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An introduction to scalar-tensor gravity: theory and experiment in the Brans-Dicke model

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Abstract

Le teorie della gravità scalare-tensore sono una classe di teorie alternative alla relatività generale in cui l'interazione gravitazionale è descritta sia dalla metrica, sia da un campo scalare. Ne costituisce un esempio caratteristico la teoria di Brans-Dicke, introdotta come estensione della relatività generale in modo da renderla conforme con il principio di Mach.

Il presente lavoro di tesi è volto a presentare un'analisi di questa teoria nei suoi aspetti principali, studiandone i fondamenti teorici e il modello cosmologico derivante, sottolineandone inoltre i limiti e le criticità; in seguito vengono esposti i risultati degli esperimenti fino ad ora svolti per verificare fondamenti e previsioni del modello.

Contents

Introduction

A theory of gravitation containing a metric and one or more dynamical scalar fields is called a scalar-tensor theory. Models of this sort yield local gravitational physics independent of the velocity of the frame, but possibly dependent on its location. Conversely, gravitational theories which only contain a metric yield local gravitational physics independent of both the velocity and the location of the frame.

Scalar-tensor theories were first introduced as a development on Dirac's large number hypothesis, which relates ratios of size scales in the universe to ratios of fundamental force scales, and which suggests a cosmology where the strength of the gravitational coupling, represented by the gravitational constant G, does not remain the same over time, but is inversely proportional to the age of the universe. This was indeed the idea behind Jordan's gravitational theory, where G takes the form of a scalar field which satisfies a general conservation equation.

The Brans-Dicke theory was introduced in the early 1960s as a modification of general relativity, in order to bring it into confirmity with Mach's principle, which is not explicitely embodied in Einstein's theory. Brans and Dicke's theory, like Jordan's, includes a non-constant gravitational coupling, represented by a variable scalar field which is the inverse of the gravitational constant employed in general relativity. In accordance with Mach's principle, gravitational coupling is determined by all matter in the universe; therefore, according to this theory, local gravitational experiments are affected by the cosmological distribution of matter.

The Brans-Dicke theory is widely regarded as the prototype for gravitational scalar-tensor theories alternative to general relativity. A variety of scalar-tensor theories have been formulated since then, expanding on the core ideas of this model, and finding relevant applications in cosmology, especially in the context of inflationary models.

The publication of Brans and Dicke's papers coincidentally happened around the time period when gravitational physics tests were beginning to be performed in space as well as in laboratories. Interest in the original formulation of the Brans-Dicke theory soon declined, as experiments continued to validate the predictions of general relativity. Despite the body of evidence supporting general relativity, alternative theories of gravity are still currently being explored via experimental testing, with scalar-tensor theories being arguably one of the most valid alternatives as for now.

In Chapter 1 of the following, the theoretical foundations and main characteristics of the Brans-Dicke theory will be laid out, along with an introduction to generalized scalar-tensor theories. Subsequently, Chapter 2 will provide an overview of the cosmological model deriving from it, including a brief description of some of its key aspects, exact solutions, and applications of generalized scalar-tensor theories in cosmology. Chapter 3 will focus on compiling and reviewing the main experimental results necessary in order to test the foundations and the predictions of this theory.

Chapter 1

The Brans-Dicke theory of gravitation

The present Chapter is dedicated to an overview of the key points of the Brans-Dicke theory of gravity, including the reasons behind its formulation and the derivation of its field equations. The motivations for the introduction of generalized scalar-tensor theories are then explored, along with a brief study of their field equations and key aspects.

1.1 History and theoretical foundations

The Brans-Dicke theory of gravitation was introduced as a modification of Einstein's general relativity in order to bring it into conformity with some formulation of Mach's principle. It expands on the work of Jordan and others, who had first formulated a self-consistent theory, alternative to general relativity, with a gravitational coupling ruled by a scalar field.

The Brans-Dicke theory is a metric theory of gravity, since it assumes that there exists a symmetric metric, test bodies follow geodesics of the metric, and the Einstein equivalence principle is valid. In addition to the metric, this model introduces a scalar field, which behaves as the inverse of the non-constant gravitational coupling, and the effect of which is a local variation of the value of the effective gravitational constant.

Like general relativity, it is a purely dynamical theory, since in both models, the gravitational fields have their structure and evolution determined by coupled partial differential field equations. In fact, the former only contains one gravitational field, the metric itself, the structure and evolution of which are ruled by Einstein's equations, which are indeed partial differential equations; in the latter, the field equation for the metric involves the scalar field (as well as ordinary matter as a source), and the equation for the scalar field involves the metric.

1.1.1 Mach's principle

Mach's principle is the hypothesis that local physical laws are determined by the large-scale distribution of matter in the universe, which consequently implies that inertial forces locally observed in an accelerated frame are to be interpreted as gravitational effects originating from distant matter accelerated relative to the frame.

Having its roots in philosophy rather than in physics, and having no unambiguous philosophical formulation, Mach's principle has no direct unambiguous mathematical interpretation. The following argument, suggested by Brans and Dicke expanding on Jordan's and Sciama's work, constitutes the starting point for the formulation of this theory [3].

From dimensional arguments, by assuming the validity of Mach's principle, the gravitational constant G can be found to be related to the mass distribution in a uniform expanding universe in the following way

$$\frac{GM}{Rc^2} \sim 1\,, \tag{1.1.1}$$

where M is the finite mass of the causally related, visible universe, and R is the radius of its boundary. This relation, despite being only significant in an order-of-magnitude manner only, suggests that either the ratio M/R should be fixed, or that the locally observed gravitational constant should be variable and determined by the mass distribution around the point in question. The latter alternative is the one explored in the Brans-Dicke theory, which is necessarily incompatible with general relativity and the strong equivalence principle (see Sect. 1.1.2). It would be expected that both distant and nearby matter should contribute to locally experienced inertial reactions.

In a linear theory, Eq. (1.1.1) could be expressed as

$$\frac{1}{G} \sim \sum_i \frac{m_i}{r_i c^2}\,, \tag{1.1.2}$$

where the sum is over all the matter contributing to the inertial reaction.

1.1.2 The principle of equivalence

There is not one unique formulation of the equivalence principle. Three formulations are currently in use, which are the weak, Einstein, and strong equivalence principle. Only the weak and Einstein formulations of the equivalence principle can be brought into agreement with Mach's principle, while the strong equivalence principle is necessarily incompatible with it.

The weak equivalence principle states that the gravitational mass of a body is proportional to its inertial mass. In other words, all gravitational accelerations are locally equal; this idea can also be expressed as the universality of free fall.

The Einstein equivalence principle states, along with the validity of the weak equivalence principle, that the outcome of local non-gravitational experiments is independent of the location, moment in time, and velocity of the freely falling reference frame where it is performed. These last two properties are respectively called local position invariance and local Lorentz invariance. It can be rephrased as stating that, in a region small enough to not exhibit any curvature effects, all the laws of physics known in special relativity are valid in local Lorentz frames.

According to Schiff's conjecture, any complete, self-consistent theory of gravitation that embodies the weak equivalence principle necessarily embodies the Einstein equivalence principle. Alternatively, it postulates that a confirmation of the weak equivalence principle is at the same time a confirmation of local Lorentz invariance and local position invariance. Although it is impossible to prove such a conjecture, and although several counterexamples have been found, it is often adopted when testing the principle of equivalence.

The strong equivalence principle expresses the validity of local Lorentz invariance and local position invariance for any local test experiment, therefore including experiments involving gravitational forces, and extends the weak equivalence principle to self-gravitating bodies as well as test bodies. It indicates that the motion of any object in a gravitational field is independent of its mass and composition, and depends only on its initial velocity and position in spacetime. As a consequence, dimensionless physical constants are independent of the location in time or space of the reference frame where they are measured, which is incompatible with the interpretation of Mach's principle that is used as a basis for the development of the Brans-Dicke theory. In fact, general relativity seems to be the only viable metric theory that is in complete accordance with the strong equivalence principle.

Thus far, experimental evidence supports the validity of all three formulations of the principle of equivalence listed above (see Sect. 3.1).

1.2 Field equations

In order to find the Brans-Dicke field equations, we start from the variational principle of general relativity (in units $c = 1$)

$$\delta \int d^4x \left(R + 16\pi G\mathcal{L}^{(m)}\right) \sqrt{-g} = 0 \,, \tag{1.2.1}$$

where g is the determinant of the metric, R is the scalar curvature, and $\mathcal{L}^{(m)}$ is the Lagrangian density of matter including all nongravitational fields. A Lagrangian density of a scalar field ϕ is then introduced, assuming G is a function of ϕ so that

$$G^{-1} \sim \phi \,. \tag{1.2.2}$$

It would be indeed reasonable to assume that G^{-1} varies as Eq. (1.2.2), so that a wave equation for ϕ with a scalar matter density source would give an equation of the form as Eq. (1.1.1). The function ϕ will therefore behave as the inverse of the gravitational coupling in the generalized variational principle.

This way, the Brans-Dicke variational principle is obtained

$$\delta \int d^4x \left(\phi R + 16\pi \mathcal{L}^{(m)} - \frac{\omega}{\phi} \phi_{,i} \phi^{,i} \right) \sqrt{-g} = 0 \,, \tag{1.2.3}$$

in which the constant dimensionless scalar parameter ω, called Brans-Dicke coupling constant, is introduced. In Eq. (1.2.3) it is divided by ϕ in order to render it, in fact, dimensionless. From a theoretical standpoint, ω should be of the order of magnitude of unity; its value can only be determined through experiment (see Sect. 3.2).

As the term involving the Lagrangian density of matter remains unchanged in the generalization of the variational principle, the equations of motion in a given metric field are the same as in general relativity.

By varying ϕ and its first derivatives in Eq. (1.2.3), one can obtain the wave equation for ϕ

$$\frac{2\omega}{\phi} \Box \phi - \frac{\omega}{\phi^2} \phi^{,i} \phi_{,i} + R = 0 \,, \tag{1.2.4}$$

in which $\Box \phi$ is the scalar d'Alembertian with respect to the metric.

The field equations for the metric field are obtained by varying the components of the metric tensor and their first derivatives in Eq. (1.2.3), which yields

$$R_{ij} - \frac{1}{2} g_{ij} R = \frac{8\pi}{\phi} T_{ij} + \frac{\omega}{\phi^2} \left(\phi_{,i} \phi_{,j} - \frac{1}{2} g_{ij} \phi_{,k} \phi^{,k} \right) + \frac{\phi_{;ij} - g_{ij} \Box \phi}{\phi} \,. \tag{1.2.5}$$

While the left side of Eq. (1.2.5) is identical to that of the Einstein field equation, the right side, in addition to the energy-momentum tensor of ordinary matter, also contains the energy-momentum tensor of the scalar field ϕ and a contribution resulting from the presence of second derivatives of the metric tensor in the scalar curvature in Eq. (1.2.3). The gravitational coupling in the right side of Eq. (1.2.5) is given by the parameter ϕ^{-1}.

By taking the trace of Eq. (1.2.5), one obtains

$$-R = \frac{8\pi}{\phi} T - \frac{\omega}{\phi^2} \phi_{,k} \phi^{,k} - \frac{3 \Box \phi}{\phi} \,, \tag{1.2.6}$$

which, combined with Eq. (1.2.4), can be used to obtain a different form of the wave equation for ϕ

$$\Box \phi = \frac{8\pi}{3 + 2\omega} T \,, \tag{1.2.7}$$

where the source term appears to be only the contracted energy-momentum tensor T.

It is important to note that, since the Brans-Dicke variational principle (1.2.3) differs from the one employed in general relativity by one term, namely the Lagrangian density of ϕ, the energy-momentum tensor representing ordinary matter remains unchanged. Therefore, as in general relativity, the energy-momentum tensor is given by

$$T^{ij} = \frac{2}{\sqrt{-g}} \frac{\partial}{\partial g_{ij}} \left(\sqrt{-g} \mathcal{L}^{(m)} \right) , \tag{1.2.8}$$

where $\mathcal{L}^{(m)}$ explicitely depends only on matter variables and the metric tensor, but has no explicit dependence upon derivatives of the metric tensor, and it is not a function of the scalar field ϕ. Consequently, the theory allows for the conservation of the energy-momentum tensor, which, as in general relativity, must have a vanishing covariant divergence,

$$T^{ij}_{;j} = 0 \,. \tag{1.2.9}$$

For an isotropic and homogeneous fluid of (matter and radiation) density ϵ and pressure p, both expressed in comoving coordinates, the energy-momentum tensor T_{ij} is given by

$$T_{ij} = -(p + \epsilon) u_i u_j + p g_{ij} \tag{1.2.10}$$

with trace T given by (the metric signature is conventionally assumed to be $- + + +$)

$$T = -\epsilon + 3p \,. \tag{1.2.11}$$

This implies that, assuming $\omega > 0$, the contribution to the scalar field ϕ from a local mass is positive.

1.2.1 Newtonian limit

The first order weak-field, slow-motion expansion, also called Newtonian approximation, of the field equations Eq. (1.2.3), is obtained by writing the metric tensor as

$$g_{ij} = \eta_{ij} + h_{ij} \,, \tag{1.2.12}$$

where η_{ij} is the Minkowskian metric tensor and h_{ij} is computed to the linear first order approximation. Likewise, for the scalar field ϕ the following form is employed

$$\phi = \phi_0 + \xi \,, \tag{1.2.13}$$

with ϕ_0 constant and ξ computed to first order in mass densities.

By substituting Eq. (1.2.13) into Eq. (1.2.7), and computing its solutions to the first order, a retarded-time solution of the form

$$\xi = -\frac{2}{3 + 2\omega} \int d^3x \frac{T}{r} \tag{1.2.14}$$

is found, in which T is to be evaluated at the retarded time.

Using Eq. (1.2.12) and employing the four-coordinate condition

$$\left(h_{ij,k} - \frac{1}{2}\eta_{ij,k}h\right)\eta^{jk} = \frac{\xi_{,i}}{\phi_0} \tag{1.2.15}$$

the field equation (1.2.5) for a stationary mass point of mass M yield first-order expressions for the scalar field

$$\phi = \phi_0 + \frac{2M}{(3+2\omega)r} \tag{1.2.16}$$

and for the components of the metric tensor

$$g_{00} \approx -1 + \left(\frac{2M}{\phi_0 r}\right)\left(1 + \frac{1}{3+2\omega}\right), \tag{1.2.17a}$$

$$g_{\alpha\alpha} \approx 1 + \left(\frac{2M}{\phi_0 r}\right)\left(1 - \frac{1}{3+2\omega}\right), \quad \alpha = 1,2,3, \tag{1.2.17b}$$

$$g_{ij} = 0, \quad i \neq j. \tag{1.2.17c}$$

From Eq. (1.2.17a), it is clear that, to the first order of approximation, the gravitational constant which is measured in experiments is not equal to the inverse of the function ϕ, instead it is given by

$$G_0 = \left(\frac{4+2\omega}{3+2\omega}\right)\frac{1}{\phi_0}, \tag{1.2.18}$$

where ϕ_0 is the asymptotic value of the scalar field ϕ.

The Newtonian limit of the Brans-Dicke field equations is accurate enough to discuss two of the three classical tests of general relativity, namely the gravitational redshift and the deflection of light, while in order to discuss the precession of the perihelion of Mercury a second-order solution for g_{00} is needed (see Sect. 1.2.2). A discussion of these tests will be provided in Sect. 3.2.

1.2.2 Parametrized post-Newtonian formalism

As anticipated in Sect. 1.2.1, not all Solar System observations can be sufficiently accurately described by a first-order approximation of Eqs. (1.2.5), (1.2.7) in the weak-field, slow motion regime. It is therefore necessary to discuss the second-order, or post-Newtonian, approximation of the Brans-Dicke equations under these conditions.

The parametrized post-Newtonian (PPN) formalism is a framework that adds second-order corrections to the Newtonian limit of a metric theory of gravity. These second-order corrections are given in terms of a set of parameters that can be (more or less directly) measured in Solar System tests.

The PPN formalism is not specific for one single theory. In fact, despite the wide variety of viable scalar-tensor theories, they are so similar at this order of approximation it is possible to construct one single post-Newtonian theory of gravity encompassing all theories, of which the Brans-Dicke model constitutes the prime example. It has been shown that, for nearly every metric theory of gravity, the metric g_{ij} has the same structure in this limit, which is found as an expansion about η_{ij} in terms of gravitational potentials of varying degrees of smallness.

The deviation of any metric theory of gravity from general relativity at the post-Newtonian level can be quantified by the value of ten parameters, each specifying a different effect, relative to general relativity. The most recent PPN notation [12] employs the parameters γ, β, ξ, α_1, α_2, α_3, ζ_1, ζ_2, ζ_3, ζ_4, which assume different values in each theory.

The amount of non-zero parameters that fully characterize a theory of gravitation at the post-Newtonian level is related to whether the theory in question is fully conservative or semi-conservative. Fully conservative theories possess both global conservation laws for angular momentum and conservation laws for total momentum; they have five non-zero PPN parameters (γ, β, ξ, α_1, α_2). Semi-conservative theories only possess conservation laws for total momentum; they are characterized by only three PPN parameters (γ, β, ξ). All metric theories of gravity predict the α_3, ξ_1, ξ_2, ξ_3, ξ_4 to be equal to zero. The physical meanings of the ten PPN parameters, along with their values in general relativity and in the Brans-Dicke model, are listed in Table 1.

PPN parameter	What it measures relative to GR	Value in GR	Value in BD
γ	How much space curvature is produced by unit rest mass?	1	$\frac{1+\omega}{2+\omega}$
β	How much nonlinearity is there in the superposition law for gravity?	1	1
ξ	Are there preferred-location effects?	0	0
α_1	Are there preferred-frame effects?	0	0
α_2		0	0
α_3		0	0
α_3	Is conservation of total momentum violated?	0	0
ζ_1		0	0
ζ_2		0	0
ζ_3		0	0
ζ_4		0	0

Table 1: Physical meaning of the ten PPN parameters and their values in general relativity (GR) and Brans-Dicke (BD) theory. The parameter α_3 is listed twice to indicate that it is a measure of two different effects.

In the special case of the Brans-Dicke model, the number of non-zero PPN parameters is reduced to two, namely γ and β. The metric in the post-Newtonian approximation is given by

$$g_{00} \approx -1 + \frac{2G_0 M}{r} + \frac{\gamma - \beta}{2}\left(\frac{2G_0 M}{r}\right)^2 , \tag{1.2.19a}$$

$$g_{\alpha\alpha} \approx 1 + \gamma\frac{2G_0 M}{r}4 . \tag{1.2.19b}$$

By measuring the values of PPN parameters it is possible to test different theories of gravitation. This will be further explored in Sect. 3.2, where a discussion of the classical tests of general relativity in terms of the PPN parameters is provided.

1.2.3 The limiting transition to general relativity

By comparing the variational principle of general relativity (1.2.1) with that of the Brans-Dicke theory (1.2.3) it is evident that, as a general rule, the larger the value of ω, the smaller the effects of the scalar field ϕ ruling the gravitational coupling. In the limit $\omega \to \infty$, (1.2.3) reduces to (1.2.1), and the predictions of the Brans-Dicke model become indistinguishable from those of general relativity.

It has been argued [8] that several exact solutions of Brans-Dicke theory do not yield the corresponding general relativistic solution in this limit. This happens in solutions where the energy-momentum tensor associated with matter has a vanishing trace.

The scalar field ϕ usually exhibits asymptotic behaviour of the sort

$$\phi = \phi_0 + O\left(\frac{1}{\omega}\right) \text{ for } \omega \to \infty , \tag{1.2.20}$$

while, in the exact solutions mentioned above, it behaves as

$$\phi = \phi_0 + O\left(\frac{1}{\sqrt{\omega}}\right) \text{ for } \omega \to \infty , \tag{1.2.21}$$

where ϕ_0 is the asymptotic value of the scalar field.

Arguably, while technically possible, generally in the Brans-Dicke model it is not physically meaningful to consider vacuum solutions, specified by an energy-momentum tensor with vanishing trace, since, according to Mach's principle, the gravitational coupling is determined by all matter in the universe. On the contrary, general relativity allows meaningful vacuum solutions, since the gravitational coupling is constant. The discrepancy between general relativity and the Brans-Dicke theory in the limit $\omega \to \infty$ under these conditions points out this conceptual difference between the two models.

It is worth remarking that the parameter ω is assumed to be strictly constant in time in the Brans-Dicke theory. The limit to general relativity discussed above is therefore not to be interpreted as a limit that is reached over time but as a mathematical tool to formally recover the predictions of general relativity from the Brans-Dicke theory.

1.3 Boundary conditions and conservation laws

It is rather evident that Eqs. (1.2.5) and (1.2.7) do not completely specify a solution, therefore imposing initial and boundary conditions is necessary. The imposition of boundary conditions is not done in a general way, but in connection with specific problems only. An exemplary situation is that of a spherically symmetric static mass shell.

Mach's principle suggests that, for a static mass shell, $\phi \to 0$ as $r \to \infty$, or that $\phi \to 0$ somewhere at some radius $r_0 < \infty$ outside all matter. However, the presence of matter inside the shell causes space to curve, so the metric in such a universe would not be flat, and it is an oversemplification to argue that $\phi \to 0$ as $r \to r_0$ (with $r_0 \leq \infty$), together with Eq. (1.2.7), yields a dependence of G of the form required by Mach's principle. In a reasonable static mass shell universe, the condition $\phi \to 0$ anywhere outside all universe requires that the pressure inside the shell be of the order of densities.

For a spherically symmetric mass distribution of mass M such that $R_1 < r < R_2$, it can be shown that if the following conditions are valid

$|\omega| > 2$,

all field quantities are time independent, spherically symmetric, single valued, and with continuous first derivatives,

for $r \to R_1^-$ the quantities ϕ, the metric components and their first derivatives have signatures that are consistent with $M > 0$ and $G > 0$,

for $R_1 < r < R_2$ one has T_i^j a diagonal tensor, $\phi > 0$, and $T_{00} \geq 0$, $T_\alpha^\alpha \leq 0$,

then there exists a constant $L > 0$ such that $\phi > L$ for all $r > R_2$.

It is also necessary to study the structure of conservation laws associated with Eqs. (1.2.5) and (1.2.7).

By assuming $\phi \neq 0$, the field equation (1.2.5) can be divided by ϕ and considered as Einstein equations having for source a modification of the energy-momentum tensor containing contributions from the scalar field. This way, an affine tensor $\mathfrak{T}_i^j$, which can be interpreted as a conserved total "mass", is conserved, yielding the following conservation law

$$(\mathfrak{T}_i^j)_{,i} = 0\,. \tag{1.3.1}$$

However, it can be found that $\mathfrak{T}_i^j$ has units of length, corresponding to some averaged gravitational constant times total mass, therefore it should rather be interpreted as a "total Schwarzschild radius". This quantity is constant for an isolated system.

From $\mathfrak{T}_i^j$ it is possible to define the affine vector

$$P_i = \int d^3x \mathfrak{T}_i^0\,, \tag{1.3.2}$$

which is constant for an isolated system (namely, a system for which the integral of T_i^α, $\alpha = 1,\ 2,\ 3$ over a bounding surface is always negligible). The affine vector P_i can be interpreted as a "total momentum".

A quantity having units of mass is also conserved. In fact, $\mathfrak{T}_i^j$ can be written as the divergence of an antisymmetric affine tensor $\mathfrak{U}_i^{jk}$; thus, another quantity $\bar{\mathfrak{T}}_i^j$ is found to be also conserved, yielding the conservation equation

$$(\bar{\mathfrak{T}}_i^j)_{,j} = 0 \quad \text{where} \quad \bar{\mathfrak{T}}_i^j = (\phi \mathfrak{U}_i^{jk})_{,k} \,. \tag{1.3.3}$$

As in the case of P_i, it is possible to define an affine vector $\bar{P}_i$ from $\bar{\mathfrak{T}}_i^j$ defined as

$$\bar{P}_i = \int d^3x \bar{\mathfrak{T}}_i^j \,, \tag{1.3.4}$$

which is constant for an isolated system and has units of mass.

1.4 Conformal transformations

The representation of the Brans-Dicke theory presented in Sect. 1.2 is only one of several possible representations. Among them, it is possible to cast this theory into a form such that the Einstein field equation is satisfied, and the gravitational coupling appears as a constant, by means of a conformal transformation.

A conformal transformation is a point-dependent rescaling of the metric tensor by a nonvanishing regular function Ω, called conformal factor, such that

$$g_{ij} \to \tilde{g}_{ij} = \Omega^2 g_{ij} \,. \tag{1.4.1}$$

It is evident that a conformal transformation is, from a mathematical standpoint, only a redefinition of units.

The new set of variables $(\tilde{g}_{ij}, \tilde{\phi})$ is called the Einstein frame, as opposed to the Jordan frame, consisting of the old set of variables (g_{ij}, ϕ). In the limit where ϕ is constant, the two frames coincide.

In the Brans-Dicke theory, the conformal factor Ω is given by

$$\Omega = \sqrt{G\phi} \,. \tag{1.4.2}$$

The metric tensor and the scalar field are then redefined as

$$\tilde{g}_{ij} = G\phi g_{ij} \tag{1.4.3}$$

and

$$\tilde{\phi}(\phi) = \sqrt{\frac{2\omega + 3}{16\pi G}} \ln\left(\frac{\phi}{\phi_0}\right) \,, \tag{1.4.4}$$

where $\phi \neq 0$, $\omega > -3/2$, and $\phi_0 = G^{-1}$, and the scalar field kinetic energy density is brought into canonical form. The possibility of performing this transformation can be seen as a reason to restrict the range of values of the coupling constant to $\omega > -3/2$.

By operating the above conformal transformation, ϕ appears in the theory as a non-gravitational field in the Einstein frame, instead of as a gravitational field in the Jordan frame. In fact, the conformal transformation moves part of the gravitational variables of the Jordan frame, namely the scalar field ϕ, into matter of the Einstein frame, represented by $\tilde{\phi}$. In the Einstein frame, it is impossible to eliminate the contribution of $\tilde{\phi}$ as a source of gravity, in agreement with the origin of ϕ, meaning the inverse gravitational coupling determined by distant matter. The conformal transformation also changes the conservation equation of the energy-momentum tensor in the two frames.

It can be shown, by means of a conformal transformation, that the non-null geodesic equation is modified by the presence of the scalar field ϕ in the Einstein frame, introducing a "fifth force" term coupling to massive test particles that causes the theory to violate the weak equivalence principle. For this reason, scalar-tensor theories of gravity are non-metric theories in the Einstein frame. On the other hand, the equation of null geodesics remains unchanged.

The possibility of operating a conformal transformation between the Jordan and Einstein frames gives rise to the question of whether the equivalence of these frames is purely mathematical or also physical. This question is all but trivial, and it is beyond the scope of this work to further explore it.

1.5 Generalized scalar-tensor theories

In the original formulation of the Brans-Dicke theory, the coupling parameter ω is assumed to be strictly constant over time. Taking into account the current lower bound for the value of this parameter, which is approximately $\omega \gtrsim 10^4$ (see Sect. 3.2.1), assuming the constancy over time of the coupling parameter renders effectively pointless to operate a distinction between this theory and general relativity. In fact, as long as experimental observations continue to validate the predictions of general relativity, it is impossible to rule out Brans-Dicke theory with a large enough ω. It would then be natural to discard the Brans-Dicke theory in favor of general relativity.

It is although possible to formulate a generalization of the Brans-Dicke theory in which the coupling parameter ω is a function of the scalar field ϕ, and is therefore allowed to vary not only in space, but also over time on a cosmological scale.

In generalized scalar-tensor theories, the variational principle employed to determine the field equations differs from Eq. (1.2.3) in that the coupling constant ω is replaced by a function of the scalar field $\omega(\phi)$. Furthermore, it is possible to introduce a potential $V(\phi)$, which plays a role analogous to that of a cosmological constant, and is therefore convenient to introduce when studying the cosmological models deriving from this class of theories.

Hence, Eq. (1.2.3) becomes

$$\delta \int d^4x \left(\phi R + 16\pi \mathcal{L}^{(m)} - \frac{\omega(\phi)}{\phi} \phi_{,i}\phi^{,i} - V(\phi) \right) \sqrt{-g} = 0 \,, \tag{1.5.1}$$

from which one can find the field equation

$$R_{ij} - \frac{1}{2} g_{ij} R = \frac{8\pi}{\phi} T_{ij} + \frac{\omega(\phi)}{\phi^2} \left(\phi_{,i}\phi_{,j} - \frac{1}{2} g_{ij} \phi_{,k}\phi^{,k} \right) + \frac{\phi_{;ij} - g_{ij}\Box\phi}{\phi} - \frac{V}{2\phi} g_{ij} \,. \tag{1.5.2}$$

By taking the trace of Eq. (1.5.2), namely

$$R = -\frac{8\pi}{\phi} T \left(R - \frac{dV}{d\phi} \right) - \frac{1}{2} \left(\phi^{,k}\phi_{,k} \right) \left(\frac{1}{\omega} \frac{d\omega}{d\phi} - \frac{1}{\phi} \right) \,, \tag{1.5.3}$$

it is possible to find the wave equation for the scalar field ϕ

$$\Box\phi = \frac{1}{2\omega(\phi) + 3} \left(8\pi T - \frac{d\omega}{d\phi} \phi_{,k}\phi^{,k} + \phi \frac{dV}{d\phi} - 2V \right) \,, \tag{1.5.4}$$

where it is implied that $\omega = \omega(\phi)$ and $V = V(\phi)$. Eqs. (1.5.2), (1.5.3), (1.5.4) are the generalized counterparts of Eq. (1.2.5), Eq. (1.2.6), and Eq. (1.2.7), respectively.

The Newtonian and post-Newtonian limits for generalized scalar-tensor theories are the same as those for Brans-Dicke; the only difference is in the value of the parameter β, which is $\beta = 1 + \phi_0\omega_0'/(3 + 2\omega_0)(4 + 2\omega_0)^2$ in generalized scalar-tensor theories, in which ϕ_0 is the asymptotic value of the scalar field, $\omega_0 = \omega(\phi_0)$, and $\omega_0' = d\omega/d\phi|_0$, as opposed to $\beta = 1$ in the Brans-Dicke theory. In addition, it is worth noting that, when studying the limit to general relativity of generalized scalar-tensor theories, the dependence of the parameter ω from the scalar field ϕ must be taken into account. The conditions under which generalized scalar-tensor theories converge to general relativity are

$$\omega \to \infty \qquad \text{and} \qquad \frac{1}{\omega^3} \frac{d\omega}{d\phi} \to 0 \,. \tag{1.5.5}$$

Only the first condition is required for convergence to general relativity in the Brans-Dicke theory, as stated in Sect. 1.2.3, since no dependence of the coupling constant from the scalar field must be taken into account.

Chapter 2

Applications in cosmology

Having discussed the foundations of the Brans-Dicke theory of gravitation in Chapter 1, it is now possible to study the cosmological model deriving from it. The present Chapter focuses on presenting the Brans-Dicke field equations for a Friedmann-Lemaître-Robertson-Walker space and its distinguishing features, as well as the analogue of these equations in the context of generalized scalar-tensor theories and some of their applications in cosmology.

2.1 Cosmological equations in Brans-Dicke theory

The Friedmann-Lemaître-Robertson-Walker (FRLW) universe is an isotropic and homogeneous universe described by the metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{2.1.1}$$

in comoving polar coordinates $(t,\, r,\, \theta,\, \phi)$, where K is the normalized curvature index and $a(t)$ is a function of the cosmic time t called scale factor. K can only assume the values 0, ± 1; we have $K = 0$ for a flat space, $K = -1$ for open, and $K = +1$ for closed with $r < 1$.

In the Brans-Dicke model, the scalar field ϕ that regulates the gravitational coupling depends only on the cosmic time, therefore in the FLRW metric one has

$$\phi^{;i}\phi_{;i} = -(\dot{\phi})^2\,, \tag{2.1.2}$$

and the wave equation for ϕ (1.2.7) becomes

$$\Box\phi = -\left(\ddot{\phi} + 3H\dot{\phi}\right) = -\frac{1}{a^3}\frac{d}{dt}\left(a^3\dot{\phi}\right)\,, \tag{2.1.3}$$

where $H = \dot{a}/a$ is the Hubble constant.

The energy-momentum tensor of ordinary matter is assumed to be described by the form corresponding to a perfect fluid with energy density ρ and pressure p

$$T_{ij}^{(m)} = (p+\rho)u_i u_j + p g_{ij} \tag{2.1.4}$$

satisfying the conservation equation (1.2.9), which takes the form

$$\dot{\rho} + 3H(\rho + p) = 0\,. \tag{2.1.5}$$

Furthermore, for the trace of the energy-momentum tensor, one has

$$T = 3p - \rho\,. \tag{2.1.6}$$

The time-time component of the field equation (1.2.5) then yields the following constraint equation, which also provides a first integral

$$H^2 = \frac{8\pi}{3\phi}\rho + \frac{\omega}{6}\left(\frac{\dot{\phi}}{\phi}\right)^2 - H\frac{\dot{\phi}}{\phi} - \frac{K}{a^2}\,. \tag{2.1.7}$$

By substituting the expression for the scalar curvature

$$R = 6\left(\dot{H} + 2H^2 + \frac{K}{a^2}\right) \tag{2.1.8}$$

into Eq. (1.2.6), one obtains

$$\dot{H} + 2H^2 + \frac{K}{a^2} = -\frac{4\pi T^{(m)}}{3\phi} - \frac{\omega}{6}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\Box\phi}{2\phi}\,. \tag{2.1.9}$$

Combining Eqs. (2.1.3), (2.1.6), (2.1.7) yields the following relation

$$\dot{H} = -\frac{8\pi}{(2\omega+3)\phi}\left((\omega+2)\rho + \omega p\right) - \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + 2H\frac{\dot{\phi}}{\phi} + \frac{K}{a^2}\,. \tag{2.1.10}$$

Furthermore, Eq. (2.1.3) reduces to

$$\ddot{\phi} + 3H\dot{\phi} = \frac{d}{dt}(a^3\dot{\phi}) = \frac{8\pi(\rho - 3p)}{2\omega+3}\,. \tag{2.1.11}$$

Given an equation of state

$$p = (\gamma - 1)\rho \tag{2.1.12}$$

with γ an assigned constant (generally $0 \le \gamma \le 2$), Eqs. (2.1.5), (2.1.10), (2.1.11) uniquely determine the value of three variables as a function of the cosmic time t, provided enough initial conditions are given. Choosing H, ϕ and ρ as dynamical variables, an initial data set of four variables (for example, the set $(H_0,\, \phi_0,\, \dot{\phi}_0,\, \rho_0)$) must be specified, in addition to the parameters K and ω. In a FLRW model where general relativity is valid, however, it is only necessary to be given the initial values of two quantities (for example, the set $(H_0,\, \rho_0)$)), along with the value of the gravitational constant G and the parameter K, in order to obtain exact solutions.

One of the four integration constants can be eliminated by imposing the additional constraint requiring that $\dot{\phi}a^3$ vanishes at the initial singularity where $a = 0$, namely

$$\lim_{a\to 0} \dot{\phi}a^3 = 0 \qquad \text{or} \qquad \lim_{t\to 0} \dot{\phi}a^3 = 0 \tag{2.1.13}$$

since $a = a(t)$ and $\phi = \phi(t)$. It was though later suggested by Dicke that the solutions relevant for cosmology may be the ones that violate this condition [6].

Moreover, integrating Eq. (2.1.5) when an equation of state (2.1.12) is specified, yields solutions for ρ of the type

$$\rho = \frac{C}{a^{3\gamma}}\,, \tag{2.1.14}$$

where C is an integration constant.

2.2 Time-dependence of the gravitational coupling

One of the most distinctive features of scalar-tensor theories is the variation of the gravitational constant G over time when the field equations are expressed in the Jordan frame. On the other hand, in the Einstein frame the gravitational coupling is strictly constant (see Sect. 1.4).

The effective gravitational coupling of the Brans-Dicke theory, evaluated at present time, as introduced in the field equation (1.2.5), is given by

$$G_0(\phi) = \frac{1}{\phi} \tag{2.2.1}$$

in accordance with (1.2.2). Furthermore, as mentioned in Sect. 1.2.1, it is possible to find an expression for the measured gravitational constant from the Newtonian approximation of the field equations, namely (1.2.18), which is valid in experiments where two mass distributions attract each other. This constant gravitational coupling is however only defined for spherically symmetric solutions, and not for cosmological ones.

The logarithmic derivative of G can be generally expressed in terms of time derivatives of the asymptotic scalar field ϕ that governs the gravitational coupling. In addition to that, the present rate of variation of G can be thought to be of the order of the expansion rate of the universe. Hence one has

$$\left(\frac{\dot{G}}{G}\right)_0 = -\left(\frac{\dot{\phi}}{\phi}\right)_0 \approx \sigma H_0\,, \tag{2.2.2}$$

where H_0 is the Hubble constant evaluated at present time and σ is a dimensionless parameter the value of which is determined by the cosmological model assumed valid. For general relativity, $\sigma = 0$, given that G is predicted to be exactly constant, while for the Brans-Dicke model it is reasonable to assume that the value of σ will be non-zero and dependent on the coupling parameter ω. It can be found that, for the Brans-Dicke theory in a spatially flat FLRW cosmology, $\sigma \approx -(2+\omega)^{-1}$ [14].

The quantity $\dot{G}/G$ is constrained by Solar System experiments, thereby providing an estimate for the lower bound of the parameter ω. The constraints thus obtained are however less stringent than the lower bounds on ω obtained through Solar System tests for the values of the PPN parameters (see Sect. 3.2.1), and therefore not useful in providing a lower bound for the value of ω at present time. A discussion on the measured rate of variation of the gravitational constant, using different methods, is provided in Sect. 3.1.4.

The requirement that scalar-tensor theories in the Jordan frame satisfy the experimental constraints on the rate of variation of the effective gravitational coupling descends from the interpretation of the scalar field ϕ as the inverse effective gravitational coupling, which is in turn based on the form in which the field equation (1.2.5) is written. It is although possible to rewrite the field equation in other forms, such that the gravitational coupling is strictly constant (see Sect. 1.4). One could therefore doubt whether it is valid to use the time variation of the effective gravitational coupling as a constraint for scalar-tensor theories, although the study of this question is beyond the scope of the present work.

2.3 Exact solutions

Studying the exact solutions of the Brans-Dicke cosmology is important in order to gain insight into the predictions of the theory and provide scenarios of the unperturbed universe. Finding such exact solutions only requires imposing a limited number of parameters as initial conditions (see Sect. 2.1); for this reason, many exact solutions can be found in the literature. Since the cosmology equations presented in Sect. 2.1 take a simpler form when a flat FLRW universe is studied, more exact solutions are found in the literature corresponding to $K = 0$ than corresponding to $K \pm 1$.

Many works on exact solutions of the Brans-Dicke cosmology in a FLRW universe assume the validity of the constraint (2.1.13), which allows to eliminate one of the four integration constants necessary to solve Eqs. (2.1.5), (2.1.10), (2.1.11), albeit with loss of generality of the solutions. Even with the imposition of this constraint, the phase space of FLRW cosmology in the Brans-Dicke model (and also in generalized scalar-tensor theories) still has a larger dimensionality than the corresponding phase space of general relativity, therefore the former has a richer variety of solutions than the latter.

Although many exact solutions are known, we will limit the present discussion to some of those of the power-law type, meaning

$$a(t) \propto t^q \,, \qquad \phi(t) \propto t^s \,, \qquad \text{with } 3q + s \geq 1 \,, \tag{2.3.1}$$

which, in the Brans-Dicke theory play a role analogous to that of the inflationary de Sitter attractor in general relativity, and are therefore employed in the study of scalar-tensor inflationary models.

Below are the most well known solutions of the Brans-Dicke cosmology for $K = 0$.

O'Hanlon-Tupper solution

This solution [13] corresponds to vacuum with $\omega > -3/2$, $\omega \neq 0, -4/3$. The scale factor and the scalar field are given by

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{q_\pm} \,, \qquad \phi(t) = \phi_0 \left(\frac{t}{t_0}\right)^{s_\pm} \,, \tag{2.3.2}$$

with

$$\begin{aligned} q_\pm &= \frac{\omega}{3(\omega+1) \mp \sqrt{3(2\omega+3)}} = \frac{1}{3\omega+4}\left(\omega + 1 \pm \sqrt{\frac{2\omega+3}{3}}\right) \,, \\ s_\pm &= \frac{1 \mp \sqrt{3(2\omega+3)}}{3\omega+4} \,, \quad \text{such that } 3q + s = 1 \,. \end{aligned} \tag{2.3.3}$$

The O'Hanlon-Tupper solution has a singularity at $t \to 0$, but does not correspond to the initial condition (2.1.13). Its limit as $\omega \to \infty$ yields $a(t) \propto t^{1/3}$, $\phi =$ constant, which does not reproduce the corresponding general relativistic solution (which is Minkowski space). By taking the values q_+ and s_- as exponents in Eq. (2.3.2), in the limit $\omega = -4/3$, the O'Hanlon-Tupper solution approaches the de Sitter space $a(t) = a_0 e^{Ht}$, $\phi(t) = \phi_0 e^{-3Ht}$, where H is the Hubble constant.

Brans-Dicke dust solution

This solution [3] corresponds to a pressureless dust fluid, namely $\gamma = 1$ in Eq. (2.1.12), and to a matter-dominated universe with $\omega \neq -4/3$. $a(t)$ and $\phi(t)$ are given by

$$a(t) = a_0 t^q \,, \qquad \phi(t) = \phi_0 t^s \,, \tag{2.3.4}$$

with

$$q = \frac{2(\omega+1)}{3\omega+4} \,, \quad s = \frac{2}{3\omega+4} \,, \quad \text{such that } 3q + s = 2 \,, \tag{2.3.5}$$

while

$$\rho = \frac{C}{a^{3\gamma}} = \rho_0 t^r \,, \quad r = -3q = \frac{-6(\omega+1)}{3\omega+4} \,, \quad \rho_0 = \frac{C}{a_0^3} \,. \tag{2.3.6}$$

Nariai solution

This [11] is a power-law solution for a flat FLRW universe with $\omega \neq -4/3\gamma(2-\gamma) < 0$, given an equation of state of the type (2.1.12). $a(t)$ and $\phi(t)$ are given by

$$a(t) = a_0(1+\delta t)^q \ , \qquad \phi(t) = \phi_0(1+\delta t)^s \, , \tag{2.3.7}$$

with

$$q = \frac{2(\omega(2-\gamma)+1)}{3\omega\gamma(2-\gamma)+4} \ , \quad s = \frac{2(4-3\gamma)}{3\omega\gamma(2-\gamma)+4} \ , \quad \text{such that } \ 3q\gamma + s = 2 \, , \tag{2.3.8}$$

while, for the matter density, one has

$$\rho(t) = \frac{C}{a^{3\gamma}} = \rho_0(1+\delta t)^r \ , \quad r = -3\gamma q \, . \tag{2.3.9}$$

In the special case of a dust fluid ($\gamma = 1$), the Nariai solution reproduces the Brans-Dicke dust solution.

A special case of the Nariai solution that is particularly relevant for cosmology is given by

$$a(t) \propto t^{\omega+1/2} \ , \qquad \phi(t) \propto t^2 \tag{2.3.10}$$

which corresponds to a universe dominated by a cosmological constant. This solution is inflationary when $\omega > 1/2$, meaning it corresponds to a scenario where the universe undergoes a spontaneous expansion, which in this case is of the power-law type.

2.4 Gravitational waves

Gravitational waves represent an important aspect of theories of gravity. The main aspects that characterize them and are most relevant to the present discussion are their speed and polarization.

Unlike the speed of electromagnetic radiation, which the Einstein equivalence principle demands to be equal in all local freely falling reference frames, the speed of gravitational waves is determined by the structure of the field equations of each metric theory of gravity. According to general relativity, in the limit of weak, short-wavelength gravitational waves, the waves propagate along null geodesics of the background spacetime, that is to say they propagate at the speed of light. By weak, short-wavelength waves, it is meant that the dimensionless amplitude that characterizes the waves is small compared to the metric of the background spacetime where the waves propagate, namely $|h_{ij}|/|g^B_{ij}| \ll 1$, and that the wavelength λ is small compared to the typical curvature radius $\mathcal{R}$ of the background spacetime, meaning $|\lambda/\mathcal{R}| \ll 1$. In other theories, the speed could be different from that of light because of the coupling of the gravitational waves to other background gravitational fields; in the Brans-Dicke model (and consequently in generalized scalar-tensor theories) it is necessary to evaluate the influence of the field ϕ on the speed of gravitational waves.

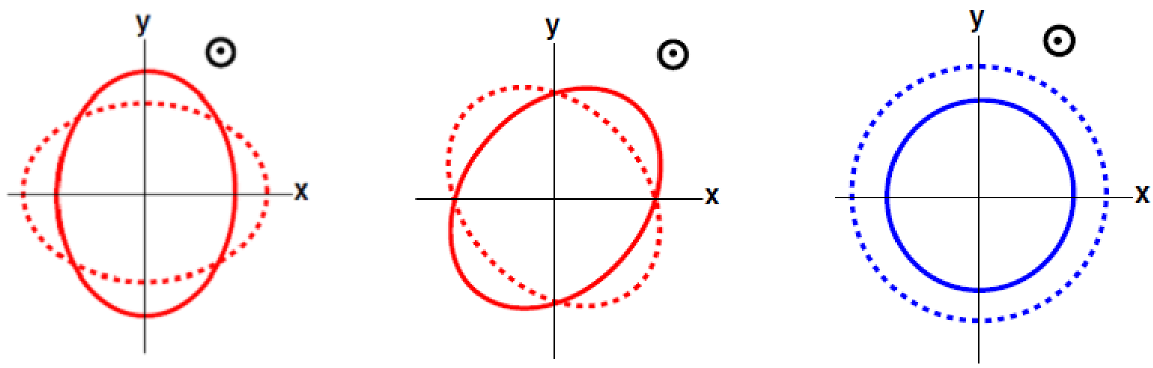
In scalar-tensor theories, gravitational waves have a component deriving from the field equation and a component deriving from the scalar field. It can be proved that the waves of the tensor field are always null, and therefore propagate at the speed of light. On the other hand, the scalar field waves are null only if no potentials $V(\phi)$ are present; in this case, the scalar field propagates like a massive field with a speed less than that of light. The tensor and scalar waves do not necessarily have the same wave vector.

Like any other type of wave, gravitational waves can be polarized. Different theories of gravity allow for different types of polarization of gravitational waves.

By examining their linearized vacuum field equations, it can be found that metric theories of gravity allow up to six modes of polarization:

1. two scalar modes, of helicity 0;
2. two vector modes, of helicity ± 1;
3. two tensor modes, of helicity ± 2.

General relativity only allows for the presence of the two tensor modes, while massless scalar-tensor theories (which have $V(\phi) = 0$) also allow for the presence of one of the two scalar modes; all three modes are transverse to the direction in which the wave propagates, while no longitudinal polarization modes are allowed by the theories in question. Figure 2.1 shows a graphic representation of the transverse polarization modes of gravitational waves.



(a) (b) (c)

Figure 2.1: The three polarization modes of gravitational waves transverse to the direction of their propagation. Shown here is the displacement that each mode produces on a ring of test particles. General relativity allows for the presence of (a) and (b), while massless scalar-tensor theories also allow (c).

Polarization modes influence the behavior of gravitational waves under a Lorentz transformation, particularly whether or not their amplitude changes during their propagation and how. Although in other metric theories of gravity it is possible to have amplitude changes following a Lorentz transformation, this effect is not predicted by general relativity or massless scalar-tensor theories.

Experimental

2.5 Cosmological equations in generalized scalar-tensor theories

As previously noted in Sect. 1.5, generalized scalar-tensor theories find valid applications in cosmology. It is possible to find the cosmology equations for generalized scalar-tensor theories in a FLRW universe by following the procedure described in Sect. 2.1. This way, it is possible to obtain the counterparts of Eqs. (2.1.5), (2.1.10), (2.1.11) for generalized scalar-tensor theories.

The time-time component (2.1.7) yields

$$H^2 = \frac{8\pi}{3\phi}\rho - H\left(\frac{\dot{\phi}}{\phi}\right) + \frac{\omega}{6}\left(\frac{\dot{\phi}}{\phi}\right)^2 - \frac{K}{a^2} + \frac{V}{6\phi}\,, \tag{2.5.1}$$

and the equation for the scalar field (2.1.3) becomes

$$\ddot{\phi} + \left(3H + \frac{\dot{\omega}}{2\omega+3}\right)\dot{\phi} = \frac{1}{2\omega+3}\left(8\pi(\rho-3p) - \phi\frac{dV}{d\phi} + 2V\right)\,. \tag{2.5.2}$$

Employing the expression for the scalar curvature (2.1.8) and substituting Eqs. (2.5.1), (2.5.2) into the trace equation (1.5.3) yields

$$6\left(\dot{H} + 2H^2 + \frac{K}{a^2}\right) = \frac{8\pi}{\phi}(\rho-3p) - \omega\left(\frac{\dot{\phi}}{\phi}\right)^2 + 3\frac{\Box\phi}{\phi} + \frac{2V}{\phi}\,. \tag{2.5.3}$$

One then obtains the analogue of Eq. (2.1.10), namely

$$\begin{aligned}\dot{H} = &-\frac{8\pi}{(2\omega+3)\phi}\left((\omega+2)\rho + \omega p\right) - \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{K}{a^2} + 2H\frac{\dot{\phi}}{\phi} \\ &+ \frac{1}{2(2\omega+3)\phi}\left(\phi\frac{dV}{d\phi} - 2V + \frac{d\omega}{d\phi}(\dot{\phi})^2\right)\,.\end{aligned} \tag{2.5.4}$$

It is implicitly assumed that $\omega = \omega(\phi)$, $\dot{\omega} = \dot{\phi}d\omega/d\phi$, and $V = V(\phi)$. Only two out of the set of Eqs. (2.5.2), (2.5.3), (2.5.4) are independent of each other.

The dependence of the coupling function ω from the scalar field ϕ must be taken into account when searching for exact solutions for the cosmology of generalized scalar-tensor theories. For this reason, finding such exact solutions often requires employing specialised techniques. Although several exact solutions of generalized scalar-tensor cosmology have been found, it is beyond the scope of this work to discuss them.

From a cosmological point of view, introducing a time-dependent coupling parameter $\omega(\phi(t))$, rather than a constant one like in the original Brans-Dicke theory, opens the door to many interesting possible scenarios. In fact, without the restriction of a constant coupling parameter, it is possible to theorise that ω would be of the order of unity in the early universe, and then tend later on to larger values so that the current experimental constraint are satisfied. This possibility would therefore allow for significant deviations from general relativity in the early universe, while still taking into account the present validation of the predictions of general relativity. This possibility is further supported by the fact that the field equations in many scalar-tensor theories can be found to contain an attractor mechanism towards general relativity [5], therefore allowing for an evolution of the value of ω towards infinity as the universe evolves.

Generalized scalar-tensor theories therefore find an interesting application in the study of scenarios where general relativity fails to produce adequate predictions, such as cosmological inflation and the present accelerated expansion of the universe.

Hyperextended inflation is an inflationary scenario based on scalar-tensor gravity. This type of scenario is introduced in order to circumvent the problems that arise in the context of the extended inflationary scenario. It can indeed be found that extended inflation based on the Nariai solution (see Sect. 2.3) places a limit on the value of ω that is obviously incompatible with the current lower bounds provided in Sect. 3.2.1. Since in generalized scalar-tensor theories $\omega = \omega(\phi)$ rather than a constant, it can vary with time without the need to introduce mass terms or other potentials, and allowing for a significant variation of the value of $\omega(\phi)$ in a short time compared to the age of the universe.

Scalar-tensor models of gravitation also find a meaningful application as a possible explanation for the present accelerated expansion of the universe. Various models of quintessence in fact make use of a variable scalar field, much like the one in scalar-tensor gravity, in order to explain the presently observed expansion rate of the universe, rather than a fixed cosmological constant like in the ΛCDM model.

Chapter 3

Tests

Chapters 1 and 2 mentioned several ways to test the predictions of the models discussed, both Brans-Dicke and generalized scalar-tensor theories. In this Chapter we provide a review of the main experimental results necessary to discuss the validity of these theories, namely the tests for the different formulations of the equivalence principle and the interpretation of the classical tests of general relativity in terms of the PPN parameters in the special case of the Brans-Dicke theory. In addition, the implications of the observation of gravitational waves in terms of scalar-tensor gravity are discussed.

3.1 Tests of the equivalence principle

Testing the principle of equivalence in its different formulations (see Sect. 1.1.2), is crucial when analysing the theoretical foundations on which a theory of gravitation is built. The Brans-Dicke theory (and, consequently, generalized scalar-tensor theories) assumes the validity of the weak and Einstein equivalence principle, while is found to be incompatible with the strong equivalence principle.

3.1.1 Weak equivalence principle

The weak equivalence principle states the equality between inertial and passive gravitational mass for all bodies. If violated, it would be possible to detect a difference between them, the entity of which would be given by the interaction a body is subjected to.

The Eötvös ratio is a measurement of the fractional difference in acceleration between two bodies, caused by the difference between the inertial mass m_I and the passive gravitational mass m_P for each body involved. It is given by

$$\eta = 2\frac{|a_1 - a_2|}{|a_1 + a_2|} \quad = \sum_A \eta^A \left(\frac{E_1^A}{m_1} - \frac{E_2^A}{m_2} \right) \quad \text{such that} \quad m_P = m_I + \sum_A \eta^A E^A \,, \qquad (3.1.1)$$

where m_1 and m_2 are inertial masses, E_1^A and E_2^A are the internal energies of the two bodies generated by interaction A, and η^A is a dimensionless parameter measuring the strength of the violation of the weak equivalence principle induced by that interaction. Experimental limits on η place therefore limits on the parameters η^A indicating (possible) violations of the weak equivalence principle. The main experimental procedure employed to measure the value of η is the Eötvös experiment, which uses a torsion balance to evaluate the difference between inertial mass and passive gravitational mass.

Presently, the weak equivalence principle has been measured to hold true within a precision of 10^{-15} by the MICROSCOPE mission.

3.1.2 Local Lorentz invariance

Local Lorentz invariance is heavily supported by experiment, as all of the experimental verifications of special relativity can be regarded as tests of this principle, including the Michelson-Morley experiment, time-dilation tests, and tests on the isotropy and independence of the velocity of the source of the speed of light. More recently, the results of the observation of gravitational waves can be regarded as experimental support of Lorentz invariance.

3.1.3 Local position invariance

Local position invariance refers to position in both time and space. The measurement of gravitational redshift can be used as a test for local position invariance, in addition to being a classical test of general relativity (see Sect. 3.2.3). Gravitational redshift is the wavelength shift between two identical frequency standards at rest at different height in a gravitational field. When testing the validity of this principle, for a static gravitational field characterized by a Newtonian gravitational potential U, the redshift can be written as

$$z = (1+\alpha)\frac{\Delta U}{c^2}\,, \tag{3.1.2}$$

where α may depend upon the nature of the clock whose shift is measured. If local position invariance is valid, $\alpha = 0$.

The first experiment to prove the existence of the gravitational redshift with high accuracy (one percent) was the Pound-Rebka experiment. More recent tests yielded an upper limit for the value of α of the order of magnitude of 10^{-5}, further confirming the validity of this principle.

Local position invariance also refers to position in time, which can be tested by studying the constancy over time of the fundamental constants of non-gravitational physics. The upper bounds on the rate of variation of these constants, of order of magnitudes ranging from 10^-16 (for the fine structure constant) to 10^{-11} (for the weak interaction constant), support the validity of this principle.

3.1.4 Strong equivalence principle

It was pointed out in Sect. 1.1.2 that the Brans-Dicke theory is predicted to violate the strong equivalence principle, as are perhaps all metric theories of gravity except general relativity. Tests for the violation of the strong equivalence principle are the Nordtvedt effect, the existence of preferred-frame and preferred-location effects, and the time-variation of the value of the gravitational constant.

If the strong equivalence principle is violated, the gravitational self-energy of a massive body is expected to contribute differently to its gravitational mass than to its inertial mass. This violation of the massive-body equivalence principle, known as Nordtvedt effect, in terms of the PPN parameters, is quantified by

$$\eta_N = 4\beta - \gamma - 3 - \frac{10}{3}\xi - \alpha_1 + \frac{2}{3}\alpha_2 - \frac{2}{3}\zeta_1 - \frac{1}{3}\zeta_2 \quad \text{such that} \quad \frac{m_P}{m} = 1 - \eta_N \frac{E_g}{m}\,, \tag{3.1.3}$$

where E_g is the negative of the gravitational self-energy of the body, and $\frac{m_p}{m}$ is the ratio between passive and active gravitational mass. General relativity predicts the absence of this effect and therefore $\eta_N = 0$, while in scalar-tensor theories $\eta_N = (2+\omega)^{-1}$.

The existence of the Nordtvedt effect would not invalidate the results of laboratory Eötvös experiments, since it would only be observable at astronomical scales. So far, Solar System tests have found no evidence of the existence of the Nordtvedt effect, although it is uncertain whether or not the scale and level of precision of these tests can be regarded as a reliable test of the strong equivalence principle.

Preferred-frame effects are quantified by the PPN parameters α_1, α_2, α_3, and some preferred-location effects are related to the value of ξ, all of which are predicted to be equal to zero in the Brans-Dicke theory, as stated in Table 1. These effects would cause anomalous changes to the orbits of celestial bodies in their eccentricity, self-acceleration, and orbital angular momentum precession, changes in the locally measured gravitational constant causing distortions of massive bodies, and precession of the spin axis of massive bodies. All tests searching for evidence of these effects have put tight bounds on the value of the parameters concerned, as listed in Table 3, thus further validating the strong equivalence principle.

The strong equivalence principle also predicts the gravitational coupling to be constant over time, in contrast with the predictions of all scalar-tensor theories. The predicted rate of variation of the gravitational constant in the special case of the Brans-Dicke model was discussed in Sect. 2.2.

The current bounds on the ratio $\dot{G}/G$ are listed in Table 2. The precision of these bounds, which is of the order of 10^{-13} yr^{-1}, is expected to be improved by the Bepi-Colombo mission to Mercury, that was launched in 2018.

Method	$\dot{G}/G$ ($10^{-13}\,\text{yr}^{-1}$)
Mars ephemeris	0.1 ± 1.6
Lunar laser ranging	4 ± 9
Binary and millisecond pulsars	-7 ± 33
Helioseismology	0 ± 16
Big Bang nucleosynthesis	0 ± 4

Table 2: Current upper bounds, determined experimentally, on the rate of variation of the gravitational constant.

In Solar System measurements, bounds on the rate of variation of the gravitational constant can be correctly obtained in a phenomenological manner by replacing G by $G_0 + \dot{G}_0(t - t_0)$ in Newton's equations of motion; this, however, does not hold true for pulsar and binary pulsar timing measurements, since in theories of gravity that are predicted to violate the strong equivalence principle the mass and moment of inertia of a gravitationally bound body may vary with the gravitational constant.

3.2 Tests of the PPN parameters

Testing the values of the ten PPN parameters is crucial when searching for experimental confirmation of any theory of gravitation. Bounds for the values of these parameters can be found through Solar System tests. In Table 3 a list of the current upper bounds on the PPN parameters is provided.

Parameter	Bound	Effect	Experiment/Remarks
$\gamma - 1$	$2.3 \cdot 10^{-5}$	time delay	Cassini tracking
	$2 \cdot 10^{-4}$	light deflection	VLBI
$\beta - 1$	$8 \cdot 10^{-5}$	perihelion shift	with $J_{2\odot} = (2.2 \pm 0.1) \cdot 10^{-7}$
	$2.3 \cdot 10^{-4}$	Nordtvedt effect	assuming $\eta_N = 4\beta - \gamma - 3$
ξ	$4 \cdot 10^{-9}$	spin precession	millisecond pulsars
α_1	10^{-4}	orbital polarization	Lunar laser ranging
	$4 \cdot 10^{-5}$		PSR J1738+0333
α_2	$2 \cdot 10^{-9}$	spin precession	millisecond pulsars
α_3	$4 \cdot 10^{-20}$	pulsar acceleration	pulsar spin-down statistics
ζ_1	$2 \cdot 10^{-2}$	—	combined PPN bounds
ζ_2	$4 \cdot 10^{-5}$	binary-pulsar acceleration	PSR 1913+16
ζ_3	10^{-8}	Newton's third law	lunar acceleration
ζ_4	—	—	not independent

Table 3: Current upper bounds, determined experimentally, for the ten PPN parameters.

Some of the parameter bounds from Table 3 have been discussed in Sect. 3.1 in relation to testing of the strong equivalence principle. The focus of the following subsections is on the so-called classical tests of general relativity, which, in their PPN formulation, can be interpreted as tests for the parameters β and γ.

In particular, the value of the parameter γ predicted by the Brans-Dicke theory is dependent on the value of ω as per Table 1, from which one finds the following expression for ω

$$\omega = \frac{2\gamma - 1}{1 - \gamma} \,. \tag{3.2.1}$$

Therefore, a measure of the upper bound of this PPN parameter corresponds to an indirect measure of the lower bound for ω.

3.2.1 Deflection and delay of light

The value of γ can be tested by measuring the gravitational deflection and delay of light.

It is predicted by general relativity that the trajectory of a photon gets deflected when it travels close to a gravitational source. As stated in Sect. 1.2.1, the deflection of light is determined by the ratio $g_{\alpha\alpha}/g_{00}$ in the Newtonian approximation of the Brans-Dicke field equations (see Eqs. (1.2.17)). The angle of deflection of light passing by a gravitational source of mass M at a distance R is given by

$$\delta\theta = \frac{4G_0 M}{R}\frac{3 + 2\omega}{4 + 2\omega} = \frac{4G_0 M}{R}\frac{1 + \gamma}{2} \tag{3.2.2}$$

and differs by the value predicted by general relativity by $(3 + 2\omega)/(4 + 2\omega)$, or, equivalently, by $(1 + \gamma)/2$, employing Eq. (3.2.1).

The gravitational deflection of light was first observed by Eddington in 1919 with an accuracy of about 30%. With the development of radio interferometry, and subsequently of very-long-baseline radio interferometry (VLBI), it was possible to find the current upper limit for the value of $\gamma - 1$, of the order of magnitude of 10^{-4} (see Table 3).

In addition to being deflected when passing through the gravitational field of a massive body, photons are also subjected to a time delay in their motion. The measurement of this effect, known as Shapiro time delay, can be used as an estimate for the value of the parameter γ. The time delay for a signal going around a massive object is given by

$$\Delta t = \frac{4G_0 M}{R}\left(1 + \frac{1+\gamma}{2}\ln\left(\frac{4r_1 r_2}{R^2}\right)\right) \,. \tag{3.2.3}$$

The current strictest upper bound on the value of $\gamma - 1$, of the order of magnitude of 10^{-5}, comes from the study of the time delay of light (see Table 3) As per Eq. (3.2.1), it yields a lower bound for the coupling constant evaluated at present time $\omega > 4 \cdot 10^4$. The implications of the measurements of this bound have been widely discussed in Sect. 1.5 in regards to the formulation of generalized scalar-tensor theories.

3.2.2 Perihelion precession of Mercury

An explanation for the anomalous perihelion shift of Mercury's orbit was first provided by general relativity. In terms of the PPN parameters, the predicted advance per orbit in the post-Newtonian limit, including a Newtonian contribution resulting from a possible solar quadrupole moment, is

$$\Delta\tilde{\omega} = \frac{6\pi G_0 m}{p}\left(\frac{2+2\gamma-\beta}{3} + \frac{2\alpha_1 - \alpha_2 + \alpha_3 + 2\zeta_2}{6}\eta + \frac{J_2 R^2}{2mp}\right) . \tag{3.2.4}$$

For the Sun-Mercury system, we can take the mass of the two body system approximately equal to the mass of the Sun ($m \approx M$), and its dimensionless reduced mass approximately equal to zero ($\eta \approx 0$). The first term in the bracket is exactly the prediction of general relativity for the Mercury perihelion shift when $\gamma = \beta = 1$ (as per Table 1), the second is a purely post-Newtonian correction, and the third is related to the solar quadrupole moment (an effect produced by the centrifugal flattening caused by the rotation of the Sun around its axis).

In scalar-tensor theories, the only PPN parameters expected to have non-zero value are γ and β, as stated in Table 1. Omitting the solar quadrupole moment term (as it is small compared to the total perihelion shift), Eq. (3.2.4) yields

$$\Delta\tilde{\omega} = \frac{6\pi G_0 M}{p}\frac{2+2\gamma-\beta}{3} = \frac{6\pi G_0 M}{p}\frac{4+3\omega}{6+3\omega} , \tag{3.2.5}$$

where, in the last part of the equation, the predictions $\beta = 1$ and $\gamma = (1+\omega)/(2+\omega)$ were employed. Therefore, in terms of ω, the prediction for the perihelion shift of Mercury differs from the value predicted by general relativity by the factor $(4+3\omega)/(6+3\omega)$.

The current bound for $\beta - 1$ listed in Table 3 was obtained through the study of the orbit of Mercury, adopting the bound on $\gamma - 1$ given by the Shapiro time delay (see Sect. 3.2.1) in order to yield an upper bound for $\beta - 1$.

The upper limits for both β and γ will be tested with higher accuracy by the BepiColombo mission set to orbit Mercury in the near future.

3.2.3 Gravitational redshift

The gravitational redshift experiment, as stated in Sect. 3.1.3, can be considered as experimental proof supporting local position invariance, which scalar-tensor theories assume to be valid.

In fact, the gravitational redshift is determined by the first order approximation of the time-time component of the metric, as given by Eq. (1.2.17a), which also determines the gravitational weight of a body. The factor containing the coupling parameter ω is absorbed into the definition of the (first order approximation of the) gravitational constant, namely Eq. (1.2.18). Thus there is no anomaly in the gravitational redshift.

3.3 Detection of gravitational waves

The possibility to detect gravitational waves opened the door to a new way to test theories of gravity. In 2017 the gravitational wave signal GW170817 was detected, which is also the first gravitational wave observation confirmed by non-gravitational means. Detection of the gravitational wave signal was in fact followed by reception of electromagnetic signals, making it possible to compare the speed of gravity and the speed of light.

The observation of GW170817 yielded a bound on the speed of gravitational waves of $-3 \cdot 10^{-15} \leq c_g/c - 1 \leq 7 \cdot 10^{-16}$ [1], therefore confirming with a high degree of accuracy that gravitational waves propagate at the speed of light, as predicted by general relativity and some alternative theories of gravity (including scalar-tensor theories, as explained in Sect. 2.4).

The measurement of the speed of gravitational waves ruled out several alternatives to general relativity, namely the ones predicting that gravity propagates at a speed different from c.

A study of the polarization of GW170817 was also carried out, finding evidence in favour of the presence of the tensor polarization modes predicted by general relativity, in comparison to vector and scalar modes.

Conclusions

The Brans-Dicke theory of gravitation, while providing an interesting theoretical framework, does not seem to be a valid alternative to general relativity in the description of the present universe. General relativity appears to withstand all experimental testing so far, and, since it is impossible to rule out a Brans-Dicke model with a large enough coupling constant, one would still prefer the more simple option as a "standard model" for gravitation. Ultimately, the extensive body of experimental evidence supporting general relativity, combined with the constraints on the value of the coupling constant ω, of several orders of magnitude higher than the theory would suggest, lead to abandoning this theory in favour of general relativity in the description of the present universe.

Although the original formulation of the Brans-Dicke theory does not appear to provide a meaningful framework for gravitation, the generalized scalar-tensor theories expanding on its core concepts can be regarded as a valid alternative to general relativity, especially in relation to models of cosmological inflation and quintessence. In fact, introducing the possible dependence of the parameter ω from the scalar field ϕ, and consequently from the cosmic time, allows for a wide variety of behaviors in the cosmological past and future that may differ greatly from general relativity, without invalidating the extensive body of evidence supporting it.

Moreover, despite general relativity having held up under extensive testing over time, performing gravitational physics tests is still important for more than one reason: firstly, because gravity is a fundamental force, and the only one that has not been unified with the others at that; and also, because the predictions of general relativity are fixed, therefore every test of the theory could potentially disprove it or find new physics.

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