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Gauge invariant coefficients in perturbative quantum gravity

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Abstract

Perturbative quantum gravity can be studied in many ways. A traditional approach is to apply covariant quantization schemes to the Einstein-Hilbert action and use heat kernel methods, as pioneered by DeWitt. An alternative approach is to consider the graviton as arising from the first quantization of particle actions, following the same methods used in string theory. An interesting model to describe the graviton is based on the so-called $\mathcal{N} = 4$ spinning particle, which has been used recently to study perturbative properties of quantum gravity, allowing in particular for the calculation of certain gauge-invariant coefficients. The latter are related to the counterterms that renormalize the one-loop effective action of pure quantum gravity with a cosmological constant. Such coefficients have already been tested in $D = 4$ dimensions. Here we study the general case of arbitrary D. We derive the gauge-invariant coefficients —the simplest one being the number of physical degrees of freedom of the graviton—using the traditional heat kernel method. We compare them with the ones obtained by using the $\mathcal{N} = 4$ spinning particle and discover that the latter fails to reproduce some of those coefficients for $D \neq 4$, suggesting the need of improving that first quantized model. This constitutes a first original result of this thesis. In the second part, we try to find an alternative worldline path integral treatment of the heat kernel, extending a previous worldline construction that was tailored to 4 dimensions only. We succeed in finding suitable worldline actions for the gauge-fixed graviton fluctuations and related ghosts. The action for the graviton fluctuations that we construct reproduces the expected Hamiltonian but does not seem to admit a perturbative path integral treatment.

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Introduction

The construction of a quantum theory for the gravitational interaction has been one of the main goals of modern theoretical physics. One approach uses the background field method with covariant gauge-fixing techniques to define the Feynman rules for computing perturbatively the effective action. A useful technique to compute and renormalize the effective action at one-loop makes use of heat kernel method, pioneered in curved space by DeWitt [1, 2, 3]. In particular, the counterterms needed to perform renormalization at one-loop are efficiently captured by heat kernel coefficients (also known as Seeley-DeWitt coefficients) related to differential operators defined by the gauge-fixed gravitational action. On-shell, these coefficients should not depend on the gauge-fixing procedure, and become gauge invariant. In this thesis we shall discuss methods for computing these gauge-invariant coefficients in the case of pure gravity with a cosmological constant, and check their consistency.

The first method we consider employs the original heat kernel approach, as developed by DeWitt. We rederive explicitly the coefficients needed to renormalize the divergences of the effective action in 4 dimensions—the heat kernel coefficients a_0, a_1, and a_2 in the notation of DeWitt—keeping the spacetime dimension D arbitrary. Some of these coefficients are reported erroneously in the literature, and we will rederive them to be sure to consider the correct ones. Evaluating them on-shell (i.e. using the metric of an Einstein manifold) produces the gauge-invariant coefficients of our interest. They constitute a benchmark for alternative approaches to quantum gravity.

Alternative methods that we wish to consider and test are related to first quantized approaches to the graviton (the heat kernel in a sense is such an approach). A first worldline path integral approach is the one developed in [4], which was tailored to 4 dimensions only, and used to check in $D = 4$ the value of the gauge-invariant coefficients we mentioned above. This model is discussed in the second part of the thesis, where attempts to extend this approach to arbitrary D are made.

A more elegant model is the one that describes the physical graviton with the $\mathcal{N} = 4$ spinning particle, developed through BRST methods in [5, 6] and within a path integral approach in [7]. It has been used to reproduce successfully the known coefficients at $D = 4$, giving at the same time a prediction at arbitrary D, see [7]. We compare these coefficients with the one we found earlier with the heat kernel technique, and find that all these methods are correct in $D = 4$, but differ at arbitrary D. We interpret these results as suggesting that the heat kernel results should be the correct ones, being derived from first principles, while the disagreement that we have found indicates the need of improving the construction of the $\mathcal{N} = 4$ spinning particle, so to match the correct results at arbitrary D.

We structure our thesis as follows. In chapter 1, after a brief introduction, we describe the heat kernel by presenting its relation with the one-loop effective action. The coefficients of the associated expansion can be written in terms of a few independent invariants constructed from the background fields of space-time. These coefficients are then specialized to the case of interest, namely perturbative quantum gravity.

In chapter 2 we describe the theory of pure quantum gravity with cosmological constant i.e. the Einstein-Hilbert action with the cosmological term. A background-quantum splitting is performed to identify the graviton fluctuations on the fixed background. The action is then expanded up to the second order in the fluctuations. The gauge symmetry of the theory requires a procedure of gauge-fixing achieved by means of BRST methods. The quadratic approximation of the action (necessary to evaluate the one-loop effective action) contains the graviton and the ghost contributions, from which we extract their invertible kinetic operators. The latter are finally used to evaluate the first three Seeley-DeWitt coefficients. These terms identify the counterterms that make the effective action finite (in 4 dimensions). When evaluated on-shell (i.e. on Einstein manifolds) they become gauge-independent, and thus define gauge-invariant coefficients. We compare them with similar ones obtained by other methods and find a mismatch at $D \neq 4$ with those obtained by first-quantizing the $\mathcal{N} = 4$ particle action, which is expected to describe the graviton. This suggests that the $\mathcal{N} = 4$ particle model needs an improvement.

In chapter 3 we try to reproduce those coefficients using an alternative approach, also based on the worldline formalism. We start with the simpler case of the ghost sector. We construct the particle action related to the ghosts and use it in a path integral, thus reproducing the corresponding Seeley-DeWitt coefficients.

In chapter 4 we try to perform the same steps for the gauge-fixed graviton fluctuations. We identify the corresponding particle action, which indeed reproduces the Hamiltonian used previously in the heat kernel approach, but find that the corresponding path integral is not easily calculable, as the standard perturbative method based on Gaussian integration is inapplicable. How to overcome this final issue is left for future research.

Chapter 1

Heat kernel expansion

After a brief introduction to the heat kernel, starting from the generating functional in path integral representation, the relation between the one-loop effective action and the heat kernel is made explicit. The utility of the heat kernel procedure lies in the possibility to write down the coefficients of the expansion taking into account a few independent invariants constructed from the background fields defined on spacetime.

The specific form of the Laplace operator, necessary for our study of the Seeley-DeWitt coefficients in quantum gravity, is presented paying particular attention to all the connections in the covariant derivative. After the introduction of the local invariants, the formulae of the heat kernel coefficients are computed for a general Hamiltonian, following Vassilevich's notes [8].

1.1 Heat kernel introduction

The heat kernel represents a powerful tool both in physics and mathematics. During the last decades, it has been extremely useful for the study of effective actions, calculations of anomalies, divergences, and asymptotics. In 1937 Fock [9] noted that it is possible to represent Green functions in terms of integrals over the so-called "proper time", an auxiliary coordinate, of a kernel that satisfies the heat equation. J. Schwinger [10] used that representation of Green functions, related to the dynamics of a particle with spacetime coordinates depending on a proper time, for studying issues such as renormalization and gauge invariance. Later B. DeWitt [1, 2, 3] applied that procedure to quantum field theory and quantum gravity, reaching important results. Other mentionable applications are calculations of the vacuum polarization, the Casimir effect, and the proof of index theorems.

In order to introduce the heat kernel, let us first define the operator $\hat{\partial}_\mu = i\hat{p}_\mu$, that reduces to the usual derivative when acting on wave functions. Consider then the following second order differential operator with a mass term

$$\hat{H}_0 = -\hat{\partial}^2 + m^2 \tag{1.1}$$

where $\hat{\partial}^2 := \hat{\partial}_\mu \hat{\partial}^\mu$ corresponds to the Laplacian in cartesian coordinates on the flat manifold $M = \mathbb{R}^D$. The heat kernel is defined as the matrix element between position eigenstates of the evolution operator in euclidean time $e^{-\beta \hat{H}_0}$, i.e.

$$K(\beta, x, y; \hat{H}_0) = \langle x| \exp\Big(-\beta \hat{H}_0\Big) |y\rangle \tag{1.2}$$

and represents the solution of the Wick-rotated (by analytic continuation $t \to -i\beta$) Schrödinger equation, the heat equation:

$$\left(\frac{\partial}{\partial \beta} + \hat{H}_0\right) K(\beta; x, y; \hat{H}_0) = 0 \tag{1.3}$$

under boundary conditions of the form

$$K(0; x, y; \hat{H}_0) = \delta^D(x - y). \tag{1.4}$$

It is quite simple to check that the explicit form of (1.2) in the case of the operator in (1.1) is given by

$$K(\beta; x, y, \hat{H}_0) = \frac{1}{(4\pi\beta)^{D/2}} \exp\left(-\frac{(x-y)^2}{4\beta} - \beta m^2\right). \tag{1.5}$$

In the case of an operator containing an arbitrary potential represented by a smooth function $V(x)$, such as

$$\hat{H} = \hat{H}_0 + V(\hat{x}) \tag{1.6}$$

the solution is not exactly computable in general, but in many cases can be treated by using a perturbative expansion of the form

$$K(\beta; x, y, \hat{H}) = K(\beta; x, y, \hat{H}_0)\Big(a_0(x, y) + a_1(x, y)\beta + a_2(x, y)\beta^2 + \dots\Big) \tag{1.7}$$

with $a_0(x, y) = 1$. The coefficients $a_0(x, y)$, $a_1(x, y)$, $a_2(x, y)$ and so on are called heat kernel coefficients (or sometimes Seeley-DeWitt coefficients), and depend on the points x, y and on the explicit form of the potential $V(x)$. Of particular interest are the heat kernel coefficients evaluated at coinciding point, $a_n(x) \equiv a_n(x, x)$.

1.2 One-loop effective action

In order to study the application of the heat kernel to quantum field theory consider the following generating functional in euclidean path integral representation

$$Z[J] = \int \mathcal{D}\phi \; e^{-S[\phi,J]} \tag{1.8}$$

which produces the correlation functions of the field ϕ, whose action is $S[\phi, J]$ which contains a "source" J (an arbitrary function which allows to compute correlation functions performing functional derivatives of the action). Since in this thesis we are interested in computing the one-loop approximation of the effective action for gravity, it is enough to expand the action up to the quadratic order in the quantum field fluctuations ϕ, namely

$$S = S_{cl} + \langle \phi, J \rangle + \langle \phi, \hat{H}\phi \rangle + \cdots \tag{1.9}$$

where S_{cl} is the action on a classical background, $\hat{H}$ is a second-order differential operator (interpreted as the Hamiltonian of a fictitious particle in heat kernel methods), and a shorthand for the integrals over the D-dimensional space-time is used:

$$\langle \phi_1, \phi_2 \rangle := \int d^D x \; \sqrt{g} \; \phi_1(x)\phi_2(x). \tag{1.10}$$

In the above equation $g = |\det(g_{\mu\nu})|$ represents the absolute value of the determinant of the metric, while the integral over the underlying space-time is the inner product on the quantum fields space. In order to dispel any doubt, it is rather significant to clarify that the classical background field which produces the action S_{cl} and the quantum field describing fluctuations are distinct and could be of totally different type. A noteworthy example is the case of a pure quantum field in the classical background of gravity.

Then, equation (1.8) with the approximation in (1.9) represents a gaussian integral solvable as follows

$$Z[J] = e^{-S_{cl}} \det{}^{-\frac{1}{2}}(\hat{H}) \exp\left(\frac{1}{4} J \hat{H}^{-1} J \right). \tag{1.11}$$

From now on we omit the hat in the Hamiltonian operator as follows $H := \hat{H}$.

Let us consider the Hamiltonian operator as in equation (1.6). By taking advantage of the heat kernel, as presented in the previous section, the propagator $H^{-1}(x, y)$ can be defined using the following integral representation

$$H^{-1}(x, y) = \int_0^\infty d\beta \; K(\beta; x, y; H) \tag{1.12}$$

which follows from equation (1.2) for the Hamiltonian of (1.6). At this stage it is convenient to introduce the effective action. For our purposes, it is enough to set the sources to zero and define the effective action in terms of the generating functional as $Z[0] = e^{-\Gamma}$, so that at one-loop it can be written as follows

$$\Gamma_{\text{1-loop}} = \frac{1}{2} \ln \det(H) \tag{1.13}$$

which represents the effects of the background fields in the one-loop approximation.

To rewrite in a useful form, let us consider the following identity valid for positive numbers λ and λ_0 (interpreted as eigenvalues of the operators H and H_0)

$$\ln \frac{\lambda}{\lambda_0} = -\int_0^\infty \frac{d\beta}{\beta} \left(e^{-\beta\lambda} - e^{-\beta\lambda_0}\right) \ . \tag{1.14}$$

We use this relation extended to the full operator H (dropping also an infinite constant) to rewrite the above effectve action as

$$\Gamma_{\text{1-loop}} = -\frac{1}{2} \int_0^\infty \frac{d\beta}{\beta} K(\beta, H) \tag{1.15}$$

with

$$K(\beta, H) = \mathrm{Tr}\big(e^{-\beta H}\big) = \int d^D x \, \sqrt{g} \, K(\beta; x, x, H) \tag{1.16}$$

and where the identity $\ln \det(H) = \mathrm{Tr} \ln(H)$ has been used.

From equations (1.15) and (1.16) we note how the one-loop effective action can be written by making use of the heat kernel and also studied in terms of the above-mentioned Seeley-DeWitt coefficients, as shown in equation (1.7). The use of the heat kernel to compute the effective action is rather convenient since the Seeley-DeWitt coefficients can be computed in terms of just few geometric invariants, as will be shown in the next section. The advantage of this procedure is based on its general validity for different gauge groups, spins, etc.

1.3 The Laplace operator and local invariants

In preparation of the computation of the general Seeley-DeWitt coefficients, that will be done in the next chapters, it is convenient to introduce the specific form of Laplace operator of interest and the so called local invariants.

For the purpose, consider a positive definite metric tensor $g_{\mu\nu}$ embedded in a Riemannian manifold M. We restrict our study to a manifold without boundary, which is the case of interest. A complete and detailed description of the heat kernel expansion in manifolds with boundaries is contained in Vassilevich's notes [8].

Let each point of the manifold M be characterized by a vector space. The latter could be considered as the representation space of a gauge group or of the symmetry group of the space-time. All these vector spaces could be seen as forming a vector bundle whose sections are functions with an index describing an internal degree of freedom.

At this stage introduce a second order differential operator of Laplace type restricted to the following specific form

$$H = -(g^{\mu\nu}\nabla_\mu\nabla_\nu + V) \tag{1.17}$$

where ∇_μ is the covariant derivative and V a matrix-valued function. The covariant derivative contains not only the Riemannian part (together with the Christoffel symbol $\Gamma_{\mu\nu}{}^{\lambda}$) but also the part related to the "gauge" connection ω_μ. Therefore, if we have for instance a scalar field transforming under a gauge group it would have also a "color" index describing the above-mentioned internal degree of freedom, and the covariant derivative would act on that index with a gauge field. Namely, the covariant derivative has the form:

$$\nabla_\mu = \nabla_\mu{}^{[R]} + \omega_\mu \tag{1.18}$$

where the Riemannian part contains the Christoffel connection

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}) \tag{1.19}$$

where the usual notation for ordinary derivative $g_{\mu\sigma,\nu} := \partial_\nu g_{\mu\sigma}$ has been used. The action of the Riemannian covariant derivative on an arbitrary vector V_ν is thus given by

$$\nabla^{[R]}_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda{}_{\mu\nu}V_\lambda. \tag{1.20}$$

One could also define the field strength of the gauge connection ω_μ as

$$\Omega_{\mu\nu} = \partial_\mu\omega_\nu + \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu], \tag{1.21}$$

and the full covariant derivative in equation (1.18) acting on the arbitrary vector V_ν as follows

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda{}_{\mu\nu}V_\lambda + \omega_\mu V_\nu. \tag{1.22}$$

where the generators of the gauge group contained in ω_μ must be chosen in the representation belonging to V_μ.

We have already anticipated that the Seeley-DeWitt coefficients can be expressed in terms of few local invariants constructed from the background fields defined on space-time. For this purpose we introduce the following invariants associated to the metric tensor $g_{\mu\nu}$ and gauge connection ω_μ. The invariants associated to the metric can be constructed using the Riemann curvature tensor whose well-known expression in terms of the Christoffel symbol is

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu{}_{\sigma\nu} - \partial_\sigma\Gamma^\mu{}_{\rho\nu} + \Gamma^\mu{}_{\rho\lambda}\Gamma^\lambda{}_{\sigma\nu} - \Gamma^\mu{}_{\sigma\lambda}\Gamma^\lambda{}_{\rho\nu}. \tag{1.23}$$

Related to the Riemann tensor one defines as usual the Ricci tensor

$$R_{\mu\nu} := R^{\sigma}{}_{\mu\sigma\nu} \tag{1.24}$$

and the Ricci scalar curvature

$$R := R^{\mu}_{\mu}. \tag{1.25}$$

Some of the invariants the we shall meet are the scalars $R, \nabla^2 R, R^2, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. Similarly, invariants associated to the gauge connection ω_μ are built from the filed strenght in (1.21), such as $\mathrm{tr}(\Omega_{\mu\nu}\Omega^{\mu\nu})$, with the trace taken over the internal gauge indices.

In the following, we shall also need the concept of "flat indices", associated to a local orthonormal frame. Consider a tangent space attached to each point of the manifold. We introduce a local orthonormal frame with a flat index described by the so called vielbein or tetrad basis $\{e_1, \ldots, e_D\}$. The vielbein components e^k_μ and the inverse e^ν_j (satisfying $e^k_\mu e^\mu_j = \delta^k_j$) connect the metric $g_{\mu\nu}$ to the flat one, according to

$$e^\mu_j e^\nu_k g_{\mu\nu} = \delta_{jk} \qquad \text{and} \qquad e^\mu_j e^\nu_k \delta^{jk} = g^{\mu\nu}. \tag{1.26}$$

One can thus write the Riemannian covariant derivative in equation (1.20) applied to a vector with a flat index:

$$\nabla_\mu v^j = \partial_\mu v^j + \sigma^{jk}_\mu v_k \tag{1.27}$$

where σ^{jk}_μ is the so-called "spin connection" used to extend the concept of covariant derivative to vectors in the tetrad basis. Its explicit expression can be found by the condition $\nabla_\mu e^k_\nu = 0$ and is

$$\sigma^{kl}_\mu = e^\nu_l \Gamma^\rho{}_{\mu\nu} e^k_\rho - e^\nu_l \partial_\mu e^k_\nu. \tag{1.28}$$

A complete and rather detailed description of the vielbein (or "vierbein") or tetrad basis, together with a review of General relativity using a vierbein (initially proposed by Einstein in 1928) is contained in J. Yepez paper [11].

1.4 General formulae for the Seeley-DeWitt coefficients

In this thesis we will not perform the computation of the general heat kernel coefficients step by step, but we will provide a synthesis of the method developed by Gilkey [12] and followed by Vassilevich [8], where the interested reader can find all the steps and some detailed references.

Consider an auxiliary smooth function $\sigma(x)$ on the manifold M, the heat kernel of equation (1.16) can be written in terms of the trace of the exponential operator as follows

$$K(\beta, \sigma, H) = \mathrm{Tr}\big(\sigma e^{-\beta H}\big) \tag{1.29}$$

where

$$K(\beta,\sigma,H)=\int_M d^Dx\sqrt{g}\,\mathrm{tr}[K(\beta;x,x;H)]\sigma(x). \tag{1.30}$$

In the last equation the trace "tr" has to be considered as over the internal indices, and $K(\beta;x,x;H)$ is the solution at coincident points of the heat equation of the form in (1.3) with boundary condition (1.4). This solution can be written in terms of a complete set of orthonormal eigenfunctions of the differential operator H $\{\phi_\lambda\}$ associated to its eigenvalues λ, as follows

$$K(\beta;x,y;H)=\sum_\lambda \phi^\dagger_\lambda(x)\phi_\lambda(y)e^{-\beta\lambda}. \tag{1.31}$$

It is possible to write the trace of equation (1.29) using an asymptotic expansion for $\beta\to 0$:

$$\mathrm{Tr}\big(\sigma e^{-\beta H}\big)\approx\sum_{k\geq 0}\beta^{k-D/2}a_k(\sigma,H) \tag{1.32}$$

where $a_k(\sigma,H)$ are the coefficients of the expansion.

At this stage it is possible to prove that this ansatz is consistent on manifolds without boundaries, with the heat kernel coefficients computable in terms of the independent invariants described earlier. Going more into detail, if we express all the possible independent invariants constructed from V, R_{ijkl} and Ω_{ij} (all introduced in the previous section, we use here flat indices) and their covariant derivatives with $I^{\mathrm{J}}_k(D)$, we have

$$a_k(f,H)=\mathrm{tr}\int_M d^Dx\sqrt{g}[\sigma(x)a_k(x;H)]=\sum_{\mathrm{J}}\mathrm{tr}\int_M d^Dx\sqrt{g}[\sigma(x)c^{\mathrm{J}}I^{\mathrm{J}}_k(D)] \tag{1.33}$$

where c^{J} are some constants. Skipping al the steps that are not necessary for the purpose of this thesis and jumping directly to the results, the general formulae for the first three heat kernel (or Seeley-DeWitt) coefficients are

$$\begin{aligned}
a_0(\sigma,H)=&\frac{1}{(4\pi)^{D/2}}\int_M d^Dx\sqrt{g}\,\mathrm{tr}[\sigma(x)]\\
a_1(\sigma,H)=&\frac{1}{(4\pi)^{D/2}}\int_M d^Dx\sqrt{g}\,\mathrm{tr}\left[\sigma(x)\left(\frac{R}{6}+V\right)\right]\\
a_2(\sigma,H)=&\frac{1}{(4\pi)^{D/2}}\int_M d^Dx\sqrt{g}\,\mathrm{tr}\Bigg\{\sigma(x)\Bigg[\frac{1}{6}\left(\frac{1}{5}R+V\right)_{;kk}+\frac{1}{2}\left(\frac{1}{6}R+V\right)^2\\
&+\frac{1}{180}\left(R^2_{ijkl}-R^2_{ij}\right)+\frac{1}{12}\Omega^2_{ij}\Bigg]\Bigg\}.
\end{aligned} \tag{1.34}$$

1.5 Summary of the formulae

Let us collect the formulae for the computation of the Seeley-DeWitt coefficients in a simple form that will be more useful in the next section for the application to the quantum gravity theory. From now on for the rest of the thesis we will indicate the first three heat kernel coefficients a_0, a_1 and a_2, using DeWitt notation that keeps them unintegrated.

Consider a second order differential operator of the form

$$\hat{H} = -\nabla^2 - V \tag{1.35}$$

where $\nabla^2 = g^{\mu\nu}\nabla_\mu\nabla_\nu$ and V is a matrix-valued function. The covariant derivative above contains both the Riemannian part (with the Levi-Civita connection) and the "gauge" connection:

$$\nabla_\mu = \nabla_\mu^{[R]} + \omega_\mu \tag{1.36}$$

where ω_μ is the gauge field whose field strength tensor $\Omega_{\mu\nu}$ is given by

$$[\nabla_\mu, \nabla_\nu]\phi = \Omega_{\mu\nu}\phi, \tag{1.37}$$

with ϕ a charged (with "color" index) scalar field. The commutation relation for the covariant derivatives of an uncharged (no "color" index) controvariant vector field V^λ is

$$[\nabla_\mu, \nabla_\nu]V^\lambda = R_{\mu\nu}{}^\lambda{}_\sigma V^\sigma. \tag{1.38}$$

Taking into account the insertion of an arbitrary smooth function $\sigma(x)$, using a perturbative approach, the heat kernel can be written as

$$\mathrm{Tr}\Big(\sigma(x)\mathrm{e}^{-\beta\hat{H}}\Big) = \int \frac{d^D x\sqrt{g}}{(4\pi\beta)^{D/2}}\mathrm{tr}[\sigma(x)(a_0(x) + a_1(x)\beta + a_2(x)\beta^2 + \mathcal{O}(\beta^3))] \tag{1.39}$$

where the trace "tr" is on the matrix indices and the coefficients are

$$a_0(x) = \mathbb{1} \tag{1.40}$$

$$a_1(x) = \frac{1}{6}R + V \tag{1.41}$$

$$a_2(x) = \frac{1}{6}\nabla^2\left(\frac{1}{5}R + V\right) + \frac{1}{2}\left(\frac{1}{6}R + V\right)^2 + \frac{1}{180}(R^2_{\mu\nu\tau\sigma} - R^2_{\mu\nu}) + \frac{1}{12}\Omega^2_{\mu\nu}. \tag{1.42}$$

Chapter 2

Perturbative quantum gravity: action and gauge fixing

In the previous chapter we have introduced the heat kernel making explicit the relation with the one-loop effective action. In the concluding section a summary has been presented, with some general formulae that will be useful soon.

The aim of this chapter is firstly to introduce the Einstein-Hilbert action in a manifold equipped with a metric tensor, where a background-fluctuations splitting is performed. The mentioned action enjoys a gauge symmetry that must be properly fixed following BRST methods.

Once that the invertible kinetic operators are isolated and reduced to a form useful for the exponentiation with a proper time, the heat kernel formulae can be applied making use of appropriate replacements. The corresponding Seeley-DeWitt coefficients a_0, a_1, and a_2 are thus computed at arbitrary dimension and at $D = 4$. At the final step they are reduced "on-shell", namely evaluated on Einstein manifolds (where the background metric satisfies Einstein field equations) and compared with B. DeWitt results and other different papers.

2.1 Quadratic approximation of quantum gravity action

Consider first a manifold M of dimension D equipped with a Riemannian metric tensor $G_{\mu\nu}(x)$ with the euclidean signature. The dynamical field of perturbative quantum gravity is the metric tensor itself, by means of which one writes the infinitesimal invariant length of space-time as follows

$$ds^2 = G_{\mu\nu}(x)dx^\mu dx^\nu. \tag{2.1}$$

The metric has a gauge symmetry known as "general change of coordinates", also referred to as diffeomorphism or reparametrization. In particular, under the change of coordinates $x \to x'(x)$, the metric has the following transformation law

$$G_{\mu\nu}(x) \to G'_{\mu\nu}(x') = G_{\sigma\tau}(x)\frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\tau}{\partial x'^\nu}. \tag{2.2}$$

which leaves the Einstein-Hilbert action invariant. The latter, using the principle of least action, yields the Einstein field equations. It is a functional of the metric tensor and can be written in the following way

$$S[G_{\mu\nu}] = -k^{-2}\int d^D x\sqrt{G}[R(G) - 2\Lambda] \tag{2.3}$$

where also the cosmological constant Λ has been inserted. $R(G)$ is the Ricci scalar curvature, function of the metric tensor, and the constant in front of the integral, tipically interpreted as the coupling constant of the theory, is $k^2 = 16\pi G_N$, where G_N is the Newtonian gravitational constant. The theory of quantum gravity constructed using the Einstein-Hilbert action has the property of being non-renormalizable. Similarly to the Fermi theory of the weak interaction, it can be interpreted as an effective field theory, namely valid up to some energy scale according to a proper cut-off dictated by the coupling constant (or the mass entering it).

At this stage it is possible to implement a background field formalism, splitting the metric tensor $G_{\mu\nu}$ into a fixed classical background $g_{\mu\nu}$ (that in general does not coincide with the Minkowski metric), and a small perturbation $h_{\mu\nu}$ which defines the quantum fluctuation of the metric:

$$G_{\mu\nu}(x) = g_{\mu\nu}(x) + h_{\mu\nu}(x). \tag{2.4}$$

The quanta of the field $h_{\mu\nu}$ identifies the so-called "gravitons" of the theory. Typically a constant k is placed in front of the quantum field $h_{\mu\nu}$, in order to control the perturbative expansion by making it as small as one wants. In this case it has been incorporated in the field.

Since we are interested in the evaluation of the one-loop effective action, we can show that, by taking advantage of the metric split, the action (2.4) can be expanded in orders of the fluctuation as follows

$$S[g+h] = \frac{1}{k^2}[S_0 + S_1 + S_2 + \sum_{n=3}^{\infty} S_n]. \tag{2.5}$$

Proceeding in this manner one can identify the linear term, which provides the Einstein field equations, and the quadratic term, precisely the one of our interest.

In order to write the explicit expression of each term up to the quadratic order in the expansion (2.5) we rewrite first the Ricci scalar in terms of the Ricci tensor as $R(G) = G^{\mu\nu}R_{\mu\nu}(G)$, where $G^{\mu\nu}$ is the inverse of the metric. It can be evaluated as

$$G^{\mu\nu}(x) = (g_{\mu\nu} + h_{\mu\nu})^{-1} = g^{\mu\nu} - h^{\mu\nu} + h^\mu_\lambda h^{\lambda\nu} + \mathcal{O}(h^3) \tag{2.6}$$

as can be easily checked by computing the following product to recover the identity:

$$G_{\mu\nu}G^{\nu\lambda} = (g_{\mu\nu} + h_{\mu\nu})(g^{\nu\lambda} - h^{\nu\lambda} + h^{\nu}_{\sigma}h^{\sigma\lambda}) = \delta^{\lambda}_{\mu} + \mathcal{O}(h^3). \tag{2.7}$$

By taking advantage of the property $\log\det A = \mathrm{tr}\log A$ and performing some logarithm and exponential expansions we can write the square root of the metric determinant in powers of $h_{\mu\nu}$ as follows

$$\sqrt{|\det G_{\mu\nu}|} = \sqrt{|\det g_{\mu\nu}|}\left(1 + \frac{1}{2}h^{\mu}_{\mu} - \frac{1}{4}h^{\mu\nu}h_{\mu\nu} + \frac{1}{8}(h^{\mu}_{\mu})^2 + \mathcal{O}(h^3)\right). \tag{2.8}$$

The action can be thus written in terms of expansions (2.6) and (2.8), by making use of the notation $h = h^{\mu}_{\mu} = g^{\mu\nu}h_{\mu\nu}$:

$$S[g+h] = -k^{-2}\int d^Dx\sqrt{g}\left(1+\frac{1}{2}h-\frac{1}{4}h^{\mu\nu}h_{\mu\nu}+\frac{1}{8}h^2+\mathcal{O}(h^3)\right)\left[\left(g^{\mu\nu}-h^{\mu\nu}+h^{\mu}_{\lambda}h^{\lambda\nu}\right)R_{\mu\nu}(g+h)-2\Lambda\right] \tag{2.9}$$

The background metric $g_{\mu\nu}$ is used to raise and lower the indices. After some algebra, integration by parts and neglect of total derivatives one gets:

$$\begin{aligned}
S_0 &= -\int d^Dx\sqrt{g}\Big\{R - 2\Lambda\Big\},\\
S_1 &= \int d^Dx\sqrt{g}\Big\{h^{\mu\nu}\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda\Big)\Big\},\\
S_2 &= -\int d^Dx\sqrt{g}\Big\{\frac{1}{4}h^{\mu\nu}(\nabla^2 + 2\Lambda)h_{\mu\nu} - \frac{1}{8}h(\nabla^2+2\Lambda)h + \frac{1}{2}\Big(\nabla^{\nu}h_{\nu\mu} - \frac{1}{2}\nabla_{\mu}h\Big)^2\\
&+ \frac{1}{2}h^{\mu\lambda}h^{\nu\sigma}R_{\mu\nu\lambda\sigma} + \frac{1}{2}\Big(h^{\mu\lambda}h^{\nu}_{\lambda} - hh^{\mu\nu}\Big)R_{\mu\nu} + \frac{1}{8}\Big(h^2 - 2h^{\mu\nu}h_{\mu\nu}\Big)R\Big\}.
\end{aligned} \tag{2.10}$$

In equations (2.10) the Ricci scalar, the Ricci tensor and the covariant derivative are constructed using the background metric tensor $g_{\mu\nu}$. For a complete and detailed calculation step by step of the Einstein-Hilbert action expansion see appendix A.1.

Following the principle of least action one can obtain the graviton equation of motion by computing $\delta S_1[h]/\delta h^{\mu\nu} = 0$. The result represents, as we have anticipated, the Einstein field equations.

$$\frac{1}{\sqrt{g}}\frac{\delta S_1}{\delta h^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 0 \tag{2.11}$$

The quadratic approximation $S_2[h]$ is the part of the action that we want to study to compute the one-loop effective action. To that end we extract the invertible kinetic operator of the graviton, whose inverse represents the propagator of the theory, and we apply the heat kernel formulae (1.40),(1.41) and (1.42) for the coefficients computation.

Befor proceeding with this pattern we should deal with the gauge symmetry of quantum gravity, performing a proper fixing with a specific gauge. This is the purpose of the next section.

2.2 Gauge fixing and invertible kinetic operators

According to what we have already briefly discussed, the Einstein-Hilbert action (2.3) originally introduced enjoys a gauge symmetry represented by general change of coordinates or diffeomorphism. Let us consider an infinitesimal change of coordinates of the following type

$$x^\mu \to x'^\mu = x^\mu - \xi^\mu(x) \tag{2.12}$$

where $\xi^\mu(x)$ is the infinitesimal vector field along which the transformation is performed. Under (2.12) the full metric tensor $G_{\mu\nu}$ has the following transformation rule

$$\begin{aligned} \delta G_{\mu\nu}(x) &= G'_{\mu\nu}(x) - G_{\mu\nu}(x) = \xi^\rho(x)\partial_\rho G_{\mu\nu}(x) + \partial_\mu \xi^\rho(x) G_{\rho\nu}(x) + \partial_\nu \xi^\rho(x) G_{\mu\rho}(x) && (2.13)\\ &= \nabla_\mu \xi_\nu(x) + \nabla_\nu \xi_\mu(x) = \pounds_\xi G_{\mu\nu}(x). && (2.14) \end{aligned}$$

The symbol $\pounds_\xi$ stands for the Lie derivative of the metric along the vector field ξ^μ. The latter, as a vector, contains D independent directions for the gauge transformation. According to the general background field formalism and in view of the metric split (2.4) one recognizes two different gauge symmetries:
(i) a quantum gauge symmetry transforming $h_{\mu\nu}$ and leaving the background inert:

$$\begin{aligned} &\delta_\epsilon g_{\mu\nu} = 0 \\ &\delta_\epsilon h_{\mu\nu} = \pounds_\epsilon(g_{\mu\nu} + h_{\mu\nu}) = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu + \epsilon^\lambda \nabla_\lambda h_{\mu\nu} + (\nabla_\mu \epsilon^\lambda) h_{\lambda\nu} + (\nabla_\nu \epsilon^\lambda) h_{\mu\lambda}; \end{aligned} \tag{2.15}$$

(ii) a classical symmetry of background diffeomorphism with $h_{\mu\nu}$ transforming as a tensor:

$$\begin{aligned} \delta_\xi g_{\mu\nu} &= \pounds_\xi g_{\mu\nu} \\ \delta_\xi h_{\mu\nu} &= \pounds_\xi h_{\mu\nu}. \end{aligned} \tag{2.16}$$

They both reproduce the gauge transformation of the full metric tensor $G_{\mu\nu}$ when acting on it. However, only the first one is a true dynamical symmetry, leaving the background field invariant. That means also that it is the only one to be "gauge-fixed". The second one instead treats the background metric as a gauge field and the quantum field $h_{\mu\nu}$ transforms as a tensor, for this reason it is called background gauge symmetry.

The reparametrization or diffeomorphism invariance of the Einstein-Hilbert action implies that the metric tensor carries some non-physical gauge degrees of freedom, which are eliminated by the ghost fields. Since we are working on a manifold of dimension D, the vector field degrees of freedom for the ghost and antighost are D for each one, while the number of independent components of a symmetric rank-2 tensor with non-vanishing trace are $D(D+1)/2$. Hence the total number of graviton polarizations is

$$N_{\text{d.o.f.}}(D) = \frac{D(D+1)}{2} - 2D = \frac{D(D-3)}{2}. \tag{2.17}$$

This value can be considered as a benchmark for the first heat kernel coefficient of one-loop quantum gravity. If reduced to dimension $D = 4$ this values becomes $N_{\text{d.o.f.}}(D) = 2$, which is the well-known number of polarizations of the graviton, namely the two physical degrees of freedom of a massless spin-2 gauge theory. One may also notice that in $D = 3$ the graviton has a null number of propagating degrees of freedom, a hint that in this case Einstein gravity has no dynamics.

For this purpose we have to perform the gauge-fixing following the BRST methods. The BRST quantization method is widely used to write the gauge-fixed action for a general non-abelian gauge theory. It can be applied to many areas such as Yang-Mills theory or other cases where the structure functions of the gauge algebra are constants and the algebra closes without employing the equation of motion. A typical example is the one provided by the theory of quantum gravity constructed from the Einstein-Hilbert action, even if it can be interpreted just as effective field theory because of its non-renomalizability problems. First we introduce a gauge-fixing function as follows

$$f^\mu := (\nabla^\nu h_{\nu\mu} - \frac{1}{2}\nabla_\mu h). \tag{2.18}$$

One might notice, at this stage, that in the quadratic approximation of the action (2.10) there is a term that coincides exactly with $\frac{1}{2} f^\mu f_\mu$. That term, that in the de Donder gauge is directly put to zero, gets removed after the gauge-fixing,

By using the function (2.18) we construct the gauge fermion $\mathbf{\Psi}$ as follows

$$\mathbf{\Psi} := b^\mu (f_\mu - \frac{i}{2}\pi_\mu) \tag{2.19}$$

with b^μ the anti-ghost (fermionic and so anti-commuting) and π_μ the auxiliary field or Nakanishi-Lautrup field in Yang-Mills theory (bosonic and so commuting). They are called non-minimal fields and are used to introduce the gauge fermion (2.19). Their BRST variation, with η the anticommuting BRST parameter, are

$$\begin{aligned} \delta_B b^\mu &= i\eta\pi^\mu \\ \delta_B \pi_\mu &= 0 \end{aligned} \tag{2.20}$$

which are obviously nilpotent. For this reason the total quadratic aproximation of the action written in the following way is manifestly BRST invariant:

$$S_{2,tot} = S_2 + s\int d^D x \sqrt{g}\mathbf{\Psi} \tag{2.21}$$

where s indicates the Slavnov variation, namely the BRST variation with the η parameter removed from the left. We are allowed to add an extra term to the quadratic action, an in eq. (2.21), because the BRST variation is nilpotent, namely $s^2 = 0$. This is a crucial point because two BRST invariant quantities, which differ by a BRST variation of a function, belong to the same cohomology class, namely they represent the same physical observable.

The integral of the Slavnov variation of the gauge fermion provides the action for the ghost and for the auxiliary field:

$$S_{gh} + S_\pi = \int d^D x \sqrt{g}\, \frac{\delta_B \mathbf{\Psi}}{\delta \eta} = \int d^D x \sqrt{g} \left[\left(\frac{\pi^2}{2} + i\pi^\mu f_\mu \right) - b^\mu s f_\mu \right] \tag{2.22}$$

where $sf_\mu = f_\mu(sh)$. The Slavnov variation of the quantum fluctuations of the metric is given by the BRST variation with the η parameter stripped off. The BRST variation can be cast from gauge transformation (2.15) just replacing the vector field ϵ_μ with the ghost field c_μ:

$$sh_{\mu\nu} = \nabla_\mu c_\nu + \nabla_\nu c_\mu + \mathcal{O}(h). \tag{2.23}$$

Using the proper equations of motion $\pi_\mu = -if_\mu$, one can integrate out the auxiliary field noting that the following identity holds

$$\frac{\pi^2}{2} + i\pi^\mu f_\mu = \frac{1}{2} f_\mu f^\mu. \tag{2.24}$$

In the total action (2.21) this term cancels the same term with opposite sign above mentioned. From the action in (2.22) what remains is the ghost action

$$S_{gh} = -\int d^D x \sqrt{g} b^\mu s f_\mu. \tag{2.25}$$

Using the Slavnov variation of the metric $h_{\mu\nu}$ (2.23) it is possible to evaluate the explicit expression of sf_μ and recognizing the Riemann tensor in terms of the commutator of covariant derivatives

$$[\nabla_\nu, \nabla_\mu] c^\nu = R_{\nu\mu}{}^\nu{}_\lambda c^\lambda \tag{2.26}$$

the ghost action reads

$$S_{gh} = -\int d^D x \sqrt{g} b^\mu (\nabla^2 c_\mu + R_{\mu\nu} c^\nu). \tag{2.27}$$

For these computations we have ignored terms like b-c-h interactions that are not relevant for one-loop calculations.

The quadratic approximation of the total action thus contains the ghost part and the graviton action:

$$S_{2,tot} = S_{gh} + S_h \tag{2.28}$$

where

$$S_h = \int d^D x\sqrt{g}\Big[-\frac{1}{4}h^{\mu\nu}\nabla^2 h_{\mu\nu} + \frac{1}{8}h\nabla^2 h - \frac{1}{2}h^{\mu\lambda}h^{\nu\sigma}R_{\mu\nu\lambda\sigma} - \frac{1}{2}\Big(h^{\mu\lambda}h^{\nu}_{\lambda} - hh^{\mu\nu}\Big)R_{\mu\nu}$$
$$- \frac{1}{8}(R-2\Lambda)h^2 + \frac{1}{4}h^{\mu\nu}h_{\mu\nu}(R-2\Lambda)\Big]\,. \tag{2.29}$$

Having found the quadratic action for the ghost and the graviton, the next step is to identify down the invertible kinetic operators. The fact that they are invertible reassures us of the correctness of the gauge-fixing procedure. The simplest case is the ghost one. The kinetic operator can be immediately found by rearranging the action as follows

$$S_{gh} = -\int d^D x\sqrt{g} b_\mu(\delta^\mu_\nu \nabla^2 + R^\mu{}_\nu)c^\nu, \tag{2.30}$$

so that one might write it as

$$S_{gh} = \int d^D x\sqrt{g} b_\mu \mathfrak{F}^\mu{}_\nu c^\nu \tag{2.31}$$

where the invertible kinetic operator is

$$\mathfrak{F}^\mu{}_\nu = -(\delta^\mu{}_\nu \nabla^2 + R^\mu{}_\nu) \tag{2.32}$$

which is an operator that acts on vector fields.

For the graviton, the invertible kinetic operator can be identified casting the action in the following form

$$S_h = \int d^D x\sqrt{g}\,\frac{1}{2}h_{\mu\nu}F^{\mu\nu,\alpha\beta}h_{\alpha\beta} \tag{2.33}$$

where the operator is

$$\begin{aligned}F^{\mu\nu\alpha\beta} =& -\frac{1}{4}\Big(g^{\mu\alpha}g^{\nu\beta} + g^{\nu\alpha}g^{\mu\beta} - g^{\mu\nu}g^{\alpha\beta}\Big)\Big(\nabla^2 - R + 2\Lambda\Big)\\ & -\frac{1}{2}\Big(R^{\mu\alpha\nu\beta} + R^{\mu\beta\nu\alpha} - g^{\mu\nu}R^{\alpha\beta} - g^{\alpha\beta}R^{\mu\nu}\Big)\\ & -\frac{1}{4}\Big(g^{\mu\alpha}R^{\nu\beta} + g^{\mu\beta}R^{\nu\alpha} + g^{\nu\alpha}R^{\mu\beta} + g^{\nu\beta}R^{\mu\alpha}\Big).\end{aligned} \tag{2.34}$$

2.3 Seeley-DeWitt coefficients

Equations (2.32) and (2.34) represent respectively the invertible kinetic operators for the ghost and the graviton. Nevertheless, the form of (2.34) is not immediately useful for the computation. For this purpose, one can introduce the following metric (sometimes called DeWitt super-metric)

$$\gamma_{\mu\nu,\alpha\beta} = g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha} - \frac{2}{D-2}g_{\mu\nu}g_{\alpha\beta}, \tag{2.35}$$

that is manifestly symmetric under the exchange of the first two indices with last two. It satisfies the equation

$$\gamma^{\rho\sigma,\mu\nu}\gamma_{\mu\nu,\alpha\beta} = \frac{1}{2}(\delta^\beta_\alpha\delta^\sigma_\beta + \delta^\sigma_\alpha\delta^\rho_\beta). \tag{2.36}$$

Eq. (2.35) can be used in order to lower the first pair of indices in expression (2.34) so that one gets the operator F in a form that can be exponentiated to yield the corresponding heat kernel:

$$\begin{aligned}
F_{\mu\nu}{}^{\sigma\tau} =& -\frac{1}{2}(\delta_\mu{}^\sigma\delta_\nu{}^\tau + \delta_\mu{}^\tau\delta_\nu{}^\sigma)(\nabla^2 + 2\Lambda) + \frac{1}{2}\left(\delta_\mu{}^\sigma\delta_\nu{}^\tau + \delta_\mu{}^\tau\delta_\nu{}^\sigma - \frac{2}{D-2}g_{\mu\nu}g^{\sigma\tau}\right)R \\
&+ \frac{2}{D-2}g_{\mu\nu}R^{\sigma\tau} + g^{\sigma\tau}R_{\mu\nu} - R_\mu{}^\sigma{}_\nu{}^\tau - R_\mu{}^\tau{}_\nu{}^\sigma \\
&- \frac{1}{2}(\delta_\mu{}^\sigma R_\nu{}^\tau + \delta_\mu{}^\tau R_\nu{}^\sigma + \delta_\nu{}^\sigma R_\mu{}^\tau + \delta_\nu{}^\tau R_\mu{}^\sigma)
\end{aligned} \tag{2.37}$$

where D is the dimension of the manifold.

Let us start with the ghost coefficients computation. By comparing the kinetic operator (2.32) with the general form (1.35) one replaces $\mathbb{1}$ with δ^μ_ν and the matrix V with R^μ_ν. Since the ghost field is a controvariant vector field, the commutation relation for the covariant derivative operators that appear in (2.32) are

$$[\nabla_\mu, \nabla_\nu]c^\sigma = R_{\mu\nu}{}^\sigma{}_\tau c^\tau. \tag{2.38}$$

The Riemann tensor in this case has to be tought as a set of $D \times D$ matrices labelled by the indices μ and ν. Therefore one also replaces $\Omega_{\mu\nu}$ with $R^\sigma{}_{\tau\mu\nu}$. Using the formulae (1.40), (1.41), (1.42) and that $\mathrm{tr}(\delta^\mu_\nu) = D$, one can check that the ghost heat kernel coefficients at arbitrary dimension D are

$$\mathrm{tr}[a_{0,gh}(x)] = D \tag{2.39}$$

$$\mathrm{tr}[a_{1,gh}(x)] = \frac{1}{6}DR + R \tag{2.40}$$

$$\mathrm{tr}[a_{2,gh}(x)] = \frac{D+5}{30}\nabla^2 R + \frac{D+12}{72}R^2 - \frac{D-90}{180}R_{\mu\nu}R^{\mu\nu} + \frac{D-15}{180}R_{\sigma\tau\mu\nu}R^{\sigma\tau\mu\nu} \tag{2.41}$$

that reduced in dimension $D=4$ are

$$\mathrm{tr}[a_{0,gh}(x)] \xrightarrow{D=4} 4 \tag{2.42}$$

$$\mathrm{tr}[a_{1,gh}(x)] \xrightarrow{D=4} \frac{5}{3}R \tag{2.43}$$

$$\mathrm{tr}[a_{2,gh}(x)] \xrightarrow{D=4} \frac{3}{10}\nabla^2 R + \frac{2}{9}R^2 + \frac{43}{90}R_{\mu\nu}R^{\mu\nu} - \frac{11}{180}R_{\sigma\tau\mu\nu}R^{\sigma\tau\mu\nu} \tag{2.44}$$

It is rather manifest that the result in (2.42) represents the correct number of degrees of freedom of the ghost vector field.

The same strategy can be applied for the graviton heat kernel coefficients. By comparing the operator (2.37) with the general form (1.35) one replaces $\mathbb{1}$ with $\delta_{\mu\nu}{}^{\sigma\tau}$ and the matrix V with $-\Xi_{\mu\nu}{}^{\sigma\tau}$, where

$$\delta_{\mu\nu}{}^{\sigma\tau} := \frac{1}{2}(\delta_\mu{}^\sigma\delta_\nu{}^\tau + \delta_\mu{}^\tau\delta_\nu{}^\sigma) \tag{2.45}$$

and

$$\begin{aligned}\Xi_{\mu\nu}{}^{\sigma\tau} := &\frac{1}{2}\left(\delta_\mu{}^\sigma\delta_\nu{}^\tau + \delta_\mu{}^\tau\delta_\nu{}^\sigma - \frac{2}{D-2}g_{\mu\nu}g^{\sigma\tau}\right)R + \frac{2}{D-2}g_{\mu\nu}R^{\sigma\tau} + g^{\sigma\tau}R_{\mu\nu}\\ &- R_\mu{}^\sigma{}_\nu{}^\tau - R_\mu{}^\tau{}_\nu{}^\sigma - \frac{1}{2}(\delta_\mu{}^\sigma R_\nu{}^\tau + \delta_\mu{}^\tau R_\nu{}^\sigma + \delta_\nu{}^\sigma R_\mu{}^\tau + \delta_\nu{}^\tau R_\mu{}^\sigma).\end{aligned} \tag{2.46}$$

One might recognize eq. (2.45) to be the symmetric Kronecker delta, with trace $\mathrm{tr}(\delta_{\mu\nu}{}^{\sigma\tau}) = \frac{1}{2}D(D+1)$. Since the kinetic operator in (2.37) acts on fields that are covariant symmetric tensors, the commutation relation of covariant derivative becomes

$$[\nabla_\mu, \nabla_\nu]h_{\sigma\tau} = R_{\sigma\tau}{}^{\rho\lambda}{}_{\mu\nu}h_{\rho\lambda} \tag{2.47}$$

where

$$R_{\sigma\tau}{}^{\rho\lambda}{}_{\mu\nu} := \frac{1}{2}(\delta_\sigma{}^\rho R_\tau{}^\lambda{}_{\mu\nu} + \delta_\sigma{}^\lambda R_\tau{}^\rho{}_{\mu\nu} + \delta_\tau{}^\rho R_\sigma{}^\lambda{}_{\mu\nu} + \delta_\tau{}^\lambda R_\sigma{}^\rho{}_{\mu\nu}) \tag{2.48}$$

that has to be considered as a set of $\frac{1}{2}D(D+1) \times \frac{1}{2}D(D+1)$ matrices labelled by symmetrized pairs of indices.

At this stage one can use the formulae (1.40), (1.41), (1.42) and proceed with a straightforward calculation with significant algebra. Some intermediate steps that might be useful for the reader are the following:
(i) For $a_1(x)$ and $a_2(x)$ computations:

$$\mathrm{tr}(\Xi_{\mu\nu}{}^{\sigma\tau}) = \Xi_{\mu\nu}{}^{\mu\nu} = \frac{1}{2}D(D-1)R - D(D+1)\Lambda\,; \tag{2.49}$$

(ii) For $a_2(x)$ computation: the graviton coefficient $a_2(x)$ contains the following contributions

$$a_2(x) \supset \frac{1}{12}\Omega_{\mu\nu}^2 \to \frac{1}{12}R_{\sigma\tau}{}^{\rho\lambda}{}_{\mu\nu}R_{\rho\lambda}{}^{\alpha\beta\mu\nu} \xrightarrow{\mathrm{tr}} \frac{1}{12}R_{\sigma\tau}{}^{\rho\lambda}{}_{\mu\nu}R_{\rho\lambda}{}^{\sigma\tau\mu\nu} = -\frac{D+2}{12}R^2_{\tau\lambda\mu\nu}; \tag{2.50}$$

$$\begin{aligned} a_2(x) \supset \frac{1}{2}\left(\frac{1}{6}R + V\right)^2 &\to \frac{1}{2}\left(\frac{1}{6}R\delta_{\mu\nu}{}^{\sigma\tau} - \Xi_{\mu\nu}{}^{\sigma\tau}\right)\left(\frac{1}{6}R\delta_{\sigma\tau}{}^{\alpha\beta} - \Xi_{\sigma\tau}{}^{\alpha\beta}\right) \\ &= \frac{1}{2}\left(\frac{R^2}{36}\delta_{\mu\nu}{}^{\alpha\beta} - \frac{1}{3}R\Xi_{\mu\nu}{}^{\alpha\beta} + \Xi_{\mu\nu}{}^{\sigma\tau}\Xi_{\sigma\tau}{}^{\alpha\beta}\right), \end{aligned} \tag{2.51}$$

where the following trace is shown

$$\begin{aligned} \mathrm{tr}(\Xi_{\mu\nu}{}^{\sigma\tau}\Xi_{\sigma\tau}{}^{\alpha\beta}) = \Xi_{\mu\nu}{}^{\sigma\tau}\Xi_{\sigma\tau}{}^{\mu\nu} =& \frac{D^3 - 5D^2 + 8D + 4}{2(D-2)}R^2 + \frac{D^2 - 8D + 4}{D-2}R^2_{\sigma\tau} \\ & 3R^2_{\mu\nu\sigma\tau} + D(D+1)\Lambda^2 - \frac{5D^3 - 17D^2 + 14D}{6(D-2)}R\Lambda. \end{aligned} \tag{2.52}$$

(*iii*) A necessary identity repeatedly used in the computation is the following one

$$\begin{aligned} R^{\alpha\beta\delta\gamma}R_{\alpha\delta\beta\gamma} &= R^{\alpha\beta\delta\gamma}(-R_{\alpha\beta\gamma\delta} - R_{\alpha\gamma\delta\beta}) = R^{\alpha\beta\delta\gamma}R_{\alpha\beta\delta\gamma} - R^{\alpha\beta\delta\gamma}R_{\alpha\gamma\delta\beta} \\ &= R^{\alpha\beta\delta\gamma}R_{\alpha\beta\delta\gamma} - R^{\alpha\beta\gamma\delta}R_{\alpha\gamma\beta\delta} = R^{\alpha\beta\delta\gamma}R_{\alpha\beta\delta\gamma} - R^{\alpha\beta\delta\gamma}R_{\alpha\delta\beta\gamma}, \end{aligned} \tag{2.53}$$

therefore

$$R^{\alpha\beta\delta\gamma}R_{\alpha\delta\beta\gamma} = \frac{1}{2}R^2_{\alpha\beta\delta\gamma}. \tag{2.54}$$

Once that the calculations are done we end up with the following graviton heat kernel coefficients at arbitrary dimension D:

$$\mathrm{tr}[a_0(x)] = \frac{1}{2}D(D+1) \tag{2.55}$$

$$\mathrm{tr}[a_1(x)] = -\frac{D(5D-7)}{12}R + D(D+1)\Lambda \tag{2.56}$$

$$\begin{aligned} \mathrm{tr}[a_2(x)] =& -\frac{D(2D-3)}{30}\nabla^2 R + \frac{25D^3 - 145D^2 + 262D + 144}{144(D-2)}R^2 \\ & - \frac{D^3 - 181D^2 + 1438D - 720}{360(D-2)}R_{\sigma\tau}R^{\sigma\tau} + \frac{D^2 - 29D + 480}{360}R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} \\ & + D(D+1)\Lambda^2 - \frac{D(5D^2 - 17D + 14)}{6(D-2)}R\Lambda. \end{aligned}$$

that reduced to $D = 4$ become

$$\mathrm{tr}[a_0(x)] \xrightarrow{D=4} 10 \tag{2.57}$$

$$\mathrm{tr}[a_1(x)] \xrightarrow{D=4} -\frac{13}{3}R + 20\Lambda \tag{2.58}$$

$$\mathrm{tr}[a_2(x)] \xrightarrow{D=4} -\frac{2}{3}\nabla^2 R + \frac{59}{36}R^2 - \frac{55}{18}R_{\sigma\tau}R^{\sigma\tau} + \frac{19}{18}R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} + 20\Lambda^2 - \frac{26}{3}R\Lambda. \tag{2.59}$$

One can recognize (2.57) to be the number of degrees of freedom of a symmetric rank-2 tensor with non-vanishing trace in dimension $D = 4$. Nevertheless, in order to have the correct number of degrees of freedom of the graviton, namely its physical polarizations, we have to sum these results to the ghost ones.

Using these coefficients we are able to write the following terms contained in the one-loop effective action Γ for the ghost and the graviton, terms that lead to divergencies and which must be renormalized away. The general expression is

$$\Gamma = \alpha \int_0^\infty \frac{d\beta}{\beta} \int \frac{d^D x\sqrt{g}}{(4\pi\beta)^{D/2}} \mathrm{tr}[a_0(x) + a_1(x)\beta + a_2(x)\beta^2 + \mathcal{O}(\beta^3)]. \tag{2.60}$$

with the value of α depending on the type of field (i.e. $\alpha = -\frac{1}{2}$ for a real boson like a real scalar, $\alpha = -1$ for a complex boson, and opposite signs for anticommuting fields like the ghosts).

In the following we write the effective action of the ghost and the graviton, specifically at $D = 4$, for a comparison with other texts and papers:

$$\Gamma_{gh} = \int_0^\infty \frac{d\beta}{\beta} \int \frac{d^4 x\sqrt{g}}{(4\pi\beta)^2} \Big[4 + \frac{5}{3}R\beta + \Big(\frac{3}{10}\nabla^2 R + \frac{2}{9}R^2 + \frac{43}{90}R_{\mu\nu}R^{\mu\nu} - \frac{11}{180}R_{\sigma\tau\mu\nu}R^{\sigma\tau\mu\nu}\Big)\beta^2 + \mathcal{O}(\beta^3)\Big]; \tag{2.61}$$

$$\Gamma_{graviton} = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta} \int \frac{d^4 x\sqrt{g}}{(4\pi\beta)^2} \Big[10 + \Big(-\frac{13}{3}R + 20\Lambda\Big)\beta + \Big(-\frac{2}{3}\nabla^2 R + \frac{59}{36}R^2 - \frac{55}{18}R_{\sigma\tau}R^{\sigma\tau} + \frac{19}{18}R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} + 20\Lambda^2 - \frac{26}{3}R\Lambda\Big)\beta^2 + \mathcal{O}(\beta^3)\Big]. \tag{2.62}$$

The total heat kernel of quantum gravity is the sum of the individual contributions of the ghost and the graviton as follows

$$\Gamma[g] = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\Big\{\mathrm{Tr}\Big[\mathrm{e}^{-\beta\hat{F}}\Big] - 2\,\mathrm{Tr}\Big[\mathrm{e}^{-\beta\hat{\mathfrak{F}}}\Big]\Big\} \tag{2.63}$$

where $\hat{\mathfrak{F}}$ and $\hat{F}$ are the second order differential operator respectively for the ghosts and the graviton. Therefore the general heat kernel coefficient of order k for quantum gravity is given by

$$\mathrm{tr}[a_{k,tot}] = \mathrm{tr}[a_k] - 2\mathrm{tr}[a_{k,gh}] \tag{2.64}$$

which yields the following coefficients

$$\mathrm{tr}[a_{0,tot}(x)] = \frac{D(D-3)}{2} \tag{2.65}$$

$$\mathrm{tr}[a_{1,tot}(x)] = -\frac{5D^2-3D+24}{12}R + D(D+1)\Lambda \tag{2.66}$$

$$\begin{aligned}\mathrm{tr}[a_{2,tot}(x)] = &-\frac{2D^2-D+10}{30}\nabla^2 R + \frac{25D^3-149D^2+222D+240}{144(D-2)}R^2 \\ &- \frac{D^3-185D^2+1806D-1440}{360(D-2)}R_{\sigma\tau}R^{\sigma\tau} + \frac{D^2-33D+540}{360}R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} \\ &+ D(D+1)\Lambda^2 - \frac{5D^3-17D^2+14D}{6(D-2)}R\Lambda\end{aligned} \tag{2.67}$$

that reduced to $D = 4$ are

$$\mathrm{tr}[a_{0,tot}(x)] \xrightarrow{D=4} 2 \tag{2.68}$$

$$\mathrm{tr}[a_{1,tot}(x)] \xrightarrow{D=4} -\frac{23}{3}R + 20\Lambda \tag{2.69}$$

$$\mathrm{tr}[a_{2,tot}(x)] \xrightarrow{D=4} -\frac{19}{15}\nabla^2 R + \frac{43}{36}R^2 - \frac{361}{90}R_{\sigma\tau}R^{\sigma\tau} + \frac{53}{45}R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} + 20\Lambda^2 - \frac{26}{3}R\Lambda. \tag{2.70}$$

As a check we recognize from eq. (2.68) the correct number of polarizations of the graviton in dimension $D = 4$. One may notice that the field equations (2.11) have not been used at any step in this calculation. Therefore these results are valid for any background field.

The total, unregulated one-loop effective action (2.63) thus reads

$$\begin{aligned}\Gamma = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\int \frac{d^4x\sqrt{g}}{(4\pi\beta)^2}\Bigg[2 + \left(-\frac{23}{3}R + 20\Lambda\right)\beta + \Bigg(-\frac{19}{15}\nabla^2 R + \frac{43}{36}R^2 \\ - \frac{361}{90}R_{\sigma\tau}R^{\sigma\tau} + \frac{53}{45}R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau} + 20\Lambda^2 - \frac{26}{3}R\Lambda\Bigg)\beta^2 + \mathcal{O}(\beta^3)\Bigg].\end{aligned} \tag{2.71}$$

As well-known, this expansion is not applicable to get the finite terms of the effective action because of the infrared divergencies that arise since the graviton is massless (these IR divergences are seen from the lack of convergence in the upper limit of the proper time integration), but it is enough to identify the UV diverging pieces (arising from the lower limit of the proper time integration) that must be renormalized away.

In general, the effective action is expected to depend on the gauge chosen in constructing the gauge-fixed action and perturbation theory. However, it becomes gauge invariant when evaluated on-shell. Thus, one can restrict the above coefficients "on-shell", namely simplify them by evaluating them on Einstein manifolds. Using the Einstein field equations (2.11) one has that the cosmological constant can be written in terms of the Ricci scalar curvature as

$$\Lambda = \frac{D-2}{2D} R \tag{2.72}$$

and the Ricci tensor is given by

$$R_{\mu\nu} = \frac{2\Lambda}{D-2} g_{\mu\nu}. \tag{2.73}$$

Using equations (2.72) and (2.73), the coefficients (2.65)-(2.67) reduce to

$$\mathrm{tr}[a_{0,tot}(x)] = \frac{D(D-3)}{2} \tag{2.74}$$

$$\mathrm{tr}[a_{1,tot}(x)] = \frac{2D^3 - 8D^2 - 66D + 72}{24(D-1)} R \tag{2.75}$$

$$\mathrm{tr}[a_{2,tot}(x)] = \frac{(D+5)(5D^2 - 42D - 144)}{720D} R^2 + \frac{D^2 - 33D + 540}{360} R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} \tag{2.76}$$

that at $D = 4$ (eqs. (2.68)-(2.70)) become

$$\mathrm{tr}[a_{0,tot}(x)] \xrightarrow{D=4} 2 \tag{2.77}$$

$$\mathrm{tr}[a_{1,tot}(x)] \xrightarrow{D=4} -\frac{8}{3} R \tag{2.78}$$

$$\mathrm{tr}[a_{2,tot}(x)] \xrightarrow{D=4} -\frac{29}{40} R^2 + \frac{53}{45} R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} \; . \tag{2.79}$$

All these terms give rise to divergences in the effective action, and must be renormalized away. As they are evaluated on-shell, they identify gauge invariant coefficients that should not depend on the gauge chosen. Any formulation of quantum gravity should be able to reproduce them independently of the scheme chosen in the calculation.

For future reference, we write them also in terms of the cosmological constant Λ, rather that in terms of the Ricci scalar curvature R

$$\mathrm{tr}[a_{0,tot}(x)] \xrightarrow{D=4} 2 \tag{2.80}$$

$$\mathrm{tr}[a_{1,tot}(x)] \xrightarrow{D=4} -\frac{32}{3} \Lambda \tag{2.81}$$

$$\mathrm{tr}[a_{2,tot}(x)] \xrightarrow{D=4} -\frac{58}{5} \Lambda^2 + \frac{53}{45} R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau}, \tag{2.82}$$

In light of the above results, we can introduce a topological invariant called Euler characteristic of the manifold that can be used to compare our results with other papers and texts. In $D = 4$ the Euler characteristic is

$$\chi(M) = \frac{1}{32\pi^2} \int d^4x \sqrt{g} (R^2_{\mu\nu\sigma\tau} - 4R^2_{\mu\nu} + R^2) = \frac{1}{32\pi^2} \int d^4x \sqrt{g} G \tag{2.83}$$

where $G = R^2_{\mu\nu\sigma\tau} - 4R^2_{\mu\nu} + R^2$ is called Gauss-Bonnet term. On Einstein manifolds $G = R^2_{\mu\nu\sigma\tau}$. Hence the part of the one-loop effective action on Einstein manifolds of dimension $D = 4$ that is logarithmically divergent, i.e.

$$\Gamma_{div} = -\frac{1}{2}\frac{1}{(4\pi)^2}\int_0^\infty \frac{d\beta}{\beta}\int d^4x\sqrt{g}\left(-\frac{58}{5}\Lambda^2 + \frac{53}{45}R^2_{\mu\nu\sigma\tau}\right) \tag{2.84}$$

can be written as

$$\Gamma_{div} = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\left(\frac{106}{45}\chi(M) - \frac{29}{40\pi^2}\Lambda^2\mathrm{vol}(M)\right). \tag{2.85}$$

where $\mathrm{vol}(M) = \int_M d^4x\sqrt{g}$ represents the volume of the manifold M.

Let us comment on these results. Neglecting the topological term, and setting the cosmological constant to zero, one recovers the well-known result of t' Hooft and Veltman [13], according to which quantum gravity is finite at one-loop (more precisely, it is free of logarithmic divergences, as given by eq. (2.85)). This result does not hold anymore at two-loops, as shown by Goroff and Sagnotti [14]. The one-loop result for quantum gravity with the cosmological constant at $D = 4$ is instead originally due to Christensen and Duff [15]. This result is also recovered by the above expression.

Let us now discuss and compare the more general heat kernel coefficients we have calculated with the literature. One may check that some of the results shown in eqs. (2.65)-(2.67) of the total quantum gravity heat kernel coefficients at arbitrary D are different from the ones reported by B. DeWitt, in particular the ones in eqs. (16.80)-(16.82) of [1] and eqs. (16.79)-(16.81) of [3]. This is exactly what we have anticipated in the introduction: some of the first three heat kernel coefficients are sometimes erroneously reported in the literature. This could lead to confusion, for example where such coefficients are used as a starting point for further purposes. In this regard we have decided to rederive them using the original heat kernel method, to make sure we consider the correct ones.

Nevertheless, our results are identically reported by I.G. Avramidi in [16], where similar heat kernel methods are used as well. The reader can compare step by step the Seeley-DeWitt coefficients for the ghost, the graviton, and the total one-loop effective action. This accordance provides an extra proof of the correctness of our results (and of theirs).

It is also worth stressing that eqs. (2.84) and (2.85) (using the Euler characteristic) are not only in agreement with Avramidi, see eq. (3.79) of [16], but also with the results of eq. (4.23) of [15], as anticipated above, while they disagree with [17], as noted by Avramidi himself. Moreover, the ghost, graviton and total coefficients at $D = 4$ are also reproduced in F. Bastianelli and R. Bonezzi [4], where they have been computed using a worldline approach to quantum gravity.

A further source of disagreement in the study of such coefficients could be related to the gauge-fixing procedure adopted. Indeed, the effective action for gauge theories in general may depend on the gauge chosen. It is expected to be gauge invariant only on-shell. For this reason, we have calculated the gauge-invariant coefficients at arbitrary D as well. To derive them, we have considered a gravitational background satisfying the Einstein field equations. These coefficients represent a benchmark for any correct construction of perturbative quantum gravity, as they should not depend on the method chosen for their calculation.

In particular, a novel method for describing the graviton makes use of the first quantization of the $\mathcal{N} = 4$ spinning particle, which applies the same strategy used in defining string theory by first quantizing a mechanical model. In this respect, we have verified that the coefficients (2.77)-(2.79) on $D = 4$ Einstein manifolds are identical to those calculated in [7], obtained precisely by using the $\mathcal{N} = 4$ spinning particle. Note that the latter can be consistently quantized only on on-shell backgrounds, while keeping the background off-shell leads to results that differ from the standard heat kernel coefficients given above. However, the more general coefficients (2.74)-(2.76) at arbitrary D differ from the ones obtained in [7] with the $\mathcal{N} = 4$ spinning particle, suggesting the need of improving the latter so to meet this benchmark in arbitrary dimensions. This fact was unexpected and provides a novel result of this thesis.

Chapter 3

Worldline formalism for the ghost

In the foregoing chapter the heat kernel expansion, with its well known formulas, has been employed for the study of perturbative quantum gravity's Seeley-DeWitt coefficients a_0, a_1 and a_2. The achieved results have been compared with the literature and sources of disagreements have been pointed out.

The aim of the second part of this thesis lies in the attempt of reproducing those results using the worldline formalism. Our approach will follow F. Bastianelli and R. Bonezzi's work [4] where the heat kernel coefficients have been computed specifically in space-time dimension $D = 4$. Our interest is to extend this approach to arbitrary dimension D and to compare results with those of the previous chapter.

The core idea of this approach resides in the possibility of constructing a worldline representation of the invertible kinetic differential operators (2.32) and (2.34) of the quadratic action. This idea should be applied separately to the ghosts and the graviton, in order to find their heat kernel coefficients and finally put them together to construct the full one-loop effective action. For this purpose the present chapter will be dedicated to the construction of a worldline model which correctly reproduces the ghost. The greater simplicity with respect to the graviton case can be used to present the theory that will be then applied also to the latter.

After a brief introduction of the effective action we proceed with the construction of a vector model that correctly describes the ghosts. This will require bosonic and fermionic phase space variables for the Hilbert space construction, where the general state contains also vector fields. All the other unecessary fields need to be projected out using an additional coupling with a worldline gauge field and an extra Chern-Simons term.

The path integral construction needs a regularization scheme to satisfy some renormalization conditions such as a chosen ordering of the associated Hamiltonian operator. The regularization procedure used (*dimensional regularization* in this thesis) will be necessary for the computation of Feynman graphs with ambiguous and divergent products of distributions.

The one-loop effective action then will be computed with a perturbative expansion in Riemann normal coordinates.

3.1 Worldline theory and effective action

In order to introduce the worldline theory let us first present the partition function for pure gravity in terms of the path integral as follows

$$Z[g] = \int DhDbDc\, e^{-S_{2,tot}}, \tag{3.1}$$

where the quadratic action is the one of eq. (2.28) and the integral is over the graviton $h_{\mu\nu}$, the ghost c_μ and the antighost b^μ. The partition function can be written using the determinants as follows

$$Z[g] \propto \mathrm{Det}_V[\mathfrak{F}^\mu{}_\nu]\mathrm{Det}_{T2}^{-1/2}[F_{\mu\nu,\alpha\beta}] \tag{3.2}$$

where V and $T2$ as subscripts represent the functional space of action of the operators, namely vectors and symmetric rank-2 tensors respectively. The operators in the last equation are the ones found in the previous chapter, for the ghost and the graviton. The one-loop effective action that we want to compute, defined by $Z[g] = e^{-\Gamma[g]}$, is given by

$$\Gamma[g] = \frac{1}{2}\{\mathrm{Tr}_{T2}\ln[F_{\mu\nu,\alpha\beta}] - 2\mathrm{Tr}_V\ln[\mathfrak{F}^\mu{}_\nu]\}. \tag{3.3}$$

If we introduce the Schwinger proper time representation for the logarithm, i.e. for a generic operator $\hat{\mathcal{O}}$

$$\mathrm{Tr}\ln\hat{\mathcal{O}} = -\int_0^\infty \frac{d\beta}{\beta}\mathrm{Tr}\left[e^{-\beta\hat{\mathcal{O}}}\right] \tag{3.4}$$

where β is the proper time parameter, the one-loop effective action can be cast as

$$\Gamma[g] = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\left\{\mathrm{Tr}\left[e^{-\beta\hat{F}}\right] - 2\mathrm{Tr}\left[e^{-\beta\hat{\mathfrak{F}}}\right]\right\}. \tag{3.5}$$

The operators $\hat{F}$ and $\hat{\mathfrak{F}}$ represent the quantum mechanical Hamiltonians of the graviton and the ghosts respectively, that acting on symmetric rank-2 tensors $\phi^{\alpha\beta}$ and vectors V^ν give

$$\begin{aligned}(\hat{F}\phi)_{\mu\nu} = F_{\mu\nu\alpha\beta}\phi^{\alpha\beta} = &-\frac{1}{2}\left(g_{\mu\alpha}g_{\nu\beta} - \frac{1}{2}g_{\mu\nu}g_{\alpha\beta}\right)\left(\nabla^2 - R + 2\Lambda\right)\phi^{\alpha\beta}\\ &-\frac{1}{2}\left(R_{\mu\alpha\nu\beta} + R_{\mu\beta\nu\alpha} - \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}\right)\phi^{\alpha\beta}\\ &-\frac{1}{2}\left(R^\lambda_\mu\phi_{\lambda\nu} + R^\lambda_\nu\phi_{\lambda\mu} - \frac{1}{2}g_{\mu\nu}R^{\alpha\beta}\phi_{\alpha\beta}\right) + \frac{1}{2}g_{\alpha\beta}R_{\mu\nu}\phi^{\alpha\beta};\end{aligned} \tag{3.6}$$

$$(\hat{\mathfrak{F}}V)_\mu = \mathfrak{F}^\mu{}_\nu V^\nu = -\frac{1}{2}\Big(\nabla^2 V_\mu + R_{\mu\nu}V^\nu\Big). \tag{3.7}$$

The aim of the next section is to study a worldline representation of the ghost differential operator when acting on a vector as shown in (3.7). Therefore, the idea is to construct the model of a particle where the associated quantum mechanical Hamiltonian $\hat{\mathfrak{F}}$ acts on a Hilbert space containing vectors.

3.2 The vector model for the ghost

In order to produce the above-mentioned model we consider first a D-dimensional space-time with metric $g_{\mu\nu}(x)$, which coordinates and related conjugate momenta are the usual real bosonic variables of the phase space $x^\mu(t)$ and $p_\mu(t)$. These variables, after the quantization, will describe the functional dependence on space-time points of the wave function which represents a state in the Hilbert space. We will see that among other fields the constructed wave function contains the studied vector field for the ghost as well. For this purpose the wave function should also have discrete indices. This is possible by constructing the phase space also with fermionic variables that are specifically worldline complex fermions, for simplicity characterized by flat Lorentz indices: $\lambda^a(t)$ and corresponding conjugate momenta $\bar{\lambda}^a(t)$.

The bosonic variables, togheter with the complex fermionic ones, define a graded phase space. We promote these variables to operators by introducing (anti)-commutation relations for the usual canonical quantization:

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu, \qquad\qquad \{\hat{\lambda}^a, \hat{\lambda}^{\dagger b}\} = \delta^{ab}. \tag{3.8}$$

where δ^{ab} is the flat metric and we put as usual $\hbar = 1$. The quantization of Grassmann odd variables produces fermionic creation and annihilation operators that give rise to a finite dimensional Hilbert space. The use of additional bosonic variables rather than fermionic ones for the discrete indices description is valid as well. The main difference, apart different signs, is that the corresponding Hilbert space would be infinite dimensional.

After the quantization we consider x^μ as the eigenvalues of $\hat{x}^\mu$, while for the fermionic sector we introduce "bra" coherent states that are eigenstates $\langle\lambda|$ of the operator $\hat{\lambda}^a$ with eigenvalues λ^a: $\langle\lambda|\,\hat{\lambda}^a = \langle\lambda|\,\lambda^a$. The corresponding bosonic and fermionic momenta are their derivatives.

Since we work in a curved space-time of metric $g_{\mu\nu}(x)$, we have to pay attention to the possible $g^{1/4}$ factors present in the bosonic momentum operator $\hat{p}_\mu$, where $g = |\text{det} g_{\mu\nu}(x)|$. This is dictated by the form of the covariant measure present in the scalar product, that is $d^D x\sqrt{g}$. Therefore, in order to guarantee hermiticity of the momentum, the latter is given by

$$p_\mu = -ig^{-1/4}\partial_\mu g^{1/4}, \tag{3.9}$$

with derivative acting through the g factor. While the momentum of the fermionic variable is

$$\bar{\lambda}_a = \frac{\partial}{\partial\lambda^a}, \tag{3.10}$$

so that $\{\bar{\lambda}_a, \lambda^b\} = \delta_a^b$.

At this juncture it is possible to construct the states of the Hilbert space using the graded variables providing the continuous and discrete indices. The generic state of the Hilbert space is represented by a wave function $|\Psi\rangle \sim \Psi(x,\lambda)$ that thanks to the coherent states is given by $\Psi(x,\lambda) = (\langle x| \otimes \langle\lambda|)\,|\Psi\rangle$. It can be Taylor expanded as follows

$$\begin{aligned} |\Psi\rangle \sim \Psi(x,\lambda) =& \Psi(x) + \Psi_a(x)\lambda^a + \frac{1}{2}\Psi_{a_1a_2}(x)\lambda^{a_1}\lambda^{a_2} + \cdots + \frac{1}{D!}\Psi_{a_1\ldots a_D}(x)\lambda^{a_1}\ldots\lambda^{a_D} \\ =& \sum_{n=0}^{D}\frac{1}{n!}\Psi_{a_1\ldots a_n}(x)\lambda^{a_1}\ldots\lambda^{a_n}. \end{aligned} \tag{3.11}$$

The upper value of the sum is D, the space-time dimension, which represents the number of independent components of a vector. Depending on the occupation number n, it is possible to identify different antisymmetric tensors. Among them there is also the vector field we want to reproduce, associated to the occupation number $n = 1$, with corresponding state of the form $|V\rangle \sim V_a(x)\lambda^a$. From now on we consider wave functions containing only $V_a(x)$ such as $|\Psi\rangle \sim V_a(x)\lambda^a$. In the next section we will describe the method to project out all the unwanted fields and to keep only the one of interest, exactly describing the ghost field of our gauge fixed quantum gravity action.

Since the aim of the chapter is to provide a worldline representation of the ghost differential operator (2.32), we need to reproduce also the Laplacian operator $\hat{\nabla}^2$. Therefore we want to construct the covariant derivative. Hence, we introduce the generators of the Lorentz group $SO(D)$

$$M^{ab} = -M^{ba} := [\lambda^a, \bar{\lambda}^b], \tag{3.12}$$

obeying the corresponding $\mathfrak{so}(D)$ algebra

$$[M^{ab}, M^{cd}] = \delta^{bc}M^{ad} + \delta^{ad}M^{bc} - \delta^{ac}M^{bd} - \delta^{bd}M^{ac}. \tag{3.13}$$

We define the covariant derivative operator as follows

$$\hat{\nabla}_\mu := \partial_\mu + \frac{1}{2}\omega_{\mu ab}M^{ab} = \partial_\mu + \omega_{\mu ab}\lambda^a\bar{\lambda}^b, \tag{3.14}$$

with $\omega_{\mu ab}$ the spin connection. It is rather important to stress that the covariant derivative operator written in the last equation acts on wave functions, and is different from covariant derivatives acting on fields. If we consider the wave function $V(x, \lambda) \sim V_a(x)\lambda^a$, the action of the operator (3.14) is given by

$$\hat{\nabla}_\mu V(x, \lambda) = (\nabla_\mu V_a)\lambda^a = (\partial_\mu V_a + \omega_{\mu a}{}^b V_b)\lambda^a, \tag{3.15}$$

where one recognizes the action of the covariant derivative on the vector field V_a. The covariant derivative can be written in a compact form by introducing the covariant momentum, which includes the momentum p_μ and the spin connection. Using eq. (3.9), the covariant derivative is given by

$$\hat{\nabla}_\mu = ig^{1/4}\pi_\mu g^{-1/4} = ig^{1/4}(p_\mu - i\omega_{\mu ab}\lambda^a\bar{\lambda}^b)g^{-1/4}, \tag{3.16}$$

where $\pi_\mu := p_\mu - i\omega_{\mu ab}\lambda^a\bar{\lambda}^b$ is exactly the covariant momentum. Once we have written the covariant derivative, we are able to write the Laplacian operator with the following ordering

$$\hat{\nabla}^2 := \frac{1}{\sqrt{g}}\hat{\nabla}_\mu g^{\mu\nu}\sqrt{g}\hat{\nabla}_\nu, \tag{3.17}$$

that using the covariant momentum is given by

$$\hat{\nabla}^2 = -g^{-1/4}\pi_\mu g^{\mu\nu}g^{1/2}\pi_\nu g^{-1/4}. \tag{3.18}$$

3.3 Worldline action for the ghost

Once we have all the ingredients for the construction of the model, we are able to provide a quantum mechanical worldline representation of the ghost invertible kinetic differential operator. The Laplacian operator (3.18) acting on the wave function gives

$$\hat{\nabla}^2 V(x, \lambda) = (\nabla^2 V_a)\lambda^a. \tag{3.19}$$

Therefore, the full Hamiltonian for the ghost is

$$H = \hat{\mathfrak{F}} := \frac{1}{2}g^{-1/4}\pi_\mu g^{\mu\nu}g^{1/2}\pi_\nu g^{-1/4} - \frac{1}{2}R_{ab}\lambda^a\bar{\lambda}^b. \tag{3.20}$$

The correctness of the latter can be easily verified by applying it to the wavefunction $V(x, \lambda) \sim V_a(x)\lambda^a$, using the identity $\{\bar{\lambda}_a, \lambda^b\} = \delta_a^b$, and checking that the result is equivalent to eq. (3.7).

Prior to writing the classical worldline action able to produce the ghost heat kernel coefficients, we have to discuss a postponed issue. In the last section we stressed that using a graded phase space it is possible to produce an Hilbert space which states can be interpreted as wave functions containing different fields. At this juncture we want to project out all the unwanted fields and maintain only the vector field, that correctly describes the ghost.

The first step to project the Hilbert space on the $n = 1$ sector is to check that λ^a and $\bar{\lambda}_a$ enjoy a $U(1)$ symmetry under the transformation laws

$$\begin{aligned} \lambda^a \to \lambda'^a &= e^{-i\alpha}\lambda^a \\ \bar{\lambda}_a \to \bar{\lambda}'_a &= e^{i\alpha}\bar{\lambda}_a. \end{aligned} \tag{3.21}$$

Therefore the kinetic term of the fermionic variables $\bar{\lambda}_a\dot{\lambda}^a$ is invariant. The corresponding conserved Noether current is of the form $\lambda^a\bar{\lambda}_a$. In order to select the occupation number $n = 1$ one can proceed coupling the variables λ^a and $\bar{\lambda}_a$ to a worldline living gauge field $a(t)$ and introducing an additional Chern-Simons term with a coupling $s = n - \frac{D}{2}$ that is quantized. In this way the free kinetic term of fermionic variables gets modified, and the associated term in the action with euclidean time is

$$S_{\lambda\bar{\lambda}} = \int_0^\beta dt[\bar{\lambda}_a\dot{\lambda}^a - ia(\lambda^a\bar{\lambda}_a - s)]. \tag{3.22}$$

The introduction of an external gauge field can be used as a Lagrange multiplier which equation of motion produces the wanted constraint. Indeed the gauge field equation of motion produces the constraint

$$C = (\lambda^a\bar{\lambda}_a - s) \tag{3.23}$$

that by setting $n = 1$ in s selects the correct physical state if applied to the full wave function as $C\Psi(x, \lambda) = 0$. The latter represents the classical constraint that upon quantization produces some ordering ambiguities. They can be resolved by a graded symmetrization in the following way

$$\hat{C} = \frac{1}{2}\left(\lambda^a\frac{\partial}{\partial\lambda^a} - \frac{\partial}{\partial\lambda^a}\lambda^a\right) - s. \tag{3.24}$$

Indeed using anti-commuting relations we have that

$$\hat{C} = \frac{1}{2}\left(\lambda^a\frac{\partial}{\partial\lambda^a} - \frac{\partial}{\partial\lambda^a}\lambda^a\right) - s = \lambda^a\frac{\partial}{\partial\lambda^a} - \frac{1}{2}\left\{\lambda^a, \frac{\partial}{\partial\lambda^a}\right\} - s = \lambda^a\frac{\partial}{\partial\lambda^a} - \frac{D}{2} - s, \tag{3.25}$$

it is possible to recognize the number operator $\hat{N} = \lambda^a\frac{\partial}{\partial\lambda^a}$ which identifies the occupation number n, and finally selecting the Chern-Simons coupling $s = n - \frac{D}{2}$, we end up with the quantum version of the wanted constraint for $n = 1$:

$$(\hat{N} - 1)\Psi(x, \lambda) = 0. \tag{3.26}$$

If the readers are interested in other applications of that procedure they may consult the papers [18] and [19].

Once the physical state describing the ghost field has been selected, we have all the ingredients to write the classical phase space action in euclidean time, using the quantum mechanical Hamiltonian of eq. (3.20):

$$S[x,p,\lambda,\bar{\lambda};a] = \int_0^\beta dt\left[-ip_\mu\dot{x}^\mu + \bar{\lambda}_a\dot{\lambda}^a + \frac{1}{2}g^{\mu\nu}\pi_\mu\pi_\nu - \frac{1}{2}R_{ab}\lambda^a\bar{\lambda}^b - ia\left(\lambda^a\bar{\lambda}_a - s\right)\right]. \quad (3.27)$$

Finding the equations of motion of the ordinary momentum p_μ

$$\frac{\delta S}{\delta p_\mu} = 0 \to p^\mu = i\dot{x}^\mu + ig^{\mu\nu}\omega_{\nu ab}\lambda^a\bar{\lambda}^b, \quad (3.28)$$

we can integrate out it and write the classical action in configuration space

$$S[x,\lambda,\bar{\lambda};a] = \int_0^\beta dt\left[\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \bar{\lambda}_a(D_t + ia)\lambda^a - \frac{1}{2}R_{ab}\lambda^a\bar{\lambda}^b + ias\right], \quad (3.29)$$

where the covariant derivative contains also the spin connection with fermionic variables

$$D_t\lambda^a = \dot{\lambda}^a + \dot{x}^\mu\omega_\mu{}^a{}_b\lambda^b. \quad (3.30)$$

3.4 Regularization procedures

In our worldline method discussion we are near to the final form of the classical action in configuration space used to recover the Seeley-DeWitt coefficients for the ghost sector. The last effort lies on the regularization scheme that one has to introduce to properly treat the path integral. What we have constructed so far with the action (3.29) represents a so-called non-linear sigma model. In general the various computations for the latter creates many problems if not properly treated. The various issues mentioned above have been widely discussed during the last decades and in this section we will mention some papers and texts for those wishing to deepen the subject. A clarifying overview of the necessity of regularization schemes and associated counterterms is contained in [20].

The presence of $g_{\mu\nu}(x)$ in the action of non linear sigma models in one dimension produces ordering ambiguities when a canonical quantization is performed. One can start with the free classical Hamiltonian of the form

$$H(x,p) = \frac{1}{2}g^{\mu\nu}(x)p_\mu p_\nu. \quad (3.31)$$

It is well known that the canonical quantization of the classical Hamiltonian produces some ordering problems of coordinates x^μ and momenta p_μ, which derive from the fact that the following commutation rule holds

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu. \quad (3.32)$$

The simplest explanation is that different quantum theories correspond just to a single classical theory described by the Hamiltonian (3.31). A general method to fix this type of ambiguities is to rely on symmetries. At this purpose one can impose general covariance (or covariance under diffeomorphisms) at the quantum level with an operatorial momentum as in eq. (3.9) and a quantum Hamiltonian operator of the form

$$\hat{H}(\hat{x}, \hat{p}) = \frac{1}{2} g^{-1/4} \hat{p}_\mu g^{1/2} g^{\mu\nu} \hat{p_\nu} g^{-1/4}, \tag{3.33}$$

that in coordinates representation reduces to

$$\hat{H} = -\frac{1}{2} \hat{\nabla}^2, \tag{3.34}$$

which Laplacian has been used with this ordering in (3.18). Although this resolves some ordering problems, it reduces the quantum Hamiltonian to a class of operators which differ from a term proportional to the scalar curvature R, the only covariant scalar object that can be constructed with two derivatives on the metric:

$$\hat{H} = -\frac{1}{2} \hat{\nabla}^2 + \frac{\xi}{2} R. \tag{3.35}$$

A common renormalization condition, that is imposed in the path integral description, requires the Hamiltonian to be covariant under general change of coordinates and the coefficient $\xi = 0$. Different values for ξ can be introduced later with an extra term in the potential $V(x)$. So far we have not mentioned the problem for requiring a renormalization scheme yet. The explanation comes directly from the form of the non linear sigma model. One can clearly see in (3.29) that the model contains double derivative interactions that would give rise to linear divergences (as seen by a power counting procedure [20],[21]). These are ultraviolet divergences, while infrared ones are not present because we are studying actions on a compact time-interval. One could implement a renomalization to remove ultraviolet linear divergences, but this is not strictly necessary. As showed in [22] one should consider the covariant measure of the path integral

$$\mathcal{D}x \sim \prod \sqrt{g} d^D x \tag{3.36}$$

where the metric dependence $\prod \sqrt{g}$ can be exponentiated introducing commuting a^μ and anticommuting b^μ, c^ν ghosts, which action is

$$S_{ghosts}[x, a, b, c] = \int_0^\beta dt \left[\frac{1}{2} g_{\mu\nu}(x)(a^\mu a^\nu + b^\mu c^\nu) \right]. \tag{3.37}$$

One can check ([20],[21]) that the above mentioned linear divergences cancel with additional linear divergences coming from local interactions of the ghosts, in such a way that the final result in the sum of diagrams is finite. This cancellation is possible only after a proper regularization of individual divergent Feynman graphs. Different regularization schemes differ by finite local counterterms, such that the value of individually regularized divergent diagrams depend on the scheme, while the final result is regularization independent. By power counting one can also check that one dimensional non linear sigma models are super-renormalizable, therefore only finite counterterms up to two-loops will appear in the regularization scheme.

There are three widely used regularization schemes that can be implemented for this purpose: *time slicing*, *mode regularization* and *dimensional regularization*.

Time slicing (TS) regularization scheme is constructed from the operatorial expression of the transition amplitude in quantum mechanics. The time-interval is discretized with N equally spaced points t_i, and the action is described only by N variables $q(t_1)\dots q(t_N)$ using a "mid-point prescription" that is connected to a Weyl ordering choice of the Hamiltonian operator $\hat{H}$. In this way one gets a discretized path integral in momentum space where integrating out the momenta and taking the continuous limit $N \to \infty$ Feynman rules are derived. We mentioned that counterterms are used to satisfy renormalization conditions, that include the symmetry of general covariance. Since the TS regularization scheme corresponds to a Weyl ordering of the Hamiltonian operator, one can check that the Weyl ordered Hamiltonian is not covariant and differ from the covariant form of eq. (3.33). Therefore this regularization scheme breaks general covariance. The only way to achieve a covariant final result is to use the counterterm

$$V_{\rm TS} = -\frac{1}{8}R + \frac{1}{8}g^{\mu\nu}\Gamma^{\beta}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\nu\beta}, \tag{3.38}$$

where the term $\Gamma\Gamma$ is non-covariant as well.

Mode regularization (MR) is derived expanding quantum fluctuations $q(t)$ around a background solution $x(t)$ in Fourier sine series. A cut-off at mode M is introduced. The problematic distributions are now under control and one can perform all the computations for Feynman graphs evaluation. Only at the end it is possible to recover the continuous limit $M \to \infty$. Also in this case the regularization scheme breaks the symmetry of general covariance, therefore, in order to guarantee the renormalization conditions, a non-covariant counterterm is necessary to restore covariance of the final result:

$$V_{\rm MR} = -\frac{1}{8}R - \frac{1}{24}g^{\mu\nu}g^{\rho\lambda}g_{\gamma\delta}\Gamma^{\gamma}{}_{\mu\rho}\Gamma^{\delta}{}_{\nu\lambda}. \tag{3.39}$$

Dimensional regularization (DR) is applied after the usual perturbative expansion of the path integral, and represents the simplest regularization in that case. The advantage of DR scheme is that the required counterterm $V_{\rm DR}$ does not need a non-covariant $\Gamma\Gamma$ term, because this approach does not break general covariance. In order to remove ambiguities of path integral distributions and their products one introduces D extra dimensions $dt \to d^{d+1}t$ and evaluates the Feynman graphs. At the end of the computation the limit $d \to 0$ can be restored. This is actually not the most practical way to use DR. Indeed one could manipulate problematic integrals using the extended space in $d+1$ dimensions, by means of useful identities which involve Green equations and integration by parts, to cast the integral that turns out to be computable at $d \to 0$. A counterterm is required also in this case, but it is the simplest one:

$$V_{\mathrm{DR}} = -\frac{1}{8}R. \tag{3.40}$$

In this thesis we will widely use dimensional regularization and corresponding manipulations of the integrals in order to compute Seeley-DeWitt coefficients. Therefore a better and detailed description will be given in the next sections, including all the computations performed. For the reader that is interested in the subject, [21] contains a complete description of the abovementioned three regularization schemes.

The counterterm $V_{\mathrm{DR}} = -\frac{1}{8}R$ for DR scheme is not the only one that our path integral constructed with (3.20) requires. Indeed that counterterm is sufficient for a bosonic path integral constructed with the Hamiltonian operator in eq. (3.34). In our model we have to deal also with fermionic variables λ^a and $\bar{\lambda}_a$. Therefore also their ordering issues have to be considered. The path integral constructed with the classical action (3.29) produces a graded-symmetric Weyl ordering for fermionic terms. If there is no meaning in the ordering of the term $\omega_{\mu ab}\lambda^a\bar{\lambda}^b$ because of antisymmetry of $\omega_{\mu ab}$, a counterterm is required for the term $-\frac{1}{2}R_{ab}\lambda^a\bar{\lambda}^b$ and is given by $-\frac{1}{4}R$. Finally the full DR counterterm to be added to the action is

$$V_{\mathrm{DR}} = -\frac{1}{8}R - \frac{1}{4}R = -\frac{3}{8}R. \tag{3.41}$$

The final action that we have to study for the ghost worldline model, including the correct counterterms, is given by:

$$S[x,\lambda,\bar{\lambda};a] = \int_0^\beta dt\left[\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \bar{\lambda}_a(D_t + ia)\lambda^a - \frac{1}{2}R_{ab}\lambda^a\bar{\lambda}^b - \frac{3}{8}R + ias\right]. \tag{3.42}$$

3.5 One-loop effective action of the ghost

As we mentioned in the previous sections the total effective action for quantum gravity comes from the individual contributions of the graviton and of the ghost:

$$\Gamma[g] \propto \Gamma_{T2} - 2\Gamma_V. \tag{3.43}$$

The vector model in worldline formalism for the ghost has been already constructed with all the ingredients. The Seeley-DeWitt coefficients $a_n(x)$ can be written by expanding the integrand in eq. (3.5) in powers of β in the following way

$$\Gamma[g] \propto \int_0^\infty \frac{d\beta}{\beta}\int \frac{d^Dx\sqrt{g(x)}}{(2\pi\beta)^{D/2}}\sum_{n=0}^\infty \beta^n a_n(x) \tag{3.44}$$

where $\beta^{-D/2}$ is the leading contribution of the free field. In this section we will compute the ghost contribution quantizing the worldline action of the vector model on the circle. The idea is to perform a standard perturbative expansion of the path integral in powers of β up to β^2 and recover the Seeley-DeWitt coefficients that constitue the diverging part of the effective action.

The path integral quantization on a circle T^1 provides the one-loop effective action

$$\Gamma_V \propto \int_0^\beta \frac{d\beta}{\beta} \int_{T^1} \frac{\mathcal{D}X}{\mathrm{Vol(Gauge)}} e^{-S_V[x,\lambda,\bar{\lambda};a]} \tag{3.45}$$

where $X = (a, x^\mu, \lambda^a, \bar{\lambda}^a)$ contains the dynamical fields that must be integrated over. The volume of the gauge group in the denominator is there to remember that one has to fix the gauge symmetry.

In order to control the perturbative expansion it is convenient to introduce an organizing parameter. It is done by rescaling the time $t = \beta\tau$. The action (3.29) can be written also rescaling the fermions as $\lambda \to \frac{1}{\sqrt{\beta}}\lambda$, $\bar{\lambda} \to \frac{1}{\sqrt{\beta}}\bar{\lambda}$ and the gauge field as $a \to \frac{1}{\beta}a$:

$$S_V[x,\lambda,\bar{\lambda};a] = \frac{1}{\beta}\int_0^1 d\tau \left[\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \bar{\lambda}_a(D_\tau + ia)\lambda^a - \frac{\beta}{2}R_{ab}\lambda^a\bar{\lambda}^b - \frac{3}{8}\beta^2 R\right] + is\int_0^1 d\tau a. \tag{3.46}$$

Proceeding like in [23] let us extract from the latter the free action for fermionic variables

$$S[x,\lambda,\bar{\lambda};a] \supset S[\lambda,\bar{\lambda};a] = \frac{1}{\beta}\int_0^1 d\tau \bar{\lambda}_a(\dot{\lambda}^a + ia\lambda^a). \tag{3.47}$$

Consider the finite gauge transformations

$$\begin{aligned} \lambda^a(\tau) \to \lambda'^a(\tau) =& e^{-i\alpha(\tau)}\lambda^a(\tau) \\ \bar{\lambda}_a(\tau) \to \bar{\lambda}'_a(\tau) =& e^{i\alpha(\tau)}\bar{\lambda}_a(\tau) \\ a(\tau) \to a'(\tau) =& a(\tau) + \dot{\alpha}(\tau) \end{aligned} \tag{3.48}$$

where $e^{-i\alpha(\tau)}$ are periodic functions on $[0,1]$. One can check that the only gauge invariant quantity that can be constructed using the gauge field $a(\tau)$ is the so called Wilson loop:

$$\omega = e^{i\int_0^1 d\tau a(\tau)}. \tag{3.49}$$

At this stage the gauge field $a(\tau)$ can be set to a constant θ by using "small" or continuously connected to the identity gauge transformation:

$$\theta = \int_0^1 d\tau a(\tau). \tag{3.50}$$

"Large" gauge transformations with $a(\tau) = 2\pi n \tau$ and n an integer allow to write the identity

$$\theta \sim \theta + 2\pi n. \tag{3.51}$$

Therefore θ represents a modular parameter ranging from 0 to 2π. After gauge fixing the action, one is left with an integral over θ which corresponds to the Wilson loop.

Since the $U(1)$ gauge group is abelian, the Faddeev-Popov determinant, being a constant, can be factorized out and absorbed in the normalization. Therefore, after these steps, the worldline path integral representation for the vector can be written as follows

$$\Gamma_V = \int_0^\beta \frac{d\beta}{\beta} \int_0^{2\pi} \frac{d\theta}{2\pi} \int_P \mathcal{D}x \int_A D\bar{\lambda} D\lambda e^{-S_V[x,\bar{\lambda},\lambda;\theta]} \tag{3.52}$$

where $S_V[x, \bar{\lambda}, \lambda; \theta]$ is the action of eq. (3.46) where the gauge field has been set to the constant value $a(\tau) = \theta$. The subscripts P (periodic) and A (antiperiodic) stand for boundary conditions prescription respectively for bosonic x^μ and fermionic (λ^a,$\bar{\lambda}^a$) variables, i.e.

$$x^\mu(0) = x^\mu(1), \qquad \lambda^a(0) = -\lambda^a(1). \tag{3.53}$$

The generally covariant measure for the bosonic path integral is a shorthand for the following expression

$$\mathcal{D}x = \prod_{0<\tau<1} \sqrt{g(x(\tau))} d^D x(\tau). \tag{3.54}$$

that is metric dependent. The measure for the fermionic path integral is flat since our fermions are vectors with flat indices.

3.6 Vector path integral and dimensional regularization

Let us proceed with the perturbative expansion of the path integral. The trajectory $x^\mu(\tau)$ of the periodic path integral can be split into a background fixed point $x_{\rm bg}(\tau)$ and fluctuations $q^\mu(\tau)$ vanishing at the boundary $q^\mu(0) = q^\mu(1) = 0$, i.e.

$$x^\mu(\tau) = x_{\rm bg}(\tau) + q^\mu(\tau). \tag{3.55}$$

The bosonic measure therefore splits as follows

$$\int_P \mathcal{D}x = \int d^D x \sqrt{g(x)} \int_D \mathcal{D}q \tag{3.56}$$

where the subscript D stands for Dirichlet boundary conditions and $\mathcal{D}q$ is given by

$$\mathcal{D}q = \prod_{0<\tau<1} \sqrt{g(q(\tau))} Dq \tag{3.57}$$

with $Dq = d^D q$. The bosonic measure, being generally covariant, is a scalar under general changes of coordinates. This introduce a field dependence not practical for the perturbative expansion with splitting (3.55). As we have anticipated, following the procedure of [24], the measure field dependence can be exponentiated introducing some commuting a^μ and anticommuting b^μ, c^μ ghosts, i.e.

$$\prod_{0<\tau<1} \sqrt{g(q(\tau))} = \int DaDbDce^{-S_{\rm gh}[x,a,b,c]} \tag{3.58}$$

where

$$S_{\rm gh}[x,a,b,c] = \frac{1}{\beta}\int_0^1 \frac{1}{2} g_{\mu\nu}(x(\tau))(a^\mu a^\nu + b^\mu c^\nu). \tag{3.59}$$

The ghost measures are translational invariant and given by

$$Da = \prod_{0<\tau<1} d^D a(\tau), \qquad Db = \prod_{0<\tau<1} d^D b(\tau), \qquad Dc = \prod_{0<\tau<1} d^D c(\tau). \tag{3.60}$$

Being auxiliary fields, the ghosts do not need boundary conditions. Since the fluctuations vanish at time boundaries we can expand them using Fourier sine series, i.e.

$$\begin{aligned} q^\mu(\tau) &= \sum_{m=1}^{\infty} q^\mu_m sin(\pi m \tau), \qquad & a^\mu(\tau) &= \sum_{m=1}^{\infty} a^\mu_m sin(\pi m \tau), \\ b^\mu(\tau) &= \sum_{m=1}^{\infty} b^\mu_m sin(\pi m \tau), \qquad & c^\mu(\tau) &= \sum_{m=1}^{\infty} c^\mu_m sin(\pi m \tau). \end{aligned} \tag{3.61}$$

where q^μ_m, a^μ_m, b^μ_m and c^μ_m are the Fourier coefficients. Therefore we can write the bosonic path integral integrating over these Fourier coeffcients, namely the measure is given by

$$DqDaDbDc \propto \prod_{m=1}^{\infty}\prod_{i=1}^{D} m\, dq^i_m da^i_m db^i_m dc^i_m. \tag{3.62}$$

The anti-periodic fermionic variables on the worldline can be expanded in half-integer modes as follows

$$\lambda^a(\tau) = \sum_{r\in\mathbb{Z}+1/2} \lambda^a_r\, e^{2\pi i r\tau}, \qquad \bar\lambda^a(\tau) = \sum_{r\in\mathbb{Z}+1/2} \bar\lambda^a_r e^{-2\pi i r\tau}. \tag{3.63}$$

At this juncture it is possible to proceed with the perturbative expansion. We divide the action that contains also the auxiliary ghosts, i.e.

$$\begin{aligned} S_V[x,\lambda,\bar\lambda;\theta] =& \frac{1}{\beta}\int_0^1 d\tau \left[\frac{1}{2} g_{\mu\nu}(x)(\dot x^\mu \dot x^\nu + a^\mu a^\nu + b^\mu c^\nu) + \bar\lambda_a(D_\tau + i\theta)\lambda^a - \frac{\beta}{2} R_{ab}\lambda^a\bar\lambda^b - \frac{3}{8}\beta^2 R\right] \\ &+ is\int_0^1 d\tau\theta \end{aligned} \tag{3.64}$$

into a free part

$$S_2 = \frac{1}{2\beta} g_{\mu\nu} \int_0^1 d\tau \bigg(\dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu \bigg) + \frac{1}{\beta} \int_0^1 d\tau \bar{\lambda}_a (\partial_\tau + i\theta) \lambda^a \tag{3.65}$$

with $g_{\mu\nu} = g_{\mu\nu}(x_{\rm bg})$ the metric in the background, and an interaction part (containing the vertices) that can be expanded in powers of β

$$\begin{aligned} S_{int} =& \frac{1}{\beta} \int_0^1 d\tau \bigg\{ \frac{1}{2} [g_{\mu\nu}(x_{\rm bg} + q) - g_{\mu\nu}] \bigg(\dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu \bigg) + \omega_{\mu ab}(x_{\rm bg} + q)\, \dot{q}^\mu \lambda^a \bar{\lambda}^b \\ & - \frac{\beta}{2} R_{ab}(x_{\rm bg} + q)\, \lambda^a \bar{\lambda}^b - \beta^2 \frac{3}{8} R(x_{\rm bg} + q) \bigg\}, \end{aligned} \tag{3.66}$$

where the Chern-Simons part has been factorized out the path integral. From the free action S_2 one can extract the propagators. Let us first plug the mode expansion (3.61) in the free action part for the bosonic fluctuations and the auxiliary ghosts, to obtain

$$\begin{aligned} S_2[q,a,b,c] =& \frac{1}{2\beta} g_{\mu\nu} \int_0^1 d\tau \bigg(\dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu \bigg) \\ =& \frac{1}{4\beta} g_{\mu\nu} \sum_{m=1}^{\infty} [(\pi m)^2 q_m^\mu q_m^\nu + a_m^\mu a_m^\nu + b_m^\mu c_m^\nu] \end{aligned} \tag{3.67}$$

since $\int_0^1 d\tau \cos^2(\pi m \tau) = \int_0^1 d\tau \sin^2(\pi m \tau) = 1/2$, being m a positive integer. The bosonic propagator for fluctuations is then given by

$$\langle q^\mu(\tau) q^\nu(\sigma) \rangle = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \langle q_m^\mu q_n^\nu \rangle \sin(\pi m \tau) \sin(\pi n \tau). \tag{3.68}$$

Introducing the sources for q_m^μ modes, performing square completion and shifting integration variable we get

$$\langle q_m^\mu q_n^\nu \rangle = \frac{\int DqDaDbDc\, q_m^\mu q_n^\nu\, e^{-S_2[q,a,b,c]}}{\int DqDaDbDc\, e^{-S_2[q,a,b,c]}} = \beta g^{\mu\nu} \delta_{mn} \frac{2}{\pi^2 m^2}. \tag{3.69}$$

A simpler method is to recognize the two-point correlation function (3.69) to be the inverse of the kinetic operator in $\exp\left[-\frac{1}{2} q_m^\mu \left(\frac{1}{2\beta} g_{\mu\nu} \pi^2 m^2 \delta_{mn} \right) q_n^\nu \right]$. The full propagator is thus given by

$$\langle q^\mu(\tau) q^\nu(\sigma) \rangle = \beta g^{\mu\nu} \sum_{m=1}^{\infty} \frac{2}{\pi^2 m^2} \sin(\pi m \tau) \sin(\pi n \sigma). \tag{3.70}$$

For the auxiliary ghosts a^μ, b^μ and c^ν propagators we can proceed in a similar way. Before writing all the propagators let us evaluate the fermionic one. First we plug the mode expansion for fermions into the fermionic part of the free action, i.e.

$$S_2[\lambda^a, \bar{\lambda}^a] = \frac{1}{\beta}\int_0^1 d\tau \bar{\lambda}_a(\partial_\tau + i\theta)\lambda^a = \frac{i}{\beta}\sum_{r\in\mathbb{Z}+1/2}(2\pi r+\theta)\bar{\lambda}_{ra}\lambda^a_r. \tag{3.71}$$

The propagator can be computed in the usual manner

$$\begin{aligned}\langle\lambda^a(\tau)\bar{\lambda}_b(\sigma)\rangle &= \sum_{r\in\mathbb{Z}+1/2}\sum_{s\in\mathbb{Z}+1/2}\langle\lambda^a_r\bar{\lambda}_{sb}\rangle\, e^{2\pi i r\tau}e^{-2\pi i s\sigma} \\ &=\beta\delta^a_b\sum_{r\in\mathbb{Z}+1/2}\frac{-i}{2\pi r+\theta}\, e^{2\pi i r(\tau-\sigma)},\end{aligned} \tag{3.72}$$

using the two-point correlation function

$$\langle\lambda^a_r\bar{\lambda}_{sb}\rangle = -i\frac{\beta}{2\pi r+\theta}\delta^b_a. \tag{3.73}$$

Let us collect all the above computed propagators providing also the continuum limit:

$$\begin{aligned}\langle q^\mu(\tau)q^\nu(\sigma)\rangle &= -\beta g^{\mu\nu}\Delta(\tau,\sigma)\,,\\ \langle a^\mu(\tau)a^\nu(\sigma) &=\beta g^{\mu\nu}\Delta_{gh}(\tau,\sigma)\,, \quad \langle b^\mu(\tau)c^\nu(\sigma) = -2\beta g^{\mu\nu}\Delta_{gh}(\tau,\sigma)\,,\\ \langle\lambda^a(\tau)\bar{\lambda^b(\sigma)}\rangle &=\beta\delta^{ab}\Delta_F(\tau-\sigma,\theta),\end{aligned} \tag{3.74}$$

where

$$\begin{aligned}\Delta(\tau,\sigma) &= \sum_{m=1}^{\infty}\left[-\frac{2}{\pi^2m^2}\sin(\pi m\tau)\sin(\pi m\sigma)\right] = (\tau-1)\sigma\theta(\tau-\sigma)+(\sigma-1)\tau\theta(\sigma-\tau)\,,\\ \Delta_{gh}(\tau,\sigma) &= \sum_{m=1}^{\infty}2\sin(\pi m\tau)\sin(\pi m\sigma) = \partial_\tau^2\Delta(\tau,\sigma) = \delta(\tau,\sigma)\,,\\ \Delta_F(\tau-\sigma,\theta) &= \sum_{r\in\mathbb{Z}+1/2}\frac{-i}{2\pi r+\theta}e^{2\pi i r(\tau-\sigma)} = \frac{e^{-i\theta(\tau-\sigma)}}{2\cos\frac{\theta}{2}}\left[e^{i\frac{\theta}{2}}\theta(\tau-\sigma)-e^{-i\frac{\theta}{2}}\theta(\sigma-\tau)\right],\end{aligned} \tag{3.75}$$

where we have computed the continuum limit and $\theta(\tau-\sigma)$ is the usual Heaviside step function. The above propagators are expressed in terms of distributions, that are defined acting on functions on the time segment $I=[0,1]$ with the above mentioned boundary conditions for bosons, fermions and auxiliary ghosts. One can also check the following identities for derivatives and equal time expression:

$$\begin{aligned}
&{}^\bullet\!\Delta(\tau,\sigma) = \sigma - \theta(\sigma-\theta)\,, \quad \Delta^\bullet(\tau,\sigma) = \tau - \theta(\tau-\sigma)\,,\\
&{}^\bullet\!\Delta^\bullet(\tau,\sigma) = 1 - \delta(\tau,\sigma)\,, \quad \Delta_{gh}(\tau,\sigma) = {}^{\bullet\bullet}\!\Delta(\tau,\sigma) = \delta(\tau,\sigma)\\
&\Delta(\tau,\tau) = \tau(\tau-1)\,, \quad {}^\bullet\!\Delta(\tau,\sigma)|_{\tau=\sigma} = \tau - \frac{1}{2}\,,
\end{aligned} \tag{3.76}$$

where the dots on the left-right hand side indicate derivatives with respect to the left-right variable. From the symmetry of the sum in the fermionic propagator we have also the following identities

$$\begin{aligned}
\Delta_F(0,\theta) =& \frac{i}{2}\tan\frac{\theta}{2}\,,\\
\Delta_F(\tau-\sigma,\theta)\Delta_F(\sigma-\tau,\theta) =& -\frac{1}{2}\cos^{-2}\frac{\theta}{2}\,, \qquad \tau\neq\sigma.
\end{aligned} \tag{3.77}$$

As anticipated, path integral computations involve products and derivatives of these distributions that are ill-defined. For this purpose it is necessary to introduce a regularization scheme. The simplest choice in this case is *dimensional regularization* (DR). In the following we present the complete procedure for its construction.

The idea of *dimensional regularization* is to extend the compact time interval $I = [0,1]$ adding d extra infinite dimensions, i.e. $I \to I \times R^d = \Omega$. We introduce $t^i \equiv (\tau, \mathbf{t})$ with $i = 0,1,...,d$ and the measure becomes $d^{d+1}t = d\tau d^d\mathbf{t}$. The action in d+1 dimensions is

$$\begin{aligned}
S_V[x,\lambda,\bar\lambda;\theta] = \frac{1}{\beta}\int_\Omega d^{d+1}t\Big[&\frac{1}{2}g_{\mu\nu}(x)(\partial_\alpha x^\mu\partial_\alpha x^\nu + a^\mu a^\nu + b^\mu c^\nu)\\
&+ \bar\lambda_a(\gamma^\alpha\partial_\alpha\lambda^a + \gamma^\alpha\partial_\alpha\dot x^\mu\omega_\mu{}^a{}_b\lambda^b + i\theta\lambda^a) - \frac{\beta}{2}R_{ab}\lambda^a\bar\lambda^b - \frac{3}{8}\beta^2 R\Big],
\end{aligned} \tag{3.78}$$

where $\partial_\alpha = \frac{\partial}{\partial t^\alpha}$ and γ^α are the gamma matrices in $d+1$ dimensions. Therefore the free action becomes

$$S_2 = \frac{1}{\beta}\int_\Omega d^{d+1}t\Big[\frac{1}{2}g_{\mu\nu}(\partial_\alpha q^\mu\partial_\alpha q^\nu + a^\mu a^\nu + b^\mu c^\nu) + \bar\lambda_a(\gamma^\alpha\partial_\alpha + i\theta)\lambda^a\Big] \tag{3.79}$$

from which one derives the following propagators in extended space Ω

$$\begin{aligned}
\Delta(t,s) =& \int\frac{d^dk}{(2\pi)^d}\sum_{m=1}^\infty\frac{-2}{(\pi m)^2+\mathbf{k}^2}\sin(\pi m\tau)\sin(\pi m\sigma)e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})}\,,\\
\Delta_{gh}(t,s) =& \int\frac{d^dk}{(2\pi)^d}\sum_{m=1}^\infty 2\sin(\pi m\tau)\sin(\pi m\sigma)e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})} = \delta(\tau,\sigma)\delta^d(\mathbf{t}-\mathbf{s})\,,\\
\Delta_F(t-s,\theta) =& -i\int\frac{d^dk}{(2\pi)^d}\sum_{r\in\mathbb{Z}+1/2}\frac{2\pi r\gamma^0 + \mathbf{k}\cdot\gamma - \theta}{(2\pi r)^2+\mathbf{k}^2-\theta^2}e^{2\pi i r(\tau-\sigma)}e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})}.
\end{aligned} \tag{3.80}$$

The above propagators, that in the $d \to 0$ limit reduce to the previous ones, obey the following relations (Green's equations)

$$\begin{aligned}\partial^\alpha \partial_\alpha \Delta(t,s) &= \Delta_{gh}(t,s) = \delta(\tau,\sigma)\delta^d(\mathbf{t}-\mathbf{s})\,,\\ \left(\gamma^\alpha \frac{\partial}{\partial t^\alpha} + i\theta\right)\Delta_F(t-s,\theta) &= \delta_A(\tau-\sigma)\delta^d(\mathbf{t}-\mathbf{s})\,,\end{aligned} \tag{3.81}$$

with δ_A which indicates the delta distribution on functions on antiperiodic boundary conditions. In the Feynman graphs computation also the following identity will be widely used

$$\left[\left(\frac{\partial^2}{\partial t^\alpha \partial s_\alpha} + \frac{\partial^2}{\partial t^\alpha \partial t_\alpha}\right)\Delta(t,s)\right]\Bigg|_{t=s} = \frac{\partial}{\partial \tau}\left[\left(\frac{\partial}{\partial \tau}\Delta(t,s)\right)\Bigg|_{t=s}\right]. \tag{3.82}$$

Propagators in (3.80) are much more complicated with respect to their $d \to 0$ limit. Indeed it is quite difficult to perform diagrams computations in the extended space Ω. However, this is not necessary. There is a better strategy which has the following steps. The idea is to start from worldline diagrams that cannot be computed safely because of ambiguous products of distributions or diverging terms, and extend them to the non-compact space Ω in DR. Problematic integrals of such diagrams then can be manipulated (using integration by parts and integrating against the delta distributions, corresponding to idetities in the extended $(d+1)$-dimensional space) to get a form that can be unambiguously computed in the $d \to 0$ limit. These expressions must not involve any products of distributions or diverging quantities. For this purpose identities such (3.81) and (3.82) are widely used. Other DM diagrams computations are present in [21],[25], [23] and [26].

Let us study the normalization of the fermionic path integral that must be replaced in the effective action (3.52). For this purpose we have to evaluate the fermionic determinant given by the free path integral for fermions with antiperiodic boundary conditions:

$$\int_A D\bar{\lambda} D\lambda e^{-S_2[\lambda^a,\bar{\lambda}^a]} = \det{}^D(\partial_\tau + i\theta). \tag{3.83}$$

Let us follow the procedures presented in [23]. First we write the Hamiltonian operator $\hat{H}_\theta$ of the D dimensional fermionic oscillator system, i.e.

$$\hat{H}_\theta = i\theta \frac{1}{2}(\hat{\lambda}^\dagger_a \hat{\lambda}^a - \hat{\lambda}^a \hat{\lambda}^\dagger_a) = i\theta\left(\hat{N} - \frac{D}{2}\right), \tag{3.84}$$

where we used the fermionic number operator $\hat{N} = \hat{\lambda}^\dagger_a \hat{\lambda}^a$. In the worldline the number operator has the only two eigenvalues 0 and 1, thus we get

$$\begin{aligned}\det{}^D(\partial_\tau + i\theta) &= \mathrm{Tr} e^{-i\theta(\hat{N}) - \frac{D}{2}}\\ &= e^{i\theta\frac{D}{2}}(1+e^{-i\theta})^D = \left(2\cos\frac{\theta}{2}\right)^D.\end{aligned} \tag{3.85}$$

This allows us to write the vector effective action as follows

$$\Gamma_V = \int_0^\beta \frac{d\beta}{\beta} \int_0^{2\pi} \frac{d\theta}{2\pi} \left(2\cos\frac{\theta}{2}\right)^D e^{-is\theta} \int \frac{d^Dx\sqrt{g(x)}}{(2\pi\beta)^{D/2}} \langle e^{-S_{int}} \rangle, \tag{3.86}$$

where $s = 1 - \frac{D}{2}$ and S_{int} refers to (3.66).

Let us now proceed with the perturbative expansion of $\langle e^{-S_{int}} \rangle$. We know that perturbative calculations can be performed in any coordinates system. Our choice are the Riemann normal coordinates togheter with the Fock-Schwinger gauge for the spin connection, centered at the background $x_{\rm bg}$. To deepen Riemann normal coordinates see [27]. In the following we have the metric, the spin connection and some curvatures expanded in these coordinates:

$$\begin{aligned}
g_{\mu\nu}(x_{\rm bg}+q) =& g_{\mu\nu} + \frac{1}{3} q^\lambda q^\sigma R_{\lambda\mu\nu\sigma} + \mathcal{O}(q^3) + q^\lambda q^\sigma q^\alpha q^\beta \left[\frac{1}{20}\nabla_\lambda\nabla_\sigma R_{\alpha\mu\nu\beta} + \frac{2}{45} R_{\tau\lambda\sigma\mu} R^\tau{}_{\alpha\beta\nu}\right], \\
\omega_{\mu ab}(x_{\rm bg}+q) =& \frac{1}{2} q^\nu R_{\nu\mu ab} + \mathcal{O}(q^2) + q^\nu q^\lambda q^\sigma \left[\frac{1}{8}\nabla_\lambda\nabla_\sigma R_{\nu\mu ab} + \frac{1}{24} R^\tau{}_{\nu\lambda\mu} R_{\sigma\tau ab}\right], \\
R_{abcd}(x_{\rm bg}+q) =& R_{abcd} + \mathcal{O}(q) + \frac{1}{2} q^\mu q^\nu \nabla_\mu\nabla_\nu R_{abcd}\,, \\
R_{ab}(x_{\rm bg}+q) =& R_{ab} + \mathcal{O}(q) + \frac{1}{2} q^\mu q^\nu \nabla_\mu\nabla_\nu R_{ab}\,, \\
R(x_{\rm bg}+q) =& R + \mathcal{O}(q) + \frac{1}{2} q^\mu q^\nu \nabla_\mu\nabla_\nu R\,,
\end{aligned} \tag{3.87}$$

where the tensors on the right hand side of the equations are computed at the fixed background $x_{\rm bg}$. For sake of simplicity we have omitted all those terms that in the path integral would give a trivial null contribution due to the odd number of fields in the correlation function. Let us replace these expansions in Riemann normal coordinates in the interacting action (3.66). The average of the euclidean exponential of the interacting action can be expanded as follows

$$\langle e^{-S_{int}} \rangle = 1 - \langle S_4 \rangle - \langle S_6 \rangle + \frac{1}{2}\langle S_4^2 \rangle + \mathcal{O}(\beta^3)\,, \tag{3.88}$$

where we used the notation where S_n indicates the term of the expansion with n quantum fields and is of order $\mathcal{O}(\beta^{n/2-1})$. After simple algebra one gets the following expressions

$$
\begin{aligned}
S_4 =& \frac{1}{6\beta} R_{\lambda\mu\nu\sigma} \int_0^1 d\tau q^\lambda q^\sigma \left(\dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu \right) + \frac{1}{2\beta} R_{\mu\nu ab} \int_0^1 d\tau q^\mu \dot{q}^\nu \lambda^a \bar{\lambda}^b \\
& - \frac{1}{2} R_{ab} \int_0^1 d\tau \lambda^a \bar{\lambda}^b - \frac{3}{8} \beta R \,, \\
S_6 =& \frac{1}{\beta} \left[\frac{1}{40} \nabla_\lambda \nabla_\sigma R_{\alpha\mu\nu\beta} + \frac{1}{45} R_{\tau\lambda\sigma\mu} R^\tau{}_{\alpha\beta\nu} \right] \int_0^1 d\tau q^\lambda q^\sigma q^\alpha q^\beta \left(\dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu \right) \\
& + \frac{1}{\beta} \left[\frac{1}{8} \nabla_\lambda \nabla_\sigma R_{\mu\nu ab} + \frac{1}{24} R^\tau{}_{\mu\lambda\nu} R_{\sigma\tau ab} \right] \int_0^1 d\tau q^\lambda q^\sigma q^\mu \dot{q}^\nu \lambda^a \bar{\lambda}^b \\
& - \frac{1}{4} \nabla_\mu \nabla_\nu R_{ab} \int_0^1 d\tau q^\mu q^\nu \lambda^a \bar{\lambda}^b - \frac{3}{16} \beta \nabla_\mu \nabla_\nu R \int_0^1 d\tau q^\mu q^\nu .
\end{aligned}
\tag{3.89}
$$

Using these expressions in (3.88) and performing all the necessary non trivial Wick contractions one gets

$$
\begin{aligned}
\langle e^{-S_{int}} \rangle =& 1 + \beta R \left(\frac{1}{3} + \frac{i}{4} \tan\frac{\theta}{2} \right) \\
& + \beta^2 \Bigg\{ \left(\frac{1}{720} - \frac{1}{192} \cos^{-2}\frac{\theta}{2} \right) R^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma} + \left(-\frac{1}{720} + \frac{1}{32} \cos^{-2}\frac{\theta}{2} \right) R^{\mu\nu} R_{\mu\nu} \\
& + \left(\frac{25}{288} - \frac{1}{32} \cos^{-2}\frac{\theta}{2} + \frac{i}{12} \tan\frac{\theta}{2} \right) R^2 + \left(\frac{7}{240} + \frac{i}{48} \tan\frac{\theta}{2} \right) \nabla^2 R \Bigg\} + \mathcal{O}(\beta^3).
\end{aligned}
\tag{3.90}
$$

All the steps and computations such as ambiguous integrals evaluation with *dimensional regularization* are contained in B.1.

The next step is to evaluate the modular integrals at arbitrary dimension D and with $s = 1 - \frac{D}{2}$, i.e.

$$
\begin{aligned}
I_1 &= \int_0^{2\pi} \frac{d\theta}{2\pi} \left(2\cos\frac{\theta}{2} \right)^2 e^{is\theta} = D \,, \\
I_2 &= \int_0^{2\pi} \frac{d\theta}{2\pi} \left(2\cos\frac{\theta}{2} \right)^2 e^{is\theta} \tan\frac{\theta}{2} = i(D-2) \,, \\
I_3 &= \int_0^{2\pi} \frac{d\theta}{2\pi} \left(2\cos\frac{\theta}{2} \right)^2 e^{is\theta} \cos^{-2}\frac{\theta}{2} = 4.
\end{aligned}
\tag{3.91}
$$

Finally replacing the results of the modular integrals in (3.91) in the expansion (3.90) one gets the ghost effective action:

$$\Gamma_V = \int_0^\infty \frac{d\beta}{\beta} \int \frac{d^D x \sqrt{g(x)}}{(2\pi\beta)^{(D/2)}} \Big\{ D + \beta R \Big(\frac{D}{12} + \frac{1}{2} \Big) + \beta^2 \Big(\frac{D+5}{120} \nabla^2 R + \frac{D+12}{288} R^2 - \frac{D-90}{720} R^{\mu\nu} R_{\mu\nu} + \frac{D-15}{720} R^{\mu\nu\sigma\tau} R_{\mu\nu\sigma\tau} \Big) \Big\}. \tag{3.92}$$

that in dimension $D = 4$ reduces to

$$\Gamma_V = \int_0^\infty \frac{d\beta}{\beta} \int \frac{d^D x \sqrt{g(x)}}{(2\pi\beta)^{(D/2)}} \Big\{ 4 + \frac{5}{6} \beta R + \beta^2 \Big(\frac{3}{40} \nabla^2 R + \frac{1}{18} R^2 + \frac{43}{360} R^{\mu\nu} R_{\mu\nu} - \frac{11}{720} R^{\mu\nu\sigma\tau} R_{\mu\nu\sigma\tau} \Big) \Big\}. \tag{3.93}$$

It is rather immediate to check that the ghost results in arbitrary dimensions and in dimension $D = 4$ coincide with the ones presented in the previous chapter computed with the heat kernel known formulas. Therefore the coefficients in $D = 4$ coincide with [4] from which we have extended the worldline formalism procedure at arbitrary dimensions.

Chapter 4

Worldline formalism for the graviton

In the last chapter of the thesis we try to find a model that is able to reproduce the gauge-fixed graviton fluctuations. In order to do this we proceed with the construction of a symmetric rank-2 tensor model by following the procedure described in the previous chapter for the ghost sector. For the purpose of reproducing the graviton fluctuations the Hilbert space of the model will require worldline complex fermionic variables that are rank-2 symmetric tensors with non-vanishing trace. As usual, the finite dimensional Hilbert space so constructed will contain, among other fields that must be projected out, the graviton fluctuations. In our work we succeed in finding the correct worldline action for the gauge-fixed graviton fluctuations, but unfortunately it seems not possible to deal with it using perturbation theory. Therefore this part will be left for future research.

4.1 The tensor model for the graviton

Let us start considering as usual a D-dimensional space-time with a metric $g_{\mu\nu}(x)$. The Hilbert space that contains, among other fields, the rank-2 symmetric tensor with non-vanishing trace can be constructed introducing the ordinary real bosonic variables $x^\mu(t)$ and $p_\mu(t)$ and worldline complex fermionic variables that are rank-2 symmetric tensors with non-vanishing trace, i.e. ψ^{ab} and $\bar{\psi}^{cd}$. Bosonic and complex fermionic variables define a graded phase space with (anti)-commuting relations upon canonical quantization:

$$[x^\mu, p_\nu] = i\delta^\mu_\nu \qquad \{\psi^{ab}, \bar{\psi}^{cd}\} = \delta^{ac}\delta^{bd} + \delta^{bc}\delta^{ad}. \tag{4.1}$$

The bosonic and fermionic momenta are represented by their derivatives, i.e.

$$p_\mu = -ig^{-1/4}\partial_\mu g^{1/4} \qquad \bar{\psi}^{ab} = \frac{\partial}{\partial\psi^{ab}}, \tag{4.2}$$

where $\bar{\psi}^{ab}$ has to be interpreted as a linear combination of ψ derivatives:

$$\frac{\partial}{\partial \psi^{ab}} \psi^{cd} = \delta^c_a \delta^d_b + \delta^c_b \delta^d_a. \tag{4.3}$$

The generic state $|\phi\rangle$ of the Hilbert space is represented by a wave function $\phi(x, \psi) = (\langle x| \otimes \langle \psi|) |\phi\rangle$ (coherent states are used, see Appendix C) Taylor expanded as

$$\begin{aligned} |\phi\rangle \sim \phi(x, \psi) =& \phi(x) + \phi_{ab}(x)\psi^{ab} + \frac{1}{2}\phi_{(ab)_1(ab)_2}(x)\psi^{(ab)_1}\psi^{(ab)_2} + \cdots + \frac{1}{N!}\phi_{(ab)_1\ldots(ab)_N}(x)\psi^{(ab)_1}\ldots\psi^{(ab)_N} \\ =& \sum_{n=0}^{N} \frac{1}{n!}\phi_{(ab)_1\ldots(ab)_n}(x)\psi^{(ab)_1}\ldots\psi^{(ab)_n}\,, \end{aligned} \tag{4.4}$$

where N is the number of independent components of a symmetric rank-2 tensor with non-vanishing trace in D dimensional space-time, namely $N = \frac{1}{2}D(D+1)$. Similarly to the vector model, with the occupation number $n = 1$ we individuate in the spectrum the symmetric rank-2 tensor $\phi_{ab}(x) \equiv h_{ab}(x)$ that correctly represents the graviton, namely the metric fluctuations. In the construction of the classical action for the tensor model we will introduce a coupling with a worldline gauge field with an additional Chern-Simons term, in order to select the state $|\phi\rangle \sim h_{ab}(x)\psi^{ab}$.

The Lorentz $SO(D)$ generators can be constructed as

$$M^{ab} = -M^{ba} := \frac{1}{2}[\psi^{ab}, \bar{\psi}^b_c] - \frac{1}{2}[\psi^{bc}, \bar{\psi}^a_c] = \psi^a \cdot \bar{\psi}^b - \psi^b \cdot \bar{\psi}^a\,, \tag{4.5}$$

where we used the shorthand notation $\psi^a \cdot \bar{\psi}^b = \psi^{ac}\bar{\psi}^b_c$. Using the Lorentz $SO(D)$ generators we are able to write the covariant derivative operator as follows

$$\hat{\nabla}_\mu := \partial_\mu + \frac{1}{2}\omega_{\mu ab}M^{ab} = \partial_\mu + \omega_{\mu ab}\psi^a \cdot \bar{\psi}^b \tag{4.6}$$

that acting on the wave function $h(x, \psi) = h_{ab}(x)\psi^{ab}$ produces

$$\hat{\nabla}_\mu h(x, \psi) = (\nabla_\mu h_{ab})\psi^{ab} = (\partial_\mu h_{ab} - \omega_\mu{}^c{}_a h_{cb} - \omega_\mu{}^c{}_b h_{ac})\psi^{ab}. \tag{4.7}$$

By means of the ordinary covariant momentum in (4.2) we cast the covariant derivative as follows

$$\hat{\nabla}_\mu = ig^{1/4}\pi_\mu g^{-1/4} = ig^{1/4}\bigg(p_\mu - i\omega_{\mu ab}\psi^a \cdot \bar{\psi}^b\bigg)g^{-1/4}, \tag{4.8}$$

where $\pi_\mu = p_\mu - i\omega_{\mu ab}\psi^a \cdot \bar{\psi}^b$ is the covariant momentum. The laplacian is constructed with the same ordering of the previous chapter, i.e.

$$\hat{\nabla}^2 := \frac{1}{\sqrt{g}}\hat{\nabla}_\mu g^{\mu\nu}\sqrt{g}\hat{\nabla}_\nu = -g^{-1/4}\pi_\mu g^{\mu\nu} g^{1/2}\pi_\nu g^{-1/4}. \tag{4.9}$$

Let us proceed with the construction of the quantum mechanical worldline representation of the invertible kinetic operator of the graviton. The action of the latter acting on a symmetric rank-2 tensor, presented in eq. (3.6), can be rearranged to recognize a term containing the left hand side of the vacuum Einstein equation, i.e.

$$\begin{aligned}F_{\mu\nu\alpha\beta}\phi^{\alpha\beta} =& -\frac{1}{2}\left(g_{\mu\alpha}g_{\nu\beta}-\frac{1}{2}g_{\mu\nu}g_{\alpha\beta}\right)\nabla^2\phi^{\alpha\beta}-\Lambda\phi^{\alpha\beta}-\frac{1}{2}\left(R_{\mu\alpha\nu\beta}+R_{\mu\beta\nu\alpha}-\frac{1}{2}g_{\mu\nu}R_{\alpha\beta}\right)\phi^{\alpha\beta}\\ &-\frac{1}{2}\left(R^{\lambda}_{\mu}\phi_{\lambda\nu}+R^{\lambda}_{\nu}\phi_{\lambda\mu}-\frac{1}{2}g_{\mu\nu}R^{\alpha\beta}\phi_{\alpha\beta}\right)+\frac{1}{2}R\phi_{\mu\nu}+\frac{1}{2}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+g_{\mu\nu}\Lambda\right)g_{\alpha\beta}\phi^{\alpha\beta}.\end{aligned} \tag{4.10}$$

The laplace operator in the eq. (4.9) acting on the wave function gives

$$\hat{\nabla}^2 h(x,\psi) = (\nabla^2 h_{ab})\psi^{ab}. \tag{4.11}$$

The quantum mechanical representation of the graviton kinetic operator, which action on a symmetric rank-2 tensor is (4.10), is given by

$$\begin{aligned}H = \hat{F} :=& -\frac{1}{2}(\hat{\nabla}^2+2\Lambda)+\frac{1}{2}R-\frac{1}{2}R_{abcd}\psi^{ac}\bar{\psi}^{bd}-\frac{1}{2}R_{ab}\psi^{a}\cdot\bar{\psi}^{b}\\ &+\frac{1}{8}(\hat{\nabla}^2+2\Lambda-R)\delta_{ac}\delta_{bd}\psi^{ac}\bar{\psi}^{bd}+\frac{1}{4}(\delta_{ac}R_{bd}+\delta_{bd}R_{ac})\psi^{ac}\bar{\psi}^{bd}.\end{aligned} \tag{4.12}$$

In order to check the correctness of the last equation, consider the term

$$\frac{1}{4}(\nabla^2+2\Lambda)g_{\mu\nu}g_{\alpha\beta}\phi^{\alpha\beta} \tag{4.13}$$

present in eq. (4.10). In order to reproduce this term by means of a quantum mechanical operator we use the operator $\hat{O}=\frac{1}{8}(\hat{\nabla}^2+2\Lambda)$ acting on the wave function $h(x,\psi)$, i.e.

$$\begin{aligned}\hat{O}h(x,\psi) =& \frac{1}{8}(\hat{\nabla}^2+2\Lambda)\delta_{ac}\delta_{bd}\psi^{ac}\bar{\psi}^{bd}(h_{ef}\psi^{ef}) = \frac{1}{8}(\hat{\nabla}^2+2\Lambda)\delta_{ac}\delta_{bd}\psi^{ac}h_{ef}\bar{\psi}^{bd}\psi^{ef}\\ =& \frac{1}{8}(\hat{\nabla}^2+2\Lambda)\delta_{ac}\delta_{bd}\psi^{ac}h_{ef}(\delta^{be}\delta^{df}+\delta^{bf}\delta^{de}) = \frac{1}{4}(\hat{\nabla}^2+2\Lambda)\delta_{ac}\delta_{bd}h^{bd}\psi^{ac}\,,\end{aligned} \tag{4.14}$$

where we used the following identity

$$\bar{\psi}^{bd}\psi^{ef} = \frac{\partial}{\partial\psi_{bd}}\psi^{ef} = \delta^{be}\delta^{df}+\delta^{bf}\delta^{de}. \tag{4.15}$$

It is rather simple to check that the last term in eq. (4.14) reproduces correctly (4.13). For all the other terms in (4.10) we can proceed in a similar fashion.

In order to write the worldline classical action for the tensor model we need to introduce a constraint able to project out all the unwanted fields in the wave function

$\phi(x,\psi)$, in order to keep only the graviton, namely the symmetric rank-2 tensor with non-vanishing trace corresponding to $n = 1$. Proceeding in the same fashion of the vector model, we introduce a $U(1)$ coupling between fermionic variables and a worldline gauge field $A(t)$, with an additional Chern-Simons term in the action. This allows us to introduce the following constraint

$$\left(\frac{1}{4}[\psi^{ab},\bar{\psi}_{ab}] - s\right)|h\rangle = 0, \tag{4.16}$$

where $\frac{1}{4}[\psi^{ab},\bar{\psi}_{ab}] = \hat{N} - \frac{1}{4}D(D+1)$ is the $U(1)$ generator, with $\hat{N}$ counting the number of ψ's in the state $\phi(x,\psi)$, and $s = 1 - \frac{1}{4}D(D+1)$.

The classical action for the tensor model in phase space and euclidean time is thus given by

$$\begin{aligned} S[x,p,\psi,\bar{\psi};A] = \int_0^\beta dt \Big[& -ip_\mu \dot{x^\mu} + \frac{1}{2}\bar{\psi}_{ab}\dot{\psi^{ab}} + \frac{1}{2}g^{\mu\nu}\pi_\mu\pi_\nu\left(1 - \frac{1}{4}\psi\bar{\psi}\right) \\ & - \Lambda\left(1 - \frac{1}{4}\psi\bar{\psi}\right) - \frac{1}{2}R_{abcd}\psi^{ac}\bar{\psi}^{bd} - \frac{1}{2}R_{ab}\psi^a\cdot\bar{\psi}^b \\ & + \frac{1}{2}R\left(1 - \frac{1}{4}\psi\bar{\psi}\right) + \frac{1}{4}\left(\delta_{ac}R_{bd} + \delta_{bd}R_{ac}\right)\psi^{ac}\bar{\psi}^{bd} \\ & - iA\left(\frac{1}{2}\psi^{ab}\bar{\psi}_{ab} - s\right)\Big], \end{aligned} \tag{4.17}$$

where we used the shorthand notation to write the fermionic traces $\psi := \delta_{ab}\psi^{ab}$ and $\bar{\psi} := \delta_{ab}\bar{\psi}^{ab}$. At this juncture, in order to write the action in configuration space we integrate out the momentum p_μ by means of its eqs. of motion

$$p_\mu = i\dot{x^\mu}\left(1 - \frac{1}{4}\psi\bar{\psi}\right)^{-1} + ig^{\mu\nu}\omega_{\nu ab}\psi^a\cdot\bar{\psi}^b\,, \tag{4.18}$$

to get

$$\begin{aligned} S[x,\psi,\bar{\psi};A] = \int_0^\beta dt \Big[& \frac{1}{2}g^{\mu\nu}\dot{x_\mu}\dot{x_\nu}\left(1 - \frac{1}{4}\psi\bar{\psi}\right)^{-1} + \frac{1}{2}\bar{\psi}^{ab}(\mathcal{D}_t + iA)\psi^{ab} \\ & - \frac{1}{2}R_{abcd}\psi^{ac}\bar{\psi}^{bd} - \frac{1}{2}R_{ab}\psi^a\cdot\bar{\psi}^b - \Lambda\left(1 - \frac{1}{4}\psi\bar{\psi}\right) \\ & + \frac{1}{2}R\left(1 - \frac{1}{4}\psi\bar{\psi}\right) + \frac{1}{4}\left(\delta_{ac}R_{bd} + \delta_{bd}R_{ac}\right)\psi^{ac}\bar{\psi}^{bd} + iAs\Big], \end{aligned} \tag{4.19}$$

where the covariant derivative contains also the spin connection, i.e. $\mathcal{D}_t\psi^{ab} = \dot{\psi}^{ab} + \dot{x}^\mu(\omega_\mu{}^a{}_c\psi^{cb} + \omega_\mu{}^b{}_c\psi^{ac})$.

The action thus obtained does not seem to admit a perturbative treatment in β. One could try to replace the term $(1-\frac{1}{4}\psi\bar{\psi})^{-1}$ with the geometric series, to get the following form of the action

$$\begin{aligned}S[x,\psi,\bar{\psi};A]=\int_0^\beta dt\bigg\{&\frac{1}{2}g^{\mu\nu}\dot{x}_\mu\dot{x}_\nu\bigg[1+\frac{1}{4}\psi\bar{\psi}+\bigg(\frac{1}{4}\psi\bar{\psi}\bigg)^2+\dots\bigg]+\frac{1}{2}\bar{\psi}^{ab}(\mathcal{D}_t+iA)\psi^{ab}\\&-\frac{1}{2}R_{abcd}\psi^{ac}\bar{\psi}^{bd}-\frac{1}{2}R_{ab}\psi^a\cdot\bar{\psi}^b-\Lambda\bigg(1-\frac{1}{4}\psi\bar{\psi}\bigg)\\&+\frac{1}{2}R\bigg(1-\frac{1}{4}\psi\bar{\psi}\bigg)+\frac{1}{4}\bigg(\delta_{ac}R_{bd}+\delta_{bd}R_{ac}\bigg)\psi^{ac}\bar{\psi}^{bd}+iAs\bigg\}.\end{aligned} \tag{4.20}$$

In order to take care of all the different orders in the perturbative approach one could rescale the time $t=\beta\tau$, and treat β as the perturbative parameter. To have a uniform β^{-1} in front of the perturbative propagators, we rescale also the fermions as $\psi\to\frac{1}{\sqrt{\beta}}\psi$ and $\bar{\psi}\to\frac{1}{\sqrt{\beta}}\bar{\psi}$ and the gauge field as $A\to\frac{1}{\beta}A$. The action thus reads

$$\begin{aligned}S[x,\psi,\bar{\psi};A]=\frac{1}{\beta}\int_0^1 d\tau\bigg[&\frac{1}{2}g^{\mu\nu}\dot{x}_\mu\dot{x}_\nu\bigg(1+\frac{1}{4\beta}\psi\bar{\psi}+\bigg(\frac{1}{4\beta}\psi\bar{\psi}\bigg)^2+\dots\bigg)\\&+\frac{1}{2}\bar{\psi}_{ab}(\mathcal{D}_t+iA)\psi^{ab}-\frac{\beta}{2}R_{abcd}\psi^{ac}\bar{\psi}^{bd}-\frac{\beta}{2}R_{ab}\psi^a\cdot\bar{\psi}^b\\&+\frac{\beta}{4}(\delta_{ac}R_{bd}+\delta_{bd}R_{ac})\psi^{ac}\bar{\psi}^{bd}+\frac{\beta}{4}\Lambda\psi\bar{\psi}-\frac{\beta}{8}R\psi\bar{\psi}+\beta^2\bigg(\frac{R}{2}-\Lambda\bigg)\\&+is\int_0^1 d\tau A.\end{aligned} \tag{4.21}$$

but we see that the vertices arising for the expansion of $(1-\frac{1}{4\beta}\psi\bar{\psi})^{-1}$ become non-perturbative. How to find ways of computing the path integral associated to this action will be left for future research.

Conclusions

The purpose of this thesis was first to rederive the heat kernel coefficients a_0, a_1 and a_2 of perturbative quantum gravity using the original approach of the heat kernel developed by B. DeWitt. After a brief introduction to the heat kernel expansion, we applied the general formulas that define these coefficients and reduced them to the specific case of pure quantum gravity, constructed perturbatively by expanding the Einstein-Hilbert action via a background-quantum splitting of the metric. Once the action has been gauge-fixed, we derived the invertible kinetic operators of the graviton fluctuations and the ghosts. The aim was to write the Seeley-DeWitt coefficients in arbitrary dimensions D and in an arbitrary gravitational background, so to compare our computations with different papers. As anticipated some of these coefficients are reported erroneously in the literature ([1],[3]). On the other hand our results are in perfect agreement with more recent works ([16]). If reduced to $D = 4$ our Seeley-DeWitt coefficients are identical to those computed via a worldline approach ([4]). These coefficients identify the counterterms that make the effective action finite (in 4 dimensions) and to be gauge-invariant they must be evaluated on-shell. For this purpose we decided to reduce our study to an Einstein manifold (which metric satisfies the Einstein field equations). The coefficients so computed, for the specific case of $D = 4$, are in agreement with the results found in [7], where the physical graviton is successfully described by a $\mathcal{N} = 4$ spinning particle model. However, our on-shell Seeley-DeWitt coefficients at arbitrary dimensions disagree with those reported in [7]. This seems to suggest the need of further studies to make that first quantized model consistent in any arbitrary dimensions.

The second part of the thesis is dedicated to the attempt of deriving the previous coefficients in an alternative way, namely using a worldline formalism. This approach has been already applied to the case of $D = 4$ perturbative quantum gravity in [4]. Starting from this result, we decided to generalize that idea to a space-time with arbitrary dimensions D. The idea was to reproduce the behaviour of the gauge-fixed graviton fluctuations and the ghosts by means of suitable particle actions, that upon quantization provide the invertible kinetic operators of the quadratic action of pure quantum gravity. Two models have been constructed: a vector model and a rank-2 tensor model, for the ghosts and the graviton, respectively. The vector model succeedes in reproducing the heat kernel coefficients associated to the ghosts. On the other hand, the tensor model, while providing a suitable worldline action for the graviton (reproducing the associated Hamiltonian used in the heat kernel approach), does not lead to a path integral that admits a perturbative expansion. How to solve that path integral it is left for future research.

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Appendix A

Perturbative quantum gravity computations

In the following appendix we will review step by step the computations for the Einstein-Hilbert action's expansion under the perturbation of the metric tensor introduced with (2.4). The main issue will concern the evaluation of the perturbed Ricci scalar curvature, up to $\mathcal{O}(h^2)$, that has to be replaced in the action. The result will be valid for an arbitrary background and in any coordinates system. The consistent algebra needed for this computation will be checked using computer algebra of *Mathematica* (*xPert* package).

A.1 Einstein-Hilbert action's expansion under metric perturbation

Since the expansion of the inverse of the metric, namely $G^{\mu\nu}$, has been already described, we will proceed with the expansion of the square root of the metric determinant. For this computation we omit the indices of the metric tensors and we make the determinant explicit when it is present in order to simplify the notation:

$$\begin{aligned}\sqrt{|\det G|} &= \sqrt{|\det(g+h)|} = \exp\log\sqrt{|\det(g+h)|} \\ &= \exp\frac{1}{2}\log\Big[|\det g\ \det\big(1+g^{-1}h\big)|\Big] \\ &= \exp\log\Big[\log\sqrt{|\det g|} + \frac{1}{2}\log\det\big(1+g^{-1}h\big)\Big].\end{aligned} \tag{A.1}$$

At this step we perform some logarithm and exponential expansions to obtain

$$
\begin{aligned}
\sqrt{|\det G|} = &= \sqrt{|\det g|}\exp\left[\frac{1}{2}\log\det|1+g^{-1}h|\right] \\
&= \sqrt{|\det g|}\exp\left[\frac{1}{2}\mathrm{tr}\log|1+g^{-1}h|\right] \\
&= \sqrt{|\det g|}\exp\left[\frac{1}{2}\mathrm{tr}\left(g^{-1}h-\frac{1}{2}(g^{-1}h)^2+\mathcal{O}(h)^3\right)\right] \\
&= \sqrt{|\det g|}\left[1+\frac{1}{2}\mathrm{tr}(g^{-1}h)-\frac{1}{4}\mathrm{tr}(g^{-1}h)^2+\frac{1}{2}\left(\frac{1}{2}\mathrm{tr}(g^{-1}h)-\frac{1}{4}(g^{-1}h)^2\right)^2+\mathcal{O}(h^3)\right] \\
&= \sqrt{|\det g|}\left[1+\frac{1}{2}\mathrm{tr}(g^{-1}h)-\frac{1}{4}\mathrm{tr}(g^{-1}h)^2+\frac{1}{8}\mathrm{tr}^2(g^{-1}h)\right]+\mathcal{O}(h^3) \\
&= \sqrt{|\det g_{\mu\nu}|}\left[1+\frac{1}{2}h^\mu_\mu-\frac{1}{4}h^{\mu\nu}h_{\mu\nu}+\frac{1}{8}(h^\mu_\mu)^2\right]+\mathcal{O}(h^3),
\end{aligned}
\tag{A.2}
$$

where in the last line the indices have been reintroduced.

By looking at the expanded action as in eq. (2.9), the next step is to study the expansion of the Ricci tensor. One can start from the evaluation of the Levi-Civita connection:

$$
\begin{aligned}
\Gamma^\rho{}_{\mu\nu}(g+h) &= \frac{1}{2}G^{\rho\sigma}(G_{\mu\sigma,\nu}+G_{\nu\sigma,\mu}-G_{\mu\nu,\sigma}) \\
&= \frac{1}{2}(g^{\rho\sigma}-h^{\rho\sigma}+h^\rho_\lambda h^{\lambda\sigma})[(g_{\mu\sigma}+h_{\mu\sigma})_{,\nu}+(g_{\nu\sigma}+h_{\nu\sigma})_{,\mu}-(g_{\mu\nu}+h_{\mu\nu})_{,\sigma}] \\
&= \Gamma^\rho{}_{\mu\nu}(0)+\Gamma^\rho{}_{\mu\nu}(1)+\Gamma^\rho{}_{\mu\nu}(2)+\mathcal{O}(h^3)
\end{aligned}
\tag{A.3}
$$

where the following shorthand has been used to simplify the notation:

$$
\begin{aligned}
\Gamma^\rho{}_{\mu\nu}(0) &: \textit{Christoffel symbol at 0-th order in h}, \\
\Gamma^\rho{}_{\mu\nu}(1) &: \textit{Christoffel symbol at 1-st order in h}, \\
\Gamma^\rho{}_{\mu\nu}(2) &: \textit{Christoffel symbol at 2-nd order in h}.
\end{aligned}
\tag{A.4}
$$

Therefore we immediately write that

$$
\Gamma^\rho{}_{\mu\nu}(0) = \frac{1}{2}g^{\rho\sigma}(g_{\mu\sigma,\nu}+g_{\nu\sigma,\mu}-g_{\mu\nu,\sigma})
\tag{A.5}
$$

and

$$
\begin{aligned}
\Gamma^\rho{}_{\mu\nu}(1) =& \frac{1}{2}g^{\rho\sigma}(h_{\mu\sigma,\nu}+h_{\nu\sigma,\mu}-h_{\mu\nu,\sigma})-\frac{1}{2}h^{\rho\sigma}(g_{\mu\sigma,\nu}+g_{\nu\sigma,\mu}-g_{\mu\nu,\sigma}) \\
=& \frac{1}{2}g^{\rho\sigma}(h_{\mu\sigma,\nu}+h_{\nu\sigma,\mu}-h_{\mu\nu,\sigma})-g_{\lambda\sigma}h^{\rho\sigma}\Gamma^\lambda{}_{\mu\nu}(0).
\end{aligned}
\tag{A.6}
$$

By using the covariant derivative applied to the fluctuation

$$\nabla_\sigma h_{\mu\nu} = \partial_\sigma h_{\mu\nu} - \Gamma^\lambda_{\sigma\mu} h_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu} h_{\mu\lambda}, \tag{A.7}$$

we can rewrite (A.6) as

$$\begin{aligned}\Gamma^\rho{}_{\mu\nu}(1) =& \frac{1}{2} g^{\rho\sigma}[\partial_\nu h_{\mu\sigma} + \partial_\mu h_{\nu\sigma} - \partial_\sigma h_{\mu\nu} - 2h_{\sigma\lambda}\Gamma^\lambda_{\mu\nu} \pm \Gamma^\lambda_{\nu\sigma} h_{\mu\lambda} \pm \Gamma^\lambda_{\mu\sigma} h_{\nu\lambda}] \\ =& \frac{1}{2} g^{\rho\sigma}(\nabla_\nu h_{\mu\sigma} + \nabla_\mu h_{\nu\sigma} - \nabla_\sigma h_{\mu\nu}),\end{aligned} \tag{A.8}$$

where the covariant derivative in the last eq. is constructed using the background metric. As one can see, the perturbation at first order in h of the Christoffel symbol is a tensor. This is manifest because it involves only covariant derivatives, so the final result is fully covariant. This information represents a powerful trick for the study of various curvature expansions under a metric perturbation, as we will see later. In the same fashion we can evaluate the term $\Gamma^\rho{}_{\mu\nu}(2)$:

$$\begin{aligned}\Gamma^\rho{}_{\mu\nu}(2) =& \frac{1}{2} h^\rho_\lambda h^{\lambda\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}) - \frac{1}{2} h^{\rho\sigma}(h_{\mu\sigma,\nu} + h_{\nu\sigma,\mu} - h_{\mu\nu,\sigma}) \\ =& - h^\rho_\lambda \Gamma^\lambda{}_{\mu\nu}(1),\end{aligned} \tag{A.9}$$

which is obviously covariant.

Once we have expanded the Christoffel symbol, we can proceed with the evaluation of the Riemann tensor, since all the other curvatures can be easily recovered by proper indices contraction. The Riemann curvature tensor can be written as

$$R^\mu{}_{\nu\rho\sigma}(g+h) = R^\mu{}_{\nu\rho\sigma}(0) + R^\mu{}_{\nu\rho\sigma}(1) + R^\mu{}_{\nu\rho\sigma}(2) + \mathcal{O}(h^3), \tag{A.10}$$

where the following shorthand notation has been used

$$\begin{aligned} &R^\mu{}_{\nu\rho\sigma}(0) : \textit{Riemann tensor at 0-th order in h}, \\ &R^\mu{}_{\nu\rho\sigma}(1) : \textit{Riemann tensor at 1-st order in h}, \\ &R^\mu{}_{\nu\rho\sigma}(2) : \textit{Riemann tensor at 2-nd order in h}. \end{aligned} \tag{A.11}$$

The order 0 in h is given by

$$R^\mu{}_{\nu\rho\sigma}(0) = \nabla_\rho \Gamma^\mu{}_{\sigma\nu}(0) - \nabla_\sigma \Gamma^\mu{}_{\rho\nu}(0), \tag{A.12}$$

where the covariant derivative is written in terms of the background.

Let us recall, as we have anticipated, that the variation of the Christoffel symbol is a tensor, therefore under the action of a covariant derivative it must satisfy a tensorial law. Hence, writing the expansion as follows

$$\begin{aligned}R^\mu{}_{\nu\rho\sigma}(1) = \partial_\rho \Gamma^\mu{}_{\sigma\nu}(1) - \partial_\sigma \Gamma^\mu{}_{\rho\nu}(1) + \Gamma^\mu{}_{\rho\lambda}\Gamma^\lambda{}_{\sigma\nu}(1) + \Gamma^\mu{}_{\rho\lambda}(1)\Gamma^\lambda{}_{\sigma\nu} \\ - \Gamma^\mu{}_{\sigma\lambda}\Gamma^\lambda{}_{\rho\nu}(1) - \Gamma^\mu{}_{\sigma\lambda}(1)\Gamma^\lambda{}_{\rho\nu},\end{aligned} \tag{A.13}$$

and adding $\pm\Gamma^{\lambda}{}_{\rho\sigma}\Gamma^{\mu}{}_{\lambda\nu}(1)$ we recognize the covariant derivatives:

$$R^{\mu}{}_{\nu\rho\sigma}(1) = \nabla_{\rho}\Gamma^{\mu}{}_{\sigma\nu}(1) - \nabla_{\sigma}\Gamma^{\mu}{}_{\rho\nu}(1). \tag{A.14}$$

In eq. (A.13) we used $\Gamma^{\mu}{}_{\rho\nu} := \Gamma^{\mu}{}_{\rho\nu}(0)$. By using eq. (A.8), eq. (A.14) can be thus expanded as follows

$$\begin{aligned} &R^{\mu}{}_{\nu\rho\sigma}(1) = \\ =&\frac{1}{2}\bigg(\nabla_{\rho}\nabla_{\nu}h^{\mu}_{\sigma} + \nabla_{\rho}\nabla_{\sigma}h^{\mu}_{\nu} + \nabla_{\sigma}\nabla^{\mu}h_{\rho\nu} - \nabla_{\sigma}\nabla_{\nu}h^{\mu}_{\rho} - \nabla_{\sigma}\nabla_{\rho}h^{\mu}_{\nu} - \nabla_{\rho}\nabla^{\mu}h_{\sigma\nu}\bigg). \end{aligned} \tag{A.15}$$

The second order perturbation of the Riemann tensor is given by

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma}(2) =&\partial_{\rho}\Gamma^{\mu}{}_{\sigma\nu}(2) - \partial_{\sigma}\Gamma^{\mu}{}_{\rho\nu}(2) + \Gamma^{\mu}{}_{\rho\lambda}\Gamma^{\lambda}{}_{\sigma\nu}(2) + \Gamma^{\mu}{}_{\rho\lambda}(2)\Gamma^{\lambda}{}_{\sigma\nu} \\ &- \Gamma^{\mu}{}_{\sigma\lambda}\Gamma^{\lambda}{}_{\rho\nu}(2) - \Gamma^{\mu}{}_{\sigma\lambda}(2)\Gamma^{\lambda}{}_{\rho\nu} + \Gamma^{\mu}{}_{\rho\lambda}(1)\Gamma^{\lambda}{}_{\sigma\nu}(1) \\ &- \Gamma^{\mu}{}_{\sigma\lambda}(1)\Gamma^{\lambda}{}_{\rho\nu}(1). \end{aligned} \tag{A.16}$$

Proceeding similarly to the previous case we end up with

$$R^{\mu}{}_{\nu\rho\sigma}(2) = \nabla_{\rho}\Gamma^{\mu}{}_{\sigma\nu}(2) - \nabla_{\sigma}\Gamma^{\mu}{}_{\rho\nu}(2) + \Gamma^{\mu}{}_{\rho\lambda}(1)\Gamma^{\lambda}{}_{\sigma\nu}(1) - \Gamma^{\mu}{}_{\sigma\lambda}(1)\Gamma^{\lambda}{}_{\rho\nu}(1), \tag{A.17}$$

that can be written in a compact form as follows

$$R^{\mu}{}_{\nu\rho\sigma}(2) = -h^{\mu}_{\beta}R^{\beta}{}_{\nu\rho\sigma}(1) - g^{\mu\alpha}g_{\beta\gamma}\bigg(\Gamma^{\gamma}{}_{\rho\alpha}(1)\Gamma^{\beta}{}_{\sigma\nu}(1) - \Gamma^{\gamma}{}_{\sigma\alpha}(1)\Gamma^{\beta}{}_{\rho\nu}(1)\bigg). \tag{A.18}$$

Since the variations at different orders in h of the Riemann tensor curvature have been found, we can proceed with the evaluation of the Ricci tensor just contracting the proper indices. Hence, we have

$$R_{\nu\sigma} = R_{\nu\sigma}(0) + R_{\nu\sigma}(1) + R_{\nu\sigma}(2) + \mathcal{O}(h^3) \tag{A.19}$$

with the usual notation, where $R_{\nu\sigma}(0) = R^{\mu}{}_{\nu\mu\sigma}(0)$,

$$R_{\nu\sigma}(1) = R^{\mu}{}_{\nu\mu\sigma}(1) = \frac{1}{2}\bigg(\nabla_{\mu}\nabla_{\nu}h^{\mu}_{\sigma} + \nabla_{\mu}\nabla_{\sigma}h^{\mu}_{\nu} - \nabla_{\sigma}\nabla_{\nu}h - \nabla^2 h_{\sigma\nu}\bigg), \tag{A.20}$$

thanks to the simplification of two indetical terms, and finally

$$R_{\nu\sigma}(2) = R^{\mu}{}_{\nu\mu\sigma}(2) = -h^{\mu}_{\beta}R^{\beta}{}_{\nu\mu\sigma}(1) - g^{\mu\alpha}g_{\beta\gamma}\bigg(\Gamma^{\gamma}{}_{\mu\alpha}(1)\Gamma^{\beta}{}_{\sigma\nu}(1) - \Gamma^{\gamma}{}_{\sigma\alpha}(1)\Gamma^{\beta}{}_{\mu\nu}(1)\bigg). \tag{A.21}$$

The Ricci scalar curvature can be computed as

$$R = R(0) + R(1) + R(2) = (g^{\nu\sigma} - h^{\nu\sigma} + h^{\nu}_{\lambda}h^{\lambda\sigma})R_{\nu\sigma}, \tag{A.22}$$

where $R(0) = g^{\nu\sigma}R_{\nu\sigma}(0)$,

$$\begin{aligned} R(1) =& - h^{\nu\sigma}R_{\nu\sigma}(0) + g^{\nu\sigma}R_{\nu\sigma}(1) \\ =& - h^{\nu\sigma}R_{\nu\sigma}(0) + \nabla_{\nu\sigma}h^{\nu\sigma} - \nabla^2 h, \end{aligned} \tag{A.23}$$

and finally

$$\begin{aligned} R(2) =& - h^{\nu\sigma}R_{\nu\sigma}(1) - g^{\nu\sigma}h^{\mu}_{\beta}R^{\beta}{}_{\nu\mu\sigma}(1) + R^{\rho\lambda}(0)h_{\alpha\rho}h^{\alpha}_{\lambda} \\ & - g^{\nu\sigma}g^{\mu\alpha}g_{\beta\gamma}\bigg(\Gamma^{\gamma}{}_{\mu\alpha}(1)\Gamma^{\beta}{}_{\sigma\nu}(1) - \Gamma^{\gamma}{}_{\sigma\alpha}(1)\Gamma^{\beta}{}_{\mu\nu}(1)\bigg). \end{aligned} \tag{A.24}$$

Let us write down some terms in (A.24) not yet expanded:

$$\begin{aligned} -g^{\nu\sigma}h^{\rho}_{\mu}R^{\mu}{}_{\nu\rho\sigma} = -\bigg(& h^{\rho}_{\mu}\nabla_{\rho}\nabla_{\nu}h^{\mu\nu} + \frac{1}{2}h^{\rho}_{\mu}\nabla^{\nu}\nabla^{\mu}h_{\rho\nu} - \frac{1}{2}h^{\rho}_{\nu}\nabla^2 h^{\mu}_{\rho} \\ & - \frac{1}{2}h^{\rho}_{\mu}\nabla^{\nu}\nabla_{\rho}h^{\mu}_{\nu} - \frac{1}{2}h^{\rho}_{\mu}\nabla_{\rho}\nabla^{\mu}h\bigg); \end{aligned} \tag{A.25}$$

$$\begin{aligned} & - g^{\nu\sigma}g^{\mu\alpha}g_{\beta\gamma}\bigg(\Gamma^{\gamma}{}_{\mu\alpha}(1)\Gamma^{\beta}{}_{\sigma\nu}(1) - \Gamma^{\gamma}{}_{\sigma\alpha}(1)\Gamma^{\beta}{}_{\mu\nu}(1)\bigg) = \\ & - \frac{1}{4}\bigg(4\nabla^{\mu}h_{\mu\beta}\nabla_{\nu}h^{\beta\nu} + \nabla_{\beta}h\nabla^{\beta}h - 4\nabla^{\mu}h_{\beta\mu}\nabla^{\beta}h - 3\nabla^{\mu}h^{\nu\beta}\nabla_{\mu}h_{\beta\nu} + 2\nabla^{\nu}h^{\mu}_{\beta}\nabla^{\beta}h_{\mu\nu}\bigg). \end{aligned} \tag{A.26}$$

The expressions in eqs. (A.25) and (A.26) can be replaced in (A.24) to get

$$\begin{aligned} R(2) =& R^{\rho\lambda}(0)h_{\alpha\rho}h^{\alpha}_{\lambda} - h^{\nu\sigma}\nabla_{\mu}\nabla_{\sigma}h^{\mu}_{\nu} - h^{\nu\sigma}\nabla_{\mu}\nabla_{\nu}h^{\mu}_{\sigma} + h^{\nu\sigma}\nabla^2 h_{\nu\sigma} + h^{\nu\sigma}\nabla_{\sigma}\nabla_{\nu}h \\ & - \nabla^{\mu}h_{\mu\beta}\nabla_{\nu}h^{\beta\nu} - \frac{1}{4}\nabla_{\beta}h\nabla^{\beta}h + \nabla^{\mu}h_{\beta\mu}\nabla^{\beta}h + \frac{3}{4}\nabla^{\mu}h^{\nu\beta}\nabla_{\mu}h_{\beta\nu} - \frac{1}{2}\nabla^{\nu}h^{\mu}_{\beta}\nabla^{\beta}h_{\mu\nu}. \end{aligned} \tag{A.27}$$

By making use of all the above written orders of expansion of the Ricci scalar, we can collect all of them in the following final expression

$$\begin{aligned} R(g+h) =& R(0) + \bigg(-h^{\nu\sigma}R_{\nu\sigma}(0) + \nabla_{\nu\sigma}h^{\nu\sigma} - \nabla^2 h\bigg) \\ & + \bigg(R^{\rho\lambda}(0)h_{\alpha\rho}h^{\alpha}_{\lambda} - h^{\nu\sigma}\nabla_{\mu}\nabla_{\sigma}h^{\mu}_{\nu} - h^{\nu\sigma}\nabla_{\nu}\nabla_{\mu}h^{\mu}_{\sigma} + h^{\nu\sigma}\nabla^2 h_{\nu\sigma} + h^{\nu\sigma}\nabla_{\sigma}\nabla_{\nu}h \\ & - \nabla^{\mu}h_{\mu\beta}\nabla_{\nu}h^{\beta\nu} - \frac{1}{4}\nabla_{\beta}h\nabla^{\beta}h + \nabla^{\mu}h_{\beta\mu}\nabla^{\beta}h + \frac{3}{4}\nabla^{\mu}h^{\nu\beta}\nabla_{\mu}h_{\beta\nu} - \frac{1}{2}\nabla^{\nu}h^{\mu}_{\beta}\nabla^{\beta}h_{\mu\nu}\bigg). \end{aligned} \tag{A.28}$$

It is worth to stress that this result is valid for any background metric and in any coordinates system. If one wants to evaluate the Ricci scalar expansion in the specific background of flat space-time they have to put background curvature to zero, so that covariant derivatives become ordinary partial derivatives with commuting property. Finally we have all the expressions to replace in the Einstein-Hilbert action as follows

$$S[g+h] = -k^{-2}\int d^Dx\sqrt{g}\left[\left(1+\frac{1}{2}h-\frac{1}{4}h^{\mu\nu}h_{\mu\nu}+\frac{1}{8}h^2\right)\Big(R(g+h)-2\Lambda\Big)+\mathcal{O}(h^3)\right]. \tag{A.29}$$

By expanding the previous eq. and perfoming some integration by parts when needed, and making use of the following relation between covariant derivatives' commutator and Riemann tensor curvature

$$[\nabla_\mu, \nabla_\beta]h^\gamma_\alpha = R^\gamma{}_{\lambda\mu\beta} + R^\lambda{}_{\alpha\beta\mu}h^\gamma_\lambda, \tag{A.30}$$

one gets the following orders of the action expansion

$$\begin{aligned}
S_0 &= -\int d^Dx\sqrt{g}\Big\{R - 2\Lambda\Big\},\\
S_1 &= \int d^Dx\sqrt{g}\Big\{h^{\mu\nu}\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda\Big) + \textit{total derivatives}\Big\},\\
S_2 &= -\int d^Dx\sqrt{g}\Big\{\frac{1}{4}h^{\mu\nu}(\nabla^2 + 2\Lambda)h_{\mu\nu} - \frac{1}{8}h(\nabla^2 + 2\Lambda)h + \frac{1}{2}\Big(\nabla^\nu h_{\nu\mu} - \frac{1}{2}\nabla_\mu h\Big)^2\\
&+ \frac{1}{2}h^{\mu\lambda}h^{\nu\sigma}R_{\mu\nu\lambda\sigma} + \frac{1}{2}\Big(h^{\mu\lambda}h^\nu_\lambda - hh^{\mu\nu}\Big)R_{\mu\nu} + \frac{1}{8}\Big(h^2 - 2h^{\mu\nu}h_{\mu\nu}\Big)R + \textit{total derivatives}\Big\}.
\end{aligned} \tag{A.31}$$

Neglecting all the total derivatives we end up with the mentioned eqs. (2.10).

The laborious and rather tedious algebra that we have done so far can be surprisingly simplified making use of the tensor computer algebra package for metric perturbation *xPert* [28] available for *Mathematica*.

Appendix B

Vector path integral computations

In order to present some examples of computation of ambiguous Feynman graphs that must be properly regularized, in this appendix we will follow all the steps for $\langle e^{-S_{int}} \rangle$ evaluation. Many terms of the expanded $\langle e^{-S_{int}} \rangle$ up to the order $\mathcal{O}(\beta^2)$ will involve 2-point functions with two derivatives, that satisfying the Green equation provide diverging Dirac delta functions. Other problems arise with products of distributions such as delta and step functions. The only way to solve these ambiguities with *dimensional regularization* is to extend the integration domain to a non-compact $(d+1)$-dimensional space Ω, where some manipulations are allowed to cast integrals in a non-ambiguous form. In the following we will see some different cases. At the end we will mention also the methods for modular integral computation.

B.1 Ambiguous integrals with DR

The present section is dedicated to the computation of $\langle S_4 \rangle$, $\langle S_6 \rangle$ and $\langle S_4^2 \rangle$, with S_4 and S_6 referring to (3.89). Let us start with $\langle S_4 \rangle$, i.e.

$$
\begin{aligned}
\langle S_4 \rangle =& \frac{1}{6\beta} R_{\lambda\mu\nu\sigma} \int_0^1 d\tau (\langle q^\lambda q^\sigma \dot{q}^\mu \dot{q}^\nu \rangle + \langle q^\lambda q^\sigma a^\mu a^\nu \rangle + \langle q^\lambda q^\sigma b^\mu c^\nu \rangle) + \frac{1}{2\beta} R_{\mu\nu ab} \int_0^1 d\tau \langle q^\mu \dot{q}^\nu \lambda^a \bar{\lambda}^b \rangle \\
&- \frac{1}{2} R_{ab} \int_0^1 d\tau \langle \lambda^a \bar{\lambda}^b \rangle - \frac{3}{8} \beta R \\
=& \frac{1}{6\beta} R_{\lambda\mu\nu\sigma} \int_0^1 d\tau (\langle q^\lambda q^\sigma \rangle \langle \dot{q}^\mu \dot{q}^\nu \rangle + \langle q^\lambda \dot{q}^\mu \rangle \langle q^\sigma \dot{q}^\nu \rangle + \langle q^\lambda \dot{q}^\nu \rangle \langle q^\sigma \dot{q}^\mu \rangle + \langle q^\lambda q^\sigma \rangle \langle a^\mu a^\nu \rangle \\
&+ \langle q^\lambda q^\sigma \rangle \langle b^\mu c^\nu \rangle) + \frac{1}{2\beta} R_{\mu\nu ab} \int_0^1 d\tau \langle q^\mu \dot{q}^\nu \rangle \langle \lambda^a \bar{\lambda}^b \rangle - \frac{1}{2} R_{ab} \int_0^1 d\tau \langle \lambda^a \bar{\lambda}^b \rangle - \frac{3}{8} \beta R.
\end{aligned}
\tag{B.1}
$$

where the Wick theorem for all the possible contractions of the fields in the correlators has been used. By means of the propagators (3.74) we get

$$\langle S_4\rangle = \frac{\beta}{6}R\int_0^1 d\tau[-\Delta|_\tau {}^\bullet\Delta^\bullet|_\tau + \Delta^\bullet|_\tau\Delta^\bullet|_\tau - \Delta|_\tau\,\Delta_{gh}|_\tau] - \frac{\beta}{2}R\int_0^1 d\tau\Delta_F|_{\tau=\sigma} - \frac{3}{8}\beta R. \tag{B.2}$$

In the first integral of the last eq. there are two diverging integrands. These terms are characterized by two derivatives, i.e. proportional to the diverging $\delta(\tau,\tau)$ as derived by the Green equation. They can be evaluated only after a proper manipulation with a regularization scheme. Using the Green equation $\Delta_{gh}(\tau,\sigma) = {}^{\bullet\bullet}\Delta(\tau,\sigma)$ in (3.76) and performing *dimensional regularization*, namely extending the integral to non-compact $d+1$ dimensions, we get

$$\begin{aligned}
&\int_0^1 d\tau(\Delta|_\tau {}^\bullet\Delta^\bullet|_\tau + \Delta|_\tau {}^{\bullet\bullet}\Delta|_\tau) \xrightarrow{d+1} \int d^{d+1}t\,\Delta|_t({}_\mu\Delta_\mu + {}_{\mu\mu}\Delta)|_t \\
&= \int d^{d+1}t\,\Delta|_\tau[{}_0({}_0\Delta|_\tau)] = -\int d^{d+1}t\,\partial_0(\Delta|_t)_0\Delta|_t \xrightarrow{d\to 0} -\int_0^1 d\tau\partial_\tau(\Delta|_\tau){}^\bullet\Delta|_\tau \\
&= -\frac{1}{2}\int_0^1 d\tau(2\tau-1)^2 = -\frac{1}{6},
\end{aligned} \tag{B.3}$$

where we used the previously mentioned identity (3.82), an integration by parts (allowed in DR) and at the end we replaced the values of $\Delta(\tau,\tau)$ and ${}^\bullet\Delta(\tau,\sigma)|_{\tau=\sigma}$ as present in (3.76). The subscript 0 indicates the derivative along the compact original dimension τ.

The other integrals, being unambiguous, can be computed without performing any manipulation, i.e.

$$\begin{aligned}
\int_0^1 d\tau\Delta^\bullet|_\tau\Delta^\bullet|_\tau &= \int_0^1 d\tau\left(\tau-\frac{1}{2}\right)^2 = \frac{1}{12}\,, \\
\int_0^1 d\tau\Delta_F|_{\tau=\sigma} &= \frac{i}{2}\tan\frac{\theta}{2}.
\end{aligned} \tag{B.4}$$

Replacing all the above results in (B.2) we finally get

$$\langle S_4\rangle - \frac{1}{3}\beta R - \frac{i}{4}\beta R\tan\frac{\theta}{2}. \tag{B.5}$$

The second step is the computation of $\langle S_6 \rangle$:

$$\langle S_6 \rangle = \frac{1}{\beta}\left[\frac{1}{40}\nabla_\lambda\nabla_\sigma R_{\alpha\mu\nu\beta} + \frac{1}{45}R_{\tau\lambda\sigma\mu}R^\tau{}_{\alpha\beta\nu}\right]\int_0^1 d\tau(\langle q^\lambda q^\sigma q^\alpha q^\beta \dot{q}^\mu \dot{q}^\nu\rangle + \langle q^\lambda q^\sigma q^\alpha q^\beta a^\mu a^\nu\rangle \tag{B.6}$$

$$+ \langle q^\lambda q^\sigma q^\alpha q^\beta b^\mu c^\nu\rangle) \tag{B.7}$$

$$+ \frac{1}{\beta}\left[\frac{1}{8}\nabla_\lambda\nabla_\sigma R_{\mu\nu ab} + \frac{1}{24}R^\tau{}_{\mu\lambda\nu}R_{\sigma\tau ab}\right]\int_0^1 d\tau\langle q^\lambda q^\sigma q^\mu \dot{q}^\nu \lambda^a\bar{\lambda}^b\rangle \tag{B.8}$$

$$- \frac{1}{4}\nabla_\mu\nabla_\nu R_{ab}\int_0^1 d\tau\langle q^\mu q^\nu \lambda^a\bar{\lambda}^b\rangle - \frac{3}{16}\beta\nabla_\mu\nabla_\nu R\int_0^1 d\tau\langle q^\mu q^\nu\rangle \tag{B.9}$$

Let us perform all the Wick contractions:

$$\begin{aligned}\langle q^\lambda q^\sigma q^\alpha q^\beta \dot{q}^\mu \dot{q}^\nu\rangle =& \langle q^\lambda q^\sigma\rangle[\langle q^\alpha q^\beta\rangle\langle \dot{q}^\mu \dot{q}^\nu\rangle + \langle q^\alpha \dot{q}^\mu\rangle\langle q^\beta \dot{q}^\nu\rangle + \langle q^\alpha \dot{q}^\nu\rangle\langle q^\beta \dot{q}^\mu\rangle]\\ &+\langle q^\lambda q^\alpha\rangle[\langle q^\sigma q^\beta\rangle\langle \dot{q}^\mu \dot{q}^\nu\rangle + \langle q^\sigma \dot{q}^\mu\rangle\langle q^\beta \dot{q}^\nu\rangle + \langle q^\sigma \dot{q}^\nu\rangle\langle q^\beta \dot{q}^\mu\rangle]\\ &+\langle q^\lambda q^\beta\rangle[\langle q^\sigma q^\alpha\rangle\langle \dot{q}^\mu \dot{q}^\nu\rangle + \langle q^\sigma \dot{q}^\mu\rangle\langle q^\alpha \dot{q}^\nu\rangle + \langle q^\sigma \dot{q}^\nu\rangle\langle q^\alpha \dot{q}^\mu\rangle]\\ &+\langle q^\lambda \dot{q}^\mu\rangle[\langle q^\sigma q^\alpha\rangle\langle q^\beta \dot{q}^\nu\rangle + \langle q^\sigma q^\beta\rangle\langle q^\alpha \dot{q}^\nu\rangle + \langle q^\sigma \dot{q}^\nu\rangle\langle q^\alpha q^\beta\rangle]\\ &+\langle q^\lambda \dot{q}^\nu\rangle[\langle q^\sigma q^\alpha\rangle\langle q^\beta \dot{q}^\mu\rangle + \langle q^\sigma q^\beta\rangle\langle q^\alpha \dot{q}^\mu\rangle + \langle q^\sigma \dot{q}^\mu\rangle\langle q^\alpha q^\beta\rangle],\end{aligned} \tag{B.10}$$

$$\begin{aligned}\langle q^\lambda q^\sigma q^\alpha q^\beta a^\mu a^\nu\rangle =& \langle q^\lambda q^\sigma\rangle\langle q^\alpha q^\beta\rangle\langle a^\mu a^\nu\rangle + \langle q^\lambda q^\alpha\rangle\langle q^\sigma q^\beta\rangle\langle a^\mu a^\nu\rangle\\ &+\langle q^\lambda q^\beta\rangle\langle q^\sigma q^\alpha\rangle\langle a^\mu a^\nu\rangle\end{aligned} \tag{B.11}$$

$$\begin{aligned}\langle q^\lambda q^\sigma q^\alpha q^\beta b^\mu c^\nu\rangle =& \langle q^\lambda q^\sigma\rangle\langle q^\alpha q^\beta\rangle\langle b^\mu c^\nu\rangle + \langle q^\lambda q^\alpha\rangle\langle q^\sigma q^\beta\rangle\langle b^\mu c^\nu\rangle\\ &+\langle q^\lambda q^\beta\rangle\langle q^\sigma q^\alpha\rangle\langle b^\mu c^\nu\rangle\end{aligned} \tag{B.12}$$

$$\begin{aligned}\langle q^\lambda q^\sigma q^\mu \dot{q}^\nu \lambda^a\bar{\lambda}^b\rangle =& \langle q^\lambda q^\sigma\rangle\langle q^\mu \dot{q}^\nu\rangle\langle \lambda^a\bar{\lambda}^b\rangle + \langle q^\lambda q^\mu\rangle\langle q^\sigma \dot{q}^\nu\rangle\langle \lambda^a\bar{\lambda}^b\rangle\\ &+\langle q^\lambda \dot{q}^\nu\rangle\langle q^\sigma q^\mu\rangle\langle \lambda^a\bar{\lambda}^b\rangle\end{aligned} \tag{B.13}$$

$$\langle q^\mu q^\nu \lambda^a\bar{\lambda}^b\rangle = \langle q^\mu q^\nu\rangle\langle \lambda^a\bar{\lambda}^b\rangle. \tag{B.14}$$

In the following we proceed with all the computations:

$$\begin{aligned} i)\, -\frac{3}{16}\beta\nabla_\mu\nabla_\nu R\int_0^1 d\tau\langle q^\mu q^\nu\rangle =& \frac{3}{16}\beta^2\nabla^2 R\int_0^1 d\tau\Delta|_\tau = \frac{3}{16}\beta^2\nabla^2 R\int_0^1 d\tau\tau(\tau-1)\\ =& -\frac{\beta^2}{32}\nabla^2 R.\end{aligned} \tag{B.15}$$

$$\begin{aligned} &ii)\, -\frac{1}{4}\nabla_\mu\nabla_\nu R_{ab}\int_0^1 d\tau\langle q^\mu q^\nu\rangle\langle \lambda^a\bar{\lambda}^b\rangle = \frac{\beta^2}{4}\nabla^2 R\int_0^1 d\tau\Delta|_\tau\Delta_F|_{\tau=\sigma}\\ &= \frac{\beta^2}{4}\nabla^2 R\int_0^1 d\tau\tau(\tau-1)\frac{i}{2}\tan\frac{\theta}{2} = -\frac{\beta^2}{48}i\nabla^2 R\tan\frac{\theta}{2}.\end{aligned} \tag{B.16}$$

$$iii)\ \frac{1}{\beta}\left[\frac{1}{8}\nabla_\lambda\nabla_\sigma R_{\mu\nu ab}+\frac{1}{24}R^\tau{}_{\mu\lambda\nu}R_{\sigma\tau ab}\right]\int_0^1 d\tau\langle q^\lambda q^\sigma q^\mu \dot{q}^\nu \lambda^a\bar{\lambda}^b\rangle = 0 \tag{B.17}$$

$$\begin{aligned} iv)\ &\frac{1}{\beta}\left[\frac{1}{40}\nabla_\lambda\nabla_\sigma R_{\alpha\mu\nu\beta}+\frac{1}{45}R_{\tau\lambda\sigma\mu}R^\tau{}_{\alpha\beta\nu}\right]\int_0^1 d\tau(\langle q^\lambda q^\sigma q^\alpha q^\beta a^\mu a^\nu\rangle + \langle q^\lambda q^\sigma q^\alpha q^\beta b^\mu c^\nu\rangle) \\ =&\beta^2\left(\frac{1}{40}\nabla^2 R+\frac{1}{20}\nabla_\lambda\nabla_\sigma R^{\lambda\sigma}-\frac{1}{45}R_{\mu\nu}R^{\mu\nu}-\frac{3}{45}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right)\mathbf{I}_1\,, \end{aligned} \tag{B.18}$$

where we used the identity (2.54) and $\mathbf{I}_1$ stands for the following integral

$$\mathbf{I}_1 = \int_0^1 d\tau\Delta|_\tau\Delta|_\tau\Delta_{gh}|_\tau. \tag{B.19}$$

$$\begin{aligned} v)\ &\frac{1}{\beta}\left[\frac{1}{40}\nabla_\lambda\nabla_\sigma R_{\alpha\mu\nu\beta}+\frac{1}{45}R_{\tau\lambda\sigma\mu}R^\tau{}_{\alpha\beta\nu}\right]\int_0^1 d\tau(\langle q^\lambda q^\sigma q^\alpha q^\beta \dot{q}^\mu \dot{q}^\nu\rangle \\ =&\beta^2\left(\frac{1}{40}\nabla^2 R-\frac{1}{45}R_{\mu\nu}R^{\mu\nu}+\frac{1}{20}\nabla_\lambda\nabla_\sigma R^{\lambda\sigma}-\frac{1}{30}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right)(\mathbf{I}_2-\mathbf{I}_3) \end{aligned} \tag{B.20}$$

where the integrals $\mathbf{I}_2$ and $\mathbf{I}_3$ stand for

$$\mathbf{I}_2 = \int_0^1 d\tau\Delta|_\tau\Delta|_\tau{}^\bullet\Delta^\bullet|_\tau\,,\quad \mathbf{I}_3 = \int_0^1 d\tau\Delta|_\tau{}^\bullet\Delta|_\tau\Delta^\bullet|_\tau. \tag{B.21}$$

Using all the above results we get

$$\langle S_6\rangle = -\frac{\beta^2}{32}\nabla^2 R-\frac{\beta^2}{48}i\tan\frac{\theta}{2}\nabla^2 R+\beta^2\left[\frac{1}{20}\nabla^2 R-\frac{1}{45}R_{\mu\nu}R^{\mu\nu}-\frac{1}{30}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right](\mathbf{I}_2+\mathbf{I}_1-\mathbf{I}_3). \tag{B.22}$$

Computing the three integrals, i.e. $\mathbf{I}_3 = -\frac{1}{120}$ and $\mathbf{I}_2+\mathbf{I}_1 = \frac{1}{30}$, and replacing the results in the above equation we get

$$\langle S_6\rangle = \beta^2\left[-\frac{7}{240}\nabla^2 R-\frac{i}{48}\tan\frac{\theta}{2}\nabla^2 R-\frac{1}{1080}R_{\mu\nu}R^{\mu\nu}-\frac{1}{720}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right] \tag{B.23}$$

The final effort is the evaluation of $\langle S_4^2\rangle$. In order to avoid confusing algebra we will use the following notation to indicate the various terms of the expandend S_4^2:

$$S_4 = \mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D}, \tag{B.24}$$

where

$$
\begin{aligned}
\mathbf{A} =& \frac{1}{6\beta} R_{\lambda\mu\nu\sigma} \int_0^1 d\tau q^\lambda q^\sigma (\dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu)\,, \\
\mathbf{B} =& \frac{1}{2\beta} R_{\mu\nu ab} \int_0^1 d\tau q^\mu \dot{q}^\nu \lambda^a \bar{\lambda}^b\,, \\
\mathbf{C} =& -\frac{1}{2} R_{ab} \int_0^1 d\tau \lambda^a \bar{\lambda}^b\,, \quad \mathbf{D} = -\frac{3}{8}\beta R.
\end{aligned} \tag{B.25}
$$

Let us proceed with the computations using this notation:

$$
\langle \mathbf{D}^2 \rangle = \frac{9}{64} \beta^2 R^2\,, \tag{B.26}
$$

$$
\langle 2\mathbf{CD} \rangle = \frac{3}{16} \beta^2 R^2 i \tan\frac{\theta}{2}\,, \tag{B.27}
$$

$$
\langle 2\mathbf{BD} \rangle = 0\,, \tag{B.28}
$$

$$
\langle 2\mathbf{AD} \rangle = -\frac{1}{32} \beta^2 R^2\,, \tag{B.29}
$$

$$
\begin{aligned}
\langle \mathbf{C}^2 \rangle =& \frac{1}{4} R_{ab} R^{cd} \int_0^1 \int_0^1 d\tau d\sigma \langle \lambda^a(\tau) \bar{\lambda}^b(\tau) \lambda_c(\sigma) \bar{\lambda}_d(\sigma) \rangle \\
=& \frac{1}{4} R_{ab} R^{cd} \int_0^1 \int_0^1 d\tau d\sigma (\langle \lambda^a(\tau) \bar{\lambda}^b(\tau) \rangle \langle \lambda_c(\sigma) \bar{\lambda}_d(\sigma) \rangle - \langle \lambda^a(\tau) \bar{\lambda}_d(\sigma) \rangle \langle \lambda_c(\sigma) \bar{\lambda}^b(\tau) \rangle) \\
=& \frac{\beta^2}{4} \left(R^2 \int_0^1 \int_0^1 d\tau d\sigma \Delta_F^2|_{\tau=\sigma} - R_{ab} R^{ab} \int_0^1 \int_0^1 d\tau d\sigma \Delta_F(\tau - \sigma) \Delta_F(\sigma - \tau) \right) \\
=& \frac{\beta^2 R^2}{4} \left(-\frac{1}{4} \cos^{-2}\frac{\theta}{2} + \frac{1}{4} \right) + \frac{\beta^2}{4} R_{ab} R^{ab} \frac{1}{4} \cos^{-2}\frac{\theta}{2}\,,
\end{aligned} \tag{B.30}
$$

where we performed Wick contractions for correlators with fermionic fields and we used the identity $\tan^2\frac{\theta}{2} = \cos^{-2}\frac{\theta}{2} - 1$;

$$
\langle 2\mathbf{BC} \rangle = -\frac{1}{\beta} R_{\mu\nu ab} R_{cd} \int_0^1 \int_0^1 d\tau d\sigma \langle q^\mu(\tau) \dot{q}^\nu(\tau) \lambda^a(\tau) \bar{\lambda}^b(\tau) \lambda^c(\sigma) \bar{\lambda}^d(\sigma) \rangle = 0\,, \tag{B.31}
$$

$$
\langle 2\mathbf{AC} \rangle = -\frac{1}{6\beta} R_{\lambda\mu\nu\sigma} R_{ab} \int_0^1 \int_0^1 d\tau d\sigma \langle q^\lambda q^\sigma (\dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu) \lambda^a \bar{\lambda}^b \rangle = -\frac{i}{48} \beta^2 R^2 \tan\frac{\theta}{2}\,, \tag{B.32}
$$

$$
\begin{aligned}
\langle \mathbf{B}^2 \rangle =& \frac{1}{4\beta^2} R_{\mu\nu ab} R_{\lambda\sigma cd} \int_0^1 \int_0^1 d\tau d\sigma \langle q^\mu(\tau) \dot{q}^\nu(\tau) \lambda^a(\tau) \bar{\lambda}^b(\tau) q^\lambda(\sigma) \dot{q}^\sigma(\sigma) \lambda^c(\sigma) \bar{\lambda}^d(\sigma) \rangle \\
=& \frac{\beta^2}{4} R_{\mu\nu ab} R^{\mu\nu ab} \mathbf{I}_4
\end{aligned}
\tag{B.33}
$$

where

$$
\mathbf{I}_4 = \int_0^1 \int_0^1 d\tau d\sigma ({\Delta^\bullet}{\Delta^\bullet} - {^\bullet\Delta}{\Delta^\bullet}) \Delta_F(\tau - \sigma) \Delta_F(\sigma - \tau). \tag{B.34}
$$

For the last integral *dimensional regularization* is needed because of the delta function contained in ${^\bullet\Delta^\bullet}$ which multiplies the step functions present in Δ_F, indeed these products of distributions are ambiguous and must be regularized. Let us extend the integral to the non-compact $d + 1$ dimensional space and perform an integration by parts:

$$
\begin{aligned}
\mathbf{I}_4 \xrightarrow{d+1} & \int d^{d+1}t \int d^{d+1}s [{}_\alpha\Delta_\beta(t,s)\Delta(t,s) - {}_\alpha\Delta(t,s)\Delta_\beta(t,s)] \mathrm{tr}[\gamma^\alpha \Delta_F(t-s) \gamma^\beta \Delta_F(s-t)] \\
=& -2 \int d^{d+1}t \int d^{d+1}s [{}_\alpha\Delta(t,s)\Delta_\beta(t,s)] \mathrm{tr}[\gamma^\alpha \Delta_F(t-s) \gamma^\beta \Delta_F(s-t)] \\
&+ \int d^{d+1}t \int d^{d+1}s \Delta_\beta(t,s)\Delta(t,s) \mathrm{tr}\Bigg[\left(\gamma^\alpha \frac{\partial}{\partial t^\alpha} \Delta_F(t-s)\right) \gamma^\beta \Delta_F(s-t) \\
&+ \Delta_F(t-s)\gamma^\beta \left(\Delta_F(s-t) \frac{\overleftarrow{\partial}}{\partial t^\alpha} \gamma^\alpha\right)\Bigg].
\end{aligned}
\tag{B.35}
$$

One can add a "mass term" $i\theta$ for free in order to obtain the Dirac eq., i.e.

$$
\left(\gamma^\alpha \frac{\partial}{\partial t^\alpha} + i\theta\right) \Delta_F(t-s) = \Delta_F(t-s) \left(-\gamma^\beta \frac{\overleftarrow{\partial}}{\partial s^\beta} + i\theta\right) = \delta_F(\tau - \sigma)\delta^d(\mathbf{t} - \mathbf{s}), \tag{B.36}
$$

to get

$$
2 \int d^{d+1}t \Delta_\beta|_t \Delta|_t \mathrm{tr}[\gamma^\beta \Delta_F|_{t=s}] \xrightarrow{d\to 0} 2 \int_0^1 d\tau \Delta_F|_{\tau=\sigma} \Delta^\bullet|_\tau \Delta|_\tau = 0. \tag{B.37}
$$

For the computation of $\mathbf{I}_4$ we need to evaluate the following term

$$
\begin{aligned}
\mathbf{I}_4 =& -2 \int d^{d+1}t \int d^{d+1}s \{[{}_\alpha\Delta(t,s)\Delta_\beta(t,s)] \mathrm{tr}[\gamma^\alpha \Delta_F(t-s) \gamma^\beta \Delta_F(s-t)]\} \\
& \xrightarrow{d\to 0} -2 \int_0^1 d\tau \int_0^1 d\sigma {^\bullet\Delta}{\Delta^\bullet} \Delta_F(\tau-\sigma)\Delta_F(\sigma-\tau) = -\frac{1}{24} \cos^{-2}\frac{\theta}{2}.
\end{aligned}
\tag{B.38}
$$

Using the above result of the integral we get

$$
\langle \mathbf{B}^2 \rangle = -\frac{\beta^2}{96} R_{\mu\nu ab} R^{\mu\nu ab} \cos^{-2}\frac{\theta}{2}. \tag{B.39}
$$

The final step is the computation of $\langle \mathbf{A}^2 \rangle$ which requires rather consistent algebra:

$$
\begin{aligned}
\langle \mathbf{A}^2 \rangle = \frac{1}{36\beta^2} R_{\lambda\mu\nu\sigma} R_{\alpha\beta\gamma\delta} \int_0^1 d\tau \int_0^1 d\sigma & \langle q^\lambda(\tau) q^\sigma(\tau) \dot{q}^\mu(\tau) \dot{q}^\nu(\tau) q^\alpha(\sigma) q^\delta(\sigma) \dot{q}^\beta(\sigma) \dot{q}^\gamma(\sigma) \rangle \\
& + \langle q^\lambda(\tau) q^\sigma(\tau) \dot{q}^\mu(\tau) \dot{q}^\nu(\tau) q^\alpha(\sigma) q^\delta(\sigma) a^\beta(\sigma) a^\gamma(\sigma) \rangle \\
& + \langle q^\lambda(\tau) q^\sigma(\tau) \dot{q}^\mu(\tau) \dot{q}^\nu(\tau) q^\alpha(\sigma) q^\delta(\sigma) b^\beta(\sigma) c^\gamma(\sigma) \rangle \\
& + \langle q^\lambda(\tau) q^\sigma(\tau) a^\mu(\tau) a^\nu(\tau) q^\alpha(\sigma) q^\delta(\sigma) \dot{q}^\beta(\sigma) \dot{q}^\gamma(\sigma) \rangle \\
& + \langle q^\lambda(\tau) q^\sigma(\tau) a^\mu(\tau) a^\nu(\tau) q^\alpha(\sigma) q^\delta(\sigma) a^\beta(\sigma) a^\gamma(\sigma) \rangle \\
& + \langle q^\lambda(\tau) q^\sigma(\tau) a^\mu(\tau) a^\nu(\tau) q^\alpha(\sigma) q^\delta(\sigma) b^\beta(\sigma) c^\gamma(\sigma) \rangle \\
& + \langle q^\lambda(\tau) q^\sigma(\tau) b^\mu(\tau) c^\nu(\tau) q^\alpha(\sigma) q^\delta(\sigma) \dot{q}^\beta(\sigma) \dot{q}^\gamma(\sigma) \rangle \\
& + \langle q^\lambda(\tau) q^\sigma(\tau) b^\mu(\tau) c^\nu(\tau) q^\alpha(\sigma) q^\delta(\sigma) a^\beta(\sigma) a^\gamma(\sigma) \rangle \\
& + \langle q^\lambda(\tau) q^\sigma(\tau) b^\mu(\tau) c^\nu(\tau) q^\alpha(\sigma) q^\delta(\sigma) b^\beta(\sigma) c^\gamma(\sigma) \rangle .
\end{aligned}
\tag{B.40}
$$

As we can see the first term is an 8-point function that using Wick theorem contains $7!! = 105$ terms. By computing all these terms and all those of the other 8-point functions, after laborious algebra we get the following result

$$
\begin{aligned}
\langle \mathbf{A}^2 \rangle = \frac{\beta^2}{36} \int_0^1 \int_0^1 d\tau d\sigma \{ & R^2 [({}^{\bullet}\Delta^{\bullet} + \Delta_{gh})|_\tau \Delta|_\tau \Delta|_\sigma ({}^{\bullet}\Delta^{\bullet} + \Delta_{gh})|_\sigma + \Delta^{\bullet}|_\tau^2 \Delta^{\bullet}|_\sigma^2 \\
& \quad - 2(\Delta^{\bullet}|_\tau^2 \Delta|_\sigma \Delta_{gh}|_\sigma + \Delta^{\bullet}|_\sigma^2 \Delta|_\tau {}^{\bullet}\Delta^{\bullet}|_\tau)] \\
& + R_{\mu\nu} R^{\mu\nu} [2\Delta|_\tau (({}^{\bullet}\Delta^{\bullet})^2 - \Delta_{gh}^2) \Delta|_\sigma + 2({}^{\bullet}\Delta^{\bullet} + \Delta_{gh})|_\tau \Delta^2 ({}^{\bullet}\Delta^{\bullet} + \Delta_{gh})|_\sigma \\
& \quad + 2\Delta|_\tau {}^{\bullet}\Delta^2 ({}^{\bullet}\Delta^{\bullet} + \Delta_{gh})|_\sigma + 2({}^{\bullet}\Delta^{\bullet} + \Delta_{gh})|_\tau (\Delta^{\bullet})^2 \Delta|_\sigma \\
& \quad - 4\Delta^{\bullet}|_\tau \Delta ({}^{\bullet}\Delta)({}^{\bullet}\Delta^{\bullet} + \Delta_{gh})|_\sigma - 4({}^{\bullet}\Delta^{\bullet} + \Delta_{gh})|_\tau \Delta (\Delta^{\bullet}) \Delta^{\bullet}|_\sigma \\
& \quad - 4\Delta|_\tau ({}^{\bullet}\Delta)({}^{\bullet}\Delta^{\bullet}) \Delta^{\bullet}|_\sigma - 4\Delta^{\bullet}|_\tau \Delta^{\bullet} ({}^{\bullet}\Delta^{\bullet}) \Delta|_\sigma \\
& \quad + 4\Delta^{\bullet}|_\tau \Delta ({}^{\bullet}\Delta^{\bullet}) \Delta^{\bullet}|_\sigma + 4\Delta^{\bullet}|_\tau \Delta^{\bullet} ({}^{\bullet}\Delta) \Delta^{\bullet}|_\sigma] \\
& + R_{\mu\nu\alpha\beta}^2 [-3\Delta^2 \Delta_{gh}^2 + 3\Delta^2 ({}^{\bullet}\Delta^{\bullet})^2 - 6(\Delta^{\bullet}) \Delta ({}^{\bullet}\Delta^{\bullet}) {}^{\bullet}\Delta + 3(\Delta^{\bullet})^2 ({}^{\bullet}\Delta)^2] \} \\
= \frac{\beta^2}{36} \left(\frac{1}{16} R^2 - \frac{1}{6} R_{\mu\nu} R^{\mu\nu} \right) . &
\end{aligned}
\tag{B.41}
$$

The final result of $\langle S_4^2 \rangle$ is thus given by

$$
\begin{aligned}
\langle S_4^2 \rangle = \beta^2 \Bigg[& \left(\frac{25}{144} + \frac{1}{6} i \tan \frac{\theta}{2} - \frac{1}{16} \cos^{-2} \frac{\theta}{2} \right) R^2 + \left(-\frac{1}{216} + \frac{1}{16} \cos^{-2} \frac{\theta}{2} \right) R_{\mu\nu} R^{\mu\nu} \\
& - \frac{1}{96} \cos^{-2} \frac{\theta}{2} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \Bigg]
\end{aligned}
\tag{B.42}
$$

Finally we have all the ingredients to compute $\langle e^{-S_{int}} \rangle = 1 - \langle S_4 \rangle - \langle S_6 \rangle + \frac{1}{2} \langle S_4^2 \rangle$, which result is exaclty (3.90).

B.2 Modular integrals

In this section we present for completeness the computation of the three modular integrals in (3.91).

Let us start with $\mathbf{I}_1$ by writing $\cos\frac{\theta}{2}$ in terms of complex exponentials $e^{i\frac{\theta}{2}}$ and performing a change of variable $z = e^{i\theta}$:

$$\mathbf{I}_1 = \int_0^{2\pi} \frac{d\theta}{2\pi} \left(2\cos\frac{\theta}{2}\right)^D e^{-i(1-\frac{D}{2})\theta} = \int_0^{2\pi} \frac{d\theta}{2\pi} (e^{i\theta}+1)^D e^{-i\theta} = -i\int_{\mathcal{C}} \frac{dz}{2\pi} \frac{(z+1)^D}{z^2} = D\,, \tag{B.43}$$

where in the last step the residue theorem has been used. We proceed similarly with the other modular integrals

$$\begin{aligned}\mathbf{I}_2 &= \int_0^{2\pi} \frac{d\theta}{2\pi} \left(2\cos\frac{\theta}{2}\right)^D e^{-i(1-\frac{D}{2})\theta} \tan\frac{\theta}{2} = -i\int_0^{2\pi} \frac{d\theta}{2\pi} (e^{i\theta}+1)^{D-1}(1-e^{-i\theta}) \\ &= -\int_{\mathcal{C}} \frac{dz}{2\pi} \frac{(z-1)(z+1)^{D-1}}{z^2} = i(D-2)\,;\end{aligned} \tag{B.44}$$

$$\begin{aligned}\mathbf{I}_3 &= \int_0^{2\pi} \frac{d\theta}{2\pi} \left(2\cos\frac{\theta}{2}\right)^D e^{-i(1-\frac{D}{2})\theta} \cos^{-2}\frac{\theta}{2} = 4\int_0^{2\pi} \frac{d\theta}{2\pi} (e^{i\theta}+1)^{D-2} \\ &= \frac{4}{2\pi i}\int_{\mathcal{C}} dz \frac{(z+1)^{D-2}}{z} = 4.\end{aligned} \tag{B.45}$$

Appendix C

Coherent states for rank-2 tensors

Consider the fermionic operators ψ^{ab} and $\bar{\psi}^{ab}$ with non-vanishing trace. Under canonical quantization they satisfy anticommutation relations, i.e.

$$\{\psi^{ab}, \bar{\psi}^{cd}\} = \delta^{ac}\delta^{bd} + \delta^{bc}\delta^{ad}\,, \tag{C.1}$$

where $a, b, c, d, = 1, \dots, D$ are flat indices. Considering the ψ's and $\bar{\psi}$'s respectively as creation and annihilation operators with respect to the vacuum $|0\rangle$, it is possible to define fermionic coherent states:

$$|\eta\rangle = e^{-\bar{\eta}_{ab}\psi^{ab}/2}\,|0\rangle\ , \quad \langle\xi| = \langle 0|\, e^{\xi^{ab}\bar{\psi}_{ab}/2}\,, \tag{C.2}$$

that obey the following relations

$$\bar{\psi}^{ab}\,|\bar{\eta}\rangle = \bar{\eta}^{ab}\,|\bar{\eta}\rangle\ , \quad \langle\xi|\,\psi^{ab} = \langle\xi|\,\xi^{ab}. \tag{C.3}$$

The above coherent states are normalized as

$$\langle\xi|\bar{\eta}\rangle = e^{\xi^{ab}\bar{\eta}_{ab}/2}. \tag{C.4}$$

For path integral construction with coherent states Lorentz invariant tensors are required. The latters are built from δ_{ab} and $\epsilon_{a_1\dots a_D}$, i.e. $Z_{(ab)_1\cdots(ab)_N}$, with $N = \frac{1}{2}D(D+1)$. The tensors $Z_{(ab)_1\cdots(ab)_N}$ so constructed are symmetric in each couple of indices $(ab)_k$ and antisymmetric by couples exchange. A simple example can be studied for $D = 2$, i.e.

$$Z_{(ab)(cd)} \propto \epsilon_{ac}\delta_{bd} + \epsilon_{ad}\delta_{bc} + \epsilon_{bd}\delta_{ac} + \epsilon_{bc}\delta_{ad}. \tag{C.5}$$

Therefore the measures for the integrals are

$$d\xi = Z_{(ab)_1\cdots(ab)_N} d\xi^{(ab)_1}\dots d\xi^{(ab)_N}\,, \quad d\bar{\eta} = Z_{(ab)_1\cdots(ab)_N} d\bar{\eta}^{(ab)_N}\dots d\bar{\eta}^{(ab)_1} \tag{C.6}$$

with the identity $d\xi d\bar{\eta} = (-1)^N d\bar{\eta} d\xi$. Using coherent states the following identities hold

$$\begin{aligned}
&\int d\xi d\bar{\eta} e^{-\frac{1}{2}\xi^{ab}\bar{\eta}_{ab}} = 1 \\
&\int d\xi d\bar{\eta} e^{-\frac{1}{2}\xi^{ab}\bar{\eta}_{ab}} |\bar{\eta}\rangle\langle\xi| = \mathbb{1} \\
&\mathrm{Tr} A = \int d\xi d\bar{\eta} e^{-\frac{1}{2}\xi^{ab}\bar{\eta}_{ab}} \langle -\xi|A|\bar{\eta}\rangle = \int d\bar{\eta} d\xi e^{\frac{1}{2}\xi^{ab}\bar{\eta}_{ab}} \langle\xi|A|\bar{\eta}\rangle.
\end{aligned} \tag{C.7}$$

Bibliography

[1] B. De Witt, "The Space-time Approach to Quantum Field Theory," in *Relativity, Groups and Topology II, Les Houches (1983)*, ed. B. De Witt and R. Stora, (Amsterdam: Elsevier, 1984) 381–738.

[2] B. De Witt, "The Global Approach to Quantum Field Theory, Volume 1", (Oxford: Oxford University Press, 2003).

[3] B. De Witt, "The Global Approach to Quantum Field Theory, Volume 2", (Oxford: Oxford University Press, 2003).

[4] F. Bastianelli and R. Bonezzi , "One-loop quantum gravity from a worldline viewpoint," JHEP **1307** (2013) 016 [arXiv:1304.7135].

[5] R. Bonezzi, A. Meyer and I. Sachs, JHEP **10** (2018), 025 doi:10.1007/JHEP10(2018)025 [arXiv:1807.07989 [hep-th]].

[6] R. Bonezzi, A. Meyer and I. Sachs, JHEP **06** (2020), 103 doi:10.1007/JHEP06(2020)103 [arXiv:2004.06129 [hep-th]].

[7] F. Bastianelli, R. Bonezzi, O. Corradini and E. Latini, JHEP **11** (2019), 124 doi:10.1007/JHEP11(2019)124 [arXiv:1909.05750 [hep-th]].

[8] D. V. Vassilevich, "Heat kernel expansion: user's manual", *Phys.Rept*, **388** (2003) 279-360 [hep-th/0306138]

[9] V. A. Fock, "The proper time in classical and quantum mechanics," *Izv. Akad. Nauk. USSR (Phys.)*, **4-5** (1937) 551-568.

[10] J. Schwinger, "On gauge invariance and vacuum polarization," *Phys. Rev.*, **82** (1951) 664-679.

[11] J. Yepez, "Einstein's vierbein field theory of curved space," USAF-TECHNICAL-REPORT-AFRL-RV-HA-TR-2011-1024 (2011) [arXiv:1106.2037].

[12] P. Gilkey, “The spectral geometry of a Riemannian manifold,” *J. Diff. Geom.* **10** (1975) 601–618.

[13] G. ’t Hooft and M. J. G. Veltman, “One loop divergencies in the theory of gravitation,” Ann. Inst. H. Poincare Phys. Theor. A **20** (1974), 69-94

[14] M. H. Goroff and A. Sagnotti, “The Ultraviolet Behavior of Einstein Gravity,” Nucl. Phys. B **266** (1986), 709-736

[15] S. M. Christensen, M. J. Duff, “Quantizing gravity with a cosmological constant,” *Nucl. Phys. B* **170** (1980) 480-506.

[16] I. G. Avramidi, S. J. Collopy, “One-Loop Quantum Gravity in the Einstein Universe,” JHEP **11** (2015) 193 [arXiv:1509.00929v2 [hep-th]].

[17] S. Hawking, “The path-integral approach to quantum gravity,” in: General Relativity, Eds. S. hawking and W. Israel, (Cambridge: Cambridge University Press, 1979), 746–789.

[18] F. Bastianelli, R. Bonezzi, O. Corradini, E. Latini , “Particles with non abelian charges,” JHEP **10** (2013) 098 [arXiv:1309.1608].

[19] F. Bastianelli, R. Bonezzi, O. Corradini, E. Latini, and K. H. Ould-Lahoucine, “A worldline approach to colored particles,” J. Phys. Conf. Ser. **1208** (2019), no. 1 012004, [arXiv:1504.03617].

[20] F. Bastianelli, “Path integrals in curved space and the worldline formalism,” proceedings of Path integrals: from quantum information to cosmology (Prague, 2005), [arXiv:hep-th/0508205].

[21] F. Bastianelli, P. van Nieuwenhuizen, “Path integrals and anomalies in curved space,” Cambridge University Press, Cambridge UK (2006).

[22] T.D. Lee and C.N. Yang,“Theory of Charged Vector Mesons Interacting with the Electromagnetic Field,” Phys. Rev. JHEP **128** (1962) 885.

[23] F. Bastianelli, P. Benincasa and S. Giombi, “Worldline approach to vector and antisymmetric tensor fields,” JHEP **0504** (2005) 010 [hep-th/0503155].

[24] F. Bastianelli,“The Path Integral for a Particle in Curved Spaces and Weyl Anomalies,” Nucl. Phys. B **376** (1992) 113 [hep-th/9112035].

[25] F. Bastianelli, O. Corradini, P. van Nieuwenhuizen,“Dimensional regularization of nonlinear sigma models on a finite time interval,” Phys. Lett. B **494** (2000) 161 [hep-th/0008045].

[26] F. Bastianelli, O. Corradini and A. Zirotti,“Dimensional regularization for N=1 supersymmetric sigma models and the worldline formalism,” Phys. Rev. D **67**, 104009 (2003) [hep-th/0211134].

[27] U. Muller, C. Schubert, A. M. E. van de Ven, “A closed formula for the Riemann normal coordinate expansion,” Gen. Rel. Grav. **31** (1999) 1759 [gr-qc/9712092].

[28] D. Brizuela, J. M. Martin-Garcia, G. A. Mena Marugan, “xPert: Computer algebra for metric perturbation theory,” Gen.Rel.Grav.41: 2415-2431 (2009) [arXiv:0807.0824].