Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
Corso di Laurea in Matematica

DIFFERENTIAL GRADED LIE ALGEBRAS IN DEFORMATION THEORY

Tesi di Laurea in Geometria

Relatore:
FRANCESCO MEAZZINI

Presentata da:
LORENZO LIPPARINI

Anno Accademico 2022-2023

Differential graded Lie algebras in Deformation Theory

Lorenzo Lipparini

Alma Mater studiorum Università di Bologna
Dipartimento di Matematica
Piazza di porta San Donato 5, 40126 Bologna, Italy
Email address, L. Lipparini: `lorenzo.lipparini9@studio.unibo.it`

Contents

Introduction

Deformation theory is simultaneously a classical subject in mathematics and a very active field of contemporary research. The first developments of the theory in its modern form date back to the 1950s, with work by Koidara, Spencer and Kuranishi on deformations of compact complex manifolds [**11, 12, 14, 13**], and by Gerstenhaber on deformations of associative algebras [**5**]. Around the same years, Nijenhuis and Richardson drew a connection between the two approaches, by recognizing that in each case the deformations are related to the solutions to a certain equation in a differential graded Lie algebra. This idea was greatly extended by Deligne in 1986 [**3**], who in a letter directed to Milson famously claimed:

> "in characteristic 0, a deformation problem is controlled by a differential graded Lie algebra, with quasi-isomorphic DG-Lie algebras giving the same deformation theory."

This set the direction for a line research in deformation theory where differential graded Lie algebras play a central role. Outstanding results have been proved in the following years. Goldman and Millson [**6**] described the local behaviour of the moduli space of flat connections on a fixed complex vector bundle; this had previously been related to the character variety of representations of the fundamental group of the manifold and to the moduli space of Higgs bundles (whose interest is renowned in non-abelian Hodge theory). Moreover, by proving formality for a certain DG-Lie algebra, they showed that the moduli space of certain representations of the fundamental group of a compact Kähler manifold admits at most quadratic singularities. Another famous result has been achieved by Kontsevich in [**15**], where he proved that every finite dimensional Poisson manifold admits a canonical deformation quantization. This was obtained by showing the Lie formality of the Hochschild cohomology of a certain algebra of smooth functions on a differentiable manifold.

This thesis aims to provide a gentle introduction to the abstract theory of the objects involved in the famous results above mentioned. In order to explain what we mean by saying that "a deformation problem is controlled by a DG-Lie algebra", we shall develop the theory concerning associative DG-algebras and DG-Lie algebras, then move to the axiomatic definition of deformation functors. Following Grothendieck, a geometric deformation problem of an object X over a field $\mathbb{K}$ can be repackaged into a functor

$$\mathrm{Def}_X\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

satisfying certain axioms, where $\mathbf{Art}_{\mathbb{K}}$ denotes the category of local Artin $\mathbb{K}$-algebras with residue field $\mathbb{K}$. Roughly speaking, the reader can think of $\mathrm{Def}_X(\mathbb{K}/(t^{n+1}))$ as the set of infinitesimal deformations of order n of the given object X. Most of the times, this should be understood as a purely geometric data. On the other hand, to any DG-Lie algebra L one can associate a functor

$$\mathrm{Def}_L\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$$

via Maurer-Cartan solutions modulo gauge action. It is worth noticing that this functor satisfies the same axioms of Def_X, and most importantly that Def_L is a purely algebraic construction depending only on the Lie structure of L. Now, it seems natural to wonder if, given a geometric deformation problem Def_X, there exists a (unique?) DG-Lie algebra L together with a natural isomorphism of functors $\mathrm{Def}_X \cong \mathrm{Def}_L$, that is to say: a DG-Lie algebra L controlling the deformation problem. The main motivation to investigate such problem is that several relevant geometric features of X (such as the local structure of some moduli space containing X) can be translated into simple algebraic properties of L (such as its cohomology).

The main result presented in this thesis is the so called "homotopy invariance theorem", which essentially tells us that the functor that associates to every DG-Lie algebra L the corresponding deformation functor Def_L factors through the homotopy category of DG-Lie algebras. This last category can be essentially understood as a localization with respect to the class of quasi-isomorphisms. As we shall better explain in the last chapter, the homotopy invariance theorem can be described by the following commutative diagram

$$\begin{tikzcd} \mathbf{DGLA} \arrow[r, "\mathrm{Def}"] \arrow[d] & \text{category of deformation functors} \\ \mathrm{Ho}(\mathbf{DGLA}) \arrow[ur, dashed] & \end{tikzcd}$$

The thesis is divided into three chapters, which can be roughly summarized as follows.

(1) We introduce the fundamental algebraic structures at play, including differential graded Lie algebras. We establish several important facts about these objects, with particular attention to their homological and homotopical properties.
(2) We set up the general theory of abstract deformation functors, and see how it applies in case of deformations of a chain complex. Then, we define the deformation functor associated to a differential graded Lie algebra, and demonstrate with a motivating example how it can be used to describe a concrete deformation problem. In particular, we enlight the general principle asserting that in characteristic 0 every deformation problem can be described as the deformation functor associated to a DG-Lie algebra.
(3) We state the homotopy invariance theorem for the deformation functor associated to a DG-Lie algebra, we discuss its implications, and finally give an outline of its proof.

CHAPTER 1

Differential graded structures

In this and all subsequent chapters, for simplicity of exposition we shall work over a field $\mathbb{K}$ of characteristic 0, even if most of the definitions can be adapted to the case of positive characteristic. By convention, unless otherwise specified, the tensor product will always be intended over the field $\mathbb{K}$.

In this chapter, we define the main algebraic structures which will be used throughout this thesis, and we explore some of their key properties. All the definitions will be carefully given without any substantial prerequisites, but when it comes to proofs we will assume that the reader has some familiarity with the basic tools of homological algebra, such as the snake lemma and the four lemma.

1. DG-vector spaces

Definition 1.1. Let I be a set. An I**-graded-vector space** V is the data of a vector space V and a direct sum decomposition $V = \bigoplus_{i\in I} V^i$.

Definition 1.2. Let V, W be I-graded vector spaces. A **morphism of** I**-graded-vector spaces** $f : V \to W$ is a linear map f which respects the grading, that is: $\forall i \in I \; f(V^i) \subseteq W^i$.

Definition 1.3. A **DG-vector space** (V,d) is the data of a $\mathbb{Z}$-graded-vector space $V = \bigoplus_{n\in\mathbb{Z}} V^n$ and a linear map $d : V \to V$, called **differential**, such that $\forall n \in \mathbb{Z} \; d(V^n) \subseteq V^{n+1}$ and $d \circ d = 0$.

Fixed a DG-vector space (V,d), by the universal property of the direct sum the differential may be decomposed as $d = \bigoplus_{n\in\mathbb{Z}} d_n$, with $d_n : V^n \to V^{n+1}$. This gives a sequence:

$$\cdots \xrightarrow{d_{-2}} V^{-1} \xrightarrow{d_{-1}} V^0 \xrightarrow{d_0} V^1 \xrightarrow{d_1} \cdots$$

In this light, the condition $d \circ d = 0$ is equivalent to $\forall n \in \mathbb{Z} \; \operatorname{Im} d_{n-1} \subseteq \ker d_n$, which makes the sequence above a cochain complex (in fact, a DG-vector space has the same data as a cochain complex of vector spaces), and allows one to consider the associated cohomology:

$$Z^n(V) = \ker d_n, \quad B^n(V) = \operatorname{Im} d_{n-1}, \quad H^n(V) = Z^n(V)/B^n(V),$$

or, more succinctly:

$$Z^*(V) = \ker d, \quad B^*(V) = \operatorname{Im} d, \quad H^*(V) = Z^*(V)/B^*(V),$$

indeed, it is easy to verify that:

$$Z^*(V) = \bigoplus_{n\in\mathbb{Z}} Z^n(V), \quad B^*(V) = \bigoplus_{n\in\mathbb{Z}} B^n(V), \quad H^*(V) \cong \bigoplus_{n\in\mathbb{Z}} H^n(V).$$

Hence, $Z^*(V)$, $B^*(V)$ and $H^*(V)$ naturally come with a $\mathbb{Z}$-graded vector space structure. One can also endow them with the differential induced by that of (V,d), but that just gives $d|_{Z^*(V)} = 0$ since $Z^*(V) = \ker d$, $d|_{B^*(V)} = 0$ since $d \circ d = 0$, and similarly $[d] = 0$ on $H^*(V)$. In other words, $Z^*(V)$, $B^*(V)$ and $H^*(V)$ are naturally seen as DG-vector spaces with trivial differential. Following the standard terminology from homological algebra, we will occasionally refer to elements of $Z^*(V)$ and $B^*(V)$ as **cocycles** and **coboundaries** respectively.

Fixed a DG-vector space (V,d), since $V = \bigoplus_{n\in\mathbb{Z}} V^n$, V is generated by the elements $v \in V^n \setminus \{0\}$ for $n \in \mathbb{Z}$; if $v \in V^n \setminus \{0\}$, we say that v is **homogeneous of degree** n, and denote its degree $\overline{v} = n$. In everything that will follow, if the notation $\overline{v}$ is used then it is *implicitly assumed* that v is a homogeneous vector. Of course, any vector $v \in V$ can be decomposed into homogeneous components, that is:

$$\exists v_{n_1} \in V^{n_1}, \dots, v_{n_k} \in V^{n_k} \quad v = v_{n_1} + \cdots + v_{n_k}.$$

Example 1.4. Let $V = \mathbb{K}[t]$ be the set of polynomials in the variable t with coefficients in $\mathbb{K}$. V can be seen as a $\mathbb{Z}$-graded vector space, where $V^n = \{p \in \mathbb{K}[t] \mid p = 0 \text{ or } p \text{ is a monomial of degree } n\}$ ($V^n = 0$ for $n < 0$). Note that if we denote by D the formal derivative operator, then $(\mathbb{K}[t], D)$ is *not* a DG-vector space, for at least two reasons:

- we have $D(V^n) \subseteq V^{n-1}$ instead of $D(V^n) \subseteq V^{n+1}$ (though this could be easily fixed by flipping the grading)
- clearly, $D \circ D \neq 0$

The correct way of seeing polynomial rings as DG-vector spaces, based on *differentials*, as opposed to *derivatives*, will be outlined in the section about DG-commutative algebras (Example 1.49).

Definition 1.5. Let (V, d_V), (W, d_W) be DG-vector spaces. A **morphism of DG-vector spaces** (or DG-linear map) $f : (V, d_V) \to (W, d_W)$ is a morphism of $\mathbb{Z}$-graded-vector spaces f that commutes with the differentials, that is: $f \circ d_V = d_W \circ f$.

Just like before, one can decompose a morphism $f : (V, d_V) \to (W, d_W)$ as $f = \bigoplus_{n \in \mathbb{Z}} f_n$; one then has the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "d_V"] & V^{-1} \arrow[r, "d_V"] \arrow[d, "f_{-1}"] & V^0 \arrow[r, "d_V"] \arrow[d, "f_0"] & V^1 \arrow[r, "d_V"] \arrow[d, "f_1"] & \cdots \\
\cdots \arrow[r, "d_W"] & W^{-1} \arrow[r, "d_W"] & W^0 \arrow[r, "d_W"] & W^1 \arrow[r, "d_W"] & \cdots
\end{tikzcd}$$

When there is no ambiguity, one can refer to a DG-vector space just as the underlying space V, and denote all the differentials of the DG-vector spaces in context with the same symbol d. For example, one can say that a morphism of DG-vector spaces $f : V \to W$ is a linear map such that $f \circ d = d \circ f$.

Remark 1.6. Of course, we have a category of I-graded-vector spaces (for any fixed I), as well as a category **DG** of DG-vector spaces. Also, both of these categories have enough structure to define exact sequences and such in the usual manner.

Remark 1.7. Any vector space V can be seen as a $\mathbb{Z}$-graded-vector space "concentrated in degree 0": $V = \bigoplus_{n \in \mathbb{Z}} V^n$, with $V^0 = V$ and $V^n = 0$ for $n \neq 0$. In turn, any $\mathbb{Z}$-graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V^n$ can be seen as a DG-vector space (V, d), with $d = 0$ being the trivial differential.

Remark 1.8. The mapping of a DG-vector space to its cohomology gives rise to a functor from the category **DG** to itself (where the cohomology is endowed with the trivial differential as usual):

$$\begin{aligned}
H^* &: \mathbf{DG} \to \mathbf{DG} \\
H^*((V, d)) &= (H^*(V), 0) \\
H^*(f) &= [v] \mapsto [f(v)]
\end{aligned}$$

Sometimes, when there is no ambiguity, we will use the shorthand notation $f : H^*(V) \to H^*(W)$ in place of $H^*(f) : H^*(V) \to H^*(W)$.

It is easy but important to observe that the category **DG** admits direct sums, which behave in much the same way they do in $\mathbf{Vect}_{\mathbb{K}}$. For completeness, we shall give the definition and state its main properties.

Definition 1.9. Let (V,d_V), (W,d_W) be DG-vector spaces. We define the **direct sum** $(V\oplus W,d)$ as follows:

$$(V\oplus W)^n=\bigoplus_{n\in\mathbb{Z}} V^n\oplus W^n,\quad d(v+w)=d_V(v)+d_W(w).$$

The direct sum $(V\oplus W,d)$ is the product of (V,d_V) and (W,d_W) in the category **DG**, with the canonical projections $\pi_V : V\oplus W\to V$, $\pi_W : V\oplus W\to W$; it is also their coproduct, with the canonical injections $\iota_V : V\to V\oplus W$, $\iota_W : W\to V\oplus W$. Moreover, we have the following identities (biproduct equations):

$$\pi_V\circ\iota_V=\mathrm{id}_V,\quad \pi_W\circ\iota_W=\mathrm{id}_W,\quad \iota_V\circ\pi_V+\iota_W\circ\pi_W=\mathrm{id}_{V\oplus W}. \tag{1.1}$$

We can easily see that the H^* functor preserves direct sums, in the sense of the following proposition.

Proposition 1.10. *Let (V,d), (W,d) be DG-vector spaces. We have an isomorphism:*

$$\begin{aligned} H^*(V\oplus W)&\cong H^*(V)\oplus H^*(W)\\ [u]&\mapsto[\pi_V(u)]+[\pi_W(u)]\\ [\iota_V(v)+\iota_W(w)]&\leftarrow\!\shortmid[v]+[w]\end{aligned}$$

PROOF. By the equations (1.1) the maps are indeed inverses of each other; we just have to show that they are well defined morphisms of DG-vector spaces.
To that end, observe that the map from left to right can be described as follows: the canonical projections $V\oplus W\to V$, $V\oplus W\to W$ induce two morphisms $H^*(V\oplus W)\to H^*(V)$, $H^*(V\oplus W)\to H^*(W)$, which in turn induce the map $H^*(V\oplus W)\to H^*(V)\oplus H^*(W)$ by the universal property of the product.
The map from right to left can be obtained similarly from the canonical injections, seeing the direct sum as a coproduct. □

Oftentimes, it is more intuitive to think of a direct sum as "internal" to some DG-vector space, for which one can equivalently use the following definitions.

Definition 1.11. Let (V,d) be a DG-vector space. A **DG-vector subspace** (W,d) of (V,d) is a $\mathbb{Z}$-graded-vector space W of the form

$$W=\bigoplus_{n\in\mathbb{Z}} W^n,\qquad \text{with } W^n\subseteq V^n,$$

such that the differential of (V,d) restricts to W, that is: $d(W)\subseteq W$, or equivalently $\forall n\in\mathbb{Z}\ d(W^n)\subseteq W^{n+1}$.

Example 1.12. Let $f:V\to W$ be a morphism of DG-vector spaces. It is easy to check that $\ker f$ id a DG-vector subspace of V, and $\operatorname{Im} f$ is a DG-vector subspace of W.

Definition 1.13. Let (V,d), (W,d) be DG-vector subspaces of (U,d). We say that (U,d) is the **direct sum** of (V,d) and (W,d) if it is their direct sum as $\mathbb{K}$-vector spaces. In that case, we write: $(U,d)=(V,d)\oplus(W,d)$.

Example 1.14. While all $\mathbb{K}$-vector subspaces admit a complement, this is not the case for DG-vector subspaces.
As a simple example, consider the $\mathbb{Z}$-graded-vector space $U=\bigoplus_{n\in\mathbb{Z}} U^n$ with $U^0=U^1=\mathbb{K}$ and $U^n=0$ for $n\neq 0$; the following are DG-vector spaces, spelled out as cochain complexes:

$$(U,d):\qquad \cdots\longrightarrow 0\longrightarrow\mathbb{K}\xrightarrow{\mathrm{id}}\mathbb{K}\longrightarrow 0\longrightarrow\cdots$$

$$(V,d):\qquad \cdots\longrightarrow 0\longrightarrow 0\longrightarrow\mathbb{K}\longrightarrow 0\longrightarrow\cdots$$

Clearly, (V,d) is a DG-vector subspace of (U,d), but it can't admit a complement (W,d) since we would necessarily have $W^0=\mathbb{K}$, $W^1=0$ and therefore $d(W^0)\nsubseteq d(W^1)$.

Among all DG-vector spaces, some particularly simple ones are those with trivial differential, or with trivial cohomology.

Definition 1.15. A DG-vector space (V,d) is called **minimal** if $d=0$.

Definition 1.16. A DG-vector space (V,d) is called **acyclic** if $H^*(V)=0$.

If (V,d) is acyclic, then by definition it has the smallest possible cohomology; on the contrary, if (V,d) is minimal then $\ker d = V$, $\operatorname{Im} d = 0$, and therefore $H^*(V)\cong V$, which is in a sense the largest possible cohomology. As it turns out, any DG-vector space is a sum of spaces of this kind.

Proposition 1.17. *Let (V,d) be a DG-vector space. Then, there exists a decomposition*

$$(V,d)=(W,d)\oplus(H,0),$$

where (W,d) is acyclic and $(H,0)$ is minimal.

PROOF. Let (V,d) be a DG-vector space. For each $n\in\mathbb{Z}$, let us choose some complements C^n and H^n:

$$V^n = Z^n(V)\oplus C^n, \quad Z^n(V)=B^n(V)\oplus H^n.$$

Consider the $\mathbb{Z}$-graded vector spaces $C=\bigoplus_{n\in\mathbb{Z}} C^n$, $H=\bigoplus_{n\in\mathbb{Z}} H^n$, $W=B^*(V)\oplus C$; observe that d restricts to $d: W\to W$ since $B^*(V)\subseteq W$, while d is trivial on H since $H\subseteq Z^*(V)$. Therefore, we have a decomposition

$$(V,d)=(W,d)\oplus(H,0).$$

It remains to show that W is acyclic; to that end, observe that because C is as a complement of $Z^*(V)$, $d: C\to B^*(V)$ is an isomorphism, and we have

$$\begin{aligned} &\forall w\in W\ \exists b\in B^*(V), c\in C \quad w=b+c, \quad d(w)=d(b)+d(c)=d(c)\\ &\implies \ker d|_W = B^*(V), \quad \operatorname{Im} d|_W = d(C)=B^*(V)\\ &\implies H^*(W)=0. \end{aligned}$$

□

It is clear from the proof that, while the decomposition exists, it is not at all canonical. Nonetheless, we will see that this proposition has some really important consequences, which we shall soon see.
Let us consider the following definition, which is a standard one from homological algebra.

Definition 1.18. A **quasi-isormorphism of DG-vector spaces** $f: V\to W$ is a morphism of DG-vector spaces f which induces an isomorphism in cohomology, that is: $f: H^*(V)\to H^*(W)$ is an isomorphism.
Given two DG-vector spaces V, W, we say that V and W are **quasi-isomorphic** if they are equivalent under the equivalence relation generated by quasi-isomorphisms. In other words, V and W are quasi-isomorphic if there exists a zigzag of finitely many quasi-isomorphisms:

$$\begin{tikzcd} & X_1 \arrow[dl] \arrow[dr] & & \dots \arrow[dl] \arrow[dr] & & X_n \arrow[dl] \arrow[dr] & \\ V & & X_2 & & X_{n-1} & & W \end{tikzcd}$$

The necessity of zigzags in the definition is ultimately due to the following: if there exists a quasi-isomorphism from V to W, it is not clear *a priori* that there is a quasi-isomorphism from W to V. We will show that actually, in the case of DG-vector spaces, a quasi-isomorphism from W to V always exists.
To start, we observe that if V and W are quasi-isomorphic DG-vector spaces, then by applying the H^* functor to the zigzag between them we can immediately see that $H^*(V)\cong H^*(W)$. Importantly, we can show that the converse is also true.

Proposition 1.19. *Let (V,d) be a DG-vector space. Then, there exist two quasi-isomorphisms of DG-vector spaces: one from $(H^*(V),0)$ to (V,d), and one from (V,d) to $(H^*(V),0)$.*

PROOF. Let (V,d) be a DG-vector space. By Proposition 1.17, there exists a decomposition $(V,d) = (W,d) \oplus (H,0)$ where (W,d) is acyclic and $(H,0)$ is minimal. By Proposition 1.10, this induces an isomorphism:

$$\begin{aligned} H^*(V) &\cong H^*(W) \oplus H^*(H) \\ [v] &\mapsto [\pi_W(v)] + [\pi_H(v)] \\ [\iota_W(w) + \iota_H(h)] &\leftarrow\!\shortmid [w] + [h] \end{aligned}$$

Since (W,d) is acyclic, we have $H^*(W) \oplus H^*(H) \cong H^*(H)$, and the isomorphism simplifies to:

$$\begin{aligned} H^*(V) &\cong H^*(H) \\ [v] &\mapsto [\pi_H(v)] \\ [\iota_H(h)] &\leftarrow\!\shortmid [h] \end{aligned}$$

Clearly, the map from left to right is the map induced in cohomology by $\pi_H : V \to H$, while the map from right to left is the one induced by $\iota_H : H \to V$; π_H and ι_H are therefore quasi-isomorphisms or DG-vector spaces. To conclude, we observe that the isomorphism above gives $H^*(V) \cong H^*(H) \cong H$ (since H is minimal), and by composing that with π_H and ι_H we get the quasi-isomorphisms we were looking for. $\square$

Since quasi-isomorphisms compose, the previous proposition immediately implies the following, which gives a really simple characterization of the quasi-isomorphism relation for DG-vector spaces.

Corollary 1.20. *Let V, W be DG-vector spaces. Then, V and W are quasi-isomorphic if and only if $H^*(V) \cong H^*(W)$. Moreover, if that is the case, there exists a quasi-isomorphism from V to W (and vice versa).*

Hence, the quasi-isomorphism class of a DG-vector space is completely determined by its cohomology, and this essentially solves the question of quasi-isomorphism for DG-vector spaces. In later sections, we will see that once we endow a DG-vector space with additional structure, and require quasi-isomorphisms to preserve the new structure, the question will become much less trivial. With that in mind, we will make some more observations about quasi-isomorphisms that will help us down the road.

To fix terminology and notation, we briefly recall the definition of pullback, and its general properties.

Definition 1.21. Let $f : L \to N$, $g : M \to N$ be morphisms in a category. A commutative square of the form

$$\begin{tikzcd} P \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"] & M \arrow[d, "g"] \\ L \arrow[r, "f"] & N \end{tikzcd}$$

is called **pullback square** (over f and g) if it satisfies the following universal property:

$$\forall Q, h : Q \to L, k : Q \to M \quad f \circ h = g \circ k \quad \Longrightarrow \quad \exists! u : Q \to P \quad h = \overline{g} \circ u, \quad k = \overline{f} \circ u.$$

If that is the case, we say that P is the fiber product of L and M over N, and denote it $P = L \times_N M$; we also say that $\overline{f}$ is the pullback of f and $\overline{g}$ is the pullback of g.

In the notation above, if one of f, g is an isomorphism, we can trivially see that the pullback exists.

Proposition 1.22. *Let $f : L \to N$, $g : M \to N$ be morphisms in a category. If f is an isomorphism, then the following is a pullback square:*

$$\begin{tikzcd} M \arrow[r, "\mathrm{id}"] \arrow[d, "f^{-1}\circ g"] & M \arrow[d, "g"] \\ L \arrow[r, "f"] & N \end{tikzcd} \tag{1.2}$$

PROOF. Let us verify the universal property directly. Given Q, $h : Q \to L$, $k : Q \to M$ such that $f \circ h = g \circ k$, there is a unique morphism u that satisfies $k = \mathrm{id} \circ u$, which is clearly $u = k$, and then we also have $h = f^{-1} \circ g \circ k = f^{-1} \circ g \circ u$. □

As usual with universal constructions, the pullback square over two morphisms is unique up to unique isomorphism. In particular, any categorical property of *some* pullback square extends to *all* pullback squares over the same morphisms. For instance, because in Diagram (1.2) the arrow $\overline{f} = \mathrm{id}$ is an isomorphism, the previous proposition immediately implies the following.

Corollary 1.23. *Isomorphisms are stable under pullbacks. That is, if $f : L \to N$ and $g : M \to N$ are morphisms in a category, and $\overline{f}$, $\overline{g}$ are their respective pullbacks:*

$$f \textit{ isomorphism} \implies \overline{f} \textit{ isomorphism}$$

Let us now focus on the categories which are interesting to us. It is straightforward to see that pullbacks exist in **DG**.

Proposition 1.24. *Let $f : L \to N$, $g : M \to N$ be morphisms of DG-vector spaces. Define*

$$s : L \times M \to N, \quad s(l, m) = f(l) - g(m).$$

Then, the following is a pullback square in **DG***:*

$$\begin{tikzcd} \ker s \arrow[r, "\pi_M"] \arrow[d, "\pi_L"] & M \arrow[d, "g"] \\ L \arrow[r, "f"] & N \end{tikzcd} \tag{1.3}$$

PROOF. The universal property of the pullback easily follows from that of the product, and from the fact that an induced map on the product $(h, k) : Q \to L \times M$ restricts to $\ker s \subseteq L \times M$ exactly when:

$$s \circ (h, k) = 0 \iff f \circ h - g \circ k = 0 \iff f \circ h = g \circ k$$

□

In particular, as a subspace of $L \times M$ the fiber product is characterized by the following exact sequence:

$$0 \longrightarrow L \times_N M \longrightarrow L \times M \xrightarrow{\ s\ } N \tag{1.4}$$

Proposition 1.25. *In the category* **DG***, surjective morphisms are stable under pullbacks.*

PROOF. We check this directly on the pullback square (1.3). If f is surjective, then for all $m \in M$ there exists an $l \in L$ such that $f(l) = g(m)$, hence $(l, m) \in \ker s$, and clearly $\overline{f}(l, m) = \pi_M(l, m) = m$. □

While we have seen that the H^* functor preserves products (direct sums), that is not the case in general with fiber products. The following proposition gives a sufficient condition.

Proposition 1.26. *Consider a pullback square in* **DG***, and the induced commutative square in cohomology:*

$$\begin{tikzcd} L \times_N M \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"] & M \arrow[d, "g"] \\ L \arrow[r, "f"] & N \end{tikzcd} \qquad \begin{tikzcd} H^*(L \times_N M) \arrow[r, "H^*(\overline{f})"] \arrow[d, "H^*(\overline{g})"] & H^*(M) \arrow[d, "H^*(g)"] \\ H^*(L) \arrow[r, "H^*(f)"] & H^*(N) \end{tikzcd}$$

Define

$$s : L \times M \to N, \quad s(l, m) = f(l) - g(m).$$

If both maps $s : L \times M \to N$, $H^*(s) : H^*(L \times M) \to H^*(N)$ are surjective, then the diagram on the right is also a pullback square. In other words, we have: $H^*(L \times_N M) \cong H^*(L) \times_{H^*(N)} H^*(M)$.

PROOF. Since s is surjective by assumption, we can extend (1.4) to a short exact sequence:

$$0 \longrightarrow L \times_N M \longrightarrow L \times M \xrightarrow{s} N \longrightarrow 0$$

By the snake lemma, the above induces a long exact sequence in cohomology:

$$\cdots \to H^i(L \times_N M) \to H^i(L \times N) \xrightarrow{H^i(s)} H^i(N) \xrightarrow{\delta_i} H^{i+1}(L \times_N M) \to H^{i+1}(L \times M) \to H^{i+1}(N) \to \cdots$$

Since each $H^i(s)$ is surjective by assumption, it follows by exactness that the connecting morphisms δ_i are all 0. Hence, the long exact sequence is equivalent to a short exact sequence of complexes:

$$0 \longrightarrow H^*(L \times_N M) \longrightarrow H^*(L \times M) \xrightarrow{H^*(s)} H^*(N) \longrightarrow 0$$

Since $H^*(L \times M) \cong H^*(L) \times H^*(M)$, this is a sequence of the form (1.4) in disguise, which implies that $H^*(L \times_N M)$ is indeed the fiber product. □

In the notation of the above proposition, suppose that f is a surjective quasi-isomorphism. Then, $s : L \times M \to N$ is surjective because $f : L \to N$ is surjective, and similarly $H^*(s) : H^*(L \times M) \cong H^*(L) \times H^*(M) \to H^*(N)$ is surjective because $H^*(f) : H^*(L) \to H^*(N)$ is bijective. Hence, the above proposition applies, and we have in particular that $H^*(\overline{f})$ is the pullback of the morphism $H^*(f)$. Now, by the previous results we have that $\overline{f}$ is surjective because f is surjective, and $H^*(\overline{f})$ is an isomorphism because $H^*(f)$ is an isomorphism; therefore $\overline{f}$ is a surjective quasi-isomorphism, and we have proven the following corollary.

Corollary 1.27. *In the category* **DG***, surjective quasi-isomorphisms are stable under pullbacks.*

There are several ways one can combine existing DG-vector spaces to define new ones; many constructions of this kind simply take existing definitions for vector spaces and enrich them with a suitable DG structure. Notably, this is the case for the tensor product.

Definition 1.28. Let (V, d_V), (W, d_W) be DG-vector spaces. We define the **tensor product** $(V \otimes W, d)$ as follows:

$$(V \otimes W)^n = \bigoplus_{i+j=n} V^i \otimes W^j, \quad d(v \otimes w) = d_V(v) \otimes w + (-1)^{\overline{v}} v \otimes d_W(w).$$

(One can verify that indeed $d \circ d = 0$, and see how that crucially depends on the $(-1)^{\overline{v}}$ sign)

As expected, the tensor product can be characterised up to isomorphism by a universal property, involving suitably defined bilinear maps.

Definition 1.29. Let (V, d_V), (W, d_W), (Z, d_Z) be DG-vector spaces. A **DG-bilinear map** $\phi : (V, d_V) \times (W, d_W) \to (Z, d_Z)$ is a function ϕ such that:

(1) $\phi : V \times W \to Z$ is a bilinear map of vector spaces
(2) $\forall i, j \in \mathbb{Z} \quad \phi(V^i, W^j) \subseteq Z^{i+j}$
(3) $\forall v, w \quad d_Z(\phi(v, w)) = \phi(d_V(v), w) + (-1)^{\overline{v}} \phi(v, d_W(w))$ (graded Leibniz rule)

Proposition 1.30. *Let V, W be DG-vector spaces. Then, we have a DG-bilinear map $\psi : V \times W \to V \otimes W$, $\psi(v,w) = v \otimes w$, satisfying the following universal property:*

$$\forall Z \text{ DG-vector space}, \phi : V \otimes W \to Z \text{ DG-bilinear } \exists! \overline{\phi} : V \otimes W \to Z \text{ DG-linear} \quad \phi = \overline{\phi} \circ \psi$$

$$\begin{tikzcd} V \times W \arrow[d, "\psi"'] \arrow[dr, "\phi"] & \\ V \otimes W \arrow[r, dashed, "\overline{\phi}"'] & Z \end{tikzcd}$$

PROOF. Observe that the DG structure of $V \otimes W$ is defined precisely in such a way that ψ is DG-bilinear. Given a DG-bilinear map ϕ, by property (1) ϕ is in particular bilinear, so by the universal property of the tensor product of vector spaces

$$\exists! \overline{\phi} : V \otimes W \to Z \text{ linear} \quad \phi = \overline{\phi} \circ \psi.$$

With that, properties (2) and (3) are respectively equivalent to the fact that $\overline{\phi}$ respects the grading and $\overline{\phi}$ commutes with the differential; hence, $\overline{\phi}$ is a morphism of DG-vector spaces. □

Looking back at Definition 1.29, while the first two requirements appear reasonable right away, the graded Leibniz rule stands out for the $(-1)^{\overline{v}}$ sign, which makes the expression "asymmetrical" in v and w. This definition turns out to be the right one (we'll see some important examples in Section 2 and Section 3), but the asymmetry does have some interesting consequences, starting from the fact that if $(v,w) \mapsto \phi(v,w)$ is DG-bilinear then by flipping the arguments one obtains a map $(w,v) \mapsto \phi(v,w)$ which is *not* DG-bilinear (in general). In fact, defining $\tilde{\phi}(w,v) = \phi(v,w)$:

$$\begin{aligned} d(\tilde{\phi}(w,v)) &= \phi(d(v),w) + (-1)^{\overline{v}} \phi(v, d(w)) \\ &= \tilde{\phi}(w, d(v)) + (-1)^{\overline{v}} \tilde{\phi}(d(w), v) \\ &\neq \tilde{\phi}(d(w), v) + (-1)^{\overline{w}} \tilde{\phi}(w, d(v)) \qquad \text{(in general).} \end{aligned}$$

A standard way to fix the problem is to introduce a sign correction, setting instead $\tilde{\phi}(w,v) = (-1)^{\overline{v}\,\overline{w}} \phi(v,w)$; indeed, with this definition we have:

$$\begin{aligned} d(\tilde{\phi}(w,v)) &= (-1)^{\overline{v}\,\overline{w}} \left(\phi(d(v),w) + (-1)^{\overline{v}} \phi(v, d(w)) \right) \\ &= (-1)^{\overline{w}} (-1)^{(\overline{v}+1)\overline{w}} \phi(d(v), w) + (-1)^{\overline{v}(\overline{w}+1)} \phi(v, d(w)) \\ &= \tilde{\phi}(d(w), v) + (-1)^{\overline{w}} \tilde{\phi}(w, d(v)). \end{aligned}$$

This is know as the **Koszul signs convention**: informally, to "get the signs right", whenever an "object of degree d passes on the other side of an object of degree h, a sign $(-1)^{dh}$ must be inserted".
The prototypical example is the DG-bilinear map $(w,v) \mapsto (-1)^{\overline{v}\,\overline{w}} v \otimes w$, which by the universal property induces the following isomorphism:

Definition 1.31. Let V, W be DG-vector spaces. The **twisting involution** is the isomorphism of DG-vector spaces defined by:

$$\mathbf{tw} : V \otimes W \to W \otimes V$$

$$\mathbf{tw}(v \otimes w) = (-1)^{\overline{v}\,\overline{w}} w \otimes v$$

(On the other hand, one can easily check that the obvious map $v \otimes w \mapsto w \otimes v$ is *not* a morphism of DG-vector spaces)

Having defined the tensor product of DG-vector spaces, it is natural to wonder if there is an internal Hom functor that goes along with it; that is, informally, one can ask if there is sensible way to think of the DG-linear maps between two spaces as elements of a DG-vector space themselves. That is the subject of the following definition.

Definition 1.32. Let (V, d_V), (W, d_W) be DG-vector space. We define the **Hom complex** $(\mathrm{Hom}^*(V, W), d)$ as follows:

$$\mathrm{Hom}^*(V,W) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^n(V,W), \quad \mathrm{Hom}^n(V,W) = \{f : V \to W \text{ linear} \mid \forall i \in \mathbb{Z}.\ f(V^i) \subseteq W^{i+n}\},$$

$$d(f)(v) = d_W(f(v)) - (-1)^{\overline{f}} f(d_V(v)).$$

Of course, all elements of $\mathrm{Hom}^*(V, W)$ are linear maps from V to W, but the inclusion $\mathrm{Hom}^*(V, W) \subseteq \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(V, W)$ is *strict* in general.

Example 1.33. Let F be an infinite dimensional vector space with $\{v_i\}_{i\in\mathbb{Z}}$ as basis. From the same underlying space F we can construct two distinct $\mathbb{Z}$-graded vector spaces V and W, whose respective gradings are defined as follows:

$$V^n = \begin{cases} F & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}, \qquad W^n = \mathrm{Span}\{v_n\}.$$

As usual, we can see V and W as DG-vector spaces with trivial differential. Of course, the identity function $\mathrm{id}_F : F \to F$ is a linear map from V to W, but we can show that it's *not* an element of $\mathrm{Hom}^*(V, W)$, from which it follows $\mathrm{Hom}^*(V, W) \subsetneq \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(V, W)$.
In fact, suppose to the contrary that $\mathrm{id}_F \in \mathrm{Hom}^*(V, W)$; then by decomposing into homogeneous components:

$$\exists f_{n_1} \in \mathrm{Hom}^{n_1}(V,W), \dots, f_{n_k} \in \mathrm{Hom}^{n_k}(V,W) \quad \mathrm{id}_F = f_{n_1} + \cdots + f_{n_k}$$
$$\implies F = \mathrm{id}_F(V^0) \subseteq f_{n_1}(V^0) + \cdots + f_{n_k}(V^0) \subseteq W^{n_1} + \cdots + W^{n_k} = \mathrm{Span}\{v_{n_1}\} + \cdots + \mathrm{Span}\{v_{n_k}\}$$
$$\implies F \subseteq \mathrm{Span}\{v_{n_1}, \dots, v_{n_k}\},$$

and that contradicts the fact that F is infinite dimensional.

Looking back at Definition 1.32, the equation for the differential can be rearranged as

$$d_W(f(v)) = d(f)(v) + (-1)^{\overline{f}} f(d_V(v)).$$

With that, we can see that the DG-structure on $\mathrm{Hom}^*(V, W)$ is defined precisely in such a way that the evaluation function $\mathrm{Hom}^*(V, W) \times V \to W, (f, v) \mapsto f(v)$ is a DG-bilinear map. In fact, we have a much more general relationship between Hom^* and the tensor product, which consists in the following adjunction.

Proposition 1.34. *Let V, W, Z be DG-vector spaces. There is a natural bijection*

$$\mathrm{Hom}_{\mathbf{DG}}(V \otimes W, Z) \cong \mathrm{Hom}_{\mathbf{DG}}(V, \mathrm{Hom}^*(W, Z)).$$

PROOF. Given V, W, Z, one can easily check that the following is a bijection:

$$\begin{aligned} \{\phi : V \times W \to Z \mid \phi \text{ is bilinear}\} &\cong \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(V, \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(W, Z)) \\ \phi &\mapsto (v \mapsto w \mapsto \phi(v, w)) \\ ((v, w) \mapsto f(v)(w)) &\leftarrow\!\shortmid f \end{aligned}$$

Importantly, for corresponding ϕ and f we have $\phi(v, w) = f(v)(w)$.
Now, given that we have the inclusions

$$\{\phi : V \times W \to Z \mid \phi \text{ is DG-bilinear}\} \subseteq \{\phi : V \times W \to Z \mid \phi \text{ is bilinear}\},$$
$$\mathrm{Hom}_{\mathbf{DG}}(V, \mathrm{Hom}^*(W, Z)) \subseteq \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(V, \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(W, Z)),$$

we want to show that the bijection above restricts to

$$\{\phi : V \times W \to Z \mid \phi \text{ is DG-bilinear}\} \cong \mathrm{Hom}_{\mathbf{DG}}(V, \mathrm{Hom}^*(W, Z)).$$

To prove that this restriction is well defined, we must show that for corresponding ϕ and f we have

$$\phi \text{ is DG-bilinear} \iff f \in \mathrm{Hom}_{\mathbf{DG}}(V, \mathrm{Hom}^*(W, Z)).$$

First, we'll verify that if ϕ is DG-bilinear, then $f \in \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(V, \mathrm{Hom}^*(W,Z))$; this amounts to checking that for all $v \in V$ we have $f(v) \in \mathrm{Hom}^*(W,Z)$. Fixed $v \in V$, decomposing into homogeneous components:

$$\begin{gathered}\exists v_{n_1} \in V^{n_1}, \dots, v_{n_k} \in V^{n_k} \quad v = v_{n_1} + \cdots + v_{n_k} \\ \implies f(v) = f(v_{n_1}) + \cdots + f(v_{n_k}).\end{gathered}$$

For each index i we have $f(v_{n_i}) \in \mathrm{Hom}^{n_i}(W,Z)$, since:

$$\forall j \in \mathbb{Z}. \quad f(v_{n_i})(W^j) \subseteq f(V^{n_i})(W^j) = \phi(V^{n_i}, W^j) \subseteq Z^{n_i+j}.$$

This, combined with the previous equation, proves that indeed $f(v) \in \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^n(W,Z) = \mathrm{Hom}^*(W,Z)$.

Now that we've established that f is a map between the right vector spaces, we can show that ϕ respects the grading if and only if f respects the grading:

$$\begin{gathered}\forall i,j \in \mathbb{Z} \quad \phi(V^i, W^i) \subseteq Z^{i+j} \\ \iff \forall i,j \in \mathbb{Z} \quad f(V^i)(W^j) \subseteq Z^{i+j} \\ \iff \forall i \in \mathbb{Z} \quad f(V^i) \subseteq \mathrm{Hom}^i(W,Z).\end{gathered}$$

If that is case, we have in particular $\overline{f(v)} = \overline{v}$ for every homogeneous vector v, and we can also see that ϕ obeys the graded Leibniz rule if and only if f commutes with the differential:

$$\begin{gathered}\forall v,w \quad d(\phi(v,w)) = \phi(d(v),w) + (-1)^{\overline{v}}\phi(v,d(w)) \\ \iff \forall v,w \quad d(f(v)(w)) = f(d(v))(w) + (-1)^{\overline{f(v)}} f(v)(d(w)) \\ \iff \forall v,w \quad f(d(v))(w) = d(f(v)(w)) - (-1)^{\overline{f(v)}} f(v)(d(w)) \\ \iff \forall v,w \quad f(d(v))(w) = d_{\mathrm{Hom}^*(W,Z)}(f(v))(w) \\ \iff \forall v \quad f(d(v)) = d_{\mathrm{Hom}^*(W,Z)}(f(v)) \\ \iff f \circ d = d_{\mathrm{Hom}^*(W,Z)} \circ f.\end{gathered}$$

From all the above, we can deduce that indeed

$$\phi \text{ is DG-bilinear} \iff f \in \mathrm{Hom}_{\mathbf{DG}}(V, \mathrm{Hom}^*(W,Z)),$$

and with that, we've managed to prove that the bijection from the start restricts to

$$\{\phi : V \times W \to Z \mid \phi \text{ is DG-bilinear}\} \cong \mathrm{Hom}_{\mathbf{DG}}(V, \mathrm{Hom}^*(W,Z)).$$

Finally, to get precisely what is written in the statement of the theorem, we just have to apply the universal property of the tensor product, according to which a DG-bilinear map $\phi : V \times W \to Z$ is equivalent to a DG-linear map $\overline{\phi} : V \otimes W \to Z$, $\overline{\phi}(v \otimes w) = \phi(v,w)$. Explicitly, we have:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{DG}}(V \otimes W, Z) &\cong \mathrm{Hom}_{\mathbf{DG}}(V, \mathrm{Hom}^*(W,Z)) \\ \overline{\phi} &\mapsto (v \mapsto w \mapsto \overline{\phi}(v \otimes w)) \\ ((v \otimes w) \mapsto f(v)(w)) &\leftarrow f\end{aligned}$$

Let us denote the bijection above $\eta(V,W,Z)$. To prove that η is natural, we regard both sides as functors $F, G : \mathbf{DG}^{\mathrm{op}} \times \mathbf{DG}^{\mathrm{op}} \times \mathbf{DG} \to \mathbf{Set}$, defined in the obvious way:

$$\begin{gathered}F(V,W,Z) = \mathrm{Hom}_{\mathbf{DG}}(V \otimes W, Z) \\ F(h_V, h_W, h_Z) = \overline{\phi} \mapsto ((v \otimes w) \mapsto h_Z(\overline{\phi}(h_V(v) \otimes h_W(w)))),\end{gathered}$$

$$\begin{gathered}G(V,W,Z) = \mathrm{Hom}_{\mathbf{DG}}(V, \mathrm{Hom}^*(W,Z)) \\ G(h_V, h_W, h_Z) = f \mapsto (v \mapsto w \mapsto h_Z(f(h_V(v))(h_W(w))))\end{gathered}$$

The naturality condition then states that, for all

$$V, W, Z, \tilde{V}, \tilde{W}, \tilde{Z} \in \mathbf{DG}, h_V \in \mathrm{Hom}_{\mathbf{DG}}(\tilde{V}, V), h_W \in \mathrm{Hom}_{\mathbf{DG}}(\tilde{W}, W), h_Z \in \mathrm{Hom}_{\mathbf{DG}}(Z, \tilde{Z}),$$

the following diagram should commute:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{DG}}(V \otimes W, Z) & \xrightarrow{\eta(V,W,Z)} & \mathrm{Hom}_{\mathbf{DG}}(V, \mathrm{Hom}^*(W, Z)) \\ {\scriptstyle F(h_V,h_W,h_Z)}\downarrow & & \downarrow{\scriptstyle G(h_V,h_W,h_Z)} \\ \mathrm{Hom}_{\mathbf{DG}}(\tilde{V} \otimes \tilde{W}, \tilde{Z}) & \xrightarrow[\eta(\tilde{V},\tilde{W},\tilde{Z})]{} & \mathrm{Hom}_{\mathbf{DG}}(\tilde{V}, \mathrm{Hom}^*(\tilde{W}, \tilde{Z})) \end{array}$$

But, if one writes down the maps explicitly, this is totally obvious:

$$\begin{array}{ccc} \overline{\phi} & \xmapsto{\eta(V,W,Z)} & ((v \otimes w) \mapsto h_Z(\overline{\phi}(h_V(v) \otimes h_W(w)))) \\ {\scriptstyle F(h_V,h_W,h_Z)}\downarrow & & \downarrow{\scriptstyle G(h_V,h_W,h_Z)} \\ (v \mapsto w \mapsto \overline{\phi}(v \otimes w)) & \xmapsto[\eta(\tilde{V},\tilde{W},\tilde{Z})]{} & (v \mapsto w \mapsto h_Z(\overline{\phi}(h_V(v) \otimes h_W(w)))) \end{array}$$

□

Another perspective on the definition of the Hom complex is the following: if $f \in \mathrm{Hom}^0(V, W)$, f is a morphism of $\mathbb{Z}$-graded vector spaces, and from the definition of the differential we have that $f \circ d = d \circ f$ if and only if $d(f) = 0$; therefore, in a sense, $d(f)$ measures how f "fails to commute" with the differentials. In particular, if $f \in \mathrm{Hom}^0(V, W)$ and $d(f) = 0$ we have that f is a morphism of DG-vector spaces, and so it induces a corresponding morphism in cohomology. In fact, this is something we can generalize.

Remark 1.35. Let V, W be DG-vector spaces. If $f \in \mathrm{Hom}^n(V, W)$ is a homogeneous cocycle of degree n, we have

$$d(f) = 0 \implies f \circ d = (-1)^n d \circ f,$$

and we can see that in particular, for all $v \in V$:

- $d(v) = 0 \implies d(f(v)) = 0$
- $f(d(v)) = (-1)^n d(f(v))$

That is, f sends cocycles to cocycles, and coboundaries to coboundaries; therefore, f induces a map in cohomology $[f] \in \mathrm{Hom}^*(H^*(V), H^*(W))$. This construction extends by linearity to any $f \in \mathrm{Hom}^*(V, W)$, and that gives a morphism

$$Z^*(\mathrm{Hom}^*(V, W)) \to \mathrm{Hom}^*(H^*(V), H^*(W)).$$

Moreover, if $f = dg$ is a coboundary in $\mathrm{Hom}^*(V, W)$ we can see that $[f] = 0$ because

$$[f] = [d(g)] = [d \circ g] - (-1)^n [g \circ d] = [d] \circ [g] - (-1)^n [g] \circ [d] = 0 \circ [g] - (-1)^n [g] \circ 0 = 0.$$

Hence, the morphism from above factors through the cohomology of $\mathrm{Hom}^*(V, W)$, to give

$$H^*(\mathrm{Hom}^*(V, W)) \to \mathrm{Hom}^*(H^*(V), H^*(W)). \tag{1.5}$$

For later use we point that, just as DG-linear maps factor though cohomology, so do DG-bilinear maps.

Remark 1.36. If $\phi : V \times W \to Z$ is a DG-bilinear map, then it induces a DG-bilinear map in cohomology:

$$H^*(V) \times H^*(W) \to H^*(Z)$$
$$([v], [w]) \mapsto [\phi(v, w)]$$

In fact, by the graded Leibniz rule we have for all $v \in V$, $w \in W$:

- $d(v) = 0, d(w) = 0 \implies d(\phi(v, w)) = 0$
- $d(w) = 0 \implies \phi(d(v), w) = d(\phi(v, w))$
- $d(v) = 0 \implies \phi(v, d(w)) = d(\phi(v, w))$

This means that ϕ restricts to cocycles, and then it is compatible with the quotient by coboundaries. Hence, the map from above is indeed well defined, and clearly DG-bilinear.

By the above remark, we have in particular that the tensor product $\otimes : V \times W \to V \otimes W$ induces a morphism

$$H^*(V) \otimes H^*(W) \to H^*(V \otimes W). \tag{1.6}$$

We want to show that both maps (1.5) and (1.6) are in fact isomorphisms.

Lemma 1.37. *Let V be a DG-vector space. Then, V is acyclic if and only if the identity is a coboundary in* $\mathrm{Hom}^*(V, V)$.

PROOF. If $\mathrm{id}_V \in \mathrm{Hom}^*(V, V)$ is a coboundary, then according to Remark 1.35 it induces the trivial map in cohomology $H^*(\mathrm{id}_V) = 0$. On the other hand, by the functoriality of H^* we have $H^*(\mathrm{id}_V) = \mathrm{id}_{H^*(V)}$. It follows $H^*(V) = 0$.

Suppose now that V is acyclic, that is $Z^*(V) = B^*(V)$. Let us choose some complements C^n:

$$V^n = Z^n(V) \oplus C^n.$$

Since $C = \bigoplus_{n\in\mathbb{Z}} C^n$ is a complement of $\ker d$, we have that $d : C \to B^*(V) = Z^*(V)$ is an isomorphism, and we may consider its inverse $d^{-1} : Z^*(V) \to C$. With that, we can define a map

$$h \in \mathrm{Hom}^{-1}(V, V), \quad h(z + c) = d^{-1}(z),$$

and we can easily see that $d(h) = \mathrm{id}_V$:

$$d(h)(z + c) = d(h(z + c)) + h(d(z + c)) = d(d^{-1}(z)) + d^{-1}(d(c)) = z + c.$$

□

Theorem 1.38 (Künneth's formulas)**.** *Let V, W be DG-vector spaces. Then, we have two natural isomorphisms*

$$H^*(V \otimes W) \cong H^*(V) \otimes H^*(W),$$
$$H^*(\mathrm{Hom}^*(V, W)) \cong \mathrm{Hom}^*(H^*(V), H^*(W)).$$

PROOF. Let use consider the case of the tensor product; the proof for the Hom complex is similar and can be found in [**17**, Theorem 5.1.6].

We start by showing that, if either V or W are acyclic, then so is $V \otimes W$. If V is acyclic, then by Lemma 1.37 the identity is a cocycle in $\mathrm{Hom}^*(V, W)$:

$$\exists h \in \mathrm{Hom}^{-1}(V, V) \quad d(h)(v) = d(h(v)) + h(d(v)) = v \tag{1.7}$$

But then, letting $\tilde{h} = h \otimes \mathrm{id}_W \in \mathrm{Hom}^{-1}(V \otimes W, V \otimes W)$ we have

$$\begin{aligned} d(\tilde{h})(v \otimes w) &= d(\tilde{h}(v \otimes w)) + \tilde{h}(d(v \otimes w)) \\ &= d(h(v) \otimes w) + \tilde{h}(d(v) \otimes w + (-1)^{\overline{v}} v \otimes d(w)) \\ &= d(h(v)) \otimes w + (-1)^{\overline{v}-1} h(v) \otimes d(w) + h(d(v)) \otimes w + (-1)^{\overline{v}} h(v) \otimes d(w) \\ &= (d(h(v)) + h(d(v))) \otimes w \\ &\overset{(1.7)}{=} v \otimes w, \end{aligned}$$

so $\mathrm{id}_{V\otimes W}$ is the coboundary of $\tilde{h}$, and hence $V \otimes W$ is acyclic.

For the general case, we apply Proposition 1.17 to decompose V and W as

$$V \cong H^*(V) \oplus A, \quad W \cong H^*(W) \oplus B,$$

where A and B are acyclic, so that we can write

$$V \otimes W \cong (H^*(V) \oplus A) \otimes (H^*(W) \oplus B) \cong (H^*(V) \otimes H^*(W)) \oplus (H^*(V) \otimes B) \oplus (A \otimes H^*(W)) \oplus (A \otimes B).$$

By the previous considerations, all terms on the right hand side are acyclic except for the first one, and it follows $H^*(V \otimes W) \cong H^*(H^*(V) \otimes H^*(W)) \cong H^*(V) \otimes H^*(W)$. Moreover, by carefully going through the chain of isomorphisms one can check that the underlying map is indeed (1.6). □

We now introduce some important constructions from homological algebra: suspensions and mapping (co)cones. The original motivation behind these definitions comes from algebraic topology, where they emerge naturally as (co)homology complexes associated to topological spaces of interest. To us, they will be useful tools because of their purely algebraic properties.

Definition 1.39. Let V be a DG-vector space. For each integer $p \in \mathbb{Z}$, we define $V[p]$ as follows:

$$V[p] = \bigoplus_{n \in \mathbb{Z}} V^{n+p}, \quad d_{V[p]} = (-1)^p d_V.$$

We call $V[-p]$ the **p-fold suspension** of V. The special case $V[-1]$ is simply called the **suspension** of V, while $V[1]$ is sometimes referred to as the **desuspension** of V. As a $\mathbb{Z}$-graded vector space, the suspension $V[-p]$ is just V with grading shifted to the right by p. In particular, the tautological map $s^p : V \to V[-p]$ is a homogeneous element of degree p in $\mathrm{Hom}^*(V, V[-p])$, and the sign in the definition of $d_{V[-p]}$ implies that it is a cocycle.

In the *Eilenberg-Moore formalism*, the elements of $V[-p]$ are seen as images of elements of V via the map s^p, which is always written explicitly; that is, we denote:

$$V[-p] = \{s^p v \mid v \in V\}.$$

This notation reminds us that passing a vector from V to $V[-p]$ involves a map of degree p, which helps to get the signs right when the Koszul signs convention is applied.

Definition 1.40. Let $f : V \to W$ be a morphism of DG-vector spaces. We define the **mapping cone** of f as follows:

$$\mathrm{cone}(f) = V[1] \oplus W, \quad d(s^{-1}v + w) = -s^{-1}d(v) + f(v) + d(w)$$

Informally, the definition of the differential of $\mathrm{cone}(f)$ is obtained by expanding symbolically $d(s^{-1}v + w) = d(s^{-1})(v) - s^{-1}d(v) + d(w)$ and setting $d(s^{-1}) = f$. As a $\mathbb{Z}$-graded vector space, $\mathrm{cone}(f)$ coincides with the direct sum $V[1] \oplus W$, but they are *not* equal as DG-vector spaces since their differentials are clearly not the same (in general). One important difference between $\mathrm{cone}(f)$ and the direct sum is that the inclusion $V[1] \to \mathrm{cone}(f)$ and the projection $\mathrm{cone}(f) \to W$ are *not* morphisms of DG-vector spaces. On the contrary, the inclusion $W \to \mathrm{cone}(f)$ and the projection $\mathrm{cone}(f) \to V[1]$ are indeed morphisms of DG-vector spaces, and they give rise to a short exact sequence of complexes:

$$0 \longrightarrow W \longrightarrow \mathrm{cone}(f) \longrightarrow V[1] \longrightarrow 0$$

which induces a long exact sequence in cohomology:

$$\cdots \to H^i(W) \to H^i(\mathrm{cone}(f)) \to H^{i+1}(V) \xrightarrow{\delta_i} H^{i+1}(W) \to H^{i+1}(\mathrm{cone}(f)) \to H^{i+2}(V) \to \cdots$$

Because the space V is "desuspended" in the sequence, we can see that the connecting morphisms $\delta_i : H^{i+1}(V) \to H^{i+1}(W)$ have the same source and target as the morphisms induced in cohomology by f. Indeed, by working through the costruction in the snake lemma one can check that the connecting morphisms are precisely $\delta_i = H^{i+1}(f)$. With that, and by the exactness of the long sequence above, we have that f is a quasi-isomorphism if and only if $H^*(\mathrm{cone}(f))$ is trivial, that is $\mathrm{cone}(f)$ is acyclic. This gives an interpretation of the mapping cone as a "detector" of quasi-isomorphisms.

The construction of cone(f) is functorial in f, in the sense that every commutative square

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[d, "f"] & E \arrow[d, "g"] \\ W \arrow[r, "\beta"] & F \end{tikzcd}$$

induces a morphism

$$\phi : \mathrm{cone}(f) \to \mathrm{cone}(g), \quad \phi(s^{-1}v + w) = s^{-1}\alpha(v) + \beta(w),$$

which is the unique one such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & W \arrow[r] \arrow[d, "\alpha"] & \mathrm{cone}(f) \arrow[r] \arrow[d, "\phi"] & V[1] \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & F \arrow[r] & \mathrm{cone}(g) \arrow[r] & E[1] \arrow[r] & 0 \end{tikzcd}$$

The above is a morphism of short exact sequences, and as such it induces a morphism of the associated long exact sequences in cohomology. By the five lemma applied to the long sequence, we see that if α and β are quasi-isomorphisms then so is ϕ.

Definition 1.41. Let $f : V \to W$ be a morphism of DG-vector spaces. We define the **mapping cocone** of f as follows:

$$C(f) = V \times W[-1], \quad d(v, sw) = (d(v), s(f(v) - d(w))).$$

The mapping cone and the mapping cocone are dual to each other, and share many formal properties. In particular, one can check that everything we have said about the mapping cone applies just as well to the mapping cocone, with obvious modifications. One can even observe that, via the natural isomorphism $(V \times W[-1])[1] \cong V[1] \oplus W$, the mapping cocone of f coincides with the suspension of the mapping cone of $-f$.

2. DG-commutative algebras

Definition 1.42. A **DG-algebra** A (short for differential graded commutative algebra) is the data of a DG-vector space (A, d) and a bilinear map

$$A \times A \to A$$
$$(a, b) \mapsto ab$$

called **product**, such that:

(1) $\overline{ab} = \overline{a} + \overline{b}$
(2) $d(ab) = d(a)b + (-1)^{\overline{a}} a d(b)$ (graded Leibniz rule)
(3) $ab = (-1)^{\overline{a}\,\overline{b}} ba$ (graded commutativity)
(4) $(ab)c = a(bc)$ (associativity)

In the above definition, properties (1) and (2) mean that the product operation in a DG-algebra A is a DG-bilinear map, property (3) then states that the corresponding DG-linear map $A \otimes A \to A$ commutes with the twisting involution **tw**.

Remark 1.43. Because we are working with fields of characteristic 0, by property (3) we have in particular

$$\overline{a} \text{ is odd} \implies a^2 = 0.$$

If one works with fields of non-zero characteristic, this is added to the definition as a separate axiom.

Remark 1.44. Of course, we have a category of **DGA** of DG-algebras, where morphisms are DG-linear maps that preserve the product.

Remark 1.45. Any DG-vector space can be seen as a DG-algebra with trivial product (the product of any two elements is zero). Similarly, any commutative associative $\mathbb{K}$-algebra can be seen as DG-algebra with trivial differential, concentrated in degree 0; in particular, the field $\mathbb{K}$ itself is a DG-algebra with trivial differential.

Definition 1.46. Let A be a DG-algebra. We say that A is **unitary** if there exists an element $1 \in A^0$ such that

$$\forall a \in A \quad 1a = a1 = a.$$

If A is a unitary DG-algebra, then $\mathbb{K}$ embeds into A as Span$\{1\}$. Indeed, the map $k \mapsto k1$ is a morphism of DG-algebras since we have

$$d(1) = d(1 \cdot 1) = d(1)1 + 1d(1) = 2d(1) \implies d(1) = 0,$$

from which it follows $\forall k \in \mathbb{K}\ d(k1) = 0$, and in particular $[1] \in H^*(A)$.

Definition 1.47. Let A be a DG-algebra. We say that A is **nilpotent** if there exists a positive integer n such that every product of n elements in A is 0.

The graded commutativity property implies that, when we multiply together homogeneous vectors, depending on their degree the product may behave either as a commutative or as an anticommutative operation.
The prototipical example of an operation of this kind is the wedge product in the exterior algebra over a vector space; fixed a vector space V, the exterior algebra $\Lambda(V)$ is seen as a $\mathbb{Z}$-graded vector space as follows:

$$\Lambda(V) = \bigoplus_{n \in \mathbb{N}} \Lambda^n(V), \quad \Lambda^n(V) = \text{Span}\{v_1 \wedge \cdots \wedge v_n \mid v_1, \ldots, v_n \in V\}.$$

The wegde product $\wedge$ is bilinear, associative, and satisfies $\overline{a \wedge b} = \overline{a} \wedge \overline{b}$. Furthermore, $\wedge$ is anticommutative on vectors: $\forall v, w \in V\ v \wedge w = -w \wedge v$; more generally, given two homogeneous elements $a = v_1 \wedge \cdots \wedge v_n$, $b = w_1 \wedge \cdots \wedge w_m$, by repeatedly applying the previous property we get

$$a \wedge b = (v_1 \wedge \cdots \wedge v_n) \wedge (w_1 \wedge \cdots \wedge w_m) = (-1)^{nm} (w_1 \wedge \cdots \wedge w_m) \wedge (v_1 \wedge \cdots \wedge v_n) = (-1)^{\overline{a}\,\overline{b}}\, b \wedge a.$$

This shows that the wedge product is indeed graded commutative.

The exterior algebra by itself is *not* a DG-algebra, because it lacks a differential d. If the reader is familiar with differential geometry, the exterior derivative may come to mind, and that leads to the following example.

Example 1.48. The de Rham complex of a smooth manifold, equipped with the wedge product, is a DG-algebra.

Related to the previous example, but of a more purely algebraic flavor, we also have the following.

Example 1.49. The de Rham complex of algebraic differential forms on the affine line is isomorphic to $\mathbb{K}[t, dt]$ with the relation $dt^2 = 0$, considered as a DG-vector space concentrated in degrees 0, 1:

$$\mathbb{K}[t, dt] = \mathbb{K}[t] \oplus \mathbb{K}[t]dt, \quad d(p(t) + q(t)dt) = p'(t)dt.$$

One can verify that this the unique DG-algebra structure on $\mathbb{K}[t, dt]$ such that $\overline{t} = 0$, $\overline{dt} = 1$, $d(t) = dt$, and the product is the usual one. Thus defined, $\mathbb{K}[t, dt]$ is a unitary DG-algebra, with a nilpotent subalgebra $\mathbb{K}[t]dt$. One can easily check that, for all $s \in \mathbb{K}$, the evaluation map at s is a morphism of DG-algebras:

$$e_s : \mathbb{K}[t, dt] \to \mathbb{K}$$
$$e_s(p(t) + q(t)dt) = p(s)$$

One can define quasi-isomorphisms of DG-algebras analogously to DG-vector spaces.

Definition 1.50. A quasi-isomorphism of DG-algebras is a quasi-isomorphism of DG-vector space which is also a morphism of DG-algebras.

Historically, the study of quasi-isomorphisms of DG-algebras has been really important. An outstanding example is the proof given in 1975 by Deligne, Griffiths, Morgan, and Sullivan [**4**] that the homotopy type of a compact Kähler manifold, over the real numbers, is a formal consequence of the real cohomology ring. Nonetheless, for the goals we have in mind it makes sense to focus our attention on the case of DG-Lie algebras, which will be treated in Section 3. It can also be observed that some of the properties we will prove there apply just as well to the case of DG-algebras (e.g. Theorem 1.65).

The following proposition about DG-algebras will be instrumental to the study of DG-Lie algebras.

Lemma 1.51. *For every $s \in \mathbb{K}$, the inclusion map $\iota : \mathbb{K} \to \mathbb{K}[t, dt]$ and the evaluation map $e_s : \mathbb{K}[t, dt] \to \mathbb{K}$ are quasi-isomorphisms of DG-algebras.*

Proof. Since $e_s \circ \iota = \mathrm{id}$, it suffices to show that ι is a quasi-isomorphism. That amounts to proving that each $[p(t) + q(t)dt] \in H^*(\mathbb{K}[t, dt])$ can be represented as $[a]$ for a unique $a \in \mathbb{K}$. If $[p(t) + q(t)dt] \in H^*(\mathbb{K}[t, dt])$, we have $d(p(t) + q(t)dt) = p'(t)dt = 0$ and hence $p(t) = a$ for some constant $a \in \mathbb{K}$; moreover, $q(t)dt$ is clearly[1] the coboundary of $\int_0^s q(s)ds$, and so we have $[p(t) + q(t)dt] = [a]$. The uniqueness of the constant a follows from the fact that every coboundary in $\mathbb{K}[t, dt]$ is of the form $q(t)dt$, so the only coboundary that is constant is 0. □

Given a DG-algebra A, by Remark 1.36 the product on A induces a product in cohomology:

$$\begin{aligned} H^*(A) \times H^*(A) &\to H^*(A) \\ ([a], [b]) &\mapsto [ab] \end{aligned}$$

With that, $H^*(A)$ is also a DG-algebra. Moreover, if A is unitary or nilpotent then so is $H^*(A)$. We can also immediately see that a unitary DG-algebra A is acyclic if and only if the identity $1 \in A$ is a coboundary.

Fixed two DG-algebras A, B, the tensor product $A \otimes B$ has a natural DG-algebra structure; the product on $A \otimes B$ is defined in the obvious way, but we have to pay attention to the Koszul signs convention:

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\overline{b_1}\,\overline{a_2}}(a_1 a_2 \otimes b_1 b_2).$$

3. DG-Lie algebras

Definition 1.52. A **DG-Lie algebra** L (short for differential graded Lie algebra) is the data of a DG-vector space (L, d) and a bilinear map

$$\begin{aligned} L \times L &\to L \\ (a, b) &\mapsto [a, b] \end{aligned}$$

called **bracket**, such that:

(1) $\overline{[a, b]} = \overline{a} + \overline{b}$
(2) $d([a, b]) = [d(a), b] + (-1)^{\overline{a}}[a, d(b)]$ (graded Leibniz rule)
(3) $[a, b] = -(-1)^{\overline{a}\,\overline{b}}[b, a]$ (graded anticommutativity)
(4) $[a, [b, c]] = [[a, b], c] + (-1)^{\overline{a}\,\overline{b}}[b, [a, c]]$ (graded Jacobi identity)

Just as with DG-algebras, properties (1) and (2) mean that the bracket in a DG-Lie algebra L is a DG-bilinear map, property (3) then states that the corresponding DG-linear map $L \otimes L \to L$ commutes with the *negative* of the twisting involution $-\mathbf{tw}$.

[1]This is the first instance where we use the assumption that $\mathbb{K}$ has characteristic 0, which is implicit in the fact that the formal antiderivative $\int_0^s q(s)ds$ is well defined.

Remark 1.53. Because we are working with fields of characteristic 0, by property (3) we have in particular

$$\overline{a} \text{ is even} \implies [a,a]=0.$$

Similarly, by properties (3) and (4) we have

$$\overline{a} \text{ is odd} \implies [a,[a,a]]=0.$$

(In fact, this extends to any a by the previous observation)
If one works with fields of non-zero characteristic, these properties are added to the definition as separate axioms.

Remark 1.54. Of course, we have a category of **DGLA** of DG-Lie algebras, where morphisms are DG-linear maps that preserve the bracket.

Remark 1.55. Any DG-vector space can be seen as a DG-Lie algebra with trivial bracket (the bracket of any two elements is zero); a DG-Lie algebra of this kind is called **abelian**. Similarly, any Lie algebra over $\mathbb{K}$ can be seen as DG-Lie algebra with trivial differential, concentrated in degree 0.

Example 1.56. Fixed a DG-vector space V, the Hom complex $\operatorname{Hom}^*(V,V)$ has a natural structure of DG-Lie algebra, where the bracket is the **graded commutator**:

$$[f,g]=f\circ g-(-1)^{\overline{f}\,\overline{g}}g\circ f.$$

Example 1.57. Let V be a DG-vector space equipped with a DG-bilinear map $\phi : V\times V\to V$. We define the DG-Lie algebra of **derivations** on V as the DG-Lie subalgebra of $\operatorname{Hom}^*(V,V)$ given by:

$$\operatorname{Der}^*(V,V)=\bigoplus_{n\in\mathbb{Z}}\operatorname{Der}^n(V,V),$$
$$\operatorname{Der}^n(V,V)=\{\delta\in\operatorname{Hom}^n(V,V)\mid \forall a,b\quad \delta(\phi(a,b))=\phi(\delta(a),b)+(-1)^{n\overline{a}}\phi(a,\delta(b))\}.$$

(One can check that the graded commutator of two derivations is again a derivation)
Observe that, since ϕ is DG-bilinear, the differential d of V is always a homogeneous derivation of degree 1.
When V is a DG-algebra it is implicitly understood that ϕ is the product; similarly, when V is a DG-Lie algebra ϕ is assumed to be the bracket.
Let us consider a DG-Lie algebra L. In the language of derivations, the Jacobi identity can be restated as

$$\forall a\in L^n\quad [a,-]\in\operatorname{Der}^n(L,L).$$

Indeed, one can verify that the following is a morphism of DG-Lie algebras:

$$\operatorname{ad}: L\to\operatorname{Der}^*(L,L)$$
$$\operatorname{ad}a=[a,-]$$

For instance, the differential of $\operatorname{Hom}^*(V,V)$ is precisely $[d,-]$, where $d\in\operatorname{Hom}^1(V,V)$ is the differential of V and $[-,-]$ is the graded commutator.

Remark 1.58. Given two morphisms of DG-Lie algebras $f : L\to N$, $g : M\to N$, the fiber product $L\times_N M$ (as seen in Proposition 1.24) has a natural structure of DG-Lie algebra, with bracket

$$[(x,y),(z,w)]=([x,y],[z,w]).$$

It is easy to check that $L\times_N M$ satisfies the universal property of the fiber product in **DGLA**; in particular, for $N=0$ we get the categorical product $L\times M$. In the case of DG-Lie algebras, we will *not* refer to $L\times M$ as the "direct sum", because it does not satisfy the universal property of the coproduct.

Contrary to the case of DG-algebras, the tensor product of two DG-Lie algebras does not make a DG-Lie algebra in any natural way. Instead, something we can do is to tensor a DG-Lie algebra with a DG-algebra, forming a so called scalar extension.

Remark 1.59. If L is a DG-Lie algebra and A is a DG-algebra, then $L\otimes A$ has a natural DG-Lie algebra structure, with bracket:

$$[x\otimes a, y\otimes b]=(-1)^{\overline{a}\,\overline{y}}[x,y]\otimes ab.$$

In this context, the tensor product by and element of A is thought of as multiplication by an "extended scalar", and it is customary to omit the symbol $\otimes$ to simplify notation. For instance, the above equation may be rewritten as

$$[xa, yb]=(-1)^{\overline{a}\,\overline{y}}[x,y]ab.$$

Example 1.60. Consider the DG-algebra of algebraic differential forms over the affine line $\mathbb{K}[t,dt]$ (Example 1.49). For any DG-Lie algebra L, we define $L[t,dt]$ as the scalar extension

$$L[t,dt]=L\otimes\mathbb{K}[t,dt].$$

As a DG-vector space, $L[t,dt]$ is generated by the elements of the form $ap(t)$, $ap(t)dt$ where $a\in L$, $p(t)\in\mathbb{K}[t]$. By applying the definitions of tensor product and scalar extension, we get the following equations for the differential and the bracket of $L[t,dt]$:

$$d(ap(t))=d(a)p(t)+(-1)^{\overline{a}}ap'(t)dt,\quad d(ap(t)dt)=d(a)p(t)dt,$$
$$[ap(t),bq(t)]=[a,b]p(t)q(t),\quad [ap(t),bq(t)dt]=[a,b]p(t)q(t)dt,\quad [ap(t)dt,bq(t)dt]=0.$$

For any fixed $s\in\mathbb{K}$, we can define the evaluation morphism at s as a function with values in L:

$$e_s : L[t,dt]\to L$$
$$e_s(a(p(t)+q(t)dt))=p(s)a$$

We can clearly see that e_s is surjective; moreover, according to Lemma 1.51 $e_s:\mathbb{K}[t,dt]\to\mathbb{K}$ is a quasi-isomorphism, and since by Künneth's formulas we have

$$H^*(L[t,dt])\cong H^*(L)\otimes H^*(\mathbb{K}[t,dt])\cong H^*(L)\otimes H^*(\mathbb{K})\cong H^*(L),$$

we can see that $e_s : L[t,dt]\to L$ is a quasi-isomorphism as well.

Any morphism of DG-Lie algebras $f:L\to M$ induces a corresponding morphism $f\otimes \mathrm{id}_{\mathbb{K}[t,dt]}$: $L[t,dt]\to M[t,dt]$, which makes the mapping $L\mapsto L[t,dt]$ a functor. Then, for every $s\in\mathbb{K}$, it is easy to see that the evaluation morphism gives a natural transformation from $L\mapsto L[t,dt]$ to the identity functor:

$$\begin{tikzcd} L[t,dt] \arrow[r, "e_s"] \arrow[d, "f\otimes\mathrm{id}_{\mathbb{K}[t,dt]}"] & L \arrow[d, "f"] \\ M[t,dt] \arrow[r, "e_s"] & M \end{tikzcd}$$

Just as we have seen in the case of DG-algebras, the cohomology of a DG-Lie algebra is again a DG-Lie algebra, with the induced bracket as in Remark 1.36.

Definition 1.61. A **quasi-isomorphism of DG-Lie algebras** $f:L\to M$ is a morphism of DG-Lie algebras f which induces an isomorphism in cohomology, that is: $f:H^*(L)\to H^*(M)$ is an isomorphism.
Given two DG-Lie algebras L, M, we say that L and M are **quasi-isomorphic** if they are equivalent under the equivalence relation generated by quasi-isomorphisms. In other words, L and M are quasi-isomorphic if there exists a zigzag of finitely many quasi-isomorphisms:

$$\begin{tikzcd}[column sep=small] & X_1 \arrow[dl] \arrow[dr] & & \dots \arrow[dl] \arrow[dr] & & X_n \arrow[dl] \arrow[dr] & \\ L & & X_2 & & X_{n-1} & & M \end{tikzcd}$$

Hence, a quasi isomorphism of DG-Lie algebras is a quasi-isomorphism of DG-vector spaces which is also a morphism of DG-Lie algebras. Clearly, some of the properties we've seen in the case of DG-vector spaces extend immediately to DG-Lie algebras (e.g. Corollary 1.27), but not all of them[2]. In particular, it is not the case that every DG-Lie algebra is quasi-isomorphic to its cohomology, and then it makes sense to introduce the following definition.

Definition 1.62. A DG-Lie algebra L is called **formal** if it is quasi-isomorphic to $H^*(L)$.

Definition 1.63. A DG-Lie algebra L is called **homotopy abelian** if it is quasi-isomorphic to an abelian DG-Lie algebra.

Since a morphism of abelian DG-Lie algebras is equivalent to a morphism of DG-vector spaces, by Proposition 1.19 any abelian DG-Lie algebra is quasi-isomorphic to its cohomology, which of course is also abelian. It follows that a DG-Lie algebra is homotopy abelian it and only if it is formal and has abelian cohomology.

Several examples of DG-Lie algebras arising from Algebraic Geometry are known to be formal or homotopy abelian. For instance, the Kodaira-Spencer algebra of a complex smooth projective variety with trivial canonical bundle is homotopy abelian (this is the so called Bogomolov-Tian-Todorov Theorem, see e.g. [**9, 10**]). On the other hand, the (homotopy class of the) DG-Lie algebra of a coherent semistable sheaf on a K3 surface is formal, but not homotopy abelian in general (this result has been named the Kaledin-Lehn formality conjecture, see e.g. [**1, 2**]).

We can characterize the quasi-isomorphism relation of DG-Lie algebras in a way which is more workable than the zigzag in Definition 1.61. This goes through the following lemma.

Lemma 1.64 (Factorization lemma)**.** *Let $f : L \to M$ be a morphism of DG-Lie algebras. Then, there exists a factorization of morphisms of DG-Lie algebras $f = g \circ i$, where g is surjective and i is the right inverse of a surjective quasi-isomorphism.*

Proof. Consider the pullback square:

$$\begin{tikzcd}
L \times_M M[t,dt] \arrow[r] \arrow[d, "\pi_L"'] & M[t,dt] \arrow[d, "e_1"] \\
L \arrow[r, "f"] & M
\end{tikzcd}$$

By the discussion in Example 1.60, e_1 is a surjective quasi-isomorphism, and by Corollary 1.27 so is its pullback π_L. We define i so that it is a right inverse to π_L:

$$i : L \to L \times_M M[t,dt], \quad i(l) = (l, f(l)).$$

Then, we define g in such a way that $f = g \circ i$:

$$g : L \times_M M[t,dt] \to M, \quad g(l, m(t)) = m(0).$$

To conclude, we can easily see that g is surjective, as for all $m \in M$ we have

$$(0, m(1-t)) \in L \times_M M[t,dt], \quad g(0, m(1-t)) = m.$$

□

Theorem 1.65. *Let L, M be DG-Lie algebras. Then, L and M are quasi-isomorphic if and only if there exists a triangle of surjective quasi-isomorphisms:*

$$\begin{tikzcd}
& K \arrow[dl] \arrow[dr] & \\
L & & M
\end{tikzcd}$$

[2] Some strong properties we have proven about DG-vector spaces in Section 1 fundamentally rely on the decomposition into minimal and acyclic subspaces, as per Proposition 1.17. That decomposition is constructed by choosing complements of vector spaces, which may fail to be closed under the Lie bracket, and that makes it unapplicable in this context.

PROOF. Let us write $L \sim M$ if there exists a triangle as in the statement of the theorem. We have to show that $\sim$ is an equivalence relation, and that it is generated by quasi-isomorphisms.

For the first part, we can clearly see that $\sim$ is reflexive and symmetric. To show that it is transitive, observe that if $L \sim M$ and $M \sim N$ we have a pair of consecutive triangles

$$\begin{tikzcd} & K \arrow[dl, two heads] \arrow[dr, two heads] & & H \arrow[dl, two heads] \arrow[dr, two heads] & \\ L & & M & & N \end{tikzcd}$$

which can be extended with a pullback square at the top, forming a larger triangle

$$\begin{tikzcd} & & K \times_M H \arrow[dl, two heads] \arrow[dr, two heads] & & \\ & K \arrow[dl, two heads] \arrow[dr, two heads] & & H \arrow[dl, two heads] \arrow[dr, two heads] & \\ L & & M & & N \end{tikzcd}$$

By Corollary 1.27, since the arrows at the bottom are surjective quasi-isomorphisms, so are the arrows at the top, and in particular the composite arrows at the sides of the larger triangle. It follows that $L \sim N$.

It remains to show that $\sim$ is generated by quasi-isomorphisms. If $f : L \to M$ is a quasi-isomorphism, then by Lemma 1.64 we have a triangle

$$\begin{tikzcd} & K \arrow[dl, two heads, "i^{-1}"'] \arrow[dr, two heads, "g"] & \\ L \arrow[rr, "f"] & & M \end{tikzcd}$$

where i^{-1} is a surjective quasi-isomorphism and g is surjective; because i and f are quasi-isomorphisms, so is g, and it follows $L \sim M$. $\square$

One application of the above characterization is to show the previously mentioned fact that not all DG-Lie algebras are quasi-isomorphic to their cohomology. For an explicit counterexample, we refer to [**17**, Example 6.2.5]. It is worth noting that the cited example also shows that a DG-Lie algebra whose bracket is trivial in cohomology is not necessarily homotopy abelian.

CHAPTER 2

Deformation functors

1. Functors of Artin rings

In what follows, we adopt the following conventions: all rings are assumed to be commutative with identity, and if R is a local ring then its maximal ideal is denoted $\mathfrak{m}_R$.

Definition 2.1. Let R be a local ring . We denote by $\mathcal{C}_R$ the category of local R-algebras with residue field $R/\mathfrak{m}_R$. An object in $\mathcal{C}_R$ is a local ring S equipped with a local morphism $\varphi_S : R \to S$, such that φ_S induces an isomorphism of residue fields:

$$\varphi_S(\mathfrak{m}_R) \subseteq \mathfrak{m}_S, \qquad \begin{tikzcd} R \arrow[r, "\varphi_S"] \arrow[d] & S \arrow[d] \\ \frac{R}{\mathfrak{m}_R} \arrow[r, "\sim"] & \frac{S}{\mathfrak{m}_S} \end{tikzcd}$$

A morphism in $\mathcal{C}_R$ from S to T is a local morphism of R-algebras $f : S \to T$:

$$f(\mathfrak{m}_S) \subseteq \mathfrak{m}_T, \qquad \begin{tikzcd} & R \arrow[dl, "\varphi_S"'] \arrow[dr, "\varphi_T"] & \\ S \arrow[rr, "f"] & & T \end{tikzcd}$$

If S is an object in $\mathcal{C}_R$, then we can identify the residue field of S with that of R through the isomorphism induced by φ_S; in this sense, $\mathcal{C}_R$ is the category of "local R-algebras with residue field $R/\mathfrak{m}_R$". In particular, it makes sense to see the canonical projections $S \to S/\mathfrak{m}_S$ for all S as functions onto the same field $R/\mathfrak{m}_R$.

Proposition 2.2. *For an object S in $\mathcal{C}_R$, denote by $\pi_S : S \to R/\mathfrak{m}_R$ the canonical projection onto the residue field, up to the identification $S/\mathfrak{m}_S \cong R/\mathfrak{m}_R$.*

If $f : S \to T$ is a morphism in $\mathcal{C}_R$, the following diagram commutes:

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "\pi_S"'] & & T \arrow[dl, "\pi_T"] \\ & \frac{R}{\mathfrak{m}_R} & \end{tikzcd}$$

PROOF. For this proof, let us use the following notation: if A/I is a quotient ring, denote the canonical projection $p_A : A \to A/I$, and if $f : A \to B$ is a morphism of rings which factors though the quotient, denote the induced morphism $[f]_A : A/I \to B$. By the universal property of the quotient, we have the following:

$$f = [f]_A \circ p_A, \tag{1.1}$$

$$p_A \text{ is an epimorphism.} \tag{1.2}$$

In the above notation, the canonical projection of S onto the residue field can be written as

$$\pi_S = [p_S \circ \varphi_S]_R^{-1} \circ p_S,$$

and we wish to prove that for every morphism $f : S \to T$ in $\mathcal{C}_R$ we have

$$\pi_S = \pi_T \circ f.$$

With all the notation in place, this is a relatively straightforward check:

$$
\begin{aligned}
&\pi_S = \pi_T \circ f \\
\iff\ & [p_S \circ \varphi_S]_R^{-1} \circ p_S = [p_T \circ \varphi_T]_R^{-1} \circ p_T \circ f \\
\overset{(1.1)}{\iff}\ & [p_S \circ \varphi_S]_R^{-1} \circ p_S = [p_T \circ \varphi_T]_R^{-1} \circ [p_T \circ f]_S \circ p_S \\
\overset{(1.2)}{\iff}\ & [p_S \circ \varphi_S]_R^{-1} = [p_T \circ \varphi_T]_R^{-1} \circ [p_T \circ f]_S \\
\iff\ & [p_T \circ \varphi_T]_R = [p_T \circ f]_S \circ [p_S \circ \varphi_S]_R \\
\overset{(1.2)}{\iff}\ & [p_T \circ \varphi_T]_R \circ p_R = [p_T \circ f]_S \circ [p_S \circ \varphi_S]_R \circ p_R \\
\overset{(1.1)}{\iff}\ & p_T \circ \varphi_T = [p_T \circ f]_S \circ p_S \circ \varphi_S \\
\overset{(1.1)}{\iff}\ & p_T \circ \varphi_T = p_T \circ f \circ \varphi_S \\
\iff\ & p_T \circ \varphi_T = p_T \circ \varphi_T,
\end{aligned}
$$

where in the last equivalence we've used the fact that f is a morphism of R-algebras. □

Hence, each object S in $\mathcal{C}_R$ comes with two maps: a local morphism $R \to S$ and a projection $S \to R/\mathfrak{m}_R$. Moreover, the morphisms in $\mathcal{C}_R$ are compatible with these maps, in the sense that every morphism $f : S \to T$ gives a commutative diagram:

$$
\begin{tikzcd}
 & R \arrow[dl, "\varphi_S"'] \arrow[dr, "\varphi_T"] & \\
S \arrow[rr, "f"] \arrow[dr, "\pi_S"'] & & T \arrow[dl, "\pi_T"] \\
 & \frac{R}{\mathfrak{m}_R} &
\end{tikzcd}
\tag{1.3}
$$

Remark 2.3. The local ring R, equipped with $\varphi_R = \mathrm{id}_R$, is an object in $\mathcal{C}_R$. By the upper triangle in Diagram (1.3) we see that R is the initial object in $\mathcal{C}_R$:

$$
\begin{tikzcd}
 & R \arrow[dl, "\mathrm{id}_R"'] \arrow[dr, "\varphi_S"] & \\
R \arrow[rr, "f"] & & S
\end{tikzcd}
\qquad \Longrightarrow \qquad f = \phi_S.
$$

The residue field $R/\mathfrak{m}_R$, equipped with $\varphi_R = \pi_R$, is also an object in $\mathcal{C}_R$. Its associated projection map is clearly $\pi_{R/\mathfrak{m}_R} = \mathrm{id}_{R/\mathfrak{m}_R}$, and then by the lower triangle in Diagram (1.3) we see that $R/\mathfrak{m}_R$ is the terminal object in $\mathcal{C}_R$:

$$
\begin{tikzcd}
S \arrow[rr, "f"] \arrow[dr, "\pi_S"'] & & \frac{R}{\mathfrak{m}_R} \arrow[dl, "\mathrm{id}_{R/\mathfrak{m}_R}"] \\
 & \frac{R}{\mathfrak{m}_R} &
\end{tikzcd}
\qquad \Longrightarrow \qquad f = \pi_S.
$$

For later use, we point out that the set-theoretic fiber product is also a fiber product in $\mathcal{C}_R$. Since by the above remark $\mathcal{C}_R$ has a terminal object, it will follow in particular that $\mathcal{C}_R$ has finite products.

Proposition 2.4. *Let $f : S \to U$, $g : T \to U$ be morphisms in $\mathcal{C}_R$. Define*

$$S \times_U T = \{(s,t) \in S \times T \mid f(s) = g(t)\}.$$

Then $S \times_U T$, equipped with the canonical projections, is a fiber product over f and g.

Proof. Let us show that $S \times_U T$ is a local ring with maximal ideal $\mathfrak{m}_S \times \mathfrak{m}_T$. It is clear that $S \times_U T$ is a ring with ideal $\mathfrak{m}_S \times \mathfrak{m}_T$, and then we just have to prove that $\mathfrak{m}_S \times \mathfrak{m}_T$ coincides with the set of non-units.

To that end, observe that for any A in $\mathcal{C}_R$:

$$a \text{ is a non-unit in } A \iff a \in \mathfrak{m}_A \iff \pi_A(a) = 0.$$

For all $(s,t) \in S \times_U T$, by the lower triangle in Diagram (1.3) we have

$$\pi_S(s) = \pi_U(f(s)) = \pi_U(g(t)) = \pi_T(t),$$

and that combined with the previous observation gives

$$s \text{ is a non-unit in } S \iff t \text{ is a non-unit in } T.$$

Now, for all $(s,t) \in S \times_U T$ we can clearly see that

$$(s,t) \text{ is a non-unit in } S \times_U T \iff s \text{ is a non-unit in } S \textbf{ or } t \text{ is a non-unit in } T,$$

and by the reasoning above, that is equivalent to

$$(s,t) \text{ is a non-unit in } S \times_U T \iff s \text{ is a non-unit in } S \textbf{ and } t \text{ is a non-unit in } T,$$

from which it follows that the set of non-units of $S \times_U T$ is indeed $\mathfrak{m}_S \times \mathfrak{m}_T$.

To conclude, we observe $S \times_U T$ equipped with $(\varphi_S, \varphi_T) : R \to S \times_U T$ is an object in $\mathcal{C}_R$, and then the universal property of the fiber product easily follows from that of the set-theoretic version. □

The case where $R = \mathbb{K}$ is a field will be the most important to us. In particular, we will be interested in those objects in $\mathcal{C}_\mathbb{K}$ which are Artin rings.

Definition 2.5. We define $\mathbf{Art}_\mathbb{K}$ as the full subcategory of Artin rings in $\mathcal{C}_\mathbb{K}$. In other words, $\mathbf{Art}_\mathbb{K}$ is the category of Artin local $\mathbb{K}$-algebras with residue field $\mathbb{K}$.

The intuition behind the definition is that, in some sense, an Artin local $\mathbb{K}$-algebra with residue field $\mathbb{K}$ is an "infinitesimal thickening" of $\mathbb{K}$. To understand this idea, it helps to keep in mind the following key examples.

Example 2.6. For a positive integer $n > 1$, consider the ring $\mathbb{K}[t]/(t^n)$, which is obtained from $\mathbb{K}$ by adjoining a nilpotent element t. With the intuition that t is "very small but not 0", $\mathbb{K}[t]/(t^n)$ can be thought of as an "infinitesimal extension" of $\mathbb{K}$: the higher the value of n, the higher the order of "infinitesimals" which exist in $\mathbb{K}[t]/(t^n)$.

More formally, we can observe that $\mathbb{K}[t]/(t^n)$ is a ring extension of $\mathbb{K}$, which as a $\mathbb{K}$-vector space splits as

$$\mathbb{K}[t]/(t^n) = \mathbb{K} \oplus (t),$$

and possesses the following properties:

- $\mathbb{K}[t]/(t^n)$ is a $\mathbb{K}$-algebra;
- $\mathbb{K}[t]/(t^n)$ is a local ring with residue field $\mathbb{K}$ and maximal ideal (t);
- $\mathbb{K}[t]/(t^n)$ is Noetherian, and its maximal ideal (t) is nilpotent.

It is a well known fact of commutative algebra that, for a local ring A, A is Artinian if and only if A is Noetherian and its maximal ideal $\mathfrak{m}_A$ is nilpotent. Hence, the above points mean that $\mathbb{K}[t]/(t^n)$ is an object in $\mathbf{Art}_\mathbb{K}$.

Importantly, we can observe that the properties discussed in the previous example are common to *all* objects in $\mathbf{Art}_\mathbb{K}$. In fact, given A in $\mathbf{Art}_\mathbb{K}$ we have a short exact sequence of $\mathbb{K}$-vector spaces:

$$0 \longrightarrow \mathfrak{m}_A \longrightarrow A \xrightarrow{\pi_A} \mathbb{K} \longrightarrow 0$$

Since $\varphi_A : \mathbb{K} \to A$ is a morphism in $\mathbf{Art}_\mathbb{K}$, by the lower triangle in Diagram (1.3) we have $\varphi_A \circ \pi_A = \pi_\mathbb{K} = \mathrm{id}_\mathbb{K}$, and by the splitting lemma it follows $A \cong \mathbb{K} \oplus \mathfrak{m}_A$; the inclusion $\mathbb{K} \to A$ is given by φ_A, which is a homomorphism of rings. Moreover, A is an Artin local $\mathbb{K}$-algebra with residue field $\mathbb{K}$, so in particular A is Noetherian and its maximal ideal $\mathfrak{m}_A$ is nilpotent.

In conclusion, every object A in $\mathbf{Art}_{\mathbb{K}}$ admits a decomposition as a $\mathbb{K}$-vector space

$$A \cong \mathbb{K} \oplus \mathfrak{m}_A \quad \text{where } \mathfrak{m}_A \text{ is nilpotent,}$$

and in this sense A can be seen as an "infinitesimal extension" of $\mathbb{K}$, where the "infinitesimals" lie in the maximal ideal $\mathfrak{m}_A$.

As a degenerate case of the above discussion, we see that the trivial extension $\mathbb{K}$ is an object in $\mathbf{Art}_{\mathbb{K}}$. Moreover, since $\mathbb{K} = \mathbb{K}/\mathfrak{m}_{\mathbb{K}}$, we have by Remark 2.3 that $\mathbb{K}$ is both the initial and terminal object in $\mathbf{Art}_{\mathbb{K}}$.

A useful characterization to keep in mind is that an object in $\mathcal{C}_{\mathbb{K}}$ belongs to $\mathbf{Art}_{\mathbb{K}}$ if and only if it is finite dimensional as a $\mathbb{K}$-vector space. For instance, this allows us to see immediately that if A, B and C are objects in $\mathbf{Art}_{\mathbb{K}}$ then so is $A \times_C B$. Another consequence of that fact is the following technical remark.

Remark 2.7. The category $\mathbf{Art}_{\mathbb{K}}$ is equivalent to a small category. In fact, for all A in $\mathbf{Art}_{\mathbb{K}}$ we may choose a basis $e_1, \dots, e_n$ of A as $\mathbb{K}$-vector space, and that gives a surjective morphism of $\mathbb{K}$-algebras $\mathbb{K}[x_1, \dots, x_n] \to A$. Hence, every object in $\mathbf{Art}_{\mathbb{K}}$ is isomorphic to some quotient algebra of $\mathbb{K}[\{x_n\}_{n\in\mathbb{N}}]$, but then it is clear that the collection of all possible quotients of $\mathbb{K}[\{x_n\}_{n\in\mathbb{N}}]$ is just a set.

Let us look again at the key example $\mathbb{K}[t]/(t^n)$. Observe that the ring $\mathbb{K}[t]/(t^n)$ can be constructed from $\mathbb{K}$ "by small steps", in the sense that we have a sequence of surjections:

$$\frac{\mathbb{K}[t]}{(t^n)} \longrightarrow \frac{\mathbb{K}[t]}{(t^{n-1})} \longrightarrow \cdots \longrightarrow \frac{\mathbb{K}[t]}{(t^2)} \longrightarrow \mathbb{K} \tag{1.4}$$

Intuitively, each of these extensions is "small" because it just adds one higher order of infinitesimals. Indeed, $\mathbb{K}[t]/(t^{k+1})$ extends $\mathbb{K}[t]/(t^k)$ by the ideal (t^k), and each element of (t^k) is an infinitesimal of "high order" because it gives 0 when multiplied with any other infinitesimal in $\mathbb{K}[t]/(t^{k+1})$. More formally, we have that the kernel of the projection $p : \mathbb{K}[t]/(t^{k+1}) \to \mathbb{K}[t]/(t^k)$ is annihilated by the maximal ideal:

$$(t)\ker p = (t^{k+1}) = 0 \quad \text{in } \mathbb{K}[t]/(t^{k+1}).$$

Moreover, each step in (1.4) can be seen as an extension of $\mathbb{K}$-vector spaces, and then it is also "small" (or rather, *principal*) in the sense that it increases the dimension by 1.

We want to generalize this idea to all objects in $\mathbf{Art}_{\mathbb{K}}$. This is useful in practice, for instance, because it gives us a setup for proofs by induction[1].

Definition 2.8. Let us consider a short exact sequence of the following form, where α is a morphism in $\mathbf{Art}_{\mathbb{K}}$ and ϕ is a morphism of B-modules:

$$0 \longrightarrow M \xrightarrow{\phi} B \xrightarrow{\alpha} A \longrightarrow 0$$

A sequence as above is called a **small extension** (and α is called a **small surjection**) if $\mathfrak{m}_B$ annihilates the kernel of α, that is: $\mathfrak{m}_B M = 0$.

Moreover, in the special case where $M = \mathbb{K}$ we call the sequence a **principal small extension**.

Now, consider any surjective morphism $\alpha : B \to A$ in $\mathbf{Art}_{\mathbb{K}}$, and let $M = \ker \alpha$. Since for some n we have $\mathfrak{m}_B^n = 0$, we can factor α as a sequence of surjections:

$$B \longrightarrow \frac{B}{\mathfrak{m}_B^{n-1} M} \longrightarrow \cdots \longrightarrow \frac{B}{\mathfrak{m}_B M} \longrightarrow A$$

In the above, the last arrow is the map induced by α on the quotient, and the rest are canonical projections; clearly, these are all small surjections.

Assume now that α is a small surjection, and let us choose a sequence of $\mathbb{K}$-vector spaces $0 \subset V_1 \subset \cdots \subset V_n \subset M$ such that $V_{k+1}/V_k \cong \mathbb{K}$. Each V_k is actually a B-module, since for all $b \in B$ and $v \in V_k$ we have

$$B = \mathbb{K} \oplus \mathfrak{m}_B \implies \exists k \in \mathbb{K}, m \in \mathfrak{m}_B \quad bv = kv + mv = kv \in V_k \quad \text{since } mv \in \mathfrak{m}_B M = 0.$$

[1] We should clarify that the importance of small extensions goes way beyond proofs by induction. For instance, they play a central role in the definition of *obstruction theories*, but that is a topic which will not be covered in this thesis.

With that, α can be factored as a sequence of surjections

$$B \longrightarrow \frac{B}{V_1} \longrightarrow \cdots \longrightarrow \frac{B}{V_n} \longrightarrow A$$

where each map in the sequence gives rise to a small extension with kernel $\frac{B}{V_k}/\frac{B}{V_{k+1}} \cong V_{k+1}/V_k \cong \mathbb{K}$, which is therefore principal.

In conclusion, any surjective morphism $\alpha : B \to A$ in $\mathbf{Art}_{\mathbb{K}}$ can be factored as a finite sequence of small surjections, each giving rise to a principal small extension. In particular, for any A in $\mathbf{Art}_{\mathbb{K}}$, by letting $\alpha = \pi_A : A \to \mathbb{K}$ we get a sequence analogous to (1.4).

Example 2.9. Let A be an object in $\mathbf{Art}_{\mathbb{K}}$, M be a finite dimensional $\mathbb{K}$-vector space. The **trivial extension** of A by M is given by:

$$0 \longrightarrow M \longrightarrow A \oplus M \longrightarrow A \longrightarrow 0$$

The product on $A \oplus M$ is defined as

$$(a, m)(b, n) = (ab, an + bm),$$

and the maximal ideal of $A \oplus M$ is $\mathfrak{m}_A \oplus M$.

Example 2.10. Consider the ring $\mathbb{K}[s,t]/(s^2,t^2,st)$. This is a local ring with maximal ideal (s,t), since we have $\mathbb{K}[s,t]/(s^2,t^2,st) = \mathbb{K} \oplus (s,t)$ where the elements in $\mathbb{K}$ are units and the elements in (s,t) are nilpotent, and then every element which doesn't belong to (s,t) is invertible. Clearly, $\mathbb{K}[s,t]/(s^2,t^2,st)$ is three-dimensional as a $\mathbb{K}$-vector space, and so it is an Artin ring. Hence, $\mathbb{K}[s,t]/(s^2,t^2,st)$ is an example of an object in $\mathbf{Art}_{\mathbb{K}}$ which is not of the form $\mathbb{K}[t]/(t^n)$; indeed, it is isomorphic to the product of $\mathbb{K}[t]/(t^2)$ with itself:

$$\frac{\mathbb{K}[s,t]}{(s^2,t^2,st)} \cong \frac{\mathbb{K}[t]}{(t^2)} \times_{\mathbb{K}} \frac{\mathbb{K}[t]}{(t^2)}$$
$$a + bs + b't \mapsto (a + bt, a + b't)$$

With regard to extensions, of course we have the following:

$$0 \longrightarrow (s,t) \longrightarrow \frac{\mathbb{K}[s,t]}{(s^2,t^2,st)} \longrightarrow \mathbb{K} \longrightarrow 0$$

The above is a small extension, but it is not principal. According to the discussion from before, it is possible to get from $\mathbb{K}$ to $\mathbb{K}[s,t]/(s^2,t^2,st)$ via a sequence of principal small extensions, for instance:

$$\frac{\mathbb{K}[s,t]}{(s^2,t^2,st)} \longrightarrow \frac{\mathbb{K}[t]}{(t^2)} \longrightarrow \mathbb{K}$$

From now on, let us denote by 0 the singleton set, or in other words the terminal object in **Set**.

Definition 2.11. A **functor of Artin rings** is a covariant functor $F : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ such that $F(\mathbb{K}) = 0$.

Since $\mathbb{K}$ is the terminal object in $\mathbf{Art}_{\mathbb{K}}$, a functor of Artin rings is just a functor from $\mathbf{Art}_{\mathbb{K}}$ to **Set** which preserves the terminal object. By Remark 2.7 $\mathbf{Art}_{\mathbb{K}}$ is equivalent to a small category, and then it is safe to consider the category of all functors of Artin rings, where morphisms are given by natural transformations.

As we'll see, in the case where F is a deformation functor (Definition 2.16) $F(\mathbb{K})$ should be thought of as the set of infinitesimal deformations of order 0, and then the property that $F(\mathbb{K})$ is a singleton corresponds to the fact there are no infinitesimal deformation of order 0, aside from the trivial one.

Example 2.12. The trivial functor $0 : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$, which sends every object to 0 and every morphism to the identity on 0, is a functor of Artin rings.

Example 2.13. Let V be a $\mathbb{K}$-vector space. For all A in $\mathbf{Art}_{\mathbb{K}}$, observe that the scalar extension $V \otimes A$ may be decomposed as

$$V \otimes A \cong V \otimes (\mathbb{K} \oplus \mathfrak{m}_A) \cong (V \otimes \mathbb{K}) \oplus (V \otimes \mathfrak{m}_A) \cong V \oplus (V \otimes \mathfrak{m}_A).$$

Intuitively, $V \otimes A$ can be thought of as an "infinitesimal thickening" of V, which extends V with an "infinitesimal part" $V \otimes \mathfrak{m}_A$. We define the **formal neighborhood** of V as the functor:

$$\widehat{V} : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}, \quad \widehat{V}(A) = V \otimes \mathfrak{m}_A.$$

We can clearly see that $\widehat{V}$ is a functor of Artin rings.

Example 2.14. For every B in $\mathbf{Art}_{\mathbb{K}}$ the Hom functor

$$h_B : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}, \quad h_B(A) = \mathrm{Hom}_{\mathbf{Art}_{\mathbb{K}}}(B, A)$$

is a functor of Artin rings. Fixed another functor of Artin rings F, according to the Yoneda lemma the natural transformations $h_B \to F$ can be characterized as follows: for every $b \in F(B)$, there exists a unique natural transformation $\phi : h_B \to F$ such that $\phi(\mathrm{id}_B) = b$.

Example 2.15. As a generalization of the previous example, consider an object R in $\mathcal{C}_{\mathbb{K}}$, and suppose in addition that R is Noetherian; in other words, R is a Noetherian local $\mathbb{K}$-algebra with residue field $\mathbb{K}$. The Hom functor $h_R : \mathcal{C}_{\mathbb{K}} \to \mathbf{Set}$ can be restricted to $\mathbf{Art}_{\mathbb{K}}$, and then it is a functor of Artin rings:

$$h_R : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}, \quad h_R(A) = \mathrm{Hom}_{\mathcal{C}_{\mathbb{K}}}(R, A).$$

Fixed another functor of Artin rings F, we wish to characterize the natural transformations $h_R \to F$. Note that because R is not an object in $\mathbf{Art}_{\mathbb{K}}$ (in general), we cannot apply the Yoneda lemma directly, so we need some kind of workaround.

The idea is to consider a sequence of "Artinian approximations" of R, given by the quotients $R/\mathfrak{m}_R^n$ for $n \in \mathbb{N}$: each $R/\mathfrak{m}_R^n$ is still an object in $\mathcal{C}_{\mathbb{K}}$, it is still Noetherian, and because the maximal ideal $\mathfrak{m}_R$ is nilpotent in $R/\mathfrak{m}_R^n$ it follows that $R/\mathfrak{m}_R^n$ is indeed an Artin ring. Hence, the quotient rings $R/\mathfrak{m}_R^n$ are objects in $\mathbf{Art}_{\mathbb{K}}$. Now, observe that for every morphism $f : R \to A$ in $\mathbf{Art}_{\mathbb{K}}$, if we choose n large enough so that $\mathfrak{m}_A^n = 0$, f can be factored through $R/\mathfrak{m}_R^n$:

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "p_n"'] & A \cong \frac{A}{\mathfrak{m}_A^n} \\ \frac{R}{\mathfrak{m}_R^n} \arrow[ur, "{[f]_n}"'] & \end{tikzcd}$$

Armed with this observation, we can show via a Yoneda-type argument that any natural transformation $\phi : h_R \to F$ is completely determined by the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ where $\phi_n = \phi(p_n) \in F(R/\mathfrak{m}_R^n)$. Indeed, for any morphism $f : R \to A$ in $\mathbf{Art}_{\mathbb{K}}$, by choosing n as in the above observation we have by naturality:

$$\begin{tikzcd} \mathrm{Hom}(R, \frac{R}{\mathfrak{m}_R^n}) \arrow[r, "\phi"] \arrow[d, "{h_R([f]_n)}"'] & F(\frac{R}{\mathfrak{m}_R^n}) \arrow[d, "{F([f]_n)}"] \\ \mathrm{Hom}(R, A) \arrow[r, "\phi"'] & F(A) \end{tikzcd} \qquad \begin{tikzcd} p_n \arrow[r, mapsto, "\phi"] \arrow[d, mapsto, "{h_R([f]_n)}"'] & \phi(p_n) \arrow[d, mapsto, "{F([f]_n)}"] \\ {[f]_n \circ p_n = f} \arrow[r, mapsto, "\phi"'] & \phi(f) = F([f]_n)(\phi(p_n)) \end{tikzcd}$$

from which it follows

$$\phi(f) = F([f]_n)(\phi_n).$$

Moreover, the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ is *consistent*, in the sense that we have $\phi_n = F(p)(\phi_{n+1})$ where $p : R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$ is the canonical projection (this is a consequence of the above, since $p = [p_n]_{n+1}$).

Conversely, any consistent sequence $\{\phi_n\}_{n\in\mathbb{N}}$ defines a natural transformation $\phi : h_R \to F$. Indeed, for every morphism $f : R \to A$ in $\mathbf{Art}_{\mathbb{K}}$, by choosing n sufficiently large we can define $\phi(f)$ as

$$\phi(f) = F([f]_n)(\phi_n),$$

and that is independent of the choice of n because the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ is consistent. Finally, ϕ is a natural transformation because for every morphism $g : A \to B$ in $\mathbf{Art}_{\mathbb{K}}$, by choosing n sufficiently large for both A and B the naturality condition amounts to:

$$\begin{tikzcd} \mathrm{Hom}(R,A) \arrow[r, "\phi"] \arrow[d, "h_R(g)"'] & F(A) \arrow[d, "F(g)"] \\ \mathrm{Hom}(R,B) \arrow[r, "\phi"'] & F(B) \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r, mapsto, "\phi"] \arrow[d, mapsto, "h_R(g)"'] & F([f]_n)(\phi_n) \arrow[d, mapsto, "F(g)"] \\ g\circ f \arrow[r, mapsto, "\phi"'] & F([g\circ f]_n)(\phi_n) = F(g)(F([f]_n)(\phi_n)) \end{tikzcd}$$

and indeed we have

$$F([g\circ f]_n)(\phi_n) = F(g\circ [f]_n)(\phi_n) = F(g)(F([f]_n)(\phi_n)),$$

where the first equality is an easy consequence of the universal property of the quotient, while the second uses the fact that F is a functor.

In conclusion, the natural transformations $h_R \to F$ can be characterized as follows: for every consistent sequence $\{\phi_n\}_{n\in\mathbb{N}}$ where $\phi_n \in F(R/\mathfrak{m}_R^n)$, there exists a unique natural transformation $\phi : h_R \to F$ such that $\phi(p_n) = \phi_n$ for all $n \in \mathbb{N}$.

Definition 2.16. Le $F : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ be a functor of Artin rings. For every pullback square in the category $\mathbf{Art}_{\mathbb{K}}$

$$\begin{tikzcd} B\times_A C \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r, "\beta"] & A \end{tikzcd}$$

denote the induced morphism in **Set**

$$\eta : F(B\times_A C) \to F(B)\times_{F(A)} F(C).$$

We say that F is a **deformation functor** if:

(1) η is surjective whenever β is surjective;
(2) η is bijective whenever $A = \mathbb{K}$.

We say that F is a **homogeneous deformation functor** if η is bijective whenever β is surjective.

In the notation of the above definition, to say that η is bijective (that is, an isomorphism in **Set**) means that F preserves the corresponding fiber product: $F(B\times_A C) \cong F(B)\times_{F(A)} F(C)$. For instance, it is a well know fact in category theory that the Hom functor preserves all limits, and it follows in particular that the functor $h_R = \mathrm{Hom}_{\mathcal{C}_{\mathbb{K}}}(R, -)$ from Example 2.15 is homogeneous.

Remark 2.17. By definition, deformation functors preserve finite products. In fact, deformation functors are in particular functors of Artin rings, which preserve the terminal object (nullary product). Moreover, in the notation of Definition 2.16, if $A = \mathbb{K}$ we have that A and $F(A)$ are terminal objects in their respective categories, and so we see that condition (2) is equivalent to the fact that F preserves binary products.

Example 2.18. We now introduce a concrete example of a deformation functor, which will help us motivate much of theory that will follow. Consider a bounded complex of $\mathbb{K}$-vector spaces (V, ∂):

$$0 \longrightarrow V^0 \xrightarrow{\partial} V^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} V^{n-1} \xrightarrow{\partial} V^n \longrightarrow 0$$

For any A in $\mathbf{Art}_{\mathbb{K}}$, we say that a deformation of (V,∂) over A is a complex of A-modules $(V\otimes A, \partial_A)$ of the form

$$0 \longrightarrow V^0\otimes A \xrightarrow{\partial_A} V^1\otimes A \xrightarrow{\partial_A} \cdots \xrightarrow{\partial_A} V^{n-1}\otimes A \xrightarrow{\partial_A} V^n\otimes A \longrightarrow 0$$

such that its residue modulo $\mathfrak{m}_A$ gives back the original complex (V,∂).
More formally, for any complex of A-modules $(V\otimes A, \partial_A)$ we can decompose $V\otimes A \cong V\oplus(V\otimes \mathfrak{m}_A)$, and then by A-linearity we have $\partial_A(V\otimes\mathfrak{m}_A) \subseteq V\otimes\mathfrak{m}_A$; hence, ∂_A is compatible with the quotient by $V\otimes\mathfrak{m}_A$, and so it

induces a map $[\partial_A]: V \to V$. We say that a complex $(V \otimes A, \partial_A)$ reduces modulo $\mathfrak{m}_A$ to (V, ∂) when $[\partial_A] = \partial$. Then, we say that two deformations $(V \otimes A, \partial_A)$, $(V \otimes A, \partial'_A)$ are equivalent if there exists an isomorphism of complexes of A-modules

$$\begin{tikzcd} 0 \arrow[r] & V^0 \otimes A \arrow[r, "\partial_A"] \arrow[d, "\phi_0"] & V^1 \otimes A \arrow[r, "\partial_A"] \arrow[d, "\phi_1"] & \cdots \arrow[r, "\partial_A"] & V^{n-1} \otimes A \arrow[r, "\partial_A"] \arrow[d, "\phi_{n-1}"] & V^n \otimes A \arrow[r] \arrow[d, "\phi_n"] & 0 \\ 0 \arrow[r] & V^0 \otimes A \arrow[r, "\partial'_A"] & V^1 \otimes A \arrow[r, "\partial'_A"] & \cdots \arrow[r, "\partial'_A"] & V^{n-1} \otimes A \arrow[r, "\partial'_A"] & V^n \otimes A \arrow[r] & 0 \end{tikzcd}$$

such that the residue of ϕ modulo $\mathfrak{m}_A$ is the identity on V, that is $[\phi] = \mathrm{id}_V$; if that is the case, we write $(V \otimes A, \partial_A) \sim (V \otimes A, \partial'_A)$. This construction yields the following functor:

$$F : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}, \qquad F(A) = \{(V \otimes A, \partial_A) \mid (V \otimes A, \partial_A) \text{ reduces modulo } \mathfrak{m}_A \text{ to } (V, \partial)\}_{/\sim}$$

To define the action of F on morphisms, observe that because any A in $\mathbf{Art}_{\mathbb{K}}$ is finite-dimensional as a $\mathbb{K}$-vector space, we have a canonical isomorphism

$$\mathrm{Hom}^*_A(V \otimes A, V \otimes A) \cong \mathrm{Hom}^*_{\mathbb{K}}(V, V) \otimes A,$$

and then we see that every morphism $f : A \to B$ in $\mathbf{Art}_{\mathbb{K}}$ induces a map

$$\overline{f} = \mathrm{id}_{\mathrm{Hom}^*_{\mathbb{K}}(V,V)} \otimes f : \mathrm{Hom}^*_{\mathbb{K}}(V, V) \otimes A \to \mathrm{Hom}^*_{\mathbb{K}}(V, V) \otimes B$$

with the following property (which is easily checked on pure tensors):

$$\forall \alpha, \beta \in \mathrm{Hom}^*_A(V \otimes A, V \otimes A) \cong \mathrm{Hom}^*_{\mathbb{K}}(V, V) \otimes A \qquad \overline{f}(\beta \circ \alpha) = \overline{f}(\beta) \circ \overline{f}(\alpha).$$

With that, to any complex of A-modules $(V \otimes A, \partial_A)$ we can associate a corresponding complex of B-modules $(V \otimes B, \overline{f}(\partial_A))$, which is still a complex because:

$$\partial_A^2 = 0 \implies \overline{f}(\partial_A)^2 = 0.$$

Moreover, this mapping is compatible with the equivalence relation $\sim$, because for every isomorphism $\phi : (V \otimes A, \partial_A) \cong (V \otimes A, \partial'_A)$ we have an induced isomorphism $\overline{f}(\phi) : (V \otimes B, \overline{f}(\partial_A)) \cong (V \otimes B, \overline{f}(\partial'_B))$; indeed, $\overline{f}(\phi)$ is a morphism of complexes because:

$$\phi \circ \partial_A = \partial'_A \circ \phi \implies \overline{f}(\phi) \circ \overline{f}(\partial_A) = \overline{f}(\partial'_A) \circ \overline{f}(\phi).$$

To see that $\overline{f}(\phi)$ is an isomorphism, we can show directly that it reduces to the identity modulo $\mathfrak{m}_B$. First, observe that ϕ reduces to the identity modulo $\mathfrak{m}_A$ if and only if:

$$\phi = \mathrm{id}_{V \otimes A} + \eta, \qquad \text{with } \eta \in \mathrm{Hom}^0_A(V \otimes A, V \otimes \mathfrak{m}_A) \cong \mathrm{Hom}^0_{\mathbb{K}}(V, V) \otimes \mathfrak{m}_A,$$

but then it follows, by the fact that $\mathrm{id}_{V \otimes A} = \mathrm{id}_V \otimes 1$, $f(1) = 1$, and $f(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$:

$$\overline{f}(\phi) = \mathrm{id}_{V \otimes B} + \overline{f}(\eta), \qquad \text{with } \overline{f}(\eta) \in \mathrm{Hom}^0_A(V \otimes B, V \otimes \mathfrak{m}_B) \cong \mathrm{Hom}^0_{\mathbb{K}}(V, V) \otimes \mathfrak{m}_B,$$

and that implies in particular that $\overline{f}(\phi)$ is an isomorphism, since $\mathrm{id}_{V \otimes B}$ is invertible and $\overline{f}(\eta) \in \mathrm{Hom}^0_{\mathbb{K}}(V, V) \otimes \mathfrak{m}_B$ is nilpotent.

Finally, it can be shown that F is a deformation functor. We defer this check to later (Example 2.46), where we'll show that F can be seen as a deformation functor associated to a DG-Lie algebra.

The construction of limits in **Set** extends immediately the functor category $\mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$, by performing the limits pointwise. For instance, the product of two functors $F, G : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ is given by $(F \times G)(A) = F(A) \times G(A)$, and one can verify that if F and G are (homogeneous) deformation functors then so is $F \times G$. The situation is more complicated in general, as for instance the pointwise fiber product of deformation functors may not be a deformation functor[2].

[2] Indeed, one can prove the stronger fact that the full subcategory of deformation functors is not closed under fiber products. See e.g. [17, Theorem 5.1.6].

Definition 2.19. Let $\phi : F \to G$ be a natural transformation of functors of Artin rings. For every morphism $f : B \to A$ in $\mathbf{Art}_{\mathbb{K}}$, consider the commutative diagram

$$\begin{tikzcd} F(B) \arrow[r, "\phi"] \arrow[d, "F(f)"] & G(B) \arrow[d, "G(f)"] \\ F(A) \arrow[r, "\phi"] & G(A) \end{tikzcd}$$

and denote the induced map

$$\mu : F(B) \to F(A) \times_{G(A)} G(B).$$

We say that ϕ is **smooth** if μ is surjective whenever f is surjective.
For a functor of Artin rings F, we say that F is **smooth** or **unobstructed** if the natural transformation $F \to 0$ is smooth; equivalently, F is smooth if $f : F(B) \to F(A)$ is surjective whenever $f : B \to A$ is surjective.

If $\phi : F \to G$ is a smooth natural transformation, then by letting $A = \mathbb{K}$ in the definition we see in particular that $\phi : F(B) \to G(B)$ is surjective for every B.

Example 2.20. If $f : B \to A$ is a morphism in $\mathbf{Art}_{\mathbb{K}}$, by the lower triangle in Diagram (1.3) we have for all $b \in B$

$$b \in \mathfrak{m}_B \iff 0 = \pi_B(b) = \pi_A(f(b)) \iff f(b) \in \mathfrak{m}_A,$$

and then if $f : B \to A$ is surjective so must be the restriction $f : \mathfrak{m}_B \to \mathfrak{m}_A$.

It follows that, for every $\mathbb{K}$-vector space V, the formal neighborhood $\widehat{V}(A) = V \otimes \mathfrak{m}_A$ (Example 2.13) is a smooth functor of Artin rings.

When a deformation functor describes a geometric deformation problem, the smoothness property has a clear interpretation. It may help at this point to give some idea, although vague and informal, of what is going on: If F is the deformation functor associated to some algebro-geometric object, then for every $n \in \mathbb{N}$ the set $F(\mathbb{K}[t]/(t^n))$ contains all possible infinitesimal deformations of order $n-1$ of the object. The canonical projection $\mathbb{K}[t]/(t^{n+1}) \to \mathbb{K}[t]/(t^n)$ induces a map $F(\mathbb{K}[t]/(t^{n+1})) \to F(\mathbb{K}[t]/(t^n))$ from the set of deformations of order n to the set of deformations of order $n-1$, which "forgets" the higher order data of the deformation. Then, a preimage of a deformation $x \in F(\mathbb{K}[t]/(t^n))$ is a higher order deformation $\widehat{x} \in F(\mathbb{K}[t]/(t^{n+1}))$ which agrees with x up to order $n-1$. A fact of life in deformation theory is that a preimage $\widehat{x}$ does not always exist, because there may be *obstructions* which prevent us from lifting x to a deformation of higher order. In the special case where F is a smooth deformation functor, we have that the induced map $F(\mathbb{K}[t]/(t^{n+1})) \to F(\mathbb{K}[t]/(t^n))$ is surjective for all n, and hence the lifting is always possible (or in other words there are no *obstructions*, which motivates the terminology "unobstructed functor").

Compared to the case of functors, the interpretation of smooth natural transformations is less immediate. Informally, we can think that they give a notion of "relative smoothness" of functors: in particular, if G is smooth and there exists a smooth natural transformation $F \to G$, then F is also smooth. From a more technical perspective, we will see that the smoothness hypothesis allows us to prove useful results which would fail to hold in full generality. As a first application, we have the following coequalizer criterion.

Theorem 2.21. *Suppose we have a commutative diagram of deformation functors*

$$\begin{tikzcd} & & F \\ F \arrow[r, "i"] \arrow[urr, bend left, "\mathrm{id}_F"] \arrow[drr, bend right, "\mathrm{id}_F"'] & R \arrow[ur, "e_0"] \arrow[dr, "e_1"'] & \\ & & F \end{tikzcd}$$

Denote by Q the colimit of this diagram in the functor category $\mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$, which is equivalently the coequalizer of e_0 and e_1, that is:

$$Q(A) = \frac{F(A)}{\text{equivalence relation generated by } e_0(x) \sim e_1(x) \text{ for } x \in R(A)}.$$

If e_0 and e_1 are smooth natural transformations, then Q is a deformation functor and the natural projection $\pi : F \to Q$ is smooth.

The previous theorem has an important application in the case of group actions.

Definition 2.22. Let $\mathcal{C}$ be a category, and let there be two functors $F : \mathcal{C} \to \mathbf{Set}$, $G : \mathcal{C} \to \mathbf{Grp}$. A **functorial action** of G on F is a natural transformation

$$\alpha : G \times F \to F$$

such that for every object A in $\mathcal{C}$, the function $G(A) \times F(A) \to F(A)$ is a group action in the usual sense. In the above setup, we define the **quotient functor** F/G as the colimit of the diagram

(1.5)
$$\begin{tikzcd} & & F \\ F \arrow[r, "i"] \arrow[urr, bend left, "\mathrm{id}_F"] \arrow[drr, bend right, "\mathrm{id}_F"'] & G \times F \arrow[ur, "\pi_2"] \arrow[dr, "\alpha"] & \\ & & F \end{tikzcd}$$

where $i(x) = (1, x)$. Note that this is the familiar quotient of a set by a group action, performed point by point: $(F/G)(A) = F(A)/G(A)$.

For a functorial group action $\alpha : G \times F \to F$, we adopt the familiar notation

$$G(A) \times F(A) \to F(A), \qquad (g, x) \mapsto gx,$$

and for every morphism in the domain category $f : B \to A$, we denote by $f : F(B) \to F(A)$ and $f : G(B) \to G(A)$ the induced maps. By naturality, we have:

(1.6)
$$\begin{tikzcd} G(B) \times F(B) \arrow[r, "\alpha"] \arrow[d, "f \times f"] & F(B) \arrow[d, "f"] \\ G(A) \times F(A) \arrow[r, "\alpha"] & F(A) \end{tikzcd} \qquad \begin{tikzcd} (g, x) \arrow[r, mapsto, "\alpha"] \arrow[d, mapsto, "f \times f"] & gx \arrow[d, mapsto, "f"] \\ (f(g), f(x)) \arrow[r, mapsto, "\alpha"] & f(g)f(x) = f(gx) \end{tikzcd}$$

Theorem 2.23. *Let $F : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ be a functor of Artin rings, $G : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Grp}$ be a group functor of Artin rings acting on F. If F and G are deformation functors and G is smooth, then F/G is a deformation functor and the projection $F \to F/G$ is smooth.*

PROOF. By Theorem 2.21, it suffices to show that the natural transformations $\pi_2, \alpha : G \times F \to F$ in Diagram (1.5) are smooth.

To show that π_2 is smooth we must prove that, for every surjective morphism $f : B \to A$, if $(h, y) \in G(A) \times F(A)$, $\tilde{x} \in F(B)$ are such that $y = f(x)$ then they are of the form in the following diagram:

$$\begin{tikzcd} G(B) \times F(B) \arrow[r, "\pi_2"] \arrow[d, "f \times f"] & F(B) \arrow[d, "f"] \\ G(A) \times F(A) \arrow[r, "\pi_2"] & F(A) \end{tikzcd} \qquad \begin{tikzcd} (g, x) \arrow[r, mapsto, "\pi_2"] \arrow[d, mapsto, "f \times f"] & x \arrow[d, mapsto, "f"] \\ (f(g), f(x)) \arrow[r, mapsto, "\pi_2"] & f(x) \end{tikzcd}$$

That is, $(h, y) = (f(g), f(x))$ and $\tilde{x} = x$ for some $g \in G(A)$, $x \in F(A)$. Clearly, we can choose $\tilde{x} = x$ and then we just have to show that $h = f(g)$ for some $g \in G(A)$, but this is obvious because f is surjective and G is smooth.

To show that α is smooth, we must do an analogous check with respect to Diagram (1.6). Let $(h, y) \in G(A) \times F(A)$, $\tilde{x} \in F(B)$ be such that $hy = f(\tilde{x})$ Then, for the same reasons as above we have $h = f(g)$ for some g, and because g is invertible we can let $\tilde{x} = gx$. The equation then becomes $f(g)y = f(gx) = f(g)f(x)$, and by applying $f(g^{-1})$ it follows $y = f(x)$. $\square$

Our next aim is to show that, given a deformation functor F, the infinitesimal deformations of first order described by F give rise to a vector space. More formally, we wish to endow the set $F(\mathbb{K}[t]/(t^2)$ with a $\mathbb{K}$-vector space structure. To get to the final definition in a elegant manner, it helps to generalize the concept of vector space to a category-theoretic setting.

Definition 2.24. Fix a field $\mathbb{K}$, and a category with binary product $\times$ and terminal object $*$. A **$\mathbb{K}$-vector space object** $(V, 0, +, -, \{\alpha\cdot\}_{\alpha \in \mathbb{K}})$ is the data of an object V equipped with morphisms

$$\begin{tikzcd} * \arrow[d, "0"] & V \times V \arrow[d, "+"] & V \arrow[d, "-"] & V \arrow[d, "\alpha\cdot"] \\ V & V & V & V \end{tikzcd}$$

such that the following diagrams commute:

$$\begin{tikzcd} V \times V \times V \arrow[r, "\mathrm{id}_V \times +"] \arrow[d, "+ \times \mathrm{id}_V"] & V \times V \arrow[d, "+"] \\ V \times V \arrow[r, "+"] & V \end{tikzcd} \qquad \begin{tikzcd} V \times V \arrow[rr, "(\pi_2, \pi_1)"] \arrow[dr, "+"'] & & V \times V \arrow[dl, "+"] \\ & V & \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "(\mathrm{id}_V, 0)"] \arrow[dr, "\mathrm{id}_V"'] & V \times V \arrow[d, "+"] \\ & V \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[r, "(\mathrm{id}_V, -)"] \arrow[dr, "0"'] & V \times V \arrow[d, "+"] \\ & V \end{tikzcd} \qquad \begin{tikzcd} V \times V \arrow[r, "+"] \arrow[d, "\alpha\cdot \times \alpha\cdot"] & V \arrow[d, "\alpha\cdot"] \\ V \times V \arrow[r, "+"] & V \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "(\alpha\cdot, \beta\cdot)"] \arrow[dr, "(\alpha+\beta)\cdot"'] & V \times V \arrow[d, "+"] \\ & V \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[r, "\alpha\cdot"] \arrow[dr, "\alpha\beta\cdot"'] & V \arrow[d, "\beta\cdot"] \\ & V \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, bend left, "1\cdot"] \arrow[r, bend right, "\mathrm{id}_V"'] & V \end{tikzcd}$$

(here, by a slight abuse of notation, we denoted by 0 the composition $V \to * \xrightarrow{0} V$).

Example 2.25. In the category **Set**, a $\mathbb{K}$-vector space object is just a $\mathbb{K}$-vector space in the usual sense. Indeed, the function $0 : * \to V$ from the singleton to V picks out the identity element $0 \in V$, the binary function $+ : V \times V \to V$ is the addition operation, $- : V \to V$ gives the additive inverse, and each function $\alpha\cdot : V \to V$ defines multiplication by the corresponding scalar $\alpha \in \mathbb{K}$. Finally, the commutative diagrams are just a reformulation of the familiar vector space axioms.

Example 2.26. In the category $\mathbf{Art}_{\mathbb{K}}$, we have a $\mathbb{K}$-vector space object given by the ring $\mathbb{K}[t]/(t^2)$ equipped with the following morphisms (recall that $\mathbb{K}$ is the terminal object in $\mathbf{Art}_{\mathbb{K}}$):

$$\begin{array}{cccc} 0 : \mathbb{K} \to \dfrac{\mathbb{K}[t]}{(t^2)} & + : \dfrac{\mathbb{K}[t]}{(t^2)} \times_{\mathbb{K}} \dfrac{\mathbb{K}[t]}{(t^2)} \to \dfrac{\mathbb{K}[t]}{(t^2)} & - : \dfrac{\mathbb{K}[t]}{(t^2)} \to \dfrac{\mathbb{K}[t]}{(t^2)} & \alpha\cdot : \dfrac{\mathbb{K}[t]}{(t^2)} \to \dfrac{\mathbb{K}[t]}{(t^2)} \\ 0(k) = k & (a + bt, a + b't) = a + (b + b')t & -(a + bt) = a - bt & \alpha \cdot (a + bt) = a + (\alpha b)t \end{array}$$

The reader is encouraged to check that these are indeed morphisms in $\mathbf{Art}_{\mathbb{K}}$, and see how that crucially depends on the fact that $t^2 = 0$; then, it is easy to see that all the relevant diagrams commute.

Hence, a $\mathbb{K}$-vector space object in a category is just an object V, plus a collection of morphisms and commutative squares involving finite products of V. Thus, it is clear that if $F : \mathcal{C} \to \mathcal{D}$ is a functor which preserves finite products, and $(V, 0, +, -, \{\alpha\cdot\}_{\alpha\in\mathbb{K}})$ is a $\mathbb{K}$-vector space object in $\mathcal{C}$, then $(F(V), F(0), F(+), F(-), \{F(\alpha\cdot)\}_{\alpha\in\mathbb{K}})$ is a $\mathbb{K}$-vector space object in $\mathcal{D}$:

$$F(0) : F(*) \cong * \to F(V), \qquad F(+) : F(V \times V) \cong F(V) \times F(V) \to F(V), \qquad F(-), F(\alpha\cdot) : F(V) \to F(V).$$

In particular, since by Remark 2.17 deformation functors preserve finite products, we can give the following definition.

Definition 2.27. Let F be a deformation functor. The **tangent space** of F is the $\mathbb{K}$-vector space

$$T^1F = F\left(\frac{\mathbb{K}[t]}{(t^2)}\right),$$

which is the image through F of the $\mathbb{K}$-vector space object described in Example 2.26.

The tangent space has many of the desirable properties one may expect. For instance, we can readily see that if $\eta : F \to G$ is a natural transformation of deformation functors, then $\eta_{\mathbb{K}[t]/(t^2)} : T^1F \to T^1G$ is a linear map.

2. Nilpotent algebras, exponentials and BCH product

The aim of this segment is to introduce some technical tools which will be essential for the subsequent section; these include exponential maps and the BCH product for Lie algebras. We will restrict ourselves to the case of nilpotent algebras, as that will be sufficient for our applications. Some proofs of the propositions we are about to state consist in a series of computations which are not particularly illuminating for the sake of this thesis, and so they have been omitted. A complete account with proofs can be found in [**17**, Chapter 2].

Definition 2.28. A **magmatic algebra** $(M, \cdot)$ is the data of a $\mathbb{K}$-vector space M and a bilinear map

$$\begin{aligned} M \times M &\to M \\ (x, y) &\mapsto x \cdot y \end{aligned}$$

The above definition is really general, and as such applies to a wide range of different structures. For instance, the bilinear operation $\cdot$ may be the product in an associative algebra, or just as well the bracket in a Lie algebra.

Definition 2.29. A **unitary associative algebra** R is the data of a $\mathbb{K}$-vector space R, an element $1 \in R$ called **unit**, and a bilinear map

$$\begin{aligned} R \times R &\to R \\ (a, b) &\mapsto ab \end{aligned}$$

called **product**, such that:

(1) $(ab)c = a(bc)$
(2) $1a = a1 = a$

In particular, the n-th power a^n is well-defined for all $a \in R$ and $n \geq 0$.

Example 2.30. For every $\mathbb{K}$-vector space V, the space of endomorphisms $\mathrm{Hom}_{\mathbb{K}}(V, V)$ is a unitary associative algebra, where function composition is the product and id_V is the unit.

The exponential map can be defined in any unitary associative algebra. As expected, this is based on the familiar Taylor series for the exponential function: this makes sense without any notion of convergence whatsoever, as long as we work with nilpotent elements (so that the series truncates to a finite sum), and over a field $\mathbb{K}$ of characteristic 0[3] (so that we don't ever divide by 0).

Definition 2.31. Let R be a unitary associative algebra, and let a be a nilpotent element of R. We define the **exponential** of a as

$$e^a = \sum_{n\geq 0} \frac{a^n}{n!} \in R.$$

One expects that the exponential function satisfies $e^{a+b} = e^a e^b$, but this is *false* in general. Importantly, we need to assume that a and b commute.

Proposition 2.32. *Let R be a unitary associative algebra, and let a, b be nilponent elements of R. If $ab = ba$, then $a+b$ is also nilpotent and we have:*

$$e^{a+b} = e^a e^b.$$

In particular, e^a is invertible with inverse e^{-a}.

PROOF. That $a+b$ is nilpotent is easily seen by considering $(a+b)^n$ for sufficiently large n, and applying the binomial formula. The identity $e^{a+b} = e^a e^b$ also follows from a straightforward computation based on the binomial expansion:

$$e^{a+b} = \sum_{n\geq 0} \frac{(a+b)^n}{n!} = \sum_{n\geq 0} \frac{1}{n!} \sum_{i+j=n} \binom{n}{i} a^i b^j = \sum_{n\geq 0} \sum_{i+j=n} \frac{a^i}{i!} \frac{b^j}{j!} = \left(\sum_{i\geq 0} \frac{a^i}{i!} \right) \left(\sum_{j\geq 0} \frac{b^j}{j!} \right) = e^a e^b.$$

Note that the assumption $ab = ba$ was required in order to use the binomial formula. □

In particular, for a vector space V and a nilpotent endomorphism $\alpha \in \mathrm{Hom}_{\mathbb{K}}(V, V)$, the exponential gives an invertible endomorphism $e^\alpha \in \mathrm{Hom}_{\mathbb{K}}(V, V)$. In the case where V is a magmatic algebra, we may wish that e^α commutes with the product; for that to be the case, we need some additional hypothesis.

Definition 2.33. Let $(M, \cdot)$ be a magmatic algebra. A **derivation** $\alpha : (M, \cdot) \to (M, \cdot)$ is a linear endomorphism of M which satisfies the Leibniz rule:

$$\forall x, y \in M \qquad \alpha(x \cdot y) = \alpha(x) \cdot y + x \cdot \alpha(y).$$

Proposition 2.34. *Let $(M, \cdot)$ be a magmatic algebra, and let α be a nilpotent derivation of M. Then, the exponential e^α satisfies:*

$$\forall x, y \in M \qquad e^\alpha(x \cdot y) = e^\alpha(x) \cdot e^\alpha(y).$$

PROOF. By inductively applying the Leibniz rule we can easily see that, for all $n \geq 0$:

$$\alpha^n(x \cdot y) = \sum_{i+j=n} \binom{n}{i} \alpha^i(x) \cdot \alpha^j(y).$$

Then, the result follows by a straightforward computation:

$$\begin{aligned} e^\alpha(x \cdot y) &= \sum_{n\geq 0} \frac{\alpha^n(x \cdot y)}{n!} = \sum_{n\geq 0} \frac{1}{n!} \sum_{i+j=n} \binom{n}{i} \alpha^i(x) \cdot \alpha^j(y) = \sum_{n\geq 0} \sum_{i+j=n} \frac{\alpha^i(x)}{i!} \cdot \frac{\alpha^j(y)}{j!} \\ &= \left(\sum_{i\geq 0} \frac{\alpha^i(x)}{i!} \right) \cdot \left(\sum_{j\geq 0} \frac{\alpha^j(y)}{j!} \right) = e^\alpha(x) \cdot e^\alpha(y). \end{aligned}$$

□

[3]This is the most prominent use of the assumption that $\mathbb{K}$ is a field of non-zero characteristic, and is the reason why the DGLA approach to deformation theory is fundamentally limited to that case: the functor Def_L, which we'll see in Section 3, is not even well-defined in positive characteristic.

Definition 2.35. Let $(M,\cdot)$ be a magmatic Lie algebra. An **ideal** $I\subseteq M$ is a vector subset of M which satisfies:

$$\forall m\in M, i\in I \qquad m\cdot i, i\cdot m\in I.$$

Consider a unitary associative algebra R. According to the above discussion, if I is a nilpotent ideal of R the exponential function is well defined on I. In much the same way, we can define the logarithm on the lateral $1+I$.

Definition 2.36. Let R be a unitary associative algebra, and let I be a nilpotent ideal of R. We define the **exponential**:

$$e\colon I\to 1+I, \qquad e^a=\sum_{n\geq 0}\frac{a^n}{n!},$$

and the **logarithm**:

$$\log\colon 1+I\to I, \qquad \log(1+a)=\sum_{n\geq 1}(-1)^{n-1}\frac{a^n}{n}.$$

Proposition 2.37. *The exponential and logarithm are inverses of each other. That is, if R is a unitary associative algebra, for all $a,b\in R$ we have:*

$$\log(e^a)=a, \qquad e^{\log(1+b)}=1+b.$$

PROOF. Because the exponential and logarithm above have been defined in terms of power series with coefficients in $\mathbb{Q}$, it suffices to show that the formal composition of the corresponding series in $\mathbb{Q}[[t]]$ is the identity series. Now, embed $\mathbb{Q}[[t]]\subseteq\mathbb{R}[[t]]$, and recall that by well known results of analysis the subring of convergent series in $\mathbb{R}[[t]]$ is isomorphic to the ring of real analytic functions locally defined around at point (germs of analytic functions). Under this isomorphism, the composition of formal series corresponds to the composition of functions, and then the thesis follows from the fact that the familiar real functions exp and log are inverses of each other. □

Consider a unitary associative algebra R with a nilpotent ideal I. Observe that I is naturally a nilpotent Lie algebra, with bracket $[a,b]=ab-ba$. Because in general $e^{a+b}\neq e^ae^b$, we may define the following binary operation on I

$$a\bullet b=\log(e^ae^b), \tag{2.1}$$

so that tautologically $e^{a\bullet b}=e^ae^b$. Now, since we know that $e^{a+b}=e^ae^b$ whenever a and b commute, we immediately see that

$$ab=ba \implies a\bullet b=a+b.$$

In light of this fact, it is not unreasonable to guess that there may exist a formula for $\bullet$ in terms of the commutator:

$$a\bullet b=a+b+\text{some expression involving }[a,b].$$

That is indeed the case. The explicit formula was first discovered by Campbell, and the operation $\bullet$ is now known as the **Baker-Campbell-Hausdorff product**. Importantly, since it can be expressed just in terms of the addition and the bracket, the operation makes sense in any nilpotent Lie algebra.

Definition 2.38. Let L be a nilponent Lie algebra. By induction, define:

$$Z_0(a,b)=b, \qquad Z_1(a,b)=a+B_1[b,a]+\frac{B_2}{2!}[b,[b,a]]+\frac{B_3}{3!}[b,[b,[b,a]]]+\cdots,$$
$$Z_{n+1}(a,b)=\frac{1}{n+1}\sum_{m\geq 0}\frac{B_m}{m!}\sum_{i_1+\cdots+i_m=n}[Z_{i_1}(a,b),[Z_{i_2}(a,b),\dots,[Z_{i_m}(a,b),a]\dots]],$$

where $B_0,B_1,\dots$ are the Bernoulli numbers, which satisfy:

$$\sum_{n\geq 0}\frac{B_n}{n!}x^n=\frac{x}{e^x-1}=1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^4}{720}+\frac{x^6}{30240}-\frac{x^8}{1209600}+\cdots.$$

We define the **Baker-Campbell-Hausdorff product** (BCH product for short) as:

$$L \times L \to L$$
$$a \bullet b = \sum_{n \geq 0} Z_n(a, b).$$

The following result guarantees that the above definition is equivalent to (2.1), in the special case where the latter applies.

Theorem 2.39. *Let R be a unitary associative algebra, and let I be a nilpotent ideal of R. Then, for all $a, b \in I$ we have*

$$e^{a \bullet b} = e^a e^b, \tag{2.2}$$

where $\bullet$ denotes the BCH product.

In the setup of the above theorem, we can see that the BCH product defines a group operation on I. Indeed, one can easily check that because I is a nilpotent ideal, $1 + I$ is a group with respect to the product $\cdot$ of the associative algebra R. Then, since the exponential map $e : I \to 1 + I$ is a bijection, equation (2.2) tells us precisely that the BCH product $\bullet$ is the unique group operation on I such that $e : (I, \bullet) \to (1 + I, \cdot)$ is an isomorphism of groups.

It can be shown that this holds in the general case: for any nilpotent Lie algebra L, the BCH product $\bullet$ is a group operation on L. For the sake of convenience, we introduce some new notation for this group.

Definition 2.40. Let L be a nilpotent Lie algebra. We define the **exponential group** $\exp(L)$ as the set of formal exponents of L:

$$\exp(L) = \{e^a \mid a \in L\}.$$

This is a group with product

$$\exp(L) \times \exp(L) \to \exp(L), \qquad e^a e^b = e^{a \bullet b},$$

with unit e^0, and inverse $(e^a)^{-1} = e^{-a}$.

Remark 2.41. It is apparent from the definition that any morphism of nilpotent Lie algebras $f : L \to M$ preserves the BCH product, and so it induces a morphism of the exponential groups

$$\exp(L) \to \exp(M), \qquad e^a \mapsto e^{f(a)}.$$

In other words, exp is a functor from the category of nilpotent Lie algebras to the category of groups.

We close this section by stating a few more properties of the exponential.

Theorem 2.42. *Let R be a unitary associative algebra, and let I be a nilpotent ideal of R. For all $a \in R$, denote by $\operatorname{ad} a$ the adjoint operator $(\operatorname{ad} a)(x) = [a, x] = ax - xa$. We have the following:*

(1) If $a \in R$ is nilpotent, then $\operatorname{ad} a$ is nilpotent, the following operator is well defined, and it is a morphism of unital associative algebras:

$$e^{\operatorname{ad} a} = \sum_{n \geq 0} \frac{(\operatorname{ad} a)^n}{n!}.$$

Moreover, $e^{\operatorname{ad} a}$ coincides with the conjugation by e^a:

$$e^{\operatorname{ad} a}(x) = e^a r e^{-a}.$$

(2) For all $a, b \in I$, we have $ab = ba$ if and only if $e^a e^b = e^b e^a$.

(3) For all $a, b \in I$, if $ab = ba$ then

$$e^{a+b} = e^a e^b = e^b e^a, \qquad \log((1 + a)(1 + b)) = \log(1 + a) + \log(1 + b)$$

In the setup of the above theorem, the combination of Item 2 and Item 3 tells us in particular that

$$\forall a, b \in I \qquad ab = ba \iff e^{a+b} = e^a e^b \text{ and } e^{b+a} = e^b e^a.$$

3. Deformation functor associated to a DG-Lie algebra

The objective of this section is to define, for every differential graded Lie algebra L, an associated deformation functor Def_L. A remarkable result, which is beyond the scope of this thesis, is that any deformation problem over a field of characteristic 0 can be described by a functor of this kind.

The first fundamental ingredient in the definition of Def_L is the **Maurer-Cartan equation**. Though initially introduced by Maurer (1879) and Cartan (1904) in the context of Lie groups, we can immediately see that the equation makes sense in any differential graded Lie algebra.

Definition 2.43. Let L be a differential graded Lie algebra. The **Maurer-Cartan equation** is:

$$d(a) + \frac{1}{2}[a, a] = 0, \qquad \text{for } a \in L^1.$$

The solutions to the equation are called **Maurer-Cartan elements** of L, and the set of all solutions is denoted $\mathrm{MC}(L) \subseteq L^1$.

The other fundamental ingredient in the definition of Def_L is the **guage action**. This is an action of the exponential group $\exp(L^0)$ on the set of Maurer-Cartan elements $\mathrm{MC}(L)$, and to define it we'll make great use of the tools from Section 2. Let L be a nilpotent differential graded Lie algebra (think of a formal neighborhood: $L = M \otimes \mathfrak{m}_A$). For $a \in L^0$, consider:

$$e^{\operatorname{ad} a} : L \to L, \qquad e^{\operatorname{ad} a} = \sum_{n \geq 0} \frac{(\operatorname{ad} a)^n}{n!}.$$

Observe that since $\bar{a} = 0$ we have $(\operatorname{ad} a)^n \in \mathrm{Hom}^0(L, L)$ and so also $e^{\operatorname{ad} a} \in \mathrm{Hom}^0(L, L)$; moreover, according to the graded Jacobi identity:

$$\begin{aligned} \forall a, b, c \qquad & [a, [b, c]] = [[a, b], c] + (-1)^{\bar{a}\bar{b}}[b, [a, c]], \\ \bar{a} = 0 \implies & (\operatorname{ad} a)([b, c]) = [(\operatorname{ad} a)(b), c] + [b, (\operatorname{ad} a)(c)], \end{aligned}$$

and then by Proposition 2.34 $e^{\operatorname{ad} a}$ preserves the bracket. Hence, $e^{\operatorname{ad} a}$ is a morphism of (non-differential) graded Lie algebras. Now, one could think to define the gauge action as the *adjoint action* $x \mapsto e^{\operatorname{ad} a}(x)$, but that'd be ill-posed because $e^{\operatorname{ad} a}$ does not send $\mathrm{MC}(L)$ to itself (indeed, the Maurer-Cartan equation depends on the differential d, but $e^{\operatorname{ad} a}$ does not commute with d in general).

The idea is then to define the gauge action as a perturbation of the adjoint action which depends on the differential d. This is best understood in the special case where d is the adjoint of some $u \in L^1$, which we'll approach first.

Example 2.44. Let L be a nilpotent differential graded Lie algebra, and suppose that the differential of L is the adjoint of some element: $d = [u, -]$, $u \in L^1$. For $a \in L^0$, define:

$$e^a * x = e^{\operatorname{ad} a}(u + x) - u. \tag{3.1}$$

We wish to show that the above is an action of $\exp(L^0)$ on $\mathrm{MC}(L)$. First, observe that $\exp(L^0)$ acts on L (as a graded Lie algebra) via the adjoint action, in fact:

$$\begin{aligned} e^{\operatorname{ad}(a \bullet b)} &= e^{(\operatorname{ad} a) \bullet (\operatorname{ad} b)} && \text{since } \operatorname{ad} : L \to \mathrm{Hom}^*(L, L) \text{ is a morphism of DGLA} \\ &= e^{\operatorname{ad} a} \circ e^{\operatorname{ad} b} && \text{since the bracket of } \mathrm{Hom}^*(L, L) \text{ is the commutator of } \circ \end{aligned}$$

Then, by writing $e^a * - = \phi^{-1} \circ e^{\operatorname{ad} a} \circ \phi$ where $\phi(x) = u + x$, we immediately see that (3.1) is also an action of $\exp(L^0)$ on L. In fact, one can prove that the action restricts to $\mathrm{MC}(L)$.

Importantly, we can also show that the definition of the gauge action is independent of the choice of u:

$$\begin{aligned} e^a * x &= e^{\operatorname{ad} a}(u+x) - u = \sum_{n\ge 0} \frac{(\operatorname{ad} a)^n}{n!}(u+x) - u = x + \sum_{n\ge 0} \frac{(\operatorname{ad} a)^n}{(n+1)!}([a,u]+[a,x]) \\ &= x + \sum_{n\ge 0} \frac{(\operatorname{ad} a)^n}{(n+1)!}([a,x] - d(a)). \end{aligned}$$

The last expression makes sense in any nilpotent DGLA, and so we can give the following definition.

Definition 2.45. Let L be a nilpotent differential graded Lie algebra. The **gauge action** is the action of $\exp(L^0)$ on $\mathrm{MC}(L)$ defined by:

$$e^a * x = x + \sum_{n\ge 0} \frac{(\operatorname{ad} a)^n}{(n+1)!}([a,x] - d(a)).$$

Example 2.46. Let us consider again the deformations of a bounded complex of $\mathbb{K}$-vector spaces (V,∂), as described in Example 2.18.

Recall that a deformation of (V,∂) over A is a complex of A-modules $(V\otimes A, \partial_A)$ which reduces to the original complex (V,∂) modulo $\mathfrak{m}_A$:

$$0 \longrightarrow V^0\otimes A \xrightarrow{\partial_A} V^1\otimes A \xrightarrow{\partial_A} \cdots \xrightarrow{\partial_A} V^{n-1}\otimes A \xrightarrow{\partial_A} V^n\otimes A \longrightarrow 0$$

The condition that ∂_A reduces to ∂ modulo $\mathfrak{m}_A$ is equivalent to

$$\partial_A = \partial + \xi, \qquad \text{with } \xi \in \operatorname{Hom}^1_A(V\otimes A, V\otimes \mathfrak{m}_A) \cong \operatorname{Hom}^1_{\mathbb{K}}(V,V)\otimes \mathfrak{m}_A,$$

where by a slight abuse of notation we use the same symbol for the differential $\partial : V \to V$ and its A-linear extension $\partial\otimes \mathrm{id}_A : V\otimes A \to V\otimes A$. Then, the condition that $(V\otimes A, \partial_A)$ is a complex becomes

$$0 = \partial_A^2 = (\partial+\xi)^2 = \partial^2 + \partial\xi + \xi\partial + \xi^2 = \partial\xi + \xi\partial + \xi^2,$$

which, in the notation of the differential graded Lie algebra $\operatorname{Hom}^*(V,V)\otimes\mathfrak{m}_A$ to which ξ belongs, is precisely the Maurer-Cartan equation:

$$d\xi + \frac{1}{2}[\xi,\xi] = 0.$$

Now, recall that two deformations $(V\otimes A, \partial_A)$, $(V\otimes A, \partial'_A)$ are deemed equivalent if there exists an isomorphism of complexes $\phi : (V\otimes A, \partial_A) \cong (V\otimes A, \partial'_A)$ which reduces to the identity on V modulo $\mathfrak{m}_A$:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & V^0\otimes A & \xrightarrow{\partial_A} & V^1\otimes A & \xrightarrow{\partial_A} & \cdots & \xrightarrow{\partial_A} & V^{n-1}\otimes A & \xrightarrow{\partial_A} & V^n\otimes A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_1} & & & & \downarrow{\scriptstyle \phi_{n-1}} & & \downarrow{\scriptstyle \phi_n} & & \\ 0 & \longrightarrow & V^0\otimes A & \xrightarrow{\partial'_A} & V^1\otimes A & \xrightarrow{\partial'_A} & \cdots & \xrightarrow{\partial'_A} & V^{n-1}\otimes A & \xrightarrow{\partial'_A} & V^n\otimes A & \longrightarrow & 0 \end{array}$$

Let $\partial_A = \partial + \xi$, $\partial'_A = \partial + \xi'$. The condition that ϕ reduces modulo $\mathfrak{m}_A$ to the identity on V is equivalent to

$$\phi = \mathrm{id}_{V\otimes A} + \eta, \qquad \text{with } \eta \in \operatorname{Hom}^0_A(V\otimes A, V\otimes\mathfrak{m}_A) \cong \operatorname{Hom}^0_{\mathbb{K}}(V,V)\otimes\mathfrak{m}_A.$$

Now, since $\operatorname{Hom}^0_{\mathbb{K}}(V,V)\otimes\mathfrak{m}_A$ is a nilpotent ideal of the unitary $\mathbb{K}$-algebra $\operatorname{Hom}^0_{\mathbb{K}}(V,V)\otimes\mathfrak{m}_A$, and $\mathbb{K}$ is a field of characteristic 0, by Proposition 2.37 we can take the logarithm and write

$$\phi = e^a, \qquad \text{with } a \in \operatorname{Hom}^0_{\mathbb{K}}(V,V)\otimes\mathfrak{m}_A.$$

Then, the condition that ϕ is a morphism of complexes becomes

$$\begin{aligned} & \phi\circ\partial_A = \partial'_A\circ\phi \\ \iff & \partial'_A = \phi\circ\partial_A\circ\phi^{-1} = e^a\circ\partial_A\circ e^{-a} = e^{\operatorname{ad} a}(\partial_A) \\ \iff & \partial + \xi' = e^{\operatorname{ad} a}(\partial+\xi) \\ \iff & \xi' = e^{\operatorname{ad} a}(\partial+\xi) - \partial, \end{aligned}$$

and since the differential of $\mathrm{Hom}^*_{\mathbb{K}}(V,V)\otimes\mathfrak{m}_A$ is the adjoint of ∂, that is precisely the equation of the gauge action:

$$\xi' = e^a * \xi.$$

Hence, we've found that the deformations of a complex (V,∂) over A can be described entirely in terms of the DGLA structure of $\mathrm{Hom}^*_{\mathbb{K}}(V,V)\otimes\mathfrak{m}_A$. More precisely, a deformation $(V\otimes A,\partial_A)$ is completely specified by a Maurer-Cartan element $\xi\in\mathrm{Hom}^1_{\mathbb{K}}(V,V)\otimes\mathfrak{m}_A$, and two elements ξ, ξ' yield the same deformation precisely when they are equivalent under the gauge action.

The above example motivates the following general definitions.

Definition 2.47. Let L be a differential graded Lie algebra. We define the following functors:

(1) the **exponential functor** $\exp_L : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Grp}$

$$\exp_L(A) = \exp(L^0\otimes\mathfrak{m}_A);$$

(2) the **Maurer-Cartan functor** $\mathrm{MC}_L(A) : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$

$$\mathrm{MC}_L(A) = \mathrm{MC}(L\otimes\mathfrak{m}_A) = \{x\in L^1\otimes\mathfrak{m}_A \mid d(x)+\tfrac{1}{2}[x,x]=0\};$$

(3) the **deformation functor associated to** L, $\mathrm{Def}_L : \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$

$$\mathrm{Def}_L(A) = \frac{\mathrm{MC}_L(L)}{\exp(L)} = \frac{\{x\in L^1\otimes\mathfrak{m}_A \mid d(x)+\tfrac{1}{2}[x,x]=0\}}{\text{gauge action}}.$$

The above are all deformation functors. Indeed: as a functor from $\mathbf{Art}_{\mathbb{K}}$ to $\mathbf{Set}$, exp is essentially the formal neighborhood of L^0, and then by the discussion in Example 2.20 it is a smooth deformation functor; also, we can immediately see that MC_L preserves fiber products, and so it is a fortiori a deformation functor; finally, by Theorem 2.23 it follows that the quotient Def_L is a deformation functor as well, and the projection $\mathrm{MC}_L \to \mathrm{Def}_L$ is smooth.

Remark 2.48. For a DG-Lie algebra L, it is important to pay attention to the difference between Def_L and the functor of deformations of L: the latter is associated to a different DG-Lie algebra!

Lemma 2.49. *Let L be a differential graded Lie algebra, and let A be an object in* $\mathbf{Art}_{\mathbb{K}}$. *If $L\otimes\mathfrak{m}_A$ is abelian, then* $\mathrm{Def}_L(A)\cong H^1(L)\otimes\mathfrak{m}_A$.

PROOF. In the case where $L\otimes\mathfrak{m}_A$ is abelian, the Maurer-Cartan equation simplifies to

$$d(x)+\frac{1}{2}[x,x]=0 \iff d(x)=0,$$

and so we immediately see that $\mathrm{MC}_L(A)=Z^1(L\otimes\mathfrak{m}_A)$. Similarly, the gauge action reduces to

$$e^a * x = x + \sum_{n\geq 0}\frac{(\mathrm{ad}\,a)^n}{(n+1)!}([a,x]-d(a)) = x - d(a),$$

and therefore we have

$$\mathrm{Def}_L(A) = \frac{Z^1(L\otimes\mathfrak{m}_A)}{d(L^0\otimes\mathfrak{m}_A)} = H^1(L\otimes\mathfrak{m}_A)\cong H^1(L)\otimes\mathfrak{m}_A.$$

□

The above lemma can be used to compute the tangent space of Def_L. Indeed, when we consider the Artin local ring $\mathbb{K}[t]/(t^2)$, because $t^2=0$ we have that $L\otimes(t)$ is abelian and then

$$T^1\mathrm{Def}_L = \mathrm{Def}_L\left(\frac{\mathbb{K}[t]}{(t^2)}\right)\cong H^1(L)\otimes(t) = H^1(L)\otimes\mathbb{K}t.$$

Because the deformation functor Def_L is defined entirely in terms of the DG-Lie structure of L, it is plain that any morphism of DG-Lie algebras $f: L\to M$ induces a corresponding morphism $f:\mathrm{Def}_L\to\mathrm{Def}_M$. A much less obvious fact is that, whenever $f: L\to M$ is a quasi-isomorphism, $f:\mathrm{Def}_L\to\mathrm{Def}_M$ is an isomorphism. That will be the subject of the next chapter.

CHAPTER 3

Homotopy invariance

We close this dissertation by stating, and proving in part, the *homotopy invariance theorem* of the deformation functor associated to a differential graded Lie algebra. This is a result of utmost importance, especially in light of the principle that any deformation problem in characteristic 0 is controlled by a DG-Lie algebra. On a basic level, the theorem implies that once we've found some DG-Lie algebra L which controls a given deformation problem, we can freely swap L with any L' quasi-isomorphic to it; by choosing L' wisely, this can create major simplifications, and ultimately lead to a better understanding of the deformation problem at hand.

A more sophisticated perspective on homotopy invariance is the following: it can be shown that **DGLA** admits a Quillen model structure where weak equivalences are quasi-isomorphisms (we refer to [7, 8] for the definitions involved), and so there exists a homotopy category Ho(**DGLA**). Roughly speaking, Ho(**DGLA**) can be understood as a localization of **DGLA** where every quasi-isomorphism admits an inverse. More formally, Ho(**DGLA**) comes with a functor **DGLA** → Ho(**DGLA**) that sends quasi-isomorphisms to isomorphisms, and has the universal property that every functor that maps out of **DGLA** and sends sends quasi-isomorphisms to isomorphisms factors uniquely through Ho(**DGLA**) (in the 2-category sense). Then, the homotopy invariance theorem states that the functor Def, mapping each differential graded Lie algebra to its associated deformation functor, factors through the homotopy category:

$$\begin{tikzcd} \mathbf{DGLA} \arrow[r, "\mathrm{Def}"] \arrow[d] & \text{category of deformation functors} \\ \mathrm{Ho}(\mathbf{DGLA}) \arrow[ur, dashed] & \end{tikzcd}$$

This interpretation of homotopy invariance paves the way to even deeper investigations of the relationship between deformation problems and differential graded Lie algebras. One can extend the notion of deformation functor in such a way that it captures so called *derived deformations*, and consider Def as a functor from **DGLA** to the category of *extended* deformation functors, see e.g. [16]; to be precise, the target of the extended functors may be also enlarged to simplicial sets. Then, a celebrated result due to many ideas and contributions (e.g. Deligne, Kontsevich, Manetti, Hinich, Pridham, Lurie) affirms that the unique functor induced by Def on Ho(**DGLA**) is an equivalence of A_∞-categories, and in that sense there is a correspondence between derived deformation problems and quasi-isomorphism classes of differential graded Lie algebras.

The proof of the homotopy invariance theorem requires a significant amount of preliminary results, some of which touch on topics that are not covered in this thesis. For instance, the proof of the following lemma relies on the theory of *obstructions* to deformation functors, and thus will be omitted.

Lemma 3.1. *Let $f : L \to M$ be a morphism of differential graded Lie algebras. Assume that the induced morphism $f : H^i(L) \to H^i(M)$ is:*

(1) surjective for $i = 1$,
(2) injective for $i = 2$.

Then, $f : \mathrm{Def}_L \to \mathrm{Def}_M$ is smooth. In particular, $f : \mathrm{Def}_L(A) \to \mathrm{Def}_M(A)$ is surjective for all A.

The following result in similar in spirit to the above, but is proven with different techniques. In this case, we are able to give at least a partial proof.

Lemma 3.2. *Let $f: L \to M$ be a morphism of differential graded Lie algebras. Assume that the induced morphism $f: H^i(L) \to H^i(M)$ is:*

(1) surjective for $i = 0$,
(2) injective for $i = 1$.

Then, $f: \mathrm{Def}_L(A) \to \mathrm{Def}_M(A)$ is injective for all A.

Proof. Denote by $p_0, p_1 : L \times L \to L$ the canonical projections. By an argument involving *Deligne grupoids* and *homotopy equalizers*, the thesis is reduced to the fact that the natural map $C(p_1 - p_0) \to C(f \circ p_1 - f \circ p_0)$ induces a smooth natural transformation $\mathrm{Def}_{C(p_1-p_0)} \to \mathrm{Def}_{C(f\circ p_1 - f\circ p_0)}$.

Now, the commutative square

$$\begin{tikzcd} L \times L \arrow[r, equal] \arrow[d, "p_1-p_0"] & L \times L \arrow[d, "f\circ p_1 - f\circ p_0"] \\ L \arrow[r, "f"] & M \end{tikzcd}$$

gives a morphism of the short exact sequences that come with the cocones:

$$\begin{tikzcd} 0 \arrow[r] & L[-1] \arrow[r] \arrow[d, "f"] & C(p_1-p_0) \arrow[r] \arrow[d] & L \times L \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M[-1] \arrow[r] & C(f\circ p_1 - f\circ p_0) \arrow[r] & L \times L \arrow[r] & 0 \end{tikzcd}$$

In turn, the above induces a morphism of long exact sequences in cohomology: we have

$$\begin{tikzcd} H^0(L) \arrow[r] \arrow[d, two heads, "f"] & H^1(C(p_1-p_0)) \arrow[r] \arrow[d] & H^1(L \times L) \arrow[r] \arrow[d, equal] & H^1(L) \arrow[d, hook, "f"] \\ H^0(M) \arrow[r] & H^1(C(f\circ p_1 - f\circ p_0)) \arrow[r] & H^1(L \times L) \arrow[r] & H^1(M) \end{tikzcd}$$

and then by the four lemma $H^1(C(p_1 - p_0)) \to H^1(C(f \circ p_1 - f \circ p_0))$ is surjective; we also have

$$\begin{tikzcd} H^1(L \times L) \arrow[r] \arrow[d, equal] & H^1(L) \arrow[r] \arrow[d, hook, "f"] & H^2(C(p_1-p_0)) \arrow[r] \arrow[d] & H^2(L \times L) \arrow[d, equal] \\ H^1(L \times L) \arrow[r] & H^1(M) \arrow[r] & H^2(C(f\circ p_1 - f\circ p_0)) \arrow[r] & H^2(L \times L) \end{tikzcd}$$

and again by the four lemma $H^2(C(p_1 - p_0)) \to H^2(C(f \circ p_1 - f \circ p_0))$ is injective. Hence, we can apply Lemma 3.1, and it follows that $\mathrm{Def}_{C(p_1-p_0)} \to \mathrm{Def}_{C(f\circ p_1 - f\circ p_0)}$ is smooth. □

Theorem 3.3. *Let $f: L \to M$ be a morphism of differential graded Lie algebras. Assume that the induced morphism $f: H^i(L) \to H^i(M)$ is:*

(1) surjective for $i = 0$,
(2) bijective for $i = 1$,
(3) injective for $i = 2$.

Then, $f: \mathrm{Def}_L \to \mathrm{Def}_M$ is a natural isomorphism.

Proof. This follows immediately from Lemma 3.1 and Lemma 3.2 combined. □

At last, as a simple corollary of the above, we have the theorem which concludes this thesis.

Theorem 3.4 (Homotopy invariance of Def_L)**.** *Let $f: L \to M$ be a morphism of differential graded Lie algebras. If $f: L \to M$ is a quasi-isomorphism, then the induced morphism $f: \mathrm{Def}_L \to \mathrm{Def}_M$ is an isomorphism.*

Bibliography

1. Ruggero Bandiera, Marco Manetti, and Francesco Meazzini, *Formality conjecture for minimal surfaces of kodaira dimension* 0, Compositio Math. **157** (2021), 215–235.
2. Nero Budur and Zyuo Zhang, *Formality conjecture for k3 surfaces*, Compositio Math. **155** (2018), 902–911.
3. Pierre Deligne, *Letter to j. millson*, April 24, 1986. Scanned copy available on J. Millson's webpage.
4. Pierre Deligne, Phillip Griffiths, John Morgan, and Dennis Sullivan, *Real homotopy theory of kähler manifolds*, Invent. Math. **29** (1975), 245–274.
5. Murray Gerstenhaber, *On the deformation of rings and algebras*, Annals of Mathematics **79** (1964), no. 1, 59–103.
6. W. M. Goldman and J. John Millson, *The deformation theory of representations of fundamental groups of compact kähler manifolds*, Publ. Math. I.H.E.S. **67** (1988), 43–96.
7. Philip S. Hirschhorn, *Model categories and their localizations*, vol. 99, American Mathematical Society Mathematical Surveys and Monographs, 2002.
8. Mark Hovey, *Model categories*, vol. 63, American Mathematical Society Mathematical Surveys and Monographs, 2007.
9. Donatella Iacono and Marco Manetti, *An algebraic proof of bogomolov-tian-todorov theorem*, Deformation Spaces. Vieweg Verlag **39** (2010), 113–133.
10. Ludmil Katzarkov, Maxim Kontsevich, and Tony Pantev, *Hodge theoretic aspects of mirror symmetry*, Providence, RI: American Mathematical Society. Proceedings of Symposia in Pure Mathematics **78** (2008), 87–174.
11. K. Kodaira and D. C. Spencer, On the Variation of Almost-Complex Structure, 139–150, Princeton University Press, Princeton, 1957, pp. 139–150.
12. K. Kodaira and D. C. Spencer, *A theorem of completeness for complex analytic fibre spaces*, Acta Mathematica **100** (1958), no. 3-4, 281 – 294.
13. K. Kodaira and D. C. Spencer, *On deformations of complex analytic structures, i*, Annals of Mathematics **67** (1958), no. 2, 328–401.
14. K. Kodaira and D. C. Spencer, *A theorem of completeness of characteristic systems of complete continuous systems*, American Journal of Mathematics **81** (1959), no. 2, 477–500.
15. Maxim Kontsevich, *Deformation quantization of poisson manifolds*, I. Letters in Mathematical Physics **66** (2003), 157–216.
16. Marco Manetti, *Extended deformation functors*, Int. Math. Res. Not. **14** (2002), 719–756.
17. Marco Manetti, *Lie methods in deformation theory*, Springer Monographs in Mathematics, 2023.