ALMA MATER STUDIORUM UNIVERSITÀ DI BOLOGNA

MUKAI MODELS FOR K3 SURFACES

in low degrees

Tesi di laurea in Geometria Algebrica

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Alla mia famiglia

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Introduzione

Uno dei principali obiettivi della geometria algebrica fin dalla sua introduzione, verso la fine dell'800, è la ricerca di una classificazione delle varietà algebriche. Questo problema è inattaccabile in tutta generalità e comunque molto arduo pur restringendoci a speciali famiglie di varietà. Nel presente lavoro studiamo la classificazione di una particolare famiglia di superfici algebriche di grande interesse geometrico, chiamate *superfici K3*. Sebbene fossero già note e studiate dai matematici durante l'800, è stato André Weil il primo ad avviarne uno studio sistematico durante gli anni '50, oltre ad avere introdotto la terminologia K3 nel 1958. Le tre "K" corrispondono rispettivamente alle iniziali di:

- Ernst Kummer, che durante l'800 ha studiato e introdotto le superfici di Kummer, che sono collegate ad un particolare esempio di superficie K3;

- Erich Kähler, difatti Weil era convinto che ogni superficie K3 ammettesse una metrica Kähler (fatto provato in seguito da Siu);

- Kunihiko Kodaira, che è stato uno dei pionieri dello studio sistematico di questa particolare famiglia di varietà algebriche e che, durante gli anni '60, ha provato che tutte le superfici K3 fanno parte di una unica famiglia.

Successivamente, lo studio delle superfici K3 è stato giustificato anche dalle più generali varietà di Calabi-Yau, di cui le K3 sono un esempio in dimensione due. Lo studio delle varietà di Calabi-Yau è importante anche da un punto di vista applicativo poiché sono uno dei modelli usati dai fisici, ad esempio, in teoria delle stringhe. Inoltre, queste superfici si possono considerare come una generalizzazione in dimensione due delle curve ellittiche, poiché la loro proprietà principale è collegata all'esistenza di una metrica piatta sulla varietà. Le curve ellittiche, o equivalentemente le superfici di Riemann compatte di genere

1, sono l'unica famiglia di curve algebriche con metrica piatta, come conseguenza della classificazione di Riemann. Il loro ruolo in dimensione due è occupato dalle superfici K3 e dai tori complessi bidimensionali, e le prime si distinguono per la proprietà di essere semplicemente connesse.

Una delle motivazioni dietro alla scrittura di questo testo è di presentare e sottolineare la ricca geometria di queste superfici attraverso esempi, calcoli e costruzioni ad esse collegati. Tuttavia, la letteratura in merito è estremamente vasta e sarebbe impossibile pensare di trattarla interamente in un unico testo. Perciò, ci concentreremo principalmente sul contributo di *Shigeru Mukai* alla teoria ed in particolare alla classificazione. Per far questo, mostreremo molti risultati che derivano dalle proprietà che definiscono una superficie K3, come ad esempio la struttura dei fibrati vettoriali o le particolari strutture assunte dalle formule in questo contesto. Questo permette di svolgere esplicitamente molti calcoli sugli invarianti algebrici, cosa che avrà grande importanza nello studio dei modelli.

Il nostro obiettivo è quello di scrivere una lista di modelli generali espliciti di superfici K3 indicizzate dal *grado* o, equivalentemente, dal *genere*. Questo invariante algebrico è definito a partire dalla struttura delle curve che giacciono sulla superficie. La versione attuale della classificazione si deve principalmente al lavoro di Mukai nella seconda metà degli anni '80 e primi anni '90, assieme ad alcuni suoi risultati degli ultimi anni. Precedentemente, alcuni di questi modelli erano già stati costruiti tramite tecniche differenti. Più precisamente, uno dei meriti del lavoro di Mukai è stato quello di vedere nella lista di modelli già esistenti un pattern che ha poi portato anche alla completa classificazione delle *Fano threefolds* (prime) di indice 1. Queste due classificazioni parallele verranno investigate all'interno della tesi e, più in generale, verrà studiata la profonda relazione che lega superfici K3 e varietà di Fano. Inoltre, un'altra importante motivazione dietro lo studio di questa classificazione è l'uso innovativo di fibrati vettoriali su grassmanniane (e altre varietà omogenee) introdotto da Mukai. Difatti, questo è stato già replicato nella ricerca su altri tipi di varietà, come ad esempio recentemente per le varietà hyperkhäler, e sottolinea una sistematicità all'interno dello studio della classificazione.

Tuttavia, la classificazione è molto lontana dall'essere conclusa e, in un certo senso, non potrà mai esserlo. Infatti, l'esistenza di una famiglia (localmente completa) di superfici K3 per un grado fissato è collegato al tipo birazionale dello spazio di moduli delle superfici K3 (polarizzate). Un importante risultato di Gritsenko, Hulek e Sankaran ci dice che, in grado alto, il tipo birazionale dello spazio di moduli non permette l'esistenza di una famiglia localmente completa all'interno di una famiglia parametrizzata, ad esempio, dallo spazio delle sezioni globali di un fibrato vettoriale. In altre parole, man mano che il genere aumenta, lo spazio di moduli smette di essere *unirazionale*, per poi diventare *di tipo generale*. Ad ogni modo, tra il genere massimo di un modello conosciuto e il primo valore in cui la parametrizzazione fallisce è presente un ampio divario. Per i valori nel mezzo è infatti ancora molto difficile trovare una descrizione esplicita del modello generale.

Uno dei principali risultati della tesi è il teorema 3.3.1, che può essere utilizzato per ottenere informazioni sullo spazio ambiente in cui immergere la superficie K3, basandosi sul grado del modello. Questi spazi ambiente sono prevalentemente varietà di Fano che posseggono una struttura di varietà omogenea. Questo permette di utilizzare la teoria della rappresentazione dei gruppi di Lie come $\mathrm{SL}(n, \mathbb{C})$ e $\mathrm{SO}(n, \mathbb{C})$ per lo studio degli invarianti algebrici delle varietà stesse.

Il Capitolo 1 è dedicato ad una introduzione delle superfici K3, con particolare attenzione al caso proiettivo. Abbiamo principalmente seguito il primo capitolo di [24], che sarà una delle principali referenze per la teoria generale per l'intero elaborato. In particolare, diamo la definizione di fibrato lineare *ampio* insieme ad alcuni esempi e calcoliamo i numeri di Hodge di una superficie K3. Inoltre, abbiamo elencato (a volte solo con una referenza per la dimostrazione) alcuni dei risultati principali riguardanti le superfici K3 e le varietà algebriche in generale. Al termine del capitolo è presente una breve discussione degli spazi di moduli di superfici K3 polarizzate, che appariranno regolarmente nel testo.

Nel Capitolo 2 comincia lo studio dei primi modelli di queste superfici, che sono noti classicamente e si ottengono come intersezioni complete di ipersuperfici in spazi proiettivi. Dimostreremo che solo tre modelli differenti si possono scrivere in questo modo. Ogni modello è presentato insieme ad alcune costruzioni (classiche e non classiche) a lui collegate. Le scelte di queste costruzioni sono state fatte con lo scopo di dare nell'insieme una lista di strumenti ed esempi interessanti riguardanti questi primi modelli. Inoltre, in alcuni casi faremo anche il calcolo esplicito del numero di parametri da cui dipendono questi modelli, che sarà poi collegato al concetto di *generalità* del modello stesso.

Il Capitolo 3 ha l'obiettivo di introdurre la teoria necessaria per il teorema principale, chiamato *vector bundle method*. Dopo aver introdotto alcuni strumenti utili allo studio dei fibrati vettoriali di rango maggiore di uno, richiamiamo la teoria degli spazi di moduli di fibrati vettoriali su superfici K3 con vettore di Mukai fissato e la nozione di stabilità (secondo Gieseker), insieme ad alcuni risultati dimostrati da Mukai negli anni '80. In seguito, enunciamo e diamo una dimostrazione del teorema che si basa su questi strumenti e su un risultato di esistenza che proviamo nell'ultima sezione usando strumenti propri della teoria di Brill-Noether.

Nel Capitolo 4 continuiamo la classificazione esponendo i modelli che sono conseguenza del vector bundle method e si ottengono come intersezioni complete di luoghi di zeri di fibrati vettoriali in grassmanniane. Per questi modelli, la classificazione è legata a quella delle 3-varietà di Fano prime di indice 1, di cui compongono una lista completa. Proviamo che solo due modelli sono intersezione completa di ipersuperfici in una grassmanniana. Ogni modello è accompagnato da alcune costruzioni ad esso collegate, con lo scopo di sottolinearne la ricchezza della geometria in cui emergono. Ad esempio, faremo grande uso della teoria degli spazi omogenei e descriveremo alcuni esempi della dualità proiettiva.

Nel Capitolo 5 portiamo una descrizione dei nuovi modelli scoperti da Mukai negli ultimi anni usando gli strumenti trattati nel corso della tesi. In particolare, questi completano la lista di modelli noti allo stato attuale dell'arte e, inoltre, evidenziano in che misura sia difficile la ricerca di nuovi modelli. Seguendo i commenti di Mukai, alcuni di questi modelli sono ispirati alle costruzioni descritte nel Capitolo 4, e questo potrebbe essere un indizio per la ricerca di nuovi modelli.

In fondo, abbiamo aggiunto alcune appendici utili al lettore. Queste riguardano teorie, strumenti e costruzioni che saranno utili nel corso della tesi e che non hanno una posizione precisa all'interno della teoria che svilupperemo. Per questo motivo, vi faremo riferimento quando sarà necessario.

Siccome la ricerca di nuovi modelli non ha prodotto nuovi modelli recentemente, altri filoni di studio sono stati portati avanti. Ad esempio, tra il 2017 e il 2019, Farkas e Verra hanno dimostrato la unirazionalità dello spazio di moduli per $g = 14, 22$. Questi risultati non sono stati ottenuti costruendo un modello generale di superficie K3 esplicito, e, d'altra parte, confermano la possibilità teorica di poterlo trovare per questi valori di g.

Introduction

Classification problems are at the core of algebraic geometry. In the present work, we restrict our attention to some special subcases of particular geometrical interest, namely a kind of algebraic surfaces called *K3 surfaces*. Although they were already known and studied in the nineteenth century, it was André Weil who started in the 50s their systematic study, also choosing the name that characterises them in 1958. The three K's stand respectively for the initials of:

- Ernst Kummer, who introduced and studied during the nineteenth century Kummer surfaces, closely related to a special case of K3 surface;
- Erich Kähler, since Weil was convinced that any K3 surface admits a Kähler metric (it was later proved by Siu);
- Kunihiko Kodaira, who was one of the pioneers of the systematic study of this special family of varieties and, in fact, during the 60s proved that all K3 surfaces are part of the same family.

The modern history of K3 surfaces is also related to that of Calabi–Yau manifolds since K3 surfaces are one the two examples of such manifolds in dimension two. The study of Calabi–Yau manifolds is interesting also from the point of view of applications, since physicists use them for their models, especially in the field of string theory.

Furthermore, the construction of K3 surfaces can be seen as a generalisation of elliptic curves in dimension two. In fact, the main property of a K3 surface is related to the existence of a flat metric on the variety. Elliptic curves, or equivalently compact Riemann surfaces of genus 1, are the only one-dimensional algebraic variety with a flat metric, as a consequence of Riemann classification. In dimension two we have either K3 surfaces or complex tori, where the first ones are distinguished by the property of being simply connected.

One of the goals of the present work is to underline the richness of the geometry of K3 surfaces through several examples, constructions and computations related to them. However, the literature concerning the world of K3 surfaces is enormous and it is beyond the scope of this text to try to cover it. Instead, we will focus mainly on the contribution of *Shigeru Mukai* to the theory and, in particular, to the classification. In order to do this, we will show how several interesting results can be derived from the conditions of a K3 surface. For instance, the structure of vector bundles or the special shapes taken by general formulae in this setting. This allows us to work out several explicit computations which will be very important in the study of examples.

Our aim is to find a list of explicit general models of K3 surfaces labelled by the *degree* or, equivalently, by the *genus*. This algebraic invariant is defined starting from the study of the generic curve that lies on the surface. The final version of this classification is due to the work of Mukai in the second half of the 80s and early 90s, along with some of his works in the last few years. Before that, some of the models were already known and were discovered by different techniques. More precisely, Mukai was able to see and establish a pattern that paved the way also for the classification of (prime) *Fano threefolds* of index 1, which was one of the main motivations behind the research. These two parallel classifications are investigated throughout this work and, more generally, the relations between K3 surfaces and Fano varieties is one of the main topics of our study.

Moreover, another prominent motivation is the innovative use of vector bundles in grassmannians (and other homogeneous varieties) developed by Mukai, which has been replicated for the research of different kinds of varieties, such as for hyperkähler manifolds in recent times, and underlines a systematic pattern behind the classification.

Nevertheless, the classification is far from being complete and, in some sense, it cannot ever be. In fact, the existence of a (locally complete) model for a K3 surface of a fixed degree is linked to the birational type of the moduli space of (polarised) K3 surfaces. A fundamental result of Gritsenko, Hulek and Sankaran asserts that, for a high degree, the birational type of the moduli space does not allow the construction of a locally complete family within a single family parameterised by the space of sections of a vector bundle.

In other words, as the degree increases, the moduli space fails to be *unirational* and, eventually, it becomes of *general type*. Anyway, there is a gap between the degree of the highest model known and the upper bound and, for the values of degree in the middle, it is an open problem to find an explicit description of the general models.

One of the main results of the thesis is theorem 3.3.1, which can be used to obtain some indications of the ambient space in which the K3 surface may lie, based on the degree. These ambient spaces will be Fano varieties, in particular grassmannians, which are endowed with a structure or homogeneous spaces. This allows us to use the representation theory of Lie groups such as $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$ for the study of the algebraic invariants of these varieties.

In Chapter 1 we provide an introduction to K3 surfaces, focusing on the projective case. We took inspiration from the first chapter of [24], which will be one of our main references for the general theory. In particular, we give the definition of *ample* line bundle with some examples and we compute the Hodge numbers of a K3 surface. Moreover, we list (sometimes only with references for the proofs) some of the main theorems concerning the world of K3 surfaces and algebraic varieties in general. We end the chapter with a brief introduction to the moduli spaces of (polarised) K3 surfaces, which will appear all along the text.

Chapter 2 marks the beginning of the study of the models, focusing on the classical ones which arise as complete intersections of hypersurfaces in projective spaces. We prove that only three different models arise in this way. Every model is presented with some classical and non-classical constructions related to it. The choices are made in order to give in the whole an interesting list of tools and constructions concerning the first examples of K3 surfaces. Moreover, in a few cases, we provide the explicit computation of the number of parameters the model depends on, which is related to the concept of *generality* of the model.

Chapter 3 is devoted to the theory behind the main theorem, the so-called *vector bundle method*. After introducing some tools for higher-rank vector bundles on an algebraic variety, we study the moduli spaces of vector bundles with fixed Mukai vector and we introduce the notion of (Gieseker) stability along with some classical results provided by Mukai in his works during the 80s. The statement of the main theorem is followed by the proof relying on these results and on an existence result proved in the last section using tools arising from Brill-Noether theory.

In Chapter 4 we continue the classification showing the models arising as a result of the vector bundle method as a complete intersection of zero loci of vector bundles in grassmannians. For these models, the classification is linked to the one of prime Fano threefolds of index 1, and it provides a complete list. We prove that only two models are complete intersections of hypersurfaces in grassmannians. Again, every model is presented together with special constructions related to it, with the aim of underlining the richness of the geometry behind them. For instance, we make strong use of the theory of homogeneous spaces and we provide some examples of projective duality.

Chapter 5 contains the descriptions of the models found for the first time by Mukai (from the 2000s to 2016) using the tools developed in the previous chapters. In particular, they complete the list of known general models of K3 surfaces at the present time and, in some sense, they point out the difficulties of the search for new models. Following Mukai's notes, most of them are inspired by the constructions explained in Chapter 4, which can be taken as a hint for the research of new ones.

At the bottom, we put some appendices for the reader. They cover tools, theories and constructions which do not have a precise place in the thesis but are useful throughout the whole text. In particular, we will often refer to them during the discussion of the models.

As the research of new general models did not provide new examples in the last years, other lines of research have brought new results in recent times. For instance, in 2017 and 2019 Farkas and Verra were able to prove the unirationality of the moduli space for genera 14 and 22. Their results do not rely on the existence of an explicit model of K3 surface and, instead, they confirm the theoretical possibility to find explicit ones corresponding to those genera.

Notations

Throughout the whole document, we will work with (smooth) projective varieties over the field of complex numbers, except when otherwise indicated. They will be denoted with F, G, X, Y and variations of these.

With $(G, \mathcal{F})$, where $\mathcal{F}$ is a globally generated vector bundle over G, we mean the zero locus $V(s) \subseteq G$ of a general global section $s \in H^0(G, \mathcal{F})$. We point out that this is a non-standard notation which turns out to be very convenient.

In $\mathbb{P}^n$ we denote by H the class (under linear equivalence) of any hyperplane section, which corresponds to the divisor of the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. In particular, H represents a generic hyperplane in $\mathbb{P}^n$.

The *Euler number* of X will be denoted by $e(X)$.

For a variety X we denote the Picard rank by $\rho(X) := \mathrm{rk}(\mathrm{NS}(X))$, where $\mathrm{NS}(X)$ is the image of $\mathrm{Pic}(X)$ in $H^2(X, \mathbb{R})$ and it is called *Néron-Severi group*.

Chapter 1

Generalities on K3 surfaces

1.1 Definition of (polarised) K3 surface

K3 surfaces are the protagonist of this work, they form a special family of algebraic surfaces which can be seen as a generalisation of elliptic curves. The general K3 surface is not projective, that is, it has Picard rank equal to 0. If we restrict our attention to the projective ones, by the Kodaira embedding theorem we can always fix a polarisation, i.e. an ample line bundle over them, and that allows us to define genus and degree.

Definition 1.1.1. Let X be a compact connected complex manifold of complex dimension two, we say that X is a *K3 surface* if:

- $H^1(X, \mathcal{O}_X) = 0$;
- the canonical bundle ω_X is trivial.

Since we are mainly interested in projective K3 surfaces, we may think of them as (complete) non-singular algebraic varieties over $\mathbb{C}$, in terms of Serre's *GAGA principle*.

Let X be a projective K3 surface, a pair (X, L) is a *polarised* K3 surface of degree d if L is an ample primitive line bundle over X such that $(L)^2 = d > 0$ (see appendix A.2 for an introduction to the intersection form). Recall that a line bundle L is said to be *ample* if there is a power $L^{\otimes m}$ that induces an embedding of X in a projective space, that is if we fix a basis $\{ s_0, ..., s_N \}$ of $H^0(X, L^{\otimes m})$ we have the embedding

$$\begin{aligned} \phi_{|L^{\otimes m}|} : X &\hookrightarrow \mathbb{P}^N \\ x &\longmapsto (s_0(x) : ... : s_N(x)). \end{aligned}$$

We can characterise ampleness on projective surfaces by asking $(L)^2 > 0$ and $(L \cdot C) > 0$ for all (irreducible) curves $C \subseteq X$ (Kleiman's Criterion, see e.g. theorem 8.1.2 [24]), while if we only have $(L \cdot C) \geq 0$ we call L *nef*.

Example 1.1.2. Fix an integer $d > 0$ and consider the line bundle $\mathcal{O}_{\mathbb{P}^2}(d)$ along with the induced morphism

$$\begin{aligned} \phi_{|\mathcal{O}_{\mathbb{P}^2}(d)|} : \mathbb{P}^2 &\longrightarrow \mathbb{P}^N \\ (z_0 : z_1 : z_2) &\longmapsto (z_0^{i_0} z_1^{i_1} z_2^{i_2})_{(i_0,i_1,i_2)}, \end{aligned}$$

where $N = \frac{(d+2)(d+1)}{2} - 1$ and (i_0, i_1, i_2) runs through all multi-indices such that $i_0 + i_1 + i_2 = d$. This is an embedding for every $d > 0$ and so $\mathcal{O}_{\mathbb{P}^2}(d)$ is ample for every $d > 0$. The image in $\mathbb{P}^N$ is called d-*Veronese* of $\mathbb{P}^2$ and for $d \geq 2$ it is not a complete intersection.

Example 1.1.3. Consider $\mathcal{O}(1, 1) := p_1^*(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$, then the induced morphism

$$\begin{aligned} \phi_{|\mathcal{O}(1,1)|} : &\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3 \\ &((z_0 : z_1), (z_0' : z_1')) \longmapsto (z_0 z_0' : z_0 z_1' : z_1 z_0' : z_1 z_1') \end{aligned}$$

is an embedding and so $\mathcal{O}(1, 1)$ is ample. The image is a quadric surface in $\mathbb{P}^3$.
On the other hand, if we take $\mathcal{O}(1, 0)$ as a line bundle then the associated map is

$$\begin{aligned} \phi_{|\mathcal{O}(1,0)|} : &\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \\ &((z_0 : z_1), (z_0' : z_1')) \longmapsto (z_0 : z_1), \end{aligned}$$

which is clearly not injective. The same happens for $\mathcal{O}(d, 0)$ and so they are all not ample. However, $\mathcal{O}(1, 0)$ is nef, as the intersection of H_1 with a generic curve $\mathcal{O}(a, b)$ with $a, b \geq 0$

is

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = b \geq 0.$$

Because the intersection pairing is even (see proposition A.2.2), the degree of a polarised K3 surface (X, L) can be written as $d = 2g - 2$, where g is called the *genus* of X and, in fact, it corresponds to the genus of any smooth curve in $|L|$[1]: by definition $g(C) = \frac{\deg(\omega_C)+2}{2}$ and by adjunction formula $\omega_C \cong L_{|C}$ (see below).
From theorem 2.4.2 of [24] (or the original proposition VIII.15 of [2]) there exists a polarised K3 surface of genus $g \geq 3$ in $\mathbb{P}^g$. It is therefore natural to try to describe the (very general) K3 surface of each genus as zero loci of equations in simple spaces, such as grassmannians. This in fact can be achieved only for low values of g, we will come back to this at the end of this chapter.

1.2 Useful theorems and results

In this section, we collect some useful results that we will recall throughout the thesis. Most of them arise naturally in the context of complex geometry, so they are stated in this more general setting. Although, we will use them only for (smooth) projective varieties. We refer to the appendices A.1 and A.9 for an introduction to vector bundles and Chern classes.

Theorem 1.2.1 (Adjunction formula)**.** *Let Y be a submanifold of a complex manifold X, then there is a natural isomorphism*

$$\omega_Y \cong (\omega_X)_{|Y} \otimes \det\left(\mathcal{N}_{Y/X}\right).$$

Proof. See proposition 2.2.17 of [23]. □

Theorem 1.2.2 (Riemann-Roch for vector bundles on surfaces)**.** *Let L be a line bundle*

[1]Note that by Bertini's theorem 1.2.3 the general curve in $|L|$ is smooth.

and E be a vector bundle of rank r on a compact complex surface S. Then

$$\chi(S,\mathcal{O}_S) = \frac{c_1(S)^2 + c_2(S)}{12} \quad \textbf{\textit{(Noether's formula)}} \; ;$$
$$\chi(S,L) = \frac{c_1(L)(c_1(L)+c_1(S))}{2} + \chi(S,\mathcal{O}_S) \, ;$$
$$\chi(S,E) = \frac{c_1(E)^2 - 2c_2(E) + c_1(E)c_1(S)}{2} + r\chi(S,\mathcal{O}_S) \, .$$

Theorem 1.2.3 (Bertini-type theorem)**.** *Let E be a vector bundle over a smooth variety. If E is generated by its global sections then a general global section of E is non-degenerate and its zero locus, if non-empty, is smooth.*

Theorem 1.2.4 (Hodge decomposition)**.** *Let X be a compact Kähler manifold of dimension n, then*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X) := H^q\left(\mathcal{A}^{p,\cdot}(X), \overline{\partial}\right)$ is (p,q)-Dolbeault cohomology, which coincides with $H^q(X,\Omega^p_X)$. Moreover, this decomposition does not depend on the chosen Kähler structure and satisfies the following symmetries:

- $\overline{H^{p,q}}(X) \cong H^{q,p}(X)$ *(conjugation);*
- $H^{p,q}(X) \cong H^{n-q,n-p}(X)$ *(Hodge $*$-operator);*
- $H^{p,q}(X) \cong H^{n-p,n-q}(X)^\vee$ *(Serre duality).*

Theorem 1.2.5 (Serre duality)**.** *Let X be a compact complex manifold of dimension n, then for every vector bundle E there are natural isomorphisms*

$$H^q(X,\Omega^p_X \otimes E) \cong H^{n-q}(X,\Omega^{n-p}_X \otimes E^\vee)^\vee.$$

Theorem 1.2.6 (Akizuki-Kodaira-Nakano vanishing)**.** *Let L be a line bundle over a compact Kähler manifold X of dimension n. Then*

$$H^q(X,\Omega^p_X \otimes L) = 0 \text{ for } p+q>n, \text{ if } L \text{ is ample;}$$
$$H^q(X,\Omega^p_X \otimes L) = 0 \text{ for } p+q<n, \text{ if } L^\vee \text{ is ample.}$$

Theorem 1.2.7 (Lefschetz hyperplane section). *Let X be a compact Kähler manifold of dimension $n+1$ and let $Y \subseteq X$ be a smooth hypersurface such that the induced line bundle $\mathcal{O}_X(Y)$ is ample, then the restriction maps $H^k(X) \longrightarrow H^k(Y)$ are isomorphisms of Hodge structures for $k \leq n-1$ and injective morphisms of Hodge structures for $k = n$.*

Proof. We have to prove that the restriction map $H^{p,q}(X) \longrightarrow H^{p,q}(Y)$ is an isomorphism for $p+q < n$ and is injective for $p+q = n$. These cohomology groups are equal to $H^q(X, \Omega^p_X)$ and $H^q(Y, \Omega^p_Y)$ respectively. For every $0 \neq s \in H^0(X, \mathcal{O}(Y))$, we have the map $\mathcal{O}_X \longrightarrow \mathcal{O}(Y)$ given by $f \longmapsto \frac{f}{s}$. The dual is in fact injective and yields the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-Y) \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y \longrightarrow 0,$$

which, after twisting by Ω^p_X, becomes

$$0 \longrightarrow \Omega^p_X \otimes \mathcal{O}_X(-Y) \longrightarrow \Omega^p_X \longrightarrow i_*\Omega^p_{X|Y} \longrightarrow 0.$$

By theorem 1.2.5 and 1.2.6, $H^q\left(X, \Omega^p_X \otimes \mathcal{O}_X(-Y)\right) = 0$ for $p+q < n+1$. Thus, the map $H^q\left(X, \Omega^p_X\right) \longrightarrow H^q\left(Y, \Omega^p_{X|Y}\right)$ is bijective for $p+q < n$ and at least injective for $p+q = n$.

On the other hand, since $\mathcal{N}_{Y/X} \cong \mathcal{O}_Y(Y)$[2], we have the following exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-Y) \longrightarrow \Omega_{X|Y} \longrightarrow \Omega_Y \longrightarrow 0.$$

We can take the p-th exterior power of the surjection, which remains surjective and yields

$$0 \longrightarrow \Omega^{p-1}_Y \otimes \mathcal{O}_Y(-Y) \longrightarrow \Omega^p_{X|Y} \longrightarrow \Omega^p_Y \longrightarrow 0.$$

Since the restriction of an ample bundle is again ample, we can apply theorem 1.2.6 to get $H^q\left(Y, \Omega^p_{X|Y}\right) \longrightarrow H^q\left(Y, \Omega^p_Y\right)$ bijective for $p+q < n$ and at least injective for $p+q = n$. The compositions of these two families of maps are the ones described in the statement. □

[2]Recall that, for a smooth subvariety, the normal bundle coincides with the restriction of the defining vector bundle to the variety.

Remark 1.2.8. (i) If we consider for example a smooth hypersurface $Y \subseteq \mathbb{P}^n$ given by a polynomial f of degree d, then we can embed $\mathbb{P}^n$ in $\mathbb{P}^N$ by the d-Veronese. We notice that the monomials of f are just scalar multiples of some of the coordinates of the Veronese map, hence Y is exactly the image of $\mathbb{P}^n$ cut by the hyperplane corresponding to the linear combination of the coordinates above with the right coefficients. In particular $\mathcal{O}_{\mathbb{P}^n}(Y) = \mathcal{O}_{\mathbb{P}^n}(d) = \mathcal{O}_{\mathbb{P}^N}(1)_{|\mathbb{P}^n}$. This shows that our statement of theorem 1.2.7 is equivalent to the original one of Lefschetz.
(ii) Clearly, theorem 1.2.7 can be extended to the case of a complete intersection simply by iterating the process for hypersurfaces.

Theorem 1.2.9 (Siu). *Every K3 surface admits a Kähler metric on it.*

Proof. We refer to the original [47]. □

Theorem 1.2.10. *All K3 surfaces are diffeomorphic to each other, i.e. they only differ for the datum of the complex structure.*

Proof. This result is due to Kodaira, see [27]. □

1.3 Hodge diamond of K3 surfaces

In this section, we want to compute the cohomology and the Hodge numbers of a K3 surface X.

Since X is connected, $h^0(X, \mathcal{O}_X) = h^{0,0} = 1$. By hypothesis, $H^1(X, \mathcal{O}_X) = 0$ and $h^2(X, \mathcal{O}_X) = h^2(X, \omega_X) = 1$, so we have $h^{1,0} = h^{0,1} = 0$ and $h^{2,0} = h^{0,2} = 1$. From this computation we get $\chi(\mathcal{O}_X) = 2$ and, by Noether's formula

$$2 = \frac{c_1(X)^2 + c_2(X)}{12} = \frac{c_2(X)}{12},$$

we obtain $c_2(X) = 24$, and we used that $c_1(X) = c_1(\mathcal{T}_X) = -c_1(\Omega_X) = -c_1(\det(\Omega_X)) = 0$.

Remark 1.3.1. From the vanishing of $H^1(X, \mathcal{O}_X)$, we have that a K3 surface admits no global vector fields. In fact, the natural alternating pairing

$$\Omega_X \times \Omega_X \longrightarrow \omega_X \cong \mathcal{O}_X$$

induces a non-canonical isomorphism $\mathcal{T}_X \cong \Omega_X$, and we know that

$$H^0(X, \Omega_X) = H^1(X, \mathcal{O}_X) = 0.$$

Using Riemann-Roch formula for $E = \Omega_X$, we obtain

$$-h^{1,1} = \chi\left(X, \Omega_X\right) = -c_2(\Omega_X) + 4 = -20.$$

Hence the last unknown Hodge number is $h^{1,1} = 20$. We summarise them in the Hodge diamond of a K3 surface:

$$\begin{array}{ccccc} 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

This also leads to a simplified Riemann-Roch formula for a K3 surface, namely

$$\chi\left(X, L\right) = 2 + \frac{c_1(L)^2}{2},$$

and to the value of the Euler number $e(X) = 24$, which can also be deduced for the Gauss-Bonnet-Chern formula A.9.3.

1.4 Moduli spaces of polarised K3 surfaces

The moduli functor of polarised K3 surfaces can be represented by a coarse moduli space which exists as a quasi-projective irreducible algebraic variety. Given that it is easy to compute the dimension, which will be 19. These results allow us to verify that our models are general, i.e. they depend exactly on 19 parameters.

Consider a polarised K3 surface (X, L) with $(L)^2 = d$, then, by 1.2.2, X has Hilbert polynomial $P(t) = \frac{d}{2}t^2 + 2$. From theorem 2.2.7 of [24], $L^{\otimes 3}$ is very ample and hence we have an embedding

$$\phi : X \hookrightarrow \mathbb{P}^N,$$

where $N = P(3) - 1$, which yields $\mathcal{O}_{\mathbb{P}^N}(1)_{|X} \cong L^{\otimes 3}$. Therefore, the corresponding Hilbert polynomial in this inclusion with respect to $\mathcal{O}_{\mathbb{P}^N}(1)_{|X}$ is $P(3t)$. We are now able to consider the Hilbert scheme $\mathrm{Hilb} := \mathrm{Hilb}^{P(3t)}_{\mathbb{P}^N}$ of all closed subschemes of $\mathbb{P}^N$ with Hilbert polynomial $P(3t)$. By proposition 5.2.1 of [24], we have an open subscheme $U \subseteq \mathrm{Hilb}$ (with a certain universal property) which parametrises polarised K3 surfaces (X, L) with $X \hookrightarrow \mathbb{P}^N$ and $\mathcal{O}_{\mathbb{P}^N}(1)_{|X} \cong L^{\otimes 3}$. Since a point $x \in U$ corresponds to an embedded K3 surface $X \subseteq \mathbb{P}^N$, the tangent space T_xU is naturally isomorphic to $\mathrm{Hom}\left(\mathcal{I}_X \big/ \mathcal{I}_X^2, \mathcal{O}_X\right)$, which, in the smooth case, can be computed as $H^0(X, \mathcal{N}_{X/\mathbb{P}^N})$[3]. Henceforth, we need the following exact sequence

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{\mathbb{P}^N|X} \longrightarrow \mathcal{N}_{X/\mathbb{P}^N} \longrightarrow 0,$$

which yields

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{T}_X) \longrightarrow H^0(X, \mathcal{T}_{\mathbb{P}^N|X}) \longrightarrow H^0(X, \mathcal{N}_{X/\mathbb{P}^N}) \longrightarrow \\ \longrightarrow H^1(X, \mathcal{T}_X) \longrightarrow H^1(X, \mathcal{T}_{\mathbb{P}^N|X}) \longrightarrow H^1(X, \mathcal{N}_{X/\mathbb{P}^N}) \longrightarrow 0. \end{aligned}$$

Since there are no global vector fields on a K3 surface, we have $H^0(X, \mathcal{T}_X) = 0$.
From the Euler sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)^{\oplus N+1} \longrightarrow \mathcal{T}_{\mathbb{P}^N|X} \longrightarrow 0,$$

we obtain

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X(1))^{\oplus N+1} \longrightarrow H^0(X, \mathcal{T}_{\mathbb{P}^N|X}) \longrightarrow 0,$$

which gives us $h^0(X, \mathcal{T}_{\mathbb{P}^N|X}) = (N+1) \cdot (N+1) - 1 = N^2 + 2N$. Since the vanishing of $H^1(X, \mathcal{O}_X(1))$ and $H^2(X, \mathcal{O}_X(1))$, we get $H^1(X, \mathcal{T}_{\mathbb{P}^N|X}) \cong H^2(X, \mathcal{O}_X) \cong \mathbb{C}$.

[3]Recall that $\mathcal{N}^\vee_{X/\mathbb{P}^N} \cong \mathcal{I}_X \big/ \mathcal{I}_X^2$, where $\mathcal{I}_X$ is the ideal of polynomials vanishing on X, given by the short exact sequence $0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^N} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0$.

It is not difficult to see that the map

$$H^1(X, \mathcal{T}_X) \longrightarrow H^1(X, \mathcal{T}_{\mathbb{P}^N|X})$$

is surjective. In fact, its Serre dual $H^1(X, \Omega_{\mathbb{P}^N|X}) \longrightarrow H^1(X, \Omega_X)$ sends $c_1(\mathcal{O}(1))$ in $c_1(L^3)$, which has degree $9d$. Thus it is not trivial.
The previous discussion leads to $h^0(X, \mathcal{N}_{X/\mathbb{P}^N}) = N^2 + 2N + 20 - 1$, therefore U is a scheme of dimension $19 + N^2 + 2N$.

The *moduli functor* of primitively polarised K3 surfaces of genus g is defined as

$$\mathcal{M}_g : (Sch/\mathbb{C})^{op} \longrightarrow (Sets),$$

which sends a scheme T of finite type over $\mathbb{C}$ to the set of pairs (f, L), where $f : \mathfrak{X} \longrightarrow T$ is a smooth proper morphism and $L \in \mathrm{Pic}_{\mathfrak{X}/T}(T)$, such that for all rational points of T the fibre, i.e. the base change, yields a K3 surface X with a primitive ample line bundle L_X of degree $2g - 2$ over it. This functor is not representable due to the existence of automorphisms of K3 surfaces. However, it is possible to prove the existence of a coarse moduli space for this functor.
A *coarse moduli space* for a moduli functor $\mathcal{M}$ is a scheme $\mathcal{F}$ with a natural transformation of functors $\phi : \mathcal{M} \longrightarrow \mathrm{Hom}(\_, \mathcal{F})$ such that:

1. for any algebraically closed field k the map $\phi(\mathrm{Spec(k)})$ is a bijection;
2. given a scheme M and a natural transformation $\psi : \mathcal{M} \longrightarrow \mathrm{Hom}(\_, M)$, there is a unique natural transformation $\chi : \mathrm{Hom}(\_, \mathcal{F}) \longrightarrow \mathrm{Hom}(\_, M)$ such that $\chi \circ \phi = \psi$.

On U we have an action of $\mathrm{PGL} := \mathrm{PGL}(N+1)$, whose dimension is $\dim(\mathrm{PGL}) = N^2+2N$. The categorical quotient $\pi : U \longrightarrow \mathcal{F}_g$ is a quasi-projective variety of dimension 19 and its rational points parameterise the orbits of the action, i.e. $\left[\underline{U}/\underline{\mathrm{PGL}}\right](\mathbb{C}) \cong \mathcal{F}_g(\mathbb{C})$. Thus, $\mathcal{F}_g$ is a quasi-projective irreducible variety of dimension 19 and a coarse moduli space for the moduli functor of primitively polarised K3 surfaces of genus g (theorem 5.2.4 of [24]). Since the local geometry of a moduli space is studied by deformation theory, we refer to appendix A.4, where we describe a possible way to prove that a model of polarised K3 surface is *general.* What we need is to compute the number of *embedded deformations*, i.e. the rank of the map $H^0(X, \mathcal{N}_{X/G}) \longrightarrow H^1(X, \mathcal{T}_X)$, and to check that it is equal to 19.

Note that it is possible to construct $\mathcal{F}_g$ as a Deligne-Mumford stack, see section 5.4 of [24]. See also section 6.4 *op. cit.* for the construction using periods and the global Torelli theorem A.2.4.

Remark 1.4.1. From the work of Mukai, we know that for $g \leq 12$ and for $g = 13$, 16, 18 and 20 the moduli space of polarised K3 surfaces is unirational[4]. Recall that a projective variety X is said to be *unirational* if there is a rational dominant morphism from a projective space to X. Furthermore, if the map admits a rational inverse, then X is said to be *rational.* In fact, for $g \neq 11$, these results of unirationality have been achieved by finding the general polarised K3 surface of each genus as the zero locus of a homogeneous vector bundle on grassmannians. This is the main subject of the present work. We will explain the tools developed by Mukai in order to reach a model for the general K3 surface of each genus.
For $g = 11$ there is no such model at the present time, and instead, unirationality is proved using the unirationality of the moduli space of curve of genus 11 along with the fact that it is birationally equivalent to a $\mathbb{P}^{11}$-fibre bundle over $\mathcal{F}_{11}$. This was achieved in [40].

Remark 1.4.2. On the other hand, the result of Gritsenko-Hulek-Sankaran in [18] affirms that the moduli space of polarised K3 surfaces is of *general type* for $g \geq 63$ and for $g = 47$, 51, 55, 58, 59 and 61. Hence, there is no hope to find a model for high genus[5]. Indeed, for a fixed genus g, if we can produce a *general* model $X = (G, \mathcal{F})$, then we also have the rational parametrisation

$$\begin{aligned} \mathbb{P}\left(H^0(G, \mathcal{F})\right) &\dashrightarrow \mathcal{F}_g \\ [s] &\longmapsto V(s) \subseteq G, \end{aligned}$$

which implies that the moduli space in genus g is unirational. Moreover, every unirational variety has Kodaira dimension equal to $-\infty$ so it is not of general type.

[4]Recently it has also been proved the case of $g = 14$ and 22 by Farkas and Verra, see [14] and [15].
[5]Moreover, if $g \geq 41$ and $g \neq 42, 45, 46, 48$ then the Kodaira dimension of $\mathcal{F}_g$ is non-negative.

Chapter 2

Classification in genus 2 to 5

If one is willing to find K3 surfaces in a reasonable ambient space the first attempt is as a hypersurface in $\mathbb{P}^3$ or as a complete intersection in a bigger projective space. The following lemma shows that only three separate cases arise in this way. They are worked out respectively in section 2.1, 2.2 and 2.3. In section 2.4, we present the model of K3 surface with genus 2, which differs from the others since it lies naturally in a weighted projective space.

Lemma 2.0.1. *Let* $X = (\mathbb{P}^{n+2}, \bigoplus_{i=1}^{n} \mathcal{O}(d_i))$ *be a K3 surface, where* $d_1 \geq d_2 \geq ... \geq d_n \geq 2$*, then either:*

1. $X = (\mathbb{P}^3, \mathcal{O}(4))$*, with genus* $g = 3$*;*

2. $X = (\mathbb{P}^4, \mathcal{O}(3) \oplus \mathcal{O}(2))$*, with genus* $g = 4$*;*

3. $X = (\mathbb{P}^5, \mathcal{O}(2)^{\oplus 3})$*, with genus* $g = 5$*.*

Proof. The condition $H^1(X, \mathcal{O}_X) = 0$ is always satisfied applying theorem 1.2.7. We only need to impose the canonical bundle ω_X to be trivial. Firstly we compute $\det\left(\mathcal{N}_{X/\mathbb{P}^{n+2}}\right)$ using Künneth formula which in this case reads

$$\bigwedge^{n} \mathcal{N}_{X/\mathbb{P}^{n+2}} \cong \bigotimes_{i=1}^{n} \mathcal{O}_X(d_i) \cong \mathcal{O}_X\left(\sum_{i=1}^{n} d_i\right)$$

and then by adjunction $\omega_X \cong (\mathcal{O}_{\mathbb{P}^{n+2}}(-n-3))_{|X} \otimes \mathcal{O}_X(d_1 + ... + d_n)$. Hence we get the condition

$$\sum_{i=1}^{n} d_i = n + 3,$$

which immediately shows that $n \leq 3$. For $n = 1$ we only have $d_1 = 4$ (2.1); for $n = 2$ we have $d_1 + d_2 = 5$, thus $d_1 = 3$ and $d_2 = 2$ (2.2); for $n = 3$ we have $d_1 + d_2 + d_3 = 6$, thus $d_1 = d_2 = d_3 = 2$ (2.3). In each case, we have the natural polarisation given by $\mathcal{O}_X(1)$, which is an ample and primitive[6] line bundle and its degree can be computed as the product of the d_i. By the relation $d = 2g - 2$ we conclude that in this way we only find K3 surfaces of genus 3, 4 and 5. In fact, the very general one of each genus (see the respective section for details). □

In the following table, we sum up the models that will appear in this chapter:

g	Projective space	Vector bundle
2	$\mathbb{P}(1,1,1,2)$	$\mathcal{O}(6)$
3	$\mathbb{P}^3$	$\mathcal{O}(4)$
4	$\mathbb{P}^4$	$\mathcal{O}(3) \oplus \mathcal{O}(2)$
5	$\mathbb{P}^5$	$\mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2)$

2.1 Genus 3

Consider a smooth quartic $X = (\mathbb{P}^3, \mathcal{O}(4))$, which is a K3 surface with a natural polarisation given by $\mathcal{O}_X(1)$. Its degree is $(\mathcal{O}_X(1))^2 = (H_{|X})^2 = (H)^2_{|X} = 4$, since a generic line intersects a quartic in four points. As the following computation will point out, this is the very general K3 surface of degree 4 or genus $\frac{4+2}{2} = 3$. We have to prove that the number of parameters in this family equals 19, that is, the dimension of the moduli space of K3 surfaces of genus 3. This number corresponds to $\dim\big(\mathrm{Im}(H^0(X, \mathcal{N}_{X/\mathbb{P}^3}) \to H^1(\mathcal{T}_X))\big)$ (see section A.4) and, in order to compute it, we need a discussion similar to the one in section 1.4 with $N = 3$. In addition, this time we have the following exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0.$$

[6]See the discussion in section 2.3.

Twisted by $\mathcal{O}_{\mathbb{P}^3}(4)$, it yields the following exact sequence in cohomology

$$0 \longrightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \longrightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4)) \longrightarrow H^0(\mathbb{P}^3, i_*\mathcal{O}_X(4)) \longrightarrow 0$$

where the last 0 is given by the vanishing of $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3})$. In particular, $h^0(X, \mathcal{O}_X(4))$ is equal to

$$h^0(\mathbb{P}^3, i_*\mathcal{O}_X(4)) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4)) - h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = \binom{3+4}{4} - 1 = 34.$$

As before, we have

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X(1))^{\oplus 4} \longrightarrow H^0(X, \mathcal{T}_{\mathbb{P}^3|X}) \longrightarrow 0,$$

which gives us $h^0(X, \mathcal{T}_{\mathbb{P}^3|X}) = 4 \cdot 4 - 1 = 15$, while from the vanishing of $H^1(X, \mathcal{O}_X(1))$ and $H^2(X, \mathcal{O}_X(1))$ we get $h^1(X, \mathcal{T}_{\mathbb{P}^3|X}) = h^2(X, \mathcal{O}_X) = 1$.
From

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{T}_X) \longrightarrow H^0(X, \mathcal{T}_{\mathbb{P}^3|X}) \longrightarrow H^0(X, \mathcal{N}_{X/\mathbb{P}^3}) \longrightarrow \\ \longrightarrow H^1(X, \mathcal{T}_X) \longrightarrow H^1(X, \mathcal{T}_{\mathbb{P}^3|X}) \longrightarrow 0, \end{aligned}$$

we obtain $\dim\left(\mathrm{Im}(H^0(X, \mathcal{N}_{X/\mathbb{P}^3}) \to H^1(X, \mathcal{T}_X))\right) = \dim\left(H^0(\mathcal{N}_{X/\mathbb{P}^3}) \big/ H^0(\mathcal{T}_{\mathbb{P}^3|X})\right)$, which is equal to $34 - 15 = 19$, as wanted.

Remark 2.1.1. This computation also shows that $\rho(X) = 1$ generically. In fact, for the universal family of K3 surfaces N, the dimension of the set of deformations

$$\{\, t \in N \mid \rho(X_t) = r \,\}$$

is $20 - r$ (see [24], section 17.1.3). Hence $1 = r = \rho(X)$ in our case. More classically, the following theorem by Lefschetz (see the original [31]) also shows that $\rho(X) = 1$ generically.

Theorem 2.1.2 (Noether-Lefschetz). *Let $X \subseteq \mathbb{P}^N$ be a general complete intersection surface, then the restriction map $\mathrm{Pic}(\mathbb{P}^N) \longrightarrow \mathrm{Pic}(X)$ is an isomorphism except in the following cases:*

- $X = (\mathbb{P}^3, \mathcal{O}(2))$;
- $X = (\mathbb{P}^3, \mathcal{O}(3))$;
- $X = (\mathbb{P}^4, \mathcal{O}(2)^{\oplus 2})$.

Corollary 2.1.3. *The general K3 surface of genus g does not contain any line.*

Proof. We know that $\mathrm{Pic}(X) = \mathbb{Z}\langle L\rangle$[7]. Since the zero locus of L is a curve of genus g, the existence of a line in X would lead to a class in the Picard group which is linearly independent to L. □

Remark 2.1.4. Nevertheless, we can produce examples of smooth quartic surfaces with higher Picard rank. For instance, the Fermat quartic $X = V(x_0^4 + x_1^4 + x_2^4 + x_3^4) \subseteq \mathbb{P}^3$ is known to have maximum Picard rank, that is $\rho(X) = 20$ (cfr. [45], page 1953). It is also known that, in the Fermat quartic, there are 48 lines (e.g. [8]) and they are all of the form

$$V(x_{i_1} - \xi_j x_{i_2}, x_{i_3} - \xi_k x_{i_4}) \subseteq X,$$

with $\xi^4 = -1$. In section 3 of [46] it has been proved that there are 20 lines among these 48 whose Gram matrix corresponds to the one associated with the Picard lattice of X. Hence all the 20 dimensions of $\mathrm{Pic}(X)$ are spanned by lines in this case (the hyperplane section is a linear combination of four of them).

Example 2.1.5. Here is another example of a K3 surface of genus 3 with $\rho(X) > 1$. Consider the vector bundles $E = \mathcal{O}_{\mathbb{P}^3}^{\oplus 4}$ and $F = \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4}$ over $\mathbb{P}^3$ and consider the morphism $\phi : E \longrightarrow F$ given by the matrix

$$\begin{pmatrix} a_{1,1}(y_1,...,y_4) & \cdots & a_{1,4}(y_1,...,y_4) \\ a_{2,1}(y_1,...,y_4) & \cdots & a_{2,4}(y_1,...,y_4) \\ \vdots & \ddots & \vdots \\ a_{4,1}(y_1,...,y_4) & \cdots & a_{4,4}(y_1,...,y_4) \end{pmatrix}$$

where the entries are generic linear forms. Then we have the degeneracy locus $\mathrm{D}_3(\phi) = \{ y \in \mathbb{P}^3 \mid \mathrm{Rk}(\phi(y)) \leq 3 \} = \{ y \in \mathbb{P}^3 \mid \det(\phi(y)) = 0 \}$ (see section A.3), which is a quartic

[7] As remark 2.1.1 points out, $\rho(X) = 1$ for the general K3 surface of every genus.

surface in $\mathbb{P}^3$. Consider on the other hand the surface X obtained by cutting $\mathbb{P}^3 \times \mathbb{P}^3$ with the system of polynomials

$$\begin{cases} \sum_{i=1}^4 a_{1,i}(y_1, ..., y_4)x_i = 0 \\ \sum_{i=1}^4 a_{2,i}(y_1, ..., y_4)x_i = 0 \\ \sum_{i=1}^4 a_{3,i}(y_1, ..., y_4)x_i = 0 \\ \sum_{i=1}^4 a_{4,i}(y_1, ..., y_4)x_i = 0 \end{cases}$$

which can be described as $X = (\mathbb{P}^3 \times \mathbb{P}^3, \mathcal{O}(1,1)^{\oplus 4})$. We can project X to one of the two factors $\mathbb{P}^3$ and we can look at the fibres. In fact, for a generic point $p = (p_1 : ... : p_4)$ the fibre is the set of points $x \in \mathbb{P}^3$ satisfying the four equations

$$\sum_{i=1}^4 a_{j,i}(p)x_i = 0 \qquad\qquad j = 1, ..., 4,$$

so it is empty in general. The locus where the fibre jumps (to a one-point set) is exactly where the matrix of ϕ is not of maximum rank, hence it is precisely $\mathrm{D}_3(\phi)$. Recall that $\mathrm{D}_2(\phi)$ has codimension 4 in $\mathbb{P}^3$ so the projection yields an isomorphism between the two K3 surfaces $\mathrm{D}_3(\phi)$ and X. Using the Koszul complex

$$0 \longrightarrow \mathcal{O}(-4,-4) \longrightarrow \mathcal{O}(-3,-3)^{\oplus 4} \longrightarrow \mathcal{O}(-2,-2)^{\oplus 6} \longrightarrow$$
$$\longrightarrow \mathcal{O}(-1,-1)^{\oplus 4} \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

it is possible to replicate the previous numerical computation to prove that this model depends on 18 parameters. Hence the Picard number of the quartic $\mathrm{D}_3(\phi)$ is equal to 2.

2.2 Genus 4

Consider $X = (\mathbb{P}^4, \mathcal{O}(3) \oplus \mathcal{O}(2))$, which is a K3 surface with a natural polarisation given by $\mathcal{O}_X(1)$. Its degree is $(\mathcal{O}_X(1))^2 = (H_{|X})^2 = (H)^2_{|X} = 6$, since in a generic plane two curves of degree 3 and 2 intersect in 6 points. Again, to demonstrate generality, we have to prove that the number of parameters in this family equals 19 (note that the dimension of the moduli space of polarised K3 surfaces doesn't depend on the genus, see section 1.4). We need the following short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-2) \oplus \mathcal{O}_{\mathbb{P}^4}(-3) \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0.$$

Twisted by $\mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(3)$, it becomes

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-3) \oplus \mathcal{O}_{\mathbb{P}^4}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-1) \oplus \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^4}(1) \longrightarrow$$
$$\longrightarrow \mathcal{O}_{\mathbb{P}^4}(2) \oplus \mathcal{O}_{\mathbb{P}^4}(3) \longrightarrow i_*\left(\mathcal{O}_X(2) \oplus \mathcal{O}_X(3)\right) \longrightarrow 0.$$

We denote the four pieces by A, B, C, D and we compute the Euler characteristic $\chi(D) = \chi(C) - \chi(B) + \chi(A) = (15+35) - (0+2+5) + (0+0) = 43$. By Kodaira vanishing, $h^0(X, \mathcal{N}_{X/\mathbb{P}^4}) = \chi(D) = 43$.
Again, in the same way as in section 1.4 and with $N = 4$, we have $h^0(X, \mathcal{T}_{\mathbb{P}^4|X}) = 5 \cdot 5 - 1 = 24$, while from the vanishing of $H^1(X, \mathcal{O}_X(1))$ and $H^2(X, \mathcal{O}_X(1))$ we get $h^1(X, \mathcal{T}_{\mathbb{P}^4|X}) = h^2(X, \mathcal{O}_X) = 1$.
Hence, $\dim\left(\mathrm{Im}(H^0(X, \mathcal{N}_{X/\mathbb{P}^4}) \to H^1(\mathcal{T}_X))\right) = \dim\left(H^0(X, \mathcal{N}_{X/\mathbb{P}^4}) \big/ H^0(X, \mathcal{T}_{\mathbb{P}^3|X})\right)$ and it is equal to $43 - 24 = 19$.

Remark 2.2.1. Now we illustrate how to associate to a K3 surface of genus 4 a symplectic manifold. Let $\ell \subseteq \mathbb{P}^4$ be a line which intersects X in two points (counted with multiplicity), and let x be a point of X. We can consider

$$V_\ell := \{\, f \in \mathbb{C}[x_0, ..., x_4]_1 \mid f_{|\ell} = 0 \,\}$$
$$F_{\ell,x} := \{\, f \in \mathbb{C}[x_0, ..., x_4]_1 \mid f_{|\ell} = 0 \text{ and } f(x) = 0 \,\}$$
$$F'_\ell := \bigcup_{x \in X} (F_{\ell,x} \times \{\, x \,\}) \subseteq V_\ell \times X$$

Since a 1-form vanishes in a line if and only if it vanishes in two of its points, V_ℓ has dimension 3. In particular, if $x \in X \setminus \ell$ then $\dim(F_{\ell,x}) = 2$. Hence F'_ℓ is a 2-vector bundle on $X \setminus \ell$. Due to a classical fact about complex surfaces, it can be extended to a vector bundle F_ℓ on X (see [22]). Moreover, F_ℓ is a stable vector bundle on X with $c_1 = c_1(\mathcal{O}(-1))$ and $c_2 = 4$, and we can consider the moduli space $\mathcal{X} := M(2, c_1(\mathcal{O}(-1)), 1)^s$

of such vector bundles on X[8]. For every $F \in \mathcal{X}$, there exists a unique line ℓ in $\mathbb{P}^4$ with $\ell \cap X = 2$ such that $F_\ell \cong F$. From this description in terms of lines, we can give to this moduli space the structure of a symplectic manifold. Since we are considering lines that intersect X in two points or that are tangent to X in one point, our moduli space is in correspondence with the points (of an open subset) of

$$\mathrm{Hilb}^2 X = \{\, \{x, y\} \mid x, y \in X, \text{ distinct or infinitely near} \,\}$$ [9]

This is the Hilbert scheme of two points of X, which is a holomorphic symplectic fourfold.

2.3 Genus 5

Consider $X = (\mathbb{P}^5, \mathcal{O}(2)^{\oplus 3})$, which is a K3 surface with a natural polarisation given by $\mathcal{O}_X(1)$. Its degree is $(\mathcal{O}_X(1))^2 = (H_{|X})^2 = (H)^2_{|X} = 8$, since in a 3-space three generic surfaces of degree 2 intersect in 8 points. In the same way as section 2.1 and 2.2, we can verify that the number of parameters of this family is 19, hence this is the very general K3 surface of degree 8 or genus 5.
Note that in this case, it could a priori exist a line bundle L over X such that $L \otimes L \cong \mathcal{O}_X(1)$, with $(L)^2 = 2$. By Lefschetz theorem 1.2.7 we have an injection $\mathrm{Pic}(\mathbb{P}^5) \hookrightarrow \mathrm{Pic}(X)$ which, since $\rho(X) = 1$, is an isomorphism of groups by Noether-Lefschetz theorem 2.1.2. Hence $\mathcal{O}_X(1)$ must be a generator and, in particular, a primitive element.

Remark 2.3.1. If we fix the three quadrics to be q_1, q_2, q_3, we can consider the incidence hypersurface $V(y_1q_1 + y_2q_2 + y_3q_3) \subseteq \mathbb{P}^2 \times \mathbb{P}^5$. Let

$$N := \mathbb{P}\left(\mathbb{C} < q_1, q_2, q_3 >\right) \cong \mathbb{P}^2$$

be the set of quadrics in $\mathbb{P}^5$ containing X. Since for every $y \in \mathbb{P}^2$ the linear combination $y_1q_1 + y_2q_2 + y_3q_3$ represents a bilinear form on V_6, we can consider $N_0 \subseteq N$ the set of the degenerate ones, i.e. the set of singular quadrics containing X. This is given by $V\left(\det(y_1q_1 + y_2q_2 + y_3q_3)\right) \subseteq \mathbb{P}^2$ which is, for a generic choice of q_1, q_2, q_3, a smooth

[8] See section 3.2 for an introduction to this object.
[9] Here by *infinitely near point* of x we mean that y is a tangent vector in the tangent space of X at x.

sextic. Thus, as we will see in the next section, we can consider the K3 surface of genus 2 which is a double cover of $\mathbb{P}^2$ ramified on this sextic. Hence to a K3 surface of genus 5 we can associate in this way a K3 surface of genus 2. Moreover, in the next section, we will provide an alternative way to view this double cover as the moduli space of a certain kind of vector bundles of rank 2 (remark 2.4.1).

2.4 Genus 2

The setting we need in order to find the very general K3 surface of genus 2 is slightly different from the previous ones. Consider the weighted projective space $\mathbb{P}(1,1,1,3)$ (Appendix A.5) cut by a polynomial f of degree 6 and denote this surface by X. Then the canonical bundle is trivial:

$$\omega_X \cong \left(\omega_{\mathbb{P}(1,1,1,3)}\right)_{|X} \otimes \det\left(\mathcal{N}_{X/\mathbb{P}}\right) \cong (\mathcal{O}_{\mathbb{P}}(-1-1-1-3))_{|X} \otimes \mathcal{O}_X(6) \cong \mathcal{O}_X.$$

However in this case the ambient space is singular at the point $(0:0:0:1)$. This is not a problem for the smoothness of X, because since f is of the form $y^2 + yf_3(x_0,x_1,x_2) + f_6(x_0,x_1,x_2)$ (where $\deg(x_i) = 1$ and $\deg(y) = 3$) and $f(0,0,0,1) = 1$, the singular point is out of X.
This surface is in fact a double cover of $\mathbb{P}^2$ ramified on a smooth sextic. After a coordinate change we can assume $f = y^2 - f_6(x_0,x_1,x_2)$, so we can consider $C = V(f_6) \subseteq \mathbb{P}^2$ and the projection $\pi : X \longrightarrow \mathbb{P}^2$ which forgets y. This is clearly a double cover ramified on C. Proposition A.5.4 is an algebraic version of the Lefschetz hyperplane theorem which works also in the weighted setting. In particular, it implies that our surface is connected and simply connected. Thus, X is a K3 surface.

Take a line $\ell \subseteq \mathbb{P}^2$, the pull-back in X is a hyperplane section which corresponds to a curve $\pi^{-1}(\ell)$ whose canonical bundle is

$$K_{\pi^{-1}(\ell)} \cong \mathcal{O}(-1-1-3) \otimes \mathcal{O}(6) \cong \mathcal{O}(1).$$

In particular, its genus is $h^0(\pi^{-1}(\ell), \mathcal{O}(1)) = 2$. In other words, any hyperplane section, i.e. a sextic in $\mathbb{P}(1,1,3)$, has genus equal to 2. In particular, X is a K3 surface of genus (and degree) 2. Now we want to compute the number of embedded deformations of this family using the Jacobian ring (see Appendix A.6). Since this number (as the Hodge numbers) does not depend on the element of the family we can choose as representative $f = y^2 - (x_0^6 + x_1^6 + x_2^6)$ and so we have as Griffiths ring

$$R_f = \mathbb{C}[x_0, x_1, x_2][y] \Big/ (2y, 6x_0^5, 6x_1^5, 6x_2^5) = \mathbb{C}[x_0, x_1, x_2] \Big/ (x_0^5, x_1^5, x_2^5).$$

The embedded deformations correspond to $\operatorname{Ker}(H^1(X, \mathcal{T}_X) \longrightarrow H^2(X, \mathcal{O}_X)) \cong [R_f]_6$. Hence $\dim\left(\operatorname{Im}(H^0(X, \mathcal{N}_{X/\mathbb{P}}) \to H^1(\mathcal{T}_X))\right) = \dim([R_f]_6) = 10 + 9 = 19$, and so this is the very general K3 surface of genus 2. This also shows that the embedded deformations are the same of C since the two Griffiths rings coincide[10]. Note that a family of plane curves of degree d depends on $\binom{d+2}{2} - 9$ parameters, while the dimension of the moduli space is $3\binom{d-1}{2} - 3$. A quick confrontation shows that these two numbers are equal only for $d = 4$ (or $g = 3$), thus only in this case the general curve is planar.

Remark 2.4.1. In [35], Mukai showed a different approach to writing down this double cover. Recall from remark 2.3.1 the net N of quadrics containing the K3 surface of genus 5 and assume that every quadric in N has at least rank 5. We have the set $N_0 \subseteq N$ of quadrics with a determinant equal to zero which corresponds to a sextic in $\mathbb{P}^2 \cong N$. Under Plücker embedding we can look at the grassmannian $\operatorname{Gr}(2,4)$ as a quadric in $\mathbb{P}^5$, that is, as the vanishing locus of $z_{12}z_{34} - z_{13}z_{24} + z_{14}z_{24}$ (cfr. section A.7). Since $\operatorname{Gr}(2,4)$ parameterises lines in $\mathbb{P}^3$, for a point $p \in \mathbb{P}^3$ we can consider $L_p \subseteq \operatorname{Gr}(2,4)$ the set of lines passing through p and for a plane $P \subseteq \mathbb{P}^3$ we can consider $L_P \subseteq \operatorname{Gr}(2,4)$ the set of lines contained in P. These are in fact planes in $\operatorname{Gr}(2,4)$ and therefore we have two distinct families of planes in $\operatorname{Gr}(2,4)$, both indexed by a $\Delta \cong \mathbb{P}^3$. Since every smooth quadric in $\mathbb{P}^5$ is isomorphic to $\operatorname{Gr}(2,4)$, this works for every $Q \in N \setminus N_0$. Now fix a quadric Q and one of the two families of planes $\{ P_t \subseteq Q \mid P_t \text{ is a plane in } \mathbb{P}^5 \}_{t \in \Delta}$, then for every $s \in Q$ let $\ell_s := \{ t \in \Delta \mid s \in P_t \}$. Let V be the 4-dimensional space of linear form on $\Delta \cong \mathbb{P}^3$, $F(s) := \{ \alpha \in V \mid \alpha_{|\ell_s} = 0 \}$ and $E(s)$ the set of linear forms on ℓ_s. We have the following

[10] This can also be deduced from the fact that choosing two (or more generally a finite number of) points does not increase the number of parameters.

exact sequence

$$0 \longrightarrow F(s) \longrightarrow V \longrightarrow E(s) \longrightarrow 0,$$

where the two maps are the inclusion and the restriction on ℓ_s. Now we define a subbundle and a quotient of the trivial bundle on X:

$$F_\Delta := \bigcup_{s \in X} F(s) \times \{ s \} \subseteq V \times X;$$
$$E_\Delta := \bigcup_{s \in X} E(s) \times \{ s \} \longleftarrow V \times X.$$

Here the point is that $\{ E_\Delta \}$, where Δ runs over all families of planes in quadrics in N, is a complete set of representatives of isomorphism classes of stable vector bundles of rank 2 on X such that $c_1 = c_1(\mathcal{O}(1))$ and $c_2 = 4$. Hence we have a map from the moduli space $M(2, c_1(\mathcal{O}(1)), 2)^s$ to N sending E_Δ to the quadric Q (see section 3.2 for the notations). Over every quadric, we have two families of planes which coincide if and only if the quadric is singular. Therefore this is the wanted double cover of $\mathbb{P}^2$ ramified over a sextic.

Chapter 3

The vector bundle method

Since the general K3 surface does not lie as a complete intersection in a projective space for a genus higher than 5, it seems reasonable to look at grassmannians (see Appendix A.7). This turns out to be the general procedure, as theorem 3.3.1 will point out. In order to state and prove the theorem, we recall some standard tools on sheaves and vector bundles on K3 surfaces such as the *Mukai vector* associated with a sheaf. Moreover, we will define the notion of *stability* for a sheaf introduced by Gieseker in [16]. In the last section, we give an alternative proof of the existence stated in the theorem, which relies on Brill-Noether theory.

3.1 The evaluation map

The following construction generalises the theory of linear systems to vector bundles of rank greater than one. Consider a variety X and define the *evaluation map* of a vector bundle E as the following morphism of sheaves

$$\begin{aligned} ev_E : H^0(X, E) \otimes \mathcal{O}_X &\longrightarrow E \\ \sum_i s_i \otimes f_i &\longmapsto \sum_i f_i s_i. \end{aligned}$$

If L is a line bundle we define the *base locus* as $\mathrm{Bs}|L| := \bigcap_{s \in H^0(X,L)} V(s)$. In the case of line bundles, we have the following

Lemma 3.1.1. *The evaluation map is surjective if and only if L is base point free, i.e.* $\mathrm{Bs}|L| = \emptyset$.

Proof. If L is base point free, then for every $x \in X$ there is at least one global section s which does not vanish at the point. Since

$$L(x) \cong L_x \bigotimes_{\mathcal{O}_{X,x}} \kappa(x) \cong L_x \big/ \mathfrak{m}_x L_x,$$

by Nakayama's lemma s forms a basis of L_x over $\mathcal{O}_{X,x}$. Hence the map is surjective on every stalk. Conversely, if x is a base point, then the evaluation at the point x is the zero map. □

We are therefore inclined to consider linear systems of higher rank, and we define a vector bundle E of rank r to be *base point free* if the evaluation map is surjective. If we tensor the surjection on the stalks over x by $\mathcal{O}_{X,x} \big/ \mathfrak{m}_x$, we obtain the fibre $E(x)$ at the point x as a r-dimensional quotient space of $H^0(E)$. Thus, its dual is a r-dimensional subspace of $H^0(E)^\vee$. Therefore, we can define the map

$$\begin{aligned} \phi_E : X &\longrightarrow \mathrm{Gr}(r, H^0(E)^\vee) \\ x &\longmapsto [E(x)^\vee]. \end{aligned}$$

The local description of ϕ_E is given by $x \longmapsto [\mathrm{span}\,\{\, t_1(x), ..., t_r(x) \,\}]$, where the t_i are dual to the $[s_i]$ which span the quotient $E(x)$. Since E is algebraic, ϕ_E is regular. Note that it satisfies $E \cong \phi_E^*(\mathcal{U}^\vee)$. By Plücker embedding, we can look at $\mathrm{Gr}(r, H^0(E)^\vee)$ inside $\mathbb{P}(\bigwedge^r H^0(E)^\vee)$ and we can consider the composite with ϕ_E.
In addition, we have the exterior power of the evaluation map[11]

$$\bigwedge^r ev_E : \bigwedge^r H^0(X, E) \otimes \mathcal{O}_X \longrightarrow \bigwedge^r E.$$

Taking global sections yields a linear map $\lambda_r : \bigwedge^r H^0(X, E) \longrightarrow H^0(\bigwedge^r E)$, and we

[11]Recall that any exterior power of a given surjection is again surjective.

obtain the following commutative triangle:

$$\begin{array}{ccc} \bigwedge^r H^0(E) \otimes \mathcal{O}_X & & \\ \downarrow{\scriptstyle \lambda_r \otimes \mathrm{id}} & \searrow^{\bigwedge^r ev_E} & \\ H^0\left(\bigwedge^r E\right) \otimes \mathcal{O}_X & \xrightarrow[ev_{\bigwedge^r E}]{} & \bigwedge^r E \end{array}$$

Proposition 3.1.2. *Let E be of rank r and base point free. If λ_r is surjective then the following diagram is commutative*

$$\begin{array}{ccc} X & \xrightarrow{\phi_E} & \mathrm{Gr}(r, H^0(E)^\vee) \\ \downarrow{\scriptstyle \phi_L} & & \downarrow \\ \mathbb{P}(H^0(L)^\vee) & \xhookrightarrow[\alpha]{} & \mathbb{P}\left(\bigwedge^r H^0(E)^\vee\right) \end{array}$$

where $L := \det(E)$ and α is the class of the linear map dual to λ_r.

Proof. We prove the statement in the case $r = 2$, but the ideas for the general case are the same. Fix a point $x \in X$. The map ϕ_E sends it to $[E(x)^\vee]$, which in the Plücker embedding corresponds to $[t_1(x) \wedge t_2(x)] \in \mathbb{P}\left(\bigwedge^2 H^0(E)^\vee\right)$ in the above notations. On the other side $\phi_L(x)$ is the class of the linear map $t_x : H^0(L) \longrightarrow \mathbb{C}$ which sends a section to its value in x. The composite with α corresponds to $\alpha(\phi_L(x)) = \alpha([t_x]) = [t_x \circ \lambda_2]$ (note that if E is base point free then $\det(E) = L$ is). So we need to confront the classes of the two maps on a basis of $H^0(E)^\vee$. We choose $\{ s_1, s_2, .., s_{s+2} \}$, where the first two elements are chosen as above. Consider the bilinear map

$$\begin{aligned} &\bigwedge^2 H^0(E)^\vee \times \bigwedge^2 H^0(E) \longrightarrow \mathbb{C} \\ &(t_1(x) \wedge t_2(x), s_{i_1} \wedge s_{i_2}) \longmapsto t_1(x)(s_{i_1}) t_2(x)(s_{i_2}) - t_2(x)(s_{i_1}) t_1(x)(s_{i_2}). \end{aligned}$$

By a standard linear algebra argument, it is non-degenerate and it induces the isomorphism $\bigwedge^2 H^0(E)^\vee \cong \left(\bigwedge^2 H^0(E)\right)^\vee$. Since $(s_1 \wedge s_2)(x)$ forms a basis for $L(x)$, $t_x(\lambda_2(s_1 \wedge s_2)) = s_1(x) \wedge s_2(x)$ and it vanishes on the other elements of the basis. Thus, under this isomorphism, $t_1(x) \wedge t_2(x)$ is sent to (a multiple of) $t_x \circ \lambda_2$. Hence the two classes must coincide in $\mathbb{P}\left(\bigwedge^r H^0(E)^\vee\right)$. □

3.2 Vector bundles on K3 surfaces

Let X be a K3 surface. In this setting, we can use specific tools introduced by Mukai to study the realm of vector bundles (or more generally coherent sheaves) on X.

Definition 3.2.1. For a vector bundle E the *Mukai vector* is defined as

$$v(E) := \mathrm{ch}(E)\sqrt{\mathrm{td}(X)},$$

using the total Chern character of E and the total Todd class of X (see Appendix A.9).

Remark 3.2.2. Since for a K3 surface $\sqrt{\mathrm{td}(X)} = 1 + \frac{c_2(X)}{24}$ and $\mathrm{ch}(E) = \mathrm{rk}(E) + c_1(E) + \frac{c_1(E)^2 - 2c_2(E)}{2}$, we have

$$v(E) = (\mathrm{rk}(E), c_1(E), \chi(E) - \mathrm{rk}(E)) \in H^*(X, \mathbb{Z}).$$

Definition 3.2.3. On $H^*(X,\mathbb{Z}) \cong H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})$ we define the Mukai pairing

$$\langle \alpha, \beta \rangle := (\alpha_2 \cdot \beta_2) - (\alpha_0 \cdot \beta_4) - (\alpha_4 \cdot \beta_0),$$

where $(\cdot)$ denotes the intersection form (see section A.2).

Remark 3.2.4. These definitions are motivated by the Hirzebruch-Riemann-Roch formula

$$\chi(E) = \int_X \mathrm{ch}(E)\mathrm{td}(X)$$

and by the equality $\chi(E,F) := \sum (-1)^i \mathrm{dim}(\mathrm{Ext}^i(E,F)) = \chi(E^\vee \otimes F)$. In fact, by using formulae in remark A.9.5 we get

$$\chi(E,F) = -\langle v(E), v(F) \rangle.$$

We use the notation $\langle v \rangle^2 := \langle v, v \rangle$. In particular, $\langle (r,h,s) \rangle^2 = (h)^2 - 2rs$. Note that if E is simple[12], then by Serre duality on the level of Ext we have $\chi(E,E) = 2 - \mathrm{Ext}^1(E,E) \leq 2$, which means $\langle v(E) \rangle^2 \geq -2$.

[12] A vector bundle E is *simple* if $h^0(E \otimes E^\vee) = 1$.

Given a sheaf E on a projective variety X with a fixed ample line bundle L we can consider its *Hilbert polynomial* defined as $P(E,m) := \chi(E \otimes L^{\otimes m})$. If (X, L) is a polarised K3 surface we can use Riemann-Roch formula to rewrite it as

$$P(E,m) = \frac{rc_1(L)^2}{2}m^2 + c_1(L)c_1(E)m + \text{const.}$$

We can divide by the leading term to get the reduced Hilbert polynomial, denoted by $p(E, M)$.

Definition 3.2.5. A coherent sheaf E on X is said to be *pure* if for every non trivial subsheaf F the dimension of $\text{supp}(F)$ equals the dimension of $\text{supp}(E)$.
A coherent sheaf E on (X, L) is said to be (*semi-*)*stable* is it is pure and

$$p(F,m) \underset{(\leq)}{<} p(E,m)$$

(with respect to the lexicographic order) for every non-trivial proper subsheaf F.

Remark 3.2.6. It is true that every stable sheaf is simple, see for example section 10.3.1 in [24].

Remark 3.2.7. By a classical result (see for example theorem 10.1.8 of [24]), the moduli space of semistable (with respect to our polarisation L of X) sheaves with fixed Mukai vector v exists as a projective variety $M(v)$. Moreover, we have the (possibly empty) open subset given by stable sheaves, denoted by $M(v)^s$. If non-empty, it is smooth of dimension $\langle v\rangle^2 + 2$ (Corollary 10.2.1 *op. cit.*). We have already seen two examples of it, namely remark 2.2.1 and 2.4.1, of dimension 4 and 2 respectively. They are both taken from [35], where Mukai more generally proved the existence of a symplectic structure on $M(v)$ (see Corollary 3.25 *op. cit.* for a more precise statement).

3.3 Main theorem

The next result is the so-called *vector bundle method* and it was stated for the first time by Mukai in [36]. The theorem can be used to find a candidate map for the embedding of our K3 surface into a grassmannian, so it is the starting point of the research of each general model of polarised K3 surface. Here we give a proof relying on classical results of Mukai on the structure of the moduli spaces introduced in the previous section. Unfortunately, the proof is not constructive, hence does not tell how but only where to find the model of our K3 surface.

Theorem 3.3.1. *Let (X, L) be a polarised K3 surface of genus $g \geq 3$ with L ample line bundle and $\mathrm{Pic}(X) \cong \mathbb{Z}\langle L \rangle$. For every pair (r, s) of positive integers with $rs = g$ there exists a (unique up to isomorphism) stable vector bundle E on X of rank r such that $\det(E) \cong L$ and $\chi(X, E) = r + s$.*

Proof. Let $M(r, c_1(L), s)^s$ be the moduli space of stable (with respect to L) sheaves on X with Mukai vector $v = (r, c_1(L), s)$. Since in this case $\langle v \rangle^2 = -2$, by Corollary 3.6 of [32] $M(v)^s$ is either empty or a reduced point. In fact, if non-empty, it coincides with $M(v)$. By the result in [28], there exists a simple sheaf E with $v(E) = (r, c_1(L), s)$ (see also the next section for an alternative way to construct E). Note that E is *rigid*, i.e. $\mathrm{Ext}^1(E, E) = 0$. Since $\mathrm{Pic}(X) \cong \mathbb{Z}\langle L \rangle$, by proposition 3.14 of [32] E is stable and hence locally free. Finally, since E is a vector bundle and $c_1(E) = c_1(L)$ we have that $\det(E) \cong L$ and it has the right invariants. □

Remark 3.3.2. For a simple sheaf, rigidness is equivalent to saying that $\mathfrak{sl}(E) := \mathcal{E}nd_0(E) \subseteq E \otimes E^\vee$ has Euler characteristic equal to 0. In fact, since on a K3 surface $E \otimes E^\vee \cong \mathfrak{sl}(E) \otimes \mathcal{O}_X$, $\chi(\mathfrak{sl}(E)) = \chi(E \otimes E^\vee) - 2 = -\dim(\mathrm{Ext}^1(E, E))$. Moreover, if E is rigid, $H^1(\mathcal{E}nd(E)) = \mathrm{Ext}^1(E, E) = 0$ and thus E does not admit any first-order deformation.

In order to describe the consequences of theorem 3.3.1, we have to compute further properties of the vector bundle E. Eventually, we will be able to study the map $\phi_E : X \longrightarrow \mathrm{Gr}(r, H^0(E)^\vee)$ associated with E and determine in which cases it is an embedding.

Lemma 3.3.3. *In addition to the hypothesis of theorem 3.3.1, assume the vector bundle E to be globally generated*[13]*. Then it satisfies $H^1(E) = H^2(E) = 0$. In particular, $h^0(E) = \chi(E) = r + s$.*

[13] If fact, this assumption is always satisfied, as we will prove using a different approach in section 3.4.

Proof. We have the short exact sequence

$$0 \longrightarrow M \longrightarrow H^0(X,E) \otimes \mathcal{O}_X \longrightarrow E \longrightarrow 0$$

which induces in cohomology the exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(M) \longrightarrow H^0(\mathcal{O}_X)^{\oplus h^0(E)} \longrightarrow H^0(E) \longrightarrow \\ \longrightarrow H^1(M) \longrightarrow H^1(\mathcal{O}_X)^{\oplus h^0(E)} \longrightarrow H^1(E) \longrightarrow \\ \longrightarrow H^2(M) \longrightarrow H^2(\mathcal{O}_X)^{\oplus h^0(E)} \longrightarrow H^2(E) \longrightarrow 0. \end{aligned}$$

Since $H^0(\mathcal{O}_X)^{\oplus h^0(E)} \longrightarrow H^0(E)$ is a isomorphism, $H^0(M) = H^1(M) = 0$ and so $\chi(M) = h^2(M)$.

If we twist the short exact sequence by $M^\vee$ and we look in cohomology it follows from $H^2(M^\vee) \cong H^0(M)^\vee = 0$ that $H^0(M \otimes E^\vee) = H^2(M^\vee \otimes E) = 0$. In order to prove that $H^2(E) = 0$, we use this information in the cohomology sequence associated to

$$0 \longrightarrow M \otimes E^\vee \longrightarrow H^0(X,E) \otimes E^\vee \longrightarrow E \otimes E^\vee \longrightarrow 0.$$

In fact, since E is simple, $H^0(E) \otimes H^0(E^\vee)$ is at most one dimensional. If $H^2(E) \neq 0$ this implies $r + s \leq h^0(E) + h^2(E) \leq 2$, which is an absurd.

Since the vanishing of $H^2(E)$, the only non-trivial part of the starting long exact sequence is

$$0 \longrightarrow H^1(E) \longrightarrow H^2(M) \longrightarrow H^2(\mathcal{O}_X)^{\oplus h^0(E)} \longrightarrow 0.$$

Now we prove that $\chi(M) = r + s$. This is obtained using Riemann-Roch formula and Whitney's formula $c(\mathcal{O}_X^{\oplus h^0(E)}) = c(E)c(M)$ (A.9) along with the facts that $c_1(M) = -c_1(E)$, $c_2(E) = rs - 1 + r - s$ and $c_2(M) = c_1(E)^2 - c_2(E)$.

The outcome of this computation is that $H^1(E) = H^2(E) = 0$, hence $h^0(E) = \chi(E) = r + s$. □

Remark 3.3.4. By Riemann-Roch formula the dimension of $H^0(L)$ is $rs + 1$ which, by an induction count, is always less or equal than $\binom{r+s}{r} = \dim\left(\bigwedge^r H^0(E)\right)$. Since our K3 surface is general, this leads us to think that the assumption of proposition 3.1.2 is always

satisfied, i.e. λ_r is surjective. In fact, this can be checked in each of the cases treated in the next chapter, see for example the discussion in section 4.2.
Moreover, if we consider λ_{r-1} instead of λ_r and we denote the kernel with M_{r-1}, then the image of the dual map α is a linear subspace of $\mathbb{P}(Z) \subseteq \mathbb{P}\left(\bigwedge^r H^0(E)^\vee\right)$, where Z is the subspace of $\left(\bigwedge^r H^0(E)\right)^\vee$ given by the linear form vanishing on $M_{r-1} \wedge H^0(E)$. This will be useful for example in section 4.3 and 4.5.

As an outcome, we can consider the map $\phi_E : X \longrightarrow \mathrm{Gr}(r, r+s)$. Note that in general ϕ_E may not be an embedding. In these terms, theorem 3.3.1 is the main tool we use in order to find general models of K3 surfaces, since it tells us where to look:

- $g = 6 = 2 \cdot 3$, hence $\mathrm{Gr}(2,5)$;
- $g = 8 = 2 \cdot 4$, hence $\mathrm{Gr}(2,6)$;
- $g = 6 = 3 \cdot 3$, hence $\mathrm{Gr}(3,6)$;
- $g = 10 = 2 \cdot 5$, hence $\mathrm{Gr}(2,7)$;
- $g = 12 = 3 \cdot 4$, hence $\mathrm{Gr}(3,7)$.

Remark 3.3.5. From the proof of theorem 3.3.1, it is clear that we are looking for Mukai vectors of the form (r, h, s) such that $(h)^2 - 2rs = -2$. However, in the case of genus 7 we have to look for $h = mc_1(L)$ for an integer $m > 1$. This leads to $12m^2 = 2rs - 2$. Already for $m = 2$, we find a solution with $r = s = 5$. This suggests that X could have a non-primitive embedding in $\mathrm{Gr}(5, 10)$, as will be the case (see section 4.3).

In the next chapter, we will produce a model for each of these genera which lies in the respective grassmannian. The existence of these models implies that the map $\phi_E : X \longrightarrow \mathrm{Gr}(r, r+s)$ is an embedding, hence we have a rational map

$$\begin{aligned} \mathbb{P}\left(H^0(\mathrm{Gr}(r, r+s), \nu)\right) &\dashrightarrow \mathcal{F}_g \\ [s] &\longmapsto V(s), \end{aligned}$$

which is a parametrisation, i.e. the image is dense, since our models are general by theorem 3.3.1. Hence the moduli space is unirational for these genera (cfr. remark 1.4.2).

3.4 An existence result

Using classical results of Brill-Noether theory on curves (listed for example in [19]), we can produce a vector bundle on our K3 surface as required in theorem 3.3.1. In fact, in [30] Lazarsfeld proved that these results hold also for our case, namely for a general curve generating the Picard group of a K3 surface. We borrow from there the following

Definition 3.4.1. Let (X, L) be a polarised K3 surface of genus g and $C \subseteq X$ a smooth general curve in $|L|$. Let A be a line bundle on C (and, at the same time, a torsion sheaf on X) such that A and $A^\vee \otimes \omega_C$ are both globally generated. We define F to be the kernel of the surjective map $H^0(A) \otimes \mathcal{O}_X \longrightarrow A$ and we call it *elementary transformation.*

Therefore, we have the following short exact sequence of sheaves on X:

$$0 \longrightarrow F \longrightarrow H^0(A) \otimes \mathcal{O}_X \longrightarrow A \longrightarrow 0.$$

Since the injection of locally free sheaves $F \longrightarrow H^0(A) \otimes \mathcal{O}_X$ is generically an isomorphism, it dualises to an injection $H^0(A) \otimes \mathcal{O}_X \longrightarrow F^\vee$. It can be proved that the cokernel is isomorphic to $A^\vee \otimes \omega_C$, i.e. we have the following exact sequence

$$0 \longrightarrow H^0(A) \otimes \mathcal{O}_X \longrightarrow F^\vee \longrightarrow A^\vee \otimes \omega_C \longrightarrow 0.$$

Lemma 3.4.2. *The elementary transformation is locally free of rank $h^0(A)$ and satisfies:*

(i) $\det(F) \cong \mathcal{O}(-C) \cong L^\vee$;

(ii) $c_2(F) = \deg(A)$.

Proof. See lemma 9.2.1 of [24]. □

The following discussion will provide a different way to construct the vector bundle E of theorem 3.3.1, which is sometimes called *Mukai-Lazarsfeld vector bundle*. Even if the construction is again non-explicit, several advantages arise from this construction. For instance, in the last remark, we are able to prove that E is globally generated (cfr. remark 3.3.3).

Recall that the Brill-Noether number is defined as $\rho(g,d,r) := g-(r+1)(g-d+r)$. If we set $g = rs$, then we can compute $\rho(rs, r-1, rs+r-s-1) = 0$. In particular, by classical Brill-Noether theory, we have a finite set of line bundles over C with $h^0 = r$ and $\deg = rs+r-s-1$. Furthermore, following the proof of corollary 1.4 of [30], we have (at least) a line bundle A on C with these invariants, which is moreover globally generated with Serre dual $A^\vee \otimes \omega_C$ also globally generated. In fact, all line bundles on C with invariants such that the Brill-Noether number vanishes are of this kind. By the previous construction, we get the following short exact sequence involving the elementary transformation

$$0 \longrightarrow F \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow A \longrightarrow 0.$$

Since all elements of $|L|$ are reduced irreducible, lemma 1.3 in [30] implies that F is simple and by Corollary 10.3.3 of [24] F is stable. Consider $E := F^\vee$, which is a stable vector bundle of rank r. By lemma 3.4.2, it has determinant isomorphic to L and $c_2(E) = \deg(A) = rs+r-s-1$ which allow us to compute $\chi(E) = r+s$ by Riemann-Roch. Hence this is a way to produce a vector bundle as required in theorem 3.3.1.

Remark 3.4.3. Note that this construction allows us to prove the results in lemma 3.3.3 more directly. In fact, using the above exacts sequences involving F and E it is easy to see that $H^0(F) = H^2(E) = 0$ and $H^1(F) = H^1(E) = 0$. Moreover, we have the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(A)^\vee \otimes \mathcal{O}_X & \longrightarrow & H^0(E)\otimes \mathcal{O}_X & \longrightarrow & H^0(A^\vee \otimes \omega_C)\otimes \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(A)^\vee \otimes \mathcal{O}_X & \longrightarrow & E & \longrightarrow & A^\vee \otimes \omega_C & \longrightarrow & 0
\end{array}$$

which, by the Five lemma, implies that E is globally generated.

The previous approach to the existence of the vector bundle in theorem 3.3.1 occurred to my mind after the study of classical Brill-Noether theory following [19]. During the writing process, I found the work of Bini-Boissière-Flamini, in which they come up with a similar construction, see e.g. theorem 3.8 in [4].

Chapter 4

Genus 6 to 10 and genus 12

For the second part of the classification, we study the models arising as a complete intersection of higher-rank vector bundles over grassmannians. In this chapter, we list the models up to genus 12, except 11. This corresponds to the classification of prime Fano threefolds of index 1. More precisely, every model of K3 surface presented in this chapter and in chapter 2 is a hyperplane section of a Fano threefold with Picard rank equal to 1 and an indivisible anticanonical bundle. From the work of Iskovskikh and later of Mukai, we know that there are no other families of Fano threefolds of this kind. However, for K3 surfaces the list is not complete and, as we will see in the next chapter, it is possible to find general models for higher genera. It is still very difficult to find new general models, as it involves computation with vector bundles which are not a direct sum of line bundles. In particular, only two distinct models arise as a complete intersection of hypersurfaces in grassmannians:

Lemma 4.0.1. *Let* $X = (\mathrm{Gr}(k, k+l), \bigoplus_{i=1}^{c} \mathcal{O}(d_i))$ *be a K3 surface, where* $d_1 \geq d_2 \geq \dots \geq d_n \geq 2$*. Then either:*

1. $X = (\mathrm{Gr}(2,5), \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 3})$*, with genus* $g = 6$*;*

2. $X = (\mathrm{Gr}(2,6), \mathcal{O}(1)^{\oplus 6})$*, with genus* $g = 8$*.*

Proof. Since we are looking for surfaces we need $c = kl - 2$, and by adjunction formula

$$k + l = \sum_{i=1}^{kl-2} d_i.$$

This time we can only require $d_i \geq 1$, hence

$$k + l \geq kl - 2 \text{ and } l \leq \frac{k+2}{k-1}.$$

In order to avoid dualities we assume $l \geq k$, therefore $2 \leq k \leq \frac{k+2}{k-1}$, which leads to $k = 2$ and $l \leq 4$. Since $\mathrm{Gr}(2,4)$ is a quadric in $\mathbb{P}^5$, we can exclude $l = 2$ as the cases arising in this way have already been considered. For $l = 3$ we have $5 = d_1 + d_2 + d_3 + d_4$ which leads to $d_1 = 2$ and $d_2 = d_3 = d_4 = 1$. For $l = 4$ we get $6 = d_1 + d_2 + d_3 + d_4 + d_5 + d_6$ which leads to $d_1 = d_2 = d_3 = d_4 = d_5 = d_6 = 1$. □

In the following table, we sum up the models that will appear in this chapter:

g	Grassmannian	Vector bundle
6	$\mathrm{Gr}(2,5)$	$\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 3}$
7	$\mathrm{OGr}^+(5,10)$	$\mathcal{O}(1/2)^{\oplus 8}$
8	$\mathrm{Gr}(2,6)$	$\mathcal{O}(1)^{\oplus 6}$
9	$\mathrm{Gr}(3,6)$	$\bigwedge^2 \mathcal{U}^\vee \oplus \mathcal{O}(1)^{\oplus 4}$
10	$\mathrm{Gr}(2,7)$	$\mathcal{Q}^\vee(1) \oplus \mathcal{O}(1)^{\oplus 3}$
12	$\mathrm{Gr}(3,7)$	$\left(\bigwedge^2 \mathcal{U}^\vee\right)^{\oplus 3} \oplus \mathcal{O}(1)$

4.1 Genus 6

By lemma 4.0.1 we can take $X = (\mathrm{Gr}(2,5), \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 3})$. Since $\deg(\mathrm{Gr}(2,5)) = 5$ and $\mathcal{O}_X(1) \cong \mathcal{O}_{\mathrm{Gr}}(1)_{|X}$, the degree of X is given by

$$(\mathcal{O}_X(1))^2 = (\mathcal{O}_{\mathrm{Gr}}(1))^2_{|X} = 5 \cdot 2 = 10.$$

Hence X is of genus 6. This is the 2-dimensional case of a special class of varieties called *Gushel-Mukai*, which are of the form $X_n = ((\mathrm{Gr}(2,5), \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 5-n})$, except for the case of dimension 6. In fact, the general form is $X_n := cG \cap \mathbb{P}(W) \cap \mathbb{Q}$, where cG denotes the (projective) cone over $\mathrm{Gr}(2,5)$, which is a seven-dimensional variety in $\mathbb{P}^{10}$, W is a linear subspace of $\bigwedge^2 V_5 \oplus \mathbb{C}$ and $\mathbb{Q}$ is a quadric hypersurface in $\mathbb{P}^{10}$.

Furthermore, we can look at our K3 surface X as the zero locus of a general section of $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 4}$ in cG. The interesting fact here is that if we require one of the four hyperplanes to pass through the vertex of the cone then what we get is again a K3 surface of genus 6. This K3 surface has been described in [21] as a double cover of a Del Pezzo surface of degree 5 branched along a curve of genus 6. Moreover, this family lies in codimension 4 inside the moduli space of K3 surfaces of genus 6, i.e. the Picard rank is greater or equal to 5.
Consider $\mathrm{Gr}(2,5)$ embedded in $\mathbb{P}^9$ via Plücker embedding map, and consider a smooth complete intersection F of $\mathrm{Gr}(2,5)$ with three hyperplanes in $\mathbb{P}^9$. This is a Fano 3-fold of index 2 and degree 5: since $F = (\mathrm{Gr}(2,5), \mathcal{O}(1)^{\oplus 3})$, by adjunction $\omega_F = \mathcal{O}_F(-2)$. This also shows that our K3 surface of genus 6 is an anticanonical divisor of F.

We present how to compute the birational type of the moduli space $\mathcal{F}_6$ (see section 1.4). Consider the natural map

$$\rho_2 : \mathrm{Sym}^2(H^0(F, \mathcal{O}_F(1))) \longrightarrow H^0(F, \mathcal{O}_F(2)),$$

which it is surjective since $\mathrm{Gr}(2,5)$ is 2-normal[14] in $\mathbb{P}^9$. In fact, $\mathcal{O}_{\mathrm{Gr}(2,5)}$ admits the following resolution in terms of $\mathcal{O}_{\mathbb{P}^9}$-modules

$$0 \longrightarrow \mathcal{O}(-5) \longrightarrow \mathcal{O}(-3)^{\oplus 5} \longrightarrow \mathcal{O}(-2)^{\oplus 5} \longrightarrow \mathcal{O}_{\mathbb{P}^9} \longrightarrow i_*\mathcal{O}_{\mathrm{Gr}(2,5)} \longrightarrow 0.$$

Using Riemann-Roch formula for Fano threefolds we get

$$h^0(\mathcal{O}_F(2)) = \chi(\omega_F^\vee) = \frac{c_1(\omega_F^\vee)^3}{2} + 3 = 23.$$

Moreover, as $\mathrm{SL}(V_2)$-module, $\mathrm{Sym}^2(H^0(F, \mathcal{O}_F(1)))$ is isomorphic to $\mathrm{Sym}^2(\mathrm{Sym}^6(V_2^\vee))$, hence the kernel of the map is a 5-dimensional $\mathrm{SL}(V_2)$-submodule. By using Littlewood-Richardson formula for Schur functors, we obtain the following decomposition

$$\mathrm{Sym}^6(V_2^\vee) \otimes \mathrm{Sym}^6(V_2^\vee) \cong \Sigma_{(12,\,0)}V_2^\vee \oplus \Sigma_{(10,\,2)}V_2^\vee \oplus \Sigma_{(8,\,4)}V_2^\vee \oplus \Sigma_{(6,\,6)}V_2^\vee.$$

[14]This means that the map $H^0(\mathcal{O}_{\mathbb{P}^9}(2)) \longrightarrow H^0(\mathcal{O}_{\mathrm{Gr}(2,5)}(2))$ is surjective.

Since $\mathrm{Sym}^2 V \oplus \bigwedge^2 V \cong V \otimes V$ and $\Sigma_{(a,\,b)} V_2^\vee \cong \Sigma_{(a-1,\,b-1)} V_2^\vee \otimes \bigwedge^2 V_2^\vee \cong \Sigma_{(a-1,\,b-1)} V_2^\vee$, we get

$$\mathrm{Sym}^2(\mathrm{Sym}^6(V_2^\vee)) \cong \mathrm{Sym}^{12}(V_2^\vee) \oplus \mathrm{Sym}^8(V_2^\vee) \oplus \mathrm{Sym}^4(V_2^\vee) \oplus \mathbb{C}.$$

In particular, for dimensional reasons, the kernel of ρ_2 (hence the vector space of quadrics containing F) is isomorphic to $\mathrm{Sym}^4(V_2^\vee)$, while $H^0(F, \omega_F^\vee)$ is isomorphic to $\mathrm{Sym}^{12}(V_2^\vee) \oplus \mathrm{Sym}^8(V_2^\vee) \oplus \mathbb{C}$. As we said above, from $|\omega_F^\vee| = \mathbb{P}\left(\mathrm{Sym}^{12}(V_2^\vee) \oplus \mathrm{Sym}^8(V_2^\vee) \oplus \mathbb{C}\right)$ starts a dominant rational morphism

$$\begin{aligned} |\omega_F^\vee| &\dashrightarrow \mathcal{F}_6 \\ s &\longmapsto V(s). \end{aligned}$$

In [34], F is presented also as the closure of the orbit of the class of $xy(x^4 - y^4)$ in $\mathbb{P}(\mathrm{Sym}^6(V_2^\vee))$ under the action of $\mathrm{PGL}(V_2)$ induced by the one of $\mathrm{SL}(2)$ on $\mathrm{Sym}^6(V_2^\vee)$. We borrow from there the following

Theorem 4.1.1. *Let X and X' be two general smooth complete intersections of F and a quadric hypersurface in $\mathbb{P}^6$. If they are projectively equivalent, then they are equivalent under the action of* $\mathrm{PGL}(V_2)$ *on F.*

Therefore, the above morphism passes through the action of $\mathrm{PGL}(2)$ and yields

$$\left(\mathrm{Sym}^{12}(V_2^\vee) \oplus \mathrm{Sym}^8(V_2^\vee)\right) /\!\!/ \mathrm{SL}(V_2) \dashrightarrow \mathcal{F}_6,$$

where the quotient is in the sense of GIT. Since both spaces have dimension 19, by the previous theorem we get the following

Corollary 4.1.2. *The moduli space $\mathcal{F}_6$ is birationally equivalent to the orbit space*

$$\mathrm{Sym}^{12}(V_2^\vee) \oplus \mathrm{Sym}^8(V_2^\vee) /\!\!/ \mathrm{SL}(V_2).$$

4.2 Genus 8

By lemma 4.0.1, we can take $X = (\mathrm{Gr}(2,6), \mathcal{O}(1)^{\oplus 6})$. Since $\deg(\mathrm{Gr}(2,6)) = 14$, the degree of X is given by

$$(\mathcal{O}_X(1))^2 = (\mathcal{O}_{\mathrm{Gr}}(1))^2_{|X} = 14.$$

Hence X is of genus 8. In this case theorem 3.3.1 can be applied concretely: the vector bundle E can be chosen to be $\mathcal{U}^{\vee}_{|X}$. We have the Koszul complex[15]

$$0 \longrightarrow \mathcal{O}(-6) \longrightarrow \mathcal{O}(-5)^{\oplus 6} \longrightarrow \mathcal{O}(-4)^{\oplus 15} \longrightarrow \mathcal{O}(-3)^{\oplus 20} \longrightarrow$$
$$\longrightarrow \mathcal{O}(-2)^{\oplus 15} \longrightarrow \mathcal{O}(-1)^{\oplus 6} \longrightarrow \mathcal{O}_G \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

We can twist it by $\mathcal{U}^{\vee}$ and by $\bigwedge^2 \mathcal{U}^{\vee} \cong \mathcal{O}(1)$ and we can compute the cohomology using the tools developed in section A.8. We summarise the results in the following

Lemma 4.2.1. *With the above notations:*

i) $H^0(X, E) \cong H^0(\mathrm{Gr}(2,6), \mathcal{U}^{\vee})$ *and it has dimension* 6*;*

ii) $0 \longrightarrow H^0(\mathrm{Gr}(2,6), \mathcal{O}^{\oplus 6}) \longrightarrow H^0(\mathrm{Gr}(2,6), \mathcal{O}(1)) \longrightarrow H^0(X, \mathcal{O}(1)) \longrightarrow 0$ *is exact;*

iii) $\bigwedge^2 H^0(X, E) \cong \bigwedge^2 H^0(\mathrm{Gr}(2,6), \mathcal{U}^{\vee}) \cong H^0(\mathrm{Gr}(2,6), \mathcal{O}(1))$.

The map λ_2 of proposition 3.1.2 corresponds to the second map in point ii) of the lemma under the isomorphism of point iii). In particular, it is surjective and it has a six-dimensional kernel.

We want to describe a projective duality construction which gives a link between this K3 surface and a cubic fourfold. Denote by $f_1, ..., f_6$ the equations of X, they span a subspace $W_6 \subseteq \bigwedge^2 V_6^{\vee}$. We define $Y := (\mathrm{Gr}(2,6) \times \mathbb{P}(W_6), \mathcal{O}(1,1))$ the incidence hypersurface cut by $\sigma := \sum_{i=1}^{6} x_i f_i$, where the x_i are coordinates on W_6. Thus, Y admits two natural projections

$$X \subseteq \mathrm{Gr}(2,6) \xleftarrow{\;p\;} Y \xrightarrow{\;q\;} \mathbb{P}^5 \supseteq \Delta$$

[15] Denote the inclusion $i : X \hookrightarrow G$. Even if all the sheaves are over G, we will omit the push-forward of $\mathcal{O}_X$ by i in the last term of the complex.

The projection on the first factor restricted to Y has a generic fibre on $\mathrm{Gr}(2,6) \setminus X$ isomorphic to $\mathbb{P}^4$, while it jumps to a $\mathbb{P}^5$ over X. Thus, Y has the structure of a stratified projective bundle over $\mathrm{Gr}(2,6)$. Now we focus on q. If we fix a point $x \in \mathbb{P}^5$, the generic fibre is a (smooth) linear section of $\mathrm{Gr}(2,6)$, which in this case corresponds to the *symplectic grassmannian* $\mathrm{SGr}(2,6) := (\mathrm{Gr}(2,6), \bigwedge^2 \mathcal{U}^\vee)$, see section 4.4. We can also consider the skew-symmetric form $\sigma(x) = \sum_{i=1}^{6} x_i f_i$[16] and in particular the locus where it degenerates, i.e. $\Delta := \{\, x \in \mathbb{P}^5 \mid \det(\sigma(x)) = 0 \,\}$. Here it is exactly where the fibre becomes singular. Since we are dealing with a 6×6 skew-symmetric matrix, Δ corresponds to the zero locus of the Pfaffian $\mathrm{Pf}(\sigma(x))$, which is a degree three polynomial[17]. Moreover, the matrix would further drop rank in codimension 6[18], hence Δ is a smooth cubic fourfold. This construction comes from the fact that $\mathrm{Gr}(2,6)$ and a Pfaffian cubic $Z \subseteq \mathbb{P}\left(\bigwedge^2 V_6^\vee\right)$ correspond to the minimal and maximal orbits respectively for the action of $\mathrm{GL}(6)$ on $\bigwedge^2 V_6^\vee$. If we intersect these two varieties with $\mathbb{P}(W_6)$ we get the K3 surface and the cubic fourfold.

This construction provides also a link between the two non-trivial Hodge structures of the two varieties, which are respectively

$$1 \qquad 20 \qquad 1$$

and

$$0 \qquad 1 \qquad 20+1 \qquad 1 \qquad 0$$

Moreover, this kind of construction has been studied and generalised in the so-called *homological projective duality* by Kuznetsov which works at the level of derived categories, see [29].

[16] Note that f_i is a skew-symmetric 2-form for every i.

[17] In fact, the determinant of a skew-symmetric matrix is a polynomial in the entries of the matrix, and the Pfaffian is the square root of it.

[18] By the principal minor theorem is enough to impose the vanishing of the 6 principal minors.

4.3 Genus 7

We pointed out in remark 3.3.5 that the general K3 surface of genus 7 could have a non-primitive embedding in $\mathrm{Gr}(5,10)$. Consider the rigid vector bundle E of rank 5 of theorem 3.3.1 and the linear map

$$\mathrm{Sym}^2 H^0(X,E) \longrightarrow H^0\left(X, \mathrm{Sym}^2 E\right)$$

arising from the evaluation map. By a Riemann-Roch computation, these two vector spaces have dimensions 55 and 54 respectively, and the map has maximal rank due to the right-exactness of symmetric powers. This implies that we have a non-zero element in the kernel and we denote it by σ. The subset of $\mathrm{Gr}(5,V)$ consisting of 5-dimensional isotropic subspaces of $V := H^0(X,E)^\vee$ with respect to σ is called *orthogonal grassmannian*, denoted by $\mathrm{OGr}(5,10) = \mathrm{OGr}(5,V)$. In particular, we have

$$\mathrm{OGr}(5,10) = (\mathrm{Gr}(5,10), \mathrm{Sym}^2(\mathcal{U}^\vee)).$$

This variety has two connected components, denoted as $\mathrm{OGr}^{\pm}(5,10)$[19]. We focus on $F := \mathrm{OGr}^{+}(5,10)$[20]. Under the composition $F \hookrightarrow \mathrm{Gr}(5,10) \hookrightarrow \mathbb{P}\left(\bigwedge^5 V\right)$, the pullback of $\mathcal{O}_{\mathbb{P}}(1)$ is $L^{\otimes 2}$ with L a primitive generator for the Picard group of F (see below). For this reason, we denote L with $\mathcal{O}(1/2)$. The vector space $H^0(F, \mathcal{O}(1/2))$ has dimension 16 and it is a half-spin representation of $\mathrm{Spin}(V)$ (which is the universal covering group of $\mathrm{SO}(V)$). In particular, we have the spinor embedding

$$|\mathcal{O}(1/2)| : F \longrightarrow \mathbb{P}\left(H^0(F, \mathcal{O}(1/2))^\vee\right) \cong \mathbb{P}^{15},$$

which is of degree 12. In fact, from the point of view of homogeneous spaces, $F \cong \mathrm{Spin}(10)/P$, where $P = P_5$ is a maximal parabolic subgroup corresponding to the root α_5. The structure of homogeneous variety (here and in the next cases) allows us to compute a lot of algebraic invariants of the variety using the representation theory of the

[19] See e.g.[38].
[20] One can equivalently choose to work in $\mathrm{OGr}^{-}(5,10)$, although the two (families of) K3 surfaces will be derived equivalent and not isomorphic.

Lie groups involved, we refer to [44] for an introduction to the subject. The Dynkin type of $\mathfrak{so}(10)$ (i.e. the Lie algebra of Spin(10)) is D_5, and we choose the following basis for the root system:

$$\Delta = \{\alpha_1 := e_1 - e_2, \alpha_2 := e_2 - e_3, \alpha_3 := e_3 - e_4, \alpha_4 := e_4 - e_5, \alpha_5 := e_4 + e_5, \}.$$

This gives us the following set of positive roots:

$$\Phi^+ = \left\{ \begin{matrix} e_1 \pm e_5 & e_2 \pm e_5 & e_3 \pm e_5 & e_4 \pm e_5 \\ e_1 \pm e_4 & e_2 \pm e_4 & e_3 \pm e_4 & \\ e_1 \pm e_3 & e_2 \pm e_3 & & \\ e_1 \pm e_2 & & & \end{matrix} \right\}$$

and the following set of P-complementary roots:

$$R_P = \left\{ \begin{matrix} e_1 + e_5 & e_2 + e_5 & e_3 + e_5 & e_4 + e_5 \\ e_1 + e_4 & e_2 + e_4 & e_3 + e_4 & \\ e_1 + e_3 & e_2 + e_3 & & \\ e_1 + e_2 & & & \end{matrix} \right\}.$$

Their sum is equal to $4(e_1 + e_2 + e_3 + e_4 + e_5) = 8\left(\frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5)\right)$ which is 8 times the fundamental weight ω associated to $\mathcal{O}(1/2)$. Next, we state the Borel-Hirzebruch theorem ([5]) which is one of the main results in the theory of homogeneous spaces.

Theorem 4.3.1 (Borel-Hirzebruch). *Let G be a simply connected semi-simple complex Lie group and P a maximal parabolic subgroup associated with a root α. Fix a basis Δ for the root system and L the positive generator of* Pic(F)*, where $F := {}^{G}/_{P}$. Then we have:*

- $n := \dim(F) = |R_P|$*, where R_P is the set of complementary roots;*
- $(L)^n = n! \prod_{\beta \in R_P} \frac{(\beta,\omega)}{(\beta,\rho)}$*, where ω is the weight corresponding to L and ρ is half of the sum of the positive roots;*
- $\sum_{\beta \in R_P} \beta = r\omega$*, where r is a positive integer such that $c_1(F) = rc_1(L)$.*

We are now able to compute some algebraic invariants of F:

$$\dim(F) = 10;$$

$$(\mathcal{O}(1/2))^{10} = 10! \prod_{\beta \in R_P} \frac{(\beta, \omega)}{(\beta, \rho)} = 12;$$

$$\omega_F^\vee \cong \mathcal{O}_F(8(1/2)).$$

Here we used that $\rho = (4, 3, 2, 1, 0)$. Therefore, if we pick 8 linear sections we get a K3 surface $X = (F, \mathcal{O}(1/2)^{\oplus 8})$ of genus 7. On the other hand, by adjunction $\omega_F \cong (\mathcal{O}_G(-10) \otimes \mathcal{O}_{\mathrm{Gr}}(6))_{|F} \cong \mathcal{O}_{\mathrm{Gr}}(-4)_{|F}$, hence $\mathcal{O}(1)_{|F} \cong \mathcal{O}_F(1/2)^{\otimes 2}$. Summing up, we have the following diagram (with the left half cartesian)

$$\begin{array}{ccccc} X & \xrightarrow{\phi_E} & \mathrm{OGr}^+(5,10) & \hookrightarrow & \mathrm{Gr}(5,10) \\ \downarrow{\scriptstyle \phi_{\det(E)}} & & \downarrow & & \downarrow \\ \mathbb{P}^7 & \hookrightarrow & \mathbb{P}^{15} & \hookrightarrow & \mathbb{P}^{251} \end{array}$$

It is worth mentioning that there are other descriptions of the general K3 surface of genus 7, which do not come from the vector bundle method (something like this happens also for genus 13, where the only known model does not follow theorem 3.3.1, see section 5.3). This description can be found e.g. in [7] or [6]. Consider the variety $M = (\mathrm{Gr}(2,5), \mathcal{O}(1)^{\oplus 2})$, which is a Del Pezzo fourfold[21] of degree 5 and $\rho = 1$. In order to find a K3 surface in M we need a rank 2 vector bundle with determinant $\mathcal{O}(2)$: it is easy to verify that $\mathcal{U}^\vee(1)$ has the right invariants. Hence we have the K3 surface $X = (\mathrm{Gr}(2,5), \mathcal{U}^\vee(1) \oplus \mathcal{O}(1)^{\oplus 2})$. In order to prove that its degree is 12 it is useful the following computation. Consider the Koszul complex

$$0 \longrightarrow \mathcal{O}(-5) \longrightarrow \mathcal{U}^\vee(-4) \oplus \mathcal{O}(-4)^{\oplus 2} \longrightarrow \mathcal{U}^\vee(-3)^{\oplus 2} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-3) \longrightarrow$$
$$\longrightarrow \mathcal{U}^\vee(-2) \oplus \mathcal{O}(-1)^{\oplus 2} \longrightarrow \mathcal{O}_G \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Since our goal is $c_1(\mathcal{O}_X(1))^2 = 12$, by Riemann-Roch we need to prove that $h^0(\mathcal{O}_X(1)) =$

[21] The index is $3 = \dim(M) - 1$.

$\chi(\mathcal{O}_X(1)) = 8$. Thus we twist the Koszul complex by $\mathcal{O}_G(1)$ to get:

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{U}^\vee(-3) \oplus \mathcal{O}(-3)^{\oplus 2} \longrightarrow \mathcal{U}^\vee(-2)^{\oplus 2} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2) \longrightarrow$$
$$\longrightarrow \mathcal{U}^\vee(-1) \oplus \mathcal{O}^{\oplus 2} \longrightarrow \mathcal{O}(1) \longrightarrow \mathcal{O}_G(1)_{|X} \longrightarrow 0.$$

It is easy to compute the cohomology of these bundles using the tools developed in section A.8. The results are listed here:

- $\mathcal{O}(-4)$, $\mathcal{U}^\vee(-3) \oplus \mathcal{O}(-3)^{\oplus 2}$ and $\mathcal{U}^\vee(-2)^{\oplus 2} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2)$ have no non-zero cohomology;
- $H^0(\mathcal{U}^\vee(-1) \oplus \mathcal{O}^{\oplus 2})$ has dimension 2 and the other cohomology groups vanish;
- $H^0(\mathcal{O}_G(1))$ has dimension 10 and the other cohomology groups vanish.

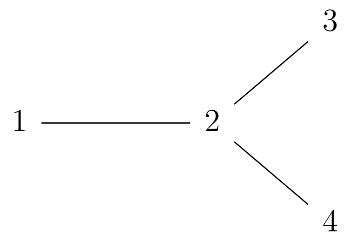
This leads to $\chi(\mathcal{O}_G(1)_{|X}) = 8$ and, in particular, $c_1(\mathcal{O}_G(1)_{|X})^2 = 12$. Since 12 cannot be written as $d^2 \cdot 2 \cdot m$ with $d > 1$, the embedding is primitive and $\mathcal{O}_G(1)_{|X} \cong \mathcal{O}_X(1)$.

Note that for the models of K3 surfaces that do not arise from the vector bundle method we need to check that the Picard rank is equal to 1. As an example, we work out the details for the previous model, namely $X = (\mathrm{Gr}(2,5), \mathcal{U}^\vee(1) \oplus \mathcal{O}(1)^{\oplus 2})$. Following remark 2.1.1, it suffices to compute that the dimension of $\mathrm{Im}(H^0(X, \mathcal{N}_{X/\mathrm{Gr}(2,5)}) \to H^1(X, \mathcal{T}_X))$ is equal 19. In doing so, we first consider the normal sequence

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{G|X} \longrightarrow \mathcal{N}_{X/G} \longrightarrow 0.$$

Then we twist the previous Koszul complex by $\mathcal{U}^\vee(1)$ in order to compute the cohomology of $\mathcal{N}_{X/G} \cong (\mathcal{U}^\vee(1) \oplus \mathcal{O}(1)^{\oplus 2})_{|X}$. It is possible to verify that in degree 0 it has dimension 45 while $H^1(\mathcal{N}_{X/G}) = 0$. We need to compute the cohomology of $\mathcal{T}_{G|X}$, so we twist the Koszul complex by $\mathcal{T}_G \cong \mathcal{U}^\vee \otimes \mathcal{Q}$. After subdividing the twisted Koszul complex into four short exact sequences we compute the cohomology of the involved vector bundles using tools developed in A.8. In this way we find that $h^0(\mathcal{T}_{G|X}) = 26$ and $h^1(\mathcal{T}_{G|X}) = 1$. Since we know that for a K3 surface $h^1(\mathcal{T}_X) = 20$, the image has dimension 19, as we wanted.

Another description of the K3 surface of genus 7 can be given in $\mathrm{OGr}^+(4,8)$. This is a Fano 6-fold of index 6 and its anticanonical bundle is given by $\mathcal{O}(6(1/2))$. Hence if we cut with $\bigwedge^3 \mathcal{U}^\vee \cong \mathcal{U}(1)$ we obtain the K3 surface $X = \left(\mathrm{Gr}(4,8), \bigwedge^3 \mathcal{U}^\vee \oplus \mathrm{Sym}^2 \mathcal{U}^\vee\right)$. As before one can show that $c_1(\mathcal{O}_X(1))^2 = 12$. The interesting fact of this model is the isomorphism between $\mathrm{OGr}^+(4,8)$ and $\mathrm{OGr}(1,8) \cong \mathbb{Q}_6 \subseteq \mathbb{P}^7$, which is a theorem of Study and it can be seen as a consequence of the *geometric triality.* Triality denotes the symmetries enjoyed by the objects that are labelled by the Dynkin diagram D_4:



In particular, the three irreducible representations of the associated Lie group Spin(8) corresponding to the leaves of the diagram are isomorphic to each other and of dimension 8. Furthermore, it is possible to prove that $\mathrm{Spin}(8)/P_1 \cong \mathrm{OGr}^+(4,8)$, $\mathrm{Spin}(8)/P_3 \cong \mathrm{OGr}(1,8)$ and $\mathrm{Spin}(8)/P_4 \cong \mathbb{Q}_6$ and they are all isomorphic, which correspond to the above-mentioned isomorphism (see e.g. [48], page 40).

4.4 Genus 9

Consider the symplectic grassmannian $\mathrm{SGr}(k,n) := \left(\mathrm{Gr}(k,n), \bigwedge^2 \mathcal{U}^\vee\right)$ (we are interested in the case of $k=3$ and $n=6$, as suggested by theorem 3.3.1), this is the locus of k-planes isotropic with respect to a non-degenerate skew-symmetric 2-form β. Consider the rigid vector bundle E over X of theorem 3.3.1. This time we use the linear map

$$\lambda_2 : \bigwedge^2 H^0(X,E) \longrightarrow H^0\left(X, \bigwedge^2 E\right).$$

A Riemann-Roch computation shows that the first space has dimension 15 while the latter 14[22]. Therefore we can take a non-zero element in the kernel as our skew-symmetric form

[22] Here we used the right-exactness of exterior power and the fact that E has cohomology only in degree zero.

β. We can associate to β a linear map

$$\neg\beta : \bigwedge^3 H^0(X,E) \longrightarrow H^0(X,E)$$
$$e_{i,j,k} \longmapsto \beta(e_i,e_j)e_k - \beta(e_i,e_k)e_j + \beta(e_j,e_k)e_i.$$

Clearly every β-isotropic 3-space W is sent to 0 by $\neg\beta$, and the linear span of $\mathrm{SGr}(3,6)$ is $\mathbb{P}(\mathrm{Ker}(\neg\beta)) \cong \mathbb{P}^{13}$ (it corresponds to the space $\mathbb{P}(Z)$ in remark 3.3.4). The situation is the following, where the left half of the diagram is cartesian:

$$\begin{array}{ccccc} X & \xrightarrow{\phi_E} & \mathrm{SGr}(3,6) & \hookrightarrow & \mathrm{Gr}(3,6) \\ {\scriptstyle \phi_{\det(E)}}\downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^9 & \hookrightarrow & \mathbb{P}^{13} & \hookrightarrow & \mathbb{P}^{19} \end{array}$$

Since the canonical bundle of $\mathrm{SGr}(3,6)$ is

$$\omega_{SG} \cong \left(\omega_{\mathrm{Gr}}(-6) \otimes \det\left(\bigwedge^2 \mathcal{U}^\vee\right)\right)_{|SG} \cong \mathcal{O}_{SG}(-4)$$

and $\mathrm{SG}(3,6)$ has dimension $9-3=6$, we obtain a K3 surface simply cutting by four linear sections. Therefore $X := \left(\mathrm{Gr}(3,6), \bigwedge^2 \mathcal{U}^\vee \oplus \mathcal{O}(1)^{\oplus 4}\right)$. Now we want to compute the degree of $(X, \det(E))$. Since E is rigid (cfr. remark 3.3.2), we obtain the two following Riemann-Roch formulae:

$$6 = \chi(E) = \frac{c_1(E)^2}{2} - c_2(E) + 6;$$
$$0 = \chi(\mathfrak{sl}(E)) = \frac{c_1(\mathfrak{sl}(E))^2}{2} - c_2(\mathfrak{sl}(E)) + 16.$$

If we play a bit with Chern classes (using formulae in section A.9), we get the linear system

$$\begin{cases} c_1(E)^2 - 3c_2(E) = -8 \\ c_1(E)^2 - 2c_2(E) = 0 \end{cases}$$

Hence $c_2(E) = 8$, $c_1(E)^2 = 16$ and $g = 9$.

Note that since $16 = 4 \cdot 4$, it is possible to have a line bundle L of degree 4 which divides

$\mathcal{O}_X(1)$. See the proof of proposition 5.3.2 for an argument which applies also to this case to prove that our polarisation is primitive.

4.5 Genus 10

As suggested by theorem 3.3.1 we have to look at a subvariety of $\mathrm{Gr}(2,7)$, which has dimension 10. This led to the study of the Cayley algebra $\mathcal{C}$[23], which is generated as a $\mathbb{C}$-algebra by a unit 1 and elements e_i with $i \in \mathbb{Z}_7$ under the following multiplication rules:

- $e_i^2 = -1$;
- $e_i e_{i+a} = -e_{i+a} e_i = e_{i+3a}$.

If we consider $\mathcal{C}_0 := \mathbb{C}\langle e_1, ..., e_7\rangle$ we have in fact our grassmannian $\mathrm{Gr}(2, \mathcal{C}_0) \cong \mathrm{Gr}(2,7)$. On $\mathrm{Gr}(2,\mathcal{C}_0)$ consider the vector bundle $\mathcal{Q}^\vee(1) \cong \bigwedge^4 \mathcal{Q}$, its space of global section is $\bigwedge^4 \mathcal{C}_0 \cong \bigwedge^3 \mathcal{C}_0^\vee$. The action of $\mathrm{SL}(\mathcal{C}_0)$ on $\bigwedge^3 \mathcal{C}_0^\vee$ has only finitely many orbits. In particular, we can choose an element $\beta \in \bigwedge^3 \mathcal{C}_0^\vee$ from the unique open orbit. For instance, a possible choice could be

$$\beta = x_{125} + x_{136} + x_{147} + x_{234} + x_{567}.$$

The stabiliser of β in $\mathrm{GL}(\mathcal{C}_0)$ is $G \subseteq \mathrm{GL}(\mathcal{C}_0)$, the simple Lie group of type G_2, see e.g. example 4.12 in [9]. In fact, G is also isomorphic to the automorphism group of $\mathcal{C}$. Moreover, we can consider the *congruence*

$$F_\beta := \{\, [W] \in \mathrm{Gr}(2,\mathcal{C}_0) \mid \beta_{|W} = 0 \,\} \subseteq \mathrm{Gr}(2,\mathcal{C}_0),$$

which, with our notations, corresponds to $F = (\mathrm{Gr}(2,\mathcal{C}_0), \mathcal{Q}^\vee(1))$.

Let V be the adjoint representation of G, whose dimension is 14. Let U be a representative of a class in F_β[24] and consider the automorphisms of G which preserves U. They form a maximal parabolic subgroup P (see [34]) and F can be realised as a closed orbit of the action of G on $\mathbb{P}(V) \cong \mathbb{P}^{13}$. Hence $F \cong {}^{G}/_{P}$ and, by Borel-Hirzebruch theorem (4.3.1), F has dimension 5 and $(\mathcal{O}_F(1))^5 = 18$, hence if we take a smooth complete intersection

[23]The Cayley algebra is sometimes referred also as octonions, denoted by $\mathbb{O}$.

[24]For instance $U := \mathrm{span}\,\{\, e_2 + 2ie_3 + e_4, -ie_5 + e_6 - ie_7 \,\}$.

of three linear sections of $\mathcal{O}_F(1)$ we get a surface of degree 18. In particular, a model for our K3 surface is given by $(\mathrm{Gr}(2,7), \mathcal{Q}^\vee(1) \oplus \mathcal{O}(1)^{\oplus 3})$.

The Lie group G_2 admits another irreducible representation, denoted by V' and of dimension 7. It is linked to the shorter root of the Lie algebra and one can prove that the corresponding parabolic subgroup P_1 induces a homogeneous variety ${}^{G}/_{P_1}$ isomorphic to a quadric $\mathbb{Q}_5 \subseteq \mathbb{P}(V') \cong \mathbb{P}^6$. Moreover, in [9] it is showed that it parameterises the lines of our Fano 5-fold F. On the other hand, one can consider the variety of lines contained in $\mathbb{Q}_5$, denoted by $F_1(\mathbb{Q}_5)$. Since it is clearly isomorphic to $\mathrm{OGr}(2,7)$, we can consider the (twisted) spinor bundle $\mathcal{S}_{2,7}(1)$ which has rank 2 and its space of global sections has dimension 8. As pointed out in the first remark of [37], the variety $(\mathrm{OGr}(2,7), \mathcal{S}_{2,7}(1))$ is isomorphic to F.

4.6 Genus 12

We can apply theorem 3.3.1 with $r = 3$ and $s = 4$. Actually, also $r = 2$ and $s = 6$ could work, but it is still unknown what is the vector bundle E_2 and what is the image of the closed embedding $\phi_E : X \hookrightarrow \mathrm{Gr}(2,8)$. Let (X, L) be a general polarised K3 surface of genus 12 and let E be the rank 3 vector bundle over X of theorem 3.3.1. As in the case of genus 9 (4.4), we consider

$$\lambda_2 : \bigwedge^2 H^0(E) \longrightarrow H^0\left(\bigwedge^2 E\right).$$

The dimension of the two space is respectively $\binom{7}{2} = 21$ and

$$h^0\left(\bigwedge^2 E\right) = \frac{c_1\left(\bigwedge^2 E\right)^2 - 2c_2\left(\bigwedge^2 E\right)}{2} + 3\cdot 2 = \frac{2c_1(E)^2 - 2c_2(E)}{2} + 6 = 18,$$

where we used that $c_2(E) = 10$. Let N be the kernel, which is of dimension 3, then the image of ϕ_E in $\mathrm{Gr}(3, H^0(E)^\vee)$ is contained in $SG_N(3, H^0(E)^\vee) := \cap_{\beta\in N} SG_\beta(3, H^0(E)^\vee)$[25]. Since $H^0\left(\bigwedge^2 \mathcal{U}^\vee\right) \cong \bigwedge^2 V_7^\vee \cong \bigwedge^2 H^0(E)$, if we fix a basis $\{\beta_1, \beta_2, \beta_3\}$ of N we get

[25] By $SG_\beta(3, H^0(E)^\vee)$ we mean the space of subspaces isotropic with respect to the form β.

three global section s_1, s_2 and s_3 of $\bigwedge^2 \mathcal{U}^\vee$. Then $SG_N(3, H^0(E)^\vee)$ is the zero locus of $(s_1, s_2, s_3) \in \left(\bigwedge^2 \mathcal{U}^\vee\right)^{\oplus 3}$ in $\mathrm{Gr}(3, H^0(E)^\vee)$. Hence, if N is generic, we can write $SG_N(3, H^0(E)^\vee)$ as $\left(\mathrm{Gr}(3, H^0(E)^\vee), \left(\bigwedge^2 \mathcal{U}^\vee\right)^{\oplus 3}\right)$ and it is a smooth threefold embedded by Plücker coordinates in $\mathbb{P}^{13}$. Indeed, the three forms β_1, β_2 and β_3 induce a (surjective) contraction $\psi : \bigwedge^3 V_7 \longrightarrow V_7^{\oplus 3}$ whose kernel M has dimension 14 and it corresponds to the set of elements of $\bigwedge^3 H^0(E)^\vee$ annihilated by the elements of N. The image F in $\mathbb{P}^{13}$ is an anti-canonical model, it has degree 22 and our K3 surface X is a hyperplane section of F. Note that the degree of F can be computed directly from the presentation $\left(\mathrm{Gr}(3, H^0(E)^\vee), \left(\bigwedge^2 \mathcal{U}^\vee\right)^{\oplus 3}\right)$ using Riemann-Roch formula for Fano threefolds:

$$\chi(\omega_F^\vee) = \frac{c_1(\omega_F^\vee)^3}{2} + 3.$$

Since $\omega_F^\vee \cong \mathcal{O}_F(1)$ and by Kodaira vanishing $\chi(\omega_F^\vee) = h^0(\omega_F^\vee)$, we get $\chi(\omega_F^\vee) = \dim(M) = 14$. Thus the equation above reads $28 = c_1(\omega_F^\vee)^3 + 6$ and tells us that $\deg(F) = 22$.

This K3 surface, or more precisely the Fano threefold F, is interesting also from a historical point of view. Firstly, it was omitted in the classification by Fano himself. When Iskovskikh completed the classification of prime Fano threefolds of index 1 in [25] and [26] he used for the construction of F the double projection from lines method introduced by Fano, which requires the existence of a line. This result was not already known and it was proved a few years later by Shokurov. Meanwhile, the work of Mukai on vector bundles which we emphasised in this thesis allowed him and Umemura to present a new construction of this manifold in [41]. Eventually, Mukai came up with the description given above in [39].

Chapter 5

Sporadic genera

In this chapter, we study the remaining known models. More precisely, these models are no more linked with the classification of prime Fano threefolds and were discovered subsequently by Mukai using the tools developed in Chapter 3. In fact, the models with genus 18 and 20, presented in [37], are consequences of the vector bundle method. The others, i.e. $g = 13$, 16, do not follow the theorem and were instead discovered by taking inspiration from the case of genus 12, as pointed out by Mukai himself in [42] and [43]. In the following table, we sum up the models that will appear in this chapter:

g	Grassmannian	Vector bundle
13	$\mathrm{Gr}(3,7)$	$\left(\bigwedge^2 \mathcal{U}^\vee\right)^{\oplus 2} \oplus \bigwedge^3 \mathcal{Q}$
16	$\mathcal{T}$	$\mathcal{E}^{\oplus 2} \oplus \mathcal{F}^{\oplus 2}$
18	$\mathrm{OGr}(3,9)$	$\mathcal{R}^{\oplus 5}$
20	$\mathrm{Gr}(4,9)$	$\left(\bigwedge^2 \mathcal{U}^\vee\right)^{\oplus 3}$

5.1 Genus 18

As in section 4.3, we consider the orthogonal grassmannian. More precisely, let F be the subvariety of $\mathrm{Gr}(3,9)$ consisting of 3-planes isotropic with respect to the quadratic form with matrix

$$\begin{pmatrix} 0 & I_3 & 0 & 0 \\ I_3 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

This is nothing but $\mathrm{OGr}(3,9) = (\mathrm{Gr}(3,9), \mathrm{Sym}^2\mathcal{U}^\vee)$. Note that, in this case, it has only one connected component and the role of our line bundle $\mathcal{O}(1/2)$ is taken by the *spinor bundle* $\mathcal{S}$ of rank 2. Let $\mathcal{R} := \mathcal{S}(1)$, the space of global sections $H^0(\mathcal{R})$ is 16-dimensional and corresponds to the *spin representation* of $\mathrm{SO}(9,\mathbb{C})$. Furthermore we have that $\det(\mathcal{R}) \cong \mathcal{O}_F(1)$.

By an easy computation we get that F is a Fano manifold of dimension 12 and canonical bundle

$$\left(\mathcal{O}_{\mathrm{Gr}}(-9) \otimes \det\left(\mathrm{Sym}^2(\mathcal{U}^\vee)\right)\right)_{|F} \cong \mathcal{O}_F(-5).$$

Consider the zero locus $X = (F, \mathcal{R}^{\oplus 5})$, which is a smooth surface by Bertini's theorem 1.2.3 since $\mathcal{R}$ is generated by its global sections. In order to show that X is a (general) K3 surface of genus 18 we need to compute several cohomology groups, and we need tools similar to the ones presented in section A.8, see remark A.8.1. Firstly we recall the Koszul complex

$$0 \longrightarrow K^{10} \longrightarrow \ldots \longrightarrow K^1 \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

where $K^i := \bigwedge^i(\mathcal{R}^{\oplus 5})^\vee \cong \bigwedge^{10-i}(\mathcal{R}^{\oplus 5}) \otimes \mathcal{O}_F(-5)$ with the isomorphism given by the perfect pairing

$$\bigwedge^{i}(\mathcal{R}^{\oplus 5}) \times \bigwedge^{10-i}(\mathcal{R}^{\oplus 5}) \longrightarrow \det(\mathcal{R}^{\oplus 5}).$$

We have $0 = H^1(\mathcal{O}_F) = H^2(K^1) = ... = H^{11}(K^{10})$ and $H^1(K^1) = H^2(K^2) = ... = H^{10}(K^{10}) \cong H^2(\mathcal{O}_F)^\vee = 0$. Hence we get

$$H^0(\mathcal{O}_F) \longrightarrow H^0(\mathcal{O}_X) \longrightarrow 0,$$

$$0 \longrightarrow H^1(\mathcal{O}_X) \longrightarrow H^2(\mathrm{Ker}(\mathcal{O}_F \longrightarrow \mathcal{O}_X)).$$

Since the latter does vanish, we have proved that X is connected and $H^1(\mathcal{O}_X) = 0$. Finally, by adjunction, $\omega_X \cong (\mathcal{O}_F(-5) \otimes \mathcal{O}_F(5))_{|X} \cong \mathcal{O}_X$ which says that X is a K3 surface.

Let $\mathcal{E} := \mathcal{U}^\vee_{|F}$, $E := \mathcal{E}_{|X}$, $R := \mathcal{R}_{|X}$ and $L := \det(R) \cong \mathcal{O}_X(1)$, thus (X, L) is a polarised K3 surface and we want to compute its degree. Since also $\det(\mathcal{E}) \cong \mathcal{O}_F(1)$, then $\det(E) \cong L$. By studying the (twisted) Koszul complex, it can be proved that $H^0(X, E) \cong H^0(F, \mathcal{E}) \cong H^0(\mathrm{Gr}(3, 9), \mathcal{U}^\vee) \cong \mathbb{C}^9$. Hence by Riemann-Roch (for a rank 3 vector bundle over a K3 surface)

$$9 = \chi(E) = \frac{(c_1(E))^2}{2} - c_2(E) + 3 \cdot 2.$$

Consider the subbundle of trace zero endomorphisms $\mathfrak{sl}(E) \subseteq E \otimes E^\vee$, it can be proved that all its cohomology groups vanish, thus $0 = \chi(\mathfrak{sl}(E)) = \chi(E \otimes E^\vee) - \chi(\mathcal{O}_X)$. Using Riemann-Roch for a rank 9 vector bundle over a K3 surface we can compute

$$\chi(E \otimes E^\vee) = -c_2(E \otimes E^\vee) + 9 \cdot 2 = 2c_1(E)^2 - 6c_2(E) + 18.$$

Therefore we have $c_1(E)^2 = 34$ (and $c_2(E) = 14$), which implies that our polarised K3 surface has genus 18.

Moreover, the cohomological computation needed above is also useful to prove the generality of the model. Since $h^0(E \otimes E^\vee) = 1$, i.e. E is simple, we can apply the following proposition to show that K3 surfaces of genus 18 arising in this way form an open subset of $\mathcal{F}_{18}$, see section 4 of the original [37] for the details.

Proposition 5.1.1. *Let E be a simple vector bundle on a K3 surface X, i.e. $h^0(\mathcal{E}nd(E))= 1$, and (X', L') a small deformation of $(X, \det(E))$. Then there is a deformation (X',E') of (X, E) such that* $\det(E') \cong L'$.

Proof. See proposition 4.1 in [37]. □

5.2 Genus 20

The following construction resembles and generalises the description of the Fano threefold given in 4.6. Consider the rank 4 vector bundle $\mathcal{U}^\vee$ on the grassmannian $\mathrm{Gr}(4, 9)$ and consider $X := \left(\mathrm{Gr}(4,9), \left(\bigwedge^2 \mathcal{U}^\vee\right)^{\oplus 3}\right)$. Since $\det\left(\bigwedge^2 \mathcal{U}^\vee\right)^{\oplus 3} \cong \mathcal{O}_{\mathrm{Gr}}(9)$, X is a smooth surface with trivial canonical bundle. As for genus 18, we can prove that the restriction map $H^0(\mathcal{O}_G) \longrightarrow H^0(\mathcal{O}_X)$ is surjective and $H^1(\mathcal{O}_X) = 0$ and we conclude that X is K3 surface.

Let $E := \mathcal{U}^\vee_{|X}$ and $L := \det(E) \cong \mathcal{O}_X(1)$, thus (X, L) is a polarised K3 surface and we want to compute its degree. After twisting the Koszul complex by $\mathcal{U}^\vee$, it can be proved that $H^0(X, E) \cong H^0(\mathrm{Gr}(4,9), \mathcal{U}^\vee) \cong \mathbb{C}^9$. Hence by Riemann-Roch (for a rank 4 vector bundle over a K3 surface)

$$9 = \chi(E) = \frac{(c_1(E))^2}{2} - c_2(E) + 4 \cdot 2.$$

Again all cohomology groups of $\mathfrak{sl}(E)$ vanish, thus $0 = \chi(\mathfrak{sl}(E)) = \chi(E \otimes E^\vee) - \chi(\mathcal{O}_X)$. Using Riemann-Roch for a rank 16 vector bundle over a K3 surface (and some Chern classes computations), we get

$$\chi(E \otimes E^\vee) = -c_2(E \otimes E^\vee) + 16 \cdot 2 = 3c_1(E)^2 - 8c_2(E) + 30.$$

Therefore, we have $c_1(E)^2 = 38$ (and $c_2(E) = 18$), which implies that our polarised K3 surface has genus 20.

The same ideas used at the end of the previous section apply here to conclude that this is the general model for a K3 surface of genus 20.

5.3 Genus 13

This is the first case where g is not the product of r and s, even with a non-primitive embedding (see section 4.3). In fact, we look at the grassmannian $\mathrm{Gr}(3,7)$ as for genus 12 and, instead of the vector bundle $\left(\bigwedge^2 \mathcal{U}^\vee\right)^{\oplus 3}$, we pick $\mathcal{V} := \left(\bigwedge^2 \mathcal{U}^\vee\right)^{\oplus 2} \oplus \bigwedge^3 \mathcal{Q}$. Hence we have the Koszul complex

$$0 \longrightarrow \bigwedge^{10} \mathcal{V}^\vee \longrightarrow \ldots \longrightarrow \bigwedge^{2} \mathcal{V}^\vee \longrightarrow \mathcal{V}^\vee \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

where $X := (\mathrm{Gr}(3,7), \mathcal{V})$ is our smooth surface ($\mathcal{V}$ is as always generated by global sections). Since $c_1(\mathcal{V}) = 7c_1(\mathcal{O}(1))$, it has a trivial canonical bundle. Using the above complex we can compute that $H^0(X, \mathcal{O}_X) \cong \mathbb{C}$ and $H^1(X, \mathcal{O}_X) = 0$, thus X is a K3 surface. Also, we have that $\chi(\mathcal{O}_X(1)) = 14$ and by Riemann-Roch formula, this implies $(\mathcal{O}_X(1))^2 = 24$. It remains to prove that this polarisation is primitive. Before that, we list some of the results obtained from the Koszul complex in a similar manner to section 5.1.

Proposition 5.3.1. *Let* $E := \mathcal{U}^{\vee}_{|X}$, *then:*

i) the restriction map $H^0(\mathrm{Gr}(3,7), \mathcal{O}_{\mathrm{Gr}}(1)) \longrightarrow H^0(X, \mathcal{O}_X(1))$ *is surjective and we have* $h^0(X, \mathcal{O}_X(1)) = 14$, $H^1(X, \mathcal{O}_X(1)) = H^2(X, \mathcal{O}_X(1)) = 0$;

ii) the restriction map $H^0(\mathrm{Gr}(3,7), \mathcal{U}^{\vee}) \longrightarrow H^0(X, E)$ *is an isomorphism and* $h^0(X, E) = 7$, $H^1(X, E) = H^2(X, E) = 0$;

iii) $H^0\left(\mathrm{Gr}(3,7), \bigwedge^2 \mathcal{U}^{\vee}\right) \longrightarrow H^0\left(X, \bigwedge^2 E\right)$ *is surjective and its kernel has dimension 2;*

iv) E is simple and semi-rigid, i.e. $H^0(\mathfrak{sl}(E)) = 0$ *and* $h^2(\mathfrak{sl}(E)) = 2$.

Note that since E is simple, proposition 5.1.1 implies that our K3 surface has Picard rank equal to one.

Proposition 5.3.2. *The polarisation* $\mathcal{O}_X(1)$ *on* X *is primitive.*

Proof. Assume that $\mathcal{O}_X(1)$ is not primitive. Since $(\mathcal{O}_X(1))^2 = 24$, there exists a line bundle L such that $\mathcal{O}_X(1) \cong L^{\otimes 2}$ and $(L)^2 = 6$. Moreover, L is a generator of $\mathrm{Pic}(X)$. By a Riemann-Roch computation and by Kodaira vanishing we have $h^0(\mathcal{O}(L^{\otimes n})) = 3n^2 + 2$ if $n \geq 1$. Consider the bundle $E \otimes L^{\vee}$, if it has a non-zero global section then there is a non-zero morphism from L to E. In particular, E contains a subsheaf isomorphic to $L^{\otimes n}$ and from the previous computation we get $n = 1$.
Consider now the quotient $Q := {}^{E}/_{L}$. Since $h^0(L) = 5 < 7 = h^0(E)$, $H^0(Q) \neq 0$. Furthermore, Q is torsion-free of rank 2 and determinant equal to L. This implies that $0 \neq H^0(Q) \cong H^0(Q^{\vee} \otimes L) \cong \mathrm{Hom}(Q, L)$, which contradicts point iv) of proposition 5.3.1.
Hence $H^0(E \otimes L^{\vee}) = 0$.
Consider the bundle $M := \left(\bigwedge^2 E\right) \otimes L^{\vee} \cong E^{\vee} \otimes L$, then $H^2(M) \cong H^0(E \otimes L^{\vee}) = 0$. Thus

$h^0(M) \geq \chi(M) = 4$[26].

Now we can pick 4 linearly independent global section of M to define the linear map

$$\varphi : \mathcal{O}_X^{\oplus 4} \longrightarrow M.$$

Since $\mathrm{Hom}(L, M) \cong \mathrm{Ext}^0(L, M) \cong \mathrm{Ext}^0(\mathcal{O}, M \otimes L^\vee) \cong H^0(E^\vee) \cong H^2(E)^\vee = 0$, it can be proved that the cokernel of φ is isomorphic to a skyscraper sheaf supported by a point and that the kernel of φ is isomorphic to $L^\vee$. Tensoring with L yields the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow L^{\oplus 4} \longrightarrow \bigwedge^2 E \longrightarrow \mathbb{C}(p) \longrightarrow 0.$$

Since $h^0(L) = 5$ and $h^0\left(\bigwedge^2 E\right) = 19$, the global section of the middle map is surjective, which is in contradiction with point iii) of proposition 5.3.1. Therefore, $\mathcal{O}_X(1)$ has to be primitive. □

5.4 Genus 16

We start considering the moduli space $\mathcal{T}$ of twisted cubics in $\mathbb{P}^3$. This space was constructed by Ellinsgrud-Piene-Strømme in [12] as the GIT quotient of $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes V_4^\vee$ under the action of $\mathrm{GL}(2) \times \mathrm{GL}(3)$. In fact, an element $t \in \mathcal{T}$ corresponds to a class of a 2×3 matrix whose entries are linear form over V_4. Hence the three 2-minors cut a (generalised) twisted cubic in $\mathbb{P}(V) \cong \mathbb{P}^3$. Moreover, one can consider the embedding

$$\begin{aligned} \mathcal{T} &\longrightarrow \mathrm{Gr}(3, \mathrm{Sym}^2 V_4^\vee) \cong \mathrm{Gr}(3, 10) \\ t &\longmapsto H^0(\mathbb{P}^3, \mathcal{O}(2-t)), \end{aligned}$$

which sends a twisted cubic (or a specialisation) to the net of quadrics in $\mathbb{P}^3$ containing it. In particular, we can consider the rank 3 vector bundle $\mathcal{E} := \mathcal{U}^\vee_{|\mathcal{T}}$ over $\mathcal{T}$. On the other hand, it is possible to embed $\mathcal{T}$ also in $\mathrm{Gr}\left(2, \Sigma_{(2,\,1)} V_4^\vee\right) \cong \mathrm{Gr}(2, 20)$ by sending a 2×3 matrix R to 2-space of linear syzygies among the three 2-minors of R. In fact, by Littlewood-Richardson formula, $\Sigma_{(2,\,1)} V_4^\vee$ is the kernel of the linear map $\mathrm{Sym}^2 V_4^\vee \otimes V_4^\vee \longrightarrow \mathrm{Sym}^3 V_4^\vee$.

[26]Here we use Riemann-Roch on M and the fact that $c_2(E) = 11$.

We denote by $\mathcal{F}$ the pull-back of $\mathcal{U}^\vee$ under this embedding.
From the description in [12], we can compute the canonical bundle of $\mathcal{T}$ as $\mathcal{O}_\mathcal{T}(-4)$. Thus, in order to find a K3 surface we need a vector bundle of rank 10 and determinant $\mathcal{O}(4)$. Since $\det(\mathcal{E}) \cong \det(\mathcal{F}) \cong \det(\mathcal{U}^\vee)_{|\mathcal{T}} \cong \mathcal{O}_\mathcal{T}(1)$, our choice is $\mathcal{E}^{\oplus 2} \oplus \mathcal{F}^{\oplus 2}$.
At this point, we have proved that $X = (\mathcal{T}, \mathcal{E}^{\oplus 2} \oplus \mathcal{F}^{\oplus 2})$ is a disjoint union of K3 surfaces and abelian surfaces. By the computations in [13], we find that the Euler number of X is equal to 24. In particular, this disjoint union contains exactly one K3 surface. Furthermore, it is possible to compute the degree of X to be equal to 30. Hence we have $g = 16$. In [43], Mukai was able to prove the connection of X through several geometric considerations. Moreover, he proved that the family $X = (\mathcal{T}, \mathcal{E}^{\oplus 2} \oplus \mathcal{F}^{\oplus 2})$ parameterises an open subset of $\mathcal{F}_{16}$, which implies that X is a general K3 surface of genus 16.

Given the description and the invariants of $\mathcal{T}$, we believe that it is possible to describe $\mathcal{T}$ as $\left(\mathrm{Gr}(3,10), \left(\bigwedge^2 \mathcal{U}^\vee\right)^{\oplus 3}\right)$. This would simplify a lot most of the computation stated above. However, the vector bundle $\mathcal{F}$ in this setting is more difficult to describe since it is a (possibly) non-homogeneous vector bundle. A first idea could be to consider the kernel of the map

$$\bigwedge^2 \mathcal{U}^\vee \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

If we prove that it is locally free, then it would have rank 2 and first Chern class $c_1 = c_1(\mathcal{O}(1))$, i.e. the right invariants. Given that, we would be able to prove that X is simply connected more easily. Hence, it could be interesting to continue the study of this special variety.

5.5 Final remark

As the classification done by Benedetti in [3] points out, there are no other general models of K3 surfaces in grassmannians (or variations of them such as orthogonal and symplectic) arising as zero loci of globally generated, homogeneous and completely reducible vector bundles. For instance, already the model in genus 16 does not appear in the classification of Benedetti, since the vector bundle is not homogeneous. For this reason, it is very difficult to find new models of K3 surfaces as we have to look for vector bundles which are not of that kind.

Instead, it is possible to shift the focus on models of K3 surfaces with Picard rank greater than one. For example, it would be useful to search in products of grassmannians or in flag varieties, since it allows to have a higher Picard rank due to the injection of integral Hodge structures. A starting point would be to generalise the work of Benedetti to a classification of surfaces with trivial canonical bundle in such ambient spaces. Then it is possible to identify the ones that admit a principal polarisation among them. As before, the existence of such models would guarantee the unirationality of subvarieties of the moduli space in each genus. In particular, the case of Picard rank equal to 2 would correspond to the study of divisors in $\mathcal{F}_g$. Therefore, this is a possible way to advance in the knowledge of the models in this special family of varieties.

Appendix A

Useful tools

A.1 Vector bundles

Definition A.1.1. Let X be a complex manifold, a *holomorphic vector bundle* on X of rank r is a complex manifold E together with a holomorphic map

$$\pi : E \longrightarrow X$$

and with a structure of complex vector space of dimension r on any fibre $\pi^{-1}(x) =: E(x)$ such that:
there exists an open covering $X = \bigcup U_i$ *with biholomorphic maps* $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ *which respect to* π *and such that*

$$\psi_{i|E(x)} : \pi^{-1}(x) \longrightarrow \mathbb{C}^r$$

is a $\mathbb{C}$*-linear isomorphism.*
If the rank is one then E is said *holomorphic line bundle*.

Remark A.1.2. Starting from a holomorphic vector bundle E over X we can consider the system of transition functions (cocycles) $\{ \psi_{i,j} \}$ with respect to the trivialising open cover $\{ U_i \}$ of X given by

$$\begin{aligned} \psi_{i,j} : & U_i \cap U_j \longrightarrow \mathrm{GL}(r, \mathbb{C}) \\ & x \longmapsto (\psi_i \circ \psi_j^{-1})(x, \cdot). \end{aligned}$$

The cocycle data determines E up to isomorphism.

Remark A.1.3. If X is algebraic, we require in addition the U_i to be Zariski-open and the $\psi_{i,j}$ to be (restriction of) rational function on X and, in this case, we call E *algebraic* vector bundle.

Remark A.1.4. Given two vector bundles E and F over X we can consider most of the classical constructions of linear algebra. We list some of them:

- direct sum $E \oplus F$, whose fibre over x is canonically isomorphic to $E(x) \oplus F(x)$;
- tensor product $E \otimes F$, whose fibre over x is canonically isomorphic to $E(x) \otimes F(x)$;
- dual $E^\vee$, whose fibre over x is canonically isomorphic to $E(x)^\vee$;
- symmetric power $\mathrm{Sym}^i E$, whose fibre over x is canonically isomorphic to $\mathrm{Sym}^i E(x)$;
- external power $\bigwedge^i E$, whose fibre over x is canonically isomorphic to $\bigwedge^i E(x)$. In particular, if E has rank r we set $\det(E) := \bigwedge^r E$, which has rank 1.

Remark A.1.5. The tangent space of a complex manifold has a natural structure of holomorphic vector bundle $\mathcal{T}_X$ of rank equal to the dimension of X. Its dual is the cotangent vector bundle Ω_X. Its sections are the 1-forms over X. The i-external power $\bigwedge^i \Omega_X =: \Omega_X^i$ has as sections the i-forms over X and, in the case of $i = \dim(X)$, we get the volume forms over X. The line bundle $\det(\Omega_X)$ is called the *canonical bundle* of X and it is denoted ω_X.

Remark A.1.6. Let Y be a submanifold of X, then we have the inclusion $\mathcal{T}_Y \subseteq \mathcal{T}_{X|Y}$, where the restriction means that we are only considering the fibre over the points of Y. We define the *normal bundle* $\mathcal{N}_{Y/X}$ as the cokernel in

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_{X|Y} \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0.$$

Definition A.1.7. To a holomorphic vector bundle E we can associate its *sheaf of section*, denoted again by E, whose datum for the open set U is

$$E(U) = \{ s : U \longrightarrow E \mid \pi \circ s = \mathrm{id}_U \} .$$

Furthermore, it has a natural structure of $\mathcal{O}_X$-module.

Proposition A.1.8. *The previous definition establishes a bijection between the set of holomorphic vector bundles of rank r over X and the set of locally free $\mathcal{O}_X$-modules of rank r.*

Proof. See proposition 2.2.19 of [23]. □

Definition A.1.9. Consider a smooth algebraic variety X, an algebraic vector bundle E is said to be *globally generated* if there exist $s_1, ..., s_r \in H^0(X, E)$ such that $\{ s_1(x), ..., s_r(x) \}$ spans $E(x)$ for every $x \in X$.

A.2 Intersection theory

Let X be a non-singular projective surface over $\mathbb{C}$, we introduce the intersection pairing on X following theorem V.1.1 of [20].

Theorem A.2.1. *There is a unique bilinear symmetric form* $\mathrm{Div}(X) \times \mathrm{Div}(X) \longrightarrow \mathbb{Z}$*, denoted by $(C \cdot D)$, such that:*

- *if C and D are two non singular curves meeting transversally, then $(C \cdot D)$ is the number of intersection points between C and D;*
- *it depends only on the class modulo linear equivalence, i.e. it is well defined at the level of* $\mathrm{Pic}(X)$*.*

This is the *intersection form* of a surface and can be written down as

$$(L_1 \cdot L_2) := \chi(X, \mathcal{O}_X) - \chi(X, L_1^\vee) - \chi(X, L_2^\vee) + \chi(X, L_1^\vee \otimes L_2^\vee).$$

If $C \subseteq X$ is a curve and L a line bundle, we write $(C \cdot L)$ instead of $(\mathcal{O}(C) \cdot L)$, and we denote the self intersection with $(C)^2$ or $(L)^2$. We list some standard properties, borrowed from section V.1 of [20]:

1. $(C \cdot L) = \deg(L_{|C})$;
2. if C and D are curves meeting only in a finite number of points then
$$(C \cdot D) = \sum_{p \in C \cap D} \dim \left(\mathcal{O}_{X,p} \Big/ (f, g) \right),$$
where f and g are the equations of C and D at p;
3. if C is a non-singular curve, then $(C)^2 = \deg \left(\mathcal{N}_{C/X} \right)$;
4. if L is an ample line bundle on X then for every curve C in X we have $(C \cdot L) > 0$.

We said that a line bundle L is *numerically trivial* if $(L \cdot L') = 0$ for every line bundle L'. We define the *Néron-Severi* group as $\mathrm{NS}(X) := \mathrm{Pic}(X) \Big/ \mathrm{Pic}^0(X)$ where $\mathrm{Pic}^0(X)$ is the Jacobian of X. The rank $\rho(X)$ of $\mathrm{NS}(X)$ is called *Picard number* of X.

Proposition A.2.2. *For a K3 surface the natural surjection*

$$\mathrm{Pic(X)} \longrightarrow \mathrm{NS}(X)$$

is an isomorphism. Moreover, the intersection pairing on Pic(X) *is even, non-degenerate and of signature* $(1, \rho(X) - 1)$.

Proof. Let L be a non-trivial line bundle in $\mathrm{Pic}^0(X)$, then L is numerically trivial, thus for an ample line bundle L' we have $(L \cdot L') = 0$. In particular, L and $L^\vee$ both have no global section[27]. Furthermore $(L)^2 = 0$, hence $\chi(X, L) = 2$ by 1.2.2. All combined, this yields $h^1(X, L) = -2$, which is absurd. Therefore, $L \cong \mathcal{O}_X$. Note that this can also be deduced from the fact that $H^1(X, \mathcal{O}_X) = 0$ using the exponential sequence.
Let $L \in$ Pic(X), again by 1.2.2, we get

$$(L)^2 = 2\chi(X, L) - 4 \equiv_2 0,$$

[27] This follows from the fact that an ample line bundle satisfies $(L' \cdot C) > 0$ for every curve C.

hence the pairing is even. The remaining facts are consequences of the Hodge index theorem (theorem V.1.9 of [20]). Note that the positive part is generated by the class of an ample line bundle. □

Under the embedding $\mathrm{Pic}(X) \hookrightarrow H^2(X, \mathbb{Z})$, the intersection form corresponds to the topological intersection form and we have the following result

Theorem A.2.3. *Let X be a complex K3 surface, then the integral cohomology $H^2(X, \mathbb{Z})$ with the intersection form $(\ \cdot\ )$ is a lattice abstractly isomorphic to*

$$E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U,$$

where U is the hyperbolic plane, i.e. the lattice of rank 2 with a basis of isotropic vectors e, f such that $(e \cdot f) = 1$, and $E_8(-1)$ is the rank 8 lattice with intersection matrix

$$\begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & 1 & & & \\ & & 1 & -2 & 0 & & & \\ & & 1 & 0 & -2 & 1 & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & 1 \\ & & & & & & 1 & -2 \end{pmatrix}$$

Theorem A.2.4 (Global Torelli)**.** *Two complex K3 surfaces X and Y are isomorphic if and only if there exists an isomorphism of integral Hodge structures between $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$ respecting the intersection pairing. Moreover, two polarised K3 surfaces (X, L) and (X', L') are isomorphic if and only if there exists an isomorphism of integral Hodge structures between $H^2(X, \mathbb{Z})$ and $H^2(X', \mathbb{Z})$ mapping $[L]$ to $[L']$.*

Proof. See section 6.3 of [24]. □

A.3 Degeneracy loci

Consider two globally generated vector bundles E and F over a smooth variety X and a morphism $\phi : E \longrightarrow F$. Then for every $x \in X$, $\phi(x)$ is a linear application between $E(x)$ and $F(x)$. We define the k^{th} *degeneracy locus* to be

$$\mathrm{D}_k(\phi) := \{\, x \in X \mid \mathrm{rk}(\phi(x)) \leq k \,\} .$$

Since the condition forces all $k+1$-minors of the matrix to vanish, we call the number

$$\dim(X) - (\mathrm{rk}(E) - k)(\mathrm{rk}(F) - k)$$

the *expected dimension* of $\mathrm{D}_k(\phi)$. For the next result, see e.g. theorem 2.3 of [49].

Lemma A.3.1. *Let X be a smooth variety and let E and F be two globally generated vector bundles over X. Then:*

(i) For a generic global section s of E, the zero locus $V(s) \subseteq X$ (if not empty) is smooth of codimension $\mathrm{rk}(E)$.

(ii) For a generic morphism $\phi : E \longrightarrow F$, the degeneracy locus $\mathrm{D}_k(\phi) \subseteq X$ (if not empty) has codimension $(\mathrm{rk}(E) - k)(\mathrm{rk}(F) - k)$ *and* $\mathrm{D}_k(\phi)_{\text{sing}} = \mathrm{D}_{k-1}(\phi)$.

We focus on the case $\mathrm{rk}(E) = \mathrm{rk}(F) = r$. Since $\phi(x)$ is a square matrix, the $r-1$-degeneracy locus is a hypersurface cut in X by the determinant of ϕ, which is a polynomial of degree r. In this case we call $\mathrm{D}_{r-1}(\phi)$ a *determinantal variety* and it is generically smooth for $\dim(X) \leq 3$. In particular, a generic determinantal cubic surface in $\mathbb{P}^3$ is smooth. Furthermore, we have a classical result of Grassmann which says that every cubic surface in $\mathbb{P}^3$ is determinantal (see the original [17], or Corollary 6.4 of [1]).

A.4 Deformation theory for K3 surfaces

Let $\mathfrak{X} \longrightarrow S$ be a smooth proper family and let X_0 be the fibre over a distinguished point $0 \in S$. Then the family is called *universal deformation* if for any other family $\mathfrak{X}' \longrightarrow S'$

with $X_0 \cong X'_{0'}$ there exists a unique map $S' \longrightarrow S$ sending $0'$ to 0 and such that

$$\begin{array}{ccccc} \mathfrak{X}' & \hookrightarrow\!\!\!\twoheadrightarrow & \mathfrak{X} \times_S S' & \longrightarrow & \mathfrak{X} \\ & \searrow & \downarrow & & \downarrow \\ & & S' & \longrightarrow & S \end{array}$$

Theorem A.4.1. *Let X_0 be a K3 surface, then it admits a universal deformation $\mathfrak{X} \longrightarrow \mathrm{Def}(X_0)$ with $\mathrm{Def}(X_0)$ smooth of dimension 20.*

Proof. The hard fact here is the result stated as theorem 6.2.5 in [24], where there are explained the relationships between the cohomology of $\mathcal{T}_{X_0}$ and $\mathrm{Def}(X_0)$. Since we know (section 1.3) that for a K3 surface

$$H^*(\mathcal{T}_{X_0}) \qquad \equiv \qquad 0 \qquad \mathbb{C}^{20} \qquad 0,$$

it follows that a smooth universal deformation exists and that

$$T_0\mathrm{Def}(X_0) \cong H^1(X_0, \mathcal{T}_{X_0}) \cong \mathbb{C}^{20}.$$

□

The elements of $H^1(X_0, \mathcal{T}_{X_0})$ parameterise the so-called *first order deformations.* A first-order deformation is a family with the dual numbers $\mathrm{Spec}\left(\mathbb{C}[x] \big/ (x)^2\right)$ as a base. Since we are dealing with polarised K3 surfaces, we are interested in deformations which preserve the polarisation, i.e. the embedding. It can be proved that if $X \hookrightarrow G$ then the (first order) *embedded deformations* are parameterised by the elements of $H^0(X, \mathcal{N}_{X/G})$, hence the number of embedded moduli corresponds to the rank of the map

$$H^0(X, \mathcal{N}_{X/G}) \longrightarrow H^1(X, \mathcal{T}_X).$$

In particular, if this number corresponds to 19 for a given model of polarised K3 surface, then the model parameterises a non-empty open subset of the moduli space $\mathcal{F}_g$.

A.5 Weighted projective spaces

On $\mathbb{C}^n \setminus \{0\}$ one can define an action of $\mathbb{C}^*$ with diagonal weights $(a_1, ..., a_n)$:

$$\lambda : (x_1, ..., x_n) \longmapsto (\lambda^{a_1} x_1, ..., \lambda^{a_n} x_n).$$

The quotient of $\mathbb{C}^n \setminus \{0\}$ under this action is denoted by $\mathbb{P}(a_1, ..., a_n)$ and is called *weighted projective space.* From the algebraic point of view, given a tuple of positive integers $(a_1, ..., a_n)$, we can consider the (graded) polynomial algebra $\mathbb{C}[x_1, ..., x_n]$ with $\deg(x_i) = a_i$. Then we define $\mathbb{P}(a_1, ..., a_n) := \mathrm{Proj}(\mathbb{C}[x_1, ..., x_n])$. We have the following

Lemma A.5.1. *$\mathbb{P}(a_1, ..., a_n) \cong \mathbb{P}(ca_1, ..., ca_n)$ for every $c \in \mathbb{N}$.*

In particular, we may assume that the greatest common divisor of the tuple is 1. Moreover, there is an important condition to satisfy: we require that the greatest common divisor of every subset of $n-1$ integer in the tuple is 1. In this case, $\mathbb{P}(a_1, ..., a_n)$ is $\mathbb{Q}$-factorial. However, in general, it is not smooth:

Proposition A.5.2. *With this hypothesis, we have:*

- *$\mathbb{P}(a_1, ..., a_n)$ is a normal irreducible projective algebraic variety;*
- *All singularities of $\mathbb{P}(a_1, ..., a_n)$ are cyclic quotient singularities;*
- *If non-singular, $\mathbb{P}(a_1, ..., a_n) \cong \mathbb{P}^{n-1}$.*

Remark A.5.3. The canonical bundle of a weighted projective space can be computed as $\omega_{\mathbb{P}} \cong \mathcal{O}(-a_1 - ... - a_n)$.

Theorem A.5.4 (Weak Lefschetz theorem)**.** *For a smooth hypersurface $X \subseteq \mathbb{P}(a_1, ..., a_n)$ we have the homomorphisms*

$$H^q(X, \Omega^p_X) \longrightarrow H^{q+1}(\mathbb{P}, \Omega^p_{\mathbb{P}}).$$

For $q > n-p-2$ it is an isomorphism while for $q = n-p-2$ it is surjective.

Proof. See theorem 4.2.2 of [10]. □

A.6 The Jacobian ring

Let $X \subseteq \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d defined by f, by Griffiths' theory of residues it is easy to compute the n-dimensional (vanishing) Hodge structure of X. Note that by theorem 1.2.7, this is the only unknown row of the Hodge diamond of X. Denote by $H_v^{\cdot,\cdot}(X)$ the *vanishing cohomology* of X, i.e. the orthogonal in $H^{\cdot,\cdot}(X)$ of the image of $H^{\cdot,\cdot}(\mathbb{P})$ under the inclusion given in theorem 1.2.7. Denote $R_f := \mathbb{C}[x_0, ..., x_{n+1}] \big/ J_f$ the *Jacobian ring*, where J_f is the ideal generated by all partial derivatives of f. We have the isomorphisms

$$H_v^{n-p+1,p-1}(X) \cong [R_f]_{pd-n-2}.$$

This is a classical result, see for example theorem 6.10 of [51]. Moreover, if $n \geq 2$, we have the following isomorphisms

$$[R_f]_d \cong \operatorname{Ker}\left(H^1(\mathcal{T}_X) \longrightarrow H^2(\mathcal{O}_X)\right),$$

which can be used to compute the embedded deformations of X.

A.7 Grassmannians

By $\operatorname{Gr}(k, k+\ell)$ we denote the (complex) *grassmannian* of k-dimensional linear subspaces of a vector space $V_{k+\ell} = V_n$. There are two important vector bundles over it:

- the *tautological bundle* $\mathcal{U}$ is the rank k bundle whose fibre over a point $[W]$ is isomorphic to the vector space W itself;

- the *quotient bundle* $\mathcal{Q}$ is the rank ℓ bundle whose fibre over a point $[W]$ is isomorphic to the vector space V/W.

Grassmannians are projective varieties: $\det(\mathcal{U}^\vee)$ induces the Plücker embedding

$$\begin{aligned} \operatorname{Gr}(k, k+\ell) &\hookrightarrow \mathbb{P}\left(\bigwedge^k V\right) \\ \operatorname{span}(w_1, ..., w_k) &\longmapsto [w_1 \wedge ... \wedge w_k]. \end{aligned}$$

In particular, $\det(\mathcal{U}) \cong \mathcal{O}_{\mathrm{Gr}(k,k+\ell)}(-1)$. Moreover, we have the Euler sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow V \otimes \mathcal{O}_G \longrightarrow \mathcal{Q} \longrightarrow 0.$$

It is not difficult to see that $\mathcal{T}_G \cong \mathrm{Hom}(\mathcal{U}, \mathcal{Q})$ (see theorem 3.5 of [11]), hence $\Omega_G \cong \mathcal{U} \otimes \mathcal{Q}^\vee$ and $\omega_G \cong \mathcal{O}_{\mathrm{Gr}}(-k-\ell)$ (where we use that $\det(\mathcal{Q}) \cong \mathcal{O}_{\mathrm{Gr}}(1)$).
Now we focus on the case of $k = 2$. Here, Plücker coordinates are given by the 4-Pfaffians of a $(\ell+2) \times (\ell+2)$ skew-symmetric matrix. For example, the grassmannian $\mathrm{Gr}(2,4)$ is a quadric in $\mathbb{P}^5$ given by vanishing of the square root of the determinant of the matrix

$$\begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{pmatrix}$$

which is $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}$.
In this case, the degree of the Plücker embedding is

$$\deg(\mathrm{Gr}(2,n)) = \frac{(2n-4)!}{(n-1)!(n-2)!},$$

while, in general, we have

$$\deg(\mathrm{Gr}(k,k+\ell)) = (k\ell)! \prod_{1 \le i < k < j \le k+\ell} (j-i)^{-1}.$$

Grassmannians are also well-known examples of homogeneous spaces. In fact, $\mathrm{Gr}(k,k+\ell)$ is a rational homogeneous variety of dimension $k\ell$ and can be written as ${}^{\mathrm{SL}(n)}/_{P}$, where P is a maximal parabolic subgroup (for example the stabiliser of the subspace given by the first k basis vector of $\mathbb{C}^n$). This point of view allows us to apply the classical theory of homogeneous variety for our purposes (see for example [44]). This is the content of the next section.

A.8 Borel-Bott-Weil theorem

We give an easy way to compute the cohomology of a special kind of vector bundles over the grassmannian $\mathrm{Gr}(k, k+\ell)$, namely the ones that are globally generated, homogeneous and completely reducible (under the action of $\mathrm{SL}(n)$, where $n = k + \ell$). They are all of the form $E := \Sigma_\alpha \mathcal{Q} \otimes \Sigma_\beta \mathcal{U}$, where Σ_α is the Schur functor associated to the weight $\alpha := (\alpha_1, ..., \alpha_\ell)$ and Σ_β is the Schur functor associated to $\beta := (\beta_1, ..., \beta_k)$. Denote their concatenation as $\gamma = (\alpha_1, ..., \alpha_\ell \mid \beta_1, ..., \beta_k)$, which is the tuple corresponding to our vector bundle E. We list some of the most frequent examples:

- $(0, ..., 0 \mid 1, ..., 0)$ corresponds to $\mathcal{U}$;
- $(1, ..., 0 \mid 0, ..., 0)$ corresponds to $\mathcal{Q}$;
- $(0, ..., 0 \mid p, ..., 0)$ corresponds to $\mathrm{Sym}^p\, \mathcal{U}$;
- $(p, ..., 0 \mid 0, ..., 0)$ corresponds to $\mathrm{Sym}^p\, \mathcal{Q}$;
- $(0, ..., 0 \mid 1, ..., 1, 0, ..., 0)$ corresponds to $\bigwedge^p \mathcal{U}$;
- $(1, ..., 1, 0, ..., 0 \mid 0, ..., 0)$ corresponds to $\bigwedge^p \mathcal{Q}$.

If we have a bundle defined by $(\alpha_1, ..., \alpha_\ell \mid \beta_1, ..., \beta_k)$ then its dual will be given by the sequence $(-\alpha_\ell, ..., -\alpha_1 \mid -\beta_k, ..., -\beta_1)$ translated by $(c, ..., c \mid c, ..., c)$ where $c := \max\{\alpha_i, \beta_j\}$. Note that the translation by an integer c is harmless, indeed it corresponds to the twist by $\mathcal{O}(c) \otimes \mathcal{O}(-c)$.
The following algorithm is a reworking of the classical theorem of Bott (theorem 11.4 of [44]). Let $\delta := (n-1, ..., 1, 0)$ and consider $\gamma + \delta$. If it has repeated entries, then for all $i \geq 0$ we have

$$H^i(\mathrm{Gr}(k, n), E) = 0.$$

Otherwise, we write $\mathrm{sort}(\gamma + \delta)$ for the tuple where the entries are rearranged in a non-increasing order and we define $\tilde{\gamma} := \mathrm{sort}(\gamma + \delta) - \delta$. Let m be the number of disorders in

$\gamma + \delta$[28], then

$$H^i(\mathrm{Gr}(k,n), E) \cong \begin{cases} \Sigma_{\tilde{\gamma}} V_n & \text{for } i = m \\ 0 & \text{otherwise} \end{cases}$$

In order to compute $h^m(\mathrm{Gr}(k,n), E)$ we recall Weyl's formula

$$\dim(\Sigma_\lambda V_n) = \prod_{1 \leq i < j \leq n} \frac{j - i - \lambda_j + \lambda_i}{j - i}.$$

Remark A.8.1. The previous discussion represents a special case of a more general theorem (the theorem of Bott) which applies to every homogeneous variety (and not only for grassmannians). For instance, the orthogonal grassmannian, treated e.g. in section 5.1, is homogeneous with respect to the action of the orthogonal group $\mathrm{SO}(n)$, and the homogeneous vector bundles that will appear are not only of the type described above. References for the general case are e.g. [44] or [48].

A.9 Chern classes

Our main reference for this section is chapter 5 of [11]. We recall the axiomatic definition and the main properties and computations of Chern classes.

Definition A.9.1. Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension n and E, E', E'' be vector bundles over X, we define the *Chern classes* $c_i(E) \in H^{2i}(X, \mathbb{Z})$ and the *total Chern class* $c(E) = c_0(E) + ... + c_r(E)$ of E as the cohomology classes satisfying the following conditions:

i) $c_0(E) = 1$;

ii) If E is a line bundle then $c(E) = 1 + c_1(E)$, where $c_1(E)$ is the divisor associated to any non-trivial section of E;

iii) Let $s_0, ..., s_{r-1}$ be global sections of E and D the locus where they are dependent, i.e.

[28] In this case the number of disorders corresponds to the number of pairs (i, j) with $1 \leq i < j \leq n$ and $\gamma_i - i < \gamma_j - j$.

the locus where the morphism

$$\phi : \mathcal{O}_X^{\oplus r} \longrightarrow E$$

loses rank. Then if i is the codimension of D we have $c_i(E) = [D] \in H^{2i}(X)$;

iv) (Functoriality) For a morphism $\phi : Y \longrightarrow X$ it holds $\phi^*(c(E)) = c(\phi^*(E))$;

v) (Whitney's formula) For a short exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0,$$

we have $c(E) = c(E')c(E'')$.

Theorem A.9.2. *There is a unique way of assigning to each vector bundle E on X a class $c(E) = c_0(E) + ... + c_r(E)$ that satisfies conditions (i)-(v) in the previous definition.*

Proof. See theorem 5.3 in [11]. □

Theorem A.9.3. *For a projective variety X of dimension n the following computations for Chern classes hold:*

- *(Trivial line bundle) $c(\mathcal{O}_X) = 1$;*
- *(Vanishing) If E has rank r, then $c_i(E) = 0$ for $i > r$;*
- *(Dual) $c_i(E^\vee) = (-1)^i c_i(E)$;*
- *(Gauss-Bonnet-Chern) $e(X) = \deg(c_n(X))$.*

Theorem A.9.4. *In the case of $\mathbb{P}^n$ we have:*

$$c(\mathbb{P}^n) = \sum_{k=0}^{n} \binom{n+1}{k} c_1(\mathcal{O}(1))^k \text{ and } e(\mathbb{P}^n) = n + 1.$$

Remark A.9.5. Here we recall the expansion of the *Chern character* and of the *Todd class* for a vector bundle E and for $\mathcal{T}_X$ respectively. Their definitions can be found e.g. in section 4.4 of [23], where they are introduced starting from a fixed connection and curvature over the (complex) vector bundle E. Here we do treat this point of view and we only need their relation with Chern classes. Note that for a surface the following expansions stop in degree 2.

$$\mathrm{ch}(E) = \mathrm{rk}(E) + c_1(E) + \frac{c_1(E)^2 - 2c_2(E)}{2} + \frac{c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)}{6} + \dots$$

$$\mathrm{Td}(X) = 1 + \frac{c_1(X)}{2} + \frac{c_1(X)^2 + c_2(X)}{12} + \frac{c_1(X)c_2(X)}{24} + \dots$$

For our computations, we use a lot of formulae involving Chern classes of tensor products, exterior and symmetric powers. We summarise here some of them[29]:

- (Tensor product)

  If E has rank r and F has rank s, then

$$c_1(E \otimes F) = sc_1(E) + rc_1(F).$$

  If E has rank r and L has rank 1, then

$$c_i(E \otimes L) = \sum_{j=0}^{i} \binom{r-j}{i-j} c_j(E)c_1(L)^{i-j}.$$

  If E has rank r and F has rank 2, then

$$c_2(E \otimes F) = c_1(E)^2 + 2c_2(E) + (2r-1)c_1(E)c_1(F) + rc_2(F) + \binom{r}{2} c_1(F)^2.$$

  If E has rank r and F has rank 3, then

$$c_2(E \otimes F) = 3c_1(E)^2 + 3c_2(E) + (3r-1)c_1(E)c_1(F) + rc_2(F) + \binom{r}{2} c_1(F)^2.$$

  If E has rank r and F has rank 4, then

$$c_2(E \otimes F) = 6c_1(E)^2 + 4c_2(E) + (4r-1)c_1(E)c_1(F) + rc_2(F) + \binom{r}{2} c_1(F)^2.$$

- (Symmetric power)

[29]They can be achieved using the so-called *splitting principle*, see e.g. chapter 5 of [11].

If E has rank r, then

$$c_1(\mathrm{Sym}^2 E) = (r+1)c_1(E),$$
$$c_1(\mathrm{Sym}^3 E) = \left(3r + \binom{r-1}{2}\right) c_1(E).$$

If E has rank 2, then

$$c(\mathrm{Sym}^2 E) = 1 + 3c_1(E) + (2c_1(E)^2 + 4c_2(E)) + 4c_1(E)c_2(E),$$
$$c(\mathrm{Sym}^3 E) = 1 + 6c_1(E) + (11c_1(E)^2 + 10c_2(E)) + (6c_1(E)^3 + 30c_1(E)c_2(E)) +$$
$$+ (18c_1(E)^2 c_2(E) + 9c_2(E)^2).$$

If E has rank 3, then

$$c_2(\mathrm{Sym}^2 E) = 5c_1(E)^2 + 5c_2(E).$$

If E has rank 5, then

$$c_2(\mathrm{Sym}^2 E) = 14c_1(E)^2 + 7c_2(E).$$

- (Exterior power)

  If E has rank r, then

$$c_1\left(\bigwedge^2 E\right) = (r-1)c_1(E),$$
$$c_1(\det(E)) = c_1(E).$$

If E has rank 2, then

$$c\left(\bigwedge^2 E\right) = 1 + c_1(E).$$

If E has rank 3, then

$$c_2\left(\bigwedge^2 E\right) = c_1(E)^2 + c_2(E).$$

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