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Complete Pick Spaces: Theory and Examples

Tesi di Laurea Magistrale in Matematica

Relatore:
Chiar.mo Prof.
Nicola Arcozzi

Correlatori:
Prof. Michael Hartz
Dott. Nikolaos Chalmoukis

Presentata da:
Miriam Abbate

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Introduction

The Nevalinna Pick problem is a complex analysis problem, named after the Finnish mathematician Rolf Nevanlinna and the Austrian mathematician Georg Pick.

The question is the following: given initial data consisting of n points $\lambda_1, \ldots, \lambda_n$ in the complex unit disc $\mathbb{D}$ and target data consisting of n points $z_1, \ldots, z_n$ in $\mathbb{D}$, can we find an analytic function from the disk into itself that interpolates the data?

The problem was independently solved independently by Pick in 1916 and by Nevalinna in 1919. Over the years, further generalizations have been studied and solved and lots of applications in several fields have been found.

In particular, among reproducing kernel Hilbert spaces (RKHS), the ones which satisfy an analog of the property above are called Pick spaces. There are many well-known examples of such spaces. Some of their properties fall within general Pick theory, while other important ones have to be proved case by case.

From this comes the idea of focusing on the study of a really simple and concrete space, that will be introduced in Chapter 3. All the results and proofs about this space which are presented here, are new.

Chapter 1 of this thesis provides an overview of the general theory: we begin by introducing the fundamental concepts of RKHS and some key examples. Furthermore, we will present all the notions needed in the other chapters of this work. The main reference for this chapter is [AgMcC2002].

Chapter 2 introduces the tree model: it is a relatively simple example of RKHS with complete Pick property constructed on a tree. The main reference for this chapter is [Ro2019]. As said before, the main results of this thesis concern the space presented in Chapter 3. It comes as a generalization of the tree example, taking the tree given by the integers and passing to the continuous case.

So, in Chapter 3, we present this complete Pick space of functions defined on the real line. This space is examined in details: many proofs of the Pick property are presented; interpolating sequences, multipliers, Carleson measures are characterized; the Corona problem is solved. Lastly, in Chapter 3 we study invariant subspaces in the Pick space under examination.

Finally, Chapter 4 provides additional information about the previous space, showing some connections with Brownian motion and the Volterra integral operator.

Some of the results from Chapter 3 where proved in collaboration with Nikolaos Chalmoukis (Università di Milano-Bicocca), my co-advisor. Some questions were posed by Michael Hartz (Universität des Saarlandes), my other co-advisor.

Contents

Chapter 1

General theory

In this first chapter we will discuss some basic notions and results that will be used later.

1.1 Reproducing kernel Hilbert spaces

Definition 1.1.1. *Let X be a set and H be a Hilbert space of functions on X. H is called a* ***reproducing kernel Hilbert space*** *(RKHS) on X if for every $z \in X$ the linear functional "evaluation at z" η_z:*

$$H \xrightarrow{\eta_z} \mathbb{C}$$

$$\eta_z(f) = f(z)$$

is bounded on H.

In this thesis all Hilbert spaces under consideration are assumed to be separable.

Remark 1.1.1. *As an application of the Riesz representation theorem, if H is a RKHS on X, for each $z \in X$, there exists a unique $k_z \in H$ such that for every $f \in H$, $f(z) = \eta_z(f) = \langle f, k_z \rangle$.*

Viceversa, if for all $z \in X$ it exists $k_z \in H$ such that for every $f \in H$, $f(z) = \langle f, k_z \rangle$, then H has the bounded point evaluation property: in fact,

$$|\eta_z(f)| = |f(z)| = |\langle f, k_z \rangle| \leq \|f\|_H \|k_z\|_H,$$

i.e. $\|\eta_z\|_{H^*} \leq \|k_z\|_H$.

Furthermore, $\|k_z\|^2 = k(z,z) = k_z(z) = \eta_z(k_z)$, *which implies* $\|\eta_z\|_{H^*} \geq \|k_z\|_H$. *We conclude that*

$$\|\eta_z\|_{H^*} = \|k_z\|_H.$$

Definition 1.1.2. *The function* k_z *is called the* ***reproducing kernel*** *for the point* z. *The function*

$$k : X \times X \mapsto \mathbb{C}$$
$$k(z,w) = k_w(z)$$

is called the reproducing kernel for H.

Proposition 1.1.1. *Let* H *be a RKHS with reproducing kernel* k. *The following properties hold:*

1. $k(z,w) = \overline{k(w,z)}$;
2. $k(z,z) = \|k_z\|^2$;
3. $\|\eta_z\|_{H^*} = \|k_z\|_H$;
4. *if* $\{e_n\}_n$ *is any orthonormal basis for* H, *then for all* $z, w \in X$
$$k(z,w) = \sum_n \overline{e_n(w)} e_n(z);$$
5. k *is positive semi-definite, i.e. for any choice of* $z_1, \dots, z_n \in X$ *and* $c_1, \dots, c_n \in \mathbb{C}$,
$$\sum_{i,j=1}^{n} \overline{c_i} c_j k(z_i, z_j) \geq 0.$$

Proof. 1. $k(z,w) = k_w(z) = \langle k_w, k_z \rangle = \overline{\langle k_z, k_w \rangle} = \overline{k_z(w)} = \overline{k(w,z)}$.

2. $k(z,z) = \langle k_z, k_z \rangle = \|k_z\|^2$.

3.
$$\|\eta_z\|_{H^*} = \sup_{\substack{f\in H\\ f\neq 0}} \frac{|\eta_z(f)|}{\|f\|_H} = \sup_{\substack{f\in H\\ f\neq 0}} \frac{|f(z)|}{\|f\|_H} = \sup_{\substack{f\in H\\ f\neq 0}} \frac{|\langle f, k_z\rangle|}{\|f\|_H} = \frac{\|k_z\|_H^2}{\|k_z\|_H} = \|k_z\|_H.$$

4. As a consequence of the Perseval's identity,
$$\begin{aligned} k(z,w) &= \langle k_w, k_z\rangle \\ &= \sum_n \langle k_w, e_n\rangle\langle e_n, k_z\rangle \\ &= \sum_n \overline{e_n(w)} e_n(z). \end{aligned}$$

5.
$$\begin{aligned} \sum_{i,j=1}^n \overline{c_i} c_j k(z_i, z_j) &= \sum_{i,j=1}^n \overline{c_i} c_j \langle k_{z_j}, k_{z_i}\rangle \\ &= \left\langle \sum_{j=1}^n c_j k_{z_j}, \sum_{i=1}^n c_i k_{z_i} \right\rangle \\ &= \left\| \sum_{i=1}^n c_i k_{z_i} \right\|^2 \geq 0. \end{aligned}$$

□

Also the viceversa of property 5 above holds, as stated in the following theorem.

Theorem 1.1.1. *If $k : X \times X \to \mathbb{C}$ is a positive semi-definite and self-adjoint function, that is not zero on the diagonal, then there exists a unique Hilbert space of functions on X with k as reproducing kernel.*

Proof. Let $k_z : X \to \mathbb{C},\ k_z(\cdot) = k(\cdot, z)$.

Define $\tilde{H} = \text{span}\{k_z : z \in X\}$ to be the pre-Hilbert space with the following inner product:
$$\langle k_x, k_y\rangle = k(y, x).$$

Therefore,
$$\|\sum a_i k_{z_i}\|^2 = \sum_{i,j} a_i \overline{a_j} k(z_j, z_i) \geq 0.$$

We claim that, for some $a_1, \dots, a_n \in \mathbb{C}$ and $z_1, \dots z_n \in X$,

$$\| \sum a_i k_{z_i} \| = 0 \Longrightarrow \sum a_i k_{z_i}(z) = 0 \quad \forall z \in X.$$

In fact, wlog let $z \in \{z_1, \dots, z_n\}$ and consider the following matrix

$$K = \Big[\langle k(z_i, z_j) \rangle \Big]_{i,j=1}^{n}$$

K can be diagonalized because k is linear and self-adjoint. So,

$$K = R^* \Delta R,$$

for some R unitary and Δ diagonal matrix given by the eigenvalues $\lambda_1, \dots, \lambda_n$ of k.

Since k is positive semi-definite, $\lambda_i \geq 0$ for every $i = 1, \dots, n$.

Suppose wlog that $\lambda_1, \dots, \lambda_m > 0$ and $\lambda_{m+1}, \dots, \lambda_n = 0$, for some $m \leq n$.

Hence, calling $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $b = Ra$,

$$\begin{aligned} 0 = a^* K a &= a^* R^* \Delta R a \\ &= b^* \Delta b \\ &= \lambda_1 |b_1|^2 + \dots + \lambda_n |b_n|^2 \\ &= \lambda_1 |b_1|^2 + \dots + \lambda_m |b_m|^2. \end{aligned}$$

It follows that $b_1 = \dots = b_m = 0$. In particular, we have

$$0 = \Delta b \Longrightarrow 0 = R^* \Delta b = K a.$$

We conclude that

$$\| \sum a_i k_{z_i} \| = 0 \Longrightarrow \sum a_i k_{z_i}(z) = 0 \quad \forall z \in X.$$

Now, it suffices to take H as the completion of $\tilde{H}$ in order to prove the existence of the Hilbert space.

To prove the uniqueness of H, let H' be another Hilbert space of function with the same reproducing kernel k. Then the map

$$U : H \to H'$$
$$\sum a_i k_{z_i} \mapsto \sum a_i k_{z_i}$$

is an isometry on a dense set. Therefore, it extends to a unitary $U : H \to H'$ such that $(Uf)(z) = f(z)$, and hence H and H' are equal. □

Definition 1.1.3. *A kernel $k : X \times X \longrightarrow \mathbb{C}$ is said to be* ***normalized*** *at $z_0 \in X$ if $k(z, z_0) = 1 \ \forall \ z \in X$.*

Remark 1.1.2. *Any non-vanishing kernel can be normalized at a point $z_0 \in X$ in the following way:*

$$\overline{k}(z,w) = \frac{k(z,w)k(z_0,z_0)}{k(z,z_0)k(z_0,w)}.$$

$\overline{k} : X \times X \to \mathbb{C}$ is again a reproducing kernel because of Theorem 1.1.1.

The RKHS with normalized kernel is equivalent to the non-normalized one.

Now, we will present some remarkable examples of RKHS.

1.1.1 The Hardy space

Definition 1.1.4. *The* ***Hardy space*** *$H^2(\mathbb{D})$ is the space of f on $\mathbb{D}$, with square summable coefficients, i.e. if $f(z) = \sum_{n=0}^{\infty} a_n z^n$,*

$$H^2(\mathbb{D}) := \left\{ f : \mathbb{D} \to \mathbb{C} : \|f\|^2_{H^2} := \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

Theorem 1.1.2. *$H^2(\mathbb{D})$ is a RKHS, with the following kernel:*

$$k(z,w) := \frac{1}{1 - z\overline{w}} \quad \forall z, w \in \mathbb{D},$$

called Szegö kernel.

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Let's prove that the evaluation functional η_z is bounded:

$$\begin{aligned}|\eta_z(f)| &= \left|\sum_{n=0}^{\infty} a_n z^n\right| \\ &\leq \sum_{n=0}^{\infty} |a_n||z|^n \\ &\leq \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} |z|^{2n}\right)^{\frac{1}{2}} = \|f\|_{H^2} \frac{1}{\sqrt{1-|z|^2}}.\end{aligned}$$

So, η_z is bounded, with $\|\eta_z\|_{(H^2)^*} \leq \frac{1}{\sqrt{1-|z|^2}}$.

It remains to prove that $k(z,w) := \frac{1}{1-z\overline{w}}$ is the reproducing kernel.

$$f(z) = \sum_{n=0}^{\infty} a_n \overline{\overline{z}^n} = \langle f, k_z \rangle,$$

with

$$k_z(w) = \sum_{n=0}^{\infty} \overline{z}^n w^n = \frac{1}{1-w\overline{z}}.$$

□

1.1.2 The Dirichlet space

Definition 1.1.5. *The* ***Dirichlet space*** $\mathcal{D}$ *is the space of holomorphic functions* f *on* $\mathbb{D}$*, for which the semi-norm*

$$\|f\|_*^2 := \int_{\mathbb{D}} |f'|^2 \, dA$$

is finite.

With dA *we denote the integration with respect to the normalized area measure,* $dA = \frac{dxdy}{\pi}$ *(if* $z = x + iy$*).*

We equip $\mathcal{D}$ *with the norm*

$$\|f\|_{\mathcal{D}}^2 = \|f\|_{H^2}^2 + \|f\|_*^2$$

The inner product is:

$$\langle f, g\rangle_{\mathcal{D}} := \langle f, g\rangle_{H^2} + \int_{\mathbb{D}} f' \ \overline{g'} dA.$$

Theorem 1.1.3. *$\mathcal{D}$ is a RKHS, with the following kernel:*

$$k(z,w) = \frac{1}{z\overline{w}} \log\left(\frac{1}{1 - z\overline{w}}\right) \qquad \forall z, w \in \mathbb{D}.$$

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Note that $k(z,w) = \frac{1}{z\overline{w}} \log\left(\frac{1}{1 - z\overline{w}}\right) = \sum_{n=0}^{\infty} \frac{z^n \overline{w}^n}{1+n}$.

Let's prove that the evaluation functional η_z is bounded. As in the proof of Theorem 1.1.2,

$$\begin{aligned} |\eta_z(f)| &\leq \|f\|_{H^2} \frac{1}{\sqrt{1 - |z|^2}} \\ &\leq \|f\|_{\mathcal{D}} \frac{1}{\sqrt{1 - |z|^2}}. \end{aligned}$$

So, η_z is bounded, with $\|\eta_z\|_{\mathcal{D}^*} \leq \frac{1}{\sqrt{1-|z|^2}}$.

It remains to prove that $k(z,w) := \frac{1}{z\overline{w}} \log\left(\frac{1}{1-z\overline{w}}\right)$ is the reproducing kernel:

$$\begin{aligned} \langle f, k_z\rangle_{\mathcal{D}} &= \langle f, k_z\rangle_{H^2} + \langle f, k_z\rangle_* \\ &= \sum_{n=0}^{\infty} a_n z^n \frac{1}{1+n} + \sum_{n=1}^{\infty} a_n z^n \frac{n}{1+n} \\ &= a_0 z^0 + \sum_{n=1}^{\infty} a_n z^n \frac{1}{1+n}(1+n) \\ &= \sum_{n=0}^{\infty} a_n z^n = f(z). \end{aligned}$$

∎

Remark 1.1.3. *The estimate* $\|\eta_z\|_{\mathcal{D}^*} \leq \frac{1}{\sqrt{1-|z|^2}}$ *obtained in the previous proof is very rough. We can easily obtain a better one, using the Remark* 1.1.1*:*

$$\|\eta_z\|^2_{\mathcal{D}^*} = \|k_z\|^2_{\mathcal{D}} = \frac{1}{|z|^2}\log\frac{1}{1-|z|^2}.$$

1.2 Pick property

Among reproducing kernel Hilbert spaces, there are some with an additional property, called (complete) Pick property.

Definition 1.2.1. *The* ***multiplier algebra*** *of a RKHS H of functions on X is*

$$Mult(H) = \{\phi : X \to \mathbb{C} \text{ s.t. } \phi \cdot f \in H \ \forall\ f \in H\}.$$

Elements of $Mult(H)$ are called multipliers.

The multiplier norm of a multiplier ϕ is

$$\|\phi\|_{Mult(H)} = \|M_\phi : f \mapsto \phi f\|_{B(H)}.$$

Remark 1.2.1. *We don't know a priori if the multiplier norm of a multiplier is finite. In fact, it is: since point evaluations are bounded, each multiplication operator M_ϕ has closed graph:*

let $f_n \to f$, $f_n\phi \to g$. Therefore, $\forall\ z$,

$$f_n(z) = \langle f_n, k_z\rangle \to \langle f, k_z\rangle = f(z)$$
$$(f_n\phi)(z) = \langle f_n\phi, k_z\rangle \to \langle g, k_z\rangle = g(z).$$

So, $(f_n\phi)(z) \to g(z)$, $(f_n\phi)(z) \to f(z)\phi(z)$, which implies that $g \equiv f\phi$.

Hence each multiplication operator M_ϕ is bounded by the closed graph theorem. Thus, $\|\phi\|_{Mult(H)} < \infty$ for all $\phi \in Mult(H)$.

Theorem 1.2.1. *Let H be a reproducing kernel Hilbert space of functions on X. Then $Mult(H) \subset L^\infty(X)$.*

Proof. It holds that, $\forall\ z$,

$$M_\phi^* k_z = \overline{\phi}(z) k_z. \tag{1.1}$$

In fact, for every $f \in H$,

$$\begin{aligned}\langle f, M_\phi^* k_z\rangle &= \langle \phi f, k_z\rangle \\ &= \phi(z) f(z) \\ &= \phi(z)\langle f, k_z\rangle \\ &= \langle f, \overline{\phi(z)} k_z\rangle.\end{aligned}$$

Then,

$$|\phi(z)| \leq \|\phi\|_{Mult}.$$

□

Definition 1.2.2. *A RKHS H of functions on X with kernel k is said to be a **Pick space** if whenever $z_1, \ldots, z_n \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$ with*

$$\left[k(z_i, z_j)(1 - \lambda_i \overline{\lambda_j})\right]_{i,j=1}^n \geq 0,$$

then there exists $\Phi \in Mult(H)$ with $\Phi(z_i) = \lambda_i$ for $1 \leq i \leq n$ and $\|\Phi\|_{Mult(H)} \leq 1$.

*We will also say that the reproducing kernel k is a **Pick kernel**.*

The matrix above is usually called Pick matrix.

Remark 1.2.2. *Note that the positivity of the Pick matrix is a necessary condition for the interpolation, i.e. if exists $\Phi \in Mult(H)$ with $\Phi(z_i) = \lambda_i$ for $1 \leq i \leq n$ and $\|\Phi\|_{Mult(H)} \leq 1$, then*

$$\left[k(z_i, z_j)(1 - \lambda_i \overline{\lambda_j})\right]_{i,j=1}^n \geq 0.$$

In fact, if such Φ exists, and we denote by M_Φ the multiplication by Φ, then M_Φ is a contraction, i.e. its operator norm is less or equal then 1. Therefore

$$I - M_\Phi M_\Phi^* \geq 0.$$

In particular, taking $v = \sum_{i=i}^{n} a_i k_{z_i}$, *we have*

$$\left\langle (I - M_\Phi M_\Phi^*) \sum_{j=i}^{n} a_j k_{z_j}, \sum_{i=i}^{n} a_i k_{z_i} \right\rangle \geq 0.$$

It holds (1.1)*:*

$$M_\Phi^* k_{z_j} = \overline{\Phi}(z_j) k_{z_j}.$$

Using the previous fact, we obtain

$$\begin{aligned}
&\left\langle (I - M_\Phi M_\Phi^*) \sum_{j=i}^{n} a_j k_{z_j}, \sum_{i=i}^{n} a_i k_{z_i} \right\rangle = \\
&= \left\langle \sum_{j=i}^{n} a_j k_{z_j}, \sum_{i=i}^{n} a_i k_{z_i} \right\rangle - \left\langle M_\Phi^* \sum_{j=i}^{n} a_j k_{z_j}, M_\Phi^* \sum_{i=i}^{n} a_i k_{z_i} \right\rangle \\
&= \sum_{i,j=i}^{n} a_j \overline{a_i} \left\langle k_{z_j}, k_{z_i} \right\rangle - \sum_{i,j=i}^{n} a_j \overline{a_i} w_i \overline{w_j} \left\langle k_{z_j}, k_{z_i} \right\rangle \\
&= \sum_{i,j=i}^{n} a_j \overline{a_i} (1 - w_i \overline{w_j}) \left\langle k_{z_j}, k_{z_i} \right\rangle \geq 0.
\end{aligned}$$

The argument in the previous remark can be generalized if one wants to do matrix interpolation. Let $\mathbb{C}^s$ and $\mathbb{C}^t$ be respectively a s-dimensional and a t-dimesional Hilbert space with a fixed basis, where t and s can be finite or countably infinite. Let $M_{s\times t}$ denote the s-by-t matrices (if $s = t$ we write simply M_s).

For the interpolation problem, instead of choosing values $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$, one can choose operators $\Lambda_1, \ldots, \Lambda_n \in B(\mathbb{C}^s, \mathbb{C}^t)$.

We will need the definition of tensor product between Hilbert spaces:

Definition 1.2.3. *Let* $(H_1, \langle \cdot, \cdot \rangle_{H_1}), (H_2, \langle \cdot, \cdot \rangle_{H_2})$ *be two Hilbert spaces. Setting*

$$\langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle = \langle f_1, g_1 \rangle_{H_1} \langle f_2, g_2 \rangle_{H_2},$$

and extending to sums, we obtain an inner product on the tensor product $H_1 \otimes H_2$*. By taking its completion, we obtain a new Hilbert space, still denoted by* $H_1 \otimes H_2$ *and called* ***Hilbert space tensor product****.*

In particular, we will consider the Hilbert space tensor product $H \otimes \mathbb{C}^n$, for some $n \in \mathbb{N}$.

Moreover, we have the following definition for multidimensional multipliers:

Definition 1.2.4. *Let H be a Hilbert space of functions on X, $\Phi \in Mult(H \otimes \mathbb{C}^t, H \otimes \mathbb{C}^s)$ is a function*

$$\Phi : X \to B(\mathbb{C}^t, \mathbb{C}^s)$$

with the property that

$$\Phi F \in H \otimes \mathbb{C}^s \quad \forall\ F \in H \otimes \mathbb{C}^t.$$

The norm of this multiplier is

$$\|\Phi\|_{Mult} = \sup_{F \in B_1(H \otimes \mathbb{C}^t)} \|\Phi F\|_{H \otimes \mathbb{C}^t},$$

where $B_1(H \otimes \mathbb{C}^t)$ is the unitary ball of the space.

In the following we will write also $\Phi \in M_{t,s(Mult(H))}$, instead of $\Phi \in Mult(H \otimes \mathbb{C}^t, H \otimes \mathbb{C}^s)$.

Definition 1.2.5. *A RKHS H of functions on X with kernel k is said to have the $M_{s \times t}$* ***Pick property*** *if, for every $z_1, \ldots, z_n \in X$ and $\Lambda_1, \ldots, \Lambda_N \in M_{s \times t}(\mathbb{C})$ with*

$$\left[k(z_i, z_j)(I - \Lambda_i \Lambda_j^*)\right]_{i,j=1}^n \geq 0,$$

then there exists $\Phi \in Mult(H \otimes \mathbb{C}^t, H \otimes \mathbb{C}^s)$ with $\Phi(z_i) = \Lambda_i$ for $1 \leq i \leq n$ and $\|\Phi\|_{Mult} \leq 1$.

Definition 1.2.6. *A RKHS H of functions on X with kernel k is said to be a* ***complete Pick space*** *if has the $M_{s \times t}$ Pick property for all positive integers s and t.*

We will also say that the reproducing kernel k is a ***complete Pick kernel***.

1.3 Complete Pick spaces characterization

There are some useful criteria to establish if a reproducing kernel Hilbert space has the complete Pick property or not.

Definition 1.3.1. *A kernel k on a set X is called **irreducible** if*

1. *k_x and k_y are linearly independent for every $x \neq y$;*

2. *$k(x, y) \neq 0$ for all $x, y \in X$.*

The following theorem by McCullough and Quiggin (see Theorem 7.6 in [AgMcC2002]), gives us a necessary and sufficient condition for our space to be Pick complete.

To prove it we will need the following Kurosh's theorem (see [Ae90]), Parrott's lemma (see [Par78]) and Schur's product theorem (see [Sch18]).

Theorem 1.3.1. *The limit of an inverse spectrum $S = \{X_\alpha, \pi^\alpha_\beta, A\}$ of non-empty compacta is a non-empty compactum.*

In plain words, what does the statement tell us? An inverse spectrum $S = \{X_\alpha, \pi^\alpha_\beta, A\}$ is given by a directed set A such that for each element $\alpha \in A$ there is a compact set X_α and for all $\beta \preceq \alpha$ there exists $\pi^\alpha_\beta : X_\alpha \to X_\beta$ continuous map with the following addictional property: if $\gamma \preceq \beta \preceq \alpha$, then $\pi^\beta_\gamma \circ \pi^\alpha_\beta = \pi^\alpha_\gamma$.

The limit of an inverse spectrum is the subset of the direct product consisting of those elements x_α satysfing $\pi^\alpha_\beta(x_\alpha) = x_\beta \ \forall \ \beta \preceq \alpha$.

The theorem tells us that this limit is non empty.

Proof. Let $X = \prod\{X_\alpha : \alpha \in A\}$ and $p_\alpha : X \to X_\alpha$ be the projection. Set

$$X^{\beta\alpha} = \{Y \in X : p_\beta(Y) = \pi^\alpha_\beta p_\alpha(Y)\}$$

and

$$\prod_\alpha = \bigcap_{\beta \prec \alpha} X^{\beta\alpha}.$$

This set is closed in X. Furthermore, the set is not empty because it coincides with the product

$$\left(\prod_{\beta\preceq\alpha}\pi^{\alpha}_{\beta}(X_\alpha)\right)\times\prod_{\beta\not\preceq\alpha}X_\beta.$$

Thus, $\left\{\prod_{\alpha}\right\}$ is a centered system of non-empty closed subsets of the compactum X, i.e. it has the finite intersection property. By definition,

$$\lim S=\bigcap_{\alpha}\{X^{\beta\alpha},\alpha,\beta\in A,\beta\preceq\alpha\},$$

so we obtain that $\lim S=\bigcap_{\alpha}\prod_{\alpha}\neq\emptyset$. □

Lemma 1.3.1. *Let H_1,K_1,H_2,K_2 be Hilbert spaces and $A\in B(H_1,K_1), B\in B(H_2,K_2), C\in B(H_1,K_2)$ be given operators. Then*

$$\inf\left\{\|W\|:W=\begin{array}{c}\\K_1\\K_2\end{array}\begin{array}{c}\begin{array}{cc}H_1&H_2\end{array}\\\begin{pmatrix}A&B\\C&D\end{pmatrix}\end{array},\ D\in B(H_2,K_2)\right\}$$

is the maximum of

$$\left\|\begin{pmatrix}A&B\end{pmatrix}\right\|$$

and

$$\left\|\begin{pmatrix}A\\C\end{pmatrix}\right\|.$$

Proof. Let ρ be the infimum of the norms of W's. The norm $D\to\|W\|$ is a continuous function of D, so there is a choice of D for which ρ is attained. Let's consider

$$W=\begin{pmatrix}A&B\\C&D\end{pmatrix}$$

such that $\|W\|=\rho$.

Let $P:H_1\oplus H_2\to H_1$, $Q:K_1\oplus K_2\to K_1$ be the orthogonal projections. We wish to prove

$$\rho=\max\{\|QW\|,\|WP\|\}.$$

Wlog suppose $\max\{\|QW\|, \|WP\|\} = 1$. Suppose by contradiction that $\rho > 1$, but $\max\{\|QW\|, \|WP\|\} \leq 1$. We divide three cases according to the size of the spaces:

1. Assume H_1, H_2, K_1, K_2 are all finite dimensional and $\dim(H_2) = 1$. Choose v unit vector in the kernel of $\rho^2 - W^*W$. Let u be any vector in H_2. For any complex number c and any $t \in K_2$,

$$\|W + c[t \otimes u]\| \geq \rho, \tag{1.2}$$

by minimality of W.

The space on which W attains its norm is $\{\mathbb{C}v\}$, because

$$\rho = \|W\| = \sup_{\|y\|=1} \|Wy\| = \sup_{\|y\|=1} \langle y, W^*Wy\rangle^{1/2}$$

if and only if $(\rho^2 - W^*W)(y) = 0$.

Therefore, looking again at (1.2),

$$\begin{aligned}\rho^2 &\leq \|W + c[t \otimes u]\|^2 \\ &= \|W\| + |c|^2\|t\|^2\|u\|^2 + 2\Re(\langle W, c[t \otimes u]\rangle)\end{aligned}$$

which could be false for c small unless

$$[t \otimes u]v \perp Wv.$$

This means that

$$\begin{aligned}0 &= \langle [t \otimes u]v, Wv\rangle \\ &= \langle \langle v, u\rangle t, Wv\rangle \\ &= \langle v, u\rangle \langle t, Wv\rangle,\end{aligned}$$

which implies that $\langle v, u\rangle = 0$ or $\langle t, Wv\rangle = 0$.

If $\langle t, Wv\rangle = 0$ for all t, then Wv would be orthogonal to K_2, and hence

$$\rho = \|Wv\| = \|QWv\| \leq 1,$$

which is a contradiction.

It follows that it must exists a t such that $\langle v, u\rangle = 0$. Remember that u was an arbitrary vector of H_2, and this forces v to be a vector of H_1, and hence

$$\rho = \|Wv\| = \|WPv\| \leq 1,$$

which is again a contradiction.

2. Assume now H_1, H_2, K_1, K_2 all finite dimensional. Let's consider an increasing sequence of subspaces of H_2

$$L_1 \subseteq L_2 \subseteq \ldots L_N = H_2$$

such that $\dim(L_i) = i$ for all $i = 1, \ldots N$. Inductively, choose $D_i \in B(L_i, K_2)$ so that

$$\left\| \begin{pmatrix} A & B|_{L_i} \\ C & D_i \end{pmatrix} \right\| = \max \left\{ \left\| \begin{pmatrix} A & B|_{L_i} \end{pmatrix} \right\|, \left\| \begin{pmatrix} A & B|_{L_{i-1}} \\ C & D_{i-1} \end{pmatrix} \right\| \right\}.$$

3. None of the spaces need to be finite dimensional. Choose increasing finite dimensional subspaces $H_1^i, H_2^i, K_1^i, K_2^i$ whose union is dense in the whole space. Define D_i inductively at each stage as before, and then take the weak* limit.

□

Theorem 1.3.2. *If A and B are both positive semi-definite N-by-N matrices, then so is $A \cdot B$.*

Here $A \cdot B$ is the Schur product of A and B, i.e. $(A \cdot B)_{ij} = A_{ij} B_{ij}$.

Proof. By decomposing A and B as sums of dyads, whose existence is guaranteed by the spectral theorem, assume $A = \sum_i a_i u_i u_i^*$ and $B = \sum_i b_i v_i v_i^*$. Then

$$A \cdot B = \sum_{ij} a_i b_j (u_i u_i^*) \cdot (v_j v_j^*) = \sum_{ij} a_i b_j (u_i \cdot v_j)(u_i \cdot v_j)^*$$

Each $(u_i \cdot v_j)(u_i \cdot v_j)^*$ is a positive semi-definite matrix. Also $a_i b_j > 0$, and we conclude that $A \cdot B$ is positive semi-definite. □

Theorem 1.3.3. *A necessary and sufficient condition for an irreducible kernel k to be a complete Pick kernel is that, for any finite set $\{\lambda_1, \ldots, \lambda_N\}$ of N distinct elements of X, the $(N-1)$-by-$(N-1)$ matrix*

$$F_N = \left(1 - \frac{k_{iN}k_{Nj}}{k_{ij}k_{NN}}\right)_{i,j=1}^{N-1}$$

is positive semi-definite.

Proof. Assume $\lambda_1, \ldots, \lambda_{N-1}$ are given points in X and $W_1, \ldots, W_{N-1}$ are $s \times t$ matrices chosen so that

$$\left[k(\lambda_i, \lambda_j)(I_{s\times s} - W_i W_j^*)\right]_{i,j=1}^{N-1} \geq 0. \tag{1.3}$$

By definition, k is Pick complete if and only if there exists a $\Phi \in M_{s,t}(Mult(H))$ with $\Phi(\lambda_i) = W_i$ for $i = 1, \ldots, N-1$ and $(*)$ $\|\Phi\|_{Mult} \leq 1$.

Condition $(*)$ holds if and only if

$$[k(\zeta, \lambda)(I_{s\times s} - \Phi(\zeta)\Phi(\lambda)^*)] \tag{1.4}$$

is a positive definite function on $X \times X$.

In fact, given T an arbitrary operator,

$$\begin{aligned} \|T\| \leq 1 &\Leftrightarrow \sup_{\|x\|=1} \|T(x)\|^2 \leq 1 \\ &\Leftrightarrow \sup_{\|x\|=1} \|T^*(x)\|^2 \leq 1 \\ &\Leftrightarrow \sup_{\|x\|=1} \langle T^*x, T^*x\rangle \leq 1 \\ &\Leftrightarrow \sup_{\|x\|=1} \langle TT^*x, x\rangle \leq 1 \\ &\Leftrightarrow \sup_{\|x\|=1} (\langle x, x\rangle - \langle TT^*x, x\rangle) \geq 0 \\ &\Leftrightarrow I - TT^* \geq 0. \end{aligned}$$

If, given (1.3), and any distinct node λ_N, one could always find a matrix W_N so that

$$\left[k(\lambda_i, \lambda_j)(I_{s\times s} - W_i W_j^*)\right]_{i,j=1}^{N} \geq 0, \tag{1.5}$$

then (1.4) could be satisfied on any finite set, and hence, by Theorem 1.3.1, one could find a Φ satisfying (1.4) on all of $X \times X$.

In fact, let A be the directed set of all finite subsets of X, partially ordered by inclusion. For each $\alpha \in A$ let Y_α be the set of all maps Φ from α to H that satisfy (1.4). Therefore Y_α is a compact set with respect to the weak topology, because for any $\lambda \in X$, $\Phi(\lambda)$ is in the closed ball of radius 1, which is weakly compact.

Finally, if $\beta \preceq \alpha$, let π^α_β be the restriction map.

Applying Theorem 1.3.1, we get that the inverse limit of this system is not empty. This means that exists a Φ satisfying (1.4) on all $X \times X$.

Let's find out which conditions garantee that for any λ_N exists W_N so that (1.4) holds.

Let $\{u^\alpha\}_{\alpha=1}^s$ be a basis of $\mathbb{C}^s$ and

$$\mathcal{M}_{N-1} := \vee\{k_i \otimes u^\alpha : 1 \leq i \leq N-1, 1 \leq \alpha \leq s\}.$$

Define

$$\begin{aligned} R : \mathcal{M}_{N-1} &\to \mathcal{M}_{N-1} \\ k_i \otimes u^\alpha &\mapsto k_i \otimes W_i^* u^\alpha \end{aligned}$$

and extend it by linearity.

Then R is a contraction if and only if, for every sequence $\{a_i^\alpha\}$ of scalars,

$$\begin{aligned} 0 \leq &\left\langle (I - R^*R) \sum_{j,\beta} a_j^\beta k_j \otimes u^\beta, \sum_{i,\alpha} a_i^\alpha k_i \otimes u^\alpha \right\rangle = \\ = &\sum_{i,j,\alpha,\beta} a_j^\beta \overline{a}_i^\alpha k_{ij} (\langle u^\beta, u^\alpha \rangle - \langle W_i W_j^* u^\beta, u^\alpha \rangle) \end{aligned}$$

(that is (1.3)).

For each choice of $W = W_N$ we get an extension R_W of R on $\mathcal{M}_N$ such that

$$R_W : k_N \otimes u^\alpha \mapsto k_N \otimes W^* u^\alpha.$$

We have to determine whether there is some W for which R_W is a contraction.

On $\mathcal{M}_N$, consider P the orthogonal projection onto $\mathcal{M}_{N-1}$. Define Q in $\vee\{k_i \otimes \mathbb{C}^t : 1 \leq i \leq N\}$ to be the orthogonal projection onto the orthogonal complement of $\mathbb{C}k_N \otimes \mathbb{C}^t$.

Decompose R_W as

$$\begin{array}{cc} & \begin{array}{cc} P & P^\perp \end{array} \\ \begin{array}{c} Q \\ Q^\perp \end{array} & \begin{pmatrix} QR_W P & QR_W P^\perp \\ R_W P & Q^\perp R_W P^\perp \end{pmatrix} \end{array} \tag{1.6}$$

To calculate $Q^\perp R_W P^\perp$, let ξ_i be a dual basis to k_i, so

$$\langle k_i, \xi_j \rangle = \delta_{ij}, \qquad \forall\, i, j = 1, \ldots, N.$$

(A dual basis exists because k is an irreducible kernel). Write

$$\xi_N = \sum_{i=1}^{N} c_i k_i.$$

Then

$$R_W P^\perp \xi_N \otimes u^\alpha = \sum_{i=1}^{N-1} c_i k_i \otimes W_i^* u^\alpha + c_N k_N \otimes W^* u^\alpha,$$

so

$$Q^\perp R_W P^\perp \xi_N \otimes u^\alpha = k_N \otimes \left[\sum_{n=1}^{N-1} c_i \frac{k_{Ni}}{k_{NN}} W_i^* + c_N W^* \right] u^\alpha.$$

As we are free to choose W (and $c_N \neq 0$ again because k is an irreducible kernel), the (2,2) entry of the matrix (1.6) can be chosen arbitrarily.

The other 3 entries do not depend on the choice of W. In fact, using the computation above,

$$\begin{aligned} QR_W P^\perp \xi_N \otimes u^\alpha &= Q \left[\sum_{i=1}^{N-1} c_i k_i \otimes W_i^* u^\alpha + c_N k_N \otimes W^* u^\alpha \right] \\ &= Q \sum_{i=1}^{N-1} c_i k_i \otimes W_i^* u^\alpha, \end{aligned}$$

so $QR_W P^\perp$ does not depend on W.

Therefore, the smallest norm of (1.6) is the same as the smallest norm of a matrix completion of

$$\begin{array}{cc} & \begin{array}{cc} P & P^{\perp} \end{array} \\ \begin{array}{c} Q \\ Q^{\perp} \end{array} & \begin{pmatrix} QR_WP & QR_WP^{\perp} \\ R_WP & * \end{pmatrix} \end{array} \tag{1.7}$$

Using Lemma 1.3.1, we obtain that

$$\begin{aligned} \inf_{W \in M_{s\times t}} \|R_W\| &= \max(\|R_WP\|, \|QR_W\|) \\ &= \max(\|R\|, \|QR_W\tilde{Q}\|), \end{aligned}$$

where $\tilde{Q}$ is the analogous operator to Q in $\mathcal{M}_N$, i.e. the projection onto the orthogonal complement of $\mathbb{C}k_N \otimes \mathbb{C}^s$. In fact,

$$R_WQ^{\perp} = Q^{\perp}R_W\tilde{Q}^{\perp}, \quad Q^{\perp} = I - Q, \quad \tilde{Q}^{\perp} = I - \tilde{Q}$$

implies that

$$QR_W = R_w - R_wQ^{\perp} = R_W - Q^{\perp}R_W\tilde{Q}^{\perp} = QR_W\tilde{Q}.$$

We initially wanted to determine whether R_W is a contraction. Having assumed that R is a contraction, using the last equality we are left to determine whether $QR_W\tilde{Q}$ is, i.e.

$$Q - \tilde{Q}R_W^*QQR_W\tilde{Q} \geq 0.$$

This is a $(N-1)s \times (N-1)s$ matrix which entries are:

$$
\begin{aligned}
&\langle [Q - \tilde{Q} R_W^* Q Q R_W \tilde{Q}] k_j \otimes u^\beta, k_i \otimes u^\alpha \rangle = \\
=&\langle Q k_j \otimes u^\beta, k_i \otimes u^\alpha \rangle - \langle \tilde{Q} R_W^* Q Q R_W \tilde{Q} k_j \otimes u^\beta, k_i \otimes u^\alpha \rangle \\
=&\langle Q k_j \otimes u^\beta, k_i \otimes u^\alpha \rangle - \langle Q R_W \tilde{Q} k_j \otimes u^\beta, Q R_W \tilde{Q} k_i \otimes u^\alpha \rangle \\
=&\left\langle \left(k_j - \frac{\langle k_j, k_N \rangle}{\|k_N\|^2} k_N \right) \otimes u^\beta, k_i \otimes u^\alpha \right\rangle \\
-&\left\langle Q R_W \left(k_j - \frac{\langle k_j, k_N \rangle}{\|k_N\|^2} k_N \right) \otimes u^\beta, Q R_W \left(k_i - \frac{\langle k_i, k_N \rangle}{\|k_N\|^2} k_N \right) \otimes u^\alpha \right\rangle \\
=&\left\langle k_j - \frac{\langle k_j, k_N \rangle}{\|k_N\|^2} k_N, k_i \right\rangle \langle u^\beta, u^\alpha \rangle - \left\langle Q R_W k_j \otimes u^\beta, Q R_W k_i \otimes u^\alpha \right\rangle \\
=&\left(k_{ij} - \frac{k_{iN} k_{Nj}}{k_{NN}} \right) \langle u^\beta, u^\alpha \rangle - \langle Q k_j, Q k_i \rangle \langle W_j^* u^\beta, W_i^* u^\alpha \rangle \\
=&\left(k_{ij} - \frac{k_{iN} k_{Nj}}{k_{NN}} \right) \langle u^\beta, u^\alpha \rangle - \left\langle k_j - \frac{k_{Nj}}{k_{NN}} k_N, k_i - \frac{k_{iN}}{k_{NN}} k_N \right\rangle \langle W_i W_j^* u^\beta, u^\alpha \rangle \\
=&\left(k_{ij} - \frac{k_{iN} k_{Nj}}{k_{NN}} \right) \langle u^\beta, u^\alpha \rangle - \left(k_{ij} + \frac{k_{Nj} k_{iN}}{k_{NN}^2} k_{NN} \right) \langle W_i W_j^* u^\beta, u^\alpha \rangle \\
=&k_{ij} (\langle u^\beta, u^\alpha \rangle - \langle W_i W_j^* u^\beta, u^\alpha \rangle) - \frac{k_{iN} k_{Nj}}{k_{NN}} (\langle u^\beta, u^\alpha \rangle - \langle W_i W_j^* u^\beta, u^\alpha \rangle) \\
=&[\langle (I - W_i W_j^*) u^\beta, u^\alpha \rangle k_{ij}] \left[1 - \frac{k_{iN} k_{Nj}}{k_{NN} k_{ij}} \right]
\end{aligned}
$$

where we have used the fact that $\tilde{Q} k_N \otimes v = 0$ for any v.

So, we have obtained that

$$
Q - \tilde{Q} R_W^* Q Q R_W \tilde{Q} = [(I_{s \times s} - W_i W_j^*) k(\lambda_i, \lambda_j)] \cdot [F_N \otimes J], \tag{1.8}
$$

where J is the s-by-s matrix all of whose entries are 1 (with respect to $\{u^\alpha\}$ and $\cdot$ denotes the Schur product between the two matrices, that is the entrywise product.

Hence, if F_N is positive, then $F_N \otimes J$ is positive and by Theorem 1.3.2, (1.8) is positive whenever (1.3) holds. In this case $Q R_W \tilde{Q}$ is a contraction.

Therefore positivity of F_N is a sufficient condition for k to be a complete Pick kernel.

It remains to prove that it is also a necessary condition to conclude the proof.

Let's do it by induction on N.

If $N = 2$,

$$F_2 = 1 - \frac{k_{12}k_{21}}{k_{11}k_{22}}$$

and $k_{12}k_{21} \leq k_{11}k_{22}$ by Cauchy Schwarz.

Let's assume now that F_{N-1} is positive and let's prove that F_N is positive too.

Let G be any positive $(N-1)$-by-$(N-1)$ matrix. Let $s = 1$ and $t = N-1$. Choose vectors $v_i \in \mathbb{C}^t$ so that

$$G_{ij} = \langle v_j, v_i \rangle,$$

and let $W_i^* \in B(\mathbb{C}, \mathbb{C}^t)$ be the matrix that sends 1 to v_i.

Then (1.3) is

$$k_{ij}(1 - G_{ij}).$$

If K is the matrix (k_{ij}), from the discussion above it follows

$$\begin{cases} G \geq 0 \\ [K \cdot (J - G)] \geq 0 \end{cases} \quad \Rightarrow \; F_N \cdot [K \cdot (J - G)] \geq 0.$$

Let H be the $(N-1)$-by-$(N-1)$ matrix whose entries are

$$H_{ij} = \frac{k_{i(N-1)}k_{(N-1)j}}{k_{(N-1)(N-1)}}$$

Note that H has rank one because the j-th row can be obtained by the i-th one by multiplication for $\frac{k_{j(N-1)}}{k_{i(N-1)}}$, given the fact that $k_{i(N-1)} \neq 0$ because of the irriducibility of k.

Let G be the $(N-1)$-by-$(N-1)$ matrix whose entries are

$$G_{ij} = 1 - \frac{H_{ij}}{k_{ij}}.$$

Therefore,

$$G = \begin{pmatrix} & & & 0 \\ & F_{N-1} & & 0 \\ & & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

that is positive by induction hypothesis. Moreover,

$$K \cdot (J - G) = H \geq 0.$$

Therefore $F_N \cdot H$ is positive.

But H has rank one, and no entries are 0, so the matrix $\frac{1}{H}$ is also positive and again using the Theorem (1.3.2) with

$$F_N = F_N \cdot H \cdot \frac{1}{H},$$

we conclude that F_N is positive. □

Another necessary and sufficient condition is given by the following theorem:

Theorem 1.3.4. *An irriducible kernel k on X has the complete Pick property if and only if, for any finite set of distinct points $\lambda_1, \ldots, \lambda_N$, the matrix*

$$L_N := \left(\frac{1}{k_{ij}}\right)_{i,j=1}^{N}$$

has exactly one positive eigenvalue.

Proof. The matrix L_N is congruent to

$$\left(\begin{array}{c|c} \frac{1}{k_{ij}} - \frac{k_{NN}}{k_{iN}k_{Nj}} & 0 \\ \hline 0 & \frac{1}{k_{NN}} \end{array}\right).$$

In fact,

$$\begin{pmatrix} & & -\frac{k_{NN}}{k_{1N}} \\ I_{N-1} & & \vdots \\ & & -\frac{k_{NN}}{k_{N-1N}} \\ 0 & \ldots \; 0 & 1 \end{pmatrix} L_N \begin{pmatrix} & & 0 \\ & I_{N-1} & \vdots \\ & & 0 \\ -\frac{k_{NN}}{k_{N1}} & \ldots - \frac{k_{NN}}{k_{NN-1}} & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} & & -\frac{k_{NN}}{k_{1N}} \\ I_{N-1} & & \vdots \\ & & -\frac{k_{NN}}{k_{N-1N}} \\ 0 & \ldots \; 0 & 1 \end{pmatrix} \begin{pmatrix} & & \frac{1}{k_{1N}} \\ & \frac{1}{k_{ij}} & \vdots \\ & & \frac{1}{k_{N-1N}} \\ \frac{1}{k_{N1}} & \ldots \; \frac{1}{k_{NN-1}} & \frac{1}{k_{NN}} \end{pmatrix} \begin{pmatrix} & & 0 \\ & I_{N-1} & \vdots \\ & & 0 \\ -\frac{k_{NN}}{k_{N1}} & \ldots - \frac{k_{NN}}{k_{NN-1}} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{k_{11}} - \frac{k_{NN}}{k_{1N}k_{N1}} & \cdots & \frac{1}{k_{1N-1}} - \frac{k_{NN}}{k_{1N}k_{NN-1}} & 0 \\ \frac{1}{k_{21}} - \frac{k_{NN}}{k_{2N}k_{N1}} & \cdots & \frac{1}{k_{2N-1}} - \frac{k_{NN}}{k_{2N}k_{NN-1}} & 0 \\ \vdots & \cdots & \vdots & \vdots \\ \frac{1}{k_{N-11}} - \frac{k_{NN}}{k_{N-1N}k_{N1}} & \cdots & \frac{1}{k_{N-1N-1}} - \frac{k_{NN}}{k_{N-1N}k_{NN-1}} & 0 \\ \frac{1}{k_{N1}} & \cdots & \frac{1}{k_{NN-1}} & \frac{1}{k_{NN}} \end{pmatrix} \begin{pmatrix} & I_{N-1} & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ -\frac{k_{NN}}{k_{N1}} & \cdots & -\frac{k_{NN}}{k_{NN-1}} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{k_{11}} - \frac{k_{NN}}{k_{1N}k_{N1}} & \cdots & \frac{1}{k_{1N-1}} - \frac{k_{NN}}{k_{1N}k_{NN-1}} & 0 \\ \vdots & \cdots & \vdots & \vdots \\ \frac{1}{k_{N-11}} - \frac{k_{NN}}{k_{N-1N}k_{N1}} & \cdots & \frac{1}{k_{N-1N-1}} - \frac{k_{NN}}{k_{N-1N}k_{NN-1}} & 0 \\ 0 & \cdots & 0 & \frac{1}{k_{NN}} \end{pmatrix}$$

The top left entry is the Shur product of F_N with the rank one negative matrix

$$-\frac{k_{NN}}{k_{iN}k_{Nj}}.$$

In fact, $(F_N)_{ij} = 1 - \frac{k_{iN}k_{Nj}}{k_{ij}k_{NN}}$ and

$$\left(1 - \frac{k_{iN}k_{Nj}}{k_{ij}k_{NN}}\right) \cdot \left(-\frac{k_{NN}}{k_{iN}k_{Nj}}\right) = \frac{1}{k_{ij}} - \frac{k_{NN}}{k_{iN}k_{Nj}}.$$

We conclude that L_N has one positive eigenvalue $(1/k_{NN})$ if and only if F_N is positive semi-definite. □

There is another characterization of complete Pick spaces given by the following Theorem (see for example [AgMcC2002]).

Theorem 1.3.5. *Suppose k is an irreducible kernel on the set X. Then k has the complete Pick property if and only if there exists a positive semi-definite function $G : X \times X \to \mathbb{D}$ and a nowhere vanishing function δ on X such that*

$$k(\zeta, \lambda) = \frac{\overline{\delta(\zeta)}\delta(\lambda)}{1 - G(\zeta, \lambda)}. \tag{1.9}$$

Proof. $\Leftarrow$) If $k(\zeta, \lambda) = \frac{\overline{\delta(\zeta)}\delta(\lambda)}{1-G(\zeta,\lambda)}$, then

$$\frac{1}{k(\zeta, \lambda)} = \frac{1}{\overline{\delta(\zeta)}\delta(\lambda)} - \frac{G(\zeta, \lambda)}{\overline{\delta(\zeta)}\delta(\lambda)}.$$

So $\frac{1}{k}$ is a rank one matrix minus a positive semi-definite matrix, and by Theorem (1.3.4) k has the complete Pick property.

⇒) Suppose now that k has the complete Pick property. Fix any point λ_0 in X. Then

$$G(\zeta,\lambda) := 1 - \frac{k(\zeta,\lambda_0)k(\lambda_0,\lambda)}{k(\zeta,\lambda)k(\lambda_0,\lambda_0)}$$

is positive semi-definite by Theorem 1.3.3. Let

$$\delta(\lambda) := \frac{k(\lambda_0,\lambda)}{\sqrt{k(\lambda_0,\lambda_0)}}.$$

Then (1.9) holds by construction.

By Cauchy-Schwarz $|k(\zeta,\lambda_0)|^2 \leq k(\zeta,\zeta)k(\lambda_0,\lambda_0)$, so it follows that $G(\zeta,\zeta)$ is in the interval $[0,1)$ for all ζ (it cannot attain the value 1 because k is irreducible). Therefore, as G is positive semi-definite, we have $|G(\zeta,\lambda)| < 1$ for all ζ and λ.

□

Now we are going to prove that the previous examples of RKHS have the complete Pick property. To do that, we are going to need some theorems.

Theorem 1.3.6. *Suppose H is a holomorphic reproducing kernel Hilbert space on $\mathbb{D}$ with*

$$k(\zeta,\lambda) = \sum_{n=0}^{\infty} a_n \overline{\lambda}^n \zeta^n.$$

Let the Taylor coefficients of $\frac{1}{k}$ at zero be

$$\frac{1}{\sum_{n=0}^{\infty} a_n t^n} = \sum_{n=0}^{\infty} c_n t^n.$$

Then H has the complete Pick property if and only if

$$c_n \leq 0 \quad \forall\, n \geq 1.$$

Proof. ⇐) Let's assume that $a_0 = 1 = c_0$, and $c_n \leq 0$ for all $n \geq 1$. Then

$$1 - \frac{1}{k(\zeta,\lambda)} = \sum_{n=1}^{\infty} (-c_n) \overline{\lambda}^n \zeta^n,$$

so

$$k(\zeta,\lambda)=\frac{1}{1-\sum_{n=1}^{\infty}(-c_n)\overline{\lambda}^n\zeta^n}$$

and we conclude that the kernel has the complete Pick property by Theorem 1.3.5 with $\delta=1$ and $G(\zeta,\lambda)=\sum_{n=1}^{\infty}(-c_n)\zeta^n\overline{\lambda}^n$.

$\Rightarrow$) Again we assume that $a_0=1=c_0$.

Suppose that exists N s.t. $c_N>0$ and $c_1,\dots,c_{N-1}\le 0$. Choose $\epsilon>0$ enough small so that

$$\sum_{N+1}^{\infty}|c_n|\epsilon^{2n}<\epsilon^{2N}c_N \tag{1.10}$$

Let

$$\omega=e^{\frac{2\pi i}{N}}$$

be a root of unity.

Select points λ_j by

$$\begin{aligned}\lambda_j&=\epsilon\omega^j, \qquad 1\le j\le N\\ \lambda_{N+1}&=0.\end{aligned}$$

Then

$$\begin{aligned}(F_{N+1})_{ij}&=1-\frac{k_{i(N+1)}k_{(N+1)j}}{k_{ij}k_{(N+1)(N+1)}}\\
&=1-k_{i(N+1)}k_{(N+1)j}\left(\sum_{n=0}^{\infty}c_n\overline{\lambda_j}^n\lambda_i^n\right)\left(\sum_{n=0}^{\infty}c_n\overline{\lambda_{(N+1)}}^n\lambda_{(N+1)}^n\right)\\
&=1-a_0^2c_0\left(\sum_{n=0}^{\infty}c_n\overline{\lambda_j}^n\lambda_i^n\right)\\
&=-\left(\sum_{n=1}^{N-1}c_n\epsilon^n\omega^{-nj}\epsilon^n\omega^{ni}+c_N\epsilon^N\omega^{-Nj}\epsilon^N\omega^{Ni}+\sum_{n=N+1}^{\infty}c_n\epsilon^n\omega^{-nj}\epsilon^n\omega^{ni}\right)\\
&=\sum_{n=1}^{N-1}(-c_n)\epsilon^{2n}\omega^{n(i-j)}-c_N\epsilon^{2N}-\sum_{n=N+1}^{\infty}(c_n)\epsilon^{2n}\omega^{n(i-j)}.\end{aligned}$$

Now note that $(F_{N+1})_{ij}$ is given by the sum for $n = 1, \ldots, N-1$ of

$$-(c_n)\epsilon^{2n} \begin{pmatrix} 1 \\ \omega^{n} \\ \omega^{2n} \\ \vdots \\ \omega^{(N-1)n} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \omega^{-n} \\ \omega^{-2n} \\ \vdots \\ \omega^{-(N-1)n} \end{pmatrix}$$

plus

$$-(c_N)\epsilon^{2N} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

plus the other terms.

Each of the first $N-1$ matrices has only one (non zero) positive eigenvalue, that is $-c_n\epsilon^{2n}N$, whose eigenvector is

$$\begin{pmatrix} \omega^{(N-1)n} \\ \omega^{(N-2)n} \\ \omega^{(N-3)n} \\ \vdots \\ 1 \end{pmatrix}.$$

The N-th matrix has only one (non zero) negative eigenvalue, that is $-c_N\epsilon^{2N}N$, whose eigenvector is

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

That eigenvectors are orthogonal. In fact,

$$\left\langle \begin{pmatrix} \omega^{(N-1)n} \\ \omega^{(N-2)n} \\ \omega^{(N-3)n} \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} \omega^{(N-1)m} \\ \omega^{(N-2)m} \\ \omega^{(N-3)m} \\ \vdots \\ 1 \end{pmatrix} \right\rangle = \omega^{N(n+m)} \sum_{k=1}^{N} w^{-k}$$

$$= \frac{1-\omega^{-(N+1)}}{1-w^{-1}} - 1$$

$$= \frac{\omega^{-N}+1}{\omega - 1} = 0.$$

Hence, it follows that the matrix given by the sum of the first N matrices has one negative eigenvalue.

By inequality (1.10), the sum of the other terms (from the (N+1)-st one) has norm less than $c_N \epsilon^{2N} N$, so adding these terms cannot eliminate the negative eigenvalue.

We conclude that F_{N+1} is not positive definite and therefore the kernel cannot have the complete Pick property. □

Corollary 1.3.6.1. *The Hardy space H^2 has the complete Pick property.*

Proof. The reciprocal of the Szegö kernel is

$$\frac{1}{k(\zeta, \lambda)} = 1 - \overline{\lambda}\zeta.$$

□

To prove that the Dirichlet space has the complete Pick property we are going to use the following Lemma, due to Kaluza (see [AgMcC2002] for a proof).

Lemma 1.3.2. *Suppose $a_0 = 1$ and a_n are strictly positive numbers satisfying*

$$\frac{a_n}{a_{n-1}} \le \frac{a_{n+1}}{a_n} \qquad \forall\ n \ge 1.$$

Then $c_n < 0$ for all $n \ge 1$, where c_n is defined by

$$\frac{1}{\sum_{n=0}^{\infty} a_n t^n} = \sum_{n=0}^{\infty} c_n t^n.$$

Proof. Consider

$$\sum_{n=0}^{\infty} c_n t^n \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} c_k t^k a_{n-k} t^{n-k}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} c_k a_{n-k} t^n.$$

For $m \geq 1$, the coefficients of t^m and t^{m+1} are

$$\sum_{k=0}^{m} c_k a_{m-k} = c_0 a_m + c_1 a_{m-1} + \cdots + c_m a_0 = 0 \tag{1.11}$$

and

$$\sum_{k=0}^{m+1} c_k a_{m+1-k} = c_0 a_{m+1} + c_1 a_m + \cdots + c_{m+1} a_0 = 0. \tag{1.12}$$

Multiplying (1.11) by $\frac{a_{m+1}}{a_m}$ and subtract from (1.12), we get

$$0 = c_0 a_{m+1} + c_1 a_m + \cdots + c_{m+1} a_0 - c_0 a_{m+1} - c_1 \frac{a_{m+1} a_{m-1}}{a_m} - c_m \frac{a_{m+1} a_0}{a_m}$$
$$= c_1 \left(a_m - \frac{a_{m+1} a_{m-1}}{a_m} \right) + \cdots + c_m \left(a_1 - \frac{a_{m+1} a_0}{a_m} \right) + c_{m+1}.$$

It follows that

$$c_{m+1} = c_m \left(\frac{a_{m+1} a_0}{a_m} - a_1 \right) + \cdots + c_1 \left(\frac{a_{m+1} a_{m-1}}{a_m} - a_m \right)$$
$$= c_m a_0 \left(\frac{a_{m+1}}{a_m} - \frac{a_1}{a_0} \right) + \cdots + c_1 a_{m-1} \left(\frac{a_{m+1}}{a_m} - \frac{a_m}{a_{m-1}} \right).$$

As $c_1 = -a_1 < 0$, and all the term in the parentheses are non-negative, it follows by induction that $c_m < 0$ for all $m \leq 1$. □

Corollary 1.3.6.2. *The Dirichlet space $\mathcal{D}$ has the complete Pick property.*

Proof. Recall that the Dirichlet kernel is

$$k(z, w) = \frac{1}{z\overline{w}} \log \left(\frac{1}{1 - z\overline{w}} \right) = \sum_{n=0}^{\infty} \frac{z^n \overline{w}^n}{1+n}$$

Therefore, $a_n = \frac{1}{1+n}$.

$$\frac{a_n}{a_{n-1}} = \frac{n}{1+n} \leq \frac{n+1}{n+2} = \frac{a_{n+1}}{a_n},$$

and by applying Lemma 1.3.2 and Theorem 1.3.6 we conclude that $\mathcal{D}$ has the complete Pick property. $\square$

We are now going to introduce the Drury-Arveson space, which in many ways generalizes the classical Hardy space H^2.

Definition 1.3.2. *Let* $\mathbb{B}_d = \{z \in \mathbb{C}^d : \|z\|_2 < 1\}$.

*The **Drury-Arveson space** is the RKHS on* $\mathbb{B}_d$ *with reproducing kernel*

$$h(z, w) = \frac{1}{1 - \overline{w} \cdot z}.$$

Remark 1.3.1. *The Drury-Arveson space has the complete Pick property. In fact, we can apply Theorem* 1.3.5*, with* $b \equiv 1$, $G(z, w) = \overline{w} \cdot z$.

The Drury-Arveson space is "universal" among those Pick spaces, in the sense of the following Theorem, due to Agler and McCarthy (see for example [AgMcC2002]).

Theorem 1.3.7. *A RKHS* H *on a set* X *with irreducible kernel* k *is a complete Pick space if and only if, for some* $1 \le d \le \infty$*, there is an injection* $F : X \to \mathbb{B}_d$ *and a nowhere-vanishing function* δ *on* X *such that*

$$k(\zeta, \lambda) = \overline{\delta(\zeta)}\delta(\lambda)h(F(\zeta), F(\lambda)). \tag{1.13}$$

Moreover, the map

$$k_\lambda \mapsto \delta(\lambda)[h]_{F(\lambda)}$$

extends to an isometric linear embedding of H *into* δH^2_d.

Proof. $\Leftarrow$) It follows by Theorem 1.3.5.

$\Rightarrow$) Represent k as in Theorem 1.3.5,

$$k(\zeta, \lambda) = \frac{\overline{\delta(\zeta)}\delta(\lambda)}{1 - G(\zeta, \lambda)}.$$

Let $d \ge \text{rank}(G)$. Then, there is a map F from X into a d-dimensional Hilbert space M so that

$$G(\zeta, \lambda) = \langle F(\zeta), F(\lambda) \rangle. \tag{1.14}$$

In fact, define $\tilde{F} : X \to M$ by $\tilde{F}(\lambda) = G_\lambda$. Fix some orthonormal basis $\{e^\alpha\}$ for M. Define the conjugate linear operator C by

$$C\left(\sum c_\alpha e^\alpha\right) = \sum \overline{c_\alpha} e^\alpha.$$

Now define $F(\lambda) = C\tilde{F}(\lambda)$. Therefore we obtain (1.14).

As G takes values in $\mathbb{D}$, it follows that the range of F is contained in the unit ball $\mathbb{B}_d$. Therefore (1.13) holds.

It says that the map

$$k_\lambda(\cdot) \mapsto \frac{\delta(\lambda)}{1 - \langle F(\lambda), F(\cdot)\rangle}$$

extends linearly to an isometry on the span of $\{k_\lambda : \lambda \in X\}$, and hence gives an isometric embedding of H into δH^2_d.

It remains to prove that F is injective: suppose that $F(\lambda_1) = F(\lambda_2)$. Then, k_{λ_1} and k_{λ_2} are linearly dependent. As complete Pick kernels are positive definite functions, it follows that $\lambda_1 = \lambda_2$.

□

Corollary 1.3.7.1. *A RKHS H on a set X with normalized kernel k is a complete Pick space if and only if, for some $1 \leq d \leq \infty$, there is an injection $F : X \to \mathbb{B}_d$ such that*

$$k(\zeta, \lambda) = h(F(\zeta), F(\lambda)). \tag{1.15}$$

Moreover, the map

$$k_\lambda \mapsto [h]_{F(\lambda)}$$

extends to an isometric linear embedding of H into H^2_d.

1.4 Interpolating sequences

In this section we are going to talk about interpolating sequences, that are sequences for which one can choose functions values independently in every point. We will first give some definitions and then some characterizing theorems.

Definition 1.4.1. *Let k be a kernel on a set X, $\{\lambda_i\}$ be a sequence of distinct points in X. Let $g_i = \frac{k_i}{\|k_i\|}$ be the normalized kernel functions. The **Grammian** associated to the sequence $\{\lambda_i\}$ is the infinite matrix*

$$G_{ij} = \langle g_j, g_i \rangle = \frac{k_{ij}}{\|k_i\|\|k_j\|}.$$

Definition 1.4.2. *Given X a space and k a kernel on X, define*

$$d(\lambda_1, \lambda_2) = \sqrt{1 - \frac{|k(\lambda_1, \lambda_2|^2}{k(\lambda_1, \lambda_1)k(\lambda_2, \lambda_2)}},$$

*which is a pseudo-metric on X, called **Gleason distance**.*

Definition 1.4.3. *Let H be a RKHS of function on X. A sequence $\{\lambda_n\}_n$ in X is **ϵ-weakly separated** if there exists a constant $\epsilon > 0$ such that, whenever $i \neq j$, there exists a function Φ_{ij} in $Mult(H)$, with $\|\Phi_{ij}\|_{Mult} \leq 1$, that satisfies $\Phi_{ij}(\lambda_i) = \epsilon, \Phi_{ij}(\lambda_j) = 0$.*

*The sequence is **d-separated** if there exists a constant $\epsilon > 0$ such that, whenever $i \neq j$, $d(\lambda_i, \lambda_j) > \epsilon$, where d is the Gleason distance.*

*The sequence is **strongly separated** if there exists a constant $\epsilon > 0$ such that, for every i there is a function Φ_i in $Mult(H)$, with $\|\Phi_i\|_{Mult} \leq 1$, that satisfies $\Phi_i(\lambda_i) = \epsilon, \Phi_i(\lambda_j) = 0$ for all $j \neq i$.*

Remark 1.4.1. *If H is a Pick space, $\{\lambda_n\}_n$ is a weakly separated sequence if and only if is a d-separated sequence.*

Definition 1.4.4. *Let H be a RKHS on X. A sequence $\{\lambda_i\}_i \subset X$ is an **interpolating sequence** for $Mult(H)$ if the map*

$$Mult(H) \to l^\infty$$
$$\phi \mapsto (\phi(\lambda_n)),$$

is surjective.

Definition 1.4.5. *Let H be a RKHS on X. A sequence $\{\lambda_i\}_i \subset X$ is a **simply interpolating sequence** for H if the range of the following map*

$$H \to l^2$$
$$f \mapsto \frac{f(\lambda_i)}{\|k_{\lambda_i}\|}$$

contains l^2.

The sequence $\{\lambda_i\}_i \subset X$ *is a* ***universally interpolating sequence*** *for* H *if the range of the same map is equal to* l^2. *It is equivalent to have the Grammian matrix bounded above and below.*

Remark 1.4.2. *Of course, a universally interpolating sequence is simply interpolating too, but the viceversa in general is not true.*

Moreover, in a Pick space the set of universally interpolating sequences and the set of multiplier interpolating sequences coincides.

Definition 1.4.6. *Let* H *be a RKHS of functions on* X. *A measure* μ *on* X *is a* ***Carleson measure*** *for* H *if there exists a constant* $M > 0$ *such that*

$$\int |f|^2 d\mu \leq M^2 \|f\|^2 \qquad \forall\ f \in H.$$

The least M *is called Carleson constant of* μ.

Definition 1.4.7. *Let* H *be a RKHS of functions on* X, *with kernel* k. *A sequence* $\{\lambda_n\}_n$ *in* X *is said to be a* ***Carleson sequence*** *if*

$$\sum_n \frac{\delta_{\lambda_n}}{\|k_n\|^2}$$

is a Carleson measure for H.

Proposition 1.4.1. *Let* $\{\lambda_i\}$ *be a sequence in* X. *The following are equivalent:*

1. *The associated Grammian is bounded;*

2. $\{\lambda_i\}$ *satisfies the Carleson measure condition.*

Proof. Note that

$$\|G\| \leq M^2 \Leftrightarrow \|\sum a_i g_i\|^2 \leq M^2 \sum |a_i|^2$$

for any sequence $\{a_i\}$ of scalars.

1. ⇒ 2. Let $f \in H$ and
$$a_i = \frac{f(\lambda_i)}{\|k_i\|}.$$

Then,

$$\begin{aligned}\sum |a_i|^2 &= \sum \frac{|f(\lambda_i)|^2}{\|k_i\|^2} \\ &= \langle f, \sum a_i g_i \rangle \\ &\leq \|f\| \|\sum a_i g_i\| \\ &\leq M\|f\| \left[\sum |a_i|^2\right]^{1/2}.\end{aligned}$$

2. ⇒ 1.

$$\begin{aligned}\left\|\sum a_i g_i\right\|^2 &= \sup_{\|f\|=1} |\langle f, \sum a_i g_i \rangle| \\ &= \sup_{\|f\|=1} \left|\sum \frac{f(\lambda_i)\overline{a_i}}{\|k_i\|}\right| \\ &\leq \sup_{\|f\|=1} \left[\sum \frac{|f(\lambda_i)|^2}{\|k_i\|^2}\right]^{\frac{1}{2}} \left[\sum |a_i|^2\right]^{\frac{1}{2}} \\ &\leq M \left[\sum |a_i|^2\right]^{\frac{1}{2}}.\end{aligned}$$

□

The following theorem gives an important characterization of interpolating sequences. The proof is quite complicated, so here is omitted (see [AHMCR17] for it).

Theorem 1.4.1. *In any normalized complete Pick space, $\{\lambda_n\}_n$ is a universally interpolating sequence if and only if is weakly separated and Carleson.*

Corollary 1.4.1.1. *In a normalized complete Pick space, if $\{\lambda_i\}$ is a weakly separated sequence whose Grammian is bounded above, then its Grammian is also bounded below.*

1.5 Invariant subspaces

By a subspace of a Hilbert space, we mean a closed linear subspace.

Definition 1.5.1. *Let H be a Hilbert space and T an operator on H. $M \subseteq H$ is an invariant subspace of T if $TM \subseteq M$.*

There is a characterization of the invariant subspaces of the Hardy space under the multiplication by the unilateral shift, due to Beurling ([Beu49], [Hel64])

Theorem 1.5.1. *Let S be the unilateral shift on H^2, i.e.*

$$(Sf)(z) = zf(z).$$

Let M be a non-zero closed subspace of H^2, invariant under S. Then there is a function ψ in H^2 of modulus 1 a.e., such that $M = \psi H^2$.

Proof. Let M be a non-zero S-invariant subspace. Hence, $SM \subseteq M$ and the containment is proper: in fact, if not, M would be invariant also by the multiplication by $\overline{z}$ and, taking any non-zero function in M and multiplying it by $\overline{z}$ a sufficient number of times, will give a function with a non-zero negative Fourier coefficient, violating the hypothesis.

Let ψ be any function of norm 1 in $M \ominus SM$, $\psi = \sum_{m=0}^{\infty} a_m z^m$. We have

$$\langle S^n\psi, \psi\rangle = \sum_{m=n}^{\infty} a_m \overline{a_{m-n}} = 0, \quad n > 0,$$

that is

$$\int_0^{2\pi} |\psi|^2 e^{-in\theta} \frac{d\theta}{2\pi}, \quad n > 0$$

and, by taking complex conjugates we get

$$\langle S^n\psi, \psi\rangle = 0, \quad \forall\, n \neq 0.$$

By Fourier analysis,

$$|\psi|^2 \sim \sum_{n=-\infty}^{\infty} \widehat{|\psi|^2}(n) e^{in\theta} = \widehat{|\psi|^2}(0).$$

Therefore, $|\psi|$ is constant a.e., and as ψ has norm one, we get $|\psi| = 1$ a.e.

Therefore, multiplication by ψ is isometric, and so

$$\vee\{S^n\psi : n \geq 0\} = \psi(\vee\{S^n 1 : n \geq 0\}) = \psi H^2.$$

Therefore, $\psi H^2 \subseteq M$.

Now, suppose $g \in M \ominus \psi H^2$. Then,

$$\langle g, S^n\psi\rangle = 0 \quad \forall\ n \geq 0,$$

and also

$$\langle \psi, S^n g\rangle = 0 \quad \forall\ n > 0.$$

Therefore $g\overline{\psi} = 0$ a.e., and hence $g = 0$. Consequently $M = \psi H^2$. □

For a general RKHS with the complete Pick property there is a theorem due to McCullough and Trent which characterizes the invariant subspaces with respect to the multipliers. We skip the proof, which is quite technical (see Theorem 0.7 of [McCTr00]).

Given a Hilbert space H, with the multiplier algebra $Mult(H)$, for all $\phi \in Mult(H)$, denote by M_ϕ the operator "multiplication by ϕ".

Given H a RKHS with normalized Pick kernel k, let $F_{(\cdot)}(\cdot)$ be the embedding of H into the Drury Arveson space defined as in Corollary 1.3.7.1.

Theorem 1.5.2. *If k is a Pick kernel, with corresponding Hilbert space H, and if $M \subseteq H$ is a subspace, then the following are equivalent:*

1. *M is invariant for each M_{F_x};*
2. *there exist $\Phi \in Mult(H \otimes L^2, H)$ s.t. M_Φ is a partial isometry (i.e. $M_\Phi|_{Ker(M_\Phi)^\perp}$ is an isometry) and $M = \Phi \cdot (H \otimes L^2)$;*
3. *M is invariant for M_ϕ for each $\phi \in Mult(H)$.*

Chapter 2

The tree model

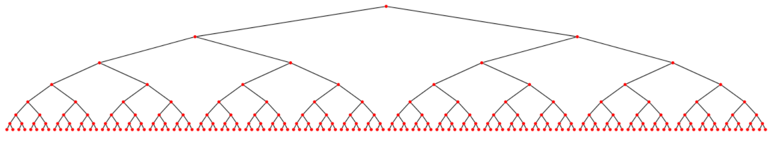
Our aim at this point is to construct a reproducing kernel Hilbert space on a tree. There are several construction: we are going to adopt the [Ro2019] one. Our Hilbert space will be Pick complete.

Let T be a rooted binary tree, assume finite for simplicity. Let $E(T)$ be the set of its edges, $V(T)$ be the set of its vertices, let o be its root.

For us a path will be a set (sequence) of vertices.

Given $x \in V(T)$, let $[o, x]$ be the unique geodesic joining o and x. According with the usual definitions, with the term geodesic we denote the shortest path from o to x.The existence and uniqueness of the geodesics follows from the fact that T is a simple connected graph.

Given $x, y \in V(T)$, we say $y < x$ if $[o, y] \subsetneq [o, x]$. Moreover let $x \wedge y$ be the unique vertex so that $[o, x] \cap [o, y] = [o, x \wedge y]$. Finally let y^- denote the immediate predecessor of y, that is $y^- < y$ and $[y^-, y]$ be an edge of T.



Let $\omega : V(T) \to \mathbb{R}^+$ such that

$$\omega(o) = 1, \ \omega(y) < \omega(x) \text{ if } y < x. \tag{2.1}$$

We will construct a Hilbert space of functions on $V(T)$ and we will prove that it is a RKHS. To do it we will need the following summation by parts formula (see [ArRoSa10]).

For a function $f : V(T) \to \mathbb{C}$, define:

$$\begin{aligned} If(x) &= \sum_{y \leq x} f(y) \\ I^*f(x) &= \sum_{y \geq x} f(y) \\ \Delta f(x) &= f(x) - f(x^-), \quad \Delta f(o) = 0. \end{aligned}$$

Lemma 2.0.1. *Let $h, f : V(T) \to \mathbb{C}$. Then,*

$$\sum_{x,y \in V(T)} h(x \wedge y) f(x) \overline{f(y)} = h(o) \left| I^* f(o) \right|^2 + \sum_{z \in V(T) \setminus o} \Delta h(z) \left| I^* f(z) \right|^2 .$$

Proof.

$$\begin{aligned} &\sum_{x,y \in V(T)} h(x \wedge y) f(x) \overline{f(y)} = \\ &= \sum_{z \in V(T)} h(z) \sum_{x \wedge y = z} f(x) \overline{f(y)} \\ &= \sum_{z \in V(T)} h(z) \Bigg[|f(z)|^2 + f(z) \left(\overline{I^* f(z) - f(z)} \right) + \\ &\quad + \overline{f(z)} \left(I^* f(z) - f(z) \right) + \sum_{t \neq s, t^-, s^- = z} I^* f(t) \overline{I^* f(s)} \Bigg] \\ &= \sum_{z \in V(T)} h(z) \left[|I^* f(z)|^2 - \sum_{t^- = z} |I^* f(t)|^2 \right]. \end{aligned}$$

□

Theorem 2.0.1. *The Hilbert space of functions $\{f : V(T) \to \mathbb{C}\}$, with norm as in the following Proposition* (2.0.1)*, is a reproducing kernel Hilbert space with reproducing kernel*

$$k : V(T) \times V(T) \to \mathbb{R}^+, \; k(x, y) = \omega(x \wedge y).$$

Moreover, H has the complete Pick property.

Proof. Apply Lemma 2.0.1, with $h = \omega$, i.e.

$$\sum_{x,y \in V(T)} \omega(x \wedge y) f(x) \overline{f(y)} = |I^* f(o)|^2 + \sum_{z \in V(T)} \Delta\omega(z) \, |I^* f(z)|^2 .$$

The r.h.s. is a positive semi-definite function, because ω is increasing. Therefore,

$$\sum_{x,y \in V(T)} k(x,y) f(x) \overline{f(y)} \geq 0$$

and k is the reproducing kernel of a Hilbert space H by Theorem 1.1.1.

Finally, H has the complete Pick property by applying Theorem 1.3.5: in fact, the function $1 - \frac{1}{k_{xy}}$ is positive semi-definite again by Lemma 2.0.1 with $h = 1 - 1/\omega$. □

There is another proof of the previous theorem that construct the embedding in the Drury Arveson space.

Theorem 2.0.2. *The reproducing kernel Hilbert space of functions* $\{f : V(T) \to \mathbb{C}\}$*, with norm as in the following Proposition* (2.0.1)*, with reproducing kernel*

$$k : V(T) \times V(T) \to \mathbb{R}^+, \;\; k(x,y) = \omega(x \wedge y).$$

is a complete Pick space.

Moreover,

$$k(x,y) = h(F(x), F(y)), \tag{2.2}$$

where h is the Drury-Arveson kernel, where $F : V(T) \to \mathbb{B}^d$, for some $1 \leq d \leq \infty$*,*

$$F(z) = \sum_{z \in [o,y]} \sqrt{\frac{1}{\omega(z^-)} - \frac{1}{\omega(z)}} e_z.$$

Proof. Note that the kernel defined in the statement is normalized at the root o. In fact, $k(x, o) = 1$ for all $x \in V(T)$.

In order to apply theorem 1.3.7.1, let's construct $F : V(T) \to \mathbb{B}^d$, for some $1 \leq d \leq \infty$, such that

$$k(x,y) = h(F(x), F(y)), \tag{2.3}$$

where h is the Drury-Arveson kernel.

We associate to $V(T)$ an orthonormal system in $\mathbb{B}^d$ in this way:

$$\begin{aligned} V(T) &\longrightarrow \{e_n\}_n \subseteq \mathbb{B}^d \\ y &\longmapsto e_y. \end{aligned}$$

Consider weights $\{c(y)\}_{y\in V(T)}$ and define

$$F(y) := \sum_{x\in[o,y]} c(x)e_x.$$

Let's choose the appropriate cofficients $\{c(y)\}$ by induction on $n(y)$, that is the number of edges in the path $[o, y]$. If $n(y) = 0$, then $y = o$ and we define $c(o) = 0$.

Suppose now to have defined $c(y)$ for every y such that $n(y) \leq N$.

Let $z \in V(T)$ be such that $n(z) = N + 1$. Then $n(z^-) = N$, and hence $F(z^-)$ is defined.

Define

$$F(z) = F(z^-) + c(z)e_z,$$

where $c(z)$ is still undefined.

From (2.3), it follows in particular that

$$\omega(z^-) = k(z^-, z^-) = h(F(z^-), F(z^-)) = \frac{1}{1 - |F(z^-)|^2},$$

which implies

$$\begin{aligned} \omega(z) &= k(z, z) \\ &= h(F(z), F(z)) \\ &= \frac{1}{1 - |F(z)|^2} \\ &= \frac{1}{1 - |F(z^-)|^2 - |c(z)|^2} \\ &= \frac{1}{\frac{1}{\omega(z^-)} - |c(z)|^2}. \end{aligned}$$

Therefore,

$$|c(z)|^2 = \frac{1}{\omega(z^-)} - \frac{1}{\omega(z)} \geq 0$$

because ω is increasing. Hence,

$$c(z) = \sqrt{\frac{1}{\omega(z^-)} - \frac{1}{\omega(z)}}$$

and

$$F(z) = \sum_{y \in [o,z]} \sqrt{\frac{1}{\omega(y^-)} - \frac{1}{\omega(y)}} e_y.$$

Now, for any $x, y \in V(T)$, is

$$\langle F(x), F(y) \rangle = 1 - \frac{1}{\omega(x \wedge y)}?$$

$$\begin{aligned}
\langle F(x), F(y) \rangle &= \left\langle \sum_{z \in [o,x]} \sqrt{\frac{1}{\omega(z^-)} - \frac{1}{\omega(z)}} e_z, \sum_{\zeta \in [o,y]} \sqrt{\frac{1}{\omega(\zeta^-)} - \frac{1}{\omega(\zeta)}} e_\zeta \right\rangle \\
&= \sum_{z \in [o, x \wedge y]} \frac{1}{\omega(z^-)} - \frac{1}{\omega(z)} \\
&= \frac{1}{\omega(o)} - \frac{1}{\omega(x \wedge y)} = 1 - \frac{1}{\omega(x \wedge y)}.
\end{aligned}$$

In conclusion, we have proved that the space of functions defined on $V(T)$ is a RKHS with the complete Pick property and complete Pick kernel k.

□

Corollary 2.0.2.1. *If $k(x, y)$ is defined in terms of ω as before and if Λ is any strictly increasing function, then $k^\Lambda(x, y) = \Lambda(k(x, y))$, associated with $\omega^\Lambda = \Lambda(\omega)$, is the kernel function of a RKHS with the complete Pick property.*

Proof. An analogous of the condition (2.1) holds and the proof of the theorem is the same. □

Remark 2.0.1. *This corollary is false in general for a RKHS H with the complete Pick property, even if Λ is just the product with a scalar λ. Is it true for $0 < \lambda \leq 1$ (see [AgMcC2002], Remark 8.10).*

We would like now to give an expression for the norm in H as before. We will do it using again the Lemma 2.0.1.

Let's compute the Gleason distance for two arbitrary vertices:

$$\begin{aligned} d(x,y)^2 &= 1 - \frac{|k(x,y)|^2}{k(x,x)k(y,y)} \\ &= 1 - \frac{\omega(x \wedge y)^2}{\omega(x)\omega(y)} \end{aligned}$$

In the case $x = y^-$,

$$\begin{aligned} d(y,y^-)^2 &= 1 - \frac{\omega(y^-)}{\omega(y)} \\ &= \frac{\omega(y) - \omega(y^-)}{\omega(y)} \\ &= \frac{\Delta\omega(y)}{\omega(y)}. \end{aligned}$$

Using the fact that $\omega(y) = \sum_{x \in (o,y]} \Delta\omega(x)$, we can see the Gleason distance between the endpoints of an edge as the ratio between the weighted length of the edge and the weighted length of the entire path $[0, y]$.

Since the span of kernel functions is H, a function $f \in H$ can be written as a finite linear combination of them:

$$f(y) = \sum \Gamma(x) k_x(y). \tag{2.4}$$

Proposition 2.0.1. *Given $f \in H$ as in* (2.4), *we have*

$$\begin{aligned} \|f\|^2 &= |I^*\Gamma(o)|^2 + \sum_{z>0} \Delta\omega(z) |(I^*\Gamma)(z)|^2 \\ &= |I^*\Gamma(o)|^2 + \sum_{z>0} \Delta\omega(z)^{-1} |(\Delta f)(z)|^2. \end{aligned}$$

Proof. The first expression is a reformulation of the summation by part formula given by Lemma 2.0.1.

In fact,

$$
\begin{aligned}
\|f\|^2 &= \left\langle \sum_x \Gamma(x)k_x(y), \sum_z \Gamma(z)k_z(y) \right\rangle \\
&= \sum_{x,z} \Gamma(x)\Gamma(z)\langle k_x(y), k_z(y)\rangle \\
&= \sum_{x,z} \Gamma(x)\Gamma(z)\omega(x \wedge z) \\
&= |I^*\Gamma(o)|^2 + \sum_{z>o} \Delta\omega(z)|I^*\Gamma(z)|^2
\end{aligned}
$$

The second follows from the fact that $(\Delta f)(z) = \Delta\omega(z)(I^*\Gamma)(z)$.

In fact, since both sides are linear functions of f, it suffices to prove it for $f = k_x$.

Select x and z. If $x < z$ or x is not comparable to z then $k_x(z) = k_x(z^-)$ and hence, with $\Gamma(y) = \delta_{xy}$, we have $Dk_x(z) = 0 = (I^*\Gamma)(z)$.

If $z < x$,

$$
\begin{aligned}
(Dk_x)(z) &= -k_x(z^-) + k_x(z) \\
&= -\omega(z^-) + \omega(z) \\
&= \Delta\omega(z) \\
&= \omega(z)(I^*\Gamma)(z).
\end{aligned}
$$

□

Chapter 3

A complete Pick space of functions on the real line

In this chapter we present another example of complete Pick space of functions on the real line, together with its main structural properties.

3.1 The space

We define the following Hilbert space of functions on $\mathbb{R}$:

$$H := \left\{ f \in AC_{loc}(\mathbb{R}) : \lim_{x \to -\infty} f(x) = 0, \; \|f\|^2 := \int_{\mathbb{R}} e^{-x} |f'(x)|^2 dx < \infty \right\},$$

with the inner product

$$\langle f, g \rangle := \int_{\mathbb{R}} e^{-x} f'(x) \overline{g'(x)} dx.$$

Lemma 3.1.1. *Norm convergence in H implies pointwise convergence.*

Proof. Let $\{f_n\}_n \subset H$ be a sequence converging to $f \in H$. Therefore,

$$\begin{aligned} |f(x) - f_n(x)|^2 &= \left| \int_{-\infty}^{x} (f'(t) - f_n'(t)) \, dt \right|^2 \\ &\leq \int_{-\infty}^{x} e^{-t} |f'(t) - f_n'(t)|^2 \, dt \cdot \int_{-\infty}^{x} e^{t} \, dt \\ &\leq \|f - f_n\|^2 \cdot e^{x} \longrightarrow 0 \quad \text{as } n \to +\infty. \end{aligned}$$

Hence $f(x) = \lim_n f_n(x)\ \forall\ x.$ □

Proposition 3.1.1. *H is a RKHS with kernel* $k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$,

$$k(x, y) = e^{x \wedge y},$$

where $x \wedge y$ *denotes the minimum between* x *and* y.

Proof. We are going to prove first of all that the evaluation functional

$$\eta_x : f \mapsto f(x) \quad \forall f \in H$$

is continuous in H.

Let $\{f_n\}_n \subset H$ be a sequence of functions converging to $f \in H$. This implies that $\{f_n\}_n$ converges also pointwise to f by Lemma 3.1.1, i.e. $\lim_{n\to\infty} f_n(x) = f(x)$ for all x.

Therefore,

$$\lim_{n\to\infty} \eta_x(f_n) = \lim_{n\to\infty} f_n(x) = f(x) = \eta_x(f)$$

and η_x is continuous.

Now, let's prove that $k_x(y) = k(x, y) = e^{x\wedge y}$ is the reproducing kernel, i.e.

$$f(x) = \eta_x(f) = \langle f,\ k_x\rangle \quad \forall f \in H.$$

Let f be a function in H. Then,

$$f(x) = \langle f, k_x\rangle = \int_{\mathbb{R}} e^{-y} f'(y)\overline{\partial_y k_x(y)} dy.$$

In fact,

$$\int_{\mathbb{R}} e^{-y} f'(y)\ \partial_y e^{x\wedge y}\ dy = \int_{-\infty}^{x} f'(y)\ dy = f(x).$$

Let $x \in \mathbb{R}$ be an arbitrary point.

$k_x : \mathbb{R} \to \mathbb{R}^+$ is a function in H:

$k_x \in AC_{loc}(\mathbb{R})$, $\lim_{y\to-\infty} k_x(y) = \lim_{y\to-\infty} e^{x\wedge y} = 0$ and

$$\begin{aligned} \|k_x\|^2 &= \int_{\mathbb{R}} e^{-y}|\partial_y e^{x\wedge y}|^2 dy \\ &= \int_{-\infty}^{x} e^{-y}|e^y|^2 dy \\ &= \int_{-\infty}^{x} e^y dy = e^x. \end{aligned}$$

□

The kernel normalized at zero is

$$\begin{aligned} L(x,y) &= \frac{k(x,y)k(0,0)}{k(x,0)k(0,y)} \\ &= \frac{e^{x\wedge y}}{e^{x\wedge 0}e^{y\wedge 0}} \end{aligned}$$

3.2 Pick property

Now, we are going to show that k is a (complete) Pick kernel and H is a (complete) Pick space.

We are going to use the following technical lemma about positive definiteness of some kind of matrices (see [BrHa1965], Theorem 3, pg. 131).

Definition 3.2.1. *A $n \times n$ matrix is said to be* ***L-shaped*** *if it has the following form:*

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_2 & a_3 & \cdots & a_n \\ a_3 & a_3 & a_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & & \vdots \\ a_n & a_n & a_n & \cdots & a_n \end{pmatrix}$$

The sequence $\{a_n\}_n$ is called determining sequence of A.

Lemma 3.2.1. *Any finite L-shaped matrix is positive definite if and only if its determining sequence is positive and decreasing.*

Theorem 3.2.1. $k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$, $k(x, y) = e^{x \wedge y}$ *is a complete Pick kernel. Therefore the RKHS* H *satisfies the complete Pick property.*

Proof. Let's construct the following matrix, for $N \in \mathbb{N}$,

$$F_N = \left(1 - \tfrac{k_{iN}k_{Nj}}{k_{ij}k_{NN}}\right)_{i,j=1}^{N-1},$$

where $x_1, \ldots, x_N \in \mathbb{R}$ are such that $x_i < x_j$ for $i < j$.

Note that is not restrictive to assume that x_N is the biggest element. In fact, if it exists j s.t. $x_N < x_j$,

$$1 - \frac{k_{iN}k_{Nj}}{k_{ij}k_{NN}} = 1 - \frac{e^{x_i}e^{x_N}}{e^{x_i}e^{x_N}} = 0.$$

Therefore, if $M = \#\{j : N < j\}$ the matrix will be of the form

$$F_N = \begin{pmatrix} F_{N-M} & 0 \\ 0 & 0 \end{pmatrix}$$

that is positive semidefinite if and only if F_{N-M} is.

In addition, note that is not restrictive to assume $x_i < x_j$ for $i < j$. The reason will be clear in the following, once the elements of F_N will be determined.

Applying our definition of k,

$$\begin{aligned} F_N &= \left(1 - \tfrac{k_{iN}k_{Nj}}{k_{ij}k_{NN}}\right)_{i,j=1}^{N-1} \\ &= \left(1 - \tfrac{e^{x_i}e^{x_j}}{e^{x_i \wedge x_j}e^{x_N}}\right)_{i,j=1}^{N-1}, \end{aligned}$$

where

$$1 - \frac{e^{x_i}e^{x_j}}{e^{x_i \wedge x_j}e^{x_N}} = \begin{cases} 1 - \frac{e^{x_j}}{e^{x_N}} & \text{if } i \le j \\ 1 - \frac{e^{x_i}}{e^{x_N}} & \text{if } i \ge j \end{cases}.$$

Hence,

$$F_N = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{N-1} \\ \alpha_2 & \alpha_2 & \alpha_3 & \cdots & \alpha_{N-1} \\ \alpha_3 & \alpha_3 & \alpha_3 & \cdots & \alpha_{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{N-1} & \alpha_{N-1} & \alpha_{N-1} & \cdots & \alpha_{N-1} \end{pmatrix},$$

where $\alpha_i = 1 - \frac{e^{x_i}}{e^{x_N}}$.

This matrix is "L-shaped" according to the definition 3.2.1. Our aim is to apply Lemma 3.2.1 to conclude the proof.

The determining sequence $\{\alpha_i\}_{i=1,\dots,N-1}$ is obviously positive and decreasing ($e^{x_i} < e^{x_j}$ if $i < j$).

We conclude that $k(x, y) = e^{x\wedge y}$ is a complete Pick kernel, by applying the McCullough-Quigging Theorem 1.3.3. □

There is an alternative proof of the complete Pick property, which uses the Agler-McCarthy Theorem 1.3.7.1. In order to present it, it is necessary to introduce a definition and make an observation.

Definition 3.2.2. *Let $\mathcal{B}(\mathbb{R})$ be the Borel sets of $\mathbb{R}$, $\mathcal{P}(Q)$ the projections on Q Hilbert space. A map*

$$\Pi : \mathcal{B}(\mathbb{R}) \longrightarrow \mathcal{P}(Q)$$

is called ***projection valued measure*** *on $\mathbb{R}$ (p.v.m.) if the following conditions hold:*

- $\Pi(\emptyset) = 0$ *and* $\Pi(\mathbb{R}) = Id_Q$
- *if $E_j \in \mathcal{B}(\mathbb{R})$ are pairwise disjoint, then*

$$\Pi\left(\cup_{j\geq 1} E_j\right) = \sum_{j\geq 1} \Pi(E_j)$$

  where $\sum_{j\geq 1} \Pi(E_j)$ is the limit of $\sum_{j=1}^{N} \Pi(E_j)$ in the strong operator topology.

Remark 3.2.1. *Let's come back to our RKHS H of functions on $\mathbb{R}$. We would like to explicitly construct an embedding into the Drury Arveson space.*

Let $x \in \mathbb{R}^+$. Note that $\forall\ y \leq x, h \geq 0$,

$$L(x + h, y) = \frac{e^{(x+h)\wedge y}}{e^{(x+h)\wedge 0} e^{y\wedge 0}} = e^y = \frac{e^{x\wedge y}}{e^{x\wedge 0} e^{y\wedge 0}} = L(x, y). \qquad (3.1)$$

Analogously, let $x \in \mathbb{R}^-$. *Note that* $\forall\ x \le y, h \le 0$,

$$L(x+h, y) = L(x, y). \tag{3.2}$$

If L *is a complete Pick kernel, using the Theorem 1.3.7.1, with* F *the embedding of* $\mathbb{R}$ *into the unit complex ball,*

$$\langle F(x), F(y) \rangle = 1 - \frac{1}{L(x,y)},$$

which implies that

$$\begin{aligned} \langle F(x+h) - F(x), F(y) \rangle &= \langle F(x+h), F(y) \rangle - \langle F(x), F(y) \rangle \\ &= \frac{1}{L(x,y)} - \frac{1}{L(x+h,y)} = 0 \quad \textit{by } (3.1), (3.2). \end{aligned}$$

This means on one hand that the future increment is independent from the past and on the other hand that the past decrement is independent from the future. This suggests to define the embedding F *using p.v.m.*

Theorem 3.2.2. $L : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$, $L(x,y) = \frac{e^{x \wedge y}}{e^{x \wedge 0} e^{y \wedge 0}}$ *is a complete Pick kernel. Therefore the RKHS* H *satisfies the complete Pick property.*

Moreover, $L(x,y) = h(F(x), F(y))$, *for* h *Drury Arveson kernel, where*

$$F : \mathbb{R} \longrightarrow \mathbb{B}_\infty$$

$$F(x) = \begin{cases} v\chi_{(0,x]} & \textit{if } x > 0 \\ v\chi_{(x,0]} & \textit{if } x \le 0 \end{cases}$$

for $v(u) = e^{-\frac{|u|}{2}}$ *and by identifying* $\mathbb{B}_\infty$ *with the unit ball of* $L^2(\mathbb{R})$.

Proof. Starting from considerations in Remark 3.2.1, let's choose $v \in L^2(\mathbb{R})$ s.t.

$$\int_{\mathbb{R}^+} |v(t)|^2 dt = \int_{\mathbb{R}^-} |v(t)|^2 dt = 1, \qquad \int_{0 \wedge x}^{0 \vee x} |v(t)|^2 dt < 1 \quad \forall\ x \in \mathbb{R}.$$

Then, let's define the following p.v.m:

$$\Pi : \mathcal{B}(\mathbb{R}) \longrightarrow \mathcal{P}(L^2(\mathbb{R}))$$

$$\Pi(E) f := \chi_E f$$

Defining

$$F : \mathbb{R} \longrightarrow \mathbb{B}_\infty$$
$$F(x) = \begin{cases} v\chi_{(0,x]} & \text{if } x > 0 \\ v\chi_{(x,0]} & \text{if } x \leq 0 \end{cases}$$

we have that

$$\langle F(x+h) - F(x), F(y) \rangle = 0 \quad \forall\, x \in \mathbb{R}^+,\; y \leq x, h \geq 0,$$

$$\langle F(x+h) - F(x), F(y) \rangle = 0 \quad \forall\, x \in \mathbb{R}^-,\; y \geq x, h \leq 0,$$

as desired.

Note that the embedding F must map $\mathbb{R}$ into $\mathbb{B}_\infty$, because we need infinite L^2-orthogonal directions in the range of F as shown in the previous computation.

Moreover,

$$\langle F(x), F(y) \rangle = \int_{0\wedge(x\vee y)}^{0\vee(x\wedge y)} |v(t)|^2 dt = 1 - \frac{1}{L(x,y)} = 1 - \frac{e^{x\wedge 0} e^{y \wedge 0}}{e^{x\wedge y}}.$$

Taking for example $x \leq y$,

$$\langle F(x), F(y) \rangle = 1 - \frac{e^{x\wedge 0} e^{y\wedge 0}}{e^x}.$$

From this it follows that, $|v(u)|^2 = e^{-u}$ if $u \geq 0$, $|v(u)|^2 = e^u$ if $u \leq 0$.

Choosing for instance $v(u) = e^{-\frac{|u|}{2}}$, that condition is satisfied and the embedding is found. □

Finally, the proof of the complete Pick property for our space could be obtained as an easy corollary of the following Theorem, due to Quiggin (see [Qui93], Theorem 6.7), requiring local hypothesis. The proof is different from the two above.

Theorem 3.2.3. *Let ρ be any real, positive, continuous and integrable function on the interval (a, b). Then the reproducing kernel Hilbert space of absolutely continuous, complex-valued functions on (a, b), which satisfy*

$\lim_{x \to a} f(x) = 0$ *and which are bounded with respect to the inner product*

$$\langle f, g \rangle = \int_a^b \frac{f'(x)\overline{g'(x)}}{\rho(x)} dx$$

has the complete Pick property.

Remark 3.2.2. *In particular, we have chosen,* $(a, b) = \mathbb{R}$, $\rho(x) = e^x$. *The theorem tells us that the complete Pick property holds also for different weights.*

In the following, we will see the case in which $\rho \equiv 1$.

3.3 Interpolating sequences

At this point the next goal is to characterize interpolating sequences for our space.

Lemma 3.3.1. *Let* $\epsilon > 0$. *The sequence* $\{x_i\}_i \subset \mathbb{R}$ *is* ϵ*-weakly separated in* H *if and only if,* $\forall\ i \neq j$,

$$|x_j - x_i| > \delta(\epsilon),$$

where $\delta(\epsilon) = \log\left(\frac{1}{1-\epsilon^2}\right)$.

Proof. To write the weak separation condition for this kernel, using the Remark 1.4.1, we compute the Gleason distance

$$d(x, y) = \sqrt{1 - \frac{|k(x, y)|^2}{k(x, x)k(y, y)}}.$$

Using our definition of L,

$$\begin{aligned} d(x, y) &= \sqrt{1 - \frac{|e^{x \wedge y}|^2}{e^x e^y}} \\ &= \begin{cases} \sqrt{1 - \frac{e^x}{e^y}} & \text{if } x \leq y \\ \sqrt{1 - \frac{e^y}{e^x}} & \text{if } x > y \end{cases} \end{aligned}$$

Therefore, the sequence $\{x_i\}_i \subset \mathbb{R}$ is ϵ-weakly separated if and only if exixts $\epsilon > 0$ such that, $\forall\ i \neq j$, $d(x_i, x_j) > \epsilon$. For example, if $x_i < x_j$,

$$\begin{aligned}\sqrt{1 - \frac{e^{x_i}}{e^{x_j}}} > \epsilon \Leftrightarrow 1 - e^{x_i - x_j} > \epsilon^2 \\ \Leftrightarrow e^{x_i - x_j} < 1 - \epsilon^2 \\ \Leftrightarrow x_j - x_i > \log\left(\frac{1}{1-\epsilon^2}\right).\end{aligned}$$

Defining $\delta(\epsilon) = \log\left(\frac{1}{1-\epsilon^2}\right)$, we conclude that $\{x_i\}_i$ is ϵ-weakly separated if and only if, $\forall\ i \neq j$,

$$|x_j - x_i| > \delta(\epsilon).$$

□

In the following proof, we are going to use the following Schur's Lemma (see for example [AgMcC2002]):

Lemma 3.3.2. *Suppose*

$$\sup_j \sum_{i=1}^{\infty} |T_{ij}| \leq c_1$$

$$\sup_i \sum_{j=1}^{\infty} |T_{ij}| \leq c_\infty.$$

Then $T : l^2 \to l^2$, *and its norm on* l^2 *is at most* $\sqrt{c_1 c_\infty}$.

Theorem 3.3.1. *Let* $\epsilon > 0$. *If* $\{x_i\}_i \subset \mathbb{R}$ *is a* ϵ*-weakly separated sequence in* H, *the Grammian is bounded above and below, i.e.*

$$c_1(\epsilon)\|x\|^2 \leq \langle Gx, x\rangle \leq c_2(\epsilon)\|x\|^2$$

for some constants $c_1(\epsilon), c_2(\epsilon) > 0$.

In particular, $\{x_i\}_i$ *is a universally interpolating sequence.*

Proof. From Lemma 3.3.1, $\{x_i\}_i$ is ϵ-weakly separated if, $\forall\ i \neq j$,

$$|x_j - x_i| > \log\left(\frac{1}{1-\epsilon^2}\right).$$

We assume wlog that $x_i < x_{i+1}$ for all i.

Choosing $2\delta := \log\left(\frac{1}{1-\epsilon^2}\right), \forall\ i \neq j$,

$$|x_j - x_i| > 2\delta|i-j|.$$

Let's consider the sum over rows and the sum over columns of the Grammian matrix G.

$$G = \left(\frac{\langle k_{x_i}, k_{x_j}\rangle}{\|k_{x_i}\|\|k_{x_j}\|}\right)_{i,j} = \left(\frac{e^{x_i \wedge x_j}}{e^{\frac{x_i}{2}} e^{\frac{x_j}{2}}}\right)_{i,j} = \left(\frac{e^{-|x_i - x_j|}}{2}\right)_{i,j}$$

The sum over the i-th row is

$$\sum_{j=1}^{\infty} e^{-\frac{|x_i - x_j|}{2}}.$$

Hence,

$$\begin{aligned}
\sum_{j=1}^{\infty} e^{-\frac{|x_i - x_j|}{2}} &= \sum_{j=1}^{\infty} \frac{1}{e^{\frac{|x_i - x_j|}{2}}} \\
&\leq \sum_{j=1}^{\infty} \frac{1}{e^{\delta|i-j|}} \\
&\leq 2\sum_{n=0}^{\infty} e^{-n\delta} \\
&= \frac{2}{1-e^{-\delta}}.
\end{aligned}$$

The matrix is symmetric, so also the sum over the j-th column is

$$\sum_{i=1}^{\infty} e^{-\frac{|x_i - x_j|}{2}} \leq \frac{2}{1-e^{-\delta}}.$$

Therefore,

$$\sup_i \left(\sum_{j=1}^{\infty} e^{-\frac{|x_i - x_j|}{2}}\right) = \sup_j \left(\sum_{i=1}^{\infty} e^{-\frac{|x_i - x_j|}{2}}\right) \leq \frac{2}{1-e^{-\delta}}.$$

By applying Shur's lemma, $G : l^2 \to l^2$ is bounded with $\|G\| \leq \frac{2}{1-e^{-\delta}}$.

To show that G is also bounded below we solve the following interpolation problem:

given $\{x_i\}_i \subset \mathbb{R}$ as before and $\{a_i\}_i \subset \mathbb{R}$ such that $\sum_i \frac{|a_i|^2}{\|k_{x_i}\|^2} < \infty$, we look for a function B in H that interpolates those sequences, i.e. s.t. $B(x_i) = a_i$ for all i.

Let's consider the following function:

$$B_i(x) = \begin{cases} \frac{a_i}{\delta}(\delta - x + x_i) & x \in [x_i, x_i + \delta] \\ \frac{a_i}{\delta}(\delta + x - x_i) & x \in [x_i - \delta, x_i) \\ 0 & \text{otherwise} \end{cases}$$

$$B_i'(x) = \begin{cases} -\frac{a_i}{\delta} & x \in (x_i, x_i + \delta) \\ \frac{a_i}{\delta} & x \in (x_i - \delta, x_i) \\ 0 & x \notin [x_i - \delta, x_i + \delta] \end{cases}$$

For all i, $B_i \in H$. In fact $B_i \in AC_{loc}(\mathbb{R})$, $\lim_{x \to -\infty} B_i(x) = 0$ and

$$\begin{aligned} \|B_i\|^2 &= \int_{\mathbb{R}} e^{-x} |B_i'(x)|^2 dx \\ &= \int_{x_i - \delta}^{x_i + \delta} e^{-x} \frac{a_i^2}{\delta^2} dx \\ &= -\frac{a_i^2}{\delta^2}(e^{-(x_i+\delta)} - e^{-(x_i-\delta)}) \\ &= 2 \sinh \delta \ \frac{a_i^2}{\delta^2} \ e^{-x_i} < \infty. \end{aligned}$$

Now, we set

$$B(x) := \sum_i B_i(x),$$

where $\sum_i B_i(x) = \lim_N \sum_{i=1}^N B_i(x)$.

Note that the definition of B is well posed because that series is absolutely convergent and hence convergent also in H norm.

Note that the supports of B_i and of B_j are disjoint for all $i \neq j$ thanks to the weak separation condition.

By construction, we obtain that $B(x_i) = a_i$ for all i.

We verify that $\|B\| < \infty$:

$$\begin{aligned}\|B\|^2 = \sum_i \|B_i\|^2 &= \sum_i 2\sinh\delta\ \frac{a_i^2}{\delta^2}\ e^{-x_i}\\ &= \frac{2\sinh\delta}{\delta^2}\sum_i a_i^2 e^{-x_i}\\ &= \frac{2\sinh\delta}{\delta^2}\sum_i \frac{|a_i|^2}{\|k_{x_i}\|^2} < \infty.\end{aligned}$$

Let's consider the following two operators:

$$\begin{aligned} R : H &\longrightarrow l^2\\ f &\mapsto \left\{\frac{f(x_i)}{\|k_{x_i}\|}\right\}\\ T : l^2 &\longrightarrow H\\ \left\{\frac{a_i}{\|k_{x_i}\|}\right\} &\mapsto B,\end{aligned}$$

where as above $B(x_i) = a_i$.

Note that $RT = Id_{l^2}$.

Therefore, using the fact that $G = RR^*$,

$$\begin{aligned}\|x\|^2 &= \|T^*R^*x\|^2\\ &\le \|T^*\|^2\|R^*x\|^2\\ &= \|T^*\|^2\langle Gx, x\rangle\\ &= \|T\|^2\langle Gx, x\rangle\\ &= \frac{2\sinh\delta}{\delta^2}\langle Gx, x\rangle.\end{aligned}$$

In conclusion,

$$c_1\|x\|^2 \le \langle Gx, x\rangle \le c_2\|x\|^2,$$

where $c_1(\epsilon) = \frac{\delta^2}{2\sinh\delta}$, $c_2(\epsilon) = \frac{2}{1-e^{-\delta}}$.

So, $\{x_i\}_i \subseteq \mathbb{R}$ is ϵ-weakly separated and Carleson, that implies $\{x_i\}_i$ is universally interpolating. □

Remark 3.3.1. *The constants $c_1(\epsilon)$ and $c_2(\epsilon)$ are approximately equal to ϵ^2 as $\epsilon \to 0$.*

In fact, $c_1(\delta) \approx c_2(\delta) \approx \delta \approx \epsilon^2$.

3.4 Multipliers and Carleson measures

Now we would like to characterize the multipliers of our complete Pick space H.

The following theorem is a version of the weighted Hardy inequality, which holds also in a more general situation (see [Muck1972]):

Theorem 3.4.1. *If μ and ν are Borel measure, ν is absolutely continuous, then there is a constant $C > 0$ for which*

$$\int_{\mathbb{R}} \left| \int_{-\infty}^{x} f(t)dt \right|^2 \, d\mu(x) \leq C \int_{\mathbb{R}} |f(x)|^2 \, d\nu(x)$$

if and only if

$$B = \sup_{r\in\mathbb{R}} \, [\mu([r,\infty))]^{\frac{1}{2}} \left[\int_{-\infty}^{r} \left(\frac{dx}{d\nu(x)} \right) dx \right]^{\frac{1}{2}} < \infty.$$

Theorem 3.4.2.

$$Mult(H) = \left\{ m \in AC_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R}) : \sup_{r\in\mathbb{R}} \; e^r \int_r^\infty e^{-x} |m'(x)|^2 dx < \infty \right\}.$$

Moreover, μ is a Carleson measure if and only if

$$\sup_{r\in\mathbb{R}} \; e^r \int_r^\infty d\mu(x) < \infty.$$

Proof. It is known that $m \in Mult(H)$ implies $m \in L^\infty$ (see remark 1.2.1).

$$\begin{aligned}
\|mf\|^2 &= \int_{\mathbb{R}} e^{-x} |(mf)'(x)|^2 dx \\
&= \int_{\mathbb{R}} e^{-x} |(m'f)(x) + (mf')(x)|^2 dx \\
&\leq 2 \int_{\mathbb{R}} e^{-x} |(m'f)(x)|^2 dx + 2 \int_{\mathbb{R}} e^{-x} |(mf')(x)|^2 dx \\
&\leq 2 \int_{\mathbb{R}} e^{-x} |(m'f)(x)|^2 dx + 2\|m\|_\infty^2 \int_{\mathbb{R}} e^{-x} |f'(x)|^2 dx \\
&\leq 2 \int_{\mathbb{R}} e^{-x} |(m'f)(x)|^2 dx + 2\|m\|_\infty^2 \|f\|^2
\end{aligned}$$

Now note that the second term is finite because $m \in L^\infty(\mathbb{R})$ and $f \in H$.

To estimate the first one, we will use the Theorem 3.4.1.

Define $d\mu(x) := e^{-x}|m'(x)|^2 dx$, $d\nu(x) = e^{-x}dx$ and let $f(x) = \int_{-\infty}^{x} F(s)ds$. Then,

$$\int_{\mathbb{R}} e^{-x}|(m'f)(x)|^2 dx = \int_{\mathbb{R}} |f(x)|^2 d\mu(x)$$

Using the Hardy inequality,

$$\begin{aligned}\int_{\mathbb{R}} |f(x)|^2 d\mu(x) &\leq C \int_{\mathbb{R}} |F(x)|^2 d\nu(x)\\ &= C \int_{\mathbb{R}} e^{-x}|F(x)|^2 dx\\ &= C\|f\|^2\end{aligned}$$

for some constant $C > 0$, if and only if the following condition holds:

$$\sup_r \left(\int_r^\infty d\mu(x)\right)^{\frac{1}{2}} \left(\int_{-\infty}^r e^x dx\right)^{\frac{1}{2}} < \infty$$

i.e.

$$\sup_r \; e^r \int_r^\infty e^{-x}|m'(x)|^2 dx < \infty.$$

We conclude that

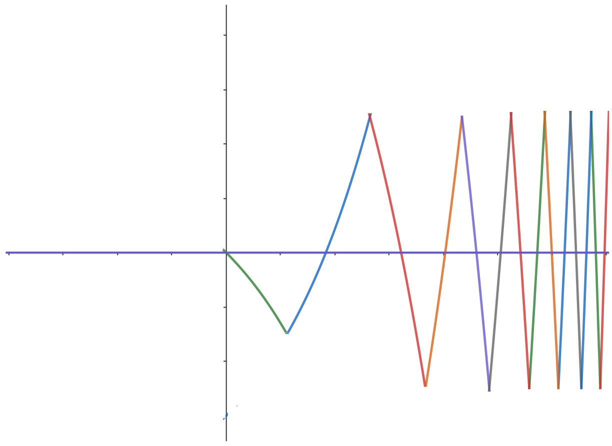
$$Mult(H) = \left\{ m \in AC \cap L^\infty(\mathbb{R}) : \sup_{r\in\mathbb{R}} \; e^r \int_r^\infty e^{-x}|m'(x)|^2 dx < \infty \right\}.$$

□

Remark 3.4.1. *One can ask if $Mult(H)$ are strictly contained in $H \cap L^\infty(\mathbb{R})$.*

The answer is yes. We can consider an absolutely continuous function f as in the following figure constructed in such a way that it remains bounded, it is in H and the derivatives, where exist, are

$$|f'(x)|^2 = \begin{cases} e^{\frac{x}{2}} & x > 0 \\ 0 & x < 0 \end{cases}$$



Therefore,

$$e^r \int_r^\infty e^{-x} e^{\frac{x}{2}} dx = e^r \left[-\frac{1}{2} e^{-\frac{x}{2}} \right]_r^\infty$$
$$= \frac{1}{2} e^{r/2} \to \infty \qquad \textit{as } r \to \infty.$$

So, we have constructed a function that is in $H \cap L^\infty(\mathbb{R})$ but not in $Mult(H)$.

Once Carleson measures have been characterized, we have another proof of Theorem 3.3.1.

Theorem 3.4.3. *If $\{x_i\}_i$ is a ϵ-weakly separated sequence, then it is also a Carleson sequence. Moreover, it is a universally interpolating sequence.*

Proof. From Lemma 3.3.1, $\{x_i\}_i$ is ϵ-weakly separated if, $\forall\ i \neq j$,

$$|x_j - x_i| > \log\left(\frac{1}{1-\epsilon^2}\right).$$

We assume wlog that $x_i < x_{i+1}$ for all i.

In addiction, $\{x_i\}_i$ is a Carleson sequence if

$$\mu = \sum_i \frac{\delta_{x_i}}{k(x_i, x_i)}$$

is a Carleson measure.

By applying the previous Theorem, ν is a Carleson measure if and only if

$$\sup_r e^r \int_r^\infty d\nu(x) < \infty.$$

With our μ,

$$\begin{aligned} e^r \int_r^\infty d\mu(x) &= e^r \sum_{x_i \geq r} \frac{1}{k(x_i, x_i)} \\ &= e^r \sum_{x_i \geq r} \frac{1}{e^{x_i}} \\ &\leq e^r \sum_{k=0}^\infty \frac{1}{e^r (\frac{1}{1-\epsilon^2})^k} \\ &= \sum_{k=0}^\infty (1-\epsilon^2)^k < +\infty \qquad \forall\, r > 0. \end{aligned}$$

□

3.5 Corona problem

The corona problem for H is to decide, given functions $f_1, \dots, f_n \in Mult(H)$, whether there exist functions $\Phi_1, \dots, \Phi_n \in Mult(H)$ such that $\sum_{i=1}^n f_i \Phi_i = 1$.

Let consider the following seminorm for $Mult(H)$:

$$\|m\|_M^2 := \sup_r \; e^r \int_r^\infty e^{-x} |m'(x)|^2 dx,$$

which is finite for the multipliers.

Theorem 3.5.1. *Let $f_1, \dots, f_n \in Mult(H)$ such that*

$$\sum_{i=1}^n |f_i|^2 \geq \epsilon > 0. \tag{3.3}$$

Then exist $\Phi_1, \dots, \Phi_n \in Mult(H)$ such that for some $C(\epsilon, i) > 0$

$$\|\Phi_i\|_M \leq C(\epsilon, i) \quad \textit{and} \quad \sum_{i=1}^n f_i \Phi_i = 1.$$

Proof. Let

$$\Phi_i(x) := \frac{\overline{f_i(x)}}{\sum_i |f_i(x)|^2} \quad \forall\, i = i, \dots, n.$$

Clearly,

$$\sum_{i=1}^{n} f_i(x)\Phi_i(x) \equiv 1.$$

$\Phi_i \in AC_{loc}(\mathbb{R})$ because f_i are and (3.4) holds.

$\Phi_i \in L^\infty(\mathbb{R})$:

$$|\Phi_i(x)| = \left| \frac{\overline{f_i(x)}}{\sum_i |f_i(x)|^2} \right| \leq \frac{|f_j(x)|}{\epsilon} < \frac{\|f_j\|_\infty}{\epsilon} < \infty.$$

$$\begin{aligned}
\|\Phi_i\|_M^2 &= \sup_r e^r \int_r^\infty e^{-x} |\Phi_i'(x)|^2 dx \\
&= \sup_r e^r \int_r^\infty e^{-x} \left| \frac{\overline{f_i'(x)} \sum_{k=1}^n |f_k(x)|^2 - 2\overline{f_i(x)} \sum_{k=1}^n f_k(x) f_k'(x)}{\left(\sum_{k=1}^n |f_k(x)|^2\right)^2} \right|^2 dx \\
&\leq 2 \sup_r e^r \int_r^\infty e^{-x} \left| \frac{\overline{f_i'(x)} \sum_{k=1}^n |f_k(x)|^2}{\left(\sum_{k=1}^n |f_k(x)|^2\right)^2} \right|^2 dx + \\
&+ 8 \sup_r e^r \int_r^\infty e^{-x} \left| \frac{\overline{f_i(x)} \sum_{k=1}^n f_k(x) f_k'(x)}{\left(\sum_{k=1}^n |f_k(x)|^2\right)^2} \right|^2 dx \\
&= 2 \sup_r e^r \int_r^\infty e^{-x} \frac{|f_i'(x)|^2}{\left(\sum_{k=1}^n |f_k(x)|^2\right)^2} dx + \\
&+ 8 \sup_r e^r \int_r^\infty e^{-x} |f_i(x)|^2 \frac{|\sum_{k=1}^n f_k(x) f_k'(x)|^2}{\left(\sum_{k=1}^n |f_k(x)|^2\right)^4} dx \\
&\leq \frac{2}{\epsilon^2} \|f_i\|_M^2 + \frac{8}{\epsilon^4} \sup_r e^r \int_r^\infty e^{-x} C^4 \left| \sum_{k=1}^n f_k'(x) \right|^2 dx \quad (C = \max \|f_k\|_\infty) \\
&= \frac{2}{\epsilon^2} \|f_i\|_M^2 + \frac{8}{\epsilon^4} C^4 \left\| \sum_{k=1}^n f_k \right\|_M^2 \\
&\leq \frac{2}{\epsilon^2} \|f_i\|_M^2 + \frac{8}{\epsilon^4} C^4 \left(\sum_{k=1}^n \|f_k\|_M \right)^2 =: C^2(\epsilon, i) < \infty.
\end{aligned}$$

□

The theorem holds also in the infinite case, both for bounded rows and bounded columns. We are going to prove it for rows, then the columns version follows as a consequence.

Theorem 3.5.2. *Let* $\{f_i\}_i \in Mult(H \otimes l^2(\mathbb{N}), H)$ *such that*

$$\sum_{i=1}^{\infty} |f_i|^2 \geq \epsilon > 0. \tag{3.4}$$

Then there exist $\{\Phi_i\}_i \in Mult(H \otimes l^2(\mathbb{N}), H)$ *such that for some* $C(\epsilon, j) > 0$

$$\|\Phi_i\|_M \leq C(\epsilon, i) \quad \textit{and} \quad \sum_{i=1}^{\infty} f_i \Phi_i = 1.$$

Proof. Construct the Φ_i as in the previous proof, i.e.

$$\Phi_i(x) := \frac{\overline{f_i(x)}}{\sum_i |f_i(x)|^2} \quad \forall\, i \in \mathbb{N}. \tag{3.5}$$

Then, as before, $\sum_i f_i \Phi_i \equiv 1, \Phi_i \in AC_{loc}(\mathbb{R})$ and $\Phi_i \in L^\infty(\mathbb{R})$.

Following the steps of the previous proof,

$$\|\Phi_i\|_M^2 = \leq \frac{2}{\epsilon^2}\|f_i\|_M^2 + \frac{8}{\epsilon^4} C^4 \left(\sum_k \|f_k\|_M\right)^2 =: C^2(\epsilon, i) < \infty.$$

where, $C = \sup \|f_k\|_\infty$.

It remains to show that $\{\Phi_i\}_i \in Mult(H \otimes l^2(\mathbb{N}), H)$, that is

$$\|\sum_k \Phi_k f\|_H^2 \leq D\|f\|_H^2 \quad \forall f \in H$$

for some $D > 0$.

Note the following fact:

$$\sup_x |\sum_k f_k(x)|^2 < \infty. \tag{3.6}$$

In fact, using

$$\|\sum_k f_k f\|_H^2 < \infty \quad \forall\, f \in H,$$

with $f = k_y$, it follows that

$$\int_{-\infty}^{y} |\sum_k f_k(x)|^2 dx < \infty \quad \forall\, y$$

and therefore (3.6).

This in particular implies also that

$$\sup_x |\sum_k \Phi_k(x)|^2 < \infty. \tag{3.7}$$

Moreover, since $\{f_k\}_k$ are multipliers, we have:

$$\int_{\mathbb{R}} e^{-x} |\sum_k f_k'(x)|^2 |f(x)|^2 dx \lesssim \int_{\mathbb{R}} e^{-x} |f'(x)|^2 dx. \tag{3.8}$$

$$\begin{aligned}
\left\|\sum_k \Phi_k f\right\|_H^2 &= \int_{\mathbb{R}} e^{-x} |\sum_k (\Phi_k f)'(x)|^2 \; dx \\
&= \int_{\mathbb{R}} e^{-x} |\sum_k \Phi_k'(x) f(x) + \sum_k \Phi_k(x) f'(x)|^2 \; dx \\
&\leq 2 \int_{\mathbb{R}} e^{-x} |\sum_k \Phi_k'(x)|^2 |f(x)|^2 \; dx + \qquad (3.9) \\
&\quad + 2 \int_{\mathbb{R}} e^{-x} |\sum_k \Phi_k(x)|^2 |f'(x)|^2 \; dx \\
&\leq 2 \int_{\mathbb{R}} e^{-x} |\sum_k \Phi_k'(x)|^2 |f(x)|^2 \; dx + \qquad (3.10) \\
&\quad + 2 \; \sup_x |\sum_k \Phi_k(x)|^2 \int_{\mathbb{R}} e^{-x} |f'(x)|^2 \; dx \\
&\lesssim \int_{\mathbb{R}} e^{-x} |\sum_k \Phi_k'(x)|^2 |f(x)|^2 \; dx + \|f\|^2 \qquad (3.11)
\end{aligned}$$

Consider now only the term

$$\int_{\mathbb{R}} e^{-x} |\sum_k \Phi_k'(x)|^2 |f(x)|^2 \; dx.$$

Using (3.5), it is

$$
\begin{aligned}
&\int_{\mathbb{R}} e^{-x} \left| \sum_k \frac{\overline{f_k'}(\sum_n |f_n|^2) - \overline{f_k}(2\sum_n f_n f_n')}{(\sum_n |f_n|^2)^2} \right|^2 |f|^2 dx \\
&= \int_{\mathbb{R}} e^{-x} \left| \sum_k \frac{\overline{f_k'}}{\sum_n |f_n|^2} - \sum_k \frac{\overline{f_k}(2\sum_n f_n f_n')}{(\sum_n |f_n|^2)^2} \right|^2 |f|^2 dx \\
&\leq 2\int_{\mathbb{R}} e^{-x} \left| \frac{\sum_k \overline{f_k'}}{\sum_n |f_n|^2} \right|^2 |f|^2 dx + 2\int_{\mathbb{R}} e^{-x} \left| \frac{\sum_k \overline{f_k} \sum_n f_n f_n'}{(\sum_n |f_n|^2)^2} \right|^2 |f|^2 dx \\
&\leq \frac{2}{\epsilon^2} \int_{\mathbb{R}} e^{-x} \left| \sum_k \overline{f_k'} \right|^2 |f|^2 dx + \frac{2}{\epsilon^4} \int_{\mathbb{R}} e^{-x} \left| \sum_k \overline{f_k} \sum_n f_n f_n' \right|^2 |f|^2 dx \\
&\overset{(3.8)}{\lesssim} \|f\|^2 + (\sup_x |\sum_k f_k(x)|^2)^2 \int_{\mathbb{R}} e^{-x} |\textstyle\sum_n f_n'|^2 |f|^2 dx \\
&\lesssim \|f\|^2 + \int_{\mathbb{R}} e^{-x} |\textstyle\sum_n f_n'|^2 |f|^2 dx \\
&\overset{(3.8)}{\lesssim} \|f\|^2.
\end{aligned}
$$

Coming back in (3.11),

$$\| \sum_k \Phi_k f \|^2 \lesssim \|f\|^2,$$

as desired. □

Corollary 3.5.2.1. *Let $\{f_i\}_i \in Mult(H, H \otimes l^2(\mathbb{N}))$ such that*

$$\sum_{i=1}^{\infty} |f_i|^2 \geq \epsilon > 0. \tag{3.12}$$

Then there exist $\{\Phi_i\}_i \in Mult(H, H \otimes l^2(\mathbb{N}))$ such that for some $C(\epsilon, j) > 0$

$$\|\Phi_i\|_M \leq C(\epsilon, i) \quad \text{and} \quad \sum_{i=1}^{\infty} f_i \Phi_i = 1.$$

3.6 Equivalent norm

This space is a weighted Sobolev space. Actually H and $W^{1,2}([0,1])$ are not equivalent. One reason is that the multipliers are different. However, as $W^{1,2}([0,1])$, H has an equivalent norm, as stated in the following theorem.

Theorem 3.6.1. *The norm* $\|\cdot\|$ *is equivalent to* $|||\cdot|||$, *where*

$$\|f\|^2 = \int_{\mathbb{R}} e^{-x}|f'(x)|^2 dx$$

and

$$|||f|||^2 = \int_{\mathbb{R}} e^{-x}[|f(x)|^2 + |f'(x)|^2]dx.$$

Proof 1. Of course $\|f\|^2 \leq |||f|||^2$ for all $f \in H$. Consider now $\int_{\mathbb{R}} e^{-x}|f(x)|^2 dx$. Set $d\mu(x) = e^{-x}dx$.

$d\mu$ is a Carleson measure:

$$e^r \int_r^{\infty} d\mu = e^r \int_r^{\infty} e^{-x}\ dx = 1 \qquad \forall\ r \geq 0.$$

Therefore,

$$\int_{\mathbb{R}} e^{-x}|f(x)|^2 dx = \int_{\mathbb{R}} |f(x)|^2 d\mu(x) \leq C\|f\|^2$$

for some $C > 0$.

In conclusion,

$$|||f|||^2 \leq (C+1)\|f\|^2.$$

□

Proof 2. Of course $\|f\|^2 \leq |||f|||^2$ for all $f \in H$.

Consider now $\int_{\mathbb{R}} e^{-x}|f(x)|^2 dx$:

$$\int_{\mathbb{R}} e^{-x}|f(x)|^2 dx = -e^{-x}|f(x)|^2\Big|_{-\infty}^{+\infty} + 2\int_{\mathbb{R}} e^{-x} f(x) f'(x) dx$$

If the first term of the r.h.s. is zero and the two other terms of the equation are finite we can conclude by applying Hölder inequality.

The $\lim_{x\to\pm\infty} e^{-x}|f(x)|^2$ is zero:

$|f(x)| = O(\|k_x\|^{\frac{1}{2}})$ because $|f(x)| = |\langle f, k_x\rangle| \leq \|f\|\|k_x\|$.

But we can pass from big o to little o, because the latter holds on a dense subspace D of H:

$$D = \operatorname{span}\{k_x(y) : x \in \mathbb{R}\}.$$

On D,

$$\lim_{x\to-\infty}\frac{|f(x)|^2}{e^x}=\lim_{x\to-\infty}\frac{|\sum_i a_ie^{x\wedge y_i}|^2}{e^x}$$
$$=\lim_{x\to-\infty}\frac{|\sum_i a_ie^{x\wedge y_i}|^2}{e^x}$$
$$=\lim_{x\to-\infty}\frac{|\sum_i a_ie^{x}|^2}{e^x}=0,$$

$$\lim_{x\to+\infty}\frac{|f(x)|^2}{e^x}=\lim_{x\to+\infty}\frac{|\sum_i a_ie^{y_i}|^2}{e^x}=0.$$

Therefore,

$$\int_{\mathbb{R}} e^{-x}|f(x)|^2dx=2\int_{\mathbb{R}} e^{-x}f(x)f'(x)dx$$

and $\int_{\mathbb{R}} e^{-x}|f(x)|^2dx<+\infty$. In fact, again on D dense subspace of H,

$$\int_{\mathbb{R}} e^{-x}|f(x)|^2dx=\int_{\mathbb{R}} e^{-x}|\sum_i a_ie^{x\wedge y_i}|^2dx$$
$$\leq 2\int_{\mathbb{R}} e^{-x}\sum_i|a_i|^2e^{2(x\wedge y_i)}dx$$
$$=2\sum_i|a_i|^2\int_{-\infty}^{y_i}e^xdx+2\sum_i|a_i|^2e^{y_i}\int_{y_i}^{+\infty}e^{-y}dx$$
$$=2\sum_i|a_i|^2(e^{y_i}+e^{y_i}e^{-y_i})$$
$$=2\sum_i|a_i|^2(e^{y_i}+1)<\infty.$$

It follows that

$$\|f\|^2\leq|||f|||^2\leq 5\|f\|^2$$

and the two norms are equivalent on all H. □

3.7 Invariant subspaces

We are going to study invariant subspaces for H. In order to do so, let's consider the equivalent space F obtained by the following change of variables:

$$\phi:\mathbb{R}\to[0,+\infty)$$
$$x\mapsto\log x$$

Therefore, defining

$$\|f\|_F^2 := \int_0^\infty |f'(t)|^2 dt$$

it follows that, if $f(x) = g(\phi(x))$

$$\begin{aligned}
\|g\|_H^2 &= \int_{-\infty}^{+\infty} e^{-x}|g'(x)|^2 dx \\
&= \int_0^{+\infty} \frac{1}{y}|g'(\log y)|^2 \frac{1}{y} dy \\
&= \int_0^{+\infty} \left| \frac{d}{dy}(g \circ \phi)(y) \right|^2 dy \\
&= \int_0^{+\infty} \left| \frac{d}{dy}(f)(y) \right|^2 dy = \|f\|_F^2.
\end{aligned}$$

Therefore, the space H is equivalent to the space

$$F = \left\{ f \in AC_{\text{loc}}([0, +\infty)) : f(0) = 0, \|f\|^2 = \int_0^\infty |f'(x)|^2 dx < \infty \right\}.$$

Theorem 3.7.1. *Every linear closed invariant subspace of F is of the form $F_A = \{f \in F : f|_A = 0\}$, where $A \subseteq [0, +\infty)$, A closed.*

Proof. F_A is invariant: for any $m \in Mult(F)$, for all $f \in F_A$, we have $mf \in F_A$.

Let $f \in F$. Define:

$$N(f) := \{x \in [0, +\infty) : f(x) = 0\}.$$

Let now $M \subseteq F$ be an invariant subspace and define:

$$N(M) := \bigcap_{f \in M} N(f).$$

Note that $N(M)$ is closed because f is continuous.

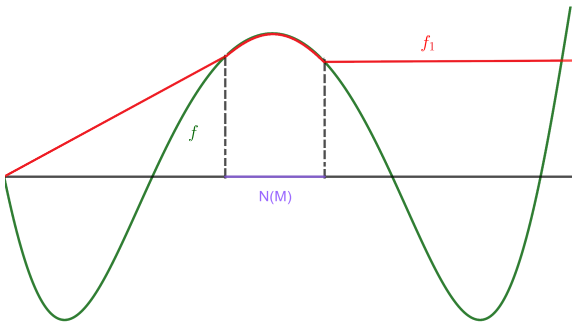
By definition, $M \subseteq F_{N(M)}$.

Let $f \in F$ and define

$$f_1(x) := \begin{cases} f(x) & \text{in } N(M) \\ \text{linear completion of } f & \text{in } [0, +\infty) \setminus N(M) \end{cases}$$

where the linear completion of f is defined in the following way: if (a,b) is a bounded component of $[0,+\infty)\setminus N(M)$, then $f_1(x)=f(a)+\frac{f(b)-f(a)}{b-a}\,x$; if $(a,+\infty)$ is a component, $f_1(x)=f(a)$.

Let $f_2=f-f_1$.



Then: $f_1\in F\ominus F_{N(M)}=F^{\perp}_{N(M)}$ and $f_2\in F_{N(M)}$.

In fact, on each component (a,b) of $[0,+\infty)\setminus N(M)$,

$$\begin{aligned}\int_a^b |f_1'(x)|^2dx &= \int_a^b \frac{[f(b)-f(a)]^2}{(b-a)^2}dx\\ &= \frac{[f(b)-f(a)]^2}{b-a}\\ &= \frac{1}{b-a}\left|\int_a^b f'(x)dx\right|^2\\ &\le \int_a^b |f'(x)|^2dx<+\infty.\end{aligned}$$

If $g\in F_{N(f)}$, then, being $N(M)\subseteq N(f)$,

$$\begin{aligned}\langle f_1,g\rangle_F &= \int_{[0+\infty)\setminus N(M)} f_1'(x)\overline{g'(x)}dx\\ &= \sum_n \int_{a_n}^{b_n} f_1'(x)\overline{g'(x)}dx\\ &= \sum_n \frac{f(b_n)-f(a_n)}{b_n-a_n}[g(b_n)-g(a_n)]=0\end{aligned}$$

where (a_n,b_n) are the bounded components of $[0,+\infty)\setminus N(M)$.

So, $f_1\in F^{\perp}_{N(f)}$, and using the fact that $F^{\perp}_{N(f)}\subseteq F^{\perp}_{N(M)}$, we conclude that $f_1\in F^{\perp}_{N(M)}$.

On the other hand, $f_2(x) = f(x) - f_1(x) = 0$ if $x \in N(M)$. Thus, $f_2 \in F_{N(M)}$.

We now show that $M^\perp \subseteq F_{N(M)}^\perp$.

Let $g \in M^\perp$. Then:

$$0 = \int_{[0,+\infty)\setminus N(f)} (mf)'(x)\overline{g'(x)}\ dx = \sum_i \int_{\alpha_i}^{\beta_i} (mf)'(x)\overline{g'(x)}\ dx$$

for all $f \in M$, $m \in Mult(F)$ and $[0,+\infty) \setminus N(f) = \bigsqcup_i (\alpha_i, \beta_i)$.

Consider m with support in (α_i, β_i). Then, we can distinguish two cases:

- If $\beta_i < +\infty$, wlog let $(\alpha_i, \beta_i) = (0, 1)$. Therefore,

$$\int_0^1 (mf)'(x)\overline{g'(x)}\ dx = 0 \tag{3.13}$$

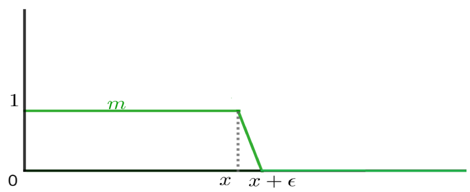
for all $f \in M$ s.t. $f(0) = f(1) = 0$ and $f(x) \neq 0$ in $(0, 1)$ and $\forall\ m \in Mult(F)$ with $\text{supp}(m) \subseteq [0, 1]$.

We want to show that g is linear on $[0, 1]$.

Now, from (3.13), it follows

$$0 = \int_0^1 m(x)f'(x)\overline{g'(x)}\ dx + \int_0^1 m'(x)f(x)\overline{g'(x)}\ dx. \tag{3.14}$$

Let m be as in the picture:



Then,

$$\int_0^1 mf'\overline{g'}\ dt = \int_0^x f'\overline{g'}\ dt + \int_x^{x+\epsilon} mf'\overline{g'}\ dt \xrightarrow{\epsilon \to 0} \int_0^x f'\overline{g'}\ dt,$$

by using Dominated Convergence Theorem, since $f'\overline{g'} \in L^1$. Moreover,

$$\int_0^1 m' f \overline{g'} dt = -\frac{1}{\epsilon} \int_x^{x+\epsilon} f \overline{g'} dt \xrightarrow{\epsilon \to 0} -f(x)\overline{g'(x)} \text{ a.e. } x.$$

Then, using (3.14),

$$\int_0^x f'(t)g'(t)dt = f(x)\overline{g'(x)} \text{ a.e. } x. \tag{3.15}$$

The l.h.s. is AC ($f'\overline{g'} \in L^1$) and f is AC and non zero. Therefore, the r.h.s is also AC and, in particular, $g'(x)$ is AC, after modifying it on a null set.

By differentiating (3.15), we obtain

$$f'(x)g'(x) = f'(x)\overline{g'(x)} + f(x)\overline{g''(x)}.$$

Since $f \neq 0$ in $(0, 1)$, we have that $g''(x) = 0$ a.e. in $(0, 1)$. Hence g' is a.e. constant in $(0, 1)$ and so g is linear and they are all AC.

- If $\beta_i = +\infty$, wlog $(\alpha_i, \beta_i) = (1, +\infty)$. Therefore,

$$\int_1^{+\infty} (mf)'(x)\overline{g'(x)}\, dx = 0$$

for all $f \in F$ s.t. $f(1) = 0$ and $f(x) \neq 0\ \forall\ x > 1$, for all $m \in Mult(F)$.

We want to show that g is constant on $[1, +\infty)$.

One could argue as in the previous case and find that g is linear on $[1, +\infty)$. But $|g'|^2$ is integrable, hence g is constant.

We conclude that g is linear on each connected component of $[0, +\infty) \setminus N(M)$. Hence, as proved before, $g \in F^{\perp}_{N(M)}$ and therefore $M = F_{N(M)}$. □

Corollary 3.7.1.1. *Every linear closed invariant subspace of H is of the form $H_A = \{f \in H : f|_A = 0\}$, where $A \subseteq \mathbb{R}$, A closed.*

Corollary 3.7.1.2. *A necessary and sufficient condition for a function f to be cyclic, i.e. $span\{f, mf, m^2f, m^3f, \dots\} = H$, is to be nowhere vanishing.*

Finally, we want to characterize invariant subspaces in terms of the Theorem 1.5.2. Firstly, we look for functions G_x ($x \in \mathbb{R}^+$) so that the invariance with respect to the multipliers is the same as the invariance with respect to G_x for all x; secondly, we look for a $\Phi \in Mult(F \otimes L^2, F)$ such that, given $M \subseteq F$ invariant, $M = \Phi \cdot (F \otimes L^2(\mathbb{R}^+))$.

Theorem 3.7.2. *Let F_A, where*

$$A = \bigcup_{n=1}^{N} [a_n, b_n],$$

be an invariant subspace of F.
For all $y \in [0, +\infty)$, let

$$G_x(y) = \begin{cases} \frac{1}{\sqrt{a_1}} \chi_{[x,a_1)}(y) & x < a_1 \\ \frac{1}{\sqrt{a_n - b_{n-1}}} \chi_{[x-b_{n-1}, a_n - b_{n-1})}(y) & b_{n-1} \leq x \leq a_n \quad (\forall\ n = 2, \ldots, N) \\ \frac{\sqrt{b_N}}{y} \chi_{[b_N, x)}(y) & x \geq b_N \\ 0 & \textit{otherwise} \end{cases}$$

Set

$$\Phi_A : F \otimes L^2(\mathbb{R}^+) \longrightarrow F$$
$$\Phi_A(f \otimes g)(x) = f\langle g, G_x \rangle$$

Then, the following are equivalent:

1. *F_A is invariant for each M_{G_x};*

2. *M_{Φ_A} is a partial isometry and $F_A = \Phi_A \cdot (F \otimes L^2)$;*

3. *F_A is invariant for M_ϕ for each $\phi \in Mult(F)$.*

Proof. To find G_x we study the kernel k^A, which is the reproducing kernel of F_A.

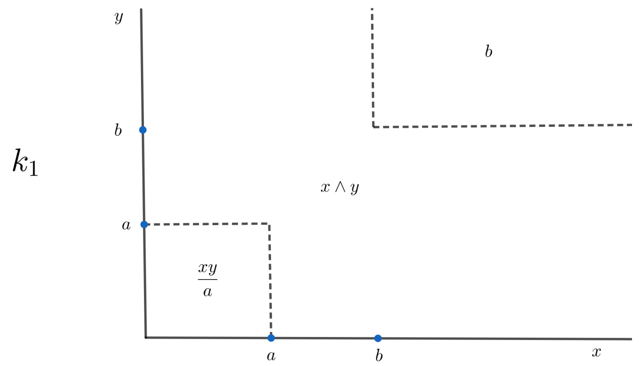
From the previous proof we know that every function $f \in F$ can be written as $f = f_1 + f_2$, where $f_1 \in F_A^{\perp}$ and $f_2 \in F_A$,

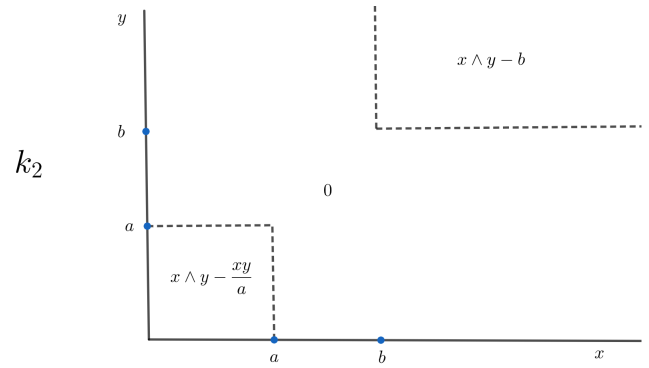
$$f_1(x) := \begin{cases} f(x) & \text{in } A \\ \text{linear completion of } f & \text{in } [0,+\infty) \setminus A \end{cases}$$

$$f_2 := f - f_1$$

Therefore, $k^A = k_2$.

The following pictures consider the most simple case $A = [a, b]$. They can be easily generalized to $A = \bigcup_{n=1}^{N} [a_n, b_n]$.





Consider now

$$\langle G_x, G_y \rangle_{L^2} = \frac{k^A}{k}(x,y) = \begin{cases} 1 - \frac{x \vee y}{a} & x, y < a \\ 0 & \text{otherwise} \\ 1 - \frac{b}{x \wedge y} & x, y \geq b \end{cases}$$

Therefore,

$$G_x(u) = \begin{cases} \frac{1}{\sqrt{a}}\chi_{[x,a)}(u) & x < a \\ \frac{\sqrt{b}}{u}\chi_{[b,x)}(u) & x \geq b \\ 0 & \text{otherwise} \end{cases}$$

Generalizing to $A = \bigcup_{n=1}^{N} [a_n, b_n]$,

$$G_x(u) = \begin{cases} \frac{1}{\sqrt{a_1}}\chi_{[x,a_1)}(u) & x < a_1 \\ \frac{1}{\sqrt{a_n - b_{n-1}}}\chi_{[x-b_{n-1}, a_n - b_{n-1})}(u) & b_{n-1} \leq x \leq a_n \quad (\forall\ n = 2, \ldots, N) \\ \frac{\sqrt{b_N}}{u}\chi_{[b_N,x)}(u) & x \geq b_N \\ 0 & \text{otherwise} \end{cases}$$

Finally, considering

$$\Phi_A : F \otimes L^2(\mathbb{R}^+) \longrightarrow F$$
$$\Phi_A(f \otimes g)(x) = f\langle g, G_x\rangle$$

we have that

$$\Phi_A^* : F \longrightarrow F \otimes L^2(\mathbb{R}^+)$$
$$\Phi_A^*(f)(x) = f \otimes G_x$$

Therefore,

$$\langle G_x, G_y\rangle_{L^2} = \Phi_A(x)\Phi_A^*(y).$$

□

Chapter 4

More about F

Recall that

$$F = \left\{ f \in AC_{\text{loc}}([0, +\infty)) : f(0) = 0, \|f\|^2 = \int_0^\infty |f'(x)|^2 dx < \infty \right\}.$$

It is immediate to see that the reproducing kernel of F is

$$m : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$$
$$m(x, y) = x \wedge y$$

It has an interesting connection with the Brownian motion. A reference is [Ka014].

Definition 4.0.1. *The* ***Brownian motion*** *is the process $B = \{B_t,\ t \geq 0\}$ such that*

1. *$B_0 = 0$;*
2. *B has independent increments: for every $t \geq 0, h \geq 0$, the future increments $B_{t+h} - B_t$ are independent from the past B_s, $s < t$;*
3. *B has Gaussian increments: $B_{t+h} - B_t \sim \mathcal{N}(0, h)$;*
4. *B has continuous paths: $t \mapsto B_t$ is continuous.*

Proposition 4.0.1. *If $B = \{B_t\}_t$ is the Brownian motion, then the covariance between B_t and B_s is given by the kernel $m(s, t)$ for all $s, t \geq 0$.*

Proof. If $s \leq t$, $B_t = B_s + (B_t - B_s) \implies \mathbb{E}[B_t] = 0$.

$$\begin{aligned} \operatorname{cov}(B_s, B_t) &= \mathbb{E}[B_s B_t] \\ &= \mathbb{E}[B_s^2] + \mathbb{E}[B_s(B_t - B_s)] \\ &= \mathbb{E}[B_s^2] \\ &= s = s \wedge t = m(s,t). \end{aligned}$$

□

Corollary 4.0.0.1. *Among centered Gaussian processes, the covariance is given by the minimum if and only if there is Brownian motion.*

In general the following theorem holds:

Theorem 4.0.1. *Let X be a stochastic process. The covariance function of X is a kernel function on T (time).*

Proof. Let $X_1, \dots, X_n$ be a sample of random variables. The matrix $(\operatorname{Cov}(X_i, X_j))$ is positive semidefinite: assuming wlog that $\mathbb{E}[X_i] = 0$ for all i, the matrix is $(\operatorname{Cov}(X_i, X_j)) = \langle X_i, X_j \rangle_{L^2} \geq 0$, because it is a Grammian matrix. □

Remark 4.0.1. *Although there is a connection between stochastic processes and RKHS, in general the properties of functions in H are not reflected in the sample paths of the process. For example, the Brownian motion has paths of unbounded variation, while F contains absolutely continuous and hence of bounded variation functions.*

Another interesting connection of the min kernel is the one with Volterra integral operator. A reference is [GMR23].

Definition 4.0.2. *The **Volterra integral operator** is $V : L^2(\mathbb{R}^+) \to \mathbb{C}$*

$$(Vg)(x) = \int_0^x g(y)dy.$$

If $S : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{C}$

$$S(x,y) = \begin{cases} 1 & 0 \le y \le x \\ 0 & x < y \end{cases}$$

then V *is the integral operator with integral kernel* S*, i.e.*

$$(Vg)(x) = \int_0^\infty S(x,y)g(y)dy.$$

Definition 4.0.3. *Given* X, Z *and* (Y, μ) *a measure space, if* $S_1 : X \times Y \to \mathbb{C}$*,* $S_2 : Y \times Z \to \mathbb{C}$ *are square-integrable functions in* Y*, we define the* ***box product*** *of* S_1 *and* S_2 *as*

$$S_1 \Box S_2 : X \times Z \to \mathbb{C}$$
$$S_1 \Box S_2(x,z) = \int_Y S_1(x,y)S_2(y,z)d\mu(y).$$

Moreover, we define the ***adjoint*** *as*

$$S_1^*(y,x) = \overline{S_1(x,y)}.$$

Coming back to the Volterra integral kernel, we have that

$$S \Box S^*(x,t) = \int_0^\infty S(x,y)S(t,y)dy = x \wedge t.$$

In general it holds the following theorem:

Theorem 4.0.2. *Let* X *be a set, let* (Y, μ) *be a measure space. If* $S : X \times Y \to \mathbb{C}$ *is square integrable in* Y*, then* $S \Box S^* : X \times X \to \mathbb{C}$ *is a kernel function.*

Proof. Set $K = S \Box S^*$. Fix points $x_1, \dots, x_n \in X$ and scalars $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. Let $g(y) = \sum_{i=1}^n \overline{\lambda_i} S(x_i, y) \in L^2(Y, \mu)$. Then,

$$\sum_{i,j=1}^n K(x_i, x_j)\overline{\lambda_i}\lambda_j = \int_Y |g(y)|^2 d\mu(y) \ge 0.$$

□

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