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Corso di Laurea Magistrale in Matematica

On the homology of configuration spaces of graphs

Tesi di Laurea Magistrale

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Introduction

In this thesis we deal with unordered configuration spaces of finite graphs. The *k-th unordered configuration space* of a topological space X is a space $B_k(X)$ that parametrizes all the possible positions of k indistinguishable particles on X. Configuration spaces are classical objects in mathematics and topological problems related to them appear in various fields such as physics and robotics, contributing with new perspectives and tools. A classical example is $B_k(\mathbb{C})$, which is the classifying space for the braid group on k strands. More in general, configuration spaces of manifolds have been largely studied, and they find applications in many areas of geometry and topology. We are interested in configuration spaces of graphs, and in particular in their homology. The main tool for the computation will be the Świątkowski complex of a graph. Described by An, Drummond-Cole and Knudsen in [1] in its algebraic version, the Świątkowski complex come essentially from the cellular chains of a cubical complex constructed by Świątkowski in [12], which is a deformation retract of the configuration space of the graph. A class in the n-degree homology of the $k-th$ configuration space can be pictured as k particles on the graph, n of which are moving at the same time without collisions. Performing these moves locally in disjoint subgraphs at the same time gives rise to higher degree classes, called *toric classes*. In this work, the focus is on the description of the generators of the homology in low degree.

In the first chapter we establish some conventions about graphs and configuration spaces. Then, we introduce the Świątkowski complex and its variants, as presented in [1]. The importance of this tool is due to Theorem 1.2.7, which states that the homology of the configuration space of a graph can be computed through the homology of the Świątkowski complex. A partial proof of this result is given in section 1.4, where we report the construction of the cubical complex due to Świątkowski. We also give several examples, in particular we define *star classes* and *loop classes* and we go through some relations they are subject to. These are classes in the first homology of the configuration space of some small graphs, which will play a key role in the remaining. Lastly, we state some results from [3] and [2] about a stabilization phenomenon observed while increasing the number of particles considered in the configurations. Stabilization maps consist in adding a still particle on an edge of the graph. This induces a $\mathbb{Z}[E]$-module structure on $H_*(B(\Gamma))$, where $\mathbb{Z}[E]$ is the ring of polynomials in the edges of Γ.

The second chapter is dedicated to the first homology group. Following [1], we prove that star and loop classes generates the first homology of the configuration space of a connected graph as a $\mathbb{Z}[E]$-module (Theorem 2.2.3). Then, we retrace the path that led Ko and Park in [9] to the proof of an explicit formula that describes the first homology in terms of combinatorial invariants of the graph. In section 1.3, we verify that the already known relations on star and loop classes form a complete set and so they give a presentation of the first homology. This fact seemed to be already understood, but not proven yet.

In the third chapter we deal with the second homology. We describe the generators in the planar case, a result that is proven in [4]. It turns out that in non-planar case the presence of "exotic" classes is related to the occurrence in the graph of certain cycles, called *pesky*. The study of these cycles reveals new kinds of classes that need to be considered among the generators for the second homology in the general case. An example was given in [4], we provide two different ones (Examples 3.3.2 and 3.3.3), which correspond to two new classes. Finally, in Definition 3.4.2, we describe a new kind of functoriality the homology of configuration spaces enjoys, which correspond to the contraction of an edge in the graph. This new tool, which is interesting in itself, could be used in the study of the generators for the second homology for non-planar graphs.

Introduzione

In questa tesi studiamo gli spazi di configurazioni su grafi finiti. Il k-*esimo spazio di configurazioni non ordinate* su uno spazio topologico X è uno spazio $B_k(X)$ che parametrizza tutte le possibili posizioni di k particelle indistinguibili su X. Gli spazi di configurazioni sono un argomento classico in matematica e problemi topologici ad essi correlati compaiono in diversi campi, come la fisica e la robotica, contribuendo con nuovi punti di vista e nuovi strumenti. Un tipico esempio è dato da $B_k(\mathbb{C})$, lo spazio di classificazione del gruppo di trecce su k stringhe. Più in generale, sono ampiamente studiati gli spazi di configurazioni su varietà, con applicazioni in svariate aree della geometria e della topologia. In questo elaborato ci occuperemo degli spazi di configurazioni su grafi e, in particolare, della loro omologia. Il principale strumento di calcolo sarà il complesso di Świątkowski associato a un grafo. Descritto nella sua versione algebrica da An, Drummond-Cole e Knudsen in [1], il complesso di Świątkowski proviene essenzialmente dal complesso di celle associato a un complesso cubico costruito da Świątkowski in [12], che è un retratto per deformazione dello spazio di configurazioni sul grafo. Una classe nell'omologia di grado n del k-esimo spazio di configurazioni può essere pensata come k particelle sul grafo, n delle quali si muovono contemporaneamente e senza collisioni. Riproducendo queste mosse localmente, in sottografi tra loro disgiunti, si ottengono classi di grado più alto, dette classi *toriche*. In questo lavoro ci concentreremo sui generatori dell'omologia in grado basso.

Nel primo capitolo stabiliamo alcune convenzioni sui grafi e sugli spazi di configurazioni. Quindi introduciamo il complesso di Świątkowski e le sue varianti, così come presentati in [1]. L'importanza di questo strumento è data dal Teorema 1.2.7, il quale stabilisce che l'omologia dello spazio di configurazioni di un grafo coincide con l'omologia del complesso di Świątkowski. Una parziale dimostrazione di questo risultato è fornita dalla sezione 1.4, dove riportiamo la costruzione del complesso cubico di Świątkowski. Proponiamo svariati esempi, in particolare definiamo le classi *star* e *loop* e alcune relazioni a cui sono soggette. Si tratta di classi nell'omologia di grado uno dello spazio di configurazione di certi piccoli grafi e ricopriranno un ruolo cruciale nel seguito. Infine, enunciamo alcuni risultati da [3] e [2] riguardo un fenomeno di stabilizzazione cui è soggetta l'omologia col crescere del numero di particelle considerate nelle configurazioni. Le mappe di stabilizzazione consistono nell'aggiungere una particella ferma su un lato del grafo. Questo induce una struttura di $\mathbb{Z}[E]$-modulo su $H_*(B(\Gamma))$, dove $\mathbb{Z}[E]$ è l'anello dei polinomi nei lati di Γ.

Il secondo capitolo è dedicato all'omologia di grado uno. Seguendo [1], dimostriamo che classi *star* e *loop* generano il primo grado di omologia dello spazio di configurazioni di un qualsiasi grafo connesso come $\mathbb{Z}[E]$-modulo (Teorema 2.2.3). Successivamente, ripercorriamo i passi che hanno portato Ko e Park in [9] alla dimostrazione di una formula esplicita che descrive l'omologia prima in termini di invarianti combinatori del grafo. Nella sezione 1.3 verifichiamo poi che alcune ben note relazioni a cui le classi *star* e *loop* sono soggette formano un sistema completo e forniscono quindi una presentazione del primo modulo di omologia. Questo fatto sembra fosse già compreso, ma non ancora dimostrato.

Nel terzo capitolo ci occupiamo invece dell'omologia in grado due. Descriviamo i generatori nel caso planare, un risultato dimostrato in [4]. Si scopre che la presenza di classi "esotiche" nel caso non planare è legata all'occorere di particolari cicli nel grafo, detti cicli *pesky* (fastidiosi). Lo studio di questi cicli rivela nuovi tipi di classi che è necessario considerare tra i generatori dell'omologia seconda per grafi generici. Un esempio viene fornito in [4], noi ne presentiamo altri due (Esempi 3.3.2 e 3.3.3), che corrispondono a due nuove classi. Infine, nella Definizione 3.4.2 descriviamo un nuovo tipo di funtorialità di cui gode l'omologia degli spazi di configurazioni, corrispondente alla contrazione di un lato del grafo. Questo nuovo strumento, che è interessante di per sé, potrà essere usato per lo studio dei generatori dell'omologia seconda per grafi non planari.

Chapter 1

The Świątkowski complex

The aim of this chapter is to establish the notations about graphs and configuration spaces and then to introduce the main tool that we will use to study the homology of configuration spaces of graphs: the Świątkowski complex. The examples that are given are very important, not only to familiarize with the subject, but also because they define some classes that generate the first homology group of the configuration space of any connected graph, as proven in the next chapter.

1.1 Configuration spaces and graphs

In this section we introduce the main objects of this work, in order to set conventions and notations.

Definition 1.1.1. Let X be a topological space. The k^{th} *ordered configuration space* of X is the set of k-tuples of pairwise distinct point:

$$C_k(X) = \{(x_1, ..., x_k) \in X^k | x_i \neq x_j \forall i \neq j\}$$

equipped with the subspace topology induced by the product topology on X^k. We are interested in the k^{th} *(unordered) configuration space* of X, which is the quotient of $C_k(X)$ by the natural action of the symmetric group $\mathfrak{S}_k$:

$$B_k(X) = C_k(X)/\mathfrak{S}_k.$$

Finally, the *(unordered) configuration space of* X is

$$B(X) = \coprod_{k \geq 0} B_k(X).$$

Configuration spaces are well-studied objects, which appear in many areas of mathematics. The most classical example is given next.

Example 1.1.2. Let us consider $X = \mathbb{C}$. Then $C_k(\mathbb{C})$ is the complement of the braid arrangement, i.e. the arrangement given by the hyperplanes $\{x_i = x_j\}_{1 \leq i < j \leq k} \subset \mathbb{C}^k$. The fundamental group of $C_k(\mathbb{C})$ is the *pure braid group* $P(k)$. We can think of a pure braid as the motion without collisions of k particles in $\mathbb{C}$, which, at the end of the motion, go back to their original positions. $B_k(\mathbb{C})$ is the orbit space of the action of $\mathfrak{S}_k$ on $C_k(\mathbb{C})$ and its fundamental group is the *braid group* $B(k)$. In fact, a braid correspond to the motion of k indistinguishable particles, each of which reaches one of the point occupied by a particles at the beginning. Because of this example, the fundamental group of the (ordered) configuration space of X is called *(pure) braid groups* of X. $C_k(\mathbb{C})$ and $B_k(\mathbb{C})$ are $K(\pi, 1)$ spaces and hence they are classifying spaces for their fundamental groups. The same is true for configuration spaces of graphs (see, for example, [12]), so we could also refer to the subject of this thesis as the homology of graph braid groups, but we will not investigate this aspect.

We are interested in unordered configurations on graphs. The definitions about graphs that follow are given following [1] and [4]. The category of graphs in which we will work, *Gph*, is not the usual one. In particular, we are going to use a non standard notion for graph morphisms.

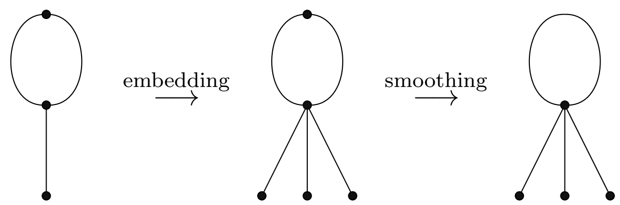
Definitions 1.1.3. A *graph* Γ is a finite 1-dimensional CW-complex. We denote $V(\Gamma)$, or simply V, the set of 0-cells, or *vertices*, and $E(\Gamma)$, or simply E, the set of 1-cells, or *edges*. A *half-edge* is the end of an edge or, more precisely, a point in the preimage of a vertex v under the attaching map of a 1-cell. We write $H(v)$ for the set of half-edges incident to v. The *degree*, or *valence*, $d(v)$ of v is the cardinality of $H(v)$.

Definition 1.1.4. A *graph morphism* is a continuous map between graph $f : \Gamma_1 \to \Gamma_2$ such that

- f is injective
- $f^{-1}(V(\Gamma_2)) \subseteq V(\Gamma_1)$

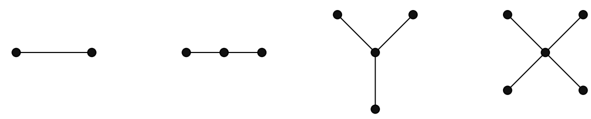
Remark 1.1.5. A graph morphism is usually called a *graph embedding* if it takes vertices to vertices and a *smoothing* if it is also a homeomorphism. Every graph morphism can be written as an embedding followed by a smoothing. We say that Γ' is a *subgraph* of Γ if there is an embedding $\Gamma' \to \Gamma$, while Γ' is said to be a *subdivision* if the map is a smoothing.

Example 1.1.6. The topological immersion of the first graph in figure into the third one is a graph morphism in the sense of Definition 1.1.4.



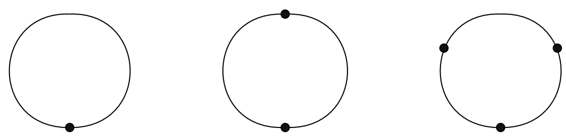
Observe that this morphism does not map vertices to vertices.

Example 1.1.7. We will often refer to the following classes of small graphs:



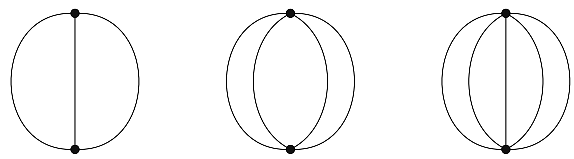
S_1 S_2 S_3 S_4

Figure 1.1: The *star graph* S_n is the cone on the discrete space of n points.



C_1 C_2 C_3

Figure 1.2: The *cycle graphs*: the loop C_1 and its subdivision



Θ_3 Θ_4 Θ_5

Figure 1.3: The *theta graphs*: Θ_n is the graph with 2 vertices, n edges and no self-loops.

The homology of the configuration space of a graph is influenced by some basic characteristics of the graph, such as the planarity or some connectivity properties. For the planarity, we use the classical characterization due to Kuratowski (see, for example, [6, Chapter 10]):

Definition 1.1.8. A graph is said to be *planar* if it does not contain as a subgraph a subdivision of the complete graph K_5 or of the complete bipartite graph $K_{3,3}$.

K_5 $K_{3,3}$

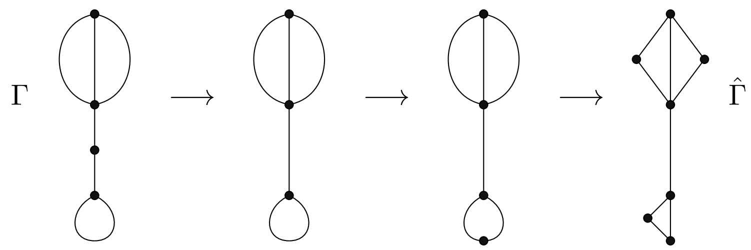


Figure 1.4: A graph Γ and its minimal simplicial model $\hat{\Gamma}$

When talking about connectedness properties, we will always assume the graph is simple:

Definition 1.1.9. A graph is said to be *simple* if it has no loops (also called self-loops, meaning edges that connect a vertex to itself) and no pair of edges between the same vertices. In particular, a simple graph is a simplicial complex. A *minimal simplicial model* of Γ is a simple graph $\hat{\Gamma}$ homeomorphic to Γ and such that the image of any non-trivial smoothing with source $\hat{\Gamma}$ is not simple.

Remark 1.1.10. Every graph admits a minimal simplicial model, which is unique up to isomorphism. Indeed, it suffices to smooth every bivalent vertex (if Γ has no components homeomorphic to S^1, in which case two bivalent vertices need to be left), add a bivalent vertex to each self-loop obtained and eventually add a bivalent vertex to all but one of each set of multiple edges. An example is given in Figure 1.4.

For our purposes, assuming that Γ is a minimal simplicial model of itself is not restrictive because the homology of the configuration space depends only on the topological space underneath the graph. On the other hand, this hypothesis is important. In particular, the notion of k-cut may not behave well on a graph with self-loop or multiple edges.

Definition 1.1.11. Let Γ be a graph. A $k-cut$ in Γ is a set of k vertices S whose removal separate at least 2 vertices. A $S-component$ of Γ is the closure in Γ of a connected component of $\Gamma \setminus S$.

Definition 1.1.12. A graph Γ is $k-connected$ if its minimal simplicial model has at least $k+1$ vertices and no $(k-1)$-cut. For $k = 1$ this equals to be connected. For $k = 2, 3$ we also use the respective terms *biconnected* and *triconnected*.

Remark 1.1.13. If Γ is k-connected, then it is also l-connected for each $1 \leq l \leq k$.

Example 1.1.14. The complete graph K_n, meaning a graph with n vertices and one edge joining each pair of vertices, is $(n-1)$-connected. In fact, K_n

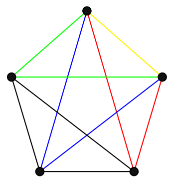
is a minimal simplicial model of itself and the removal of any $n-2$ vertices leaves two vertices connected by an edge. K_n is not n-connected because of a matter of size.

Example 1.1.15. The complete bipartite graph $K_{m,n}$ is a graph with $m+n$ vertices partitioned in two sets of size m and n, and every possible edge that could connect two vertices belonging to different sets. Suppose $m \leq n$, then $K_{m,n}$ is m-connected. Indeed, $K_{m,n}$ is a minimal simplicial model of itself and the removal of $m-1$ vertices leaves at least one vertex for each set, which guarantees the connection of the complement. $K_{m,n}$ is not $m+1$ connected because the m-cut given by one of the two sets the vertices are partitioned in, disconnects the graph.

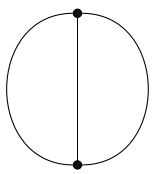
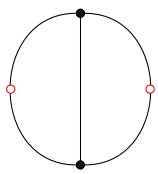
An equivalent description of k-connectedness is provided by the following classical result, due to Menger [11] (see also [6]):

Theorem 1.1.16. *A graph Γ is k-connected if and only if, for distinct vertices x and y in a minimal simplicial model $\hat{\Gamma}$, there exist k paths in $\hat{\Gamma}$ joining x and y, disjoint excepts at endpoints.*

Example 1.1.17. In K_5 any pair of vertices is connected by 4 disjoint paths, as shown in the figure:



Example 1.1.18. Even if in Θ_3 the two vertices are connected by three disjoint path, Θ_3 is only 2-connected. In fact, a minimal simplicial model of Θ_3 does not enjoy this proprety:

Θ_3 :  $\hat{\Theta}_3$: 

The same hold for any Θ_n with $n \geq 2$.

We state another useful property of k-connected graphs (see [6, Proposition 9.4]):

Proposition 1.1.19. *Let Γ be a k-connected graph, any two set of vertices in a minimal simplicial model $\hat{\Gamma}$, each of size k, can be joined by k disjoint paths.*

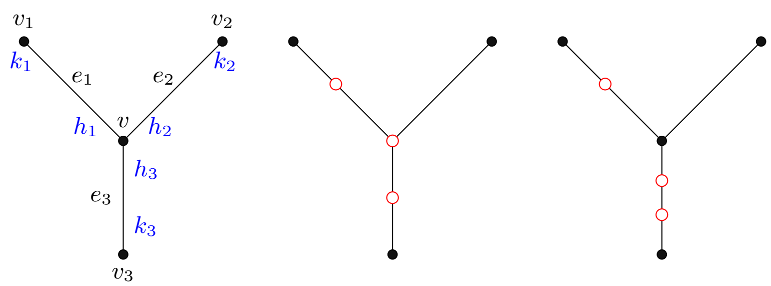


Figure 1.5: The graph S_3 with labeled vertices, edges and half-edges and two configuration on it. Red circles represent particles.

1.2 The Świątkowski complex

The main tool we will use in the study of the homology of the configuaration space of a graph is the Świątkowski complex. This complex was introduced, in the algebraic form we present it, by An, Drummond-Cole and Knudsen in [1]. However, it turns out to be isomorphic to the cellular chains of a cubical complex described in an earlier work of Świątkowski ([12]), as we will see in section 1.4. The importance of this object is given by Theorem 1.2.7.

Given a graph Γ we denote $\mathbb{Z}[E]$ the ring of polynomials in the set of variables $E = E(\Gamma)$. For each $v \in V = V(\Gamma)$, we consider the free $\mathbb{Z}$-module generated by the symbol $\varnothing$, v and the half-edges incident to v:

$$S(v) = \mathbb{Z}\langle \varnothing, v, h \in H(v)\rangle$$

Then we define the $\mathbb{Z}[E]$-module

$$S(\Gamma) = \mathbb{Z}[E] \otimes \bigotimes_{v \in V} S(v)$$

Remark 1.2.1. The idea is that a monomial in $\mathbb{Z}[E]$ represents a configuration on the edges of Γ. In $S(v)$ the generator v corresponds to a particle in v and we can think of a half-edge $h \in H(V)$ as a particles moving from v in the direction of h. In this perspective, the generator $\varnothing$ means that there are no particles around v.

Example 1.2.2. With the notation of Figure 1.5, the Świątkowski complex of S_3 is

$$\begin{aligned} S(S_3) = \mathbb{Z}[E] &\otimes \mathbb{Z}\langle \varnothing, v, h_1, h_2, h_3\rangle \otimes \mathbb{Z}\langle \varnothing_1, v_1, k_1\rangle \\ &\otimes \mathbb{Z}\langle \varnothing_2, v_2, k_2\rangle \otimes \mathbb{Z}\langle \varnothing_3, v_3, k_3\rangle \end{aligned}$$

The first configuration correspond to the elementary tensor

$$e_1 e_3 \otimes v \otimes \varnothing_1 \otimes \varnothing_2 \otimes \varnothing_3,$$

the second one to

$$e_1 e_3^2 \otimes \varnothing \otimes \varnothing_1 \otimes \varnothing_2 \otimes \varnothing_3.$$

The tensor $e_1 e_3 \otimes h_3 \otimes \varnothing_1 \otimes \varnothing_2 \otimes \varnothing_3$ represent in some sense the transition between the two states, with two stationary particles respectively on e_1 and e_3 and one moving from v in the direction of h_3.

We now endow $S(\Gamma)$ with the unique bigrading additive for the tensor product given by

$$|\varnothing| = (0,0), \ |e| = |v| = (0,1), \ |h| = (1,1)$$

for any $e \in E$, $v \in V$ and for any half-edge h. According to the interpretation described above, the second grading, also said *weight*, states the number of particles involved. The first grading, our *degree*, indicates how many particles are moving. We denote $S_k(\Gamma)_n$ the part of degree n and weight k of the Świątkowski complex.

In order to make $S(\Gamma)$ into a chain complex, we need a differential ∂. We define it on generators by $\partial(v) = \partial(\varnothing) = 0$ for any $v \in V$ and

$$\partial(h) = e(h) - v(h)$$

for any half-edge h, where $e(h)$ and $v(h)$ are the edge and the vertex corresponding to h. The differential of a tensor product is given by the rule

$$\partial(a \otimes b) = \partial(a) \otimes b + (-1)^{deg(a)} a \otimes \partial(b)$$

where $deg(a)$ is given by the first grading, the degree, which turns out to be the natural homological grading, since the differential is a map of bidegree $(-1, 0)$.

Remark 1.2.3. This is consistent with the interpretation of h as a moving particle, $\partial(h)$ is indeed the difference of the endpoints of its path.

For the sake of brevity, we omit the tensor symbol and all the factors $\varnothing$ writing elements of $S(\Gamma)$, using some sort of multiplicative notation. This could be misleading because not all the products among the generators are accepted. More specifically, an elementary tensor in $S(\Gamma)$ correspond to a product in which appears at most one factor for each $S(v)$, because at most one particle can sit in or move from or towards a vertex v.

Example 1.2.4. Referring to Figure 1.5 and Example 1.2.2, we will write just $e_1 e_3 v$ for the element of $S(S_3)$ representing the first configuration and $e_1 e_3^2$ for the second one. These have bidegree $(0,3)$, while $e_1 e_3 h_3$ has bidegree $(1,3)$ and it holds

$$\partial(e_1 e_3 h_3) = e_1 e_3^2 - e_1 e_3 v$$

Note that, for example, the writing $e_1 e_3 v h_3$ does not represent any element of $S(S_3)$.

Definition 1.2.5. The *Świątkowski complex* of Γ is the bigraded $\mathbb{Z}[E]$-module

$$S(\Gamma) = \mathbb{Z}[E] \otimes \bigotimes_{v \in V} S(v)$$

together with the differential ∂. We write $S_k(\Gamma)_n$ for the degree n and weight k component of $S(\Gamma)$.

This is part of the construction of a functor from $\mathcal{G}ph$ to the category of Abelian groups. In fact, a graph morphism $f : \Gamma_1 \to \Gamma_2$ induces a map $S(f) : S(\Gamma_1) \to S(\Gamma_2)$ which takes

- an edge $e \in E(\Gamma_1)$ to the edge of Γ_2 which contains $f(e)$;
- S_v to $S_{f(v)}$ in the obvious way if v is a vertex in Γ_1 such that $f(v)$ is a vertex of Γ_2;
- S_v to $\mathbb{Z}[e]$ when $f(v)$ is contained in the inner part of the edge $e \in E(\Gamma_2)$, by sending $\varnothing$ to 1, v to e and h to 0 for any $h \in H(v)$.

Remark 1.2.6. This map preserves the $\mathbb{Z}[E]$-module structure and induces a map in the homology of the Świątkowski complex

Our interest in the Świątkowski complex is given by the following result:

Theorem 1.2.7. *For any graph* Γ

$$H_*(B(\Gamma)) \cong H_*(S(\Gamma))$$

as functors from the category of graphs Gph *to the category of bigraded abelian groups.*

In section 1.4 we outline a geometric proof of the isomorphism at the level of objects. A complete proof of the theorem can be found in [1, Theorem 4.5]. A slightly more general statement is reported in section 1.5 and proven in [3, Theorem 2.10].

In the following, we will use a simplified version of the Świątkowski complex:

Definition 1.2.8. Let Γ be a graph and let v be a vertex, we define $\tilde{S}_v$ as the submodule of S_v spanned by $\varnothing$ and the differences $h_{ij} = h_i - h_j$ of half-edges at v. The *reduced Świątkowski complex of* Γ *relative to* U is

$$S^U(\Gamma) = \mathbb{Z}[E] \otimes \bigotimes_{v \in V \setminus U} S_v \otimes \bigotimes_{v \in U} \tilde{S}_v$$

When $U = V(\Gamma)$ we denote it by $\tilde{S}(\Gamma)$ and we call it just the *reduced Świątkowski complex of* Γ.

Remark 1.2.9. The reduced complexes are subcomplex of $S(\Gamma)$. The differential is given by $\partial(h_{ij}) = e(h_i) - e(h_j)$. There are no canonical generators for $\tilde{S}_v$ unless v has valence 1, in this case $\tilde{S}_v = \langle\varnothing\rangle$.

Example 1.2.10. The reduced complex of S_3 is

$$\begin{aligned}\tilde{S}(S_3) &= \mathbb{Z}[E] \otimes \mathbb{Z}\langle\varnothing, h_2 - h_1, h_3 - h_1\rangle \otimes \mathbb{Z}\langle\varnothing\rangle \otimes \mathbb{Z}\langle\varnothing\rangle \otimes \mathbb{Z}\langle\varnothing\rangle \\ &\cong \mathbb{Z}[E] \otimes \mathbb{Z}\langle\varnothing, h_2 - h_1, h_3 - h_1\rangle \\ &= \mathbb{Z}[E] \otimes \mathbb{Z}\langle\varnothing, h_{21}, h_{31}\rangle\end{aligned}$$

Theorem 1.2.11. *For any graph Γ without isolated vertices and for any $U \subset V(\Gamma)$, the complexes $S^U(\Gamma)$ and $S(\Gamma)$ have the same homology. In particular, for any graph Γ*

$$H_*(B(\Gamma)) \cong H_*(\tilde{S}(\Gamma))$$

Proof. Let us prove the result for $U = \{v\}$, the theorem follows by induction. Let h_v be a half edge incident to v, then

$$S^U(\Gamma) = \mathbb{Z}[E] \otimes \bigotimes_{w \in V\setminus\{v\}} S_w \otimes \mathbb{Z}\langle\varnothing, h - h_v | h \in H(v)\rangle$$

Let e be $e(h_v)$, then by changing the basis of $S(\Gamma)$ as a $\mathbb{Z}[E]$-module we have the following decomposition:

$$S(\Gamma) = S^U(\Gamma) \;\oplus\; \left(\mathbb{Z}[E] \otimes \bigotimes_{w \in V\setminus\{v\}} S_w \otimes \mathbb{Z}\langle e - v, h_v\rangle\right)$$

This is a decomposition in subcomplexes because $\partial(h - h_v) = (e(h) - e)\varnothing$ and $\partial(h_v) = e - v$. Let $a \otimes h_v + b \otimes (e - v)$ be a generic element of the second subcomplex. Suppose

$$\begin{aligned}\partial(a \otimes h_v + b \otimes (e - v)) &= \partial a \otimes h_v + (-1)^{deg(a)} a \otimes (e - v)\varnothing + \partial b \otimes (e - v)\varnothing \\ &= 0\end{aligned}$$

Then $\partial a = 0$ and $(-1)^{deg(a)} a + \partial b = 0$. In particular, $a = (-1)^{deg(a)+1} \partial b$ and $deg(b) = deg(a) + 1$. It follows that

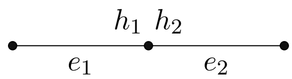
$$\begin{aligned}\partial((-1)^{deg(b)} b \otimes h_v) &= (-1)^{deg(b)} \partial b \otimes h_v + (-1)^{2deg(b)} b \otimes (e - v)\varnothing \\ &= a \otimes h_v + b \otimes (e - v)\varnothing\end{aligned}$$

This proves that the second subcomplex is exact and hence that $S(\Gamma)$ and $S^U(\Gamma)$ have the same homology. □

1.3 Some important examples

In this section we look at the degree zero homology group and we go trough some examples of classes in the first homology group of $B(\Gamma)$ for some elementary graphs Γ. We will also discuss some relations these classes are subject to. The importance of these examples is that the Świątkowski complex enjoy functoriality for graph embeddings, so we can find classes and relations of this kind every time these small graphs are embedded in a bigger one.

Example 1.3.1. Using the reduced complex, it is evident that $H_0(B(S_2))$ is generated as $\mathbb{Z}[E]$-module by the empty configuration $\varnothing \in \mathbb{Z}[E] \otimes \mathbb{Z}\langle \varnothing, h_1 - h_2 \rangle$.



In addiction, it holds $[e_1 \otimes \varnothing] = [e_2 \otimes \varnothing]$ in $H_0(B(S_2))$, that is $[e_1] = [e_2]$ in the simplified notation. Indeed, $\partial(h_2 - h_1) = e_2 - e_1$.

We also observe that $H_1(B(S_2)) = 0$, since $\partial(h_2 - h_1) \neq 0$.

Proposition 1.3.2. *Let Γ be a connected graph, then $H_0(B_k(\Gamma)) \cong \mathbb{Z}$ for any k and $H_0(B(\Gamma)) \cong \mathbb{Z}[e]$ for any fixed edge e.*

Proof. Using the reduced Świątkowski complex, $H_0(B(\Gamma))$ is generated as $\mathbb{Z}[E]$-module by the empty configuration. One can conclude by the repeated use of the relation found in Example 1.3.1. More specifically, the image of the differential in $B(\Gamma)$ is generated by the element of the form $e - e'$ for any pair of adjacent edges e and e'. By connectivity we conclude that in $H_0(B(\Gamma))$ all the edges are equivalent and hence that for any k the zeroth homology of $B_k(\Gamma)$ is generated by e^k for some fixed edge e. $\square$

Definition 1.3.3. Let Γ be a graph and $i : S_2 \to \Gamma$ a graph morphism. The relation in $H_0(B(\Gamma))$ induced by i by the relation described in Example 1.3.1 is called *I-relation.*

Example 1.3.4. For any graph Γ, $B_1(\Gamma) = \Gamma$. It follows that $H_1(B_1(\Gamma)) \cong H_1(\Gamma)$. Let us consider for example the graph C_1 with one edge e, one vertex v and two half-edges h_1 and h_2. Then, $b = h_1 - h_2 \in \tilde{S}_1(C_1)$ is a representative for the unique non zero class in $H_1(B_1(C_1)) \cong H_1(C_1)$. In fact, $\partial(b) = e(h_1) - e(h_2) = e - e = 0$ and b is clearly not trivial since there are no degree 2 element in $\tilde{S}(C_1)$. This class, in effect, generates the whole $H_1(B(C_1))$ as $\mathbb{Z}[E]$-module, since there are no more half-edges. More generally, there is such a class in any subdivision of C_1, that is in any C_n. A representative is given by an alternating sum of the half-edges. We denote by β this class. It is clear that β generates the whole $H_1(B(C_n))$, since the smoothing $s : C_n \to C_1$ is a quasi isomorphism by Theorem 1.2.7 and it takes β to the class of b in C_1. In addiction, in $H_1(B(C_n))$ it holds

$$e_i\beta = e_j\beta \ \ \forall e_i, e_j \in E(C_n) \tag{1.1}$$

because s takes any edge of C_n to the only edge of C_1.

Definition 1.3.5. Let Γ be a graph with a loop, meaning such that there is a graph morphism $i : C_n \to \Gamma$ for some n. Then a *loop class* in Γ is the image of the class β of Example 1.1 through the map $H_1(\tilde{S}(C_n)) \to H_1(\tilde{S}(\Gamma))$ induced by i. Moreover, any loop class satisfy the relation 1.1, which is called *O-relation.*

Remark 1.3.6. For any graph Γ, the loop classes in $H_1(B(\Gamma))$ satisfy also the usual relations on loop cycles coming from the isomorphism $H_1(B_1(\Gamma)) \cong H_1(\Gamma)$.

Example 1.3.7. Consider S_3 labeled as in Figure 1.5. The first homology group of $\tilde{S}_1(S_3)$ is clearly trivial. However, there is a non trivial class α if we consider two particles instead of just one. A representative is given by the chain

$$\begin{aligned} a_{123} &= e_1(h_2 - h_3) + e_2(h_3 - h_1) + e_3(h_1 - h_2) \\ &= e_1 h_{23} + e_2 h_{31} + e_3 h_{12} \end{aligned}$$

in $\tilde{S}_2(S_3)$, where the subscript records the order in which the edges are considered. This chain represents two particles moving in S_3 as shown in Figure 1.6 and it is closed because

$$\partial(a_{123}) = e_1(e_2 - e_3) + e_2(e_3 - e_1) + e_3(e_1 - e_2) = 0$$

We consider unordered configuration, so it is clear that the image represent a cycle in $B_2(S_3)$. We denote the class of a_{123} by α_{123}. Note that $\alpha_{123} = \alpha_{231} = \alpha_{312}$, while the class $\alpha_{132} = \alpha_{321} = \alpha_{213}$ has opposite sign. Since the projection $C_2(S_3) \to B_2(S_3)$ is a covering map, α_{123} is not trivial because if it were trivial the representative a_{123} would lift to some loop in $C_2(S_3)$. But it is clear that there are no loop in $C_2(S_3)$ whose projection is a_{123} because, as shown in Figure 1.6, the two particles are switched by a_{123}.

Definition 1.3.8. Let Γ be a graph with a vertex of valence at least 3, that is such that there is a graph morphism $i : S_3 \to \Gamma$, then a *star class* in Γ is the image of the class α of the last example through the map $H_1(\tilde{S}(S_3)) \to H_1(\tilde{S}(\Gamma))$ induced by i.

In the second chapter, section 2.2, we will prove that star classes and loop classes generate the first homology of $B(\Gamma)$ as a $\mathbb{Z}[E]$-module for any

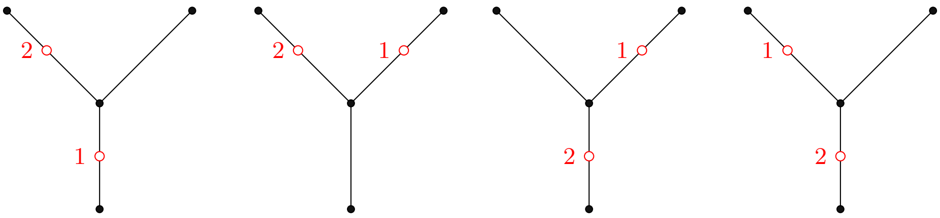


Figure 1.6: A non trivial class in in the first homology of $\tilde{S}_2(S_3)$

connected graph Γ. Nevertheless, these classes are important also in higher degree homology. As exposed in section 1.2, we think of a class in the first homology in terms of particles moving, one at a time, around the graph. We can imagine to perform the basic moves corresponding to a star or a loop class locally in disjoint region of a bigger graph at the same time. This induces a higher degree classe, which corresponds, at the level of the Świątowski complex, to a tensor product of star and loop classes that involves different vertices of the graphs. These new classes are called *toric classes*.

The following examples exhibit some important relations star classes and loop classes are subject to.

Example 1.3.9. We now consider the graph S_4 with edges $\{e_i\}_{i=1,\dots,4}$. Using the same convention on subscripts of Example 1.3.7, one can easly find that

$$a_{234} - a_{341} + a_{412} - a_{123} = 0 \tag{1.2}$$

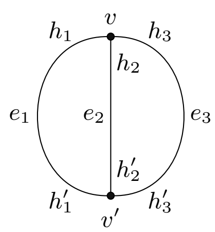
and

$$e_1 a_{234} - e_2 a_{341} + e_3 a_{412} - e_4 a_{123} = 0 \tag{1.3}$$

Then the same relations holds for the respective classes in $H_1(B(S_4))$.

Definition 1.3.10. Let Γ be a graph with a vertex v of valence at least 4, that is such that there is a graph morphism $S_4 \to \Gamma$. The star classes in Γ coming from the embedded S_4 respect the relations 1.2 and 1.3, which are called respectively *S-relation* and *X-relation*.

Example 1.3.11. Let us consider the graph Θ labeled as follow:



Let α_{123} and α'_{123} be the star classes respectively at v and v' with the usual convention on subscripts. Then

$$\alpha_{123} = -\alpha'_{123} \tag{1.4}$$

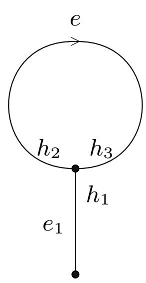
In fact:

$$\begin{aligned}
&\partial((h_1 - h_2)(h'_1 - h'_3) - (h_1 - h_3)(h'_1 - h'_2))\\
=&(e_1 - e_2)(h'_1 - h'_3) - (h_1 - h_2)(e_1 - e_3) - (e_1 - e_3)(h'_1 - h'_2)\\
&+ (h_1 - h_3)(e_1 - e_2)\\
=&e_1(h_2 - h_3) + e_2(h_3 - h_1) + e_3(h_1 - h_2) + e_1(h'_2 - h'_3) + e_2(h'_3 - h'_1)\\
&+ e_3(h'_1 - h'_2)\\
=&\alpha_{123} + \alpha'_{123}
\end{aligned}$$

The same relation hols by considering a subdivision of Θ, meaning a graph Θ' such that there is a smoothing $\Theta' \to \Theta$, because the smoothing is a quasi-isomorphism that takes the corresponding star classes of Θ' in α_{123} and α'_{123}.

Definition 1.3.12. Let Γ be a graph, Θ' a subdivision of Θ and $i : \Theta' \to \Gamma$ a graph morphism. Then the star classes coming from Θ' respect the relation 1.4, which is called Θ*-relation*.

Example 1.3.13. Let L be the so called lollipop graph:



The loop class $\beta = [h_2 - h_3]$ and the star class α_{123} satisfy

$$\alpha_{123} = (e_1 - e)\beta \tag{1.5}$$

In fact the usual representative of α_{123} at the chain level in L is

$$a_{123} = e_1(h_2 - h_3) + e(h_3 - h_1) + e(h_1 - h_2) = (e_1 - e)(h_2 - h_3)$$

The same holds for any graph L_n obtained from L subdividing e into n edges. In Equation 1.5 e can be replaced by any edge of the cycle subgraph by the O-relation.

Definition 1.3.14. Let Γ be a graph with a loop and $i : L_n \to \Gamma$ a graph morphism for some n. Then the loop class and the star class coming from L_n are subject to the relation 1.5 which is said Q*-relation*.

1.4 The cubical complex

The aim of this section is to clarify the connection between the Świątkowski complex and the configuration space of a graph. We report the construction of a cubical complex $K_k(\Gamma)$ that embed as a deformation retract into $B_k(\Gamma)$ and whose chain complex is a reduction of $S_k(\Gamma)$. This construction is due to Świątkowski and can be found in [12].

Definition 1.4.1. Let Γ be a graph and let $\tilde{V}$ be the set of its branched vertices, meaning the set of the vertices of valence greater than 3. We define the graded poset $P_k\Gamma = (P_k^{(0)}\Gamma, ..., P_k^{(n)}\Gamma, ...)$, where $P_k^{(n)}\Gamma$ denotes the collection of n-faces of $P_k\Gamma$, that is the set of pairs (f, S) such that

- $S = \{h_1, ..., h_n\}$ is a set consisting of exactly n half-edges of Γ with the property that $v(h_i)$ is branched for each $h_i \in S$ and $v(h_i) \neq v(h_j)$ for $i \neq j$;

- f is a function $f : E \cup \tilde{V} \to \mathbb{N}$ such that $f(v) \in \{0, 1\}$ and $f(v(h_i)) = 0$ for $i = 1, ..., n$;

- $\sum_{x\in E\cup\tilde{V}} f(x) = k - n$.

The order is generated by the following rule: $(f_1, S) \prec (f_2, S \cup \{h\})$ if $h \notin S$ and one of the following two conditions holds:

- $f_1(x) = \begin{cases} f_2(x) + 1 & \text{if } x = e(h), \\ f_2(x) & \text{otherwise;} \end{cases}$

- $f_1(x) = \begin{cases} f_2(x) + 1 & \text{if } x = v(h), \\ f_2(x) & \text{otherwise.} \end{cases}$

Remark 1.4.2. By analogy with the interpretation of the Świątkowski complex, a 0-face correspond to a configuration of k particles. The function f determines the number of particles lying in each part of the graph. A 1-face represent an elementary move, meaning a configuration with one particle moving from a vertex to an edge, in the direction given by the unique half-edge in S. The position of the still particles is given by f. Finally, n-faces correspond to configurations with n moving particles, represented by the elements of S. The conditions $v(h_i) \neq v(h_j)$ and $f(v(h_i)) = 0$ for any $h_i, h_j \in S$ avoid collisions. Regarding the order, $(f_1, S) \prec (f_2, S \cup \{h\})$ when (f_1, S) is obtained by $(f_2, S \cup \{h\})$ by "freezing" the moving particle given by h at one of the two endpoints of its path.

Lemma 1.4.3. *$P_k(\Gamma)$ is the face poset of a uniquely defined cubical complex $K_k(\Gamma)$.*

Proof. For each $F \in P_k^{(n)}$ the subposet $\{A | A \prec F\}$ is isomorphic to the face poset of the n-dimensional standard cube. In fact, $F = (f, S)$ has exactly $2n$ $(n-1)$-faces: 2 for each half-edge in S. □

Remark 1.4.4. It is clear that, assuming Γ has no bivalent vertices, the chain complex coming from the cubical complex K_k is isomorphic to the reduced Świątkowski complex $S_k^U(\Gamma)$ relative to the set $U = V \setminus \tilde{V}$ of the non-branched vertices. The face $F = (f, S) \in P_k^{(n)}\Gamma$ correspond to the monomial

$$\prod_{e \in E} e^{f(e)} \prod_{v \in \tilde{V}} v^{f(v)} \prod_{h \in S} h \in S_k^U(\Gamma)_n.$$

The differential of the Świątkowski complex corresponds to the alternating sum of the two $(n-1)$-dimensional faces given to define the order in $P_k\Gamma$. The aim of the remainder of this section is to prove Theorem 1.4.6 and so that $K_k(\Gamma)$ is homotopically equivalent to $B_k(\Gamma)$. It follows that the homology of the configuration space $B_k(\Gamma)$ can be computed through $S_k^U(\Gamma)$ and hence, thanks to Theorem 1.2.11, through any version of the Świątkowski complex.

Proposition 1.4.5. *There is an embedding $i : K_k(\Gamma) \to B_k(\Gamma)$.*

Proof. Let $F = (f, S)$ be a $|S|$-cell of the cubical complex $K_k(\Gamma)$. We think of the set $[0,1]^S$ of all the functions $S \to [0,1]$ as the standard $|S|$-dimensional cube. Then, let $\tau : F \to [0,1]^S$ be the unique isomorphism of cubes such that, for any vertex $p = (\psi, \varnothing)$ of F, the function $\tau(p) \in [0,1]^S$ is given by:

$$\tau(p)(h) = 1 - \psi(v(h)) \text{ for all } h \in S.$$

For each $x \in F$, $\tau(x)$ is a function $S \to [0,1]$. Let H be the set of all half-edges of Γ. Then $\tau(x)$ can be extended to a function $\tau_0 : H \to [0,1]$ by:

$$\tau_0(x)(h) = \begin{cases} \tau(x) & \text{if } h \in S; \\ 1 & \text{otherwise.} \end{cases}$$

We view elements of $B_k(\Gamma)$ as subsets of Γ consisting of k elements. The embedding $i_F : F \to B_k(\Gamma)$ takes a point $x \in F$ to a collection of k points of Γ containing:

- all the vertices $v \in \tilde{V}$ such that $f(v) = 1$;
- for any $e \in E$, m_e points lying on e, where $m_e = f(e) + |\{h \in S \text{ s.t. } e(h) = e\}|$.

The idea is that a point $x \in F$ corresponds to the configuration given by (f, S) in which the "moving particle" corresponding to a half-edge $h \in S$ lies on $e(h)$ the closer to the vertex $v(h)$, the closer the point is to the face of F obtained removing h from S. More precisely, we fix a length metric d on

Γ for which each edge has length 1. Let h, h' be the two half edges incident to the edge e and let m be the integer $f(e) + |\{h \in S \text{ s.t. } e(h) = e\}|$. If $m = 0$ there are no particles of the configurations lying on e. Otherwise, we consider on e the orientation for which h "is before" h' . The image of $x \in F$ through the embedding i_F is the subset $\{A_1, ..., A_m\}$ of e which satisfy:

(i) $v(h) \leq A_1 < ... < A_m \leq v(h')$ with respect to the natural order on e given by the orientation described above;

(ii) $d(A_1, A_2) = d(A_2, A_3) = ... = d(A_{n-1}, A_m) =: d$;

(iii) if $m \geq 2$, $d(v(h), A_1) = \tau_0(x)(h) \cdot d$ and $d(v(h'), A_1) = \tau_0(x)(h') \cdot d$;

(iv) if $m = 1$, $\tau_0(x)(h') \cdot d(v(h), A_1) = \tau_0(x)(h) \cdot d(v(h'), A_1)$.

Note that the last condition makes sense because $\tau_0(x)(h)$ and $\tau_0(x)(h')$ can not be both zero. The maps i_F are continuous and injective, moreover they agree on common faces, therefore the family $\{i_F | F \in P_k\Gamma\}$ defines an embedding $i : K_k(\Gamma) \to B_k(\Gamma)$. □

The following result is part of the construction due to Świątkowski ([12]). In [10], Lütgehetmann observed that the retraction proposed in [12] does not work. The map used in the following proof is essentially the one described by Chettih-Lütgehetmann in [7].

Theorem 1.4.6. *There is a deformation retraction of $B_k(\Gamma)$ into the image of $K_k(\Gamma)$ through the embedding i.*

Proof. First, we need to define a retraction $r : B_k(\Gamma) \to i(K_k(\Gamma))$, where i is the embedding of the previous Proposition. We use the same length metric d on Γ such that each edge has length 1. Let $C \in B_k(\Gamma)$, that is a subset of Γ consisting of k elements. The aim is to send continuously C to a configuration that, on each edge, is of the form described by items (i)-(iv) in the proof of the previous Proposition. For an edge e let n_e^C be the number of particles of C lying on the inner part of e or in the non-branched vertices of e. Then $r(C)$ will be a configuration given by the particles of C lying on branched vertices, meaning $C \cap \tilde{V}$, and for each edge e a set $\{A_1, ..., A_{n_e^C}\}$ of n_e^C particles lying on e. We need to determine the exact position of these last particles. Let e be an edge with $n_e^C \neq 0$, and let h and h' be the half-edges on e, then $\{A_1, ..., A_{n_e^C}\}$ has to satisfy

(i) $v(h) \leq A_1 < ... < A_{n_e^C} \leq v(h')$ with respect to the natural order on e given by the orientation;

(ii) $d(A_1, A_2) = d(A_2, A_3) = ... = d(A_{n_e^C - 1}, A_{n_e^C}) =: d$;

(iii) if $n_e^C \geq 2$, $d(v(h), A_1) = \tau_h^C \cdot d$ and $d(v(h'), A_1) = \tau_{h'}^C \cdot d$;

(iv) if $n_e^C = 1$, $\tau_{h'}^C \cdot d(v(h), A_1) = \tau_h^C \cdot d(v(h'), A_1)$.

for some choice of $\tau_h^c \in [0,1]$ for each half-edge h. The value τ_h^c determines the distance from $v(h)$ to the closest particle of the configuration $r(C)$. Let $\{B_1, ..., B_{n_e^c}\}$ be the positions of the particles of C on e. Then let $d_1(C,h) = d(v(h), B_1)$ be the length of the first segment determined by C on e the with respect to the orientation given by h, and let $d_2(C,h)$ be the length of the second one. Note that if $n_e^C = 1$ then $d_2(C,h) = d(B_1, v(h'))$, otherwise $d_2(C,h) = d(B_1, B_2) = d$. Then for any half-edge h we define

$$\delta_h^C = \min\{1, \frac{d_1(C,h)}{d_2(C,h)}\}$$

We observe that the only chance for this ratio to be greater than 1 is when $n_e^C = 1$ and the particles is closer to $v(h')$. Then we are ready to define τ_h^C:

$$\tau_h^C = \begin{cases} 1 \text{ if } v(h) \text{ is not branched or } v(h) \in C, \\ \min\{\ 1,\ \delta_h^C/\min\{\delta_k^C | k \in H(v(h)) \setminus h\}\ \} \ \text{ otherwise.} \end{cases}$$

If $v(h)$ is not branched or there is a particle of C on it, i.e. $v(h) \in C$, then τ_h^C has to be 1 in order to obtain a configuration in $i(K_k(\Gamma))$. That is because of the properties of the function τ_0 described in the previous proof. In the second case the result is less then 1 if and only if h is the half-edge incident to v with the lowest value of δ^C. In fact the value of τ_h can differ from 1 for at least one half-edge for each vertex to obtain a configuration in the image of the embedding of the cubical complex. The function r maintains the structure of the configuration C, meaning that $C \cap \tilde{V} = r(C) \cap \tilde{V}$ and C and $r(C)$ have the same number of particles on each edge. In addiction the parameters d_1 and d_2 depend continuously on C. It follows that r is continuous.

Let us show that r is a retraction, i.e. r is the identity on $K_k(\Gamma)$. Let $x \in K_k(\Gamma)$ be such that $i(x) = C$, we need to prove that, for each edge e with $n_e^C > 0$, the A_is coincide with the B_is. In other terms we need that $\tau(x)(h) = \tau_h^C$ and $\tau(x)(h') = \tau_h'^C$, where h and h' are the half-edges of e. If $\tau(x)(h) = 1$ it is clear that $d_1(C,h) \geq d_2(C,h)$, and hence $\delta_h^C = 1$ and also $\tau_h^C = 1$, since $\delta_k^C \leq 1$ for any other k. Suppose now $\tau(x)(h) \neq 1$, then all the other half-edges k incidents to $v(h)$ are such that $\tau(x)(k) = 1$ because just one half edge can have value different from 1 for each vertex. It follows that $\delta_k^C = 1$ for any other k incident to v and hence $\tau_h^C = \delta_h^C$. Now we distinguish the two cases:

- if $n_e^C > 1$, then $\tau_h^C = \delta_h^C = \frac{d_1(C,h)}{d}$ and hence τ_h^C coincide with the parameter $\tau(x)(h)$ with the property that $d_1(C,h) = d(v(h), B_1) = \tau(x)(h) \cdot d$.

- if $n_e^C = 1$, then $\tau(x)(h')$ has to be 1 since there is only one particles on e and it is moving in the direction of h, so $h' \notin S$. It follows that $d_1(C,h) > d_2(C,h)$ and $\tau_h^C = \delta_h^C = \frac{d_1(C,h)}{d_2(C,h)} = \frac{d(v(h),B_1)}{d(v(h),B_2)}$. Then τ_h^C coincides with the parameter $\tau(x)(h)$ with the property that $\tau_{h'}^C \cdot d(v(h), B_1) = \tau_h^C \cdot d(v(h'), B_1)$.

To conclude, we need to construct a homotopy H between r and the identity of $B_k(\Gamma)$. As observed, r does not change the structure of a configuration C. It follows that for each edge e there is a family $\{C_e(t)|t \in [0,1]\}$ of configurations on e that depends continuously on t and that connects $C \cap e$ and $r(C) \cap e$. Then

$$H(t, C) = (C \cap \tilde{V}) \cup \bigcup_{e \in E} C_e(t)$$

is the required homotopy. □

1.5 Edge stabilization

Before we deal with the low-degree homology in the following chapters, we report in this section some result about a stabilization phenomenon observed with the rise of the number of the particles considered. As seen in section 1.2, the Świątkowski chain complex comes with an action of $\mathbb{Z}[E]$. This action has a pretty clear topological counterpart. The multiplication by and edge e corresponds to a map $\sigma_e : B(\Gamma) \to B(\Gamma)$ that basically add a particle on the edge e by replacing the particles on the open edge e with the collection of averages of consecutive particles or endpoints. More precisely, parametrizing the interior of e by the interval $(0,1)$, σ_e replace the subconfiguration of the particles on e corresponding to the points $\{x_1, ..., x_j\} \subset (0,1)$ with particles in the following positions

$$\{\frac{x_1}{2}, \frac{x_1 + x_2}{2}, ..., \frac{x_j + 1}{2}\}$$

This map, which is equivalent to the one defined in [3], is continuous and independent on the parametrization of the graph up to homotopy. In this way, we obtain an action of the graded ring $\mathbb{Z}[E]$ on the homology $H_*(B(\Gamma))$, which became an object in the following category:

Definition 1.5.1. Let *Mod* be the category whose objects are pair (R, M) with R a graded commutative ring with unit and M a differential bigraded R-module. A morphism $(R_1, M_1) \to (R_2, M_2)$ is a pair (f, g) where $f : R_1 \to R_2$ is a graded ring morphism and $g : M_1 \to M_2$ is a differential bigraded R_1-module morphism, by considering on M_2 the R_1-module structure given by $rm = f(r)m$ for any $r \in R_1$, $m \in M_2$.

The action of $\mathbb{Z}[E]$ on $H_*(B(\Gamma))$ is natural in the sense that it commutes with the maps induced by graph morphisms. It follows that $H_*(B(-))$ is a functor *Gph*$\to$*Mod*. Clearly, also the Świątkowski complex $S(\Gamma)$ is a bi-graded differential $\mathbb{Z}[E]$-module, and the maps induced by graph morphisms respect this structure that descend to homology. In other words, $H_*(S(-))$ is also a functor *Gph*$\to$*Mod*.

Similarly, the maps σ_e induce an action of $R[E]$ on $H_*(B(\Gamma);R)$ for any commutative ring R. Also an analogous definition of the Świątkowski complex can be given with coefficients in R:

$$S(\Gamma;R) = R[E] \otimes_{\mathbb{Z}} \bigotimes_{v\in V} S(v)$$

where $S(v) = \mathbb{Z}\langle \varnothing, v, h \in H(v)\rangle$ are the $\mathbb{Z}$-module described in section 1.2. The following generalization of Theorem 1.2.7 is proven in [3]:

Theorem 1.5.2. *For any graph* Γ

$$H_*(B(\Gamma);R) \cong H_*(S(\Gamma;R))$$

as functors Gph$\to$Mod*.*

Since $S(\Gamma)$ is finitely generated as $\mathbb{Z}[E]$-module and $\mathbb{Z}[E]$ is Noetherian, we have

Corollary 1.5.3. *For any* $i \geq 0$*,* $H_i(B(\Gamma))$ *is finitely generated as a* $\mathbb{Z}[E]$*-module.*

The same is true with coefficient in a field $\mathbb{K}$. Then we are in the hypothesis of the Hilbert-Serre Theorem (see, for example, [5, Theorem 11.1, Corollary 11.2]) and we have:

Theorem 1.5.4. *Let* $\mathbb{K}$ *be a field. For any* $i \geq 0$ *and for* k *large enough,* $dim_{\mathbb{K}}\ H_i(B_k(\Gamma);\mathbb{K})$ *coincides with a polynomial in* k*.*

The degree of this polynomial is described in [3] in terms of certain connectivity invariants of the graph. In [2] there is also an explicit formula and it is proven, with a dimensional argument, that $H_i(B_k(\Gamma);\mathbb{K})$ is asymptotically generated by toric classes, for $k \to \infty$.

Chapter 2

The first homology group

A lot is known about the first homology group of configuration spaces of graphs. In [9] is proven an explicit formula in terms of combinatorial invariants of the graph. With the new language furnished by the Świątkowski complex, in [1], generators for the first homology as a $\mathbb{Z}[E]$-module are identified. In this chapter we describe some tools useful in the study of the homology in all degrees and we collect these results about the first homology. In addiction, in section 1.3, we prove that some already known relations the generators are subject to, form a complete set and they provide a presentation of the first degree homology.

2.1 Tools

In this section we follow [1] to develop some technical tools that will be helpful to investigate the homology of the configuration space of a graph using the Świątkowski complex. The first result is the computation of the Euler characteristic, which is immediate thanks to the reduced complex.

Theorem 2.1.1. *The Euler characteristic of $B_k(\Gamma)$ is given by*

$$\chi(B_k(\Gamma)) = \sum_{U \subset V} (-1)^{|U|} \binom{k - |U| + |E| - 1}{|E| - 1} \prod_{v \in U} (d(v) - 1)$$

where $d(v)$ is the valence of v.

Proof. By Theorem 1.2.11, we need to compute the rank of the $\mathbb{Z}$-module $\tilde{S}_k(\Gamma)_n$. We fix a basis for $\tilde{S}_v$ for each v of the form

$$\tilde{S}_v = \mathbb{Z}\langle \varnothing,\ h - h_v \mid h \in H(v) \setminus \{h_v\} \rangle$$

for some half-edges h_v in $H(v)$, when it is non empty. Then a basis for $\tilde{S}_k(\Gamma)_n$ is given by the choice of n pair of half-edges incidents to distinct vertices and a monomial in $\mathbb{Z}[E]$ of degree $k - n$. To count the elements

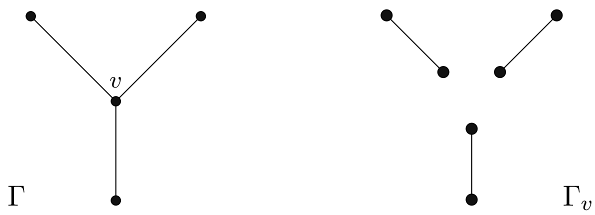


Figure 2.1: Vertex explosion

of this basis first we need to choose a set U of n vertices; then there are $d(v)-1$ non trivial generators of $\tilde{S}_v$ for each $v \in U$. Finally, there are

$$\binom{k-|U|+|E|-1}{|E|-1}$$

monomials of degree $k-n=k-|U|$ in $\mathbb{Z}[E]$. It follows that the rank of the $\mathbb{Z}$-module $\tilde{S}_k(\Gamma)_n$ is

$$\sum_{|U|=n} \binom{k-|U|+|E|-1}{|E|-1} \prod_{v \in U} (d(v)-1)$$

□

One of the most useful tool provided by the use of the Świątkowski complex is the long exact sequence of Theorem 2.1.4, which facilitates the computation of the homology of $B(\Gamma)$ using simpler graphs coming from Γ through vertices explosions.

Definition 2.1.2. Let v be a vertex of Γ. We define the vertex explosion of Γ at v as the graph Γ_v obtained by replacing the vertex v with $|H(v)|$ vertices $\{v_h\}_{h\in H(v)}$ and modifying the attaching maps so that each half-edge $h \in H(v)$ becomes incident to the corresponding new vertex v_h.

Remark 2.1.3. There is a graph morphism $i : \Gamma_v \to \Gamma$ which takes the new vertex v_h to a point of the edge $e(h)$ and is the identity on the other vertices. This construction depends on the choice of the point of $e(h)$, but the map is unique at the level of the Świątkowski complex.

Theorem 2.1.4. *Let v be a vertex of Γ and $h_0 \in H(v)$. There is a long exact sequence of differential bigraded $\mathbb{Z}[E]$-modules*

$$\ldots \longrightarrow H_{n+1}(B_k(\Gamma_v)) \xrightarrow{i_*} H_{n+1}(B_k(\Gamma)) \xrightarrow{\psi} \bigoplus_{h\in H(v)\setminus h_0} H_n(B_{k-1}(\Gamma_v)) \longrightarrow$$

$$\xrightarrow{\delta} H_n(B_k(\Gamma_v)) \xrightarrow{i_*} H_n(B_k(\Gamma)) \xrightarrow{\psi} \bigoplus_{h\in H(v)\setminus h_0} H_{n-1}(B_{k-1}(\Gamma_v)) \longrightarrow \ldots$$

where

- *i_* is the map induced by the graph morphism of Remark 2.1.3;*
- *ψ is induced by the map $\tilde{S}_k(\Gamma)_{n+1} \to \bigoplus_{h\in H(v)\smallsetminus h_0} \tilde{S}_{k-1}(\Gamma_v)_n$ that takes the element $b + \sum(h-h_0)a_h$, where b does not involve half-edges at v, to $(a_h)_{h\in H(v)\smallsetminus h_0}$;*
- *on the summand indexed by h, δ is the multiplication for $e(h)-e(h_0)$.*

Proof. We consider the subcomplex of $\tilde{S}(\Gamma)$

$$F^0\tilde{S}(\Gamma) = \mathbb{Z}[E] \otimes \bigotimes_{w\in V\smallsetminus\{v\}} \tilde{S}_w$$

consisting of the chains without half-edges incident to w. We consider the following exact sequence of complexes

$$0 \to F^0\tilde{S}(\Gamma) \to \tilde{S}(\Gamma) \to \tilde{S}(\Gamma)/F^0\tilde{S}(\Gamma) \to 0$$

The resulting long exact sequence in homology gives the result. It is immediate that $F^0\tilde{S}(\Gamma) \cong \tilde{S}(\Gamma_v)$ and that the inclusion of $F^0\tilde{S}(\Gamma)$ in $\tilde{S}(\Gamma)$ correspond to the map induced by the graph morphism of Remark 2.1.3. We can think of $\tilde{S}(\Gamma)/F^0\tilde{S}(\Gamma)$ as the complex of the chains of $\tilde{S}(\Gamma)$ with a factor in $\tilde{S}_v$, exactly one since products with more than one half-edge per vertex are not allowed. It follows that the subcomplex of the quotient of weight k and degree n is isomorphic to the sum of copies of $\tilde{S}_{k-1}(\Gamma_v)_n$ indexed by the choice of a generator of $\tilde{S}_v$, meaning by the choice of h in $H(v) \smallsetminus \{h_0\}$. It is clear that the projection to the quotient of an element $b + \sum(h-h_0)a_h$ is the class of $\sum(h-h_0)a_h$ which is identified, through the isomorphism described, to the collection $(a_h)_{h\in H(v)\smallsetminus\{h_0\}}$. Let us study how the connecting homomorphism δ behaves. Through the identification described above the scheme is the following:

$$\begin{tikzcd}
\tilde{S}_k(\Gamma_v)_{n+1} \arrow[r] \arrow[d] & \tilde{S}_k(\Gamma)_{n+1} \arrow[r] \arrow[d] & \bigoplus_{h\in H(v)\smallsetminus\{h_0\}} \tilde{S}_{k-1}(\Gamma_v)_{n-1} \arrow[d] \arrow[dll, dotted] \\
\tilde{S}_k(\Gamma_v)_n \arrow[r] & \tilde{S}_k(\Gamma)_n \arrow[r] & \bigoplus_{h\in H(v)\smallsetminus\{h_0\}} \tilde{S}_{k-1}(\Gamma_v)_{n-2}
\end{tikzcd}$$

By following the diagram we have the result about δ:

$$\begin{tikzcd}
 & \sum(h-h_0)c_h \arrow[r] \arrow[d, "\partial"] & (c_h)_{h\in H(v)\smallsetminus\{h_0\}} \\
\sum(e(h)-e(h_0))c_h \arrow[r, hook, "i"] & \sum(e(h)-e(h_0))c_h &
\end{tikzcd}$$

the class of a boundary $(c_h)_{h\in H(v)\smallsetminus\{h_0\}}$ is sent by δ to $\sum(e(h)-e(h_0))c_h$. $\square$

Proposition 2.1.5. *For any edge e of Γ, the multiplication by e is injective on $H_*(B(\Gamma))$.*

Proof. In this proof we use the regular Świątkowski complex, not its reduced version. We think of an element of $S(\Gamma)$ as a polynomial in variables $E \cup V \cup H$ (but not all the polynomials are possible). By fixing an edge e, we can change the system of generators and write such a polynomial in variables $e, e-e', e-v, h$ for any edge e' different from e, for any vertex v and for any half-edge h. We write U for the set of variables $\{e-e', e-v, h\}_{e \neq e' \in E, h \in H}$ other than e. Then an arbitrary chain in $S(\Gamma)$ can be written in the form

$$b = \sum_{i=0}^{M} e^i b_i(U)$$

for some polynomials $b_i \in \mathbb{Z}[U]$. Let $c = \sum_{j=0}^{M-1} e^j c_j(U)$ be a cycle in $S(\Gamma)$ such that $ec = \partial b$. Let us show that c is a boundary. We observe that the differential $b_i' = \partial b_i(U)$ is still a polynomial in U, i.e.

$$\partial b = \sum_{i=0}^{M} e^i b_i'(U)$$

So the assumption becomes:

$$ec = \sum_{i=1}^{M} e^i c_{i-1}(U) = \sum_{i=0}^{M} e^i b_i'(U).$$

It follows that $b_0' = 0$, that is $\partial b_0 = 0$ and

$$ec = \partial b = \partial(b - b_0) = \partial(\sum_{i=1}^{M} e^i b_i(U)) = e\ \partial(\sum_{i=0}^{M} e^i b_{i+1}).$$

Then c is a boundary:

$$c = \partial(\sum_{i=0}^{M} e^i b_{i+1}).$$

□

Proposition 2.1.6. *Let Γ be a connected graph with a bivalent vertex v whose removal disconnects Γ. Let e_1 and e_2 be the two edges adjacent to v. There is an isomorphism of $\mathbb{Z}[E]$-modules*

$$H_*(B(\Gamma)) \cong H_*(B(\Gamma_v))/e_1 \sim e_2.$$

Proof. Applying Theorem 2.1.4 to our case we obtain the long exact sequence

$$... \to H_n(B_{k-1}(\Gamma_v)) \xrightarrow{\delta} H_n(B_k(\Gamma_v)) \xrightarrow{i_*} H_n(B_k(\Gamma)) \xrightarrow{\psi} H_{n-1}(B_{k-1}(\Gamma_v)) \xrightarrow{\delta} ...$$

We claim that δ is injective. It follows that i_* is surjective and so

$$H_n(B_k(\Gamma_v)) \cong H_n(B_k(\Gamma))/ker(i_*) \cong H_n(B_k(\Gamma))/im(\delta)$$

and we conclude because in our case δ is the multiplication by $e_1 - e_2$ (up to a sign, depending on the choice of the foxed half-edge at v). Let us show that δ is injective. Suppose $\delta\beta = 0$ for some $\beta \in H_{n-1}(B_{k-1}(\Gamma_v))$. We write $\beta = \sum_{j=0}^{k-1} \beta_j$ where β_j has j particles lying in the connected component Γ' of Γ_v containing (the image through i of) e_1. Then $\delta\beta = 0$ means $(e_1 - e_2)\beta = 0$, i.e. $\sum e_1\beta_j = \sum e_2\beta_j$. Comparing in the last equation the summands of each term with the same number of particles on Γ', we obtain the following relations:

$$e_1\beta_{k-1} = e_2\beta_0 = 0 \text{ and } e_1\beta_j = e_2\beta_{j+1} \text{ for } 0 \leq j < k-1$$

By repeated applications of Proposition 2.1.5, $\beta_{k-1} = \beta_0 = 0$ and so all the β_j are zero. □

2.2 Generators for the homology in degree one

The purpose of this section is to prove that star classes and loop classes generate the first homology of the configuration space of a connected graph. This result is proven in [1].

Lemma 2.2.1. *$H_1(B_k(S_n))$ is generated by star classes.*

Proof. We proceed by induction on n. The claim is true for S_1 and S_2, which are topologically intervals. The statement holds also for S_3, since $H_1(B(S_3))$ is freely generated by one star class α. In fact, the Euler characteristic in weight k is given by:

$$\chi(B_k(S_3)) = \binom{k+2}{2} - \binom{k+1}{2} \cdot 2$$

By Proposition 1.3.2, $H_0(B_k(S_3))$ has rank 1. It follows that

$$\mathrm{rk}(H_1(B_k(S_3)) = 1 - \chi(B_k(S_3)) = \frac{k^2 - k}{2}$$

which corresponds to the rank of $\mathbb{Z}[E]\langle\alpha\rangle$, i.e. to the number of monomials in 3 variables of degree $k - 2$.

We may assume $n \geq 4$. We use the reduced Świątkowski complex relative to the set U of 1-valent vertices. We write $e_1, ..., e_n$ for the edges of S_n and h_i for the half-edge at n-valent vertex v lying on e_i. An element $a \in S^U(S_n)_1 \cong \mathbb{Z}[E] \otimes \mathbb{Z}\langle h_1, ..., h_n \rangle$ has the form $a = \sum_{i=1}^n p_i h_i$ for some $p_i \in \mathbb{Z}[E]$. Suppose a is a cycle, i.e. that $\partial a = \sum_{i=1}^n p_i(e_i - v) = 0$. The aim is to write a as a sum in which a term is generated by star cycles and the other summand does not involve e_n, so that we can conclude by the inductive hypotesis. In particular, we partially rewrite a in therms of the star cycles

$$a_{i,n-1,n} = (e_n - e_{n-1})h_i + (e_i - e_n)h_{n-1} + (e_{n-1} - e_i)h_n.$$

For $i \leq n-2$, we write $p_i = p_i'(e_n - e_{n-1}) + r_i$, where r_i does not involve the variable e_n. Then

$$\begin{aligned} a &= \sum_{i=1}^{n-2}(p_i'(e_n - e_{n-1})h_i + r_i h_i) + p_{n-1}h_{n-1} + p_n h_n \\ &= \sum_{i=1}^{n-2}(p_i' a_{i,n-1,n-2} + r_i h_i) + q_{n-1}h_{n-1} + q_n h_n \end{aligned}$$

where

$$q_{n-1} = p_{n-1} - \sum_{i=1}^{n-2}(e_i - e_n)p_i' \quad \text{and} \quad q_n = p_n - \sum_{i=1}^{n-2}(e_{n-1} - e_i)p_i'.$$

The claim is that

$$a - \sum_{i=1}^{n-2} p_i' a_{i,n-1,n-2} = \sum_{i=1}^{n-2} r_i h_i + q_{n-1}h_{n-1} + q_n h_n$$

does not involve neither e_n or h_n and so it comes from the S_{n-1} that misses the nth edge. Then the inductive hypotesis completes the proof. We write $q_{n-1} = e_n q_{n-1}' + r_{n-1}$ and $q_n = e_n q_n' + r_n$, where r_n and r_{n-1} do not involve the variable e_n. Since the star cycles have differential zero, the hypotesis that a is a cycle means that

$$\partial a = \sum_{i=1}^{n-2} r_i(e_i - v) = +q_{n-1}(e_{n-1} - v) + q_n(e_n - v) = 0$$

which is equivalent to require

$$\begin{cases} \sum r_i + q_{n-1} + q_n = 0 \\ \sum e_i r_i + e_{n-1}q_{n-1} + e_n q_n = 0 \end{cases}$$

By taking the terms involving e_n we obtain the two conditions

$$\begin{cases} e_n(q'_{n-1} + q'_n) = 0 \\ e_{n-1}q'_{n-1} + e_n q'_n + r_n = 0 \end{cases}$$

The first equation implies $q'_{n-1} = -q_n$ and then the second one becomes $(e_n - e_{n-1})q'_n + r_n = 0$. Since r_n does not involve e_n, $q'_n = r_n = 0$, whence $q_n = 0$. Then by the first equation also $q'_{n-1} = 0$, and so q_{n-1} does not involve the variable e_n. This conclude the proof of the claim and hence of the Proposition. □

The Koszul complex provides a simpler proof of this fact, as well as a presentation of the $\mathbb{Z}[E]$-module $H_1(B(S_n))$. The result about the relations will be useful in section 2.4:

Lemma 2.2.2. *In the $\mathbb{Z}[E]$-module $H_1(B(S_n))$ the star classes are subject only to the S-relation and the X-relation.*

Proof. We consider the $\mathbb{Z}[E]$-module $M := \tilde{S}(B(S_n))_1 \cong \mathbb{Z}[E]\langle h_i - h_0\rangle_{i=1,\dots,n-1}$ and the $\mathbb{Z}$-linear map $\partial : \tilde{S}(B(S_n))_1 \to \mathbb{Z}[E]$, $h_i - h_0 \mapsto e_i - e_0$. Consider the Koszul complex

$$0 \to \bigwedge^{n-1} M \xrightarrow{d_n} \dots \to \bigwedge^{3} M \xrightarrow{d_3} \bigwedge^{2} M \xrightarrow{d_2} M \xrightarrow{\partial} \mathbb{Z}[E] \xrightarrow{0}$$

where, for any $a_i \in M$,

$$d_k(a_1 \wedge \dots \wedge a_k) = \sum_{i=1}^{k} (-1)^{i+1} \partial(a_i) a_1 \wedge \dots \wedge \hat{a}_i \wedge \dots a_k$$

Since $e_1 - e_0, \dots, e_{n-1} - e_0$ is a regular sequence, the higher homology modules of the Koszul complex are all zero and we get a free presentation of $\ker(\partial) \cong H_1(B(S_n))$:

$$\bigwedge^{3} M \xrightarrow{d_3} \bigwedge^{2} M \xrightarrow{d_2} \ker(\partial) \to 0$$

Since $\bigwedge^2 M$ has rank $\binom{n-1}{2}$, we choose the star classes

$$\alpha_{0,i,j} = (e_i - e_0)(h_j - h_0) - (e_j - e_0)(h_i - h_0), \quad \text{for } 1 \le i < j \le n-1$$

as generators for the first homology, which are enough thanks to the S-relation. The rank of $\bigwedge^3 M$ is $\binom{n-1}{3}$. It follows that the following relations, coming from the X-relation, are sufficient:

$$r_{0ijk} = (e_i - e_0)\alpha_{0jk} - (e_j - e_0)\alpha_{0ik} + (e_k - e_0)\alpha_{0ij}, \quad \text{for } 1 \le i < j < k \le n-1$$

□

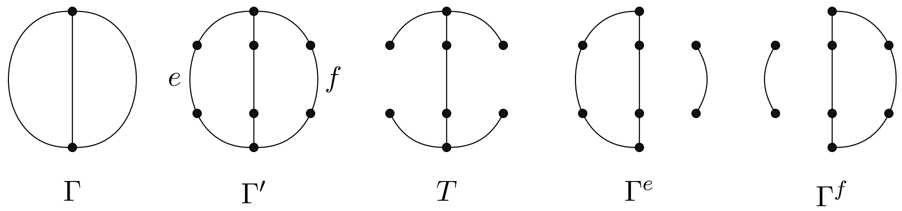


Figure 2.2: An example of the construction in the proof of Theorem 2.2.3. In this case $E_0 = \{e, f\}$

Theorem 2.2.3. *If* Γ *is connected, then* $H_1(B(\Gamma))$ *is generated as a* $\mathbb{Z}[E]$-*module by star classes and loop classes.*

Proof. Assume Γ is a tree. Then we proceed by induction on the number of vertices of valence at least 3. The base case is given by Lemma 2.2.1. The inductive step is given by reducing the computation to smaller trees using Proposition 2.1.6. In the general case we consider the subdivision Γ' of Γ obtained adding two vertices on every edge of Γ. Then each edge of Γ corresponds to three edges of Γ'. Let E_{mid} be the set of the middle edges of Γ', that is the edges none of whose vertices are vertices of Γ. Let T be a spanning tree of Γ', meaning a subgraph which is a tree and contain all the vertices. We denote E_0 the set of the edges not included in T, which is a subset of E_{mid}. For $e \in E_0$ we define the subgraph Γ^e of Γ' as the disjoint union of the cycle subgraph C_e contained in $e \cup T$ and the remaining edges in $E_{mid} \setminus E(C_e)$ (see Figure 2.2). We now consider the complex

$$\hat{S}(\Gamma) = \tilde{S}(T) \oplus \bigoplus_{e \in E_0} \tilde{S}(\Gamma^e).$$

The embedding of T and Γ^e in Γ' and the smoothing $\Gamma' \to \Gamma$ induce a map $\phi : \hat{S} \to \tilde{S}$. It is clear that the first homology of $B(\Gamma^e)$ is generated as a $\mathbb{Z}[E]$-module by its loop class and we have already seen that the result holds for T. Therefore, it suffices to show that ϕ induces a surjective map on the first homology. First, we observe that at the chain level ϕ is clearly surjective, in fact $\phi|_{\tilde{S}(T)_1}$ is already surjective. It suffices to show that for any $b \in \tilde{S}(T)$ with image $\phi(b)$ closed, there exists a $b' \in \hat{S}(\Gamma)$ closed with the same image. For each $e \in E_0$, let e' and e'' are the edges of Γ' such that $\phi(e) = \phi(e') = \phi(e'')$. Let $c_e \in \tilde{S}(C_e)$ be a cycle representing the loop class in C_e. There exists an element $b_e \in \tilde{S}(T)_1$ with $\phi(b_e) = \phi(c_e)$ and $\partial b_e = \pm(e' - e'')$. Then $\phi(b_e - c_e) = 0$ and $\partial(b_e - c_e) = \pm(e' - e'')$. Let $b \in \tilde{S}(T)$ be such that $\partial\phi(b) = 0$, then ∂b is in the kernel of ψ, which is the the $\mathbb{Z}[E]$-submodule generated by the elements $(e' - e'')$, i.e.

$$\partial b = \sum_{e \in E_0} p_e(e' - e'')$$

for some $p_e \in \mathbb{Z}[E]$. We consider

$$b' = b + \sum_{e \in E_0} \mp p_e(b_e - c_e) \in \hat{S}(\Gamma).$$

Then $\phi(b) = \phi(b')$ and $\partial b' = 0$. $\square$

2.3 An explicit formula

In [9] is given an explicit formula for the first homology of the configuration space of a connected graph in terms of some combinatorial invariants related to the connectivity of the graph. In [1] some of the results which lead to the formula are revised in terms of the machinery provided by the Świątkowski complex. In this section we report these results and the proof of the formula (Theorem 2.3.6).

In the remainder we assume Γ is a simplicial model of itself, since it is convenient dealing with connectivity issues.

Definition 2.3.1. Let v be a 1-cut of a connected graph Γ, we define

$$N(k, \Gamma, v) = \binom{k + \mu(v) - 2}{k - 1}(\nu(v) - 2) - \binom{k + \mu(v) - 2}{k} - (\nu(v) - \mu(v) - 1)$$

where $\mu(v)$ is the number of v-components of Γ and $\nu(v)$ is the valence of v in Γ. Let $V_1(\Gamma)$ denote a set of 1-cuts that decomposes Γ into biconnected components and copies of topological segments. Then

$$N_1(k, \Gamma) = \sum_{v \in V_1(\Gamma)} N(k, \Gamma, v).$$

Definition 2.3.2. Let Γ be a biconnected graph and $\{x, y\}$ a 2-cut. A $\{x, y\}$-*decomposition* consists in a pair of subgraphs $\{\Gamma_1, \Gamma_{-1}\}$ where

- Γ_1 is the union of a $\{x, y\}$-component G_1 and a path in the closure of its complement joining x and y.

- Γ_{-1} is the union of the closure G_{-1} of the complement of G_1 and a path in G_1 joining x and y.

Remark 2.3.3. This definition is given in [1, Notation C.11]. Γ_1 is a subdivision of what in [9] is called a *marked* $\{x, y\}$-*component*, i.e. a graph obtained by an $\{x, y\}$-component (in the sense of Definition 1.1.11), by adding an edge between x and y. One can iterate the construction of Definition 2.3.2 a number of times in order to obtain a marked component for each $\{x, y\}$-component in Γ.

By following the argument in [8], for any biconnected graph it is possible to find a sequence V_2 of 2-cuts that decompose Γ into triconnected graphs and copies of topological circles.

Definition 2.3.4. For a biconnected graph Γ, let V_2 be a set of 2-cuts that decomposes Γ into triconnected components and copies of topological circles. We define

$$N_2(\Gamma) = \sum_{\{x,y\}\in V_2} \frac{(\mu(\{x,y\})-1)(\mu(\{x,y\})-2)}{2}$$

where $\mu(\{x,y\})$ is the number of $\{x,y\}$-components in Γ. For any connected graph Γ, we define

$$N_2(\Gamma) = \sum_{i=1}^{j} N_2(\Gamma_i)$$

where $\Gamma_1, ..., \Gamma_j$ are the biconnected components of Γ obtained after a number of 1-cuts as in Definition 2.3.1.

By the 1-cuts in V_1 and the 2-cuts in V_2, it is possible to decompose any connected graph Γ in triconnected graphs, cycles and segments.

Definition 2.3.5. For a connected graph Γ, given the decomposition provided by V_1 and V_2, we denote by $N_3(\Gamma)$ the number of triconnected components of Γ that are planar and by $N_3'(\Gamma)$ the number of those non-planar.

We are now ready to state the formula for the first homology:

Theorem 2.3.6. *Let Γ be a connected graph. Then*

$$H_1(B_k(\Gamma)) \cong \mathbb{Z}^{N_1(k,\Gamma)+N_2(\Gamma)+N_3(\Gamma)+\beta_1(\Gamma)} \oplus \mathbb{Z}_2^{N_3'(\Gamma)}.$$

In order to prove Theorem 2.3.6 we need some lemmas which can be found both in [9] and [1]. In [1], they are proven using the tools provided by the Świątkowski complex. We only report one of these proofs that will be useful next.

Lemma 2.3.7 ([9, Lemma 3.12], [1, Lemma C.8])**.** *Let Γ be a biconnected graph. For any edge e and for any $k \geq 2$, multiplication by e^{k-2} induces an isomorphism*

$$H_1(B_2(\Gamma)) \xrightarrow{\cong} H_1(B_k(\Gamma))$$

Proof. By Lemma 2.1.5, it suffices to show the surjectivity. In particular, we need to show that $p(E)\gamma$ is divisible by e for any monomial p in E and for any star or loop class γ, then we conclude by induction. We may assume $p(E)$ is divisible by an edge $e' \neq e$, otherwise there is anything to prove. Then we write $p(E)\gamma = [q(E)e'c]$, where $[c] = \gamma$.

If γ is a star class, we may assume c is a star cycle with support v. By biconnectivity, there is a path between e and e' in $\Gamma \smallsetminus \{v\}$. Then, by the I-relation, $[q(E)e'a] = [q(E)ea]$ because their difference is the differential of $q(E)ha$, where h is the alternating sum of the half-edges involved in the path found.

If γ is a loop, take a path between e and e', which always exists because Γ is connected. If this path is disjoint from the support of c, the same argument of the star case works. If the whole path is contained in c, the same conclusion follows by the O-relation. We may assume e' lie on the support of c, while e does not. By Q-relation, $[q(E)e'c] = [\pm q(E)ec \pm q(E)a]$, where $[a]$ is a star class. This concludes the proof because the case of $[a]$ is already proven. ☐

Remark 2.3.8. The proof of this Lemma only uses I-relation, O-relation and Q-relation.

Remark 2.3.9. Lemma 2.3.7 provides and example of the stabilization phenomenon described in section 1.5.

Lemma 2.3.10 ([9, Lemma 3.15], [1, Lemma C.9]). *Let Γ be a 3-connected graph, then*

$$H_1(B_2(\Gamma)) \cong \mathbb{Z}^{\beta_1(\Gamma)} \oplus K$$

where

$$K = \begin{cases} \mathbb{Z} & \text{if } \Gamma \text{ is planar,} \\ \mathbb{Z}_2 & \text{otherwise} \end{cases}$$

Lemma 2.3.11 ([9, Lemma 3.11], [1, Lemma C.10]). *Let Γ be a graph, v a 1-cut and $\{\Gamma_i\}_{i=1}^{\mu(v)}$ the set of v-components of Γ. Then,*

$$H_1(B_k(\Gamma)) \cong \left(\bigoplus_{i=1}^{\mu(v)} H_1(B_k(\Gamma_i)) \right) \oplus \mathbb{Z}^{N(k,\Gamma,v)}$$

where $N(k, \Gamma, v)$ is defined as in Definition 2.3.1.

Lemma 2.3.12 ([9, Lemma 3.13], [1, Lemma C.14]). *Let $\{x, y\}$ be a $2-cut$ of a biconnected graph Γ and let $\{\Gamma_1, \Gamma_{-1}\}$ be a $\{x, y\}$-decomposition. There is a split exact sequence*

$$0 \to H_1(B_2(C)) \to H_1(B_2(\Gamma_1)) \oplus H_1(B_2(\Gamma_{-1})) \to H_1(B_2(\Gamma)) \to 0$$

where C is the union of the two disjoint paths between x and y of Definition 2.3.2. The maps are defined by the respective inclusions, with the map $H_1(B_2(\Gamma_{-1})) \to H_1(B_2(\Gamma))$ twisted by an overall sign. In particular,

$$H_1(B_2(\Gamma)) \oplus \mathbb{Z} \cong H_1(B_2(\Gamma_1)) \oplus H_1(B_2(\Gamma_{-1})).$$

In addiction to these results, it will be needed the following computation on the graphs Θ_n:

Lemma 2.3.13. *It holds*

$$H_1(B_2(\Theta_n)) \cong \mathbb{Z}^{\frac{(n-1)(n-2)}{2}+n-1}.$$

Proof. By applying Theorem 2.1.4 exploding one of the two n-valent vertices of Θ_n, we obtain the following exact sequence

$$0 \to H_1(B_2(S_n)) \to H_1(B_2(\Theta_n)) \to \bigoplus_{n-1} H_0(B_1(S_n)) \to \\ \to H_0(B_2(S_n)) \to H_0(B_2(\Theta_n)) \to 0$$

Since both S_n and Θ_n are connected, we have

$$\bigoplus_{n-1} H_0(B_1(S_n)) \cong \mathbb{Z}^{n-1} \quad \text{and} \quad H_0(B_2(S_n)) \cong H_0(B_2(\Theta_n)) \cong \mathbb{Z}$$

Also $H_1(B_2(S_n))$ is free, because there are no degree 2 chains in $S(S_n)$. By Theorem 2.1.1

$$\chi(B_2(S_n)) = \frac{3n - n^2}{2}.$$

Since $H_0(B_2(S_n)) \cong \mathbb{Z}$,

$$rk(H_1(B_2(S_n))) = 1 - \frac{3n - n^2}{2} = \frac{(n-1)(n-2)}{2}.$$

It follows by looking at the exact sequence that also $H_1(B_2(\Theta_n))$ is free and its rank is $\frac{(n-1)(n-2)}{2} + n - 1$. $\square$

We have now all the tools needed to prove Theorem 2.3.6:

Proof of Theorem 2.3.6. Let V_1 be a sequence of 1-cuts which decompose Γ in biconnected components and segments. By applying Lemma 2.3.11 for each 1-cut in V_1, we obtain

$$H_1(B_k(\Gamma)) \cong \bigoplus_{i \in I} H_1(B_k(\Gamma_i)) \oplus \mathbb{Z}^{N_1(k,\Gamma)}$$

where $\{\Gamma_i\}_{i\in I}$ are the biconnected components of Γ. By Lemma 2.3.7, $H_1(B_k(\Gamma_i)) \cong H_1(B_2(\Gamma_i))$. Since $N_2(\Gamma)$, $N_3(\Gamma)$, $N_3'(\Gamma)$ and $\beta_1(\Gamma)$ are the sum of those for Γ_i, it suffices to show that

$$H_1(B_2(\Gamma)) \cong \mathbb{Z}^{N_2(\Gamma)+N_3(\Gamma)+\beta_1(\Gamma)} \oplus \mathbb{Z}_2^{N_3'(\Gamma)}$$

for any biconnected graph Γ. Let V_2 be a set of 2-cuts that decomposes Γ into triconnected graphs and graphs homeomorphic to circles. Let $\{x, y\}$

be a 2-cut in V_2. We apply Lemma 2.3.12 repeatedly in order to obtain a marked component for each $\{x, y\}$-component. In particular, we have to make $\mu(\{x, y\}) =: \mu$ $\{x, y\}$-decompositions. During each decomposition, a marked component is detached and a simple path between x and y is added to the graph which will be decomposed in the next step. In the last decomposition, the $\mu - th$ one, when the last marked component is isolated, it remains a graph made just of μ simple paths between x and y, i.e. a subdivision of Θ_μ. Let $\{\Gamma_h^{\{x,y\}}\}_{h=1}^{\mu}$ be the set of marked $\{x, y\}$-components of Γ, then by Lemma 2.3.12

$$H_1(B_2(\Gamma)) \oplus \mathbb{Z}^\mu \cong \bigoplus_{h=1}^{\mu} H_1(B_2(\Gamma_h^{\{x,y\}})) \oplus H_1(B_2(\Theta_\mu))$$

Since $H_1(B_2(\Theta_\mu)) \cong \mathbb{Z}^{(\mu-1)(\mu-2)/2+(\mu-1)}$, we obtain

$$H_1(B_2(\Gamma)) \oplus \mathbb{Z} \cong \bigoplus_{h=1}^{\mu} H_1(B_2(\Gamma_h^{\{x,y\}})) \oplus \mathbb{Z}^{(\mu-1)(\mu-2)/2}$$

Let $\{\hat{\Gamma}_j\}_{j\in J}$ be the set of marked components of Γ obtained by cutting along V_2. By proceeding in the same way for each 2-cut in V_2 we have

$$H_1(B_2(\Gamma)) \oplus \mathbb{Z}^{|V_2|} \cong \bigoplus_{j\in J} H_1(B_2(\hat{\Gamma}_j)) \oplus \mathbb{Z}^{N_2(\Gamma)}$$

By Lemma 2.3.10,

$$H_1(B_2(\hat{\Gamma}_j)) \cong \begin{cases} \mathbb{Z}^{\beta_1(\Gamma)} \oplus \mathbb{Z} & \text{if } \hat{\Gamma}_j \text{ is a planar triconnected graph;} \\ \mathbb{Z}^{\beta_1(\Gamma)} \oplus \mathbb{Z}_2 & \text{if } \hat{\Gamma}_j \text{ is a nonplanar triconnected graph;} \\ \mathbb{Z} & \text{if } \hat{\Gamma}_j \text{ is homeomorphic to a cycle.} \end{cases}$$

It follows that

$$\bigoplus_{j\in J} H_1(B_2(\hat{\Gamma}_j)) \cong \mathbb{Z}^{N_3(\Gamma)+\sum_j \beta_1(\hat{\Gamma}_j)} \oplus \mathbb{Z}_2^{N_3'(\Gamma)}$$

To conclude, it suffices to show that $\sum_{j\in J} \beta_1(\hat{\Gamma}_j) = \beta_1(\Gamma) + |V_2|$. Recall that $\beta_1(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + 1$. In the computation of β_1 we can pretend that a marked $\{x, y\}$-component is obtained by a $\{x, y\}$-component by adding just an edge and no vertices. Indeed, in the alternating sum, this give the same result as the insertion of a simple path. Then

$$\sum_{j\in J}\beta_1(\hat{\Gamma}_j) = \sum_{j\in J}\Big(|E(\hat{\Gamma}_j)| - |V(\hat{\Gamma}_j)| + 1\Big)$$
$$= \sum_{j\in J}|E(\hat{\Gamma}_j)| - \sum_{j\in J}|V(\hat{\Gamma}_j)| + |J|$$
$$= |E(\Gamma)| + \sum_{\{x,y\}\in V_2}\mu(\{x,y\}) - \left(|V(\Gamma)| + 2(\sum_{\{x,y\}\in V_2}(\mu(\{x,y\}) - 1))\right) + |J|$$
$$= |E(\Gamma)| + |V(\Gamma)| - \sum_{\{x,y\}\in V_2}\mu(\{x,y\}) + 2|V_2| + |J|$$

But $|J| = \sum_{\{x,y\}\in V_2}\mu(\{x,y\}) - |V_2| + 1$, because by summing up the number of connected components for each 2-cut we are counting $|V_2| - 1$ components twice. Thus $\sum_{j\in J}\beta_1(\hat{\Gamma}_j) = |E(\Gamma)| + |V(\Gamma)| + 1 + |V_2| = \beta_1(\Gamma) + |V_2|$ and

$$H_1(B_k(\Gamma)) \cong \mathbb{Z}^{N_1(k,\Gamma)+N_2(\Gamma)+N_3(\Gamma)+\beta_1(\Gamma)} \oplus \mathbb{Z}_2^{N_3'(\Gamma)}.$$

□

2.4 Relations

For any connected graph Γ, loop classes and star classes generate $H_1(B(\Gamma))$ as a $\mathbb{Z}[E]$-module. The relations they are subject to are described in Section 1.3. The aim of this section is to prove that these form a complete set of relations, which is an new result.

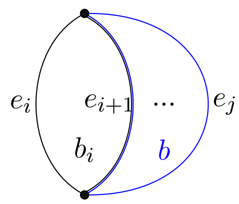
Theorem 2.4.1. *$H_1(B(\Gamma))$ is presented as a $\mathbb{Z}[E]$-module by loop classes and star classes subject to the following relations:*

- *The relations arising by the I-relation (Definition 1.3.3): if γ is a star class or a loop class, then $[e\gamma] = [e'\gamma]$ as long as e and e' are connected by a path disjoint from the support of γ;*
- *The O-relation (Definition 1.1);*
- *The usual relations on loop cycles coming from $H_1(\Gamma)$;*
- *The S-relation (Definition 1.3) and the usual rule on signs for star classes: $\alpha_{ijk} = (-1)^{sgn(\sigma)}\alpha_{\sigma(i)\sigma(j)\sigma(k)}$;*
- *The X-relation (Definition 1.3);*
- *The Θ-relation (Definition 1.4);*
- *The Q-relation (Definition 1.5).*

In order to prove Theorem 2.4.1, we go over the steps that lead to the proof of the formula in Theorem 2.3.6. As seen, the strategy is to reduce the problem to simpler graphs by cutting Γ at some vertices. Lemma 2.4.2 and 2.4.3 provide the base cases, while 2.4.4 describe what happens in term of relations during a 1-cut.

Lemma 2.4.2. *Consider the graph Θ_n. The $\mathbb{Z}[E]$-module $H_1(B_2(\Theta_n))$ is generated by star and loop classes which are subject only to the relations of Theorem 2.4.1.*

Proof. By Lemma 2.3.13, $H_1(B_2(\Theta_n))$ is a free $\mathbb{Z}$-module of rank $\frac{(n-1)(n-2)}{2}+n-1$. Let us show that this result can be obtained by taking the quotient of the star and the stabilized loop classes for the relations already known. By the Q-relation it suffices to consider the stabilized loop classes, meaning the classes obtained by loop cycles multiplied by an edge. Clearly we may consider only the loop cycles in a family of generators for $H_1(\Theta_n)$, for example the set $\{b_i\}_{i=1}^{n-1}$, where b_i is the loop cycle given by the $i-th$ and the $(i+1)$-th edge. Now we can distinguish two kind of generators for $H_1(B_2(\Theta_n))$: the loop cycles multiplied by an edge belonging to its support and those which are not. By O-relation, for the first kind of generators, it does not matter what edge in the support we pick. Thus there are essentially $n-1$ generators of the first kind. For the second kind, if suffices to choose two indexes in $\{1, ..., k-1\}$: the first one corresponding to the edge and the second one to the loop cycle in the family above. We need just to show that we may assume the edge has index smaller than the loop cycle. In fact, suppose $1 \le i \le n-1$ and $i+2 \le j \le n$

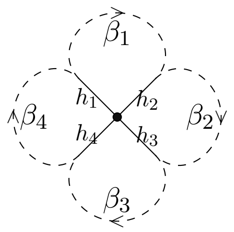


Let b be the blue cycle in the figure, then by Q-relation, $[(e_i - e_{i+1})b]$ and $[(e_j - e_{i+1})b_i]$ correspond to the same star class. It follows that $[e_j b_i] = [(e_i - e_{i+1})b + e_{i+1}b_i]$ and, since $b = b_{i+1} + ... + b_{j-1}$, the class of $e_j b_i$ can be written in terms of generators already considered. This concludes the proof. $\square$

The proof of the following Lemma is based on results that can be found in [1].

Lemma 2.4.3. *Let G be a triconnected graph. The $\mathbb{Z}[E]$-module $H_1(B_2(G))$ is generated by star and loop classes which are subject only to the relations of Theorem 2.4.1.*

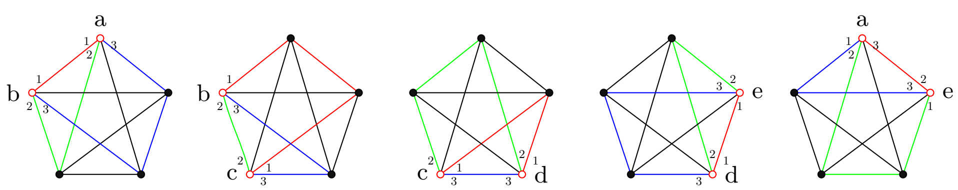
Proof. By Lemma 2.3.10, $H_1(B_2(G)) \cong \mathbb{Z}^{\beta_1(G)} \oplus K$, where $K = \mathbb{Z}$ if G is planar and $K = \mathbb{Z}_2$ otherwise. The first term is generated by stabilized loop classes. Thanks to the O-relation the choice of the edge does not matter, as long it lies on the support of the loop class. By the Q-relation and I-relation, a loop class multiplied by and edge which is not in its support is homologous to a linear combination of a star class and a loop class multiplied by an edge in its support. Thus it suffices to prove that in a triconnected graph any two star classes coincide up to sign and that, in the non-planar case, any star class has torsion 2. By triconnectivity, every pair of vertices is joined by three disjoint paths. In other words, a subdivision of Θ_3 can be embedded in Γ between any pair of vertices. It follows that, fixed a vertex v, any star class of Γ coincides up to sign to a star class at v. Let us show that any two star classes at v coincide up to sign. We may assume they differ only for one half edge and then proceed inductively, so we are now looking at an embedded S_4 at v. Since Γ_v is biconnected, there are two disjoint path joining $\{h_1, h_2\}$ and $\{h_3, h_4\}$ and the same for $\{h_1, h_3\}$ and $\{h_2, h_4\}$. By relabeling if necessary, we may suppose there is the following four-leaf graph embedded:



Then, by the Q-relation and the I-relation we have for example:

$$\begin{aligned}
\alpha_{123} &= (e_3 - e_1)\beta_1 = (e_4 - e_1)\beta_1 = \alpha_{124} \\
&= (e_2 - e_4)\beta_4 = (e_3 - e_4)\beta_4 = \alpha_{134} \\
&= (e_1 - e_3)\beta_3 = (e_2 - e_3)\beta_3 = \alpha_{234}
\end{aligned}$$

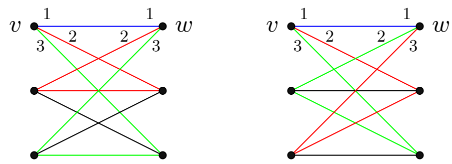
This proves that in a triconnected graph all the star classes coincide up to sign. Now suppose Γ is non-planar, then it admits K_5 or $K_{3,3}$ as a minor. Since every star class coincide up to sign, it suffices to prove the star classes in K_5 and $K_{3,3}$ are 2-torsion. Consider the graph K_5 in the figure and the star class α^a_{123} at the vertex a.



There are subdivisions of Θ_3 embedded between each of the five pairs of consecutive vertices, represented in red in each of the copy of the graph in the figure. By applying the Θ-relation for each of them, we get

$$\alpha^a_{123} = -\alpha^b_{123} = \alpha^c_{123} = -\alpha^d_{123} = \alpha^e_{123} = -\alpha^a_{312}$$

Since $\alpha^a_{123} = \alpha^a_{312}$, we get the result for K_5. The case of $K_{3,3}$ is similar. Consider two vertices v and w directly joined by an edge. There are two different ways to embed a subdivision of Θ_3 between them, as shown in the figure:



It follows that the star class at v equals to two star classes at w that have opposite sign. In fact, using the first embedding we get $\alpha^v_{123} = -\alpha^w_{123}$, while using the second $\alpha^v_{123} = -\alpha^w_{132} = \alpha^w_{123}$. $\square$

Lemma 2.4.4. *Let v be a 1-cut of valence ν of the graph Γ and let $\{\Gamma_i\}_{i=1}^{\mu}$ be the set of v-components of Γ. Cosider the formula given by Lemma 2.3.11:*

$$H_1(B_k(\Gamma)) \cong \left(\bigoplus_{i=1}^{\mu} H_1(B_k(\Gamma_i))\right) \oplus \mathbb{Z}^{N(k,\Gamma,v)}$$

The additional term $\mathbb{Z}^{N(k,\Gamma,v)}$ is generated by star classes which are subject only to the relations of Theorem 2.4.1.

In particular, by assuming that Theorem 2.4.1 holds for the terms coming from the v-components, it follows that the result holds for Γ too.

Proof. We need to verify that the kernel of the obvious map $H_1(B_k(\Gamma)) \to \bigoplus_{i=1}^{\mu} H_1(B_k(\Gamma_i))$ is generated by $N(k,\Gamma,v)$ stabilized star classes subject only to the relations of Theorem 2.4.1. It is clear that the kernel is generated by star classes at v, we need to study how many of them are needed by considering the relations known. Let ν_i denote the valence of v in Γ_i, then the half edges of Γ at v can be enumerated in the following way:

$$\underbrace{h^1_1, ..., h^1_{\nu_1}}_{\in\Gamma_1}, \underbrace{h^2_1, ..., h^2_{\nu_2}}_{\in\Gamma_2}, ..., \underbrace{h^{\mu}_1, ..., h^{\mu}_{\nu_\mu}}_{\in\Gamma_\mu}$$

We consider the lexicographic order, meaning that all the half-edges lying in the n-th component come before those in the $(n+1)$-th component. With respect to this order, we use letters such as i, j or l as a subscript instead of both the subscript and the superscript to indicate a generic half-edge and we denote, for example, by e_i the edge corresponding to h_i. By the S-relation we may consider only the star classes of the form α_{1ij}, meaning those involving the half edge h_1^1. Indeed, for any α_{lij} we may consider the embedded S_4 involving the edges e_1, e_l, e_i and e_j and, by S-relation, we can express α_{lij} in terms of the other star classes. Furthermore, the order of the indices only affect the sign. We may also consider only the star classes whose support does not lie in only one of the components. Indeed, suppose α_{1ij} is a star class whose support lie entirely in Γ_1 and e_l does not. Then by the X-relation, the I-relation and the S-relation, in the homology we have:

$$\begin{aligned} e_l\alpha_{1ij} &= e_1\alpha_{ijl} - e_i\alpha_{1jl} + e_j\alpha_{1il} = e_1(\alpha_{ijl} - \alpha_{1jl} + \alpha_{1il}) \\ &= e_1(\alpha_{jl1} - \alpha_{1il} + \alpha_{1ij} - \alpha_{1jl} + \alpha_{1il}) = e_1\alpha_{1ij} \end{aligned}$$

It follows that $e_l\alpha_{1ij} \in H_1(B(\Gamma_1))$ and so does any stabilized star class whose support lie entrely on Γ_1.

Let $M = \{h_1^1, ..., h_1^\mu\}$ denote the set of the first half-edge of each component. Then it suffices to consider the following three families of star classes:

i) α_{1ij}, with $h_i \in \Gamma_1$ and $h_j \in M \setminus \{h_1^1\}$;

ii) α_{1ij}, with $h_i \notin \Gamma_1 \cup M$ and $h_j \in M \setminus \{h_1^1\}$;

iii) α_{1ij}, with $h_i, h_j \in M \setminus \{h_1^1\}$.

The first family generates all the star classes with two half-edges lying in Γ_1. In fact, if h_i lies in Γ_1 and h_j, h_l lies in the same component, in $\Gamma \setminus \{v\}$ there are a path joining h_1^1 and h_i and a path joining h_j and h_l. Let β denote the loop cycle determined by the first path, then by the Q-relation and the I-relation

$$\alpha_{1ij} = [(e_j - e_1)\beta] = [(e_l - e_1)\beta] = \alpha_{1il}$$

The last case to discuss is that of the classes α_{1ij} in which neither h_i nor h_j is in M. If h_i and h_j belong to the same component Γ_m, there is a loop cycle β in Γ_m involving h_i and h_j and $\alpha_{1ij} = [(e_i - e_1)\beta]$. Let $h_m \in M$ be the first half-edge of Γ_m. The loop β can be written as the sum of a loop β' joining h_i and h_m and another one, β'', joining h_1^m and h_j. Then by the O-relation and the Q-relation

$$\alpha_{1ij} = [(e_i - e_1)\beta] = [(e_i - e_1)(\beta' + \beta'')] = [(e(h_1^m) - e_1)(\beta' + \beta'')] = \alpha_{1im} \pm \alpha_{1jm}$$

Finally, suppose h_i, h_j are not M and they lie in different components: $h_i \in \Gamma_m$ and $h_j \in \Gamma_n$. Let $h_m \in M$ and $h_n \in M$ be the first half-edges in Γ_m and Γ_n respectively. Then by by applying two times the S-relation we get

$$\alpha_{1ij} = \alpha_{1im} - \alpha_{1jm} + \alpha_{ijm} \tag{2.1}$$

By the same reasoning above, since h_i and h_m lie in the same component, thanks to the Q-relation and the I-relation, $\alpha_{ijm} = \alpha_{inm}$. Then by apllying the S relation again we get

$$\alpha_{inm} = \alpha_{1nm} - \alpha_{1im} + \alpha_{1in} \tag{2.2}$$

Then by equations 2.1 and 2.2, we can write α_{1ij} in terms of star classes already considered. This proves that the families of star classes *i)-iii)* give generators for the kernel in wight 2. In weight k we need to multiply them for a monomial in $\mathbb{Z}[E]$ of degree $k-2$. The aim is to show that the stabilized generators are $N(k, \Gamma, v)$. First, we observe that the stabilized classes of the family iii) generates a module isomorphic to $H_1(B_k(S_\mu))$. Using the Euler characteristic, for example, one can check that it has rank

$$(\mu - 1)\binom{k+\mu-2}{k-1} - \binom{k+\mu-1}{k} + 1 \tag{2.3}$$

Let us consider the star classes of i) and ii). It suffices to consider α_{1ij} with a coefficient that is a monomial of degree $k-2$ in the variables $\{e_l \in M : l \leq j\}$. In fact, we can restrict to M thanks to the I-relation. Let us prove that it suffices to consider edges with indices smaller than j:

(i) if α_{1ij}, with $h_i \in \Gamma_1$ and $h_j \in M \smallsetminus \{h_1^1\}$, suppose $l > j$. Then, by the X-relation, the I-relation and the S-relation, in the homology we have:

$$\begin{aligned} e_l\alpha_{1ij} &= e_1\alpha_{ijl} - e_i\alpha_{1jl} + e_j\alpha_{1il} = e_1\alpha_{ijl} - e_1\alpha_{1jl} + e_j\alpha_{1il} \\ &= e_1(\alpha_{1jl} - \alpha_{1il} + \alpha_{1ij}) - e_1\alpha_{1jl} + e_j\alpha_{1il} \\ &= -e_1\alpha_{1il} + e_1\alpha_{1ij} + e_j\alpha_{1il} \end{aligned}$$

where all the summands of the last line satisfy the request.

(ii) if α_{1ij}, with $h_i \notin \Gamma_1 \cup M$ and $h_j \in M \smallsetminus \{h_1^1\}$, suppose $l > j$. Then, by the X-relation and the S-relation, in the homology we have:

$$\begin{aligned} e_l\alpha_{1ij} &= e_1\alpha_{ijl} - e_i\alpha_{1jl} + e_j\alpha_{1il} \\ &= e_1(\alpha_{1jl} - \alpha_{1il} + \alpha_{1ij}) - e_i\alpha_{1jl} + e_j\alpha_{1il} \end{aligned}$$

The only term that can be problematic is $e_i\alpha_{1jl}$. Let $h_m \in M$ be the first half-edge in the component of h_i. Then, by the I-relation, $e_i\alpha_{1jl} = e_m\alpha_{ijl}$. The only case to discuss is when $m > l$. Then by the X-relation and the S-relation:

$$\begin{aligned} e_m\alpha_{1jl} &= e_1\alpha_{jlm} - e_j\alpha_{1lm} + e_l\alpha_{1jm} \\ &= e_1(\alpha_{1lm} - \alpha_{1jm} + \alpha_{1jl}) - e_j\alpha_{1lm} + e_l\alpha_{1jm} \end{aligned}$$

where all the summands of the last line satisfy the request, since $m > l > j$.

To conclude, we need to count the stabilized star classes of $i)$ and $ii)$. For the first family, we need to choose h_i in $\Gamma_1 \smallsetminus \{h_1^1\}$, which has cardinality $\nu_1 - 1$. Then for each choice of $h_j \in M \smallsetminus \{h_1^1\}$, we have to pick a monomial of degree $k-2$ in the variables $\{m \in M : m \leq j\}$. Thus the generators of the first kind are

$$(\nu_1 - 1)\sum_{j=2}^{\mu}\binom{k+j-3}{j-1} = (\nu_1 - 1)(\sum_{h=0}^{\mu}\binom{k+h-2}{h} - 1)$$

$$= (\nu_1 - 1)(\binom{k+\mu-2}{\mu-1} - 1) = (\nu_1 - 1)(\binom{k+\mu-2}{k-1} - 1) \tag{2.4}$$

For the family $ii)$, we need to chose $h_i \notin \Gamma_1 \cup M$, meaning in a set of cardinality $\nu - \nu_1 - \mu + 1$. Then, as above, for each choice of $h_j \in M \smallsetminus \{h_1^1\}$, we have to pick a monomial of degree $k-2$ in the variables $\{m \in M : m \leq j\}$. Thus the generators of the second kind are

$$(\nu - \nu_1 - \mu + 1)(\binom{k+\mu-2}{k-1} - 1) \tag{2.5}$$

Finally, summing up the values in 2.3, 2.4 and 2.5 we get

$$(\nu - 1)\binom{k+\mu-2}{k-1} - \binom{k+\mu-1}{k} + \mu - \nu + 1$$

$$= \binom{k+\mu-2}{k-1}(\nu - 2) - \binom{k+\mu-2}{k} - (\nu - \mu - 1) = N(k, \Gamma, v)$$

□

We have now all the tools to prove Theorem 2.4.1:

Proof of Theorem 2.4.1. By a series of 1-cut we can decompose the graph Γ into biconnected components and segments. Then by lemma 2.4.4, the problem of finding a complete system of relations can be reduced to the biconnected case. Lemma 2.3.7 is proven only applying I-relation, O-relation and Q-relation and it further reduces the study to the 2-particles case. By Lemma 2.3.12, for a biconnected graph Γ with a 2-cut $\{x, y\}$, $H_1(B_2(\Gamma))$ can be found from the homology of the components of an $\{x, y\}$-decomposition by taking their direct sum and considering that a loop class is counted twice. Then, as in the proof of Theorem 2.3.6, one can repeat this process several times in order to get a marked component for each of the μ $\{x, y\}$-component. At the end of this process we get an extra graph homeomorphic to Θ_μ. By Lemma 2.4.2, $H_1(B_2(\Theta_n))$ is presented only using the relations known. Thus the same computation of the prove of Theorem 2.3.6 applies and the problem of finding the relations can be reduced to the $\{x, y\}$-components of Γ. A series of 2-cuts permits to decompose Γ into triconnected graphs and cycles. The case of the cycle is trivial, then Lemma 2.4.3 gives the result. □

Chapter 3

The second homology group

In this chapter we deal with the second homology of configuration spaces of graphs. In the first two sections, we report a result about the generators in the planar case and the main tools used in [4] to prove it. In particular, it turns out that the occurrence of "exotic" classes is connected to the presence in the graph of certain cycles, called *pesky*. These cycles are useful to study the non-planar case. In section 3.4 we suggest a new kind of funtoriality for edge contraction that configuration spaces enjoy. We believe that this tool, which is interesting in itself, could facilitate the study of the non-planar case.

3.1 The planar case and a reformulation of the problem

As mentioned before, taking the tensor product of classes in the first homology involving distinct vertices produces classes in higher degree. Unfortunately, toric classes do not generate the entire second homology as a $\mathbb{Z}[E]$-module. A new sort of class appear in the second homology of the configuration space of Θ_4 with 3 particles. Classes of this kind are said *theta classes.* In [4] it is proven that theta classes together with toric classes generate the second homology for any planar graph. In this paragraph, following this work, we states this result (Theorem 3.1.4), we report a reformulation (Theorem 3.1.5) and we prove them equivalent.

Proposition 3.1.1. *The Abelian group $H_2(B_3(\Theta_4))$ is free of rank* 1.

Proof. By applying Theorem 2.1.4 exploding one of the two 4-valent vertices of Θ_4, we obtain the following exact sequence

$$0 \to H_2(B_3(\Theta_4)) \xrightarrow{\psi} \bigoplus_{3} H_1(B_2(S_4)) \xrightarrow{\delta} H_1(B_3(S_4)) \to ...$$

We have already seen in Lemma 2.3.13 that $H_1(B_2(S_4)) \cong \mathbb{Z}^3$. By the same argument one can found $H_1(B_3(S_4)) \cong \mathbb{Z}^{11}$. In particular, using the usual notation, by the S-relation $H_1(B_2(S_4))$ is generated by the classes α_{1jk} for $2 \leq j < k \leq 4$. On the other hand, $H_1(B_3(S_4))$ is generated by the classes $(e_i - e_1)\alpha_{1jk}$ and $e_1\alpha_{1jk}$ for $2 \leq j < k \leq 4$, subject to the relation

$$(e_4 - e_1)\alpha_{123} - (e_3 - e_1)\alpha_{124} + (e_2 - e_1)\alpha_{134} = 0$$

which is essentially the X-relation. By privileging the half-edge h_1, it follows by the explicit formula for δ given in Theorem 2.1.4 that its image is free of rank 8. Thus $H_2(B_3(\Theta_4))$ has to be free of rank 1. □

Remark 3.1.2. In [4] an explicit representative A is given for a generator of $H_2(B_3(\Theta_4))$. We report the formula, which can be useful for computations:

$$A = \sum_{\sigma \in \Sigma_4} \operatorname{sgn}(\sigma)(e_1 - e_{\sigma(3)})(h_{1,\sigma(1)} - h_{1,1})(h_{2,\sigma(2)} - h_{2,1})$$

Definition 3.1.3. Let Γ be a graph, Θ_4' a subdivision of Θ_4 and $i : \Theta_4' \to \Gamma$ a graph morphism. A *theta class* in $H_2(B_3(\Gamma))$ is the image of a generator of $H_2(B_3(\Theta_4))$ under the map induced by i.

Theorem 3.1.4. *For a planar graph Γ, the second homology $H_2(B(\Gamma))$ is generated as a $\mathbb{Z}[E]$-module by theta classes and tensor products of loop and star classes.*

We write $M(\Gamma)$ for the submodule of $H_2(B(\Gamma))$ spanned by theta classes and tensor products of loop and star classes and $M_k(\Gamma)$ for its wight k part. The following Theorem is equivalent to Theorem 3.1.4, as shown in Proposition 3.1.6

Theorem 3.1.5. *For any connected planar graph Γ and any bivalent vertex v, consider the long exact sequence of Theorem 2.1.4:*

$$\ldots \to H_2(B_k(\Gamma_v)) \xrightarrow{i_*} H_2(B_k(\Gamma)) \xrightarrow{\psi} H_1(B_{k-1}(\Gamma_v)) \xrightarrow{\delta} H_1(B_k(\Gamma_v)) \to \ldots \tag{3.1}$$

Then the restriction $\psi : M_k(\Gamma) \to ker(\delta)$ is surjective.

Proposition 3.1.6. *Theorem 3.1.4 and 3.1.5 are equivalent.*

Proof. Clearly $M(\Gamma) = H_2(B(\Gamma))$ implies Theorem 3.1.5 by exactness of the sequence. For the reverse implication, we proceed by induction on $\beta_1(\Gamma)$.

If $\beta_1(\Gamma) = 0$, Γ is a tree. Let us prove that in this case Theorem 3.1.4 holds. We proceed by induction on the number of vertices of valence greater than 2. If there is only one branched vertex, then Γ is homeomorphic to some S_n and $H_2(B(\Gamma)) = 0$. Suppose Γ has n vertices of valence greater than 2, we can, after a subdivision if necessary, explode a bivalent vertex

v of Γ such that Γ_v has two connected components, Γ_v^1 and Γ_v^2, that are tree with less than n branched vertices. $B(\Gamma_v) = B(\Gamma_v^1) \times B(\Gamma_v^2)$ and, by Theorem 2.3.6, $H_1(B(\Gamma_v^i))$ are torsion-free because planar. It follows by Künneth isomorphism that Theorem 3.1.4 holds for Γ_v because it holds by induction for its connected components. The result follows by Proposition 2.1.6.

If $\beta_1(\Gamma) \geq 1$, by subdividing if necessary, one can find a bivalent vertex v such that Γ_v is connected. By induction, Theorem 3.1.4 holds for Γ_v. Then, looking at the long exact sequence 3.1, $\ker(\psi) = i_*(H_2(B(\Gamma_v)) = i_*(M(\Gamma_v)) \subset M(\Gamma)$ and ψ induces the isomorphism

$$\frac{H_2(B(\Gamma))}{M(\psi)} \cong \frac{\ker(\delta)}{\psi(M(\Gamma))}$$

But $\ker(\delta) = \psi(M(\Gamma))$ by assumption, thus $H_2(B(\Gamma)) = M(\Gamma)$ as desired. □

3.2 Pesky cycles

In light of Proposition 3.1.6, the aim of this section is to study the quotient $\ker(\delta)/\psi(M(\Gamma))$ arising from the explosion of a bivalent vertex in a connected graph. In the following, Γ is a connected graph and e is a fixed edge. We subdivide e by adding a bivalent vertex v. We will write v' and v'' for the resulting vertices in Γ_v and e' and e'' for the corresponding edges. We may also assume that Γ_v is connected because this is the only case needed to prove Theorem 3.1.4.

Lemma 3.2.1. *Let γ be a path between v' and v'' in Γ_v. A loop or star class lies in $\psi(M(\Gamma))$ if it admits a representing cycle with support disjoint from γ.*

Proof. Suppose a is a cycle in $H_1(B_{k-1}(\Gamma_v))$ with support disjoint from γ. The path γ correspond to a loop cycle in Γ. The we can take the tensor product of the image of the cycle a in Γ and this loop cycle. Then, by definition of ψ, the class of this tensor product lies in $M(\Gamma) \cap \psi^{-1}([a])$. □

Proposition 3.2.4 describes the quotient coming from this explosion in terms of some special cycles said *pesky*:

Definition 3.2.2. Let $c \in S_k(\Gamma_v)_1$ be a 1-cycle of the form

$$c = \sum_{i=1}^{s} p_i b_i$$

where the b_i are loop cycles and the p_i are polynomials in the edges of Γ_v. the cycle c is said to be *pesky* if it has the following properties:

P) Every path between v and v' in Γ_v intersects the support of each b_i;

E) Every edge involved in p_i is contained in the support of b_i;

S) No 1-cut of Γ_v separating e' and e'' are contained in the support of c;

K) The class of c lies in the kernel of δ.

Lemma 3.2.3. *Let c be a representative for a class in ker(δ), meaning a cycle that satisfies K), of the form*

$$c = \sum_i p_i b_i + \sum_j q_j a_j$$

where the b_j are loop cycles and the a_j are star cycles. Then c is homologous to a cycle which satisfies also S) and E), where the condition E) on star summands imposes that the edges in q_i has to be adjacent to the support of a_i.

Proof. Let W be the set of 1-cuts of Γ_v separating e' and e''. Let us show that c lies in the image of $H_1(B((\Gamma_v)_W)))$, where $(\Gamma_v)_W$ is the result of the consecutive explosions of the vertices $W \cup \{v\}$. Let $w \in W$ and consider the exact sequence

$$... \to H_1(B_k(\Gamma_w)) \overset{i_*^w}{\to} H_1(B_k(\Gamma_v)) \overset{\psi^w}{\to} \bigoplus_{d(w)-1} H_0(B_{k-1}((\Gamma_v)_w)) \overset{\delta^w}{\to} ...$$

Since e' and e'' lies in distinct components of $(\Gamma_v)_W$, the multiplication by $e' - e''$ is injective on the third term of the sequence. The fact that $\delta([c]) = 0$ means that $[(e' - e'')c] = 0$. Thus c has to be in the kernel of ψ^w, that is the image of i^w. By repeating the same argument for each 1-cut in W, we conclude that $[c]$ lies in the image of $H_1(B((\Gamma_v)_W)))$ and hence it admits a representative which satisfies S). The Q-relation and connectedness of Γ_v imply that we may take a representative which satisfies also E). □

Proposition 3.2.4. *The quotient ker(δ)/$\psi(M(\Gamma))$ is generated by the images of pesky cycles.*

Proof. Let $c = \sum_i p_i b_i + \sum_j q_j a_j$ be a representative for a class in ker(δ). By Lemma 3.2.3, we may assume c satisfies K), E) and S). Let I (resp. J) be the set of indices i (resp. j) such that there is a path between v' and v'' avoiding the support of b_i (resp. a_j). We write

$$c_1 = \sum_{i \in I} p_i b_i + \sum_{j \in J} q_j a_j \quad \text{and} \quad c_2 = c - c_1.$$

By Lemma 3.2.1, $[c_1] \in \psi(M(\Gamma))$, while c_2 satisfy P). In addiction,

$$\delta([c]) = \delta([c_1]) + \delta([c_2]) = \delta([c_2])$$

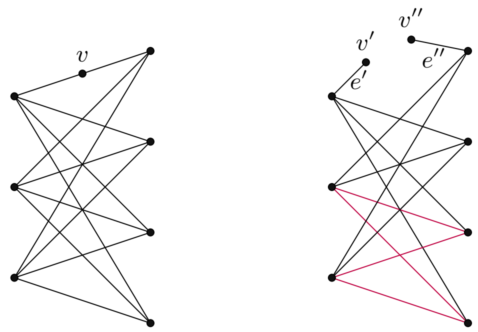
because $\delta \circ \psi = 0$ by exactness. It follow that c_2 satisfies K), and hence E) and S), because c does. Note that c and c_2 lies in the same class in the quotient $\ker(\delta)/\psi(M(\Gamma))$ and c_2 satisfy also P), by construction. We conclude by observing that a cycle satisfying both P) and S) can not have star summands, since the support of a star cycle consists only of a vertex. □

3.3 Counterexamples in non-planar graphs

The proof of Theorem 3.1.5 exhibited in [4] proceeds by two induction. The first one is on the number of 1-cut of the minimal simplicial model of the graph and it reduces the study of the pesky cycles to the biconnected case. The second induction further reduces the problem to the triconnected case. This inductive argument uses a combinatorial parameter that, in some sense, counts the number of 2-cut, for instance the N_2 defined in section 2.3. The triconnected case is addressed by combinatorial argument. More specifically, any pesky cycle in a triconnected planar graph is part of an embedded *angelic graph* (see [4, Section 7]). The study of these graphs, which are essentially just of three kinds, leads to the conclusion that any pesky cycle lies in the image of $M(\Gamma)$.

Unfortunately, in non-planar graphs there could be pesky cycles that do not come from known classes in the second homology, as Examples 3.3.1, 3.3.2 and 3.3.3 show. The first counterexample is exhibited in [4], while the other two are original. They provide new types of classes that need to be considered in the study of the generators for the second homology in the general case.

Example 3.3.1 ([4, Example 6.8])**.** Let $K_{3,4}$ be the complete bipartite graph in figure. The red cycle c is pesky with respect to the explosion of the vertex v that subdivide the edge e.



It is clear that the support of c does not contain any 1-cut separating v' and v'' and that it does not intersects any path joining them. Let us show that $[c]$ lies in the kernel of δ. Let e_0 be any edge in the support of c, then

$$\delta([c]) = [(e' - e'')c] = [(e' - e_0)c] - [(e'' - e_0)c]$$

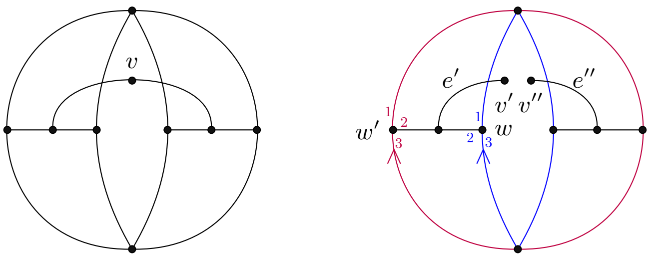
By the Q-relation, these two summands are star classes with support contained in $K_{3,4}\smallsetminus\{e\}$. Since $K_{3,4}\smallsetminus\{e\}$ is triconnected and non-planar, any star classes coincide up to sign and is 2-torsion. It follows that $\delta([c])$ is necessarily null. This proves that c is pesky. It reflects a new class in $H_2(B_2(K_{3,4}))$. In fact, $M_2(K_{3,4})$ is empty because it may contain only classes of tensor product of disjoint cycles, since star and theta classes require more particles, but there are no pairs of disjoint cycles in $K_{3,4}$.

By symmetry, one can found several other pesky cycles. By direct computation one can check that the new classes in this graph are (at least) two. Indeed, by Theorem 2.1.1, $\chi(B_2(K_{3,4})) = -3$. But

$$\chi(B_2(K_{3,4})) = 1 - \mathrm{rk}(H_1(B_2(K_{3,4}))) + \mathrm{rk}(H_2(B_2(K_{3,4})))$$

Since $K_{3,4}$ is non-planar and triconnected and $\beta_1(K_{3,4}) = |E| - |V| + 1 = 12 - 7 + 1 = 6$, by Theorem 2.3.6, $H_1(B_2(K_{3,4})) = \mathbb{Z}^6 \oplus \mathbb{Z}_2$. It follows that $\mathrm{rk}(H_2(B_2(K_{3,4}))) = 2$.

Example 3.3.2. Let Γ be the graph in figure and let c and c' be the loop cycles shown respectively in blue and red:



Let us show that $c + c'$ is pesky with respect to the explosion of the vertex v. Neither c nor c' contains 1-cuts separating v' and v'', and both clearly intersect every path between them. Let $\gamma = [c + c']$ and let e_0 and e_0' be edges in the support of c and c' respectively, then

$$\delta(\gamma) = [(e' - e'')(c + c')] = [(e' - e_0)c] - [(e'' - e_0)c] + [(e' - e_0')c'] - [(e'' - e_0')c']$$

By I-relation and Q-relation

$$[(e' - e_0)c] = \alpha^w_{213} = -\alpha^w_{123} \quad \text{and} \quad [(e' - e_0')c'] = \alpha^{w'}_{213} = -\alpha^{w'}_{123}$$

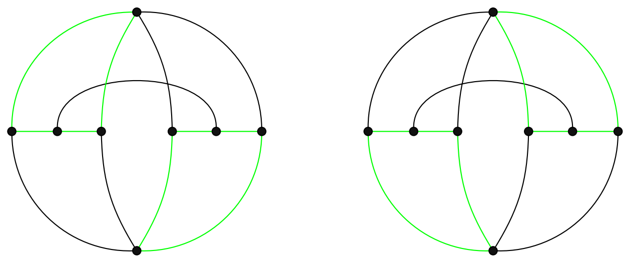
with the usual notation for the star classes. By Θ-relation $\alpha^w_{123} = -\alpha^{w'}_{123}$, then $[(e' - e_0)c] + [(e' - e_0')c'] = 0$. By the same argument, $[(e'' - e_0)c] + [(e'' - e_0')c'] = 0$. It follows that $\delta(\gamma) = 0$. This shows that $c + c'$ is pesky. By Theorem 2.1.1, $\chi(B_2(\Gamma)) = -2$. Since Γ is non-planar and triconnected and $\beta_1(\Gamma) = |E| - |V| + 1 = 13 - 8 + 1 = 6$, by Theorem 2.3.6, $H_1(B_2(\Gamma)) = \mathbb{Z}^6 \oplus \mathbb{Z}_2$. We have

$$\chi(B_2(\Gamma)) = 1 - \mathrm{rk}(H_1(B_2(\Gamma))) + \mathrm{rk}(H_2(B_2(\Gamma)))$$

and hence

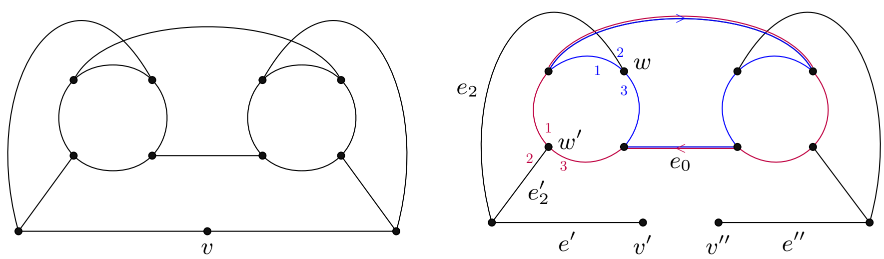
$$\mathrm{rk}(H_2(B_2(\Gamma))) = 3$$

But there are only two pairs of disjoint loop cycles in Γ (none of them involving the half edges at v):



This means that in $H_2(B_2(\Gamma))$ there is a new class which is reflected by the pesky cycle described. Clearly this class is different by the one of Example 3.3.1, since $K_{3,4}$ and Γ can not be embedded one in the other.

Example 3.3.3. Let G be the graph in figure and let c and c' be the loop cycles shown respectively in blue and red.



Let us show that, exploding the vertex v, $c + c'$ is pesky. Clearly every path between v' and v'' intersect the support of both c and c' and these supports do not contain any 1-cut separating the two vertices. It remains to show that the class $\gamma = [c + c']$ is in the kernel of δ.

$$\delta(\gamma) = [(e' - e'')(c + c')] = [(e' - e_0)(c + c')] - [(e'' - e_0)(c + c')]$$

By I-relation and Q-relation,

$$[(e' - e_0)c] = [(e_2 - e_0)c] = \alpha^w_{213} = -\alpha^w_{123}$$

with the usual notation for the star classes and where e_0 is an edge in the support of both c and c'. Using the same notation,

$$[(e' - e_0)c'] = [(e'_2 - e_0)c'] = \alpha^{w'}_{213} = -\alpha^{w'}_{123}.$$

Since the half edges involved in these two star classes lie on three disjoint path between w and w', by the Θ-relation $\alpha_{123}^{w} = -\alpha_{123}^{w'}$. It follows that $[(e'-e_0)(c+c')] = 0$ and by a symmetric argument also $[(e''-e_0)(c+c')] = 0$.

Thus $c+c'$ is pesky and it determines a new class in $H_2(B_2(G))$. Indeed, this class can not be the image through ψ of a tensor product of loop classes. The argument is similar to the one in Example 3.3.2, essentilly in G there are no pairs of disjoint loop cycles involving the half-edges at v. This class is also different from the one described in Example 3.3.1, since G_v is planar and it is not possible to obtain a planar graph by exploding some edge of $K_{3,4}$. This class is also different from the one of Example 3.3.2, since Γ can not be embedded in our graph G. Nevertheless, in the next section we explain why this two classses are closely related.

3.4 Edge contraction

In this section we observe that the contraction of one edge induces a contravariant map a the level of the Świątkowski complex, and hence in the homology of configuration spaces. This is a new kind of functoriality that can be helpful in the study of the generators for the second homology in the non-planar case. However, we believe that this is a tool interesting in itself, which could have further applications in future.

Definition 3.4.1. Let e be an edge of Γ that is not a self-loop and let v and v' be its endpoints. We denote by Γ^e the result of the *contraction* of the edge e, meaning the quotient of $\Gamma \setminus e$ by the identification of the two vertices v' and v''.

Note that the contraction of an edge does not lead to a graph morphism in the sense of Definition 1.1.4. Nevertheless, we think it as some kind of map $\Gamma \dashrightarrow \Gamma^e$ because it induces a contravariant map in the homology of the configuration spaces:

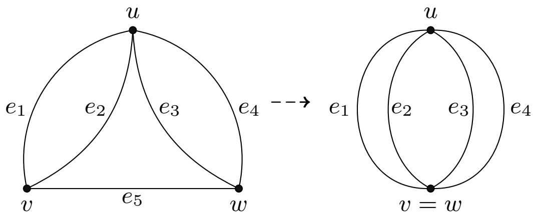
Definition 3.4.2. Let Γ^e be the graph obtained from Γ by the contraction of the edge e that is not a self-loop. Let v be the vertex of Γ^e that comes from the identification of the endpoints v' and v'' of e and let h' and h'' be the half edges of Γ which lie on e and are incident on v' and v'' respectively. We abuse notation in using the same names for vertices, edges and half-edges that are both in Γ and Γ^e. We consider the map $\phi^e : \tilde{S}(\Gamma^e) \to \tilde{S}(\Gamma)$ that is the obvious map on $\tilde{S}_w(\Gamma^e) = \tilde{S}_w(\Gamma)$ for $w \neq v$ and that on $\tilde{S}_v$ is given by:

$$\phi^e(h_i - h_j) = \begin{cases} h_i - h' + h'' - h_j & \text{if in } \Gamma \ h_i \in H(v') \text{ and } h_j \in H(v''); \\ h_i - h_j & \text{otherwise.} \end{cases}$$

Since in all the cases $\partial(\phi^e(h_i - h_j)) = \partial(h_i - h_j)$, ϕ^e induces a map in the homology $\varphi^e : H_n(B(\Gamma^e)) \to H_n(B(\Gamma))$.

First, we observe that this map is not always trivial:

Example 3.4.3. The graph Θ_4 can be obtained by contracting the edge e_5 of the graph G in figure:



This induces a map $\varphi^{e_5} : H_2(B_3(\Theta_4)) \to H_2(B_3(G))$. By direct computation one can check that the theta class is sent to the sum of the two toric classes of $H_2(B_3(G))$ given by the tensor product of a cycle and a star class. In particular, let A be the representative of the theta class given in Remark 3.1.2, then

$$\phi^{e_5}(A) = a^v_{125}b_2 + a^w_{345}b_1 + \partial(h^u_{41}h^v_{21}h^w_{35} + h^u_{21}h^v_{51}h^w_{43} - h^u_{31}h^v_{21}h^w_{45})$$

where b_1 and b_2 are the cycles

$$b_1 = h^u_{21} - h^v_{21} \quad \text{and} \quad b_2 = h^u_{43} - h^w_{43} = h^u_{41} - h^u_{31} - h^w_{45} + h^w_{35}$$

while a^v_{125} and a^w_{345} are the usual representative for the star classes at v and w respectively:

$$a^v_{125} = (e_1 - e_5)h^v_{21} - (e_1 - e_2)h^v_{51} \quad \text{and} \quad a^w_{345} = (e_5 - e_4)h^w_{35} - (e_5 - e_3)h^w_{45}.$$

The maps coming from the contraction of an edge give rise to a different kind of functoriality for the homology of the configuration space other than the maps coming from immersions. Clearly this operation can be iterated as long as the new edge to contract is not a self-loop. This means that any forest can be contracted.

Remark 3.4.4. Consider an edge contraction $\Gamma \dashrightarrow \Gamma^e$, and let v be a vertex e is not incident to. Then we can consider also the edge contraction $\Gamma_v \dashrightarrow \Gamma^e_v$. It will be useful to notice that the maps induced in the homology by these edge contractions commute with the maps of the long exact sequence of Theorem 2.1.4, so that we have the following commutative diagram:

$$\begin{tikzcd}
\ldots \arrow[r, "i^e_*"] & H_n(B_k(\Gamma^e))) \arrow[r, "\psi^e"] \arrow[d] & \bigoplus\limits_{|H(v)|-1} H_{n-1}(B_{k-1}(\Gamma^e_v))) \arrow[r, "\delta^e"] \arrow[d] & H_{n-1}(B_k(\Gamma^e_v))) \arrow[r, "i^e_*"] \arrow[d] & \ldots \\
\ldots \arrow[r, "i_*"] & H_n(B_k(\Gamma))) \arrow[r, "\psi"] & \bigoplus\limits_{|H(v)|-1} H_{n-1}(B_{k-1}(\Gamma_v))) \arrow[r, "\delta"] & H_{n-1}(B_k(\Gamma_v))) \arrow[r, "i_*"] & \ldots
\end{tikzcd} \tag{3.2}$$

The commutativity is trivial for i_*, which is just the map coming form the inclusion, and δ, which is on each component the multiplication by the difference of two edges different from e. Recall that, on Γ^e, ψ is induced by the map $\tilde{S}_k(\Gamma^e)_{n+1} \to \bigoplus_{h\in H(v)\smallsetminus h_0} \tilde{S}_{k-1}(\Gamma^e_v)_n$ that takes $b + \sum(h-h_0)a_h$ to $(a_h)_{h\in H(v)\smallsetminus h_0}$, where b and a_h do not involve half-edges at v. It suffice to notice that

$$\phi^e(b + \sum(h-h_0)a_h) = \phi^e(b) + \sum_{h\in H(v)\smallsetminus\{h_0\}} (h-h_0)\phi^e(a_h)$$

and that $\phi^e(b)$ and $\phi^e(a_h)$ do not involve half-edges at v.

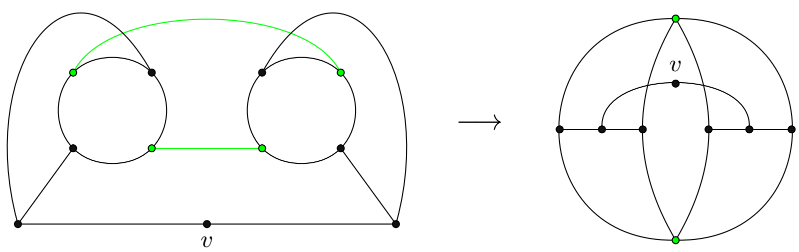
The commutativity of diagram 3.2 can be used to study what happens to the classes that are not in $M(\Gamma)$ when contracting an edge. In particular, we wish to describe an "exotic" class γ in $H_2(B(\Gamma))$ as the image through φ^e of a class in $H_2(B(\Gamma^e))$. Suppose γ is revealed, by exploding a bivalent vertex v, by a pesky cycle c. This means that $\psi(\gamma) = [c]$.

$$\begin{tikzcd}
\ldots \arrow[r, "i^e_*"] & H_2(B_k(\Gamma^e))) \arrow[r, "\psi^e"] \arrow[d] & H_1(B_{k-1}(\Gamma^e_v))) \arrow[r, "\delta^e"] \arrow[d, "\varphi^e"] & H_1(B_k(\Gamma^e_v))) \arrow[r, "i^e_*"] \arrow[d] & \ldots \\
\ldots \arrow[r, "i_*"] & H_2(B_k(\Gamma))) \arrow[r, "\psi"] & H_1(B_{k-1}(\Gamma_v))) \arrow[r, "\delta"] & H_1(B_k(\Gamma_v))) \arrow[r, "i_*"] & \ldots
\end{tikzcd} \tag{3.3}$$

By exactness and commutativity, if the class of c can be lifted to something in the kernel of δ^e, then γ comes from some class in $H_2(B_k(\Gamma^e))$. We observe that the a loop class in $H_1(B_k(\Gamma))$ is always in the image of φ^e. To be precise, the only classes that do not lie in the image fo φ^e are the star classes involving an half-edge in e when e is a tail. Then the problem is when c admits a preimage in the kernel of δ^e.

This reasoning suggests that it can be convenient to describe the homology of the configuration space of a graph in terms of classes coming not only from embedded subgraphs but also from graphs obtained by the contraction of some forest. In fact, allowing this new kind of functoriality, the "exotic" class found in Example 3.3.3 descends from the one of Example 3.3.2:

Example 3.4.5. Consider the graph G of Example 3.3.3. By contracting two edges one obtain the graph Γ of Example 3.3.2.



This induces a diagram like diagram 3.3. The vertices exploded correspond and the pesky cycle $c + c'$ of G lifts to the one of Γ, for which we used the same notation and which clearly lies in the kernel of δ. By the commutativity of the diagram, the new class in $H_2(B_2(G))$ of Example 3.3.3 is the image of the new class in $H_2(B_2(\Gamma))$ determined in Example 3.3.2 through the map $\varphi : H_2(B_2(\Gamma)) \to H_2(B_2(G))$ induced by the contraction of the two green edges.

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