ij} \ll Y_i. \tag{6.79}$$

We must remember that we need to express our results in terms of canonically normalized fields. Formally, we obtain the same results as in (6.39), with the identification of s with i.
Finally, we can study the structure of minima of this potential. Clearly, we start from derivatives

$$\begin{aligned}\frac{\partial V_\theta}{\partial \theta_b^c} &= a_b b_b \sum_{i=2}^{h^{1,1}} X_i \sin\left(a_i\theta_i - a_b\theta_b\right) - a_b b_b Z \sin\left(a_b\theta_b\right), \\ \frac{\partial V_\theta}{\partial \theta_i^c} &= -a_i b_i X_i \sin\left(a_i\theta_i - a_b\theta_b\right) - \sum_{j\neq i} a_i b_i X_{ij} \sin\left(a_i\theta_i - a_j\theta_j\right) + \\ &\quad - a_i b_i \left\{ Y_i \sin\left(a_i\theta_i\right) + \sum_{j\neq i}\left[Y_{ji}\sin\left(a_i\theta_i\right)\right]\right\}\end{aligned} \tag{6.80}$$

and we set them equal to 0. Under (6.79), we find that the only extrema are such that

$$\sin\left(a_b\theta_b\right) = \sin\left(a_i\theta_i\right) = 0 \Rightarrow \left(\theta_b, \theta_i\right) = \left(\frac{n_b\pi}{a_b}, \frac{n_i\pi}{a_i}\right) \quad , \quad i = 2, \ldots, h^{1,1} \tag{6.81}$$

Still in regime (6.79), the Hessian matrix on the extrema is

$$\left.\frac{\partial^2 V_\theta}{\partial\theta_i^c \partial\theta_j^c}\right|_{(n_b,n_i)} = \begin{pmatrix} (-1)^{n_b+1} a_b^2 b_b^2 Z & 0 \\ 0 & (-1)^{n_i+1} a_i^2 b_i^2 Y_i \end{pmatrix} \tag{6.82}$$

We have minima for odd n_b and n_i, then the Hessian looks like

$$\left.\frac{\partial^2 V_\theta}{\partial\theta_i^c \partial\theta_j^c}\right|_{(n_b,n_i)} = \begin{pmatrix} a_b^2 b_b^2 Z & 0 \\ 0 & a_i^2 b_i^2 Y_i \end{pmatrix} \tag{6.83}$$

Finally, the axion squared masses are simply half of the eigenvalues of this matrix, so

$$m^2_{\theta_b^c} = \frac{4}{3}\left(a_b^3 A_b\right)|W_0|\, e^{-a_b\tau_b} \sim e^{-\mathcal{V}^{2/3}}, \tag{6.84a}$$

$$m^2_{\theta_i^c} = \frac{8}{3\lambda_i}\left(a_i^3 A_i\right)|W_0|\,\frac{\tau_i^{3/2}}{\tau_b^{3/2}} e^{-a_i\tau_i} \sim \frac{(\ln\mathcal{V})^{3/2}}{\mathcal{V}^2}. \tag{6.84b}$$

Notice the huge hierarchy between these masses: the big modulus is exponentially suppressed compared to the others $h^{1,1}-1$, which share a similar order of magnitude due to their same asymptotic behaviour.

6.3 Axion masses for fibred Calabi-Yau's

6.3.1 Fibred Calabi-Yau: $h^{1,1} = 3$

We aim to generalize these calculations to more arbitrary Calabi-Yau manifolds, where both large and small moduli are present in varying numbers. In order to pursue this goal, we start with the simpler case of a *Fibred Calabi-Yau*. In this scenario, we have more than 1 large modulus. We start with just 2 of them and 1 small modulus in order to have the hierarchy (6.1) and so a competition between perturbative and non-perturbative effects. Now, the volume looks like

$$\mathcal{V} = \alpha \left(\sqrt{\tau_1}\tau_2 - \lambda_s \tau_s^{3/2} \right) \tag{6.85}$$

The Kähler potential and the superpotential are similar to the previous case

$$K = -2 \ln \left(\mathcal{V} + \xi' \right) \tag{6.86}$$

$$W = W_0 + \sum_{j=1}^{3} A_j \, e^{-a_j T_j}. \tag{6.87}$$

The calculations are conceptually the same as before: the derivatives of K, the Kähler metric and its inverse are

$$K_1 = -\frac{\tau_2}{2\sqrt{\tau_1}\left(\mathcal{V} + \xi'\right)} \simeq -\frac{1}{2\tau_1} \tag{6.88}$$

$$K_2 = -\frac{\tau_1^{1/2}}{\mathcal{V} + \xi'} \simeq -\frac{1}{\tau_2} \tag{6.89}$$

$$K_s = \frac{3}{2}\lambda_s \frac{\tau_s^{1/2}}{\left(\mathcal{V} + \xi'\right)} \simeq \frac{3}{2}\lambda_s \frac{\tau_s^{1/2}}{\sqrt{\tau_1}\tau_2} \tag{6.90}$$

$$K_{ij} = \begin{pmatrix} \frac{1}{4\tau_1^2} & \frac{\lambda_s \tau_s^{3/2}}{4\tau_1^{3/2}\tau_2^2} & -\frac{3\lambda_s \tau_s^{1/2}}{8\tau_1^{3/2}\tau_2} \\ \frac{\lambda_s \tau_s^{3/2}}{4\tau_1^{3/2}\tau_2^2} & \frac{1}{2\tau_2^2} & -\frac{3\lambda_s \tau_s^{1/2}}{4\tau_1^{1/2}\tau_2^2} \\ -\frac{3\lambda_s \tau_s^{1/2}}{8\tau_1^{3/2}\tau_2} & -\frac{3\lambda_s \tau_s^{1/2}}{4\tau_1^{1/2}\tau_2^2} & \frac{3\lambda_s}{8\tau_1^{1/2}\tau_2\tau_s^{1/2}} \end{pmatrix} \tag{6.91}$$

$$K^{ij} = \begin{pmatrix} 4\tau_1^2 & 4\lambda_s \tau_1^{1/2}\tau_s^{3/2} & 4\tau_1\tau_s \\ 4\lambda_s \tau_1^{1/2}\tau_s^{3/2} & 2\tau_2^2 & 4\tau_2\tau_s \\ 4\tau_1\tau_s & 4\tau_2\tau_s & \frac{8}{3\lambda_s}\tau_1^{1/2}\tau_2\tau_s^{1/2} \end{pmatrix} \tag{6.92}$$

The scalar potential is given by the usual terms:

$$
\begin{aligned}
V_{\rm np1} &= e^K \left[\sum_{i,j}^{3} K^{ij}\partial_i W \partial_j \bar{W}\right] = \\
&= 4\left(a_1^2A_1^2\right)\frac{\tau_1}{\tau_2^2}e^{-2a_1\tau_1} + 2\left(a_2^2A_2^2\right)\frac{1}{\tau_1}e^{-2a_2\tau_2} + \frac{8}{3\lambda_s}\left(a_s^2A_s^2\right)\frac{\tau_s^{1/2}}{\tau_1^{1/2}\tau_2}e^{-2a_s\tau_s} \\
&+ 8\lambda_s\left(a_1A_1a_2A_2\right)\frac{\tau_s^{3/2}}{\tau_1^{1/2}\tau_2^2}e^{-(a_1\tau_1+a_2\tau_2)}\cos\left(a_1\theta_1 - a_2\theta_2\right) + \\
&+ 8\left(a_1A_1a_sA_s\right)\frac{\tau_s}{\tau_2^2}e^{-(a_1\tau_1+a_s\tau_s)}\cos\left(a_1\theta_1 - a_s\theta_s\right) + \\
&+ 8\left(a_2A_2a_sA_s\right)\frac{\tau_s}{\tau_1\tau_2}e^{-(a_2\tau_2+a_s\tau_s)}\cos\left(a_2\theta_2 - a_s\theta_s\right),
\end{aligned}
\tag{6.93}
$$

the linear term in $|W_0|$

$$
\begin{aligned}
V_{np2} &= e^K\left[W_0\left(\sum_{i,j=1}^{3} K^{i\bar{j}}K_i\partial_{\bar{j}}\overline{W} + \text{ h.c. }\right)\right] = \\
&= |W_0|\left[4\left(a_1A_1\right)\left(\frac{1}{\tau_2^2} - \lambda_s\frac{\tau_s^{3/2}}{\tau_1^{1/2}\tau_2^2}\right)e^{-a_1\tau_1}\cos\left(a_1\theta_1\right) + \right. \\
&+ 4\left(a_2A_2\right)\left(\frac{1}{\tau_1\tau_2} - 2\lambda_s\frac{\tau_s^{3/2}}{\tau_1^{3/2}\tau_2^2}\right)e^{-a_2\tau_2}\cos\left(a_2\theta_2\right) + \\
&\left. +4\left(a_sA_s\right)\frac{\tau_s}{\tau_1\tau_2^2}e^{-a_s\tau_s}\cos\left(a_s\theta_s\right)\right]
\end{aligned}
\tag{6.94}
$$

and the last, quadratic in $|W_0|$

$$
V_{\alpha'} = \frac{3\xi'\left|W_0\right|^2}{4\tau_1^{3/2}\tau_2^3}. \tag{6.95}
$$

The potential for the axions can be extracted

$$
\begin{aligned}
V_\theta = 8\lambda_s\left(a_1A_1a_2A_2\right)\frac{\tau_s^{3/2}}{\tau_1^{1/2}\tau_2^2}e^{-(a_1\tau_1+a_2\tau_2)}\cos\left(a_1\theta_1-a_2\theta_2\right)+\\
+8\left(a_1A_1a_sA_s\right)\frac{\tau_s}{\tau_2^2}e^{-(a_1\tau_1+a_2\tau_2)}\cos\left(a_1\theta_1-a_s\theta_s\right)+\\
+8\left(a_2A_2a_sA_s\right)\frac{\tau_s}{\tau_1\tau_2}e^{-(a_2\tau_2+a_s\tau_s)}\cos\left(a_2\theta_2-a_s\theta_s\right)+\\
+4|W_0|\left[\left(a_1A_1\right)\frac{1}{\tau_2^2}e^{-a_1\tau_1}\cos\left(a_1\theta_1\right)+\left(a_2A_2\right)\frac{1}{\tau_1\tau_2}e^{-a_2\tau_2}\cos\left(a_2\theta_2\right)+\right.\\
\left.+\left(a_sA_s\right)\frac{\tau_s}{\tau_1\tau_2^2}e^{-a_s\tau_s}\cos\left(a_s\theta_s\right)\right]\equiv\\
\equiv\sum_{i=1}^{3}\left[X_i\cos\left(a_i\theta_i\right)+\sum_{j\neq i}X_{ij}\cos\left(a_i\theta_i-a_j\theta_j\right)\right].
\end{aligned}
\tag{6.96}
$$

In order to canonically normalize the fields, we define new fields from the kinetic term

$$
\begin{aligned}
\sum_{i,j=1}^{3}K_{ij}\partial_\mu T_i\partial\overline{T}_j\supset\frac{1}{4\tau_1^2}\left(\left(\partial_\mu\tau_1\right)^2+\left(\partial_\mu\theta_1\right)^2\right)+\frac{1}{2\tau_2^2}\left(\left(\partial_\mu\tau_2\right)^2+\left(\partial_\mu\theta_2\right)^2\right)+\\
+\frac{3\lambda_s}{8\tau_1^{1/2}\tau_2\tau_s^{1/2}}\left(\left(\partial_\mu\tau_s\right)^2+\left(\partial_\mu\theta_s\right)^2\right)
\end{aligned}
\tag{6.97}
$$

The simplest redefinitions are

$$
\tau_1=e^{\sqrt{2}\chi},\quad \tau_2=e^{\phi} \tag{6.98}
$$

but we prefer to mix them as

$$
\tau_1=e^{\sqrt{\frac{2}{3}}\chi+\frac{2}{\sqrt{3}}\phi},\quad \tau_2=e^{\sqrt{\frac{2}{3}}\chi-\frac{1}{\sqrt{3}}\phi} \tag{6.99}
$$

in order to rewrite the overall volume and the ratio between the original moduli as

$$
\mathcal{V}=\sqrt{\tau_1}\tau_2=e^{\sqrt{\frac{3}{2}}\chi},\quad u=\frac{\tau_1}{\tau_2}=e^{\sqrt{3}\phi}. \tag{6.100}
$$

The canonically normalized axions in terms of the original moduli are

$$
\theta_1=\sqrt{2}\left\langle\tau_1\right\rangle\theta_1^c,\quad \theta_2=\left\langle\tau_2\right\rangle\theta_2^c,\quad \theta_3=\frac{2}{\sqrt{3}}\left\langle\tau_1\right\rangle^{1/4}\left\langle\tau_2\right\rangle^{1/2}\left\langle\tau_s\right\rangle^{1/4}\theta_s^c. \tag{6.101}
$$

Now, the extrema of (6.96) are given by the vanishing of the first derivatives

$$
\frac{\partial V_\theta}{\partial\theta_i^c}=-X_ia_ib_i\sin\left(a_i\theta_i\right)-\sum_{j\neq i}X_{ij}a_ib_i\sin\left(a_i\theta_i-a_j\theta_j\right)=0 \tag{6.102}
$$

that in the decompactification limit are

$$\sin(a_i\theta_i) = 0 \Rightarrow \theta_i = \left(\frac{n_i\pi}{a_i}\right). \tag{6.103}$$

The Hessian in these points and in this regime is

$$\left.\frac{\partial^2 V_\theta}{\partial\theta_i^c\partial\theta_j^c}\right|_{n_i} = \begin{pmatrix} (-1)^{n_1+1}X_1a_1^2b_1^2 & 0 & 0 \\ 0 & (-1)^{n_2+1}X_2a_2^2b_2^2 & 0 \\ 0 & 0 & (-1)^{n_s+1}X_sa_s^2b_s^2 \end{pmatrix}. \tag{6.104}$$

On the minima, when the n_i are odd, the squared axion masses are

$$m_{\theta_1^c}^2 = \frac{1}{2}X_1a_1^2b_1^2 = 4\left|W_0\right|\left(a_1^3A_1\right)\left(\frac{\tau_1}{\tau_2}\right)^2 e^{-a_1\tau_1}, \tag{6.105a}$$

$$m_{\theta_2^c}^2 = \frac{1}{2}X_2a_2^2b_2^2 = 2\left|W_0\right|\left(a_2^3A_2\right)\frac{\tau_2}{\tau_1}e^{-a_2\tau_2}, \tag{6.105b}$$

$$m_{\theta_s^c}^2 = \frac{1}{2}X_sa_s^2b_s^2 = \frac{8}{3}\left|W_0\right|\left(a_s^3A_s\right)\frac{\tau_s^{3/2}}{\tau_1^{1/2}\tau_2}e^{-a_s\tau_s}. \tag{6.105c}$$

Apart from axion stabilization, moduli stabilization must also be considered. On the axion minimum, the potential for the moduli at leading order in the volume is

$$\begin{aligned} V_{\rm LVS} &= \frac{8}{3\lambda_s}\left(a_s^2A_s^2\right)\frac{\tau_s^{1/2}}{\tau_1^{1/2}\tau_2}e^{-2a_s\tau_s} - 4\left|W_0\right|\left(a_sA_s\right)\frac{\tau_s}{\tau_1\tau_2^2}e^{-a_s\tau_s} + \frac{3\xi'}{4\tau_1^{3/2}\tau_2^3}\left|W_0\right|^2 = \\ &= \frac{8}{3\lambda_s}\left(a_s^2A_s^2\right)\frac{\tau_s^{1/2}}{\mathcal{V}}e^{-2a_s\tau_s} - 4\left|W_0\right|\left(a_sA_s\right)\frac{\tau_s}{\mathcal{V}^2}e^{-a_s\tau_s} + \frac{3\xi'}{4\mathcal{V}^3}\left|W_0\right|^2 \end{aligned} \tag{6.106}$$

The minus sign in front of the second term is crucial for ensuring the existence of a minimum, which is guaranteed by working on the axion minimum. We notice that it is completely analogous to the potential (6.3), thus we automatically know that it stabilizes the small modulus and the overall volume, leaving one flat direction (only 2 of the 3 moduli are stabilized). As we know from 6.1, full stabilization can be achieved by adding subleading V_{F4} quantum corrections, written in terms of the volumes of the 2-cycles t_i, obtaining

$$V_{\rm LVS} = \lambda\frac{\tau_s^{1/2}}{\tau_1^{1/2}\tau_2}e^{-2a_s\tau_s} - \mu\frac{\tau_s}{\tau_1\tau_2^2}e^{-a_s\tau_s} + \frac{\nu}{\tau_1^{3/2}\tau_2^3} + \frac{\rho}{\tau_1^2\tau_2^4}\left(\Pi_1t_1 + \Pi_2t_2\right) \tag{6.107}$$

where all numerical factors are included in the constants λ, μ, ν and ρ.
We are considering the fibred CY $\mathbb{C}P^4_{[1,1,2,2,6]}$ with a blow-up mode, as described in [8]. The 2-cycles t_i can then be written in terms of the 4-cycles as

$$t_2 = \sqrt{\frac{\tau_1}{c}}, \quad t_1 = \frac{\tau_2}{2\sqrt{c\tau_1}}, \tag{6.108}$$

where the constant c appears in the expression for the volume as $\mathcal{V} = ct_1t_2^2 - ...$, where we have omitted the blow-up moduli. The expressions (6.108) follow simply by inverting the expression $\tau_i = \partial\mathcal{V}/\partial t_i$, which also fixes c as $1/4\alpha^2$. Instead of substituting them directly into potential (6.107), we rewrite everything in terms of the canonically normalized fields, using the relations

$$\mathcal{V}u = \tau_1^{3/2}, \qquad \frac{\mathcal{V}^2}{u} = \tau_2^3. \tag{6.109}$$

Then, the potential (6.107) is

$$\begin{aligned} V_{\text{LVS}} = \lambda\frac{\tau_s^{1/2}}{\mathcal{V}}e^{-2a_s\tau_s} - \mu\frac{\tau_s}{\mathcal{V}^2}e^{-a_s\tau_s} + \frac{\nu}{\mathcal{V}^3} + \\ + \frac{\rho}{\sqrt{c}\mathcal{V}^{11/3}}u^{1/3}\left(\frac{\Pi_1}{2u} + \Pi_2\right) \end{aligned} \tag{6.110}$$

where the first line represents the leading term, while the second one is a subleading correction that stabilizes the mode u. Clearly, the modes $\mathcal{V}$ and u must be considered functions of the real canonically normalized fields χ and ϕ. It should be noted that τ_s must also be understood in terms of the canonically σ

$$\tau_s = \frac{2}{\sqrt{3}}\langle\tau_1\rangle^{1/4}\langle\tau_2\rangle^{1/2}\langle\tau_s\rangle^{1/4}\sigma = \frac{2}{\sqrt{3}}\mathcal{V}^{1/2}\langle\tau_s\rangle^{1/4}\sigma \tag{6.111}$$

Now, the modulus u is simply extremized by

$$\frac{\partial V_{\text{LVS}}}{\partial\phi} = \frac{\partial V_{\text{LVS}}}{\partial u}\frac{du}{d\phi} = \frac{\rho u^{1/3}}{\sqrt{3c}\mathcal{V}^{11/3}}\left(\Pi_2 - \frac{\Pi_1}{u}\right) = 0. \tag{6.112}$$

Thus, (6.112) simply gives

$$\langle u\rangle = e^{\sqrt{3}\langle\phi\rangle} = \frac{\Pi_1}{\Pi_2} = \frac{\tau_1}{\tau_2}, \tag{6.113}$$

which is the same result obtained in (6.19). The subleading corrections cause a small shift in the minimum of the leading term, which is the same as the one found in 6.1 and that we report here

$$\begin{cases} \dfrac{\partial V_{\text{LVS}}}{\partial\chi} = \dfrac{\partial V_{\text{LVS}}}{\partial\mathcal{V}}\dfrac{d\mathcal{V}}{d\chi} = 0 \\ \dfrac{\partial V_{\text{LVS}}}{\partial\sigma} = \dfrac{\partial V_{\text{LVS}}}{\partial\tau_s}\dfrac{d\tau_s}{d\sigma} = 0 \end{cases} \Leftrightarrow \quad \begin{cases} \langle\mathcal{V}\rangle = e^{\sqrt{\frac{3}{2}}\langle\chi\rangle} \simeq W_0e^{a_s\langle\tau_s\rangle} \\ \langle\tau_s\rangle \simeq \xi'^{2/3} \end{cases} \tag{6.114}$$

We have used the non-canonical modulus τ_s, but in terms of σ we have

$$\langle\sigma\rangle \simeq \frac{1}{\langle\mathcal{V}\rangle^{1/2}} \tag{6.115}$$

These relations indeed define a minimum, which can be confirmed by evaluating the Hessian. The Hessian can also be used to extract the masses of the moduli.
It is useful to note that

$$\frac{\partial^2 V_{\rm LVS}}{\partial\phi\partial\sigma} = 0, \tag{6.116}$$

everywhere and

$$\frac{\partial^2 V_{\rm LVS}}{\partial\phi\partial\chi} \propto \frac{\partial V_{\rm LVS}}{\partial\phi} = 0. \tag{6.117}$$

on the extrema defined by (6.113). The other derivatives can be evaluated using the chain rule, which simplifies on the extrema as follows

$$\left.\frac{\partial^2 V_{\rm LVS}}{\partial\chi^2}\right|_{\langle\ldots\rangle} = \left.\frac{\partial^2 V_{\rm LVS}}{\partial\mathcal{V}^2}\left(\frac{d\mathcal{V}}{d\chi}\right)^2\right|_{\langle\ldots\rangle} \tag{6.118}$$

$$\left.\frac{\partial^2 V_{\rm LVS}}{\partial\sigma^2}\right|_{\langle\ldots\rangle} = \left.\frac{\partial^2 V_{\rm LVS}}{\partial\tau_s^2}\left(\frac{d\tau_s}{d\sigma}\right)^2\right|_{\langle\ldots\rangle} \tag{6.119}$$

$$\left.\frac{\partial^2 V_{\rm LVS}}{\partial\chi\partial\sigma}\right|_{\langle\ldots\rangle} = \left.\frac{\partial^2 V_{\rm LVS}}{\partial\mathcal{V}\partial\tau_s}\frac{d\mathcal{V}}{d\chi}\frac{d\tau_s}{d\sigma}\right|_{\langle\ldots\rangle} \tag{6.120}$$

These quantities can be easily evaluated from definitions (6.100)

$$\frac{d\mathcal{V}}{d\chi} = \sqrt{\frac{3}{2}}\mathcal{V}, \quad \frac{d\tau_s}{d\sigma} = \frac{2}{\sqrt{3}}\mathcal{V}^{1/2}\langle\tau_s\rangle\,. \tag{6.121}$$

The derivatives of $V_{\rm LVS}$, ignoring numerical prefactors, are

$$\frac{\partial^2 V_{\rm LVS}}{\partial\mathcal{V}^2} = \frac{|W_0|^2\,\nu}{\mathcal{V}^5}, \tag{6.122}$$

$$\frac{\partial^2 V_{\rm LVS}}{\partial\tau_s^2} = \frac{|W_0|^2\,\nu}{\mathcal{V}^3}, \tag{6.123}$$

$$\frac{\partial^2 V_{\rm LVS}}{\partial\tau_s\partial\mathcal{V}} = -\frac{|W_0|^2\,\nu}{\mathcal{V}^4}. \tag{6.124}$$

Because of (6.116) and (6.117), the 3×3 Hessian on the extrema reduces to a 2×2 block and $\partial^2 V_{\rm LVS}/\partial\phi^2$. The eigenvalues of the former can be easily approximated as

$$m_\sigma^2 \sim m_{\tau_s}^2 \sim \mathrm{tr}\,(H_{2\times2}) \sim \frac{1}{\mathcal{V}^2}, \tag{6.125a}$$

$$m_\chi^2 \sim m_{\mathcal{V}}^2 \sim \frac{\det\,(H_{2\times2})}{\mathrm{tr}\,(H_{2\times2})} \sim \frac{1/\mathcal{V}^5}{1/\mathcal{V}^2} \sim \frac{1}{\mathcal{V}^3}, \tag{6.125b}$$

$$m_\phi^2 \sim \frac{\partial^2 V_{\rm LVS}}{\partial\phi^2} \sim \frac{u^{1/3}}{\mathcal{V}^{11/3}}\Pi_1. \tag{6.125c}$$

All the eigenvalues are positive, so we have indeed a minimum.
We can use these information also to rewrite the axion masses (6.105a)-(6.105c) as

$$m^2_{\theta_1^c} \sim \left(\frac{\Pi_1}{\Pi_2}\right)^2 e^{-a_1\left(\frac{\Pi_1}{\Pi_2}\right)^{2/3}\mathcal{V}^{2/3}} \tag{6.126a}$$

$$m^2_{\theta_2^c} \sim \frac{\Pi_2}{\Pi_1} e^{-a_2\left(\frac{\Pi_2}{\Pi_1}\right)^{1/3}\mathcal{V}^{2/3}} \tag{6.126b}$$

$$m^2_{\theta_s^c} \sim \frac{1}{\mathcal{V}^2} \tag{6.126c}$$

Let's take a closer look at these formulas. We impose the volumes of the 4-cycles are greater than 10, so

$$\begin{aligned} \tau_1 &= (\mathcal{V}u)^{2/3} > 10 \quad \Rightarrow \quad u > \frac{10^{3/2}}{\mathcal{V}} \\ \tau_2 &= \frac{(\mathcal{V}u)^{2/3}}{u} > 10 \quad \Rightarrow \quad u < \frac{\mathcal{V}^2}{10^3} \end{aligned} \tag{6.127}$$

thus, the range of the variable $x := \langle u \rangle$ is

$$\frac{10^{3/2}}{\mathcal{V}} < x < \frac{\mathcal{V}^2}{10^3}. \tag{6.128}$$

The overall volume $\mathcal{V}$ will be fixed at constant values in a range where $\mathcal{V} \geq 100$ so that the effective field theory is under control and the axions are not effectively massless. However, we aim to avoid the cosmological moduli problem by ensuring that the masses of the moduli are $\gtrsim 30$ TeV. Consequently, we impose a bound on the volume by constraining the lightest modulus ϕ to be heavier:

$$m^2_\phi \sim M_p^2 \frac{u^{1/3}}{\mathcal{V}^{11/3}}\Pi_1 \gtrsim (30\text{TeV})^2 \quad \Rightarrow \quad u \gtrsim \frac{\mathcal{V}^{11}}{\Pi_1^3}\left(\frac{30\text{TeV}}{M_p}\right)^6 = \frac{25\mathcal{V}^{11}}{\Pi_1^3}10^{-96}. \tag{6.129}$$

We have that for $\mathcal{V} \lesssim 10^{10.3}\Pi_1^{1/3}$, the effective range is simply (6.128).
On the other hand, axion masses are not constrained by cosmological problems in the same way as the moduli masses. Therefore, they can be arbitrarily small. However, their spectrum is of interest due to the potential role they can play as candidates for Dark Matter. So, it is important to distinguish whether axions are massive or massless. We say that an axion is massless if its mass is smaller than the current value of Dark Energy

$$m < 10^{-32}\,\text{eV}. \tag{6.130}$$

The constants that appear in the expressions of non-perturbative effects can be written as

$$a_1 = a_2 = \frac{2\pi}{N}, \tag{6.131}$$

$N = 1$	θ_1^c	θ_2^c
CMB rotation	$2.66 < x < 3.02$	–
Cold DM	$1.98 < x < 2.66$	–
BH superradiance	$1.48 < x < 2.24$	$0.32 < x < 0.46$

Table 6.1: Values of x that sustain the effects named in the left column for $N = 1$.

$N = 2$	θ_1^c	θ_2^c
CMB rotation	$7.61 < x < 8.65$	–
Cold DM	$5.67 < x < 7.61$	–
BH superradiance	$4.26 < x < 6.43$	–

Table 6.2: Values of x that sustain the effects named in the left column for $N = 2$. Notice that from θ_2^c there are no effects now.

where the parameter N is set to be equal in both cases.
An axion is massless if

$$\frac{m_{\theta_1^c}^2}{M_p^2} = x^2 e^{-\frac{2\pi}{N}(x\mathcal{V})^{2/3}} \lesssim 10^{-120}. \tag{6.132}$$

In Fig. 6.3 and Fig. 6.4 we find the plots of the axion masses for different values of N and volumes fixed at 100 and 200 respectively.

By examining these plots, we learn that:

(i) there is not a fixed hierarchy between θ_1^c and θ_2^c: for small x, θ_1^c is generically heavier than θ_2^c and for $N = 1$, this hierarchy is satisfied for any x. However, for $N = 2$ and $N = 10$ there are x where θ_2^c becomes heavier. This follows from their different behaviours: $m_{\theta_1^c}$ decreases with x, while $m_{\theta_2^c}$ increases;

(ii) increasing $\mathcal{V}$ makes all spectra lighter, while increasing N makes the axions heavier;

(iii) these data can be combined with the observable constraints of the axiverse presented in Section 5.2.3. For instance, with $\mathcal{V} = 100$ and $N = 10$, both axions are so massive that they are incompatible with the windows of any effects for all values of x. Instead, for $N = 1, 2$ θ_1^c could potentially give rise to cosmological effects. Using the data in Section 5.2.3, the ranges for x that sustain the various phenomena are summarized in Tables 6.1 and 6.2. No contributions come from θ_2^c (with the exception of a very small window for black hole superradiance with $N = 1$) because with $\mathcal{V} = 100$, $x > 0.32$ by (6.128).

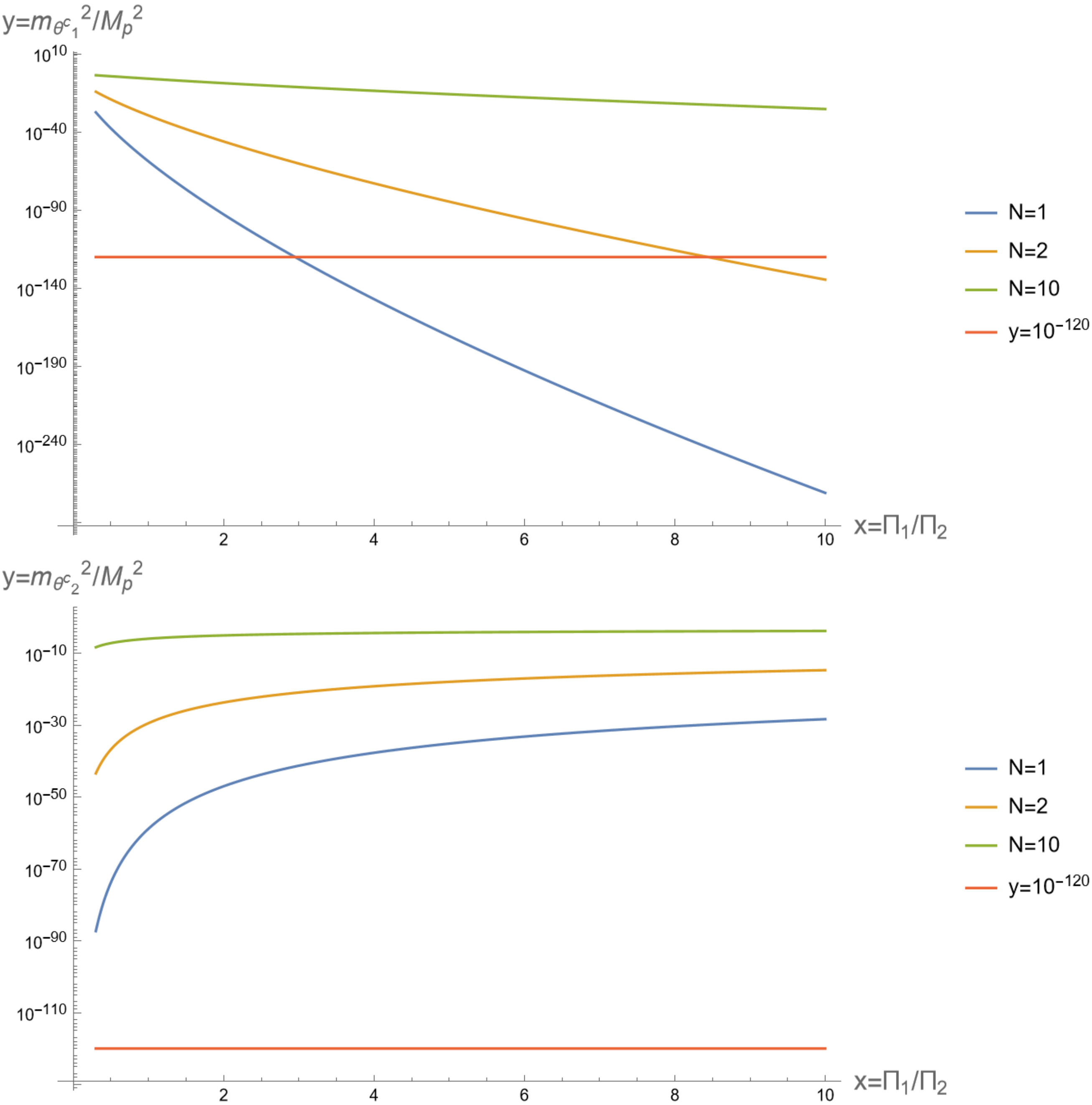

Figure 6.3: The squared masses of θ_1^c and θ_2^c are plotted as functions of $x = \Pi_1/\Pi_2$, with respect to the squared Planck mass M_p^2. We consider 3 values for N: 1, 2 and 10. $\mathcal{V}$ is fixed at value of 100, so the range of x is $0.32 \lesssim x < 10$. The horizontal red line represents the boundary between the effectively massless axions below it and the massive axions above it. To enhance clarity, the y-axis is logarithmically rescaled.

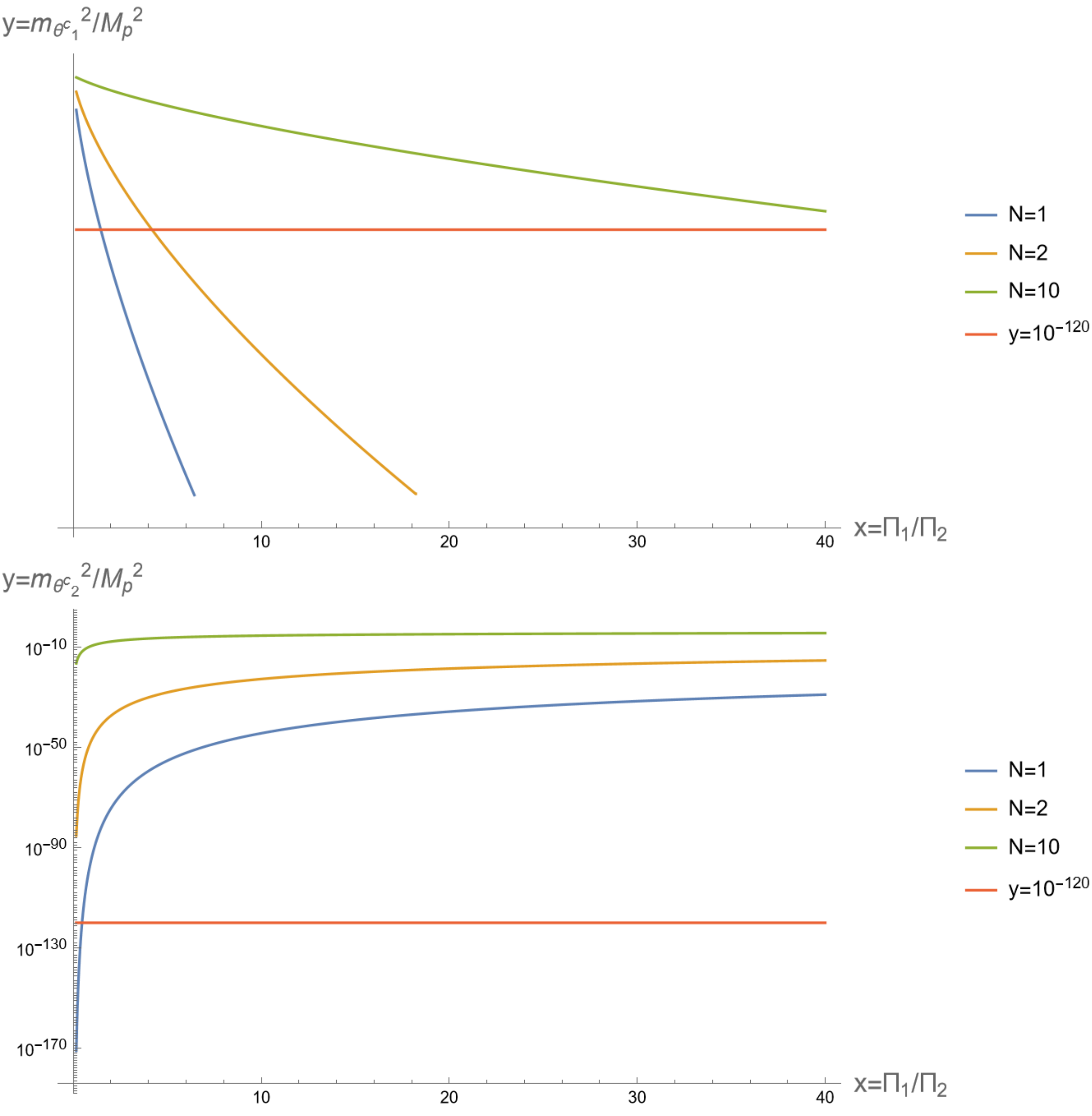

Figure 6.4: The squared masses of θ_1^c and θ_2^c are plotted as functions of $x = \Pi_1/\Pi_2$, with respect to the squared Planck mass M_p^2. We consider the same values of N as before. $\mathcal{V}$ is fixed at value of 200, so the range of x is now $0.16 \lesssim x < 40$.

6.3.2 Fibred Calabi-Yau: $h^{1,1} = 4$

We would to generalize the previous analysis of a fibred CY with $h^{1,1} = 3$ to a case with $h^{1,1} = 4$, studied in more detail in [10]. The volume is now

$$\mathcal{V} = \alpha \left(\sqrt{\tau_1 \tau_2 \tau_3} - \lambda_s \tau_s^{3/2} \right) , \tag{6.133}$$

where $\alpha = c_a$ and $\lambda_s = c_b / c_a$ are constants introduced to rewrite the volume in [10] in an expression that is more coherent with respect to this work.
As always, we can express the Kähler and the superpotential. Derivatives of Kähler potential and Kähler metric are

$$K_i \simeq -\frac{1}{2\sqrt{\tau_i}} \tag{6.134}$$

$$K_s \simeq \frac{3}{2} \lambda_s \frac{\tau_s^{1/2}}{\sqrt{\tau_1 \tau_2 \tau_3}} \tag{6.135}$$

$$K_{ij} = \begin{pmatrix} \frac{1}{4\tau_1^2} & 0 & 0 & -\frac{3}{8}\lambda_s \frac{\tau_s^{1/2}}{\tau_1^{3/2}\sqrt{\tau_2\tau_3}} \\ 0 & \frac{1}{4\tau_2^2} & 0 & -\frac{3}{8}\lambda_s \frac{\tau_s^{1/2}}{\tau_2^{3/2}\sqrt{\tau_1\tau_3}} \\ 0 & 0 & \frac{1}{4\tau_3^2} & -\frac{3}{8}\lambda_s \frac{\tau_s^{1/2}}{\tau_3^{3/2}\sqrt{\tau_1\tau_2}} \\ -\frac{3}{8}\lambda_s \frac{\tau_s^{1/2}}{\tau_1^{3/2}\sqrt{\tau_2\tau_3}} & -\frac{3}{8}\lambda_s \frac{\tau_s^{1/2}}{\tau_2^{3/2}\sqrt{\tau_1\tau_3}} & -\frac{3}{8}\lambda_s \frac{\tau_s^{1/2}}{\tau_3^{3/2}\sqrt{\tau_1\tau_2}} & \frac{3}{8}\lambda_s \frac{1}{\sqrt{\tau_1\tau_2\tau_3\tau_s}} \end{pmatrix} \tag{6.136}$$

$$K^{ij} = \begin{pmatrix} 4\tau_1^2 & 6\lambda_s \frac{\tau_s^{3/2}\tau_1^{1/2}\tau_2^{1/2}}{\tau_3^{1/2}} & 6\lambda_s \frac{\tau_s^{3/2}\tau_1^{1/2}\tau_3^{1/2}}{\tau_2^{1/2}} & 4\tau_s\tau_1 \\ 6\lambda_s \frac{\tau_s^{3/2}\tau_1^{1/2}\tau_2^{1/2}}{\tau_3^{1/2}} & 4\tau_2^2 & 6\lambda_s \frac{\tau_s^{3/2}\tau_2^{1/2}\tau_3^{1/2}}{\tau_1^{1/2}} & 4\tau_s\tau_2 \\ 6\lambda_s \frac{\tau_s^{3/2}\tau_1^{1/2}\tau_3^{1/2}}{\tau_2^{1/2}} & 6\lambda_s \frac{\tau_s^{3/2}\tau_2^{1/2}\tau_3^{1/2}}{\tau_1^{1/2}} & 4\tau_3^2 & 4\tau_s\tau_3 \\ 4\tau_s\tau_1 & 4\tau_s\tau_2 & 4\tau_s\tau_3 & \frac{8}{3\lambda_s}\sqrt{\tau_1\tau_2\tau_s\tau_s} \end{pmatrix} \tag{6.137}$$

V_θ is given by V_{np_1} and V_{np_2} as always and in the end it looks like

$$\begin{aligned}
V_\theta = {} & 12\lambda_s \sum_{\substack{i=1\\ j\neq k}}^{3} (a_i A_i a_j A_j)\, \frac{r_s^{3/2}}{\sqrt{\tau_i \tau_j}\tau_k} e^{-(a_i\tau_i + a_j\tau_j)} \cos(a_i\theta_i - a_j\theta_j) + \\
& + 8 \sum_{\substack{j=1\\ j<k}}^{3} (a_i A_i a_s A_s)\, \frac{\tau_s}{\tau_j \tau_k} e^{-(a_i\tau_i + a_s\tau_s)} \cos(a_i\theta_i - a_s\theta_s) + \\
& + 4\,|W_0| \sum_{\substack{j=1\\ j<k}}^{3} (a_i A_i)\, \frac{1}{\tau_j\tau_k} e^{-a_i\tau_i} \cos(a_i\theta_i) + \\
& + 4\,|W_0|\,(\theta_s A_s)\, \frac{\tau_s}{\tau_1\tau_2 r_3} e^{-a_s\tau_s} \cos(a_s\theta_s) \equiv \\
& \equiv \sum_{i=1}^{3} \Bigg[\sum_{j<k\neq i} X_{ijk} \cos(a_i\theta_i) + \sum_{j\neq k\neq i} Y_{ijk} \cos(a_i\theta_i - a_j\theta_j) \\
& + \sum_{j<k\neq i} Z_{ijk} \cos(a_i\theta_i - a_s\theta_s) \Bigg] + X_s \cos(a_s\theta_s)\,.
\end{aligned} \tag{6.138}$$

The canonical fields are given by

$$\sum_{i=1}^{4} K_{ij}\partial_u T_i \partial_u T_j \supset \sum_{i=1}^{3} \frac{1}{4\tau_i^2} \left((\partial_u\tau_i)^2 + (\partial_u\theta_i)^2 \right) + \tag{6.139}$$

$$+ \frac{3}{8}\lambda_s \frac{1}{\sqrt{\tau_1\tau_2\tau_3\tau_s}} \left((\partial_u\tau_s)^2 + (\partial_u\theta_s)^2 \right). \tag{6.140}$$

The simplest redefinition involves the fields χ_i as

$$\tau_1 = e^{\sqrt{2}\chi_1} \quad \tau_2 = e^{\sqrt{2}\chi_2} \quad \tau_3 = e^{\sqrt{2}\chi_3}. \tag{6.141}$$

However, a more adapted choice for our case involves a rotation of these fields like

$$\begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{2}\chi \\ \phi_1 \\ \phi_2 \end{pmatrix}. \tag{6.142}$$

The big Kähler moduli become

$$\begin{cases} \tau_1 = e^{\sqrt{\frac{2}{3}}\chi + \frac{2}{\sqrt{3}}\phi_1 - \frac{1}{\sqrt{3}}\phi_2} \\ \tau_2 = e^{\sqrt{\frac{2}{3}}\chi - \frac{1}{\sqrt{3}}\phi_1 + \frac{2}{\sqrt{3}}\phi_2} \\ \tau_3 = e^{\sqrt{\frac{2}{3}}\chi - \frac{1}{\sqrt{3}}\phi_1 - \frac{1}{\sqrt{3}}\phi_2} \end{cases} \tag{6.143}$$

The overall volume and the ratios of the moduli over the third one are simply

$$\mathcal{V} = e^{\sqrt{6}\chi}, \tag{6.144a}$$

$$u_1 = \frac{\tau_1}{\tau_3} = e^{\sqrt{3}\phi_1}, \tag{6.144b}$$

$$u_2 = \frac{\tau_2}{\tau_3} = e^{\sqrt{3}\phi_2}. \tag{6.144c}$$

The small modulus gives rise to the canonical field σ

$$\tau_s = \frac{2}{\sqrt{3\lambda_s}} \mathcal{V}^{1/2} \left\langle \tau_s \right\rangle^{1/4} \sigma. \tag{6.145}$$

Finally, the canonical axions are

$$\theta_i = \sqrt{2} \left\langle \tau_i \right\rangle \theta_i^c, \quad \theta_s = \frac{2}{\sqrt{3\lambda_s}} \mathcal{V}^{1/2} \langle \tau_s \rangle^{1/4} \theta_s^c. \tag{6.146}$$

Now, in the decompactification limit, the extrema take the usual form

$$\begin{cases} \frac{\partial V_\theta}{\partial \theta_s^c} = 0 \\ \frac{\partial V_\theta}{\partial \theta_i^c} = 0 \end{cases} \Leftrightarrow \begin{cases} \theta_s = \frac{n_s \pi}{a_s} \\ \theta_i = \frac{n_i \pi}{a_i} \end{cases}, \quad n_i, n_s \in \mathbb{N} , \tag{6.147}$$

which is indeed a minimum looking the Hessian, that again in the decompactification limit is diagonal and the positive eigenvalues give us the axion mass spectrum

$$m_{\theta_s^c}^2 = \frac{1}{2} X_s a_s^2 b_s^2 = \frac{2}{3\lambda_s} \left(a_s^3 A_s \right) \frac{\tau_s^{3/2}}{\sqrt{\tau_1 \tau_2 \tau_3}} e^{-a_s \tau_s} \tag{6.148a}$$

$$m_{\theta_i^c}^2 = \frac{1}{2} X_{ijk} a_i^2 b_i^2 = 4 \left| W_0 \right| \left(a_i^3 A_i \right) \frac{\tau_i^2}{\tau_j \tau_k} e^{-a_i \tau_i}, \quad j < k \neq i. \tag{6.148b}$$

On an axion minimum, the potential $V_{\rm LVS}$

$$V_{\rm LVS} = \lambda \frac{\sqrt{\tau_s}}{\mathcal{V}} e^{-2a_s \tau_s} - \mu \frac{\tau_s}{\mathcal{V}^2} e^{-\alpha_s \tau_s} + \frac{\nu}{\mathcal{V}^3} + \frac{\rho}{\mathcal{V}^4} \sum_{i=1}^{3} \Pi_i t_i \tag{6.149}$$

where the first 3 therms define a standard LVS potential and the last subleading term represents the quantum corrections that stabilize the flat directions ϕ_1 and ϕ_2, corresponding to u_1 and u_2. In order to perform the stabilization, we need to write the volumes of the 2-cycles in terms of the canonical fields. From $\mathcal{V} = c t_1 t_2 t_3$

$$t_1 = \sqrt{\frac{\tau_2 \tau_3}{c \tau_1}} = \frac{\mathcal{V}^{1/3} u_2^{1/3}}{\sqrt{c} u_1^{2/3}}, \tag{6.150a}$$

$$t_2 = \sqrt{\frac{\tau_2 \tau_3}{c \tau_1}} = \frac{\mathcal{V}^{1/3} u_1^{1/3}}{\sqrt{c} u_2^{2/3}}, \tag{6.150b}$$

$$t_3 = \sqrt{\frac{\tau_1 \tau_2}{c \tau_3}} = \frac{\mathcal{V}^{1/3} \left(u_1 u_2 \right)^{1/3}}{\sqrt{c}}. \tag{6.150c}$$

In order to match with the expression (6.133), $c = 1/\alpha^2$. Now, by plugging the expressions for t_i into (6.149), V_{LVS} becomes

$$\begin{aligned} V_{\text{LVS}} = & \lambda \frac{\tau_s^{1/2}}{\mathcal{V}} e^{-2a_s\tau_s} - \mu \frac{\tau_s}{\mathcal{V}^2} e^{-a_s\tau_s} + \frac{\nu}{\mathcal{V}^4} + \\ & + \frac{\rho}{\sqrt{c}\mathcal{V}^{11/3}} \left(\Pi_1 \frac{u_2^{1/3}}{u_1^{2/3}} + \Pi_2 \frac{u_1^{1/3}}{u_2^{2/3}} + \Pi_3 (u_1 u_2)^{1/3} \right) \end{aligned} \tag{6.151}$$

By extremising with respect to ϕ_i (corresponding to u_i), we obtain

$$\langle u_1 \rangle = \frac{\Pi_1}{\Pi_3}, \quad \langle u_2 \rangle = \frac{\Pi_2}{\Pi_3}, \tag{6.152}$$

as we already expected from the general analysis in 6.1. The Hessian shows that it is a minimum and it takes on a block structure

$$\begin{array}{cc} \mathcal{V}\tau_s & \phi_1\phi_2 \end{array} \\ \left(\begin{array}{c|c} H_{2\times 2} & \\ \hline & \Phi_{2\times 2} \end{array} \right) \tag{6.153}$$

where

$$\Phi_{2\times 2} = \frac{\rho \Pi_3}{\sqrt{c} V^{11/3}} (u_1 u_2)^{1/3} \begin{pmatrix} 2 & -1/3 \\ -1/3 & 2 \end{pmatrix}. \tag{6.154}$$

We can easily extract the masses. The ones corresponding to $\mathcal{V}$ and τ_s are unchanged and we report them here for completeness. Together with the axion masses, the full mass spectrum is

$$m_\chi^2 \sim m_{\mathcal{V}}^2 \sim \frac{\det (H_{2\times 2})}{\text{tr} (H_{2\times 2})} \sim \frac{1}{\mathcal{V}^3}, \tag{6.155a}$$

$$m_\sigma^2 \sim m_{\tau_s}^2 \sim \text{tr} (H_{2\times 2}) \sim \frac{1}{\mathcal{V}^2}, \tag{6.155b}$$

$$m_{\phi_1}^2 = \frac{5\rho}{6\sqrt{c}} \frac{\Pi_3}{\mathcal{V}^{11/3}} (u_1 u_2)^{1/3} \sim \frac{\Pi_3}{\mathcal{V}^{11/3}} (u_1 u_2)^{1/3}, \tag{6.155c}$$

$$m_{\phi_2}^2 = \frac{7\rho}{6\sqrt{c}} \frac{\Pi_3}{\mathcal{V}^{11/3}} (u_1 u_2)^{1/3} \sim \frac{\Pi_3}{\mathcal{V}^{11/3}} (u_1 u_2)^{1/3}, \tag{6.155d}$$

$$m_{\theta_1^c}^2 = 4 |W_0| \left(a_1^3 A_1\right) \frac{u_1^2}{u_2} e^{-a_1 \frac{(u_1\mathcal{V})^{2/3}}{u_2^{1/3}}} \sim \frac{u_1^2}{u_2} e^{-a_1 \frac{(u_1\mathcal{V})^{2/3}}{u_2^{1/3}}}, \tag{6.155e}$$

$$m_{\theta_2^c}^c = 4 |W_0| \left(a_2^3 A_2\right) \frac{u_2^2}{u_1} e^{-a_2 \frac{(u_2\mathcal{V})^{2/3}}{u_1^{1/3}}} \sim \frac{u_2^2}{u_1} e^{-a_2 \frac{(u_2\mathcal{V})^{2/3}}{u_1^{1/3}}}, \tag{6.155f}$$

$$m_{\theta_3^c}^2 = 4 |W_0| \left(a_3^3 A_3\right) \frac{1}{u_1 u_2} e^{-a_3 \frac{\mathcal{V}^{2/3}}{(u_1 u_2)^{1/3}}} \sim \frac{1}{u_1 u_2} e^{-a_3 \frac{\mathcal{V}^{2/3}}{(u_1 u_2)^{1/3}}}, \tag{6.155g}$$

$$m_{\theta_s^c}^2 = \frac{2}{3\lambda_s} \left(a_s^3 A_s\right) \frac{\tau_s^{3/2}}{\mathcal{V}} e^{-a_s\tau_s} \sim \frac{1}{\mathcal{V}^2}. \tag{6.155h}$$

Now, in order to study this spectrum in terms of the topological quantities, we demand that the 4-cycle volumes are bigger than 10, so

$$\begin{aligned}
\tau_1 &= \frac{(\mathcal{V}u_1)^{2/3}}{u_2^{1/3}} > 10 \Rightarrow u_2 < \frac{(\mathcal{V}u_1)^2}{10^3} \Rightarrow u_1 > \frac{10^{3/2}u_2^{1/2}}{\mathcal{V}}, \\
\tau_2 &= \frac{(\mathcal{V}u_2)^{2/3}}{u_1^{1/3}} > 10 \Rightarrow u_1 < \frac{(\mathcal{V}u_2)^2}{10^3} \Rightarrow u_1 < \frac{\mathcal{V}^2u_2^2}{10^3}, \\
\tau_3 &= \frac{\mathcal{V}^{2/3}}{(u_1u_2)^{1/3}} > 10 \Rightarrow u_1u_2 < \frac{\mathcal{V}^2}{10^3} \Rightarrow u_1 < \frac{\mathcal{V}^2}{u_1 10^3}.
\end{aligned} \tag{6.156}$$

We need then to distinguish two cases:

(i) for u_2 smaller than 1, we have

$$\frac{10^3}{\mathcal{V}^2} < u_2 < 1, \qquad \frac{10^{3/2}u_2^{1/2}}{\mathcal{V}} < u_1 < \frac{(\mathcal{V}u_2)^2}{10^3}; \tag{6.157}$$

(ii) for u_2 greater than 1, we obtain

$$1 < u_2 < \frac{\mathcal{V}^2}{10^3}, \qquad \frac{10^{3/2}u_2^{1/2}}{\mathcal{V}} < u_1 < \frac{\mathcal{V}^2}{u_2 10^3}. \tag{6.158}$$

In order to avoid the cosmological moduli problem, we fix the lightest Kähler moduli ϕ_i to be heavier than 30 TeV

$$m_{\phi_i}^2 \sim M_p^2 \frac{(u_1u_2)^{1/3}}{\mathcal{V}^{11/3}} \Pi_3 \gtrsim (30\text{TeV})^2 \quad \Rightarrow \quad u_1u_2 \gtrsim \frac{\mathcal{V}^{11}}{\Pi_3^3}\left(\frac{30\text{TeV}}{M_p}\right)^6, \tag{6.159}$$

and thus, we have the interval

$$\frac{\mathcal{V}^{11}}{\Pi_3^3}\left(\frac{30\text{TeV}}{M_p}\right)^6 \lesssim u_1u_2 < \frac{(\mathcal{V}u_2)^2}{10^3}. \tag{6.160}$$

In order to be meaningful, the left quantity must be smaller than the right one, so

$$\mathcal{V} \lesssim \left(\frac{\Pi_3}{10}\right)^{1/3}\left(\frac{M_p}{30\text{TeV}}\right)^{2/3} = (1.6\Pi_3)^{1/3}10^{9.3}. \tag{6.161}$$

The effective ranges for u_1 and u_2 are determined by the intersections between the ones in (i) and (ii) with the bounds coming from the moduli problem. Namely

$$\mathcal{V}_1 := \left(10^4\Pi_3^2\right)^{1/10}\left(\frac{M_p}{30\text{TeV}}\right)^{2/5} \simeq 1.7\Pi_3^{1/5}10^6, \tag{6.162a}$$

$$\mathcal{V}_2 := \left(10\Pi_3^2\right)^{1/8}\left(\frac{M_p}{30\text{TeV}}\right)^{1/2} \simeq 2.7\Pi_3^{1/4}10^7, \tag{6.162b}$$

$$\mathcal{V}_3 := \left(\frac{\Pi_3}{10}\right)^{1/3}\left(\frac{M_p}{30\text{TeV}}\right)^{2/3} \simeq 2.5\Pi_3^{1/3}10^9, \tag{6.162c}$$

we can write the intervals as

(i) when $u_2 < 1$ the cases are

(i)-(a) with $\mathcal{V} < \mathcal{V}_1$,

$$\frac{10^3}{\mathcal{V}^2} < u_2 < 1, \quad \frac{10^{3/2} u_2^{1/2}}{\mathcal{V}} < u_1 < \frac{(\mathcal{V} u_2)^2}{10^3}; \tag{6.163}$$

(i)-(b) for $\mathcal{V}_1 < \mathcal{V} < \mathcal{V}_2$,

$$\frac{\mathcal{V}^8}{10\Pi_3^2} \left(\frac{30\text{TeV}}{M_p} \right)^4 < u_2 < 1, \quad \frac{10^{3/2} u_2^{1/2}}{\mathcal{V}} < u_1 < \frac{(\mathcal{V} u_2)^2}{10^3}; \tag{6.164}$$

(i)-(c) again with $\mathcal{V}_1 < \mathcal{V} < \mathcal{V}_2$, we can consider

$$\frac{10^3}{\mathcal{V}^2} < u_2 < \frac{\mathcal{V}^8}{10\Pi_3^2} \left(\frac{30\text{TeV}}{M_p} \right)^4, \quad \frac{\mathcal{V}^{11}}{u_2 \Pi_3} \left(\frac{30\text{TeV}}{M_p} \right)^6 < u_1 < \frac{(\mathcal{V} u_2)^2}{10^3}; \tag{6.165}$$

(i)-(d) finally, $\mathcal{V}_2 < \mathcal{V} < \mathcal{V}_3$ and

$$\frac{10^3}{\mathcal{V}^2} < u_2 < 1, \quad \frac{\mathcal{V}^{11}}{u_2 \Pi_3^3} \left(\frac{30\text{TeV}}{M_p} \right)^6 < u_1 < \frac{(\mathcal{V} u_2)^2}{10^3}; \tag{6.166}$$

(ii) for $u_2 > 1$ the cases are

(ii)-(a) with $\mathcal{V} < \mathcal{V}_2$,

$$1 < u_2 < \frac{\mathcal{V}^2}{10^3}, \quad \frac{10^{3/2} u_2^{1/2}}{\mathcal{V}} < u_1 < \frac{\mathcal{V}^2}{u_2 10^3}; \tag{6.167}$$

(ii)-(b) for $\mathcal{V}_2 < \mathcal{V} < \mathcal{V}_3$, we have

$$1 < u_2 < \frac{\mathcal{V}^8}{10\Pi_3^2} \left(\frac{30\text{TeV}}{M_p} \right)^4, \quad \frac{\mathcal{V}^{11}}{u_2 \Pi_3^3} \left(\frac{30\text{TeV}}{M_p} \right)^6 < u_1 < \frac{\mathcal{V}^2}{u_2 10^3}; \tag{6.168}$$

(ii)-(c) finally, still for $\mathcal{V}_2 < \mathcal{V} < \mathcal{V}_3$,

$$\frac{\mathcal{V}^8}{10\Pi_3^2} \left(\frac{30\text{TeV}}{M_p} \right)^4 < u_2 < \frac{\mathcal{V}^2}{10^3}, \quad \frac{10^{3/2} u_2^{1/2}}{\mathcal{V}} < u_1 < \frac{\mathcal{V}^2}{u_2 10^3}. \tag{6.169}$$

In order to realize some plots of the axion mass spectrum, we consider $\mathcal{V} < \mathcal{V}_1$, which determines the relevant intervals and we set

$$a_1 = a_2 = a_3 = \frac{2\pi}{N}. \tag{6.170}$$

$N=1$	θ_1^c	θ_2^c	θ_3^c
CMB rotation	$1.89 < u_1 < 2.15$	–	$0.22 < u_1 < 0.28$
Cold DM	$1.41 < u_1 < 1.89$	–	$0.28 < u_1 < 0.51$
BH superradiance	$1.05 < u_1 < 1.60$	–	$0.40 < u_1 < 0.91$

Table 6.3: Intervals for u_1 correspondent to the effect named in the left column. Note that θ_2^c is too massive to contribute. The ranges are found for $\mathcal{V} = 100, u_2 = 0.5$ and $N = 1$.

Several cases are shown in Fig. 6.5, 6.6 and 6.7.

As with $h^{1,1} = 3$, here we have no fixed hierarchy between the 3 axions. The spectra become lighter as $\mathcal{V}$ increases, but heavier as N increases. Again, $N = 10$ gives rise to axions that are too heavy, beyond phenomenological windows. For the values of $u_2 > 1$ considered, axions are either heavier or lighter than the detectable scale (roughly between 10^{-122} and 10^{-76}). However, in the region $u_2 < 1$, cosmological effects could be present. For instance, taking $\mathcal{V} = 100$ and $u_2 = 0.5$, for $N = 1$ there are different intervals of u_1 for θ_1^c and θ_3^c, where observables are achievable. They are shown in Table 6.3. Notice that the windows for θ_1^c are now smaller than in the previous case. For $N = 2$ they are too heavy for any effect.

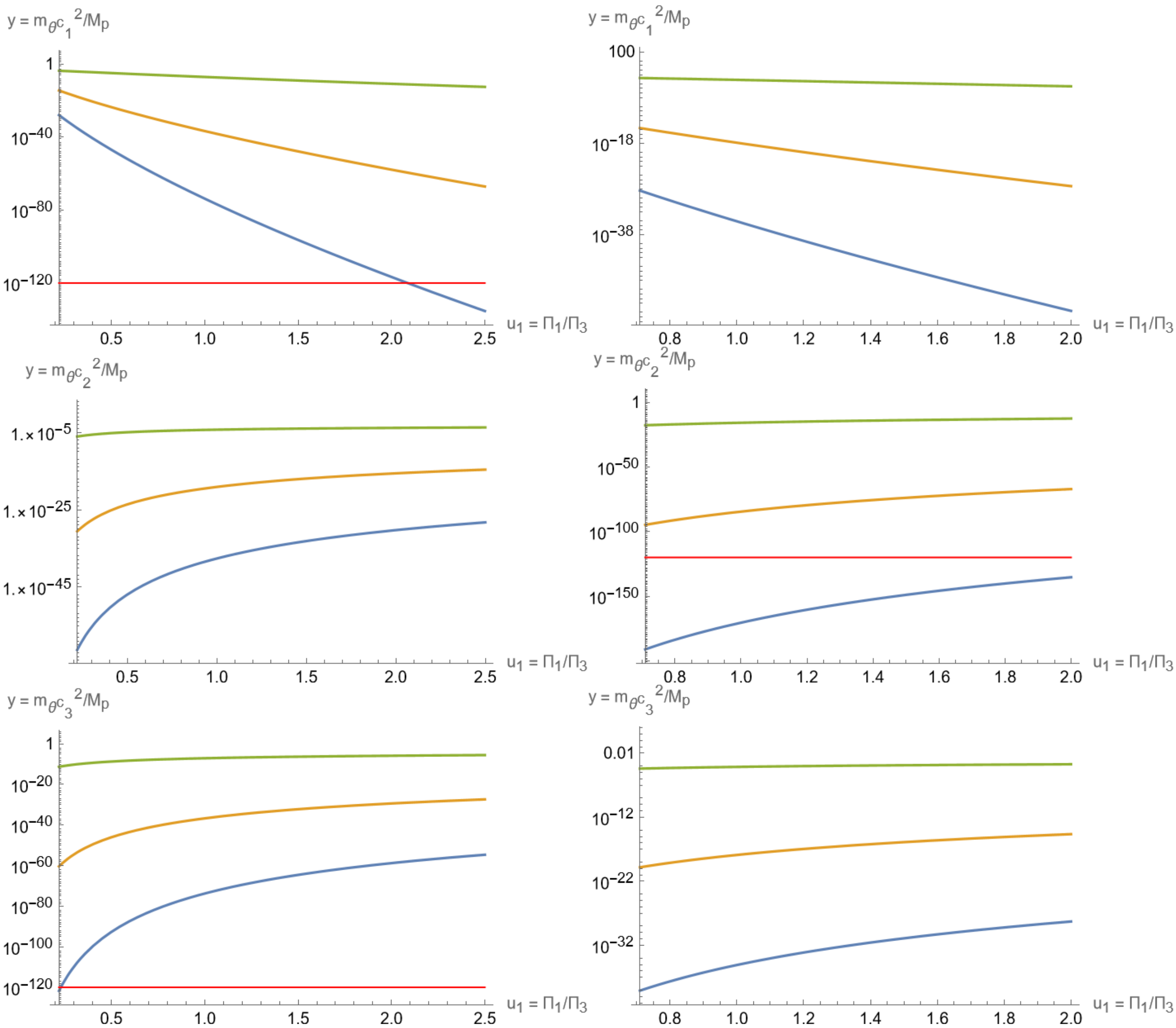

Figure 6.5: Plots of the axion masses for $\mathcal{V} = 100$. For this value, the parameter u_2 varies in $0.1 < u_2 < 10$. The plots on the left hand side are realized for $u_2 = 0.5$, then we have $0.22 < u_1 < 2.5$. On the right hand side, $u_2 = 5$ and so $0.71 < u_1 < 2$. The blue line corresponds to $N = 1$. The yellow line corresponds to $N = 2$. The green line corresponds to $N = 10$. The horizontal red line corresponds to the boundary between massive and effectively massless axions.

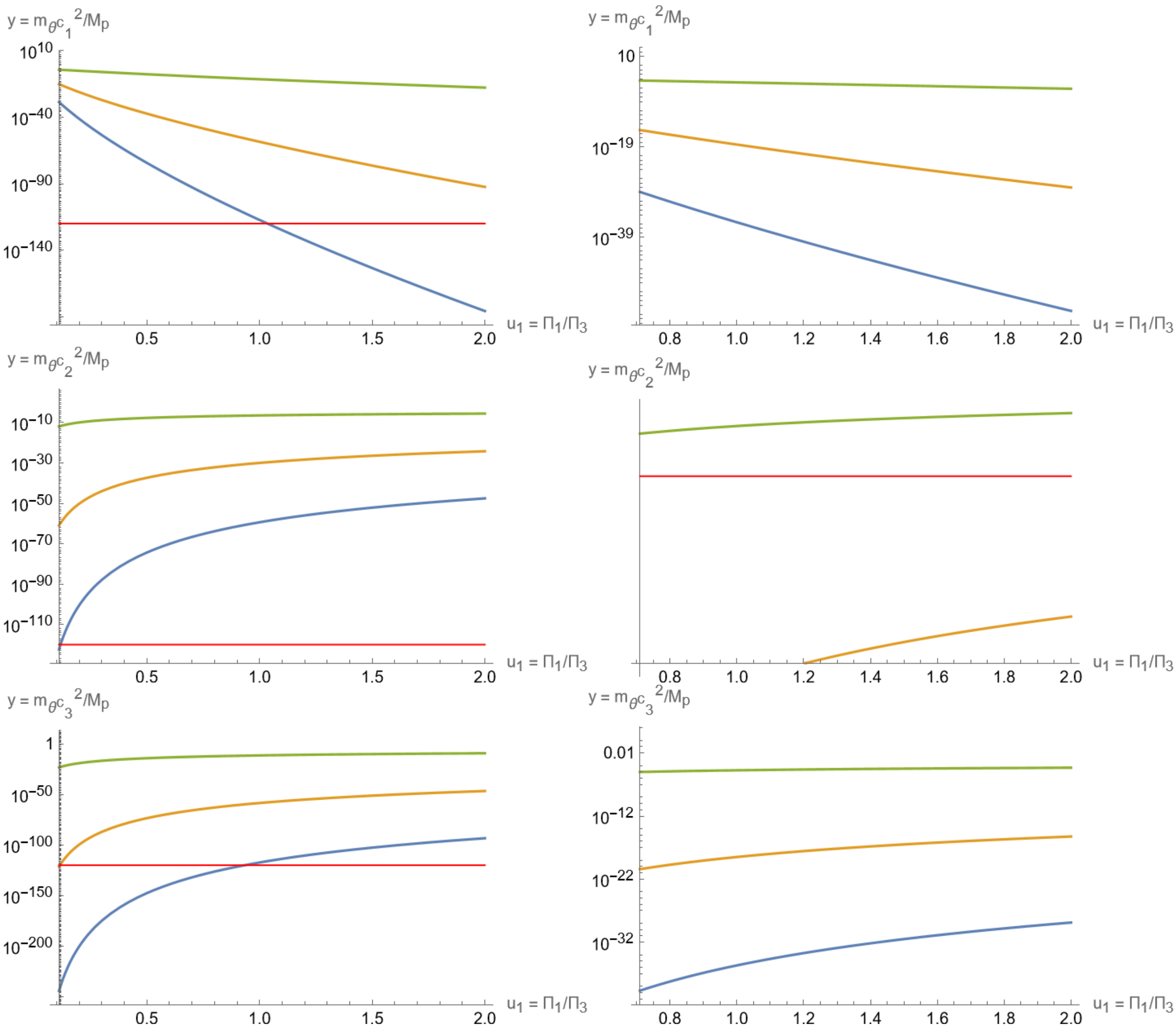

Figure 6.6: Plots of the axion masses for $\mathcal{V} = 200$. For this value, the parameter u_2 varies in $0.025 < u_2 < 40$. The plots on the left hand side are realized for $u_2 = 0.5$, then we have $0.11 < u_1 < 20$. On the right hand side, $u_2 = 20$ and so $0.71 < u_1 < 2$. The blue line corresponds to $N = 1$. The yellow line corresponds to $N = 2$. The green line corresponds to $N = 10$. The horizontal red line corresponds to the boundary between massive and effectively massless axions.

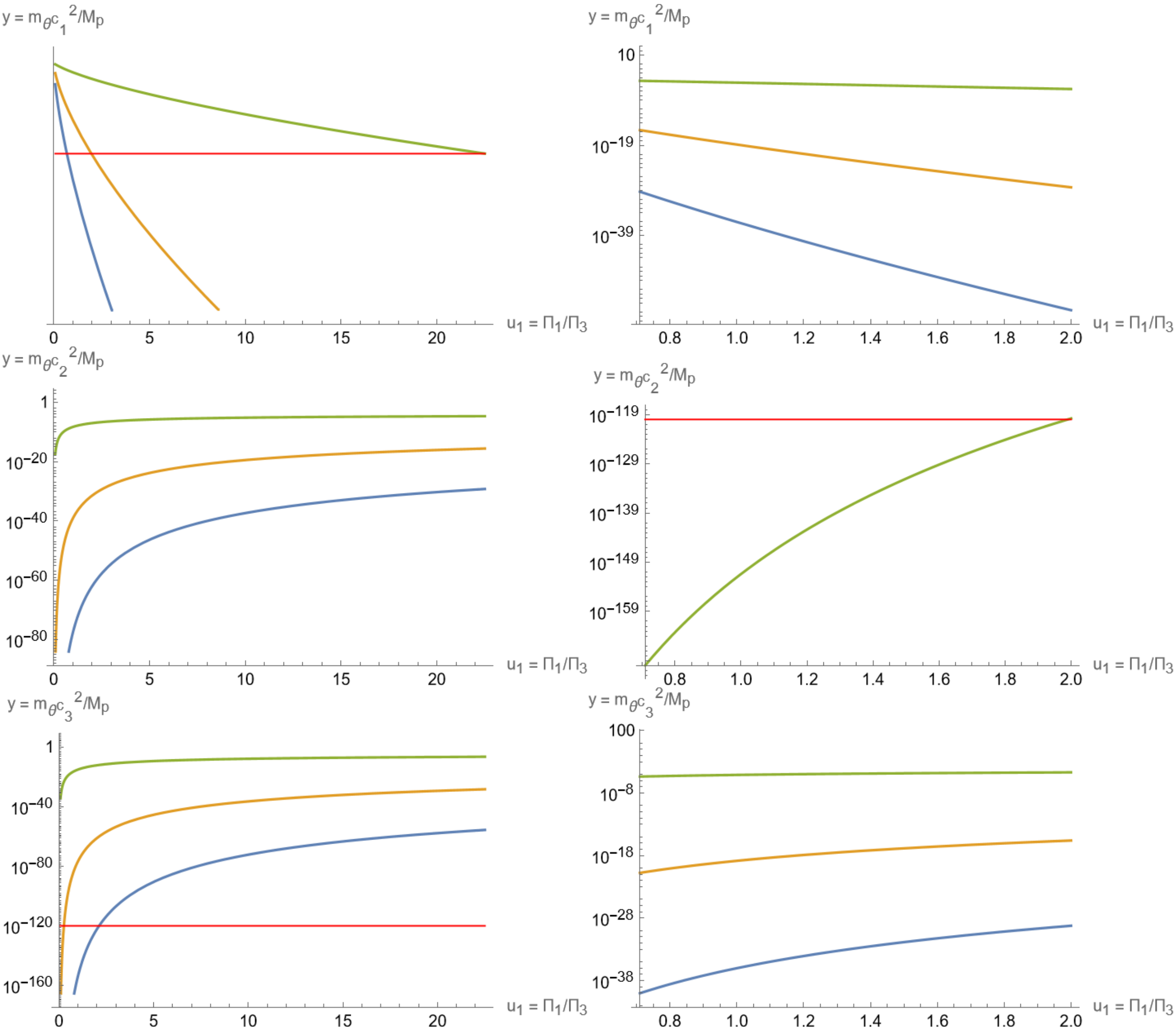

Figure 6.7: Plots of the axion masses for $\mathcal{V} = 300$. For this value, the parameter u_2 varies in $0.001 < u_2 < 90$. The plots on the left hand side are realized for $u_2 = 0.5$, then we have $0.08 < u_1 < 22.5$. On the right hand side, $u_2 = 45$ and so $0.71 < u_1 < 2$. The blue line corresponds to $N = 1$. The yellow line corresponds to $N = 2$. The green line corresponds to $N = 10$. The horizontal red line corresponds to the boundary between massive and effectively massless axions.

Chapter 7

Conclusions

In this thesis, we address the problem of moduli stabilization for the axiverse in string theory. After discussing some intriguing proposals beyond current models, such as SUSY and extra dimensions, we present the fundamentals of string theory. We cover both its initial bosonic formulation and the superstring theories, with the ultimate goal of extracting the final spacetime spectrum, including fermions, which are essential for constructing a realistic model of matter.

Apart from theoretical issues, moduli stabilization needs to considered also in a cosmological perspective. To understand this, some basic notions of General Relativity are recalled, as well as its usage in the description of the dynamics of the Universe. This discussion naturally leads to the challenges associated with the ΛCDM model and the attempts to fix them, such as inflation, a period of accelerated expansion during the Early Universe.

In order to consistently realize inflation or, more generically, any cosmological solution within the framework of string theory, we must extract low-energy effective field theories. This process is complex and involves various choices, resulting in significant arbitrariness in the final model. These topics are presented in Chapter 5. Compactification in string theory is much more subtle than dimensional reduction in ordinary QFT, due to the specific requirements of the framework (e.g. a non-trivial topology for the compact dimensions, the request of a Calabi-Yau manifold to preserve SUSY). We also introduce higher-dimensional objects such as Dp-branes and Op-planes, which are necessary for formulating a physically viable theory. Furthermore, we summarize some novel results related to the topology of Calabi-Yau manifolds (in particular for the huge class of toric Calabi-Yau threefolds) involving a large number of moduli and axions. These findings include the evaluation of the volume and others topological quantities combined with physical information, such as the mass scale of the lightest axion in the theory. These discoveries are of significant interest because they contribute to a better understanding of Calabi-Yau manifolds in physically realistic scenarios, which involve hundreds of moduli and axions. Additionally, they provide quantitative control of the theory in this region of the moduli space. These formal aspects must be considered when attempting to incorporate cosmology into string theory. The string degrees of freedom can play many roles, including supporting inflation in the Early Universe, potentially serving as the inflaton field. However, these degrees of freedom must be consistent with the observational evidence of standard cosmology, leading to constraints like the cosmological moduli problem, which imposes a lower bound on the moduli mass of order 30 TeV. Furthermore, in Section 5.2.3, it is discussed one of the most intriguing scenarios of string cosmology, the axiverse, a plethora of axions, as predicted by string compactification. Its potential cosmological effects, such as CMB rotation or black hole superradiance, are presented with a particular focus on the axion mass ranges that sustain them.

Subsequently, the main results of this thesis concern exactly these particles. By performing moduli stabilization and incorporating constraints from cosmology, axion mass spectra can be extracted. They depend on the topological property of the Calabi-Yau considered. In this work, two kinds of them are studied: the Swiss-Cheese and the Fibred ones, each with varying or even arbitrary number of moduli and axions. The resulting spectra can be combined with the data presented in Section 5.2.3, enabling us to translate them into constraints on topological properties of Calabi-Yau manifolds, identifying which topology of the extra-dimensions leads to specific cosmological observations.

In the future, these computations can be extended to more complex Calabi-Yau manifolds, with the goal of establishing a systematic connection between Calabi-Yau topology and phenomenological insights derived from upcoming cosmological observations. Even the absence of detecting these phenomena would result in a more constrained set of allowed string theory solutions.

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