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Quantum Riemannian Geometry on Graphs for Gauge Field Theory

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Abstract

In questo lavoro estendiamo concetti classici della geometria Riemanniana al fine di risolvere le equazioni di Maxwell sul gruppo delle permutazioni S_3.

Cominciamo dando la strutture algebriche di base e la definizione di calcolo differenziale quantico con le principali proprietà. Generalizziamo poi concetti della geometria Riemanniana, quali la metrica e l'algebra esterna, al caso quantico. Tutto ciò viene poi applicato ai grafi dando la forma esplicita del calcolo differenziale quantico su $\mathbb{K}(V)$, della metrica e Laplaciano del secondo ordine e infine dell'algebra esterna.

A questo punto, riscriviamo le equazioni di Maxwell in forma geometrica compatta usando gli operatori e concetti della geometria differenziale su varietà che abbiamo generalizzato in precedenza. In questo modo, considerando l'elettromagnetismo come teoria di gauge, possiamo risolvere le equazioni di Maxwell su gruppi finiti oltre che su varietà differenziabili. In particolare, noi le risolviamo su S_3.

Contents

Introduction

The Maxwell's theory of Electromagnetism played a pivotal role in Physics. In fact, it is the first unifying theory, treating electrical and magnetic phenomena at once. The very first step towards the theory of General Relativity came by realising that Maxwell's equation are not invariant under Galilei's transformation: the Lorentz's transformations were introduced and then the Special Relativity followed. This gave the foundations for a more complete theory of gravity. The geometrical framework underlying such a theory was different from the three Euclidean dimensions. Time was not just a parameter anymore, but rather a real dimension. Einstein published the theory of General Relativity in 1915 using differential geometry. What is special about this formalism is that it can be generalised to be applied to discrete frameworks. This allows us to rewrite Maxwell's theory on graphs and develop a theory of gravity on a discrete setup.

In this work, we aim to solve Maxwell's equations on graphs. To do so, we follow three main steps. First, we begin by giving the fundamental algebraic structures on which we develop the discussion.

Second, we rewrite differential geometry in a discrete framework, we call it a quantum geometry. Starting from the generalisation of the concept of differential calculus, we then extend the metric, the connection and the second-order differential operator to the quantum framework. In addition, we discuss how the exterior algebra changes when applied to the new setup. Finally, we write these objects and concepts on graphs, giving their explicit form too.

In conclusion, we solve Maxwell's theory on the permutation group S_3, as a concrete example of graph. To do so, we first rewrite Maxwell's equations in the formalism of differential geometry. We then note that seeing electromagnetism as a gauge theory is much more immediate in this notation, and that it is a valuable method to solve Maxwell's equations. In conclusion, after recalling the results of the previous chapters, we solve those very equations on a S_3 by finding the gauge potential.

Chapter 1

Algebraic structures

In this first chapter we introduce the mathematical language required for our further discussion. The starting point is the concept of algebra, which is the foundation for the rest of this work. We will then discuss modules, ideals, groups and actions. For more details we invite the reader to consult [6], [8].

1.1 Algebras

In this section we introduce the most important structure: the algebra. An algebra is a vector space together with an additional operator, the multiplication, with conditions compatible with the vector space structure. In this work, whenever we refer to a field $\mathbb{K}$ we imply that $\mathbb{K}$ is either the field of real numbers $\mathbb{R}$ or of complex numbers $\mathbb{C}$. For more details see [6] chapter 4 section 7.

Definition 1.1.1. Let $\mathbb{K}$ be a field. An *algebra* A over $\mathbb{K}$ is a vector space over $\mathbb{K}$ equipped with a product operator $\bullet$ defined as follows

$$\begin{aligned} \bullet: \ A \times A &\longrightarrow \ A \\ a, b \ &\longrightarrow \ a \cdot b \qquad \forall\, a, b \in A \end{aligned}$$

such that

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (a + b) \cdot c = a \cdot c + b \cdot c, \quad \forall\, a, b, c \in A.$$

From now on, we assume the existence of the multiplicative identity 1_A that is an element of the vector space A which does not affect the algebra product, i.e.

$$1_A \cdot a = a = a \cdot 1_A, \quad \forall\, a \ \in \ A$$

Here are some examples of algebras to better understand the concept. This is the starting point on which we will build further structures.

Example 1.1.2. 1. *The algebra of the matrices.* Consider the vector space of $n \times n$ matrices with entries in the field $\mathbb{K}$, i.e.

$$M_{n,n}(\mathbb{K}) = \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,n} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n,1} & m_{n,2} & \cdots & m_{n,n} \end{pmatrix}$$

where the $m_{i,j} \in \mathbb{K}$. The algebra product is the usual rows by columns product, defined by

$$\begin{aligned} \bullet : M_{n,n}(\mathbb{K}) \times M_{n,n}(\mathbb{K}) &\longrightarrow M_{n,n}(\mathbb{K}) \\ M_1, M_2 &\longrightarrow M_1 \cdot M_2 \qquad \forall\ M_1, M_2 \in M_{n,n}(\mathbb{K}). \end{aligned}$$

The vector space $M_{n,n}(\mathbb{K})$ of the square matrices over the field $\mathbb{K}$, together with the product rows by columns described above, forms an algebra called the *algebra of the matrices.*

In this algebra, we define the multiplicative identity as the matrix identity $\mathbb{I}$, i.e.

$$\mathbb{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

2. *The polynomial algebra.* Given the field $\mathbb{K}$, consider the vector space $A = \mathbb{K}[X]$ of polynomials of the form

$$p = p_0 + p_1 X + p_2 X^2 + \cdots + p_n X^n + \cdots$$

where $p_i \in \mathbb{K}$. The product that we define in this case is the usual product of polynomials, i.e.

$$\begin{aligned} \bullet : \ A \times A &\longrightarrow \ A \\ a, b &\longrightarrow \ a \cdot b \qquad \forall\ a, b \in A \end{aligned}$$

such that, given

$$a = a_0 + a_1X + a_2X^2 + \cdots + a_nX^n + \cdots$$
$$b = b_0 + b_1X + b_2X^2 + \cdots + b_nX^n + \cdots$$

we have

$$a \cdot b = a_0 \times b_0 + (a_1 \times b_0 + a_0 \times b_1)X + (a_0 \times b_2 + a_1 \times b_1 + a_2 \times b_0)X^2 + \cdots$$

The infinite dimensional vector space $A = \mathbb{K}[X]$, equipped with the product defined above, forms an algebra called the *polynomial algebra*.

In this algebra, we define the multiplicative identity as the degree zero polynomials with coefficient 1, i.e.

$$1_A = 1$$

In general, the product is not necessarily commutative. Nevertheless, it is useful to analyse the commutative case to first familiarise with the concept of algebra. For this reasons, we provide definition and examples for the commutative case.

Definition 1.1.3. An algebra A is called *commutative* if the product operator is commutative, i.e.

$$a \cdot b = b \cdot a$$

for all $a, b \in A$.

Example 1.1.4. We analyse the commutativity condition given above in the two cases in example 1.1.2.

1. *The algebra of the matrices.* The product rows by columns that we defined is clearly not commutative.

2. *The polynomial algebra.* The polynomial algebra is a commutative algebra. In fact, by looking at the equation

$$a \cdot b = a_0 \times b_0 + (a_1 \times b_0 + a_0 \times b_1)X + (a_0 \times b_2 + a_1 \times b_1 + a_2 \times b_0)X^2 + \cdots$$

   we see that each of the coefficients of the polynomial involves commutative operations. In fact, the multiplication is the usual multiplication between scalars, hence commutative. For this reason, the product $a \cdot b$ is commutative, as well as the polynomial algebra.

1.2 Subalgebras and homomorphisms

In this section, we define the notion of subalgebra and homomorphism between algebras.

Definition 1.2.1. A vector subspace S of an algebra A is called *subalgebra* if it is itself an algebra equipped with the same product as A.

This definition is intuitive and highlights that in some cases we can extract - from the algebra - a vector subspace with the same properties as the algebra. Here are some examples based on those in the previous section.

Example 1.2.2.

1. *The algebra of the matrices.* Consider the algebra of the square matrices, there are a lot of examples of subalgebras with the same rows by columns product. Some of them are
   (a) Diagonal matrices;
   (b) Upper (or lower) triangular matrices.
2. *The algebra of polynomials.* We give some examples of subalgebras in the algebra of polynomials with the usual product.
   (a) Take S as the degree zero polynomials. In this case, S coincides exactly with $\mathbb{K}$;
   (b) $S = \{\textit{even degree polynomials}\}$. Polynomials of even degree are a vector subspace of $\mathbb{K}[X]$ and the product as defined before preserves the property of the powers being even, therefore this is a subalgebra of the polynomial algebra.

Now, we define a morphism between algebras that preserve the algebraic structure.

Definition 1.2.3. Let A and B be algebras over the same field $\mathbb{K}$. A function $F : A \longrightarrow B$ is called *homomorphism* between A and B if it has the following properties:

1. $F(x +_A y) = F(x) +_B F(y)$;
2. $F(kx) = kF(x), \quad \forall k \in \mathbb{K}$;
3. $F(x \cdot_A y) = F(x) \cdot_B F(y)$

for all $x, y \in A$.

In the definition above, the subscripts $_A$ and $_B$ have been assigned to the sum and product operators to highlight that they are either in the algebra A or B, and thus different in general. The first two properties define F as a $\mathbb{K}$-linear map, while the third provides a relation between the product in A and the product in B. For more details see [5], [6], [8].

Although in this chapter we decided to use the notion of algebra as a starting point, most of the resources cited here base their definitions on rings. The two approaches are equivalent and additional details about this equivalence can be found in appendix A.

1.3 Modules

In this section, we give the definitions of module, submodule and homomorphism between modules, together with some examples based on what we previously studied for algebras.

Definition 1.3.1. Let A be an algebra. A (left) A-*module* E is a vector space over the same field $\mathbb{K}$ equipped with a function

$$\begin{aligned} A \times E &\longrightarrow E \\ a, e &\longrightarrow a \cdot e \end{aligned}$$

such that:

1. $a \cdot (e_1 + e_2) = a \cdot e_1 + a \cdot e_2, \quad e_i \in E;$
2. $(a + b) \cdot e = a \cdot e + b \cdot e;$
3. $a(b \cdot e) = (ab) \cdot e;$

for all $a, b \in A$ and $e \in E$.

The definition above can easily be changed to give the definition of a right A-module. In that case, the elements of the algebra A are multiplied to the elements of the module E from the right. Despite this similarity, the two types are intrinsically different.

We note that the points from (1) to (3) equip the module with the distributive and associative properties with respect to the multiplication by an element of the algebra. In a way, this shows that the module is not only a vector space with the multiplication by a scalar, but also with the multiplication by an element of the algebra. With this interpretation in mind, one needs to consider the necessary precautions regarding the side from which we multiply. For more details about modules see [6] chapter 4.

Recall there is the multiplicative identity (1_A) in the algebras we will work with. Let E be a A-module and $1_A \in A$. We call E a *unitary, or unital* module if multiplying 1_A by an element of the module does not affect the element, i.e.

$$1_A \cdot e = e$$
$$e \cdot 1_A = e$$

for all $e \in E$ (left or right module).

Observation 1.3.2. In general, algebras are not necessarily commutative. However, when this is the case, a module acquire some interesting further properties. Let A be a commutative algebra, i.e. $ab = ba$ for all $a, b \in A$. Therefore a left A-module E has the following additional property

$$a(b \cdot e) = (ab) \cdot e = (ba) \cdot e = b(a \cdot e).$$

We can then naturally define a right A-module using the left A-module, i.e.

$$a \cdot e := e \cdot a$$

for all $a \in A$ and $e \in E$. This definition and the properties of the left A-module are enough to show that the right A-module fulfils the requirements in definition 1.3.1. We provide a proof for the property (3) only. We can write

$$(e \cdot a)b = (a \cdot e)b = b(a \cdot e) = (ba) \cdot e = (ab) \cdot e = e \cdot (ab)$$

for all $a, b \in A$ and $e \in E$. One could also start from a right A-module and define the left with an analogous reasoning. However, this is not the only way in which we can define the missing module. For further details we refer the reader to [1] chapter 2 and [6] chapter 5.

Here are some examples of module defined on the algebras of the previous section's examples.

Example 1.3.3. 1. *The algebra of matrices.*

a) *Left module.* Let $A = M_{n,n}(\mathbb{K})$ be the algebra. We define a left A-module as the vector space V of the column vectors of length n with the usual product rows by columns.

$$\begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,n} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n,1} & m_{n,2} & \cdots & m_{n,n} \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v'_1 \\ v'_2 \\ \vdots \\ v'_n \end{pmatrix}$$

where $m_{i,j}, v_i, v'_i \in \mathbb{K}$. The explicit values of v'_i are

$$v'_i = \sum_{j=1}^{n} m_{ij}v_j = m_{ij}v_j$$

b) *Right module* Let $A = M_{n,n}(\mathbb{K})$ be the algebra. We define the right A-module as the vector space U of the row vectors of length n with the usual product rows by columns.

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix} \times \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,n} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n,1} & m_{n,2} & \cdots & m_{n,n} \end{pmatrix} = \begin{pmatrix} u'_1 & u'_2 & \cdots & u'_n \end{pmatrix}$$

where $m_{i,j}, u_i, u'_i \in \mathbb{K}$. Again, the explicit values of u'_i are

$$u'_i = \sum_{j=1}^{n} u_j m_{ji} = u_j m_{ji}$$

2. *The algebra of polynomials.* Let $A = \mathbb{K}[X]$ be the algebra, thus commutative. For this reason, we can naturally define the right module from the left module. Let us take the example of E_n as the polynomials of minimum grade n with the usual multiplication. The product of a generic polynomial by an element of E_n gives another element of E_n. Therefore, E_n is an A-module with the usual product in the algebra of polynomials.

We now define a case in which the module has both a right and a left action of the algebra. We are using the word 'action' to refer to the multiplication of the algebra to the module from one side. At the end of this chapter we briefly explain why the two words are interchangeable. From now on, we will use 'action' for the sake of simplicity, but remember that the concept of action is much more general.

Definition 1.3.4. Let A be an algebra. An A-*bimodule* E over the algebra is a module on which A has a well-defined action both from the left and from the right, i.e.

$$(a \cdot e) \cdot b = a \cdot (e \cdot b)$$

for all $a, b \in A$ and $e \in E$.

From this definition, it is easy to see that every algebra is a bimodule over itself.

Another rather important type of a module is the free-module. It is a module that admits a basis. Along the line of [6], the definition of a free module is given by a theorem, proven on [6], page 181. Before, we give the definition of cyclic module, an object involved in the theorem.

Definition 1.3.5. Let A be an algebra. Let C be a left (right) A-module. C is called *cyclic module* if it is generated by one element.

We now have all the necessary ingredients to state the theorem.

Theorem 1.3.6. *Let A be an algebra. The following conditions on a unitary A-module F are equivalent:*

1. *F has nonempty basis;*
2. *F is the internal direct sum of a family of cyclic A-modules, each of which is isomorphic as a left A-module to A;*
3. *F is A-module isomorphic to a direct sum of copies of the left A-module A;*
4. *there exists a nonempty set X and a function $\iota : X \longrightarrow F$ with the following property: given any unitary A-module E and a function $f : X \longrightarrow E$, there exists a unique A-module homomorphism $\bar{f} : F \longrightarrow E$ such that $\bar{f}\iota = f$.*

Definition 1.3.7. Let A be an algebra. An A-module E is a *free A-module* if it satisfies the equivalent conditions of theorem 1.2.4.

Observation 1.3.8. We can define two different dimensions for a module. Let E be a free module over an algebra A. The dimension of E as a module over A is given by the number of elements in the basis defined in the theorem above. On the other hand, we can consider E as a vector space over a field $\mathbb{K}$, same as A. In this case, the dimension of E is intrinsically different and depends solely on the type of the elements of E. To distinguish between the two, we refer to the first as

$$dim_A E$$

and to the second as

$$dim_{\mathbb{K}} E.$$

An example of this duplicity can be found at the end of the next chapter.

Now we define the submodule, the homomorphism between modules and finally the concept of ideal.

1.4 Submodules, homomorphisms and ideals

In this part, we provide the definition of submodule together with some examples following the examples we gave for the module. In addition, we give the definition of homomorphism between modules, and of the ideal.

Definition 1.4.1. Let A be an algebra and E be a left (right) A-module. A *submodule* S of E is a vector subspace of E such that

$$a \cdot s \in S$$

for all $a \in A$ and $s \in S$.

Here we give some examples for the submodule based on those we gave for the module, example 1.3.3.

Example 1.4.2. 1. *The algebra of matrices.* The null vector

$$\underline{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \end{pmatrix}$$

is a trivial example of submodule.

2. *The algebra* $\mathbb{K}[X]$ *of polynomials.* Let $\mathbb{K}_2[X]$ be the vector space of polynomials of minimum grade 2 and bimodule of $\mathbb{K}[X]$. All the vector subspaces $\mathbb{K}_j[X]$ with $j \in \{3, 4, \cdots\}$ are submodules of $\mathbb{K}_2[X]$.

We now provide the notion of homomorphism between modules. This is a morphism that preserves the module structure.

Definition 1.4.3. Let A be an algebra and E and E' be (left) A-modules over A. A function $F : E \longrightarrow E'$ is an *A-module homomorphism* if

$$F(e_1 + e_2) = F(e_1) + F(e_2)$$

and

$$F(a \cdot e_1) = a \cdot F(e_1)$$

for all $e_1, e_2 \in E$ and $a \in A$

For more details on modules, submodules and homomorphisms between modules see [6] chapter 5. We then introduce a really important object in algebra: the ideal.

Definition 1.4.4. Let A be an algebra. Let I be a vector subspace of A. I is called *left ideal* if

$$a \cdot x \in I$$

for all $a \in A, x \in I$.

An analogous definition can be given for right ideals and for biideals as well.

Although this definition looks similar to the one of submodule, there is a substantial difference. The submodule is a vector subspace of a module, and in general it can be of a different nature than the algebra. On the other hand, an ideal is a vector subspace of the algebra itself. If one recalls that an algebra is a bimodule over itself, then one can use the algebra as a module in the definition 1.4.1 and it will become equivalent to the definition 1.4.4.

In addition, it is worth highlighting that the concept of ideal is stronger than the one of subalgebra. In fact, the subalgebra requires the product between its elements to be an element of the subalgebra. The ideal adds the requirement that the product of its element with other elements of the algebra (not necessarily in the ideal) is an element of the ideal.

1.5 Groups and actions

In this paragraph, we introduce the notion of group, subgroup and homomorphism between groups. Then we provide a definition for action. We add these topics because group theory and actions are largely used in theoretical physics. Moreover, the definition of graphs is based on the concept of group. In addition, these objects are some of the most important remaining tools in a discussion of algebraic structures. For more details we refer to [6] chapter 1 and 2 and [8] chapter 1.

Definition 1.5.1. Let G be a nonempty set equipped with the following binary operation on G

$$\begin{aligned} \bullet : G \times G &\to G \\ (g_1, g_2) &\to g_1 g_2 \end{aligned}$$

G is a *group* if it satisfies the following properties.

1. The binary operation is associative, i.e.

   $$(g_1 g_2) g_3 = g_1 (g_2 g_3)$$

   for all $g_i \in G$;

2. There exists a (two-sided) identity element $e \in G$ such that

   $$ge = eg = g$$

   for all $g \in G$;

3. There exists a (two-sided) inverse element $g^{-1} \in G$ such that

$$g^{-1}g = gg^{-1} = e$$

for all $g \in G$.

If a nonempty set satisfies only the property (1), we call it a *monoid*. If a monoid satisfies the property (2), we call it a *semigroup*. Those two more general structures are useful to provide some result in this field without losing generality. In addition, we define the commutative group.

Definition 1.5.2. Let G be a group. G is called *commutative* if its binary operation is commutative, i.e.

$$g_1g_2 = g_2g_1$$

for all $g_1, g_2 \in G$.

We then proceed to define the subgroup and the homomorphism between groups.

Definition 1.5.3. Let S be a subset of a group G with the same binary operation. S is a *subgroup* of G if it satisfies the properties in the definition 1.5.1.

As proved in [6] chapter 1 page 24, the identity element in a group is unique. Therefore, given a group G with identity element e, then if S is a subgroup of G, $e \in S$, and vice versa.

Definition 1.5.4. Let G and G' be groups. A *homomorphism* of G into G' is a mapping $F : G \longrightarrow G'$ that satisfies the following properties.

1. It preserves the group binary operation, i.e.

$$F(xy) = F(x)F(y)$$

for all $x, y \in G$;

2. It maps the identity element of G into the identity element of G', i.e.

$$F(e) = e'$$

for $e \in G$ and $e' \in G'$;

3. It maps the inverse of an element of G in the inverse of the image of that element, i.e.

$$F(x^{-1}) = F(x)^{-1}$$

for all $x \in G$.

We now introduce the concept of group action. The concept of a group acting on a set is incredibly useful to analyse algebraic objects. In the scope of our discussion, the concept of action generalises the role of the modules, which we will discuss in the following chapter.

Definition 1.5.5. Let G be a group and A be a set. A *group action* of G on A is a map

$$\begin{aligned} a : G \times A &\longrightarrow A \\ (g, a) &\longrightarrow g \cdot a \end{aligned}$$

for all $g \in G$ and $a \in A$, that satisfies the following properties.

1. $g_1(g_2 \cdot a) = (g_1 g_2) \cdot a$;
2. $e \cdot a = a$;

for all $g_1, g_2 \in G$ and $a \in A$. e is the identity element of the group G.

In this chapter, we saw many algebraic structures as well as some recurrent patterns. Let us consider the definition of module (1.3.1) and the definition of action (1.5.5). In the first case, we have a vector space with the algebra "acting" on one side. In the second case, we have a set with a group acting on one side. One can express a vector space as a set by adding some required properties, as well as express an algebra as a group with some additional properties. Hence, we show that the concept of algebraic side multiplication and action are interchangeable - with some precautions -. This justifies the usage of the word 'act' when referring to an algebra multiplied to a module from one side. For more details on algebraic structures see [5], [6] and [8].

Chapter 2

Quantum Differential Calculus

In this chapter, we want to introduce the generalisation of differential structures on manifolds. We start from the differential calculus, and the concept of algebra of functions on an ordinary differential manifold $\mathcal{M}$, focusing on the example $\mathcal{M} = \mathbb{R}^n$. We then proceed to the module of differential forms on $\mathcal{M}$ and we conclude with the definition of Quantum Differential Calculus.

2.1 Differentiable functions on a manifold

Let $\mathcal{M}$ be a differentiable manifold, and $A = \mathcal{C}^\infty(\mathcal{M})$ the algebra of differentiable functions on $\mathcal{M}$. For the sake of simplicity, we pick $\mathbb{R}^n$ as the manifold, i.e. $\mathcal{M} = \mathbb{R}^n$, the general case being a small variation.

This rather simple framework allows us to introduce the concept of first-order differential calculus in a familiar way.

Definition 2.1.1. We define d, called the *exterior derivative*, as a linear map

$$\begin{aligned} d : \mathcal{C}^\infty(\mathbb{R}^n) &\longrightarrow \Omega^1 \\ f &\longrightarrow df \end{aligned}$$

for all $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, satisfying the Leibniz identity, i.e.

$$d(fg) = (df)g + f(dg)$$

for all $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$. Ω^1 denotes the space of first-order differential forms, i.e.

$$\Omega^1 = \Big\{ \sum_{i\in[n]} a^i dx_i \mid a^i \in \mathcal{C}^\infty(\mathbb{R}^n) \Big\}$$

This map, together with some additional conditions, is uniquely defined and takes the form

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

for all $f \in \mathcal{C}^\infty(\mathbb{R}^n)$. For more on the exterior derivative and its uniqueness see [14] section 1.2.

We recall an example.

Example 2.1.2. *Differential 1-forms on* $\mathbb{R}^2$. The pair (Ω^1, d) with Ω^1 and d defined as follows

$$\Omega^1 = \left\{ f(x,y)dx + g(x,y)dy,\ \forall f, g \in \mathcal{C}^\infty(\mathbb{R}^2) \right\}$$

$$d : \mathcal{C}^\infty(\mathbb{R}^2) \to \Omega^1 \quad \text{such that} \quad d(f) = \partial_x f \cdot dx + \partial_y f \cdot dy$$

The way in which we defined Ω^1 allows us to show that it is isomorphic to

$$\left\{ (f,g) |\ f, g \in \mathcal{C}^\infty(\mathbb{R}^2) \right\}$$

In other words, we showed Ω^1 to be isomorphic to a free-module by providing a basis. Ω^1 is also isomorphic to

$$(\mathcal{C}^\infty(\mathbb{R}^2))^2 = \mathcal{C}^\infty(\mathbb{R}^2) \otimes \mathcal{C}^\infty(\mathbb{R}^2)$$

The property of being a free A-module allows us to define a dimension for Ω^1 as a A-bimodule. In this specific case the dimension is 2.

Now, we highlight the pivotal properties of this example. Firstly, we see that functions of $\mathcal{C}^\infty(\mathbb{R}^n)$ can be multiplied to elements of Ω^1 from both sides, using the usual multiplication. Then, we recall that the map d satisfies the Leibniz rule, and that every element of Ω^1 can be written as a linear combination of elements fdg with $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$. These very properties are characteristic of a first-order differential calculus, as we will see. When we will abstract the concept of first-order differential calculus, they will constitute core requirements. Overall, this example is one to keep in mind throughout the discussion of Quantum Differential Calculus.

2.2 First-order differential calculus

In this section, we introduce the generalised concept of first-order differential calculus, taking as a guidance the notion of exterior derivative. It is helpful to keep in mind the first section as a motivating example for the properties discussed below. Let A be an algebra, not necessarily commutative.

Definition 2.2.1. A *first-order differential calculus* (FODC) over A is a pair (Ω^1, d) such that

- Ω^1 is a A-bimodule;
- A linear map $d : A \to \Omega^1$ satisfies the Leibniz rule, i.e.

$$d(ab) = (da)b + a(db), \quad \forall\ a, b \in A;$$

- $\Omega^1 = AdA = \text{span}\,\{adb|a, b \in A\}$, surjectivity condition;
- $\ker d = \mathbb{K}.1$, connectedness condition.

A first-order differential calculus without the surjectivity condition is called *generalised*.

The third property allows us to 'move' the element of A in $A \cdot dA \cdot A$ from the right to the left side. In other words, the surjectivity condition states that an element of Ω^1 of the form

$$da \cdot b, \quad \forall a, b \in A$$

can be rewritten as

$$da \cdot b = \sum_i a_i' db_i', \quad \forall a_i', b_i' \in A.$$

Since A is not necessarily commutative, a left module is not necessarily a right module, or vice versa. Actually, even if A is commutative, the right and left modules may be defined differently from one another, recall observation 1.3.2. For these reasons, the surjectivity property is non-trivial.

Observation 2.2.2. Here we show the first relevant difference between the commutative and non-commutative cases. In the usual differential geometry, the left and the right modules coincide, i.e. $a \cdot db = db \cdot a$ for all $a, b \in A$. However, it is not reasonable to impose this when A is non-commutative. In addition, applying d to the identity element of the algebra gives $d1 = 0$.

We now provide the definition of a differentiable algebra map, as this is an important tool to work with FODCs.

Definition 2.2.3. Let $\Phi : A \to B$ be an algebra map with A, B algebras equipped with first-order differential calculi. Φ is called *differentiable* if there exists a bimodule map $\Phi_* : \Omega^1_A \to \Omega^1_B$ such that we have the following commuting square.

$$\begin{tikzcd} \Omega^1_A \arrow[r, "\Phi_*"] & \Omega^1_B \\ A \arrow[u, "d"] \arrow[r, "\Phi"] & B \arrow[u, "d"] \end{tikzcd}$$

From the surjectivity assumption on differential calculi, this is the same as saying that $\Phi_*(xdy) := \Phi(x)d\Phi(y)$ gives a well-defined map from Ω^1_A to Ω^1_B.

2.3 Inner and Universal Calculus

In this section, we define the two most powerful concepts of this work: inner and universal differential calculi. A discussion after the definitions of these two objects explains their respective importance.

In addition, we will see why we referred to differential calculi as quantum in the first place. We use the word "quantum" to mean that non-commutative operations are involved. Although the two fields of quantum algebras and non-commutative algebras do not overlap perfectly, in this section we assume interchangeability. As a matter of fact, the content of this section is meaningful only if a quantum framework is taken into consideration.

We begin with the inner differential calculus.

Definition 2.3.1. A FODC (Ω^1, d) over an algebra A as in definition 2.2.1 is called *inner differential calculus* if there exists a $\theta \in \Omega^1$ such that

$$da = [\theta, a] = \theta \cdot a - a \cdot \theta$$

for all $a \in A$.

This is a type of calculus with which it is easier to work. In fact, to calculate the differential of an element of the algebra, one only needs to calculate the commutator between that element with the θ. Note that there is a right and a left action in the commutator. This calculation replaces the formal application of d to the element of the algebra.

In addition, when a differential calculus is inner, it highlights another difference between the commutative and non-commutative cases. Indeed, if we reconsider $A = \mathcal{C}^\infty(\mathcal{M})$ and $\Omega^1 = \{\text{differential forms}\}$, then the da in the definition above is vanishing, i.e. $da = 0$. We now give the definition of Universal Calculus through a proposition.

Proposition 2.3.2. *Let A be an algebra over a field $\mathbb{K}$ with the multiplicative unity.*

1. *There exists a universal quantum differential calculus (Ω^1_{uni}, d) given by*

$$\Omega^1_{uni} = \ker(\cdot) \subseteq A \otimes A$$

where $\cdot$ is the usual multiplication in the algebra A, and d is defined by

$$\begin{aligned} d:\ A &\longrightarrow \Omega^1_{uni} \\ a &\longrightarrow 1 \otimes a - a \otimes 1 \end{aligned}$$

2. *Any other quantum differential calculus is isomorphic to* $\Omega^1_{uni}/\mathcal{N}$*, for some sub-bimodule* $\mathcal{N} \subseteq \Omega^1_{uni}$*;*

3. *If* Ω^1_{uni} *is finite-dimensional then* Ω^1_{uni} *is left and right-parallelisable.*

Proof. Here we show that d satisfies the Leibniz rule.

$$d(ab) = 1 \otimes ab - ab \otimes 1$$
$$(da)b + adb = (1 \otimes a - a \otimes 1)b + a(1 \otimes b - b \otimes a)$$

by making the product explicit we see

$$(da)b + adb = 1 \otimes ab - a \otimes b + a \otimes b - ab \otimes 1$$

the two central terms cancel out, proving the Leibniz rule

$$(da)b + adb = 1 \otimes ab - ab \otimes 1 = d(ab)$$

For the proof of the other points see [3] page 5. □

We provide a further discussion on the meaning of the first two points of the proposition. The first point defines the differential calculus itself. The definition of d is such that the result of applying d to any element of the algebra gives an element of the Ω^1_{uni}, i.e. $da \in \ker(\cdot)$. The " $\cdot$ " product refers to the algebra product defined as

$$\begin{aligned} \cdot :\ A \otimes A &\longrightarrow\ A \\ a \otimes b\ &\longrightarrow\ ab \end{aligned}$$

for all $a, b \in A$. We use the tensor product to highlight the linearity of this product whereas one may have expected bi-linearity if we used the usual multiplication. Here are the two properties used in the proof for the Leibniz rule

$$a(b \otimes c) = ab \otimes c$$
$$(b \otimes c)d = b \otimes cd$$

for all $a, b, c, d \in A$.

The second point is the core property of the universal calculus. It explains why we call this specific differential calculus "universal". This proposition states that for any algebra, there exists a quantum differential calculus from which we can derive all the other quantum differential calculi.

The third point will not be discussed in further details, but we provide a side note. The dimension in point (3) is the dimension of Ω^1_{uni} as a vector space over $\mathbb{K}$. This is intrinsically different from the dimension of Ω^1_{uni} as a free A-bimodule. In section 2.1, the dimension of the differential calculus as free A-bimodule is

$$dim_A \Omega^1 = 2$$

whereas the dimension as a vector space over $\mathbb{K}$ is

$$dim_{\mathbb{K}}\, \Omega^1 = \infty.$$

Chapter 3

Introduction to Quantum Riemannian Geometry

In the previous chapter, we introduced the concept of quantum differential calculus with in mind the motivating example of differential calculus on a manifold $\mathbb{R}^n$. Along the same lines, we want to define a generalisation of the concepts of Riemannian geometry. In this chapter, we provide a generalised definition of metric g, connection ∇ and the second order operator Δ, i.e. Laplacian. We will motivate these definitions keeping in mind the example $A = \mathbb{C}^\infty(\mathbb{R}^n)$, as we did before. Our aim is to give the tools to build a Quantum Riemannian geometry framework on graphs. For the whole discussion about Riemannian Geometry we invite the reader to consult [14] and [15], in particular the chapters 1, 2 and 3 of the second. For a complete discussion about Quantum Riemannian geometry see the chapter 1 of [3] and the whole paper [10].

3.1 Metric

The metric on a Riemannian manifold has an important role, it is used to measure the length of a vector. In tensor notation, after fixing local coordinate, we write

$$|U| = g_{ij}U^iU^j$$

where g_{ij} is the metric tensor and U^i are coordinates of a vector defined in the tangent space of a manifold. For more details about our notation see chapter 1 of [14]. In addition, the metric tensor is the foundation of the Einstein Field Equation. For further physical details see [11, 4] and, for further mathematical details see [15]. In conclusion, one should regard the metric tensor as an object that encodes the geometry of a manifold.

We are interested in showing how one can express the metric tensor in the quantum framework of FODC. The first important tool is the notion of bimodule inner product.

Definition 3.1.1. Let A be an algebra and Ω^1 a quantum differential calculus over A. We define the bimodule inner product $(,) : \Omega^1 \otimes_A \Omega^1 \to A$ as a bilinear map such that

$$(\omega \cdot a, \eta) = (\omega, a \cdot \eta), \quad a(\omega, \eta) = (a \cdot \omega, \eta), \quad (\omega, \eta)a = (\omega, \eta \cdot a)$$

for all $a \in A$ and $\omega, \eta \in \Omega^1$. We call (A, Ω^1, d) a differential algebra.

Definition 3.1.2. Let (A,Ω^1,d) be a differential calculus. An element $g \in \Omega^1 \otimes_A \Omega^1$ is a *quantum metric* if it is invertible in the following sense. There exist a bimodule inner product

$$(,) : \Omega^1 \otimes_A \Omega^1 \longrightarrow A$$

such that

$$((\omega, \bullet) \otimes id)g = \omega = (id \otimes (\bullet, \omega))g$$

for all $\omega \in \Omega^1$.

We now provide a motivating example to explain the inverse property.

Observation 3.1.3. *Euclidean differential Geometry.* In this example, we first interpret the role of the bimodule map by looking at its correspondent in Euclidean differential geometry. Then, we motivate the inverse property by showing how it holds in the framework of $\mathbb{C}^\infty(\mathbb{R}^n)$.

1. Let $A = \mathbb{C}^\infty(\mathcal{M})$. One could use the local coordinates for a generic manifold, but here we pick $\mathcal{M} = \mathbb{R}^n$ for the sake of simplicity. In this way, we can simply consider global coordinates. We define the metric as

$$\begin{aligned} g: \ T\mathcal{M} \otimes T\mathcal{M} &\longrightarrow \ \mathcal{C}^\infty(\mathbb{R}^n) \\ u, v &\longrightarrow \ g(u, v) = g_{ij} \end{aligned}$$

   where $T\mathcal{M}$ is the tangent bundle. We assume that the tangent space is isomorphic to its dual, i.e. $T\mathcal{M} \cong T^*\mathcal{M}$. Therefore, we induce the dual map

$$g^* : T^*\mathcal{M} \otimes T^*\mathcal{M} \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n)$$

   where $T^*\mathcal{M}$ is the dual of the tangent bundle. This environment provides an interpretation of the bimodule map $(,)$. In fact, if $\partial_1, \cdots, \partial_n$ is a basis at each point for the tangent space $T_P\mathcal{M}$, 1-forms $dx^1, \cdots, dx^n$ is a basis for the dual $T_P^*\mathcal{M}$ at each point. Hence we have an identification between $T_P^*\mathcal{M}$ and Ω^1. For this reason, we identify the bimodule map with the dual metric map, i.e.

$$\begin{array}{rcccccc} g^*: & T^*\mathcal{M} & \otimes & T^*\mathcal{M} & \longrightarrow & \mathbb{C}^\infty(\mathbb{R}^n) \\ (,): & \Omega^1 & \otimes & \Omega^1 & \longrightarrow & A \end{array}$$

where A is the algebra on which we define Ω^1. Finally, we interpret the bimodule map as the inverse metric, i.e.

$$(,) \Longleftrightarrow g^*$$

2. We now show the explicit form of the inverse condition when $A = \mathbb{C}^\infty(\mathcal{M})$. This process will further justify why we interpret the bimodule map as the inverse metric. Firstly, we recall that $\omega \in \Omega^1$ can be written as $\omega = \omega_k dx^k$.

   Then, we rewrite the inverse condition and put this ω into it.

$$((\omega, \bullet) \otimes \mathrm{id})g = \omega, \quad ((\omega_k dx^k, \bullet) \otimes \mathrm{id})g = \omega_k dx^k$$

   Here, the $\bullet$ plays the role of a placeholder. We also recall the form of a metric when $A = \mathcal{C}^\infty(\mathbb{R}^n)$ and we choose canonical global coordinates $x^1, \cdots, x^n$, i.e. $g = g_{rs} dx^r \otimes dx^s$ with $r, s \leq n$. We proceed with the calculation.

$$\begin{aligned} ((\omega_k dx^k, \bullet) \otimes id)(g_{rs} dx^r \otimes dx^s) &= \omega_k dx^k \\ (\omega_k dx^k, g_{rs} dx^r)(dx^s) &= \omega_k dx^k \\ \omega_k g_{rs} (dx^k, dx^r) dx^s &= \omega_k dx^k \end{aligned}$$

   Now, we recall the correspondence between $(,)$ and g^* in this framework, i.e.

$$(dx^k, dx^r) = g^*_{kr}.$$

   Hence we can write

$$\begin{aligned} \omega_k g_{rs} (dx^k, dx^r) dx^s &= \omega_k dx^k \\ \omega_k g_{rs} g^*_{kr} dx^s &= \omega_k dx^k \end{aligned}$$

   The term g^* is the inverse metric of g, therefore multiplying them together gives the Kronecker delta with the two non-dummy indices, i.e. $g_{rs} g^*_{kr} = \delta_{ks}$. Finally, this gives us the wanted result

$$\begin{aligned} \omega_k g_{rs} g^*_{kr} dx^s &= \omega_k dx^k \\ \omega_k \delta_{ks} dx^s &= \omega_k dx^k \\ \omega_k dx^k &= \omega_k dx^k \end{aligned}$$

   where in the last step we simply did $\delta_{ks} dx^s = dx^k$.

3.2 Connection and Second-order differential operator

In this section, we define the notion of connection and the Laplace-Beltrami operator in the Quantum Riemannian Geometry framework.

Definition 3.2.1. Let (A, Ω^1, d) be a quantum differential calculus. We call *linear connection* a linear map

$$\nabla : \Omega^1 \longrightarrow \Omega^1 \otimes_A \Omega^1$$

such that

$$\nabla(f\omega) = df \otimes \omega + f\nabla\omega$$

for all $\omega \in \Omega^1$ and $f \in A$.

This can be directly compared to the usual connection in differential geometry. We refer to [15], chapters 6 and 9, for the Riemannian counterpart.

Then, we give the definition of the second-order differential operator.

Definition 3.2.2. Let (A, Ω^1, d) be a quantum differential calculus and $(,)$ a bimodule map as in definition 3.1.2. We call *second-order differential operator* a linear map

$$\Delta : A \longrightarrow A$$

such that

$$\Delta(fg) = (\Delta f)g + f\Delta g + 2(df, dg)$$

for all $f, g \in A$.

To interpret the definition above, we provide an example in the motivating framework $A = \mathcal{C}^\infty(\mathbb{R}^n)$.

Example 3.2.3. Let A be $\mathcal{C}^\infty(\mathbb{R}^n)$. Then the Laplace-Beltrami operator, i.e.

$$\Delta f = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j f)$$

satisfies the property required in the definition above. The proof is the simple calculation, see Appendix C.

In addition, we extend the definition so that we are able to define the quantum Laplace-Beltrami operator.

Definition 3.2.4. Let (A, Ω^1, d) be a differential calculus, ∇ a bimodule connection and $(,)$ a bimodule map as in definition 3.1.1. We define a *quantum Laplace-Beltrami operator*

$$\Delta^{\mathcal{L}} = (,)\nabla d$$

We provide an additional feature of this operator.

Proposition 3.2.5. *Let $\Delta^{\mathcal{L}}$ be a quantum Laplacian-Beltrami operator. Then $\Delta^{\mathcal{L}}$ is a second-order differential operator according to definition 3.2.2.*

Proof. We explicitly do the calculation. Recall that we have a bimodule connection, so the formula in definition 3.2.1 is well-defined on the right product too, i.e.

$$\nabla(\omega f) = \nabla(\omega)f + \omega \otimes df.$$

We will write $\Delta = \Delta^{\mathcal{L}}$ to ease the notation. Therefore we write

$$\begin{aligned}
\Delta(fg) =& (,)\nabla(df \cdot g + f \cdot dg) \\
=& (,)\Big(\nabla(df)g + df \otimes dg + df \otimes dg + f\nabla(dg)\Big)
\end{aligned}$$

Then, we distribute the bimodule linear map over the terms and we obtain

$$\begin{aligned}
\Delta(fg) =& (,)\nabla(df)g + (,)df \otimes dg + (,)df \otimes dg + (,)f\nabla(dg) \\
=& \cancelto{\Delta f}{(,)\nabla(df)} \cdot g + (,)df \otimes dg + (,)df \otimes dg + f \cdot \cancelto{\Delta g}{(,)\nabla(dg)}
\end{aligned}$$

Finally, recall the action of $(,)$, i.e.

$$(,) : \Omega^1 \otimes \Omega^1 \longrightarrow A$$

therefore we can conclude the proof, i.e.

$$\begin{aligned}
\Delta(fg) =& \Delta f \cdot g + (,)df \otimes dg + (,)df \otimes dg + f \cdot \Delta g \\
=& \Delta f \cdot g + 2(df, dg) + f \cdot \Delta g
\end{aligned}$$

This shows that Δ fulfils the requirements in definition 3.2.2, which means that the quantum Laplace-Beltrami operator is a second-order differential operator. □

We continue by giving an interesting observation regarding the inner case.

Proposition 3.2.6. *Let (A, Ω^1, d) be a inner differential calculus (see definition 2.3.1) via an element θ and $(,)$ a bimodule map. There exist two different second-order differential operators that we call associated Laplacians, i.e.*

$$
\begin{aligned}
{}_\theta\Delta f &= -2(df, \theta) \\
\Delta_\theta f &= 2(\theta, df)
\end{aligned}
$$

for all $f \in A$.

Proof. The proof of the proposition above is the simple calculation to show that these two differential operators satisfy the requirement in definition 3.2.2. Here we prove that ${}_\theta\Delta$ is a second-order differential, the proof for Δ_θ can be found [3] page 15.

$$
\begin{aligned}
{}_\theta\Delta(fg) =& -2(d(fg), \theta) \\
=& -2(df \cdot g + f \cdot dg, \theta) \\
=& -2(df \cdot g, \theta) - 2(f \cdot dg, \theta) \\
=& -2(df, g\theta) - 2f(dg, \theta) \\
=& -2(df, g\theta) - 2f(dg, \theta) + 2(df, \theta g) - 2(df, \theta g) \\
=& -2f(dg, \theta) - 2(df, \theta g) + 2(df, \theta g - g\theta)
\end{aligned}
$$

Now consider ${}_\theta\Delta f \cdot g$ and $f \cdot {}_\theta\Delta g$. They can be respectively written as

$$
{}_\theta\Delta(f) \cdot g = -2(df, \theta)g = -2(df, \theta g)
$$

and

$$
f \cdot {}_\theta\Delta g = f \cdot \big(-2(dg, \theta)\big) = -2f(dg, \theta).
$$

Therefore, we can identify them in the expression above and rewrite

$$
{}_\theta\Delta(fg) = f \cdot {}_\theta\Delta g + {}_\theta\Delta(f) \cdot g + 2(df, \theta g - g\theta)
$$

Finally, recall that for an inner calculus $dg = [\theta, g]$. This concludes the proof, i.e.

$$
{}_\theta\Delta(fg) = f \cdot {}_\theta\Delta g + {}_\theta\Delta(f) \cdot g + 2(df, dg)
$$

□

Above, we provided the definition and motivating examples of some generalised differential tools.

3.3 Exterior Algebra

The differential machinery described above and in the previous chapter has a key application in the construction of the de Rham complex. A formal discussion of the de Rham complex is not in the scope of the dissertation. Therefore we invite the reader to consult [15] and [16] for further details. A de Rham complex can be thought of as a chain

$$\mathcal{C}^\infty(\mathcal{M}) \to \Omega^1(\mathcal{M}) \to \cdots \to \Omega^n(\mathcal{M}) \to 0$$

with the exterior derivative $d : \Omega^i(\mathcal{M}) \longrightarrow \Omega^{i+1}(\mathcal{M})$. In particular we consider $\Omega^0(\mathcal{M}) = \mathcal{C}^\infty(\mathcal{M})$ and $\Omega^n(\mathcal{M}) = \mathcal{C}^\infty(\mathcal{M})dx^1\wedge\cdots\wedge dx^n$. The space of all differential forms $\Omega(\mathcal{M}) = \oplus_{i=1}^n \Omega^i(\mathcal{M})$ forms a graded algebra with the exterior product $\wedge$.

Definition 3.3.1. A *differential graded algebra* or DGA on an algebra A is

1. A graded algebra $\Omega = \oplus_{n\geq 0}\Omega^n$ with $\Omega^0 = A$;
2. A map $d : \Omega^n \longrightarrow \Omega^{n+1}$ such that $d^2 = 0$ and
$$d(\omega \wedge \rho) = (d\omega) \wedge \rho + (-1)^n \omega \wedge d\rho, \quad \forall\ \omega, \rho \in \Omega,\ \omega \in \Omega^n;$$
3. A, dA generate Ω (*surjectivity condition*).

When the surjectivity condition holds we call it an *exterior algebra* on A.

We call a DGA *non-degenerate* if the wedge products $\wedge : \Omega^m \otimes \Omega^1 \longrightarrow \Omega^{m+1}$ and $\wedge : \Omega^1 \otimes \Omega^m \longrightarrow \Omega^{m+1}$ are non-degenerate. For any DGA we define the *volume dimension* as the largest n such that $\Omega^n \neq 0$. This can be infinite. We will have a finite volume dimension n with Ω^n free with one generator both in the case we will discuss later as well as in the case of $\Omega^k(\mathbb{R}^n)$ differential forms on $\mathbb{R}^n$. We will refer to that element as Vol. A DGA of volume dimension n is *inner* if there exists a $\theta \in \Omega^1$ such that for all $m < n$

$$d\omega = [\theta, \omega] := \theta \wedge \omega - (-1)^m \omega \wedge \theta, \quad \forall\ \omega \in \Omega^m.$$

In the definition of DGA we identify Ω^1 to be the first-order differential calculus described previously. We would like to build the higher order differential calculi by knowing the first-order only. This can be done by asking some additional properties to the $\Omega^n,\ n > 1$ to have consistency in their definition.

Proposition 3.3.2. *Every first-order calculus Ω^1 on an algebra A has a 'maximal prolongation' Ω_{max} to an exterior algebra, where for every relation $\sum_i a_i \cdot db_i = \sum_j r_j \cdot ds_j$ in Ω^1 with $a_i, b_i, r_j, s_j \in A$ we impose the relation*

$$\sum_i da_i \wedge db_i + \sum_j dr_j \wedge ds_j = 0$$

in Ω^2.

The relations above are for Ω^2 only. However, they can immediately be extended to higher degrees by computing the exterior product of elements of Ω^1 with the relations for Ω^2, i.e. we can write $a \wedge (\sum_i da_i \wedge db_i + \sum_j dr_j \wedge ds_j) = 0$ when $a \wedge (\sum_i a_i db_i) = a \wedge (\sum_j r_j ds_j)$. This process can be repeated to build a DGA with consistency in the definition of higher order differential forms.

Observation 3.3.3. A maximal prolongation Ω_{max} defined as in 3.3.2 is such that Ω^1 is embedded in Ω_{max}.

This observation is non-trivial and the proof of that can be found in [3] at page 24. For more details about the concepts presented in this chapter we recommend [3] section 1.5 page 22.

There is also an additional condition that we naturally ask. In the example of $\mathcal{M} = \mathbb{R}^n$ with general canonical coordinates $x^1, \cdots, x^n$, the metric tensor can be written as $g = g_{ij} dx^i \otimes dx^j$. We impose $\wedge(g) = \sum_{i,j} g_{ij} dx^i \wedge dx^j + g_{ji} dx^j \wedge dx^i$ and we recall that the metric tensor is symmetric, i.e. $g_{ij} = g_{ji}$. Thus we can rewrite the symmetry condition as $\wedge(g) = 0$. We will ask that these relations hold even for the quantum metric in definition 3.1.2.

In the following chapters, we cover the applications of these concepts into the framework of graphs. This will be useful both to understand the definitions above and to introduce the necessary tools to develop a naive theory of electromagnetism on an unusual framework.

Chapter 4

Quantum Riemannian Geometry on Graphs

In this chapter, we introduce the concept of graph and we apply the machinery developed in the previous chapters. We do this in order to describe how the quantum differential calculus is the perfect framework to introduce key concepts of Riemannian geometry on graphs. The majority of the work here is taken from [3]. Thus, we invite the reader to consult it for further details and additional content. We begin with a small introduction to graphs, we define an algebra and a quantum differential calculus on graphs, and then we give the definition of metric and Laplacian on graphs. Finally, we provide the concept of the exterior algebra on a Cayley graph. We will use this very setup when developing Maxwell's theory on discrete sets.

4.1 Graphs

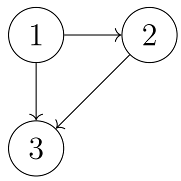
In this section, we introduce the notion of graph and the notation that we will use in this chapter. We finish with the definition of digraph, defining the framework in which we will discuss quantum differential geometry.

Definition 4.1.1. We call *graph* a pair (V, E) where V is a set of vertices and E is the set of distinct edges that connect pair of vertices. The edges in E are then given by pairs of vertices. We refer to the graph as (V, E).

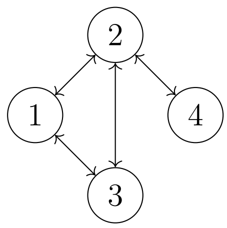
A graph (V, E) is said *directed* if the edges are ordered pairs of vertices.

Example 4.1.2. 1. Example of a directed graph.

The vertices and edges are $V = \{1, 2, 3\}$ and $E = \{(1, 2), (2, 3), (1, 3)\}$.



2. The following is an example of undirected graph. In this case the direction of the edges is not defined, and we use the notation of a simple line. Sometimes, we can use a double arrow instead. In this example the vertices are $V = \{1, 2, 3, 4\}$ and the edges are $E = \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2), (2, 4), (4, 2)\}$.



From now on we will work with directed graphs, and we will shorten it to *digraphs*.

4.2 Algebra of functions on a set

In this section, we define an algebra on a digraph and we provide a basis for it. We are going to use this very algebra to create a differential structure on digraphs. Let us start from the definition of this algebra. From now on we assume all our graphs to be digraphs.

Definition 4.2.1. Let $\mathbb{K}$ be a field and V a finite set. We define the algebra $\mathbb{K}(V)$ as follows.

$$\mathbb{K}(V) = \{f : V \longrightarrow \mathbb{K}\}$$

In other words, $\mathbb{K}(V)$ is the algebra of all the functions that associate to each element of V a value in $\mathbb{K}$. An interesting property of this algebra is that it is finite dimensional.

Proposition 4.2.2. *Let $A = \mathbb{K}(V)$ be an algebra as defined above. It is a finite dimensional vector space whose dimension is equal to the number of vertices of the graph, i.e. dim(A) = number of elements of V.*

This claim allows us to define a finite canonical basis for A as the next step shows.

Proposition 4.2.3. *Let $A = \mathbb{K}(V)$ for a graph (V, E). We have*

$$\mathcal{B} = \{\delta_x, \forall x \in V\}, \qquad \delta_x(y) = \begin{cases} 1 & x = y \\ 0 & \textit{otherwise} \end{cases}$$

for all $x, y \in V$, is a basis for A.

Given this proposition, we can write every f as a linear combination of all the δ_x, i.e.

$$f = f_1\delta_1 + f_2\delta_2 + \cdots + f_n\delta_n = \sum_{i=1}^{n} f_i\delta_i$$

where $f_i \in \mathbb{K}$ and δ_i is the basis function correspondent to the i^{th} vertex.

We will use the algebra described above to define a quantum differential calculus on digraphs.

4.3 Quantum Differential Calculus on $\mathbb{K}(V)$

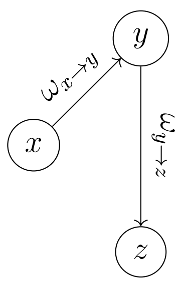
In this section, we start from the most important theorem that relates a quantum differential calculus with digraphs. Then in its proof, we give an explicit expression for such a calculus and finally we show that it is inner.

Theorem 4.3.1. *Let V be a finite set and $A = \mathbb{K}(V)$. There is a one-to-one correspondence between Ω^1, quantum differential calculus over A, and digraphs over V.*

$$\Omega^1 \ \textit{FODC on } A = \mathbb{K}(V) \Longleftrightarrow \textit{digraphs } (V, E),\ \Omega^1 = \mathbb{K}(E)$$

Before the proof, we make some comments. In [10] this theorem is presented as an equivalence between categories. This means that the relation between quantum differential calculi and digraphs is much more than a one-to-one correspondence.

First, we define the objects involved in the proof of the theorem. Given a pair of vertices x, y, we formally define $\omega_{x\to y}$ as the edge that connects them, i.e.



We can equivalently view $\omega_{x\to y}$ as the function

$$\begin{array}{rccl} \omega_{x\to y}: & E & \longrightarrow & \mathbb{K} \\ & (x,y) & \longrightarrow & 1 \\ & (x',y') & \longrightarrow & 0 \quad \forall (x',y') \neq (x,y) \end{array}$$

so that $\mathbb{K}(E) = \text{span}\{\omega_{x\to y}\}$ for all $(x,y) \in E$.

Now we define the action of elements of the algebra on the $\mathbb{K}(E)$. We want $\mathbb{K}(E)$ to be Ω^1, hence we need an A-bimodule additional structure.

Definition 4.3.2. Let $f \in A$ be $f : V \longrightarrow \mathbb{K}$. We now define the left and right action of f over a generic edge $\omega_{x\to y} \in \mathbb{K}(E)$ as

$$f \cdot \omega_{x\to y} = f(x)\omega_{x\to y}$$

$$\omega_{x\to y} \cdot f = f(y)\omega_{x\to y}$$

where $f(x), f(y) \in \mathbb{K}$.

Sketch of the proof.

($\Longleftarrow$) We define $\Omega^1 = \mathbb{K}(E)$ as above. Given $A = \mathbb{K}(V)$ (we already have a set of vertices, and a set E of edges between the vertices). Given the two definitions above for the left and right action of the algebra, the Ω^1 is an A-bimodule.

We now define the exterior derivative d for Ω^1 to be a quantum differential calculus. Let us define $d : A \longrightarrow \Omega^1$ as follows

$$df := \sum_{x\to y}(f(y) - f(x))\omega_{x\to y}$$

where $x \to y$ under the summation symbol means for all the edges, i.e. $\forall\, \omega_{x\to y} \in E$. We now check that this is a well-defined differential. To do so we check that it satisfies the Leibniz rule. From the definition we have

$$d(fg) = \sum_{x\to y}(f(y)g(y) - f(x)g(x))\omega_{x\to y}$$

and on the other hand we have

$$
\begin{aligned}
d(f)g + f(dg) &= \Big(\sum_{x\to y}(f(y)-f(x))\omega_{x\to y}\Big)g + f\Big(\sum_{x\to y}(g(y)-g(x))\omega_{x\to y}\Big) \\
&= \sum_{x\to y}(f(y)-f(x))(\omega_{x\to y}\cdot g) + \sum_{x\to y}(g(y)-g(x))(f\cdot\omega_{x\to y}) \\
&= \sum_{x\to y}(f(y)-f(x))(g(y)\omega_{x\to y}) + \sum_{x\to y}(g(y)-g(x))(f(x)\omega_{x\to y}) \\
&= \sum_{x\to y}(f(y)g(y) - \cancel{f(x)g(y)} + \cancel{g(y)f(x)} - f(x)g(x))\omega_{x\to y}) \\
&= d(fg)
\end{aligned}
$$

The surjectivity condition for Ω is trivially obtained from how we defined the left and right action and Ω itself. It is interesting to show what the effects of d over a basis element of $A = \mathbb{K}(V)$ are. Recalling proposition 4.2.3 we see

$$
d\delta_x = \sum_{w\to z}(\delta_x(z) - \delta_x(w))\ \omega_{w\to z}
$$

the only way for this expression to be non-trivial is to consider the edges, whose either starting or ending vertex is x. Therefore the summation becomes

$$
d\delta_x = \sum_{w\to x}\omega_{w\to x} - \sum_{x\to z}\omega_{x\to z}
$$

Given this result, an interesting quantity to consider is the following

$$
\delta_y d\delta_x = \delta_y \cdot \Big(\sum_{w\to x}\omega_{w\to x} - \sum_{x\to z}\omega_{x\to z}\Big)
$$

where we assumed $x \neq y$. By explicitly computing the action of δ_y with the two terms we obtain

$$
\delta_y d\delta_x = \sum_{w\to x}\delta_y\cdot\omega_{w\to x} - \sum_{x\to z}\delta_y\cdot\omega_{x\to z}
$$

Recalling the definition of δ_y and the effect of the left action of the algebra we have

$$
\delta_y d\delta_x = \sum_{w\to x}\delta_y(w)\ \omega_{w\to x} - \sum_{x\to z}\cancelto{0}{\delta_y(x)}\ \omega_{x\to z}
$$

The only way for the first term not to vanish is for w to be equal to y. In that case, $\delta_y(w) = \delta_y(y) = 1$, therefore we can write

$$\delta_y d\delta_x = \omega_{y\to x}$$

We need this last result to prove the theorem the other way around. In other words, this expression allows us to find whether there is an edge between two given vertices or not. This depends solely on the elements of the quantum differential calculus expressed in terms of the basis of the algebra. For the proof of the other way around, see [3] page 18.

$\square$

When we are equipped with an algebra $A = \mathbb{K}(V)$, there exists a universal quantum differential calculus. Thanks to the theorem above, we can derive a digraph from that quantum differential calculus that for now we call universal digraph. Since all the other possible quantum differential calculus are derivable from the universal one, it means all the possible digraphs can be derived from the universal digraph. For this reason, the universal digraph is the digraph with all the possible edges, i.e. each vertex is once connected to all the others, also known as *complete* digraph.

We now see a really interesting property of the quantum differential calculus as defined above.

Proposition 4.3.3. *Let (V, E) be a digraph. Given the algebra $A = \mathbb{K}(V)$ and the quantum differential calculus $\Omega^1 = \mathbb{K}(E)$ equipped with*

$$df = \sum_{x\to y} (f(y) - f(x))\omega_{x\to y}$$

for all $x, y \in V$. This quantum differential calculus is inner. In other words, there exists $\theta \in \Omega^1$ such that we can write

$$df = [\theta, f] = \theta \cdot f - f \cdot \theta$$

This proposition shows that a quantum differential calculus over a digraph is intrinsically non commutative because an element $df \in \Omega^1$ when Ω^1 is inner, is non-trivial if and only if the action of the algebra is non commutative. We now provide a proof for the proposition.

Proof. Let us define θ as the sum of all the possible edges.

$$\theta = \sum_{x\to y} \omega_{x\to y}$$

Now we need to prove that

$$df = \theta \cdot f - f \cdot \theta$$

By writing down the right hand side explicitly we see

$$\begin{aligned}
\theta \cdot f - f \cdot \theta &= \Big(\sum_{x\to y} \omega_{x\to y}\Big) \cdot f - f \cdot \Big(\sum_{x\to y} \omega_{x\to y}\Big) \\
&= \sum_{x\to y} \omega_{x\to y} \cdot f - \sum_{x\to y} f \cdot \omega_{x\to y} \\
&= \sum_{x\to y} f(y)\omega_{x\to y} - f(x)\omega_{x\to y} \\
&= \sum_{x\to y} (f(y) - f(x))\omega_{x\to y} \\
&= df
\end{aligned}$$

□

It is worth remarking that in this example we built a quantum differential calculus that is a bimodule of a commutative algebra. What is non-commutative is the action of the algebra on Ω^1. The inner property of this quantum differential calculus will come in handy when we will define further differential structures.

In this last proposition we summarise the results obtained in this chapter to provide a consistent environment on which we will define further differential tools.

Proposition 4.3.4. *Let X be a finite set. Differential calculi $\Omega^1(X)$ on the algebra $A = \mathbb{K}(X)$ are inner and correspond to directed graphs on X, with*

$$\Omega^1 = \mathrm{span}_{\mathbb{K}}\, \omega_{x\to y}, \quad f \cdot \omega_{x\to y} = f(x)\omega_{x\to y}, \quad \omega_{x\to y} \cdot f = \omega_{x\to y} f(y)$$

$$df = \sum_{x\to y} (f(y) - f(x))\omega_{x\to y}, \quad \theta = \sum_{x\to y} \omega_{x\to y}$$

Now, to be able to do Riemannian geometry on graphs we need some additional differential operators. Thus we give an explicit form to some of the objects introduced in chapter 3. Then, we provide the machinery to introduce physical theories on this very framework.

4.4 Metric and Second-order Laplacian

We now introduce two fundamental objects: the metric and the second-order Laplacian. We give their explicit form in a proposition, motivating the definition we have given. However, we first define some notation to make the discussion clearer. Recall the one-to-one correspondence between a digraph and a quantum differential calculus.

Definition 4.4.1. Let Ω^1 be a FODC over a finite set. We call Ω^1 *symmetric* if the correspondent graph is an undirected graph which means that we have arrows in both directions for every edge. An undirected graphs can then be regarded as 'bidirected'.

To introduce the metric along the lines of chapter 3, we first define the bimodule inner product $(\,,\,)$.

Proposition 4.4.2. *Let Ω^1 be a FODC over a finite set. Any bimodule inner product $(\,,\,) : \Omega^1 \otimes \Omega^1 \to A$ takes the form*

$$(\omega_{x\to y}, \omega_{y'\to x'}) = \lambda_{x\to y}\delta_{x,x'}\delta_{y,y'}\delta_x$$

for some numbers $\lambda_{x\to y}$ called arrow weights.

For the proof that any of the inner products can be written in the form above see [3]. We now establish a necessary and sufficient condition for a metric to exist in this setting.

Proposition 4.4.3. *Let Ω^1 be a FODC over a finite set and $(\,,\,)$ be a bimodule inner product as for definition 4.4.2. There exists a generalised quantum metric g if and only if Ω^1 is symmetric and the weights $\lambda_{x\to y}$ are all nonzero. In that case, we can write*

$$g = \sum_{x\to y} g_{x\to y}\omega_{x\to y} \otimes_A \omega_{y\to x}$$

where

$$g_{x\to y} = \frac{1}{\lambda_{y\to x}}.$$

In appendix D, we prove that this definition for g fulfils the requirement in definition 3.1.2. We now define the Second-order Laplacian in the framework of digraphs.

Proposition 4.4.4. *Let Ω^1 be a FODC over a finite set and $(\,,\,)$ be a bimodule inner product. The induced Second-order Laplacian is given by*

$$(\Delta_\theta f)(x) = ({}_\theta\Delta f)(x) = 2\sum_{y|x\leftrightarrow y}(f(x) - f(y))$$

with $x, y \in X$. We are summing over all y such that there is an arrow with y as one of the edges.

Proof. Recall the previous result,

$$\Delta_\theta f = {}_\theta\Delta f = -2(df, \theta).$$

Recall *df* from proposition 4.3.4, i.e.

$$df = \sum_{x \to y} (f(y) - f(x))\omega_{x \to y}$$

then we write

$$\begin{aligned}
-2(df, \theta) &= -2\Big(\sum_{x \to y}(f(y) - f(x))\omega_{x \to y}, \sum_{y' \to x'} \omega_{y' \to x'}\Big) \\
&= -2\sum_{x \to y}\sum_{y' \to x'} (f(y) - f(x))\Big(\omega_{x \to y}, \omega_{y' \to x'}\Big) \\
&= -2\sum_{x \to y}\sum_{y' \to x'} (f(y) - f(x))\lambda_{x \to y}\delta_{x,x'}\delta_{y,y'}\delta_x \\
&= 2\sum_{x \to y}\sum_{y \to x} \lambda_{x \to y}(f(x) - f(y))\delta_x
\end{aligned}$$

Therefore

$$(\Delta_\theta f)(x) = ({}_\theta\Delta f)(x) = -2(df, \theta)(x) = 2\sum_{x \leftrightarrow y} \lambda_{x \to y}(f(x) - f(y))\delta_x(x)$$

where the $\delta_x(x)$ fixes the value of x. We have

$$(\Delta_\theta f)(x) = ({}_\theta\Delta f)(x) = -2(df, \theta)(x) = 2\sum_{y|x \leftrightarrow y} \lambda_{x \to y}(f(x) - f(y))$$

□

4.5 Exterior Algebra of Finite Groups

In this section, we see the application of the concept of exterior algebra on finite groups. Indeed, as we saw in section 4.3, a quantum differential calculus on a discrete set is associated to a directed graph. In addition, a Ω^1 on a graph is always inner by $\theta = \sum_{x \to y} \omega_{x \to y}$. We now see how to construct the maximal prolongation of the first-order calculus.

Proposition 4.5.1. *Let $\Omega^1(X) = \mathbb{K}(E)$. Its maximal prolongation $\Omega_{max}(X)$ has relations*

$$\sum_{y:p\to y\to q} \omega_{p\to y} \wedge \omega_{y\to q} = 0$$

for all $p \neq q$ with $p \not\to q$ and

$$d\omega_{p\to q} = \sum_{y:y\to p} \omega_{y\to p} \wedge \omega_{p\to q} + \sum_{y:q\to y} \omega_{p\to q} \wedge \omega_{q\to y}$$

Again, the proof can be found on [3] page 29. As we saw at the end of the previous chapter, it is natural to impose further relations so that the edge symmetric condition holds, i.e. $\wedge(g) = 0$.

4.6 Exterior Algebra on Cayley graphs

We now look at another type of graphs, the Cayley graphs.

Definition 4.6.1. Let G be a discrete group. A *Cayley graph* is a graph (G, E) in which the edges have the form $x \to ax$ for all $x \in G$ and $a \in \mathcal{C}$ for a fixed subset $\mathcal{C} \subseteq G \setminus e$ of the group not containing the identity. This set $\mathcal{C}$ is the set of generators of the graph.

Recall the results in section 4.3, when G is at least finite and the first-order calculus over G is a free module over a vector space Λ^1 with basis

$$e_a = \sum_{x\in G} \omega_{x\to xa},$$

such that $\Omega^1 = \mathbb{K}(G) \cdot \Lambda^1$.

Recall $\omega_{x\to xa} = \delta_x d\delta_{xa}$, we assume that the group G itself can act on the algebra of functions from the left and from the right. We define the left action as $x \triangleright \delta_y = \delta_{xy}$ and the right action as $\delta_x \triangleleft y = \delta_{xy}$ for all $x, y \in G$ and δ_x basis element of $\mathbb{K}(G)$. A calculus is called *(right) left covariant* if the action described above is extended to Ω^1 in a way that commutes with d, the exterior derivative.

We now proceed to an important result.

Proposition 4.6.2. *Let G be a finite group. Left-covariant calculi $\Omega^1(G)$ on $\mathbb{K}(G)$ correspond to Cayley directed graphs based on subsets $\mathcal{C} \subseteq G \setminus e$, with Λ^1 the space of left-invariant 1-forms, and with relations and exterior derivative*

$$e_a \cdot f = R_a(f)e_a, \quad df = \sum_{a\in\mathcal{C}} (R_a(f) - f)e_a$$

for $f \in \mathbb{K}(G)$. The calculus is inner by $\theta = \sum_a e_a$ and bicovariant if and only if $\mathcal{C}$ is stable under conjugation. Here $R_a(f) = f((\bullet)a)$.

We do not provide a proof for this proposition, however it can be found in [3] section 1.7. Moreover, a precise explanation for what we mean by stable under conjugation is not provided here. Whenever we will state the bicovariance of a calculus we just assume the covariance under group action from both sides. For more details, we invite the reader to consult [3].

Now we turn to exterior algebra on Cayley graphs.

Proposition 4.6.3. *Every left-covariant calculus Ω^1 on a finite group G has a natural left-covariant exterior algebra $\Omega_L(G)$ generated by $\mathbb{K}(G)$ and an algebra Λ_L of left-invariant forms where the latter is generated by e_a with the quadratic relations*

$$\sum_{a,b\in\mathcal{C}:ab=z} e_a \wedge e_b = 0$$

for all $z \in G \setminus \{e\}$. This is also inner with the same θ as before. The exterior derivative on degree 1 is given by

$$de_c = \theta \wedge e_c + e_c \wedge \theta - \sum_{a,b\in\mathcal{C}:ab=c} e_a \wedge e_b$$

for all $c \in \mathcal{C} \subseteq G \setminus \{e\}$

As we saw in the previous section, the possibly non symmetric Euclidean metric on any symmetric graph is $g = \sum_{a\in\mathcal{C}} e_a \otimes e_{a^{-1}}$ and hence has $\wedge(g) = \sum e_a \wedge e_{a^{-1}} = \theta \wedge \theta$. We will set this to zero as an additional condition. Namely, this is equivalent to extending the quadratic relations above to $z \in G$.

Now we provide the necessary notation to then discuss electromagnetism on S_3 in the following chapter.

Proposition 4.6.4. *Let $\Omega(G)$ be the canonical bicovariant exterior algebra on a finite group, as defined by a Cayley graph with generators $\mathcal{C}$. Generalised quantum metrics g exist if and only if $\mathcal{C}$ has inverses and takes the form*

$$g = \sum_{a\in\mathcal{C}} c_a e_a \otimes {e_a}^{-1}, \quad (e_a, e_b) = \frac{\delta_{a^{-1},b}}{R_a(c_{a^{-1}})}$$

where $c_a \in \mathcal{K}(G)$ are nowhere zero. This is edge-symmetric if and only if $c_a = R_a(c_{a^{-1}})$ for all $a \in \mathcal{C}$ and a quantum metric if and only if $c_a = c_{a^{-1}}$ for all $a \in \mathcal{C}$. The inner element Laplacians are

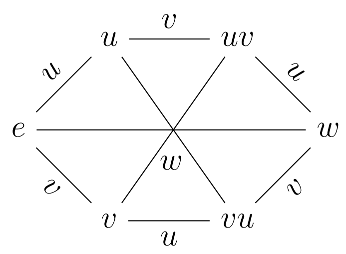
$${}_\theta\Delta = \Delta_\theta = -2\sum_a \frac{1}{R_a(c_a^{-1})}\partial^a$$

This includes the canonical graph Euclidean metric where every edge has unit weight, which for $\Omega(G)$ comes out as

$$g = \sum_a e_a \otimes e_{a^{-1}}, \quad (e_a, e_b) = \delta_{a^{-1},b}, \quad {}_\theta\Delta = \Delta_\theta = -2\sum_a \partial^a$$

Both of the last propositions are reported here without a proof. However, an interested reader can consult [3] for them, as well as further details and examples. We now provide an example of all the concepts discussed above applied to the permutation group S_3.

Example 4.6.5. Let $G = S_3$ with generators u, v, w and relations $u^2 = v^2 = e$ and $uvu = vuv = w$. To build the calculus we take $\mathcal{C} = \{u, v, w\}$, the set of 2-cycles, that result in the following Cayley graph.



From the graph, one can clearly see that $u^2 = v^2 = e$. This is reasonable because they are permutations and repeating the same twice gives back the starting order. Then we use the proposition 4.6.3 to express the relations between the left-invariant forms and their differential. From the graph we see, $w = uvu$ or $w = vuv$. We apply the first equation in the proposition for $z = uv$ and we obtain

$$e_u \wedge e_v + e_v \wedge e_w + e_w \wedge e_u = 0$$

because $vw = v^2uv = uv$ and $wu = uvu^2 = uv$. Similarly for $z = vu$ we have

$$e_v \wedge e_u + e_w \wedge e_v + e_u \wedge e_w = 0$$

.

From the extended conditions $\theta \wedge \theta = 0$ we pick $z = e = u^2 = v^2 = w^2$ and therefore we get

$$e_u^2 = e_v^2 = e_w^2 = 0$$

Then we calculate the first derivatives.

$$\begin{aligned} de_u =& \theta \wedge e_u + e_u \wedge \theta - \sum_{a,b \in \mathcal{C}: ab=u} e_a \wedge e_b \\ =& \cancelto{0}{e_u \wedge e_u} + e_v \wedge e_u + e_w \wedge e_u + \cancelto{0}{e_u \wedge e_u} + e_u \wedge e_v + e_u \wedge e_w \\ =& (e_w \wedge e_u + e_u \wedge e_v) + (e_v \wedge e_u + e_u \wedge e_w) \\ =& - e_v \wedge e_w - e_w \wedge e_v \end{aligned}$$

therefore we have

$$de_u + e_v \wedge e_w + e_w \wedge e_v = 0.$$

Similarly,

$$de_v + e_u \wedge e_w + e_w \wedge e_u = 0$$

$$de_w + e_v \wedge e_u + e_u \wedge e_v = 0$$

From the equations above, applying the wedge operator with another element e_i with $i = \{u, v, w\}$ we can extract the dimensions of the Ωs in the DGA. From the calculations we got $1, 3, 4, 3, 1$ for $\Omega^0, \Omega^1, \cdots, \Omega^4$. The volume element is

$$\mathrm{Vol} := e_u \wedge e_v \wedge e_u \wedge e_w = e_v \wedge e_u \wedge e_v \wedge e_w = -e_w \wedge e_u \wedge e_v \wedge e_u = -e_w \wedge e_v \wedge e_u \wedge e_v.$$

The most general quantum metric, such that $\wedge(g) = 0$, has the form

$$g = c_u e_u \otimes e_u + c_v e_v \otimes e_v + c_w e_w \otimes e_w$$

with the coefficients always non-vanishing. The edge-symmetric condition is for $c_u = R_u(c_{u^{-1}}) = R_u(c_u)$ and can be rewritten in the form

$$R_u(c_u) - c_u = \partial^u c_u = 0$$

and similarly $\partial^v c_v = \partial^w c_w = 0$. In this example the canonical Laplacian is

$$\Delta_\theta = -2\left(\frac{1}{c_u}\partial^u + \frac{1}{c_v}\partial^v + \frac{1}{c_w}\partial^w\right)$$

This operator has three eigenvalues that are respectively $0, 6$ and 12. The identity permutations 1 for example, gives

$$\partial^i 1 = R_i 1 - 1 = 1 - 1 = 0$$

for any permutation $i = \{u, v, w\}$. This means that 1 is an eigenvector of Δ_θ with eigenvalue 0. Furthermore, if we consider the function that gives back the sign of the permutation of three elements we see

$$\partial^i \mathrm{sign} = R_i \mathrm{sign} - \mathrm{sign} = -2\mathrm{sign}$$

where the operator R_i is just applying an additional permutation $i = \{u, v, w\}$ before extracting the sign. Thus, it changes the sign. We can see that this function is an eigenvalue of Δ_θ with eigenvalue 12.

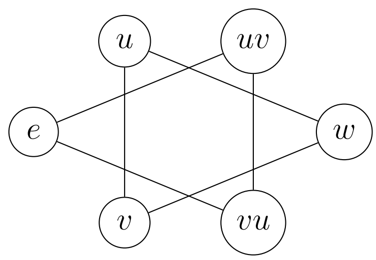
Finally, we analyse the following interesting function

$$\psi_x = 2\delta_x - \delta_{xuv} - \delta_{xvu}$$

We apply Δ_θ to it and we obtain

$$\begin{aligned}\Delta_\theta \psi_x = -2\Big[&2\delta_{xu} - 2\delta_x + 2\delta_{xv} - 2\delta_x + 2\delta_{xw} - 2\delta_x \\ &- \delta_{xw} + \delta_{xuv} - \delta_{xu} + \delta_{xuv} - \delta_{xv} + \delta_{xuv} \\ &- \delta_{xv} + \delta_{xvu} - \delta_{xw} + \delta_{xvu} - \delta_{xu} + \delta_{xvu}\Big] \\ = -2\Big[&- 6\delta_x + 3\delta_{xuv} + 3\delta_{xvu}\Big] \\ = 6&\psi_x\end{aligned}$$

Hence, ψ_x is an eigenfuction of Δ_θ with eigenvalue 6. Consider the elements of S_3 being in order $\{e, u, v, w, uv, vu\}$, then one may expect 6 possible ψ_x. The six functions are related by $\psi_e + \psi_{uv} + \psi_{vu} = 0$ and $\psi_u + \psi_v + \psi_w = 0$, that reduce the number of independent functions to four. This can be visualised in the following figure.



We are now equipped with all the necessary tools to develop a Maxwell's theory of electromagnetism on graphs.

Chapter 5

Geometric form of Maxwell's equation

In this chapter, we start with the local form of Maxwell's equations for a fixed frame of reference, and then rewrite them in the formalism of differential geometry. This approach is convenient for two reasons. First, the geometrical formulation highlights some symmetries and elegantly shows their invariance under gauge transformations. Second, we express them in terms of operators that are immediately extended to the graph setting we have introduced in the previous chapter.

We will first recall some notions of differential geometry that are necessary. Then, we will introduce two operators to develop a Maxwell's theory on the graph framework. Finally, we will rewrite the equations in a geometric and compact way. For full discussion about differential geometry on manifolds, we invite the reader to consult [14] and [15].

5.1 Forms and Wedge Product

We first introduce the *exterior forms* on $\mathbb{R}^n$.

Let $\mathcal{M}$ be a manifold, and $\mathcal{M} = \mathbb{R}^n$ for simplicity. Exterior one-forms at a point $x \in \mathcal{M}$ are linear maps from the tensor space to $\mathcal{M}$ into real numbers $\mathbb{R}$, i.e.

$$\begin{array}{rccc} \omega\colon & T_xM & \longrightarrow & \mathbb{R} \\ & v & \longrightarrow & \omega(v) \end{array}$$

We call the dual vector space T_x^*M, the cotangent space of $\mathcal{M}$ at x.

We also introduce the notation for the partial derivative. The partial derivative is in general defined as follows.

$$df = \sum_i \partial_i f dx^i$$

In case of f smooth function on $\mathbb{R}^n$ and $\{x_1, x_2, \cdots, x_n\}$ a set of global coordinates, the partial derivative by the component x^i is given by $\partial_i f = \frac{\partial}{\partial x^i} f$.

If the smooth n-manifold $\mathcal{M}$ is not euclidean, then an atlas of local charts must be constructed. In our discussion, an euclidean manifold is enough, however we recommend the reader to consult [14] and [15] for the general case.

Given the definition of directional derivative of a smooth function f over $\mathbb{R}^n$, i.e.

$$df(v)|_x = v(f)(x) = \sum_{i=1}^{n} v^i \partial_i f|_x$$

we rewrite the vector v on $\mathbb{R}^n$ as $v = \sum_{i=1}^{n} v^i \partial_i$. Moreover, a vector field V on $\mathbb{R}^n$ can be written as $\sum_i v^i(x)\partial_i$. Thus, ∂_i form a basis for $T_x M$.

Using the directional derivative, we see that dx^i form a basis for the dual vector space with respect to ∂_i, i.e. $T_x^* M = \text{span}\{dx^i\}$ for $i = [n]$. For this reason we write a one-form $\omega \in T_x^* M$ as

$$\omega = \sum_{i=1}^{n} \omega_i dx^i$$

A one-form ω is said to be smooth if it is defined on all points of $\mathcal{M}$ and if $\omega(V)$ is smooth for all vector fields $V \in \mathcal{V}(\mathcal{M})$.

The one-form can be extended to k-forms, but we need an additional object. There exists a skew-symmetric, associative produce of exterior forms that we call *exterior product* or *wedge product*. A simplified way to introduce it is by its action on vectors. For example,

$$(dx^i \wedge dx^j)(v, w) = v^i w^j - v^j w^i = \det \begin{pmatrix} v^i & w^i \\ v^j & w^j \end{pmatrix}$$

In this way, it can easily be extended to three or more factors, i.e.

$$(dx^i \wedge dx^j \wedge dx^k)(u, v, w) = \det \begin{pmatrix} u^i & v^i & w^i \\ u^j & v^j & w^j \\ u^k & v^k & w^k \end{pmatrix}.$$

The set of $dx^i \wedge dx^j$ for all combinations of i and j such that $i \neq j$ provides a basis for arbitrary smooth 2-forms, i.e.

$$\omega = \sum_{i<j} \omega_{ij}(x) dx^i \wedge dx^j.$$

The skew-symmetry limits the number of basis elements to $\binom{n}{2}$. This construct can easily be extended to an arbitrary (finite) number of factors, i.e.

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \cdots, i_k}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad k = 3, ..., n$$

where $\omega_{i_1,\cdots,i_k}(x)$ are again smooth functions on $\mathbb{R}^n$. We usually denote the space of the k-forms over a manifold $\mathcal{M}$ as $\Lambda^k(\mathcal{M})$, i.e. $\omega \in \Lambda^k(\mathcal{M})$. The dimension of this space is given by the combinations of elements dx^i without repeating them. Therefore

$$\dim \Lambda^k(\mathcal{M}) = \binom{n}{k}$$

Using the notation of the previous chapters, one can see that $\omega \in \Omega^1$.

5.2 Exterior Derivative and Wedge product

The electric and magnetic fields in the Maxwell's equation will be rewritten as k-forms and therefore we now need to recall the action of the exterior derivative on k-forms. The exterior derivative d maps k-forms to $(k+1)$-forms, i.e.

$$\begin{array}{rccc} d: & \Lambda^k(\mathcal{M}) & \longrightarrow & \Lambda^{k+1}(\mathcal{M}) \\ & \omega & \longrightarrow & d\omega \end{array}$$

In addition, the exterior derivative fulfils a graded Leibniz rule. When applied to the wedge product of a r-form ω and a s-form σ $(r, s \leq n)$ the result is

$$d(\omega \wedge \sigma) = (d\omega) \wedge \sigma + (-1)^r \ \omega \wedge (d\sigma).$$

As we will see, the exterior derivative is a generalisation of the total derivative for functions, and of the gradient, the curl and the divergence for vector fields in $\mathbb{R}^n$. Thus, it is the object we need to generalise the Maxwell's equations.

The remarkable result, which is vital for the rest of the discussion, is that the exterior derivative applied twice always yields zero, i.e.

$$d \circ d = 0$$

Finally, we introduce some notation.

Let ω be a k-form. We call it *closed* if the application of the exterior derivative to it yields zero, i.e.

$$d\omega = 0, \quad \omega \in \Lambda^k(\mathcal{M})$$

Let η be a $(k+1)$-form. We say η is *exact* if it is the exterior derivative of a k-form, i.e.

$$\eta = d\omega, \quad \eta \in \Lambda^{k+1}(\mathcal{M}),\ \omega \in \Lambda^k(\mathcal{M}).$$

Clearly, every exact form is also a closed form. For further details and proofs on differential geometry see [14] and [15].

Finally, in the following example we briefly show how the exterior derivative replaces the divergence and curl of a vector field in $\mathbb{R}^3$.

5.3 Hodge operator

We now introduce the hodge operator. Let V be a finite-dimensional vector space equipped with a non-degenerate scalar product $(,)$. We recall from linear algebra that the scalar product on V induces a scalar product on its dual V^* through a linear map $\mathcal{L}$ such that

$$\begin{array}{rccc} \mathcal{L}\colon & V & \longrightarrow & V^* \\ & v & \longrightarrow & \mathcal{L}(v) = \mathcal{L}_v \end{array}$$

where $\mathcal{L}_v w = (v, w)$ for all $v, w \in V$. The non-degeneracy of $(,)$ implies $\ker \mathcal{L} = 0$. Because $\dim V = \dim V^*$, then $\mathcal{L}$ is an isomorphism. Since we assume the scalar product on V to be non-degenerate we define a scalar product on V^* as

$$(v^*, w^*)_{V^*} = (\mathcal{L}^{-1}v, \mathcal{L}^{-1})_V$$

for every $v, w \in V$ given $\mathcal{L}(v) = v^*$ and $\mathcal{L}(w) = w^*$. Consider now the space $\Lambda^k(V^*)$ where its elements are a linear combination of elements of the type $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k$. There exists a unique scalar product defined on $\Lambda^k(V^*)$ such that

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k, \gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_k) = \det \Big((\alpha_i, \gamma_i) \Big).$$

(see [2], page 670). There is an additional notion that we need to introduce to define the Hodge operator, the orientation of a basis. Let V be a n-dimensional vector space equipped with a non-degenerate scalar product. Let also $\mathcal{B}_1$ and $\mathcal{B}_2$ be two orthonormal basis for V. There exists a unique linear map L from $\mathcal{B}_1$ to $\mathcal{B}_2$ such that $\det L$ is either $+1$ or -1. This means that we can identify two equivalence classes. Explicitly, if the map between two basis has $\det L = +1$ then the two basis are in the same equivalence class. On the other hand, if $\det L = -1$ the two basis are in different equivalence classes. Now we define the *orientation* of the basis by assigning $+1$ to an equivalence class and -1 to the other, arbitrarily. Consider now the space $\Lambda^n(V^*)$ with two basis $\mathcal{B}_1 = e_1 \wedge \cdots \wedge e_n$ and $\mathcal{B}_2 = f_1 \wedge \cdots \wedge f_n$. They differ by a value $\det L = \pm 1$ where L is the change of basis matrix.

Finally, a non-degenerate scalar product together with the choice of orientation of V determines a unique $e_1 \wedge \cdots \wedge e_n \in \Lambda^n(V^*)$. We refer to this element as σ.

Consider now differential forms on the manifold $\mathbb{R}^n$ with global coordinates $x^1, \cdots, x^n$. The conclusion above implies that for each point $x \in \mathbb{R}^n$, we get a unique element of $\Lambda^n(T_x^* M)$, namely a n-form. We define

$$\mathrm{Vol} = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

Now we have all the ingredients necessary to introduce the $\circledast$-operator, usually known as Hodge operator.

First, we show that a wedge operator, together with the choice of Vol $\in \Lambda^n(V^*)$, assigns to each $\lambda \in \Lambda^k(V^*)$ a linear function on $\Lambda^{n-k}(V^*)$. If we pick $\omega \in \Lambda^{n-k}(V^*)$ then $\lambda \wedge \omega$ is an element of $\Lambda^n(V^*)$, hence a multiple of Vol. This means we can write

$$\lambda \wedge \omega = f(\omega)\mathrm{Vol}$$

Therefore, each $\lambda \in \Lambda^k(V^*)$ identifies a linear map

$$\begin{array}{cccc} \lambda : & \Lambda^{n-k}(V^*) & \longrightarrow & \mathbb{R} \\ & \omega & \longrightarrow & f(\omega) \end{array}$$

There is a unique element, that we shall denote $\circledast\lambda$, which determines the same function $f(\omega)$ from $\Lambda^{n-k}(V^*)$ to $\mathbb{R}$ via the scalar product. Thus, given $\lambda \in \Lambda^k(V^*)$, we uniquely define $\circledast\lambda \in \Lambda^{n-k}(V^*)$ by the condition

$$\lambda \wedge \omega = (\circledast\lambda, \omega)\mathrm{Vol}$$

for all $\omega \in \Lambda^{n-k}(V^*)$. Note that if we change orientation the sign of Vol changes and hence the sign of $\circledast\lambda$ as well. In the cases of $k = 0, n$, we define 1 as the basis element of $\Lambda^0(V^*)$, with scalar product $(1, 1) = 1$ and the trivial wedge product $1 \wedge \omega = \omega = \omega \wedge 1$.

The general way to calculate it is to apply the condition above using basis elements of $\Lambda^{n-k}(V^*)$ as ω in turns. In fact, once λ is a basis element of $\Lambda^k(V^*)$ then the calculation is easier.

We now provide an example in the $\mathbb{R}^4$ Minkowski space, thus with the Lorentz scalar product.

Example 5.3.1. Consider the four-dimensional spacetime with the Lorentz scalar product. The basis for $\Lambda^1(V)$ is $\{dt, dx, dy, dz\}$ with $(dt, dt) = 1$ and $(dx, dx) = (dy, dy) = (dz, dz) = -1$, and $\sigma = dt \wedge dx \wedge dy \wedge dz$. From the definition we see

$$\circledast(dt \wedge dx \wedge dy \wedge dz) = 1.$$

In addition, using the equation for a scalar product between differential forms in the previous page we have

$$(\mathrm{Vol}, \mathrm{Vol}) = \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -1$$

hence

$$\circledast 1 = -\mathrm{Vol}.$$

Now we proceed to the other differential forms. Recall that any switch of two factors in a differential form comes with a change of sign, i.e.

$$dx^1 \wedge dx^2 \wedge \cdots \wedge dx^i \wedge dx^{i+1} \wedge \cdots \wedge dx^n = (-1) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{i+1} \wedge dx^i \wedge \cdots \wedge dx^n$$

this is true in the specific case of the usual differential geometry. However the wedge product is much more general as we will see later when we develop differential geometry on a graph.

$$\begin{array}{rcl}
(dt \wedge dx \wedge dy) \wedge dz = (\circledast(dt \wedge dx \wedge dy), dz)\sigma & \Longrightarrow & \circledast(dt \wedge dx \wedge dy) = -dz \\
\text{Similarly} & \Longrightarrow & \circledast(dt \wedge dx \wedge dz) = dy \\
& \Longrightarrow & \circledast(dt \wedge dy \wedge dz) = -dx \\
& \Longrightarrow & \circledast(dx \wedge dy \wedge dz) = -dt
\end{array}$$

$$\begin{array}{rcl}
(dt \wedge dx) \wedge (dy \wedge dz) = (\circledast(dt \wedge dx), (dy \wedge dz))\sigma & \Longrightarrow & \circledast(dt \wedge dx) = dy \wedge dz \\
\text{Similarly} & \Longrightarrow & \circledast(dt \wedge dy) = -dx \wedge dz \\
& \Longrightarrow & \circledast(dt \wedge dz) = dx \wedge dy \\
(dx \wedge dy) \wedge (dt \wedge dz) = (\circledast(dx \wedge dy), (dt \wedge dz))\sigma & \Longrightarrow & \circledast(dx \wedge dy) = -dt \wedge dz \\
& \Longrightarrow & \circledast(dx \wedge dz) = dt \wedge dy \\
& \Longrightarrow & \circledast(dy \wedge dz) = -dt \wedge dx
\end{array}$$

$$\begin{array}{rcl}
dt \wedge (dx \wedge dy \wedge dz) = (\circledast dt, dx \wedge dy \wedge dz)\sigma & \Longrightarrow & \circledast dt = -dx \wedge dy \wedge dz \\
\text{Similarly} & \Longrightarrow & \circledast dx = -dt \wedge dy \wedge dz \\
& \Longrightarrow & \circledast dy = dt \wedge dx \wedge dz \\
& \Longrightarrow & \circledast dz = -dt \wedge dx \wedge dy
\end{array}$$

5.4 Codifferential and Laplace-de Rham operator

In this section, we introduce the codifferential and the Laplace-de Rham operator. They are a combination of the exterior derivative and the Hodge operator. The codifferential and the exterior derivative d play a pivotal role in the geometric expression of Maxwell's equations.

Definition 5.4.1. Let d denote the exterior derivative and $\circledast$ denote the Hodge dual on $\mathbb{R}^n$. The *codifferential* is a linear map from Λ^k to Λ^{k-1}, i.e.

$$\begin{array}{rccc}
\delta : & \Lambda^k & \longrightarrow & \Lambda^{k-1} \\
& \omega & \longrightarrow & \delta\omega
\end{array}$$

The action of this operator is defined by

$$\delta = (-1)^{n(k+1)+1} \circledast d \circledast .$$

In addition, we call the *Laplace-de Rham* operator the sum of the combined operations $d \circ \delta$ and $\delta \circ d$, i.e.

$$\Delta_{LdR} := d \circ \delta + \delta \circ d$$

Recall that the exterior derivative raises the order of the differential form, whereas the codifferential lowers it. Thus, the Laplace-de Rham operator doesn't affect the order of the differential form, i.e.

$$\Delta_{LdR} : \Lambda^k \longrightarrow \Lambda^k$$

In the next section, we show how we can use the formalism of differential geometry and the operator just defined to rewrite the Maxwell's equations in a geometric form.

5.5 Classical Maxwell's equations in compact geometric form

Now we turn to Maxwell's equations. Our aim is to write the Maxwell's equations in a geometric form. We begin with their integral and local form for a fixed frame of reference. Then, by expressing the Lorentz force law as a relativistic Newton's second law, we will introduce a field strength tensor $F^{\mu\nu}$. We will use that tensor to express the Maxwell's field equations in tensorial notation. Finally, we will use differential geometry to highlight the geometric structure of the equations.

Let S be a surface and V a volume embedded in $\mathbb{R}^n$.

Electronic Gauss' law	$\oiint_{\partial V} (\mathbf{D}(t, \mathbf{x}) \cdot \mathbf{n}) d\sigma = f_G \iiint_V \rho(t, \mathbf{x}) d^3x$
Faraday's law	$\oint_{\partial S} \mathbf{E}(t, \mathbf{x}) \cdot d\mathbf{s} = -f_F \frac{\partial}{\partial t} \iint_S \mathbf{B}(t, \mathbf{x}) \cdot \mathbf{n}(t, \mathbf{x}) d\sigma$
Magnetic Gauss' law	$\oiint_S (\mathbf{B}(t, \mathbf{x}) \cdot \mathbf{n}) d\sigma = 0$
Biot-Savart's law	$\oint_{\partial S} \mathbf{H}(t, \mathbf{x}) \cdot d\mathbf{s} - \frac{f_{BS}}{f_G} \frac{\partial}{\partial t} \iint_S (\mathbf{D}(t, \mathbf{x}) \cdot \mathbf{n}) d\sigma = f_{BS} \iint_S (\mathbf{j}(t, \mathbf{x}) \cdot \mathbf{n}) d\sigma$

Recall the Maxwell's equations in local form. We will show that introducing the exterior derivative allows us to rewrite the equations in a more elegant way.

Electronic Gauss' law	$\nabla \cdot \mathbf{D}(t, \mathbf{x}) = f_G \rho(t, \mathbf{x})$
Faraday's law	$\nabla \times \mathbf{E}(t, \mathbf{x}) = -f_F \frac{\partial}{\partial t} \mathbf{B}(t, \mathbf{x})$
Magnetic Gauss' law	$\nabla \cdot \mathbf{B}(t, \mathbf{x}) = 0$
Biot-Savart's law	$\nabla \times \mathbf{H}(t, \mathbf{x}) - \frac{f_{BS}}{f_G} \frac{\partial}{\partial t} \mathbf{D}(t, \mathbf{x}) = f_{BS} \mathbf{j}(t, \mathbf{x})$

The constants f_G, f_F and f_{BS} depend on the system of unit. If we pick the SI-system then their fixed values are $f_G = f_F = f_{BS} = 1$. On the other hand, with Gaussian units the constants' values are $f_F = 1/c$, $f_G = 4\pi$ and $f_{BS} = 4\pi/c$. The question of units is no further discussed here, we invite the interested reader to consult [7, 12]. From now on we will use Gaussian units.

Consider now the integral Faraday's law. The closed curve ∂S is a one dimensional manifold embedded in $\mathbb{R}^3$. Consider the path integral, it must be the integral of a one-form along ∂S, and consequently on $\mathbb{R}^3$. It seems natural to associate the electric field to a one-form, i.e.

$$\omega_{\mathbf{E}} := E_1(t,\mathbf{x})dx^1 + E_2(t,\mathbf{x})dx^2 + E_3(t,\mathbf{x})dx^3.$$

Similarly, we naturally associate the field $\mathbf{B}$ to a two-forms, i.e.

$$\omega_{\mathbf{B}} := B_1(t,\mathbf{x})dx^2 \wedge dx^3 + B_2(t,\mathbf{x})dx^3 \wedge dx^1 + B_3(t,\mathbf{x})dx^1 \wedge dx^2.$$

Following a similar argument but applied to the electronic Gauss' law, we write

$$\omega_{\mathbf{D}} := D_1(t,\mathbf{x})dx^2 \wedge dx^3 + D_2(t,\mathbf{x})dx^3 \wedge dx^1 + D_3(t,\mathbf{x})dx^1 \wedge dx^2,$$

and lastly, for the Biot-Savart law we express $\mathbf{H}$ as a one-form, i.e.

$$\omega_{\mathbf{H}} := H_1(t,\mathbf{x})dx^1 + H_2(t,\mathbf{x})dx^2 + H_3(t,\mathbf{x})dx^3.$$

In conclusion, we express the source terms in the inhomogeneous equations as k-forms as well. Namely,

$$\omega_\rho := \rho(t,\mathbf{x})dx^1 \wedge dx^2 \wedge dx^3$$
$$\omega_{\mathbf{j}} := j_1(t,\mathbf{x})dx^2 \wedge dx^3 + j_2(t,\mathbf{x})dx^3 \wedge dx^1 + j_3(t,\mathbf{x})dx^1 \wedge dx^2.$$

Given this notation, we can express the Maxwell's equations in terms of k-forms. By doing that we unify the expression of the curl and divergence under the exterior derivative.

Electronic Gauss' law	$d\omega_{\mathbf{D}} = 4\pi\omega_\rho$
Faraday's law	$d\omega_{\mathbf{E}} + \frac{1}{c}\frac{\partial}{\partial t}\omega_{\mathbf{B}} = 0$
Magnetic Gauss' law	$d\omega_{\mathbf{B}} = 0$
Biot-Savart's law	$d\omega_{\mathbf{H}} - \frac{1}{c}\frac{\partial}{\partial t}\omega_{\mathbf{D}} = \frac{4\pi}{c}\omega_{\mathbf{j}}$

The right hand side of Maxwell's equations is either zero or a source term. Those very source terms undergo motion according to Lorentz force law. In fact, one needs that force law to have a complete theory of electromagnetism. We then rewrite that equation in a tensorial way.

5.6 Field Strength Tensor

In this section, we introduce the Field Strength tensor via a discussion on relativistic kinematics. This tensor will be useful to give a compact form to Maxwell's equations. This section follows the discussion in [12] section 2.3.

Consider a particle with mass m. As far as the Standard Model is concerned, all the charged particles are massive. We define two Frames of Reference, a first one in which the particle is at rest, $\mathbf{K}_0$, and a second one in which the particle is moving at constant speed $\mathbf{v}$, that we call $\mathbf{K}$.

The four-momentum in $\mathbf{K}_0$ is $\mathbf{P}|_{\mathbf{K}_0} = (mc, \mathbf{0})^T$, whereas $\mathbf{K}$ is $\mathbf{P}|_{\mathbf{K}} = (\frac{1}{c}E, \mathbf{p})^T$, with $\mathbf{p} = m\gamma v$. The two momenta are related by a boost operator $\mathbf{L_P}$, i.e.

$$\mathbf{L_P} = \frac{1}{mc^2}\begin{pmatrix} E & c\mathbf{p}^T \\ c\mathbf{p} & mc^2\mathbb{I}_3 + \frac{c^2}{E+mc^2}\mathbf{p}\mathbf{p}^T \end{pmatrix}.$$

The force in $\mathbf{K}_0$ is given by $m\ddot{\mathbf{x}} = \mathbf{F}_N(\mathbf{x})$ whereas in $\mathbf{K}$ is $f(\mathbf{x}) = \mathbf{L_P}\mathbf{F}_N(\mathbf{x})$. Therefore, a Lorentz covariant version of the Newton's second law reads

$$m\frac{d^2}{d\tau^2}\mathbf{x}(\tau) = m\frac{d}{d\tau}u(\tau) = f(\mathbf{x})$$

where the components of $f(\mathbf{x})$ are

$$\begin{pmatrix} f^0 \\ \mathbf{f} \end{pmatrix} = \begin{pmatrix} \gamma & \frac{1}{c}\gamma\mathbf{v}^T \\ \frac{1}{c}\gamma\mathbf{v} & \mathbb{I}_3 + \frac{\gamma^2}{c^2(\gamma+1)}\mathbf{v}\mathbf{v}^T \end{pmatrix}\begin{pmatrix} 0 \\ \mathbf{F}_N \end{pmatrix} = \begin{pmatrix} \frac{1}{c}\gamma(\mathbf{v}\cdot\mathbf{F}_N) \\ \mathbf{F}_N + \frac{\gamma^2}{c^2(\gamma+1)}(\mathbf{v}\cdot\mathbf{F}_N)\mathbf{v} \end{pmatrix}$$

Recall $u(\tau)|_{\mathbf{K}} = (\gamma c, \gamma\mathbf{p})^T$, the equations of motion above, that can be written explicitly as

$$mc\frac{d\gamma}{dt} = \frac{1}{c}(\mathbf{F}_N\cdot\mathbf{v})$$

$$\cancel{\gamma}\frac{d}{dt}\mathbf{p} = \cancel{\gamma}\mathbf{F}_N + \frac{1}{c}\mathbf{v}\times\left(\frac{\gamma\cancel{\gamma}}{c(\gamma+1)}(\mathbf{v}\times\mathbf{F}_N)\right)$$

If we consider the force on a non-moving charged particle, i.e. $\mathbf{F}_N = q\mathbf{E}$, then the boosted Lorentz force equations read

$$m\gamma\frac{d}{dt}(\gamma c) = \frac{\gamma}{c}\mathbf{v}\cdot\frac{d\mathbf{p}}{dt} = \frac{\gamma}{c}\mathbf{v}\cdot\left(q\mathbf{E}(t,\mathbf{x}) + \frac{q}{c}\mathbf{v}\times\mathbf{B}(t,\mathbf{x})\right) = \gamma\frac{1}{c}q\mathbf{E}\cdot\mathbf{v}$$

$$m\gamma\frac{d}{dt}(\gamma\mathbf{v}) = \gamma\left(q\mathbf{E}(t,\mathbf{x}) + \frac{q}{c}\mathbf{v}\times\mathbf{B}(t,\mathbf{x})\right)$$

the left hand side can be written covariantly $m\frac{du^\mu}{d\tau}$, the right hand side can also be expressed in term of the four-velocity. Let us introduce

$$F^{\mu\nu}(x) := \begin{pmatrix} 0 & -E^1(x) & -E^2(x) & -E^3(x) \\ E^1(x) & 0 & -B^3(x) & B^2(x) \\ E^2(x) & B^3(x) & 0 & -B^1(x) \\ E^3(x) & -B^2(x) & B^1(x) & 0 \end{pmatrix}, \quad x = (t, \mathbf{x})^T$$

and let this field act on $u_\nu = g_{\nu\mu}u^\mu = \gamma(c, -\mathbf{x})^T$. In this way, we generalise the equations of motion to a relativistic compact expression, i.e.

$$m\frac{du^\mu}{d\tau} = \frac{q}{c}F^{\mu\nu}u_\nu.$$

5.7 Maxwell's equations in compact tensorial form

The field strength tensor is extremely useful to express Maxwell's equations in a more compact way. In fact, we can directly rewrite the homogeneous equations as follow

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0, \quad \mu \neq \nu \neq \lambda \in (0,1,2,3)$$

One may also introduce a Levi-Civita symbol in 4 dimensions to make the equation even more compact. Let $\epsilon_{\mu\nu\sigma\tau}$ have the following properties,

$$\epsilon_{\mu\nu\sigma\tau} = \begin{cases} +1 & \mu\nu\sigma\tau = 0123 \\ +1 & \text{even permutations of } 0123 \\ -1 & \text{odd permutations of } 0123 \end{cases}$$

In this case, the homogeneous Maxwell's equations become

$$\epsilon_{\mu\nu\sigma\tau}\partial^\nu F^{\sigma\tau}(x) = 0, \quad \mu = 0,1,2,3$$

A similar discussion can be done for the inhomogeneous equations. However, we first need to translate the source terms in a single Lorentz invariant vector. Multiple approaches can be taken. Recall the charge conservation in local form, i.e.

$$\frac{\partial\rho(x)}{\partial t} + \nabla \cdot \mathbf{j}(x) = 0, \quad x = (t, \mathbf{x})^T.$$

Consider the operator $\partial_\nu = (\frac{\partial}{\partial(ct)}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, we can include all the information of the source terms in a single tensor, i.e.

$$j^\nu = (c\rho(x), \mathbf{j}(x))^T.$$

Hence, we write the conservation equation as

$$\partial_\nu j^\nu = 0.$$

A different and deeper approach to this matter can be found on [12] page 129.

Now, to rewrite the left hand side, we introduce a new tensor similarly to what we did for $F^{\mu\nu}$ but using the fields $\mathbf{D}$ and $\mathbf{H}$. We define

$$\mathcal{F}^{\mu\nu}(x) := \begin{pmatrix} 0 & -D^1(x) & -D^2(x) & -D^3(x) \\ D^1(x) & 0 & -H^3(x) & H^2(x) \\ D^2(x) & H^3(x) & 0 & -H^1(x) \\ D^3(x) & -H^2(x) & H^1(x) & 0 \end{pmatrix}, \quad x = (t, \mathbf{x})^T$$

Then, the inhomogeneous Maxwell's equations in Gaussian units can be written as

$$\partial_\mu \mathcal{F}^{\mu\nu}(x) = \frac{4\pi}{c} j^\nu, \quad \nu = 0, 1, 2, 3$$

The explicit calculation to verify that those are the Maxwell's equations can be found in [12] section 2.3.

5.8 Compact geometric form of Maxwell's equations

In the previous paragraph, we saw that the formulation with the two tensors $F^{\mu\nu}$ and $\mathcal{F}^{\mu\nu}$ is convenient. We aim to incorporate them in the geometric expression of the Maxwell's equations.

First, we set ourselves in the $\mathbb{R}^4$ Minkowski space with metric $\mathbf{g} = \text{diag}(1, -1, -1, -1)$. In the previous section, we simply extended the operators, now a more formal approach is taken. A basis for k-forms is given by wedge products of elements dx^μ with $\mu = 0, 1, 2, 3$. We define the two-form

$$\omega_F := \sum_{\mu<\nu} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

where the covariant components $F_{\mu\nu}$ can be obtained by contracting $F^{\mu\nu}$ twice with the Minkowski metric tensor. If one sticks to the frame of reference in which we defined $F^{\mu\nu}$, the explicit expression for $F_{\mu\nu}$ is

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}.$$

Recall the k-forms associated to $\mathbf{E}$ and to $\mathbf{B}$. It naturally follows that we express ω_F in terms of them, i.e.

$$\omega_F = dx^0 \wedge \omega_{\mathbf{E}} - \omega_{\mathbf{B}}.$$

Similarly, we define the two-form correspondent to $\mathcal{F}^{\mu\nu}$.

$$\omega_{\mathcal{F}} := \sum_{\mu<\nu} \mathcal{F}_{\mu\nu} dx^\mu \wedge dx^\nu$$

At this point, we have all the necessary ingredients to rewrite the Maxwell's equations. We recall the result of applying the exterior derivative to a two-form. We can write the homogeneous Maxwell's equations in terms of the simple exterior derivative of the ω_F two-form, i.e.

$$d\omega_F = 0$$

For the inhomogeneous equations we need some additional steps. First, we pick $dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ as a basis for Λ^4. Let us apply the Hodge operator to $\omega_{\mathcal{F}}$,

$$\begin{aligned}\circledast\omega_{\mathcal{F}}(x) = \sum_{\mu<\nu} \mathcal{F}_{\mu\nu}(x)\Big(\circledast (dx^\mu \wedge dx^\nu)\Big) &= \sum_{\mu<\nu} \mathcal{F}_{\mu\nu}(x)\left(\frac{1}{2} g^{\mu\lambda} g^{\nu\rho} \epsilon_{\lambda\rho\sigma\tau} dx^\sigma \wedge dx^\tau\right) \\ &= \frac{1}{2}\sum_{\mu<\nu} \mathcal{F}^{\lambda\rho}(x)\epsilon_{\lambda\rho\sigma\tau} dx^\sigma \wedge dx^\tau,\end{aligned}$$

then we apply the exterior derivative,

$$d(\circledast\omega_{\mathcal{F}})(x) = \frac{1}{4}\epsilon_{\lambda\rho\sigma\tau}\partial_\alpha \mathcal{F}^{\lambda\rho}(x)(dx^\alpha \wedge dx^\sigma \wedge dx^\tau).$$

Note that the index α needs to be different from both σ and τ. In addition, thanks to the presence of the term $\epsilon_{\lambda\rho\sigma\tau}$, λ and ρ also need to be different from both σ and τ. Therefore we can either have $\lambda = \alpha$ or $\rho = \alpha$. This is made clearer by applying the second star operator to the last result, i.e.

$$\begin{aligned}\circledast d \circledast \omega_{\mathcal{F}}(x) = \frac{1}{4}\epsilon_{\lambda\rho\sigma\tau}\partial_\alpha \mathcal{F}^{\lambda\rho}(x)\Big(\circledast (dx^\alpha \wedge dx^\sigma \wedge dx^\tau)\Big) &= \frac{1}{4}\epsilon_{\lambda\rho\sigma\tau}\partial_\alpha \mathcal{F}^{\lambda\rho}(x)\Big(\epsilon_{\beta\gamma\delta\eta} g^{\beta\alpha} g^{\gamma\sigma} g^{\delta\tau} dx^\eta\Big) \\ &= \frac{1}{4}\partial_\alpha \mathcal{F}^{\lambda\rho}(x)\epsilon_{\lambda\rho\sigma\tau}\epsilon^{\alpha\sigma\tau\delta} g_{\delta\eta} dx^\eta\end{aligned}$$

Consider now the two Levi-Civita symbols, we can write

$$\epsilon_{\lambda\rho\sigma\tau}\epsilon^{\alpha\sigma\tau\delta} = \epsilon_{\sigma\tau\lambda\rho}\epsilon^{\sigma\tau\alpha\delta} = -2(\delta^\alpha_\lambda\delta^\delta_\rho - \delta^\alpha_\rho\delta^\delta_\lambda).$$

Using this expression in the result before yields

$$\begin{aligned}\circledast d \circledast \omega_{\mathcal{F}} = -\frac{1}{2}\partial_\alpha \mathcal{F}^{\lambda\rho}(x)(\delta^\alpha_\lambda\delta^\delta_\rho - \delta^\alpha_\rho\delta^\delta_\lambda) g_{\delta\eta} dx^\eta &= -\frac{1}{2}\Big(\partial_\lambda \mathcal{F}^{\lambda\delta}(x) - \partial_\alpha \mathcal{F}^{\delta\alpha}(x)\Big) g_{\delta\eta} dx^\eta \\ &= -\partial_\lambda \mathcal{F}^{\lambda\delta}(x) g_{\delta\eta} dx^\eta\end{aligned}$$

Recall the definition 5.4 of codifferential, and consider $n = 4$ and $k = 2$ for that definition. Then we can write the last result simply as

$$\delta\omega_{\mathcal{F}} = \partial_\lambda \mathcal{F}^{\lambda\delta}(x) g_{\delta\eta} dx^\eta.$$

Now we discuss the right hand side of the inhomogeneous equation. In a previous result we were able to write the source terms as a single tensor, i.e.

$$j^\nu = (c\rho(x), \mathbf{j}(x))^T.$$

On the other hand, we wrote the source terms as a three-form for the density of charge and a two-form for the current density. We can immediately associate a three-form to the source term by extending the current two-forms including the time differential

$$\omega_j = \frac{1}{3!} \epsilon_{\mu\alpha\beta\gamma} j^\mu dx^\alpha \wedge dx^\beta \wedge dx^\gamma$$

If we take the dual to this expression we obtain

$$\circledast\omega_j = \frac{1}{3!} \epsilon_{\mu\alpha\beta\gamma} j^\mu \epsilon^{\alpha\beta\gamma\delta} g_{\delta\eta} dx^\eta,$$

then we consider the two Levi-Civita symbols contraction as before

$$\epsilon_{\mu\alpha\beta\gamma}\epsilon^{\alpha\beta\gamma\delta} = 3!\delta^\delta_\mu.$$

Finally, we can write

$$\circledast\omega_j = j^\mu g_{\delta\eta} dx^\eta$$

Recall the inhomogeous Maxwell's equations in tensorial formalism from the previous section and consider the latter result and the previous expression obtained for the left hand side. We can finally write the inhomogeous Maxwell's equations as

$$\delta\omega_{\mathcal{F}} = \frac{4\pi}{c} \circledast \omega_j.$$

We can improve the Maxwell's equations when we express them in vacuum. In this case, the relations between the fields **D**, **H** and **E**, **B** in Gaussian units are simply $\mathbf{D} = \mathbf{E}$ and $\mathbf{H} = \mathbf{B}$. For this reason, the $F^{\mu\nu}$ and $\mathcal{F}^{\mu\nu}$ have the same expression. If we change slightly the notation so that $\omega_F \equiv F$ and $\omega_j \equiv j$, we can express the Maxwell's equations in vacuum in an elegant and compact form.

$$\begin{array}{rl} \text{Homogeneous} & dF = 0 \\ \text{Inhomogenous} & \delta F = J \end{array}$$

where $J = \frac{4\pi}{c} \circledast j$. This is the most compact geometric form that the Maxwell's equations can get when expressed in vacuum. The tools of differential geometry help us to highlight that gauge invariance of those equations. This will be covered in the next chapter.

Chapter 6

Maxwell's electromagnetism as a gauge theory on graphs

In this chapter, we will discuss the gauge invariance of Maxwell's equations (ME). This gives an alternative method to solve them. We will then rewrite the equations in the framework of graphs. Finally, we will provide the example of Maxwell's theory on the permutation group S_3.

We begin with a brief recall of gauge invariance of ME in local form. Then we will follow the same path as the previous chapter: first use tensor expression and finally k-forms.

6.1 The classical Vector and scalar potential

Here we recall the vector and scalar potentials in the classical electromagnetic theory. For more details see [7] sections 6.2 and 6.3 and [12] section 1.6. We begin by recalling the ME in local form in vacuum expressed in Gaussian units.

$$\nabla \times \mathbf{E}(t,\mathbf{x}) = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}(t,\mathbf{x}) \qquad \nabla \cdot \mathbf{E}(t,\mathbf{x}) = 4\pi\rho(t,\mathbf{x})$$

$$\nabla \cdot \mathbf{B}(t,\mathbf{x}) = 0 \qquad \nabla \times \mathbf{B}(t,\mathbf{x}) - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}(t,\mathbf{x}) = \frac{4\pi}{c}\mathbf{j}(t,\mathbf{x})$$

Recall that the divergence of the curl of a vector gives zero. From the magnetic homogeneous equation we can say $\mathbf{B} = \nabla \times \mathbf{A}$ where $\mathbf{A}$ is a vector potential with no additional requirements so far. The other homogeneous equation can be rewritten as $\nabla \times (\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}) = 0$. Recall now that the curl of the gradient of a vector is vanishing, then we can say $\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$ where the minus in front of the gradient is a convention.

We now rewrite the ME in terms of $\mathbf{A}$ and Φ. The two homogeneous equations are simply vanishing. The two inhomogeneous ones can be rewritten as

$$\nabla^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot\mathbf{A} = -4\pi\rho$$
$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right) = -\frac{4\pi}{c}\mathbf{j}$$

Hence, we reduced the number of equations to two. However they are still coupled. In order to relax this coupling we can make a further consideration. The field $\mathbf{B}$ is invariant with respect to the sum of a gradient of a scalar function, i.e.

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\Lambda.$$

To leave $\mathbf{E}$ unchanged, we immediately transform the scalar potential as follows

$$\Phi \to \Phi' = \Phi + \frac{\partial\Lambda}{\partial t}.$$

As long as the vector and scalar potential are modified with a scalar function via the two *gauge transformations* above, the Maxwell's equations remain unchanged. This gives us freedom to define the potentials without changing the electromagnetic fields. In other words, we can freely pick a $\Lambda \in \mathcal{C}^\infty(\mathbb{R}^3)$ without affecting the physics as long as the potentials transform as above.

We call *gauge* the condition that we ask the two potentials to satisfy. The two most common gauge choices are the Lorenz Gauge and the Coulomb gauge. The first asks the potentials to satisfy the condition

$$\nabla\cdot\mathbf{A}' + \frac{1}{c}\frac{\partial\Phi'}{\partial t} = 0$$

That is equivalent to asking the scalar function Λ, defined as above, to satisfy

$$\nabla^2\Lambda - \frac{1}{c}\frac{\partial^2\Lambda}{\partial t^2} = -\left(\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right).$$

The second requires the vector potential to satisfy

$$\nabla\cdot\mathbf{A}' = 0$$

That is equivalent to asking

$$\nabla^2\Lambda = -\nabla\cdot\mathbf{A}.$$

These two different gauges are used in different setups and for different reasons. For more details consult [7] section 6.3. We provide them here as a familiar reference. In the next two sections we will discuss the look of these very same steps when we use the tensor (and geometric) form of ME rather than the local one.

6.2 Vector and scalar potential in tensor form

Recall the vacuum Maxwell's equations in tensor compact form, i.e.

$$\epsilon_{\mu\nu\sigma\tau}\partial^{\nu}F^{\sigma\tau}(x) = 0$$

$$\partial_{\nu}F^{\nu\mu}(x) = \frac{4\pi}{c}j^{\mu}$$

for $\mu = 0, 1, 2, 3$. In this notation, we need to proceed with the two potentials as we did with the sources. We combine them in a single tensor, i.e.

$$A(x) = (\Phi(x), \mathbf{A}(x))^T.$$

Recall the partial derivative operator in Minkowski space as defined in the previous section, i.e.

$$\partial^{\mu} = (\frac{1}{c}\frac{\partial}{\partial t}, -\nabla).$$

In this way, the field strength tensor can easily be defined in terms of the four-potential

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}.$$

This is an equivalent but more compact formulation than what we had before. As a matter of fact, for $\nu = 0$ we have

$$\begin{aligned} F^{i0} = E^{i} &= \partial^{i}A^{0} - \partial^{0}A^{i} \\ &= -\partial_{i}\Phi - \frac{1}{c}\frac{\partial A}{\partial t} \end{aligned}$$

where we used $\partial^{i} = g^{ij}\partial_{j}$ with $g^{ij} = \text{diag}(+, -, -, -)$. Note that for $\mu = 3$ and $\nu = 2$ we have

$$\begin{aligned} F^{32} = B^{1} =& \partial^{3}A^{2} - \partial^{2}A^{3} \\ =& -\partial_{3}A^{2} + \partial_{2}A^{3} = (\nabla \times \mathbf{A})^{1} \end{aligned}$$

and similarly we obtain the other components of $\mathbf{B} = \nabla \times \mathbf{A}$.

Both gauge transformations for scalar and for vector potentials can be written with a single expression, i.e.

$$A^{\mu} \rightarrow A'^{\mu} = A^{\mu} - \partial^{\mu}\Lambda$$

where Λ is a well-behaved scalar field. We can see from the definition of $F^{\mu\nu}$ and the commutativity of the partial derivative that the field is invariant to this gauge transformation of A.

When we consider the tensor potential, the homogeneous ME are simply vanishing, whereas the inhomogeneous read

$$\Box A^\nu + \partial^\nu(\partial_\mu A^\mu) = \frac{4\pi}{c} j^\nu$$

where $\Box = \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial}{\partial t} - \Delta$. This is equivalent to the result obtained in the previous section.

With this notation, the Lorenz gauge condition is expressed by

$$\partial_\mu A^\mu = 0, \quad \mu = 0, 1, 2, 3$$

whereas the Coulomb condition is simply

$$\partial_i A^i = 0, \quad i = 1, 2, 3.$$

6.3 Vector and scalar potentials in geometric form

The previous step from usual to tensor notation was pretty much a rewriting. On the other hand, expressing something in geometric form is a deeper conceptual step. In fact, we will be able to use the operators of differential geometry defined on graphs to develop the theory on that very framework.

Recall the vacuum Maxwell's equation in geometric form, i.e.

$$dF = 0$$

$$\delta F = J$$

Now, we write the potential as a 1-form by using its covariant components $A_\nu = g_{\nu\lambda} A^\lambda$, i.e.

$$\omega_A := A_\nu dx^\nu$$

using the Einstein convention for the sum. By applying the exterior derivative we get

$$\begin{aligned} d\omega_A &= \partial_\mu A_\nu dx^\mu \wedge dx^\nu \\ &= \sum_{\mu<\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu. \end{aligned}$$

We now recognise $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and therefore we obtain

$$\omega_F = d\omega_A \iff F = dA$$

This means that we see F as an exact 2-form.

Recall $d \circ d = 0$, in this formulation the homogeneous ME are trivially satisfied. On the other hand, rewriting the inhomogeneous equations is more laborious. The explicit calculation can be found in [12] pages 148-149. The final form of the inhomogeneous equations is

$$\Delta_{LdR}\omega_A - d \circ \delta\omega_A = J \Longleftrightarrow \Delta_{LdR}A - d \circ \delta A = J$$

where $J = \frac{4\pi}{c} j_\nu$. If we explicit the coordinates we get

$$\Box A_\nu - \partial_\nu(\partial^\mu A_\mu) = \frac{4\pi}{c} j_\nu$$

that can be recovered to the equations in the previous section by just applying the inverse metric appropriately to match the indices.

In this framework, the gauge transformations can easily be written as

$$\omega_A \to \omega'_A = \omega_A + d\Lambda.$$

One can immediately see that $\omega_F = d\omega_A$ remains invariant to an addition of an exact form to the gauge potential ω_A.

The gauge condition that is usually picked in this setup is the Coulomb gauge. With this notation, it takes the form $\delta\omega_A = 0$. Thus, the conditions on Λ are

$$\delta\omega'_A = 0 = \delta\omega_A + \delta d\Lambda$$

In the latter section we developed the tools to solve the Maxwell's equations in geometric form using gauge theory. In the next section, we will use these tools to find the expression for the gauge potential when we solve the Maxwell's equations in the graphs framework.

6.4 Maxwell's theory on graphs

The methods developed so far in this chapter can be used to extend Maxwell's theory on discrete groups and graphs. Recall the Maxwell's equations in geometric compact form

$$dF = 0, \quad \delta F = J$$

There are various methods to solve these equations. In our case, we decide to consider the electromagnetic field F as a modulo exact form, i.e. $F = d\alpha$ with $\alpha \in \Omega^1$. With some precautions, we can already work this theory in nice cases armed only with an exterior algebra over an algebra A and a quantum metric, and the calculus will typically be inner. We will look at the case where $A = \mathbb{C}(G)$ in a discrete group G and $\Omega(G)$

bicovariant calculus which is inner with $\theta = \sum_a e_a$, as we saw previously. We consider the Euclidean metric with coefficients in the basis denoted by $\eta^{a,b} = \delta_{a,b^{-1}}$, i.e.

$$g = \sum_{a\in\mathcal{C}} \delta_{a,a^{-1}} e_a \otimes e_{a^{-1}}$$

where $\mathcal{C} \subseteq G \setminus \{e\}$ as in proposition 4.6.4. We assume that there is up to scale a unique top form Vol, which we take as a n-fold product of the e_a basis elements. For the volume form to be central we need the degrees of these elements to multiply to e in the group. In this setting we define the antisymmetric tensor ϵ by

$$e_{a_1} \wedge \cdots \wedge e_{a_n} = \epsilon_{a_1,\cdots,a_n}\mathrm{Vol}$$

which is either zero or $a_1a_2\cdots a_n = e$ in G by centrality. From this tensor we define the Hodge star by

$$\begin{aligned}\circledast(e_{a_1} \wedge \cdots \wedge e_{a_m}) &= \sum_{b,c} d_m^{-1}\epsilon_{a_1\cdots a_m b_{m+1}\cdots b_n}\eta^{b_{m+1}c_{m+1}}\cdots\eta^{b_nc_n} e_{c_n}\wedge\cdots\wedge e_{c_{m+1}}\\ &= \sum_{a_{m+1},\cdots,a_n} d_m^{-1}\epsilon_{a_1\cdots a_n} e_{a_n^{-1}}\wedge\cdots\wedge e_{a_{m+1}^{-1}}.\end{aligned}$$

for some normalisation constants d_m. From this definition, the Hodge operator extends as a bimodule map. In nice cases, we can choose the constants so that $\circledast^2 = \epsilon_m\mathrm{id}$, where $\epsilon_m = \pm1$, depending on the degree m. For the rest of the discussion, we assume this is true. We also write $a^{\circledast} = \circledast\, a$ to ease the notation. Note that by definition we have $\mathrm{Vol}^{\circledast} = 1$ and that

$$e_a^{\circledast} \wedge e_b = \epsilon_1 d_{n-1}\eta^{a,b}\mathrm{Vol}, \quad e_a \wedge e_b^{\circledast} = (-1)^{n-1}\epsilon_1 d_{n-1}\eta^{a,b}\mathrm{Vol}$$

holds in our discussion. In this case we find

$$de_a^{\circledast} = \theta e_a^{\circledast} - (-1)^{n-1}e_a^{\circledast}\theta = 0$$

Therefore the e_a are coclosed, i.e. $e_a^{\circledast}$ are closed. We write an $n-1$-forms as $\beta = \sum_a \beta^a e_a^{\circledast}$, and find

$$d\beta = \sum_{a,b}\partial^b\beta^a e_b \wedge e_a^{\circledast} = \epsilon_1 d_{n-1}(-1)^{n-1}(\sum_a \partial^{a^{-1}}\beta^a)\mathrm{Vol}$$

From this and from $\partial^{a^{-1}}\partial^a = -\partial^a - \partial^{a^{-1}}$ and $\Delta_\theta = -2\sum_a \partial^a$, it follows that

$$\circledast d \circledast d = \epsilon_1 d_{n-1}(-1)^{n-1}\Delta_\theta$$

on functions. When G is finite, we define $\int f = \sum_{x\in G} f(x)$ as the analogue of integration. We extend this to n-forms by $\int f\mathrm{Vol} = \int f$ and by our assumptions we find that $\int d\beta = 0$

for any $n-1$-form β. This can be shown by the explicit calculation, recall $x \in G$ and $a \in \mathcal{C}$ the set of generators of the group,

$$\int d\beta = \sum_x d\beta = \sum_x \epsilon_1 d_{n-1}(-1)^{n-1} \sum_a \partial^{a^{-1}} \beta(x)$$

we ignore the coefficients and consider the sum only

$$\sum_{x,a} \partial^{a^{-1}} \beta(x) = \sum_{x,a} (R_{a^{-1}} - id)\beta(x) = \sum_{x,a} \beta(xa^{-1}) - \beta(x).$$

The argument of the sum doesn't necessarily vanish, however since xa^{-1} and x are all the elements in the G, the total sum gives zero. Thus $\int d\beta = 0$.

Proposition 6.4.1. *Assume the above nice properties of* $\circledast$ *, that* $H^{n-1}_{dR}(G) = \mathbb{C}\theta^{\circledast}$*, and that* G *is finite. Then* $J = \sum_a J^a e_a \in \Omega^1$ *is coexact if and only if*

$$\sum_a \partial^{a^{-1}} J^a = 0, \quad \int \sum_a J^a = 0$$

Any solution ψ *of the wave equation* $\Delta_\theta \psi = m^2 \psi$ *generates such a source*

$$J^a = 2(\partial^a \bar{\psi})\psi - \partial^a(\bar{\psi}\psi) + \frac{m^2}{|G||\mathcal{C}|} \int \bar{\psi}\psi$$

J_a *being the 'current associated to* ψ*.*

Proof. We write here the proof for the first claim. A complete proof can be found in [3] page 67.

By our cohomology assumptions on H^{n-1}_{dR}, if J is coexact, i.e. $dJ^{\circledast} = 0$, then we can write $\mu\theta^{\circledast} + d\beta$ for some complex constant μ and some $n-2$-form β. We have $\theta^{\circledast} \wedge \theta = \epsilon_1 d_{n-1}|\mathcal{C}|\mathrm{Vol}$ by the nice properties of $\circledast$ as defined before, while $\int d\beta \wedge \theta = \int d(\beta \wedge \theta) = 0$, so imposing $\int J^{\circledast} \wedge \theta = 0$ forces $\mu = 0$. Hence $J^{\circledast}$ is exact. So the requirements

$$dJ^{\circledast} = 0, \quad \int J^{\circledast} \wedge \theta = 0$$

imply $J^{\circledast}$ is exact. The converse is clear, therefore these two conditions are equivalent to J being coexact. Under our assumptions, we can rewrite these requirements to match the expression in the proposition by writing

$$dJ^{\circledast} = \epsilon_1 d_{n-1}(-1)^{n-1}(\sum_a \partial^{a^{-1}} J^{\circledast\, a})\mathrm{Vol}, \quad J^{\circledast} \wedge \theta = \epsilon_1 d_{n-1}(\sum_a J^a)\mathrm{Vol}.$$

□

J being coexact is necessary to solve the Maxwell's equations and it means that $J^{\circledast} = d(F^{\circledast})$ for some 2-form F. If this is exact, then we will have a 'gauge potential' α such that $\circledast d \circledast d\alpha = J$, as desired. These steps mirror the classical treatment of electromagnetism. Now, we impose the 'Coulomb gauge fixing' $\delta\alpha = 0$ that however does not completely fix the freedom of the freedom in α. Parallel to our treatment of J, it makes sense to ask the stronger version, i.e. that α is coexact.

$$\circledast d \circledast d : \Omega^1_{\text{coexact}} \to \Omega^1_{\text{coexact}}$$

then becomes an operator on the space of coexact 1-forms to itself. This is what we aim to diagonalise to solve Maxwell's theory. This discussion and more details can be found in [3] section 1.8.

6.5 Maxwell's theory on the Permutation Group S_3

In this section. we discuss the Maxwell's theory applied to S_3. More details can be found in [3] page 68 and in [13]. Recall the result of example 4.6.5. We consider the group $G = \{e, u, v, w, uv, vu\}$ with $|G| = 6$ and the generators $\mathcal{C} = \{u, v, w\}$ with $|\mathcal{C}| = 3$. There we have a DGA with 12 total non zero elements. In particular, the basis picked for each degree of the DGA are

$$\Omega^1 = \text{span}\{e_u, e_v, e_w\}, \quad \Omega^2 = \text{span}\{e_u \wedge e_v,\ e_v \wedge e_u,\ e_v \wedge e_w,\ e_w \wedge e_v\}$$
$$\Omega^3 = \text{span}\{e_w \wedge e_u \wedge e_v,\ e_u \wedge e_v \wedge e_w,\ e_v \wedge e_w \wedge e_u\}$$

We also have 12 non zero values for the $\epsilon_{a_1,\cdots,a_n}$ introduced in the previous section, i.e.

$$\epsilon_{uvuw} = \epsilon_{vuvw} = 1, \quad \epsilon_{wuvu} = \epsilon_{wvuv} = -1$$

and their cyclic rotations under $u \to v \to w \to u$. We also set

$$d_0 = 12, \quad d_1 = 4, \quad d_2 = \sqrt{3}, \quad d_3 = 1, \quad d_4 = 1$$

to give the Hodge star as $\circledast 1 = -\text{Vol}$, $\circledast\text{Vol} = 1$ and

$$\circledast e_u = e_w \wedge e_u \wedge e_v, \quad \circledast e_v = e_u \wedge e_v \wedge e_w, \quad \circledast e_w = e_v \wedge e_w \wedge e_u,$$

$$\circledast(e_u \wedge e_v) = -\frac{1}{\sqrt{3}}(e_u \wedge e_v + 2e_v \wedge e_w), \quad \circledast(e_v \wedge e_w) = \frac{1}{\sqrt{3}}(e_v \wedge e_w + 2e_u \wedge e_v)$$

$$\circledast(e_v \wedge e_u) = \frac{1}{\sqrt{3}}(e_v \wedge e_u + 2e_w \wedge e_v), \quad \circledast(e_w \wedge e_v) = -\frac{1}{\sqrt{3}}(e_w \wedge e_v + 2e_v \wedge e_u)$$

$$\circledast(e_w \wedge e_u \wedge e_v) = -e_u, \quad \circledast(e_u \wedge e_v \wedge e_w) = -e_v, \quad \circledast(e_v \wedge e_w \wedge e_u) = -e_w$$

The natural normalisation constants are such that $\circledast\circledast = -\mathrm{id}$. The nice properties we assumed before hold. In addition, we assume that the cohomology condition in proposition 6.4.1 holds for the Maxwell's theory. We are solving $\circledast d \circledast d\alpha = J$ and we ask α to be coexact. From the results in [3] page 69, the eigenvalues of $\circledast d \circledast d$ on $\Omega^1_{\text{coeaxct}}$ are $-6, -12, -18$, each with 4-dimensional eigenspaces, the total space being 12-dimensional. The 18 dimensions for the choice of the three components of α are reduced to 12 by the coexactedness 'gauge fixing'.

Now we pick ψ to be the scalar wave function. If ψ is an eigenfunction of Δ_θ with eigenvalues $-m^2$, then

$$J^a = 2(\partial^a \bar{\psi})\psi - \partial^a(\bar{\psi}\psi) + \frac{m^2}{18}\int \bar{\psi}\psi$$

obeys the source conservation conditions. Recall the example 4.6.5, the constant function and the sign function both generate zero source. On the other hand, the $m^2 = 6$ modes do not. In particular, the 'point source' wave function ψ_x at any point $x \in S_3$ gives a source

$$J^a_x = 1 - 3\delta_x - 3\delta_{xa}$$

as an element of $\Omega^1_{\text{coexact}}$. This can be obtained by substituing ψ_x in the equation of J^a_x as a function of ψ, i.e.

$$J^a = \cancelto{0}{2(R_a)(\bar{\psi})\psi} - 2\,\mathrm{id}(\bar{\psi})\psi - R_a(\bar{\psi}\psi) + \bar{\psi}\psi + \frac{6}{18}\cancelto{6}{\int \bar{\psi}\psi} = 2 - R_a(\bar{\psi}\psi) - \bar{\psi}\psi$$

Though we can define the source on the six different points of the graph, the six functions are not independent. In fact, we immediately see that $J_{xu} + J_{xv} + J_{xw} = 0$ from which we derive

$$J_u + J_v + J_w = 0, \quad J_e + J_{uv} + J_{vu} = 0$$

that shows a destructive interference. Thus, we have 4 possible independent sources. From the direct calculation we see that J_x is an eigenfunction of $\circledast d \circledast d$ with eigenvalue -12, and hence we can write a gauge potential $\alpha = -\frac{1}{12}J_x$, whose components are $\alpha^a = J^a_x$.

Recall that $F = d\alpha$ and the choice for the basis of Ω^2. We then calculate the explicit components of F.

$$d\alpha = d(\sum_a \alpha^a e_a) = \sum_a d(\alpha^a e_a)$$

where $\alpha^a \in \Omega^0$ and $e_a \in \Omega^1$. The exterior derivative fulfils a graded Leibniz rule as the differential geometry case. Thus, we write

$$d\alpha = \sum_a \Big[(d\alpha^a) \wedge e_a + \alpha^a de_a\Big].$$

Then, we recall

$$d\alpha^a = \partial^b \alpha^a e_b = (R_b - \mathrm{id})\alpha^a e_b = (R_b \alpha^a - \alpha^a) e_b$$

and

$$de_a = \theta \wedge e_a + e_a \wedge \theta = \sum_{b|b\neq a} \Big(e_b \wedge e_a + e_a \wedge e_b \Big)$$

By plugging these results in the equation above for $d\alpha$ we get

$$\begin{aligned}
\alpha =& \sum_{a,b} \Big[(R_b\alpha^a - \alpha^a) e_b \wedge e_a + \alpha^a (e_b \wedge e_a + e_a \wedge e_b) \Big] \\
=& \sum_{a,b} \Big[(R_b\alpha^a - \cancel{\alpha^a} + \cancel{\alpha^a}) e_b \wedge e_a + \alpha^a e_a \wedge e_b \Big] \\
=& \sum_{a,b} (R_b\alpha^a) e_a \wedge e_b + \alpha^b e_b \wedge e_a) \\
=& \sum_{a,b} (R_b\alpha^a + \alpha^b) e_b \wedge e_a
\end{aligned}$$

Therefore we have $F^{ba} = (R_b\alpha^a + \alpha^b)$. However, the choice of Ω^2 basis implies the following consideration.

$$\begin{aligned}
F =& \sum_{a,b} F^{ab} e_a \wedge e_b \\
=& F^{uv} e_u \wedge e_v + F^{uw} e_u \wedge e_w + F^{vu} e_v \wedge e_u + F^{vw} e_v \wedge e_w + F^{wu} e_w \wedge e_u + F^{wv} e_w \wedge e_v \\
=& (F^{uv} - F^{wu}) e_u \wedge e_v + (F^{vu} - F^{uw}) e_v \wedge e_u + (F^{vw} - F^{wu}) e_v \wedge e_w + (F^{wv} - F^{uw}) e_w \wedge e_v
\end{aligned}$$

Therefore, the components of F are

$$F^{uv} = R_u\alpha^v + \alpha^u - R_w\alpha^u - \alpha^w, \quad F^{vu} = R_v\alpha^u + \alpha^v - R_u\alpha^w - \alpha^u$$

$$F^{vw} = R_v\alpha^w + \alpha^v - R_w\alpha^u - \alpha^w, \quad F^{wv} = R_w\alpha^v + \alpha^w - R_u\alpha^w - \alpha^u$$

Recall $\alpha^a = -\frac{1}{12}(1 - 3\delta_x - 3\delta_{xa})$. Finally, the components of F on S_3 with our assumptions are

$$F^{uv} = \frac{1}{2}(\delta_{xu} - \delta_{xw}), \quad F^{vu} = \frac{1}{2}(\delta_{xv} - \delta_{xu}), \quad F^{vw} = \frac{1}{2}(\delta_{xv} - \delta_{xw}), \quad F^{wv} = \frac{1}{2}(\delta_{xw} - \delta_{xu})$$

Consider now F in differential geometry. The electric field and the magnetic field components are related via the Hodge operator, i.e. $\circledast F_E = F_B$. Following the same considerations we can naturally divide the F on S_3 accordingly.

$$F_E = (F^{uv}, F^{vu}), \quad F_B = (F^{vw}, F^{wv})$$

Appendix A

Alternative definition of module and algebra

Almost all the resources cited in the first chapter use slightly different definitions that rely on other algebraic structures (rings or monoids). For this reason a quick comparison is given here.

A.1 Introduction

Most of the citations use rings as a starting point for the definitions we gave above, for this reason the direct comparison may appear doubtful. In the following section a brief extension on this regard is provided.

A *group* is a set together with a binary operation that has associativity, the identity element and the inverse element. For more see section 1.4 and [6] chapter 1,[8] chapter 1.

A *ring* is a nonempty set R with two binary operations (addition and multiplication) such that $(R, +)$ is an abelian group and the multiplication is associative and distributive on both sides, see [6] page 115. A definition of module may be given starting from rings.

Definition A.1.1 (Alternative)**.** Let A be a ring. A (left) A-*module* E is an additive abelian group together with a function $A \times E \longrightarrow E$ (the image of (a,e) being denoted by $a.e$) such that

1. $a.(e_1 + e_2) = a.e_1 + a.e_2$
2. $(a + b).e_1 = a.e_1 + b.e_1$
3. $a(b.e_1) = (ab).e_1$ for all $a, b \in A$ and $e_i \in E$.

If A has an identity element 1_A and $1_A.e = e$ for all $e \in E$ then E is said to be an *unitary A-module*.

The unitary module over a ring structure is really close to a vector field. All the properties for the addition are already present because E is an additive abelian group and the properties for the multiplication by an element of the ring (scalar) are provided by the definition above. The only property that is left to be added to the definition to make E a vector field is the inverse of the element in the ring A.

Definition A.1.2. Let A be a unitary ring. A is called *division ring* if there exists a^{-1} for all $a \in A \setminus \{0\}$. Where 0 is the addition identity (assumed $0 \neq 1$, 1 being the multiplicative identity).

If A is a commutative division unitary ring then it is a *field*, and finally a unitary (left) A-module is a (left) *vector space* over a field A.

A.2 Algebra

We now see how the definition of algebra changes once the notions above are used as a starting point.

Definition A.2.1. Let K be a commutative ring with identity. A *K-algebra* A is a ring A such that:

1. $(A, +)$ is a unitary (left) K-module
2. $k(ab) = a(kb)\ \forall k \in K,\ a, b \in A$

For more see [6] page 227.

Firstly, in the definition 1.1.1, K was a field (or a commutative division unitary ring), therefore unitary (left) K-module (A) is a vector space. Secondly, A was a vector space in definition 1.1.1. Recall A being a ring, the second property in the definition above asks for the multiplication between the field and the vector space (and ring) to be compatible with the product of the ring. This concludes the comparison between the two definitions and shows that they perfectly overlap considering the constraints that apply for our discussion.

A.3 Module

We have given the definition of algebra based on the quickly introduced concept of rings and groups. Now we see that the definition A.1.1 is equivalent to 1.3.1 once A is an algebra.

First of all, the additional properties that a ring needs to have in order to become an algebra are expressed in the definition A.2.1. Looking now at E, it is a vector space in 1.3.1 and an additive abelian group in A.1.1. The abelian additive group shares the same properties for the addition in a vector space, see [9] chapter 1 for vector space and [6, 8] for groups. The properties that are missing regard the product. We will work with unitary algebras, in which case there exists a 1_A (identity element for the product). Recall the definition A.1.1 point 4 and the algebra being a vector space, then

$$\lambda 1_A . e = \lambda \ ? \ e$$

is valid for all $\lambda \in \mathbb{K}$ and $e \in E$. The question mark highlights the fact that we need to define the product between an element of $\mathbb{K}$ with an element of E. Once defined, the module (additive abelian group) acquires the properties that were missing.

Proof. We want to prove that the following properties are inherited by an additive abelian group once we define a valid product for all the elements of a field. Let $\mathbb{K}$ be a field and V be a vector space. The properties are

1. $a(b\vec{v}) = (ab)\vec{v}$
2. $1_{\mathbb{K}}\vec{v} = \vec{v}$
3. $(a + b)\vec{v} = a\vec{v} + b\vec{v}$
4. $a(\vec{v} + \vec{u}) = a\vec{v} + a\vec{u}$

for all $a, b \in \mathbb{K}$ and $\vec{v}, \vec{u} \in V$. We now proceed to prove that they are valid for an A-module over an algebra as in definition A.1.1.

1. Recall the property 3 in definition A.1.1 and the algebra A being a vector space over $\mathbb{K}$, we can write:
$$\lambda(\mu e) = \lambda 1_A(\mu 1_A . e) = (\lambda 1_A \mu 1_A).e = (\lambda\mu)e$$
2. Recall the property 4 in definition A.1.1 and the algebra being a vector space over $\mathbb{K}$, then trivially:
$$1_{\mathbb{K}} e = 1_{\mathbb{K}} 1_A . e = 1_A . e = e$$
3. Recall the property 2 in definition A.1.1 and the algebra A being a vector space over $\mathbb{K}$, we can write:
$$(\lambda + \mu)e = (\lambda 1_A + \mu 1_A).e = \lambda 1_A . e + \mu 1_A . e = \lambda e + \mu e$$
4. Similarly to the previous point but recalling the property 1 instead of 2.

□

This simple proof concludes the comparison between the two definitions of module that have been given.

Appendix B

Einstein Cosmological Field Equations

The following are the field equations for cosmological general relativity

$$G_{ij} + \Lambda g_{ij} = R_{ij} - \frac{1}{2} R g_{ij} + \Lambda g_{ij} = \kappa T_{ij}$$

where R_{ij} is the Ricci tensor and T_{ij} the energy-momentum tensor. We see how the left hand side of the equation, usually called geometrical side, depends solely on the metric tensor and its first and second derivative. One should recall the definition of the Ricci tensor to see its dependency on the metric tensor, see [4] chapter 3 and 4.

Appendix C

Laplace-Beltrami in $\mathcal{C}^\infty(\mathbb{R}^n)$

The Laplace-Beltrami operator in $\mathcal{C}^\infty(\mathbb{R}^n)$ satisfies the condition for a second-order differential operator in the general case of a quantum differential algebra, as in definition 3.2.2. Here we give the explicit calculation.

Proof. Given the Laplace-Beltrami operator Δf, i.e.

$$\Delta f = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j f)$$

we apply it to two functions $f, h \in \mathcal{C}^\infty(\mathbb{R}^n)$.

$$\begin{aligned}
\Delta fh =& \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j(fh)\right)\\
=& \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}((\partial_j f)h + f\partial_j h)\right)\\
=& \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}(\partial_j f)h\right) + \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}f\partial_j h\right)\\
=& \frac{1}{\sqrt{|g|}}\partial_i\left(h\cdot\sqrt{|g|}g^{ij}\partial_j f\right) + \frac{1}{\sqrt{|g|}}\partial_i\left(f\cdot\sqrt{|g|}g^{ij}\partial_j h\right)\\
=& \frac{1}{\cancel{\sqrt{|g|}}}\partial_i h\cdot\cancel{\sqrt{|g|}}g^{ij}\partial_j f + h\cdot\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j f\right) + \frac{1}{\cancel{\sqrt{|g|}}}\partial_i f\cdot\cancel{\sqrt{|g|}}g^{ij}\partial_j h +\\
&+ f\cdot\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j h\right)\\
=& \cancelto{\Delta f}{\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j f\right)}\cdot h + f\cdot\cancelto{\Delta h}{\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j h\right)} + g^{ij}\partial_i h\partial_j f + g^{ij}\partial_j h\partial_i f\\
=& (\Delta f)h + f\Delta h + 2g^{ij}\partial_i h\partial_j f
\end{aligned}$$

Let us consider the last term. Recall $g^*_{ij} := g^{ij}$ and the correspondence between $(,)$ and g^* shown in section 3.1. Therefore we can write

$$g^{ij}\partial_i h \partial_j f = g^*_{ij}\partial_i h \partial_j f = (dx^i, dx^j)\partial_i h \partial_j f.$$

Now, we recall the properties of the bimodule map, as for definition 3.1.2. Therefore we write

$$\begin{aligned}
g^{ij}\partial_i h \partial_j f =&(dx^i, dx^j)\partial_i h \partial_j f \\
=&(dx^i, dx^j \partial_i h \partial_j f) \\
=&(dx^i, \partial_i h dx^j \partial_j f) \\
=&(dx^i \partial_i h, dx^j \partial_j f) \\
=&(\partial_i h dx^i, \partial_j f dx^j) \\
=&(dh, df)
\end{aligned}$$

Finally, we can write

$$\Delta(fh) = (\Delta f)h + f\Delta h + 2g^{ij}\partial_i h \partial_j f$$

$$\Delta(fh) = (\Delta f)h + f\Delta h + 2(dh, df)$$

Although the commutativity of the product in this framework makes calculation easy, we formally show all the meaningful steps. $\square$

Appendix D

Inverse condition of the metric on bidirected graphs

Proof. Recall the metric in proposition 4.4.3, i.e.

$$g = \sum_{x\to y} g_{x\to y}\omega_{x\to y} \otimes_A \omega_{y\to x}$$

where

$$g_{x\to y} = \frac{1}{\lambda_{y\to x}} \in \mathbb{K}$$

and the inverse condition in definition 3.1.2, i.e.

$$((\omega, \bullet) \otimes id)g = \omega = (id \otimes (\bullet, \omega)).$$

For the sake of simplicity we only prove the left hand side, but the other side can be similarly proven. First, we recall $\omega \in \Omega^1(X)$ therefore we can write

$$\omega = \omega_{x'\to y'}$$

where $x', y' \in X$ fixed.

Then we do the calculation

$$\begin{aligned}
\omega =&((\omega, \bullet) \otimes id)g \\
\omega_{x'\to y'} =&((\omega_{x'\to y'}, \bullet) \otimes id)\Big(\sum_{x\to y} g_{x\to y}\omega_{x\to y} \otimes_A \omega_{y\to x}\Big) \\
=&\sum_{x\to y}(\omega_{x'\to y'}, g_{x\to y}\omega_{x\to y}) \cdot \omega_{y\to x} \\
=&\sum_{x\to y} g_{x\to y}(\omega_{x'\to y'}, \omega_{x\to y}) \cdot \omega_{y\to x}
\end{aligned}$$

Recall proposition 4.4.2, i.e.

$$(\omega_{x\to y}, \omega_{y'\to x'}) = \lambda_{x\to y}\delta_{x,x'}\delta_{y,y'}\delta x$$

therefore we can write

$$\omega_{x'\to y'} = \sum_{x\to y} g_{x\to y}\lambda_{x'\to y'}\delta_{x',y}\delta_{y',x}\delta_{x'} \cdot \omega_{y\to x}$$

The deltas with two indices are true Kronecker deltas, therefore they fix the free variables y and x to the fixed value x' and y', respectively. The sum is thus not a sum anymore and we obtain

$$\begin{aligned}
\omega_{x'\to y'} &= g_{y'\to x'}\lambda_{x'\to y'}\delta_{x'} \cdot \omega_{x'\to y'} \\
&= g_{y'\to x'}\lambda_{x'\to y'}\cancelto{1}{\delta_{x'}(x')}\omega_{x'\to y'} \\
&= \frac{1}{\cancel{\lambda_{x'\to y'}}}\cancel{\lambda_{x'\to y'}}\omega_{x'\to y'} \\
&= \omega_{x'\to y'}
\end{aligned}$$

□

Bibliography

[1] Michael Atiyah and Ian G. MacDonald. *Introduction to Commutative Algebra*. Addison-Wesley Publishing Company, 1994.

[2] Paul Bamberg and Shlomo Sternberg. *A course in mathematics for students of physics: 2*. Cambridge University Press, 1990.

[3] Edwin J. Beggs and Shahn Majid. *Quantum Riemaniann Geometry*. Springer, 2020.

[4] Sean M. Carroll. Lecture notes on general relativity, 1997.

[5] Richard M. Foote David S. Dummit. *Abstract algebra*. John Wiley & Sons, Inc., 2004.

[6] Thomas W. Hungerford. *Algebra*. Springer, 1980.

[7] John David Jackson. *Classical Electrodyamics*. John Wiley 'I&' Sons, Inc., 3 edition, 1999.

[8] Serge Lang. *Algebra*. Springer, 2002.

[9] Serge Lang. *Linear Algebra*. Springer, 3 edition, 2004.

[10] Shahn Majid. Noncommutative riemaniann geometry on graphs. *Journal of Geometry and Physics*, 69:74–93, 2013.

[11] Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler. *Gravitation*. W. H. Freeman and company, New York, 1972.

[12] Florian Scheck. *Classical Field Theory*. Springer, 2 edition, 2018.

[13] E. Raineri Shahn Majid. Electromagnetism and gauge theory on the permutation group S_3. *Journal of Geometry and Physics*, 44:129–155, 2002.

[14] Loring W. Tu. *An introduction to manifolds*. Springer, 2010.

[15] Loring W. Tu. *Differential Geometry: Connection, Curvature, and Characteristic Classes*. Springer Cham, 2017.

[16] Franck Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Springer, 1971.