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Emerging Dark Matter from corpuscular Dark Energy

Supervisor:
Prof. Roberto Casadio

Co-supervisor:
Dr. Andrea Giusti

Submitted by:
Giuseppe Ciocia

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Abstract

In the last years, the standard model of cosmology has been corroborated by a wide number of astrophysical observations. Despite its undeniable success, nowadays there is little knowledge about the true nature of dark matter and dark energy. In this thesis we use a different approach to give an intriguing answer to these open problems, in the light of the corpuscular model of gravity. We give a general overview on the reasons behind the need for a corpuscular theory of the gravitational interaction. Then, we show that if the same picture is extended to cosmological spaces, dark energy naturally emerges as a quantum state of the gravitational dynamics, and it is described as a Bose-Einstein condensate of very soft and virtual gravitons without the necessity of introducing an exotic dark fluid. Besides, the cosmic condensate responds locally to the presence of baryonic matter, and the back-reaction manifests itself in the emergence of a dark force that mimics a dark matter behavior. In particular, at galactic scales the MOND formula for the acceleration is recovered. Then, a first attempt of estimating the back-reaction is proposed within the framework of Bootstrapped Newtonian gravity, that allows for an effective field description where Newtonian theory is "bootstrapped" introducing post-Newtonian corrections, providing a useful tool for calculations. Finally, we show that a logarithmic potential arises as a solution of the Bootstrapped field equation, in accordance with MOND prediction.

Sommario

Negli ultimi anni, il modello standard della cosmologia è stato corroborato da un ampio numero di osservazioni astrofisiche. Nonostante il suo innegabile successo, ancora oggi la vera natura della materia oscura e dell'energia oscura appare poco chiara. In questa tesi abbiamo usato un approccio differente per dare una risposta intrigante a questi problemi aperti, alla luce del modello corpuscolare della gravità. Qui diamo una panoramica generale sulle ragioni che stanno dietro alla necessità di una teoria corpuscolare per l'interazione gravitazionale. Successivamente, mostriamo che se la stessa prospettiva è estesa a spazi cosmologici, l'energia oscura emerge naturalmente come uno stato quantistico della dinamica gravitationale, ed è descritta come un condensato di Bose-Einstein di gravitoni virtuali, senza il bisogno di introdurre un fluido oscuro esotico. Inoltre, il condensato cosmico risponde localmente alla presenza di materia barionica, e questa reazione si manifesta nell'emergenza di una forza oscura che imita un comportamento tipico della materia oscura. In particolare, a scale galattiche si ritrova la formula dell'accelerazione MOND. In seguito, viene proposto un primo tentativo di stimare la forza di reazione utilizzando la teoria Newtoniana Bootstrapped, che permette una descrizione di campo effettiva in cui la teoria Newtoniana è modificata introducendo correzione post-Newtoniane, e fornisce un utile strumento di calcolo. Infine, viene mostrato che dall'equazione di campo Bootstrapped è possibile ritrovare un potenziale logaritmico, in accordo con le previsioni della teoria MOND.

Contents

Chapter 1

Introduction

In the first part of this thesis we will present the main motivation underlying the quest for a quantum theory of the gravitational interaction. After a quick review of Einstein's theory of gravity, we will address the problem of finding a quantum realization of General Relativity, that inevitably leads to the perturbative non-renormalizability of the theory. In this context the corpuscular model of gravitational interaction takes its place, since it offers a fascinating and intriguing perspective. In the last part of the chapter, the fundamental aspects of the model will be presented.

1.1 Einstein's General Relativity

The gravitational interaction is successfully described by Einstein's theory of General Relativity (GR). GR is a metric theory of gravity, delineating how the geometry of a four-dimensional pseudo-Riemannian space-time metric manifold is determined by the presence of energy-momentum. The extraordinary intuition of Einstein was that gravity is not a force acting on objects, as the Newtonian model prescribes. All the objects always follow a straight path, but the presence of mass-energy curves the space-time geometry itself. In Einstein's theory of gravity the gravitational field is represented by a symmetric rank-2 tensor $g_{\mu\nu}$. The formalism is based on the definition of the Einstein-Hilbert action,

$$S_{\mathrm{EH}} = \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,\left[-\frac{R}{16\pi G_{\mathrm{N}}}\right] , \tag{1.1}$$

where $G_{\mathrm{N}} = 6.67 \times 10^{-11}\,\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$ is Newton's gravitational constant, the integral covers a region $\mathcal{M}$ of the space-time manifold, we indicate by g the determinant of the metric, and $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar. Moreover one introduces a matter action S_M for non-gravitational fields that gives rise to the energy-momentum tensor defined as

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\rm M}}{\delta g^{\mu\nu}} \,. \tag{1.2}$$

The energy-momentum tensor acts as a source for the gravitational field. The total action has the form

$$S = S_{\rm EH} + S_{\rm M} \,, \tag{1.3}$$

and from its variation with respect to the metric tensor the Einstein's field equations are obtained,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_{\rm N} T_{\mu\nu} \,. \tag{1.4}$$

The left-hand side of (1.4) is also denoted as the Einstein tensor $G_{\mu\nu}$, and it is constructed entirely with the metric tensor and its first and second derivatives, and is linear in the second derivatives. While in Newton's gravity the gravitational potential is described by a single scalar function, in the Einstein's formulation there are 10 independent components for a tensor potential to be found. We expect that the field equations in General Relativity comprise 10 algebraically independent equations, at least in principle. Besides, the Einstein tensor satisfies four Bianchi identities,

$$\nabla_\mu G^{\mu\nu} = 0 \,. \tag{1.5}$$

As a consequence, there are not 10 independent equations for the field, but only 6. We are left with four degrees of freedom, corresponding to the fact that General Relativity is a *generally covariant* theory. As a matter of fact, if $g_{\mu\nu}(x)$ is a solution of the field equations, another solution $g'_{\mu\nu}(x)$ can always be found by performing a general coordinate transformation

$$x^\mu \longrightarrow x'^\mu = x'^\mu(x^\nu) \tag{1.6}$$

involving four arbitrary functions. Since Einstein's equations are highly non-linear, finding out an exact solution for the space-time metric is not an easy task. One usually tries to impose symmetry conditions to the metric *a priori* , related to the physical system to be studied. The first exact solution was discovered by Schwarzschild in 1916, who proposed a solution of the vacuum Einstein's equations (where $T^{\mu\nu} = 0$) for the space-time metric in the region outside a point-like source under the hypothesis of spherical symmetry. The assumption of spherical symmetry reduces the unknown independent functions to be determined to only 2. The Schwarzschild metric in standard coordinates reads,

$$ds^2 = -\left(1 - \frac{2G_{\rm N}M}{r}\right) dt^2 + \left(1 - \frac{2G_{\rm N}M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \,, \tag{1.7}$$

where M is the mass of the source and r is the areal radius, defined such that the physically measured area of a surface of constant r and t is given by $A = 4\pi r^2$. The Schwarzschild or gravitational radius is defined as

$$R_{\mathrm{S}} = 2\, G_{\mathrm{N}} M\,. \tag{1.8}$$

Although it corresponds to a coordinate singularity of (1.7), the gravitational field is regular in R_S. The gravitational radius represents the location of the event horizon of a Schwarzschild black hole. Once a test particle has crossed the surface $r = R_S$, the only allowed physical trajectories are those that involve a decrease of the coordinate r. Therefore the test particle has no other possibility but to fall towards $r = 0$, which corresponds to a true gravitational singularity.
At the present GR has the best agreement with experiments and observations. In the last century a plethora of experimental tests have been passed by the theory, starting from the first measurement of the bending of light in 1919 by Eddington. One of the most recent and striking evidence was the detection of gravitational waves [11].

1.2 Motivation for a quantum theory of gravity

Despite the success of Einstein's theory of gravity, nowadays it is clear that GR cannot be a valid description of gravitational interactions at the most fundamental level. The *singularity theorems* by Hawking and Penrose [36] state that, under very general conditions, singularities in space-time are unavoidable, and this signals the breakdown of the theory. Moreover, it should be noticed that GR remains a classical theory, which is not unified in a consistent way within a quantum framework. We know that the success of quantum theories in describing Nature at fundamental scales is supported by a huge number of experimental tests with very high precision. As an example, Quantum Electrodynamics (QED) is one of the most accurate physical theory of the history of science: the agreement between measurements and calculations for the anomalous magnetic dipole moment of the electron is around 1 part in 1 billion [23].
Different motivations for quantizing gravity can be identified. First of all, there are *conceptual* reasons. From a historical point of view, in several cases the evolution of physical theories has led to the unification of different models into a unique framework. All the non-gravitational interactions, for example, are united in the Standard Model of Particle Physics, which is a quantum field theory. The quantization of gravity can certainly pave the way for a "theory of everything", so that a complete unification of all fundamental interactions could be achieved.

In addition, Quantum Gravity should be able to remove inconsistencies between quantum physics and GR. In General Relativity the gravitational field is represented by the space-time metric. Therefore, a quantization of the gravitational field would correspond to a quantization of the space-time metric itself. The quantum dynamics of the gravitational field would correspond to a dynamical quantum space-time. But Quantum Field Theories presuppose a fixed, non-dynamical background space for the dynamics of quantum fields. In particular, the problem of time arises: in quantum theories, time is an external parameter not described by any operator. In GR space-time is a dynamical object. The two concepts are incompatible, so one expects that a fully quantum gravity theory must lead to modifications in the way we conceive time. Besides, there are *unsolved problems.* The existence of astrophysical black holes is one of the most famous predictions of General Relativity. However, what happens in the vicinity of the singularity is not clear yet. One could expect that quantum effects show up leading to a general avoidance of space-time singularities [13]. Furthermore, the final stages of the black hole evolution requires a more general theory. Singularities are also present in cosmological models, and a quantum theory of gravity should allow us to achieve a fundamental understanding of the early Universe. It must be noticed that if gravity is quantized, this implies that the whole Universe must be described in quantum terms, leading to the concept of quantum cosmology.

In addition, astrophysical observations show that a large part of our Universe is not completely understood yet. The explanation of galaxy rotation curves requires the existence of an exotic form of non-baryonic matter known as dark matter; the accelerated cosmological expansion is provided by dark energy. Since a conclusive answer is not achieved yet, this can be interpreted as a hint that a more fundamental theory of gravity is necessary, and can hopefully enlighten the true nature of the dark sector of the Universe. In this thesis dark matter and dark energy have a central role, and will be discussed extensively in Chapters 3 and 4.

1.2.1 The scale of Quantum Gravity

In a quantum theory of gravity, quantum effects arise at any scale, since classical properties are only an emergent phenomenon. However, there exists a scale at which we expect a quantum behavior of the gravitational interaction to be non-negligible with respect to other interactions. In 1899 Planck introduced his famous Planck units, that are unique combinations of the fundamental constants $(c, G_{\mathrm{N}}, \hbar)$ that yield the

Planck mass, length and time,

$$m_{\mathrm{P}} = \sqrt{\frac{\hbar c}{G_{\mathrm{N}}}} \approx 1.22 \times 10^{19}\ \mathrm{GeV}\,, \tag{1.9}$$

$$\ell_{\mathrm{P}} = \sqrt{\frac{\hbar G_{\mathrm{N}}}{c^3}} \approx 1.62 \times 10^{-35}\ \mathrm{m}\,, \tag{1.10}$$

$$t_{\mathrm{p}} = \sqrt{\frac{\hbar G_{\mathrm{N}}}{c^5}} \approx 5.40 \times 10^{-44}\ \mathrm{s}\,. \tag{1.11}$$

For an elementary particle whose mass is m_{P}, its Compton length ℓ_p is equal (apart from a factor of 2) to its Schwarzschild radius. Therefore the space-time curvature of that particle is no longer negligible. If the Planck scale truly represents the minimal length of Nature or not depends on the approach of quantum gravity one prefers the most. In causal set theory and Loop Quantum Gravity the existence of a minimal scale is a consequence of spacetime discreteness. The Planck scale is understood most generally as an operational limit, meaning nothing smaller can be probed.
Performing an experiment at the Planck scale would mean to investigate physical phenomena far beyond the current accepted models. The Standard Model of particle physics is in very good agreement with experiments at the particle accelerators. The LHC at CERN is the biggest and most effective particle accelerator, it has a diameter of $\sim$ 27 km and reaches an energy for particle collisions in the center of mass of $\sim$ 14 TeV. Investigating that energy scale led to the discovery of the Higgs boson in 2012 [1]. However, in order to probe the Planck energy, an accelerator with the size of several thousand light years would be needed.
The major task is to build up a consistent quantum theory of gravity that can be subjected to experimental tests. The experimental inaccessibility of the Planck scale does not imply that every possible fundamental theory cannot produce observational predictions. The trans-Planckian collision picture is only one way for introducing a quantum gravity phenomenology, but other possible means can be afforded: for example high-precision low-energy measurements involving cold atoms [29].

1.2.2 Semi-classical gravity

Since the major part of the experimental results leads to a confirmation of classical GR as a good theory of gravity, one can argue that the gravitational field could remain classical, while the matter fields should obey quantum mechanical equations. Therefore one can look for a semi-classical theory of gravity, that is an *exact* theory in which quantum degrees of freedom are coupled to classical degrees of freedom.

In principle, in a fully quantum description, one expects that all observable quantities are given by the expectation values of a suitable operator $\hat{O}$ acting on a quantum state $|\psi\rangle$. Since many such observables appear to satisfy classical equations to a very good approximation, the quantum state $|\psi\rangle$ must be such that

$$\langle\psi|\hat{O}|\psi\rangle \simeq O_{\rm cl}\,, \tag{1.12}$$

where $O_{\rm cl}$ satisfies the classical equations. Let us assume that all the observables can be written as the sum of a background part and a perturbation,

$$O = O_{\rm cl} + o\,, \tag{1.13}$$

where the perturbation o is assumed to be "small" compared to the background. In the semi-classical theory, the perturbation part alone will be promoted to an operator, while the background remains classical, namely

$$O \longrightarrow O_{\rm cl} + \hat{o}\,. \tag{1.14}$$

This procedure is known as the background field method, and once accepted it can be applied to all the fields. In particular, the metric tensor can be split into

$$g_{\mu\nu} \longrightarrow g^{\rm cl}_{\mu\nu} + \epsilon\,\hat{h}_{\mu\nu}\,, \tag{1.15}$$

while for matter fields,

$$\phi \longrightarrow \phi_{\rm cl} + \epsilon\,\hat{\phi}\,, \tag{1.16}$$

where we introduced the parameter ϵ in order to formally keep track of the expansion. The next assumption we make is that the classical parts of the metric and of the matter fields satisfy the Einstein's field equations. This will generate an external and fixed classical space-time as a background for the quantum fields $\hat{h}_{\mu\nu}$ and $\hat{\phi}$. In this case the quantum gravitational perturbations $\hat{h}_{\mu\nu}$ do not affect the causal structure of space-time, which is completely determined by the classical background metric, but it simply interacts with the other matter fields.

Quantum field theories on a fixed curved space-time have been extensively studied starting from the middle of the last century. Their development is not only of fundamental theoretical interest, but could also lead to observational consequences. The presence of quantum fields modifies the notions of vacuum and particles [32]. Since the vacuum is only invariant with respect to Poincaré transformations, observers that are not related by inertial motion refer to different types of vacua. As a consequence, the particle creation phenomenon can occur. A significant example is the result obtained by Hawking: a late time observer detects a thermal flux of particles radiated by a black hole, with a temperature given by

$$T_{\rm BH} = \frac{\hbar k}{2\pi k_B c}\,, \tag{1.17}$$

where k is the surface gravity of the black hole. Through the Hawking mechanism, the black hole loses mass because of the emission of radiation, until it reaches a mass comparable to $m_{\rm P}$. Since Hawking radiation was found in the semi-classical approximation, at this point the derivation itself breaks down, and cannot be trusted anymore.

Another notable problem with semi-classical theories of gravity is the co-existence of quantum and classical systems, coupled together in a hybrid dynamics. It is not so clear if this approach can lead to internal inconsistencies [25].

1.3 Linearized gravity and the graviton

Let us proceed with the attempt of quantizing the metric theory following the background field procedure. Consider a suitable choice of reference frame such that the metric tensor is decomposed into a fixed non-dynamical background and a perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{1.18}$$

where we choose $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ as the Minkowski metric. The perturbation $h_{\mu\nu}$ is supposed to be small, in the sense that its components are small compared to the flat metric ones in standard Cartesian coordinates. Neglecting terms of higher order than the linear one in the perturbations, the linearized version of the Ricci tensor is obtained,

$$R_{\mu\nu} = \frac{1}{2}\left(-\Box h_{\mu\nu} - \partial_\mu\partial_\nu h + \partial_\sigma\partial_\mu h^\sigma_\nu + \partial_\sigma\partial_\nu h^\sigma_\mu\right) + \mathcal{O}(h^2)\,, \tag{1.19}$$

where $h = \eta_{\mu\nu}h^{\mu\nu}$ because the indices are raised and lowered by means of the flat metric. Every equation for $h_{\mu\nu}$ will not determine it in a unique way, because any solution can always generate other solutions by performing an infinitesimal change of coordinates of the form

$$x^\mu \longrightarrow x'^\mu = x^\mu + \varepsilon^\mu(x)\,. \tag{1.20}$$

The invariance of the full theory under general coordinate transformations leads to the invariance of the linear theory under the gauge transformation of the field, namely

$$h_{\mu\nu} \longrightarrow h'_{\mu\nu} = h_{\mu\nu} - \partial_\nu\varepsilon_\mu - \partial_\mu\varepsilon_\nu\,. \tag{1.21}$$

In order to simplify the expressions, one can fix the gauge by introducing the harmonic condition (or the de Donder gauge), that is

$$\partial_\mu h^\mu_{\ \nu} = \frac{1}{2}\partial_\nu h \,. \tag{1.22}$$

It should be underlined that this corresponds to the choice of a particular class of reference frames. Using (1.19) together with (1.22) in Einstein's equations (1.4), one obtains the linearized Einstein's field equations,

$$\Box h_{\mu\nu} = -16\pi G_{\mathrm{N}} \left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T \right) , \tag{1.23}$$

where $T = \eta^{\mu\nu} T_{\mu\nu}$ is the trace of the energy-momentum tensor. One usually defines the transverse and traceless perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\lambda_{\ \lambda} \,, \tag{1.24}$$

so that (1.23) simply assumes the form

$$\Box \bar{h}_{\mu\nu} = -16\pi G_{\mathrm{N}} T_{\mu\nu} \,. \tag{1.25}$$

Notice that the harmonic condition now reads $\partial_\nu \bar{h}_\mu^{\ \nu} = 0$ in direct analogy to the Lorenz gauge condition for the electromagnetic vector potential $\partial_\mu A^\mu = 0$. In particular, the harmonic gauge condition is consistent with the vanishing of the derivative of the energy-momentum tensor $\partial_\nu T^{\mu\nu} = 0$, but not with the vanishing of its covariant derivative $\nabla_\nu T^{\mu\nu} = 0$. Therefore, although $T_{\mu\nu}$ acts as a source for $h_{\mu\nu}$, in the linear approximation there is no exchange of energy between matter and the gravitational field. In other words, at the linear level we are completely neglecting the back reaction of the gravitational field onto matter, as expected.

In the vacuum case, imposing $T_{\mu\nu} = 0$, the simplest solutions to (1.23) are plane waves,

$$h_{\mu\nu}(x) = e_{\mu\nu}\, \mathrm{e}^{i k_\alpha x^\alpha} + e^*_{\mu\nu}\, \mathrm{e}^{-i k_\alpha x^\alpha} \,, \tag{1.26}$$

introducing $e_{\mu\nu}$ as the symmetric polarization tensor, and k^μ as the light-like wave vector. With $h_{\mu\nu}$ still obeying the de Donder gauge (1.22) one is free to perform another coordinate transformation of the form (1.20) by fixing the four functions $\varepsilon^\mu(x)$ in such a way that the additional condition $\Box\, \varepsilon^\mu(x) = 0$ is satisfied, and the harmonic gauge still holds. At the end of the story, we start with 10 components of the symmetric rank-2 tensor $h_{\mu\nu}$, then we impose 4 conditions introducing a gauge fixing, and 4 more conditions that leave the gauge fixing condition invariant. We are

left with only 2 independent degrees of freedom for the gravitational field in the linear approximation. This is equivalent to say that gravitational waves carry 2 transverse polarizations, as it happens for electromagnetic waves.
Moreover, if a plane wave Ψ transforms as $\Psi \longrightarrow \Psi' = e^{ih\theta}\,\Psi$ under a rotation, one defines h as the wave *helicity*. It can be shown that the helicity of a gravitational wave is equal to ± 2. The absolute value of the helicity represents the spin for a massless particle. In the quantum theory, these states will correspond to the helicity states of the graviton.

It is worth noting that the field equations of linearized gravity can be obtained through the Euler-Lagrange field equations from the Fierz-Pauli action,

$$\begin{aligned} S_{\mathrm{FP}} = & \frac{1}{32\pi G_{\mathrm{N}}} \int \mathrm{d}^4x \, \left(\frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\lambda h_{\mu\nu}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h \right) \\ & + \frac{1}{2}\int \mathrm{d}^4x \, h_{\mu\nu} T^{\mu\nu} \, . \end{aligned} \tag{1.27}$$

The first line describes pure linear gravity, while the last part contains the coupling with the source. The action (1.27) is gauge invariant up to a total derivative. For the interaction sector, the invariance is guaranteed by the continuity equation for $T^{\mu\nu}$.
It is now clear that a helicity-2 field theory with the Fierz-Pauli action cannot be a candidate for a fundamental theory of gravity, for the simple reason that it doesn't account for the back reaction of the gravitational field, as mentioned before. One could try to artificially insert the canonical energy-momentum tensor of gravity, defined as

$$t^{\mu\nu} = \frac{\delta L}{\delta\, \partial_\mu h^{\alpha\beta}}\, \partial^\nu h^{\alpha\beta} - \eta^{\mu\nu} L \, , \tag{1.28}$$

by adding it to the right-hand side of linearized Einstein equations (1.25). This modification would, however, lead to a Lagrangian cubic in the fields, which in turn would give a new contribution to $t_{\mu\nu}$ and so on. This series continues indefinitely, and sums exactly to the Einstein's equations, and to the full Einstein-Hilbert action [14]. Once the iteration is begun it must be continued to all orders, since conservation only holds for the full series. Thus, the theory is either left in its free linear form, or it must be an infinite series. This argument shows that at the classical level the Fierz-Pauli action (1.27) inevitably leads to General Relativity.

Let us proceed with the quantization of the theory. One starts form a superposition of plane-wave solutions (1.26) and formally turns this into an operator,

$$\hat{h}_{\mu\nu}(x) = \sum_{\sigma=\pm 2} \int \frac{\mathrm{d}^3 k}{\sqrt{2|\mathbf{k}|}} \left[\hat{a}(\mathbf{k},\sigma)\, e_{\mu\nu}(\mathbf{k},\sigma)\, \mathrm{e}^{ik_\alpha x^\alpha} + \hat{a}^\dagger(\mathbf{k},\sigma)\, e^*_{\mu\nu}(\mathbf{k},\sigma)\, \mathrm{e}^{-ik_\alpha x^\alpha}\right]_{k_0=|\mathbf{k}|} \tag{1.29}$$

The operators $\hat{a}(\mathbf{k},\sigma)$ and $\hat{a}^\dagger(\mathbf{k},\sigma)$ are interpreted as annihilation/creation operators: they respectively annihilate/create a particle called graviton, with momentum $\hbar\mathbf{k}$ and helicity σ. They obey the usual canonical commutation relations,

$$\left[\hat{a}(\mathbf{k},\sigma), \hat{a}^\dagger(\mathbf{k}',\sigma')\right] = \delta_{\sigma,\sigma'}\,\delta(\mathbf{k}-\mathbf{k}') \tag{1.30}$$

while the other commutators vanish. The graviton is the analogous of the photon for the electromagnetic field. It is a massless spin-2 particle, whose concept depends on the presence of a flat Minkowski space-time as a background, so that Poincaré symmetry is ensured. The concept of a particle itself is strictly related to the classification of irreducible representations of the Poincaré group. The vanishing mass of the graviton is a consequence of the infinite range nature of the gravitational interaction, and of the fact that gravitational waves in vacuum propagates at the speed of light (being $k_\mu k^\mu = 0$).

1.3.1 The Newtonian limit

Every theory of gravity must reproduce the Newtonian dynamics in the appropriate Newtonian limit. The first step is to consider a weak gravitational field, or equivalently the case of small curvature, in the sense that the curvature radius is much larger than the characteristic wavelength of the test particle. As a matter of fact, Newtonian gravity is a very good approximation far from strong gravitational fields. Once again we assume there exists a reference frame such that the full metric is written as a Minkowskian background plus a small perturbation $h_{\mu\nu}$, as we introduced in (1.18). By substituting in Einstein's equations, we obtained the linearized form of the field equations (1.23).
In addition to the weak field limit, we assume that all matter in the system is non-relativistic, moving with a characteristic velocity much slower than the speed of light in the chosen coordinate frame $x^\mu = (t, \vec{x})$. Therefore the time-time component of the metric h_{00} is dominant. Moreover, we suppose a static gravitational field, so that

$$\partial_t h_{00} = 0\,. \tag{1.31}$$

Looking at the harmonic gauge (1.22), for a static configuration $h_{00} = h_{00}(\vec{x})$, and the condition (1.31) is automatically satisfied . One has to attain a non-relativistic version of the energy momentum tensor. Since it is defined as

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{M}}}{\delta g^{\mu\nu}} = 2 \frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}} - g_{\mu\nu} \mathcal{L}_{\mathrm{M}} \,, \tag{1.32}$$

the only contribution we need to consider in the Newtonian limit is the mass density of the source $\rho(\vec{x})$. The energy-momentum tensor is approximated by

$$T_{\mu\nu} \simeq u_\mu u_\nu \, \rho(\vec{x}) \,. \tag{1.33}$$

The four-velocity of the source is given by

$$u^\mu = \left(\frac{1}{\sqrt{1-v^2}} \,, \frac{\vec{v}}{\sqrt{1-v^2}} \right) \,. \tag{1.34}$$

In the non-relativistic limit $|v| \ll 1$, and $u^\mu \simeq \delta^\mu_0$. The time-time component of the linear equation (1.23) now reads

$$\triangle h_{00}(\vec{x}) = -8\pi G_{\mathrm{N}} T_{00} \simeq -8\pi G_{\mathrm{N}} \rho(\vec{x}) \,. \tag{1.35}$$

where $\triangle$ is the Laplace operator. The last expression must be compared with the Newton equation for the gravitational potential V sourced by a density ρ through the Poisson equation

$$\triangle V(\mathbf{x}) = 4\pi G_{\mathrm{N}} \, \rho(\mathbf{x}) \,. \tag{1.36}$$

The comparison leads to identify $h_{00} = -2\,V + \mathrm{const.}$ The integration constant is uniquely fixed by the boundary conditions. In particular, if the space-time is asymptotically flat, the Newtonian potential vanishes at infinity, and so does h_{00}, therefore the constant vanishes. Thus the curved space-time metric and the Newtonian potential are related by

$$g_{00} = -(1 + 2\,V) \,. \tag{1.37}$$

1.4 Non renormalizability of General Relativity

Starting from the Fierz-Pauli action (1.27) one can ask whether it could be quantized in a way similar to electrodynamics, where one arrives to a very successful theory like QED. In the framework of Quantum Field Theory, the standard way of proceeding is to perform a quantum perturbation theory of General Relativity.

Let us review some notions of quantum theories of fields. Such theories use local field operators $\phi(x)$, where x is a point in the background (flat) space-time. This leads to the occurrence of arbitrarily small distances, that translate to arbitrarily large

momenta in the momentum space. In particular, in computing transition amplitudes in perturbation theory using Feynman diagrams, it is not surprising that integrating all the way up to infinity in momentum space, divergences can show up. This is not a problem of a restricted class of theories: even one of the simplest model one can formulate, the self-interacting scalar theory $\lambda\phi^4$, seems to be useless because the answer to almost any calculation is infinity.
A theory is called *renormalizable* if these divergences can all be removed by a redefinition of a finite number of physical constants (mass, charges, etc.) and fields. The value of these constants can only be determined experimentally. A non-renormalizable theory needs an infinite number of parameters to be determined by the experiments, which corresponds to a complete lack of predictability.
It turns out that the mass dimensionality of the coupling constant for the interaction (in natural units $\hbar = c = 1$) decides about renormalizability. This is also known as the power-counting criterion. The dimension is given by a coefficient called superficial degree of divergence, defined as

$$\Delta = 4 - d - \sum_f n_f(s_f + 1)\,, \tag{1.38}$$

where d is the number of derivatives, n_f the number of fields of type f, and $s_f = 0, 1/2, 1, 0$ for scalars, fermions, massive vector fields, and photons and gravitons, respectively. A theory is renormalizable only if the superficial degree of freedom is non negative.
Let us see what happens for GR. Considering once again the expansion of the general curved metric into a flat background plus a small perturbation (1.18), if one proceeds in a perturbative expansion of the Einstein-Hilbert action (1.1), the first non-vanishing contributions are schematically given by

$$S_{\mathrm{EH}} \sim \frac{1}{16\pi G_{\mathrm{N}}} \int \mathrm{d}^4x \left[(\partial h)^2 + (\partial h)^2 h + \cdots\right] \tag{1.39}$$

where the first order is represented by the Fierz-Pauli action (1.27) in the case of pure gravity, while all the other infinite terms contain 2 derivatives of h because the Ricci scalar contains 2 derivatives of the metric. Let us rescale $h = \sqrt{8\pi G_{\mathrm{N}}}\,\tilde{h}$ so that the field has the usual canonical dimensions, and the previous expression reads

$$S_{\mathrm{EH}} \sim \frac{1}{2} \int \mathrm{d}^4x \left[(\partial\tilde{h})^2 + \sqrt{8\pi G_{\mathrm{N}}}\,(\partial\tilde{h})^2\tilde{h} + \cdots\right] \tag{1.40}$$

Each interaction vertex brings a factor of $\sqrt{G_{\mathrm{N}}}$. Considering the first interaction term in (1.40) the superficial degree of freedom is $\Delta = 4 - 2 - 3(0 + 1) = -1$, that is negative. This is consistent with the fact that in natural units ($\hbar = c = 1$) Newton's constant is

$$G_{\rm N} = \frac{1}{m_{\rm P}^2}\,. \tag{1.41}$$

We conclude that General Relativity is a perturbatively non-renormalizable theory. Having a closer look to the action (1.40), the first term in the expansion gives rise to the linear theory, described by the introduction of gravitons. The successive terms in the expansion are a manifestation of the non-linearity of the full Einstein's theory of gravity, which in turn is interpreted as a self-interaction between gravitons.
As a final remark, it should be underlined that the non-renormalizability of GR in the perturbative approach does not imply that GR is non-renormalizable in a non-perturbative way. In other words, one could not exclude that the incurable presence of divergences is an artifact only of the perturbative method. One might think that objects like black holes or the Universe should not be treated perturbatively.

1.4.1 General Relativity as an Effective Field Theory

A non-renormalizable theory must not be discarded as useless. In a sense, the non-renormalizability implies that the quantum field theory should break down at some energy scale, which in general is higher enough with respect to the scale we can probe with experiments. One is led to the introduction of a high energy cut-off scale where we think the theory stops being valid, and consequently we allow only for momenta smaller than the cut-off to run in the Feynman integrals.
The low energy behavior of many systems in Nature turns out to be mostly independent from higher energy scales. In the description of the propagation of waves in water, for example, one can largely ignore the underlying theory of molecules and atoms. The actual physical theory describing interactions and constituents at small scales would be useful in determining various coefficients in the low energy theory, such as the viscosity coefficient. However, once this quantity has been measured, the low energy theory is enough for making predictions for the phenomena it refers to, as long as one focuses on the correct energy scale. We refer to the descriptions applicable at low energies as *effective field theories*. In general, non-renormalizable theories are considered effective theories, valid far below a certain high energy scale cut-off, and thus evading the problem of infinitely many loop contributions. In other words, they are predictive as long as low energy scales are considered.
For such theories one could search for a *ultraviolet (UV) completion*. An UV-complete theory is formally predictive up to all possible high energies. As we have pointed out, perturbative quantum General Relativity falls into the not-UV-complete class of

theories, with the cut-off in energy equal to the Planck energy. The effective gravitational coupling in the graviton-graviton scattering increases with the energy as $G_N E^2 = E^2/m_P^2$ in natural units: while approaching the Planck scale, the coupling becomes of order 1, and since the scattering amplitude increases in the trans-Planckian region ($E \gg m_P$) the violation of unitarity is an inevitable consequence.

It is useful to define a dimensionless quantum self-coupling of gravitons. In the following, we will use Planck units with $c = 1$, so that $\hbar = \ell_P m_P$ and $G_N = \ell_P / m_P$. Let us start from the well-known Newtonian potential energy that describes the gravitational attraction of 2 bodies with same mass m,

$$U_N(r) = -\frac{G_N m^2}{r} = -\frac{\ell_P\, m^2}{m_P\, r} \,. \tag{1.42}$$

Introducing the characteristic wavelength $\lambda = \hbar/m = \ell_P m_P/m$, the potential is rewritten as,

$$U_N(r) = -\ell_P m_P \, \frac{\alpha_{gr}}{r} \,, \tag{1.43}$$

where the gravitational coupling is defined

$$\alpha_{gr} = \frac{\ell_P^2}{\lambda^2} \,. \tag{1.44}$$

From a relativistic generalization of the formula above, one is led to interpret (1.44) as the strength of graviton-graviton interaction, but since gravitons are massless, the role of the Compton wavelength is replaced by the actual wavelength.
The coupling is equal the ratio between the quantum Planck length and the classical wavelength. If the wavelength is far from the Planck scale ($\lambda \gg \ell_P$) the coupling is very small and the theory is in a weakly coupled regime. But approaching the Planck scale ($\lambda \sim \ell_P$) we enter a strongly coupled regime ($\alpha_{gr} \sim 1$) and GR is no longer valid.
The standard approach to fix the issue is the *Wilsonian UV completion.* The idea is to integrate-in new weakly coupled degrees of freedom above the breakdown threshold. One hopes that the introduction of new physics is enough to allow the calculations of gravitational amplitudes at arbitrarily energy scales.

1.4.2 Self-completeness of gravity

In this section we shall review the main aspects of a non-Wilsonian approach to UV-completion of gravity and, more generally, a class of perturbatively non-renormalizabile

theories, first introduced by Dvali and Gomez. The following sections refer to the main concepts of [19], [20] and [16].
The starting point is that in Einstein's theory of gravity below the Planck scale the only propagating degree of freedom is the massless spin-2 graviton. In the Wilsonian view, new degrees of freedom in the trans-Planckian region ($E \gg m_{\rm P}$) should be introduced to restore perturbative unitarity. However, it seems that in General Relativity the Planck length $\ell_{\rm P}$ has the characteristics of the shortest length in Nature, and is the lower bound on any distance that can be resolved.
Let us suppose to perform an experiment in order to probe a distance $L \ll \ell_{\rm P}$. The measurement can be thought to be a scattering process, in which one has to localize an energy at the order of $E \sim 1/L$ into a small portion of space of radius L. The Schwarzschild radius corresponding to the localized energy is

$$R_{\rm S}(L) \sim G_{\rm N} E \sim \frac{\ell_{\rm P}^2}{L} \,. \tag{1.45}$$

In the trans-Planckian region $L \gg \ell_{\rm P}$, and one can easly see that the Schwarzschild radius is bigger than the distance we want to probe, and it also exceeds the Planck length. Any attempt of probing a length scale beyond the Planck length will require the localization of energy within a smaller radius than the corresponding Schwarzschild radius. Therefore, the measurement should lead to the formation of a macroscopic black hole. Once a black hole is formed, the maximal information that can be extracted is equal to the information encoded at the black hole horizon, preventing any measurement in the trans-Planckian scale.
Equivalently, according to Dvali and Gomez, the physics at sub-Planckian distances is equal to the physics at macroscopic distances, that manifests itself in the correspondence

$$L \Longleftrightarrow \frac{\ell_{\rm P}^2}{L} \,. \tag{1.46}$$

Black holes are classical objects, entirely governed by the infrared (IR) sector of the theory. Dvali and Gomez claim a correspondence between UV gravity and IR gravity. The previous argument shows that, as opposed to the Wilsonian point of view, propagating degrees of freedom in the trans-Planckian region cannot exist, on the contrary they are mapped onto non-propagating classical states, that are fully described in terms of IR propagating degrees of freedom, the gravitons. This immediately restores unitarity, since the probability for a scattering involving an exchange in momentum of the order of $\hbar/\ell_{\rm P}$ is exponentially suppressed, and the scattering amplitude is dominated by the black hole formation.

In this sense, General Relativity is self-complete in the UV sector. The procedure of solving the unitarity violation by means of the formation of classical configurations is called *classicalization*. A classical state in a quantum field theory is no longer an independent entity, and from the quantum point of view it can be described as a coherent superposition of fundamental degrees of freedom with large occupation numbers.

1.5 Black hole quantum picture

Let us consider a spherically symmetric source of a gravitational field, with mass M, homogeneous density and a characteristic radius R. Suppose that the source is static, thus its radius is much bigger than the Schwarzschild radius $R_{\rm S} = 2\,G_{\rm N}M$. At this level the gravitational field is considered weak, and the approximation of linear gravity is valid everywhere. The time-time component of the metric is dominant and outside the source it gives rise to the Newtonian potential,

$$\phi_{\rm N}(r) = -\frac{G_{\rm N}M}{r} \tag{1.47}$$

From a quantum field theory point of view, the linearized metric represents a superposition of quantum states, that are non-propagating longitudinal gravitons. One can introduce a measure of the *classicality* of a system: a configuration is said to be classical if the occupation number of gravitons is very high,

$$N \gg 1\,. \tag{1.48}$$

An estimate of the occupation number proceeds as follows. The gravitational part of the energy is

$$E_{\rm grav} \simeq \frac{G_{\rm N}M^2}{R} = \frac{\ell_{\rm P}M^2}{m_{\rm P}R}\,. \tag{1.49}$$

and it can be interpreted as the sum of the energies of the individual gravitons with wavelength λ and occupation number N_λ,

$$E_{\rm grav} \simeq \sum_\lambda N_\lambda \varepsilon_\lambda\,, \tag{1.50}$$

where the Compton-de Broglie relation yields the energy for a single graviton,

$$\varepsilon_\lambda = \frac{\ell_{\rm P} m_{\rm P}}{\lambda}\,. \tag{1.51}$$

The reason why the total gravitational energy is well approximated by a simple sum of energies is that, first of all, we expect the wavelength distribution to be peaked on the value $\lambda = R$, corresponding to the characteristic size of the system. The contribution from the shorter wavelengths is exponentially suppressed. Since $R \gg \ell_{\mathrm{P}}$ the graviton-graviton coupling (1.44) is very small, and the gravitons should be considered as weakly interacting. We therefore neglect the interactions between individual gravitons in the calculation of the energy.
Besides, gravitons couple to all sources of energy, and in particular to (1.50). But since $R \gg R_{\mathrm{S}}$, the gravitational energy is negligible with respect to the mass-energy of the source, thus for the moment we can forget about the interaction between each individual graviton and the collective gravitational energy. The graviton occupation number N is roughly given by the total gravitational energy divided by the characteristic energy of a single quantum, and it scales as

$$N \sim \frac{M^2}{m_{\mathrm{P}}^2}\,. \tag{1.52}$$

Gravitons are bosons, therefore their occupation number can be very high and they can condensate. One can suppose that the ensemble of gravitons composing the system represents a Bose-Einstein condensate. However, if we neglect gravitational self-sourcing, the condensate cannot be self-sustained. The situation changes dramatically if, instead of a static source, one studies a gravitational collapse. In this case the radius of the source decreases until it reaches the Schwarzschild radius $R = R_{\mathrm{S}}$. As it was pointed out before, since $R \gg \ell_{\mathrm{P}}$ the gravitons are weakly interacting once again, and we don't account for the graviton-graviton interaction. But now the gravitational energy (1.50) is of the same order of magnitude of the mass-energy of the source, in fact $E_{\mathrm{grav}} \sim M$. At this point the self-sourcing by the collective gravitational energy becomes important and the condensate becomes self-sustained.
It is worth noting that even when the radius of the system is equal to the Schwarzschild radius, the length scale of the source is far from the Planck scale, and $R_{\mathrm{S}} \gg \ell_{\mathrm{P}}$. Thus, the scaling of the occupation number (1.52) still holds.

The situation we have just delineated corresponds to the formation of a classical black hole. In this view the classical geometric interpretation is completely discarded, and black holes appear intrinsically quantum. The usual background geometry should be an emergent property of the system, which is represented at the fundamental level as a Bose-Einstein condensate of weakly interacting gravitons with occupation number

N. The latter quantity can be recast as the inverse of the graviton coupling,

$$N \sim \frac{\lambda^2}{\ell_{\rm P}^2} = \frac{1}{\alpha_{\rm gr}} \, . \tag{1.53}$$

The previous relation defines the critical occupation number at which the graviton condensate becomes self-sustained. From this perspective, since the system is described as a self-sustained bound state which exhibits all the properties of a Bose-Einstein gas at the *critical point* of a quantum phase transition, the condition $\alpha_{\rm gr} N = 1$ defines the criticality [17]. As a matter of fact, for $\alpha_{\rm gr} N < 1$ the gravitational attraction would not be strong enough to keep the gravitons together. On the contrary, for $\alpha_{\rm gr} N > 1$ the gravitational attraction among the constituents would induce instability producing a collapse. The criticality condition can be obtained by the balance of the kinetic energies of individual gravitons and the collective binding potential, that is

$$K_{\rm G}(R) + U_{\rm G}(R) \simeq (1 - \alpha_{\rm gr} \, N) \, \frac{\ell_{\rm P} m_{\rm P}}{R} = 0 \, . \tag{1.54}$$

It turns out that among all possible sources of given characteristic size R, the occupation number N is maximized by a black hole. In other words, for maximal N the wavelength of the occupying quanta λ saturates the length that classically would be defined as the gravitational Schwarzschild radius of the configuration. This is the reason why the critical point coincides with the *maximal packing*: once the gravitons are overpacked, increasing N would necessarily mean increasing the characteristic length R.

1.5.1 Universality of the occupation number

The occupation number does not depend on the physical composition of the source, and this gives to N the attribute of *universality*. If the presented picture is correct, all the properties of black holes should be related to the occupation number, in such a way that the number N is the unique characteristic of a black hole. As a matter of fact, from the considerations above, the following scaling law have been presented,

- mass: $M \sim m_{\rm P} \sqrt{N}$,
- interaction coupling: $\alpha_{\rm gr} \sim N$,
- wavelength: $\lambda \sim \ell_{\rm P} \sqrt{N}$,
- characteristic length: $R \sim \ell_{\rm P} \sqrt{N}$.

The universality of N plays a central role in the classicalization of gravity in the trans-Planckian regime. In fact, let us propose once again the ideal experiment of probing a distance comparable to the Planck length: the scattering will produce a black hole, which is understood as a N-particle state. Therefore, for any high energy scattering process involving two particles with an energy in the center of mass of the order of m_{P},

$$2 \longrightarrow \text{black hole}\,, \tag{1.55}$$

and from a quantum point of view,

$$2 \longrightarrow N\,. \tag{1.56}$$

We conclude that all gravitational states with a trans-Planckian energy in the center of mass are actually N-particle states. Now, since the composition of the source is not important in determining the occupation number of gravitons, for classical/semiclassical sources composed by a large amount of particles of relatively long wavelength particles, and for quantum sources composed by few highly energetic particles, in the deep UV the result will be always a large-N state. Regardless whether in the absence of gravity the source would be quantum or classical, with gravity it always classicalizes as long as $M \gg m_{\mathrm{P}}$, and $R \sim R_S$. At this point, any source becomes a gravitationally self-sustained N-particle bound state, or equivalently a Bose-Einsteins condensate of N soft weakly interacting gravitons.

Chapter 2

Bootstrapped Newtonian gravity

Up to this point, we have discussed general information about the non renormalizability of General Relativity, and we have underlined the main aspects of UV completion of gravity via classicalization. This allows us to introduce the corpuscular model of gravity. The underlying idea is that the classical geometric description of gravitational interaction emerges from a fundamental fully quantum picture in terms of constituent self-interacting gravitons. Black holes represent in this sense an interesting arena for testing the model, since they appear naturally composed by a large number of gravitons, which superimpose in the same quantum state and realizing a critical Bose-Einstein condensate on the verge of a quantum phase transition. The aim of this Chapter is to extend and formalize the corpuscular picture beginning with the study of a gravitational collapse, which represents the physical process that can eventually lead to a black hole formation. Following the work of Casadio *at al.* [6, 7, 8, 9], we will review the derivation of "bootstrapped Newtonian gravity", an effective non-linear scalar theory for the gravitational field that includes post-Newtonian corrections to the usual Newtonian behavior.

2.1 The gravitational collapse from a corpuscular perspective

Suppose to consider an isolated compact source composed by N_B identical components, that we shall call baryons with an individual mass μ. Let us assume spherical symmetry, and let R be the radius of the object. Neglecting radiative emissions, the total energy is conserved and equals the Arnowitt-Deser-Misner (ADM) mass M of the system. This is always true in the Newtonian description, since the notion of energy is well-defined. However, the same result can be achieved in General Relativity, where time reparametrization invariance leads to the Hamiltonian constraint, that in an asymptotically flat space-time reads

$$H = H_{\mathrm{B}} + H_{\mathrm{G}} = M \,, \tag{2.1}$$

where H_{G} and H_{B} are the Hamiltonians of gravity and baryonic matter respectively. In order to get an energy balance of the system, suppose that in the initial state the baryons are very far apart at infinity so that they do not interact at all. Thus, the total energy corresponds to the baryonic rest mass,

$$E_{\mathrm{B}} = \mu \, N_{\mathrm{B}} \simeq M \,. \tag{2.2}$$

Afterwards, the baryons become closer and closer and the radius of the system decreases. When the N_{B} constituents start interacting, one needs to take into account their gravitational interaction, which is always attractive. Moreover, since the final configuration is supposed to be at equilibrium, an additional interaction among baryons must be introduced. The baryon-baryon interaction must be repulsive in such a way that the gravitational attraction is compensate, and the collapse can end. The baryons acquire kinetic energy K_B as well. The baryon energy at a given radius R is given by

$$E_{\mathrm{B}}(R) = M + K_{\mathrm{B}}(R) + U_{\mathrm{BG}}(R) + U_{\mathrm{BB}}(R) \,, \tag{2.3}$$

where $U_{\mathrm{BG}} < 0$ is the gravitational potential energy for the baryons, and $U_{\mathrm{BB}} \geq 0$ accounts for the repulsive baryonic interaction. In the classical theory the right-hand side of (2.3) contains all the energy contributions and, because of energy conservation, it must be equal to the mass M. We therefore find the condition

$$K_{\mathrm{B}}(R) + U_{\mathrm{BG}}(R) + U_{\mathrm{BB}}(R) = 0 \,, \tag{2.4}$$

from which one recovers the classical equations of motion. Supposing that the classical dynamics is well approximated by the Newtonian theory, the field V_{N} has the usual Newtonian form (1.47), and the baryonic gravitational energy coincides with

$$U_{\mathrm{BG}}(R) \simeq N_{\mathrm{B}} \, \mu \, V_{\mathrm{N}}(R) = -\frac{\ell_{\mathrm{P}} M^2}{m_{\mathrm{P}} R} \,, \tag{2.5}$$

which is negative as expected and falls like $1/R$. However, in a quantum theory the classical field can be seen as emergent from an underlying corpuscular description in terms of gravitons. Discussing the quantum corpuscular picture of black holes, we introduced the graviton occupation number $N_{\rm G}$, and we found a quadratic scale with the source mass, namely

$$N_{\rm G} \simeq \frac{M^2}{m_{\rm P}^2} \, . \tag{2.6}$$

In addition, let us assume a typical energy of a graviton given by the Compton-de Broglie relation, with a wavelength coinciding to the size R of the system,

$$\epsilon_{\rm G}(R) \simeq -\frac{\ell_{\rm P} m_{\rm P}}{R} \, , \tag{2.7}$$

which is negative. It is an indication that the gravitons of a static potential are off-shell. The expression (2.5) has a straightforward interpretation in terms of the corpuscular model. As a matter of fact, one can write

$$U_{\rm BG}(R) \simeq N_{\rm G} \, \epsilon_{\rm G}(R) \, . \tag{2.8}$$

It is well known that gravitons self-interact, therefore a term describing the graviton-graviton interaction should be added to the energy balance. The graviton self-interaction potential has the form

$$U_{\rm GG}(R) \simeq N_{\rm G} \, \epsilon_{\rm G}(R) \, V_{\rm N}(R) \simeq \frac{M^2 \ell_{\rm P}^2}{m_{\rm P}^2 R^2} \, . \tag{2.9}$$

The latter quantity is positive and depends on the size of the source like $1/R^2$. It is worth noting that these are characteristics typical of the post-Newtonian correction to the Newtonian potential. This establishes a connection between the corpuscular model and the standard post-Newtonian expansion of the Schwarzschild metric. The equation (2.4) now yields,

$$K_{\rm B}(R) + U_{\rm BG}(R) + U_{\rm BB}(R) + U_{\rm GG}(R) = 0 \, . \tag{2.10}$$

Comparing the graviton-graviton energy to the gravitational baryon energy, and introducing the Schwarzschild radius $R_{\rm S} = 2\ell_{\rm P} M / m_{\rm P}$, one finds that

$$\left| \frac{U_{\rm GG}}{U_{\rm BG}} \right| \simeq \frac{R_{\rm S}}{R} \, . \tag{2.11}$$

For non-marginally bound object, such as compact stars, the radius is much grater than the Schwarzschild radius, and the graviton self-interaction is negligible. However, the situation changes if the collapse continues until $R \simeq R_{\rm S}$, and classically a black hole forms. When the radius of the object reaches its Schwarzschild radius, the contributions of $U_{\rm BG}$ and $U_{\rm GG}$ become at the same order of magnitude of the mass-energy of the source, since

$$U_{\rm GG}(R_{\rm S}) \simeq -U_{\rm BG}(R_{\rm S}) \simeq M\,. \tag{2.12}$$

Here we recover the maximal packing condition (1.54) first introduced in the black hole quantum portrait, corresponding to the criticality condition for a Bose-Einstein condensate. The remarkable difference is that in the above description we considered baryonic matter in addition to the gravitons, although the effective number of soft gravitons in the black hole is much larger than the number of baryons. For example, for a solar mass black hole ($M = M_\odot \simeq 10^{38}\, m_{\rm P}$) made of neutrons ($\mu = 10^{-19}\, m_{\rm P}$), the number of baryons in the system is $N_{\rm B} = M/\mu \simeq 10^{57}$, while the gravitons are $N_{\rm G} \simeq M^2/m_{\rm P}^2 \simeq 10^{76} \gg N_B$. This is consistent with the corpuscular model of gravitons.
However, taking into account baryonic matter has a notable consequence. The Hamiltonian constraint for the baryonic source now reads

$$K_{\rm B}(R_{\rm S}) + U_{\rm BB}(R_{\rm S}) \simeq 0\,, \tag{2.13}$$

and since we assumed $U_{BB} \geq 0$, the only possible solution for (2.13) allowing a black hole formation is

$$K_{\rm B}(R_{\rm S}) \simeq U_{\rm BB}(R_{\rm S}) \simeq 0\,. \tag{2.14}$$

This somehow signals a possible end of the collapse. This first result shall be handled carefully, since a more detailed model is needed, that, for example, considers the spatial distribution of the source.

2.2 Quantum origin of the classical field

We shall deal with the problem of finding the connection between the classical Newtonian potential and its quantum interpretation in terms of gravitons. In order to set the topic, we can refer to an useful analogy concerning a static electric field. In particular, the Coulomb potential generated by an electric charge is conceptually equivalent to the Newtonian gravitational potential sourced by a massive object, whose mass defines its "gravitational charge". In the same way, one can address the question of interpreting the electric potential in the light of the quantum theory, in other words, photons. It has been shown [2] that the Coulomb potential can be described in terms of a coherent state of virtual photons. Coherent states are eigenstates of field annihilation operators, and in general are those states that most closely correspond to classical field configurations.

Going back to gravity, one can reasonably assume that the Newtonian potential can be represented by the expectation value of a suitable real quantum scalar field $\hat{\Phi}$ on a coherent state $|g\rangle$ of virtual scalar gravitons, namely

$$V_{\rm N} \sim \langle\, g \,|\, \hat{\Phi} \,|\, g \,\rangle \ . \tag{2.15}$$

As a matter of fact, the Newtonian potential for a spherically symmetric static source is a solution of the Poisson equation,

$$\triangle V_{\rm N}(r) = 4\pi \, \frac{\ell_{\rm P}}{m_{\rm P}} \, \rho(r) \, , \tag{2.16}$$

where $\rho(r)$ represents the matter density. Let us rescale the field and the source in such a way that they have the usual canonical dimensions,

$$\Phi(r) \equiv \sqrt{\frac{m_{\rm P}}{\ell_{\rm P}}} \, V(r) \, , \qquad J(r) \equiv 4\pi \, \sqrt{\frac{\ell_{\rm P}}{m_{\rm P}}} \, \rho(r) \, , \tag{2.17}$$

so that $[\Phi] = [\text{mass}]^{\frac{1}{2}}[\text{length}]^{-\frac{1}{2}}$ and $[J] = [\text{mass}]^{\frac{1}{2}}[\text{length}]^{-\frac{5}{2}}$. The Poisson equation for the field now yields

$$\triangle \Phi_{\rm N}(r) = J(r) \, , \tag{2.18}$$

and in the momentum space the classical solution is

$$\tilde{\Phi}_{\rm N}(k) = -\frac{\tilde{J}(k)}{k^2} \, . \tag{2.19}$$

Introducing a suitable set of normal modes, the quantum field $\hat{\Phi}$ can be split into a negative and a positive frequency part, and an appropriate set of annihilation/creation operators $\{\hat{a}_k \, , \hat{a}_k^\dagger\}$ can be introduced, satisfying the usual bosonic commutation relations. A coherent state $|g\rangle$ is defined as follows,

$$\hat{a}_k \, |g\rangle = e^{i\gamma_k(t)} g_k \, |g\rangle \ . \tag{2.20}$$

If one chooses the state in such a way that

$$g_k = \sqrt{\frac{k}{2\,\ell_{\rm P}\, m_{\rm P}}} \, \tilde{\Phi}_{\rm N}(k) = -\frac{\tilde{J}(k)}{\sqrt{2\,\ell_{\rm P}\, m_{\rm P}\, k^3}} \, , \qquad \gamma_k(t) = -kt \, , \tag{2.21}$$

and assuming the normalization $\langle g|g\rangle = 1$, the classical solution of the Poisson equation is recovered exactly as

$$\Phi_{\rm N}(r) = \langle\, g \,|\, \hat{\Phi} \,|\, g \,\rangle \ . \tag{2.22}$$

Moreover, the form of the coherent state in terms of the vacuum of the model, defined as the state annihilated by any annihilation operator, reads

$$|g\rangle = e^{-N_{\rm G}/2} \exp\left\{ \int_0^\infty \frac{k^2 {\rm d}k}{2\,\pi^2}\, g_k\, \hat{a}_k^\dagger \right\} |0\rangle \ . \tag{2.23}$$

The normalization factor $N_{\rm G}$ turns out to be the total occupation number of the scalar gravitons in the state $|g\rangle$, and imposing the state normalization one finds

$$N_{\rm G} = \int_0^\infty \frac{k^2 {\rm d}k}{2\,\pi^2}\, g_k^2\,. \tag{2.24}$$

The latter quantity is usually ill-defined. It should be regularized by means of suitable cut-offs in order to remove the divergences. In the case of a source with finite size R and mass M, the assumption that the volume of the universe is infinite or, equivalently, that the source is eternal and its static gravitational field extends to infinite distances, produces an infrared divergence. One is led to bring in a finite length R_∞ representing the size of the universe within which the gravitational field is static. At the end of the calculation [9], one finds that the occupation number scales quadratically with the source mass in accordance with (2.6), with a weak logarithmic dependence on the ratio R/R_∞. This last correction disappears in the ideal limit where $R_\infty \to \infty$.

2.3 Effective scalar theory

In the following, we are going to derive an effective field theory for the gravitational potential up to the first post-Newtonian correction in terms of a scalar field. For the sake of simplicity we assume a static and spherically symmetric configuration. One starts from the classical Einstein-Hilbert action coupled with matter,

$$S = S_{\rm EH} + S_{\rm M} = \int {\rm d}^4x \sqrt{-g} \left(-\frac{m_{\rm P}}{16\,\pi\,\ell_{\rm P}}\, R + \mathcal{L}_{\rm M} \right) , \tag{2.25}$$

where R is the Ricci scalar and $\mathcal{L}_M$ is the Lagrangian for the baryonic matter that sources the field.

The first step consists in performing an expansion of the action in the weak field limit, that is assuming an appropriate reference frame in which the metric can be decomposed as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the flat Minkowski metric, and $|h_{\mu\nu}| \ll 1$. If the characteristic velocity of matter is many orders of magnitude smaller than the speed of light, the energy-momentum tensor for the non-relativistic matter component is given in terms of the energy density by $T_{\mu\nu} \simeq u_\mu u_\nu \, \rho$. One can show that this tensor follows from the simple Lagrangian

$$\mathcal{L}_{\mathrm{M}} \simeq -\rho \, . \tag{2.26}$$

The only relevant component of the metric is $h_{00} = -2\, V_{\mathrm{N}}$, where the Newtonian potential V_{N} is known to satisfy the Poisson equation,

$$\triangle V_{\mathrm{N}}(r) = 4\pi \, \frac{\ell_{\mathrm{P}}}{m_{\mathrm{P}}} \, \rho(r) \, . \tag{2.27}$$

Here r denotes the radial coordinate, and both the density and the potential depend only on r because of spherical symmetry. It is now straightforward to introduce an effective scalar field theory for the gravitational potential. In the formulation of the model we will start from the massless Fierz-Pauli action, that replaces the Einstein-Hilbert action to avoid inconsistencies in the matter coupling. Thus,

$$\begin{aligned} L_{\mathrm{N}}[V] &= \int \mathrm{d}^3 x \, (\mathcal{L}_{\mathrm{FP}} + \mathcal{L}_{\mathrm{M}}) \\ &\simeq 4\pi \int_0^\infty r^2 \mathrm{d}r \left(\frac{m_{\mathrm{P}}}{32\,\pi\,\ell_{\mathrm{P}}} \, h_{00} \, \triangle h_{00} + \frac{1}{2} \, h_{00} \, T_{00} \right) \\ &= 4\pi \int_0^\infty r^2 \mathrm{d}r \left(\frac{m_{\mathrm{P}}}{8\,\pi\,\ell_{\mathrm{P}}} \, V \triangle V - \rho \, V \right) \\ &= -4\pi \int_0^\infty r^2 \mathrm{d}r \left[\frac{m_{\mathrm{P}}}{8\,\pi\,\ell_{\mathrm{P}}} \, (V')^2 + \rho \, V \right] , \end{aligned} \tag{2.28}$$

where the prime denotes differentiation with respect to r, and in the last line we integrated by part. The variation of the previous Lagrangian with respect to the field V_{N} entails exactly the Poisson equation (2.27). However, let us modify the effective model in order to go beyond the Newtonian level. The idea is to add non-linear terms for taking into account the gravitational self-interaction. From (2.28) we derive the Hamiltonian,

$$H_{\mathrm{N}}[V] = -L_{\mathrm{N}}[V] \simeq 4\pi \int_0^\infty r^2 \mathrm{d}r \left(-\frac{m_{\mathrm{P}}}{8\,\pi\,\ell_{\mathrm{P}}} \, V \triangle V + \rho \, V \right) . \tag{2.29}$$

Evaluating the expression on-shell employing the equation of motion (2.27), the Newtonian potential energy is obtained,

$$\begin{aligned} U_{\mathrm{N}}(R) &= 2\pi \int_0^R r^2 \mathrm{d}r \, \rho(r) \, V_{\mathrm{N}}(r) \\ &= -\frac{m_{\mathrm{P}}}{2\,\ell_{\mathrm{P}}} \int_0^R r^2 \mathrm{d}r \, \left[V_{\mathrm{N}}'(r) \right]^2 , \end{aligned} \tag{2.30}$$

where we used once again the Poisson equation and an integration by part. This represents the interaction energy of the matter distribution with the gravitational field inside a sphere of radius R. We can now define a self-gravitational source $J_{\rm V}$ proportional to the gravitational energy $U_{\rm N}$ per unit volume, namely,

$$J_{\rm V}(r) = \frac{4}{4\pi r^2}\,\frac{\mathrm{d}U_{\rm N}(r)}{\mathrm{d}r} = -\frac{m_{\rm P}}{8\,\pi\,\ell_{\rm P}}\,\left[V'(r)\right]^2 . \tag{2.31}$$

The appearance of the above contribution can in fact be found at the next-to-leading order (NLO) in the expansion of the full relativistic theory. Proceeding in the expansion of the matter Lagrangian, the analogous higher order term for the coupling with matter yields the current

$$J_\rho(r) = -2V^2(r)\,. \tag{2.32}$$

Moreover, one is led to introduce the inner pressure of the source. As a matter of fact, in absence of pressure the system is necessarily collapsing and a stable configuration cannot be reached. Besides, for compact sources the pressure can be very large, and can become the dominant contribution. We must therefore add a corresponding potential energy $U_{\rm B}$, associated with the work done by the force responsible for the pressure p, so that

$$p \simeq \frac{1}{4\pi r^2}\,\frac{\mathrm{d}U_{\rm B}(r)}{\mathrm{d}r} = J_{\rm B}(r)\,. \tag{2.33}$$

The total Lagrangian must contain the couplings of the potential field with the aforementioned currents $J_{\rm V}$, J_ρ and $J_{\rm B}$. Introducing three apposite dimensionless coupling parameters $q_{\rm V}$, q_ρ and $q_{\rm B}$ to keep track of the coupling strength, we are left with the Bootstrapped Newtonian Lagrangian,

$$\begin{aligned} L[V] &= L_{\rm N}[V] - 4\pi\int_0^\infty r^2\mathrm{d}r\,\left[\,q_{\rm V}\,J_{\rm V}\,V + q_{\rm B}\,J_{\rm B}\,V + q_\rho\,J_\rho\,(\rho+p)\right] \\ &= -4\pi\int_0^\infty r^2\mathrm{d}r\,\left[\,\frac{m_{\rm P}}{8\,\pi\,\ell_{\rm P}}(1-4q_{\rm V}V)\,(V')^2 + V(\rho+q_{\rm B}\,p) - 2\,q_\rho V^2\,(\rho+p)\right] . \end{aligned} \tag{2.34}$$

Considering the expression of the Laplace operator in the case of spherical symmetry, that is $\triangle f(r) = r^{-2}\,(r^2 f')'$, the corresponding Euler-Lagrange equation for the field V is given by

$$(1-4q_{\rm V}\,V)\,\triangle V = \frac{4\,\pi\,\ell_{\rm P}}{m_{\rm P}}\,(\rho+q_{\rm B}\,p) - \frac{16\,\pi\,\ell_{\rm P}}{m_{\rm P}}\,q_\rho\,V\,(\rho+p) + 2\,q_{\rm V}\,(V')^2\,. \tag{2.35}$$

A closer look to the previous equation shows that, with the presence of non-linear terms, the freedom to shift the potential by an arbitrary constant, as it happens in the Newtonian theory, is lost. In addition, the source conservation equation must be considered,

$$p' = -V' (\rho + p) , \tag{2.36}$$

which in turn allows for the determination of the pressure. It can be seen as a correction to the usual Newtonian formula that accounts for the contribution of the pressure to the energy density, or as an approximation for the Tolman-Oppenheimer-Volkoff equation in General Relativity. Furthermore, it is useful noticing that the Lagrangian (2.34) and the field equation (2.35) reduce to the usual Newtonian ones if all the coupling parameters are set to zero, namely $q_V = q_\rho = q_B = 0$.

Before going forward and presenting possible solutions of the field equation, some conceptual considerations should be underlined. The Bootstrapped Newtonian Lagrangian (2.34) stems from a truncation of the full Einstein-Hilbert action up to a post-Newtonian order. The introduction of non-linearities will eventually lead to a modification of the usual Newtonian behavior, and one expects that the well-known post-Newtonian corrections are recovered. Since including those non-linear terms shall be viewed as the first step in the perturbative reconstruction of General Relativity, the main assumption is that the post-Newtonian perturbations are small compared to the Newtonian leading order. This consistency condition imposes a limitation on the physical systems one is allowed to study. Very compact objects and, in particular, black holes fall outside the domain of validity of the model.
Inspired by the link between the corpuscular model and the post-Newtonian corrections, the authors proceeded in studying the non-linear equation of the effective theory at face value, without requiring that the corrections it introduces with respect to the Newtonian potential remain small. This provides an extension of the Newtonian theory that, at least in principle, should apply to stronger field regimes. This is what the authors called "bootstrapping" the Newtonian gravity. In other words, one shall disregard the geometric origin of the Bootstrapped potential, and look at it as an effective field theory in a flat space-time.

2.4 Vacuum solution

Finding a solution for the Bootstrapped Newtonian field equation (2.35) for a generic source is not an easy task. Detailed calculations for the inner potential in the case of simple source models, such as homogeneous or Gaussian matter distributions, can be found in [8]. The important result we want to stress is that no equivalent of the

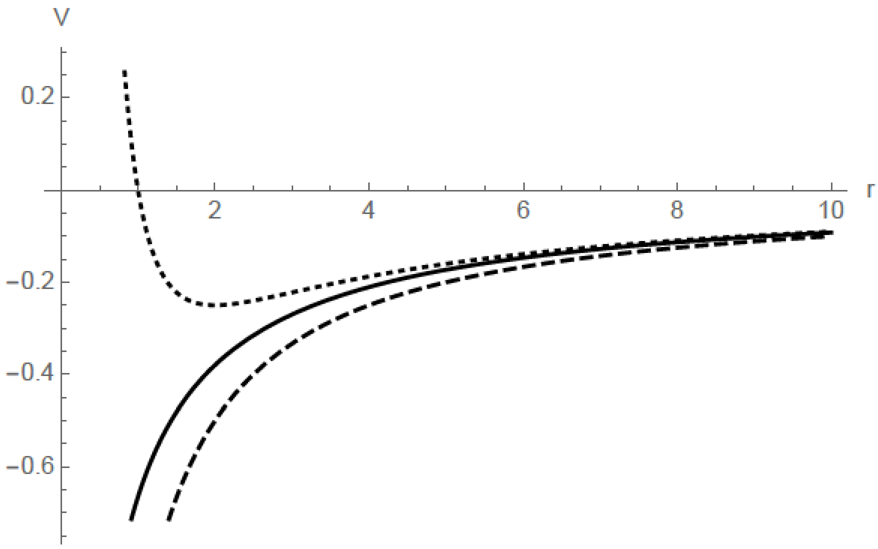

Figure 2.1: Potential $V_{\rm BN}$ (solid line) *vs* Newtonian potential (dashed line) *vs* order $G_{\rm N}^2$ expansion of $V_{\rm BN}$ (dotted line) for $r > 0$ (in units of $G_{\rm N}M$).

Buchdahl limit is found for an isotropic and homogeneous star, so that the source can be at equilibrium for any value of compactness, thus preventing the collapse and the singularity formation.
Besides, in absence of matter and pressure ($\rho = p = 0$) the equation can be easly integrated. The vacuum field equation for the Bootstrapped potential V reads,

$$\triangle V(r) = \frac{2\, q_{\rm V}\, [V'(r)]^2}{1 - 4\, q_{\rm V}\, V(r)} \,. \tag{2.37}$$

The general *vacuum solution* assumes the following form,

$$V_{\rm BN}(r) = \frac{1}{4\, q_{\rm V}} \left[1 - c_1 \left(1 + \frac{c_2}{r} \right)^{2/3} \right] . \tag{2.38}$$

The integration constants are fixed by requiring that the solution reproduces the usual Newtonian behavior for the mass M in the limit $r \to \infty$. One finds that $c_1 = 1$ while $c_2 = 6\, q_{\rm V}\, G_{\rm N}\, M$, yielding the vacuum Bootstrapped potential,

$$V_{\rm BN}(r) = \frac{1}{4q_{\rm V}} \left[1 - \left(1 + \frac{6q_{\rm V}G_{\rm N}M}{r} \right)^{2/3} \right] . \tag{2.39}$$

As a matter of fact, if we take the large r expansion of the potential, and considering for the moment $q_{\rm V} = 1$, the first terms read

$$V_{\rm BN}(r) \simeq \underbrace{-\frac{G_{\rm N}\, M}{r}}_{\text{Newtonian}} \; \underbrace{+\frac{G_{\rm N}^2\, M^2}{r^2}}_{\text{post-Newtonian}} \; -\frac{8G_{\rm N}^3\, M^3}{3\, r^3} \,. \tag{2.40}$$

At the order $G_{\rm N}^2$ we recover the expected post-Newtonian correction to the Newtonian potential. The behavior of the vacuum potential $V_{\rm BN}$ is not so different from the Newtonian one. As one can infer from (2.1), $V_{\rm BN}$ is an increasing function of the distance, but it remains negative. The two curves distance themselves more evidently in the limit $r \to 0$, and since in this limit

$$\frac{V_{\rm BN}(r)}{V_{\rm N}(r)} \simeq \left(\frac{r}{G_{\rm N} M}\right)^{1/3} , \tag{2.41}$$

the Bootstrapped potential diverges slowly, showing that the presence of a non-linear term in the right-hand side behaves as a partial regulator.

Chapter 3

Corpuscular cosmology

So far, the discussion focused on systems involving collapsing matter and black holes, characterized by a strong regime of gravity. This is not by chance, since the existence of a singularity in the interior of a black hole is considered a problematic prediction of classical General Relativity. Therefore, one hopes that in the view of the corpuscular model, a general avoidance of a full gravitational collapse is achieved. In particular, the Bootstrapped Newtonian effective theory shows interesting results in that direction. Our next step will be to extend the microscopic picture for describing gravitational interaction at large scales. In this Chapter we will briefly review the standard cosmological model. Specifically, we will show that there are hints that the corpuscular model can give a simple explanation to the existence of a "dark sector" in our Universe.

3.1 The ΛCDM model

Einstein's theory of gravity can be used to build up a cosmological model. We refer to the cosmological problem as the issue of determining a large-scale metric and a corresponding large-scale mass-energy distribution satisfying Einstein's equations (1.4). Once the problem is resolved, it will give the boundary conditions to be used at large distances for local models, like Schwarzschild solution.
There is little hope that the entire universe can be described by an exact mass-energy and momentum distribution $T_{\mu\nu}$. One usually employs the reasonable assumption that, on sufficiently large scales, matter distribution is *homogeneous* and *isotropic*. This statement is known as the Cosmological Principle, an extension of the Copernican Principle stating that all spatial positions in the Universe are essentially equivalent. Homogeneity and isotropy allow us to describe the geometry and the evolution

of the Universe in terms of two cosmological quantities: the spatial curvature parameter k and the scale factor a, accounting for the expansion (or contraction) of the Universe. These simple assumptions uniquely identify a 4-dimensional metric with a 3-dimensional maximally symmetric subspace, know as the Friedmann-Robertson-Walker-Lemaître (FRWL) metric,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right], \tag{3.1}$$

where the coordinates employed are the comoving coordinates, and in particular t denotes the cosmic time. With a suitable rescaling of the radial coordinate, the curvature parameter can be chosen to have the value $0, +1, -1$, corresponding to a flat, closed and open Universe respectively. All the astronomical observations agree with the fact that, since the distance between galaxies increases in time, the Universe is undergoing a cosmological expansion. The expansion rate depends on the present-day value of the Hubble constant, one of the last estimates yields $H_0 = 67.4\,\mathrm{km/s/Mpc}$ [10].

The cosmological equations of motion are derived from Einstein's equations. Taking a closer look to the field equations (1.4), one notices that, actually, the most general form of Einstein's field equations reads

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu}\,. \tag{3.2}$$

where Λ is an arbitrary constant with dimension of the square of an inverse length, usually indicated as *cosmological constant*. It was originally introduced by Einstein with the motivation to allow for a static homogeneous universe in the presence of matter. Immediately after the first astronomical observations proving that our Universe is expanding, the cosmological constant term became unnecessary, until it was redeemed in the modern cosmological model, as we shall see further on.

The standard cosmological model considers the Universe as filled with a homogeneous perfect fluid of matter and energy described in terms of its energy density $\rho = \rho(t)$ and its pressure $p = p(t)$, with energy-momentum tensor given by

$$T_{\mu\nu} = (\rho + p)\,u_\mu u_\nu + p\,g_{\mu\nu}\,, \tag{3.3}$$

where u_μ is the fluid 4-velocity, and equation of state of the form

$$p = \omega\rho\,, \tag{3.4}$$

where ω is a constant. Requiring that the FRWL space-time is a solution of Einstein's field equations with cosmological constant, the latter reduce to two Friedmann equations, namely

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_\mathrm{N}}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}\,, \tag{3.5a}$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{4\pi G_\mathrm{N}}{3}\left(\rho + 3p\right), \tag{3.5b}$$

where the dot indicates the total time derivative. Since the first Friedmann equation (3.5a) contains only first order time derivatives, it is not dynamical, rather it represents a constraint. A third equation comes from the covariant conservation of the energy-momentum tensor and reads,

$$\dot{\rho} = -3H(\rho + p)\,, \tag{3.6}$$

where the Hubble parameter is defined as $H(t) = \dot{a}/a$. Introducing the density parameter,

$$\Omega = \frac{8\pi G_\mathrm{N}}{3H^2}\rho = \frac{\rho}{\rho_{critical}}\,, \tag{3.7}$$

and substituting into the first Friedmann equation, a strict relation between the curvature parameter k and the Universe density ρ can be found. In particular, for a spatially flat Universe ($k = 0$) the density coincides with the critical value $\rho = \rho_{critical}$. From recent observations it turns out that the current average density of the Universe is estimated at

$$\rho_0 \simeq 10^{-29}\,\mathrm{g/cm^3} = \rho_{critical}\,, \tag{3.8}$$

leading us to picture a flat Universe, with the total density parameter that almost equals the unity and it's expressed as the sum of the density parameters of each species composing our Universe,

$$\Omega_{tot} = \sum_{\mathrm{species}} \Omega_i \simeq 1\,. \tag{3.9}$$

The currently accepted concordance cosmological model is the ΛCDM model, that states the existence of three main contributions to the Universe density:

- *Ordinary matter* ($\Omega_b \simeq 0.05$), comprising everything made up of baryonic matter like stars, planets, gas, galaxies.
- *Cold dark matter* ($\Omega_c \simeq 0.26$), a form of non-baryonic, non-relativistic and almost neutral matter necessary in order to account for gravitational effects observed in large-scale structures, such as galaxy rotation curves.
- *Dark energy* ($\Omega_\Lambda \simeq 0.69$), introduced to explain the direct evidence of an accelerating cosmic expansion, and composing the larger part of the Universe.

3.1.1 Dark energy and the cosmological constant

Let us discuss the Einstein's equations (3.2) in the vacuum, imposing $T_{\mu\nu} = 0$. The equations can be suitably recast as

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \,. \tag{3.10}$$

In the following we assume $\Lambda > 0$. The first important feature is that there is no flat space solution for such an empty universe. In other words, if the cosmological constant does not vanish, in absence of matter Minkowski metric is no longer a solution of Einstein's equations. Imposing spherical symmetry and staticity *a priori* for the metric, the most general line element reads

$$\mathrm{d}s^2 = -e^{\nu(r)}\mathrm{d}t^2 + e^{\lambda(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \tag{3.11}$$

where the functions $\nu(r)$ and $\lambda(r)$ has to be determined. Substituting in the field equations, at the end of the calculations one finds that

$$e^{\nu(r)} = e^{-\lambda(r)} = 1 - \frac{\Lambda r^2}{3} + \frac{c}{r}\,, \tag{3.12}$$

where c is an arbitrary integration constant. If we set $c = 0$ *de Sitter solution* in static coordinates is obtained,

$$\mathrm{d}s^2 = -\left[1 - \frac{\Lambda r^2}{3}\right]\mathrm{d}t^2 + \left[1 - \frac{\Lambda r^2}{3}\right]^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,, \tag{3.13}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$. The cosmological horizon is located at

$$r_{\mathrm{H}} = \sqrt{\frac{3}{\Lambda}} \equiv L\,. \tag{3.14}$$

Since the metric (3.13) is static, it is not modified by the transformation $t \to t+\epsilon$. In terms of differential geometry, there exists a Killing vector associated to time translational invariance, whose coordinate representation is $\xi^\mu = (1,0,0,0)$. Considering that the norm of the latter vector is $g_{\mu\nu}\,\xi^\mu\,\xi^\nu = -1+\frac{\Lambda}{3}r^2$, we conclude that the Killing vector is space-like in the region $r > L$, time-like for $r < L$, and null when $r = L$. In the region $r > L$ trajectories with constant radial coordinate are not allowed: an observer that crosses the cosmological horizon is dragged away by the accelerating expansion of the universe, with no hope of coming back.

In de Sitter spacetime L, which is proportional to $1/\sqrt{\Lambda}$, represents a characteristic length, that one can interpret as the radius of the observable universe. As a matter of

fact, any signal emitted outside the cosmological horizon can only move at increasing r, so that an observer inside the horizon will never receive any information from the outside.
In view of the fact that in GR we have the freedom of choosing any suitable set of coordinates, one can employ a particular coordinate transformation [34],

$$r' = \frac{r\, e^{-t/R}}{\sqrt{1-\frac{r^2}{L^2}}}\,, \qquad t' = t + \frac{L}{2}\log\left(1-\frac{r^2}{L^2}\right)\,, \tag{3.15}$$

for rewriting the de Sitter metric in a different fashion, namely

$$\mathrm{d}s^2 = -\mathrm{d}t'^2 + e^{2Ht'}\left(\mathrm{d}r'^2 + r'^2\mathrm{d}\Omega^2\right)\,, \tag{3.16}$$

where $H = 1/L$ is the inverse of the Hubble radius. Comparing (3.16) with (3.1) one realizes that the de Sitter metric can be recast in a FRWL form. In particular, it is equivalent to a spatially flat FRWL space-time with an exponential scale factor, thus describing an exponentially expanding Universe. Besides, the inverse of the cosmological horizon coincides with the Hubble constant. Same results are achieved by solving the two Friedmann equations for the scale factor in the case of a flat Universe filled only with a cosmological constant.
In the standard model of cosmology, dark energy is fully accounted for by the cosmological constant Λ. In particular, dark energy is modeled as a an exotic cosmic fluid fulfilling a vacuum equation of state, the energy density being constant and the pressure being negative,

$$\rho_\Lambda = -p_\Lambda = \frac{\Lambda}{8\pi G_\mathrm{N}}\,, \qquad \omega = -1\,. \tag{3.17}$$

The current best estimate of the cosmological constant is $\Lambda_0 = 2.84 \times 10^{-122}\,\ell_\mathrm{P}^{-2}$ [10]. One shall notice that the conclusion that the accelerated expansion requires a new energy component beyond matter and radiation relies on the assumption that General Relativity is the correct description of gravity on cosmological scales.

3.1.2 Schwarzschild-de Sitter space-time

It is now clear that de Sitter space-time does not describe our Universe. The de Sitter universe in completely empty of matter, and dark energy fills all space. Let us consider a slightly different physical situation: suppose that a point-like matter source with mass M is located in the origin $r = 0$ of our coordinate reference frame inside a de Sitter universe. Once more we impose spherical symmetry and staticity.

The gravitational field generated by the spherical source in the vacuum is given by a metric tensor $g_{\mu\nu}$ solution of the Einstein's equation (3.10). One can notice that if $\Lambda = 0$ (or analogously in the limit $L \to \infty$) the solution must reproduce Schwarzschild space-time. The integration constant in Eq. (3.12) is now fixed in such a way that $c = -2G_{\rm N}M$, and the *Schwarzschild-de Sitter solution* is obtained,

$$\mathrm{d}s^2 = -\left(1 - \frac{2G_{\rm N}M}{r} - \frac{r^2}{L^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2G_{\rm N}M}{r} - \frac{r^2}{L^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,. \qquad (3.18)$$

The metric represents the gravitational field outside a point-like source that, at large distances, recovers a cosmological behavior typical of the de Sitter space-time. On the contrary, at small distances the cosmological term is negligible, thus retrieving a Schwarzschild-like behavior. Actually, there exists a generalization of Birkhoff's theorem which states that any spherically symmetric solution of Einstein's equations with cosmological constant in absence of other stress-energy sources is isometric to Schwarzschild-de Sitter spacetime [37].
The condition $g^{rr} = 0$ allows us to study the eventual presence of horizons. Schwarzschild-de Sitter space-time contains two separate horizons only if

$$M < \frac{L}{3\sqrt{3}\,G_{\rm N}}\,. \qquad (3.19)$$

As a consequence, the mass of a black hole in a de Sitter universe has an upper limit, depending on the value of the cosmological constant. The explicit expressions for the horizons are,

$$R_{\rm H/L} = \frac{2\,L}{\sqrt{3}}\cos\left[\frac{\pi}{3} \pm \frac{1}{3}\arccos\left(\frac{3\sqrt{3}\,G_{\rm N}M}{L}\right)\right]\,. \qquad (3.20)$$

Since $R_{\rm H} < R_{\rm L}$, one identifies $R_{\rm H}$ as the black hole horizon and, with the assumption that $G_{\rm N}M \ll L$, one finds that it is not so different from the usual Schwarzschild radius,

$$R_{\rm H} \simeq 2\,G_{\rm N}M\left[1 + \left(\frac{G_{\rm N}M}{L}\right)^2\right]\,. \qquad (3.21)$$

Moreover, $R_{\rm L}$ is the cosmological horizon, that in the same limit approaches the de Sitter Hubble radius,

$$R_{\rm L} \simeq L\left(1 - \frac{G_{\rm N}M}{2\,L}\right)\,. \qquad (3.22)$$

3.2 Corpuscular picture of de Sitter space-time

In Chapter 1 we discussed how the classical geometry of space-time could be conceived as an emerging property of a coherent state of a very large number of gravitons. Since gravitons are bosons and owing to the self-interacting nature of gravity, the system of gravitons can eventually lead to the formation of a condensate. We have seen that black holes have a relatively simple interpretation in terms of N gravitons composing a self-sustained Bose-Einstein condensate at the critical point. In this perspective, all the classical geometric properties characterizing a black hole in GR emerge from a single quantum feature, that is the graviton occupation number. In this section we are going to discuss the possibility that not only black holes, but also other configurations such as cosmological spaces, and in particular de Sitter space, represent composite entities of quantum constituent gravitons, whose wavelength is set by the characteristic classical size of the system.
Let us start by considering the first Friedmann equation (3.5a) for a spatially flat space in the sole presence of a cosmological constant, that is

$$H^2 = \frac{\Lambda}{3} = \frac{8\pi G_{\rm N}}{3}\rho_\Lambda \,, \tag{3.23}$$

where we used the expression (3.17) for the vacuum energy density. One can now integrate on the volume inside the Hubble radius $L = 1/H$ obtaining

$$L \simeq G_{\rm N} L^3 \rho_\Lambda \simeq \frac{\ell_{\rm P}}{m_{\rm P}} M_\Lambda \,, \tag{3.24}$$

where M_Λ should be interpreted as a sort of energy contribution of the cosmological constant inside a sphere with the size of the Hubble radius. The previous relation that correlates the Hubble radius with M_Λ resembles very closely the well-known relation between the Schwarzschild radius of a black hole and its ADM mass. The suggestion is to extend the quantum picture from black holes to de Sitter space-time, and that can eventually lead to a fundamental quantum interpretation of de Sitter geometry in the context of the corpuscular model of gravity.
Tracing back the argument for the description of a gravitational collapse, we recall that starting from the Hamiltonian constraint and the subsequent energy balance one recovers interesting results such as the maximal packing condition for black holes. Accordingly, let us consider the Hamiltonian constraint in FRWL cosmology, that can be separated into a contribution from the purely gravitational part of the system, plus a matter term,

$$H = H_{\rm G} + H_{\rm M} = 0 \,. \tag{3.25}$$

In an Universe filled only with the cosmological constant, the matter content is absent, and therefore we set $H_{\rm M} = 0$. The Universe is a classical object, the most classical of all in some sense, but one expects that the underlying description is fully accounted by a fundamental quantum theory. The main purpose is to interpret dark energy as a result of the self-interaction of virtual and very soft gravitons only, with occupation number N_Λ and eventually forming a Bose-Einstein condensate. The typical energy of a single graviton in the condensate is approximately constant and given by the usual de Broglie-Compton relation,

$$\epsilon_\Lambda \simeq \frac{m_{\rm P}\,\ell_{\rm P}}{L}\,. \tag{3.26}$$

Following the classicality criterion, one expects $N_\Lambda \gg 1$, and that the classical behavior is recovered as a coherent state of the constituent quanta. This gives rise to a Newtonian potential energy of the form

$$U_{\rm N} = M_\Lambda\,\phi_{\rm N}(L_\Lambda) \simeq N_\Lambda \epsilon_\Lambda\,. \tag{3.27}$$

Since the graviton wavelength is of the order of the cosmological horizon, the graviton self-interaction is negligible, while from the interaction between each graviton with the collective state, a post-Newtonian contribution to the energy follows,

$$U_{\rm PN} \simeq N_\Lambda\,\epsilon_\Lambda\,\phi_{\rm N}(L_\Lambda) \simeq N_\Lambda^{3/2}\,\frac{m_{\rm P}\,\ell_{\rm P}^2}{L^2}\,, \tag{3.28}$$

where we used the Newtonian potential $\phi_{\rm N}$ in (3.27). The Hamiltonian constraint now reads

$$H_{\rm G} = U_{\rm N} + U_{\rm PN} = 0\,, \tag{3.29}$$

and after the substitutions one directly finds the scaling law for the graviton occupation number, that is

$$N_\Lambda \sim \frac{L^2}{\ell_{\rm P}^2} = \frac{m_\Lambda^2}{m_{\rm P}^2}\,. \tag{3.30}$$

This shows that for describing the whole Universe a very large number of gravitons is necessary. Moreover, one gets

$$M_\Lambda \simeq m_{\rm P}\sqrt{N_\Lambda}\,. \tag{3.31}$$

One is tempted to compare these results to the typical corpuscular scaling laws met in the black hole quantum picture, therefore we will pursue the analogy and interpret the de Sitter universe as a Bose-Einstein condensate of N_Λ soft and weakly-interacting gravitons with characteristic wavelength $\lambda \simeq L$. This corresponds to an intriguing picture, since in the view of the corpuscular model of gravity, the underlying quantum description in terms of microscopic constituent gravitons is sufficient to give an explanation to the behavior commonly attributed to dark energy, with the main advantage being that the introduction of a cosmological constant, as well as the assumption of an exotic cosmic dark fluid, turn out to be unnecessary.

3.2.1 Emergence of a curved geometry

In the corpuscular model of gravity the classical notion of curved geometry is substituted by the quantum notion of large graviton occupation number on a flat space-time. Therefore, the curved metric must appear as an effective emerging description. Considering a free falling test particle in a gravitational field sourced by a generic energy-momentum source, in a geometric perspective the particle follows a geodesic motion in the background field. One should address the problem of interpreting the same phenomenon in the framework of the underlying quantum theory, that is a corpuscular description in which the constituent microscopic quanta are gravitons. In particular, the aim is to show how a graviton Bose-Einstein condensate can reproduce de Sitter space-time in the classical limit.
Let us consider a metric tensor $g_{\mu\nu}$ representing a gravitational field sourced by the energy-momentum tensor $T_{\mu\nu}$ via classical Einstein's equations. For the sake of simplicity, let us assume the case of small curvature, taking up the weak field limit of the metric in the form of

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon\, h_{\mu\nu} + \epsilon^2\, h'_{\mu\nu} + \mathcal{O}(\epsilon^3)\,, \tag{3.32}$$

where we introduced a flat Minkowski background and a suitable small parameter ϵ for formally proceed in the expansion. The leading order perturbation $h_{\mu\nu}$ corresponds to the linear level $h_{\mu\nu}$, the successive sub-leading term $h'_{\mu\nu}$ is instead the first non-linear correction, and so on.
In Section (1.3) we pointed out that the metric perturbation $h_{\mu\nu}$ is a solution of the linearized Einstein's equations, while $h'_{\mu\nu}$ shall be obtained by iteration introducing in the right-hand side of the equations of linear gravity the canonical gravitational energy-momentum tensor evaluated on the first order perturbation. This leads to the first correction to the linear level, stemming from a gravitational cubic self-interaction. The process could be continued to arbitrarily high order, recovering conclusively the fully non-linear Einstein field equations.
Besides, assume that a probe of energy-momentum $\tau_{\mu\nu}$ is in motion in the classical gravitational field generated by $T_{\mu\nu}$. The probe can in principle be a classical or

a quantum particle: in the latter case the whole system would be treated as semi-classical. At first linear order in the perturbation, the geodesic motion of the probe can be evaluated from its coupling in the action, that at linear level reads

$$S_{int} = \int \mathrm{d}^4x \, h_{\mu\nu} \tau^{\mu\nu} \, . \tag{3.33}$$

In an iterative way, the non-linear corrections to the motion are attained at each order in the weak field expansion.
However, the underlying description of the theory shall be quantum. We therefore promote the metric perturbation into a quantum field, after a suitable rescaling for canonical dimensionality,

$$h_{\mu\nu} \longrightarrow \sqrt{\frac{\ell_{\rm P}}{m_{\rm P}}} \, \hat{h}_{\mu\nu} \, . \tag{3.34}$$

A quantum way for treating the problem of the motion of the probe particle is to imagine graviton exchanges between the probe source $\tau_{\mu\nu}$ and the field source $T_{\mu\nu}$. Thus, the process shall involve the absorption or the emission of microscopic quanta. The leading order corresponds to the exchange of a single graviton, and the related scattering amplitude at the linear level yields

$$\mathcal{S}^{(1)} = \frac{\ell_{\rm P}}{m_{\rm P}} \int \mathrm{d}^4x_1 \int \mathrm{d}^4x_2 \, \tau^{\mu\nu}(x_1) \, \Delta_{\mu\nu,\alpha\beta}(x_1 - x_2) \, T^{\alpha\beta}(x_2) \, , \tag{3.35}$$

where $\Delta_{\mu\nu,\alpha\beta}(x_1 - x_2)$ is the graviton propagator, and we notice that the amplitude is first order in Newton's constant. Again, one can proceed in the same way at every order in the weak field expansion, obtaining subsequent corrections to the motion associated to different order scattering amplitude $\mathcal{S}^{(n)}$ at increasing order n in Newton's constant.
Suppose that the probe particle is quantum, and it is associated to a quantum field ϕ. This can be a matter field, or the electromagnetic field for photons. In order to study the transition amplitude for the scattering, the needful tools are the matrix elements

$$\langle \, f \, | \, \mathcal{S}^{(n)} \, | \, i \, \rangle \, , \tag{3.36}$$

where $|i\rangle$ and $|f\rangle$ are the initial and the final states in which the probe is prepared, respectively.
The remarkable point is that both perspectives, one purely classical and the other one concerning the scattering with constituent quanta, are equivalent for the purpose of describing the motion of the probe particle. The equivalence holds to all orders in the weak field expansion.

Let us use this argument in the particular case of de Sitter geometry. Classical de Sitter space-time is a solution of the field equations (3.10), where the source of the gravitational field is now the cosmological constant Λ. Since $L = 1/H = \sqrt{3/\Lambda}$ is the cosmological horizon, it represents the characteristic curvature radius, therefore for the sake of a weak field expansion we shall consider the evolution of a probe with wavelength smaller that the Hubble radius. The linearized equations read

$$R^{(1)}_{\mu\nu} = \Lambda\, \eta_{\mu\nu}\,, \tag{3.37}$$

where in the left-hand side there is the linearized version of Ricci tensor. The latter equation admits several gauge-equivalent solutions for describing de Sitter space in the case of short distances and short time scales. One possible choice is

$$h_{00} = h_{0i} = 0\,, \qquad h_{ij} = H^2\,(t^2\delta_{ij} + n_i n_j r^2)\,, \tag{3.38}$$

where $n_i = x_i/r$ and $r = \sqrt{\sum_k x_k^2}$. This corresponds to the linearized de Sitter metric in closed slicing form up to a numerical factor set to one. In other words, we are considering an approximation of the full non-linear metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \cosh(Ht)\left(\frac{\mathrm{d}r^2}{1-H^2r^2} + r^2\mathrm{d}\Omega^2\right)\,, \tag{3.39}$$

when $Hr \ll 1$ and $Ht \ll 1$. The classical metric perturbation can be expanded as

$$h_{ij}(x) = \int \mathrm{d}^4k\,\left[u_{ij}\,(b_{u,k}\,e^{ik_\mu x^\mu} + b^*_{u,k}\,e^{-ik_\mu x^\mu}) + v_{ij}\,(b_{v,k}\,e^{ik_\mu x^\mu} + b^*_{v,k}\,e^{-ik_\mu x^\mu})\right]\,, \tag{3.40}$$

where the orthogonal projectors $u_{ij} = \delta_{ij} - n_i n_j$ and $v_{ij} = n_i n_j$ have been introduced, while b, b^* are complex coefficients. It shall be stressed that the field h_{ij} does not satisfy any free wave equation, and as a consequence the dispersion relation for the wave vector k_μ is not the massless one, but it is determined by the source, in this case the cosmological constant.

In the quantum version of the theory the field is promoted to an operator. In details, the expansion coefficients become a set of creation $\{\hat{a}^\dagger_{u,k}\,, \hat{a}^\dagger_{v,k}\}$ and annihilation $\{\hat{a}_{u,k}\,, \hat{a}_{v,k}\}$ operators fulfilling the usual bosonic commutation relations

$$\left[\hat{a}_{u,k}\,, \hat{a}^\dagger_{u,k'}\right] = \delta^4(k-k')\,, \qquad \left[\hat{a}_{v,k}\,, \hat{a}^\dagger_{v,k'}\right] = \delta^4(k-k')\,, \tag{3.41}$$

and the other commutators vanish. One can proceed in the construction of the Fock space: each vector corresponds to a state with a given occupation number of longitudinal off-shell gravitons. Instead of using a description in terms of coherent states, as we did in the discussion on the Newtonian potential, here we shall employ the number-eigenstate representation, so that $|N_k\rangle$ is the quantum state of N quanta with four-momentum k. Thus, classical weak-field de Sitter space-time must be interpreted as a quantum state with a very large occupation number of gravitons whose wavelength distribution is strongly peaked at the characteristic curvature radius of the system. This allows us to write

$$|\text{de Sitter}\rangle = |N_H\rangle \ . \tag{3.42}$$

Consider once again a probe particle ϕ with energy-momentum $\tau_{\mu\nu}(\phi)$. As we pointed out, at the classical level the propagation of the particle in de Sitter space-time follows from the coupling with the metric perturbation in the form

$$\int \mathrm{d}^4x \, h_{\mu\nu}\tau^{\mu\nu}(\phi) \, . \tag{3.43}$$

However, in the quantum picture the particle evolution is interpreted as the interaction between the probe and the graviton condensate. Moreover, the classical picture should emerge as a large-N limit of a quantum transition from an initial state $|\phi_{in}, N_H\rangle$ to a final state $|\phi_f, N_H \pm 1\rangle$ in which the occupation number of gravitons in the condensate (with wavelength L) changes by one. These correspond to scattering processes where a graviton is emitted or exchanged so that the probe evolves from the initial to the final state. The matrix elements that enter the scattering description are

$$\langle N_H + 1|\hat{a}_k^\dagger|N_H\rangle \ , \qquad \langle N_H - 1|\hat{a}_k|N_H\rangle \ . \tag{3.44}$$

The explicit calculation can be found in [18]. Comparing the amplitudes from the scattering processes with (3.43) one recovers the following effective classical metric,

$$h_{ij} \simeq \frac{\ell_{\mathrm{P}}\sqrt{N_H}}{L} \left[u_{ij} \, r^2 + v_{ij} \left(t^2 + r^2\right) \right] H^2 \, . \tag{3.45}$$

Now, since the condensate is on the verge of a phase transition, the criticality condition corresponds to the scaling law (3.30) for the occupation number up to $1/N$ corrections, in such a way that the prefactor in (3.45) is one, and the previous expression reproduces the weak-field de Sitter metric (3.38) in the large-N limit.

3.3 Corpuscular dark force

The standard ΛCDM cosmological model offers an acceptable explanation for a wide range of astrophysical observations, including the CMB spectrum and the formation

of large-scale cosmic structures, and there is little doubt on the accordance between the model calculations and the experimental data.
However, the agreement comes with the cost of supposing the existence of a “dark sector”, composing around 95% of our Universe. One of the problems of this well-established approach is that it fits the phenomenological data extremely fine as long as one does not presuppose anything about the true physical nature of the exotic components. As we have seen, dark energy turns out to be necessary in order to account for the accelerated expansion of the Universe. A complete satisfactory justification for the presence of such an energy is not achieved yet. For example, the interpretation in terms of a vacuum energy appears in contrast with the calculations of zero-point energy of quantum fields from QFT: the gap between the measured value of the cosmological constant and the theoretical prediction is enormous, more than 120 orders of magnitude [33]. A similar argument arises for dark matter, composing more than 83% of matter in the Universe according to the standard model. After a brief review of the phenomenological origin of the dark matter hypothesis and alternative descriptions in the framework of modified gravity, in this Section we will address the problem in the corpuscular model of gravity.

3.3.1 Dark matter and MOND theories

What astronomical observations tell us is that the dynamics of galactic and extra-galactic structures do not correspond to the observed visible matter and energy. The most significant piece of evidence concerns rotation curves of spiral galaxies [27]. A rotation curve consists in a plot of the velocities of visible matter (mostly stars and gas) as a function of the distance from the galaxy center. Most of the stars are located in the inner part of the galaxy, in a region around 10 kpc from the center. A system obeying Newton’s law of gravity should exhibit a rotation curve that, after having reached a maximum, declines in a Keplerian manner, as it happens for example in the Solar system: in this case the rotation velocity decreases as $r^{-1/2}$.
On the contrary, observations suggest that the rotation velocity tends to remain approximately constant at large radii, far outside the central core of galaxies. Since the total mass inside a radius R of a spiral galaxy is approximately given by $M_{\rm tot} = v_{\rm rot}^2 R/G_{\rm N}$ where $v_{\rm rot}$ is the rotation velocity, we infer that at large radii the expected mass that gives rise to the measured rotation speed is much larger than the observed mass. This gives rise to the missing mass problem.
The flatness of rotation curves is a common characteristic of spiral galaxies and was measured for a wide number of cosmic structures, such as dwarf spheroidal galaxies,

giant elliptical galaxies and clusters of galaxies, although the amount of missing mass can be very different for dissimilar systems. The possible solutions to the missing mass problem fall in one of the following approaches:

- the introduction of an unseen and yet unknown form of matter-energy;
- the failure of our current theory of gravity;
- both possibilities.

The most popular explanation, and thus included in the concordance model, relies on the assumption that a non-baryonic and electromagnetic close-to-neutral form of matter exists. For these reasons it is known as dark matter, and in principle it should supply for the missing mass. There are strong evidences that dark matter should exist indeed, including cosmic structure formation, observations of the anisotropy power spectrum of the CMB and galaxy clusters. However, from a theoretical perspective the introduction of a new exotic matter contribution seems unnatural. In addition, the question on what are the microscopic components of dark matter remains open: in spite of a huge number of possible theoretical candidates (WIMPs, axions, sterile neutrinos, dark photons, primordial black holes,...) at the present the quest for a laboratory evidence or an astrophysical detection hasn't led to significant results.

It cannot go unnoticed that the ΛCDM model is not completely satisfactory: the missing satellite problem, the cusp-core issue and the too-big-too-fail problem represent open questions. One of the challenges for the standard approach is to explain the baryonic Tully-Fisher relation [28]: a strong empirical correlation can be found connecting the amount of visible baryonic matter $m_{\rm B}$ with the measurable asymptotic rotation velocity v_f of galaxies,

$$m_{\rm B} = \frac{v_f^4}{G_{\rm N}\, a_0}\,, \tag{3.46}$$

where $a_0 \simeq 1.2 \times 10^{-10}\,{\rm m\,s^{-2}}$ has the dimension of an acceleration. The concordance model cannot furnish a complete explication, since it predicts a different exponent for the rotation velocity and a different normalization. The disagreement can be solved only by a fine tuning of the parameter, supposing for example that, because of an unknown mechanism, the detected baryonic mass is only a fraction of the total baryonic mass available in the galaxy [21].
A second line of thought paves the way for some kind of modification in the gravitational theory. This is the case of *modified Newtonian dynamics* (MOND) [30]. The

central point is that the standard description of the gravitational interaction in terms of GR is modified when the acceleration of the system falls below a_0. It is various orders of magnitude smaller that the typical accelerations of planets in the Solar system, but it becomes comparable with the acceleration of stars in the external region of a galaxy.
In MOND, a_0 is promoted to a fundamental physical constant, and its value is surprisingly related to the cosmological constant, since $a_0 \simeq H/6 = 1/6L$. Then, for baryonic matter in the inner core of a galaxy, Newtonian gravity is still valid and the acceleration at a given distance r is

$$a_{\rm B}(r) = \frac{G_{\rm N}\, m_{\rm B}(r)}{r^2}\,, \tag{3.47}$$

where $m_B(r)$ is the amount of matter enclosed in a sphere of radius r. Differently, in the region outside the core the gravitational acceleration is modified with respect to the Newtonian form and reads

$$a_{\rm MOND}(r) = \sqrt{a_0\, a_{\rm B}(r)} \simeq \sqrt{\frac{a_{\rm B}(r)}{6L}}\,. \tag{3.48}$$

This simple formula is in excellent agreement with galaxy rotation curves, although it cannot provide a complete description for all the observations for which dark matter is invoked: for example it fails when used for the missing mass of galaxy clusters. Despite the difficulties, the MOND framework fits perfectly with the idea that the current theory of gravity, in particular General Relativity, is not a fundamental theory of Nature and should be modified. It shall not be surprising that in the last years many attempts of reproducing a MOND-like behavior in terms of emerging theories of gravity have been proposed. An important contribution follows from Verlinde's entropic approach [41], that interprets the laws of gravity as an emergent phenomenon whose origin lies in quantum entanglement of microscopic degrees of freedom. Following the argument, it can be shown that an additional gravitational force pops up at galactic scales, so to explain the MONDian acceleration.

Besides, let us underline a curious coincidence that establishes an unexpected link between MOND theories, and thus dark matter phenomenology, and dark energy interpreted as a cosmological constant. Since $a_0 \sim \sqrt{\Lambda}$, this gives rise to the question if it is a numerical coincidence or it's physically relevant. In the latter case, a fundamental theory should be able to explain such a relation between the two constants. As we shall see in the following, a connection between the two main components of the Universe dark sector, as well as a possible explanation for the MOND behavior at galactic scales can be found in the corpuscular model of gravity.

3.3.2 Emergence of a dark force

Seemingly different systems such as black holes and de Sitter space-time are characterized by several orders of magnitude of difference in their typical size. If $R_{\rm S} = 2\,G_{\rm N} m_B$ is the Schwarzschild radius of a black hole with mass $m_{\rm B}$ and L is the cosmological horizon, we identify the hierarchy $R_{\rm S} \ll L$. Nevertheless, in the framework of corpuscular gravity both systems have a fundamental quantum interpretation in terms of very soft and virtual constituent gravitons that form a Bose-Einstein condensate at the critical point. As we have seen, the criticality condition implies a definite scaling law of graviton occupation number, that is very similar in both cases and depends on the characteristic curvature radius or length scale. In the study of a gravitational collapse from a corpuscular perspective the scaling relation (2.6), first found for black holes, was shown to hold with a good approximation for a general compact source with typical size $R_{\rm B}$, whose gravitational field outside the matter distribution is well described by Newtonian theory. In this case, the hierarchy one should consider is $R_{\rm S} \lesssim R_{\rm B} \ll L$.
We can summarize the results by noticing that for $r \simeq L$, defining the cosmological scale, and for $r \simeq R_{\rm B}$, that is associated to the Newtonian scale, the graviton scaling reads

$$N_{\rm G}(r) \sim \frac{m^2(r)}{m_{\rm P}^2} \sim \frac{r^2}{\ell_{\rm P}^2}\,, \tag{3.49}$$

where $m(r)$ gives the mass (with the appropriate definition depending on the system) inside a sphere of radius r. For example, $m(L) = m_\Lambda$ representing the total mass-energy enclosed in the Hubble horizon, and $N_{\rm G}(L)$ can be interpreted, in a sense, as the total number of gravitons inside the visible universe. Besides, $N_{\rm G}(R_{\rm B})$ represents the number of gravitons that react locally to the presence of the baryonic source, and from which the usual Newtonian dynamics emerges. It is important to stress that the graviton number is not a measurable physical quantity, since all the gravitons are off-shell and thus non-propagating.
The scaling law defines two *holographic regimes* of gravity, one at small scales and the other at very large length scales, since the number of gravitons depends on the square of the characteristic size. Although the cosmological and the Newtonian regimes are similar according to this point of view, there are significant differences between them. The Newtonian potential energy is indeed negative,

$$U_{\rm N}(r) = -\frac{G_{\rm N}\, m_{\rm B}^2}{r} \simeq N_{\rm G}\,\epsilon_{\rm G}(r)\,, \tag{3.50}$$

and implies a negative typical energy for single gravitons, namely

$$\epsilon_{\rm G}(r) \simeq -\frac{m_{\rm P}\,\ell_{\rm P}}{r}\,. \tag{3.51}$$

On the contrary, the cosmological mass-energy is positive, and since

$$m_\Lambda = \frac{m_{\rm P}\,L}{\ell_{\rm P}} \simeq N_\Lambda\,\epsilon_\Lambda\,, \tag{3.52}$$

one can infer a positive typical energy for a graviton in the cosmological condensate, that is

$$\epsilon_\Lambda \simeq \frac{m_{\rm P}\,\ell_{\rm P}}{L}\,. \tag{3.53}$$

The switch in sign for the graviton energy in the two holographic regimes is a strong indication that they are indeed different. Ultimately, this is not so surprising, since the small scale regime is related to objects at astrophysical size, while the large scale regime describes the expansion of the Universe. Moreover, one expect that they must be connected by a mesoscopic regime, in which the gravitational interaction is no longer holographic, but deviates in a new dissimilar behavior.
For this purpose let us consider an empty de Sitter universe whose dynamics is driven only by dark energy. From a corpuscular point of view, it emerges from the underlying cosmological Bose-Einstein condensate that we shall call DEC (abbreviation for dark energy condensate). However, locally, the DEC cannot be considered as a whole, but just as a background medium with positive constant energy density ρ_Λ given by (3.17). The total graviton energy in a spherical region of radius r is then,

$$M(r) \simeq \frac{4\pi}{3}\rho_\Lambda r^3 \sim \frac{m_{\rm P}}{L^2\ell_{\rm P}}r^3 \simeq N_{\rm G}(r)\,\epsilon_\Lambda\,. \tag{3.54}$$

One directly deduces that the scaling law for the graviton occupation number becomes

$$N_{\rm G}(r) \simeq \frac{r^3}{L\,\ell_{\rm P}^2}\,. \tag{3.55}$$

This shows that in the intermediate region between the Newtonian and the cosmological ones, and neglecting any presence of baryonic matter, the graviton number scales as the cube of the characteristic length, and thus proportional to the volume. In this region, gravity manifests its *extensive* behavior, that in turn must be interpolated somehow between the two holographic regimes.
Now, let us suppose to introduce $N_{\rm B}$ baryonic particles, each one with mass μ, in the cosmic condensate. As a first approximation, suppose that the introduced matter

is very diluted, in the sense that the DEC does not react locally to its presence. Moreover, let us assume that baryonic matter is in very small quantity compared to cosmological mass-energy. In classical geometric terms, the original de Sitter space turns into a Schwarzschild-de Sitter space-time, whose metric is thus given by (3.18), where the source mass is now $m_{\rm B} = N_{\rm B}\,\mu$. In Section (3.1.2) it is shown that the dimension of the visible Universe for a Schwarzschild-de Sitter solution decreases. In fact, imposing the condition for calculating the cosmological horizon, and denoting it with L_H, one finds

$$H^2 L_H^2 = 1 - \frac{2\,G_{\rm N} m_{\rm B}}{L_H} = 1 - \frac{m_{\rm B}}{m_{\rm tot}}\,, \tag{3.56}$$

where the ADM mass of the whole Universe $m_{tot} = L_H/2G_{\rm N}$ has been defined. If $m_{\rm B} \ll m_{tot}$ the location of the horizon, according to (3.20) as well, reads

$$L_H \simeq L - G_{\rm N} N_{\rm B}\,\mu\,. \tag{3.57}$$

The effect of the baryonic source is to reduce the size of the cosmological horizon with respect to a pure de Sitter case, although the correction is small in general. In the corpuscular model, this implies that baryonic matter influences the graviton condensate, in the sense that a fraction of gravitons in the DEC responds locally to the presence of matter by modifying their energy and thus dropping off the condensate. As a result, the number of gravitons in the DEC decreases and the cosmological horizon shrinks.

Furthermore, let us consider the next step in the subsequent approximations. Assume that baryonic matter starts to gather together in a clumped distribution of mass $m_{\rm B}(r) = N_{\rm B}(r)\,\mu$ that, for the sake of simplicity, we suppose spherically symmetric. Since $m_{\rm B} \ll m_{\rm tot}$ as well, the major part of gravitons remains in the condensate phase. If N_Λ is the graviton occupation number in the DEC, then

$$\begin{aligned} N_\Lambda &\sim \frac{(m_{\rm tot} - m_{\rm B})^2}{m_{\rm P}^2} = \frac{m_{\rm tot}^2\left(1 - \frac{m_{\rm B}}{m_{\rm tot}}\right)^2}{m_{\rm P}^2} = \\ &= \frac{m_{tot}^2\, H^4 L_H^4}{m_{\rm P}^2} \sim \frac{H^4 L_H^6}{\ell_{\rm P}^2}\,. \end{aligned} \tag{3.58}$$

The latter quantity is indeed different from the total number of gravitons, that is,

$$N_{\rm tot} \sim \frac{m_{\rm tot}^2}{m_{\rm P}^2}\,. \tag{3.59}$$

This implies that $N_{\rm tot} - N_\Lambda$ gravitons are not in the condensate phase, and their number amounts to,

$$\begin{aligned}\delta N = N_{\text{tot}} - N_{\Lambda} \sim \frac{m_{\text{tot}}^2 - (m_{\text{tot}} - m_{\text{B}})^2}{m_{\text{P}}^2} = \\ = \frac{2m_{\text{B}}\, m_{\text{tot}}}{m_{\text{P}}^2} - \frac{m_{\text{B}}^2}{m_{\text{P}}^2} \sim \frac{m_{\text{B}}\, L_H}{m_{\text{P}} l_p} - \frac{m_{\text{B}}^2}{m_{\text{P}}^2}\,.\end{aligned} \tag{3.60}$$

At this point it shall be noticed that the occupation number of gravitons no longer in the DEC is bigger than the simple number of gravitons closely bound to the source, that enforce the usual Newtonian dynamics and reads

$$N_{\text{B}} \sim \frac{m_{\text{B}}^2}{m_{\text{P}}^2}\,. \tag{3.61}$$

In fact, only a small fraction of the DEC gravitons is affected by local matter. In particular, the cosmic condensate does not react to matter at the full cosmological scale, but only at a local scale.

Now, we are led to identify three classes of gravitons: N_{Λ} quanta compose the DEC and reproduce the behavior that classically is described by the cosmological constant; N_{B} gravitons are not in the condensed phase and are strictly connected to the baryon matter source reproducing the local Newtonian dynamics; the remaining non-condensed gravitons have been interpreted as the constituent quanta that mediate the interaction between baryonic matter and the dark energy condensate. From the hierarchy $R_{\text{S}} \ll L_H$ it follows that the contribution of the negative term in the last line of (3.60) is negligible, and the number of gravitons responsible for the mediation is approximately

$$N_{\text{DF}} \sim \frac{m_{\text{B}}\, L_H}{\ell_{\text{P}}\, m_{\text{P}}}\,. \tag{3.62}$$

The authors in [5] claim that the main effect of the gravitons (3.62) is the emergence of a dark force, at which one associates a dark acceleration that can mimic MOND acceleration at galactic scales and thus dark matter phenomenology.

First of all, one has to interpret the radial acceleration felt by a test particle at a distance r from the center of the baryonic source, that we suppose to be a galaxy, in terms of the effective number of gravitons that contribute to the aforementioned acceleration. In [4] the following formula for the corpuscular acceleration (in modulus) has been proposed,

$$a(r) \sim \frac{\epsilon_{\text{G}}^2(r)}{m_{\text{P}}^2\, \ell_{\text{P}}}\, \sqrt{N_{\text{eff}}(r)}\,. \tag{3.63}$$

This expression was first obtained for the non-condensed gravitons which generate the Newtonian acceleration, but it will be taken for granted for all kinds of gravitons previously mentioned. Therefore, for every population of gravitons, an effective number of gravitons $N_{\rm eff}(r)$ with interaction energy $\epsilon_{\rm G}(r)$ contribute with a radial acceleration $a(r)$ given by (3.63) to the total acceleration of a test particle. At galactic scales, observations of flat rotation curves allow us to write down the total acceleration as the sum of three contributions,

$$a(r) = a_{\rm B}(r) + a_{\rm DE}(r) + a_{\rm DF}(r)\,, \tag{3.64}$$

each one associated to a different behavior of constituent gravitons. In particular, $a_{\rm B}$ is the usual Newtonian acceleration while $a_{\rm DE}$ is the acceleration felt by the particle because of the accelerated expansion of the Universe,

$$a_{\rm B}(r) = -\frac{\ell_{\rm P}\, m_{\rm B}}{m_{\rm P}\, r^2}\,, \qquad a_{\rm DE}(r) = H^2\, r\,. \tag{3.65}$$

Before using the corpuscular acceleration formula for evaluating the dark force acceleration, one has to estimate the number of *effective* gravitons producing the dark force. A detailed derivation follows from an energy balance argument, and can be found in [5]. Here we notice that, if the dark force arises from the gravitational interaction between baryonic matter of mass $m_{\rm B}$ and the gravitons in the DEC only contained inside a sphere of a given radius r, the dark force energy contribution can be evaluated as

$$H_{\rm DF} = -\frac{G_{\rm N}\, m_{\rm B}\, M(r)}{r} = -\frac{G_{\rm N}\, m_{\rm DF}^2}{r}\,, \tag{3.66}$$

where in the first equality $M(r)$ is the dark mass-energy enclosed in the sphere, already introduced in (3.54), while in the right-hand side the expression of the energy can be recast as a function of an effective dark mass $m_{\rm DF}$. It is worth noticing that we are considering the extensive gravitational regime, so that the graviton occupation number scales as (3.55). Assuming an interaction energy for the gravitons in the form $\epsilon_{\rm G} \sim m_{\rm P}\ell_{\rm P}/r$, from $M(r) = N_{\rm G}\epsilon_{\rm G}(r)$ it follows that,

$$M(r) \simeq \frac{m_{\rm P}\, r^2}{\ell_{\rm P}\, L}\,. \tag{3.67}$$

At the end of the calculation one finds that the effective dark force gravitons are

$$N_{\rm DF}(r) \sim \frac{m_{\rm DF}^2}{m_{\rm P}^2} = \frac{m_{\rm B}\, M(r)}{m_{\rm P}^2} = \frac{m_{\rm B}\, r^2}{\ell_{\rm P}\, m_{\rm P}\, L}\,. \tag{3.68}$$

In conclusion, from (3.63) the acceleration provided by the gravitons that mediate the interaction of baryonic matter and cosmic condensate is,

$$a_{\rm DF}(r) \sim \frac{\epsilon_{\rm G}^2(r)}{m_{\rm P}^2\, \ell_{\rm P}}\sqrt{N_{\rm DF}(r)} = \sqrt{\frac{\ell_{\rm P}\, m_{\rm B}}{m_{\rm P}\, r^2\, L}} = \sqrt{\frac{a_{\rm B}(r)}{L}}\,. \tag{3.69}$$

A part of a numerical factor, the result reproduces the MOND formula (3.48). In [5] this result is claimed to be a consequence of the tension between the holographic regime and the extensive behavior of gravity, that leads to deviations from the local Newtonian dynamics, so that the response of the graviton condensate to the presence of baryonic matter makes both regimes important at galactic scales.

3.4 Long-range Quantum Gravity

We began this thesis by introducing motivations for the need of a fundamental quantum theory of the gravitational interaction. In all successive chapters the construction of a possible model has been shown, relying on the corpuscular gravity. However, as we pointed out in Section (1.2.1), a common shared starting point is that gravity should manifest its quantum nature at the Planck scale. This brings together a notable series of problems, starting from the fact that building up a real experiment which can probe $\ell_{\rm P}$ is almost an impossible task at the present. Nevertheless, the corpuscular model of gravity provides a different perspective concerning the scale of Quantum Gravity. Dark matter phenomenology has a natural explanation as a dark force emerging from virtual and very soft gravitons pulling out from the cosmic dark condensate, therefore the description of gravity at galactic scales becomes a measurable quantum phenomenon. In other words, in the corpuscular approach a dark matter behavior is interpreted as a *long-range quantum gravity* effect [40].
To clarify the point, let us consider once again a de Sitter universe with cosmological horizon L, in which we introduce a baryonic source of mass $m_{\rm B}$ that, for simplicity, is assumed point-like. Considering that pure de Sitter space-time geometry has an intrinsic scale invariance, the presence of matter generates a characteristic length scale r_0. A natural choice is to identify the new scale with the Schwarzschild radius of the source $r_0 = R_{\rm S} = \ell_{\rm P} m_{\rm B}/m_{\rm P}$. Whether quantum effects emerge or not at the scale length r_0 strictly depends on the de Broglie-Compton characteristic wavelength of matter distribution, that is

$$\lambda \simeq \frac{m_{\rm P}\,\ell_{\rm P}}{m_{\rm B}}\,. \tag{3.70}$$

In close analogy with optics, we can infer that quantum effects are completely negligible if the considered wavelength is much smaller than the scale ($\lambda \ll r_0$), while they become important in the case both quantities are comparable ($\lambda \simeq r_0$). It is now clear that if r_0 coincides with the gravitational radius of the source, Quantum Gravity shows up when $m_{\rm B} \simeq m_{\rm P}$, thus recovering the usual Planck scale, with the inevitable conclusion that quantum effects are confined in the microscopic realm.

However, in the previous section it was shown that, according to the corpuscular model, the cosmic graviton condensate reacts to the presence of matter. In this case it seems reasonable that r_0 will also depend on the cosmological scale L, thus becoming macroscopic. To evaluate the de Broglie-Compton wavelength associated to the baryonic mass, one shall consider a test particle of mass m at distance r and its classical Newtonian energy,

$$U_{\rm N}(r) = -\frac{G_{\rm N}\, m\, m_{\rm B}}{r}\,. \tag{3.71}$$

The typical scale of quantum gravity effects is now determined by the condition

$$\lambda \simeq \frac{\hbar}{|U_{\rm N}|} \simeq r_0\,. \tag{3.72}$$

This implies that $r \simeq r_0\, R_{\rm S}\, m/\hbar$, and therefore quantum effects become relevant at r_0 when the test particle has $m \sim \hbar/R_{\rm S}$, that is of the same order of the Compton mass of a black hole with mass $m_{\rm B}$. As a consequence, quantum gravity effects can arise at mesoscopic and macroscopic scales, for example of the order of magnitude of galactic radii. A good estimate for r_0 can be guessed in different ways depending on the particular model one is referring to.

It is fascinating that the competition between the holographic and the extensive regimes of gravity leads to an interesting prediction for r_0. As we have seen, one can identify two regimes: the baryonic matter dominated regime and a dark energy dominated regime. For the first one, the graviton scaling law is holographic, and from (3.60) one can guess the number of gravitons subtracted to the cosmological condensate in a sphere of radius r,

$$\delta N(r) \sim \frac{m_{\rm B}\, r}{m_{\rm P} \ell_{\rm P}} = \frac{R_{\rm S}\, r}{\ell_{\rm P}^2}\,. \tag{3.73}$$

In fact, for $r = L$ (3.62) is recovered. The second regime exhibits an extensive behavior, as it is shown in (3.55). Looking at the transition region, imposing that the respective numbers of gravitons become comparable, one finds that

$$r_0 \simeq \sqrt{R_{\rm S}\, L}\,, \tag{3.74}$$

that is, the geometric mean of the black hole horizon and the cosmological horizon. For a given mass distribution, the previous formula sets the scale at which dark matter phenomena are no longer negligible, and thus quantum gravity effects become macroscopic. For example, for a typical spiral galaxy with $m_{\rm B} = 10^{11}\, M_\odot$ one finds $r_0 \simeq 6\,$Kpc, whereas for a typical dwarf galaxy with $m_{\rm B} = 10^7\, M_\odot$ if follows that $r_0 \simeq 80\,$pc.

Chapter 4

Cosmological Bootstrapped Newtonian potential

4.1 Motivation

The corpuscular model of gravitational interaction provides an intriguing point of view on the fundamental quantum nature of gravity. The idea that gravitational systems are interpreted as quantum systems of microscopic gravitons finds a first possible realization in the interpretation of black holes as quantum composite self-sustained condensates at the critical point. In the previous Chapter it has been shown that the corpuscular interpretation also extends to cosmological spaces. In particular dark energy arises naturally as a quantum state of the gravitational dynamics, and the cosmological acceleration has its justification in terms of the interaction with the soft gravitons of the cosmological condensate. In other words, there is no need of an exotic cosmic fluid, since the accelerated expansion is fully accounted by the self-interacting gravitons in the DEC.
Another interesting prediction of the model comes up with the introduction of baryonic matter: the local back-reaction of the cosmic condensate to the matter source seems to reproduce a dark matter behavior. The response to the presence of matter emerges as a dark force that produces a modification of the Newtonian dynamics at galactic scales, thus recovering the MOND acceleration.
The aim of this Chapter is to formally evaluate the back-reaction, that mimics a dark matter behavior, by means of the Bootstrapped Newtonian gravity. We have seen that the Bootstrapped Newtonian theory allows for an effective field description of the gravitational potential that can be interpreted as a modification of the Newtonian dynamics, encoded in Poisson equation, with the introduction of post-Newtonian

order terms. It provides an useful tool for calculations, and our target is to extend the same model to the cosmological systems, and then a MOND-like potential is expected to arise.
Let us remember that, assuming staticity and spherical symmetry, the Bootstrapped Newtonian potential in the region outside a source with (ADM) mass M is

$$V_{\rm BN}(r) = \frac{1}{4q_{\rm V}}\left[1-\left(1+\frac{6\,q_{\rm V}G_{\rm N}M}{r}\right)^{2/3}\right], \tag{4.1}$$

and we recall that $q_{\rm V}$ is a dimensionless coupling constant. As we have seen, for $q_{\rm V} = 1$ the classical Newtonian potential and its post-Newtonian correction are recovered. Since the Bootstrapped Newtonian potential (4.1) was derived in the case of asymptotically flat space, it vanishes in the limit $r \to \infty$, and cannot describe a cosmological behavior at large distances. The aforementioned potential is valid only locally, in the range of distances at which any cosmological expansion can be neglected. Besides, the corrections to the Newtonian potential becomes significant only in the vicinity of the black hole horizon.

First of all, the question we are going to address is to find a possible solution of the field equation (2.35) of Bootstrapped Newtonian gravity that can reproduce a de Sitter regime at distant r, and at the same time that reproduces the local dynamics by means of a potential of the form (4.1). This is similar to the Einstein-Straus problem in General Relativity, that consists in finding a continuous matching between the local metric and the cosmological regime.
In GR the space-time metric in the neighborhood of a static spherically symmetric star coincides with the Schwarzschild one. For a real star in our Universe, since space-time undergoes a cosmological expansion, far from the source mass one expects to recover a FRWL cosmology. The boundary conditions on which the original Schwarzschild solution is obtained, in particular the asymptotic flatness, are not valid anymore. For the sake of simplicity let us assume that the cosmological metric coincides with de Sitter solution (3.13). If $2m$ is the Schwarzschild radius of the mass distribution M, with $m = G_{\rm N}M$, and L the Hubble horizon, the outside gravitational potential is now described by the Schwarzschild-de Sitter metric,

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}-\frac{r^2}{L^2}\right)\mathrm{d}t^2 + \left(1-\frac{2m}{r}-\frac{r^2}{L^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,. \tag{4.2}$$

The field is sourced by the baryonic mass and by a dark fluid with vacuum equation of state, with cosmological constant $\Lambda = 3/L^2$. For a realistic source, the strong gravity regime near the black hole horizon weakens increasing the distance and when $r \gg 2m$ the gravitational field is described quite well by Newtonian gravity. The turnaround radius $R_{\mathrm{T}} \sim (L^2 m)^{1/3}$ represents the radius where the repulsive effect of the cosmological constant starts to be relevant. Near the cosmological horizon the gravitational effects become stronger, space-time curvature is no longer negligible since the Universe is described as a whole system, and the Newtonian limit looses its validity. Therefore one identifies two regimes of strong gravity near the two horizons, when $r \sim 2m$ and $r \sim L$. The Newtonian regime corresponds to a length scale R satisfying the hierarchy

$$2m \ll R \ll L. \tag{4.3}$$

4.2 Harmonic coordinates

In the derivation of the Newtonian limit for a metric theory of gravity the first step consists in the choice of a suitable reference frame. The identification of a scalar potential V as the Newtonian limit of a metric potential follows from the selection of a coordinate system, associated to a particular physical observer. As we have pointed out, the basic ingredients consist in the non-relativistic motion of matter together with the weak stationary field approximation. Then one sets a gauge fixing condition. In Section 1.3.1 the harmonic gauge was employed. A set of space-time coordinates X^μ are said to be harmonic if

$$\Box X^\lambda = g^{\mu\nu}\nabla_\mu\nabla_\nu X^\lambda = 0\,, \tag{4.4}$$

where the covariant derivative ∇ has been introduced. Let us restrict to the particular case of a static and isotropic gravitational field. This means that there exists a set of space-time coordinates $(t, \mathbf{x})$ in such a way that the metric tensor representing the field does not depend on the coordinate t and the invariant line element contains only spatial rotational invariant forms constructed with $\mathbf{x}$ and $\mathrm{d}\mathbf{x}$. The assumption of spherical symmetry leads naturally to set aside the spatial coordinate $\mathbf{x}$ in favor of polar coordinates $(\bar{r}, \theta, \phi)$ where the usual angular coordinates are introduced, and $\bar{r}$ has the meaning of the *areal coordinate*: the physically measured area of a surface of constant $\bar{r}$ and t is given by $A = 4\pi\bar{r}^2$. We refer to this set of coordinates as the *standard* one, and the generic metric has the form

$$\mathrm{d}s^2 = -A(\bar{r})\,\mathrm{d}t^2 + B(\bar{r})\,\mathrm{d}\bar{r}^2 + \bar{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,. \tag{4.5}$$

A line element in standard coordinates is said to be in a Schwarzschild-like form if $A(\bar{r})\,B(\bar{r}) = 1$. Setting $A(\bar{r}) = f(\bar{r})$,

$$\mathrm{d}s^2 = -f(\bar{r})\,\mathrm{d}t^2 + \frac{\mathrm{d}\bar{r}^2}{f(\bar{r})} + \bar{r}^2\,\mathrm{d}\Omega^2\,. \tag{4.6}$$

Starting from the standard static coordinates, one can construct a new set of coordinates $(t, \mathbf{X})$ that fulfill the harmonic condition (4.4). By parametrizing

$$X_1 = r(\bar{r})\sin\theta\cos\phi\,, \qquad X_2 = r(\bar{r})\sin\theta\sin\phi\,, \qquad X_3 = r(\bar{r})\cos\theta\,, \tag{4.7}$$

we introduce r as the *harmonic radius*, or harmonic coordinate. Imposing (4.4) one finds that the harmonic radial coordinate, as a function of the areal coordinate, must satisfy a second order differential equation [42] that reads,

$$\frac{\mathrm{d}}{\mathrm{d}\bar{r}}\left[\bar{r}^2 f(\bar{r})\frac{\mathrm{d}r(\bar{r})}{\mathrm{d}\bar{r}}\right] - 2\,r(\bar{r}) = 0\,. \tag{4.8}$$

Once a solution of (4.8) is found and assuming that $r(\bar{r})$ is invertible, at least in the domain of interest, the metric function $f(\bar{r})$ can be expressed in terms of the harmonic radius, and we denote it with $F(r)$. Finally, the general metric in harmonic coordinates yields,

$$\mathrm{d}s^2 = -F(r)\,\mathrm{d}t^2 + \frac{\bar{r}^2}{r^2}\,\mathrm{d}\mathbf{X}^2 + \frac{1}{r^2}\left[\frac{1}{F(r)\,r'^2} - \frac{\bar{r}^2}{r^2}\right](\mathbf{X}\cdot\mathrm{d}\mathbf{X})^2\,, \tag{4.9}$$

where the prime denotes differentiation with respect to the areal coordinate.
The equation (1.37) expresses the connection between the Newtonian potential and the time-time component of the metric tensor. Since it is derived from the assumption of a harmonic reference frame, the correct definition of the Newtonian limit must refer to the metric (4.9) in harmonic coordinates. We therefore set

$$V(r) = -\frac{1}{2}\left[1 - F(r)\right]\,. \tag{4.10}$$

A very simple case is Schwarzschild space-time outside a source of mass M, whose metric function is $f(\bar{r}) = 1 - 2m/\bar{r}$. Solving (4.8) reveals that the harmonic coordinate is related to the areal coordinate by a linear relation,

$$r(\bar{r}) = \bar{r} - m\,. \tag{4.11}$$

Therefore, far from the black hole horizon the areal and the harmonic coordinates are essentially the same. The corresponding scalar potential reads,

$$V(r) = -\frac{m}{r}\left(1 + \frac{m}{r}\right)^{-1} = -\frac{m}{r} + \frac{m^2}{r^2} + \cdots \tag{4.12}$$

The potential we obtain contains the usual Newtonian potential as a first order term in Newton's constant [1], but shows an infinite series of post-Newtonian corrections. Thus, the choice of the correct reference frame is an unavoidable step in the derivation of the Newtonian limit: adopting the standard form of Schwarzschild metric, instead of the harmonic one, would produce a different potential [2]. In the following we will repeat the same reasoning for cosmological spaces.

4.2.1 de Sitter

Let us start by considering de Sitter space-time. Denoting with L the Hubble radius of the de Sitter universe, the second order differential equation for the harmonic coordinate has the form,

$$\frac{\mathrm{d}}{\mathrm{d}\bar{r}}\left[\bar{r}^2\left(1-\frac{\bar{r}^2}{L^2}\right)\frac{\mathrm{d}r(\bar{r})}{\mathrm{d}\bar{r}}\right]-2\,r(\bar{r})=0\,. \tag{4.13}$$

The general analytic solution reads,

$$r(\bar{r})=a_1\left[\frac{L}{\bar{r}}+\left(1+\frac{L^2}{\bar{r}^2}\right)\left(a_2-\operatorname{arctanh}\frac{\bar{r}}{L}\right)\right]\,, \tag{4.14}$$

where a_1, a_2 are two integration constants. Let us fix the constants with the following argument. In close analogy with the Schwarzschild case, one expects that the harmonic coordinate r should almost coincide with the areal coordinate $\bar{r}$ at large distance from the horizon. Therefore we investigate the case $\bar{r} \ll L$, or equivalently, introducing the dimensionless parameter $y = \bar{r}/L$, in the limit $y \ll 1$. The solution (4.14) is singular for $y = 0$. However, the limit for $y \to 0$ of the harmonic coordinate exists and it is finite at the condition that the constant a_2 vanishes. In fact, setting $a_2 = 0$ one finds that

$$\lim_{y\to 0} r(y) = 0\,. \tag{4.15}$$

The first derivative in the same limit is constant,

$$\lim_{y\to 0} r'(y) = -\frac{4\,a_1}{3}\,. \tag{4.16}$$

This uniquely allows us to choose $a_1 = -3L/4$. In fact the relation between the harmonic coordinate and the areal one for de Sitter space assumes now the form

$$r(\bar{r})=\frac{3L}{4}\left[-\frac{L}{\bar{r}}+\left(1+\frac{L^2}{\bar{r}^2}\right)\operatorname{arctanh}\frac{\bar{r}}{L}\right]\,. \tag{4.17}$$

[1] since the quantity m is first order in G_{N}.

[2] Considering Schwarzschild metric in standard coordinates, which depends on the areal radius, one obtains the usual result of a potential $V = -m/r$ without post-Newtonian corrections.

The first terms of a series expansion around the coordinate origin of the solution yields,

$$r \simeq \bar{r} \left[1 + \frac{2\,\bar{r}^2}{5L^2} + \mathcal{O}\left(\frac{\bar{r}^4}{L^4}\right)\right] . \tag{4.18}$$

The result explicitly shows that the two coordinates are indeed different only if $\bar{r}$ approaches the Hubble horizon. Let us proceed in the derivation of the Newtonian limit for de Sitter metric. Although the inverse function of (4.17) has no closed form, from its series expansion one deduces that

$$\bar{r} \simeq r \left[1 - \frac{2\,r^2}{5L^2} + \mathcal{O}\left(\frac{r^4}{L^4}\right)\right] . \tag{4.19}$$

Then, we use the relation just found to recast the de Sitter metric function $f(\bar{r}) = 1 - \bar{r}^2/L^2$, originally containing the areal coordinate, in the form

$$F(r) = 1 - \frac{r^2}{L^2}\left[1 - \frac{2\,r^2}{5L^2} + \mathcal{O}\left(\frac{r^4}{L^4}\right)\right]^2 . \tag{4.20}$$

Finally, the scalar potential corresponding to the Newtonian limit of de Sitter metric is now,

$$V(r) \simeq -\frac{r^2}{2\,L^2}\left(1 - \frac{2\,r^2}{5L^2}\right)^2 \simeq -\frac{r^2}{2\,L^2} + \frac{2\,r^4}{5L^4} \tag{4.21}$$

up to r^4/L^4 terms. It is worth noting that the result (4.21) contains a quadratic potential, which is usually referred to when dealing with the Newtonian limit of de Sitter. This allows for a Newtonian interpretation of the accelerated expansion of the universe: a test particle in such a universe is subjected to a force that increases linearly with the distance, and that pushes the particle towards the cosmological horizon. However, the same result contains also a quartic post-Newtonian correction that becomes dominant near the Hubble horizon, when $r \sim L$.

4.2.2 Schwarzschild-de Sitter

The situation changes in the Schwarzschild-de Sitter case. The corresponding differential equation for the harmonic radius depends on two length scales, defined by the Schwarzschild radius $2m$ and the Hubble radius L. The second order differential equation for the harmonic coordinate now reads,

$$\frac{\mathrm{d}}{\mathrm{d}\bar{r}}\left[\bar{r}^2\left(1 - \frac{2m}{\bar{r}} - \frac{\bar{r}^2}{L}\right)\frac{\mathrm{d}r(\bar{r})}{\mathrm{d}\bar{r}}\right] - 2r(\bar{r}) = 0\,. \tag{4.22}$$

Let us introduce the dimensionless quantities

$$x = \frac{\bar{r}}{m}\,, \qquad \lambda = \frac{L}{m}\,. \tag{4.23}$$

The equation (4.22) can be recast in a different fashion, namely,

$$\left[-x + \frac{x^2}{2} - \frac{x^4}{2\lambda^2}\right]\frac{\mathrm{d}^2 r(x)}{\mathrm{d}x^2} + \left[-1 + x - \frac{2x^3}{\lambda^2}\right]\frac{\mathrm{d}r(x)}{\mathrm{d}x} - r(x) = 0\,. \tag{4.24}$$

We obtain a second order linear differential equation with polynomial coefficients, a particular case of a class of differential equations of the following form,

$$p(x)\,F''(x) + p'(x)\,F'(x) + \mu F(x) = 0\,, \tag{4.25}$$

where the prime is a derivative with respect to the variable x, $F(x)$ is the unknown function, μ is a real constant and $p(x)$ is a polynomial in x with real coefficients and degree n,

$$p(x) = \sum_{j=0}^{n} b_j x^j\,. \tag{4.26}$$

It seems natural to find a power series representation of $F(x)$ by assuming the existence of a set of real coefficients $\{a_k\}$ such that

$$F(x) = \sum_{k=0}^{\infty} a_k x^k\,, \tag{4.27}$$

and subsequently its derivatives read,

$$F'(x) = \sum_{k=1}^{\infty} k\,a_k x^{k-1}\,, \qquad F''(x) = \sum_{k=2}^{\infty} k(k-1)\,a_k x^{k-2}\,. \tag{4.28}$$

After the substitution in the general equation (4.25) and equating terms with same power of x, one can find a set of recursive relations that allows for the determination of the coefficients a_k, given the coefficients b_j. Then, one compares (4.25) with (4.24) and, after setting the polynomial degree $n = 4$ and noticing that

$$b_0 = b_3 = 0\,, \qquad b_1 = \mu = -1\,, \qquad b_2 = -\frac{1}{2}\,, \qquad b_4 = -\frac{1}{2\lambda^2}\,, \tag{4.29}$$

the calculation yields,

$$a_k = \frac{1}{2}\left(1 - \frac{k-2}{k^2}\right) a_{k-1} + \left(\frac{3-k}{2k\lambda^2}\right) a_{k-3}\,, \tag{4.30}$$

and we use the assumption $a_{-2} = a_{-3} = 0$. The coefficient a_0 remains arbitrary and must be fixed with an additional condition in order to select the more convenient solution. Using the recursive formula for the equation (4.24) one is left with the following power series of x,

$$r(x) = a_0 \left[1 - x + \frac{x^4}{8\lambda^2} + \mathcal{O}(x^5)\right] . \tag{4.31}$$

One expects that in the limit $\lambda \to \infty$, corresponding to the vanishing of the cosmological constant, the Schwarzschild harmonic coordinate (4.11) is recovered. We therefore choose $a_0 = -m$ and,

$$r(x) = -m \left[1 - x + \frac{x^4}{8\lambda^2} + \mathcal{O}(x^5)\right] \simeq -m + \bar{r} - \frac{\bar{r}^4}{8L^2 m} + \cdots \tag{4.32}$$

The first two terms of the expansion reproduces the Schwarzschild case, then we find a series of corrections. Looking at the first x^4 correction, it becomes comparable with the preceding one when $x \sim \lambda^{2/3}$. For a typical spiral galaxy, like the Milky Way, one can reasonably assume $\lambda \sim 10^{12}$, so that the harmonic radius deviates considerably from the Schwarzschild one for $x \sim 10^8$. Since the radius of the Milky Way is around $x \sim 10^4$, we conclude that as long as one is dealing with systems at galactic scales, the harmonic radius is almost equal to the areal one.
Differently from Schwarzschild and de Sitter case, here there is no closed solution in terms of a finite polynomial. In fact, the necessary condition for a polynomial solution of degree $N > 0$ can be found to be

$$N(N+3)\, b_4 = 0 \,, \tag{4.33}$$

and since $b_4 \neq 0$ there is no natural non-trivial solution for the polynomial degree. For the sake of completeness, one is free to introduce a different rescaled variable in terms of the Hubble horizon,

$$y = \frac{\bar{r}}{L} \,. \tag{4.34}$$

The differential equation for the harmonic radius now reads,

$$\left[-\frac{y}{\lambda} + \frac{y^2}{2} - \frac{y^4}{2}\right] \frac{\mathrm{d}^2 r(y)}{\mathrm{d}y^2} + \left[-\frac{1}{\lambda} + y - 2y^3\right] \frac{\mathrm{d}r(y)}{\mathrm{d}y} - r(y) = 0 \,. \tag{4.35}$$

Making use of the Ansatz $r = \sum_k c_k y^k$, with the same argument of this Section one arrives to the very same solution (4.32).

4.3 Bootstrapped potential with cosmological constant

Let us start from the Bootstrapped Newtonian equation (2.35) for an effective gravitational field $V(r)$. We make use of the assumption that the field is sourced by a dark energy perfect fluid with constant energy density,

$$\rho = \frac{m_{\mathrm{P}}\,\Lambda}{8\,\pi\,\ell_{\mathrm{P}}}\,. \tag{4.36}$$

The non-linear field equation assumes the following form,

$$\triangle V = -\Lambda + \frac{2\,q_{\mathrm{V}}\,(V')^2}{1-4\,q_{\mathrm{V}}\,V}\,. \tag{4.37}$$

One can perceive that incorporating the dimensionless parameter q_{V} into the definition of the field by means of a transformation $q_{\mathrm{V}}\,V \to V$, the field equation is recast as

$$\triangle V = -q_{\mathrm{V}}\Lambda + \frac{2\,(V')^2}{1-4\,V}\,, \tag{4.38}$$

and this results in a rescaling of Λ by the same factor. It should be noticed that if the coupling parameter q_{V} is set to zero, the equation reduces to Poisson equation in the present of a cosmological constant with no matter density,

$$\triangle V_0 = -\Lambda\,. \tag{4.39}$$

Let us recall that the general solution of (4.39) for a spherically symmetric system is

$$V_0(r) = -\frac{\Lambda}{6}r^2 - \frac{c_1}{r} + c_2\,. \tag{4.40}$$

with c_1, c_2 integration constants. Then, the equation must be supplied with suitable boundary conditions in order to fix the constants, according to the physical system under study. In particular we can distinguish two cases of interest:

- an universe in which matter is completely absent, and the only source of the gravitational field is the constant Λ. In this case the field diverges when $r \to \infty$ but it remains regular in $r = 0$. Given that an observer in the origin of the coordinate frame $r = 0$ experiences no acceleration, we can safely choose $c_1 = c_2 = 0$, and

$$V_0(r) = -\frac{\Lambda r^2}{6}\,. \tag{4.41}$$

- the gravitational field outside a spherically symmetric source of mass M. The condition we currently impose is that, turning off the cosmological term, the field has the usual Newtonian behavior. In other words, in the limit $\Lambda \to 0$ the r^{-1} Newtonian potential is recovered. Thus, we choose $c_1 = G_N M$, $c_2 = 0$, and

$$V_0(r) = -\frac{G_N M}{r} - \frac{\Lambda r^2}{6} \,. \tag{4.42}$$

The Bootstrapped equation (4.37) appears as a non-linear modification of the Poisson equation, and one expects that it can describe both the two aforementioned different cases. In this Section we will set the boundary conditions accurately in order to derive the Bootstrapped gravitational potential for an universe empty of matter first, then we will focus on the Bootstrapped potential in the region external to a matter source.

4.3.1 Absence of matter

For reproducing the gravitational field responsible for an accelerated expansion in an empty universe, we use the conditions

$$V(0) = 0 \,, \qquad V'(0) = 0 \,. \tag{4.43}$$

The solution can be found numerically and it is shown in Figure 4.1 for $q_V = 1$. The field reproduces the Newtonian behavior almost precisely for small r, while, increasing the distance, the Bootstrapped potential diverges more slowly.
We can introduce a Newtonian definition of cosmological horizon according to a notion analogous to the escape velocity. A test particle at a distance R is subjected to a force $F = -\mathrm{d}V/\mathrm{d}r$ pointing in the direction of increasing r. The minimum speed that the test particle must possess in order to move from its location and reaching $r = 0$ against the cosmological expansion is calculated from energy conservation,

$$\frac{1}{2} v_{\min}^2 + V(R) = V(0) = 0 \,. \tag{4.44}$$

Since the particle speed cannot exceed the speed of light $c = 1$, there exists a distance $r = L_H$ such that, if the test particle is located outside this radius, the minimum speed for escaping from the expansion in bigger than c. If one defines L_H as the cosmological horizon, the condition

$$2\, V(L_H) = -1 \tag{4.45}$$

allows us of to determine such horizon. Figure 4.2 shows how the ratio L_H/L varies as a function of different values of the constant Λ. The cosmological horizon for the

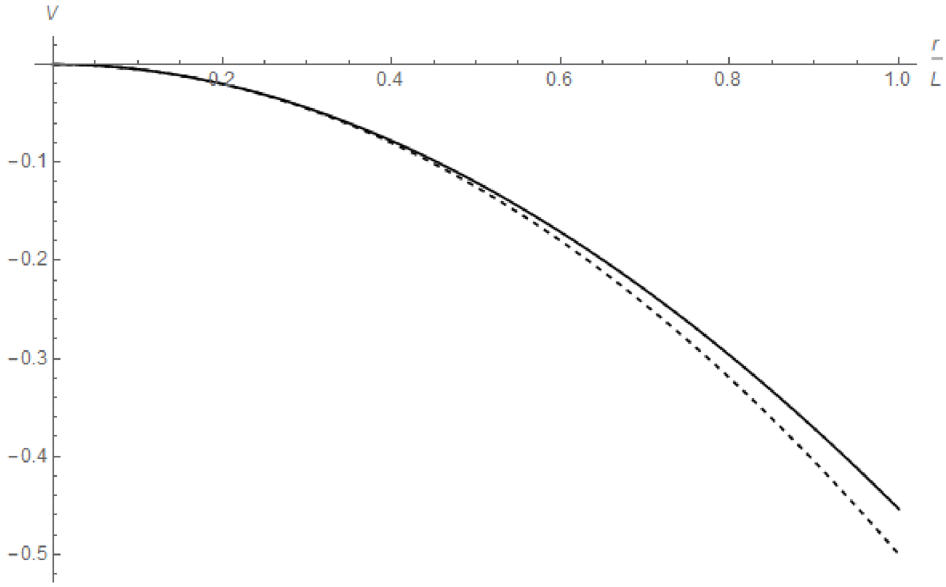

Figure 4.1: Comparison between numerical solution of Bootstrapped Newtonian field equation (4.37) in the case of an empty universe (solid line) and the quadratic potential V_0 in (4.41) (dashed line), in units of Hubble radius L and $q_V = 1$.

Bootstrapped potential appears to slightly overcome the de Sitter horizon L. One finds that

$$L \simeq 0.95\, L_{\rm H}\,. \tag{4.46}$$

4.3.2 Exterior region of a static source

Consider now the solution outside a source of mass M. As we have pointed out, locally the Bootstrapped potential should reproduce the vacuum solution (4.1). Therefore, one expects that at a fairly short length scale $r_* \ll L$ the following equalities are fulfilled with good approximation,

$$V(r_*) = V_{\rm BN}(r_*)\,, \qquad V'(r_*) = V'_{\rm BN}(r_*)\,. \tag{4.47}$$

If $R_{\rm S} = 2G_{\rm N}M$ is the Schwarzschild radius of the source, a reasonable choice is $r_* = 10\, R_{\rm S}$. In Figure 4.3 numerical solutions with these conditions are presented, for different values of Λ and, equivalently, of the Hubble radius L.
Although the field equation has no closed solution, we will try to obtain an analytic approximation of the potential under some particular limits. Let us introduce the following Ansatz,

$$V(r) \equiv V_0(r) + \psi(r)\,, \tag{4.48}$$

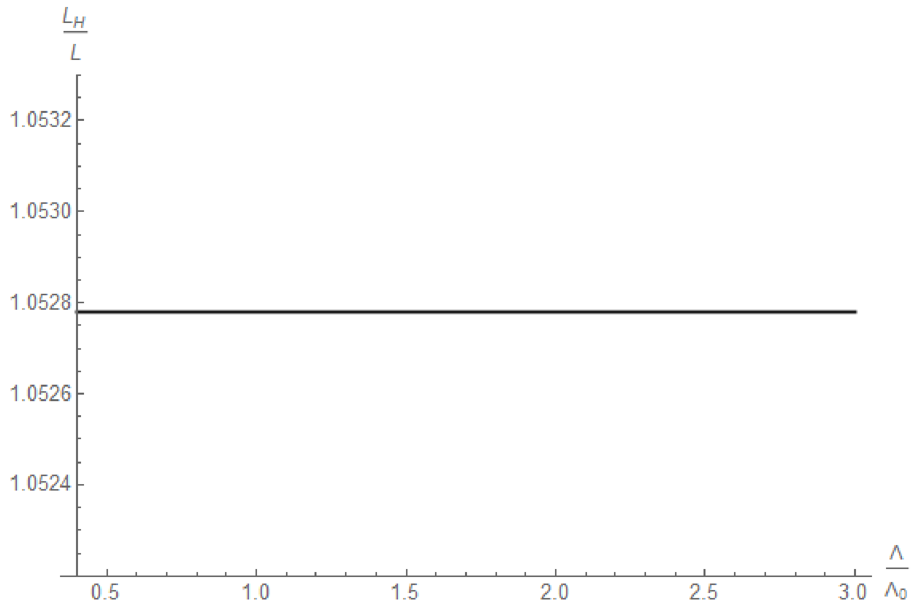

Figure 4.2: Ratio between the cosmological horizon $L_{\rm H}$ ($q_{\rm V} = 1$) and the de Sitter Hubble radius L as a function of the constant Λ. We take as a reference value the current best estimate for the cosmological constant $\Lambda_0 = 2.84 \times 10^{-122}\, \ell_{\rm P}^{-2}$ [10].

where V_0 is a solution of modified Poisson equation (4.39) and $\psi(r)$ is an unknown function. One should notice that if $\Lambda \to 0$ the solution potential must show the usual Newtonian behavior at large distances. This is equivalent of looking for a solution $V(r)$ that satisfies the following boundary condition,

$$\lim_{r\to+\infty} V(r)\bigg|_{\Lambda=0} = 0\,. \tag{4.49}$$

Making use of the Ansatz, the boundary condition on $V(r)$ implies two requirements which must hold together, namely,

$$\lim_{r\to+\infty} V_0(r)\bigg|_{\Lambda=0} = 0 \qquad \text{and} \qquad \lim_{r\to+\infty} \psi(r)\bigg|_{\Lambda=0} = 0\,. \tag{4.50}$$

By choosing $V_0(r)$ as (4.41), the first condition is automatically fulfilled. Therefore, in the following, we impose that, in the large r limit and having set aside the cosmological constant, the function $\psi(r)$ alone reproduces the Newtonian potential. The substitution of the Ansatz in the field equation (4.37) leads to a differential equation for ψ,

$$\triangle\psi = \frac{2\, q_{\rm V}\, (V_0' + \psi')^2}{1 - 4\, q_{\rm V}\, (V_0 + \psi)}\,, \tag{4.51}$$

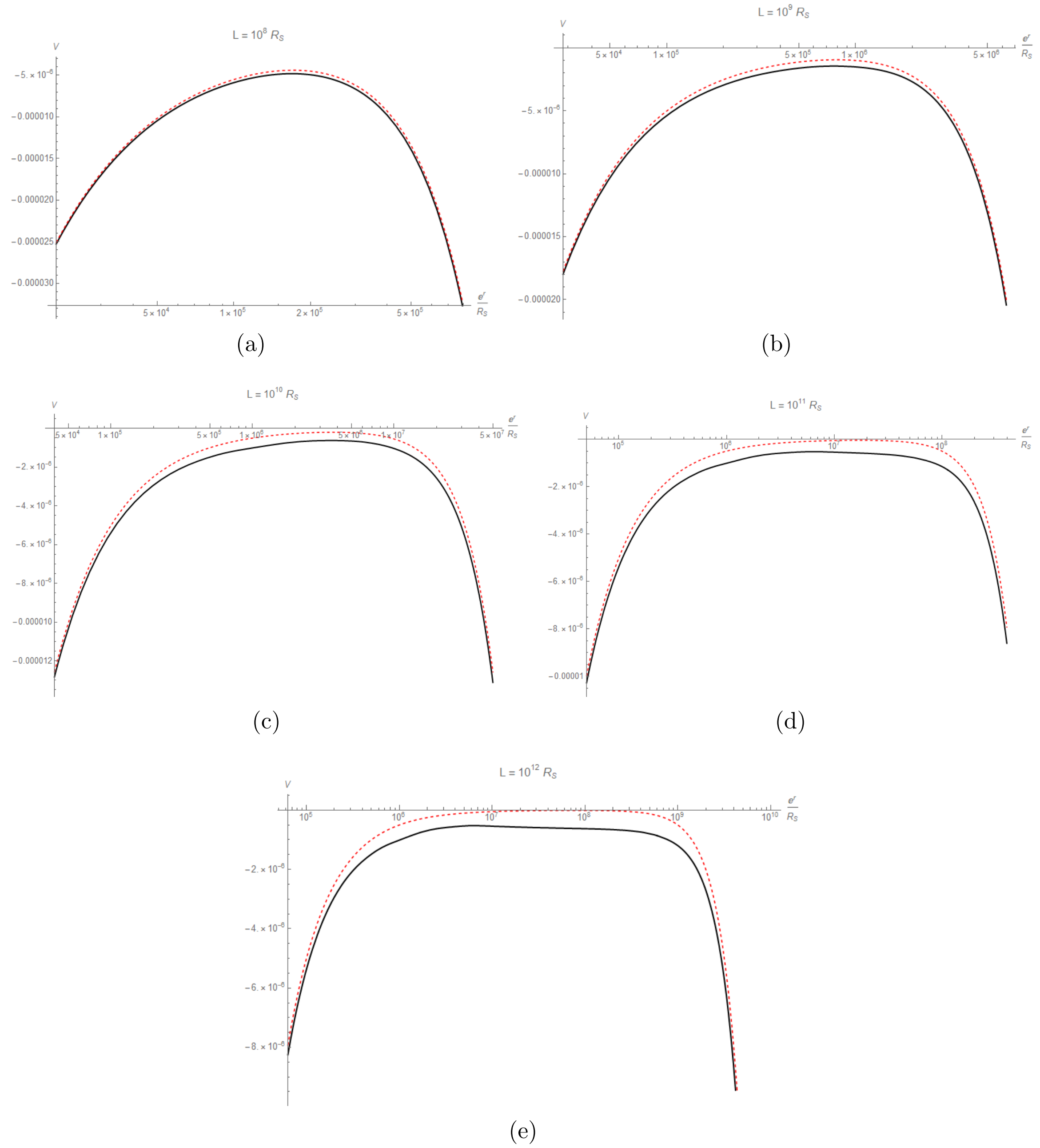

Figure 4.3: Numerical solutions for the Bootstrapped potential outside a static source of mass M with $q_{\mathrm{V}} = 1$ (solid line) compared to Newtonian potential (4.42) (dashed line) in units of gravitational radius $R_{\mathrm{S}} = 2G_{\mathrm{N}}M$ for different values of the Hubble horizon in units of R_{S}.

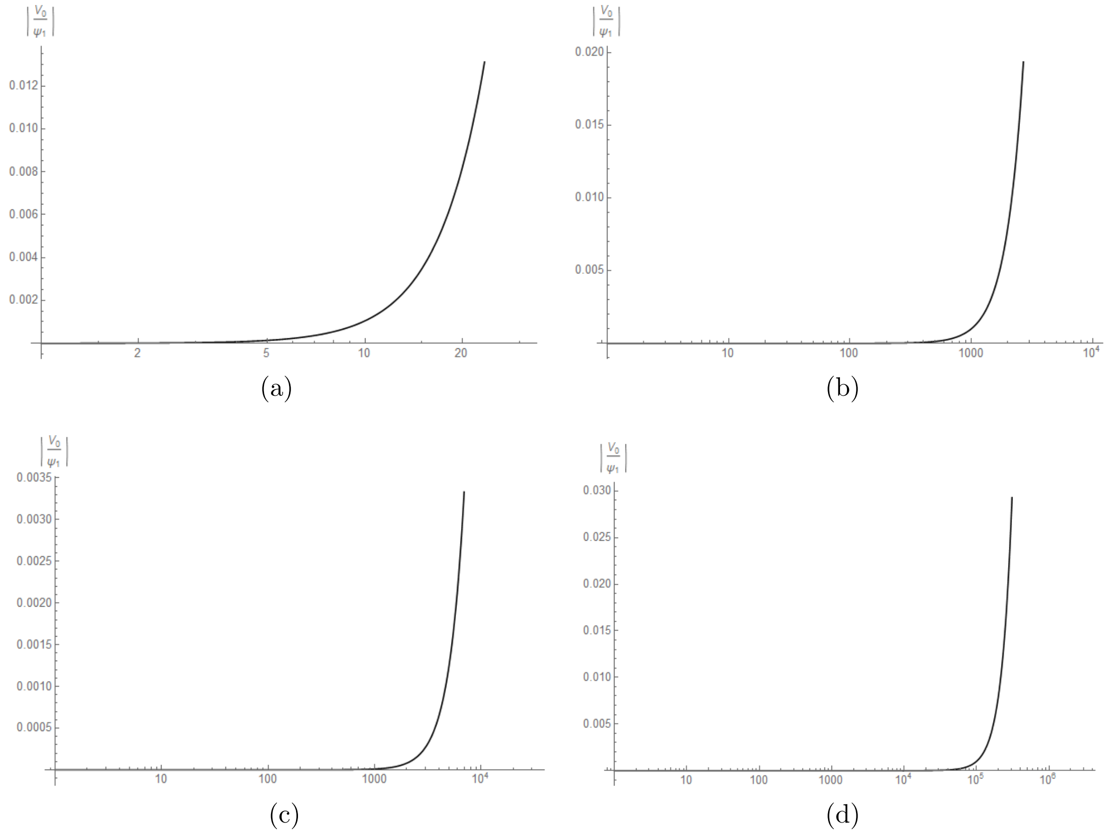

Figure 4.4: Ratios between the potential V_0 and the function ψ_1 in absolute value with $q_{\rm V} = 1$ as a function of the radial distance r, in units of gravitational radius $R_{\rm S} = 2G_{\rm N}M$ for different values of the parameter: (a) $L = 10^3\,R_{\rm S}$, (b) $L = 10^6\,R_{\rm S}$, (c) $L = 10^7\,R_{\rm S}$, (d) $L = 10^9\,R_{\rm S}$.

where we make use of the fact that V_0 solves the modified Poisson equation. We investigate two possible cases. In the small distance limit one has $|V_0(r)| \ll |\psi(r)|$ and the equation (4.51) reduces to

$$\triangle\psi = \frac{2\,q_{\rm V}\,(\psi')^2}{1-4\,q_{\rm V}\,\psi}\,. \tag{4.52}$$

Therefore, $\psi(r)$ satisfies the very same equation for the Bootstrapped Newtonian vacuum potential in the asymptotically flat case, presented in (2.37). Since the general integral is known,

$$\psi_1(r) = \frac{1}{4\,q_{\rm V}}\left[1 - c_1\left(1+\frac{c_2}{r}\right)^{2/3}\right], \tag{4.53}$$

one can fix the integration constants by comparing the behavior at infinity to the Newtonian potential, giving $c_1 = 1$ and $c_2 = 6\, q_{\rm V}\, G_{\rm N} M$. Finally the Bootstrapped potential in this limit reads,

$$V_1(r) \simeq \psi_1(r) + V_0(r) = \frac{1}{4\, q_{\rm V}} \left[1 - \left(1 + \frac{6\, q_{\rm V}\, G_{\rm N} M}{r} \right)^{2/3} \right] - \frac{r^2}{2L^2} \, . \tag{4.54}$$

The potential appears as a sort of modification of the Schwarzschild-de Sitter Newtonian potential: the quadratic term enforces the cosmological expansion, but far away from the cosmological horizon the first term dominates. Performing the expansion in r under the condition that $G_{\rm N} M \ll r \ll L$, one finds the expected positive post-Newtonian contribution to the Newtonian potential for $q_{\rm V} = 1$, namely

$$V_1(r) \simeq -\frac{G_{\rm N} M}{r} - \frac{r^2}{2L^2} + q_{\rm V} \frac{G_{\rm N}^2 M^2}{r^2} + \cdots \tag{4.55}$$

The proposed solution $V_1(r)$ was found assuming that the ratio between V_0 and ψ_1 (in absolute value) remains very small. In Figure 4.4 the latter ratio has been plotted for different values of L.

Let us analize a second possibility, that is the large r limit corresponding to the asumption that $|V_0(r)| \gg |\psi(r)|$. The vacuum field equation now reads,

$$\triangle \psi = \frac{2\, q_{\rm V}\, (V_0')^2}{1 - 4\, q_{\rm V}\, V_0} \, . \tag{4.56}$$

The equation can be easly integrated and the general solution yields,

$$\psi_2(r) = c_1 - \frac{c_2}{r} + \frac{r^2}{6L^2} - \frac{1}{4 q_{\rm V}} \left[\log\left(1 + q_{\rm V} \frac{2r^2}{L^2} \right) + \sqrt{\frac{2}{q_{\rm V}}} \, \frac{\arctan\left(\sqrt{2\, q_{\rm V}}\, r/L \right)}{r/L} \right] . \tag{4.57}$$

Form the boundary condition (4.50) we deduce that $c_1 = 1/2$ and $c_2 = G_{\rm N} M$. The Bootstrapped potential in this case is

$$V_2(r) \simeq -\frac{G_{\rm N} M}{r} - \frac{r^2}{3L^2} - \frac{1}{4 q_{\rm V}} \left[-2 + \log\left(1 + q_{\rm V} \frac{2r^2}{L^2} \right) + \sqrt{\frac{2}{q_{\rm V}}} \, \frac{\arctan\left(\sqrt{2\, q_{\rm V}}\, r/L \right)}{r/L} \right] . \tag{4.58}$$

A series expansion for $r \ll L$ reveals that,

$$V_2(r) \simeq -\frac{G_{\rm N} M}{r} - \frac{r^2}{2L^2} + q_{\rm V} \frac{r^4}{10 L^4} + \cdots \tag{4.59}$$

4.3.3 Expansion in terms of the coupling parameter

The Bootstrapped equation (4.37) contains the dimensionless coupling parameter q_{V}, which in turn implements the gravitational self-coupling of the field. We introduce an expansion of the solution potential in terms of the self-interaction parameter q_{V},

$$V(r) = V^{(0)}(r) + q_{\mathrm{V}} V^{(1)}(r)\,. \tag{4.60}$$

After a substitution in the field equation, we can separate it into an equation for the zero order potential

$$\triangle V^{(0)} + \Lambda = 0\,, \tag{4.61}$$

and a second equation for the first order potential,

$$\triangle V^{(1)} = 2\left[V^{(0)\prime}\right]^2\,. \tag{4.62}$$

From (4.61) one exactly recovers the modified Poisson equation. Making use of (4.42), the first order equation can be integrated,

$$V^{(1)}(r) = c_1 - \frac{c_2}{r} + \frac{G_{\mathrm{N}}^2 M^2}{r^2} + \frac{r^4}{10L^4} - \frac{2G_{\mathrm{N}} M r}{L^2}\,. \tag{4.63}$$

The overall potential up to first order contributions in q_{V} can be written as

$$V(r) = -\frac{G_{\mathrm{N}} M + q_{\mathrm{V}}\, c_2}{r} - \frac{r^2}{2L^2} + q_{\mathrm{V}}\left(c_1 + \frac{G_{\mathrm{N}}^2 M^2}{r^2} + \frac{r^4}{10L^4} - \frac{2G_{\mathrm{N}} M r}{L^2}\right)\,. \tag{4.64}$$

In the $1/r$ term the mass of the source appears shifted by an arbitrary constant. This is not problematic, since one can redefine $G_{\mathrm{N}} M + q_{\mathrm{V}}\, c_2 \to G_{\mathrm{N}} M$ and then one neglects higher order than one in q_{V}. Besides, we fix $c_1 = 0$ for the boundary conditions. In conclusion the Bootstrapped potential reads,

$$V(r) = -\frac{G_{\mathrm{N}} M}{r} - \frac{r^2}{2L^2} + q_{\mathrm{V}}\left(\frac{r^4}{10L^4} - \frac{2G_{\mathrm{N}} M r}{L^2} + \frac{G_{\mathrm{N}}^2 M^2}{r^2}\right)\,. \tag{4.65}$$

The solution (4.65) shows that at order zero in the coupling parameter the potential concides with the Poissonian one. The first order terms have the form of post-Newtonian corrections: by a comparison with (4.21) we identify a r^4 correction to the de Sitter potential, as well as the already mentioned r^{-2} second order in G_{N} correction.

Conclusion and outlook

In this work an approach to Quantum Gravity based on the corpuscular model of gravitational interaction was presented. After an introduction on the problem of a quantum theory of gravity, paying particular attention to the non-renormalizability of General Relativity, beginning with Section 1.5 we discussed the main motivation underlying the corpuscular interpretation of gravitational systems. The fist interesting results were found in the context of black hole physics. The corpuscular picture of a black hole has the merit of describing it in pure quantum terms, as a critical Bose-Einstein condensate of microscopic self-interacting gravitons, whose wavelength distribution is peaked on the characteristic length of the system. This picture is naturally extended to a cosmological context. In fact, the relation (3.24) between the cosmological horizon and the mass-energy enclosed in the de Sitter horizon shows an analogy with the definition of a Schwarzschild radius of a black hole. This leads to a corpuscular interpretation of de Sitter space-time, and gives a quantum description of dark energy in terms of a critical Bose-Einstein condensate of gravitons. The corpuscular model predicts that the presence of baryonic matter causes the dark energy condensate to respond locally, since a small part of cosmic gravitons modify their interaction energy and are pulled out of the condensate. One can distinguish three populations of quanta: the gravitons that remain in the condensate phase, responsible for the accelerated cosmic expansion; the gravitons that interact with the baryonic source, enforcing the Newtonian gravitational dynamics; the gravitons that give rise to a dark force. Using (3.63) the contribution to the radial acceleration from these dark gravitons has been calculated, and reproduces the MOND result (3.48) up to a numerical factor. To be more precise, MOND theory predicts that at galactic scales the gravitational acceleration is no longer the Newtonian one (that scales as r^{-2}) but goes like r^{-1}. This indicates that the gravitational potential manifests a logarithmic behavior. In Chapter 4 we presented a possible approach for the derivation of the gravitational potential that can encode the back-reaction of the dark energy condensate to local matter. In particular we made use of the Bootstrapped Newtonian gravity, that has been presented as an effective field theory of the gravitational potential in Chapter 2. The Bootstrapped vacuum solution (4.1) refers to an asymptotically flat space and cannot include a cosmological behavior. We considered the Boostrapped Newtonian field equation in the case the field is sourced by a constant vacuum energy density and assuming staticity and spherical symmetry. The resulting equation (4.37) can be interpreted as a non-linear extension of the classical Poisson equation with cosmological constant. We found that the Bootstrapped potential shows a cosmological behavior at large distances, and with a suitable choice of the boundary conditions, it can describe two different systems: an expanding universe completely empty of matter, and the gravitational field outside a massive source.

The first result follows from the condition that the solution potential and its derivative are regular on $r = 0$, and that an observer in the coordinate origin is not subjected to the cosmological acceleration. The numerical solution (Figure 4.1) differs from the Newtonian solution in the vicinity of the Hubble horizon. In addition, a Newtonian definition of the cosmological horizon was introduced, thus the Bootstrapped universe is found to be slightly larger than the de Sitter one.

For the potential in the presence of matter we evaluated the numerical solution (Figure 4.3) and we propose two analytic forms. With the function (4.54) we recover the local Bootstrapped potential plus a quadratic de Sitter-like contribution. Besides, the most interesting result follows from the solution (4.58), which shows that from the Bootstrapped field equation a logarithmic potential can arise. This result is significant since, according to MOND theory, in order to mimic a dark matter phenomenology the gravitational potential should scale logarithmically. In the light of the corpuscular model, the presence of a logarithmic expression for the potential can be interpreted as a dark matter behavior that emerges from the back-reaction of the dark energy condensate.

One weakness of the result found in this work is that it is not clear if the logarithmic potential can be the dominant contribution at galactic scales. Although it is interesting that a suitable choice of the source in Bootstrapped Newtonian theory eventually leads to the desired form of the potential, one future challenge will concern a deep analysis of the non-linear Bootstrapped field equation. Moreover, one has to link the effective field description of the potential to the corpuscular model by performing a quantum uplifting of the classical theory. In particular, the classical potential should come up as the expectation value of a suitable scalar quantum field on a coherent state.

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