Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
Corso di Laurea in Matematica

TUTTE'S 5-FLOW CONJECTURE

Tesi di Laurea magistrale in Teoria dei Grafi

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Abstract

L'obiettivo di questa tesi è quello di mostrare e collegare i risultati, ottenuti fino ad oggi, utili ad affrontare una congettura della teoria dei grafi, proposta nel 1954 da William Thomas Tutte. La congettura in questione è la Tutte's 5-flow conjecture, la quale afferma che ogni grafo senza ponti ammette un nowhere-zero 5-flow, cioè un flusso a valori interi non nulli compresi tra -4 e 4. Inizieremo dando delle nozioni di base sulla teoria dei grafi, utili per i teoremi successivi, e dimostrando alcuni risultati riguardo i flussi sui grafi orientati e in particolare sul polinomio di flusso. Successivamente tratteremo due casi: i grafi immergibili nel piano $\mathbb{R}^2$ e i grafi immergibili nel piano proiettivo $\mathbb{P}^2$. Nel primo caso vedermo la correlazione tra flussi e colorazioni e dimostreremo un teorema ancora più forte della congettura di Tutte, utilizzando il teorema dei 4 colori. Nel secondo caso invece vedremo come nel 1984 Richard Steinberg ha utilizzato lo Splitting Lemma di Fleischner per mostrare che non può esistere un controesempio minimale della congettura nel caso di grafi immergibili nel piano proiettivo. Nel quarto capitolo vedremo invece i teoremi di François Jaeger (1976) e Paul D. Seymour (1981). Il primo dimostrò che ogni grafo senza ponti ammette un nowhere-zero 8-flow, il secondo riuscì a scendere ulteriormente mostrando che oni grafo senza ponti ammette anche un nowhere-zero 6-flow. Nel quinto e ultimo capitolo invece ci sarà una piccola introduzione al polinomio di Tutte e verrà mostrato come quest'ultimo è collegato al polinomio di flusso tramite il Recipe Theorem. Infine vedremo alcune applicazioni dei flussi tramite lo studio dei network e delle loro proprietà.

Contents

Introduction

The aim of this thesis is to show and put together the results, obtained so far, useful to tackle a conjecture of graph theory proposed in 1954 by William Thomas Tutte. The conjecture in question is Tutte's 5-flow conjecture, which states that every bridgeless graph admits a nowhere-zero 5-flow, namely a flow with non-zero integer values between -4 and 4. We will start by giving some basics on graph theory, useful for the followings, and proving some results about flows on oriented graphs and in particular about the flow polynomial. Next we will treat two cases: graphs embeddable in the plane $\mathbb{R}^2$ and graphs embeddable in the projective plane $\mathbb{P}^2$. In the first case we will see the correlation between flows and colorings and prove a theorem even stronger than Tutte's conjecture, using the 4-color theorem. In the second case we will see how in 1984 Richard Steinberg used Fleischner's Splitting Lemma to show that there can be no minimal counterexample of the conjecture in the case of graphs in the projective plane. In the fourth chapter we will look at the theorems of François Jaeger (1976) and Paul D. Seymour (1981). The former proved that every bridgeless graph admits a nowhere-zero 8-flow, the latter managed to go even further showing that every bridgeless graph admits a nowhere-zero 6-flow. In the fifth and final chapter there will be a short introduction to the Tutte polynomial and it will be shown how it is related to the flow polynomial via the Recipe Theorem. Finally we will see some applications of flows through the study of networks and their properties.

Chapter 1

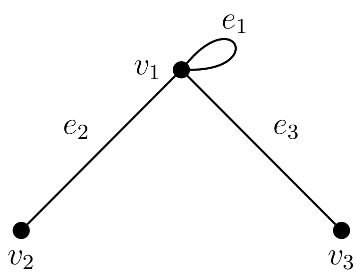
Graph theory preliminaries

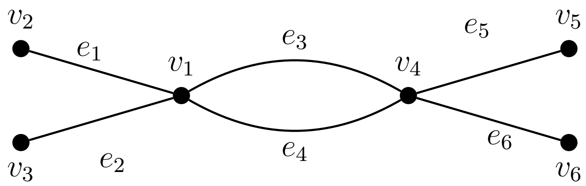
In this first chapter, I will give the basic notions and results needed to fully understand the thesis.

1.1 First definitions

We start with some basic definitions.

Definition (Simple graph, multigraph and digraph)**.** *A **simple graph**, or undirected simple graph, is a pair $G = (V, E)$, where V is a set of elements named vertices, and E is a subset of $\{(v, w)|v, w \in V, v \neq w\}$. We will say that $(v, w) \in E$ is an edge e joining v and w, and we will write $e = vw$.*
*We will also say that two vertices v, w are adjacent if there exists an edge e joining them. In this definition multiple edges joining two vertices and edges joining a vertex to itself, named loops, are not allowed. If we allow these kind of edges in our graph, we will call it an undirected multigraph, or simply a graph. A **digraph**, or directed graph, is a graph where each edge has an orientation from one vertex to the other.*





These two graphs are not simple, the first one has a loop edge, e_1, while the second has two multiple edges, e_3, e_4.

We will always consider graphs with a finite number of vertices and edges.
The **degree** of a vertex v, $d(v)$, is the number of edges incident with v.

Definition. *Let $G = (V, E)$ be a graph.*
We define the complement graph of G, $\bar{G} = (V, A)$, as the graph on the same set of vertices V of G such that two vertices of $\bar{G}$ are adjacent if and only if they are not adjacent in G.

Definition (Graph isomorphism)**.** *Let $G = (V, E), H = (U, A)$ be graphs.*
G and H are isomorphic if there exists a bijection $f : V \longrightarrow U$ such that any two vertices $u, v \in V$ are adjacent if and only if $f(u), f(v) \in U$ are adjacent.

Definition (Subgraph)**.** *Let $G = (V, E)$ be a graph. A* ***spanning subgraph*** *of G is a graph (V, A) with $A \subseteq E$. We will denote it by $G(E)$.*
An ***induced subgraph*** *of G is a graph (U, A) with $U \subseteq V$ and*
$A = \{(v, w) \in E | v, w \in U\}$. We will denote it by $G(U)$.

Let $G = (V, E)$ be a graph.
The **rank** of G is $r(G) = |V| - c(G)$, where $c(G)$ denotes the number of connected components of G, and the **nullity** of G is $n(G) = |E| - r(G)$.
Notation. If $G = (V, E)$ is a graph and $A \subseteq E$, we will denote the rank (nullity) of the spanning subgraph $G(A)$ by $r_G(A)$ ($n_G(A)$).

Let $e = vw \in E$ be an edge of G:
The **deletion** of e forms the spanning subgraph $G \backslash e = (V, E \backslash \{e\})$, the **contraction** of e forms the graph G/e obtained deleting e from G and identifying the vertices v and w.
The edge e is a **bridge** if $c(G \backslash e) > c(G)$.
A **stable set**, or **independent set**, on G is a set of its vertices with no adjacency.
$v \in V$ is an articulation point, or cutvertex, of G if the induced subgraph $G' = (U, A)$, with $U = V \backslash \{v\}$, has $c(G') > c(G)$.

Definition (Trail and path)**.** *Let $G = (V, E)$ be a graph. A* ***trail*** *on G is a sequence of distinct edges joining a sequence of vertices, with two consecutive edges having at least one vertex in common. If the vertices are also distinct, we call it a* ***path****.*

Definition (Tree and forest)**.** *A* ***tree*** *is a connected graph with no closed trail in it. Every tree with n vertices has n+1 edges.*
A ***forest*** *is a graph where every connected component is a tree. Every forest F has $r(F)$ edges.*

Example 1
The **complete graph** K_n is the graph with n vertices and an edge for each pair of

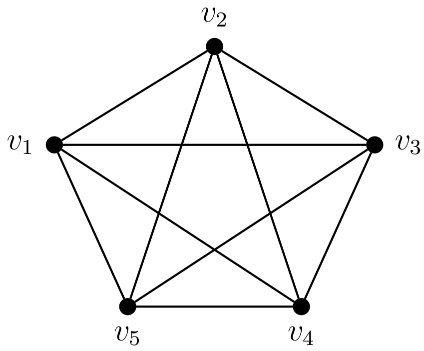


Figure 1.1: K_5

distinct vertices.

Example 2
A graph is **bipartite** if we can divide his vertices into two disjoint stable sets U and W.
If $|U| = n$, $|W| = m$, and each vertex of U is adjacent to each vertex of W, we call it a **complete bipartite graph** and denote it by $K_{n,m}$.

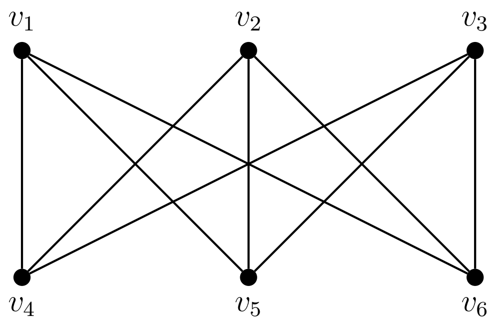


Figure 1.2: $K_{3,3}$

1.2 Flows on digraphs

As we saw on the very first definition, if $G = (V, E)$ is an undirected graph, we can provide its edges with an orientation ω to transform it into a digraph.
ω assigns a direction to each edge $uv \in E$, that could be either $u \to v$ or $v \to u$.
We will denote G with the orientation ω as G^ω.

For each $v \in V$, $\delta^+(\{v\})$ denotes the set of edges directed out of v, and $\delta^-(\{v\})$ the set of edges directed into v.

Definition (Nowhere-zero A-flow)**.** *Let A be an abelian group, a map $\phi : E \longrightarrow A$ that, $\forall v \in V$, satisfies:*

$$\sum_{e\in\delta^+(\{v\})} \phi(e) = \sum_{e\in\delta^-(\{v\})} \phi(e)$$

*is called **A-circulation**.*
*$Supp(\phi) := \{e \in E | \phi(e) \neq 0\}$ is named the **support** of ϕ.*
*If $Supp(\phi) = E$, we call ϕ a **nowhere-zero A-flow**.*
We say that an undirected graph $G = (V, E)$ admits a nowhere-zero A-flow if there exists an orientation ω on G where we can define a nowhere-zero A-flow.
We say that G admits a nowhere-zero k-flow if there exists an orientation ω on G where we can define a nowhere-zero $\mathbb{Z}$-flow ϕ with $0 < |\phi(e)| < k, \forall e \in E$.

Theorem. *Let $G = (V, E)$ be a digraph, $k \geq 2$.*
There exists a nowhere-zero $\mathbb{Z}_k$-flow on G $\Leftrightarrow$ there exists a nowhere-zero k-flow on G.

Proof. ($\Leftarrow$) If ϕ is a k-flow on G, we can just consider the natural group homomorphism $\sigma_k : \mathbb{Z} \longrightarrow \mathbb{Z}_k$ to obtain a $\mathbb{Z}_k$-flow on G, namely $\sigma_k \circ \phi := \psi$.
($\Rightarrow$) We can assume that G has no loops.
Let ψ be a nowhere-zero $\mathbb{Z}_k$-flow on G, and let F be the set of all functions $\phi : E \longrightarrow \mathbb{Z}$ with $|\phi(e)| < k$, $\forall e \in E$, and $\sigma_k \circ \phi = \psi$.
Note that each ϕ is nowhere-zero.
It is easy to see that $F \neq \emptyset$.
We want to find $\phi \in F$ such that $\phi(v, V) := \sum\limits_{e\in\delta^+(\{v\})} \phi(e) - \sum\limits_{e\in\delta^-(\{v\})} \phi(e) = 0, \forall v \in V$, where the sum is performed in the group $(\mathbb{Z}, +)$.
Consider $U, W \subseteq V$ and define $\phi(U, W) := \sum\limits_{u\in U} \phi(u, W)$.
Now, consider $\phi \in F$ for which the sum

$$K(\phi) := \sum_{v\in V} \left| \sum_{e\in\delta^+(\{v\})} \phi(e) - \sum_{e\in\delta^-(\{v\})} \phi(e) \right| = \sum_{v\in V} |\Phi(v, V)|$$

is least possible.
If $K(\phi) = 0$, then ϕ is clearly a nowhere-zero k-flow.
Suppose $K(\phi) \neq 0$. Since $\sum\limits_{v\in V} \phi(v, V) = 0$, there exists a vertex $u \in V$ with $\phi(u, V) > 0$.
Let $v_1, v_2 \in V$ be two adjacent vertices in G, $e = (v_1, v_2)$, define:

$$\bar{\phi}(v_1, v_2) := \begin{cases} \phi(e) & \text{if } e \in \delta^+(\{v_1\}) \\ -\phi(e) & \text{otherwise} \end{cases}$$

Let $U \subseteq V$ be the set of all vertices u' for which G contains a path $P = P(u, u') = u_0u_1u_2...u_{t-1}u_t$ from u to u' such that, for all $e_i = (u_i, u_{i+1})$ in the path P, we have $\bar{\phi}(u_i, u_{i+1}) > 0$.
Let $U' := U\backslash\{u\}$, we want to prove that there exists $u' \in U'$ with $\phi(u', V) < 0$.
By definition of U, $\bar{\phi}(u', v) \leq 0$ for all $e = (u', v)$ such that $u' \in U, v \in V\backslash U \Rightarrow \phi(U, V\backslash U) \leq 0 \Rightarrow \phi(U', V\backslash U) \leq 0$.
In particular, this holds for $u' = u$ and, since $\phi(u, V) > 0$, then $\phi(u, U') > 0$ and $\phi(U', U') = 0$.
Therefore $\sum\limits_{u \in U'} \phi(u', V) = \Phi(U', V) = \phi(U', V\backslash U) - \phi(u, U') + \phi(U', U') < 0$,
so there exists $u' \in U'$ with $\phi(u', V) < 0$.
We have that $u' \in U$, so there is a path $P' = P(u, u') = u_0u_1...u_{t-1}u_t$ from u to u' such that, for all $e_i = (u_i, u_{i+1})$ in the path P', we have $\bar{\phi}(u_i, u_{i+1}) > 0$.
We will now define a new function $\phi' : E \longrightarrow \mathbb{Z}$ as follows:

$$\phi'(e) := \begin{cases} \phi(e) + k & \text{if } e = e_i = u_iu_{i+1} \text{ and } u_i \text{ is directed toward } u_{i+1} \\ \phi(e) - k & \text{if } e = e_i = u_iu_{i+1} \text{ and } u_{i+1} \text{ is directed toward } u_i \\ \phi(e) & \text{otherwise} \end{cases}$$

By definition of P', $|\phi'(e)| < k$ for all $e \in E$, so $\phi' \in F$.
We now want to calculate $K(\phi')$.
Note that $\phi'(v, V) = \phi(v, V)$ for all $v \in V\backslash\{u, u'\}$.
However, for u and u', we have $\phi'(u, V) = \phi(u, V) - k$ and $\phi'(u', V) = \phi(u', V) + k$.
Since ψ is a $\mathbb{Z}_k$-flow and
1) $\sigma_k(\phi(u, V)) = \psi(u, V) = 0 \in \mathbb{Z}_k$
2) $\sigma_k(\phi(u', V)) = \psi(u', V) = 0 \in \mathbb{Z}_k$,
we have that $\phi(u, V)$ and $\phi(u', V)$ are multiple of k, but $\phi(u, V) > 0 \Rightarrow \phi(u, V) \geq k$ and $\phi(u', V) < 0 \Rightarrow \phi(u', V) \leq -k$.
Then $|\phi'(u, V)| < |\phi(u, V)|$ and $|\phi'(u', V)| < |\phi(u', V)|$.
So we have $K(\phi') < K(\phi)$, that is a contradiction on the choice of ϕ, and therefore $K(\phi) = 0$ and ϕ is a nowhere-zero k-flow. $\square$

Tutte also proved the following theorem.

Theorem. *Let $G = (V, E)$ be a graph, H, H' abelian groups of the same order $k \in \mathbb{N}$. G has a nowhere-zero H-flow $\Leftrightarrow$ G has a nowhere-zero H'-flow.*

Here we will focus on k-flows, and in particular we will see Tutte's 5-flow conjecture which states:

Conjecture (Tutte's 5-flow conjecture)**.** *Every bridgeless graph G has a nowhere-zero 5-flow.*

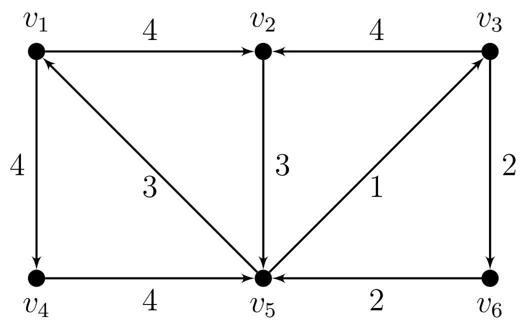


Figure 1.3: An example of a $\mathbb{Z}_5$ flow on a graph.

We will return on it later.

Now we want to define the flow polynomial, which calculates the number of nowhere-zero flows on a graph, but first we need to introduce the notion of signed characteristic vectors.

Definition (Cycle, circuit, cutset and bond)**.** *A* ***cycle*** *is a spanning subgraph of a graph G where all vertices have an even degree.*
A ***circuit*** *is a cycle that is minimal with respect to inclusion.*
A ***cutset*** *of $G = (V, E)$ is a subset of edges defined by a partition $(U, V \backslash U)$ of V as follows:*
Consider a partition $(U, V \backslash U)$ of V, the cutset defined by this partition is
$K := \{uv \in E | u \in U, v \in V \backslash U\}$.
A ***bond*** *is a cutset that is minimal with respect to inclusion.*
Note that if $U \subseteq V$, the set $\delta^+(U) \cup \delta^-(U)$ is a bond.
If $k \in \mathbb{N}$, a k-bond is a bond containing k edges.

Observation. A cutset with a single edge is always a 1-bond.

Definition (Spanning tree)**.** *Let $G = (V, E)$ be a graph.*
A spanning tree T of G is a spanning subgraph of G such that:

- *T is a tree.*
- *If $e \in E$ is not an edge of T, then $T \cup \{e\}$ is not a tree.*

Observation. If we have a graph G and a spanning tree T, adding an edge e to T creates a cycle.
If T is a spanning tree of a graph G, for each $e \in E \backslash T$, there is a unique circuit of G

contained in $T \cup \{e\}$. We'll denote it by C_{T+e}.

Let C be a circuit on a digraph G with an orientation ω. We can give it two possible cyclic orientations. Choose one of them and call it $\tilde{C}$, we say that $\tilde{C}$ is a **signed circuit**. Define C^+ to be the set of edges in $\tilde{C}$ with the same orientation of G^ω and C^- to be the set of the remaining edges of $\tilde{C}$.

Definition (Signed characteristic vector). *Let G^ω be a digraph and $\tilde{C}$ a signed circuit of G^ω.*
*The **signed characteristic vector** of $\tilde{C}$ is defined by*

$$\chi_C(e) = \begin{cases} 1 & \text{if } e \in C^+ \\ -1 & \text{if } e \in C^- \\ 0 & \text{otherwise} \end{cases}$$

We can similarly define the signed characteristic vector χ_B of a signed bond $\tilde{B}$.

Here we can see an important relationship between these two vectors.

Theorem. *Let $G = (V, E)$ be a graph, C a circuit and B a bond on G, then:*

$$\sum_{e \in E} \chi_B(e)\chi_C(e) = 0$$

Proof. Let $(U, V \backslash U)$ be a partition of V.
If $B = \{uv \in E | u \in U, v \in V \backslash U\}$, then $\sum_{e \in E} \chi_B(e)\chi_C(e)$ is the number of oriented edges of C going from U to $V \backslash U$, minus the number of oriented edges of C going from $V \backslash U$ to U. This number is equal to zero, because, in a closed trail, for each edge from U to $V \backslash U$, there is one edge from $V \backslash U$ to U. □

1.3 Incidence matrix and another way to define flows

Definition (Incidence matrix). *Let $G^\omega = (V, E)$ be an oriented graph and A be an abelian group.*
*The **incidence matrix** of G^ω is $D = (d_{v,e})$ such that*

$$d_{v,e} = \begin{cases} 1 & \text{if } e \text{ is directed out of } v \\ -1 & \text{if } e \text{ is directed into } v \\ 0 & \text{if } e \text{ is a loop on } v \text{, or is not incident with } v \end{cases}$$

For instance, an incidence matrix for the graph 1.3 is:

$$\begin{pmatrix} 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

The matrix D defines a homomorphism $D : A^E \longrightarrow A^V$ as follows. If ϕ is a map from E to A, then:

$$D\phi(v) = \sum_{\{e=uv|v\to u\}} \phi(e) - \sum_{\{e=uv|u\to v\}} \phi(e).$$

Note that an A-flow is an element of $kerD$!

Theorem. *Let G be a graph and $V_1, V_2, ..., V_{c(G)}$ the vertex sets of its connected components, then:*

$$Im(D) = \{f : V \longrightarrow A | \sum_{v\in V_i} f(v) = 0 \ \forall i \leq c(G)\} \cong A^{r(g)}.$$

Proof. If $\phi : E \longrightarrow A$, then

$$\sum_{v\in V_i} D\phi(v) = \sum_{v\in V_i}\sum_{e\in E} d_{v,e}\phi(e) = \sum_{e\in E}\phi(e)\sum_{v\in V_i} d_{v,e} = 0$$

The last equality holds because the integers $d_{v,e}$ in the inner sum are all zeros, if e is not an edge in V_i, or are all zeros but $a+1$ and $a-1$.
Suppose now that $f : V \longrightarrow A$ is such that $\sum_{v\in V_i} f(v) = 0 \ \forall i \leq c(G)$
Fix $i \leq c(G)$, $u \in V_i$, and define $f_i(w) := \sum_{v\in V_i} f(v)\delta_v(w) = \sum_{v\in V_i\setminus\{u\}} f(v)(\delta_v(w) - \delta_u(w))$ to be the restriction of f on V_i, where δ_v is the Kronecker delta.
Since every V_i is connected, $\forall v \in V_i$ we can find a path from u to v:
$u = v_0 \to v_1 \to ... \to v_{k-1} \to v_k = v$ with v_j and v_{j+1} joined by the edge e_j.
We have $\delta_v - \delta_u = (\delta_{v_k} - \delta_{v_{k-1}}) + ... + (\delta_{v_1} - \delta_{v_0}) = D(\pm\delta_{e_{k-1}}) + ... + D(\pm\delta_{e_0})$, with the signs chosen in accord with the orientation ω.
But $\delta_v - \delta_u \in Im(D)\ \forall V_i \Rightarrow f_i \in Im(D)\ \forall i \Rightarrow f \in Im(D)$. □

For the first isomorphism theorem, we have that $Im(D) \cong A^E/ker(D)$, and for the previous theorem we can see that $ker(D) \cong A^{n(G)}$.
We now see another definition for A-flows:
If $U \subseteq V$ and $e = uv \in E$ with $u, v \in U$, then $e \in \delta^+(\{u\}) \cap \delta^-(\{v\})$, or

$e \in \delta^+(\{v\}) \cap \delta^-(\{u\})$, so $\chi_{\delta^+(\{u\}) \cup \delta^-(\{v\})} + \chi_{\delta^+(\{v\}) \cup \delta^-(\{u\})} = 0$.
We then have

$$\sum_{u \in U} \chi_{\delta^+(\{u\}) \cup \delta^-(\{u\})} = \chi_{\delta^+(U) \cup \delta^-(U)}.$$

For the same reason we have:

$$\sum_{u \in U} \left(\sum_{e \in \delta^+(\{u\})} \phi(e) - \sum_{e \in \delta^-(\{u\})} \phi(e) \right) = \sum_{e \in \delta^+(U)} \phi(e) - \sum_{e \in \delta^-(U)} \phi(e).$$

We can then define an A-flow ϕ to be a map $\phi : E \longrightarrow A$ such that for every bond B of G, the following holds $\sum_{e \in B^+} \phi(e) - \sum_{e \in B^-} \phi(e) = 0$.

Let $\langle \phi, \psi \rangle := \sum_{e \in E} \phi(e)\psi(e)$ and let $\mathbf{B}$ and $\mathbf{C}$ be the sets of signed bonds and circuits, then $\langle \chi_B, \chi_C \rangle = 0$ for each $B \in \mathbf{B}$ and $C \in \mathbf{C}$.

We can then state that ϕ is an A-flow if and only if

$$\langle \phi, \chi_B \rangle = 0 \ \forall B \in \mathbf{B}.$$

Since the characteristic vectors of bonds and circuits are orthogonal, we can say that ϕ is an A-flow if and only if for each $C \in \mathbf{C}$ there exists an $a_C \in A$ such that:

$$\phi = \sum_{C \in \mathbf{C}} a_C \chi_C.$$

We can then define the set of A-flows of G as:

$$F_A := \left\{ \sum_{C \in \mathbf{C}} a_C \chi_C | a_C \in A \right\}$$

1.4 The flow polynomial

Observation. Note that if G is a graph and T is a spanning tree of G, then $\{\chi_{C_{T+e}} | e \in E \backslash T\}$ form a basis for F_A.
In fact, $e \in E \backslash T$ is an edge of C_{T+f} if and only if $f = e$, so the characteristic vectors $\{\chi_{C_{T+e}} | e \in E \backslash T\}$ are linearly independent and their number is $n(G) = |E \backslash T|$.

Lemma. *Let $G = (V, E)$ be a connected graph and T a spanning tree. Let A be an abelian group and $\psi : E \backslash T \longrightarrow A$.*
There is a unique A-flow ϕ on G such that $\phi(e) = \psi(e) \ \forall e \in E \backslash T$.

Proof. Consider the vector $\phi := \sum\limits_{e \in E \setminus T} \psi(e)\chi_{C_{T+e}}$.
ϕ is a linear combination of vectors in F_A, so is an A-flow and, obviously, $\phi(e) = \psi(e)$ if $e \in E \setminus T$.
It is unique because any vector in F_A has a unique expression as a linear combination of a fixed base. $\square$

We are now ready to define the flow polynomial of a graph.

Theorem. *Let A be an abelian group, $|A| = k$ and G a digraph, then* $F(G, k) := \sum\limits_{F \subseteq E} (-1)^{|E|-|A|} k^{n(F)}$ *counts the number of nowhere-zero A-flows of G.*

Proof. Since a spanning forest T of a subgraph (V, F) of G has $r(F)$ edges, by the previous lemma it is easy to see that the number of A-flows of (V, F) is equal to $k^{|F|-r(F)} = k^{n(F)}$. For the inclusion-exclusion principle, the theorem follows. $\square$

We call $F(G, k)$ the **flow polynomial** of G.
The flow polynomial has an important recurrence property, know as the deletion-contraction recurrence.

Theorem (Deletion-contraction for the flow polynomial)**.**

$$F(G, k) = \begin{cases} F(G/e, k) - F(G \setminus e, k) & \textit{if } e \textit{ is ordinary} \\ (k-1)F(G \setminus e, k) & \textit{if } e \textit{ is a loop} \\ 0 & \textit{if } e \textit{ is a bridge} \\ 1 & \textit{if } E = \emptyset \end{cases}$$

Proof. If $E = \emptyset$, it is trivial.
If G has a bridge e, it can not have a nowhere-zero A-flow, because $\{e\}$ is a cut of G, so $F(G, k) = 0$.
If G has a loop e, we can assign to it an arbitrary value from 1 to $k - 1$ and this does not change the number of nowhere-zero k-flows on $G \setminus e$.
Finally, if e is ordinary we have two bijections, one from nowhere-zero k-flows of G/e and the set of k-flows of G that can be zero only at e, and one from nowhere-zero k-flows of $G \setminus e$ and the set of flows of G that are zero only at e. Thus, obviously, $F(G, k) = F(G/e, k) - F(G \setminus e, k)$. $\square$

Chapter 2

The planar subcase

In this chapter we will see the first and easiest result found on the way to demonstrate the 5-flow conjecture, that is the fact that every planar graph admits a nowhere-zero 4-flows.
In order to do this, we need to know some theory about planar graphs and colorings.

2.1 Planar graphs

Definition (Graph embedding). *An **embedding** of a graph $G(V, E)$ into a surface S a set of points and arcs, (P, A), on S such that:*

- *Every point $s_v \in P$ is associated to a vertex $v \in V$ and every arc $a_e \in A$ is associated with an edge $e \in E$*
- *The endpoints of an arc a_e are points in P associated with the vertices incident with e.*
- *Two arcs never intersect at a point which is interior to either of the arcs.*

Definition (Planar graph). *A graph G is **planar** if there exists an embedding of G into $\mathbb{R}^2$.*

Obviously, if a graph is a planar, the representation of G on $\mathbb{R}^2$ is not unique.
We will call these representations *plane graph.*
An important property of plane graphs is the existence of the dual graph.

Definition (Faces and dual graph). *Let G be a plane graph on $\mathbb{R}^2$.*
*A **face** of G is a connected component of $\mathbb{R}^2 \backslash G$.*
*The **dual graph** $G^* = (V^*, E^*)$ of $G = (V, E)$ is the graph where every vertex is a face of G, every pair of vertices (v, u) has an edge joining them if the two faces associated*

with v and u are separated by an edge in E (multiple edges if the faces are separated by multiple edges), and every vertex v has a loop if the face of G associated with v appears on both sides of an edge in E.

Not every graph is planar, to see this we need to introduce the Euler's formula.

Theorem (The Euler's Formula)**.** *Let $G = (V, E)$ be a plane graph with $|V|=n$, $|E|=l$ and #faces=f.*
Then $n - l + f = c(G) + 1$.

Proof. Suppose G is connected. We proceed by induction on the number n of vertices.
If G has only 1 vertex and k loops, then $n = 1, l = k, f = k + 1$ and
$n - l + f = 1 - k + k + 1 = 2$
Suppose now the theorem is true for graphs with $|V| = n - 1$ and let $n \geq 2$.
Since G is connected, it has at least one edge e that is not a loop, so G/e has $n - 1$ vertices, $l - 1$ edges and f faces, hence, by the induction hypothesis,
$2 = (n - 1) - (l - 1) + f = n - l + f$.
If G has k connected components $G_1, ..., G_k$, each one with n_i vertices, l_i edges and f_i faces, we have $\sum_{i=1}^{k}(n_i - l_i + f_i) = \sum_{i=1}^{k} 2 = 2k$.
Note now that $\sum_{i=1}^{k} n_i = n$, $\sum_{i=1}^{k} l_i = l$ and $\sum_{i=1}^{k} f_i = f + k - 1$,
so $n + l - f = 2k - k + 1 = k + 1 = c(G) + 1$. □

With Euler's Formula we can easily deduce that if two plane graphs are isomorphic, then they have the same number of faces.

Theorem. *If $G = (V, E)$ is a simple planar graph with $|V| = n \geq 3$ and $|E| = l$, then $l \leq 3n - 6$*

Proof. If $n = 3$, the only connected graph is the tree with 2 edges, and $2 = l \leq 3n-6 = 3$.
If $n \geq 4$, let $G^{'}$ be a plane representation of G with f faces, and $G^{'*}$ its dual graph.
Let d_i be the degree of the vertex v_i in $G^{'*}$, then $2l = \sum_{i=1}^{f} d_i \geq 3f \Rightarrow f \leq \frac{2}{3}l$. Hence,

$$2 = n - l + f \leq n - l + \frac{2}{3}l \Rightarrow l \leq 3n - 6.$$

□

With the same method, it is easy to see that if we add the hypothesis that G can not have any circuit of length three, then we have $l \leq 2n - 4$.
By these two results, we can prove that the complete graph K_5 and the complete bipartite graph $K_{3,3}$ are not planar.
In fact, in K_5 we have $n = 5, l = 10$, but $10 = l > 3n - 6 = 9$, and in $K_{3,3}$ we have

$n = 6, l = 9$ and no circuits of length three, but $9 = l > 2n - 4 = 8$.
Kuratowski gave a necessary and sufficient condition for graph planarity in 1930, known as the Kuratowski theorem.

Theorem. *A graph G is planar if and only if it doesn't contain any subgraph that is a subdivision of either K_5 or $K_{3,3}$.*
Here a subdivision of an edge $e = uv$ of G is the operation of deleting the edge e, and adding to G a new vertex w and the edges (u, w) and (w, v), and the subdivision of a graph G is a finite sequence of subdivisions of edges of G.

The proof of this theorem needs a lot of preliminaries, and is not essential within the scope of this thesis.

2.2 Graph coloring

Intuitively, a graph coloring is an assignment of a color to each vertex such that no adjacent vertices never share the same color.

Definition (Vertex coloring)**.** *Let $G = (V, E)$ be a graph.*
*A **vertex k-coloring** of G is a function $f : V \longrightarrow \mathbb{Z}_k$ such that:*
$\forall u, v \in V$, u adjacent to v $\Rightarrow f(u) \neq f(v)$.

Note that, for a fixed k, not every graph G is k-colorable, but for every graph G, there exists a natural number k such that G is k-colorable.

Definition. *Let $G = (V, E)$ be a graph, we will define the **chromatic number** of G as:*

$$\chi(G) := min\{k \in \mathbb{N} | \exists f : V \longrightarrow \mathbb{Z}_k,\ f \text{ is a } k\text{-coloring of } G\}$$

We will give some examples.
Example 1. The complete graph K_n has chromatic number $\chi(K_n) = n$.
That is obvious because each vertex is joined to every other vertex.
Example 2. A graph G has chromatic number equal to two $\Leftrightarrow$ G is bipartite.
That it true because, if G is bipartite, we can assign a color to each partition of G and it would be a coloring of G.
Conversely, if $f : V \longrightarrow \mathbb{Z}_2$ is a coloring of G, we can create a partition of the vertices of G into two stable subsets U, W, with $U = \{v \in V | f(v) = 0\}, W = \{v \in V | f(v) = 1\}$.
Example 3. A generic tree T is also a bipartite graph, so it has $\chi(T) = 2$.
Observation. If $\Delta(G)$ is the maximum degree of a vertex in G, $\chi(G) \leq \Delta(G) + 1$.
That can be seen by introducing the greedy coloring algorithm.

Definition (Greedy coloring algorithm)**.** *Let $G = (V, E)$ be a graph.*
*A **greedy coloring** of G is a coloring f obtained by ordering its vertices $v_0, v_1, ..., v_n$, setting $f(v_0) = 0$ and, for every $i = 1, ..., n$, $f(v_i) = f(v_{i-1})$ if possible, and $f(v_i) = f(v_{i-1}) + 1$ otherwise.*

Theorem. *Let $G = (V, E)$ be a graph.*
Then $\chi(G) \leq \Delta(G) + 1$.

Proof. Start by defining an ordering on V. Take a vertex v with minimum degree, repeat the process recursively on the induced graph generated by G removing v, and place v as the last vertex of the ordering.
The maximum degree of a vertex removed during the process is called the **degeneracy** of G, denoted with d.
If we now apply the greedy coloring algorithm, it will use at most d+1 colors, but $d \leq \Delta(G) \Rightarrow \chi(G) \leq \Delta(G) + 1$. □

We will see that the number of k-coloring of a planar graph G is strictly related to the number of nowhere-zero k-flows of G.

2.3 The chromatic polynomial

In the first chapter, we defined the flow polynomial to be an invariant which calculates the number of nowhere-zero flows. The same can be done for the colorings of G with the chromatic polynomial.

Theorem (Chromatic polynomial)**.** *Let $G = (V, E)$ be a graph.*
$P(G, k) := \sum\limits_{F \subseteq E} (-1)^{|F|} k^{c(F)}$ counts the number of k-colorings of G.

Proof. Let $k \in \mathbb{N}, uv = e \in E$.
Define $A_e = \{f : V \longrightarrow \mathbb{Z}_k | f(u) = f(v)\}$, A_e is the set of k-colorings of A/e.
The set of k-colorings of G can then be expressed as $\bigcap\limits_{e \in E} A_e^c$.
Now, $|\bigcap\limits_{e \in E} A_e^c| = \sum\limits_{F \subseteq E} (-1)^{|F|} |\bigcap\limits_{e \in F} A_e|$ by Sylvester's formula, but $|\bigcap\limits_{e \in F} A_e| = |\{f : V \longrightarrow \mathbb{Z}_k | f(u) = f(v) \forall u, v$ joined by an edge in F$\}| = k^{c(F)}$ because $\bigcap\limits_{e \in F} A_e$ contains the maps f that are constant when restricted on a connected component of $V, F)$.
Thus we have the assertion. □

Observation. Note that if G is not connected, with $G = G_1 \sqcup G_2$, then $P(G, x) = P(G_1, x)P(G_2, x)$.

The chromatic polynomial could be seen in another way.
Let $G = (V, E)$ be a graph, $|V| = n$, and set S_k to be the number of partition of vertices in V in k subsets such that every block is a stable set.
The chromatic polynomial of G can be seen as:

$$P(G, x) = \sum_{k=\chi(G)}^{n} S_k x(x-1)\cdots(x-k+1)$$

The drawback of this definition is that it is often difficult to compute S_k for arbitrarily large graphs.
As we said, the chromatic polynomial is related to the flow polynomial, for instance it obeys a similar recurrence formula as the flow one.

Theorem. *Let $G = (V, E)$ be a graph, $|V| = n$ and $e \in E$:*

$$P(G, x) = \begin{cases} 0 & \textit{if e is a loop} \\ (x-1)P(G/e, x) & \textit{if e is a bridge} \\ P(G\backslash e, x) - P(G/e, x) & \textit{if e is not a loop} \\ x^n & \textit{if } E = \emptyset \end{cases}$$

So if e is a bridge we have that
$P(G, x) = (x-1)P(G/e, x) = P(G\backslash e, x) - P(G/e, x) \Leftrightarrow P(G\backslash e, x) = xP(G/e, x).$

Proof. If G contains a loop or $E = \emptyset$, it is trivial.
If $e = uv$ is not a loop, then a coloring of $G\backslash e$ can either assign to u and v two different colors or the same one.
Conversely, in G/e, u and v are the same vertex, so they share the same color in any coloring, so $P(G\backslash e, x) = P(G, x) + P(G/E, x) \Rightarrow P(G, x) = P(G\backslash e, x) - P(G/E, x)$.
If e is a bridge, then $G\backslash e = G^{'} \sqcup G^{''}$, with u vertex of $G^{'}$ and v vertex of $G^{''}$.
The number of k-colorings of $G^{'}$ with a fixed color assigned to u are $\frac{1}{k}P(G^{'}, k)$ (the same holds for $G^{''}$ and v), so we have that the number of k-coloring of $G\backslash e$ that assign the same fixed color to u and v are $\frac{1}{k^2}P(G^{'}, k)P(G^{''}, k)$, but we can assign k different colors to u and v, so:
$P(G/e, k) = \frac{1}{k}P(G^{'}, k)P(G^{''}, k) = \frac{1}{k}P(G\backslash e, k)\forall k \Rightarrow xP(G/e, x) = P(G\backslash e, x).$ □

2.4 Colorings of planar graphs

We will now consider the notion of k-coloring applied to planar graphs.

Lemma. *Let $G = (V, E)$ be a planar graph.*
There exists a vertex $v \in V$ with degree $d(v) \leq 5$.

Proof. $|V| = n, |E| = l$.
If $\forall v \in V$, $d(v) > 5$, then $2l = \sum_{i=1}^{n} d(v_i) \geq 6n \Rightarrow l \geq 3n \geq 3n - 6$ and this is impossible, since G is planar. □

Theorem (Five color theorem)**.** *Every simple planar graph G has $\chi(G) \leq 5$.*

Proof. Suppose $G = (V, E)$ be the smallest planar graph that can not be colored with less than six colors, if it exists.
G is planar, so there exists $v \in V$ with $d(v) \leq 5$.
1) $d(v) < 5$:
We can consider $G \backslash v$, that is 5-colorable for the minimality of G, and then add back v to G and label it with one of the five colors not already assigned to its adjacent vertices.
2) $d(v) = 5$:
Again, $G \backslash v$ is 5-colorable. Fix a coloring of it.
Consider the five vertices, v_1, v_2, v_3, v_4, v_5, adjacent to v in clockwise order. Each one must be labeled with a different color, otherwise we could assign to v one of the remaining colors.
Suppose that the color i is assigned to the vertex v_i.
Consider G_{ij} to be the induced subgraph of $G \backslash v$, having vertices labeled by colors i and j. v_i and v_j must be in the same connected components of G_{ij}, if not, we could switch the two colors in the connected component containing v_i and the assign to v the color i.
Obviously, if $i, j \neq t, k$, then $G_{ij} \cap G_{tk} = \emptyset$.
Consider G_{13}, G_{24} and take two paths $v_1 \rightarrow v_3$ and $v_2 \rightarrow v_4$. They must intersect in a vertex $w \in G_{13} \cap G24$, contradiction. □

We can finally see the correlation between k-coloring and nowhere-zero k-flows.

Theorem (Tutte)**.** *Let $G = (V, E)$ be a planar bridgeless graph.*
G is k-colorable $\Leftrightarrow$ G admits a nowhere-zero k-flow.

Proof. Consider $G^* = (V^*, E^*)$ to be the dual graph of a plane representation of G.
Let $f : V^* \longrightarrow \mathbb{Z}_k$ be a k-coloring of G^* and ω be an arbitrary orientation of G.
Define $\phi : E \longrightarrow \mathbb{Z}_k$ as $\phi(e) := f(g_L^\omega(e)) - f(g_R^\omega(e))$, where $g_L^\omega(e)$ ($g_R^\omega(e)$) is equal to the vertex of G^* associated with the face of G^ω on the left (right) of e.
The map ϕ is a nowhere-zero k-flow on G.
Conversely, let $\phi : E \longrightarrow \mathbb{Z}_k$ be a nowhere-zero k-flow on G^ω. Define a k-coloring $f : V^* \longrightarrow \mathbb{Z}_k$ as follows (here vertices of the dual graph G^* will be considered as faces of G).
Assign to the unlimited face $\bar{F}$ of G the color 0. Consider now a path from each other face F to $\bar{F}$, define $f(F)$ to be a sum defined as follows:
For each edge e traversed on the path $F \rightarrow \bar{F}$, add $\phi(e)$ to $f(F)$ if the arc $\omega(e)$ points on the right of the directed path, and subtract it otherwise.
We have constructed a k-coloring. □

Every planar bridgeless graph admits a nowhere-zero 5-flow, so Tutte's conjecture is already proven to be true.
In reality, there is a stronger theorem for planar graphs, which says

Theorem (4-color theorem). *Every simple planar graph G is 4-colorable.*

This theorem implies that Tutte's conjecture for planar graphs is even stronger, because such graphs admits a nowhere-zero 4-flow.
Many attempts were made to prove the theorem since it was first conjectured, in 1852, but it remained unsolved for many years.
Ultimately, it was proven in 1976 by Kenneth Appel and Wolfgang Haken, using a new approach.
They found a set of around 1500 graphs and demonstrated that if the theorem holds for every graph in the set, it also holds for all graphs in general.
Finally, using a computer, in about 1200 hours of work, they found a 4-coloring for each graph on the set.

Chapter 3

The projective subcase

In the previous chapter we saw a particular type of graphs, planar graphs.
We will now generalize this concept introducing the genus of a graph.

3.1 Graph genus

Definition (Genus and demigenus of a 2-manifold)**.** *Let S be a 2-manifold, or surface, and $\chi(S)$ be the Euler characteristic of S.*
If S is orientable, we will say that it has genus $g = \frac{2-\chi(S)}{2}$. Otherwise, we will say that it has demigenus $g = 2 - \chi(S)$.

Definition (Genus of a graph)**.** *Let $G = (V, E)$ be a graph.*
The genus of G is the minimum $n \in \mathbb{N}$ such that there exists an embedding of G into a 2-manifold S of genus, or demigenus, n.
We will denote it by γ_G.

The definition makes sense and every graph has a genus because if G is a graph, then we can place its vertices on the sphere S^2 and add a handle for each edge of G, so if G has m edges, $\gamma_G \leq m$.

Example 1. A planar graph G has genus $\gamma_G = 0$.
Example 2. $\gamma_{K_5} = \gamma_{K_{3,3}} = 1$ because they can be embedded on the torus, so $\gamma_{K_5} \leq 1$ and $\gamma_{K_{3,3}} \leq 1$, but they are not planar graph, so $\gamma_{K_5} \neq 0 \neq \gamma_{K_{3,3}}$.

In this chapter we will focus on graphs that can be embedded in the projective plane.

3.2 Useful informations

We will now see some graph theory information that will be useful in the future.

Definition (k-connected and k-edge-connected graphs)**.** *Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$.*
*We will say that G is k-**connected** if $|V| > k$ and remains connected whenever less than k vertices are removed.*
*We will say that G is k-**edge-connected** if $|E| > k$ remains connected whenever less than k edges are removed.*

Theorem. *Let $G = (V, E)$ be a k-edge-connected graph.*
Then $d(v) \geq k, \forall v \in V$.

Proof. Let $v \in V$ be a vertex of G such that $d(v) < k$.
We can disconnect G by removing all edges incident with v, so G is not k-edge connected. □

Definition (k-regular graphs)**.** *Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$.*
*G is a k-**regular** if $d(v) = k, \forall v \in V$.*

Definition (Cubic graph)**.** *A graph G is **cubic** if it is a 3-regular graph.*

Definition (k-gon)**.** *Let G be a graph embedded on a surface S.*
*We will say that G contains a k-**gon** if it has a face having precisely k edges in its boundary.*

Theorem. *Let $G = (V, E)$ be a graph, $e' \in E$.*
Then e' is a bridge $\Leftrightarrow$ e' is a 1-bond.

Proof. Let e' be a bridge of G, then we can consider C_1, C_2 to be the two connected components of $G \backslash e'$.
Now, (C_1, C_2) is a partition of V and the cutset associated to it is $\{e'\}$, but a cutset with a single edge is a 1-bond, so e' is a 1-bond.
Conversely, let $e' = u'v'$ be not a bridge and suppose that it is a 1-bond.
We would have a partition of V, $(U, V \backslash U)$, such that
$K := \{e = uv \in E | u \in U, v \in V \backslash U\} = \{e'\}$.
Let $u' \in U, v' \in V \backslash U$.
e' is not a bridge, so there exists a path $P = P(u', v')$ from u' to v' not passing through e', but that is impossible, because P would start in U and arrive in $V \backslash U$, so the cutset contains at least another edge, and $\{e'\}$ can not be a 1-bond. □

Corollary. *Let $G = (V, E)$ be a connected graph, B be a bond of G.*
If we remove all edges of B from G, we disconnect G.

Definition (2-cell embedding)**.** *Let G be a graph embedded on a surface S.*
*G is a **2-cell embedding, or cellular embedding**, if every face of G is homeomorphic to an open disk.*

Definition (Simplest embeddings)**.** *Let G be a graph, M be a surface and let G be embeddable into M.*
Denote by $G(M)$ the embedding of G into M
*. We will say that the embedding $G(M)$ is **simplest** if, for every surface N such that there exists an embedding $G(N)$ of G into N, we have $\chi(M) \geq \chi(N)$.*

Definition. *Maximal embeddings Let G be a graph, M be a surface.*
Let $\|G(M)\|$ be the number of connected components of $M \backslash G(M)$.
*We will say that the embedding $G(M)$ is **maximal** if, $\forall N$ surface such that there exists an embedding $G(N)$ of G into N, we have $\|G(M)\| \geq \|G(N)\|$.*

Theorem (Characterization theorem)**.** *Let G be a graph, M be a surface.*
The embedding $G(M)$ is simplest $\Leftrightarrow$ $G(M)$ is a maximal cellular embedding.

The proof of the Characterization Theorem is omitted, it can be found in [1], [2] and [3].

Definition. *Let $G = (V, E)$ be a graph.*
*A **block** of G is a maximal connected subgraph of G that has no articulation point.*
*G is a **block graph** if it has no articulation points.*

Theorem. *Let $G = (V, E)$ be a graph, $B_1 = (V_1, E_1), B_2 = (V_2, E_2)$ two blocks of G.*
$|V_1 \cap V_2| \leq 1$.

Proof. Suppose that $|V_1 \cap V_2| \geq 2$, $v_1, v_2 \in V_1 \cap V_2$.
By definition, if we delete a vertex v from B_i it remains connected.
We want to prove that $(B_1 \cup B_2) \backslash v$ is still connected $\forall v \in V_1 \cup V_2$, so $B_1 \cup B_2$ is a block of G, which contradicts the hypothesis of maximality of B_1 and B_2.
It is easy to see that $B := (B_1 \cup B_2) \backslash v$ is still connected, because at least one of v_1, v_2 is a vertex of B, so $\forall u, w \in B$, we have two paths $P' = P(u, v_i), P" = P(v_i, w)$, and joining them together we get a path $P = P(u, w)$ in B, so B is connected. □

Theorem. *Let $z_1, z_2, ..., z_r \in \mathbb{Z}_k, k \geq 3$ with $r \leq k - 2$.*
Then there exists $z_0 \in \mathbb{Z}_k \backslash \{0\}$ such that
$z_1 + z_0, z_2 + z_0, ..., z_r + z_0$ are all nonzero.

Proof. Simply choose z_0 in $\mathbb{Z}_k \backslash \{0, -z_1, -z_2, ..., -z_r\}$. □

3.3 The splitting lemma

Here we will see Herbert Fleischner's splitting lemma, a classic result in graph theory that will be used in many proofs.

Lemma (Splitting lemma)**.** *Let $G = (V, E)$ be a connected bridgeless graph, $e_1, e_2, e_3 \in E$ three edges incident with a vertex $v \in V$ with $d(v) \geq 4$.*
Let e_1 and e_3 belong to two different blocks if v is an articulation point.
Define two new graphs, G_{12}, G_{13} as follows:
Split v into two vertices, v', v_{12}, such that e_1, e_2 are now incident with v_{12} and the other edges incident with v are now incident with v'. Leave the remainder unchanged.
This is a new graph: G_{12}.
Define G_{13} analogously.
Then at least one of G_{12} and G_{13} is a connected bridgeless graph.

Proof. We will divide the proof in two distinct parts:
(1) Let G_{12} be disconnected with 2 connected components C_1, C_2 with $e_1, e_2 \in C_1$, $e_3 \in C_2$.
e_1, e_2, e_3 are not bridges in C_1, C_2, otherwise $G'_{12} := G_{12} \backslash e_i$ would have three connected components. Identifying v' and v_{12} in G'_{12} we would have $G \backslash e_i$ not connected, so e_i would be a bridge, but G is bridgeless.
Any other edge $e = (r, s) \in C_1$ is not a bridge in G_{12} too, otherwise there would be a edge minimal path $P = P(r, s)$ in G from r to s not passing through e (because G is bridgeless).
This path would start from $r \in C_1$, then go to C_2 (it can not remain in C_1 all the time, otherwise e would not be a bridge in G_{12}) and finally return in C_1 and arrive to s.
However, C_1 and C_2 are connected only by v in G, so P would pass many times through v, but it is minimal, so e can not be a bridge in G_{12}.
We will now prove that G_{13} is connected by showing that every vertex u is connected to v'.
(a) $v_{13} \neq u \in C_1 \subset G_{12}$.
There is a path $P = P(u, v_{12})$ in G_{12}.
(a.1) If P passes through e_2, it is equivalent to a path $P' = P(u, v')$ in G_{13}.
(a.2) If P passes through e_1, it is equivalent, instead, to a path $P' = P(u, v_{13})$ in G_{13}.
But G_{12} is bridgeless, so there is cycle $K \subset C_2$ such that $e_3 \in K$.
K in G_{13} is equivalent to a path $P" = P(v_{13}, v')$ such that $P' \cap P" = \emptyset$, because $C_1 \cap C_2 = \emptyset$.
(b) $v_{13} \neq u \in C_2$.
C_2 is bridgeless, so there is path $P = P(u, v')$ that does not contain e_1, e_2, e_3, so it can be seen in G_{13} as the same path.
(c) $u = v_{13}$.

As in (a), just consider a cycle $K \subset C_2 \subset G_{12}$ with $e_3 \in K$ and notice that it correspond to a path $P = P(v_{13}, v')$ in G_{13}.
We now have to prove that G_{13} is bridgeless.
e_1, e_2, e_3 are not bridges of G_{13} (it can be seen by the same reasoning as above for G_{12}).
Let e be an edge of G_{12}, $e \neq e_i, i = 1, 2, 3$. G_{12} is bridgeless, so there exists a cycle $K_{12} \subset G_{12}$ with $e \in K_{12}$.
If $e_i \notin K_{12}, i = 1, 2, 3$,K_{12} is also a cycle in G_{13}.
Otherwise, if $e_1 \in K_{12}$, then $e_2 \in K_{12}$ and $e_3 \notin K_{12}$.
K_{12} can be seen in G_{13} as a path $P = P(v', v_{13})$, then we can take the path $P^{"} = P(v_{13}, v')$ found in point **(a.2)** and join them together to form a cycle in G_{13} containing e.
Finally, if $e_1 \notin K_{12}, e_3 \in K_{12}$, let K'_{12} be a cycle in G_{12} that pass through e_1, e_2.
$K_{12} \cap K'_{12} = \emptyset$ and again, joining them together in G_{13} will give us a cycle containing e.
(2) Let G_{12}, G_{13} be connected, G_{12} not bridgless.
We will prove that G_{13} is bridgeless.
Observation. If G_{1j} is connected but not bridgeless, then every path $P(v_{1j}, v')$ contains every bridge of G_{1j}.
Otherwise, every bridge of G_{1j} that is not contained in a fixed path P_0, would be a bridge of G, but G is bridgeless.
(2.1) v articulation point.
e_1, e_3 belong to two different blocks by hypothesis and there exist two cycles C_1, C_3 with $e_1 \in C_1, e_3 \in C_3, C_1 \cap C_3 = \{v\}$.
C_r correspond to a path P_r in G_{13} $(r = 1, 3)$ with $P_1 \cap P_3 = \{v_{1j}, v'\}$.
Then G_{13} is bridgeless for the observation.
(2.2) v not an articulation point.
$G \backslash v$ is connected. Let v_i be the vertex incident with e_i, with $v_i \neq v$.
There exists a path $P(v_1, v_2) \subset G \backslash v$, so $P(v_1, v_2) \cup \{e_1, e_2\} := C_{12}$ is a cycle in G and G_{12}.
Consider a path $P = P(v_{12}, v') \subset G_{12}$. We have that P contains all bridges of G_{12}.
Starting a run through P in v_{12}, let $v^{"}$ be the last vertex adjacent to a bridge $e_{v^{"}}$ of G_{12}.
$v^{"} \neq v'$, otherwise v would be an articulation point of G.
Consider now the two connected components $C' \supset \{v_{12}\}, C^{"} \supset \{v'\}$ of $G \backslash v^{"}$.
e_3 is not a bridge of $C^{"}$ (same preceding argument with e_3 in place of $e_{v^{"}}$).
There exists a cycle $C_3 \supset \{e_3\}$ in $C^{"}$ and it is also a cycle in G and G_{12}.
However, $C_{12} \cap C_3 = \{v'\}$, so, proceeding as in case **(2.1)**, we conclude the proof of the lemma. $\square$

3.4 The conjecture in the projective plane

We can now see Tutte's 5-flow theorem for graphs embeddable in the projective plane, we will prove it by contradiction.

Theorem (Tutte's 5-flow theorem in the projective case). *Let $G = (V, E)$ be a bridgeless graph embeddable into the projective plane.*
G has a nowhere-zero 5-flow.

Let G' be a bridgeless graph embedded in the projective plane that does not have a nowhere-zero 5-flow, minimal with respect to number of edges plus vertices.
We will prove that such graph can not exist.

Proposition. *G' is connected.*

Proof. If G' is not connected, its connected component would not have a nowhere-zero 5-flow, contrary to the hypothesis of minimality of G'. □

Proposition. *G' is 2-connected and has no loops.*

Proof. Suppose that G' contains a loop e.
Since G' is bridgeless, $G' \backslash e$ is bridgeless too.
Now, $G' \backslash e$ has a nowhere-zero 5-flow by minimality of G', but loops are irrelevant during the construction of nowhere-zero flows, so G' has a nowhere-zero 5-flow too.
Suppose now that G' is not 2-connected, then it is not a block graph and will contain at least 2 bridgeless blocks, which, by minimality of G', would have a nowhere-zero 5-flow .
The union of such blocks would then have a nowhere-zero 5-flow of G', because two blocks always share at most one vertex, so they never share the same edge. □

Proposition. *The embedding of G' is a 2-cell embedding.*

Proof. For the characterization theorem, if the embedding is not a cellular embedding it is not a simplest embedding, so G' is a planar graph and hence has a nowhere-zero 5-flow, contradiction. □

Proposition. *G' is a simple cubic graph.*

Proof. $G' = (V, E)$ is 2-edge-connected, so $\forall v \in V$, $d(v) \neq 1$.
Suppose there exists a 2-bond $\{e_1, e_2\}$ of G'.
We have that G'/e_1 is bridgeless and has a nowhere-zero 5-flow, but that is impossible (it is easy to see that G' would have a nowhere-zero 5-flow too), so G' is 3-edge-connected.
Suppose now that G' contains a vertex v with $d(v) > 3$.
Denote its incident edges by $e_1, e_2, ..., e_{d(v)}$, ordered clockwise.
By the splitting lemma, there are two edges e_i, e_{i+1} incident with v and two other vertices u_i, u_{i+1} so that if we eliminate e_i, e_{i+1} from G' and join u_i, u_{i+1} by a new edge, we obtain a new bridgeless graph G" that has a nowhere-zero 5-flow.
Once again, it is not possible, because then we could easily obtain a nowhere-zero 5-flow for G'. Thus G' is cubic, and since it is 3-edge-connected, it is also simple.

□

Proposition. *G' contains a 3-gon, a 4-gon or a 5-gon.*

Proof. Let n, m, f, f_k be, respectively, the number of vertices, edges, faces and k-gons of G'.
G' is cubic, so we have $3n = 2m$. Combined with the Euler's Formula we have $n - m + f = 1 \Rightarrow n + f = m + 1 \Rightarrow 3n + 3f = 3m + 3 \Rightarrow 3f = m + 3$ We also have $f = \sum_{k \geq 3} f_k$, because G' is simple, and $2m = \sum_{k \geq 3} k f_k$.
Then $6f = 2m + 6 \Rightarrow \sum_{k \geq 3} f_k = \sum_{k \geq 3} k f_k + 6 \Rightarrow \sum_{k \geq 3} (6 - k) f_k = 6$.
The last equation implies that at least one of f_3, f_4, f_5 must be greater then zero. $\square$

Proposition. *G' can not contain a 3-gon, a 4-gon or a 5-gon*

Proof. **1)** Suppose G' contains a 3-gon.
Let $v_1, v_2, v_3, e_1, e_2, e_3$ be the three vertices and three edges of the 3-gon of G', and let l_1, l_2, l_3 be the three edges not in the 3-gon, incident with v_1, v_2, v_3, respectively .
Consider $G_1 := G' \backslash e_1$ and notice that it is isomorphic to the graph obtained from G' by contracting e_1, e_2, e_3 and subdividing l_1 and l_3.
We have that G_1 is bridgeless, so it has a nowhere-zero 5-flow, from which we can obtain a nowhere-zero 5-flow of G' as in figure 3.1.
2) Suppose G' contains a 4-gon.
Let $e_1, e_2, e_3, e_4, v_1, v_2, v_3, v_4, l_1, l_2, l_3, l_4$ be edges and vertices defined as in the 3-gon case.
Let G_{13}, (G_{24}), be the graph obtained from G' by removing edges e_1 and e_3, (e_2 and e_4).
The graph G_{13}, (G_{24}), is isomorphic to the graph obtained from G' by contracting e_1, e_2, e_3, e_4 and splitting the resulting vertex, as in the splitting lemma, so that l_1 and l_2 share a vertex, and l_3 and l_4 share a vertex, (l_1 and l_4 share a vertex, and l_2 and l_3 share a vertex), and finally subdividing l_1 and l_4, (l_1 and l_2).
By the splitting lemma, one between G_{13} and G_{24} is a bridgeless graph, so has a nowhere-zero 5-flow.
Again, we can obtain a nowhere-zero 5-flow of G', as in figure 3.2.
3) Suppose G' contains a 5-gon.
Let $e_1, e_2, e_3, e_4, e_5, v_1, v_2, v_3, v_4, v_5, l_1, l_2, l_3, l_4, l_5$ be edges and vertices defined as in the 3-gon and 4-gon cases.
Let G_{24}, (G_{35}), be the graph obtained as in the previous case.
Again, G_{24}, (G_{35}), is isomorphic to the graph obtained by G' by contracting the edges of the 5-gon, splitting the resulting vertex so that l_2 and l_3 share a vertex, and l_1, l_4 and l_5 share a vertex, (l_3 and l_4 share a vertex, and l_1, l_2 and l_5 share a vertex), and finally subdividing l_1, l_2 and l_4, (l_2, l_3 and l_5).
By the splitting lemma, at least one of the two graphs G_{13} and G_{24} is bridgeless, so it has a nowhere-zero 5-flow.
We can conclude, as in cases **a)** and **b)**, obtaining the nowhere-zero 5-flow for G' as in figure 3.3. $\square$

Obviously we have a contradiction, because G' must contain a 3-gon, a 4-gon or a 5-gon, but it can not contain any one of them, so G' can not exists.

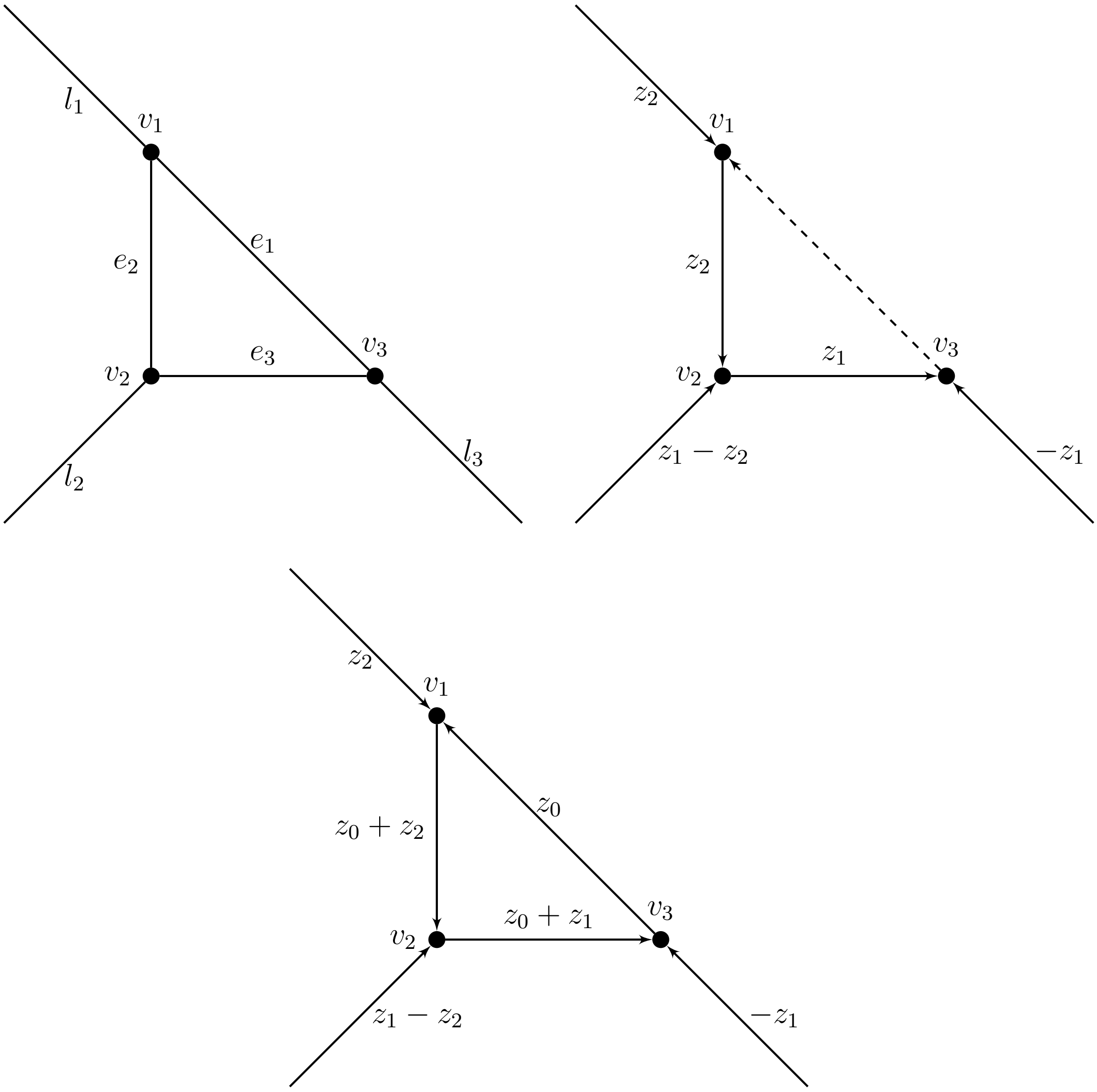


Figure 3.1: 5-flow on the 3-gon

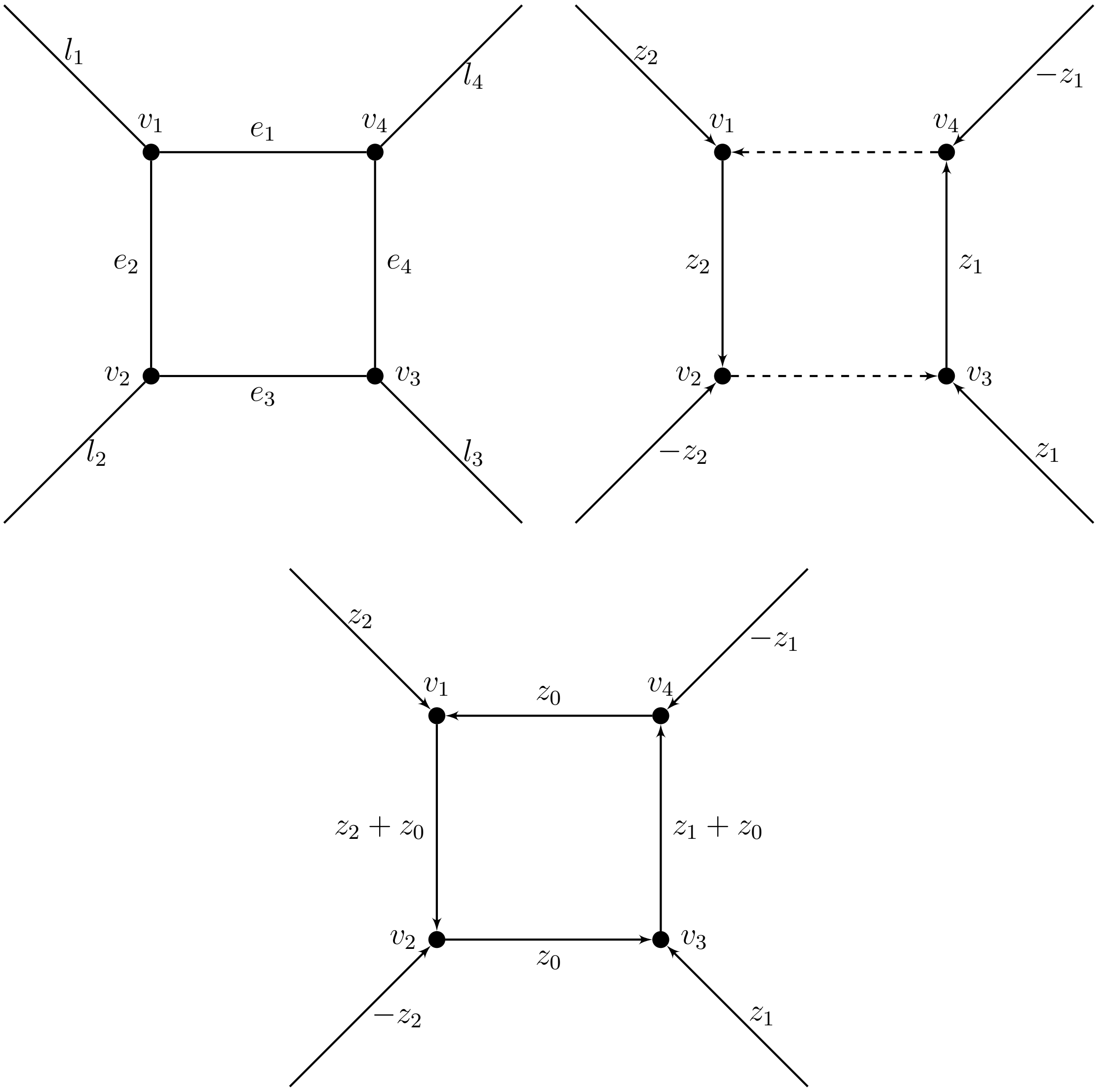
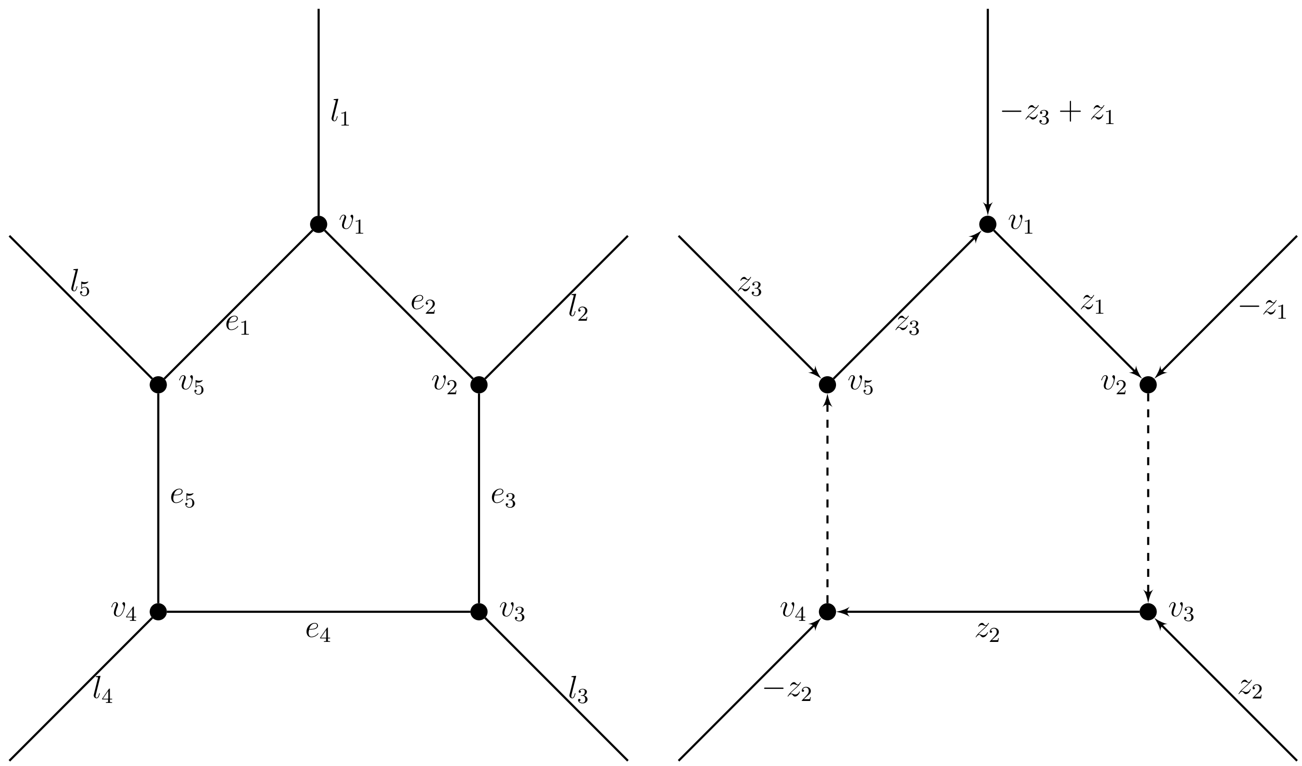


Figure 3.2: 5-flow on the 4-gon



l_1
v_1
l_5
e_1
e_2
l_2
v_5
v_2
e_5
e_3
v_4
v_3
e_4
l_4
l_3
$-z_3 + z_1$
v_1
z_3
z_3
z_1
$-z_1$
v_5
v_2
v_4
v_3
z_2
$-z_2$
z_2

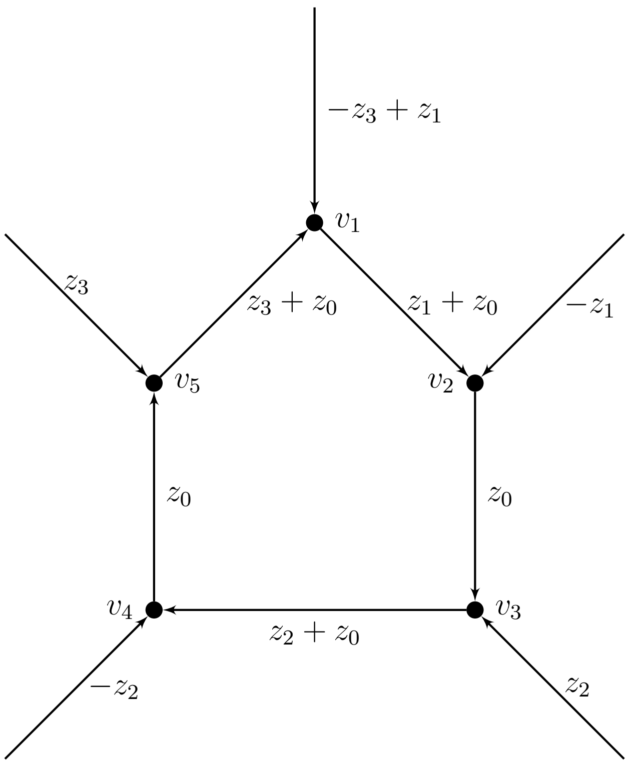


Figure 3.3: 5-flow on the 5-gon

Chapter 4

Two upper bounds for nowhere-zero k-flows

We have seen some type of graphs where the conjecture is proven to be true, in this chapter we will focus on two weaker theorems, one proven by F. Jaeger, and the other by P. D. Seymour.
Let G be a bridgeless graph, we define the **flow number** of G as:

$$k(G) := min\{k \in \mathbb{N} |\ G \text{ has a nowhere-zero } k\text{-flow } \phi\}.$$

Jaeger proved that $k(G)$ has an upper bound, and that $\forall G$ bridgeless graph, $k(G) \leq 8$. Later Seymour improved the result by proving that $k(G) \leq 6$. This was a huge step toward Tutte's conjecture, which states that $k(G) \leq 5$.

4.1 François Jaeger's theorems

Theorem. *Let $G = (V, E)$ be a graph.*
If G is bridgeless, then it has a nowhere-zero 8-flow.

We need three preliminary results in order to prove the theorem.

Lemma. *Let $G = (V, E)$ be a graph, $p \in \mathbb{N}$.*
G has a nowhere-zero 2^p-flow $\Leftrightarrow$ there exist p subsets $F_1, F_2, ..., F_p \subset E$ with F_i even $\forall i$ (namely, every vertex of F_i has even degree) and $\bigcup_{i=1}^{p} F_i = E$.

Proof. Remember that the existence of an A-flow depend exclusively on the cardinality of A, so G has a nowhere-zero 2^p-flow $\Leftrightarrow$ G has a $\underbrace{(\mathbb{Z}_2 \times ... \times \mathbb{Z}_2)}_{p}$-flow.

($\Rightarrow$) Let ϕ be a nowhere-zero $\underbrace{(\mathbb{Z}_2 \times ... \times \mathbb{Z}_2)}_{p}$-flow on G.
Define $F_i := \{e | e \in E, (\phi(e))_i = 1\}$.
Obviously F_i is even $\forall i = 1, ..., p$, and $\bigcup_{i=1}^{p} F_i = E$.
($\Leftarrow$) Define ϕ as follows:

$$(\phi(e))_i := \begin{cases} 1 & \text{if } e \in F_i \\ 0 & \text{otherwise} \end{cases}$$

We have that ϕ is a nowhere-zero $(\mathbb{Z}_2 \times ... \times \mathbb{Z}_2)$-flow because $E = \bigcup_{i=1}^{p} F_i$. □

Proposition. *Let $G = (V, E)$ be a graph, T be a spanning tree of G. There exists $F \subset T$ such that $(E \backslash T) \cup F$ is even.*

Proof. We will describe an algorithm to find such F.
Define $U := E \backslash T, F = \emptyset, M = T$. Then **Step 1.** Take a leaf vertex v of M and control its degree in U, if $d(v)$ is even go to step 2, otherwise add the edge of M into U and F and go to step 2.
Step 2. Update M deleting v, then, if M has more then one vertex left, return to step 1, otherwise stop.
At the end of the algorithm, U is even and is equal to $(E \backslash T) \cup F$. □

We need a final theorem, but we will not see the proof here.

Theorem (Nash-Williams)**.** *Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ and P a partition of V. Define G/P to be the graph on the set P with an edge joining two vertices $U, W \in P$ for every edge of G with an end in U and the other in W.*
Then:
G contains k pairwise edge-disjoint spanning trees $\Leftrightarrow$ for every partition P of V, G/P has at least $k(|P| - 1)$ edges.

The proof can be found in [4].

Corollary. *Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$.*
G is $2k$-edge-connected $\Leftrightarrow$ G contains k pairwise edge-disjoint spanning trees.

Proof of Jaeger's theorem. We will divide the proof in two distinct cases:
1) Suppose G not 3-edge-connected, we will prove it by induction on the number of edges.
Base case: G has only two vertices, so it is planar and has a nowhere-zero 4-flow.
Inductive case: G is 2-edge-connected, so there exists a cutset $\{e_1, e_2\}$ and, by induction on the number of vertices, $G' = G/e_1$ has a nowhere-zero 8-flow ϕ.

Define:

$$\phi'(e) := \begin{cases} \phi(e) & \text{if } e \neq e_1 \\ \phi(e_2) & \text{if } e = e_1 \end{cases}$$

ϕ' is a nowhere-zero 8-flow for G.

2) Suppose G is 3-edge-connected.
Create a new 6-edge-connected graph $G' = (V, E')$ by duplicating every edge of G.
By Nash-Williams Theorem, there exist three pairwise disjoint spanning trees $T_1', T_2', T_3' \subset E'$ with $T_1' \cup T_2' \cup T_3' = E'$.
Define $T_i := \{e|e$ is an edge of T_i' or e has a duplicate that is an edge of $T_i'\}$.
We have that $T_1 \cup T_2 \cup T_3 = E$ and, for every $e \in E$, there exists i such that $e \notin T_i$.
We can now use the previously defined algorithm to find three set $F_i \subset T_i$ such that $G_i := (E \backslash T_i) \cup F_i$ is even.
We then have that $G_1 \cup G_2 \cup G_3 = E$ and, by the first proposition of this section, the theorem is proven. □

Is worth mentioning that there is another important theorem, due to François Jaeger, on the existence of k-flows.

Theorem. *Let $G = (V, E)$ be a graph.*
G is 4-edge-connected $\Rightarrow$ G has a nowhere-zero 4-flow.

Proof. G is 4-edge-connected, so there exist two pairwise disjoint threes $T_1, T_2 \subset E$ with $T_1 \cup T_2 = E$.
As in the last proof, we can find $F_i, i = 1, 2, 3$ such that $G_1 \cup G_2 \cup G_3 = E$ with $G_i := (E \backslash T_i) \cup F_i$ and, by the first proposition, the theorem is proven. □

4.2 Nowhere-zero 6-flows

In 1980, Paul D. Seymour improved Jaeger's result proving that:

Theorem. *Let $G = (V, E)$ be a graph.*
If G is bridgeless, then it has a nowhere-zero 6-flow.

We will start with a little variation on a theorem already seen in Chapter 3, the proof is the same.

Theorem. *Let $G = (V, E)$ be a bridgeless graph, $k \in \mathbb{N}, k > 2$ and let $\chi(G)$ be the chromatic number of G.*
If G is the minimal graph, with respect to $|V| + |E|$, such that $\chi(G) > k$, then G is simple, cubic and 3-connected.

Observation. The idea now is to demonstrate the 6-flow theorem for simple, cubic, 3-connected graphs and, similarly to the projective case, demonstrate that there can not be a minimum counterexample of the theorem, with respect to the sum of vertices plus edges.

Theorem. *Let $G = (V, E)$ be a directed graph, $k > 0$.*
If ϕ is a $\mathbb{Z}$-flow on G, then there exists a k-flow ϕ' such that $\phi(e) \equiv \phi'(e)$ $(mod\ k)$, $\forall e \in E$.

Proof. Just define $\phi'(e) := k_e \equiv \phi(e)$ $(mod\ k)$. $\square$

We will now define a useful tool.

Definition. *Let $G = (V, E)$ be a graph, $X \subseteq E, k \in \mathbb{N}, k > 0$.*
Define $\langle X \rangle_k$ as the smallest subset $Y \subseteq E$ such that:
1) *$X \subseteq Y$.*
2) *There is no circuit C in G such that $0 < |C \backslash Y| \leq k$.*
Here C is treated as a set of edges rather than a subgraph.

Observation 1. $\langle X \rangle_k$ is uniquely defined, if both Y' and $Y"$ satisfy **1)** and **2)**, then so does $Y' \cap Y"$.

Definition. *A **closure operator** on a set X is a function $f : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X, such that f satisfies the following conditions $\forall A, B \subseteq X$:*
1) *$A \subseteq f(A)$*
2) *$A \subseteq B \Rightarrow f(A) \subseteq f(B)$*
3) *$f(f(A)) = f(A)$*

Observation 2. $X \longrightarrow \langle X \rangle_k$ is a closure operator.
We will now use this fact to prove the following theorem.

Theorem. *Let $G = (V, E)$ be a graph, $k \in \mathbb{N}, k > 1$.*
If there exists $X \subseteq E$ such that $\langle X \rangle_{k-1} = E$, then G has a k-flow ϕ with $E \backslash X \subseteq Supp(\phi)$.

Proof. We will proceed by induction on $n = |E \backslash X|$.
If $n = 0$, then $|E \backslash X| = \emptyset \subseteq Supp(\phi)$.
Suppose now that the theorem holds for $n - 1, n \neq 0$.
We have $\langle X \rangle_{k-1} \neq X$, so there is a cycle C with $0 < |C \backslash X| \leq k - 1$.
$X \longrightarrow \langle X \rangle_k$ is a closure operator, so $\langle X \cup C \rangle_{k-1} = E$, and by induction there is a k-flow ϕ with $E \backslash (X \cup C) \subseteq Supp(\phi)$.
Consider now a $\mathbb{Z}_3$-circulation ϕ' with $Supp(\phi') = C$.
Since $|C \backslash X| \leq k - 1$, we can find an integer m such that $0 \leq m \leq k - 1$ and
$m \not\equiv -\frac{\phi(e)}{\phi'(e)}, \forall e \in C \backslash X$.
Define now $\psi := \phi + m\phi'$. Then:

$$\psi(e) = \begin{cases} \phi(e) & \text{if } e \in E \backslash (X \cup C) \\ \phi(e) + m\phi'(e) & \text{if } e \in C \backslash X \end{cases}.$$

By definition of m, $\psi(e) \not\equiv 0$ $(mod\ k)$ $\forall e \in E \backslash X$.
The result follows from the previous theorem. $\square$

Theorem. *Let $G = (V, E)$ be a graph, $V \neq \emptyset$.*
If $d(v) \geq 2, \forall v \in V$, then there exists a 2-connected subgraph $H = (V', E')$ of G with $|V'| \geq 3$ such that at most one vertex of H is adjacent in G to some vertices in $G \backslash H$.

Proof. We just need to prove the theorem for each connected component of G, so we can suppose that G is connected.
If G is a block graph, it is trivial. Suppose G is not 2-connected.
Let $\mathfrak{B}$ be the collection of blocks of G, and let $\mathfrak{D}$ be the collection of vertices of G which are in at least two blocks of $\mathfrak{B}$.
We will define a new bipartite graph G' on $\mathfrak{B} \cup \mathfrak{D}$ by saying that $B \in \mathfrak{B}, D \in \mathfrak{D}$ are adjacent in G' if D is a vertex of B.
G' has no circuits, otherwise, we would have an induced circuit $C := D_1 B_1 ... D_k B_k D_1$, $k \geq 2$, and $\bigcup_{i=1}^{k} B_i$ would be a block, contradicting the maximality of each B_i.
Then G' has at least a vertex v with $d(v) = 1$, but each $D \in \mathfrak{D}$ has $d(D) \geq 2$, so there at least a block $B \in \mathfrak{B}$ that shares at most one vertex with other blocks.
We will prove that $B =: H$ is the subgraph we are looking for.
G has no isolated vertices, so H has at least 2 vertices.
Note now that every edge $e \in E$ is in a block of $\mathfrak{B}$, so at most one vertex of H is adjacent in G to vertices not in H.
Suppose now u is a vertex in H not adjacent to vertices not in H, then, since G is simple and each vertex v has $d(v) \geq 2$, H has at least three vertices. □

Lemma. *Let $G = (V, E)$ be a simple, 3-connected graph.*
$|V| \geq 3 \Rightarrow$ there are disjoint circuits $C_1, ..., C_k, k \in \mathbb{N}$, such that $\langle C_1 \cup ... \cup C_k \rangle_2 = E$.

Proof. G is 3-connected, so there exists a circuit C.
$\langle C \rangle_2$ is connected, because G is simple, so we can choose a maximum $k \in \mathbb{N}$ such that there exist disjoint circuits $C_1, ..., C_k$ with $\langle C_1 \cup ... \cup C_k \rangle_2$ connected.
Define $C' := C_1 \cup ... \cup C_k, X := \langle C' \rangle_2$, $U := \{v \in V | v \text{ is incident with edges in } H\}$ and $G' := G \backslash U$.
We want to prove that G' has no vertices. In this case we have that $U = V$ and $\langle X \rangle_2 = E \Rightarrow \langle C' \rangle_2 = \langle C_1 \cup ... \cup C_k \rangle_2 = E$ and the theorem is proven.
Suppose by contradiction that G' has at least one vertex.
There is no vertex v in G' adjacent in G to two vertices $v_1, v_2 \in U$, otherwise there would be a path $P(v_1, v_2)$ from v_1 to v_2 using only vertices in X, since X is connected.
Then, if $e_1 := (v_1, v), e_2 := (v_2, v)$, $P(v_1, v_2) \cup \{e_1, e_2\}$ would give a circuit, which contradicts the definition of $X = \langle C' \rangle_2$.
Since G is 3-connected, we have that each vertex of G' has $d(v) \geq 2$.
By the previous theorem, there is a 2-connected subgraph H of G' with at most one vertex adjacent in G' to vertices of $G' \backslash H$, and with at least three vertices.
Since H has at least three vertices, G is 3-connected and $U \neq \emptyset$, there are at least three

vertices of H adjacent in G to vertices in $G\backslash H$.
Then, there are at least two vertices u_1, u_2 in H, both adjacent in G to vertices in U, but since H is 2-connected and has more than two vertices, there exists a circuit C_{k+1} in H containing both u_1 and u_2.
Then $C_1, ..., C_{k+1}$ are disjoint, and, if e_1, e_2 are edges joining, respectively, u_1, u_2 to U, then $\{e_1, e_2\} \subseteq \langle C_1 \cup ... \cup C_{k+1}\rangle_2$, and so $\langle C_1 \cup ... \cup C_{k+1}\rangle_2$ is connected, contradicting the hypothesis of maximality of k. □

We can finally prove the 6-flow theorem.

Proof. Suppose by contradiction that the theorem is false.
Then there exists a minimal counterexample G, with respect to the number of vertices plus edges.
By the first theorem of this chapter, G is simple, cubic and 3-connected, so we just need to prove the theorem for such kind of graphs.
By the previous lemma, there exist $k \in \mathbb{N}$ disjoint circuits, $C_1 \cup ... \cup C_k := X$, with $\langle X\rangle_2 = E$.
We can find a 3-flow ϕ_3 of G with $E\backslash X \subseteq Supp(\phi_3)$.
Now, let ϕ_2 be a 2-flow of G with $Supp(\phi_2) = X$, and set $\phi = \phi_3 + 3\phi_2$.
We have that if $e \in E\backslash X$, $|\phi(e)| \in \{1, 2\}$, and if $e \in X$, $\phi(e) = \phi_3(e) \pm 3$.
In both cases, $0 < |\phi(e)| \leq 6$, so ϕ is a nowhere-zero 6-flow of G. □

Chapter 5

Tutte Polynomial

Tutte graph polynomial is arguably the most important and studied polynomial in graph theory. It is related with many aspects of graph theory, like the chromatic polynomial and the flow polynomial.
Note that we will refer to an edge that is neither a bridge nor a loop as **ordinary**.

5.1 Tutte Polynomial and Recipe Theorem

Definition (Tutte polynomial)**.** *Let $G = (V, E)$ be a graph.*
*The **Tutte polynomial** of G can be defined recursively by:*

$$T_G(x, y) := \begin{cases} T_{G/e}(x, y) + T_{G\backslash e}(x, y) & \textit{if } e \textit{ is an ordinary edge} \\ xT_{G/e}(x, y) & \textit{if } e \textit{ is a bridge} \\ yT_{G\backslash e}(x, y) & \textit{if } e \textit{ is a loop} \\ 1 & \textit{if } E = \emptyset \end{cases}$$

It is important to see that the polynomial is well defined, because it does not matter which order the edge e is chosen in the recursive formula.

Theorem. *Let $G = (V, E)$ be a graph.*
If $e, e' \in E$, then applying the recurrence with respect to edge e and then with respect to edge e' is the same as with the reverse order.

Proof. We need some observations:
(1.a) If e is a bridge in G, it remains a bridge in G/e' and $G\backslash e'$.
(1.b) If e is a loop in G, it remains a loop in G/e' and $G\backslash e'$.
(2.a) If e is ordinary in G and there is a cutset K containing e but not e', then e is ordinary in G/e'.

(2.b) If e is ordinary in G and there is a cycle containing e but not e', then e is ordinary in $G \backslash e'$.
(1.a),**(1.b)** and **(2.b)** are easy to see, **(2.a)** is true because the cutset K containing e but not e' remains a cutset in G/e', so e is not a bridge in G/e'.
e is not even a loop too in G/e' because the existence of K implies that e and e' are not parallel edges.
For each possible combination of edges e and e' in **(1.a)**,**(1.b)**,**(2.a)** and **(2.b)**, it is easy to see that swapping the order in which e and e' are deleted or contracted does not matter.
We have two other cases to consider:
(3.a) If e is ordinary in G and any cutset containing e contains also e', then e' is an ordinary edge in G and a loop in G/e, and e is a loop in G/e'.
(3.b) If e is ordinary in G and any cycle containing e contains also e', then e' is an ordinary edge in G and a bridge in $G \backslash e$, and e is a bridge in $G \backslash e'$.
Note that **(3.a)**$\Leftrightarrow$ e and e' are two parallel edges.
In this case we have that e is a loop in G/e' and e' is a loop in G/e. This symmetry implies that the order in which we take e and e' does not matter.
Finally, we can conclude the proof for case **(3.b)** by the same argument. $\square$

So the Tutte Polynomial is well defined.
Now, we want to see an important theorem, which states that the Tutte Polynomial can describe all the other graph invariants with a particular property.

Theorem (Recipe Theorem). *Let $\mathfrak{G}$ be a minor-closed class of graphs. There exists a unique graph invariant $U : \mathfrak{G} \longrightarrow \mathbb{Z}[x, y, z, s, t]$, known as* ***Universal Polynomial*** *of G, such that, if $G = (V, E) \in \mathfrak{G}$:*

$$U(G) := \begin{cases} xf(G/e) & \text{if } e \text{ is a bridge in } G \\ yf(G \backslash e) & \text{if } e \text{ is a loop in } G \\ zf(G/e) + sf(G \backslash e) & \text{if } e \text{ is an ordinary edge in } G \\ t^{|V|} & \text{if } E = \emptyset \end{cases}$$

We also have $U(G) = t^{c(G)} z^{r(G)} s^{n(G)} T_G\left(\frac{x}{z}, \frac{y}{s}\right)$.

Proof. Uniqueness follows by induction on the number of edges, so we just need to prove the formula.
If $G = \bar{K}_n$, then $U(G) = t^n, c(G) = n, r(G) = n(G) = 0$ and $T_G\left(\frac{x}{z}, \frac{y}{s}\right) = 1$, so the formula is true.
If G consists only of k bridges and l loops, we have that $U(G) = t^{c(G)} x^k y^l$.
We also have $r(G) = k$, $n(G) = l$ and $T_G\left(\frac{x}{z}, \frac{y}{s}\right) = \left(\frac{x}{z}\right)^k \left(\frac{y}{s}\right)^l$, so we have
$U(G) = t^{c(G)} x^k y^l = t^{c(G)} z^{r(G)} s^{n(G)} \left(\frac{x}{z}\right)^{r(G)} \left(\frac{y}{s}\right)^{n(G)} = t^{c(G)} z^{r(G)} s^{n(G)} T_G\left(\frac{x}{z}, \frac{y}{s}\right)$.

Now, let e be an ordinary edge of G.
We have that $c(G) = c(G/e) = c(G\backslash e) \Rightarrow r(G/e) = r(G) - 1, r(G\backslash e) = r(G)$, $n(G/e) = n(G)$ and $n(G\backslash e) = n(G) - 1$.
Then, we can proceed by induction on the number of ordinary edges to obtain:
$U(G) = zU(G/e)+sU(G\backslash e) = z{\cdot}t^{c(G)}z^{r(G)-1}s^{n(G)}T_{G/e}\left(\frac{x}{z},\frac{y}{s}\right)+s{\cdot}t^{c(G)}z^{r(G)}s^{n(G)-1}T_{G\backslash e}\left(\frac{x}{z},\frac{y}{s}\right)$
$= t^{c(G)}z^{r(G)}s^{n(G)}T_G\left(\frac{x}{z},\frac{y}{s}\right)$. □

We will now connect the Tutte Polynomial with the Chromatic Polynomial and the Flow Polynomial:

Theorem. *Let $G = (V, E)$ be a graph, $P(G, k)$ the chromatic polynomial of G and $F(G, k)$ the flow polynomial of G.*
We have that:
(1) $P(G, k) = (-1)^{r(G)}k^{c(G)}T_G(1 - k, 0)$
(2) $F(G, k) = (-1)^{n(G)}T_G(0, 1 - k)$.

Proof. We can use the Recipe Theorem to prove both equivalences.
Remember the recursive definitions of $P(G, k)$ and $F(G, k)$:

$$P(G,k) = \begin{cases} P(G\backslash e, k) - P(G/e, k) & \text{if } e \text{ is ordinary} \\ 0 & \text{if } e \text{ is a loop} \\ (k-1)P(G/e, k) & \text{if } e \text{ is a bridge} \\ k^{|V|} & \text{if } E = \emptyset \end{cases}$$

$$F(G,k) = \begin{cases} F(G/e, k) - F(G\backslash e, k) & \text{if } e \text{ is ordinary} \\ (k-1)F(G\backslash e, k) & \text{if } e \text{ is a loop} \\ 0 & \text{if } e \text{ is a bridge} \\ 1 & \text{if } E = \emptyset \end{cases}$$

Let $U(x, y, z, s, t)$ be the Universal Polynomial of G, it is easy to see that
$P(G, k) = U(k - 1, 0, -1, 1, k) = (-1)^{r(G)}k^{c(G)}T_G(1 - k, 0)$ and
$F(G, k) = U(0, k - 1, 1, -1, 1) = (-1)^{n(G)}T_G(0, 1 - k)$. □

There is also a closed formula for the Tutte Polynomial:

Theorem. *Let $G = (V, E)$ be a graph. We have:*

$$T_G(x, y) = \sum_{A \subseteq E} (x - 1)^{r(G)-r_G(A)}(y - 1)^{n_G(A)}.$$

Proof. Start by setting $R_G(x, y) := \sum\limits_{A \subseteq E} x^{r(G)-r_G(A)}y^{n_G(A)}$.
We want to prove that $T_G(x, y) = R_G(x - 1, y - 1)$.

We need to verify that:
(1) $R_G(x,y) = 1$ if $E = \emptyset$
(2) $R_G(x,y) = (x+1)R_{G\backslash e}(x,y)$ if e is a bridge
(3) $R_G(x,y) = (y+1)R_{G/e}(x,y)$ if e is a loop
(4) $R_G(x,y) = R_{G\backslash e}(x,y) + R_{G/e}(x,y)$ if e is ordinary.
If $E = \emptyset$, we have $r(G) = n(G) = 0$, so $R_G(x,y) = 1$.
Consider now $A \subseteq E$, then $e \notin A \Rightarrow r_G(A) = r_{G\backslash e}(A)$.
On the other hand, if $e \in A$, then:

$$r_{G\backslash e}(A\backslash e) = \begin{cases} r_G(A) - 1 & \text{if } e \text{ is a bridge} \\ r_G(A) & \text{if } e \text{ is a loop} \end{cases}$$

and $r_{G/e}(A\backslash e) = r_G(A) - 1$ if e is ordinary or a bridge.
Let $e \in E$ be a bridge. Then:

$$\begin{aligned} R_G(x,y) &= \sum_{A\subseteq E\backslash e} x^{r(G)-r_G(A)} y^{n_G(A)} + \sum_{e\in A\subseteq E} x^{r(G)-r_G(A)} y^{n_G(A)} \\ &= x \sum_{A\subseteq E\backslash e} x^{r_{G\backslash e}(E\backslash e)-r_{G\backslash e}(A)} y^{|A|-r_{G\backslash e}(A)} + \sum_{H=A\backslash e} x^{r_{G\backslash e}(E\backslash e)+1-(r_{G\backslash e}(H)+1)} y^{|H|+1-(r_{G\backslash e}(H)+1)} \\ &= (x+1)R_{G\backslash e}(x,y). \end{aligned}$$

If e is a loop we use the same argument.
If e is ordinary, then:

$$\begin{aligned} R_G(x,y) &= \sum_{A\subseteq E\backslash e} x^{r(G)-r_G(A)} y^{n_G(A)} + \sum_{e\in A\subseteq E} x^{r(G)-r_G(A)} y^{n_G(A)} \\ &= \sum_{A\subseteq E\backslash e} x^{r_{G\backslash e}(E\backslash e)-r_{G\backslash e}(A)} y^{|A|-r_{G\backslash e}(A)} + \sum_{H=A\backslash e} x^{r_{G/e}(E\backslash e)+1-(r_{G/e}(H)+1)} y^{|H|+1-(r_{G/e}(H)+1)} \\ &= R_{G\backslash e}(x,y) + R_{G/e}(x,y). \end{aligned}$$

So we have $T_G(x,y) = R_G(x-1, y-1) = \sum_{A\subseteq E} (x-1)^{r(G)-r_G(A)}(y-1)^{n_G(A)}$. □

There are several interesting specialisations of the Tutte Polynomial, we will give some examples:
(i) $T_G(2,1)$ evaluates the number of forests in G.
(ii) $T_G(1,1)$ evaluates the number of spanning forests of G.
(iii) $T_G(1,2)$ evaluates the number of spanning subgraphs of G with $c(G)$ connected components.
(iv) $T_G(2,2)$ evaluates the number of spanning subgraphs of G.
All these properties can be proven by the previous theorem.

Appendix A

Flow network and Max-Flow Min-Cut theorem

Flows are often studied on a particular type of digraph, called network, defined as follows:

Definition. *Flow Network and Feasible Flow A* ***network*** *is a connected digraph* $G = (V, E)$ *with two vertices* $s, t \in V$*, named respectively* ***source*** *and* ***sink****, such that* $\delta^-(\{s\}) = \emptyset$ *and* $\delta^+(\{t\}) = \emptyset$*, together with a* ***capacity function*** $c : E \longrightarrow \mathbb{R}^+ \cup \{0\}$.
A ***feasible flow*** *on* G *is a function* $\phi : E \longrightarrow \mathbb{R}^+ \cup \{0\}$ *such that* $\phi(e) \leq c(e), \forall e \in E$ *and* $\sum_{e \in \delta^+(\{v\})} \phi(e) = \sum_{e \in \delta^-(\{v\})} \phi(e), \forall v \in V \backslash \{s, t\}$.
A ***flow network*** *is a network* G *with a feasible flow* ϕ.

Observation. Consider a network $G = (V, E)$ with source s, sink t and a feasible flow ϕ. By the previous definition we have that

$$\sum_{v \in V} \left(\sum_{e \in \delta^+(\{v\})} \phi(e) - \sum_{e \in \delta^-(\{v\})} \phi(e) \right) =$$

$$= \sum_{e \in \delta^+(\{s\})} \phi(e) - \sum_{e \in \delta^-(\{t\})} \phi(e) + \sum_{v \in V \backslash \{s,t\}} \left(\sum_{e \in \delta^+(\{v\})} \phi(e) - \sum_{e \in \delta^-(\{v\})} \phi(e) \right) =$$

$$= \sum_{e \in \delta^+(\{s\})} \phi(e) - \sum_{e \in \delta^-(\{t\})} \phi(e) = 0$$

$$\Rightarrow \sum_{e \in \delta^+(\{s\})} \phi(e) = \sum_{e \in \delta^-(\{t\})} \phi(e).$$

We will say that $v(\phi) := \sum_{e \in \delta^+(\{s\})} \phi(e) = \sum_{e \in \delta^-(\{t\})} \phi(e)$ is the **flow value** of the flow ϕ.

In the following, for every cutset (S, S^c) defined on a network G, we suppose that $s \in S$ and $t \in S^c$.
Moreover, if $U \subseteq V$, $u \in V$, we define:
(i) $\delta^+(U) := \bigcup_{v \in U} \delta^+(\{v\})$
(ii) $\delta^-(U) := \bigcup_{v \in U} \delta^-(\{v\})$
(iii) $\delta^+(u, U) = \delta^+(\{u\}) \cap \delta^-(U) = \delta^-(U, u)$
(iv) $\delta^+(U, u) = \delta^+(U) \cap \delta^-(\{u\}) = \delta^-(u, U)$.

Definition (Cutset capacity)**.** *Let $G = (V, E)$ be a network with capacity c, (S, S^c) be a cutset of G.*
The ***cutset capacity*** *of (S, S^c) is defined as:*
$c(S, S^c) := \sum_{v \in S} \sum_{e \in \delta^+(v, S^c)} c(e)$.

Definition. *Let $G = (V, E)$ be a network with capacity c.*
A ***maximal flow*** *of G is a feasible flow ϕ on G of maximal flow value $v(\phi)$.*
A ***mimal cut*** *of G is a cutset (S, S^c) on G of minimal cutset capacity $c(S, S^c)$.*

We want to define an algorithm to find a maximal flow on a network G.

Definition (Ford-Fulkerson algorithm)**.** *Let $G = (V, E)$ be a network with capacity c, source s and sink t, and let ϕ be a feasible flow on G.*
Step 1. *Define $H = \{s\}$.*
Step 2. *Consider all $v \in H$. For each $e = (v, w)$ incident with v, add w to H if one of the following holds:*
(a) *$e \in \delta^+(\{v\})$ and $\phi(e) < c(e)$*
(b) *$e \in \delta^-(\{v\})$ and $\phi(e) > 0$.*
Repeat step 2 until you can not add vertices to H anymore.
Step 3. *If $t \notin H$ stop the algorithm, otherwise there exists an undirected path $s = u_0u_1...u_{k-1}u_k = t$ such that for all $e_i = (u_i, u_{i+1}), i = 0, ..., k-1$, we have that either $e_i \in \delta^+(\{u_i\})$ and $c(e_i) > \phi(e_i)$ or $e_i \in \delta^-(\{u_i\})$ and $\phi(e_i) > 0$.*
Define:

$$\epsilon_i := \begin{cases} c(e_i) - \phi(e_i) & \text{if } e_i \in \delta^+(\{u_i\}) \\ \phi(e_i) & \text{if } e_i \in \delta^-(\{u_i\}) \end{cases}$$

and $\epsilon = \min_{i=0,...,k-1} \epsilon_i$.
Step 4. *Define a new flow*

$$\bar{\phi}(e) := \begin{cases} \phi(e_i) + \epsilon & \text{if } e = e_i \text{ and } e_i \in \delta^+(\{u_i\}) \\ \phi(e_i) - \epsilon & \text{if } e = e_i \text{ and } e_i \in \delta^-(\{u_i\}) \\ \phi(e) & \text{otherwise} \end{cases}$$

and restart from step 1 replacing ϕ with $\bar{\phi}$.

The Ford-Fulkerson algorithm gives a new feasible flow (it is easy to see) on G that is a maximal flow, we will prove it during the proof of the next theorem.

Theorem (Max-Flow Min-Cut)**.** *Let $G = (V, E)$ be a flow network with capacity c. Let ϕ be a maximal flow on G and (S, S^c) be a minimal cut of G, then $v(\phi) = c(S, S^c)$.*

Proof. Consider a feasible flow ϕ and a cutset (S, S^c), and let s, t be the source and sink of G.
We have that:

$$\begin{aligned}
v(\phi) &= \sum_{e\in\delta^+(\{s\})} \phi(e) + \sum_{v\in S\setminus\{s\}} \underbrace{\left(\sum_{e\in\delta^+(\{v\})} \phi(e) - \sum_{e\in\delta^-(\{v\})} \phi(e)\right)}_{=0} \\
&= \sum_{v\in S}\left(\sum_{e\in\delta^+(v,S^c)} \phi(e) - \sum_{e\in\delta^-(v,S^c)} \phi(e)\right) \\
&\leq \sum_{v\in S}\sum_{e\in\delta^+(v,S^c)} \phi(e) \\
&\leq \sum_{v\in S}\sum_{e\in\delta^+(v,S^c)} c(e) = c(S, S^c).
\end{aligned}$$

So, we have that $v(\phi) \leq c(S, S^c)$.
We need to prove that there exist a flow ϕ and a cutset (S, S^c) such that $v(\phi) = c(S, S^c)$, in this case ϕ is a maximal flow and (S, S^c) is a minimal cut.
Let ϕ be the flow obtained by the Ford-Fulkerson algorithm and define
$S := \{s\}\cup\{v \in V|$ there exists an undirected path $s = u_0u_1, ..., u_{k-1}u_k = v$ such that, for all $e_i = (u_i, u_{i+1}), i = 0, ..., k-1$, we have that $e_i \in \delta^+(\{u_i\})$ and $c(e_i) > \phi(e_i)$, or $e_i \in \delta^-(\{u_i\})$ and $\phi(e_i) > 0\}$.
Then $t \notin S$ and (S, S^c) is a cutset of G.
Furthermore, we have $c(S, S^c) = \sum_{v\in S}\sum_{e\in\delta^+(v,S^c)} c(e) = \sum_{v\in S}\sum_{e\in\delta^+(v,S^c)} \phi(e)$.

Finally, since $\sum_{v\in S}\sum_{e\in\delta^-(v,S^c)} \phi(e) = 0$, we have $v(\phi) = \sum_{v\in S}\left(\sum_{e\in\delta^+(v,S^c)} \phi(e) - \sum_{e\in\delta^-(v,S^c)} \phi(e)\right) = \sum_{v\in S}\sum_{e\in\delta^+(v,S^c)} \phi(e) = c(S, S^c)$. □

Flow network theory has many applications, like the construction of water pipes or electrical networks, and the study of other graph properties, for example:

Theorem (Menger)**.** *Let $G = (V, E)$ be a connected graph, $s, t \in V$. The minimal cardinality of a cutset (S, S^c) with $s \in S, t \in S^c$ is equal to the maximum number of edge-disjoint paths from s to t.*

Proof. Suppose that the maximum number of edge-disjoint paths from s to t is equal to $k \in \mathbb{N}$.
We want to define a flow network, by giving G an orientation, a capacity c and a feasible flow ϕ.
For each $e \in E$ incident with s or t, direct e so that $\delta^-(\{s\}) = \delta^+(\{t\}) = \emptyset$.
Consider now k edge-disjoint paths $P_1(s,t), ..., P_k(s,t)$ from s to t. For each $e \in P_i(s,t)$, direct e with the same orientation of the path.
Direct every other edge of G in any way.
Finally, define $c(e) = 1$ and $\phi(e) = 0$,$\forall e \in E$, and apply the Ford-Fulkerson algorithm.
We have that every iteration of the algorithm takes a path $P_i(s,t)$ and changes the flow only on its edges, namely $\phi(e) = 1, \forall e \in P_i(s,t)$.
After k iterations the algorithm stops, because there are no more paths from s to t that are edge-disjoint with every $P_i(s,t)$, so the maximal flow has value $v(\phi) = k$.
We can conclude by the max-flow min-cut theorem. $\square$

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