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DECIDABILITY OF STRONG EQUIVALENCES FOR FINITE PETRI NETS

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Sommario

I sistemi concorrenti e distribuiti, a partire da piccoli progetti fatti in casa fino a backend estremamente complessi basati su microservizi, compongono la maggior parte dei prodotti software odierni. Ci sono qualità rilevanti che li distinguono e ne fanno un interessante oggetto di studio: ogni sistema è composto da una serie di unità più piccole, ciascuna delle quali esegue delle computazioni locali senza che le altre possano prendervi parte, e che possono cooperare (di solito attraverso meccanismi di sincronizzazione) o competere per le risorse. La teoria della concorrenza mostra che non è facile definire quando due sistemi sono equivalenti: alcune nozioni, come l'isomorfismo, sono troppo restrittive ed altre, come l'equivalenza di tracce, troppo lasche. In ambito distribuito il problema peggiora ulteriormente perché, ad esempio, non è possibile interagire con un sottoinsieme del sistema per inferirne il comportamento globale.

Tra i vari modelli matematici proposti, le reti di Petri finite sono uno dei più studiati ed adatti alla descrizione di questo tipo di sistemi, poiché ne ricalcano le qualità più distintive. Infatti, lo stato globale di una Rete di Petri è formato da una collezione (detta marking) di stati locali (detti token). L'esecuzione di una transizione è una trasformazione locale che riguarda solo una parte dei token in un marking. Pertanto, è semplice interpretare un token come un processo sequenziale ed un marking come un sistema distribuito che porta avanti una computazione più complessa all'interno della quale i processi possono cooperare o competere. Le equivalenze interessanti su reti di Petri quindi devono considerare vari aspetti del modello, quali concorrenza (e relazioni di causalità), scelte e ramificazioni, invarianti. Queste non sono considerate dalla maggior parte delle equivalenze per altri modelli, ad esempio interleaving bisimilarity. Dalla loro introduzione del 1962 da parte di C.A. Petri, sono state studiate diverse varianti e classi di reti, ciascuna con differente potere espressivo.

Nella prima parte del lavoro, si studiano equivalenze su reti Place/Transition (finite) safe e bounded, i.e. reti composte solo da posti e transizioni, nelle quali il numero massimo di token in ogni posto è rispettivamente 1 oppure finito. Sono riportati o provati alcuni risultati riguardanti due equivalenze simili tra loro, fully-concurrent bisimilarity e causal-ney bisimilarity, che posseggono buona parte (o tutte, nel caso della seconda) le proprietà elencate sopra. Queste equivalenze considerano ogni diversa esecuzione concorrente della rete come un oggetto matematico a sè stante, detto processo. Le prove sono basate su una generalizzazione, per mezzo di indici, della tecnica di prova a marking ordinati usata da Vogler per dimostrare la decidibilità di fully-concurrent bisimilarity su reti safe.

Nella seconda parte del lavoro, si studiano equivalenze su reti Place/Transition (finite) ma con archi inibitori e senza limiti al numero di token che possono occupare un posto, dette reti PTI. In questo modello, i token possono non solo permettere l'esecuzione di una transizione, ma anche bloccarla. Si formula una equivalenza per reti PTI, chiamata pti-place bisimilarity, che estende in modo conservativo place-bisimilarity (una equivalenza decidibile per reti Place/Transition, proposta recentemente), e se ne prova la decidibilità. Questa è la prima volta che una equivalenza è provata essere decidibile per reti PTI. Il risultato è rilevante per due ragioni: la prima è che le reti PTI sono un modello Turing-completo, dove molte altre proprietà (ad esempio, la reachability) sono indecidibili; la seconda è che pti-place-bisimilarity è più discriminante di causal-net bisimilarity ma meno dell'isomorfismo, consentendo per esempio lo svolgimento dei loop o la condivisione di risorse. Tuttavia, rispetto alle altre equivalenze note, ha la particolarità di non essere coinduttiva: non è detto che l'unione di pti-place bisimilarity sia a sua volta una pti-place bisimilarity.

Abstract

Distributed and concurrent systems, ranging from small-sized hobby projects to extremely complex application backends based on microservices, are arguably the largest part of present-day software artifacts. These are defined by distinctive features: the system is composed by a collection of smaller units, each one performing local computations and executing as a black-box to others, which can cooperate (usually, by synchronization) or compete for resources. When studying concurrency, it is not easy to define what does it mean for two systems to be equivalent: some notions, such as isomorphism, are too fine to be useful and others, such as trace equivalence, might be too abstract and consider equivalent systems which intuitively should not be. In a distributed setting, the scenario gets even worse, because it is not possible to, e.g., interact with a part of the system to infer the general behavior.

Among many mathematical modeling languages, finite Petri Nets are one of the most studied and suitable for the description of distributed and concurrent systems, as they mirror the two former features. Indeed, a Petri Net describes the global state of a system as composed of a collection, called marking, of local states, called tokens. The execution of a transition is a local transformation involving some tokens of a marking. A token can thus be interpreted as a sequential process and a marking as a distributed system where a complex task is being executed, with possibility of cooperation or competition between processes. Sensible equivalences on Petri Nets must then take into account many aspects of the model, such as concurrency (causality relations), choices and branching, inevitability of transitions, invariants (w.r.t. substitutions or execution time). Indeed, most of the usual equivalences on other concurrency models, such as interleaving bisimilarity, do not hold a number of the former properties. Since the introduction of the model in 1962 by C.A. Petri, many different flavors and classes of nets have been studied, each with different expressiveness and properties.

The first part of this work is concerned with equivalences on safe and bounded finite Place/Transition nets, i.e. nets composed only of places and transitions where the number of tokens in each place can be at most 1 or finite respectively. Some results are adapted or proved for two similar equivalences, namely fully-concurrent bisimilarity and causal-net bisimilarity, holding most (even all, in the case of the second) of the interesting properties from above. These equivalences take into account a different object, called process, for each possible concurrent run of the net. The proofs are developed using an index-based generalization of the ordered marking proof technique that Vogler used to demonstrate that fully-concurrent bisimilarity (or, equivalently, history-preserving bisimilarity) is decidable on finite safe nets.

In the second part of this work, finite Place/Transition Petri nets with inhibitor arcs (PTI nets for short) and without bound on the number of tokens are studied. In this model, tokens can not only allow a transition to execute, but also inhibit it. PTI nets are equipped with a behavioral equivalence, called pti-place bisimilarity, that conservatively extends place bisimilarity (a decidable Place/Transition net equivalence, recently proposed), and is the first decidable equivalence for this model. The decidability result is truly relevant for two reasons: first, PTI nets are a Turing-complete model of computation and many properties, such as reachability, are not decidable; second, pti-place-bisimilarity is finer than causal-net bisimilarity but coarser than isomorphism, allowing e.g. loop unwinding and resource sharing, thus being an actually sensible and useful equivalence. However, differently from standard concurrent equivalences, it is not coinductive: the union of pti-place bisimilarities might not be a pti-place bisimilarity.

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Chapter 1

Introduction

Petri nets are one one of the most studied and largely used mathematical modeling languages for the description of concurrent and distributed systems. The model was first proposed in 1962 by C. A. Petri [Pet62] as a generalization of automata, where the basic idea is to describe the global state of a system as composed of a collection, called *marking*, of local states, called *tokens*. The execution of a *transition* is a local transformation involving some tokens of a marking. A token can thus be interpreted as a sequential process and a marking as a distributed system where a complex task is being executed, with possibility of cooperation or competition between processes.

This work focuses on two types of Petri nets: the original place/transition (P/T) nets [Pet62] and P/T nets with inhibitor arcs (PTI), first introduced in [AF73]. The latter is an extension of the former, where a token (actually, the *presence* of it) may not only allow, but also prevent the execution of a transition. Despite the seemingly small difference, the introduction of inhibitor arcs greatly increase the expressive power of P/T nets by reaching Turing-completeness [Age74]. However, both these flavors of Petri nets have been extensively used in the study of concurrent and distributed systems, e.g. in [Gor17; Rei85; Pet81] for P/T nets and in [Hac76; Mar+98; BG09] for PTI nets.

As stated above, P/T nets are not Turing complete, and indeed, not surprisingly, the reachability problem (i.e., checking whether a given marking is reachable from the initial one) is decidable. Nonetheless, only recently the decidability of *place bisimilarity* [Old91; ABS91; Gor21], a behavioral equivalence, coarser than net isomorphism (which is decidable but too discriminating) and finer than interleaving bisimilarity (which is undecidable [Jan95]) has been proved [Gor21]. Yet, interleaving bisimilarity is decidable for finite bounded nets (i.e. nets with a finite number of reachable markings) and it has been conjectured (e.g., in [Vog91]) that also truly concurrent behavioral equivalences are decidable for this restricted class of P/T nets. Truly concurrent equivalences are ”truly” concurrent in the sense that they capture all causal links between actions, and thereby all concurrency. While some decidability results [Vog91; JM96] for these are available in the setting of finite safe nets (i.e., nets that can hold one token at most in any place), to the best of our knowledge, very little decidability proofs were provided for the case of bounded nets [MP97; Val93].

In [Gla15] some interesting properties of a ”good” equivalence are outlined, such as:

- *concurrency*: the equivalence should fully capture causality relations and concurrency (and the interplay between causality and branching time);
- being a *branching time* equivalence: the equivalence should take into account at which point the choice between two executions is made, and not only consider the set of possible executions of a process. This allows to capture phenomena like deadlock behaviour;
- preserving *inevitability*: if two nets are equivalent, and in one the occurrence of a certain action is inevitable, then so is it in the other;
- being *preserved under action refinement*: if in two equivalent nets the same substitutions of nets for actions are made, the resulting nets are still equivalent;
- being *real-time consistent*: for every association of execution times to actions, assuming that actions happen as soon as they can, the running times associated with computations in equivalent systems are the same.

There are many equivalences (e.g., trace equivalence, interleaving bisimilarity, partial order trace equivalence) which do not hold a relevant number of these properties [Gla15]. In the first part of this work, we will focus on equivalences that hold most of the five: causal-net bisimilarity and fully-concurrent bisimilarity.

Both equivalences are based on the notion of *process* of a net, which is essentially a way to model a run of the net using a new, conflict-free, acyclic net (called *causal net*) together with a mapping function to the original one. Given a net, a process can be obtained by unrolling it, choosing one of the alternatives in case of conflict: as a matter of fact, conflicts in the original are represented by the existence of multiple processes, each of which models a different run. Note that this notion of process differs from the one used in process algebra, where a "process" refers to the entire behaviour of a system, including all its choices. The acyclic nature of the process net gives rise to a preorder on its transitions (called *events*).

Causal-net bisimilarity, proposed in [Gor20a], coincides with structure-preserving bisimilarity [Gla15] and holds all of the five former properties. Intuitively, two marked nets are causal-net equivalent if their processes share the same causal net. Moreover, nets that differ only in their unreachable parts are nonetheless considered causal-net equivalent.

Fully-concurrent bisimilarity was defined in [Bes+91], and it is an adaptation to P/T nets of history-preserving bisimilarity, defined in [GG89] over event structure [Win87], but first proposed in [RT88] under the name of behavior-structure bisimilarity, and independently defined also by [DDM89] (who named it mixed ordering bisimilarity). The definition of fully-concurrent bisimilarity we actually use is that in [Gor20a], which is a slight adaptation of the original one in [Bes+91]. It holds most of the properties stated above (including, *ça va sans dire*, being truly concurrent), but does not preserve inevitability (see [Gla15]). However, it has been longer studied (e.g. [Vog91; JM96]) and is slightly coarser than causal-net bisimilarity, taking into account only the ordering of events and not the whole causal net. As a matter of fact, decidability of fully-concurrent bisimilarity on bounded nets was proved by Montanari and Pistore in [MP97]; however we use a different approach, defined directly on the net, and argue that this leads to a better complexity in the general case, thus making it an interesting addition to this work.

For the decidability of fully-concurrent and causal-net bisimilarity on finite safe nets, we follow the approach used by Vogler in [Vog91] and [Vog95], i.e. defining a preorder that keeps information on the most recently generated tokens. Then the same proof technique is generalized to bounded nets by providing a refined semantics based on the individual, rather than collective, token interpretation [Gla05]. This refined semantics is, to the best of our knowledge, new. A similar idea was followed in [Val93]; however, some significant details of that work are wrong, first and foremost the individual treatment of tokens.

On the other hand, also PTI nets are a well-known (see, e.g., [Bus02; JK95; Pet81]) distributed model of computation, and moreover a Turing-complete one [Age74]. They have been largely exploited, e.g., for modeling systems with priorities [Hac76], for performance evaluation of distributed systems [Mar+98] and to provide π-calculus [MPW92; SW01] with a net semantics [BG09].

As finite PTI nets constitute a Turing-complete model of computation, essentially all the properties of interest are undecidable, notably the reachability problem, and so even termination: it is undecidable whether a deadlock marking is reachable from the initial one. Also interleaving bisimulation equivalence is undecidable for finite PTI nets, as it is already undecidable [Jan95] on the subclass of finite P/T nets [Rei85]. Similarly, one can prove that also well-known truly-concurrent behavioral equivalences, such as *fully-concurrent* bisimilarity [Bes+91], are undecidable [Esp98] for finite PTI nets. Despite this, in the second part of this work, we show that it is possible to define a *sensible*, behavioral equivalence which is actually *decidable* on finite PTI nets. This equivalence, we call *pti-place bisimilarity*, is a conservative extension of *place bisimilarity on finite P/T nets*, introduced in [ABS91] as an improvement of *strong bisimulation* [Old91], and recently proved decidable in [Gor21].

Place bisimilarity is an equivalence over markings, based on relations over the *finite set of net places*, rather than over the (possibly infinite) set of net markings. This equivalence is very natural and intuitive: as a place can be interpreted as a sequential process type (and each token in this place as an instance of a sequential process of that type), a place bisimulation states which kinds of sequential processes (composing the distributed system represented by the finite P/T net) are to be considered as equivalent. Moreover, this equivalence does respect the expected causal behavior of P/T nets, as it is slightly finer than causal-net bisimilarity and fully-concurrent bisimilarity.

We extend this idea in order to be applicable to PTI nets. Informally, a binary relation R over the set S of places is a *pti-place bisimulation* if for all markings m_1 and m_2 which are *bijectively* related via R (denoted by $(m_1, m_2) \in R^{\oplus}$, where $R^{\oplus}$ is called the *additive closure* of R), if m_1 can perform transition t_1, reaching marking m_1', then m_2 can perform a transition t_2, reaching m_2', such that

- the pre-sets of t_1 and t_2 are related by $R^{\oplus}$, the label of t_1 and t_2 is the same, the post-sets of t_1 and t_2 are related by $R^{\oplus}$, and also $(m_1', m_2') \in R^{\oplus}$, as required by a place bisimulation [ABS91; Gor21], but additionally it is required that
- the inhibiting sets of t_1 and t_2 are related by $R^{\oplus}$, and, finally, that whenever $(s, s') \in R$, s belongs to the inhibiting set of t_1 if and only if s' belongs to the inhibiting set of t_2;

and symmetrically if m_2 moves first. Two markings m_1 and m_2 are pti-place bisimilar, denoted by $m_1 \sim_p m_2$, if a pti-place bisimulation R exists such that $(m_1, m_2) \in R^{\oplus}$.

We prove that pti-place bisimilarity is an equivalence, but it is not coinductive as the union of pti-place bisimulations may be not a pti-place bisimulation; so, in general, there is not the largest pti-place bisimulation, rather many maximal pti-place bisimulations. In fact, pti-place bisimilarity is the relation on markings given by the union of the additive closure of each maximal pti-place bisimulation. We also prove that $\sim_p$ is sensible, as it respects the causal semantics of PTI nets. As a matter of fact, following the approach in [BP99; BP00], we define a process-oriented, bisimulation-based, behavioral semantics for PTI nets, called *causal-net bisimilarity*, and we prove that this is slightly coarser than pti-place bisimilarity.

We also show that $\sim_p$ is decidable for finite PTI nets. As a place relation $R \subseteq S \times S$ is finite if the set S of places is finite, there are finitely many place relations for a finite net. We can list all these place relations, say $R_1, R_2, \ldots R_n$. It is possible to decide whether R_i is a pti-place bisimulation by checking two *finite* conditions over a *finite* number of marking pairs: this is a non-obvious observation, as a pti-place bisimulation requires that the pti-place bisimulation game holds for the infinitely many pairs (m_1, m_2) belonging to $R_i^{\oplus}$. Hence, to decide whether $m_1 \sim_p m_2$, it is enough to check, for $i = 1, \ldots n$, whether R_i is a pti-place bisimulation and, in such a case, whether $(m_1, m_2) \in R_i^{\oplus}$.

The work is organized as follows: in Chapter 2, some basic definitions about P/T nets are recalled, together with some from the causality-based sematics known in literature; in Chapter 3, we provide an ordered marking semantics for safe nets and relate it to the causal one, leading to some decidability results for safe nets (this chapter is only partially original: we adapt Vogler's work [Vog91] and provide a new result for causal-net bisimilarity); in Chapter 4, we refine the classic and ordered marking semantics by means of indexing, to show that results of Chapter 3 can be generalized to the class of bounded nets; in Chapter 5, we recall basic definitions about PTI nets and introduce a causality-based semantics and PTI causal-net bisimilarity; in Chapter 6, we introduce pti-place bisimilarity and prove that it is finer than PTI causal-net bisimilarity; in Chapter 7 we prove the decidability of pti-place bisimilarity; finally, in Chapter 8 we discuss related works and some possible further developments.

Part I

P/T Nets

Time runs out for any finite observer.
There are no closed systems.
Even I only stretch the finite matrix.

LETO II, THE GOD EMPEROR

Chapter 2

Basic Definitions

2.1 Petri nets

Definition 2.1.1. (Multiset) *Let $\mathbb{N}$ be the set of natural numbers. Given a finite set S, a* multiset *over S is a function $m : S \to \mathbb{N}$. Its* support *set $dom(m)$ is $\{s \in S \mid m(s) \neq 0\}$. The set $\mathscr{M}(S)$ of all multisets over S is ranged over by m. We write $s \in m$ if $m(s) > 0$. The* multiplicity *of s in m is the number $m(s)$. The* size *of m, denoted by $|m|$, is the number $\sum_{s \in S} m(s)$, i.e., the total number of its elements. A multiset m such that $dom(m) = \emptyset$ is called* empty *and is denoted by θ. We write $m \subseteq m'$ if $m(s) \leq m'(s)$ for all $s \in S$.*

Multiset union $\_ \oplus \_$ *is defined as follows: $(m \oplus m')(s) = m(s) + m'(s)$; it is commutative, associative and has θ as neutral element.* Multiset difference $\_ \ominus \_$ *is defined as follows: $(m_1 \ominus m_2)(s) = max\{m_1(s) - m_2(s), 0\}$. The* scalar product *of a number j with m is the multiset $j \cdot m$ defined as $(j \cdot m)(s) = j \cdot (m(s))$. By s_i we also denote the multiset with s_i as its only element. Hence, a multiset m over $S = \{s_1, \ldots, s_n\}$ can be represented as $k_1 \cdot s_1 \oplus k_2 \cdot s_2 \oplus \ldots \oplus k_n \cdot s_n$, where $k_j = m(s_j) \geq 0$ for $j = 1, \ldots, n$.* □

Definition 2.1.2. (Finite P/T net) *A* labeled finite P/T net *is a tuple $N = (S, A, T)$, where*

- *S is the finite set of* places*, ranged over by s (possibly indexed),*
- *A is the finite set of* labels*, ranged over by ℓ (possibly indexed), and*
- *$T \subseteq (\mathscr{M}(S) \setminus \{\emptyset\}) \times A \times \mathscr{M}(S)$ is the finite set of* transitions*, ranged over by t (possibly indexed).*

The size *of a net is the total number of its places and transitions.*
Given a transition $t = (m, \ell, m')$ we use the notation:

- *$^\bullet t$ to denote its* pre-set *m (which is a nonempty multiset) of tokens to be consumed;*
- *$l(t)$ for its* label *ℓ, and*
- *$t^\bullet$ to denote its* post-set *m' (which is a multiset) of tokens to be produced.*

Hence, transition t can be also represented as $^\bullet t \xrightarrow{l(t)} t^\bullet$. We also define pre-sets and post-sets for places as follows: $^\bullet s = \{t \in T \mid s \in t^\bullet\}$ and $s^\bullet = \{t \in T \mid s \in {}^\bullet t\}$. Note that the pre-set (post-set) of a place is a set. □

***Remark* 1.** *The definition of T as a set of triples ensures that the net is transition simple, i.e., for any $t_1, t_2, \in T$, if $^\bullet t_1 = {}^\bullet t_2$ and $t_1^\bullet = t_2^\bullet$ and $l(t_1) = l(t_2)$, then $t_1 = t_2$. Note also that we are assuming that each transition has a nonempty pre-set: in our interpretation of net models a transition can only be performed by some sequential processes. For the same reason we assume that a transition can have an empty post-set: some sequential process might terminate its execution successfully, modelled as the elimination of the token.*

Definition 2.1.3. (Marking, net system) *A multiset over S is called a* marking*. Given a marking m and a place s, we say that the place s contains $m(s)$* tokens*, graphically represented by $m(s)$ bullets inside place s. A* net system *$N(m_0)$ is a tuple (S, A, T, m_0), where (S, A, T) is a net and m_0 is a marking over S, called the* initial marking*. We also say that $N(m_0)$ is a* marked *net.* □

In the graphical description of finite P/T nets, places (represented as circles) and transitions (represented as boxes) are connected by directed arcs. The arcs may be labeled with the natural number representing the number of tokens of that type that are to be removed from (or produced into) that place; no label on the arc is interpreted as the number one, i.e., one token flowing on the arc. This numerical label of the arc is called its *weight*.

Static properties of a net can be computed by just considering its definition (e.g. its *size*, defined as $|S|+|T|$), whereas dynamic properties, such as reachability, can be determined only when a net is marked.

The sequential semantics of a marked net is defined by the so-called *token game* representing the flow of tokens through it. There are several possible variants and interpretations (see [Gla05]); in the following we present the collective interpretation i.e., tokens are not treated as individual units. According to this interpretation, multiple tokens on the same place are indistinguishible. Moreover, we use an interleaving semantics: only a single transition at a time can fire.

Definition 2.1.4. (Token game) *A transition t is* enabled *at m, denoted $m[t\rangle$, if ${}^\bullet t \subseteq m$. The execution, or* firing, *of t enabled at m produces the marking $m' = (m \ominus {}^\bullet t) \oplus t^\bullet$, written $m[t\rangle m'$.* □

Definition 2.1.5. (Firing sequence, reachable marking) *A* firing sequence *starting at m is defined inductively as follows:*

- *$m[\varepsilon\rangle m$ is a firing sequence (where ε denotes an empty sequence of transitions) and*
- *if $m[\sigma\rangle m'$ is a firing sequence and $m'[t\rangle m''$, then $m[\sigma t\rangle m''$ is a firing sequence.*

If $\sigma = t_1 \dots t_n$ (for $n \geq 0$) and $m[\sigma\rangle m'$ is a firing sequence, then there exist $m_1, \dots, m_{n+1}$ such that $m = m_1[t_1\rangle m_2[t_2\rangle \dots$ $\dots m_n[t_n\rangle m_{n+1} = m'$, and $\sigma = t_1 \dots t_n$ is called a transition sequence *starting at m and ending at m'. The set of* reachable markings *from m is $[m\rangle = \{m' \mid \exists \sigma . m[\sigma\rangle m'\}$. Note that the set of reachable markings may be countably infinite for finite P/T nets.* □

Definition 2.1.6. (Classes of finite P/T Nets) *A* finite marked *P/T net $N = (S, A, T, m_0)$ is:*

- safe *if every place contains at most one token under every reachable marking, i.e. $\forall s \in S$, $m(s) \leq 1$ for all $m \in [m_0\rangle$.*
- bounded *if the number of token in any place is bounded by some k for any reachable marking, i.e. $\exists k \in \mathbb{N}, \forall s \in S$ such that $m(s) \leq k$ for all $m \in [m_0\rangle$. If this is the case, we say that the net is k-bounded.*

Note that a safe net is just a 1-bounded net. Thus, rather than a multiset, a marking of a safe net can be written a set*, namely the set of places containing a token.* □

In [Gor20a] is defined a place-counting multiset lifting of a relation on places, called *additive closure*, and some of its properties, which will be used in the next chapters.

Definition 2.1.7. (Additive closure) *Given a net $N = (S, A, T)$ and a* place relation *$R \subseteq S \times S$, we define a* marking relation *$R^\oplus \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$, called the* additive closure *of R, as the least relation induced by the following axiom and rule.*

$$\frac{}{(\theta, \theta) \in R^\oplus}\ \textbf{(Emp)} \qquad \frac{(s_1, s_2) \in R \quad (m_1, m_2) \in R^\oplus}{(s_1 \oplus m_1, s_2 \oplus m_2) \in R^\oplus}\ \textbf{(Clo)}$$

□

***Remark* 2.** *Note that:*

- *two markings are related by $R^\oplus$ only if they have the same size;*
- *R is an equivalence relation, then its additive closure $R^\oplus$ is also an equivalence relation;*
- *if $R_1 \subseteq R_2$, then $R_1^\oplus \subseteq R_2^\oplus$, i.e., the additive closure is monotonic.*

An alternative way to define that two markings m_1 and m_2 are related by $R^\oplus$ is to state that m_1 can be represented as $s_1 \oplus s_2 \oplus \dots \oplus s_k$, m_2 can be represented as $s'_1 \oplus s'_2 \oplus \dots \oplus s'_k$ and $(s_i, s'_i) \in R$ for $i = 1, \dots, k$.

2.2 Causality-based semantics

We outline some definitions, adapting them from literature (cf. e.g. [GR83; BD87; Vog91; Gor20a]).

Definition 2.2.1. (Acyclic net) *A net $N = (S, A, T)$ is* acyclic *if there exists no sequence $x_1 x_2 \dots x_n$ such that $n \geq 3$, $x_i \in S \cup T$ for $i = 1, \dots, n$, $x_1 = x_n$, $x_1 \in S$ and $x_i \in {}^\bullet x_{i+1}$ for $i = 1, \dots, n-1$, i.e., the arcs of the net do not form any cycle.* □

The concurrent semantics of a transition sequence in a marked net is defined by a class of particular safe nets where places are not branched (therefore they are essentially deterministic) and all arcs have weight 1. This type of net is called *causal net* [BD87; Old91; Gla15; Gor20a]. We use B to denote its places (called *conditions*), E to denote its transitions (called *events*), and L to denote its labels.

Definition 2.2.2. (Causal net) *A causal net is a finite marked net $C(\mathsf{m}_0) = (B, L, E, \mathsf{m}_0)$ satisfying the following conditions:*

1. *C is acyclic;*
2. *$\forall b \in B \ \ |{}^\bullet b| \leq 1 \wedge |b^\bullet| \leq 1$ (i.e., the places are not branched);*
3. *$\forall b \in B \ \ \mathsf{m}_0(b) = \begin{cases} 1 & \text{if } {}^\bullet b = \emptyset \\ 0 & \text{otherwise;} \end{cases}$*
4. *$\forall e \in E \ \ {}^\bullet e(b) \leq 1 \wedge e^\bullet(b) \leq 1$ for all $b \in B$ (i.e., all the arcs have weight 1).*

We denote by $Min(C)$ the set m_0, and by $Max(C)$ the set $\{b \in B \mid b^\bullet = \emptyset\}$.
A sequence of events $\sigma \in E^*$ is maximal *(or* complete*) if it contains all events in E, each taken once only.* □

Note that a causal net is finite; since it is acyclic, it represents a finite computation.

Note also that any reachable marking of a causal net is a set, i.e., this net is *safe*; in fact, the initial marking is a set and, assuming by induction that a reachable marking m is a set and enables e, i.e., $\mathsf{m}[e\rangle\mathsf{m}'$, then also $\mathsf{m}' = (\mathsf{m} \ominus {}^\bullet e) \oplus e^\bullet$ is a set, because the net is acyclic and because of the condition on the shape of the post-set of e (weights can only be 1).

As the initial marking of a causal net is fixed by its shape (according to item 3 of Definition 2.2.2), in the following, in order to make the notation lighter, we often omit the indication of the initial marking (also in their graphical representation), so that the causal net $C(\mathsf{m}_0)$ is denoted by C.

Definition 2.2.3. (Moves of a causal net) *Given two causal nets $C = (B, L, E, \mathsf{m}_0)$ and $C' = (B', L, E', \mathsf{m}_0)$, we say that C moves in one step to C' through e, denoted by $C[e\rangle C'$, if ${}^\bullet e \subseteq Max(C)$, $E' = E \cup \{e\}$ and $B' = B \cup e^\bullet$; in other words, C' extends C by one event e.* □

Definition 2.2.4. (Folding and Process) *A* folding *from a causal net $C = (B, L, E, \mathsf{m}_0)$ into a net system $N(m_0) = (S, A, T, m_0)$ is a function $\rho : B \cup E \to S \cup T$, which is type-preserving, i.e., such that $\rho(B) \subseteq S$ and $\rho(E) \subseteq T$, satisfying the following:*

- *$L = A$ and $\mathsf{l}(e) = l(\rho(e))$ for all $e \in E$;*
- *$\rho(\mathsf{m}_0) = m_0$, i.e., $m_0(s) = |\rho^{-1}(s) \cap \mathsf{m}_0|$;*
- *$\forall e \in E, \rho({}^\bullet e) = {}^\bullet \rho(e)$, i.e., $\rho({}^\bullet e)(s) = |\rho^{-1}(s) \cap {}^\bullet e|$ for all $s \in S$;*
- *$\forall e \in E, \rho(e^\bullet) = \rho(e)^\bullet$, i.e., $\rho(e^\bullet)(s) = |\rho^{-1}(s) \cap e^\bullet|$ for all $s \in S$.*

A pair (C, ρ), where C is a causal net and ρ a folding from C to a net system $N(m_0)$, is a process *of $N(m_0)$, written also as π.* □

Definition 2.2.5. (Partial orders of events from a process) *From a causal net $C = (B, L, E)$, we can extract the* partial order of its events *$\mathsf{E}_C = (E, \preceq)$, where $e_1 \preceq e_2$ iff there exists a sequence $x_1 x_2 x_3 \ldots x_n$ such that $n \geq 3$, $x_i \in B \cup E$ for $i = 1, \ldots, n$, $e_1 = x_1, e_2 = x_n$, and $x_i \in {}^\bullet x_{i+1}$ for $i = 1, \ldots, n-1$; in other words, $e_1 \preceq e_2$ if there is a path from e_1 to e_2. Given a process $\pi = (C, \rho)$, we denote $\preceq$ as $\leq_\pi$, i.e. given $e_1, e_2 \in E$, $e_1 \leq_\pi e_2$ if and only if $e_1 \preceq e_2$.* □

Definition 2.2.6. (Moves of a process) *Let $N(m_0) = (S, A, T, m_0)$ be a net system and let (C_i, ρ_i), for $i = 1, 2$, be two processes of $N(m_0)$. We say that (C_1, ρ_1) moves in one step to (C_2, ρ_2) through e, denoted by $(C_1, \rho_1) \xrightarrow{e} (C_2, \rho_2)$, if $C_1[e\rangle C_2$ and $\rho_1 \subseteq \rho_2$.*
If $\pi_1 = (C_1, \rho_1)$ and $\pi_2 = (C_2, \rho_2)$, we denote the move as $\pi_1 \xrightarrow{e} \pi_2$. □

Proposition 2.2.7. *Assume $\pi = (C, \rho)$ a process of N such that $\pi \xrightarrow{e} \pi' = (C', \rho')$. Then $\forall b \in Max(C), \exists b'' \in Max(C), \forall b' \in Max(C')$, where $b' \in e^\bullet$ and $b'' \in {}^\bullet e$, it is true that if ${}^\bullet b \leq_{\pi'} {}^\bullet b'$, then ${}^\bullet b \leq_\pi {}^\bullet b''$.*

Proof. By Definition 2.2.5, ${}^\bullet b \leq_{\pi'} {}^\bullet b'$ means that there exists a path in C' starting from ${}^\bullet b$ and ending at ${}^\bullet b'$. Let us choose b'' to be the condition immediately before ${}^\bullet b'$ in that path. It follows that there exists a path in C starting from ${}^\bullet b$ and ending at ${}^\bullet b''$: then, by Definition 2.2.5, we get the thesis. □

Note that, given a complete transition sequence of C' containing ${}^\bullet b$ and ${}^\bullet b'$, then also ${}^\bullet b''$ is contained in the sequence and the relative position of these three events respects the preorder $\leq_{\pi'}$.

2.2.1 Causal-net bisimilarity

Causal-net bisimilarity [Gor20a; Gla15] is defined in terms of processes of a net. Intuitively, two markings of a net are causal-net bisimilar if two processes based on the same associated causal net are bisimilar.

Definition 2.2.8. (Causal-net bisimulation) *Let $N = (S,A,T)$ be a finite P/T net. A* causal-net bisimulation *is a relation R, composed of triples of the form (ρ_1, C, ρ_2), where, for $i = 1,2$, (C,ρ_i) is a process of $N(m_{0_i})$ for some m_{0_i}, such that if $(\rho_1, C, \rho_2) \in R$ then*

i) $\forall t_1, C', \rho_1'$ such that $(C,\rho_1) \xrightarrow{e} (C',\rho_1')$, where $\rho_1'(e) = t_1$, $\exists t_2, \rho_2'$ such that $(C,\rho_2) \xrightarrow{e} (C',\rho_2')$, where $\rho_2'(e) = t_2$, and $(\rho_1', C', \rho_2') \in R$;

ii) symmetrically, $\forall t_2, C', \rho_2'$ such that $(C,\rho_2) \xrightarrow{e} (C',\rho_2')$, where $\rho_2'(e) = t_2$, $\exists t_1, \rho_1'$ such that $(C,\rho_1) \xrightarrow{e} (C',\rho_1')$, where $\rho_1'(e) = t_1$, and $(\rho_1', C', \rho_2') \in R$.

Two markings m_1 and m_2 of N are cn-bisimilar (or cn-bisimulation equivalent), denoted by $m_1 \sim_{cn} m_2$, if there exists a causal-net bisimulation R containing a triple $(\rho_1^0, C^0, \rho_2^0)$, where C^0 contains no events and $\rho_i^0(Min(C^0)) = \rho_i^0(Max(C^0)) = m_i$ for $i = 1,2$. □

In [Gor20a] it is proved that causal-net bisimulation has the following properties, shared also by other good equivalences. Let us denote by $\sim_R^{cn} = \{(m_1,m_2) \mid m_1$ is cn-bisimilar to m_2 thanks to $R\}$:

- cn-bisimilarity $\sim_{cn}$ can be seen as $\bigcup\{\sim_R^{cn} \mid$ R is a causal-net bisimulation$\} = \sim_{\mathscr{R}}^{cn}$, where $\mathscr{R} = \bigcup\{R \mid R$ is a causal-net bisimulation$\}$ is the largest causal-net bisimulation.
- the identity relation is a causal-net bisimulation;
- the inverse relation of a causal-net bisimulation is a causal-net bisimulation;
- the relational composition, up to net isomorphism, of two causal-net bisimulations is a causal-net bisimulation;
- the union of causal-net bisimulations is a causal-net bisimulation.

The following lemma shows a property of causal-net bisimulation that will be used in the next chapters.

Lemma 2.2.9. *Given two markings m_1 and m_2 of N, if $m_1 \sim_{cn} m_2$ then $|m_1| = |m_2|$.*

Proof. If $m_1 \sim_{cn} m_2$, then there exists a causal-net bisimulation R such that $(\rho_1^0, C^0, \rho_2^0) \in R$, where where C^0 contains no events and $\rho_i^0(Min(C^0)) = \rho_i^0(Max(C^0)) = m_i$ for $i = 1,2$. Then, since ρ_i^0 maps conditions to places, $|m_1| = |m_2|$. □

2.2.2 Fully-concurrent bisimilarity

Fully-concurrent bisimilarity, also known as *history-preserving* bisimilarity, was first introduced in [RT88], adapted to Petri Nets in [Bes+91] and independently defined in [DDM89]. It is similar to causal-net bisimilarity but slightly coarser: two nets are said to be fully-concurrent bisimilar if their causal nets have isomorphic event structures. In the following, we outline the definition given in [Gor20a].

Definition 2.2.10. (Fully-concurrent bisimilarity) *Given a finite P/T net $N = (S,A,T)$, a* fully-concurrent bisimulation *is a relation R, composed of triples of the form (π_1, f, π_2) where, for $i = 1,2$, $\pi_i = (C_i, \rho_i)$ is a process of $N(m_{0i})$ for some m_{0i} and f is an isomorphism between E_{C_1} and E_{C_2}, such that if $(\pi_1, f, \pi_2) \in R$ then:*

i) $\forall t_1, \pi_1'$ such that $\pi_1 \xrightarrow{e_1} \pi_1'$, where $\rho_1'(e_1) = t_1$, there exist e_2, t_2, π_2', f' such that

1. *$\pi_2 \xrightarrow{e_2} \pi_2'$ where $\rho_2'(e_2) = t_2$,*
2. *$f' = f \cup \{e_1 \mapsto e_2\}$,*
3. *$(\pi_1', f', \pi_2') \in R$;*

ii) symmetrically, if π_2 moves first.

Two markings m_1, m_2 of N are fc-bisimilar (or fc-bisimulation equivalent), denoted by $m_1 \sim_{fc} m_2$ if a fully-concurrent bisimulation R exists, containing a triple $(\pi_1^0, \emptyset, \pi_2^0)$ where $\pi_i^0 = (C_i^0, \rho_i^0)$ such that C_i^0 contains no events and $\rho_i^0(Min(C_i^0)) = \rho_i^0(Max(C_i^0)) = m_i$ for $i = 1,2$. □

In [Gor20a] it is proved that also fully-concurrent bisimilarity has the following properties. Let us denote by $\sim_R^{fc} = \{(m_1, m_2) \mid m_1 \text{ is fc-bisimilar to } m_2 \text{ thanks to } R\}$:

- fc-bisimilarity $\sim_{fc}$ can be seen as $\bigcup\{\sim_R^{fc} \mid \text{R is a fully-concurrent bisimulation}\} = \sim_{\mathscr{R}}^{fc}$, where $\mathscr{R} = \bigcup\{R \mid R \text{ is a fully-concurrent bisimulation}\}$ is the largest fully-concurrent bisimulation.
- the identity relation is a fully-concurrent bisimulation;
- the inverse relation of a fully-concurrentbisimulation is a fully-concurrent bisimulation;
- the relational composition, up to net isomorphism, of two fully-concurrent bisimulations is a fully-concurrent bisimulation;
- the union of fully-concurrent bisimulations is a fully-concurrent bisimulation.

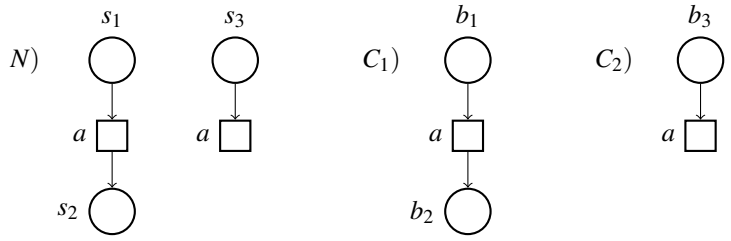

Figure 2.1: A finite P/T net N and two causal nets: C_1 corresponds to the maximal process of $N(s_1)$ and C_2 corresponds to the maximal process of $N(s_3)$.

Example 1. In Figure 2.1 a simple finite P/T net N is given. It is easy to see that C_1 (resp. C_2) corresponds to a process π_1 (resp. π_2) of $N(s_1)$ (resp. $N(s_3)$), where ρ_1 (resp. ρ_2) maps each condition net to place net having same subscript and each event to the transition having same label.
Consider places s_1 and s_3: we have $s_1 \sim_{fc} s_3$ and this is proved by relation

$$R = \{(((b_1, \{a\}, \emptyset, b_1), b_1 \mapsto s_1), \emptyset, ((b_3, \{a\}, \emptyset, b_3), b_3 \mapsto s_3))), (\pi_1, e_{a_1} \mapsto e_{a_2}, \pi_2)\}.$$

Indeed, $((b_1, \{a\}, \emptyset, b_1), b_1 \mapsto s_1)$ is a process of $N(s_1)$ and $((b_3, \{a\}, \emptyset, b_3), b_3 \mapsto s_3)$ is a process of $N(s_3)$, as both processes contain no events and are such that minimal and maximal conditions are the same and mapped on the right initial markings. If $((b_1, \{a\}, \emptyset, b_1), b_1 \mapsto s_1)$ moves first by $((b_1, \{a\}, \emptyset, b_1), b_1 \mapsto s_1) \xrightarrow{e_{a_1}} \pi_1$, $((b_3, \{a\}, \emptyset, b_3), b_3 \mapsto s_3)$ can respond with $((b_3, \{a\}, \emptyset, b_3), b_3 \mapsto s_3) \xrightarrow{e_{a_2}} \pi_2$, and $(\pi_1, e_{a_1} \mapsto e_{a_2}, \pi_2) \in R$. The case where $((b_3, \{a\}, \emptyset, b_3), b_3 \mapsto s_3)$ moves first is symmetrical.
However, it is not true that $s_1 \sim_{cn} s_3$, because C_1 and C_2 are not isomorphic and therefore it is not possible to build a causal-net bisimulation. This example shows how causal-net bisimilarity is finer than fully-concurrent bisimilarity. □

Chapter 3

Decidability Results for Finite Safe Petri Nets

In this chapter, in order to prove that some truly concurrent bisimilarities are decidable for safe nets, we adapt a proof technique developed by Vogler in [Vog91] that he used to prove that history-preserving bisimilarity ([RT88; GG89; DDM89]) is decidable for safe nets. This technique is based on the concept of ordered marking, i.e., a safe marking (hence a set) equipped with a preorder on its elements (i.e., places) that reflects the precedence in the generation of the tokens. We define some bisimulations on ordered markings which are clearly decidable, and prove that they coincide with the corresponding truly concurrent bisimulation.

3.1 Ordered marking semantics

We outline some definitions from [Vog91].

Definition 3.1.1. (Ordered marking) *Given a safe net $N = (S,A,T,m_0)$, a safe ordered marking is a pair $(m,\leq)$ such that $m \subseteq S$ and $\leq \subseteq m \times m$ is a preorder, i.e. a reflexive and transitive relation. The set of all safe ordered markings of N is denoted by $OM(N)$. If $N(m_0)$ is a safe net, we define the* initial ordered marking *$init(N)$ as $(m_0, m_0 \times m_0)$. If the initial marking is not clear from the context, we write $init(N(m_0))$ instead, to denote the initial ordered marking.* □

Definition 3.1.2. (Token game with OM) *Let $N = (S,A,T,m_0)$ be a safe net, let m be a reachable safe marking and $t \in T$ a transition enabled at m. Given the ordered marking $(m,\leq)$ we say that t is* enabled *at $(m,\leq)$, denoted $(m,\leq)[t\rangle$. The firing of t produces the ordered marking $(m',\leq')$ where $m' = m \ominus {}^\bullet t \oplus t^\bullet$ and $\leq'$ is defined by:*
for all $s,s' \in m'$, we have $s \leq' s'$ if and only if:

1. *$s,s' \in m \ominus {}^\bullet t$ and $s \leq s'$, or*
2. *$s,s' \in t^\bullet$, or*
3. *$s \in m \ominus {}^\bullet t$, $s' \in t^\bullet$ and there exists $s'' \in {}^\bullet t$ with $s \leq s''$.*

This is denoted as $(m,\leq)[t\rangle(m',\leq')$. □

Definition 3.1.3. (Firing sequence with OM) *Given a safe net $N(m_0)$ and a reachable marking m, a firing sequence starting at $(m,\leq)$ is defined inductively as follows:*

- *$(m,\leq)[\varepsilon\rangle(m,\leq)$ is a firing sequence (where ε denotes an empty sequence of transitions) and*
- *if $(m,\leq)[\sigma\rangle(m',\leq')$ is a firing sequence and $(m',\leq')[t\rangle(m'',\leq'')$, then $(m,\leq)[\sigma t\rangle(m'',\leq'')$ is a firing sequence.*

If $\sigma = t_1 \ldots t_n$ (for $n \geq 0$) and $(m,\leq)[\sigma\rangle(m',\leq')$ is a firing sequence, then there exist $(m_1,\leq_1),\ldots,(m_{n+1},\leq_{n+1})$ such that $(m,\leq) = (m_1,\leq_1)[t_1\rangle(m_2,\leq_2)[t_2\rangle \ldots\ldots (m_n,\leq_n)[t_n\rangle(m_{n+1},\leq_{n+1}) = (m',\leq')$, and $\sigma = t_1 \ldots t_n$ is called a transition sequence *starting at $(m,\leq)$ and ending at $(m',\leq')$.*
The set of reachable ordered markings *from $(m,\leq)$ is $[(m,\leq)\rangle = \{(m',\leq') \mid \exists\sigma.(m,\leq)[\sigma\rangle(m',\leq')\}$. We denote by $[init(N)\rangle$ the set of all the ordered markings reachable from $(m_0, m_0 \times m_0)$.* □

Proposition 3.1.4. *Given a safe net $N = (S,A,T,m_0)$, the set $OM(N)$ and the set $[init(N)\rangle$ are finite.*

Proof. In finite safe nets, a marking is a subset of S, which is finite by definition. Therefore $\mathscr{P}(S)$ is finite i.e., the set of safe markings of N is finite. Since there is at most one token in every place $s \in S$, the set of possible preorders for a safe marking is finite, because $\leq \subseteq S \times S$. Therefore $OM(N)$ is finite.
Since $[init(N)\rangle \subseteq OM(N)$, also $[init(N)\rangle$ is finite. □

3.2 Ordered marking and causality-based semantics

Again inspired by [Vog91], in the following we offer a preliminary result to the decidability proof: the link between ordered markings and causal nets. It is possible to establish a partial order based on processes, and that partial order is coherent with the one generated from the token game semantics on ordered markings.

Lemma 3.2.1. (A minimality condition for $\leq$) *Given $\pi = (C,\rho)$ a process of $N(m_0)$ and σ a complete transition sequence of C (i.e., such that π is obtained by extending the causal net with a transition in σ at a time), and $init(N)[\rho(\sigma)\rangle(m,\leq)$. Then, $\forall b,b' \in Max(C)$, if $b \in Min(C)$ then $\rho(b)$ is minimal for $\leq$, i.e. $\forall \rho(b') \in \rho(Max(C))$, $\rho(b) \leq \rho(b')$.*

Proof. By induction on $|\sigma|$.

- Case 0: $\pi = (C^0,\rho^0)$, where C^0 contains no transitions and $\rho^0(Max(C^0) = \rho^0(Min(C^0)) = m_0$.
  Since $init(N)[(C^0,\rho^0)\rangle init(N)$, then $\leq = m_0 \times m_0$; therefore $\forall \rho^0(b') \in \rho^0(Max(C))$, $\rho^0(b) \leq \rho^0(b')$.
- Case $n+1$: The inductive hypotesis is $\pi = (C,\rho)$ and $init(N)[\pi\rangle(m,\leq)$, where the thesis holds. The inductive step is $\pi \xrightarrow{e} \pi' = (C',\rho')$ where $C[e\rangle C'$, $\rho'(e) = t$ and $(m,\leq)[t\rangle(m',\leq')$.

  By cases on $\rho'(b')$:

  - if $\rho'(b') \in m' \ominus {}^\bullet t$: then $\rho'(b') = \rho(b')$, the thesis follows from the inductive hypotesis on $(m,\leq)$.
  - if $\rho'(b') \in t^\bullet$: then, since transitions have nonempty pre-set, there exists $\rho'(b'') \in {}^\bullet t$, meaning $\rho'(b'') \leq' \rho'(b')$. Since $\rho'(b'') \in m$, it is true that $\rho'(b'') = \rho(b'')$. By inductive hypotesis on $(m,\leq)$, we have $\rho(b) \leq \rho(b'')$. By conservative extension of ρ' w.r.t ρ, $\rho'(b) \leq' \rho'(b'')$, and by transitivity $\rho'(b) \leq' \rho'(b')$.
  - if $\rho'(b') \in {}^\bullet t$: absurd, because $\rho'(b') \in \rho'(Max(C'))$ and $\pi \xrightarrow{e} \pi'$ with $\rho'(e) = t$. □

Proposition 3.2.2. *Given a net $N = (S,A,T)$ such that $N(m_{01})$ and $N(m_{02})$ are both safe nets, two processes $\pi_i = (C,\rho_i)$ of $N(m_{0i})$ for $i = 1,2$, a complete transition sequence σ of C and two markings m_1,m_2 such that $init(N(m_{0i}))[\rho_i(\sigma)\rangle(m_i,\leq_i)$ for $i = 1,2$, we have that $|m_1| = |m_2|$.*

Proof. By induction on the length of σ.

- Case 0: $\sigma = \varepsilon$.
  The two processes are $\pi_i^0 = (C^0,\rho_i^0)$ where C^0 contains no events and $\rho_i^0(Min(C^0)) = \rho_i^0(Max(C^0)) = m_i$ for $i = 1,2$. Then, since ρ_i^0 maps conditions to places, $|m_{01}| = |m_{02}|$.
- Case n+1 : $\sigma = \delta\, e$.
  The inductive hypotesis is $init(N(m_{0i}))[\rho_i(\sigma))\rangle(m_i,\leq_i)$, such that $|m_1| = |m_2|$, for $i = 1,2$. The induction step is $(C,\rho_i) \xrightarrow{e} (C',\rho_i')$ for $i = 1,2$. Since each ρ_i' maps e to a transition t_i, $|{}^\bullet t_1| = |{}^\bullet e| = |{}^\bullet t_2|$ and $|t_1^\bullet| = |e^\bullet| = |t_2^\bullet|$. By inductive hypotesis, $|m_1| = |m_2|$; since $|m_1'| = |m_1| - |{}^\bullet t_1| + |t_1^\bullet|$ and $|m_2'| = |m_2| - |{}^\bullet t_2| + |t_2^\bullet|$, by transitivity we get $|m_1'| = |m_2'|$. □

Definition 3.2.3. ($\leq$ from process) *Let $N(m_0)$ be a safe net. Consider $\pi = (C,\rho)$ a process of $N(m_0)$ and an ordered marking $(m,\leq)$. We write $init(N)[\pi\rangle(m,\leq)$ if ρ is a bijection from $Max(C)$ to m such that for all $b,b' \in Max(C)$:*

$$\rho(b) \leq \rho(b') \iff \begin{cases} b \in Min(C) & \text{(1), or} \\ {}^\bullet b \neq \emptyset \wedge {}^\bullet b' \neq \emptyset \wedge {}^\bullet b \leq_\pi {}^\bullet b' & \text{(2)} \end{cases}$$

□

The following lemma, a slight modification of Lemma 3.4 in [Vog91], shows the relation between the order derived from a process and the operational preorder.

Lemma 3.2.4. (Link between process and operational preorder) *Let $N(m_0)$ be a safe net and let $\pi = (C,\rho)$ be a process of $N(m_0)$ such that $init(N)[\pi\rangle(m,\leq)$. Then $(m,\leq)[t\rangle(m',\leq')$ if and only if $\pi \xrightarrow{e} \pi'$ where $\rho'(e) = t$ and $init(N)[\pi'\rangle(m',\leq')$.*

Proof $\Rightarrow$). By induction on the length of σ, complete transition sequence of C such that $init(N)[(\rho(\sigma)\rangle(m,\leq)$.

- Case 0 : $\sigma = \varepsilon$.
  Then $\pi = \pi^0 = (C^0,\rho^0)$ where C^0 contains no transitions and $\rho^0(Max(C^0) = \rho^0(Min(C^0)) = m_0$, and $init(N)[\pi^0\rangle init(N)$. Extend C^0 with $e = ({}^\bullet e, l(t), e^\bullet)$ such that $\rho'(e) = t$ and $C^0[e\rangle C'$. Then by definition $\pi^0 \xrightarrow{e} \pi'$.

  Now we prove that $\forall b,b' \in Max(C'), \rho'(b) \leq' \rho'(b') \implies (b \in Min(C')) \vee ({}^\bullet b \neq \emptyset \wedge {}^\bullet b' \neq \emptyset \wedge {}^\bullet b \leq_{\pi'} {}^\bullet b')$ by cases on the definition of $\rho'(b) \leq' \rho'(b')$.

  - if $\rho'(b),\rho'(b') \in m_0 \ominus {}^\bullet t$ and $\rho'(b) \leq \rho'(b')$:
    then since $m_0 = \rho^0(Min(C^0))$ and $\rho'(b) = \rho^0(b)$, by transitivity $\rho'(b) \in \rho'(Min(C'))$ and therefore $b \in Min(C')$, satisfying condition 1 of Definition 3.2.3.
  - if $\rho'(b),\rho'(b') \in t^\bullet$:
    then $\rho'(b),\rho'(b') \in \rho'(e)^\bullet$, thus $b,b' \in e^\bullet$; this means that ${}^\bullet b = {}^\bullet b' \neq \emptyset$ and, since they are generated from the same event e, $e = {}^\bullet b \leq_{\pi'} {}^\bullet b' = e$, satisfying condition 2 of Definition 3.2.3.
  - if $\rho'(b) \in m_0 \ominus {}^\bullet t$ and $\rho'(b') \in t^\bullet$ and $\exists \rho'(b'') \in {}^\bullet t$ such that $\rho'(b) \leq_{\pi^0} \rho'(b'')$:
    then since $m_0 = \rho^0(Min(C^0))$ and $\rho'(b) = \rho^0(b)$, by transitivity $\rho'(b) \in \rho'(Minc(C'))$ and therefore $b \in Min(C')$, satisfying condition 1 of Definition 3.2.3.

- Case n+1 : $\sigma = \delta e$.
  The inductive hypotesis is that there exist $\pi = (C,\rho)$ and $(m,\leq)$ such that the thesis holds. The induction step is $(m,\leq)[t\rangle(m',\leq')$.

  By definition, $C = (B,L,E)$ and $\rho(Max(C)) = m$. We extend C and ρ by an event $e = ({}^\bullet e, l(t), e^\bullet)$ such that $\rho'(e) = t$. Since $(m,\leq)[t\rangle, \rho(Max(C)) = m \supseteq \rho'({}^\bullet e) = {}^\bullet t$. Let $C[e\rangle C'$, since $\rho'(Max(C')) = \rho'(Max(C)) \ominus {}^\bullet \rho'(e) \oplus \rho'(e)^\bullet = \rho(Max(C)) \ominus \rho'({}^\bullet e) \oplus \rho'(e^\bullet) = m \ominus {}^\bullet t \oplus t^\bullet = m'$, then $\pi \xrightarrow{e} \pi'$.

  Now we prove that $\forall b,b' \in Max(C'), \rho'(b) \leq' \rho'(b') \implies (b \in Min(C')) \vee ({}^\bullet b \neq \emptyset \wedge {}^\bullet b' \neq \emptyset \wedge {}^\bullet b \leq_{\pi'} {}^\bullet b')$ by cases on the definition of $\rho'(b) \leq' \rho'(b')$.

  - if $\rho'(b),\rho'(b') \in m \ominus {}^\bullet t$ and $\rho'(b) \leq \rho'(b')$:
    then since $\rho'(b),\rho'(b') \in m \ominus {}^\bullet t$, this follows from the inductive hypotesis on $(m,\leq)$.
  - if $\rho'(b),\rho'(b') \in t^\bullet$:
    then $\rho'(b),\rho'(b') \in \rho'(e)^\bullet$, thus $b,b' \in e^\bullet$; this means that ${}^\bullet b = {}^\bullet b' \neq \emptyset$ and, since they are generated from the same event e, $e = {}^\bullet b \leq_{\pi'} {}^\bullet b' = e$, satisfying condition 2 of Definition 3.2.3.
  - if $\rho'(b) \in m \ominus {}^\bullet t$ and $\rho'(b') \in t^\bullet$ and $\exists \rho'(b'') \in {}^\bullet t$ such that $\rho'(b) \leq \rho'(b'')$:
    + if $b \in Min(C')$: trivially from condition 1 of Definition 3.2.3;
    + if $b \notin Min(C')$: then ${}^\bullet b \neq \emptyset$, and since $\rho'(b') \in t^\bullet$ then $b' \in e^\bullet$, meaning ${}^\bullet b' \neq \emptyset$. Also, $\rho'(b'') \in {}^\bullet t$ implies $b'' \in {}^\bullet e$. Since $b,b'' \in Max(C)$ and $\rho(b) \leq \rho(b'')$, then by inductive hypotesis ${}^\bullet b \leq_\pi {}^\bullet b''$. Then ${}^\bullet b'' \leq_{\pi'} {}^\bullet b'$ and by transitivity ${}^\bullet b \leq_{\pi'} {}^\bullet b'$, satisfying condition 2 of Definition 3.2.3.

  Therefore, $init(N)[\pi'\rangle(m',\leq')$.

Proof $\Leftarrow$). By induction on the length of σ, complete transition sequence of C such that $init(N)[\pi\rangle(m,\leq)$.

- Case 0 : $\sigma = \varepsilon$.
  Then $\pi = \pi^0 = (C^0,\rho^0)$ where C^0 contains no transitions and $\rho^0(Max(C^0) = \rho^0(Min(C^0)) = m_0$, and $init(N)[\pi^0\rangle init(N)$. Since $\pi^0 \xrightarrow{e} \pi'$ with $\rho'(e) = t$, then $m_0 \supseteq {}^\bullet t$, i.e. $m_0[t\rangle$. Since $\rho'(Max(C')) = \rho^0(Max(C^0)) \ominus {}^\bullet \rho'(e) \oplus \rho'(e)^\bullet = \rho^0(Max(C^0)) \ominus \rho'({}^\bullet e) \oplus \rho'(e^\bullet) = m_0 \ominus {}^\bullet t \oplus t^\bullet = m'$, then $m_0[t\rangle m'$.

  Now we prove that $\forall b,b' \in Max(C'), (b \in Min(C')) \vee ({}^\bullet b \neq \emptyset \wedge {}^\bullet b' \neq \emptyset \wedge {}^\bullet b \leq_{\pi'} {}^\bullet b') \implies \rho'(b) \leq' \rho'(b')$ by enumeration.

  - if $b \in Min(C')$:
    then, by Lemma 3.2.1, $\rho'(b)$ is minimal for $\leq$ and by transitivity it is minimal for $\leq'$. Therefore condition 1 of Definition 3.1.2 is satisfied.

- if ${}^\bullet b \neq \emptyset \wedge {}^\bullet b' \neq \emptyset \wedge {}^\bullet b \leq_{\pi'} {}^\bullet b'$:
  Then there are four possible combinations of $\rho'(b), \rho'(b')$:
  + if $\rho'(b), \rho'(b') \in t^\bullet$: trivially by condition 2 of Definition 3.1.2.
  + other cases : absurd, since ${}^\bullet b \neq \emptyset$ and ${}^\bullet b' \neq \emptyset$, so neither $\rho'(b)$ nor $\rho'(b')$ can be an element of m_0.

- Case n+1 : $\sigma = \delta\, e$.
  The inductive hypotesis is that there exist $\pi = (C, \rho)$ and $(m, \leq)$ such that the thesis holds. The induction step is $\pi \xrightarrow{e} \pi'$ with $init(N)[\pi'\rangle(m', \leq')$.

  By definition, $\rho'(Max(C')) = m'$ and $\rho(Max(C)) = m$; since $\pi \xrightarrow{e} \pi'$ where $\rho'(e) = t$, $m \supseteq {}^\bullet t$ i.e., $m[t\rangle$. For the same reasons, $\rho'(Max(C')) = \rho'(Max(C)) \ominus \rho'({}^\bullet e) \oplus \rho'(e^\bullet) = m \ominus {}^\bullet t \oplus t^\bullet = m'$ thus $m[t\rangle m'$.
  Now we prove that $\forall b, b' \in Max(C'), (b \in Min(C')) \vee ({}^\bullet b \neq \emptyset \wedge {}^\bullet b' \neq \emptyset \wedge {}^\bullet b \leq_{\pi'} {}^\bullet b')) \Longrightarrow \rho'(b) \leq' \rho'(b')$ by inspection of the condition.

  - if $b \in Min(C)$ then:
    + if $\rho'(b), \rho'(b') \in m \ominus {}^\bullet t$: then $\rho'(b) = \rho(b)$ and $\rho'(b') = \rho(b')$. By Lemma 3.2.1 $\rho(b)$ is minimal for $\leq$ and therefore $\rho(b) \leq \rho(b')$; since $\rho'(b), \rho'(b') \in m \ominus {}^\bullet t$, condition 1 of Definition 3.1.2 is satisfied;
    + if $\rho'(b), \rho'(b') \in t^\bullet$: trivially by condition 2 of Definition 3.1.2.
    + if $\rho'(b) \in m \ominus {}^\bullet t$ and $\rho'(b') \in t^\bullet$: since $\rho'(b) = \rho(b)$ because $\rho'(b) \in m \ominus {}^\bullet t$, by Lemma 3.2.1 there exists $s'' \in {}^\bullet t$ such that $\rho(b) \leq s''$, and thus $\rho'(b) \leq' \rho'(b')$ by condition 3 of Definition 3.1.2.
    + if $\rho'(b') \in m \ominus {}^\bullet t$ and $\rho'(b) \in t^\bullet$: absurd, since $b \in Min(C)$.
  - if ${}^\bullet b \neq \emptyset \wedge {}^\bullet b' \neq \emptyset \wedge {}^\bullet b \leq_{\pi'} {}^\bullet b'$ then:
    + if $\rho'(b), \rho'(b') \in m \ominus {}^\bullet t$: by inductive hypotesis, $\rho'(b) = \rho(b) \leq \rho(b') = \rho'(b')$ and, since $\rho'(b), \rho'(b') \in m \ominus {}^\bullet t$, condition 1 of Definition 3.1.2 is satisfied.
    + if $\rho'(b), \rho'(b') \in t^\bullet$: trivially by condition 2 of Definition 3.1.2.
    + if $\rho'(b) \in m \ominus {}^\bullet t$ and $\rho'(b') \in t^\bullet$: since $\rho'(b) = \rho(b) \in m \ominus {}^\bullet t$ and $\pi \xrightarrow{e} \pi'$, By Proposition 2.2.7, there exists $b'' \in {}^\bullet e$ such that ${}^\bullet b \leq_\pi {}^\bullet b''$. Since $\rho(b'') \in \rho'({}^\bullet e) = {}^\bullet \rho'(e) = {}^\bullet t \supseteq m$, then by inductive hypotesis $\rho(b) \leq \rho(b'')$ and thus $\rho'(b) \leq' \rho'(b')$ by condition 3 of Definition 3.1.2.
    - if $\rho'(b') \in m \ominus {}^\bullet t$ and $\rho'(b) \in t^\bullet$: absurd, since ${}^\bullet b \leq_\pi {}^\bullet b'$.

  Therefore, $(m, \leq)[t\rangle(m', \leq')$.

□

Example 2. In Figure 3.1(a) a simple finite P/T net N is given, with initial marking $m_0 = s_1 \oplus s_2$. Figure 3.1(b,c,d) shows how the process corresponding to the transition sequence $t_1 t_2 t_3$ grows. For simplicity's sake, in the following each condition will be mapped to the place having same subscript and each event will be mapped to the transition having same label. We will denote each process π_i as the one thus corresponding to causal net C_i.

Before any transition fires, we have $init(N) = (m_0, \leq_0)$ where $\leq_0 = \{(s_1, s_1), (s_2, s_2), (s_1, s_2), (s_2, s_1)\}$ by Definition 3.1.1. Not surprisingly, both b_1 (mapped to s_1) and b_2 (mapped to s_2) are minimal for $\leq_{\pi_0}$.

After the firing of transition t_1, labeled by t, we have $\leq_1 = \{(s_3, s_3), (s_2, s_2), (s_2, s_3)\}$ because $s_2 \leq_0 s_1$ and s_1 is deleted when the transition generates s_3. At the same time, we have that b_2 is minimal for $\leq_{\pi_1}$, but b_3 is not.

After the firing of transition t_2, labeled by u, we have $\leq_1 = \{(s_3, s_3), (s_4, s_4)\}$ because there is no causal relationship between s_3 and s_4. Indeed, b_3 and b_4 are not related by $\leq_{\pi_2}$, but neither of them is minimal.

Finally, after the firing of transition t_3, labeled by c, we have $\leq_2 = \{(s_5, s_5)\}$ because it is a generated token. As expected, $b_5 \leq_{\pi_2} b_5$ because it is maximal. □

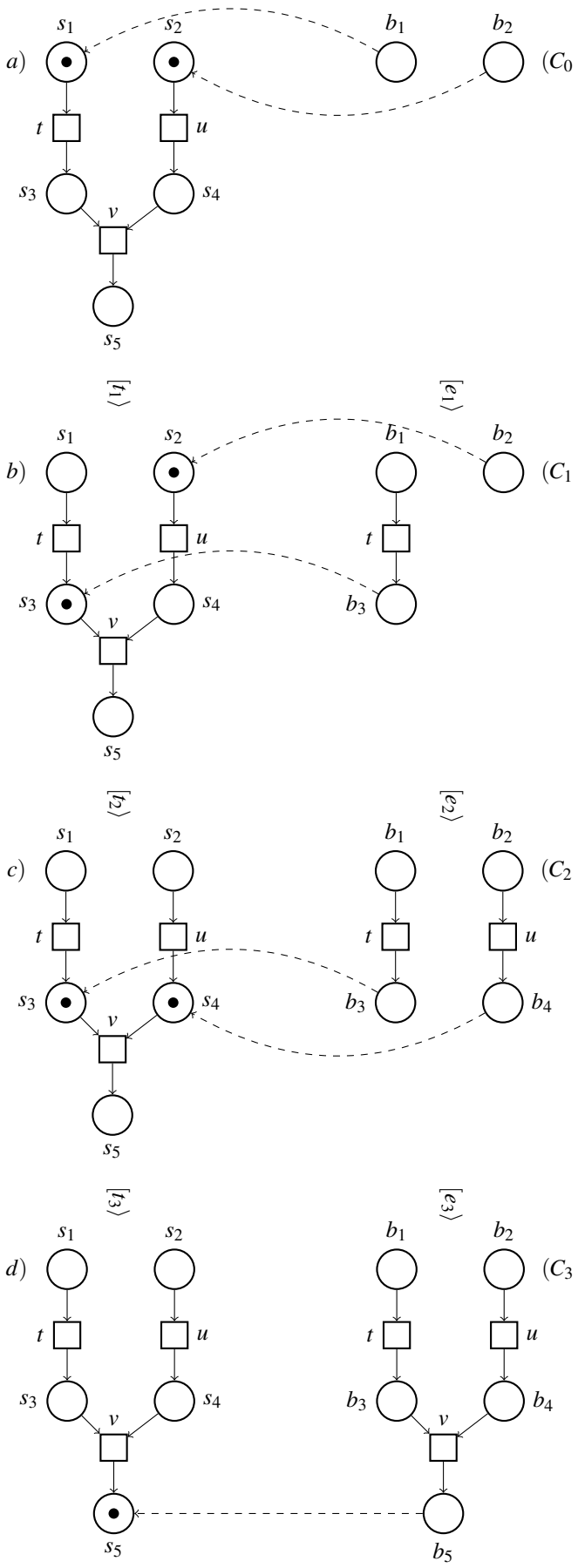

Figure 3.1: Execution of the transitions labeled by t, u and v on a safe finite net with initial marking $m_0 = s_1 \oplus s_2$ and corresponding process (only the mapping of maximal conditions to tokens is displayed).

3.3 Decidability of causal-net bisimilarity for safe nets

We define a new bisimulation based on orderd markings, OMC-bisimulation. Intuitively, two ordered markings are related by an OMC-bisimulation if the corresponding markings are bisimilar and the places of each preorder have the same history (starting from the initial configuration).

Definition 3.3.1. (OMC-bisimulation) *Let $N = (S,A,T)$ be a net. An OMC-bisimulation is a relation $\mathfrak{B} \subseteq OM(N) \times OM(N) \times \mathscr{P}(S \times S)$ such that if $((m_1,\leq_1),(m_2,\leq_2),\beta) \in \mathfrak{B}$, then:*

- $|m_1| = |m_2|$;
- *$\forall t_1$ such that $(m_1,\leq_1)[t_1\rangle(m'_1,\leq'_1)$, there exist $t'_2, m'_2, \leq'_2$ such that for β' defined as $\forall s_1 \in m'_1, \forall s_2 \in m'_2$:*

$$s_1\ \beta'\ s_2 \iff \begin{cases} s_1 \in m_1 \ominus {}^\bullet t_1\,,\ s_2 \in m_2 \ominus {}^\bullet t_2 \text{ and } s_1\ \beta\ s_2 \\ \text{or} \\ s_1 \in t_1^\bullet\,,\ s_2 \in t_2^\bullet \end{cases}$$

  the following hold:

  - ${}^\bullet t_1\ \beta^\oplus\ {}^\bullet t_2$,
  - $(m_2,\leq_2)[t_2\rangle(m'_2,\leq'_2)$ *where* $((m'_1,\leq'_1),(m'_2,\leq'_2),\beta') \in \mathfrak{B}$ *and* $l(t_1) = l(t_2)$;
- *symmetrically, if $(m_2,\leq_2)$ moves first.*

If $N(m_1)$ and $N(m_2)$ are both safe nets, we say that m_1, m_2 are OMC-bisimilar, denoted by $m_1 \sim_{omc} m_2$, if there exists an OMC-bisimulation $\mathfrak{B}$ containing the triple $(init(N(m_1)), init(N(m_2)), m_1 \times m_2)$. □

Next, we show that causal-net bisimilarity and OMC-bisimilarity coincide on finite safe nets.

Theorem 3.3.2. (CN-bisimilarity implies OMC-bisimilarity) *Let $N(m_{01})$ and $N(m_{02})$ be safe nets. If $m_{01} \sim_{cn} m_{02}$, then $m_{01} \sim_{omc} m_{02}$.*

Proof. If $m_{01} \sim_{cn} m_{02}$, then there exists a CN-bisimulation R_1 containing the triple $(\rho_1^0, C^0, \rho_2^0)$, where C^0 contains no events and $\rho_i^0(Min(C^0)) = \rho_i^0(Max(C^0)) = m_{0i}$ for $i = 1,2$. Let us consider

$$\begin{aligned} R_2 \stackrel{def}{=} \{((m_1,\leq_1),(m_2,\leq_2),\beta) \mid & (\rho_1, C, \rho_2) \in R_1 \text{ and} \\ & init(N(m_{01}))[(C,\rho_1)\rangle(m_1,\leq_1) \text{ and} \\ & init(N(m_{02}))[(C,\rho_2)\rangle(m_2,\leq_2) \text{ and} \\ & \forall s_1 \in \rho_1(Max(C)), \forall s_2 \in \rho_2(Max(C)), \\ & \quad s_1 \beta s_2 \text{ iff } \rho_1^{-1}({}^\bullet s_1) = \rho_2^{-1}({}^\bullet s_2)\}. \end{aligned}$$

If we prove that R_2 is an OMC-bisimulation, then since $init(N(m_{0i}))[(C^0,\rho_i^0)\rangle init(N(m_{0i}))$ and $\rho_i^0(Min(C^0)) = = \rho_i^0(Max(C^0)) = m_{0i}$ for $i = 1,2$, it follows that $(init(N(m_{01})), init(N(m_{02})), m_{01} \times m_{02}) \in R_2$ and so $m_{01} \sim_{omc} m_{02}$.

We consider a tuple $((m_1,\leq_1),(m_2,\leq_2),\beta) \in R_2$. If $init(N(m_{01}))[(C,\rho_1)\rangle(m_1,\leq_1)$, and $(m_1,\leq_1)[t_1\rangle(m'_1,\leq'_1)$ then by Lemma 3.2.4 $(C,\rho_1) \xrightarrow{e} (C',\rho'_1)$ with $\rho'_1(e) = t_1$ and $init(N(m_{01}))[(C',\rho'_1)\rangle(m'_1,\leq'_1)$. Since $(\rho_1,C,\rho_2) \in R_1$, it is also true that $(C,\rho_2) \xrightarrow{e} (C',\rho'_2)$ with $\rho'_2(e) = t_2$ and $(\rho'_1,C',\rho'_2) \in R_1$. Then, since by definition $init(N(m_{02}))[(C,\rho_2)\rangle (m_2,\leq_2)$ and $init(N(m_{02}))[(C',\rho'_2)\rangle(m'_2,\leq'_2)$, by Lemma 3.2.4 $(m_2,\leq_2)[t_2\rangle(m'_2,\leq'_2)$.

Now we prove that the definition of β' arising from R_2 is coherent with the one of Definition 3.3.1, i.e. it implies both

1. $\forall s_1 \in m'_1\,.\, s_2 \in m'_2$

$$s_1\ \beta'\ s_2 \iff \begin{cases} s_1 \in m_1 \ominus {}^\bullet t_1\,,\ s_2 \in m_2 \ominus {}^\bullet t_2 \text{ and } s_1\ \beta\ s_2 & \text{(i)} \\ \text{or} & \\ s_1 \in t_1^\bullet\,,\ s_2 \in t_2^\bullet & \text{(ii)} \end{cases}$$

and

2. ${}^\bullet t_1 \; \beta^\oplus \; {}^\bullet t_2$.

Proof 1) The two implications are proved separately.

- If $s_1 \beta' s_2$ iff $\rho_1'^{-1}({}^\bullet s_1) = \rho_2'^{-1}({}^\bullet s_2)$:
  Consider the event e such that $\rho_1'(e) = t_1$ and $\rho_2'(e) = t_2$:
  + if $s_1 \in t_1^\bullet$ and $s_2 \in t_2^\bullet$: (ii) is trivial.
  + if $s_1 \in m_1 \ominus {}^\bullet t_1$ and $s_2 \in m_2 \ominus {}^\bullet t_2$: then since $((m_1, \leq_1), (m_2, \leq_2), \beta) \in R_2$, the hypotesis $s_1 \beta s_2$ holds. Moreover, since s_1 and s_2 don't move, (i) holds.
  + other cases: absurd, since $\rho_1^{-1}({}^\bullet s_1) = \rho_2^{-1}({}^\bullet s_2)$.
- If (i) or (ii) holds:
  Consider the transitions t_1, t_2 such that $\rho_1'(e) = t_1$ and $\rho_2'(e) = t_2$:
  + if (i): then since $((m_1, \leq_1), (m_2, \leq_2), \beta) \in R_2$, the hypotesis $s_1 \beta s_2$ holds. Moreover, since s_1 and s_2 don't move, $s_1 \beta' s_2$.
  + if (ii): then $\rho_1'^{-1}({}^\bullet s_1) = \rho_1'^{-1}(t_1) = e$, and $\rho_2'^{-1}({}^\bullet s_2) = \rho_2'^{-1}(t_2) = e$, therefore by transitivity $s_1 \beta' s_2$.

Proof 2) Since by definition $\rho_1^{-1}({}^\bullet s_1) = \rho_1^{-1}(t_1) = e = \rho_2^{-1}(t_2) = \rho_2^{-1}({}^\bullet s_2)$, then $|t_1^\bullet| = |e^\bullet| = |t_2^\bullet|$.

The proof of ${}^\bullet t_1 \; \beta^\oplus \; {}^\bullet t_2$ is done by induction on $|{}^\bullet t_1|$: note that the base case is 1, because transitions have nonempty preset.

- Case $|{}^\bullet t_1| = |{}^\bullet t_2| = 1$: let ${}^\bullet t_1 = s_1'$ and ${}^\bullet t_2 = s_2'$.
  For $i = 1,2$, $s_i' \in \rho_i(Max(C))$, and since t_1 and t_2 are mapped by ρ_1 and ρ_2 respectively on the same event e, s_1' and s_2' are mapped on the same place $s \in B$. Therefore $s_1' \beta s_2'$ and By rule **(Clo)**, ${}^\bullet t_1 \; \beta^\oplus \; {}^\bullet t_2$.
- Case $|{}^\bullet t_1| = |{}^\bullet t_2| = n+1$: let ${}^\bullet t_1 = s_1' \oplus n_1$ and ${}^\bullet t_2 = s_2' \oplus n_2$.
  By inductive hypotesis, $n_1 \beta^\oplus n_2$. For $i = 1,2$, $s_i' \in \rho_i(Max(C))$, and since t_1 and t_2 are mapped by ρ_1 and ρ_2 respectively on the same event e, s_1' and s_2' are mapped on the same place $s \in B$. Therefore $s_1' \beta s_2'$. By rule **(Clo)**, ${}^\bullet t_1 \; \beta^\oplus \; {}^\bullet t_2$.

Thus $((m_1', \leq_1'), (m_2', \leq_2'), \beta') \in R_2$.

Next, we prove that $|m_1'| = |m_2'|$. We know that, with the definitions from above, (ρ_1, C, ρ_2), $(\rho_1', C', \rho_2') \in R_1$ and $((m_1, \leq_1), (m_2, \leq_2), \beta)$, $((m_1', \leq_1'), (m_2', \leq_2'), \beta') \in R_2$. Then, since for $i = 1,2$ it is true that $init(N(m_{0i}))[(C, \rho_i)\rangle (m_i, \leq_i)$ and $init(N(m_{0i}))[(C, \rho_i)\rangle (m_i', \leq_i')$, by Proposition 3.2.2, $|m_1'| = |m_2'|$.

The case in which $(m_2, \leq_2)$ moves first is symmetrical.

Therefore, R_2 is an OMC-bisimulation and $m_{01} \sim_{omc} m_{02}$. □

Theorem 3.3.3. (OMC-bisimilarity implies CN-bisimilarity) *Let $N(m_{01})$ and $N(m_{02})$ be safe nets If $m_{01} \sim_{omc} m_{02}$, then $m_{01} \sim_{cn} m_{02}$.*

Proof. If $m_{01} \sim_{omc} m_{02}$ there exists an OMC-bisimulation R_1 containing the tuple $(init(N(m_{01})), init(N(m_{02})), m_{01} \times m_{02})$. Let us consider

$$
\begin{aligned}
R_2 \stackrel{def}{=} \{(\rho_1, C, \rho_2) \mid & ((m_1, \leq_1), (m_2, \leq_2), \beta) \in R_1 \text{ and} \\
& \text{for } i = 1,2,\ (C, \rho_i) \text{ is a process of } N(M_{0i}) \textit{and} \\
& \text{for } i = 1,2,\ init(N(M_{0i}))[(C, \rho_i)\rangle (m_i, \leq_i) \text{ and} \\
& \forall s_1 \in \rho_1(Max(C)), \forall s_2 \in \rho_2(Max(C)), \\
& \quad s_1 \beta s_2 \text{ iff } \rho_1^{-1}({}^\bullet s_1) = \rho_2^{-1}({}^\bullet s_2)\}.
\end{aligned}
$$

If we prove that R_2 is a CN-bisimulation, then we have that $m_{01} \sim_{cn} m_{02}$, because $(\rho_1^0, C^0, \rho_2^0) \in R_2$, where C^0 contains no transitions and, for $i = 1,2$, $\rho_i^0(Min(C^0)) = \rho_i^0(Max(C^0)) = m_{0i}$. Indeed, since $(init(N(m_{01})), init(N(m_{02})), m_{01} \times m_{02}) \in R_1$ and (C^0, ρ_i^0) is a process of $N(m_{0i})$ and $init(N(m_{0i}))[(C^0, \rho_i^0)\rangle init(N(m_{0i}))$, we have $(\rho_1^0, C^0, \rho_2^0) \in R_2$, and therefore $m_{01} \sim_{cn} m_{02}$.

Assume $(\rho_1, C, \rho_2) \in R_2$. If $(C, \rho_1) \xrightarrow{e} (C', \rho_1')$ where $\rho_1'(e) = t_1$, since by hypotesis $init(N(m_{01}))[(C, \rho_1)\rangle (m_1, \leq_1)$ and $init(N(m_{01}))[(C', \rho_1')\rangle (m_1', \leq_1')$, by Lemma 3.2.4 $(m_1, \leq_1)[t_1\rangle (m_1', \leq_1')$. Since we have $((m_1, \leq_1), (m_2, \leq_2), \beta) \in R_1$ then there exist $t_2', m_2', \leq_2'$ such that $(m_2, \leq_2)[t_2\rangle (m_2', \leq_2')$ where $((m_1', \leq_1'), (m_2', \leq_2'), \beta') \in R_1$. By hypotesis $init(N(m_{02}))\ [(C, \rho_2)\rangle (m_2, \leq_2)$, so by Lemma 3.2.4, $(C, \rho_2) \xrightarrow{e} (C', \rho_2')$ where $\rho_2'(e) = t_2$ and $init(N(m_{02}))[(C', \rho_2')\rangle (m_2', \leq_2')$. Note that, for $i = 1,2$, (C, ρ_i) is a process of $N(m_{0i})$.

Now we prove that the definition of β' specified in R_2 is coherent with the one induced by Definition 3.3.1, i.e. $\forall s'_1 \in \rho'_1(Max(C'))\ \forall s'_2 \in \rho'_2(Max(C'))$ we have $s'_1\ \beta'\ s'_2$ if and only if $\rho_1'^{-1}({}^\bullet s_1) = \rho_2'^{-1}({}^\bullet s_2)$. By induction on the length of σ, complete transition sequence of C such that $init(N(m_{01})[(C',\rho'_1\rangle(m'_1,\leq'_1)$. Note that also $init(N(m_{02})[(C',\rho'_2\rangle$
$(m'_2,\leq'_2)$ with the same complete transition sequence.

- Case 1 : $\sigma = e$.

  Proof $\Rightarrow$*)* by cases on the definition of $s'_1\ \beta' s'_2$:

  - if $s'_1 \in t_1^\bullet$ and $s'_2 \in t_2^\bullet$:
    By hypotesis we know that ${}^\bullet s'_1 = t_1$ and ${}^\bullet s'_2 = t_2$. Since $(C,\rho_i) \xrightarrow{e} (C',\rho'_i)$ where $\rho'_i(e) = t_i$ for $i = 1,2$, then t_1 and t_2 are mapped on the same transition e. Therefore $\rho_1^{-1}({}^\bullet s'_1) = \rho_1^{-1}(t_1) = e$ and $\rho_2^{-1}({}^\bullet s'_2) = \rho_1^{-1}(t_2) = e$, and by transitivity we get $\rho_1^{-1}({}^\bullet s'_1) = \rho_2^{-1}({}^\bullet s'_2)$.
  - if $s'_1 \in m_1 \ominus {}^\bullet t_1$ and $s'_2 \in m_2 \ominus {}^\bullet t_2$ and $s'_1\ \beta s'_2$:
    then $\rho_1^{-1}({}^\bullet s'_1) = \emptyset$ and $\rho_2^{-1}({}^\bullet s'_2) = \emptyset$. Since the tokens did not move, $\rho'_i(s'_i) = \rho_i(s'_i)$ for $i = 1,2$ and by transitivity $\rho_1^{-1}({}^\bullet s'_1) = \rho_2^{-1}({}^\bullet s'_2)$.

  Proof $\Leftarrow$*)* Consider the event e: we know that e is a transition of C' and not of C, such that $\rho'_1(e) = t_1$ and $\rho'_2(e) = t_2$. there are four possibilities for s'_1, s'_2:

  - if $s'_1 \in t_1^\bullet$ and $s'_2 \in t_2^\bullet$:
    then $s'_1\ \beta'\ s'_2$ by the second condition of β'.
  - if $s'_1 \in m_1 \ominus {}^\bullet t_1$ and $s'_2 \in m_2 \ominus {}^\bullet t_2$:
    We know that $\rho_1'^{-1}({}^\bullet s_1) = \rho_2'^{-1}({}^\bullet s_2)$; since the tokens did not move, $\rho'_i(s'_i) = \rho_i(s'_i)$ for $i = 1,2$ and therefore $\rho_1^{-1}({}^\bullet s'_1) = \rho_2^{-1}({}^\bullet s'_2)$. Moreover, because $\sigma = e$ it must be that ${}^\bullet s'_1 = \emptyset$ and ${}^\bullet s'_2 = \emptyset$. By the fact that, at the beginning, $\beta = k_{01} \times k_{02}$, we have that $s'_1\,\beta\,s'_2$. Thus, the first condition of β' holds.
  - other cases:
    absurd since $\rho_1'^{-1}({}^\bullet s'_1) = \rho_2'^{-1}({}^\bullet s'_2)$.

- Case n+1: $\sigma = \delta\, e$.

  Proof $\Rightarrow$*)* by cases on the definition of $s'_1\ \beta'\ s'_2$:

  - if $s'_1 \in t_1^\bullet$ and $s'_2 \in t_2^\bullet$:
    By hypotesis we know that ${}^\bullet s'_1 = t_1$ and ${}^\bullet s'_2 = t_2$. Since $(C,\rho_i) \xrightarrow{e} (C',\rho'_i)$ where $\rho'_i(e) = t_i$ for $i = 1,2$, then t_1 and t_2 are mapped on the same transition e. Therefore $\rho_1^{-1}({}^\bullet s'_1) = \rho_1^{-1}(t_1) = e$ and $\rho_2^{-1}({}^\bullet s'_2) = \rho_1^{-1}(t_2) = e$, and by transitivity we get $\rho_1^{-1}({}^\bullet s'_1) = \rho_2^{-1}({}^\bullet s'_2)$.
  - if $s'_1 \in m_1 \ominus {}^\bullet t_1$ and $s'_2 \in m_2 \ominus {}^\bullet t_2$ and $s'_1\ \beta s'_2$:
    By inductive hypotesis on $(m_1,\leq_1)$ and $(m_2,\leq_2)$, we have $s'_1\ \beta\ s'_2 \Rightarrow \rho_1^{-1}({}^\bullet s'_1) = \rho_2^{-1}({}^\bullet s'_2)$.
    Since the tokens did not move, $\rho'_i(s'_i) = \rho_i(s'_i)$ for $i = 1,2$; therefore by transitivity $\rho_1'^{-1}({}^\bullet s'_1) = \rho_2'^{-1}({}^\bullet s'_2)$.

  Proof $\Leftarrow$*)* Consider the event e: we know that e is a transition of C' and not of C, such that $\rho'_1(e) = t_1$ and $\rho'_2(e) = t_2$. there are four possibilities for s'_1, s'_2:

  - if $s'_1 \in t_1^\bullet$ and $s'_2 \in t_2^\bullet$:
    then $s'_1\ \beta'\ s'_2$ by the second condition of β'.
  - if $s'_1 \in m_1 \ominus {}^\bullet t_1$ and $s'_2 \in m_2 \ominus {}^\bullet t_2$:
    By inductive hypotesis on $(m_1,\leq_1)$ and $(m_2,\leq_2)$, we have $\rho_1^{-1}({}^\bullet s'_1) = \rho_2^{-1}({}^\bullet s'_2) \Rightarrow s'_1\ \beta\ s'_2$ Since the tokens did not move, $\rho'_i(s'_i) = \rho_i(s'_i)$ for $i = 1,2$; therefore, the first condition of β' holds.
  - other cases:
    absurd since $\rho_1'^{-1}({}^\bullet s'_1) = \rho_2'^{-1}({}^\bullet s'_2)$.

Thus $(\rho'_1, C', \rho'_2) \in R_2$.
The case in which (C_2,ρ_2) moves first is symmetrical.
Therefore, R_2 is a CN-bisimulation and $m_{01} \sim_{cn} m_{02}$. □

Corollary 3.3.4. (OMC-bisimilarity and CN-bisimilarity coincide) *Let $N(m_{01})$ and $N(m_{02})$ be safe nets. $m_1 \sim_{omc} m_2$ if and only if $m_1 \sim_{cn} m_2$.*

Proof. By theorems 3.3.2 and 3.3.3, we get the thesis. □

Corollary 3.3.5. (CN-bisimilarity is decidable for finite safe nets) *Given $N(m_1)$ and $N(m_2)$ safe nets, it is decidable to check whether $m_1 \sim_{cn} m_2$.*

Proof. By Corollary 3.3.4 we have to check whether there exists an OMC-bisimulation $\mathfrak{B}$ for the given net N and marking m_1, m_2. By Proposition 3.1.4, $OM(N)$, $[init(N(m_1))\rangle$ and $[init(N(m_2)\rangle$ are finite. Since S is finite, also $\mathscr{P}(S \times S)$ is finite, thus there are only finitely many objects that are possible elements of $\mathfrak{B}$.

Since $OM(N)$ is finite, the set of all possible tuples $((m_1, \leq_1), (m_2, \leq_2), \beta)$ is finite. Since $\mathfrak{B}$ is a set of said tuples, there are only finitely many possible sets $\mathfrak{B}$. Therefore we can check by exhaustive search whether one of them is an OMC-bisimulation. □

Note that in general the complexity of this decision procedure is prohibitive. Already the number of reachable markings of a safe net can be exponential in the size of the net. It could be conjectured that the complexity is complete for DEXPTIME, since this equivalence is similar to history-preserving bisimilarity (i.e. fully-concurrent bisimilarity), which (as shown in [JM96]) is in that class.

3.4 Decidability of fully-concurrent bisimulation for safe nets

In the following chapter we will extend the work of [Vog91] by showing that fully-concurrent bisimilarity is decidable for finite bounded nets, too. In order to help the reader with the (rather complicated) bounded net proof, we display also a proof for the safe case which is slightly different and more detailed than Vogler's sketched one in [Vog91].

We outline the definition of OM-bisimilarity from [Vog91]. Intuitively, two ordered markings are related by an OM-bisimulation if the corresponding markings are bisimilar and the places of each preorder related. Note that we do note require that tokens produced/consumed by a transition are bijectively related to those consumed/produced by the corresponding transition, as in OMC-bisimulation (cf. Definition 3.3.1). It is enough that a token s_1 precedes some s_1', which is consumed by t_1, and that s_1' is related to some token s_2' consumed by t_2.

Definition 3.4.1. (OM-bisimulation) *Let $N = (S, A, T)$ be a net. An OM-bisimulation is a relation $\mathfrak{B} \subseteq OM(N) \times OM(N) \times \mathscr{P}(S \times S)$ such that if $((m_1, \leq_1), (m_2, \leq_2), \beta) \in \mathfrak{B}$, then:*

- *$\forall t_1$ such that $(m_1, \leq_1)[t_1\rangle(m_1', \leq_1')$, there exist $t_2', m_2', \leq_2'$ such that for β' defined as $\forall s_1 \in m_1', \forall s_2 \in m_2'$:*

$$s_1 \; \beta' \; s_2 \iff \begin{cases} s_1 \in m_1 \ominus {}^\bullet t_1 \,,\, s_2 \in m_2 \ominus {}^\bullet t_2 \text{ and } s_1 \; \beta \; s_2 \\ \text{or} \\ s_1 \in t_1^\bullet \,,\, s_2 \in t_2^\bullet \end{cases}$$

  the following hold:

  - *$(m_2, \leq_2)[t_2\rangle(m_2', \leq_2')$ where $((m_1', \leq_1'), (m_2', \leq_2'), \beta') \in \mathfrak{B}$ and $l(t_1) = l(t_2)$;*
  - *$\forall s_1 \in {}^\bullet t_1$, $\exists s_1' \in {}^\bullet t_1$, $s_2' \in {}^\bullet t_2$ such that $s_1 \leq_1 s_1' \wedge s_1' \; \beta \; s_2'$, and symmetrically $\forall s_2 \in {}^\bullet t_2$, $\exists s_2' \in {}^\bullet t_2$, $s_1' \in {}^\bullet t_1$ such that $s_2 \leq_2 s_2' \wedge s_1' \; \beta \; s_2'$.*

- *symmetrically, if $(m_2, \leq_2)$ moves first.*

If $N(m_1)$ and $N(m_2)$ are both safe nets, we say that m_1, m_2 are OM-bisimilar, denoted by $m_1 \sim_{om} m_2$, if there exists an OM-bisimulation $\mathfrak{B}$ containing the triple $(init(N(m_1)), init(N(m_2)), m_1 \times m_2)$. □

This definition is aimed at relating $\sim_{fc}$ and $\sim_{om}$ so that tokens are related by β if the transition generating one is mapped to the transition generating the other by f. Next, we show that fully-concurrent bisimilarity and OM-bisimilarity coincide on finite safe nets.

Theorem 3.4.2. (FC-bisimilarity implies OM-bisimilarity) *Let $N(m_{01})$ and $N(m_{02})$ be safe nets. If $m_{01} \sim_{fc} m_{02}$, then $m_{01} \sim_{om} m_{02}$.*

Proof. If $m_{01} \sim_{fc} m_{02}$, then there exists a FC-bisimulation R_1 containing the triple (π_1^0, π_2^0, f), where C_i^0 contains no events and $\rho_i^0(Min(C_i^0)) = \rho_i^0(Max(C_i^0)) = m_{0i}$ for $i = 1,2$. Let us consider

$$
\begin{aligned}
R_2 \stackrel{def}{=} \{((m_1,\leq_1),(m_2,\leq_2),\beta) | & (\pi_1, f, \pi_2) \in R_1 \text{ and} \\
& init(N(m_{01}))[\pi_1\rangle(m_1,\leq_1) \text{ and} \\
& init(N(m_{02}))[\pi_2\rangle(m_2,\leq_2) \text{ and} \\
& \forall s_1 \in S, \forall s_2 \in S: \\
& s_1 \beta s_2 \text{ iff} \\
& \quad s_1 \in m_1, s_2 \in m_2, \\
& \quad \exists b_1 \in Max(C_1) \text{ such that } \rho_1(b_1) = s_1, \\
& \quad \exists b_2 \in Max(C_2) \text{ such that } \rho_2(b_2) = s_2, \\
& \quad \text{either } b_1 \in Min(C_1) \wedge b_2 \in Min(C_2) \\
& \quad \text{or } {}^\bullet b_1 \neq \emptyset \wedge {}^\bullet b_2 \neq \emptyset \wedge f({}^\bullet b_1) = {}^\bullet b_2\}.
\end{aligned}
$$

If we prove that R_2 is an OM-bisimulation, then since $init(N(m_{0i}))[(C_i^0, \rho_i^0)\rangle init(N(m_{0i}))$ and $\rho_i^0(Min(C_i^0)) = = \rho_i^0(Max(C_i^0)) = m_{0i}$ for $i = 1,2$, it follows that $(init(N(m_{01})), init(N(m_{02})), m_{01} \times m_{02}) \in R_2$ and so $m_{01} \sim_{om} m_{02}$.

We consider a tuple $((m_1,\leq_1),(m_2,\leq_2),\beta) \in R_2$. If $init(N(m_{01}))[\pi_1\rangle(m_1,\leq_1)$, and $(m_1,\leq_1)[t_1\rangle(m_1',\leq_1')$ then by Lemma 3.2.4 $\pi_1 \xrightarrow{e} \pi_1'$ with $\rho_1'(e_1) = t_1$ and $init(N(m_{01}))[\pi_1'\rangle(m_1',\leq_1')$. Since $(\pi_1, f, \pi_2) \in R_1$, it is also true that $\pi_2 \xrightarrow{e_2} \pi_2'$ with $\rho_2'(e_2) = t_2$ and $(\pi_1', f', \pi_2') \in R_1$. Then, since by definition $init(N(m_{02}))[\pi_2\rangle(m_2,\leq_2)$ and $init(N(m_{02}))[\pi_2'\rangle$
$(m_2',\leq_2')$, by Lemma 3.2.4 we have $(m_2,\leq_2)[t_2\rangle(m_2',\leq_2')$.

Now we prove that the definition of β' arising from R_2 is coherent with the one of Definition 3.4.1, i.e. it implies both

1. $\forall s_1 \in m_1', s_2 \in m_2'$

$$
s_1 \beta' s_2 \iff \begin{cases} s_1 \in m_1 \ominus {}^\bullet t_1, s_2 \in m_2 \ominus {}^\bullet t_2 \text{ and } s_1 \beta s_2 & \text{(i)} \\ \text{or} & \\ s_1 \in t_1^\bullet, s_2 \in t_2^\bullet & \text{(ii)} \end{cases}
$$

   and

2. $\forall s_1 \in {}^\bullet t_1$, $\exists s_1' \in {}^\bullet t_1, s_2' \in {}^\bullet t_2$ such that $s_1 \leq_1 s_1' \wedge s_1' \beta s_2'$ and symmetrically $\forall s_2 \in {}^\bullet t_2$, $\exists s_2' \in {}^\bullet t_2, s_1' \in {}^\bullet t_1$ such that $s_2 \leq_2 s_2' \wedge s_1' \beta s_2'$.

   Proof 1) The two implications are proved separately.

   Proof $\Rightarrow$: assume $s_1 = \rho'(b_1)$ and $s_2 = \rho'(b_2)$. Then:

   - If $b_1 \in Min(C_1')$ and $b_2 \in Min(C_2')$:
     then for $i = 1,2$, $s_i \in \rho_i'(Min(C_i')) = m_{0i}$ and $s_i \in m_i \ominus {}^\bullet t_i$. Since the initial β is $m_{01} \times m_{02}$, we have $s_1 \beta s_2$, satisfying condition (i).
   - if ${}^\bullet b_1 \neq \emptyset \wedge {}^\bullet b_2 \neq \emptyset \wedge f({}^\bullet b_1) = {}^\bullet b_2$:
     Let us consider events e_1', e_2' such that $b_1 \in e_1'^\bullet$ and $b_2 \in e_2'^\bullet$. There are four possible cases:
     + if $e_1' = e_1$ and $e_2' = e_2$: since $s_1 = \rho_1'(b_1)$ and $t_1 = \rho_1'(e_1) = \rho_1'(e_1')$, we have $s_1 \in t_1^\bullet$. For the same reason, $s_2 \in t_2^\bullet$ and therefore condition (ii) holds.
     + if $e_1' \neq e_1$ and $e_2' \neq e_2$: then $s_1 = \rho_1'(b_1) = \rho_1(b_1) \in m_1 \ominus {}^\bullet t_1$ because e_1' has occurred before e_1. For the same reason, $s_2 \in m_2 \ominus {}^\bullet t_2$. Since $((m_1,\leq_1),(m_2,\leq_2),\beta) \in R_2$ and $f({}^\bullet b_1) = {}^\bullet b_2$, then $s_1 \beta s_2$, therefore condition (i) holds.
     + other cases: absurd since $f'(e_1) = e_2$.

   Proof $\Leftarrow$: Since $s_1 \in m_1'$ and $s_2 \in m_2'$, there exist b_1 and b_2 such that $s_1 = \rho_1'(b_1)$ and $s_2 = \rho_2'(b_2)$. Consider the complete transition sequence $\sigma_1 = \delta_1 e_1$ such that $init(N(m_{01}))[\rho_1'(\sigma_1)\rangle(m_1',\leq_1')$, and the complete transition sequence $\sigma_2 = \delta_2 e_2$ such that $init(N(m_{02}))[\rho_2'(\sigma_2)\rangle(m_2',\leq_2')$, and $f'(\sigma_1) = \sigma_2$. It is easy to see that $|\sigma_1| = |\sigma_2|$; we prove the thesis by induction on the length of σ_1:

   + Case 1: $\sigma_1 = e_1$ and $\sigma_2 = e_2$. By cases on the condition:

- if $s_1 \in m_1 \ominus {}^\bullet t_1$, $s_2 \in m_2 \ominus {}^\bullet t_2$ and $s_1 \ \beta \ s_2$:
  then $s_1 \in m_{01}$ and since $m_{01} = \rho_1'(Min(C'))$ we have ${}^\bullet b_1 = \emptyset$. For the same reason, also ${}^\bullet b_2 = \emptyset$.
- if $s_1 \in t_1^\bullet$, $s_2 \in t_2^\bullet$:
  then ${}^\bullet b_1 = e_1$ and ${}^\bullet b_2 = e_2$: therefore, we have by construction that $f'({}^\bullet b_1) = {}^\bullet b_2$.

+ Case n+1: $\sigma_1 = \delta_1 e_1$ and $\sigma_2 = \delta_2 e_2$. The inductive hypotesis is $init(N(m_{0i}))[\rho_i(\delta_i)\rangle(m_i, \leq_i)$ where the thesis holds and the inductive step is $(m_i, \leq_i)[\rho_i'(e_i)\rangle(m_i', \leq_i')$, where $\rho_i'(e_i) = t_i$, for $i = 1, 2$. By cases on the condition:
  - if $s_1 \in m_1 \ominus {}^\bullet t_1$, $s_2 \in m_2 \ominus {}^\bullet t_2$ and $s_1 \ \beta \ s_2$:
    We need to separate two cases for s_1.
    + If $s_1 \in m_{01}$: since $s_1 = \rho_1'(b_1)$ we have $b_1 \in Min(C_1')$; moreover since $s_1 \beta s_2$, it is true that $s_2 \in m_{02}$, and the same reasoning applies, therefore $b_1 \in Min(C_1')$ and $b_2 \in Min(C_2')$.
    + If $s_1 \notin m_{01}$: then ${}^\bullet b_1 \neq \emptyset$ and ${}^\bullet b_2 \neq \emptyset$. Since $s_1 \in m_1 \ominus {}^\bullet t_1$, $s_2 \in m_2 \ominus {}^\bullet t_2$, and $((m_1, \leq_1), (m_2, \leq_2), \beta) \in R_2$, then by inductive hypotesis we have $f({}^\bullet b_1) = {}^\bullet b_2$, and by conservative extension of f we get the thesis.
  - if $s_1 \in t_1^\bullet$, $s_2 \in t_2^\bullet$:
    then $s_1 \in t_1^\bullet$ and so ${}^\bullet b_1 = e_1$, therefore ${}^\bullet b_1 \neq \emptyset$. The same applies to s_2, and since $f'(e_1) = e_2$, we have $f'({}^\bullet b_1) = {}^\bullet b_2$.

Proof 2) Let us consider events e_1, e_2 such that $\pi_1 \xrightarrow{e_1} \pi_1'$ with $\rho'(e_1) = t_1$, $\pi_2 \xrightarrow{e_2} \pi_2'$ with $\rho'(e_2) = t_2$, and $f'(e_1) = e_2$. We assume $s_1 \in {}^\bullet t_1$ where there exists b_1 such that $\rho_1(b_1) = s_1$. Note that, since $\pi_1 \xrightarrow{e_1} \pi_1'$ with $\rho'(e_1) = t_1$, it is true that $b_1 \in Max(C_1)$. We are to prove that $\exists s_1' \in {}^\bullet t_1$, $\exists s_2' \in {}^\bullet t_2$ such that $s_1 \leq_1 s_1'$ and $s_1' \ \beta \ s_2'$. In the following, in some cases we have $s_1 = s_1'$: if that is true, then $s_1 \leq_1 s_1'$ by reflexivity of $\leq_1$.

There are two possible cases for b_1:

- if $b_1 \in Min(C_1)$:
  There are two possible subcases:
  + $\exists b_2' \in \rho_2^{-1}({}^\bullet t_2)$ such that $b_2' \in Min(C_2)$:
    Then by definition of ρ_2 there exists $s_2' = \rho_2(b_2')$, and by definition of β we have $s_1 \ \beta \ s_2'$.
  + otherwise:
    Since $\pi_2 \xrightarrow{e_2} \pi_2'$ with $\rho'(e_2) = t_2$, there exists $b_2' \in Max(C_2)$ such that $\rho_2(b_2') = s_2'$ and $s_2' \in {}^\bullet t_2$.
    Let us consider $t_2' = {}^\bullet s_2'$ and the related event $e_2' \in \mathsf{E}_{C_2}$ such that $\rho_2(e_2') = t_2'$. Since f is an isomorphism between E_{C_1} and E_{C_2}, there exists event $e_1' \in \mathsf{E}_{C_1}$ such that $f(e_1') = e_2'$. Then, by definition of ρ_1, there exists t_1' such that $\rho_1(e_1') = t_1'$. Therefore there exists s_1' such that $t_1' = {}^\bullet s_1'$ and a b_1' such that $\rho_1(b_1') = s_1'$, i.e. $e_1' = {}^\bullet b_1'$. Since $b_1 \in Min(C_1)$, by Lemma 3.2.1 it is true that s_1 is minimal for $\leq_1$, and therefore $s_1 \leq_1 s_1'$.
    Finally, since $e_1' = {}^\bullet b_1'$, $e_2' = {}^\bullet b_2'$ and $f(e_1') = e_2'$, we have $s_1' \ \beta \ s_2'$.
- if $b_1 \notin Min(C_1)$:
  Assume $t_1' = {}^\bullet s_1$ and $e_1' \in \mathsf{E}_{C_1}$ such that $\rho_1(e_1') = t_1'$. Since f is an isomorphism between E_{C_1} and E_{C_2}, there exists $e_2' \in \mathsf{E}_{C_2}$ such that $e_2' = f(e_1')$. Then, by definition of ρ_2, there exists also t_2' such that $\rho_2(e_2') = t_2'$.
  From this, we get that there exists s_2' such that $t_2' = {}^\bullet s_2'$ and b_2' such that $\rho_2(b_2') = s_2'$. Since e_1' is an immediate predecessor of e_1 in E_{C_1}, by definition of f it is true that e_2' is an immediate predecessor of e_2 in E_{C_2}. Therefore it is possible to choose s_2' not only such that $t_2' = {}^\bullet s_2'$, but also $s_2' \in {}^\bullet t_2$.
  Finally, we have that $e_1' = {}^\bullet b_1$, $e_2' = {}^\bullet b_2'$ and $f(e_1') = e_2'$, therefore $s_1 \ \beta \ s_2'$.

The proof of the case for $s_2 \in {}^\bullet t_2$ is symmetrical and therefore omitted.

Thus, $(\pi_1', f', \pi_2') \in R_2$.
The case in which π_2 moves first is symmetrical.
Therefore, R_2 is an OM-bisimulation and $m_{01} \sim_{om} m_{02}$. □

Theorem 3.4.3. (OM-bisimilarity implies FC-bisimilarity) *Let $N(m_{01})$ and $N(m_{02})$ be safe nets. If $m_{01} \sim_{om} m_{02}$, then $m_{01} \sim_{fc} m_{02}$.*

Proof. If $m_{01} \sim_{om} m_{02}$ then there exists an OM-bisimulation R_1 containing the tuple $(init(N(m_{01})), init(N(m_{02})), m_{01} \times m_{02})$. Let us consider

$$\begin{aligned} R_2 \stackrel{def}{=} \{(\pi_1, f, \pi_2) \mid\ & ((m_1, \leq_1), (m_2, \leq_2), \beta) \in R_1 \text{ and} \\ & \text{for } i = 1,2,\ \pi_i = (C_i, \rho_i) \text{ is a process of } N(M_{0i}) \text{ and} \\ & init(N(M_{0i}))[\pi_i\rangle(m_i, \leq_i) \text{ and} \\ & f \text{ is an isomorphism } \mathsf{E}_{C_1} \to \mathsf{E}_{C_2} \text{ and} \\ & \forall b_1 \in Max(C_1), \forall b_2 \in Max(C_2), \\ & \rho_1(b_1)\ \beta\ \rho_2(b_2) \text{ iff} \\ & \quad \text{either } {}^\bullet b_1 = \emptyset \wedge {}^\bullet b_2 = \emptyset \\ & \quad \text{or } {}^\bullet b_1 \neq \emptyset \wedge {}^\bullet b_2 \neq \emptyset \wedge f({}^\bullet b_1) = {}^\bullet b_2\}. \end{aligned}$$

If we prove that R_2 is a FC-bisimulation, then we have that $m_{01} \sim_{fc} m_{02}$, because $(\pi_1^0, \emptyset, \pi_2^0) \in R_2$, where, for $i = 1,2$, each $\pi_i^0 = (C_i^0, \rho_i^0)$ is such that C_i^0 contains no transitions and $\rho_i^0(Min(C_i^0)) = \rho_i^0(Max(C_i^0)) = m_{0i}$. Indeed, since $(init(N(m_{01})), init(N(m_{02})), m_{01} \times m_{02}) \in R_1$ and (C_i^0, ρ_i^0) is a process of $N(m_{0i})$ and $init(N(m_{0i}))[(C_i^0, \rho_i^0)\rangle$ $init(N(m_{0i}))$, we have $(\pi_1^0, \emptyset, \pi_2^0) \in R_2$, and therefore $m_{01} \sim_{fc} m_{02}$.

Assume $(\pi_1, f, \pi_2) \in R_2$. If $\pi_1 \xrightarrow{e_1} \pi_1'$ where $\rho_1'(e_1) = t_1$, since by hypotesis $init(N(m_{01}))[\pi_1\rangle(m_1, \leq_1)$ and $init(N(m_{01}))[\pi_1'\rangle(m_1', \leq_1')$, by Lemma 3.2.4 $(m_1, \leq_1)[t_1\rangle(m_1', \leq_1')$. Since $((m_1, \leq_1), (m_2, \leq_2), \beta) \in R_1$ then there exist $t_2', m_2', \leq_2'$ such that $(m_2, \leq_2)[t_2\rangle(m_2', \leq_2')$ where $((m_1', \leq_1'), (m_2', \leq_2'), \beta') \in R_1$. Since by hypotesis $init(N(m_{02}))[\pi_2\rangle$ $(m_2, \leq_2)$, then by Lemma 3.2.4, $\pi_2 \xrightarrow{e_2} \pi_2'$ where $\rho_2'(e_2) = t_2$ and $init(N(m_{02}))[\pi_2'\rangle(m_2', \leq_2')$. Note that, for $i = 1,2$, (C_i', ρ_i') is a process of $N(m_{0i})$.

We extend f with the mapping $f'(e_1) = e_2$: since inductively f is an isomorphism between E_{C_1} and E_{C_2}, and $\pi_1 \xrightarrow{e_1} \pi_1'$, $\pi_2 \xrightarrow{e_2} \pi_2'$, then f' is an isomorphism between $\mathsf{E}_{C_1'}$ and $\mathsf{E}_{C_2'}$.

Now we need to check that the definition of β' from Definition 3.4.1 is coherent with the one obtained from R_2, i.e. the following condition holds:
$\forall b_1 \in Max(C_1'), b_2 \in Max(C_2'), \forall s_1 \in m_1', s_2 \in m_2'$ such that $\rho_1'(b_1) = s_1$ and $\rho_2'(b_2) = s_2$,

$$\rho_1'(b_1)\ \beta'\ \rho_2'(b_2) \iff \begin{cases} s_1 \in m_1 \ominus {}^\bullet t_1,\ s_2 \in m_2 \ominus {}^\bullet t_2 \text{ and } s_1\ \beta\ s_2 & \text{(i)} \\ \text{or} & \\ s_1 \in t_1^\bullet,\ s_2 \in t_2^\bullet & \text{(ii)} \end{cases}$$

Let us consider the complete transition sequence $\sigma_1 = \delta_1\ e_1$ of C_1', where:

- $init(N(m_{01}))[\rho_1'(\sigma_1)\rangle(m_1', \leq_1')$, and
- there exists $\sigma_2 = \delta_2\ e_2$ obtained by mapping each event in σ_1 with f', such that $init(N(m_{02}))[\rho_2'(\sigma_2)\rangle(m_2', \leq_2')$.

It is trivial that $|\sigma_1| = |\sigma_2|$, therefore we prove the thesis by induction on the length of σ_1:

- Case 1: $\sigma_1 = e_1$, $\sigma_2 = e_2$. We prove the two implications separately.

  Proof $\Rightarrow$) by cases on the definition of β':

  - if ${}^\bullet b_1 = \emptyset \wedge {}^\bullet b_2 = \emptyset$:
    then tokens b_1, b_2 did not move, so neither s_1, s_2 did. Therefore $s_1 \in m_1 \ominus {}^\bullet t_1$ and $s_2 \in m_2 \ominus {}^\bullet t_2$, i.e. $s_1 \in m_{01}$ and $s_2 \in m_{02}$. Since the initial β is $m_{01} \times m_{02}$, we have $s_1\ \beta\ s_2$, satisfying condition (i).
  - if ${}^\bullet b_1 \neq \emptyset \wedge {}^\bullet b_2 \neq \emptyset \wedge f'({}^\bullet b_1) = {}^\bullet b_2$:
    then ${}^\bullet b_1 = e_1$ and ${}^\bullet b_2 = e_2$, therefore $s_1 \in t_1^\bullet$ and $s_2 \in t_1^\bullet$, satisfying condition (ii).

  Proof $\Leftarrow$) by cases:

  - if $s_1 \in m_1 \ominus {}^\bullet t_1$ and $s_2 \in m_2 \ominus {}^\bullet t_2$ and $s_1\ \beta\ s_2$:
    then since the only events in σ_1, σ_2 are respectively e_1, e_2 and $\rho_1'(e_1) = t_1, \rho_2'(e_2) = t_2$, we have ${}^\bullet b_1 = \emptyset$ and ${}^\bullet b_2 = \emptyset$ because each $s_i \in \rho_i'(Min(C_i'))$.
  - if $s_1 \in t_1^\bullet$ and $s_2 \in t_2^\bullet$:
    then since the only events in σ_1, σ_2 are respectively e_1, e_2 and $\rho_1'(e_1) = t_1, \rho_2'(e_2) = t_2$, and for each $s_i \in \rho_i'(Max(C_i')$ we have ${}^\bullet b_1 = e_1$ and ${}^\bullet b_2 = e_2$. Moreover, $f'(e_1) = e_2$.

- Case $n+1$: $\sigma_1 = \delta_1\ e_1$ and $\sigma_2 = \delta_2\ e_2$. We prove the two implications separately.

  Proof $\Rightarrow$*)* by cases on the definition of β':

  - if ${}^\bullet b_1 = \emptyset \wedge {}^\bullet b_2 = \emptyset$:
    Then $b_1 \in Min(C_1')$ and $b_2 \in Min(C_2')$. For this reason, $s_1 \in m_{01}$ and $s_2 \in m_{02}$, therefore $s_1 \in m_1 \ominus {}^\bullet t_1$ and $s_2 \in m_2 \ominus {}^\bullet t_2$, i.e. $s_1 \in m_{01}$ and $s_2 \in m_{02}$. Since the initial β is $m_{01} \times m_{02}$, we have $s_1\ \beta\ s_2$, satisfying condition (i).
  - if ${}^\bullet b_1 \neq \emptyset \wedge {}^\bullet b_2 \neq \emptyset \wedge f'({}^\bullet b_1) = {}^\bullet b_2$:
    There are two cases for the event which generates b_1:
    + if ${}^\bullet b_1 = e_1$: then since $f'({}^\bullet b_1) = e_2$, we have ${}^\bullet b_2 = e_2$; therefore $s_1 \in t_1^\bullet$ and $s_2 \in t_2^\bullet$, satisfying condition (ii).
    + if ${}^\bullet b_1 \neq e_1$: then since ${}^\bullet b_1 \neq \emptyset$, there exists $e_1' \in \delta_1$ such that ${}^\bullet b_1 = e_1'$. By the fact that f' is an isomorphism between $\mathsf{E}_{C_1'}$ and $\mathsf{E}_{C_2'}$, and that $f'(e_1) = e_2$, there exists also $e_2' \in \delta_2$ where $f^{-1}(e_1') = e_2'$ such that ${}^\bullet b_2 = e_2'$. By inductive hypotesis on δ_1 we have $s_1\ \beta\ s_2$, satisfying condition (i).

  Proof $\Leftarrow$*)* by cases:

  - if $s_1 \in m_1 \ominus {}^\bullet t_1$ and $s_2 \in m_2 \ominus {}^\bullet t_2$ and $s_1\ \beta\ s_2$:
    then there are two possible cases for b_1:
    + if $b_1 \in Min(C_1')$:
      then, since $Min(C_1') = Min(C_1)$ and $Min(C_2') = Min(C_2)$, the same reasoning of the base case of induction applies.
    + if $b_1 \notin Min(C_1')$:
      then ${}^\bullet b_1 \neq \emptyset$; since however b_1 does not move, because $s_1 \in m_1 \ominus {}^\bullet t_1$, it is possible to apply the induction hypotesis on δ_1, therefore ${}^\bullet b_2 \neq \emptyset$ and $f({}^\bullet b_1) = {}^\bullet b_2$, and by conservative extension of f, $f'({}^\bullet b_1) = {}^\bullet b_2$.
  - if $s_1 \in t_1^\bullet$ and $s_2 \in t_2^\bullet$:
    then since $\rho_1'(e_1) = t_1, \rho_2'(e_2) = t_2$, and for each $s_i \in \rho_i'(Max(C_i')$ we have ${}^\bullet b_1 = e_1$ and ${}^\bullet b_2 = e_2$. Moreover, $f'(e_1) = e_2$.

  Thus $(\pi_1', f', \pi_2') \in R_2$.
  The case in which π_2 moves first is symmetrical.
  Therefore, R_2 is an FC-bisimulation and $m_{01} \sim_{fc} m_{02}$. □

Corollary 3.4.4. (OM-bisimilarity and FC-bisimilarity coincide) *Let $N(m_{01})$ and $N(m_{02})$ be safe nets. $m_1 \sim_{om} m_2$ if and only if $m_1 \sim_{fc} m_2$.*

Proof. By theorems 3.4.2 and 3.4.3, we get the thesis. □

Corollary 3.4.5. (FC-bisimilarity is decidable for finite safe nets) *Given $N(m_1)$ and $N(m_2)$ safe nets, it is decidable to check whether $m_1 \sim_{fc} m_2$.*

Proof. By Corollary 3.4.4 we have to check whether there exists an OM-bisimulation $\mathfrak{B}$ for the given net N and marking m_1, m_2. By Proposition 3.1.4, $OM(N)$, $[init(N(m_1))\rangle$ and $[init(N(m_2)\rangle$ are finite. Since S is finite, also $\mathscr{P}(S \times S)$ is finite, thus there are only finitely many objects that are possible elements of $\mathfrak{B}$.

Since $OM(N)$ is finite, the set of all possible tuples $((m_1, \leq_1), (m_2, \leq_2), \beta)$ is finite. Since $\mathfrak{B}$ is a set of said tuples, there are only finitely many possible sets $\mathfrak{B}$. Therefore we can check by exhaustive search whether one of them is an OM-bisimulation. □

Note that also in this case the complexity of this decision procedure is prohibitive. Already the number of reachable markings of a safe net can be exponential in the size of the net. However, in [JM96] it is shown that the complexity of checking whether two safe markings are history-preserving bisimilar (i.e. fully-concurrent bisimilar) is complete for DEXPTIME. This is done using a proof technique which keeps a preorder of most recently fired transitions instead of most recent tokens.

Example 3. Consider again the simple finite P/T net net N in Figure 2.1. As expected, we have $s_1 \sim_{om} s_3$ and this is proved by relation

$$R = \{((s_1, s_1 \leq s_1), (s_3, s_3 \leq s_3), s_1\ \beta\ s_3), ((s_2, s_2 \leq s_2), (\theta, \emptyset), \emptyset)\}.$$

Indeed, the two initial ordered markings are related by R and, since they are initial, tokens s_1 and s_3 are related by β. If $(s_1, s_1 \leq s_1)$ moves first by $(s_1, s_1 \leq s_1)[t_{a_1}\rangle(s_2, s_2 \leq s_2)$, $(s_3, s_3 \leq s_3)$ can respond with $(s_3, s_3 \leq s_3)[t_{a_2}\rangle(\theta, \emptyset)$, and $((s_2, s_2 \leq s_2), (\theta, \emptyset), \emptyset) \in R$. Note that, since θ is empty, also relation β in the final OM-triple is empty. The case where $(s_3, s_3 \leq s_3)$ moves first is symmetrical.
Not surprisingly, $s_1 \not\sim_{omc} s_3$, because $|s_2| \neq |\theta|$ and therefore an OMC-bisimulation cannot be built. □

Chapter 4

Decidability Results for Finite Bounded Petri Nets

For bounded nets, we cannot assume the useful coincidence between a token and its current place, because there can be a (bounded) set of tokens in each place. The use of ordered markings makes sense on sets, but not on multisets: for that reason we distinguish each individual token by introducing *indexed markings*, which are sets that can be abstracted to multisets by forgetting the indexing information. The index, unique for every token in a place, allows tokens to be treated as a single individual unit: this leads us to define a token game according to the individual token philosophy in a way that is, to the best of our knowledge, original. Since multisets are turned into sets, Vogler's proof technique [Vog91] can be generalized to bounded nets by using *ordered* indexed markings.

4.1 Indexed marking semantics

Definition 4.1.1. (Indexed marking) *Given a finite net $N = (S,A,T)$ and a marking m of N, an* indexed marking *is a function $k : S \to \mathscr{P}(\mathbb{N})$ such that the associated (de-indexed) marking m is obtained as $m(s) = |k(s)|$ for each $s \in S$. In this case, we write $\alpha(k) = m$. The support set $dom(k)$ is $\{s \in S \mid k(s) \neq \emptyset\}$.*
The set of the indexed markings with support set S is denoted $\mathfrak{K}(S)$.
We define the set of indexed places as $\{(s_1,i_1),\ldots,(s_n,i_n)\} \in \mathscr{P}(S \times \mathbb{N})$ where each $s_j \in S$, $i_j \in \mathbb{N}$ and $\nexists j_1, j_2$ such that $s_{j_1} = s_{j_2} \wedge i_{j_1} = i_{j_2}$ (this last condition guarantees that each token on a place s has an index different from the index of any other token on s). Note that an indexed marking can also be represented as a set of indexed places, i.e. a subset of $S \times \mathbb{N}$. In the same fashion, the set of all indexed marking $\mathfrak{K}(S)$ can also be seen as a subset of the set of indexed places, i.e. $\mathfrak{K}(S) \subseteq \mathscr{P}(S \times \mathbb{N})$. Each element of an indexed marking, i.e. each indexed place, is a token.
An indexed marking $k \in \mathfrak{K}(S)$ is closed *if $k(s) = \{1,\ldots,|k(s)|\}$ for all $s \in dom(k)$. If there exists a bounded marked net $N(m_0)$ and a closed indexed marking k_0 such that $\alpha(k_0) = m_0$, we say that k_0 is an* initial indexed *marking of N. The application of such marking k_0 to N is denoted by $N(k_0)$.*
We define the difference between an indexed marking and a marking with same support as $\boxminus : \mathfrak{K}(S) \to \mathscr{M}(S) \to \mathscr{P}(\mathfrak{K}(S))$

$$
\begin{aligned}
k \boxminus \theta &= \{k\} \\
k \boxminus (s \oplus m) &= (k \boxminus s) \boxminus m \\
\{k_1,\ldots k_n\} \boxminus m &= k_1 \boxminus m \cup \ldots \cup k_n \boxminus m \\
k \boxminus s = \{k' \mid k'(s') &= k(s') \text{ if } s' \neq s \text{ , or} \\
&= k(s) \setminus \{n\} \text{ if } s' = s \text{ and } n \in k(s)\}
\end{aligned}
$$

And the union of an indexed marking and a marking with same support as $\boxplus : \mathfrak{K}(S) \to \mathscr{M}(S) \to \mathfrak{K}(S)$

$$
\begin{aligned}
k \boxplus \theta &= k \\
k \boxplus (s \oplus m) &= (k \boxplus s) \boxplus m \\
k \boxplus s &= k'
\end{aligned}
$$

where for all $s' \in S$, $k'(s')$ is defined as:

$$k'(s') = \begin{cases} k(s') & \text{if } s' \neq s \\ k(s) \cup \{n\} & \text{if } s' = s,\ n = min(\overline{k(s)}) \text{ where } \overline{k(s)} = \mathbb{N} \setminus k(s) \end{cases}$$

Note that we use $min(H)$, where $H \in \mathscr{P}(\mathbb{N})$, to denote the least element of H. □

Note that the difference between an indexed marking and a marking is a *set* of indexed markings: since it makes no sense to prefer a single possible execution over another, all possible choices for n are to be considered.

The token game is modified accordingly, taking into account the individual token interpretation.

Definition 4.1.2. (Token game with indexed markings) *Given a net $N = (S,A,T)$ and an indexed marking $k \in \mathfrak{K}(S)$ such that $m = \alpha(k)$, we say that a transition $t \in T$ is* enabled *at k if ${}^\bullet t \subseteq m$, denoted $k[\![t\rangle$. If t occurs, the firing of t enabled at k produces the indexed marking k', denoted $k[\![t\rangle k'$, if*

- *$\exists k'' \in k \boxminus {}^\bullet t$ and*
- *$k' = k'' \boxplus t^\bullet$.*

Note that there can be more than a single marking produced from the firing of t (the transition relation is nondeterministic), but for all k' such that $k[\![t\rangle k'$, it is true that $\alpha(k') = m \ominus {}^\bullet t \oplus t^\bullet$. Therefore, if $m[t\rangle m'$ then $k[\![t\rangle k'$, where $\alpha(k') = m'$. □

In the following, we prefer to use the interpretation of indexed markings as set of indexed places i.e., we denote an indexed marking $k \in \mathfrak{K}(S)$ as the set $\{(s_1,n_1)\dots(s_i,n_i)\}$ where $|k| = i$.

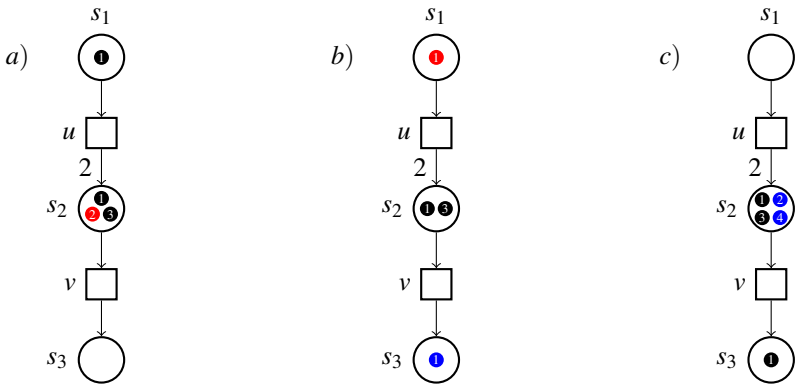

Figure 4.1: Execution of the transition labeled by v, then of the transition labeled by u, on a bounded net with initial marking $m_0 = s_1 \oplus 3s_2$. Tokens to be consumed are in red, generated ones in blue.

Example 4. In Figure 4.1(a) a simple net N is given. The initial marking is $m_0 = s_1 \oplus 3s_2$, therefore the marked net $N(m_0)$ is 5-bounded. The initial indexed marking is $k_0 = \{(s_1,1),(s_2,1),(s_2,2),(s_2,3)\}$.
Let us suppose that transition t_2, labeled by v, occurs. There are three possible ways to remove a token from s_2: removing $(s_2,1)$, or removing $(s_2,2)$, or removing $(s_2,3)$. Indeed, the operation $k_0 \boxminus {}^\bullet t_2$ yields a set of three possible indexed markings, each one a possible result of the difference: $\{\{(s_1,1),(s_2,2),(s_2,3)\},\{(s_1,1),(s_2,1),(s_2,3)\}$, $\{(s_1,1),(s_2,1),(s_2,2)\}\}$. Let us choose, for the sake of the argument, that the token deleted by t_2 is $(s_2,2)$, i.e. choose $k' = \{(s_1,1),(s_2,1),(s_2,3)\}$. The union $k' \boxplus t_2^\bullet$ easily yields the indexed marking $k_1 = \{(s_1,1),(s_2,1),(s_2,3),(s_3,1)\}$, as depicted in Figure 4.1(b). Note that this choice was arbitrary and two other values of k_1 are possible. Indeed, from Definition 4.1.2, we know that the transition relation on indexed markings is nondeterministic. However, the resulting marked net is the same for all three cases, that is, the same of Figure 4.1(b) without indexes.
Now we suppose that (given the indexed marking k_1 from above) transition t_1, labeled by u, occurs. In that case, $k_1 \boxminus {}^\bullet t_1$ yields the singleton set $\{\{(s_2,1),(s_2,3),(s_3,1)\}\}$, therefore we choose $k'' = \{(s_2,1),(s_2,3),(s_3,1)\}$. Since $t_1^\bullet = s_2 \oplus s_2$, we show in detail how $k'' \boxplus t_1^\bullet$ is computed. First, we apply the definition for union with non-singleton multisets: $k'' \boxplus (s_2 \oplus s_2) = (k'' \boxplus s_2) \boxplus s_2$. Then, we compute $k'' \boxplus s_2$: since the least free index for the place s_2 is 2, $k'' \boxplus s_2 = \{(s_2,1),(s_2,2)(s_2,3),(s_3,1)\}$. Now we apply again the definition: note that this time the least free index for s_2 is 4, and the final result is $k_2 = \{(s_2,1),(s_2,2)(s_2,3),(s_2,4),(s_3,1)\}$. The resulting marked net is depicted in Figure 4.1(c). □

	generated	deleted	untouched
$m[t\rangle m'$	$t^\bullet$	${}^\bullet t$	$m \ominus {}^\bullet t$
$k[\![t\rangle k'$	$k' \setminus k''$	$k \setminus k''$	k''

Table 4.1: Different notation for tokens in the token game. On the first line, the collective case. On the last, the individual case.

The notation for tokens in the token game has become slightly more unintuitive, so in Table 4.1 we provide a comparison between the one displayed in previous sections and the one we will use in the following part of this work. Given a transition t such that $k[\![t\rangle k'$ and $m[t\rangle m'$, assume $k'' \in k \boxminus {}^\bullet t$ such that $k' = k'' \boxplus t^\bullet$.

Definition 4.1.3. (Firing sequence with IM) *Given a finite net $N = (S,A,T)$ and an indexed marking k, a* firing sequence *starting at k is defined inductively as follows:*

- *$k[\![\varepsilon\rangle k$ is a firing sequence (where ε denotes an empty sequence of transitions) and*
- *if $k[\![\sigma\rangle k'$ is a firing sequence and $k'[\![t\rangle k''$, then $k[\![\sigma t\rangle k''$ is a firing sequence.*

If $\sigma = t_1 \dots t_n$ (for $n \geq 0$) and $k[\![\sigma\rangle k'$ is a firing sequence, then there exist $k_1, \dots k_{n+1}$ such that $k = k_1[\![t_1\rangle k_2 \dots$ $\dots k_n[\![t_n\rangle k_{n+1} = k'$ and $\sigma = t_1 \dots t_n$ called a transition sequence *starting at k and ending at k'. The set of reachable indexed markings from k is $[\![k\rangle = \{k' \mid \exists \sigma . k[\![\sigma\rangle k'\}$. Given a bounded net $N(k_0)$, we call $[\![k_0\rangle$ the set of reachable indexed marking of N, denoted by $IM(N)$.* □

Proposition 4.1.4. *Given a finite bounded net $N = (S,A,T,m_0)$, the set $IM(N) \subseteq \mathfrak{K}(S)$ of reachable indexed markings is finite.*

Proof. By definition, S is finite and since $N(m_0)$ is bounded, there exists a $h \in \mathbb{N}$ such that $\forall s \in S, \forall m \in [m_0\rangle, m(s) \leq h$, i.e. the net is h-bounded. The set of all indexed markings of N is infinite, because $\mathscr{P}(S \times \mathbb{N})$ is infinite. However, the index of each token $(s,n) \in \mathfrak{K}(S)$ in a reachable indexed marking cannot grow infinitely. Indeed, for all $s \in S$ and for all $m \in [m_0\rangle$, the initial index of a token is at most $|m(s)|$ because the initial indexed marking is closed, and since $N(m_0)$ is h-bounded, by definition of $\boxplus$ (we choose always the least possible one), the index of a token in a reachable indexed marking is always less or equal than h. Therefore, $IM(N)$ is finite. □

If $N(m_0)$ is h-bounded, we have that $IM(N) \subseteq \mathscr{P}(S \times 1, \dots, h)$, i.e. all reachable indexed markings are finitely many.
To define some of the bisimulations used in the following proofs, a generalization of the ones from Chapter 3 using ordered *indexed* markings, a generalization of additive closure [Gor20a] is needed. Instead of places, we use tokens.

***Remark* 3. (Additive closure on tokens)** *In the following, we will use the definition of additive closure [Gor20a] also on* token relations*. Note that, since the indexed marking semantics considers* sets *of tokens, it would make no sense to consider multisets obtained by additive closure of tokens, even if technically possible.*

As in the safe case, two indexed markings are related by $R^\oplus$ only if they have the same size.

4.2 Ordered indexed marking semantics

Following the approach of the previous chapter, we define a semantics based on *ordered indexed markings*, where the preorder reflects the precedence in the generation of tokens.

Definition 4.2.1. (Ordered indexed marking) *Given a net $N = (S,A,T)$ and an indexed marking $k \in \mathfrak{K}(S)$, the pair $(k, \leq)$ is an ordered indexed marking if $\leq \subseteq k \times k$ is a preorder, i.e. a reflexive and transitive relation. The set of all possible ordered indexed markings of N is denoted by $OIM(N)$. If k_0 is the initial marking of N, which is assumed to be closed (cf. Definition 4.1.1) we define the* initial ordered indexed marking *$init(N)$ as $(k_0, k_0 \times k_0)$. If the initial indexed marking is not clear from the context, we write $init(N(k_0))$ to denote the initial ordered indexed marking.* □

Definition 4.2.2. (Token game with ordered indexed markings) *Given a net $N = (S,A,T)$ and an ordered indexed marking $(k, \leq)$, we say that a transition $t \in T$ is* enabled *at $(k, \leq)$ if $k[\![t\rangle$; this is denoted by $(k, \leq)[\![t\rangle$.*

The firing of t enabled at $(k, \leq)$ may produce an ordered indexed marking $(k', \leq')$ - and we denote this by $(k, \leq)[\![t\rangle(k', \leq')$ - where:

- $k'' \in k \boxminus {}^\bullet t$ *such that* $k' = k'' \boxplus t^\bullet$
- *for all* $(s_h,i_h),(s_j,i_j) \in k'$, $(s_h,i_h) \leq' (s_j,i_j)$ *if and only if:*
  - $(s_h,i_h),(s_j,i_j) \in k''$ *and* $(s_h,i_h) \leq (s_j,i_j)$*, or*
  - $(s_h,i_h),(s_j,i_j) \in k' \setminus k''$*, or*
  - $(s_h,i_h) \in k''$, $(s_j,i_j) \in k' \setminus k''$ *and there exists* $(s_l,i_l) \in k \setminus k''$ *such that* $(s_h,i_h) \leq (s_l,i_l)$.

Note that, as for indexed markings, many different ordered indexed markings are produced from the firing of t. This means that also the transition relation for ordered indexed markings is nondeterministic. □

Example 5. Consider again the net in Figure 4.1 and the first part of the execution of Example 4, i.e. $k_0[\![t_2\rangle k_1$. According to Definition 4.2.1, the initial ordered indexed marking is $(k_0,\leq_0)$, where $\leq_0 = k_0 \times k_0$. When t_2 fires, token $(s_2,2)$ is removed and token $(s_3,1)$ is generated, while all other tokens are untouched. Let us denote the preorder induced by the firing of t_2 as $\leq_1$. According to the first item of Definition 4.2.2, since $(s_3,1)$ is generated by the firing of t_2, we have $(s_3,1) \leq_1 (s_3,1)$. According to the second item of Definition 4.2.2, the preorder on all tokens untouched by t_2 remains the same, therefore e.g. $(s_2,3) \leq_1 (s_1,1)$ and viceversa. Furthermore, consider $(s_1,1)$ and $(s_3,1)$: we have that t_2 generates $(s_3,1)$, deletes $(s_2,2)$ and leaves $(s_1,1)$ untouched. Since $(s_1,1) \leq_0 (s_2,2)$, by the third item of Definition 4.2.2 we have $(s_1,1) \leq_1 (s_3,1)$. The same reasoning applies to all untouched tokens. Summing up, we have $(k_0,\leq_0)[\![t_2\rangle(k_1,\leq_1)$ where $\leq_1 = \leq_0 \setminus \{((s_i,n_i),(s_j,n_j)) \in k_0 \mid (s_i,n_i) = (s_2,2) \vee (s_j,n_j) = (s_2,2)\} \cup \{((s_1,1),(s_3,1)),((s_2,1),(s_3,1)),((s_2,3),(s_3,1)),((s_3,1),(s_3,1))\}$. □

Definition 4.2.3. (Firing sequence with OIM) *A* firing sequence *starting at* $(k,\leq)$ *is defined inductively as follows:*

- $(k,\leq)[\![\varepsilon\rangle(k,\leq)$ *is a firing sequence (where* ε *denotes an empty sequence of transitions) and*
- *if* $(k,\leq)[\![\sigma\rangle(k',\leq')$ *is a firing sequence and* $(k',\leq')'[\![t\rangle(k'',\leq'')$*, then* $(k,\leq)[\![\sigma t\rangle(k'',\leq'')$ *is a firing sequence.*

If $\sigma = t_1 \dots t_n$ *(for* $n \geq 0$*) and* $(k,\leq)[\![\sigma\rangle(k',\leq')$ *is a firing sequence, then there exist* $(k_1,\leq_1),\dots,(k_{n+1},\leq_{n+1})$ *such that* $(k,\leq) = (k_1,\leq_1)[\![t_1\rangle(k_2,\leq_2)\dots(k_n,\leq_n)[\![t_n\rangle(k_{n+1},\leq_{n+1}) = (k',\leq')$ *and* $\sigma = t_1 \dots t_n$ *called a* transition sequence *starting at* $(k,\leq)$ *and ending at* $(k',\leq')$*. The set of reachable ordered indexed markings from* $(k,\leq)$ *is* $[\![(k,\leq)\rangle = \{(k',\leq') \mid \exists \sigma \,.\, (k,\leq)[\![\sigma\rangle(k',\leq')\}$.
Given a closed indexed marking k_0*, the set of all the reachable ordered indexed markings of* $N(k_0)$ *is denoted by* $[\![init(N)\rangle$. □

Proposition 4.2.4. *Given a bounded net* $N = (S,A,T,m_0)$, $[\![init(N)\rangle$ *is finite.*

Proof. The set $IM(N) \subseteq \mathfrak{K}(S)$ of reachable indexed markings is finite because of Proposition 4.1.4. The set of possible preorders for a reachable indexed marking $k = \{(s_1,n_1)\dots(s_m,n_m)\} \in IM(N)$ is finite, because $\leq \subseteq k \times k$. Therefore, $[\![init(N)\rangle$ is finite. □

4.3 Ordered indexed marking and causality-based semantics

In Chapter 3, given a process $\pi = (C,\rho)$, the preorder induced on a net N by π, i.e. the preorder on tokens of a marking derived by applying ρ to $Max(C)$, was obtained just by looking at the *structure* of C, because tokens were identified with their respective places. Since on bounded nets it is possible to have more than one token in a place, the *dynamics* of the causal net must be taken into consideration to compute the correct index for each token in the current marking.

Given a transition sequence σ, there is an operational preorder on tokens obtained by Definition 4.2.3, and a preorder derived from the process $\leq_\pi$ obtained from the causal net C corresponding to the transition sequence σ. In the following, we relate the two.

If $\pi = (C,\rho)$ is a process of a marked net $N(m_0)$ and k_0 is the initial indexed marking for $N(m_0)$, i.e. $\alpha(k_0) = m_0$ (cf. Definition 4.1.1), we say that π is a process of $N(k_0)$ as a shorthand for π is a process of $N(\alpha(k_0))$.

Lemma 4.3.1. (A minimality condition for $\leq$**)** *Let* $\pi = (C,\rho)$ *be a process of* $N(k_0)$ *and* σ *a complete transition sequence of* C *such that* $init(N)[\![\rho(\sigma)\rangle(k,\leq)$*. For all* $b \in Max(C)$ *such that* $(\rho(b),i) \in k$*, if* $b \in Min(C)$*, then:*

i) $(\rho(b),i) \in k_0$*; and*

ii) for all $b' \in Max(C)$ *such that* $(\rho(b'),i') \in k$*,* $(\rho(b),i) \leq (\rho(b'),i')$.

In other words, if $b \in Max(C)$ and also $b \in Min(C)$, then the current token $(\rho(b),i) \in k$ was actually already present in the initial indexed marking k_0 and is also minimal for the preorder $\leq$.

Proof. By induction on the length of σ.

- Case 0: $\sigma = \varepsilon$.
  For the first statement, we have that since $Min(C) = Max(C) = \alpha(k_0)$ because the causal net does not contain any event, then $(\rho(b),i) \in k_0$.

  For the second statement, since for all $(\rho(b),i) \in k_0$ it is true that ${}^\bullet b = \emptyset$, and $\leq = k_0 \times k_0$, then $(\rho(b),i) \leq (\rho(b'),i')$.

- Case n+1: $\sigma = \alpha\, e$ where $e \notin \alpha$.
  The induction hypotesis is $init(N)[\![\rho(\alpha)\rangle(k,\leq)$ where the thesis holds for $(k,\leq)$.
  The step is $(k,\leq)[\![\rho'(e)\rangle(k',\leq')$ and $(C,\rho) \xrightarrow{e} (C',\rho')$. Assume $k'' \in k \boxminus {}^\bullet\rho'(e)$ such that $k' = k'' \boxplus \rho'(e)^\bullet$.
  For the first statement, consider the token $(\rho'(b),i)$:

  - if $(\rho'(b'),i') \in k''$: then $(\rho'(b),i) = (\rho(b),i)$ because it has not been touched by transition t; by inductive hypotesis $(\rho(b),i) \in k_0$ and therefore $(\rho'(b),i) \in k_0$.
  - if $(\rho'(b'),i') \in k' \setminus k''$: absurd, because it would mean that ${}^\bullet b \neq \emptyset$, and a causal net cannot be cyclic (cf. Definition 2.2.2), contradicting the hypotesis $b \in Min(C)$.
  - $(\rho'(b'),i') \in k \setminus k''$: absurd, because $b \in Max(C')$ by hypotesis.

  Note that this means that token $(\rho(b),i)$ - which is the same as $(\rho'(b),i)$ - never moved.

  For the second statement, consider the token $(\rho'(b'),i')$:

  - if $(\rho'(b'),i') \in k''$: since the inductive hypotesis on $(k,\leq)$ holds, i.e. $(\rho(b),i) \leq (\rho(b'),i')$ and b,b' are not touched by event e, i.e. $(\rho(b),i) = (\rho'(b),i)$ and $(\rho(b'),i') = (\rho'(b'),i')$, then $(\rho'(b),i) \leq' (\rho'(b'),i')$;
  - if $(\rho'(b'),i') \in k' \setminus k''$: then $(\rho'(b'),i')$ must have been generated from a $(\rho(b''),i'') \in k \setminus k''$. Since the inductive hypotesis on $(k,\leq)$ holds for all $(\rho(b''),i'') \in k \setminus k''$, then $(\rho'(b),i) \leq' (\rho'(b'),i')$.
  - if $(\rho'(b'),i') \in k \setminus k''$: absurd, because $b' \in Max(C')$. □

Proposition 4.3.2. *Given a net $N = (S,A,T)$ such that $N(k_{01})$ and $N(k_{02})$ are both bounded nets, two processes $\pi_i = (C,\rho_i)$ of $N(k_{0i})$ for $i = 1,2$, a complete transition sequence σ of C and two indexed markings k_1,k_2 such that $init(N(k_{0i}))[\![\rho_i(\sigma)\rangle(k_i,\leq_i)$ for $i = 1,2$, we have that $|k_1| = |k_2|$.*

Proof. Analogous to Proposition 3.2.2, as indexing and net boundedness do not change the argument. □

In order to state the next theorem, we use the following notation: given a transition sequence $\delta = t_1,\dots t_{i-1},t_i,\dots,t_n$, we denote the cut of δ at t_i as $\delta \setminus t_i = t_1,\dots,t_{i-1}$.

Theorem 4.3.3. (Coherence of $\leq$ and process)
Let $\pi = (C,\rho)$ be a process of $N(k_0)$ and σ a complete transition sequence of C such that $init(N)[\![\rho(\sigma)\rangle(k,\leq)$.
$\forall b,b' \in Max(C)$, $\forall (\rho(b),i),(\rho(b'),i') \in k$ we have:

$$(\rho(b),i) \leq (\rho(b'),i') \iff \begin{cases} b \in Min(C) \wedge (\rho(b),i) \in k_0 & (1) \\ or & \\ {}^\bullet b \neq \emptyset \wedge {}^\bullet b' \neq \emptyset \wedge {}^\bullet b \leq_\pi {}^\bullet b' \wedge \Phi & (2) \end{cases}$$

where

$$\Phi = \begin{cases} init(N)[\![\rho(\sigma) \setminus \rho({}^\bullet b)\rangle(H,\leq_H)[\![\rho({}^\bullet b)\rangle(J,\leq_J), \\ \quad where\ (\rho(b),i) \in J\ and\ let\ L \in H \boxminus {}^\bullet\rho({}^\bullet b)\ ,\ (\rho(b),i) \notin L, \\ \quad such\ that\ J = L \boxplus \rho({}^\bullet b)^\bullet \quad \wedge \\ init(N)[\![\rho(\sigma) \setminus \rho({}^\bullet b')\rangle(H',\leq_{H'})[\![\rho({}^\bullet b')\rangle(J',\leq_{J'}), \\ \quad where\ (\rho(b'),i') \in J'\ and\ let\ L' \in H' \boxminus {}^\bullet\rho({}^\bullet b')\ ,\ (\rho(b'),i') \notin L', \\ \quad such\ that\ J' = L' \boxplus \rho({}^\bullet b')^\bullet \end{cases}$$

Proof. We prove the implication in the two directions.

Proof $\Rightarrow$*).* By induction on the length of σ.

- case 0: $\sigma = \varepsilon$.
  Since $init(N)[\![\varepsilon\rangle init(N)$, $(\rho(b),i) \in k_0$. Moreover since $\rho(Max(C^0)) = \rho(Min(C^0)) = k_0$ and b did not move because $\sigma = \varepsilon$, then $b \in Min(C^0)$. The condition (1) is satisfied.

- case n+1: $\sigma = \delta.e$ where $e \notin \delta$.
  The induction hypotesis is $init(N)[\![\rho(\delta)\rangle(k,\leq)$ where the thesis holds for $(k,\leq)$.
  The step is $(k,\leq)[\![\rho'(e)\rangle(k',\leq')$ and $\pi = (C,\rho) \xrightarrow{e} (C',\rho') = \pi'$.
  Let $k'' \in k \boxminus {}^\bullet\rho'(e)$ such that $k' = k'' \boxplus \rho'(e)^\bullet$, then, by cases on the definition of $(\rho'(b),i) \leq' (\rho'(b'),i')$:

  - if $(\rho'(b),i),(\rho'(b'),i') \in k''$ and $(\rho'(b),i) \leq (\rho'(b'),i')$:
    then $(\rho'(b),i),(\rho'(b'),i')$ are not touched by transition $\rho'(e)$, therefore the thesis follows from inductive hypotesis on $(k,\leq)$.
  - if $(\rho'(b),i),(\rho'(b'),i') \in k' \setminus k''$:
    then ${}^\bullet b = {}^\bullet b' = e \neq \emptyset$ and thus ${}^\bullet b \leq_{\pi'} {}^\bullet b'$. Φ is trivial since $(\rho'(b),i)$ and $(\rho'(b'),i')$ are generated by the same transition $\rho'(e)$ and $(k,\leq)[\![\rho'(e)\rangle(k',\leq')$. Condition (2) is satisfied.
  - if $(\rho'(b),i) \in k''$, $(\rho'(b'),i') \in k' \setminus k''$ and there exists $(\rho'(b''),i'') \in k \setminus k''$ such that $(\rho'(b),i) \leq (\rho'(b''),i'')$:
    + if $(\rho'(b),i)$ is such that $\rho'(b) \in \rho'(Min(C'))$:
      then $b \in Min(C')$ and $(\rho'(b),i) \in k_0$, therefore the condition (1) is satisfied.
    + if $(\rho'(b),i)$ is such that $\rho'(b) \notin \rho'(Min(C'))$:
      then ${}^\bullet b' \neq \emptyset$ by hypotesis, ${}^\bullet b \neq \emptyset$ because $(\rho'(b),i) \notin \rho'(Min(C))$ and, since $(\rho'(b),i) \leq (\rho'(b''),i'')$, by induction we have ${}^\bullet b \leq_\pi {}^\bullet b''$. Since $b'' \in {}^\bullet e$ and $b' \in e^\bullet$, it is true that ${}^\bullet b'' \leq_{\pi'} {}^\bullet b'$, and by transitivity ${}^\bullet b \leq_{\pi'} {}^\bullet b'$. In the case of $(\rho'(b),i)$, Φ is trivial as in the case above.
      In the case of $(\rho'(b'),i')$, since $\rho'({}^\bullet b') = {}^\bullet\rho'(b')$, we have $init(N)[\![\rho(\delta)\rangle(k,\leq)[\![\rho'({}^\bullet b') = \rho'(e)\rangle$ $(k',\leq')$ and $(\rho'(b'),i') \in k' \setminus k''$, i.e. given any $k'' \in k \boxminus {}^\bullet\rho'(b')$ such that $k' = k'' \boxplus \rho'(b')^\bullet$ $(\rho'(b'),i') \notin k''$. Therefore, condition (2) is satisfied.

Proof $\Leftarrow$*).* By induction on the length of σ.

- case 0: $\sigma = \varepsilon$.
  Since $(\rho(b),i) \in k_0$ and ${}^\bullet b = \emptyset$, then by Lemma 4.3.1, $(\rho(b),i) \leq (\rho(b'),i')$ for all $(\rho(b'),i') \in k_0$, i.e. $\leq = k_0 \times k_0$.

- case n+1: $\sigma = \delta.e$ where $e \notin \delta$.
  The induction hypotesis is $init(N)[\![\rho(\delta)\rangle(k,\leq)$ where the thesis holds for $(k,\leq)$.
  The step is $(k,\leq)[\![\rho'(e)\rangle(k',\leq')$ and $\pi = (C,\rho) \xrightarrow{e} (C',\rho') = \pi'$.
  Let $k'' \in k \boxminus {}^\bullet\rho'(e)$ such that $k' = k'' \boxplus \rho'(e)^\bullet$, then by inspection on the hypotesis:

  - if condition (1) holds:
    since for Lemma 4.3.1 $(\rho'(b),i)$ is minimal for $\leq'$, the thesis follows.
  - if condition (2) holds:
    there are 4 possible combinations of $(\rho'(b),i),(\rho'(b'),i')$:
    + if $(\rho'(b),i),(\rho'(b'),i') \in k''$:
      The inductive hypotesis on $(k,\leq)$ holds, i.e. $(\rho'(b),i) \leq (\rho'(b'),i')$ and since b,b' are not touched by event e, $(\rho'(b),i) \leq' (\rho'(b'),i')$.
    + if $(\rho'(b),i),(\rho'(b'),i') \in k' \setminus k''$:
      Then, since the two tokens are generated by the same transition $\rho'(e)$, $(\rho'(b),i) \leq' (\rho'(b'),i')$.
    + if $(\rho'(b),i) \in k''$ and $(\rho'(b'),i') \in k' \setminus k''$:
      Then, since $(\rho'(b'),i') \in k' \setminus k''$, by Proposition 2.2.7, it is true that there exists $b'' \in {}^\bullet e$ such that ${}^\bullet b \leq_\pi {}^\bullet b''$, and by Φ we have that $(\rho'(b''),i'') \in k \setminus k''$. Note that Φ holds for $(\rho'(b''),i'')$, too.
      Then, by inductive hypotesis, $(\rho'(b),i) \leq (\rho'(b''),i'')$. Therefore, since $(\rho'(b'),i') \in k' \setminus k''$ and $(\rho'(b''),i'') \in k \setminus k''$, we have $(\rho'(b),i) \leq' (\rho'(b'),i')$.
    + if $(\rho'(b'),i') \in k''$ and $(\rho'(b),i) \in k' \setminus k''$:
      absurd, since ${}^\bullet b \leq_{\pi'} {}^\bullet b'$.

□

Now we introduce the following notation as a shorthand: given a process $\pi = (C, \rho)$ of a net $N(k_0)$ and σ a complete transition sequence of C, we denote $init(N)[\![\pi\rangle(k, \leq)$ if $init(N)[\![\rho(\sigma)\rangle(k, \leq)$. Note that this is just notation, whereas in Chapter 3 we had Definition 3.2.3. The equivalent of Lemma 3.2.4 follows directly from Theorem 4.3.3.

Lemma 4.3.4. *Let $\pi = (C, \rho)$ a process of $N(k_0)$ such that $init(N)[\![\pi\rangle(k, \leq)$. Then $(k, \leq)[\![t\rangle(k', \leq')$ if and only if $\pi \xrightarrow{e} \pi'$ where $\rho'(e) = t$ and $init(N)[\![\pi'\rangle(k', \leq')$.*

Proof. Consider the sequence $\sigma.e$ where σ is a complete transition sequence of C. The thesis follows from Theorem 4.3.3. □

Example 6. In Figure 4.2(a), the same 5-bounded P/T net N as Figure 4.1 is depicted, together with its empty process. Figure 4.2(b,c) shows how the process corresponding to the transition sequence $t_2 t_1$ grows. We consider the same execution as Example 4, i.e. $k_0[\![t_2\rangle k_1[\![t_1\rangle k_2$. For simplicity's sake, in the following each condition will be mapped to the place having same subscript and each event will be mapped to the transition having same label. We will denote each process π_i as the one thus corresponding to causal net C_i.

Before any transition fires, we have $init(N) = (k_0, \leq_0)$ where $\leq_0 = k_0 \times k_0$ by Definition 4.2.1. Not surprisingly, all conditions b_i^j are minimal in the causal net C_0 and mapped to tokens in the initial indexed ordered marking.

The firing of t_2 deletes token $(s_2, 2)$ and generates token $(s_3, 1)$; moreover, since $(s_2, 1) \leq_0 (s_2, 2)$ we have $(s_2, 1) \leq_1 (s_3, 1)$. Note that $b_2^1 \in Min(C_1)$ but $b_3 \notin Min(C_1)$.

After the firing of t_1, there are four tokens in place s_2. However, since $(s_2, 2)$ and $(s_2, 4)$ are generated by t_1, they are greater in $\leq_2$ than $(s_2, 1)$ and $(s_2, 3)$. This can also be seen at the process level: b_2^1 and b_2^3 are minimal conditions of C_2, while b_2^4 and b_2^5 are not. On the other hand, note that, just as b_2^4 and b_3 are not minimal in C_2 but also not related by $\leq_{\pi_2}$, also $(s_2, 2)$ and $(s_3, 1)$ are not related by $\leq_2$. □

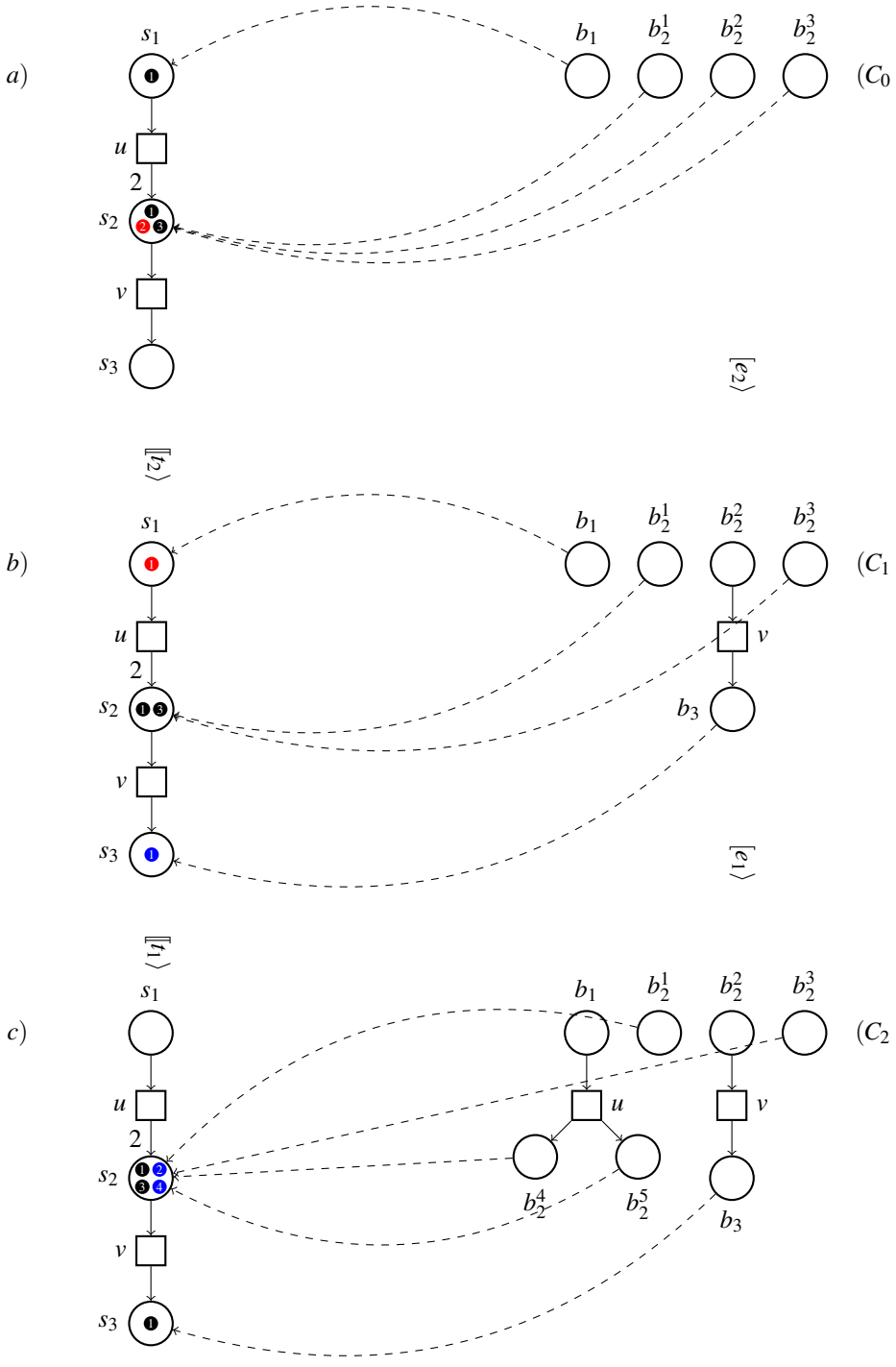

Figure 4.2: Execution of the transition labeled by v, then u, on the net of Figure 4.1 and corresponding process (only the mapping of maximal conditions to tokens is displayed). Tokens to be consumed are red, generated ones blue.

4.4 Decidability of causal-net bisimilarity for bounded nets

We adapt the theorems of Section 3.3 to the bounded case.

Definition 4.4.1. (OIMC-bisimulation) *Let $N = (S,A,T)$ be a net. An OIMC-bisimulation is a relation $\mathfrak{B} \subseteq OIM(N) \times OIM(N) \times \mathscr{P}((S \times \mathbb{N}) \times (S \times \mathbb{N}))$ such that if $((k_1, \leq_1), (k_2, \leq_2), \beta) \in \mathfrak{B}$, then:*

- $|k_1| = |k_2|$
- *$\forall t_1\, k'_1, \leq'_1$ such that $(k_1, \leq_1)[\![t_1\rangle\!\rangle(k'_1, \leq'_1)$ (where we assume that $k''_1 \in k_1 \boxminus {}^\bullet t_1$ such that $k'_1 = k''_1 \boxplus t_1^\bullet$), then there exist $t'_2, k'_2, \leq'_2$ (where we assume $k''_2 \in k_2 \boxminus {}^\bullet t_2$ such that $k'_2 = k''_2 \boxplus t_2^\bullet$), and for β' defined as $\forall (s_1,n_1) \in k'_1, \forall (s_2,n_2) \in k'_2$:*

$$(s_1,n_1)\ \beta'\ (s_2,n_2) \iff \begin{cases} (s_1,n_1) \in k''_1,\ (s_2,n_2) \in k''_2 \text{ and } (s_1,n_1)\ \beta\ (s_2,n_2) \\ \text{or} \\ (s_1,n_1) \in k'_1 \setminus k''_1,\ (s_2,n_2) \in k'_2 \setminus k''_2 \end{cases}$$

  the following hold:

  - $(k_1 \setminus k''_1)\ \beta^{\oplus}\ (k_2 \setminus k''_2)$,
  - $(k_2, \leq_2)[\![t_2\rangle\!\rangle(k'_2, \leq'_2)$ *where* $((k'_1, \leq'_1), (k'_2, \leq'_2), \beta') \in \mathfrak{B}$ *and* $l(t_1) = l(t_2)$.

- *symmetrically, if $(k_2, \leq_2)$ moves first.*

Two markings m_1 and m_2 of N are OIMC-bisimilar, denoted $m_1 \sim_{oimc} m_2$, if there exists an OIMC-bisimulation $\mathfrak{B}$ containing the triple $(init(N(k_1)), init(N(k_2), k_1 \times k_2)$ where, for $i = 1,2$,, k_i is the closed indexed marking such that $m_i = \alpha(k_i)$. □

Theorem 4.4.2. (CN-bisimilarity implies OIMC-bisimilarity) *Let $N = (S,A,T)$ be a net. Given two markings m_{01}, m_{02} of N, if $m_{01} \sim_{cn} m_{02}$, then $m_{01} \sim_{oimc} m_{02}$.*

Proof. If $m_{01} \sim_{cn} m_{02}$, then there exists a CN-bisimulation R_1 containing a triple $(\rho_1^0, C^0, \rho_2^0)$, where C^0 contains no events and $\rho_i^0(Min(C^0)) = \rho_i^0(Max(C^0)) = m_{0i}$ for $i = 1,2$. Given k_{0i} closed indexed marking such that $m_{0i} = \alpha(k_{0i})$ for $i = 1,2$, let us consider

$$\begin{aligned} R_2 \stackrel{def}{=} \{((k_1, \leq_1), (k_2, \leq_2), \beta) \mid & (\rho_1, C, \rho_2) \in R_1 \text{ and} \\ & init(N(k_{01}))[\![(C,\rho_1)\rangle\!\rangle(k_1, \leq_1) \text{ and} \\ & init(N(k_{02}))[\![(C,\rho_2)\rangle\!\rangle(k_2, \leq_2) \text{ and} \\ & \forall s_1 \in \rho_1(Max(C)),\ (s_1,n_1) \in k_1, \\ & \forall s_2 \in \rho_2(Max(C)),\ (s_2,n_2) \in k_2, \\ & \quad (s_1,n_1)\beta(s_2,n_2) \text{ iff } \rho_1^{-1}({}^\bullet s_1) = \rho_2^{-1}({}^\bullet s_2)\}. \end{aligned}$$

If we prove that R_2 is an OIMC-bisimulation, then since $init(N(k_{0i}))[\![(C^0, \rho_i^0)\rangle\!\rangle init(N(k_{0i}))$ and $\rho_i^0(Min(C^0)) = \rho_i^0(Max(C^0)) = m_{0i}$ and $m_{0i} = \alpha(k_{0i})$ for $i = 1,2$, it follows that $(init(N(k_{01})), init(N(k_{02})), k_{01} \times k_{02}) \in R_2$ and therefore $m_{01} \sim_{oimc} m_{02}$.

Assume $((k_1, \leq_1), (k_2, \leq_2), \beta) \in R_2$. If $(k_1, \leq_1)[\![t_1\rangle\!\rangle(k'_1, \leq'_1)$, since $init(N(k_{01}))[\![(C,\rho_1)\rangle\!\rangle(k_1, \leq_1)$, by Lemma 4.3.4 $(C,\rho_1) \xrightarrow{e} (C',\rho'_1)$ where $\rho'_1(e) = t_1$ and $init(N(k_{01}))[\![(C',\rho'_1)\rangle\!\rangle(k'_1, \leq'_1)$. Since $((k_1, \leq_1), (k_2, \leq_2), \beta) \in R_1$, it's also true that $(C,\rho_2) \xrightarrow{e} (C',\rho'_2)$ where $\rho'_2(e) = t_2$ and $((k'_1, \leq'_1), (k'_2, \leq'_2), \beta') \in R_1$. Then, since $init(N(k_{02}))[\![(C,\rho_2)\rangle\!\rangle(k_2, \leq_2)$ and $init(N(k_{02}))[\![(C',\rho'_2)\rangle\!\rangle(k'_2, \leq'_2)$ because (C,ρ_2) and (C',ρ'_2) are processes of N, by Lemma 4.3.4, $(k_2, \leq_2)[\![t_2\rangle\!\rangle(k'_2, \leq'_2)$.

Now we prove that the definition of β' arising from R_2 is coherent with the one of Definition 4.4.1: denoting $k''_1 \in k_1 \boxminus {}^\bullet t_1$ such that $k'_1 = k''_1 \boxplus t_1^\bullet$ and $k''_2 \in k_2 \boxminus {}^\bullet t_2$ such that $k'_2 = k''_2 \boxplus t_2^\bullet$, we prove that it implies the following two propositions.

1. $\forall (s_1,n_1) \in k'_1, \forall (s_2,n_2) \in k'_2$

$$(s_1,n_1)\ \beta'\ (s_2,n_2) \iff \begin{cases} (s_1,n_1) \in k''_1,\ (s_2,n_2) \in k''_2 \text{ and } (s_1,n_1)\ \beta\ (s_2,n_2) & \text{(i)} \\ \text{or} \\ (s_1,n_1) \in k'_1 \setminus k''_1,\ (s_2,n_2) \in k'_2 \setminus k''_2 & \text{(ii)} \end{cases}$$

   and

2. $(k_1 \setminus k_1'')\ \beta^{\oplus}\ (k_2 \setminus k_2'')$.

Proof 1)
The two implications are proved separately.

- if $(s_1,n_1)\,\beta'\,(s_2,n_2) \iff \rho_1'^{-1}({}^\bullet s_1) = \rho_2'^{-1}({}^\bullet s_2)$:
  Consider the event e such that $\rho_1'(e) = t_1$ and $\rho_2'(e) = t_2$. There are four possibilities for $(s_1,n_1),(s_2,n_2)$:
  - if $(s_1,n_1) \in k_1' \setminus k_1''$ and $(s_2,n_2) \in k_2' \setminus k_2''$: condition (ii) is trivial.
  - if $(s_1,n_1) \in k_1''$ and $(s_2,n_2) \in k_2''$: since $((k_1,\leq_1),(k_2,\leq_2),\beta) \in R_2$, the hypotesis $(s_1,n_1)\,\beta\,(s_2,n_2)$ holds. Then, condition (i) is satisfied.
  - other cases: absurd, because $\rho_1'^{-1}({}^\bullet s_1) = \rho_2'^{-1}({}^\bullet s_2)$.
- if (i) or (ii) hold:
  Consider the transitions t_1,t_2 such that $\rho_1'(e) = t_1$ and $\rho_2'(e) = t_2$:
  - if (i) holds: then since $((k_1,\leq_1),(k_2,\leq_2),\beta) \in R_2$, $(s_1,n_1)\,\beta\,(s_2,n_2)$ holds; since $(s_1,n_1),(s_2,n_2)$ don't move, $(s_1,n_1)\,\beta'\,(s_2,n_2)$.
  - if (ii) holds: then $\rho_1'^{-1}({}^\bullet s_1) = \rho_1'^{-1}(t_1) = e$, and $\rho_2'^{-1}({}^\bullet s_2) = \rho_2'^{-1}(t_2) = e$, therefore $(s_1,n_1)\,\beta'\,(s_2,n_2)$.

Proof 2)
Since by definition $\rho_1^{-1}({}^\bullet s_1) = \rho_1^{-1}(t_1) = e = \rho_2^{-1}(t_2) = \rho_2^{-1}({}^\bullet s_2)$, then $|{}^\bullet t_1| = |{}^\bullet e| = |{}^\bullet t_2|$ and $|t_1^\bullet| = |e^\bullet| = |t_2^\bullet|$. Therefore, since $|k_1 \setminus k_1''| = |{}^\bullet t_1|$ and $|k_2 \setminus k_2''| = |{}^\bullet t_2|$, by transitivity $|k_1 \setminus k_1''| = |k_2 \setminus k_2''|$. The proof of $(k_1 \setminus k_1'')\ \beta^{\oplus}\ (k_2 \setminus k_2'')$ is then done by induction on $|k_1 \setminus k_1''|$. Note that the base case is 1, since transitions have nonempty preset.

- Case $|k_1 \setminus k_1''| = |k_2 \setminus k_2''| = 1$:
  Assume $\{(s_1,n_1)\} = k_1 \setminus k_1''$, $\{(s_2,n_2)\} = k_2 \setminus k_2''$. Since $s_1 \in \rho_1(Max(C))$, $s_2 \in \rho_2(Max(C))$ and $\rho_1'(t_1) = e = \rho_2'(t_2)$, then s_1,s_2 are mapped on the same $b \in B$: therefore, $\rho_1^{-1}({}^\bullet s_1) = \rho_2^{-1}({}^\bullet s_2)$ and thus $(s_1,n_1)\,\beta\,(s_2,n_2)$. By rule **(Clo)**, $(s_1,n_1)\ \beta^{\oplus}\ (s_2,n_2)$.
- Case $|k_1 \setminus k_1''| = |k_2 \setminus k_2''| = n+1$:
  Assume $k_1 \setminus k_1'' = (s_1',n_1') \cup k_1^n$ and $k_2 \setminus k_2'' = (s_2',n_2') \cup k_2^n$. By inductive hypotesis, $k_1^n\ \beta^{\oplus}\ k_2^n$. Since $s_1' \in \rho_1(Max(C))$, $s_2' \in \rho_2(Max(C))$ and $\rho_1'(t_1) = e = \rho_2'(t_2)$, then s_1',s_2' are mapped on the same $b' \in B$: therefore, $\rho_1^{-1}({}^\bullet s_1') = \rho_2^{-1}({}^\bullet s_2')$ and thus $(s_1',n_1')\,\beta\,(s_2',n_2')$. By rule **(Clo)**, $k_1 \setminus k_1''\ \beta^{\oplus}\ k_2 \setminus k_2''$.

Thus $((k_1',\leq_1'),(k_2',\leq_2'),\beta') \in R_2$.

Next, we prove that $|k_1'| = |k_2'|$. We know that, with the definitions from above, (ρ_1,C,ρ_2) , $(\rho_1',C',\rho_2') \in R_1$ and $((k_1,\leq_1),(k_2,\leq_2),\beta)$, $((k_1',\leq_1'),(k_2',\leq_2'),\beta') \in R_2$. Then, since for $i = 1,2$ by definition $init(N(k_{0i}))[\![(C,\rho_i)\rangle (k_i,\leq_i)$ and $init(N(k_{0i}))[\![(C',\rho_i')\rangle (k_i',\leq_i')$, by Lemma 4.3.2, $|m_1'| = |m_2'|$. Moreover, since $\alpha(k_1') = m_1'$ and $\alpha(k_2') = m_2'$, we have $|k_1'| = |k_2'|$.

The case in which $(k_2,\leq_2)$ moves first is symmetrical.

Therefore, R_2 is an OIMC-bisimulation, and thus $m_{01}\ \sim_{oimc}\ m_{02}$. □

Theorem 4.4.3. (OIMC-bisimilarity implies CN-bisimilarity) *Let $N = (S,A,T)$ be a net. Given two markings m_{01},m_{02} of N, if $m_{01}\ \sim_{oimc}\ m_{02}$, then $m_{01}\ \sim_{cn}\ m_{02}$.*

Proof. If $m_{01}\ \sim_{oimc}\ m_{02}$ there exists an OIMC-bisimulation R_1 containing the tuple $(init(N(k_{01})),init(N(k_{02})),k_{01} \times k_{02})$, where $\alpha(k_{01}) = m_{01}$, $\alpha(k_{02}) = m_{02}$, and k_{01},k_{02} are closed.

Let us consider

$$
\begin{aligned}
R_2 \stackrel{def}{=} \{(\rho_1,C,\rho_2) \mid\ & ((k_1,\leq_1),(k_2,\leq_2),\beta) \in R_1 \text{ and} \\
& (C,\rho_1) \text{ is a process of } N(m_{01}) \text{ and} \\
& (C,\rho_2) \text{ is a process of } N(m_{02}) \text{ and} \\
& init(N(k_{01}))[(C,\rho_1)\rangle (k_1,\leq_1) \text{ and} \\
& init(N(k_{02}))[(C,\rho_2)\rangle (k_2,\leq_2) \text{ and} \\
& \forall s_1 \in \rho_1(Max(C))\,,\,(s_1,n_1) \in k_1, \\
& \forall s_2 \in \rho_2(Max(C))\,,\,(s_2,n_2) \in k_2, \\
& \qquad (s_1,n_1)\beta(s_2,n_2) \text{ iff } \rho_1^{-1}({}^\bullet s_1) = \rho_2^{-1}({}^\bullet s_2)\}.
\end{aligned}
$$

If we prove that R_2 is a CN-bisimulation, then we prove that $(\rho_1^0, C^0, \rho_2^0) \in R_2$, where C^0 contains no transitions and, for $i = 1,2$, $\rho_i^0(Min(C^0)) = \rho_i^0(Max(C^0)) = m_{0i}$, because $(init(N(m_{01})), init(N(m_{02})), m_{01} \times m_{02}) \in R_1$ and (C^0, ρ_i^0) is a process of $N(m_{0i})$ and $init(N(m_{0i}))[(C^0, \rho_i^0)\rangle init(N(m_{0i}))$. Then, since $(\rho_1^0, C^0, \rho_2^0) \in R_2$, we have $m_{01} \sim_{cn} m_{02}$.

Assume $(\rho_1, C, \rho_2) \in R_2$. If $(C, \rho_1) \xrightarrow{e} (C', \rho_1')$ where $\rho_1'(e) = t_1$, since by hypotesis $init(N(k_{01}))[\![(C, \rho_1)\rangle (k_1, \leq_1)$ and $init(N(k_{01}))[\![(C', \rho_1')\rangle (k_1', \leq_1')$, then by Lemma 4.3.4, $(k_1, \leq_1)[\![t_1\rangle (k_1', \leq_1')$. Since $((k_1, \leq_1), (k_2, \leq_2), \beta) \in R_1$, it's also true that $(k_2, \leq_2)[\![t_2\rangle (k_2', \leq_2')$ where $((k_1', \leq_1'), (k_2', \leq_2'), \beta') \in R_1$; because by hypotesis $init(N(k_{02}))[\![(C, \rho_2)\rangle (k_2, \leq_2)$, then by Lemma 4.3.4, $init(N(k_{02}))[\![(C', \rho_2')\rangle (k_2', \leq_2')$ and $(C, \rho_2) \xrightarrow{e} (C', \rho_2')$ where $\rho_2'(e) = t_2$. Note that each (C', ρ_i') is a process of N.

Now we prove that the definition of β' of R_1 is coherent with the one specified in R_2, i.e. $\forall s_1' \in \rho_1'(Max(C'))$, $(s_1', n_1') \in k_1', \forall s_2' \in \rho_2'(Max(C'))$, $(s_2', n_2') \in k_2'$, we have $s_1' \; \beta' \; s_2'$ if and only if $\rho_1'^{-1}({}^\bullet s_1) = \rho_2'^{-1}({}^\bullet s_2)$. This is proved by induction on the length of σ, complete transition sequence of C' such that $init(N(k_{01}))[\![(C', \rho_1')\rangle (k_1', \leq_1')$. Note that also $init(N(k_{02}))[\![(C', \rho_2')\rangle (k_2', \leq_2')$ with the same complete transition sequence.

- Case 1 : $\sigma = e$.

  Proof $\Rightarrow$*)* by cases on the definition of $(s_1', n_1') \; \beta'(s_2', n_2')$:

  - if $(s_1', n_1') \in k_1' \setminus k_1''$ and $(s_2', n_2') \in k_2' \setminus k_2''$:
    By hypotesis we know that ${}^\bullet s_1' = t_1$ and ${}^\bullet s_2' = t_2$. Since $(C, \rho_i) \xrightarrow{e} (C', \rho_i')$ where $\rho_i'(e) = t_i$ for $i = 1,2$, then t_1 and t_2 are mapped on the same transition e. Therefore $\rho_1^{-1}({}^\bullet s_1') = \rho_1^{-1}(t_1) = e$ and $\rho_2^{-1}({}^\bullet s_2') = \rho_1^{-1}(t_2) = e$, and by transitivity we get $\rho_1^{-1}({}^\bullet s_1') = \rho_2^{-1}({}^\bullet s_2')$.
  - if $(s_1', n_1') \in k_1''$ and $(s_2', n_2') \in k_2''$ and $(s_1', n_1') \; \beta \; (s_2', n_2')$:
    then $\rho_1^{-1}({}^\bullet s_1') = \emptyset$ and $\rho_2^{-1}({}^\bullet s_2') = \emptyset$. Since the tokens did not move, $\rho_i'(s_i') = \rho_i(s_i')$ for $i = 1,2$ and by transitivity $\rho_1^{-1}({}^\bullet s_1') = \rho_2^{-1}({}^\bullet s_2')$.

  Proof $\Leftarrow$*)* Consider the event e: we know that e is a transition of C' and not of C, such that $\rho_1'(e) = t_1$ and $\rho_2'(e) = t_2$. there are four possibilities for $(s_1', n_1'), (s_2', n_2')$:

  - if $(s_1', n_1') \in k_1' \setminus k_1''$ and $(s_2', n_2') \in k_2' \setminus k_2''$:
    then $(s_1', n_1') \; \beta' \; (s_2', n_2')$ by the second condition of β'.
  - if $(s_1', n_1') \in k_1''$ and $(s_2', n_2') \in k_2''$:
    We know that $\rho_1'^{-1}({}^\bullet s_1') = \rho_2'^{-1}({}^\bullet s_2')$; since the tokens did not move, $\rho_i'(s_i') = \rho_i(s_i')$ for $i = 1,2$ and therefore $\rho_1^{-1}({}^\bullet s_1') = \rho_2^{-1}({}^\bullet s_2')$. Moreover, because $\sigma = e$ it must be that ${}^\bullet s_1' = \emptyset$ and ${}^\bullet s_2' = \emptyset$. By the fact that, at the beginning, $\beta = k_{01} \times k_{02}$, we have that $(s_1', n_1') \; \beta \; (s_2', n_2')$. Thus, the first condition of β' holds.
  - other cases:
    absurd since $\rho_1'^{-1}({}^\bullet s_1') = \rho_2'^{-1}({}^\bullet s_2')$.

- Case n+1: $\sigma = \delta \, e$.

  Proof $\Rightarrow$*)* by cases on the definition of $(s_1', n_1') \; \beta' \; (s_2', n_2')$:

  - if $(s_1', n_1') \in k_1' \setminus k_1''$ and $(s_2', n_2') \in k_2' \setminus k_2''$:
    By hypotesis we know that ${}^\bullet s_1' = t_1$ and ${}^\bullet s_2' = t_2$. Since $(C, \rho_i) \xrightarrow{e} (C', \rho_i')$ where $\rho_i'(e) = t_i$ for $i = 1,2$, then t_1 and t_2 are mapped on the same transition e. Therefore $\rho_1^{-1}({}^\bullet s_1') = \rho_1^{-1}(t_1) = e$ and $\rho_2^{-1}({}^\bullet s_2') = \rho_1^{-1}(t_2) = e$, and by transitivity we get $\rho_1^{-1}({}^\bullet s_1') = \rho_2^{-1}({}^\bullet s_2')$.
  - if $(s_1', n_1') \in k_1''$ and $(s_2', n_2') \in k_2''$ and $(s_1', n_1') \; \beta \; (s_2', n_2')$:
    By inductive hypotesis on $(k_1, \leq_1)$ and $(k_2, \leq_2)$, we have $(s_1', n_1') \; \beta \; (s_2', n_2') \Rightarrow \rho_1^{-1}({}^\bullet s_1') = \rho_2^{-1}({}^\bullet s_2')$. Since the tokens did not move, $\rho_i'(s_i') = \rho_i(s_i')$ for $i = 1,2$; therefore by transitivity $\rho_1'^{-1}({}^\bullet s_1') = \rho_2'^{-1}({}^\bullet s_2')$.

  Proof $\Leftarrow$*)* Consider the event e: we know that e is a transition of C' and not of C, such that $\rho_1'(e) = t_1$ and $\rho_2'(e) = t_2$. there are four possibilities for $(s_1', n_1'), (s_2', n_2')$:

  - if $(s_1', n_1') \in k_1' \setminus k_1''$ and $(s_2', n_2') \in k_2' \setminus k_2''$:
    then $(s_1', n_1') \; \beta' \; (s_2', n_2')$ by the second condition of β'.
  - if $(s_1', n_1') \in k_1''$ and $(s_2', n_2') \in k_2''$:
    By inductive hypotesis on $(k_1, \leq_1)$ and $(k_2, \leq_2)$, we have $\rho_1^{-1}({}^\bullet s_1') = \rho_2^{-1}({}^\bullet s_2') \Rightarrow . \; (s_1', n_1') \; \beta \; (s_2', n_2')$
    Since the tokens did not move, $\rho_i'(s_i') = \rho_i(s_i')$ for $i = 1,2$; therefore, the first condition of β' holds.

- other cases:
  absurd since $\rho_1'^{-1}({}^\bullet s_1') = \rho_2'^{-1}({}^\bullet s_2')$.

Thus $(\rho_1', C', \rho_2') \in R_2$.
The case in which (C_2, ρ_2) moves first is symmetrical.
Therefore, R_2 is a CN-bisimulation and $m_{01} \sim_{cn} m_{02}$. □

Theorem 4.4.4. (OIMC-bisimilarity and CN-bisimilarity coincide) *Let $N = (S, A, T)$ be a net and m_1, m_2 two markings of N. $m_1 \sim_{oimc} m_2$ if and only if $m_1 \sim_{cn} m_2$.*

Proof. By theorems 4.4.2 and 4.4.3, we get the thesis. □

Corollary 4.4.5. (CN-bisimilarity is decidable for finite bounded nets) *Given $N(m_1)$ and $N(m_2)$ bounded nets, it is decidable to check whether $m_1 \sim_{cn} m_2$.*

Proof. By Theorem 4.4.4 we have to check whether there exists an OIMC-bisimulation $\mathfrak{B}$ for the given net N and initial markings m_1, m_2.

If we restrict $\mathfrak{B}$ to $\mathfrak{B}' = \{((k_1, \leq_1), (k_2, \leq_2), \beta) \in \mathfrak{B} \mid (k_i, \leq_i) \in [\![init(N(k_{0i}))]\!\rangle$, where k_{0i} is closed and $\alpha(k_{0i}) = m_i$ for $i = 1, 2\}$, we have that $\mathfrak{B}'$ is still an OIMC-bisimulation for m_1, m_2.

Indeed, by definition $init(N(k_{0i})) \in [\![init(N(k_{0i}))]\!\rangle$; if $((k_1, \leq_1), (k_2, \leq_2), \beta) \in \mathfrak{B}'$ and $(k_1, \leq_1)[\![t_1 \rangle (k_1', \leq_1')$, then it is true that $(k_2, \leq_2)[\![t_2 \rangle (k_2', \leq_2')$ and $(k_i', \leq_i')$ is reachable from $init(N(k_{0i}))$ for $i = 1, 2$, and since k_i' is reachable from k_{0i}, it is also true that β' relates only tokens reachable from k_{01} and k_{02}.

Then, to state that $m_1 \sim_{oimc} m_2$, it is enough to consider the ordered indexed markings contained in $[\![init(N(k_{01}))]\!\rangle$ and $[\![init(N(k_{02}))]\!\rangle$ and by Proposition 4.2.4 these are finitely many.

By Proposition 4.1.4, the reachable indexed marking from k_{01} and k_{02} are finite, and therefore there are finitely many relations β to consider.

Therefore we can check by exhaustive search whether one of the finitely many possible sets of triples is an OIMC-bisimulation.

□

We conclude this section with some comments on the complexity of the decision procedure. Assume that the considered net has (less than) s places, t transitions and it is h-safe. Then there will be at most hs tokens in every reachable marking, and since the possible preorders on hs elements are $2^{O(hs \cdot log(hs))}$, there are at most $2^{O(hs \cdot log(hs))}$ ordered indexed markings. Since β is a binary relation on tokens, it contains at most $O((hs)^2)$ elements; therefore there are at most $2^{O(hs \cdot log(hs))}$ possible elements of $\mathfrak{B}$. Note that, according to Definition 4.2.2, it is possible to construct a labeled transition system where states are ordered indexed markings and transitions are derived from T. Therefore, it is possible to construct an OIMC-bisimulation starting from the labeled transistion system containing $init(N(k_{01}))$ and $init(N(k_{02}))$. The algorithm consumes all reachable states of the transition system; for each pair of triples, it requires scanning $O(t^2 (hs)^2)$ transitions for the bisimulation game (because the transition relation on ordered indexed markings is nondeterministic) and $O((hs)^2)$ tokens for the condition on β. Therefore the upper bound for our decision procedure is $2^{O(hs \cdot log(hs) + log(t))}$. Thus, our proposed algorithm has complexity in DEXPTIME, just as known algorithms to decide fully-concurrent bisimilarity on safe and bounded nets [JM96; MP97].

4.5 Decidability of fully-concurrent bisimulation for bounded nets

In the following we extend the work of [Vog91] by using the ordered indexed marking proof technique to show that fully-concurrent bisimilarity is decidable for finite bounded nets, in the same fashion as what we did for causal-net bisimilarity in Section 4.4.

Definition 4.5.1. (OIM-bisimulation) *Let $N = (S, A, T)$ be a net. An OIM-bisimulation is a relation $\mathfrak{B} \subseteq OIM(N) \times OIM(N) \times \mathscr{P}((S \times \mathbb{N}) \times (S \times \mathbb{N}))$ such that if $((k_1, \leq_1), (k_2, \leq_2), \beta) \in \mathfrak{B}$, then:*

- *$\forall t_1, k_1', \leq_1'$ such that $(k_1, \leq_1)[\![t_1 \rangle (k_1', \leq_1')$, (where we assume that $k_1'' \in k_1 \boxminus {}^\bullet t_1$ such that $k_1' = k_1'' \boxplus t_1^\bullet$), there exist $t_2', k_2', \leq_2'$ (where we assume $k_2'' \in k_2 \boxminus {}^\bullet t_2$ such that $k_2' = k_2'' \boxplus t_2^\bullet$),*
  and for β' defined as $\forall (s_1, n_1) \in k_1', \forall (s_2, n_2) \in k_2'$:

$$(s_1, n_1)\ \beta'\ (s_2, n_2) \iff \begin{cases} (s_1, n_1) \in k_1'', (s_2, n_2) \in k_2'' \text{ and } (s_1, n_1)\ \beta\ (s_2, n_2) \\ \text{or} \\ (s_1, n_1) \in k_1' \setminus k_1'', (s_2, n_2) \in k_2' \setminus k_2'' \end{cases}$$

the following hold:

- $(k_2,\leq_2)[\![t_2\rangle(k_2',\leq_2')$ *where* $((k_1',\leq_1'),(k_2',\leq_2'),\beta') \in \mathfrak{B}$ *and* $l(t_1)=l(t_2)$*;*
- $\forall(s_1,n_1) \in k_1 \setminus k_1''$, $\exists(s_1',n_1') \in k_1 \setminus k_1''$, $(s_2',n_2') \in k_2 \setminus k_2''$ *such that* $(s_1,n_1) \leq_1 (s_1',n_1') \wedge (s_1',n_1')\ \beta\ (s_2',n_2')$ *and symmetrically* $\forall(s_2,n_2) \in k_2 \setminus k_2''$, $\exists(s_2',n_2') \in k_2 \setminus k_2''$, $(s_1',n_1') \in k_1 \setminus k_1''$ *such that* $(s_2,n_2) \leq_2 (s_2',n_2') \wedge (s_1',n_1')\ \beta\ (s_2',n_2')$

- *symmetrically, if* $(k_2,\leq_2)$ *moves first.*

Two markings m_1 and m_2 of N are OIM-bisimilar, denoted $m_1 \sim_{oim} m_2$, if there exists an OIM-bisimulation $\mathfrak{B}$ containing the triple $(init(N(k_1)), init(N(k_2), k_1 \times k_2)$ where, for $i=1,2$, k_i is the closed indexed marking such that $m_i = \alpha(k_i)$.

As for safe nets (cf. Section 3.4.5), this definition is aimed at relating $\sim_{fc}$ and $\sim_{oim}$ so that tokens are related by β if the transition generating one is mapped to the transition generating the other by f. Next, we show that fully-concurrent bisimilarity and OIM-bisimilarity coincide on finite bounded nets.

Theorem 4.5.2. (FC-bisimilarity implies OIM-bisimilarity) *Let $N=(S,A,T)$ be a net. Given two markings m_{01}, m_{02} of N, if $m_{01} \sim_{fc} m_{02}$, then $m_{01} \sim_{oim} m_{02}$.*

Proof. If $m_{01} \sim_{fc} m_{02}$, then there exists a FC-bisimulation R_1 containing the triple $(\pi_1^0, \emptyset, \pi_2^0)$, where $\pi_i^0 = (C_i^0, \rho_i^0)$ is such that C_i^0 contains no events and $\rho_i^0(Min(C_i^0)) = \rho_i^0(Max(C_i^0)) = m_{0i}$ for $i=1,2$. Given k_{0i} closed indexed marking such that $m_{0i} = \alpha(k_{0i})$ for $i=1,2$, let us consider

$$
\begin{aligned}
R_2 \stackrel{def}{=} \{((k_1,\leq_1),(k_2,\leq_2),\beta) \mid & (\pi_1,f,\pi_2) \in R_1 \text{ and} \\
& init(N(k_{01}))[\pi_1\rangle(k_1,\leq_1) \text{ and} \\
& init(N(k_{02}))[\pi_2\rangle(k_2,\leq_2) \text{ and} \\
& \forall(s_1,n_1) \in k_1, \forall(s_2,n_2) \in k_2 : \\
& (s_1,n_1)\ \beta\ (s_2,n_2) \text{ iff} \\
& \quad \exists b_1 \in Max(C_1) \text{ such that } \rho_1(b_1) = s_1, \\
& \quad \exists b_2 \in Max(C_2) \text{ such that } \rho_2(b_2) = s_2, \\
& \quad \text{either } b_1 \in Min(C_1) \wedge b_2 \in Min(C_2) \\
& \quad \text{or } {}^\bullet b_1 \neq \emptyset \wedge {}^\bullet b_2 \neq \emptyset \wedge f({}^\bullet b_1) = {}^\bullet b_2\}.
\end{aligned}
$$

If we prove that R_2 is an OIM-bisimulation, then since $init(N(k_{0i}))[\![\pi_i^0\rangle init(N(k_{0i}))$ and $\rho_i^0(Min(C_i^0)) = = \rho_i^0(Max(C_i^0)) = k_{0i}$ where $\alpha(k_{0i}) = m_{0i}$ for $i=1,2$, it follows that $(init(N(k_{01})), init(N(k_{02})), k_{01} \times k_{02}) \in R_2$ and therefore $m_{01} \sim_{oim} m_{02}$.

Assume $((k_1,\leq_1),(k_2,\leq_2),\beta) \in R_2$. If $init(N(k_{01}))[\![\pi_1\rangle(k_1,\leq_1)$, and $(k_1,\leq_1)[\![t_1\rangle(k_1',\leq_1')$ then by Lemma 4.3.4 $\pi_1 \xrightarrow{e} \pi_1'$ with $\rho_1'(e_1) = t_1$ and $init(N(k_{01}))[\![\pi_1'\rangle(k_1',\leq_1')$. Since $(\pi_1,f,\pi_2) \in R_1$, it is also true that $\pi_2 \xrightarrow{e_2} \pi_2'$ with $\rho_2'(e_2) = t_2$ and $(\pi_1',f',\pi_2') \in R_1$. Then, since by definition $init(N(k_{02}))[\![\pi_2\rangle(k_2,\leq_2)$ and $init(N(k_{02}))[\![\pi_2'\rangle (k_2',\leq_2')$, by Lemma 4.3.4 we have $(k_2,\leq_2)[\![t_2\rangle(k_2',\leq_2')$.

Now we prove that the definition of β' arising from R_2 is coherent with the one of Definition 4.5.1, i.e. it implies both

1. $\forall(s_1,n_1) \in k_1'$, $(s_2,n_2) \in k_2'$

$$
(s_1,n_1)\ \beta'\ (s_2,n_2) \iff \begin{cases} (s_1,n_1) \in k_1'', (s_2,n_2) \in k_2'' \text{ and } (s_1,n_1)\ \beta\ (s_2,n_2) & \text{(i)} \\ \text{or} & \\ (s_1,n_1) \in k_1' \setminus k_1'', (s_2,n_2) \in k_2' \setminus k_2'' & \text{(ii)} \end{cases}
$$

and

2. $\forall(s_1,n_1) \in k_1 \setminus k_1''$, $\exists(s_1',n_1') \in k_1 \setminus k_1''$, $(s_2',n_2') \in k_2 \setminus k_2''$ such that $(s_1,n_1) \leq_1 (s_1',n_1') \wedge (s_1',n_1')\ \beta\ (s_2',n_2')$, and symmetrically $\forall(s_2,n_2) \in k_2 \setminus k_2''$, $\exists(s_2',n_2') \in k_2 \setminus k_2''$, $(s_1',n_1') \in k_1 \setminus k_1''$ such that $(s_2,n_2) \leq_2 (s_2',n_2') \wedge (s_1',n_1')\ \beta\ (s_2',n_2')$.

Proof 1) The two implications are proved separately.

Proof $\Rightarrow$: assume $s_1 = \rho'(b_1)$ and $s_2 = \rho'(b_2)$. Then:

- If $b_1 \in Min(C_1')$ and $b_2 \in Min(C_2')$:
  then for $i = 1,2$, $s_i \in \rho_i'(Min(C_i')) = m_{0i}$, therefore $(s_i,n_i) \in k_{0i}$, thus $(s_i,n_i) \in k_i''$. Since the initial β is $k_{01} \times k_{02}$, and k_{01}, k_{02} are closed, we have $(s_1,n_1)\beta(s_2,n_2)$, satisfying condition (i).
- if ${}^\bullet b_1 \neq \emptyset \wedge {}^\bullet b_2 \neq \emptyset \wedge f({}^\bullet b_1) = {}^\bullet b_2$:
  Let us consider events e_1', e_2' such that $b_1 \in e_1'^\bullet$ and $b_2 \in e_2'^\bullet$. There are four possible cases:
  + if $e_1' = e_1$ and $e_2' = e_2$: since $s_1 = \rho_1'(b_1)$ and $t_1 = \rho_1'(e_1) = \rho_1'(e_1')$, we have $(s_1,n_1) \in k_1' \setminus k_1''$. For the same reason, $(s_2,n_2) \in k_2' \setminus k_2''$ and therefore condition (ii) holds.
  + if $e_1' \neq e_1$ and $e_2' \neq e_2$: then $s_1 = \rho_1'(b_1) = \rho_1(b_1)$ and therefore $(s_1,n_1) \in k_1''$ because e_1' has occurred before e_1. For the same reason, $(s_2,n_2) \in k_2''$. Since $((k_1,\leq_1),(k_2,\leq_2),\beta) \in R_2$ and $f({}^\bullet b_1) = {}^\bullet b_2$, then $(s_1,n_1)\beta(s_2,n_2)$, therefore condition (i) holds.
  + other cases: absurd since $f'(e_1) = e_2$.

Proof $\Leftarrow$: Since $(s_1,n_1) \in k_1'$ and $(s_2,n_2) \in k_2'$, there exist b_1 and b_2 such that $s_1 = \rho'(b_1)$ and $s_2 = = \rho'(b_2)$. Consider the complete transition sequence $\sigma_1 = \delta_1 e_1$ such that $init(N(k_{01}))[\![\rho_1'(\sigma_1)\rangle (k_1',\leq_1')$, and the complete transition sequence $\sigma_2 = \delta_2 e_2$ such that $init(N(k_{02}))[\![\rho_2'(\sigma_2)\rangle (k_2',\leq_2')$, and $f'(\sigma_1) = \sigma_2$. It is easy to see that $|\sigma_1| = |\sigma_2|$; we prove the thesis by induction on the length of σ_1:

+ Case 1: $\sigma_1 = e_1$ and $\sigma_2 = e_2$. By cases on the condition:
  - if $(s_1,n_1) \in k_1''$, $(s_2,n_2) \in k_2''$ and $(s_1,n_1)\ \beta\ (s_2,n_2)$:
    then $(s_1,n_1) \in k_{01}$, where $\alpha(k_{01}) = m_{01}$, and since $m_{01} = \rho_1'(Min(C'))$ we have ${}^\bullet b_1 = \emptyset$. For the same reason, also ${}^\bullet b_2 = \emptyset$.
  - if $(s_1,n_1) \in k_1' \setminus k_1''$, $(s_2,n_2) \in k_2' \setminus k_2''$:
    then ${}^\bullet b_1 = e_1$ and ${}^\bullet b_2 = e_2$: therefore, we have by construction that $f'({}^\bullet b_1) = {}^\bullet b_2$.
+ Case n+1: $\sigma_1 = \delta_1 e_1$ and $\sigma_2 = \delta_2 e_2$. The inductive hypotesis is $init(N(k_{0i}))[\![\rho_i(\delta_i)\rangle (k_i,\leq_i)$ where the thesis holds and the inductive step is $(k_i,\leq_i)[\![\rho_i'(e_i)\rangle (k_i',\leq_i')$, where $\rho_i'(e_i) = t_i$, for $i = 1,2$. By cases on the condition:
  - if $(s_1,n_1) \in k_1''$, $(s_2,n_2) \in k_2''$ and $(s_1,n_1)\ \beta\ (s_2,n_2)$:
    We need to separate two cases for s_1.
    + if $(s_1,n_1) \in k_{01}$: since $\alpha(k_{01}) = m_{01}$ and $s_1 = \rho_1'(b_1)$, then $b_1 \in Min(C_1')$; moreover since $(s_1,n_1)\ \beta\ (s_2,n_2)$, it is true that $(s_2,n_2) \in k_{02}$, and the same reasoning applies, therefore $b_1 \in Min(C_1')$ and $b_2 \in Min(C_2')$.
    + if $(s_1,n_1) \notin k_{01}$: then ${}^\bullet b_1 \neq \emptyset$ and ${}^\bullet b_2 \neq \emptyset$. Since $(s_1,n_1) \in k_1''$, $(s_2,n_2) \in k_2''$, and $((k_1,\leq_1),(k_2,\leq_2),\beta) \in R_2$, then by inductive hypotesis $f({}^\bullet b_1) = {}^\bullet b_2$, and by conservative extension of f we get the thesis.
  - if $(s_1,n_1) \in k_1' \setminus k_1''$, $(s_2,n_2) \in k_2' \setminus k_2''$:
    then since $\rho_1'(b_1) = s_1$ we have ${}^\bullet b_1 = e_1$, therefore ${}^\bullet b_1 \neq \emptyset$. The same applies to s_2, and since $f'(e_1) = e_2$, we have $f'({}^\bullet b_1) = {}^\bullet b_2$.

Proof 2) Let us consider events e_1, e_2 such that $\pi_1 \xrightarrow{e_1} \pi_1'$ with $\rho'(e_1) = t_1$, $\pi_2 \xrightarrow{e_2} \pi_2'$ with $\rho'(e_2) = t_2$, and $f'(e_1) = e_2$. We assume a token $(s_1,n_1) \in k_1 \setminus k_1''$ where $s_1 \in {}^\bullet t_1$ and there exists b_1 such that $\rho_1(b_1) = s_1$. Note that, since $\pi_1 \xrightarrow{e_1} \pi_1'$ with $\rho'(e_1) = t_1$, it is true that $b_1 \in Max(C_1)$. We are to prove that $\exists (s_1',n_1') \in k_1 \setminus k_1''$, $\exists (s_2',n_2') \in k_2 \setminus k_2''$ such that $(s_1,n_1) \leq_1 (s_1',n_1')$ and $(s_1',n_1')\ \beta\ (s_2',n_2')$. In the following, in some cases we have $(s_1,n_1) = (s_1',n_1')$: if that is true, then $(s_1,n_1) \leq_1 (s_1',n_1')$ by reflexivity of $\leq_1$.

There are two possible cases for b_1:

- if $b_1 \in Min(C_1)$:
  There are two possible subcases:
  + $\exists b_2' \in \rho_2^{-1}({}^\bullet t_2)$ such that $b_2' \in Min(C_2)$:
    Then by definition of ρ_2 there exists $s_2' = \rho_2(b_2')$, and a token (s_2',n_2'). By definition of β we have $(s_1,n_1)\ \beta\ (s_2',n_2')$.
  + otherwise:
    Since $\pi_2 \xrightarrow{e_2} \pi_2'$ with $\rho'(e_2) = t_2$, there exists $b_2' \in Max(C_2)$ such that $\rho_2(b_2') = s_2'$ and $(s_2',n_2') \in k_2 \setminus k_2''$.
    Let us consider $t_2' = {}^\bullet s_2'$ and the related event $e_2' \in \mathsf{E}_{C_2}$ such that $\rho_2(e_2') = t_2'$. Since f is an isomorphism between E_{C_1} and E_{C_2}, there exists event $e_1' \in \mathsf{E}_{C_1}$ such that $f(e_1') = e_2'$. Then, by definition of ρ_1, there exists t_1' such that $\rho_1(e_1') = t_1'$. Therefore there exists (s_1',n_1') such that $t_1' = {}^\bullet s_1'$ and a b_1' such that $\rho_1(b_1') = s_1'$, i.e. $e_1' = {}^\bullet b_1'$. Since $b_1 \in Min(C_1)$, by Lemma 4.3.1 it is true that (s_1,n_1) is minimal for $\leq_1$, and therefore $(s_1,n_1) \leq_1 (s_1',n_1')$.

Finally, since $e'_1 = {}^\bullet b'_1$, $e'_2 = {}^\bullet b'_2$ and $f(e'_1) = e'_2$, we have $(s'_1, n'_1)\ \beta\ (s'_2, n'_2)$.

- if $b_1 \notin Min(C_1)$:
  Assume $t'_1 = {}^\bullet s_1$ and $e'_1 \in \mathsf{E}_{C_1}$ such that $\rho_1(e'_1) = t'_1$. Since f is an isomorphism between E_{C_1} and E_{C_2}, there exists $e'_2 \in \mathsf{E}_{C_2}$ such that $e'_2 = f(e'_1)$. Then, by definition of ρ_2, there exists also t'_2 such that $\rho_2(e'_2) = t'_2$.
  From this, we get that there exists (s'_2, n'_2) such that $t'_2 = {}^\bullet s'_2$ and b'_2 such that $\rho_2(b'_2) = s'_2$. Since e'_1 is an immediate predecessor of e_1 in E_{C_1}, by definition of f it is true that e'_2 is an immediate predecessor of e_2 in E_{C_2}. Therefore it is possible to choose (s'_2, n'_2) not only such that $t'_2 = {}^\bullet s'_2$, but also $s'_2 \in {}^\bullet t_2$.
  Finally, we have that $e'_1 = {}^\bullet b_1$, $e'_2 = {}^\bullet b'_2$ and $f(e'_1) = e'_2$, therefore $(s_1, n_1)\ \beta\ (s'_2, n'_2)$.

The proof of the case for $(s_2, n_2) \in k_2 \setminus k''_2$ is symmetrical and therefore omitted.

Thus, $(\pi'_1, f', \pi'_2) \in R_2$.
The case in which π_2 moves first is symmetrical.
Therefore, R_2 is an OIM-bisimulation and $m_{01} \sim_{oim} m_{02}$. □

Theorem 4.5.3. (OIM-bisimilarity implies FC-bisimilarity) *Let $N = (S, A, T)$ be a net. Given two markings m_{01}, m_{02} of N, if $m_{01} \sim_{oim} m_{02}$, then $m_{01} \sim_{fc} m_{02}$.*

Proof. If $m_{01} \sim_{oim} m_{02}$ there exists an OIM-bisimulation R_1 containing the tuple $(init(N(k_{01})), init(N(k_{02})), k_{01} \times k_{02})$, where $\alpha(k_{01}) = m_{01}$, $\alpha(k_{02}) = m_{02}$, and k_{01}, k_{02} are closed.
Let us consider

$$
\begin{aligned}
R_2 \stackrel{def}{=} \{(\pi_1, f, \pi_2) \mid\ & ((k_1, \leq_1), (k_2, \leq_2), \beta) \in R_1 \text{ and} \\
& \text{for } i = 1, 2,\ \pi_i = (C_i, \rho_i) \text{ is a process of } N(k_{0i}) \text{ and} \\
& init(N(k_{0i}))[\![\pi_i\rangle(k_i, \leq_i) \text{ and} \\
& f \text{ is an isomorphism } \mathsf{E}_{C_1} \to \mathsf{E}_{C_2} \text{ and} \\
& \forall b_1 \in Max(C_1), b_2 \in Max(C_2) \text{ such that} \\
& (\rho_1(b_1), n_1) \in k_1, (\rho_2(b_2), n_2) \in k_2, \\
& (\rho_1(b_1), n_1)\ \beta\ (\rho_2(b_2), n_2) \text{ iff} \\
& \qquad \text{either } {}^\bullet b_1 = \emptyset \wedge {}^\bullet b_2 = \emptyset \\
& \qquad \text{or } {}^\bullet b_1 \neq \emptyset \wedge {}^\bullet b_2 \neq \emptyset \wedge f({}^\bullet b_1) = {}^\bullet b_2\}.
\end{aligned}
$$

If we prove that R_2 is an FC-bisimulation, then we have that $m_{01} \sim_{fc} m_{02}$, because $(\pi^0_1, \pi^0_2, \emptyset) \in R_2$, where, for $i = 1, 2$, each $\pi^0_i = (C^0_i, \rho^0_i)$ is such that C^0_i contains no transitions and $\rho^0_i(Min(C^0_i)) = \rho^0_i(Max(C^0_i)) = m_{0i}$. Indeed, since $(init(N(k_{01})), init(N(k_{02})), k_{01} \times k_{02}) \in R_1$ and (C^0_i, ρ^0_i) is a process of $N(k_{0i})$ and $init(N(k_{0i}))[\![(C^0_i, \rho^0_i)\rangle init(N(k_{0i}))$, and $\alpha(k_{0i}) = m_{0i}$ we have $(\pi^0_1, \pi^0_2, \emptyset) \in R_2$, and therefore $m_{01} \sim_{fc} m_{02}$.

Assume $(\pi_1, f, \pi_2) \in R_2$. If $(C_1, \rho_1) \xrightarrow{e_1} (C'_1, \rho'_1)$ where $\rho'_1(e_1) = t_1$, since by hypotesis $init(N(k_{01}))[\![\pi_1\rangle (k_1, \leq_1)$ and $init(N(m_{01}))[\![\pi'_1\rangle(k'_1, \leq'_1)$, by Lemma 4.3.4 $(k_1, \leq_1)[\![t_1\rangle(k'_1, \leq'_1)$. Since $((k_1, \leq_1), (k_2, \leq_2), \beta) \in R_1$ then there exist $t'_2, k'_2, \leq'_2$ such that $(k_2, \leq_2)[\![t_2\rangle(k'_2, \leq'_2)$ where $((k'_1, \leq'_1), (k'_2, \leq'_2), \beta') \in R_1$. Since by hypotesis $init(N(k_{02}))[\![\pi_2\rangle(k_2, \leq_2)$, then by Lemma 4.3.4, $\pi_2 \xrightarrow{e_2} \pi'_2$ where $\rho'_2(e_2) = t_2$ and $init(N(k_{02}))[\![\pi'_2\rangle(k'_2, \leq'_2)$. Note that, for $i = 1, 2$, (C'_i, ρ'_i) is a process of $N(k_{0i})$.

We extend f with the mapping $f'(e_1) = e_2$: since inductively f is an isomorphism between E_{C_1} and E_{C_2}, and $\pi_1 \xrightarrow{e_1} \pi'_1$, $\pi_2 \xrightarrow{e_2} \pi'_2$, then f' is an isomorphism between $\mathsf{E}_{C'_1}$ and $\mathsf{E}_{C'_2}$.

Now we need to check that the definition of β' from Definition 4.5.1 is coherent with the one obtained from R_2, i.e. the following condition holds:
$\forall b_1 \in Max(C'_1), b_2 \in Max(C'_2)$ such that $(s_1, n_1) \in k'_1$, $(s_2, n_2) \in k'_2$ where $\rho'_1(b_1) = s_1$ and $\rho'_2(b_2) = s_2$,

$$
(s_1, n_1)\ \beta'\ (s_2, n_2) \iff \begin{cases} (s_1, n_1) \in k''_1, (s_2, n_2) \in k''_2 \text{ and } (s_1, n_1)\ \beta\ (s_2, n_2) & \text{(i)} \\ \text{or} & \\ (s_1, n_1) \in k'_1 \setminus k''_1, (s_2, n_2) \in k'_2 \setminus k''_2 & \text{(ii)} \end{cases}
$$

Let us consider the complete transition sequence $\sigma_1 = \delta_1\ e_1$ of C'_1, such that:

- $init(N(k_{01}))[\![\rho'_1(\sigma_1)\rangle(k'_1, \leq'_1)$, and

- there exists $\sigma_2 = \delta_2\, e_2$ obtained by mapping each event in σ_1 with f', where $init(N(k_{02}))[\![\rho'_2(\sigma_2)\rangle(k'_2, \leq'_2)$.

It is trivial that $|\,\sigma_1\,| = |\,\sigma_2\,|$, therefore we prove the thesis by induction on the length of σ_1:

- Case 1: $\sigma_1 = e_1$, $\sigma_2 = e_2$. We prove the two implications separately.

  Proof $\Rightarrow$) by cases on the definition of β':

  - if ${}^\bullet b_1 = \emptyset \wedge {}^\bullet b_2 = \emptyset$:
    then b_1, b_2 did not move, so neither tokens $(s_1,n_1),(s_2,n_2)$ did. Therefore $(s_1,n_1) \in k''_1$ and $(s_2,n_2) \in k''_2$. Since the initial β is $k_{01} \times k_{02}$, we have $(s_1,n_1)\ \beta\ (s_2,n_2)$, satisfying condition (i).
  - if ${}^\bullet b_1 \neq \emptyset \wedge {}^\bullet b_2 \neq \emptyset \wedge f'({}^\bullet b_1) = {}^\bullet b_2$:
    then ${}^\bullet b_1 = e_1$ and ${}^\bullet b_2 = e_2$, therefore $(s_1,n_1) \in k'_1 \setminus k''_1$ and $(s_2,n_2) \in k'_2 \setminus k''_2$, satisfying condition (ii).

  Proof $\Leftarrow$) by cases:

  - if $(s_1,n_1) \in k''_1$ and $(s_2,n_2) \in k''_2$ and $(s_1,n_1)\ \beta\ (s_2,n_2)$:
    then since the only events in σ_1, σ_2 are respectively e_1, e_2 and $\rho'_1(e_1) = t_1, \rho'_2(e_2) = t_2$, we have ${}^\bullet b_1 = \emptyset$ and ${}^\bullet b_2 = \emptyset$ because each $s_i \in \rho'_i(Min(C'_i))$.
  - if $(s_1,n_1) \in k'_1 \setminus k''_1$ and $(s_2,n_2) \in k'_2 \setminus k''_2$:
    then since the only events in σ_1, σ_2 are respectively e_1, e_2 and $\rho'_1(e_1) = t_1, \rho'_2(e_2) = t_2$, and for each $s_i \in \rho'_i(Max(C'_i)$ we have ${}^\bullet b_1 = e_1$ and ${}^\bullet b_2 = e_2$. Moreover, $f'(e_1) = e_2$.

- Case $n+1$: $\sigma_1 = \delta_1\, e_1$ and $\sigma_2 = \delta_2\, e_2$. We prove the two implications separately.

  Proof $\Rightarrow$) by cases on the definition of β':

  - if ${}^\bullet b_1 = \emptyset \wedge {}^\bullet b_2 = \emptyset$:
    Then $b_1 \in Min(C'_1)$ and $b_2 \in Min(C'_2)$. For this reason, $(s_1,n_1) \in k_{01}$ and $(s_2,n_2) \in k_{02}$, therefore $(s_1,n_1) \in k''_1$ and $(s_2,n_2) \in k''_2$. Since the initial β is $k_{01} \times k_{02}$, we have $(s_1,n_1)\ \beta\ (s_2,n_2)$, satisfying condition (i).
  - if ${}^\bullet b_1 \neq \emptyset \wedge {}^\bullet b_2 \neq \emptyset \wedge f'({}^\bullet b_1) = {}^\bullet b_2$:
    There are two cases for the event which generates b_1:
    + if ${}^\bullet b_1 = e_1$: then since $f'({}^\bullet b_1) = e_2$, we have ${}^\bullet b_2 = e_2$; therefore $(s_1,n_1) \in k'_1 \setminus k''_1$ and $(s_2,n_2) \in k'_2 \setminus k''_2$, satisfying condition (ii).
    + if ${}^\bullet b_1 \neq e_1$: then since ${}^\bullet b_1 \neq \emptyset$, there exists $e'_1 \in \delta_1$ such that ${}^\bullet b_1 = e'_1$. By the fact that f' is an isomorphism between $\mathsf{E}_{C'_1}$ and $\mathsf{E}_{C'_2}$, and that $f'(e_1) = e_2$, there exists also $e'_2 \in \delta_2$ where $f^{-1}(e'_1) = e'_2$ such that ${}^\bullet b_2 = e'_2$. By inductive hypotesis on δ_1 we have $(s_1,n_1)\ \beta\ (s_2,n_2)$, satisfying condition (i).

  Proof $\Leftarrow$) by cases:

  - if $(s_1,n_1) \in k''_1$ and $(s_2,n_2) \in k''_2$ and $(s_1,n_1)\ \beta\ (s_2,n_2)$:
    then there are two possible cases for b_1:
    + if $b_1 \in Min(C'_1)$:
      then, since $Min(C'_1) = Min(C_1)$ and $Min(C'_2) = Min(C_2)$, the same reasoning of the base case of induction applies.
    + if $b_1 \notin Min(C'_1)$:
      then ${}^\bullet b_1 \neq \emptyset$; since however b_1 does not move, because $(s_1,n_1) \in k''_1$, it is possible to apply the induction hypotesis on δ_1, therefore ${}^\bullet b_2 \neq \emptyset$ and $f({}^\bullet b_1) = {}^\bullet b_2$, and by conservative extension of f, $f'({}^\bullet b_1) = {}^\bullet b_2$.
  - if $(s_1,n_1) \in k'_1 \setminus k''_1$ and $(s_2,n_2) \in k'_2 \setminus k''_2$:
    then since $\rho'_1(e_1) = t_1, \rho'_2(e_2) = t_2$, and for each $s_i \in \rho'_i(Max(C'_i)$ we have ${}^\bullet b_1 = e_1$ and ${}^\bullet b_2 = e_2$. Moreover, $f'(e_1) = e_2$.

Thus $(\pi'_1, f', \pi'_2) \in R_2$.
The case in which π_2 moves first is symmetrical.
Therefore, R_2 is an FC-bisimulation and $m_{01} \sim_{fc} m_{02}$. □

Corollary 4.5.4. (OIM-bisimilarity and FC-bisimilarity coincide) *Let $N = (S,A,T)$ be a net and m_1, m_2 two markings of N. $m_1 \sim_{oim} m_2$ if and only if $m_1 \sim_{fc} m_2$.*

Proof. By theorems 4.5.2 and 4.5.3, we get the thesis. □

Corollary 4.5.5. (FC-bisimilarity is decidable for finite bounded nets) *Given $N(m_1)$ and $N(m_2)$ bounded nets, it is decidable to check whether $m_1 \sim_{fc} m_2$.*

Proof. By Corollary 4.5.4 we have to check whether there exists an OIM-bisimulation $\mathfrak{B}$ for the given net N and initial markings m_1, m_2. The proof is then analogous to the one outlined in Corollary 4.4.5, and therefore omitted. □

As in Section 4.4, we conclude with some remarks on the complexity of this decision procedure. The reasoning of OIMC-bisimilarity can be adapted to OIM-bisimilarity, yielding a DEXPTIME complexity.

4.5.1 Related work on fully-concurrent bisimilarity

Decidability of fully-concurrent bisimilarity for bounded nets was already proved by Montanari and Pistore [MP97]. However, thir approach is not defined directly on Petri nets, rather it exploits an encoding of Petri nets into so-called *causal automata*, a model of computation designed for handling dependencies between transitions by means of names. In addition to this, the encoding in [MP97] works modulo isomorphisms, so that, in order to handle correctly the dependency names, at each step of the construction costly renormalizations are required. On the contrary, our construction is very concrete and works directly on the net. Thus, we conjecture that, even if the worst-case complexity is roughly the same, our algorithm performs generally better.

During the review process of a paper based on our work, an anonymous referee suggested that decidability of fully-concurrent bisimilarity for bounded nets using the ordered indexed marking idea was already proved by Valero-Ruiz in his PhD thesis [Val93]. However, Valero-Ruiz's approach differs from ours both in how the proof is conducted and, most importantly, in accuracy.

In his work, ordered indexed markings are defined in such a way that they are always closed. Even if the definition of token game admits that one can choose an arbitrary token (i.e. a token with an arbitrary index), the indexing of the target ordered marking is not clear. Depending on the chosen token to remove, there may appear an hole in the indexing (cf. Example 5), and therefore it is stated that the resulting ordered indexed marking may be subject to renaming to be again closed. This definition does not ensure the individuality of tokens: one token not used in a transition can be renamed, so that (even if it is not taking part to the transition) its index before and after the transition is different. Moreover, isomorphism of ordered indexed marking is defined only on closed ones, therefore is not clear how the renaming is carried on. Also the proof of the foundational theorem relating the operational preorder on tokens obtained by the ordered indexed marking semantics and the preorder derived by a process (similar to Lemma 4.3.4) is flawed. Despite allowing arbitrary tokens to be choosen when performing a transition, it is assumed that for each place the chosen token is always the one with index 1, as if there is some form of renormalization applied at some point to ensure that one can simply choose always that token. However, it is not clarified why the chosen token must always have index 1, and, if a renormalization is applied, it is not defined nor stated explicitly. Moreover, in the proof of that theorem, the author does not consider the case of a transition that needs more than one token on the same place to fire. In the same fashion, the definition of token game with ordered indexed markings does not consider the case of transitions generating more than one token on the same place. Another critical point is in the definition of the indexed ordered marking-based bisimulation (similar to Definition 4.5.1), where the possible renaming of tokens between transition steps is not taken into account.

These reasons are enough to conclude that Valero-Ruiz's proof of decidability of fully-concurrent bisimilarity for bounded nets is deeply flawed. Therefore our work can be considered the first one to have proved it using the ordered indexed marking approach.

Part II

PTI Nets

It is not the present which influences the future, thou fool, but the future which forms the present. You have it all backwards. Since the future is set, an unfolding of events which will assure that future is fixed and inevitable.

LETO II ATREIDES

Chapter 5

Basic Definitions

5.1 Petri nets with inhibitor arcs

Definition 5.1.1. (Place/Transition net with inhibitor arcs) *A finite Place/Transition net* with inhibitor arcs *(PTI net for short) is a tuple $N = (S,A,T,I)$, where*

- *(S,A,T) is a finite P/T net where ${}^\bullet t \neq \emptyset$ for all $t \in T$;*
- *$I \subseteq S \times T$ is the* inhibiting relation.

Given a transition $t \in T$, we use the notation ${}^\circ t$ to denote its inhibiting set $\{s \in S \mid (s,t) \in I\}$ *of places to be tested for absence of tokens.* □

We assume the usual graphical convention for Petri nets (see Section 2.1). In particular, the inhibiting relation I is graphically represented by arcs ending with a small circle on the transition side.

Definition 5.1.2. (Marking, PTI net system) *A* PTI net system *$N(m_0)$ is a tuple (S,A,T,I,m_0), where (S,A,T,I) is a PTI net and m_0 is a multiset over S, called the* initial marking. *We also say that $N(m_0)$ is a* marked *net.* □

Definition 5.1.3. (Token game) *A transition t is* enabled *at m, denoted $m[t\rangle$, if ${}^\bullet t \subseteq m$ and ${}^\circ t \cap dom(m) = \emptyset$. The execution, or* firing*, of t enabled at m produces the marking $m' = (m \ominus {}^\bullet t) \oplus t^\bullet$, written $m[t\rangle m'$.* □

Definition 5.1.4. (Firing sequence, reachable marking, safe net) *A* firing sequence *starting at m is defined inductively as follows:*

- *$m[\varepsilon\rangle m$ is a firing sequence (where ε denotes an empty sequence of transitions) and*
- *if $m[\sigma\rangle m'$ is a firing sequence and $m'[t\rangle m''$, then $m[\sigma t\rangle m''$ is a firing sequence.*

The set of reachable markings *from m is $[m\rangle = \{m' \mid \exists \sigma . m[\sigma\rangle m'\}$. A PTI system $N = (S,A,T,I,m_0)$ is* safe *if for each marking $m \in [m_0\rangle$, we have that $m(s) \leq 1$ for all $s \in S$.* □

5.2 Causality-based semantics

Following [BP99; BP00], we define here a possible causal semantics for PTI nets. In order to maintain the pleasant property that a process univocally determines the causal dependencies among its event, it is not enough to just enrich causal P/T nets with inhibitor arcs. Indeed, the *reason* why a condition is empty may influence the causal relation of events. To solve the problem, in [BP99; BP00] inhibitor arcs are partitioned into two sets: *before* inhibitor arcs and *after* inhibitor arcs. If a condition is connected to an event by a before inhibitor arc, the event fires because the condition has not held yet; if they are connected by an after inhibitor arc, the event fires because the condition does not hold anymore.

Definition 5.2.1. (Causal PTI net) *A causal PTI net is a tuple $C(\mathsf{m}_0) = (B,L,E,Y^{be},Y^{af},\mathsf{m}_0)$ satisfying the following conditions, denoting the flow relation of C by F:*

1. *(B,L,E,m_0) is a causal P/T net;*
2. *$(B,L,E,Y^{be} \cup Y^{af},\mathsf{m}_0)$ is a marked PTI net;*

3. before *and* after *requirements are met, i.e.*
   (a) *If $b\, Y^{be}\, e$, then there exists $e' \in E$ such that $e'\, \mathsf{F}\, b$, and*
   (b) *If $b\, Y^{af}\, e$, then there exists $e' \in E$ such that $b\, \mathsf{F}\, e'$;*
4. *the relation $\mathsf{F} \cup \prec_{af} \cup \prec_{be}$ is acyclic, where $\prec_{af} = \mathsf{F}^{-1} \circ Y^{af}$ and $\prec_{be} = (Y^{be})^{-1} \circ \mathsf{F}^{-1}$.*

We denote by $Min(C)$ the set m_0, and by $Max(C)$ the set $\{b \in B \mid b^\bullet = \emptyset\}$. □

Relation $\prec_{af} \subseteq E \times E$ states that $e \prec_{af} e'$ if e consumes the token in a place b inhibiting e': this is clearly a casual dependency. Instead, relation $\prec_{be} \subseteq E \times E$ states that $e \prec_{be} e'$ if e' produces a token in a place b inhibiting e: this is clearly a temporal precedence, because the two events can be causally independent, yet they cannot occur in any order, as if e' occurs, then e is disabled.

Definition 5.2.2. (Folding and PTI process) *A* folding *from a causal PTI net $C = (B, L, E, Y^{be}, Y^{af}, \mathsf{m}_0)$ into a PTI net system $N(m_0) = (S, A, T, I, m_0)$ is a function $\rho : B \cup E \to S \cup T$, which is type-preserving, i.e., such that $\rho(B) \subseteq S$ and $\rho(E) \subseteq T$, satisfying the following:*

- *ρ is a P/T folding from (B, L, E, m_0) into (S, A, T, m_0);*
- *for all $s \in S$ and $e \in E$, if $(s, \rho(e)) \in I$ then for all $b \in B$ such that $\rho(b) = s$, it holds $(b, e) \in Y^{be} \cup Y^{af} \cup \mathsf{F}^{-1}$, and*
  for all $b \in B$ and $e \in E$, if $(b, e) \in Y^{be} \cup Y^{af}$ then $(\rho(b), \rho(e)) \in I$.

A pair (C, ρ), where C is a causal PTI net and ρ a folding from C to a PTI net system $N(m_0)$, is a PTI process *of $N(m_0)$.* □

Each inhibitor arc in the causal net has a corresponding inhibitor arc in the net system. The only case where a condition b is not connected to an event e is when b is in the post-set of e: as b starts to hold only after e occurs, the only possibility is to put a before arc. This would make the relation $\prec_{be}$ reflexive, invalidating item 4 of Definition 5.2.1. However, since b is in the post-set of e, we are sure that e happens before b is fulfilled, hence making useless the presence of a before inhibitor arc. For this reason, with the requirement $(b, e) \in Y^{be} \cup Y^{af} \cup \mathsf{F}^{-1}$, we ask for the presence of an inhibitor arc only if there exists no flow from e to b.

Definition 5.2.3. (Moves of a PTI process) *Let $N(m_0) = (S, A, T, I, m_0)$ be a PTI net system and let (C_i, ρ_i), for $i = 1, 2$, be two PTI processes of $N(m_0)$, where $C_i = (B_i, L, E_i, Y_i^{be}, Y_i^{af}, \mathsf{m}_0)$. We say that (C_1, ρ_1) moves in one step to (C_2, ρ_2) through e, denoted by $(C_1, \rho_1) \xrightarrow{e} (C_2, \rho_2)$, if the following hold:*

- *${}^\bullet e \subseteq Max(C_1)$, $E_2 = E_1 \cup \{e\}$, $B_2 = B_1 \cup e^\bullet$, $\rho_1 \subseteq \rho_2$, i.e. the P/T process of (C_1, ρ_1) moves in one step through e to the P/T process of (C_2, ρ_2).*
- *Given two relations $\mathscr{B}$ and $\mathscr{A}$, defined as*
  - *$\forall b \in e^\bullet$, $\forall e' \in E_1$ we have $b\, \mathscr{B}\, e'$ if and only if $(\rho_2(b), \rho_2(e')) \in I$,*
  - *$\forall b \in B_2$ such that $b^\bullet \neq \emptyset$, we have $b\, \mathscr{A}\, e$ if and only if $(\rho_2(b), \rho_2(e)) \in I$,*

  we have $\{b \in B_2 \mid b\, \mathscr{A}\, e\} \cap Max(C_1) = \emptyset$.
- *Finally, $Y_2^{be} = Y_1^{be} \cup \mathscr{B}$ and $Y_2^{af} = Y_1^{af} \cup \mathscr{A}$.* □

The item $\{b \in B_2 \mid b\, \mathscr{A}\, e\} \cap Max(C_1) = \emptyset$ models the fact that a transition can fire only if all its inhibiting places are free. Indeed, an event can fire only if its (so far known) inhibiting conditions are not maximal. Note that, by construction, before arcs can connect only new inhibiting conditions to past events and in particular we do not allow before arcs connecting a condition in the post-set of a newly added event e with the event e itself. Moreover, after arcs can only connect old inhibiting conditions to the new event e and since $\{b \in B_2 \mid b\, \mathscr{A}\, e\} \cap Max(C_1) = \emptyset$, the old inhibiting conditions cannot be in the pre-set of the newly added event e. Therefore, both relations $\prec_2^{be}$ and $\prec_2^{af}$ are acyclic, and since F_2 is acyclic too, (C_2, ρ_2) is truly a process of $N(m_0)$.

Example 7. Consider the three nets in Figure 5.1, where we use the graphical convention that before inhibitor arcs and after inhibitor arcs are represented by lines between a condition and an event: the former labeled by b, the latter labeled by a. The initial marking of N is $m_0 = s_1 \oplus s_3$. The shape of a process generated by $N(m_0)$ may depend on the order of transitions in a given transition sequence. As a matter of fact, transition sequences containing the same transitions but in a different order may generate different processes, e.g. C_1 and C_2. Indeed, C_1 represents the transition sequence $t_1\, t_3\, t_2$, while C_2 represents the transition sequence $t_2\, t_1\, t_3$. Note that the underlying causal P/T net of these two processes is the same, but before and after inhibitor arcs are different.

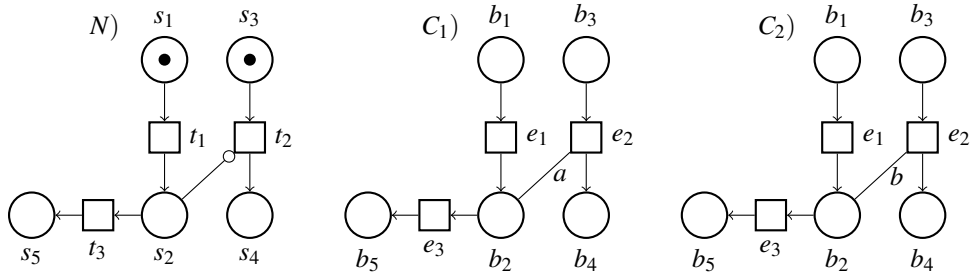

Figure 5.1: A marked PTI net and two PTI causal nets corresponding to its two maximal processes.

5.2.1 Causal-net bisimilarity for PTI nets

Definition 5.2.4. (Causal-net bisimulation) *Let $N = (S,A,T,I)$ be a PTI net. A* causal-net bisimulation *is a relation R, composed of triples of the form (ρ_1, C, ρ_2), where, for $i = 1,2$, (C,ρ_i) is a process of $N(m_{0i})$ for some m_{0i}, such that if $(\rho_1, C, \rho_2) \in R$ then*

i) $\forall t_1, C', \rho_1'$ such that $(C,\rho_1) \xrightarrow{e} (C',\rho_1')$, where $\rho_1'(e) = t_1$, $\exists t_2, \rho_2'$ such that $(C,\rho_2) \xrightarrow{e} (C',\rho_2')$, where $\rho_2'(e) = t_2$, and $(\rho_1', C', \rho_2') \in R$;

ii) symmetrically, $\forall t_2, C', \rho_2'$ such that $(C,\rho_2) \xrightarrow{e} (C',\rho_2')$, where $\rho_2'(e) = t_2$, $\exists t_1, \rho_1'$ such that $(C,\rho_1) \xrightarrow{e} (C',\rho_1')$, where $\rho_1'(e) = t_1$, and $(\rho_1', C', \rho_2') \in R$.

Two markings m_1 and m_2 of N are cn-bisimilar (or cn-bisimulation equivalent), denoted by $m_1 \sim_{cn} m_2$, if there exists a causal-net bisimulation R containing a triple $(\rho_1^0, C^0, \rho_2^0)$, where C^0 contains no events and $\rho_i^0(Min(C^0)) = \rho_i^0(Max(C^0)) = m_i$ for $i = 1,2$. □

If $m_1 \sim_{cn} m_2$, then these two markings have the same causal PTI nets, so that the executions originating from the two markings have the same causal dependencies (determined by F and $\prec_{af}$) and the same temporal dependencies (determined by $\prec_{be}$). Causal-net bisimilarity $\sim_{cn}$ is an equivalence relation, as proved in the following propositions.

Proposition 5.2.5. *For each PTI net $N = (S,A,T,I)$, the following hold:*

1. *the identity relation $\mathscr{I} = \{(\rho, C, \rho) \mid \exists m \in \mathscr{M}(S).(C,\rho)$ is a process of $N(m)\}$ is a causal-net bisimulation;*
2. *the inverse relation $R^{-1} = \{(\rho_2, C, \rho_1) \mid (\rho_1, C, \rho_2) \in R\}$ of a causal-net bisimulation R is a causal-net bisimulation;*
3. *the relational composition, up to net isomorphism, $R_1 \circ R_2 = \{(\rho_1, C, \rho_3) \mid \exists \rho_2.\, (\rho_1, C, \rho_2) \in R_1 \wedge (\overline{\rho}_2, \overline{C}, \overline{\rho}_3) \in R_2 \wedge (C,\rho_2)$ and $(\overline{C},\overline{\rho}_2)$ are isomorphic processes via $f \wedge \rho_3 = \overline{\rho}_3 \circ f\}$ of two causal-net bisimulations R_1 and R_2 is a causal-net bisimulation;*
4. *the union $\bigcup_{i\in I} R_i$ of causal-net bisimulations R_i is a causal-net bisimulation.*

Proof. Trivial for 1, 2 and 4. For case 3, assume that $(\rho_1, C, \rho_3) \in R_1 \circ R_2$ and that $(C,\rho_1) \xrightarrow{e} (C',\rho_1')$ with $\rho_1'(e) = t_1$. Since R_1 is a causal-net bisimulation and $(\rho_1, C, \rho_2) \in R_1$, we have that there exist t_2, ρ_2' such that $(C,\rho_2) \xrightarrow{e} (C',\rho_2')$, with $\rho_2'(e) = t_2$, and $(\rho_1', C', \rho_2') \in R_1$. Since (C,ρ_2) and $(\overline{C},\overline{\rho}_2)$ are isomorphic via f, it follows that $(\overline{C},\overline{\rho}_2) \xrightarrow{e'} (\overline{C}',\overline{\rho}_2')$, with $\overline{\rho}_2'(e') = t_2$, where (C',ρ_2') and $(\overline{C}',\overline{\rho}_2')$ are isomorphic via f', where f' extends f in the obvious way (i.e., by mapping event e to e').

As $(\overline{\rho}_2, \overline{C}, \overline{\rho}_3) \in R_2$ and R_2 is a causal-net bisimulation, for $(\overline{C},\overline{\rho}_2) \xrightarrow{e'} (\overline{C}',\overline{\rho}_2')$, with $\overline{\rho}_2'(e') = t_2$, there exist $t_3, \overline{\rho}_3'$ such that $(\overline{C},\overline{\rho}_3) \xrightarrow{e'} (\overline{C}',\overline{\rho}_3')$, with $\overline{\rho}_3'(e') = t_3$, and $(\overline{\rho}_2', \overline{C}', \overline{\rho}_3') \in R_2$. As $\rho_3 = \overline{\rho}_3 \circ f$, it follows that (C,ρ_3) and $(\overline{C},\overline{\rho}_3)$ are isomorphic via f. Therefore, $(C,\rho_3) \xrightarrow{e} (C',\rho_3')$ is derivable, too, where $\rho_3'(e) = t_3$ and $\rho_3' = \overline{\rho}_3' \circ f'$, so that (C',ρ_3') and $(\overline{C}',\overline{\rho}_3')$ are isomorphic via f'.

Summing up, if $(\rho_1, C, \rho_3) \in R_1 \circ R_2$ and $(C,\rho_1) \xrightarrow{e} (C',\rho_1')$, with $\rho_1'(e) = t_1$, then $\exists t_3, \rho_3'$ such that $(C,\rho_3) \xrightarrow{e} (C',\rho_3')$, with $\rho_3'(e) = t_3$, and $(\rho_1', C', \rho_3') \in R_1 \circ R_2$.

The symmetric case when (C,ρ_3) moves first is analogous, hence omitted. Therefore,

$R_1 \circ R_2$ is a causal-net bisimulation, indeed. □

Proposition 5.2.6. *For each PTI net $N = (S,A,T,I)$, relation $\sim_{cn} \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$ is an equivalence relation.*

Proof. Standard, by exploiting Proposition 5.2.5. □

Chapter 6

Pti-place bisimilarity

We now present pti-place bisimilarity, which conservatively extends *place bisimilarity* [ABS91; Gor21] to the case of PTI nets. First, we list some more auxiliary properties of additive closure [Gor20b].

6.1 Additive closure and its properties

As PTI nets are P/T nets extended with a set of inhibitor arcs, the definition of additive closure still applies. We add some remarks to Definition 2.1.7, which we will use in the following chapters.

Remark* 4.** *[Gor20b](Some properties of additive closure)** *For each place relation $R \subseteq S \times S$, the following hold:*

1. *If R is an equivalence relation, then $R^{\oplus}$ is an equivalence relation.*
2. *If $R_1 \subseteq R_2$, then $R_1^{\oplus} \subseteq R_2^{\oplus}$, i.e., the additive closure is monotone.*
3. *If $(m_1, m_2) \in R^{\oplus}$ and $(m_1', m_2') \in R^{\oplus}$, then $(m_1 \oplus m_1', m_2 \oplus m_2') \in R^{\oplus}$, i.e., the additive closure is additive.*

Now we list some useful, and less obvious, properties of additively closed place relations that will be useful in the following.

Proposition 6.1.1. *[Gor20b]***(Some more properties of additive closure)** *For each family of place relations $R_i \subseteq S \times S$, the following hold:*

1. *$\emptyset^{\oplus} = \{(\theta, \theta)\}$, i.e., the additive closure of the empty place relation yields a singleton marking relation, relating the empty marking to itself.*
2. *$(\mathscr{I}_S)^{\oplus} = \mathscr{I}_M$, i.e., the additive closure of the identity relation on places $\mathscr{I}_S = \{(s,s) \mid s \in S\}$ yields the identity relation on markings $\mathscr{I}_M = \{(m,m) \mid m \in \mathscr{M}(S)\}$.*
3. *$(R^{\oplus})^{-1} = (R^{-1})^{\oplus}$, i.e., the inverse of an additively closed relation R equals the additive closure of its inverse R^{-1}.*
4. *$(R_1 \circ R_2)^{\oplus} = (R_1^{\oplus}) \circ (R_2^{\oplus})$, i.e., the additive closure of the composition of two place relations equals the compositions of their additive closures.* □

Finally, we consider the problem of checking if $(m_1, m_2) \in R^{\oplus}$ [Gor20b; Gor21]. The naive algorithm for checking whether $(m_1, m_2) \in R^{\oplus}$ would simply consider m_1 represented as $s_1 \oplus s_2 \oplus \ldots \oplus s_k$, and then would scan all the possible permutations of m_2, each represented as $s_1' \oplus s_2' \oplus \ldots \oplus s_k'$, to check that $(s_i, s_i') \in R$ for $i = 1, \ldots, k$. Of course, this algorithm is in $O(k!)$, but a computationally better algorithm can be adapted from graph theory.

Definition 6.1.2. (Bipartite graph) *A* bipartite graph *$G = (V, E)$ is a graph with at least two vertices such that V can be split into two disjoint subsets V_1 and V_2, both nonempty, where every edge $uv \in E$ is such that $u \in V_1$ and $v \in V_2$, or $v \in V_1$ and $u \in V_2$.* □

Definition 6.1.3. (Matching, maximum matching, perfect matching) *Given a bipartite graph $G = (V_1 \cup V_2, E)$, a set of edges $M \subseteq E$ is a* matching *if no vertex $v \in V$ is incident with more than one edge $e \in M$, i.e. there do not exist two distinct edges $vu, vw \in M$ where $u \neq w$. A matching of maximum cardinality is called a* maximum matching*. A matching is* perfect *if every vertex of the graph is incident to an edge of the matching.*
Note that checking if a maximum matching is perfect reduces to checking whether the size of the matching equals the number of nodes in each partition. □

The algorithm proposed by Hopcroft, Karp and Karzanov [HK73] computes the maximum matching in a bipartite graph in $O(h\sqrt{k})$, where h is the number of edges in the bipartite graph and k is maximum number of nodes in a partition (therefore, $h \leq k^2$).

Given two markings m_1, m_2 and a place relation R, checking whether $(m_1, m_2) \in R^{\oplus}$ can be reduced to checking if there exists a perfect matching in a bipartite graph in a straightforward way.

First of all, a bipartite graph must be generated from m_1, m_2, R:

- Define an indexing i for each token in m_1 (resp. m_2) in such a way that each pair $(s_n, i(s_n))$ is unique for m_1 (resp. m_2).
- Define V_1 (resp. V_2) as the set of resulting tokens, each tagged with l (resp. r);
- Define $E = \{((s_1, n_1, l), (s_2, n_2, r)) \mid s_1 \, R \, s_2 \wedge (s_1, n_1, l) \in V_1 \wedge (s_2, n_2, r) \in V_2\}$.

As $V_1 \cap V_2 = \emptyset$ because of the tags, $(V_1 \cup V_2, E)$ is a bipartite graph. For the sake of simplicity, we will drop the tags if clear from the context.

The application of the Hopcroft-Karp-Karzanov algorithm [HK73] on the resulting graph yields a maximum matching M; moreover, if each vertex is incident to an edge of M, the matching is perfect.

If M is a perfect matching and $|M| = |m_1| = |m_2|$, then by construction each arc $m \in M$ relates two individual tokens, one from m_1 and one from m_2, which are not related by any other arc $m' \in M$. Therefore, each arc in the matching can also be seen as an application of the additive closure rule **(Clo)** in the derivation of $(m_1, m_2) \in R^{\oplus}$ (see Definition 2.1.7).

The definition of the bipartite graph takes $O(k^2)$ time (where k is the maximum number of tokens), then, since $|E| \leq k^2$, we have that the maximum matching algorithm requires $O(k^2\sqrt{k})$, and finally, checking whether the matching M is perfect and if $|M| = |m_1| = |m_2|$ takes linear time. To sum up, the complexity of checking whether $(m_1, m_2) \in R^{\oplus}$ is in $O(k^2\sqrt{k})$.

Example 8. Consider two markings $m_1 = s_1 \oplus 2 \cdot s_2$, $m_2 = s_1 \oplus s_2 \oplus s_3$ and a place relation $R = \{(s_1, s_1), (s_2, s_2), (s_2, s_3)\}$. Figure 6.1 shows the bipartite graph derived from m_1, m_2, R. Note that, for the sake of simplicty, tags are left implicit: vertices tagged with l are displayed on the left and the ones tagged with r on the right. To uniquely identify each token, two closed indexed markings (see Definition 4.1.1) are used. In red, we show a perfect matching corresponding to a proof of $(m_1, m_2) \in R^{\oplus}$.

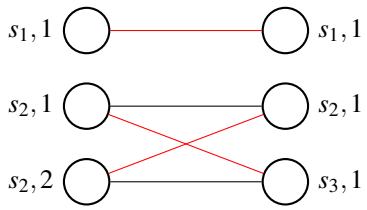

Figure 6.1: Checking $(s_1 \oplus 2 \cdot s_2, s_1 \oplus s_2 \oplus s_3) \in R^{\oplus}$ as maximum matching (in red) on a bipartite graph.

6.2 Pti-place bisimulation and its properties

We are now ready to introduce pti-place bisimulation, which is a non-interleaving behavioral relation defined over the net places. Note that for P/T nets, place bisimulation [ABS91; Gor21] and pti-place bisimulation coincide because $I = \emptyset$.

Definition 6.2.1. (Pti-place bisimulation) *Let $N = (S, A, T, I)$ be a PTI net. A* pti-place bisimulation *is a relation $R \subseteq S \times S$ such that if $(m_1, m_2) \in R^{\oplus}$ then*

1. *$\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists t_2$ such that $m_2[t_2\rangle m_2'$ and*
   - *(a) $({}^\bullet t_1, {}^\bullet t_2) \in R^{\oplus}$, $({}^\circ t_1, {}^\circ t_2) \in R^{\oplus}$, $(t_1^\bullet, t_2^\bullet) \in R^{\oplus}$, $l(t_1) = l(t_2)$, $(m_1', m_2') \in R^{\oplus}$,*
   - *(b) $\forall s, s' \in S. (s, s') \in R \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$.*
2. *$\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists t_1$ such that $m_1[t_1\rangle m_1'$ and*
   - *(a) $({}^\bullet t_1, {}^\bullet t_2) \in R^{\oplus}$, $({}^\circ t_1, {}^\circ t_2) \in R^{\oplus}$, $(t_1^\bullet, t_2^\bullet) \in R^{\oplus}$, $l(t_1) = l(t_2)$, $(m_1', m_2') \in R^{\oplus}$,*

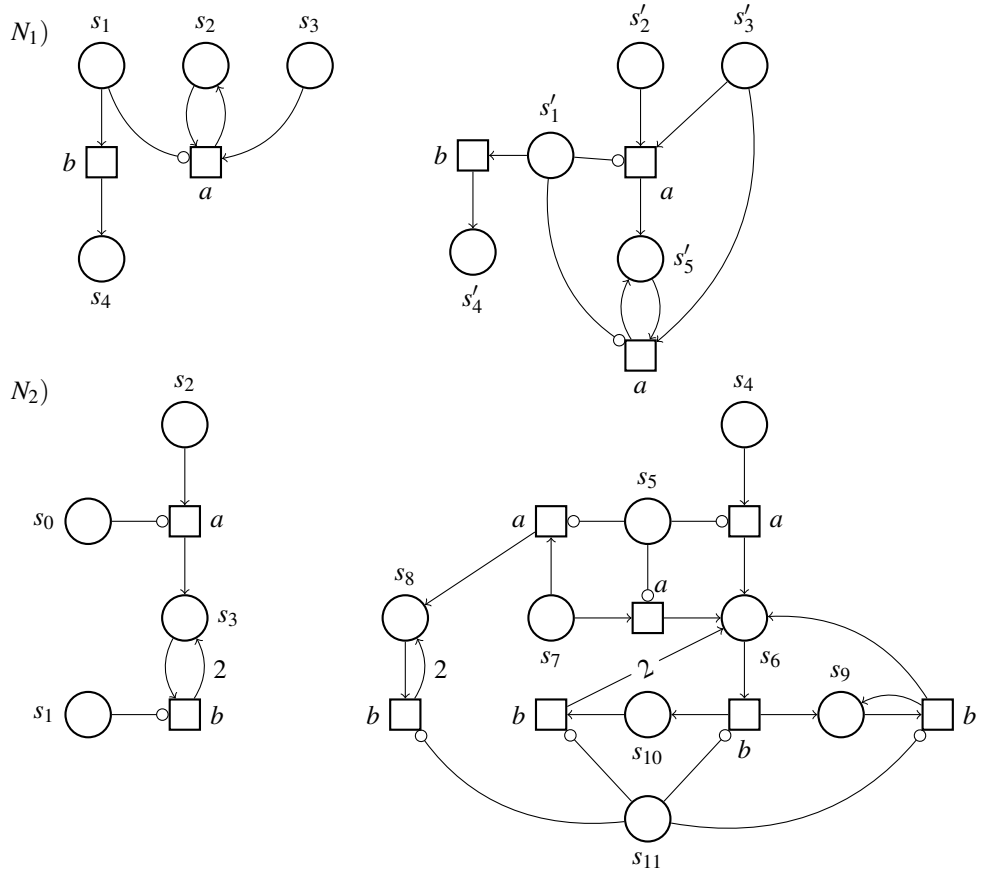

Figure 6.2: Two PTI nets, whose transitions are labeled either by a or by b.

(b) $\forall s,s' \in S.(s,s') \in R \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$.

Two markings m_1 and m_2 are pti-place bisimilar*, denoted by $m_1 \sim_p m_2$, if there exists a pti-place bisimulation R such that $(m_1,m_2) \in R^\oplus$.* □

Conditions 1(b) and 2(b) make sure that the relation R respects the inhibiting behavior of places. Thus, not only the sets ${}^\circ t_1$ and ${}^\circ t_2$ must be bijectively related, but also an inhibiting place for one of the two transitions cannot be related via R to a non-inhibiting place for the other transition.

Example 9. Consider the PTI net N_1 in Figure 6.2, where the right part is an unwinding of the left one. The relation $R = \{(s_1,s'_1),(s_2,s'_2),(s_3,s'_3),(s_4,s'_4),(s_2,s'_5)\}$ is a pti-place bisimulation; and so, e.g., $2 \cdot s_2 \oplus 2 \cdot s_3 \sim_p s'_2 \oplus s'_5 \oplus 2 \cdot s'_3$.

Now consider the PTI net N_2 in Figure 6.2. Not only the loop labeled by b on the left is unwinded on the right, but also the a-labeled transition on the left is replicated three times on the right. The relation $R' = \{(s_0,s_5),(s_1,s_{11}),(s_2,s_4),(s_2,s_7),(s_3,s_6),(s_3,s_8),(s_3,s_9),(s_3,s_{10})\}$ is a pti-place bisimulation and so, e.g., $2 \cdot s_2 \oplus s_3 \sim_p s_4 \oplus s_7 \oplus s_9$.

In order to prove that $\sim_p$ is an equivalence relation, we now list some useful properties of pti-place bisimulation relations.

Proposition 6.2.2. *For each PTI net $N = (S,A,T,I)$, the following hold:*

1. *The identity relation $\mathscr{I}_S = \{(s,s) \mid s \in S\}$ is a pti-place bisimulation;*
2. *the inverse relation $R^{-1} = \{(s',s) \mid (s,s') \in R\}$ of a pti-place bisimulation R is a pti-place bisimulation;*
3. *the relational composition $R_1 \circ R_2 = \{(s,s'') \mid \exists s'.(s,s') \in R_1 \wedge (s',s'') \in R_2\}$ of two pti-place bisimulations R_1 and R_2 is a pti-place bisimulation.*

Proof. The proof is almost standard, due to Proposition 6.1.1.

(1) $\mathscr{I}_S$ is a pti-place bisimulation as for each $(m,m) \in \mathscr{I}_S^{\oplus}$ whatever transition t the left (or right) marking m performs a transition (say, $m[t\rangle m'$), the right (or left) instance of m in the pair does exactly the same transition $m[t\rangle m'$ and, of course, $({}^\bullet t, {}^\bullet t) \in \mathscr{I}_S^{\oplus}$, $({}^\circ t, {}^\circ t) \in \mathscr{I}_S^{\oplus}$, $(t^\bullet, t^\bullet) \in \mathscr{I}_S^{\oplus}$, $l(t) = l(t)$, $(m',m') \in \mathscr{I}_S^{\oplus}$, by Proposition 6.1.1(2), and, also, $\forall s \in S.(s,s) \in \mathscr{I}_S \Rightarrow (s \in {}^\circ t \Leftrightarrow s \in {}^\circ t)$, as required by the pti-place bisimulation definition.

(2) Suppose $(m_2,m_1) \in (R^{-1})^{\oplus}$ and $m_2[t_2\rangle m_2'$. By Proposition 6.1.1(3) $(m_2,m_1) \in (R^{\oplus})^{-1}$ and so $(m_1,m_2) \in R^{\oplus}$. Since R is a pti-place bisimulation, item 2 of the bisimulation game ensures that there exist t_1 and m_1' such that $m_1[t_1\rangle m_1'$, with $({}^\bullet t_1, {}^\bullet t_2) \in R^{\oplus}$, $({}^\circ t_1, {}^\circ t_2) \in R^{\oplus}$, $l(t_1) = l(t_2)$, $(t_1^\bullet, t_2^\bullet) \in R^{\oplus}$ and $(m_1',m_2') \in R^{\oplus}$; moreover, $\forall s,s' \in S.(s,s') \in R \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$. Summing up, if $(m_2,m_1) \in (R^{-1})^{\oplus}$, to the move $m_2[t_2\rangle m_2'$, m_1 replies with the move $m_1[t_1\rangle m_1'$, such that (by Proposition 6.1.1(3)) $({}^\bullet t_2, {}^\bullet t_1) \in (R^{-1})^{\oplus}$, $({}^\circ t_2, {}^\circ t_1) \in (R^{-1})^{\oplus}$, $l(t_2) = l(t_1)$, $(t_2^\bullet, t_1^\bullet) \in (R^{-1})^{\oplus}$, $(m_2',m_1') \in (R^{-1})^{\oplus}$ and, moreover, $\forall s,s' \in S.(s',s) \in R^{-1} \Rightarrow (s' \in {}^\circ t_2 \Leftrightarrow s \in {}^\circ t_1)$, as required. The case when m_1 moves first is symmetric and so omitted.

(3) Suppose $(m,m'') \in (R_1 \circ R_2)^{\oplus}$ and $m[t_1\rangle m_1$. By Proposition 6.1.1(4), we have that $(m,m'') \in R_1^{\oplus} \circ R_2^{\oplus}$, and so there exists m' such that $(m,m') \in R_1^{\oplus}$ and $(m',m'') \in R_2^{\oplus}$. As $(m,m') \in R_1^{\oplus}$ and R_1 is a pti-place bisimulation, if $m[t_1\rangle m_1$, then there exist t_2 and m_2 such that $m'[t_2\rangle m_2$ with $({}^\bullet t_1, {}^\bullet t_2) \in R_1^{\oplus}$, $({}^\circ t_1, {}^\circ t_2) \in R_1^{\oplus}$, $l(t_1) = l(t_2)$, $(t_1^\bullet, t_2^\bullet) \in R_1^{\oplus}$ and $(m_1,m_2) \in R_1^{\oplus}$; moreover, $\forall s,s' \in S.(s,s') \in R_1 \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$. But as $(m',m'') \in R_2^{\oplus}$ and R_2 is a pti-place bisimulation, we have also that there exist t_3 and m_3 such that $m''[t_3\rangle m_3$ with $({}^\bullet t_2, {}^\bullet t_3) \in R_2^{\oplus}$, $({}^\circ t_2, {}^\circ t_3) \in R_2^{\oplus}$, $l(t_2) = l(t_3)$, $(t_2^\bullet, t_3^\bullet) \in R_2^{\oplus}$ and $(m_2,m_3) \in R_2^{\oplus}$; moreover, $\forall s',s'' \in S.(s',s'') \in R_2 \Rightarrow (s' \in {}^\circ t_2 \Leftrightarrow s'' \in {}^\circ t_3)$. Summing up, for $(m,m'') \in (R_1 \circ R_2)^{\oplus}$, if $m[t_1\rangle m_1$, then there exist t_3 and m_3 such that $m''[t_3\rangle m_3$ and (by Proposition 6.1.1(4)) $({}^\bullet t_1, {}^\bullet t_3) \in (R_1 \circ R_2)^{\oplus}$, $({}^\circ t_1, {}^\circ t_3) \in (R_1 \circ R_2)^{\oplus}$, $l(t_1) = l(t_3)$, $(t_1^\bullet, t_3^\bullet) \in (R_1 \circ R_2)^{\oplus}$ and $(m_1,m_3) \in (R_1 \circ R_2)^{\oplus}$; moreover, $\forall s,s'' \in S.(s,s'') \in R_1 \circ R_2 \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s'' \in {}^\circ t_3)$, as required. The case when m'' moves first is symmetric and so omitted. □

Proposition 6.2.3. *For each PTI net $N = (S,A,T,I)$, relation $\sim_p \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$ is an equivalence relation.*

Proof. Direct consequence of Proposition 6.2.2. □

By Definition 6.2.1, pti-place bisimilarity can be defined in the following way:

$\sim_p = \bigcup\{R^{\oplus} \mid R \text{ is a pti-place bisimulation}\}$.

By monotonicity of the additive closure (Remark 4(2)), if $R_1 \subseteq R_2$, then $R_1^{\oplus} \subseteq R_2^{\oplus}$. Hence, we can restrict our attention to maximal pti-place bisimulations only:

$\sim_p = \bigcup\{R^{\oplus} \mid R \text{ is a } \textit{maximal} \text{ pti-place bisimulation}\}$.

However, it is not true that

$\sim_p = (\bigcup\{R \mid R \text{ is a } \textit{maximal} \text{ pti-place bisimulation}\})^{\oplus}$

because the union of pti-place bisimulations may be not a pti-place bisimulation (as already observed for place bisimulation in [ABS91; Gor21]), so that its definition is not coinductive.

Example 10. Consider the net in Figure 6.3. Clearly, R and R', defined as follows, are both pti-place bisimulations.

$$R = \{(s_1,s_1),(s_2,s_2),(s_3,s_3),(s_4,s_4),(s_5,s_5)\}$$
$$R' = \{(s_1,s_1),(s_2,s_3),(s_3,s_2),(s_4,s_4),(s_5,s_5)\}$$

It would be easy to make R,R' maximal by adding all possible combinations of s_1,s_4,s_5 (since they are all stuck places), however it would not be meaningful for this example, so we prefer to keep it simple.

Note that the union $R \cup R'$ is not a pti-place bisimulation as, for example, $2 \cdot s_2 \not\sim_p s_2 \oplus s_3$. Indeed, if $2 \cdot s_2$ moves first by $2 \cdot s_2[t_1\rangle s_1 \oplus s_2$, then $s_2 \oplus s_3$ can only respond with $s_2 \oplus s_3[t_2\rangle s_5$ since t_1 and t_3 are inhibited. However, $s_1 \oplus s_2 \not\sim_p s_5$, because the former can perform transition t_1, while the latter is stuck. □

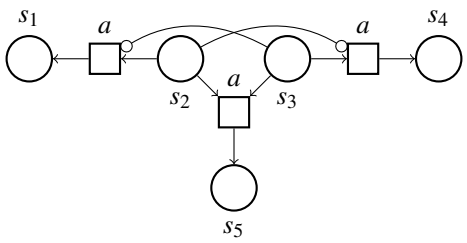

Figure 6.3: A PTI net.

6.3 Pti-place bisimilarity is finer than causal-net bisimilarity

Theorem 6.3.1. (Pti-place bisimilarity implies causal-net bisimilarity) *Let $N = (S,A,T,I)$ be a PTI net and m_1, m_2 two of its markings. If $m_1 \sim_p m_2$, then $m_1 \sim_{cn} m_2$.*

Proof. If $m_1 \sim_p m_2$, then there exists a pti-bisimulation R_1 such that $(m_1,m_2) \in R_1^{\oplus}$. Let us consider

$$R_2 \stackrel{def}{=} \{(\rho_1,C,\rho_2) | (C,\rho_1) \text{ is a PTI process of } N(m_1) \text{ and } (C,\rho_2) \text{ is a PTI process of } N(m_2) \text{ and } \forall b \in B\ (\rho_1(b),\rho_2(b)) \in R_1\}.$$

We want to prove that R_2 is a causal-net bisimulation. First of all, consider a triple of the form $(\rho_1^0, C^0, \rho_2^0)$, where C^0 is the causal PTI net without events and ρ_1^0, ρ_2^0 are such that $\rho_i^0(Min(C^0)) = \rho_i^0(Max(C^0)) = \rho_i^0(B^0) = m_i$ for $i = 1,2$, and $(\rho_1^0(b), \rho_2^0(b)) \in R_1$ for all $b \in B^0$. Then $(\rho_1^0, C^0, \rho_2^0)$ must belong to R_2, because (C^0, ρ_i^0) is a process of $N(m_i)$, for $i = 1,2$ and, by hypothesis, $(m_1,m_2) \in R_1^{\oplus}$. Hence, if R_2 is a causal-net bisimulation, then the triple $(\rho_1^0, C^0, \rho_2^0) \in R_2$ ensures that $m_1 \sim_{cn} m_2$.

Assume $(\rho_1,C,\rho_2) \in R_2$. In order for R_2 to be a cn-bisimulation, we must prove that

1. $\forall t_1, C', \rho_1'$ such that $(C,\rho_1) \xrightarrow{e} (C',\rho_1')$, where $\rho_1'(e) = t_1$, $\exists t_2, \rho_2'$ such that $(C,\rho_2) \xrightarrow{e} (C',\rho_2')$, where $\rho_2'(e) = t_2$, and $(\rho_1', C', \rho_2') \in R_2$;

2. symmetrical, if (C,ρ_2) moves first.

Assume $(C,\rho_1) \xrightarrow{e} (C',\rho_1')$ with $\rho_1'(e) = t_1$. Since $(\rho_1,C,\rho_2) \in R_2$, for all $b \in Max(C)$ we have $(\rho_1(b),\rho_2(b)) \in R_1$ and therefore $(\rho_1(Max(C)),\rho_2(Max(C))) \in R_1^{\oplus}$. As $\rho_1(Max(C))[t_1\rangle \rho_1'(Max(C'))$ and R_1 is a pti-place bisimulation, there exist t_2, m_2 such that $\rho_2(Max(C))[t_2\rangle m_2$ with $({}^\bullet t_1, {}^\bullet t_2) \in R_1^{\oplus}$, $({}^\circ t_1, {}^\circ t_2) \in R_1^{\oplus}$, $l(t_1) = l(t_2)$, $(t_1^\bullet, t_2^\bullet) \in R_1^{\oplus}$, $(\rho_1'(Max(C')), m_2) \in R_1^{\oplus}$ and, moreover, $\forall s,s' \in S.(s,s') \in R_1 \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$.

Therefore, since t_1 and t_2 have the same pre-sets/post-sets up to R_1, it is possible to derive $(C,\rho_2) \xrightarrow{e} (C'',\rho_2')$, where ρ_2' is such that $\rho_2'(e) = t_2$ and $(\rho_1'(b),\rho_2'(b)) \in R_1$ for each $b \in e^\bullet$ (which is really possible because $(t_1^\bullet, t_2^\bullet) \in R_1^{\oplus}$). Now we prove that $C' = C''$. The underlying P/T parts of C' and C'' are obviously the same (so C' and C'' have the same events, the same conditions and the same flow relation), therefore we have to check that also the newly added (after/before) inhibitor arcs are the same, i.e.,

- $\forall b \in B'$ such that $b^\bullet \neq \emptyset$ we have $b \mathscr{A}_1 e \iff b \mathscr{A}_2 e$, and
- $\forall b \in e^\bullet\ \forall e' \in E$ we have $b \mathscr{B}_1 e' \iff b \mathscr{B}_2 e'$,

where we denote $\mathscr{A}_1$ (resp. $\mathscr{B}_1$) the after (before) inhibitor arcs obtained by extending C to C' and $\mathscr{A}_2$ (resp. $\mathscr{B}_2$) the after (before) inhibitor arcs obtained by extending C to C''. However, these additional requests are trivially satisfied because we know that $\forall s,s' \in S.(s,s') \in R_1 \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$. In fact, if $b \mathscr{A}_1 e$, then, by Definition 5.2.2, there is an inhibitor arc from $\rho_1(b)$ to t_1, i.e., $\rho_1(b) \in {}^\circ t_1$. Since $(\rho_1(b),\rho_2(b)) \in R_1$, this implies that $\rho_2(b) \in {}^\circ t_2$ and so $b \mathscr{A}_2 e$. The implication on the other side is symmetrical, and therefore omitted. The argument for relations $\mathscr{B}_1, \mathscr{B}_2$ is the same, and therefore omitted.

To conclude, we have that $C' = C''$. Thus, $(C,\rho_2) \xrightarrow{e} (C',\rho_2')$ with $\rho_2'(e) = t_2$ and $(\rho_1'(b),\rho_2'(b)) \in R_1$ for each $b \in e^\bullet$. Therefore, for all $b' \in B'$ it holds that $(\rho_1'(b'),\rho_2'(b')) \in R_1$, because for all $b' \in B$ this holds by hypothesis and for all $b' \in e^\bullet$ this follows by construction (thanks to the fact that $(t_1^\bullet, t_2^\bullet) \in R_1^{\oplus}$). As a consequence $(\rho_1', C', \rho_2') \in R_2$.

The case where (C,ρ_2) moves first is symmetrical and therefore omitted. Thus, R_2 is a causal-net bisimulation and, since $(\rho_1^0, C^0, \rho_2^0) \in R_2$, we have $m_1 \sim_{cn} m_2$. □

There are at least the following three important technical differences between causal-net bisimilarity and pti-place bisimilarity.

1. A causal-net bisimulation is a very complex relation – composed of cumbersome triples of the form (ρ_1,C,ρ_2) – that must contain infinitely many triples if the net system offers a never-ending behavior. On the contrary, a pti-place bisimulation is always a very simple finite relation over the finite set S of places.

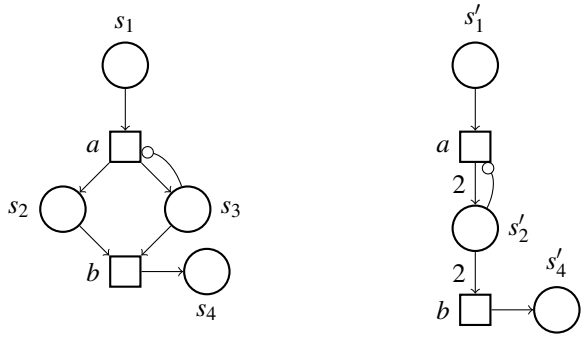

Figure 6.4: Two PTI nets.

2. A causal net bisimulation proving that $m_1 \sim_{cn} m_2$ is a relation specifically designed for showing that m_1 and m_2 generate the same causal nets, step by step. If we want to prove that, e.g., $n \cdot m_1$ and $n \cdot m_2$ are causal-net bisimilar (which may not hold!), we have to construct a new causal-net bisimulation to this aim. Instead, a pti-place bisimulation R relates those places which are considered equivalent under all the possible R-related contexts. Hence, if R justifies that $m_1 \sim_p m_2$ as $(m_1, m_2) \in R^{\oplus}$, then for sure the same R justifies that $n \cdot m_1$ and $n \cdot m_2$ are pti-place bisimilar, as also $(n \cdot m_1, n \cdot m_2) \in R^{\oplus}$.

3. Finally, while pti-place bisimilarity is decidable (see the next section), it is not known whether causal-net bisimilarity is decidable on finite PTI nets.

However, these technical advantages of pti-place bisimilarity over causal-net bisimilarity are balanced by an increased discriminating power of the former over the latter, that, in a few cases, might appear even excessive, as the following intriguing example shows.

Example 11. Consider the net in Figure 6.4. First of all, note that $s_2 \sim_{cn} s'_2$, because both are stuck markings. However, we have that $2 \cdot s_2 \not\sim_{cn} 2 \cdot s'_2$ because $2 \cdot s_2$ is stuck, while $2 \cdot s'_2$ can perform b. This observation is enough to conclude that $s_2 \not\sim_p s'_2$, because a pti-place bisimulation R relates places that are equivalent under any R-related context: if $(s_2, s'_2) \in R$ then $(2 \cdot s_2, 2 \cdot s'_2) \in R^{\oplus}$, but these two markings do not satisfy the pti-place bisimulation game, so R is not a pti-place bisimulation.

Nonetheless, it is interesting to observe that $s_1 \sim_{cn} s'_1$, because they generate the same causal PTI nets, step by step; moreover, even for any $n \geq 1$ we have $n \cdot s_1 \sim_{cn} n \cdot s'_1$. However, $s_1 \not\sim_p s'_1$ because it is not possible to build a pti-place bisimulation R containing the pair (s_1, s'_1). The problem is that it would be necessary to include, into the candidate pti-place relation R, also the pair (s_2, s'_2), which is not a pti-place bisimulation pair, as discussed above. Therefore, no pti-place bisimulation R can relate s_1 and s'_1.

Chapter 7

Pti-place bisimilarity is decidable

In order to prove that $\sim_p$ is decidable, we first need a technical lemma which states that it is decidable to check whether a place relation $R \subseteq S \times S$ is a pti-place bisimulation.

Lemma 7.0.1. *Given a finite PTI net $N = (S,A,T,I)$ and a place relation $R \subseteq S \times S$, it is decidable whether R is a pti-place bisimulation.*

Proof. We want to prove that R is a pti-place bisimulation if and only if the following two finite conditions are satisfied:

1. $\forall t_1$ such that ${}^\bullet t_1[t_1\rangle$, $\forall m$ such that $({}^\bullet t_1, m) \in R^\oplus$, $\exists t_2$ such that ${}^\bullet t_2[t_2\rangle$ and
   (a) ${}^\bullet t_2 = m$,
   (b) $({}^\circ t_1, {}^\circ t_2) \in R^\oplus$, $(t_1^\bullet, t_2^\bullet) \in R^\oplus$, $l(t_1) = l(t_2)$,
   (c) $\forall s,s' \in S.(s,s') \in R \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$.
2. $\forall t_2$ such that ${}^\bullet t_2[t_2\rangle$, $\forall m$ such that $(m, {}^\bullet t_2) \in R^\oplus$, $\exists t_1$ such that ${}^\bullet t_1[t_1\rangle$ and
   (a) ${}^\bullet t_1 = m$,
   (b) $({}^\circ t_1, {}^\circ t_2) \in R^\oplus$, $(t_1^\bullet, t_2^\bullet) \in R^\oplus$, $l(t_1) = l(t_2)$,
   (c) $\forall s,s' \in S.(s,s') \in R \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$.

First we prove the implication from left to right, only for condition 1, as the other is symmetrical. If R is a pti-place bisimulation and $({}^\bullet t_1, m) \in R^\oplus$, then from ${}^\bullet t_1[t_1\rangle t_1^\bullet$ it follows that there exists t_2 such that ${}^\bullet t_2[t_2\rangle t_2^\bullet$ with ${}^\bullet t_2 = m$, $({}^\circ t_1, {}^\circ t_2) \in R^\oplus$, $(t_1^\bullet, t_2^\bullet) \in R^\oplus$, $l(t_1) = l(t_2)$ and, moreover, $\forall s,s' \in S.(s,s') \in R \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$. Therefore, conditions (a), (b) and (c) are trivially satisfied.

Now we prove the implication from right to left, i.e., if conditions 1 and 2 hold for R, then R is a pti-place bisimulation. Suppose $(m_1, m_2) \in R^\oplus$ and $m_1[t_1\rangle m_1'$. Let $q = \{(s_1, s_1'), (s_2, s_2'), \ldots, (s_k, s_k')\}$ be any set of associations that can be used to prove that $(m_1, m_2) \in R^\oplus$. So this means that $m_1 = s_1 \oplus s_2 \oplus \ldots \oplus s_k$, $m_2 = s_1' \oplus s_2' \oplus \ldots \oplus s_k'$ and that $(s_i, s_i') \in R$ for $i = 1, \ldots, k$. If $m_1[t_1\rangle m_1'$, then $m_1' = m_1 \ominus {}^\bullet t_1 \oplus t_1^\bullet$. Consider the set of associations $p = \{(\overline{s}_1, \overline{s}_1'), \ldots, (\overline{s}_h, \overline{s}_h')\} \subseteq q$, with $\{\overline{s}_1, \ldots, \overline{s}_h\} = {}^\bullet t_1$.

Note that $({}^\bullet t_1, \{\overline{s}_1', \ldots, \overline{s}_h'\}) \in R^\oplus$ and that ${}^\bullet t_1[t_1\rangle$. Hence, by condition 1, there exists a transition t_2 such that ${}^\bullet t_2[t_2\rangle$, ${}^\bullet t_2 = \{\overline{s}_1', \ldots, \overline{s}_h'\}$, $({}^\circ t_1, {}^\circ t_2) \in R^\oplus$, $l(t_1) = l(t_2)$, $(t_1^\bullet, t_2^\bullet) \in R^\oplus$, and $\forall s,s' \in S.(s,s') \in R \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$. By hypothesis, ${}^\circ t_1 \cap dom(m_1) = \emptyset$, so since $(m_1, m_2) \in R^\oplus$ and condition (c) holds, we have that ${}^\circ t_2 \cap dom(m_2) = \emptyset$. Therefore, since ${}^\bullet t_2 \subseteq m_2$, also $m_2[t_2\rangle m_2'$ is firable, where $m_2' = m_2 \ominus {}^\bullet t_2 \oplus t_2^\bullet$, and we have that $({}^\bullet t_1, {}^\bullet t_2) \in R^\oplus$, $({}^\circ t_1, {}^\circ t_2) \in R^\oplus$, $(t_1^\bullet, t_2^\bullet) \in R^\oplus$, $l(t_1) = l(t_2)$, $(m_1', m_2') \in R^\oplus$ and, moreover, $\forall s,s' \in S.(s,s') \in R \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$, as required, where $(m_1', m_2') \in R^\oplus$ holds as, from the set q of matching pairs for m_1 and m_2, we have removed those in p and we have added those justifying $(t_1^\bullet, t_2^\bullet) \in R^\oplus$.

If $m_2[t_2\rangle m_2'$, then we have to use an argument symmetric to the above, where condition 2 is used instead. Hence, we have proved that conditions 1 and 2 are enough to prove that R is a pti-place bisimulation.

Finally, the complexity of this procedure is as follows. For condition 1, we have to consider all the net transitions, and for each t_1 we have to consider all the markings m that $({}^\bullet t_1, m) \in R_i^\oplus$, and for each pair (t_1, m) we have to check whether there exists a transition t_2 such that $m = {}^\bullet t_2$, $l(t_1) = l(t_2)$, $({}^\circ t_1, {}^\circ t_2) \in R_i^\oplus$, $(t_1^\bullet, t_2^\bullet) \in R_i^\oplus$ and, moreover, that $\forall s,s' \in S.(s,s') \in R \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$. And the same for condition 2. Therefore, this procedure has worst-case time complexity $O(q \cdot \frac{(n+p-1)!}{(n-1)! \cdot p!} \cdot p^2 \sqrt{p} \cdot q \cdot (p^2 \cdot \sqrt{p} + n^2 \cdot p))$, where $q = |T|$, $n = |S|$, p

is the least number such that $|{}^\bullet t| \leq p$, $|{}^\circ t| \leq p$ and $|t^\bullet| \leq p$ for all $t \in T$. As a matter of fact, the distribution of p tokens over n places is given by the binomial coefficient $\binom{n+p-1}{p} = \frac{(n+p-1)!}{(n-1)!\cdot p!}$; the size of R is at most n^2; checking Subcondition (a) is in $O(p^2\sqrt{p}\cdot q)$; checking Subcontidion (b) is in $O(p^2\cdot\sqrt{p})$; finally, checking Subcondition (c) is in $O(n^2\cdot p)$. □

Theorem 7.0.2. (Pti-place bisimilarity is decidable) *Given a PTI net $N = (S,A,T,I)$, for each pair of markings m_1 and m_2, it is decidable whether $m_1 \sim_p m_2$.*

Proof. If $|m_1| \neq |m_2|$, then $m_1 \nsim_p m_2$ by Proposition 2. Otherwise, we can assume that $|m_1| = k = |m_2|$. Since S is finite, the set of all the place relations over S is finite as well. Let us list all the place relations as follows: $R_1, R_2, \ldots, R_n$. Therefore, for $i = 1,\ldots,n$, by Lemma 7.0.1 we can decide whether the place relation R_i is a pti-place bisimulation and, in such a case, we can check whether $(m_1,m_2) \in R_i^\oplus$ in $O(k^2\sqrt{k})$ time. As soon as we have found a pti-place bisimulation R_i such that $(m_1,m_2) \in R_i^\oplus$, we stop concluding that $m_1 \sim_p m_2$. If none of the R_i is a pti-place bisimulation such that $(m_1,m_2) \in R_i^\oplus$, then we can conclude that $m_1 \nsim_p m_2$. □

***Remark* 5. (Time complexity of the decision procedure)** *First of all, we note that if $|S| = n$, then the set of all the place relations over S has cardinality 2^{n^2}. Moreover, the procedure for checking if a place relation R_i in this set is a pti-place bisimulation has worst-case complexity $O(q\cdot\frac{(n+p-1)!}{(n-1)!\cdot p!}\cdot p^2\sqrt{p}\cdot q\cdot(p^2\cdot\sqrt{p}+n^2\cdot p))$, Finally, if R_i is a place bisimulation, the cost of checking if $(m_1,m_2) \in R_i^\oplus$ is $O(k^2\sqrt{k})$ if the two markings have size k. Summing up, the procedure is exponential in the size of the net n.*

Chapter 8

Conclusions

Petri nets are one one of the most studied and largely used mathematical modeling languages for the description of concurrent and distributed systems. The main advantage of the model is is to describe the global state of a system as composed of a collection of local states, where the execution of a *transition* is a local transformation.

The focus of this work was on two types of Petri nets: the original place/transition (P/T) nets [Pet62] and P/T nets with inhibitor arcs (PTI), first introduced in [AF73]. The latter is an extension of the former, where local state may also disallow execution, leading (somewhat surprisingly) PTI nets to be a Turing-complete model of computation.

In the first part of this work, we have extended Vogler's proof technique in [Vog91], based on ordered markings, that he used to prove decidability of (strong) fully-concurrent bisimilarity for safe nets, to bounded nets by means of indexed ordered markings. The extension is flexible enough to be applicable also to another similar equivalence, namely causal-net bisimilarity. While decidability of fully-concurrent bisimilarity for bounded nets was already proved by Montanari and Pistore [MP97], our result for causal-net bisimilarity is new. However, the approach of [MP97] is not defined directly on Petri nets, rather it exploits an encoding of Petri nets into so-called *causal automata*, a model of computation designed for handling dependencies between transitions by means of names. In addition to this, their encoding works modulo isomorphisms, so that, in order to handle correctly the dependency names, at each step of the construction costly renormalizations are required. On the contrary, our construction is very concrete and works directly on the net. Thus, we conjecture that, even if the worst-case complexity is roughly the same, our algorithm performs generally better. Decidability of fully-concurrent bisimilarity using indexed ordered markings was claimed to be proved also by Valero-Ruiz [Val93], however some significant flaws and inaccuracies in his work (which was never published in English) regarding the individual handling of tokens are enough to invalidate such a claim.

A possible extension of this work regards unbounded P/T nets. While Esparza proved [Esp98] that all behavioral equivalences ranging from interleaving bisimilarity to fully-concurrent bisimilarity are undecidable on unbounded P/T nets, the proof of undecidability by Jančar [Jan95] does not apply to causal-net bisimilarity, so that the problem of its decidability over unbounded P/T nets is open. However, one may conjecture that since infinitely many tokens in a net are mapped to infinitely many maximal conditions in a process, also causal-net bisimilarity might be undecidable on unbounded P/T nets.

As a future work, we plan to extend Vogler's results in [Vog95] about decidability of weak fully-concurrent bisimilarity on safe nets with silent moves, to bounded nets with silent moves. As Vogler's approach in that case relies on ordering of transitions instead of places, it could be possible to generalize our idea to sets of indexed transitions. However, the work presents some non-trivial technical details.

In the second part of this work, we introduced pti-place bisimilarity, a decidable and sensible behavioral equivalence for finite PTI nets. The decidability result is based on the fact that the net is finite, even if the associated reachability graph may not be so. To decide pti-place bisimilarity one has to check a large (but finite!) number of local conditions, which are surprisingly enough sum up the (possibly infinite) behavior of the net. As a matter of fact, those local conditions can be checked on finite elements of the net, namely places and transitions. When the checking is done, the combination of conditions is sufficient to express the net's whole behavior.

Future work will be devoted to see whether the pti-place bisimulation idea can be extended to other, even larger classes of nets, such as *lending* Petri nets, where transitions are allowed to consume tokens from a place even if it does not contain enough tokens (thus enabling negative-valued markings) [BCP15].

To the best of our knowledge, the decidability of a behavioral equivalence for a Turing-complete formalism has been proved only once before. In fact, in [Lan+11] it is proved that (interleaving) bisimilarity is decidable

for a small process calculus, called HOcore, with higher-order communication (but without restriction), that is, nonetheless, Turing-complete.

Turing-completeness of HOcore is proved by providing an encoding of Minsky machines, which are a model based on increment and test for zero of registers. The ability to increment and test for zero are the key also to show Turing-completeness of PTI nets. Moreover, it is well-known that another process calculus, the π-calculus, is Turing-complete and can encode higher-order behavior [MPW92; SW01]; it is also possible to provide a PTI semantics for it [BG09]. These three facts lead to a suggestive question: in a concurrent or distributed setting, are higher-order capabilities and test-for-zero on the same level, or is one more foundational than the other? Petri nets have often been called ”the assembly language of concurrency”: which higher-order process calculus is ”the λ-calculus of concurrency”, and is it possible to encode one in the other?

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