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Analytical Map between EPRL Spin Foam Models in Loop Quantum Gravity

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Don't panic

Abstract

The quantization of the gravitational interaction is a fundamental problem of modern Physics. Loop Quantum Gravity (LQG), and its covariant formulation, Spin Foam theory, are one of the many theoretical frameworks which attempt to build a quantum theory of gravitation.

Spin Foam theory provides a regularized, background-independent, and Lorentz covariant path integral for Quantum Gravity on a fixed triangulation (a discretization of the space-time manifold). Spin Foam models assign transition amplitudes to LQG kinematical states.

The state of the art of the theory is the Engle-Pereira-Rovelli-Livine (EPRL) model, formulated with the Euclidean and the Lorentzian signatures. The two models differ by their gauge group structures, which are respectively $SO(4, \mathbb{R})$ and $SO^+(1, 3)$.

The first is a compact gauge group: it has finite-dimensional unitary irreducible representations, and the integral on the group manifold is simple. The second is non-compact. Therefore, the computations in the Lorentzian EPRL model are notably more complicated than the Euclidean one. The Euclidean model is the preferred choice for physical calculations. Given their similarities it has been so far assumed, as a strong hypothesis, that the results obtained in the simpler Euclidean model also hold for the Lorentzian one.

This work's primary goal is to present the principal characteristics of the models and a set of prescriptions that, once followed, map the structure and, at least in a qualitative way, the results obtained with the Euclidean model into the Lorentzian one.

Chapter 1 provides an overview of the basic ingredients of the discussion, namely General Relativity, BF theories and how LQG provides transition amplitudes between quantum states of spacetime.

Chapters 2 and 3 present respectively a description of the Euclidean and Lorentzian EPRL models, from the representation theory of their relative gauge groups to the construction of the EPRL transition amplitudes.

Chapter 4 portrays the current state of research in EPRL Spin Foam theory, with a qualitative description of the main results achieved in both models.

The main topic of the thesis and my original work is contained in Chapter 5, in which, from a set of prescriptions, the group structure of the Euclidean model is mapped into the Lorentzian one, allowing a comparison between the transition amplitudes.

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Chapter 1

Introduction to Loop Quantum Gravity

1.1 The Quest for Quantum Gravity

The scientific revolution of the XXth century gave birth to two extremely powerful theories to describe the world around us, Quantum Mechanics (QM) and General Relativity (GR). With these tools we have been able to investigate Nature from its microscopic properties up to the greatest structures of our universe, with an ever growing number of empirical support and without a single experimental evidence disproving their fundamental principles found so far. Despite their descriptive and predictive power, the two frameworks are irreconcilable one with each other, failing when trying to describe phenomena in which both paradigms are needed.
Quantum Mechanics, later developed in the modern Quantum Field Theory (QFT), led to the formulation of the Standard Model (SM) of particle physics, which unifies the strong, weak and electromagnetic interaction under one coherent description.
General Relativity, on the other hand, not only provides an incredibly precise description of the gravitational interaction, but more importantly gives us a revolutionary new way to formulate the fundamental concepts of Space and Time, questioning the very definitions of inertial observers and coordinate frames.
The incompatibility between the two frameworks arises from the fact that they essentially describe two different worlds. In the world of QFT, the fundamental building blocks of Nature are quantum fields, which interact on a flat spacetime described by Minkowski metric, according to Einstein's theory of Special Relativity. Conversely, GR describes a world in which matter and radiation do not have quantum properties, but spacetime is a dynamical entity. Its metric is not necessarily flat, as it is related to the distribution of matter and energy, which in turn evolves according to the metric of spacetime itself. In the words of J.A. Wheeler: *"Spacetime tells matter how to move; matter tells spacetime how to curve"*.
The world we live in, however, shows both quantum features and the effects of a dynamical spacetime, meaning that neither of the two theories offer a complete description of our reality, yet providing a valid approximation of it in their respective contexts. For these reasons, a theory of Quantum Gravity is needed. Once formulated, it should be able to describe, coherently with the principles of Quantum

Mechanics, the properties of a dynamical spacetime predicted by General Relativity. This goal is still far to be achieved, as none of the models proposed in the last century have ever been supported by any experimental evidence. However, progress has been made, leading to the formulation of a vast number of competing theories, some perfected, other discarded, over the course of the years.
Loop Quantum Gravity (LQG) is one of the many theoretical frameworks that attempt to achieve this goal. It is based upon the idea of describing spacetime as a fundamental quantum field, *with* which, and not *on* which, the other quantum fields interact and evolve. In its most recent formulation, called Spin Foam theory, spacetime regions are discretized through triangulation, and quantum states are assigned to space-like boundaries of said regions. Spin Foam theory assigns transition amplitudes to such states.
The state of the art model is the EPRL model [16], which is formulated with both Euclidean and Lorentzian signature. The two models are built with the same principles and differ only for their gauge group structure: $SO(4,\mathbb{R})$ for the Euclidean signature and $SO^+(1,3)$ for the Lorentzian one. Both models provide an expression for the transition amplitude between LQG states.

1.2 General Relativity and Holst Action

The most common expression of General Relativity, also called the *tensor field formulation*, describes spacetime as a locally flat 4-dimensional Riemannian manifold. The components $g_{\mu\nu}$ of its metric tensor are determined by the stress-energy tensor $T_{\mu\nu}$ of the local distribution of matter and energy through Einstein's equations, expressed in natural units ($8\pi G = c = 1$) as:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} \tag{1.1}$$

Where $R_{\mu\nu}$ and R, the Ricci tensor and Ricci scalar, are related to the components of the Riemann tensor $R^{\mu}_{\nu\rho\sigma}$, which can be expressed as functions of the metric once a Levi-Civita connection has been established on the manifold. In vacuum, the right hand side vanishes, and the resulting equations can be obtained through Hamilton's principle from the *Einstein-Hilbert action*:

$$S[g] = \int d^4x\sqrt{-g}R \tag{1.2}$$

Where g is the determinant of the metric.
The same results of the tensor field formulation can be achieved introducing the *tetrad formalism*, in which the tangent space at any point x of the manifold is mapped to Minkowski space. The tetrads, denoted by $e^I_\mu(x)$, provide a map from the manifold to the local reference frame. The Minkowski metric of the local reference frame is mapped to the manifold metric by the tetrad field:

$$g_{\mu\nu}(x) = e^I_\mu(x)e^J_\nu(x)\eta_{IJ} \tag{1.3}$$

This description is more redundant than the metric one, as now the metric is invariant under a new local Lorentz gauge transformation:

$$e^I_\mu(x) \to \Lambda^I_J e^J_\mu(x) \tag{1.4}$$

The inverse tetrads determine a set of four orthonormal vector fields at each point x, related as follows to the Minkowski metric:

$$e^\mu_I(x)e^\nu_J(x)g_{\mu\nu} = \eta_{IJ} \tag{1.5}$$

Given a connection ω^{IJ}_μ defined in the Lie algebra of the Lorentz group, and introducing the formalism of differential forms:

$$e^I = e^I_\mu dx^\mu \tag{1.6}$$
$$\omega^{IJ} = \omega^{IJ}_\mu dx^\mu \tag{1.7}$$

The torsion and the curvature of the connection are defined through first and second Cartan equation by:

$$T^I = de^I + \omega^I_J \wedge e^J \tag{1.8}$$
$$F^I_J = d\omega^I_J + \omega^I_K \wedge \omega^K_J \tag{1.9}$$

The Levi-Civita connection of General Relativity is recovered as the unique solution of the equation:

$$de^I + \omega^I_J \wedge e^J = 0 \tag{1.10}$$

That implicitly gives the relation between a connection with a vanishing torsion and the tetrad field. Under such assumptions, the Riemann tensor is related to the curvature through:

$$R^\mu_{\nu\rho\sigma} = e^\mu_I e^J_\nu F^I_{J\rho\sigma} \tag{1.11}$$

Which in the end allows to write the Einstein-Hilbert action as a functional of the tetrad field:

$$S[e] = \frac{1}{2}\int e^I \wedge e^J \wedge F^{KL}\epsilon_{IJKL} \tag{1.12}$$

Omitting the contracted indices and defining $\star F_{IJ} \equiv \frac{1}{2}F^{KL}\epsilon_{IJKL}$, it can be reformulated as:

$$S[e] = \int e \wedge e \wedge \star F \tag{1.13}$$

Once (1.10) has been imposed, F depends only on the tetrad field, that is then the only variable upon which the action depends, from which one recovers, through Hamilton's Principle, Einstein's equations.
It is possible to formulate General Relativity in a more general formalism, called *first-order formulation*, in which the connection is considered a variable as well as the tetrad field, resulting in the *Palatini action*:

$$\mathcal{S}[e,\omega] = \int e \wedge e \wedge \star F[\omega] \tag{1.14}$$

In this formulation, (1.10) is recovered performing the variation of the action in respect to ω, while Einstein's equations arise again from the variation in respect to the tetrad field. The Palatini action describes then a theory that *on-shell* reduces to General Relativity.

It is possible to show that there is only another possible combination of the tetrads and the curvature that shares the same symmetries of the Palatini action, which is given by:

$$\int e \wedge e \wedge F \tag{1.15}$$

This, called the *Holst term*, can be added to the Palatini action, obtaining:

$$\mathcal{S}[e, \omega] = \int e \wedge e \wedge \star F[\omega] + \frac{1}{\gamma} \int e \wedge e \wedge F[\omega] \tag{1.16}$$

The coupling constant is chosen as the inverse of the *Barbero-Immirzi* parameter, defined in LQG as a free parameter of the theory, and denoted by γ. This term does not affect the equations of motion, since it can be shown to be vanishing on-shell. The two terms can be collected into:

$$\mathcal{S}[e, \omega] = \int (\star e \wedge e + \frac{1}{\gamma} e \wedge e) \wedge F[\omega] \tag{1.17}$$

Which is called the *Holst Action*. This formulation is equivalent to General Relativity in the classical framework, but has been relevant in the building of 4-dimensional Quantum Gravity.

1.3 BF Theories

The bridge between classical and quantum General Relativity in covariant LQG is provided by the features of a family of topological theories called *BF theories*. To define a BF theory it is sufficient to consider a Lie group G with a Lie algebra $\mathfrak{g}$, and a d-dimensional manifold $\mathcal{M}$. Denoting with ω a connection and with B a general $(d-2)$-form, both valued in the $\mathfrak{g}$ algebra, the action of a BF theory is defined as:

$$S[B, \omega] = \int_{\mathcal{M}} B \wedge F[\omega] \tag{1.18}$$

Where $F[\omega]$ is the curvature of the connection, defined by the second Cartan equation (1.9), and adopting the differential form formalism as well as the same conventions of (1.14) on contracted indices. The equations of motion of this action are given by:

$$F = 0 \tag{1.19}$$

$$d_\omega B = 0 \tag{1.20}$$

The first equation characterizes the solutions with flat connections, while the seconds imposes to the external covariant derivative of B to be vanishing on-shell.

The action is invariant under gauge transformations and, given a $(d-3)$-form η, has an additional symmetry given by:

$$\omega \rightarrow \omega \tag{1.21}$$

$$B \rightarrow B + d_\omega \eta \tag{1.22}$$

This is sufficient to state that all solutions of the theory are the same up to gauge transformations, meaning that it describes a system with no local excitations. However, systems with local degrees of freedom can be described through the BF formalism, adding some constraint to the action. This is the case of General Relativity.

1.4 General Relativity as a Constrained BF Theory

Given a generic BF action on a 4-dimensional manifold, we can add the term:

$$\int_{\mathcal{M}} \lambda_{IJKL} B^{IJ} \wedge B^{KL} = \int_{\mathcal{M}} \lambda B \wedge B \tag{1.23}$$

Where λ_{IJKL} is a completely antisymmetric Lagrange multiplier. We obtain:

$$\mathcal{S}[e, \omega, \lambda] = \int_{\mathcal{M}} [B \wedge F[\omega] + \lambda B \wedge B] \tag{1.24}$$

The variation of the action in respect to the Lagrange multiplier λ constrains the field B to satisfy:

$$B_{[IJ} \wedge B_{KL]} = 0 \tag{1.25}$$

This equation is solved by the simple bivectors:

$$B = \pm \star (e \wedge e) \tag{1.26}$$

$$B = \pm e \wedge e \tag{1.27}$$

Therefore (1.25) is usually referred to as the *simplicity constraint*. The most general solution of the simplicity constraint is a linear combination of (1.26) and (1.27):

$$B = \star(e \wedge e) + \frac{1}{\gamma}(e \wedge e) \tag{1.28}$$

Where $\frac{1}{\gamma}$ is a free parameter. The action reduces to the Holst action for General Relativity:

$$\mathcal{S}[e, \omega] = \int (\star e \wedge e + \frac{1}{\gamma} e \wedge e) \wedge F[\omega] \tag{1.29}$$

1.5 The Linear Simplicity Constraint

The simplicity constraint allows to reformulate GR as a constrained BF theory. Given a space-like hypersurface Σ of $\mathcal{M}$, we can fix the internal gauge to be the *time gauge*. With this choice the normal to Σ is:

$$n_I = (1, 0, 0, 0) \tag{1.30}$$

This gauge fixing breaks the Lorentz symmetry of the theory, reducing it to an $SO(3)$ rotational symmetry, and it is equivalent to choose a preferred Lorentzian frame. On Σ, $n_I e^I = 0$, and the components of B^{IJ} can be separated in their boost and rotational parts, defined as:

$$K^I \equiv n_J B^{IJ} \tag{1.31}$$

$$L^I \equiv n_J (\star B)^{IJ} \tag{1.32}$$

Taking into account (1.28) and the relation between the normal vector to Σ and the tetrad field, the two parts can be computed explicitly, obtaining:

$$K^I = n_J B^{IJ} = n_J (\star e \wedge e)^{IJ} \tag{1.33}$$

$$L^I = n_J (\star B)^{IJ} = \frac{1}{\gamma} n_J (\star e \wedge e)^{IJ} \tag{1.34}$$

Hence:

$$K^I = \gamma L^I \tag{1.35}$$

Moreover, the antisymmetric properties of B^{IJ} are sufficient to deduce that the components of K^I and L^I normal to Σ, namely K^0 and L^0 are vanishing, thus K and L can be considered as three-dimensional vectors in the space (x_1, x_2, x_3), and the previous relation reduces to:

$$\vec{K} = \gamma \vec{L} \tag{1.36}$$

Where

$$K^i = B^{i0} \tag{1.37}$$

$$L^i = \frac{1}{2}\epsilon^i_{jk} B^{jk} \tag{1.38}$$

The proportionality relation between the rotational and boost parts of B^{IJ} just obtained is called the *Linear Simplicity Constraint* (LSC), and its implementation is fundamental in order to establish a bridge between LQG models and General Relativity. The reason of expressing GR as a constrained BF theory is given by the fact that the latter has a precise discretization procedure, summarized in the following section, which in the end can be interpreted as the transition amplitude between quantized boundary states.

1.6 Discretized BF Partition Function

Given a BF theory, it is possible to define its partition function as a path integral:

$$Z \equiv \int D[B]D[\omega]e^{iS[B,\omega]} \tag{1.39}$$

Which, once integrated over the measure $D[B]$, results in:

$$Z = \int D[\omega]\delta(F(\omega)) \tag{1.40}$$

So that Z can be interpreted as the volume, in the phase space of all possible connections, of the region characterized by flat connections, i.e. the solutions to the classical equations of motion of the BF action.
As any functional integration, the one expressed in (1.39) is a formal expression which needs to be regularized through a discretization in order to be computed. This is achieved by defining a triangulation Δ of the manifold $\mathcal{M}$ and assigning a dual graph $\Delta^\star$ to it. The triangulation of a d-dimensional smooth manifold is achieved by dividing it into discrete geometrical units of dimension d, called d-cells (triangles are the most intuitive choice in the 2-dimensional case, hence the name).
The dual graph $\Delta^\star$ is constructed by assigning to each d-cell a point-like object, called *vertex*. The vertices are connected one to each other through *edges*, one-dimensional objects dual to the $(d-1)$-cells of the triangulation. Finally, a certain number of edges, depending on the geometrical dimension of the triangulation, bounds the *faces* of $\Delta^\star$, two-dimensional objects dual to the $(d-2)$-cells of the triangulation.

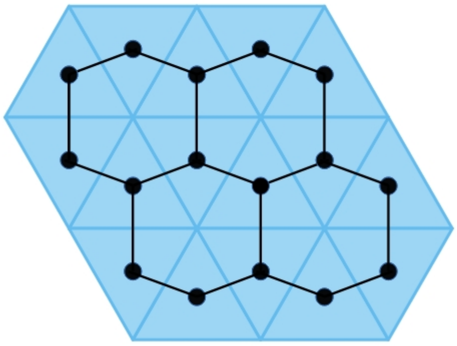

Figure 1.1. A 2-dimensional triangulation (blue) with its corresponding dual graph (black).

This process can be easily visualized for a 2-dimensional triangulation, as shown in Figure 1.1. Here we provide a schematic view for triangulations of dimension 2,3, and 4, which is the relevant case in the discretization of the General Relativity BF Amplitude[1]:

$\Delta^\star$	2-Dimensional	3-Dimensional	4-Dimensional
Vertices v	Triangles	Tetrahedra	4-Simplices
Edges e	Segments	Triangles	Tetrahedra
Faces f	Points	Segments	Triangles

Once a triangulation has been fixed, the regularization of the partition function proceeds by discretizing B and ω as follows:

$$B \to B_f \equiv \int_f B \tag{1.41}$$

$$\omega \to g_e \equiv \mathbb{P}e^{-\int_e \omega} \tag{1.42}$$

That is, assigning to each face of $\Delta^\star$ the object B_f, defined as the surface integral of B over f, and to each edge the *holonomy* (path-ordered integrated exponentiation) along e of the connection ω, denoted by g_e. Since ω is valued in the $\mathfrak{g}$ algebra of the gauge group G, by construction the objects g_e are elements of G. The measures in the functional integral are taken to be the invariant measures over the algebra and the groups:

$$D[B] \to \prod_f dB_f \tag{1.43}$$

$$D[\omega] \to \prod_e dg_e \tag{1.44}$$

[1] 4-Simplices are the 4-dimensional homologue of tetrahedra. They are defined as the convex region of $\mathbb{R}^4$ delimited by five points.

The elements g_e can be collected to define at each face the objects[2]:

$$U_f \equiv \prod_{e \in f} g_e \tag{1.45}$$

Finally, the partition function (1.39) can be rewritten as:

$$Z(\Delta) = \int \prod_f dB_f \prod_e dg_e e^{iTr[B_f U_f]} \tag{1.46}$$

The integration over dB_f gives the Dirac delta of the group centered at U_f for each face of $\Delta^\star$, leaving the integrations over the group measures dg_e:

$$Z(\Delta) = \int_G \prod_e dg_e \prod_f \delta(U_f) \tag{1.47}$$

Since $g_e \in G$, by construction $U_f \in G$ as well. The Dirac delta of U_f can then be expanded using Peter-Weyl theorem, obtaining:

$$\delta(U_f) = \sum_\rho d_\rho Tr[\rho(U_f)] \tag{1.48}$$

Where ρ denotes a generic *unitary irreducible representation* (irrep, for short) of G, d_ρ the dimension of the vector space assigned to it[3] and $\rho(U_f)$ is the representation matrix of the group element U_f. Then, the partition function depends on all possible configurations of irreps assigned to the faces of $\Delta^\star$, and it is given by:

$$Z(\Delta) = \sum_{\{\rho_f\}} \prod_f d_{\rho_f} \int_G \prod_e dg_e Tr[\rho(g_{e_1} ... g_{e_n})] \tag{1.49}$$

As each edge bounds exactly d faces, the same group element g_e appears in d traces. This allows to manipulate the matrices inside the trace obtaining a final expression for the regularized BF partition function:

$$Z(\Delta) = \sum_{\{\rho_f\}} \prod_f d_{\rho_f} \prod_e \int_G dg_e \prod_{f \ni e} \rho_f(g_e) \tag{1.50}$$

1.7 From Discrete to Quantum

The discreteness of the BF partition function just derived comes from the regularization of a functional integral, it does not bring insights on the quantum properties of spacetime. In LQG such properties arise from a different context.
Given a spacetime region bounded by space-like hypersurfaces, once time gauge is chosen, it is possible to perform a triangulation of the 3-dimensional boundary, and assign a dual graph to it. This results in a discretization of the physical space

[2]In the mathematicians' community, the holonomy of a connection is defined only on closed paths. Hence in the LQG literature the term holonomy is used sometimes to denote the elements g_e and other times the newly defined objects U_f

[3]In the case of non-compact groups which admit only infinite-dimensional irreps, d_ρ denotes the Plancherel measure of $L^2(G)$ with the Haar measure of G.

geometry, called *Spin Network*, characterized by *nodes* dual to tetrahedra connected by *links* dual to the triangles shared by adjacent tetrahedra.
In this framework it is possible to define an Hilbert space of $SU(2)$ invariant states, in which geometrical properties as Length, Area and Volume, are generalized into hermitian operators acting on such states. In particular, an Area operator is assigned to each link of the network, while a Volume operator is defined at each node. From the study of the eigenvalues of such operators, a discrete spectrum emerges, with non-vanishing minimum eigenvalues. In particular, the minimum eigenvalue of the Area operator, called the *area gap*, provides the physical scale of LQG:

$$A_g = \gamma \frac{\sqrt{3}\hbar G}{2c^3} \approx 10^{-70} m^2 \tag{1.51}$$

We stress at this point that the discreteness of the spectrum of the geometric operators is not an artefact of the regularization based on a triangulation. However it is an effect of the compactness of the $SO(3) \approx SU(2)$ group that represents the residual Lorentz gauge invariance restricted to the space-like hypersurface.
Spin network states diagonalize simultaneously the Area and Volume operators. However, they do not diagonalize the metric of 3-dimensional physical space. The presence of a discrete spectrum of Area and Volume, with a non vanishing ground-state, combined with the uncertainty in the determination of the geometry, reveals the quantum nature of geometry, in which spin network states are interpreted as *quanta of space*.
The study of spin network states is usually referred to as the *kinematic* of quantum space, as it describes quantized geometry at a given time. A *dynamic* of space-time is then needed to describe how such states evolve in time, and how a probability amplitude can be assigned to any transition between two kinematic states. This is achieved by Spin Foam theory, in which the dual graph assigned to a given triangulation of 4-dimensional spacetime is called, in fact, a *Spin Foam*, bounded by spin network states, and where the transition amplitude between such states is obtained as a path integral, which in the end coincides with the discretized partition function of a 4-dimensional BF theory. As early mentioned, the state of the art of Spin Foam theory is given by the two EPRL models, which are going to be discussed in the next chapters.

Chapter 2

The Euclidean EPRL Model

In this chapter we explore the Euclidean EPRL model. In the first section we present a summary of its group theory features, namely the properties of $SO(4,\mathbb{R})$ and its representations. The second section focuses on the imposition of the quantum Linear Simplicity Constraint, followed by the computation of the EPRL amplitude.

2.1 Representation Theory of $SO(4,\mathbb{R})$

The structure of the Euclidean EPRL model is $SO(4,\mathbb{R})$, that is the group of rotations in a 4-dimensional Euclidean space, i.e. the group of linear transformations under which the quadratic form $x_\mu x^\mu$ is invariant. The Lie algebra of such group is generated by a set of six independent differential operators $D_{\alpha\beta}$, acting as generators of the rotations on the generic plane (x_α, x_β):

$$D_{\alpha\beta} = -i\left(x_\alpha \partial_\beta - x_\beta \partial_\alpha\right) \tag{2.1}$$

For which hold the following commutation relations:

$$[D_{\alpha\beta}, D_{\gamma\delta}] = i\delta_{\alpha\gamma}D_{\beta\delta} + i\delta_{\beta\delta}D_{\alpha\gamma} + i\delta_{\alpha\delta}D_{\gamma\beta} + i\delta_{\beta\gamma}D_{\delta\alpha} \tag{2.2}$$

These operators can be grouped as follows:

$$\vec{L} = (D_{23}, D_{31}, D_{12}) \tag{2.3}$$

$$\vec{A} = (D_{14}, D_{24}, D_{34}) \tag{2.4}$$

These two vectors act respectively as generators of rotations in the subspace (x_1, x_2, x_3) and rotations on planes (x_i, x_4), with $i = 1, 2, 3$. The latter can be called, although improperly, the generator of "boosts" along the i-th axis. They form the closed algebra given by the following commutation relations:

$$[L_i, L_j] = i\epsilon_{ijk}L_k \tag{2.5}$$

$$[L_i, A_j] = i\epsilon_{ijk}A_k \tag{2.6}$$

$$[A_i, A_j] = i\epsilon_{ijk}L_k \tag{2.7}$$

To find the Casimirs of this algebra it is sufficient to look at all the independent combinations of generators which are invariant under coordinates transformations. There are only two of such objects, given by:

$$C_1 \equiv \frac{1}{2} D_{\alpha\beta} D^{\alpha\beta} = L^2 + A^2 \tag{2.8}$$

$$C_2 \equiv \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} D^{\alpha\beta} D^{\gamma\delta} = \vec{L} \cdot \vec{A} \tag{2.9}$$

Being a compact group, $SO(4, \mathbb{R})$ admits unitary irreducible representations with finite dimension. These irreps can be constructed by exploiting the known fact that such group is locally isomorphic to $SU(2) \times SU(2)$, usually referred to as $Spin(4)$, which Lie algebra is given by: $spin(4) = su(2) \oplus su(2)$.
As a consequence, the generators of $so(4)$ can be combined in such a way they generate two $su(2)$ commuting algebras. This is achieved by defining the operators:

$$\vec{J_1} \equiv \frac{1}{2} \left(\vec{L} + \vec{A} \right) \tag{2.10}$$

$$\vec{J_2} \equiv \frac{1}{2} \left(\vec{L} - \vec{A} \right) \tag{2.11}$$

For which holds:

$$\left[J_1^i, J_1^j \right] = i\epsilon_{ijk} J_1^k \tag{2.12}$$

$$\left[J_2^i, J_2^j \right] = i\epsilon_{ijk} J_2^k \tag{2.13}$$

$$\left[J_1^i, J_2^j \right] = 0 \tag{2.14}$$

The irreps of $Spin(4)$ are thereby given by the tensor product of two independent irreps of $SU(2)$. We recall that J^2 is the Casimir operator of $su(2)$, and it is possible to define the basis $|j, m\rangle$ of simultaneous eigenstates of it and J_3, in which their action is given by:

$$J^2 |j, m\rangle = j(j+1) |j, m\rangle \tag{2.15}$$

$$J_3 |j, m\rangle = m |j, m\rangle \tag{2.16}$$

The Hilbert space of any $Spin(4)$ irrep is then denoted by $\mathcal{H}_{j_1, j_2} = \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$, with a basis that diagonalizes J_{1_3} and J_{2_3} given by the states:

$$|j_1, m_1, j_2, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle \tag{2.17}$$

It is easily shown that the elements of this basis are eigenstates of the $so(4)$ Casimirs. In fact:

$$C_1 = L^2 + A^2 = 2 \left(J_1^2 + J_2^2 \right) \tag{2.18}$$

$$C_2 = \vec{L} \cdot \vec{A} = J_1^2 - J_2^2 \tag{2.19}$$

From which it follows:

$$C_1 |j_1, m_1, j_2, m_2\rangle = 2j_1(j_1+1) + 2j_2(j_2+1) |j_1, m_1, j_2, m_2\rangle \tag{2.20}$$

$$C_2 |j_1, m_1, j_2, m_2\rangle = j_1(j_1+1) - j_2(j_2+1) |j_1, m_1, j_2, m_2\rangle \tag{2.21}$$

As Biedenharn suggested in [12], these eigenvalues can be expressed in terms of two new parameters, $p \equiv j_1 + j_2 + 1$[1] and $q \equiv j_1 - j_2$. In this parametrization the Casimirs' eigenvalues are given by:

$$L^2 + A^2 = p^2 + q^2 - 1 \tag{2.22}$$

$$\vec{L} \cdot \vec{A} = pq \tag{2.23}$$

This choice of parameters is key to display the parallelism between the representation theories of $SO(4,\mathbb{R})$ and $SL(2,\mathbb{C})$, as we show later in Chapter 5.
Equations (2.22) and (2.23) suggest that any irrep of $SO(4,\mathbb{R})$ can be labelled by a couple of half-integer numbers (p,q), with $p > 0$ and $p \geq q$. We denote such irreps with $\mathcal{H}^{(p,q)}$.
From the theory of angular momentum we also know that the tensor product between two $SU(2)$ irreps can be decomposed into a direct sum of the irreps labelled by the eigenvalues of the square modulus of the total angular momentum operator $\vec{L} \equiv \vec{J}_1 + \vec{J}_2$. That is:

$$\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \mathcal{H}_j \tag{2.24}$$

With $Dim(\mathcal{H}_{j_1,j_2}) = (2j_1+1)(2j_2+1)$. By construction, we can then decompose $\mathcal{H}^{(p,q)}$ as follows:

$$\mathcal{H}^{(p,q)} = \bigoplus_{j=|q|}^{p-1} \mathcal{H}_j \tag{2.25}$$

With j labelling the eigenvalues of the square modulus of $\vec{L}$, and a natural basis given by the eigenstates of C_1, C_2, L^2 and L_3, denoted by $|p,q,j,m\rangle$. The dimension of such representations expressed in term of p and q is equal to $p^2 - q^2$. Other relevant operators in this decomposition are $L_\pm \equiv L_1 \pm iL_2$ and $A_\pm \equiv A_1 \pm iA_2$. Explicit action of such operators on the basis elements is given in detail in [25].[2]
To construct the representation matrix in this space, we first parametrize a generic $SO(4,\mathbb{R})$ element using the Cartan decomposition:

$$g = e^{i\vec{\alpha}\cdot\vec{L}} e^{iA_3 t} e^{i\vec{\beta}\cdot\vec{L}} \equiv u e^{itA_3} v^\dagger \tag{2.26}$$

With:

$$\vec{\alpha}, \vec{\beta} \in S(3) \tag{2.27}$$

$$0 \leq t \leq \pi \tag{2.28}$$

Since the operators L_i act as generators of rotations in the 3-dimensional subspace (x_1, x_2, x_3), the first and the last exponential factors in (2.26) can be seen as elements of $SO(3)$, which we know being locally isomorphic to $SU(2)$.
Any element of $SO(4,\mathbb{R})$ can be then parametrized with a triad $(u,v,t) \in SU(2) \times SU(2) \times [0,\pi]$, with u and v defined as $u \equiv e^{i\alpha\cdot\vec{L}}$ and $v \equiv e^{-i\beta\cdot\vec{L}}$.

[1] We use a slightly different definition from the original paper, in which $p \equiv j_1 + j_2$.
[2] Although expressed in terms of j_1, j_2, j and m, and expressing the "boost" generators with K_i.

This parametrization is redundant since it provides seven parameters and the Lie algebra generating the group elements is six-dimensional. This ambiguity is totally harmless. Two parametrizations of the same group element differs by a $U(1)$ action: the freedom consists then on re-defining u and v by multiplying them both on the right by an arbitrary $e^{-i\phi L_3}$. We choose this parametrization because it induces a simple decomposition of the representation matrix. This is also true for $SL(2,\mathbb{C})$, as we show later in 3.1.
To compute the representation matrices of $SO(4,\mathbb{R})$ it is useful to consider both the relation of the basis elements $|p,q,j,m\rangle$ with the $SU(2)$ irreps and the expression for the resolution of the identity in this basis, given by:

$$I = \sum_{j=|q|}^{p-1} \sum_{m=-j}^{j} |p,q,j,m\rangle \langle p,q,j,m| \tag{2.29}$$

Minding these considerations, we obtain:

$$\begin{aligned} D^{pq}_{jmln}(g) &\equiv \langle p,q,j,m| u e^{itA_3} v^\dagger |p,q,l,n\rangle \\ &= \sum_{j',m',l',n'} \langle p,q,j,m| u |p,q,j',m'\rangle \langle p,q,j',m'| e^{itA_3} |p,q,l',n'\rangle \langle p,q,l',n'| v^\dagger |p,q,l,n\rangle \\ &= \sum_{j',m',l',n'} \delta_{jj'}\delta_{ll'} D^j_{mm'}(u) \langle p,q,j',m'| e^{itA_3} |p,q,l',n'\rangle D^l_{n'n}(v^\dagger) \\ &= \sum_{m',n'} D^j_{mm'}(u) \langle p,q,j,m'| e^{itA_3} |p,q,l,n'\rangle D^l_{n'n}(v^\dagger) \\ &\equiv \sum_{m',n'} D^j_{mm'}(u) d^{pq}_{jm'ln'}(t) D^l_{n'n}(v^\dagger) \end{aligned} \tag{2.30}$$

Where $D^j_{mm'}(u)$ and $D^l_{n'n}(v^\dagger)$ are the well-known Wigner matrices of $SU(2)$, while the term defined as $d^{pq}_{jm'ln'}(t)$ is the element of a matrix which properties are better understood when it is expressed in the basis $|j_1,m_1,j_2,m_2\rangle$. This is achieved by expanding the basis elements as follows:

$$|p,q,j,m\rangle = \sum_{m_1,m_2} |j_1,m_1,j_2,m_2\rangle \langle j_1,m_1,j_2,m_2|p,q,j,m\rangle \tag{2.31}$$

For the sake of compactness, we now express the states $|p,q,j,m\rangle$ simply as $|j,m\rangle$, since the relevant information we care to stress in the computation is them being diagonal in respect to L^2 and L_3. We obtain:

$$
\begin{aligned}
d^{pq}_{jm'ln'}(t) &= \sum_{m_i,m'_i} \langle j,m'|j_1,m_1,j_2,m_2\rangle \langle j_1,m_1,j_2,m_2| e^{it\left(J_{1_3}-J_{2_3}\right)} |j_1,m'_1,j_2,m'_2\rangle \langle j_1,m'_1,j_2,m'_2|l,n'\rangle \\
&= \sum_{m_i,m'_i} \langle j,m'|j_1,m_1,j_2,m_2\rangle \langle j_1,m_1| e^{itJ_{1_3}} |j_1,m'_1\rangle \langle j_2,m_2| e^{-itJ_{2_3}} |j_2,m'_2\rangle \langle j_1,m'_1,j_2,m'_2|l,n'\rangle \\
&= \sum_{m_i,m'_i} \delta_{m_1m'_1}\delta_{m_2m'_2} \langle j,m'|j_1,m_1,j_2,m_2\rangle e^{i(m_1-m_2)t} \langle j_1,m'_1,j_2,m'_2|l,n'\rangle \\
&= \sum_{m_1,m_2} \langle j,m'|j_1,m_1,j_2,m_2\rangle e^{i(m_1-m_2)t} \langle j_1,m_1,j_2,m_2|l,n'\rangle \\
&= \sum_{m_1,m_2} C^{jm'}_{j_1m_1j_2m_2} e^{i(m_1-m_2)t} C^{ln'}_{j_1m_1j_2m_2}
\end{aligned}
\tag{2.32}
$$

Where $C^{jm'}_{j_1m_1j_2m_2}$ and $C^{ln'}_{j_1m_1j_2m_2}$ are $SU(2)$ Clebsh-Gordan coefficients. From their properties follows that (2.32) is non-zero only if both m' and n' are equal to m_1+m_2. We can make this condition explicit by defining the $SO(4,\mathbb{R})$ *reduced Wigner matrix* or, for short, *d-matrix*.

$$
d^{pq}_{jlm'}(t) \equiv \sum_{m_1,m_2} C^{jm'}_{j_1m_1j_2m_2} e^{i(m_1-m_2)t} C^{lm'}_{j_1m_1j_2m_2} \tag{2.33}
$$

Such that:

$$
d^{pq}_{jm'ln'}(t) = \delta_{m'n'} d^{pq}_{jlm'}(t) \tag{2.34}
$$

From which follows immediately:

$$
D^{pq}_{jmln}(g) = \sum_{m'} D^{j}_{mm'}(u) d^{pq}_{jlm'}(t) D^{l}_{m'n}(v^\dagger) \tag{2.35}
$$

2.2 Construction of the Euclidean EPRL Model

As we have seen in 1.5, to classically describe GR as a BF theory, the 2-form B appearing in the Holst action must satisfy the Linear Simplicity Constraint on every space-like hypersurface, that is equation (1.36). Since the EPRL models aim to qualify as quantized versions of GR interpreted as a discretized BF theory, such constraint has to be imposed in them too. The main difference from the classical theory is that $\vec{L}$ and $\vec{A}$ are not vector fields but operators on an Hilbert space. Since they do not commute, it is not possible to find an Hilbert subspace on which $\vec{A}-\gamma\vec{L}=0$ strongly, then the LSC has to be imposed weakly in a different way. This is achieved by interpreting $\vec{A}=\gamma\vec{L}$ as a relation in terms of the action of the Casimirs on the eigenstates of the theory, that is:

$$
L^2 + A^2 |p,q,j,m\rangle = (\gamma^2+1)L^2 |p,q,j,m\rangle \tag{2.36}
$$

$$
\vec{L}\cdot\vec{A} |p,q,j,m\rangle = \gamma L^2 |p,q,j,m\rangle \tag{2.37}
$$

Leading to:

$$
p^2+q^2-1 = (\gamma^2+1)j(j+1) \tag{2.38}
$$

$$
pq = \gamma j(j+1) \tag{2.39}
$$

For the purposes of the theory, it is sufficient to satisfy such conditions in the limit of large quantum numbers, allowing us to rewrite them as:

$$p^2 + q^2 = (\gamma^2 + 1)j^2 \tag{2.40}$$
$$pq = \gamma j^2 \tag{2.41}$$

The solutions must be coherent with the conditions $p \geq 0$ and $p > q$. The choice is also dictated by the fact that the value of the Barbero-Immirzi parameter have to be set greater than 1 in order to have results which are relevant in the LQG framework, as explained in [25]. Following such prescriptions one obtains:

$$p = \gamma j \tag{2.42}$$
$$q = j \tag{2.43}$$

The second equation fixes the convention $j_1 > j_2$, which from now one we consider implicitly. The relevant irreps of $SO(4,\mathbb{R})$ in the Euclidean EPRL model are then $\mathcal{H}^{(\gamma j, j)}$, with $j \in \frac{\mathbb{N}}{2}$ and $Dim\left(\mathcal{H}^{(\gamma j,j)}\right) = p^2 - q^2 = (\gamma^2 - 1)j^2$.
This constraint on the states of the theory is equivalent to impose weakly the LSC, that is requiring that the expectation value of $\vec{A} - \gamma\vec{L}$ over any of such states vanishes at the classical limit $j \to \infty$ and (reintroducing physical dimensions) $\hbar \to 0$. The one-to-one correspondence between $SU(2)$ representations an the $SO(4,\mathbb{R})$ ones relevant for the theory is made explicit by the map:

$$Y_\gamma : \mathcal{H}_j \to \mathcal{H}^{(\gamma j, j)} \tag{2.44}$$

Applied independently on each edge of the dual graph $\Delta^\star$. The complete transition amplitude in the Euclidean model is then obtained by implementing the Y_γ map accordingly into the BF partition function, as expressed in equation (1.49), obtaining:

$$Z^E_{eprl}(\Delta) = \sum_{\{j_f\}} \prod_f d_{j_f} \int_{SO(4,\mathbb{R})} \prod_e dg_{ev} Tr\left[D^{\gamma j_f, j_f}(g_{11}Y_\gamma^\dagger Y_\gamma g_{12}^{-1} g_{22} Y_\gamma^\dagger Y_\gamma ... Y_\gamma^\dagger Y_\gamma g_{n1}^{-1})\right] \tag{2.45}$$

The group element inside the Wigner matrix of $SO(4,\mathbb{R})$ is constructed following the connectivity of the graph. In detail, to each face of the dual graph is assigned an $SO(4,\mathbb{R})$ irrep. To each half-edge is assigned a group element g_{ev} with the two indices denoting to which edge and to which of the two vertices sharing such edge it is referring. The Y_γ map is applied at the boundary of every vertex, namely between the two group elements of different vertices sharing the same edge. The presence of the inverse element is due to the orientation of the graph.
Another notable difference between the EPRL Euclidean amplitude and the one computed in the generic $SO(4,\mathbb{R})$ BF theory is the value assigned to d_{j_f}. In the latter this factor is given by the dimension of the representation assigned to each face, meaning that with the LSC it would read $(\gamma^2 - 1)j_f^2$. In the EPRL model instead, d_{j_f} is set to be equal to the dimension of the $SU(2)$ representation $\mathcal{H}_{j_f}$ mapped onto $\mathcal{H}^{(\gamma j_f, j_f)}$, hence, $d_{j_f} = 2j_f + 1$. A brief summary of the debate over this topic and the motivation that led to such choice is given in [10].
The equation for the EPRL amplitude can be manipulated to be more easily computed and understood. We know that the representation matrix of two or more group

elements can be decomposed in a product of two or more representation matrices. The Y_γ map projects the sum (2.25) over the minimum spin subspace $j = q$. Hence, the matrix elements inside the trace can be written as:

$$D^{\gamma j,j}_{jmjm}(g_{11}Y^\dagger_\gamma Y_\gamma g^{-1}_{12} g_{22} Y^\dagger_\gamma Y_\gamma ... Y^\dagger_\gamma Y_\gamma g^{-1}_{n1}) = \\ = D^{\gamma j,j}_{jmjm'}(g^{-1}_{n1} g_{11}) D^{\gamma j,j}_{jm'jm''}(g^{-1}_{12} g_{22}) ... D^{\gamma j,j}_{jm^{(n)}jm}(g^{-1}_{n-1,n} g_{nn}) \quad (2.46)$$

With an implicit sum over the primed magnetic indices. Each matrix element on the right hand side is then decomposed without further application of the Y_γ map, resulting in:

$$D^{\gamma j,j}_{jmjm'}(g^{-1}_{n1} g_{11}) = \sum_{l=j}^{\gamma j-1} \sum_{n=-l}^{l} D^{\gamma j,j}_{jmln}(g^{-1}_{n1}) D^{\gamma j,j}_{lnjm'}(g_{11}) \quad (2.47)$$

And so on. In a 4-dimensional triangulation, each edge of the dual graphs bounds exactly 4 faces. Hence, once decomposed as shown, the representation matrices of each group element appears in exactly 4 traces, and the EPRL amplitude can be factorized as follows:

$$Z^E_{eprl}(\Delta) = \sum_{\{j_f\}} \left(\prod_f (2j_f + 1) \right) \sum_{l_f=j_f}^{\gamma j_f - 1} \prod_v \prod_{e \in v} \int_{SO(4,\mathbb{R})} dg_{ev} \prod_{a=1}^{4} D^{\gamma j_{f_a}, j_{f_a}}_{j_{f_a} m_{f_a} l_{f_a} n_{f_a}}(g_{ev}) \quad (2.48)$$

With an implicit sum over the magnetic indices m and n. Once written in this form, it is clear that the amplitude can be obtained summing the contributions of two distinct factors over all possible configurations of spins j_f. One depends on the already discussed values d_{j_f} which define a *Face Amplitude*, denoted with $A_f(j_f) = 2j_f + 1$, whereas the other is built by integrals over the group measure of the product of four representation matrices. Through equations (2.26) and (2.35) it is possible to write such integrals as follows:

$$\int_{SO(4,\mathbb{R})} dg \prod_a D^{\gamma j_a, j_a}_{j_a m_a l_a n_a}(g) = \sum_{m'_a} \int dudvd\mu(t) \prod_a D^{j_a}_{m_a m'_a}(u) d^{\gamma j_a, j_a}_{j_a l_a m'_a}(t) D^{l_a}_{m'_a n_a}(v^\dagger) \quad (2.49)$$

Where du and dv are the Haar measures of $SU(2)$ and $\mu(t) = \frac{2}{\pi} \sin^2 t dt$, defined as in [19], in order to be $\int_0^\pi d\mu(t) = 1$. Each integral in the sum over the magnetic indices m'_a can be factorized, resulting in:

$$\int dudvd\mu(t) \prod_a D^{j_a}_{m_a m'_a}(u) d^{\gamma j_a, j_a}_{j_a l_a m'_a}(t) D^{l_a}_{m'_a n_a}(v^\dagger) = \\ = \int_{SU(2)} du \prod_a D^{j_a}_{m_a m'_a}(u) \int_{SU(2)} dv \prod_a D^{l_a}_{m'_a n_a}(v^\dagger) \int_0^\pi d\mu(t) \prod_a d^{\gamma j_a, j_a}_{j_a l_a m'_a}(t) \quad (2.50)$$

The integrals in du and dv can be explicitly computed recalling the definitions and results of $SU(2)$ recoupling theory summarized in Appendix A. After computing both $SU(2)$ integrals appearing in it, (2.49) can be written as:

$$\sum_{m'_a} \sum_{i,k} d_i d_k \begin{pmatrix} j_a \\ m_a \end{pmatrix}^{(i)} \begin{pmatrix} j_a \\ m'_a \end{pmatrix}^{(i)} \begin{pmatrix} l_a \\ n_a \end{pmatrix}^{(k)} \begin{pmatrix} l_a \\ m'_a \end{pmatrix}^{(k)} \int_0^\pi d\mu(t) \prod_a d^{\gamma j_a, j_a}_{j_a l_a m'_a}(t) \quad (2.51)$$

Which can be expressed in a more compact form by defining a new object, called *Euclidean Booster function*:

$$B_4^E(j_a, l_a, i, k) \equiv \sum_{m'_a} \begin{pmatrix} j_a \\ m'_a \end{pmatrix}^{(i)} \begin{pmatrix} l_a \\ m'_a \end{pmatrix}^{(k)} \int_0^\pi d\mu(t) \prod_a d^{\gamma j_a, j_a}_{j_a l_a m'_a}(t) \tag{2.52}$$

So that:

$$\int_{SO(4,\mathbb{R})} dg \prod_a D^{\gamma j_a, j_a}_{j_a m_a l_a n_a}(g) = \sum_{i,k} d_i d_k \begin{pmatrix} j_a \\ m_a \end{pmatrix}^{(i)} \begin{pmatrix} l_a \\ n_a \end{pmatrix}^{(k)} B_4^E(j_a, l_a, i, k) \tag{2.53}$$

This allows us to write explicitly the remaining factors of the amplitude as:

$$\sum_{m_{f_a}, n_{f_a}} \prod_v \prod_{e \in v} \sum_{i_e, k_e} d_{i_e} d_{k_e} \begin{pmatrix} j_{f_a} \\ m_{f_a} \end{pmatrix}^{(i_e)} \begin{pmatrix} l_{f_a} \\ n_{f_a} \end{pmatrix}^{(k_e)} B_4^E(j_{f_a}, l_{f_a}, i_e, k_e) \tag{2.54}$$

Now, notice that for each half-edge of any vertex, the spins j_a and the magnetic indices m_a are shared with the correspondent half-edge of the neighbouring vertex. For each edge of the dual graph it is possible to define an *Edge Amplitude* $A_e(i_e)$, factoring out a contribution given by:

$$\sum_{m_{f_a}} d_{i_e} d_{i_{e'}} \begin{pmatrix} j_{f_a} \\ m_{f_a} \end{pmatrix}^{(i_e)} \begin{pmatrix} j_{f_a} \\ m_{f_a} \end{pmatrix}^{(i_{e'})} = \delta_{i_e, i_{e'}} d_{i_e} \equiv \delta_{i_e, i_{e'}} A_e(i_e) \tag{2.55}$$

Where the result is obtained applying (A.24). Then, each vertex contributes separately to the amplitude as follows:

$$\prod_{e \in v} \sum_{n_{f_a}} \sum_{k_e} d_{k_e} \begin{pmatrix} l_{f_a} \\ n_{f_a} \end{pmatrix}^{(k_e)} B_4^E(j_{f_a}, l_{f_a}, i_e, k_e) \tag{2.56}$$

It is possible to identify in the contraction between the $4jm$-symbols appearing in this expression one of the many existing $SU(2)$ invariant objects, that is the Wigner $15j$-symbol defined in (A.32). This is sufficient to define one last contribution to $Z^E_{eprl}(\Delta)$, the *Vertex Amplitude* $A_v(j_f, i_e)$, given by:

$$A_v(j_f, i_e) = \sum_{l_f, k_e} \left(\prod_e d_{k_e} B_4^E(j_f, l_f, i_e, k_e) \right) \{15j\}(l_f, k_e) \tag{2.57}$$

Finally, the Euclidean EPRL Amplitude is obtained:

$$Z^E_{eprl}(\Delta) = \sum_{\{j_f\}} \sum_{i_e} \prod_f A_f(j_f) \prod_e A_e(i_e) \prod_v A_v(j_f, i_e) \tag{2.58}$$

Chapter 3

The Lorentzian EPRL Model

As a parallel to the previous chapter, here we discuss the structure and properties of the algebraic structure of the Lorentzian EPRL model, followed by the explicit computation of its EPRL amplitude.

3.1 Representation Theory of $SL(2,\mathbb{C})$

The structure of the Lorentzian EPRL model is the *Lorentz spin group* $SL(2,\mathbb{C})$, that is the universal cover of $SO^+(1,3)$, the group of Lorentz transformations in a 4-dimensional spacetime. Its Lie algebra is generated by the components of the anti-symmetric tensor:

$$J_{\alpha\beta} = i\left(x_\alpha \partial_\beta - x_\beta \partial_\alpha\right) \tag{3.1}$$

For which the commutation relations are given by:

$$[J_{\alpha\beta}, J_{\gamma\delta}] = -i\delta_{\alpha\gamma}J_{\beta\delta} - i\delta_{\beta\delta}J_{\alpha\gamma} + i\delta_{\alpha\delta}J_{\gamma\beta} + i\delta_{\beta\gamma}J_{\delta\alpha} \tag{3.2}$$

In the signature $(-,+,+,+)$ the tensor $J_{\alpha\beta}$ can be decomposed in two vector operators which components are defined as:

$$L^i \equiv -\frac{1}{2}\epsilon^i_{jk}J^{jk} \tag{3.3}$$

$$K^i \equiv J^{0i} \tag{3.4}$$

Acting respectively as the generators of spatial rotations and boosts. From (3.2) follows that they form a closed algebra generated by the commutation relations:

$$[L_i, L_j] = i\epsilon_{ijk}L_k \tag{3.5}$$

$$[L_i, K_j] = i\epsilon_{ijk}K_k \tag{3.6}$$

$$[K_i, K_j] = -i\epsilon_{ijk}L_k \tag{3.7}$$

This algebra can be manipulated similarly of what we have shown in 2.1, combining its generators in such a way they generate two commuting $su(2)$ algebras. This procedure is also called the "complexification" of $sl(2,\mathbb{C})$, and it suggests the existence of a link between the representations of $SL(2,\mathbb{C})$ and $SU(2)$. Although this link exists and has a fundamental role in the construction of the EPRL model, we choose

to keep as relevant operators $\vec{L}$ and $\vec{K}$.
The Casimirs of this algebra are given by the Lorentz-invariant combinations of the generators $J_{\alpha\beta}$. Since they are defined up to a multiplication constant, we decide to use the following convention:

$$C_1 \equiv -\frac{1}{2} J_{\alpha\beta} J^{\alpha\beta} = K^2 - L^2 \tag{3.8}$$

$$C_2 \equiv -\frac{1}{8} \epsilon^{\gamma\delta}_{\alpha\beta} J_{\gamma\delta} J^{\alpha\beta} = \vec{L} \cdot \vec{K} \tag{3.9}$$

The unitary irreducible representations of $SL(2, \mathbb{C})$ are parametrized by couples of numbers $(\chi, \eta) \in \mathbb{R} \times \mathbb{Z}$. The various realizations of such irreps are well summarized in [23]. Most importantly, there exists a realization over $SU(2)$ functions that allows to decompose any $SL(2, \mathbb{C})$ irrep into a direct sum of an infinite number of $SU(2)$ ones. In such realization, the $SL(2, \mathbb{C})$ irreps are labelled by two parameters $(\rho, k) \in \mathbb{R} \times \mathbb{Z}/2$ defined as $\rho = \frac{\chi}{2}$ and $k = \frac{\eta}{2}$, so that the generic Hilbert space can be written as:

$$\mathcal{H}^{(\rho,k)} = \bigoplus_{j=|k|}^{\infty} \mathcal{H}_j \tag{3.10}$$

It is possible to show that for any couple $\rho, k > 0$, the representations labelled by the couples (ρ, k) and $(-\rho, -k)$ are unitarily equivalent. For this reason it is sufficient for our purposes to focus only on representations parametrized by $(\rho, k) \in \mathbb{R}_+ \times \mathbb{N}/2$, in which:

$$\mathcal{H}^{(\rho,k)} = \bigoplus_{j=k}^{\infty} \mathcal{H}_j \tag{3.11}$$

The most natural basis in this decomposition is given by the eigenstates of the operators L^2 and L_3, denoted by $|\rho, k, j, m\rangle$ and such that:

$$L^2 |\rho, k, j, m\rangle = j(j+1) |\rho, k, j, m\rangle \tag{3.12}$$

$$L_3 |\rho, k, j, m\rangle = m |\rho, k, j, m\rangle \tag{3.13}$$

The action of the operators $L_\pm \equiv L_1 \pm iL_2$, $K_\pm \equiv K_1 \pm iK_2$, and K_3 is given explicitly in [23]. Starting from those expressions it is possible to compute the eigenvalues of the Casimirs acting on the elements of $|\rho, k, j, m\rangle$:

$$(K^2 - L^2) |\rho, k, j, m\rangle = (\rho^2 - k^2 + 1) |\rho, k, j, m\rangle \tag{3.14}$$

$$\vec{L} \cdot \vec{K} |\rho, k, j, m\rangle = \rho k |\rho, k, j, m\rangle \tag{3.15}$$

The $SL(2, \mathbb{C})$ representation matrices, or $SL(2, \mathbb{C})$ *Wigner matrices*, are given, for any $h \in SL(2, \mathbb{C})$, by:

$$D^{(\rho,k)}_{jmln}(h) \equiv \langle \rho, k, j, m | h | \rho, k, l, n \rangle \tag{3.16}$$

In order to write explicitly these functions of h, it is useful to consider the Cartan decomposition of $SL(2, \mathbb{C})$, which allows us to parametrize any group element with a non-unique triad $(u, v, r) \in SU(2) \times SU(2) \times \mathbb{R}_+$ as follows:

$$h = u e^{-irK_3} v^\dagger \tag{3.17}$$

Moreover, since $SU(2) \subset SL(2,\mathbb{C})$, $SL(2,\mathbb{C})$ Wigner matrices can be defined also for elements of $SU(2)$. Not surprisingly, such matrices are trivially related to the original $SU(2)$ ones, as it can be shown that:

$$D^{(\rho,k)}_{jmln}(u) = \delta_{jl} D^{j}_{mn}(u) \tag{3.18}$$

For any $u \in SU(2)$. This property can be exploited, together with the resolution of the identity:

$$I = \sum_{j=k}^{\infty} \sum_{m=-j}^{j} |\rho,k,j,m\rangle \langle \rho,k,j,m| \tag{3.19}$$

To manipulate (3.16) and obtain:

$$D^{(\rho,k)}_{jmln}(h) = \sum_{m'} D^{j}_{mm'}(u) d^{(\rho,k)}_{jlm'}(r) D^{l}_{m'n}(v^{\dagger}) \tag{3.20}$$

Where m' spans in the interval $[-min(j,l), min(j,l)]$, and defining the *reduced Wigner matrix*, or *d-matrix* of $SL(2,\mathbb{C})$ as:

$$d^{(\rho,k)}_{jlm}(r) \equiv D^{(\rho,k)}_{jmlm}(e^{-irK_3}) \tag{3.21}$$

This matrix has been already studied and it is possible to find various different expressions for it in the literature. The most recent choice on how to express it in topics related to LQG is given in [30] by:

$$\begin{aligned} d^{(\rho,k)}_{jlm}(r) = (-1)^{\frac{j-l}{2}} & \frac{\Gamma(j+i\rho+1)}{\Gamma(l+i\rho+1)} \left| \frac{\Gamma(l+i\rho+1)}{\Gamma(j+i\rho+1)} \right| \frac{\sqrt{(2j+1)(2l+1)}}{(j+l+1)!} \\ & \sqrt{(j+k)!(j-k)!(j+m)!(j-m)!(l+k)!(l-k)!(l+m)!(l-m)!} e^{(i\rho-k-m-1)r} \\ & \sum_{s,t} (-1)^{s+t} e^{-2tr} \frac{(k+s+m+t)!(j+l-k-m-s-t)!}{t!s!(j-k-s)!(j-m-s)!(k+m+s)!(l-k-t)!(l-m-t)!(k+m+t)!} \\ & {}_2F_1 \left[\{l-i\rho+1, k+m+s+t+1\}, \{j+l+2\}; 1-e^{-2r} \right] \end{aligned} \tag{3.22}$$

With the ranges of summation dictated by the existence conditions of the factorials. We focus deeply on the properties of this matrix elements in 5.2. The details about the mathematical objects involved in its discussion are summarized in Appendix B.

3.2 Construction of the Lorentzian EPRL Model

The representation theory of $SL(2,\mathbb{C})$ is not sufficient to build a BF model of Quantum Gravity since it does not automatically satisfy the Linear Simplicity Constraint for large quantum numbers. As in the Euclidean one, the Lorentzian model is built by imposing such constraints on each face of the dual graph. The procedure is rather similar to the former model, as it consists in looking for the representations for which (1.36) can be imposed weakly as a relation between the Casimir operators, which translates into:

$$K^2 - L^2 |\rho,k,j,m\rangle = (\gamma^2 - 1) L^2 |\rho,k,j,m\rangle \tag{3.23}$$

$$\vec{L} \cdot \vec{K} |\rho,k,j,m\rangle = \gamma L^2 |\rho,k,j,m\rangle \tag{3.24}$$

This can be interpreted as a set of conditions for the eigenvalues, namely:

$$\rho^2 - k^2 + 1 = (\gamma^2 - 1)j(j+1) \tag{3.25}$$
$$\rho k = \gamma j(j+1) \tag{3.26}$$

Once again, it is sufficient to solve such equations in the limit of large quantum numbers, allowing us to rewrite them as:

$$\rho^2 - k^2 = j^2(\gamma^2 - 1) \tag{3.27}$$
$$\rho k = \gamma j^2 \tag{3.28}$$

Easily solved by:

$$\rho = \gamma j \tag{3.29}$$
$$k = j \tag{3.30}$$

The irreps of $SL(2,\mathbb{C})$ relevant in the Lorentzian model are then $\mathcal{H}^{(\gamma j,j)}$. Hence, there is a one-to-one correspondence between $SU(2)$ irreps and the $SL(2,\mathbb{C})$ ones used in the theory. As in the Euclidean model, this correspondence can be implemented by a map, defined as follows:

$$Y_\gamma : \mathcal{H}_j \rightarrow \mathcal{H}^{(\gamma j,j)} \tag{3.31}$$

Applying this map independently on each edge of the dual graph, the construction of the Lorentzian EPRL transition amplitude is straightforward and is given by:

$$Z^L_{eprl}(\Delta) = \sum_{\{j_f\}} \prod_f d_{j_f} \int_{SL(2,\mathbb{C})} \prod_e dh_{ev} Tr \left[D^{(\gamma j_f, j_f)}(h_{11} Y^\dagger_\gamma Y_\gamma h^{-1}_{12} h_{22} Y^\dagger_\gamma Y_\gamma ... Y^\dagger_\gamma Y_\gamma h^{-1}_{n1}) \right] \tag{3.32}$$

Such expression is obtained from the $SL(2,\mathbb{C})$ classical BF amplitude minding the same considerations given in 2.2 on how to write correctly the group elements inside the Wigner matrices. The following can be seen as a parallel to what has been shown in that section, with a focus on the main differences that arise due to the different properties of the gauge groups involved. First, the matrix elements inside the trace are decomposed as in equation (2.46), with the Y_γ maps projecting each sum over the minimum spin subspaces and an implicit sum over the primed magnetic indices. Then, without further application of the Y_γ map, one obtains terms of the form:

$$D^{(\gamma j,j)}_{jmjm'}(h^{-1}_{n1} h_{11}) = \sum_{l \geq j} \sum_{n=-l}^{l} D^{(\gamma j,j)}_{jmln}(h^{-1}_{n1}) D^{(\gamma j,j)}_{lnjm'}(h_{11}) \tag{3.33}$$

Being obtained from a 4-dimensional triangulation, in this dual graph each edge bounds exactly 4 faces as in the Euclidean case, allowing to write the following expression for the EPRL amplitude:

$$Z^L_{eprl}(\Delta) = \sum_{\{j_f\}} \left(\prod_f (2j_f + 1) \right) \sum_{l_f \geq j_f} \prod_v \prod_{e \in v} \int_{SL(2,\mathbb{C})} dh_{ev} \prod_{a=1}^{4} D^{(\gamma j_{f_a}, j_{f_a})}_{j_{f_a} m_{f_a} l_{f_a} n_{f_a}}(h_{ev}) \tag{3.34}$$

With an implicit sum over the magnetic indices m and n. As expected, the amplitude can be then interpreted as the sum over all possible configurations of spins attached to the faces of the dual graph of the product between two distinct factors, depending respectively on the Face Amplitudes $A_f(j_f) = (2j_f + 1)$ and on integrals over the group measure which we want to compute explicitly.
There are two main differences between the integrals appearing in the computation of the Euclidean and Lorentzian EPRL amplitudes. The first and more obvious is that the latter integration is performed over a different measure and involves matrix elements that are defined in a different way. Another subtler difference is that, as a direct consequence of the non-compactness of $SL(2,\mathbb{C})$, they give a divergent contribution. As shown in [17], using the properties of the Haar measure, among the five integrals associated to the edges of each vertex, one is redundant and gives a contribution proportional to the volume of $SL(2,\mathbb{C})$ which is infinite due to the already mentioned non-compactness of the group. Then, in order to have a non-divergent amplitude, it is sufficient to drop one integral for each vertex. The choice is arbitrary since the result is independent from it. In the following, the expression $\prod_{e\in v}$ has then to be understood as the product over all the edges attached to the vertex v but one. Using $SL(2,\mathbb{C})$ Cartan decomposition (3.17) each integral can be written as:

$$\int_{SL(2,\mathbb{C})} dh \prod_a D^{(\gamma j_a, j_a)}_{j_a m_a l_a n_a}(h) = \sum_{m'_a} \int dudvd\mu(r) \prod_a D^{j_a}_{m_a m'_a}(u) d^{(\gamma j_a, j_a)}_{j_a l_a m'_a}(r) D^{l_a}_{m'_a n_a}(v^\dagger) \tag{3.35}$$

Where du and dv have the same meaning of the Euclidean amplitude, and the measure over r is given by $\mu(r) = \frac{2}{\pi}\sinh^2(r)dr$. This measure is not normalizable since the integral of $\sinh^2 r$ diverges as $r \to \infty$, so it can be defined up to an arbitrary constant we choose to put equal to $\frac{2}{\pi}$. The integrals can then be factorized as:

$$\begin{aligned}\int dudvd\mu(r) \prod_a D^{j_a}_{m_a m'_a}(u) d^{(\gamma j_a, j_a)}_{j_a l_a m'_a}(r) D^{l_a}_{m'_a n_a}(v^\dagger) = \\ = \int_{SU(2)} du \prod_a D^{j_a}_{m_a m'_a}(u) \int_{SU(2)} dv D^{l_a}_{m'_a n_a}(v^\dagger) \int_0^\infty d\mu(r) \prod_a d^{(\gamma j_a, j_a)}_{j_a l_a m'_a}(t)\end{aligned} \tag{3.36}$$

The integrals over $SU(2)$ are computed as in the Euclidean model, obtaining, for equation (3.35):

$$\sum_{m'_a}\sum_{i,k} d_i d_k \begin{pmatrix} j_a \\ m_a \end{pmatrix}^{(i)} \begin{pmatrix} j_a \\ m'_a \end{pmatrix}^{(i)} \begin{pmatrix} l_a \\ n_a \end{pmatrix}^{(k)} \begin{pmatrix} l_a \\ m'_a \end{pmatrix}^{(k)} \int_0^\infty d\mu(r) \prod_a d^{(\gamma j_a, j_a)}_{j_a l_a m'_a}(r) \tag{3.37}$$

It is possible to define, similarly to the Euclidean case, a *Lorentzian Booster function*:

$$B^L_4(j_a, l_a, i, k) \equiv \sum_{m'_a} \begin{pmatrix} j_a \\ m'_a \end{pmatrix}^{(i)} \begin{pmatrix} l_a \\ m'_a \end{pmatrix}^{(k)} \int_0^\infty d\mu(r) \prod_a d^{(\gamma j_a, j_a)}_{j_a l_a m'_a}(r) \tag{3.38}$$

So that:

$$\int_{SL(2,\mathbb{C})} dh \prod_a D^{(\gamma j_a, j_a)}_{j_a m_a l_a n_a}(h) = \sum_{i,k} d_i d_k \begin{pmatrix} j_a \\ m_a \end{pmatrix}^{(i)} \begin{pmatrix} l_a \\ n_a \end{pmatrix}^{(k)} B^L_4(j_a, l_a, i, k) \tag{3.39}$$

This allows us to define, likewise to 2.2, the Lorentzian Edge and Vertex Amplitudes:

$$A_e(i_e) = 2i_e + 1 \tag{3.40}$$

$$A_v(j_f, i_e) = \sum_{l_f, k_e} \left(\prod_e d_{k_e} B_4^L(j_f, l_f, i_e, k_e) \right) \{15j\}(l_f, k_e, i') \tag{3.41}$$

Recalling the prescription of excluding one arbitrary edge from the product, and with i' labelling the intertwiner attached to that edge. In the end the Lorentzian EPRL Amplitude is given by:

$$Z^L_{eprl}(\Delta) = \sum_{\{j_f\}} \sum_{i_e} \prod_f A_f(j_f) \prod_e A_e(i_e) \prod_v A_v(j_f, i_e) \tag{3.42}$$

Chapter 4

Main Results of the EPRL Models

The EPRL models provide the basic tools of a quantum theory, that is a description of the Hilbert space in which the states are defined, and a formal procedure to compute transition amplitudes between them. Since they have been formalized, such tools have been used by the community to test the models for theoretical consistency and find practical uses for it to describe physical phenomena and make predictions. Here we present a non exhaustive overview of the main results obtained in the EPRL framework.

4.1 Classical Limit of the Vertex Amplitude

As we mentioned in 1.7, spin network boundary states are characterized by the eigenvalues of the Area and Volume operators assigned respectively to each link and node of the network. In general the operators describing geometric quantities, e.g. dihedral angles, do not commute with each other. It is impossible to find a spin network state that is an eigenvalue of all geometric operators and represents a classical geometry in this strict sense.
As the spin network states provide a basis of an Hilbert space, it is possible to build, through superposition, states for which the relative uncertainty of all geometrical operators is minimal. These *coherent states* describe a quantum geometry peaked on a classical geometry in euclidean space. Coherent states play a central role in the computation of the classical limit of the theory, that is the limit for large quantum numbers we mentioned in 2.2 and 3.2.
In [4] the classical limit of the vertex amplitude is computed with a coherent boundary state peaked on a 4-simplex. In the limit of large spins, such amplitude contains the exponential of the Regge action, which is defined as a classical discretization of the Einstein-Hilbert action of General Relativity. This result has been achieved both in the Euclidean and Lorentzian signature and can be summarized as follows: The spin foam vertex amplitudes describe the properties of a theory that is both discrete and quantized. In the limit of large spins, it reduces to a classical theory described by the Regge action, which is still discrete. General Relativity is then regained taking the continuum limit of this theory, which is performed by refining the triangulation

indefinitely.
This is a fundamental result, however not conclusive, as it has not yet been generalized in the case of an arbitrary number of vertices. As the research on this topic is currently ongoing, we mention two approaches proposed in the community, namely the *Effective Spin Foam Models* [1][2] and the *Micro-Local Analysis* of Spin Foam partition functions [20], which has been developed only in the Euclidean framework.

4.2 Continuum Limit

As we already stated, Spin Foam theory describes systems that are both quantum and discrete. To restore a classical theory with an infinite number of degrees of freedom is then necessary to perform both the classical and the continuum limit separately. In the previous section, we summarized one possible approach, that is, taking first the limit of large quantum numbers to obtain a discrete classical theory, performing the continuum limit on the resulting model.
Another possible approach is to perform first the continuum limit, obtaining a quantum theory with an infinite number of degrees of freedom, then study the classical limit for large quantum numbers. The introduction of a triangulation to regularize the path integral breaks the diffeomorphism invariance, which is a fundamental feature of General Relativity. Moreover, the results obtained in Spin Foam depend both on the choice of triangulation and a set of arbitrary prescriptions, such as the definition of the Face Amplitude and the method through which the LSC is imposed.
The general idea behind the continuum limit of Spin Foam amplitudes is to define a renormalization group flow for the vertex amplitudes, refining triangulations. The main goal is to study the existence of fixed points. At a fixed point, we expect that the diffeomorphism invariance is restored and that we should achieve independence from the various ambiguities in the quantization procedure. An overview of the current state of research on this topic can be found in [31].

4.3 Numerical Computation

Although well defined, the EPRL amplitudes face the problem of explicit computation even for relatively simple configurations. This is in part due to the intricate combinatorial structure dwelling in its very definition of sum over all possible configurations of spins. Moreover, the Lorentzian model, being built over a non-compact group, presents an infinite sum over representation indices. This makes arduous, if not impossible, to obtain results through analytical computation without making strong approximations, as for example the ones leading to the large spins classical limit.
For general cases, results have to be computed numerically. Even for numerical computations however, the task is far from being easy. For instance, the already mentioned infinite sum in the Lorentzian vertex amplitude can be performed only by truncating it with an arbitrary cut-off. The choice of this cut-off is not dictated by any theoretical assumption, yet it is strongly limited by computing power. As an example of the ongoing state of research in this topic, we cite `sl2cfoam`, a C-coded

library for the evaluation of the Lorentzian EPRL vertex amplitude, which current state of development is presented in [14].

4.4 n-point Correlation Function

One of the first approaches ever made to Quantum Gravity is the formulation of GR as a Quantum Field theory, in which the metric $g_{\mu\nu}$ acts as a field operator, with on-shell conditions given by Einstein's equations, which excitations over vacuum state can be interpreted as *gravitons*, boson particles responsible for the mediation of the gravitational force. However, this approach fails once one realizes that this leads to the definition of a non-renormalizable theory, with non-removable divergences arising at the two-loop level and above.
As non-renormalizable theories can act as effective theories at low perturbative orders, a first-order 2-point correlation function, or *graviton propagator*, has been defined and computed, and it is considered a candidate for consistency checks of other Quantum Gravity theories. For this reason, the computation of an n-point correlation function for Spin Foam LQG has been object of research. The main result of such research has been the computation of a 2-point correlation function with coherent boundary conditions, which has been shown to coincide to the 2-point graviton propagator. However, this result has been achieved only in the Euclidean signature. More details about this result and the current state of research in this topic can be found in [9],[29] and [7].

4.5 Cosmology

A first approach to cosmology in the Spin Foam framework has been presented in [11], and refined in [8] with the addition of the cosmological constant. In these papers, *Spin Foam Cosmology* is formalized as a theory able to compute transition amplitudes between boundary states describing a 3-sphere with expanding radius. Transition amplitudes are computed in the Euclidean signature, and the results obtained so far suggest that, choosing coherent boundary states peaked on homogeneous and isotropic geometries, and in the approximation of large volumes, the transition amplitude tends, at the classical limit, to the evolutionary dynamics predicted by the Friedmann-Robertson-Walker metric.

Chapter 5

Analytical Map between Models

In this chapter we discuss the main topic of the thesis, that is the possibility to map the structure and, at least in a qualitative way, the results, of the Euclidean EPRL model into the Lorentzian one through an analytic continuation of its parameters. This mapping involves the generators of the algebra, the eigenvalues of the Casimirs, and the group elements, inducing a map between both the representation matrices and the structure of the EPRL amplitudes of the two models.

5.1 Mapping of the Algebras

We start by considering the generators of $so(4)$, $\vec{L}$ and $\vec{A}$, and performing the following transformation:

$$\vec{L} \rightarrow \vec{L} \tag{5.1}$$

$$\vec{A} \rightarrow -i\vec{A} \tag{5.2}$$

We can now use equations (2.5) to (2.7) to compute the commutation relations between this new set of operators, obtaining:

$$[L_i, L_j] = i\epsilon_{ijk}L_k \tag{5.3}$$

$$[L_i, iA_j] = i\epsilon_{ijk}iA_k \tag{5.4}$$

$$[iA_i, iA_j] = -i\epsilon_{ijk}L_k \tag{5.5}$$

Defining $\vec{K} \equiv -i\vec{A}$ we recover:

$$[L_i, L_j] = i\epsilon_{ijk}L_k \tag{5.6}$$

$$[L_i, K_j] = i\epsilon_{ijk}K_k \tag{5.7}$$

$$[K_i, K_j] = -i\epsilon_{ijk}L_k \tag{5.8}$$

That are precisely the commutation relations between the generators of $sl(2,\mathbb{C})$. We can then interpret (5.1) and (5.2) as a map from $so(4)$ to $sl(2,\mathbb{C})$. This map induces a correspondence between the eigenvalues of the Casimir operators of the two algebras. Given two representations $\mathcal{H}^{(p,q)}$ and $\mathcal{H}^{(\rho,k)}$ we see that:

$$K^2 - L^2 = \rho^2 - k^2 + 1 \tag{5.9}$$

$$K^2 - L^2 = -A^2 - L^2 = -p^2 - q^2 + 1 \tag{5.10}$$

And:

$$\vec{K} \cdot \vec{L} = \rho k \tag{5.11}$$

$$\vec{K} \cdot \vec{L} = -i\vec{A} \cdot \vec{L} = -ipq \tag{5.12}$$

The two resulting equations:

$$\rho^2 - k^2 + 1 = -p^2 - q^2 + 1 \tag{5.13}$$

$$\rho k = -ipq \tag{5.14}$$

Have four independent solutions, namely: $(p, q) = (\pm k, \pm i\rho) \vee (\pm i\rho, \pm k)$. This means that (5.1) and (5.2) can also be seen as a map from irreps of $SO(4, \mathbb{R})$ to either of the two unitarily equivalent irreps of $SL(2, \mathbb{C})$ parametrized by the couples (ρ, k), $(-\rho, -k)$ through an analytic continuation of the parameters, chosen among:

$$p \to i\rho \wedge q \to k \tag{5.15}$$

$$p \to -i\rho \wedge q \to -k \tag{5.16}$$

$$q \to i\rho \wedge p \to k \tag{5.17}$$

$$q \to -i\rho \wedge p \to -k \tag{5.18}$$

These options are all equivalent, hence for our purposes we can focus only on the first option, without losing generality.
Recalling the Cartan decomposition of the group elements of $SO(4, \mathbb{R})$ and $SL(2, \mathbb{C})$ we have:

$$g \in SO(4, \mathbb{R}) \to g = ue^{itA_3}v^\dagger \tag{5.19}$$

$$h \in SL(2, \mathbb{C}) \to h = ue^{-irK_3}v^\dagger \tag{5.20}$$

The action of (5.2) implies then $g = ue^{-tK_3}v^\dagger$. Meaning that, once chosen the same $SU(2)$ elements to decompose g and h, our map induces a correspondence between them, through an analytic continuation of the boost parameter:

$$t \to ir \tag{5.21}$$

5.2 Representation Matrices

After listing the prescriptions through which one can map the group structure of the Euclidean model into the Lorentzian one, the next relevant objects are the representation matrices. Recalling equations (2.35) and (3.20):

$$D^{pq}_{jmln}(g) = \sum_{m'} D^j_{mm'}(u) d^{pq}_{jlm'}(t) D^l_{m'n}(v^\dagger)$$

$$D^{(\rho,k)}_{jmln}(h) = \sum_{m'} D^j_{mm'}(u) d^{(\rho,k)}_{jlm'}(r) D^l_{m'n}(v^\dagger)$$

It is clear that, once a correspondence has been established between the irreps and the group elements, the only difference remaining between the two relations is in the functional form of the respective d-matrix elements.

To obtain an equivalence between them we proceed by starting from the reduced Wigner matrix element of $SL(2,\mathbb{C})$, manipulating it in such a way it reduces to the $SO(4,\mathbb{R})$ one under the action of the map we defined. Here and in the following computation, whenever the argument x of a factorial is not a non-negative integer, it has to be intended as the Gamma function evaluated at $x+1$. The choice of using extensively the factorial notation has been made for the sake of compactness and readability of the computation. Starting from (3.22), we modify the factors at the beginning of the expression, using (B.5) and (B.4), obtaining:

$$(-1)^{\frac{j-l}{2}}\frac{\Gamma(j+i\rho+1)}{\Gamma(l+i\rho+1)}\left|\frac{\Gamma(l+i\rho+1)}{\Gamma(j+i\rho+1)}\right| = (-1)^{j-l}\frac{\sqrt{(i\rho-j-1)!\,(j+i\rho)!}}{\sqrt{(i\rho-l-1)!\,(l+i\rho)!}} \tag{5.22}$$

So that, putting $z=e^{-r}$:

$$\begin{aligned} d^{(\rho,k)}_{jlm}(z) = (-1)^{j-l}&\sqrt{\frac{(i\rho-j-1)!\,(j+i\rho)!}{(i\rho-l-1)!\,(l+i\rho)!}}\frac{\sqrt{(2j+1)(2l+1)}}{(j+l+1)!}z^{-(i\rho-k-m-1)}\\ &\sqrt{(j+k)!(j-k)!(j+m)!(j-m)!(l+k)!(l-k)!(l+m)!(l-m)!}\\ &\sum_{s,t}(-1)^{s+t}z^{2t}\frac{(k+s+m+t)!(j+l-k-m-s-t)!}{t!s!(j-k-s)!(j-m-s)!(k+m+s)!(l-k-t)!(l-m-t)!(k+m+t)!}\\ &{}_2F_1\left[\{l-i\rho+1,k+m+s+t+1\},\{j+l+2\};1-z^2\right] \end{aligned} \tag{5.23}$$

Using (B.12), one can express the ${}_2F_1$ of $1-z^2$ as the sum of two Hypergeometric functions evaluated at z^{-2}, obtaining:

$$\begin{aligned} &{}_2F_1\left[\{l-i\rho+1,k+m+s+t+1\},\{j+l+2\};1-z^2\right] =\\ &\frac{(j+l+1)!(l-m-i\rho-k-s-t-1)!}{(l-i\rho)!(j+l-m-k-s-t)!}z^{-2(k+m+s+t+1)}\\ &{}_2F_1\left[\{j+i\rho+1,k+m+s+t+1\},\{m+i\rho+k+s+t-l+1\};z^{-2}\right]\\ &+\frac{(j+l+1)!(m+i\rho+k+s+t-l-1)!}{(j+i\rho)!(k+m+s+t)!}z^{-2(l-i\rho+1)}\\ &{}_2F_1\left[\{j+l-m-k-s-t+1,l-i\rho+1\},\{l-m-i\rho-k-s-t+1\};z^{-2}\right] \end{aligned} \tag{5.24}$$

Substituting it in (5.23) we obtain:

$$
\begin{aligned}
d_{jlm}^{(\rho,k)}(z) = (-1)^{j-l}&\sqrt{\frac{(i\rho-j-1)!\,(j+i\rho)!}{(i\rho-l-1)!\,(l+i\rho)!}}\frac{\sqrt{(2j+1)(2l+1)}}{(j+l+1)!}z^{-(i\rho-k-m-1)}\\
&\sqrt{(j+k)!(j-k)!(j+m)!(j-m)!(l+k)!(l-k)!(l+m)!(l-m)!}\\
&\sum_{s,t}(-1)^{s+t}z^{2t}\frac{(k+s+m+t)!(j+l-k-m-s-t)!}{t!s!(j-k-s)!(j-m-s)!(k+m+s)!(l-k-t)!(l-m-t)!(k+m+t)!}\\
&\left\{\frac{(j+l+1)!(l-m-i\rho-k-s-t-1)!}{(l-i\rho)!(j+l-m-k-s-t)!}z^{-2(k+m+s+t+1)}\right.\\
&{}_2F_1\left[\{j+i\rho+1,k+m+s+t+1\},\{m+i\rho+k+s+t-l+1\};z^{-2}\right]\\
&+\frac{(j+l+1)!(m+i\rho+k+s+t-l-1)!}{(j+i\rho)!(k+m+s+t)!}z^{-2(l-i\rho+1)}\\
&\left.{}_2F_1\left[\{j+l-m-k-s-t+1,l-i\rho+1\},\{l-m-i\rho-k-s-t+1\};z^{-2}\right]\right\}
\end{aligned}
\tag{5.25}
$$

We can write the two resulting Hypergeometric functions explicitly, and, exploiting (B.8), change the sign of any parameter. This leads to some cancellation, and brings us to:

$$
\begin{aligned}
d_{jlm}^{(\rho,k)}(z) = (-1)^{j-l}&\sqrt{(2j+1)(2l+1)}\sqrt{\frac{(i\rho-j-1)!}{(l+i\rho)!\,(j+i\rho)!}}\\
&\sqrt{(j+k)!(j-k)!(j+m)!(j-m)!(l+k)!(l-k)!(l+m)!(l-m)!}\\
&\left\{\frac{1}{(l-i\rho)!\sqrt{(i\rho-l-1)!}}\sum_{s,t,n}(-1)^{s+t}(-1)^{n}z^{-(i\rho+k+m+2s+2n+1)}\right.\\
&\frac{(j+i\rho+n)!(k+m+s+t+n)!(l-m-i\rho-k-s-t-1-n)!}{t!s!n!(j-k-s)!(j-m-s)!(k+m+s)!(l-k-t)!(l-m-t)!(k+m+t)!}\\
&+\sqrt{(i\rho-l-1)!}\sum_{s,t,n}(-1)^{s+t}z^{2t+i\rho+k+m-2l-2n-1}\\
&\left.\frac{(j+l-m-k-s-t+n)!(m+i\rho+k+s+t-l-1-n)!}{t!s!n!(j-k-s)!(j-m-s)!(k+m+s)!(l-k-t)!(l-m-t)!(k+m+t)!(i\rho-l-1-n)!}\right\}
\end{aligned}
\tag{5.26}
$$

The contributions in s and t in both sums can be decoupled shifting the index n. We perform $n \to n-s-k-m$ in the first sum and $n \to n+t+k-l$ in the second. Such shifts, as well as the ones performed later in the discussion, keep the summed index on integer values. This is guaranteed by the fact that, even if k,j, l and m might be half-integers, the sum or difference of any two of them takes integer values.

We obtain:

$$\begin{aligned} d^{(\rho,k)}_{jlm}(z) = (-1)^{j-l}\sqrt{(2j+1)(2l+1)}\sqrt{\frac{(i\rho-j-1)!}{(l+i\rho)!\,(j+i\rho)!}} \\ \sqrt{(j+k)!(j-k)!(j+m)!(j-m)!(l+k)!(l-k)!(l+m)!(l-m)!} \\ \left\{\frac{1}{(l-i\rho)!\sqrt{(i\rho-l-1)!}}\sum_{s,t,n}(-1)^{s+t}(-1)^{n-m-s-k}z^{-(i\rho-k+1-m+2n)}\right. \\ \frac{(i\rho-k+j+n-m-s)!}{s!(-k+n-m-s)!(j-k-s)!(j-m-s)!(k+m+s)!} \\ \frac{(n+t)!(-i\rho-1-n-t+l)!}{t!(l-k-t)!(l-m-t)!(k+m+t)!} \\ +\sqrt{(i\rho-l-1)!}\sum_{s,t,n}(-1)^{s+t}z^{i\rho-k-1+m-2n} \\ \frac{(j-m+n-s)!(i\rho+m-n-1+s)!}{s!(j-k-s)!(j-m-s)!(k+m+s)!} \\ \left.\frac{1}{t!(l-k-t)!(l-m-t)!(k+m+t)!(i\rho-k-1-n-t)!(k-l+n+t)!}\right\} \\ \equiv \mathcal{F}_1(z)+\mathcal{F}_2(z) \end{aligned} \tag{5.27}$$

To study the two contributions to $d^{(\rho,k)}_{jlm}(z)$ separately, we have defined the functions $\mathcal{F}_1(z)$ and $\mathcal{F}_2(z)$ as:

$$\begin{aligned} \mathcal{F}_1(z) = (-1)^{j-l}\frac{\sqrt{(2j+1)(2l+1)}}{(l-i\rho)!}\sqrt{\frac{(i\rho-j-1)!}{(l+i\rho)!\,(j+i\rho)!(i\rho-l-1)!}} \\ \sqrt{(j+k)!(j-k)!(j+m)!(j-m)!(l+k)!(l-k)!(l+m)!(l-m)!} \\ \sum_{s,t,n}(-1)^{t+n-m-k}z^{-(i\rho-k+1-m+2n)} \\ \frac{(i\rho-k+j+n-m-s)!}{s!(j-k-s)!(j-m-s)!(k+m+s)!(-k+n-m-s)!} \\ \frac{(n+t)!(-i\rho-1-n-t+l)!}{t!(l-k-t)!(l-m-t)!(k+m+t)!} \end{aligned} \tag{5.28}$$

$$\begin{aligned} \mathcal{F}_2(z) = (-1)^{j-l}\sqrt{(2j+1)(2l+1)}\sqrt{\frac{(i\rho-l-1)!\,(i\rho-j-1)!}{(l+i\rho)!\,(j+i\rho)!}} \\ \sqrt{(j+k)!(j-k)!(j+m)!(j-m)!(l+k)!(l-k)!(l+m)!(l-m)!} \\ \sum_{s,t,n}(-1)^{s+t}z^{i\rho-k-1+m-2n} \\ \frac{(j-m+n-s)!(i\rho+m-n-1+s)!}{s!(j-k-s)!(j-m-s)!(k+m+s)!} \\ \frac{1}{t!(l-k-t)!(l-m-t)!(k+m+t)!(i\rho-k-1-n-t)!(k-l+n+t)!} \end{aligned} \tag{5.29}$$

In the following discussion, $\mathcal{F}_2(z)$ plays a more relevant role, so we focus on it first. After performing another shift in the sum indices, namely $t \to t+l-k-n$, we can rearrange the terms to visualize better the contribution of each quantity:

$$
\begin{aligned}
\mathcal{F}_2(z) = &\sum_{s,t,n} z^{i\rho-k-1+m-2n} \\
&\sqrt{\frac{(2j+1)(i\rho-j-1)!(j+k)!(j-k)!(j+m)!(j-m)!}{(j+i\rho)!}} \\
&(-1)^{j-k-n}(-1)^s \frac{(j-m+n-s)!(i\rho+m-n-1+s)!}{s!(j-k-s)!(j-m-s)!(k+m+s)!} \\
&\sqrt{\frac{(2l+1)(i\rho-l-1)!(l+k)!(l-k)!(l+m)!(l-m)!}{(l+i\rho)!}} \\
&(-1)^t \frac{1}{t!(i\rho-l-1-t)!(k-m+n-t)!(n-t)!(l+m-n+t)!(l-k-n+t)!}
\end{aligned}
\tag{5.30}
$$

In the last two rows we can recognize the structure of the analytically continued Clebsh-Gordan coefficients expressed as in (A.9), though the correspondence is not trivial, as they are given by:

$$
\begin{aligned}
&\left\langle \left(\frac{i\rho+k-1}{2}, m-n+\frac{i\rho-k-1}{2}\right), \left(\frac{i\rho-k-1}{2}, n-\frac{i\rho-k-1}{2}\right) \middle| l, m \right\rangle = \\
&\sqrt{(i\rho+m-n-1)!(k-m+n)!n!(i\rho-k-1-n)!} \\
&\sqrt{\frac{(2l+1)(i\rho-l-1)!(l+k)!(l-k)!(l+m)!(l-m)!}{(i\rho+l)!}} \\
&\sum_t (-1)^t \frac{1}{t!(i\rho-l-t-1)!(k-m+n-t)!(n-t)!(l+m-n+t)!(l-k-n+t)!}
\end{aligned}
\tag{5.31}
$$

And the first four factors are missing in the expression of $\mathcal{F}_2(z)$. These factors are useful to prove that in fact the contribution that appears inside the sum, after the power of z, can be seen as the product of two analytically continued Clebsh-Gordan coefficients. Such product gives:

$$
\begin{aligned}
&\left\langle \left(\frac{i\rho+k-1}{2}, m-n+\frac{i\rho-k-1}{2}\right), \left(\frac{i\rho-k-1}{2}, n-\frac{i\rho-k-1}{2}\right) \middle| j,m \right\rangle \\
&\left\langle \left(\frac{i\rho+k-1}{2}, m-n+\frac{i\rho-k-1}{2}\right), \left(\frac{i\rho-k-1}{2}, n-\frac{i\rho-k-1}{2}\right) \middle| l,m \right\rangle = \\
&(i\rho+m-n-1)!(k-m+n)!n!(i\rho-k-1-n)! \\
&\sqrt{\frac{(2j+1)(i\rho-j-1)!(j+k)!(j-k)!(j+m)!(j-m)!}{(i\rho+j)!}} \\
&\sum_s (-1)^s \frac{1}{s!(i\rho-j-s-1)!(k-m+n-s)!(n-s)!(j+m-n+s)!(j-k-n+s)!} \\
&\sqrt{\frac{(2l+1)(i\rho-l-1)!(l+k)!(l-k)!(l+m)!(l-m)!}{(i\rho+l)!}} \\
&\sum_t (-1)^t \frac{1}{t!(i\rho-l-t-1)!(k-m+n-t)!(n-t)!(l+m-n+t)!(l-k-n+t)!}
\end{aligned}
\tag{5.32}
$$

We now focus only on the terms that are different from what appears in (5.30), namely the first four factorials and the sum over s. Multiplying and dividing for the missing terms, the latter can be seen as the definition of a Generalized Hypergeometric function of the family ${}_3F_2$. We write them together obtaining:

$$
\begin{aligned}
&\sum_s (-1)^s \frac{(i\rho+m-n-1)!(k-m+n)!n!(i\rho-k-1-n)!}{s!(i\rho-j-s-1)!(k-m+n-s)!(n-s)!(j+m-n+s)!(j-k-n+s)!} = \\
&\frac{(i\rho+m-n-1)!(i\rho-k-1-n)!}{(i\rho-j-1)!(j+m-n)!(j-k-n)!} \\
&{}_3F_2\left[\{-n, m-k-n, j-i\rho+1\}, \{j-k-n+1, j+m-n+1\}; 1\right]
\end{aligned}
\tag{5.33}
$$

We now apply identity (B.15) to ${}_3F_2$, obtaining:

$$
\begin{aligned}
&{}_3F_2\left[\{-n, m-k-n, j-i\rho+1\}, \{j-k-n+1, j+m-n+1\}; 1\right] = \\
&\frac{\Gamma(j+m-n+1)\Gamma(k-j)\Gamma(m-j)\Gamma(k-i\rho+n+1)}{\Gamma(j-i\rho+1)\Gamma(k+n-j)\Gamma(m-n-j)\Gamma(k+m+1)} \\
&{}_3F_2\left[\{m-j, k-j, i\rho+m-n\}, \{m-n-j, k+m+1\}; 1\right] \\
&-\frac{\Gamma(j+m-n+1)\Gamma(k-j)\Gamma(m-j)\Gamma(k-i\rho+n+1)\Gamma(j-n-k)}{\Gamma(-n)\Gamma(m-k-n)\Gamma(j-i\rho+1)\Gamma(k+n-j)\Gamma(k+m+1)} \\
&{}_3F_2\left[\{k-j, m-j, k-i\rho+n+1\}, \{k+n-j+1, k+m+1\}; 1\right]
\end{aligned}
\tag{5.34}
$$

Where the existence conditions become:
$Re(j+1-i\rho) > 0 \wedge Re(j+1+i\rho) > 0$, both true since $i\rho$ is a purely imaginary number and $j > 0$. Moreover, for the final aim of this discussion it is sufficient to work in the condition $n > 0$, meaning that the term $\Gamma(-n)$ diverges, hence the second term of the sum does not contribute. Keeping only the first term of the sum

and writing explicitly ${}_3F_2$ in the right-hand side, we obtain:

$$\begin{aligned}&{}_3F_2\left[\{-n, m-k-n, j-i\rho+1\}, \{j-k-n+1, j+m-n+1\}; 1\right] = \\ &\frac{(k-i\rho+n)!(j+m-n)!(k-j-1)!(m-j-1)!(j-m)!(j-k)!}{(i\rho+m-n-1)!(j-i\rho)!(k+n-j-1)!(m-n-j-1)!(j-m+n)!} \\ &\sum_s (-1)^s \frac{(i\rho+m-n-1+s)!(j-m+n-s)!}{s!(j-m-s)!(j-k-s)!(k+m+s)!}\end{aligned} \tag{5.35}$$

Using repeatedly (B.4), we can manipulate the factorials outside the sum so that they cancel out among themselves or with the ones appearing in (5.33). Accounting for all the arising phases, and the fact that both n and $j-k$ are integers, we obtain:

$$\begin{aligned}&\frac{(i\rho+m-n-1)!(i\rho-k-1-n)!}{(i\rho-j-1)!(j+m-n)!(j-k-n)!} \\ &{}_3F_2\left[\{-n, m-k-n, j-i\rho+1\}, \{j-k-n+1, j+m-n+1\}; 1\right] = \\ &(-1)^{j-k-n} \sum_s (-1)^s \frac{(i\rho+m-n-1+s)!(j-m+n-s)!}{s!(j-m-s)!(j-k-s)!(k+m+s)!}\end{aligned} \tag{5.36}$$

Hence:

$$\begin{aligned}&\left\langle \left(\frac{i\rho+k-1}{2}, m-n+\frac{i\rho-k-1}{2}\right), \left(\frac{i\rho-k-1}{2}, n-\frac{i\rho-k-1}{2}\right) \middle| j, m \right\rangle \\ &\left\langle \left(\frac{i\rho+k-1}{2}, m-n+\frac{i\rho-k-1}{2}\right), \left(\frac{i\rho-k-1}{2}, n-\frac{i\rho-k-1}{2}\right) \middle| l, m \right\rangle = \\ &\sqrt{\frac{(2j+1)(i\rho-j-1)!(j+k)!(j-k)!(j+m)!(j-m)!}{(i\rho+j)!}} \\ &(-1)^{j-k-n} \sum_s (-1)^s \frac{(i\rho+m-n-1+s)!(j-m+n-s)!}{s!(j-m-s)!(j-k-s)!(k+m+s)!} \\ &\sqrt{\frac{(2l+1)(i\rho-l-1)!(l+k)!(l-k)!(l+m)!(l-m)!}{(i\rho+l)!}} \\ &\sum_t (-1)^t \frac{1}{t!(i\rho-l-t-1)!(k-m+n-t)!(n-t)!(l+m-n+t)!(l-k-n+t)!}\end{aligned} \tag{5.37}$$

As claimed, we then have:

$$\begin{aligned}\mathcal{F}_2(z) = \sum_n z^{i\rho-k-1+m-2n} &\left\langle \left(\frac{i\rho+k-1}{2}, m-n+\frac{i\rho-k-1}{2}\right), \left(\frac{i\rho-k-1}{2}, n-\frac{i\rho-k-1}{2}\right) \middle| j, m \right\rangle \\ &\left\langle \left(\frac{i\rho+k-1}{2}, m-n+\frac{i\rho-k-1}{2}\right), \left(\frac{i\rho-k-1}{2}, n-\frac{i\rho-k-1}{2}\right) \middle| l, m \right\rangle\end{aligned} \tag{5.38}$$

We now focus again on $\mathcal{F}_1(z)$. From its definition in (5.28), we immediately spot in the sums over s and t the structure of two ${}_3F_2$ functions.
Multiplying and dividing for the missing terms we obtain:

$$\begin{aligned}\mathcal{F}_1(z) = (-1)^{j-l}&\frac{\sqrt{(2j+1)(2l+1)(i\rho-j-1)!}}{\sqrt{(i\rho+j)!(i\rho+l)!(i\rho-l-1)!}}\\ &\sqrt{\frac{(j+k)!(j+m)!(l+k)!(l+m)!}{(j-k)!(j-m)!(l-k)!(l-m)!}}\frac{1}{(l-i\rho)!}\left[\frac{1}{(k+m)!}\right]^2\\ &\sum_n(-1)^{n-m-k}z^{-(i\rho-k+1-m+2n)}\frac{(i\rho-k+j+n-m)!n!(l-i\rho-1-n)!}{(n-k-m)!}\\ &{}_3F_2\left[\{k-j,k-n+m,m-j\},\{k-i\rho-j-n+m,k+m+1\};1\right]\\ &{}_3F_2\left[\{k-l,n+1,m-l\},\{i\rho-l+n+1,k+m+1\};1\right]\end{aligned} \tag{5.39}$$

The second Hypergeometric function can be manipulated into the same form of the first. To achieve this, we apply the identity (B.16), obtaining:

$$\begin{aligned}&{}_3F_2\left[\{k-l,n+1,m-l\},\{i\rho-l+n+1,k+m+1\};1\right]=\\ &\frac{\Gamma(k-i\rho-n)\Gamma(m-n-i\rho)}{\Gamma(l-n-i\rho)\Gamma(k-i\rho-n+m-l)}\\ &{}_3F_2\left[\{k-l,k-n+m,m-l\},\{k-i\rho-l-n+m,k+m+1\};1\right]\\ &+\frac{\Gamma(k-i\rho-n)\Gamma(1+l-i\rho)\Gamma(m-n-i\rho)\Gamma(i\rho-l+n+1)\Gamma(k+m+1)}{\Gamma(k-l)\Gamma(n+1)\Gamma(m-l)\Gamma(l-i\rho-n+1)\Gamma(k-i\rho+l-n+m+1)}\\ &{}_3F_2\left[\{k-i\rho-n,1+l-i\rho,m-n-i\rho\},\{l-i\rho-n+1,k-i\rho+l-n+m+1\};1\right]\end{aligned} \tag{5.40}$$

Here the existence condition of the identity are given by:
$Re(l+1-i\rho)>0 \wedge Re(l+1+i\rho)>0$, both true since $i\rho$ is a purely imaginary number and $l>0$. Moreover, since $l>k$ and $m<l$, the terms $\Gamma(k-l)$ and $\Gamma(m-l)$ diverge, hence the second term of the sum does not contribute. Keeping only the first term of the sum, we can re-write $\mathcal{F}_1(z)$ as:

$$\begin{aligned}\mathcal{F}_1(z) = (-1)^{j-l}&\frac{\sqrt{(2j+1)(2l+1)(i\rho-j-1)!}}{\sqrt{(i\rho+j)!(i\rho+l)!(i\rho-l-1)!}}\\ &\sqrt{\frac{(j+k)!(j+m)!(l+k)!(l+m)!}{(j-k)!(j-m)!(l-k)!(l-m)!}}\frac{1}{(l-i\rho)!}\left[\frac{1}{(k+m)!}\right]^2\\ &\sum_n(-1)^{n-m-k}z^{-(i\rho-k+1-m+2n)}\frac{(i\rho-k+j+n-m)!n!(k-i\rho-n-1)!(m-n-i\rho-1)!}{(n-m-k)!(k-i\rho-n+m-l-1)!}\\ &{}_3F_2\left[\{k-j,k-n+m,m-j\},\{k-i\rho-j-n+m,k+m+1\};1\right]\\ &{}_3F_2\left[\{k-l,k-n+m,m-l\},\{k-i\rho-l-n+m,k+m+1\};1\right]\end{aligned} \tag{5.41}$$

Identity (B.8) allows to have more than one equivalent expression for the same Hypergeometric function. Exploiting this freedom of choice, we express $\mathcal{F}_1(z)$

writing the two ${}_3F_2$ explicitly as follows:

$$\begin{aligned}\mathcal{F}_1(z) = (-1)^{j-l}\frac{\sqrt{(2j+1)(2l+1)(i\rho-j-1)!}}{\sqrt{(i\rho+j)!(i\rho+l)!(i\rho-l-1)!}}\\ \sqrt{\frac{(j+k)!(j+m)!(l+k)!(l+m)!}{(j-k)!(j-m)!(l-k)!(l-m)!}}\frac{1}{(l-i\rho)!}\left[\frac{1}{(k+m)!}\right]^2\\ \sum_n(-1)^{n-m-k}z^{-(i\rho-k+1-m+2n)}\frac{(i\rho-k+j+n-m)!n!(k-i\rho-n-1)!(m-n-i\rho-1)!}{(n-m-k)!(k-i\rho-n+m-l-1)!}\\ \sum_s(-1)^s\frac{(j-k)!(n-m-k)!(j-m)!(k-i\rho-j-n+m-1)!(k+m)!}{s!(j-k-s)!(j-m-s)!(n-m-k-s)!(k-i\rho-j-n+m-1+s)!(k+m+s)!}\\ \sum_t(-1)^t\frac{(l-k)!(n-m-k)!(l-m)!(k-i\rho-l-n+m-1)!(k+m)!}{t!(l-k-t)!(l-m-t)!(n-m-k-t)!(k-i\rho-l-n+m-1+t)!(k+m+t)!}\end{aligned} \tag{5.42}$$

Simplifying the common terms we obtain:

$$\begin{aligned}\mathcal{F}_1(z) = (-1)^{j-l}\frac{\sqrt{(2j+1)(2l+1)}}{(l-i\rho)!}\sqrt{\frac{(i\rho-j-1)!}{(j+i\rho)!(l+i\rho)!(i\rho-l-1)!}}\\ \sqrt{(j+k)!(j-k)!(j+m)!(j-m)!(l+k)!(l-k)!(l+m)!(l-m)!}\\ \sum_n(-1)^{n-m-k}z^{-(i\rho-k+1-m+2n)}\\ n!(k-i\rho-j-n+m-1)!(i\rho-k+j+n-m)!(n-m-k)!(k-i\rho-n-1)!(m-n-i\rho-1)!\\ \sum_s(-1)^s\frac{1}{s!(j-k-s)!(j-m-s)!(n-m-k-s)!(k-i\rho-j-n+m-1+s)!(k+m+s)!}\\ \sum_t(-1)^t\frac{1}{t!(l-k-t)!(l-m-t)!(n-m-k-t)!(k-i\rho-l-n+m-1+t)!(k+m+t)!}\end{aligned} \tag{5.43}$$

This expression can be further simplified applying (B.4) to $(i\rho-k+j+n-m)!$ and $(l-i\rho)!$. After that, the expression presents a total prefactor given by: $(-1)^{2n+2j-2m-2l-2k+2i\rho}$. This reduces to $(-1)^{2i\rho}$, since n, $j-m$ and $l+k$ all take integer values. Hence, we have:

$$\begin{aligned}\mathcal{F}_1(z) = (-1)^{2i\rho}\sqrt{(2j+1)(2l+1)}\sqrt{\frac{(i\rho-j-1)!(i\rho-l-1)!}{(j+i\rho)!(l+i\rho)!}}\\ \sqrt{(j+k)!(j-k)!(j+m)!(j-m)!(l+k)!(l-k)!(l+m)!(l-m)!}\\ \sum_n z^{-(i\rho-k+1-m+2n)}n!(n-m-k)!(k-i\rho-n-1)!(m-n-i\rho-1)!\\ \sum_s(-1)^s\frac{1}{s!(j-k-s)!(j-m-s)!(n-m-k-s)!(k+m+s)!(k-i\rho-j-n+m-1+s)!}\\ \sum_t(-1)^t\frac{1}{t!(l-k-t)!(l-m-t)!(n-m-k-t)!(k+m+t)!(k-i\rho-l-n+m-1+t)!}\end{aligned} \tag{5.44}$$

We now apply the following change of variables:

$$\begin{aligned} s &\to n-m-q-s \\ t &\to n-m-q-t \end{aligned}$$

So that:

$$\begin{aligned} \mathcal{F}_1(z) &= (-1)^{2i\rho}\sqrt{(2j+1)(2l+1)}\sqrt{\frac{(i\rho-j-1)!(i\rho-l-1)!}{(j+i\rho)!(l+i\rho)!}} \\ &\sqrt{(j+k)!(j-k)!(j+m)!(j-m)!(l+k)!(l-k)!(l+m)!(l-m)!} \\ &\sum_n z^{-(i\rho-k+1-m+2n)} n!(n-m-k)!(k-i\rho-n-1)!(m-n-i\rho-1)! \\ &\sum_s (-1)^s \frac{1}{s!(-i\rho-j-1-s)!(n-s)!(n-m-k-s)!(j-n+k+s)!(j-n+m+s)!} \\ &\sum_t (-1)^t \frac{1}{t!(-i\rho-l-1-t)!(n-t)!(n-m-k-t)!(l-n+k+t)!(l-n+m+t)!} \end{aligned} \tag{5.45}$$

In here we can again recognize the structure of (A.9), with a different set of parameters than the one found in $\mathcal{F}_2(z)$. In particular, what we obtain is:

$$\begin{aligned} \mathcal{F}_1(z) = \sum_n & z^{-(i\rho-k+1-m+2n)} \\ &\left\langle \left(-\frac{i\rho+k+1}{2}, -\frac{i\rho-k+1}{2}+m-n\right), \left(\frac{k-i\rho-1}{2}, n-\frac{k-i\rho-1}{2}\right) \middle| j,m \right\rangle \\ &\left\langle \left(-\frac{i\rho+k+1}{2}, -\frac{i\rho-k+1}{2}+m-n\right), \left(\frac{k-i\rho-1}{2}, n-\frac{k-i\rho-1}{2}\right) \middle| l,m \right\rangle \end{aligned} \tag{5.46}$$

Where, as a result of a manipulation on the factorials, performed to match the ones in the previous expression, the phase $(-1)^{2i\rho}$ vanishes. Putting together what we have shown for $\mathcal{F}_1(z)$ and $\mathcal{F}_2(z)$, we can express the $SL(2,\mathbb{C})$ reduced Wigner matrix as:

$$\begin{aligned} d^{(\rho,k)}_{jlm}(r) = \sum_n & e^{-(-i\rho+k-1+m-2n)r} \\ &\left\langle \left(-\frac{i\rho+k+1}{2}, -\frac{i\rho-k+1}{2}+m-n\right), \left(\frac{k-i\rho-1}{2}, n-\frac{k-i\rho-1}{2}\right) \middle| j,m \right\rangle \\ &\left\langle \left(-\frac{i\rho+k+1}{2}, -\frac{i\rho-k+1}{2}+m-n\right), \left(\frac{k-i\rho-1}{2}, n-\frac{k-i\rho-1}{2}\right) \middle| l,m \right\rangle \\ &+ \sum_n e^{-(i\rho-k-1+m-2n)r} \\ &\left\langle \left(\frac{i\rho+k-1}{2}, \frac{i\rho-k-1}{2}+m-n\right), \left(\frac{i\rho-k-1}{2}, n-\frac{i\rho-k-1}{2}\right) \middle| j,m \right\rangle \\ &\left\langle \left(\frac{i\rho+k-1}{2}, \frac{i\rho-k-1}{2}+m-n\right), \left(\frac{i\rho-k-1}{2}, n-\frac{i\rho-k-1}{2}\right) \middle| l,m \right\rangle \end{aligned} \tag{5.47}$$

Which is manifestly invariant under the exchange $(\rho, k) \leftrightarrow (-\rho, -k)$, as expected. We now show how the $SO(4,\mathbb{R})$ reduced matrix elements can be written in a similar form. First, we recall equation (2.33):

$$d^{pq}_{jlm}(t) \equiv \sum_{m_1,m_2} C^{jm}_{j_1 m_1 j_2 m_2} e^{i(m_1-m_2)t} C^{lm}_{j_1 m_1 j_2 m_2}$$

One of the two sums can be suppressed exploiting the fact that both of the Clebsh-Gordan coefficients appearing in it are non-zero only if $m_1 + m_2 = m$. Imposing $m_1 = m - m_2$, it can be written as:

$$d^{pq}_{jlm}(t) \equiv \sum_{m_2} C^{jm}_{j_1 (m-m_2) j_2 m_2} e^{i(m-2m_2)t} C^{lm}_{j_1 (m-m_2) j_2 m_2} \tag{5.48}$$

Using the definitions $p = j_1 + j_2 + 1$ and $q = j_1 - j_2$ and defining a new summing index $n \equiv j_2 + m_2 = \frac{p-q-1}{2} + m_2$, we obtain:

$$\begin{aligned} d^{pq}_{jlm}(t) = &\sum_{n=0}^{p-q-1} e^{i(p-q-1+m-2n)t} \\ &\left\langle \left(\frac{p+q-1}{2}, \frac{p-q-1}{2} + m - n\right), \left(\frac{p-q-1}{2}, n - \frac{p-q-1}{2}\right) \middle| jm \right\rangle \\ &\left\langle \left(\frac{p+q-1}{2}, \frac{p-q-1}{2} + m - n\right), \left(\frac{p-q-1}{2}, n - \frac{p-q-1}{2}\right) \middle| lm \right\rangle \end{aligned} \tag{5.49}$$

The upper bound of the sum can be left unspecified since no contribution arise from the terms with $n > p - q - 1$, due to the properties of the Clebsh-Gordan coefficients. Moreover, we can artificially add a non contributing term, symmetric in the exchange $(p, q) \leftrightarrow (-p, -q)$, which automatically vanishes since for this term the triangular inequalities for the Clebsh-Gordan coefficients of $SU(2)$ would read:

$$j > j_1 - j_2 \tag{5.50}$$
$$j < -j_1 - j_2 - 2 < 0 \tag{5.51}$$

Being $j > 0$, the second inequality is not satisfied, hence the added term does not contribute. This allows us to write:

$$\begin{aligned} d^{p,q}_{jlm}(t) = &\sum_n e^{i(-p+q-1+m-2n)t} \\ &\left\langle \left(-\frac{p+q+1}{2}, -\frac{p-q+1}{2} + m - n\right), \left(\frac{q-p-1}{2}, n - \frac{q-p-1}{2}\right) \middle| j, m \right\rangle \\ &\left\langle \left(-\frac{p+q+1}{2}, -\frac{p-q+1}{2} + m - n\right), \left(\frac{q-p-1}{2}, n - \frac{q-p-1}{2}\right) \middle| l, m \right\rangle \\ &+ \sum_n e^{i(p-q-1+m-2n)t} \\ &\left\langle \left(\frac{p+q-1}{2}, \frac{p-q-1}{2} + m - n\right), \left(\frac{p-q-1}{2}, n - \frac{p-q-1}{2}\right) \middle| j, m \right\rangle \\ &\left\langle \left(\frac{p+q-1}{2}, \frac{p-q-1}{2} + m - n\right), \left(\frac{p-q-1}{2}, n - \frac{p-q-1}{2}\right) \middle| l, m \right\rangle \end{aligned} \tag{5.52}$$

Written in this form, under (5.15) and (5.21), we recover exactly expression (5.47). We have thus been able to prove that the $SL(2,\mathbb{C})$ reduced Wigner matrices elements can be seen as the analytical continuation of the $SO(4,\mathbb{R})$ ones. The mapping from $SL(2,\mathbb{C})$ to $SO(4,\mathbb{R})$ is more trivial since, without any further manipulation other than the application of the inverse map, $\rho \to -ip$, $k \to q$, $r \to -it$, (5.47) automatically reduces to (5.49), since $\mathcal{F}_1(z)$ vanishes once it is expressed as a product of regular $SU(2)$ Clebsh-Gordan coefficients. As a direct consequence, we can state that the $SO(4,\mathbb{R})$ d-matrix can be equivalently expressed as:

$$
\begin{aligned}
d^{pq}_{jlm}(t) = {} & (-1)^{j-l} \frac{\sqrt{(p-j-1)!\,(j+p)!}}{\sqrt{(p-l-1)!\,(l+p)!}} \frac{\sqrt{(2j+1)(2l+1)}}{(j+l+1)!} e^{-i(p-q-m-1)t} \\
& \sqrt{(j+q)!(j-q)!(j+m)!(j-m)!(l+q)!(l-q)!(l+m)!(l-m)!} \\
& \sum_{s,s'} e^{2its'} \frac{(-1)^{s+s'}(q+s+m+s')!(j+l-q-m-s-s')!}{s!s'!(j-q-s)!(j-m-s)!(q+m+s)!(l-q-s')!(l-m-s')!(q+m+s')!} \\
& {}_2F_1\left[\{l-p+1, q+m+s+s'+1\}, \{j+l+2\}; 1-e^{2it}\right]
\end{aligned}
\tag{5.53}
$$

5.3 Mapping of the Amplitudes

We have proved that, under our prescriptions, the representation matrices of $SO(4,\mathbb{R})$ and $SL(2,\mathbb{C})$ are mapped one into each other. The next step is to see how this equivalence is reflected into the construction of the EPRL models, starting from the imposition of the LSC. In the Euclidean model, this is achieved by setting p to γj and q to j, as shown in (2.42) and (2.43). Implementing our map, such relations become:

$$i\rho = \gamma j \tag{5.54}$$

$$k = j \tag{5.55}$$

Reducing to (3.29) and (3.30) if we perform the transformation $\gamma \to i\gamma$.
We can then conclude that, once the map $\mathcal{H}^{(p,q)} \leftrightarrow \mathcal{H}^{(\rho,k)}$ is established, the imposition of the LSC in both models is fulfilled performing an analytic continuation on the Barbero-Immirzi parameter. This is not sufficient to prove that the general expressions of the amplitudes, given by equations (2.45) and (3.32), are formally equivalent, as they differ by their integration domain, which is not automatically mapped after the analytic continuation of the parameters. In their final expressions, given in equations (2.58) and (3.42), their difference becomes implicit as it depends only on the definitions of the Vertex Amplitudes or, more precisely, of the Booster functions B_4^E and B_4^L.
The next step in our discussion is to show how they become proportional one to each other after the application of our map. In the following we summarize two possibilities explored in our research. The first, which has been inconclusive so far, is to compute both functions explicitly and prove that one can be analytically continued into the other. The second and more conclusive proof we present consists into a direct manipulation of the integration domain of the Euclidean Booster functions.

5.3.1 Computation of the Euclidean Booster Function

To compute explicitly the Euclidean Booster function we first derive the coefficients of the Clebsh-Gordan series for $SO(4,\mathbb{R})$. This series is involved in the recoupling of two different unitary irreducible representations of $SO(4,\mathbb{R})$. As their tensor product can be decomposed into the direct sum of single irreps, the canonical basis $|p_1,q_1,j,m\rangle \otimes |p_2,q_2,j',m'\rangle$ can be expanded as:

$$|(p_1,q_1,j,m),(p_2,q_2,j',m')\rangle = \sum_{p_3,q_3}\sum_{l,n}\sqrt{p_3^2-q_3^2}\,C^{p_3q_3ln}_{p_1q_1jm,p_2q_2j'm'}\,|p_3,q_3,l,n\rangle \quad (5.56)$$

So that:

$$\langle p_3,q_3,l,n|(p_1,q_1,j,m),(p_2,q_2,j',m')\rangle = \sqrt{p_3^2-q_3^2}\,C^{p_3q_3ln}_{p_1q_1jm,p_2q_2j'm'} \quad (5.57)$$

The choice of separating the dimensional factor $\sqrt{p_3^2-q_3^2}$ from the definition of the $SO(4,\mathbb{R})$ Clebsh-Gordan coefficients is purely conventional. Recalling equation (2.31), we can expand any element of the basis that diagonalizes L^2 and L_3 in term of the basis given by the tensor product of the eigenstates of $J^2_{(1,2)}$ and $J_{(1,2)_3}$:

$$|j,m\rangle = \sum_{m_1,m_2} |j_1,m_1,j_2,m_2\rangle\,\langle j_1,m_1,j_2,m_2|j,m\rangle \quad (5.58)$$

Where the factors $\langle j_1,m_1,j_2,m_2|j,m\rangle$ appearing in the sum are $SU(2)$ Clebsh-Gordan coefficients. Using this expansion, we obtain:

$$\begin{aligned}|(p_1,q_1,j,m),(p_2,q_2,j',m')\rangle = \sum_{m_a,m'_a} &\left\langle \frac{p_1+q_1-1}{2},m_1,\frac{p_1-q_1-1}{2},m_2\middle| j,m\right\rangle\\ &\left\langle \frac{p_2+q_2-1}{2},m'_1,\frac{p_2-q_2-1}{2},m'_2\middle| j',m'\right\rangle\\ &\left|\frac{p_1+q_1-1}{2},m_1,\frac{p_1-q_1-1}{2},m_2\right\rangle\\ &\left|\frac{p_2+q_2-1}{2},m'_1,\frac{p_2-q_2-1}{2},m'_2\right\rangle\end{aligned} \quad (5.59)$$

We can now recouple together the first spins appearing in the two tensor product states, obtaining:

$$\left|\frac{p_1+q_1-1}{2},m_1\right\rangle\otimes\left|\frac{p_2+q_2-1}{2},m'_1\right\rangle = \sum_{k_1,r_1}\left\langle k_1,r_1\middle|\frac{p_1+q_1-1}{2},m_1,\frac{p_2+q_2-1}{2},m'_1\right\rangle|k_1,r_1\rangle \quad (5.60)$$

Similarly:

$$\left|\frac{p_1-q_1-1}{2},m_2\right\rangle\otimes\left|\frac{p_2-q_2-1}{2},m'_2\right\rangle = \sum_{k_2,r_2}\left\langle k_2,r_2\middle|\frac{p_1-q_1-1}{2},m_2,\frac{p_2-q_2-1}{2},m'_2\right\rangle|k_2,r_2\rangle \quad (5.61)$$

So that the left hand side of (5.59) reads:

$$\sum_{m_a,m'_a}\sum_{k_a,r_a}\left\langle \frac{p_1+q_1-1}{2},m_1,\frac{p_1-q_1-1}{2},m_2\middle| j,m\right\rangle\left\langle \frac{p_2+q_2-1}{2},m'_1,\frac{p_2-q_2-1}{2},m'_2\middle| j',m'\right\rangle$$
$$\left\langle k_1,r_1\middle|\frac{p_1+q_1-1}{2},m_1,\frac{p_2+q_2-1}{2},m'_1\right\rangle\left\langle k_2,r_2\middle|\frac{p_1-q_1-1}{2},m_2,\frac{p_2-q_2-1}{2},m'_2\right\rangle$$
$$|k_1,r_1,k_2,r_2\rangle \tag{5.62}$$

Before performing the scalar product with $|p_3,q_3,l,n\rangle$ we expand it as:

$$|p_3,q_3,l,n\rangle=\sum_{n_1,n_2}\left\langle \frac{p_3+q_3-1}{2},n_1,\frac{p_3-q_3-1}{2},n_2\middle| l,n\right\rangle\left|\frac{p_3+q_3-1}{2},n_1,\frac{p_3-q_3-1}{2},n_2\right\rangle \tag{5.63}$$

So that:

$$\langle p_3,q_3,l,n|(p_1,q_1,j,m),(p_2,q_2,j',m')\rangle=\sum_{m_a,m'_a,n_a}\left\langle \frac{p_1+q_1-1}{2},m_1,\frac{p_1-q_1-1}{2},m_2\middle| j,m\right\rangle$$
$$\left\langle \frac{p_2+q_2-1}{2},m'_1,\frac{p_2-q_2-1}{2},m'_2\middle| j',m'\right\rangle$$
$$\left\langle \frac{p_3+q_3-1}{2},n_1,\frac{p_3-q_3-1}{2},n_2\middle| l,n\right\rangle$$
$$\left\langle \frac{p_3+q_3-1}{2},n_1\middle|\frac{p_1+q_1-1}{2},m_1,\frac{p_2+q_2-1}{2},m'_1\right\rangle$$
$$\left\langle \frac{p_3-q_3-1}{2},n_2\middle|\frac{p_1-q_1-1}{2},m_2,\frac{p_2-q_2-1}{2},m'_2\right\rangle \tag{5.64}$$

Using (A.31) and the orthogonality relations for the $SU(2)$ Clebsh-Gordan coefficients, we obtain:

$$\langle p_3,q_3,l,n|(p_1,q_1,j,m),(p_2,q_2,j',m')\rangle=$$
$$\sqrt{\left(2\frac{p_3+q_3-1}{2}+1\right)\left(2\frac{p_3-q_3-1}{2}+1\right)}\sqrt{2j+1}\sqrt{2j'+1}\,\langle l,n|j,m,j',m'\rangle\begin{Bmatrix}\frac{p_1+q_1-1}{2} & \frac{p_2+q_2-1}{2} & \frac{p_3+q_3-1}{2}\\ \frac{p_1-q_1-1}{2} & \frac{p_2-q_2-1}{2} & \frac{p_3-q_3-1}{2}\\ j & j' & l\end{Bmatrix}$$
$$=\sqrt{p_3^2-q_3^2}\sqrt{2j+1}\sqrt{2j'+1}\,\langle l,n|j,m,j',m'\rangle\begin{Bmatrix}\frac{p_1+q_1-1}{2} & \frac{p_2+q_2-1}{2} & \frac{p_3+q_3-1}{2}\\ \frac{p_1-q_1-1}{2} & \frac{p_2-q_2-1}{2} & \frac{p_3-q_3-1}{2}\\ j & j' & l\end{Bmatrix} \tag{5.65}$$

Confronting this result with equation (5.57) we obtain the explicit formula for the $SO(4,\mathbb{R})$ Clebsh-Gordan coefficients:

$$C^{p_3q_3ln}_{p_1q_1jm,p_2q_2j'm'}=\sqrt{2j+1}\sqrt{2j'+1}\,\langle l,n|j,m,j',m'\rangle\begin{Bmatrix}\frac{p_1+q_1-1}{2} & \frac{p_2+q_2-1}{2} & \frac{p_3+q_3-1}{2}\\ \frac{p_1-q_1-1}{2} & \frac{p_2-q_2-1}{2} & \frac{p_3-q_3-1}{2}\\ j & j' & l\end{Bmatrix} \tag{5.66}$$

Which, defining the $SO(4,\mathbb{R})$ *χ-functions* as:

$$\chi(j,j',l) \equiv \sqrt{2j+1}\sqrt{2j'+1}\begin{Bmatrix} \frac{p_1+q_1-1}{2} & \frac{p_2+q_2-1}{2} & \frac{p_3+q_3-1}{2} \\ \frac{p_1-q_1-1}{2} & \frac{p_2-q_2-1}{2} & \frac{p_3-q_3-1}{2} \\ j & j' & l \end{Bmatrix} \tag{5.67}$$

Becomes:

$$C^{p_3q_3ln}_{p_1q_1jm,p_2q_2j'm'} = \langle l,n|j,m,j',m'\rangle\,\chi(j,j',l) \tag{5.68}$$

Now we show how the integral of four representation matrices elements can be written in terms of these coefficients. First, we recall two properties that the representation matrices must satisfy, namely the composition rule:

$$D^{p_1q_1}_{j_1m_1l_1n_1}(g)\,D^{p_2q_2}_{j_2m_2l_2n_2}(g) = \sum_{p_3,q_3}\sum_{j_3,m_3,l_3,n_3}\left(p_3^2-q_3^2\right)C^{p_3q_3j_3m_3}_{p_1q_1j_1m_1,p_2q_2j_2m_2}C^{p_3q_3l_3n_3}_{p_1q_1l_1n_1,p_2q_2l_2n_2}D^{p_3q_3}_{j_3m_3l_3n_3}(g) \tag{5.69}$$

And the orthogonality relation:

$$\int dg\overline{D^{p_1q_1}_{j_1m_1l_1n_1}(g)}D^{p_2q_2}_{j_2m_2l_2n_2}(g) = \frac{\delta_{p_1p_2}\delta_{q_1q_2}}{p_1^2-q_1^2}\delta_{j_1j_2}\delta_{m_1m_2}\delta_{l_1l_2}\delta_{n_1n_2} \tag{5.70}$$

Where:

$$\overline{D^{pq}_{jmln}(g)} = (-1)^{j-m}(-1)^{l-n}D^{pq}_{j-ml-n}(g) \tag{5.71}$$

From these we can compute:

$$\begin{aligned}
&\int dgD^{p_1q_1}_{j_1m_1l_1n_1}(g)\,D^{p_2q_2}_{j_2m_2l_2n_2}(g)\,\overline{D^{p_3q_3}_{j_3m_3l_3n_3}(g)} = \\
&= \sum_{p_4q_4j_4m_4l_4n_4}\left(p_4^2-q_4^2\right)C^{p_4q_4j_4m_4}_{p_1q_1j_1m_1\;p_2q_2j_2m_2}C^{p_4q_4l_4n_4}_{p_1q_1l_1n_1\;p_2q_2l_2n_2}\int dgD^{p_4q_4}_{j_4m_4l_4n_4}(g)\,\overline{D^{p_3q_3}_{j_3m_3l_3n_3}(g)} \\
&= \sum_{p_4q_4j_4m_4l_4n_4}\left(p_4^2-q_4^2\right)C^{p_4q_4j_4m_4}_{p_1q_1j_1m_1\;p_2q_2j_2m_2}C^{p_4q_4l_4n_4}_{p_1q_1l_1n_1\;p_2q_2l_2n_2}\frac{\delta_{p_3p_4}\delta_{q_3q_4}}{p_4^2-q_4^2}\delta_{j_3j_4}\delta_{m_3m_4}\delta_{l_3l_4}\delta_{n_3n_4} \\
&= C^{p_3q_3j_3m_3}_{p_1q_1j_1m_1\;p_2q_2j_2m_2}C^{p_3q_3l_3n_3}_{p_1q_1l_1n_1\;p_2q_2l_2n_2} \\
&= \langle j_3,m_3|j_1m_1j_2m_2\rangle\,\langle l_3,n_3|l_1n_1l_2n_2\rangle\,\chi(j_1,j_2,j_3)\,\chi(l_1,l_2,l_3)
\end{aligned} \tag{5.72}$$

And, similarly:

$$\begin{aligned}
\int dgD^{p_1q_1}_{j_1m_1l_1n_1}(g)\,D^{p_2q_2}_{j_2m_2l_2n_2}(g)\,D^{p_3q_3}_{j_3m_3l_3n_3}(g) &= (-1)^{j_1+j_2-j_3}(-1)^{l_1+l_2-l_3}\sqrt{2j_3+1}\sqrt{2l_3+1} \\
&\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ n_1 & n_2 & n_3 \end{pmatrix}\chi(j_1,j_2,j_3)\chi(l_1,l_2,l_3)
\end{aligned} \tag{5.73}$$

Combining this last relation with (5.69) we obtain the following expression for the integral of the product of four representation matrices elements:

$$\int dg \prod_{a=1}^{4} D^{p_a q_a}_{j_a m_a l_a n_a}(g) = \sum_{p_{12},q_{12}} \sum_{j_{12},m_{12},l_{12},n_{12}} (p_{12}^2 - q_{12}^2)(-1)^{j_{12}+j_3-j_4}(-1)^{l_{12}+l_3-l_4}$$
$$\sqrt{2j_4+1}\sqrt{2l_4+1}\,\langle j_{12},m_{12}|j_1,m_1,j_2,m_2\rangle\,\langle l_{12},n_{12}|l_1,n_1,l_2,n_2\rangle$$
$$\begin{pmatrix} j_{12} & j_3 & j_4 \\ m_{12} & m_3 & m_4 \end{pmatrix}\begin{pmatrix} l_{12} & l_3 & l_4 \\ n_{12} & n_3 & n_4 \end{pmatrix}$$
$$\chi(j_1,j_2,j_{12})\chi(l_1,l_2,l_{12})\chi(j_{12},j_3,j_4)\chi(l_{12},l_3,l_4) \tag{5.74}$$

Using the properties of the $3jm$ and $4jm$-symbols and performing the sums over the magnetic indices m_{12} and n_{12} one obtains:

$$\int dg \prod_{a=1}^{4} D^{p_a q_a}_{j_a m_a l_a n_a}(g) = \sum_{p_{12},q_{12}} \sum_{j_{12},l_{12}} (p_{12}^2 - q_{12}^2)(-1)^{j_1-j_2+j_3-j_4}(-1)^{l_1-l_2+l_3-l_4}$$
$$\sqrt{2j_{12}+1}\sqrt{2l_{12}+1}\sqrt{2j_4+1}\sqrt{2l_4+1}\begin{pmatrix} j_a \\ m_a \end{pmatrix}^{(j_{12})}\begin{pmatrix} l_a \\ n_a \end{pmatrix}^{(l_{12})}$$
$$\chi(j_1,j_2,j_{12})\chi(l_1,l_2,l_{12})\chi(j_{12},j_3,j_4)\chi(l_{12},l_3,l_4) \tag{5.75}$$

Writing explicitly this integral is crucial to determine the Euclidean Booster function, as the latter can be obtained contracting the former with two $4jm$-symbols:

$$\sum_{m_a,n_a}\begin{pmatrix} j_a \\ m_a \end{pmatrix}^{(i)}\begin{pmatrix} l_a \\ n_a \end{pmatrix}^{(k)}\int dg \prod_{a=1}^{4} D^{p_a q_a}_{j_a m_a l_a n_a}(g) =$$
$$= \sum_{m'_a,m_a,n_a}\begin{pmatrix} j_a \\ m_a \end{pmatrix}^{(i)}\begin{pmatrix} l_a \\ n_a \end{pmatrix}^{(k)}\int_{SU(2)} du \prod_a D^{j_a}_{m_a m'_a}(u)\int_{SU(2)} dv \prod_a D^{l_a}_{m'_a n_a}(v^\dagger)\int_0^\pi d\mu(t)\prod_a d^{p_a q_a}_{j_a l_a m'_a}(t)$$
$$= \sum_{m'_a,m_a,n_a}\sum_{i',k'} d_{i'}d_{k'}\begin{pmatrix} j_a \\ m_a \end{pmatrix}^{(i)}\begin{pmatrix} l_a \\ n_a \end{pmatrix}^{(k)}\begin{pmatrix} j_a \\ m_a \end{pmatrix}^{(i')}\begin{pmatrix} l_a \\ n_a \end{pmatrix}^{(k')}\begin{pmatrix} j_a \\ m'_a \end{pmatrix}^{(i')}\begin{pmatrix} l_a \\ m'_a \end{pmatrix}^{(k')}\int_0^\pi d\mu(t)\prod_a d^{p_a q_a}_{j_a l_a m'_a}(t)$$
$$= \sum_{m'_a}\sum_{i',k'} d_{i'}d_{k'}\begin{pmatrix} j_a \\ m'_a \end{pmatrix}^{(i')}\begin{pmatrix} l_a \\ m'_a \end{pmatrix}^{(k')}\frac{\delta_{ii'}}{d_i}\frac{\delta_{kk'}}{d_k}\int_0^\pi d\mu(t)\prod_a d^{p_a q_a}_{j_a l_a m'_a}(t)$$
$$= \sum_{m'_a}\begin{pmatrix} j_a \\ m'_a \end{pmatrix}^{(i)}\begin{pmatrix} l_a \\ m'_a \end{pmatrix}^{(k)}\int_0^\pi d\mu(t)\prod_a d^{p_a q_a}_{j_a l_a m'_a}(t)$$
$$= B_4^E(j_a,l_a,i,k) \tag{5.76}$$

Confronting this result with (5.75) we obtain the following explicit formula for the Euclidean Booster function:

$$B_4^E(j_a, l_a, j_{12}, l_{12}) = \sum_{p_{12}, q_{12}} (p_{12}^2 - q_{12}^2)(-1)^{j_1-j_2+j_3-j_4}(-1)^{l_1-l_2+l_3-l_4} \frac{\sqrt{(2j_4+1)(2l_4+1)}}{\sqrt{(2j_{12}+1)(2l_{12}+1)}} \chi(j_1, j_2, j_{12})\chi(l_1, l_2, l_{12})\chi(j_{12}, j_3, j_4)\chi(l_{12}, l_3, l_4) \tag{5.77}$$

5.3.2 Computation of the Lorentzian Booster Function

The process to obtain an explicit formula for the Lorentzian Booster function follows the same logic of the previous paragraph, with a detailed computation given in [30]. Here we recall the main results, starting from the expansion of the canonical basis for the tensor product of two $SL(2,\mathbb{C})$ irreps:

$$|(\rho_1, k_1, j_1, m_1), (\rho_2, k_2, j_2, m_2)\rangle = \int d\rho \sum_{k,j,m} C^{\rho k j m}_{\rho_1 k_1 j_1 m_1, \rho_2 k_2 j_2 m_2} |\rho, k, j, m\rangle \tag{5.78}$$

The $SL(2,\mathbb{C})$ Clebsh-Gordan coefficients can be factorized in terms of $SU(2)$ ones:

$$C^{\rho k j m}_{\rho_1 k_1 j_1 m_1, \rho_2 k_2 j_2 m_2} = \langle j, m | j_1, m_1, l_1, n_1 \rangle \, \chi(j_1, j_2, j) \tag{5.79}$$

Where the $SL(2,\mathbb{C})$ χ-*function* is not defined in the same way of the $SO(4,\mathbb{R})$ one. To go forward we need the composition rule and orthogonality relation between the $SL(2,\mathbb{C})$ Wigner matrices, given by:

$$D^{(\rho_1,k_1)}_{j_1 m_1 l_1 n_1}(h) D^{(\rho_2,k_2)}_{j_2 m_2 l_2 n_2}(h) = 4\int_{-\infty}^{\infty} d\rho \sum_k (\rho^2+k^2) \sum_{j,l,m,n} \overline{C^{\rho k j m}_{\rho_1 k_1 j_1 m_1, \rho_2 k_2 j_2 m_2}} C^{\rho k l n}_{\rho_1 k_1 l_1 n_1, \rho_2 k_2 l_2 n_2} D^{(\rho,k)}_{jmln}(h) \tag{5.80}$$

And:

$$\int dh \overline{D^{(\rho_1,k_1)}_{j_1 m_1 l_1 n_1}(h)} D^{(\rho_2,k_2)}_{j_2 m_2 l_2 n_2}(h) = \frac{\delta(\rho_1-\rho_2)\delta_{k_1 k_2}}{4(\rho_1^2+k_1^2)} \delta_{j_1 j_2}\delta_{l_1 l_2}\delta_{m_1 m_2}\delta_{n_1 n_2} \tag{5.81}$$

Where:

$$\overline{D^{(\rho,k)}_{jmln}(h)} = (-1)^{j-m}(-1)^{l-n} D^{(\rho,k)}_{j-ml-n}(h) \tag{5.82}$$

From which one can compute the following integrals:

$$\int dh D^{(\rho_1,k_1)}_{j_1 m_1 l_1 n_1}(h) D^{(\rho_2,k_2)}_{j_2 m_2 l_2 n_2}(h) \overline{D^{(\rho_3,k_3)}_{j_3 m_3 l_3 n_3}(h)} = \langle j_3, m_3 | j_1, m_1, j_2, m_2 \rangle \langle l_3, n_3 | l_1, n_1, l_2, n_2 \rangle \bar{\chi}(j_1, j_2, j_3)\chi(l_1, l_2, l_3) \tag{5.83}$$

$$\int dh \prod_{a=1}^{3} D^{(\rho_a,k_a)}_{j_a m_a l_a n_a}(h) = (-1)^{j_1-j_2+j_3}(-1)^{l_1-l_2+l_3}\sqrt{2j_3+1}\sqrt{2l_3+1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \bar{\chi}(j_1, j_2, j_3)\chi(l_1, l_2, l_3) \tag{5.84}$$

$$\int dh \prod_{a=1}^{4} D^{(\rho_a,k_a)}_{j_a m_a l_a n_a}(h) = (-1)^{j_1-j_2+j_3-j_4}(-1)^{l_1-l_2+l_3-l_4}\sqrt{2j_3+1}\sqrt{2l_3+1}$$
$$4\int_{-\infty}^{\infty} d\rho_{12} \sum_{k_{12}}(\rho_{12}^2+k_{12}^2) \sum_{j_12,l_12} \sqrt{2j_{12}+1}\sqrt{2l_{12}+1} \begin{pmatrix} j_a \\ m_a \end{pmatrix}^{(j_{12})} \begin{pmatrix} l_a \\ n_a \end{pmatrix}^{(l_{12})}$$
$$\bar{\chi}(j_1,j_2,j_{12})\bar{\chi}(j_{12},j_3,j_4)\chi(l_1,l_2,l_{12})\chi(l_{12},l_3,l_4) \tag{5.85}$$

These results are sufficient to obtain the following explicit formula for the Lorentzian Booster function:

$$B_4^L(j_a,l_a,j_{12},l_{12}) = 4\int_{-\infty}^{\infty} d\rho_{12} \sum_{k_12}(\rho_{12}^2+k_{12}^2)(-1)^{j_1-j_2+j_3-j_4}(-1)^{l_1-l_2+l_3-l_4}\frac{\sqrt{(2j_4+1)(2l_4+1)}}{\sqrt{(2j_{12}+1)(2l_{12}+1)}}$$
$$\bar{\chi}(j_1,j_2,j_{12})\bar{\chi}(j_{12},j_3,j_4)\chi(l_1,l_2,l_{12})\chi(l_{12},l_3,l_4) \tag{5.86}$$

Confronting this result with (5.76) we immediately notice a correspondence, at least formal, between the two Booster functions. This reduces the problem to the difficult, yet possibly feasible, task of proving if and how the $SO(4,\mathbb{R})$ χ-functions, or equivalently, the $SU(2)$ $9j$-symbols can be analytically continued in order to match the $SL(2,\mathbb{C})$ χ-functions, defined as in [30]:

$$\chi(j_1,j_2,j_3) = \frac{(-1)^{\frac{K+J}{2}}}{4\sqrt{2\pi}} x(\rho_a,k_a)\Gamma\left(\frac{1-iP+K}{2}\right)\Gamma\left(\frac{1-iP-K}{2}\right)$$
$$\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}\kappa(\rho_a,k_a,j_a) \tag{5.87}$$

With:

$$J = \sum_a j_a \tag{5.88}$$
$$K = \sum_a k_a \tag{5.89}$$
$$P = \sum_a \rho_a \tag{5.90}$$
$$e^{i\Phi_j^\rho} = \frac{\Gamma(j+i\rho+1)}{|\Gamma(j+i\rho+1)|} \tag{5.91}$$

And:

$$
\begin{aligned}
x(\rho_a,k_a) = & \frac{\Gamma(\frac{1+iP-K}{2})}{\left|\Gamma(\frac{1+iP-K}{2})\right|}\frac{\Gamma(\frac{1-i(\rho_1+\rho_2+\rho_3)-k_1+k_2+k_3}{2})}{\left|\Gamma(\frac{1-i(\rho_1+\rho_2+\rho_3)-k_1+k_2+k_3}{2})\right|}\frac{\Gamma(\frac{1-i(\rho_1-\rho_2+\rho_3)+k_1-k_2+k_3}{2})}{\left|\Gamma(\frac{1-i(\rho_1-\rho_2+\rho_3)+k_1-k_2+k_3}{2})\right|} \\
& \frac{\Gamma(\frac{1-i(-\rho_1-\rho_2+\rho_3)-k_1-k_2+k_3}{2})}{\left|\Gamma(\frac{1-i(-\rho_1-\rho_2+\rho_3)-k_1-k_2+k_3}{2})\right|}
\end{aligned} \tag{5.92}
$$

$$
\begin{aligned}
\kappa(\rho_a,k_a,j_a) = & \frac{(-1)^{j_1-j_2+j_3-k_1-k_2}}{\sqrt{2j_3+1}}e^{-i(\Phi^{\rho_1}_{j_1}+\Phi^{\rho_2}_{j_2}-\Phi^{\rho_3}_{j_3})}\sqrt{\frac{(j_1-k_1)!(j_2+k_2)!}{(j_1+k_1)!(j_2-k_2)!}} \\
& \sum_{n=-j_1}^{\min[k_3+j_2]}\sum_{s_1=\max[k_1,n]}^{j_1}\sum_{s_2=\max[-k_2,n-k_3]}^{j_2}\sqrt{\frac{(j_1-n)!(j_2+k_3-n)!}{(j_1+n)!(j_2-k_3+n)!}} \\
& \begin{pmatrix} j_1 & j_2 & j_3 \\ n & k_3-n & -k_3 \end{pmatrix}\frac{(-1)^{s_1+s_2-k_1+k_2}(j_1+s_1)!(j_2+s_2)!}{(j_1-s_1)!(s_1-k_1)!(s_1-n)!(j_2-s_2)!(s_2+k_2)!(k_3-n+s_2)!} \\
& \frac{\Gamma(\frac{1-i(\rho_1-\rho_2-\rho_3)-K+2s_1}{2})\Gamma(\frac{1-i(\rho_1-\rho_2+\rho_3)+K+2s_2}{2})\Gamma(\frac{1-i(\rho_1+\rho_2-\rho_3)-k_1+k_2+k_3-2n+2s_1+2s_2}{2})}{\Gamma(1-i\rho_1+s_1)\Gamma(1-i\rho_2+s_2)\Gamma(1+i\rho_3+s_1+s_2)\Gamma(\frac{1-iP-k_1+k_2+k_3-2n}{2})}
\end{aligned} \tag{5.93}
$$

In these definitions one could spot some similarities with the combinatorial structure of the Wigner $9j$-symbols, such as the explicit presence of one $3jm$-symbol (out of six) and of three independent finite sums, which could suggest that there is in fact a correspondence between the two objects. However, this correspondence is yet to be found, leaving its search to future purposes. Now we take a few steps back and show how, with the results obtained in 5.2, it is possible to define an analytic continuation of the Euclidean Booster function, which can be shown to be proportional to the Lorentzian one.

5.3.3 Analytic Continuation of the Euclidean Booster Function

Before the imposition of the Linear Simplicity Constraint, both B_4^E and B_4^L can be defined as functions of the representation parameters attached to each face, such that:

$$
B_4^E(p_a,q_a,j_a,l_a,i,k) = \sum_{m_a}\begin{pmatrix} j_a \\ m_a \end{pmatrix}^{(i)}\begin{pmatrix} l_a \\ m_a \end{pmatrix}^{(k)}\int_0^\pi \prod_{a=1}^4 d^{p_a q_a}_{j_a l_a m_a}(t)d\mu(t) \tag{5.94}
$$

$$
B_4^L(\rho_a,k_a,j_a,l_a,i,k) = \sum_{m_a}\begin{pmatrix} j_a \\ m_a \end{pmatrix}^{(i)}\begin{pmatrix} l_a \\ m_a \end{pmatrix}^{(k)}\int_0^\infty \prod_{a=1}^4 d^{(\rho_a,k_a)}_{j_a l_a m_a}(r)d\mu(r) \tag{5.95}
$$

With:

$$
d\mu(t) = \frac{2}{\pi}\sin^2(t)dt \tag{5.96}
$$

$$
d\mu(r) = \frac{2}{\pi}\sinh^2(r)dr \tag{5.97}
$$

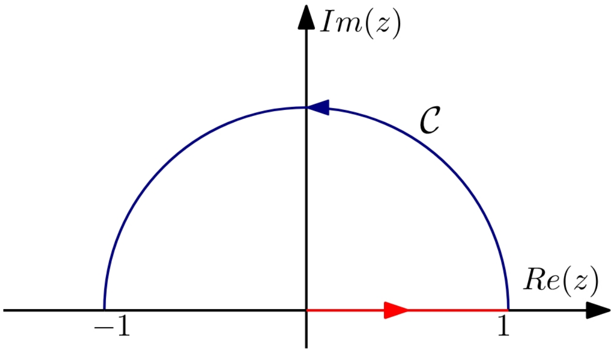

Figure 5.1. Integration paths for $z = e^{it}$ (blue) and $z = e^{-r}$ (red)

It is possible to formulate the integrals as complex integrals on some integration path. First, we focus on the measures. For $z = e^{it}$ one has:

$$\sin^2(t) = -\frac{(z^2-1)^2}{4z^2} \tag{5.98}$$

$$dt = -\frac{i}{z}dz \tag{5.99}$$

From which follows:

$$d\mu(t) = \frac{i}{2\pi}\frac{(z^2-1)^2}{z^3}dz \equiv d\mu(z) \tag{5.100}$$

So that:

$$\int_0^\pi d\mu(t) = \int_{\mathcal{C}} d\mu(z) \tag{5.101}$$

Where $\mathcal{C}$ denotes the unitary semicircle in the upper complex semi-plane, as shown in Figure 5.1.
Similarly, from $z = e^{-r}$ we obtain:

$$\sinh^2(r) = \frac{(z^2-1)^2}{4z^2} \tag{5.102}$$

$$dr = -\frac{1}{z}dz \tag{5.103}$$

Followed by:

$$d\mu(r) = -\frac{1}{2\pi}\frac{(z^2-1)^2}{z^3}dz = i d\mu(z) \tag{5.104}$$

And:

$$\int_0^\infty d\mu(r) = -i\int_0^1 d\mu(z) \tag{5.105}$$

To avoid confusion with the reduced representation matrices of the two models we now define a class of auxiliary functions:

$$\tilde{d}^{\alpha,\beta}_{jlm}(z) \equiv (-1)^{j-l} \frac{\sqrt{(\alpha-j-1)!\,(j+\alpha)!}}{\sqrt{(\alpha-l-1)!\,(l+\alpha)!}} \frac{\sqrt{(2j+1)(2l+1)}}{(j+l+1)!} z^{-(\alpha-\beta-m-1)}$$
$$\sqrt{(j+\beta)!(j-\beta)!(j+m)!(j-m)!(l+\beta)!(l-\beta)!(l+m)!(l-m)!}$$
$$\sum_{s,s'} z^{2s'} \frac{(-1)^{s+s'}(\beta+s+m+s')!(j+l-\beta-m-s-s')!}{s!s'!(j-\beta-s)!(j-m-s)!(\beta+m+s)!(l-\beta-s')!(l-m-s')!(\beta+m+s')!}$$
$${}_2F_1\left[\{l-\alpha+1,\beta+m+s+s'+1\},\{j+l+2\};1-z^2\right] \tag{5.106}$$

These functions are are well defined for half-integer values of β, j, l, m and either for half-integer or purely imaginary values of α. In particular from (5.23) and (5.53) we have that, for $\alpha = i\rho \in i\mathbb{R}$, $\beta = k \in \frac{\mathbb{N}}{2}$, $z = e^{-r}$ one recovers:

$$\tilde{d}^{i\rho,k}_{jlm}(z) = d^{(\rho,k)}_{jlm}(r) \tag{5.107}$$

While for $\alpha = p \in \frac{\mathbb{N}}{2}$, $\beta = q \in \frac{\mathbb{N}}{2}$, $z = e^{it}$:

$$\tilde{d}^{p,q}_{jlm}(z) = d^{pq}_{jlm}(t) \tag{5.108}$$

I.e. the reduced representation matrices respectively of $SL(2,\mathbb{C})$ and $SO(4,\mathbb{R})$. Furthermore, they satisfy the following rule under parity transformation:

$$\tilde{d}^{\alpha,\beta}_{jlm}(-z) = (-1)^{(\beta-\alpha+m)}\tilde{d}^{\alpha,\beta}_{jlm}(z) \tag{5.109}$$

We now define a new set of functions, given by:

$$\mathcal{D}^{\alpha_a,\beta_a}_{j_a l_a;ik}(z) \equiv \sum_{m_a} \begin{pmatrix} j_a \\ m_a \end{pmatrix}^{(i)} \begin{pmatrix} l_a \\ m_a \end{pmatrix}^{(k)} \prod_{a=1}^{4} \tilde{d}^{\alpha_a,\beta_a}_{j_a l_a m_a}(z) \tag{5.110}$$

Which allow to write both the Euclidean and Lorentzian Booster functions compactly as:

$$B_4^E(p_a,q_a,j_a,l_a,i,k) = \int_{\mathcal{C}} \mathcal{D}^{p_a,q_a}_{j_a l_a;ik}(z)d\mu(z) \tag{5.111}$$
$$B_4^L(\rho_a,k_a,j_a,l_a,i,k) = -i\int_0^1 \mathcal{D}^{i\rho_a,k_a}_{j_a l_a;ik}(z)d\mu(z) \tag{5.112}$$

We can use this redefinition at our advantage. The first thing we notice is that, expressed in term of the $\mathcal{D}$-functions just defined, the analytic continuation of the Euclidean Booster function under our map is well defined, as it is given by:

$$B_4^E(i\rho_a,k_a,j_a,l_a,i,k) = \int_{\mathcal{C}} \mathcal{D}^{i\rho_a,k_a}_{j_a l_a;ik}(z)d\mu(z) \tag{5.113}$$

Then, we can manipulate the integral appearing in the Lorentzian Booster function. We achieve this by defining the following path on the complex plane:

$$\mathcal{C}_T \equiv \mathcal{C} \cup [-1,1] \equiv \mathcal{C} \cup \mathcal{I} \tag{5.114}$$

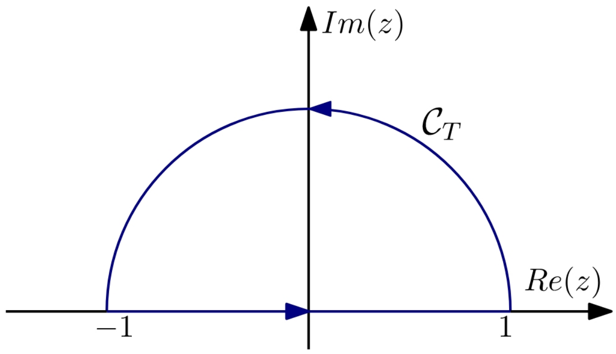

Figure 5.2

This path, shown in Figure 5.2, encloses a region of the complex plane inside which the $\mathcal{D}$-functions, computed with $\alpha \in i\mathbb{R}$, have no residues. Thus, applying Cauchy theorem we obtain (omitting upper and lower indices for brevity):

$$\int_{\mathcal{C}_T} \mathcal{D}(z) d\mu(z) = 0 \tag{5.115}$$

Which implies:

$$\int_{\mathcal{C}} \mathcal{D}(z) d\mu(z) = -\int_{\mathcal{I}} \mathcal{D}(z) d\mu(z) \tag{5.116}$$

Moreover, from the properties of the $\tilde{d}$-functions earlier defined, it follows that:

$$\mathcal{D}(-z) = (-1)^{\sum_a (\beta_a - \alpha_a)} \mathcal{D}(z) \tag{5.117}$$

As the sum of the magnetic indices vanishes due to the properties of the $4jm$-symbols. Now, given that $d\mu(-z) = d\mu(z)$, one has:

$$\int_{-1}^{0} \mathcal{D}(z) d\mu(z) = \int_{1}^{0} \mathcal{D}(-z) d\mu(-z) = -(-1)^{\sum_a (\beta_a - \alpha_a)} \int_{0}^{1} \mathcal{D}(z) d\mu(z) \tag{5.118}$$

We can use such properties to rewrite B_4^L as:

$$B_4^L(\rho_a, k_a, j_a, l_a, i, k) = -\frac{i}{1 - (-1)^{\sum_a (k_a - i\rho_a)}} \int_{\mathcal{I}} \mathcal{D}^{i\rho_a, k_a}_{j_a, l_a; ik}(z) d\mu(z) \tag{5.119}$$

At last, from (5.113) and (5.116) we recover:

$$B_4^L(\rho_a, k_a, j_a, l_a, i, k) = \frac{i}{1 - (-1)^{\sum_a (k_a - i\rho_a)}} B_4^E(i\rho_a, k_a, j_a, l_a, i, k) \tag{5.120}$$

This proves that, under the right prescriptions, the Euclidean Booster function can be analytically continued through our map in such a way the Lorentzian Booster function becomes proportional to it. The proportionality term, once the Linear Simplicity Constraint is imposed, depends only on the spins j_f attached to the faces, so it factors out, leading to a proportionality between the EPRL Vertex Amplitudes of the two models. This result could be naively inverted, defining the analytic continuation of the Lorentzian Booster function as $B_4^L(-ip_a, q_a, j_a, l_a, i, k)$ obtaining:

$$B_4^E(p_a, q_a, j_a, l_a, i, k) = \frac{1-(-1)^{\sum_a (q_a - p_a)}}{i} B_4^L(-ip_a, q_a, j_a, l_a, i, k) \tag{5.121}$$

However, there are at least two (solvable) issues with this inversion. The first thing one notices of this last relation is that the proportionality factor could vanish for certain combinations of the parameters q_a and p_a. Another problem with this definition is that, for integer values of α and β, the $\tilde{d}$-functions diverge for $z = 0$. To properly invert the map from the Euclidean Booster to the Lorentzian one it is then needed to define a regularization of the integral with some arbitrary prescription *"à la Feynmann"*.

Chapter 6

Conclusions and Further Perspectives

Spin Foam theory constitutes the edge of research in the Loop Quantum Gravity framework. It provides a Lorentz-covariant and background-independent rigorous way to compute transition amplitudes between LQG states. The two signatures in which the theory is formulated, presented in Chapters 2 and 3, despite being built on fundamentally different gauge structures, lead to transition amplitudes which share a formally identical expression, differing only for the definition of the Vertex Amplitude. Moreover, defining the two auxiliary parameters p and q in the Euclidean case, it is possible to build it in such a way the resemblance to the Lorentzian one is even more evident, as in the decomposition of the Hilbert space, as in the form taken by the Linear Simplicity Constraint relations.

The main motivation behind this thesis was the curiosity about how much this resemblance could be exploited to relate the results obtained in the two models, and how far the parallelism between them could be carried forward. In Chapter 5 we showed how, through a set of prescriptions, namely the analytical continuation of the algebra's generators, Casimirs' eigenvalues and boost parameters of the group elements, the gauge structure of the Euclidean model can be mapped into the Lorentzian one. Moreover, we proved an equivalence between the reduced representation matrices of $SO(4,\mathbb{R})$ and $SL(2,\mathbb{C})$, that is summarized in equation (5.53). This equivalence allowed us to define the auxiliary $\tilde{d}$-functions, which were later used to construct the analytic continuation of the Booster functions.

The main result of the thesis is then summarized by equation (5.120), that shows the proportionality between the Lorentzian Booster function and the analytically continued Euclidean one. Since the resulting factor depends only on the representation indices, this result can be extended to the Vertex Amplitudes, which, under our map, become proportional one to each other. Further research is needed to regularize the procedure for the inverse mapping and verify the expected result, presented in equation (5.121). Moreover, a consistency check for this results could be provided by finding an explicit correspondence, under analytic continuation, between the Euclidean and the Lorentzian χ-functions, defined respectively by (5.67) and (5.87).

As we discussed in Chapter 5, despite the Lorentzian signature describes the physical model of the theory, some of the current results achieved in Spin Foam are obtained

only in the Euclidean signature. This is motivated by the fact that computations in the Euclidean model are easier to perform and sometimes are the only possibility. The results presented in this work suggest a way to generalize, at least in a qualitative way, these achievements into the Lorentzian framework. For example, the proportionality between the Euclidean and Lorentzian Vertex Amplitudes could be implemented into the many-vertex analysis of the classical limit, in the derivation of the n-point correlation function, or in cosmological arguments, providing a more efficient procedure to approach numerical computations.
As we mentioned in the introduction to this thesis, the goal of Quantum Gravity is far to be achieved, yet progress is being made by the research community. We hope that this work, and its further developments, may be a source of inspiration for members of our community, providing a small contribution to such progress.

Appendix A

$SU(2)$ Recoupling Theory

Here we summarize all the aspects in the recoupling theory of $SU(2)$ which have been relevant in our discussion. The main source for this appendix is [23].

$SU(2)$ Wigner Matrix

The representation matrices of any unitary irreducible representation of $SU(2)$ are called *Wigner matrices.* In the basis $|j,m\rangle$ the generic element is given by:

$$D^j_{mn}(g) = \langle j,m|g|j,n\rangle \tag{A.1}$$

Which satisfies the orthogonality relation:

$$\int_{SU(2)} dg\overline{D^j_{mn}(g)}D^{j'}_{m'n'} = \frac{1}{2j+1}\delta_{jj'}\delta_{mm'}\delta_{nn'} \tag{A.2}$$

And, among others, the symmetry property:

$$D^j_{mn}(g^\dagger) = \overline{D^j_{nm}(g)} = (-1)^{n-m}D^j_{-n,-m}(g) \tag{A.3}$$

Clebsh-Gordan Coefficients

In the combination of two $SU(2)$ irreps, the recoupling basis elements $|j_1,j_2,j,m\rangle$ are obtained starting from the tensor product basis $|j_1,m_1,j_2,m_2\rangle$ through the relation:

$$|j_1,j_2,j,m\rangle = \sum_{m_1,m_2} C^{jm}_{j_1m_1j_2m_2}\,|j_1,m_1,j_2,m_2\rangle \tag{A.4}$$

Where:

$$C^{jm}_{j_1m_1j_2m_2} \equiv \langle j_1,m_1,j_2,m_2|j_1,j_2,j,m\rangle \tag{A.5}$$

Are the *Clebsh-Gordan coefficients.* It is possible to demonstrate that $C^{jm}_{j_1m_1j_2m_2} \in \mathbb{R}$, moreover, they are non-zero if and only if:

$$|j_1-j_2| \leq j \leq j_1+j_2 \tag{A.6}$$

$$m = m_1+m_2 \tag{A.7}$$

The Clebsh-Gordan coefficients satisfy an orthogonality relation given by:

$$\sum_{m_1,m_2} C^{jm}_{j_1m_1j_2m_2} C^{j'm'}_{j_1m_1j_2m_2} = \delta_{jj'}\delta_{mm'} \tag{A.8}$$

There are many explicit expressions for the Clebsh-Gordan coefficients.
In 5.2 we have used Van Der Waerden's Formula [35]:

$$C^{jm}_{j_1m_1j_2m_2} = \delta_{m,m_1+m_2}\sqrt{2j+1}\sqrt{\frac{(j_1+j_2-j)!(j_1-j_2+j)!(-j_1+j_2+j)!}{(j_1+j_2+j+1)!}}$$
$$\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j+m)!(j-m)!}$$
$$\sum_t (-1)^t \frac{1}{t!(j_1+j_2-j-t)!(j_1-m_1-t)!(j_2+m_2-t)!(j-j_2+m_1+t)!(j-j_1-m_2+t)!} \tag{A.9}$$

Where the range of summation is given by the existence conditions of the factorials. This expression can be also used to define an analytic continuation of the Clebsh-Gordan coefficients, with complex spins involved [26]. It is also possible to show the following relation involving the product of two Wigner matrix elements:

$$D^{j_1}_{m_1n_1}(g)D^{j_2}_{m_2n_2}(g) = \sum_{j\in\mathbb{N}/2}\sum_{m,m'} C^{jm}_{j_1m_1j_2m_2} C^{jm'}_{j_1n_1j_2n_2} D^j_{mm'}(g) \tag{A.10}$$

$3jm$-Symbol

Given the tensor product of three $SU(2)$ irreps $\mathcal{H}_{j_1}\otimes\mathcal{H}_{j_2}\otimes\mathcal{H}_{j_3}$ its $SU(2)$-invariant subspace is not empty only if the triangular relation $|j_1-j_2|\leq j_3\leq j_1+j_2$ is satisfied. In that case, it is one-dimensional and is generated by the unit vector:

$$|0\rangle = \sum_{m_1,m_2,m_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix}\bigotimes_{k=1}^{3}|j_k,n_k\rangle \tag{A.11}$$

Where:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix} \equiv \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} C^{j_3,-m_3}_{j_1m_1j_2m_2} \tag{A.12}$$

Are called the *Wigner $3jm$-symbols*.
From their relation with the Clebsh-Gordan coefficients it is evident that they are real numbers an non-zero only if $m_1+m_2+m_3=0$, moreover from the triangular relation it easily follows that the sum of the spins is always an integer.
Furthermore, they have a clockwise cyclic symmetry:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2\end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1\end{pmatrix} \tag{A.13}$$

And two symmetries involving a phase factor:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix} = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3\end{pmatrix} \tag{A.14}$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix} = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3\end{pmatrix} \tag{A.15}$$

The orthogonality relations between the Clebsh-Gordan coefficients imply two similar relations between the $3jm$-symbols, given by:

$$\sum_{j,m}(2j+1)\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{pmatrix} = \delta_{m_1 m_1'}\delta_{m_2 m_2'} \tag{A.16}$$

$$\sum_{m_1,m_2}(2j+1)\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{pmatrix} = \delta_{jj'}\delta_{mm'} \tag{A.17}$$

The projector over the $SU(2)$ invariant subspace of the tensor product of any number of representations is obtained by integrating the product of the same number of Wigner matrices. On the other hand, once one has the explicit expression of a basis of this subspace, the projector can be computed as any projector on an Hilbert space, hence it follows the equivalence:

$$\int_{SU(2)} dg \bigotimes_{k=1}^{n} D^{j_k}(g) = \sum_j |j\rangle\langle j| \tag{A.18}$$

In the case of the recoupling of three spins the invariant subspace is one-dimensional and it follows that:

$$\int_{SU(2)} dg \bigotimes_{k=1}^{3} D^{j_k}(g) = |0\rangle\langle 0| \tag{A.19}$$

Taking the matrix elements of both sides in the magnetic basis it is straightforward the following result:

$$\int_{SU(2)} dg D^{j_1}_{m_1 n_1}(g) D^{j_2}_{m_2 n_2}(g) D^{j_3}_{m_3 n_3}(g) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \tag{A.20}$$

$4jm$-Symbol

In the case of the tensor product of four $SU(2)$ irreps, the $SU(2)$ invariant subspace is not empty only if the sum of the four spins takes integer values. In that case, its dimension depends on the values of the spins involved and there is a non-unique choice to construct its basis. An orthonormal basis is given by the states:

$$|i\rangle_{12} = \sum_{m_1,m_2,m_3,m_4} \sqrt{2i+1}\begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(i)} \bigotimes_{k=1}^{4} |j_k, m_k\rangle \tag{A.21}$$

With: $i \in [Min(|j_1 - j_2|, |j_3 - j_4|)], ..., Max(j_1 + j_2, j_3 + j_4)]$.

The coefficients appearing in the definition of the basis element are called the *Wigner $4jm$-symbols*, and are defined as:

$$\begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(i)} \equiv \sum_{m=-i}^{i} (-1)^{i-m}\begin{pmatrix} j_1 & j_2 & i \\ m_1 & m_2 & m \end{pmatrix}\begin{pmatrix} i & j_3 & j_4 \\ -m & m_3 & m_4 \end{pmatrix} \tag{A.22}$$

For such objects it holds an orthogonality relation given by:

$$\sum_{m_1,m_2,m_3} \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(i)} \begin{pmatrix} j_1 & j_2 & j_3 & j'_4 \\ m_1 & m_2 & m_3 & m'_4 \end{pmatrix}^{(i')} = \frac{\delta_{ii'}}{d_i} \frac{\delta_{j_4 j'_4} \delta m_4 m'_4}{d_{j_4}} \tag{A.23}$$

From which follows that:

$$\sum_{m_1,m_2,m_3,m_4} \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(i)} \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(i')} = \frac{\delta_{ii'}}{d_i} \tag{A.24}$$

In our discussion we adopted for the $4jm$-symbol the same compact notation presented in [30]:

$$\begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(i)} = \begin{pmatrix} j_i \\ m_i \end{pmatrix}^{(i)} \tag{A.25}$$

Applying (A.18) we obtain that the integration over the Haar measure of four Wigner matrices can be expressed in term of $4jm$-symbols as:

$$\int_{SU(2)} du D^{j_1}_{m_1 n_1}(u) D^{j_2}_{m_2 n_2}(u) D^{j_3}_{m_3 n_3}(u) D^{j_4}_{m_4 n_4}(u) = \sum_j (2j+1) \begin{pmatrix} j_i \\ m_i \end{pmatrix}^{(i)} \begin{pmatrix} j_i \\ n_i \end{pmatrix}^{(i)} \tag{A.26}$$

This holds also for $D^{j_i}_{m_i n_i}(u^\dagger)$, due properties of the Haar measure of $SU(2)$.

$6j$-Symbol

However its dimension is fixed by the four spins involved, the basis of the $SU(2)$-invariant subspace of $\bigotimes_{k=1}^4 \mathcal{H}_{j_k}$ described in the previous section is not unique. Other basis can be built contracting the spins in a different order, for instance:

$$|i'\rangle_{23} = \sum_{m_1,m_2,m_3,m_4} \begin{pmatrix} j_4 & j_1 & j_2 & j_3 \\ m_4 & m_1 & m_2 & m_3 \end{pmatrix}^{(i')} \prod_{k=1}^{4} |j_k, m_k\rangle \tag{A.27}$$

The coefficients of the basis transformation matrix are given by:

$$_{12}\langle i|i'\rangle_{23} = \sqrt{2i+1}\sqrt{2i'+1}(-1)^{j_1+j_2+j_3-j_4-2i-2i'} \begin{Bmatrix} j_1 & j_2 & i \\ j_3 & j_4 & i' \end{Bmatrix} \tag{A.28}$$

Where:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \equiv \sum_{m_k} (-1)^{\sum_k (j_k - m_k)} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_5 & j_6 \\ m_1 & -m_5 & m_6 \end{pmatrix} \begin{pmatrix} j_4 & j_2 & j_6 \\ m_4 & m_2 & -m_6 \end{pmatrix} \begin{pmatrix} j_3 & j_4 & j_5 \\ m_3 & -m_4 & m_5 \end{pmatrix} \tag{A.29}$$

Is the definition of the *Wigner $6j$-symbol*. Its relevance in LQG is due to its appearance in the Ponzano-Regge semiclassical limit of General Relativity and in the 3-dimensional $SU(2)$ BF theory. Three notable symmetries of such symbol are given by:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_2 & j_1 & j_3 \\ j_5 & j_4 & j_6 \end{Bmatrix} = \begin{Bmatrix} j_3 & j_2 & j_1 \\ j_6 & j_5 & j_4 \end{Bmatrix} = \begin{Bmatrix} j_4 & j_2 & j_3 \\ j_1 & j_5 & j_6 \end{Bmatrix} \tag{A.30}$$

$9j$-Symbol

Differently from the $3jm$ and $4jm$, the $6j$-symbol does not have magnetic indices attached to its spins as they are all summed over in its definition. It is the first of a numerous family of invariant objects of $SU(2)$. It is followed by the *Wigner $9j$-symbol*, which can be defined as:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix} = \sum_{m_i} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_6 \\ m_4 & m_5 & m_6 \end{pmatrix} \begin{pmatrix} j_7 & j_8 & j_9 \\ m_7 & m_8 & m_9 \end{pmatrix} \begin{pmatrix} j_1 & j_4 & j_7 \\ m_1 & m_4 & m_7 \end{pmatrix} \begin{pmatrix} j_2 & j_5 & j_8 \\ m_2 & m_5 & m_8 \end{pmatrix} \begin{pmatrix} j_3 & j_6 & j_9 \\ m_3 & m_6 & m_9 \end{pmatrix}$$

It is possible to write it in a more explicit way as a sum of products between Clebsh-Gordan coefficients, such that:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix} = \frac{1}{\sqrt{(2j_3+1)(2j_6+1)(2j_7+1)(2j_8+1)}} \frac{1}{2j_9+1} \sum_{m_i} C^{j_3 m_3}_{j_1 m_1 j_2 m_2} C^{j_6 m_6}_{j_4 m_4 j_5 m_5} C^{j_9 m_9}_{j_7 m_7 j_8 m_8} C^{j_7 m_7}_{j_1 m_1 j_4 m_4} C^{j_8 m_8}_{j_2 m_2 j_5 m_5} C^{j_9 m_9}_{j_3 m_3 j_6 m_6} \tag{A.31}$$

This formula has been useful to define the coefficients of the Clebsh-Gordan series of $SO(4)$.

$15j$-Symbol

Another invariant object of $SU(2)$ is the *Wigner $15j$-symbol*. There are five different families of $15j$-symbols, all of which can be defined in more than one equivalent way. The relevant object of the LQG is the $15j$-symbols of the first kind [35], which can be expressed as the contraction over their magnetic indices of the product of five $4jm$-symbols:

$$\{15j\}(i_k, j_a) \equiv \sum_{m_a} \begin{pmatrix} j_1 & j_2 & j_3 & j_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}^{(i_1)} \begin{pmatrix} j_1 & j_5 & j_6 & j_7 \\ m_1 & m_5 & m_6 & m_7 \end{pmatrix}^{(i_2)} \begin{pmatrix} j_7 & j_2 & j_8 & j_9 \\ m_7 & m_2 & m_8 & m_9 \end{pmatrix}^{(i_3)} \begin{pmatrix} j_9 & j_6 & j_3 & j_{10} \\ m_9 & m_6 & m_3 & m_{10} \end{pmatrix}^{(i_4)} \begin{pmatrix} j_{10} & j_8 & j_5 & j_4 \\ m_{10} & m_8 & m_5 & m_4 \end{pmatrix}^{(i_5)} \tag{A.32}$$

These objects appear in the computation of the Vertex amplitude of both the Euclidean and the Lorentzian EPRL model.

Appendix B

Complex Combinatorial Objects

This second appendix summarizes the mathematical objects and relations which have been used extensively in 5.2.

Gamma Function and Pochammer Symbols

The Gamma function $\Gamma(z)$ is a mathematical object defined as the analytic continuation of the factorial from $\mathbb{N}$ to the complex plane. Its most know property is its relation with the factorial when its argument is a positive integer:

$$\Gamma(n) = (n-1)! \tag{B.1}$$

$\Gamma(z)$ is analytic everywhere apart from 0 and the negative integers.
An useful identity satisfied by the Gamma function is:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{B.2}$$

This relation is not well defined for $z \in \mathbb{N}$, and for such values both sides of the equation are divergent. However, this divergence can be "cured" if, for example, it appears on both sides of a fraction. As a consequence, one could rephrase it as the "soft" identity:

$$\Gamma(n) = \frac{\epsilon(-1)^{1-n}}{\Gamma(1-n)} \tag{B.3}$$

Where ϵ is a divergent term ensuring the finiteness of $\Gamma(n)$. This is particularly useful in order to define in a weak sense the factorial of a negative integer. In fact, switching the position of the Gamma functions and expressing them as factorials one obtains:

$$(-n)! = \frac{\epsilon(-1)^{1-n}}{(n-1)!} \tag{B.4}$$

In our discussion we used multiple times this identity, ensuring each time to reabsorb the divergent term between numerators and denominators. The Gamma function satisfies another identity related to its modulus, which can be expressed as:

$$|\Gamma(a+ib)| = \sqrt{\Gamma(a+ib)\Gamma(a-ib)} \tag{B.5}$$

Another combinatorial object directly related to the factorials and the Gamma functions is the *Pochammer symbol*, often referred to as the *rising factorial*, defined as:

$$(a)_n = \prod_{k=0}^{n-1}(a+k) = \frac{(a+n-1)!}{(a-1)!} \tag{B.6}$$

It can also be expressed in terms of Gamma functions:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{B.7}$$

This allows to generalize the Pochammer symbol to complex arguments, and, through (B.4), obtain the identities:

$$\begin{aligned}(-a)_n &= (-1)^n(a-n+1)_n \\ &= (-1)^n\frac{\Gamma(a+1)}{\Gamma(a+1-n)} \\ &= (-1)^n\frac{a!}{(a-n)!}\end{aligned} \tag{B.8}$$

Hypergeometric Function

One of the practical uses for the Pochammer symbols is in the definition of the *Hypergeometric function*, a complex function involved in the definition of the reduced $SL(2,\mathbb{C})$ Wigner matrix and defined by three parameters, given by:

$${}_2F_1\left[\{a,b\},\{c\};z\right] \equiv \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \tag{B.9}$$

The Hypergeometric function is analytic in z on the entire complex plane, and it is evidently symmetric in respect of the exchange $a \leftrightarrow b$. Among the numerous identities it satisfies, the following two have been useful in our discussion:

$${}_2F_1\left[\{a_1,a_2\},\{b_1\};z\right] = (1-z)^{-a_2}\,{}_2F_1\left[\{b_1-a_1,a_2\},\{b_1\};\frac{z}{z-1}\right] \tag{B.10}$$

$$\begin{aligned}{}_2F_1\left[\{a_1,a_2\},\{b_1\};z\right] = &\frac{\Gamma(b_1)\Gamma(b_1-a_1-a_2)}{\Gamma(b_1-a_1)\Gamma(b_1-a_2)}\,{}_2F_1\left[\{a_1,a_2\},\{a_1+a_2-b_1+1\};1-z\right] \\ &+\frac{\Gamma(b_1)\Gamma(a_1+a_2-b_1)}{\Gamma(a_1)\Gamma(a_2)}(1-z)^{b_1-a_1-a_2} \\ &{}_2F_1\left[\{b_1-a_1,b_1-a_2\},\{b_1-a_1-a_2+1\};1-z\right]\end{aligned} \tag{B.11}$$

Moreover, applying them in sequence, one obtains the relation:

$$\begin{aligned}{}_2F_1\left[\{a_1,a_2\},\{b_1\};z\right] = &\frac{\Gamma(b_1)\Gamma(a_1-a_2)}{\Gamma(a_1)\Gamma(b_1-a_2)}(1-z)^{-a_2}\,{}_2F_1\left[\{b_1-a_1,a_2\},\{a_2-a_1+1\};\frac{1}{1-z}\right] \\ &+\frac{\Gamma(b_1)\Gamma(a_2-a_1)}{\Gamma(a_2)\Gamma(b_1-a_1)}(1-z)^{-a_1}\,{}_2F_1\left[\{b_1-a_2,a_1\},\{a_1-a_2+1\};\frac{1}{1-z}\right]\end{aligned} \tag{B.12}$$

Generalized Hypergeometric Function

The concept of Hypergeometric function can be extended to an arbitrary number of parameters, there exists a virtually infinite number of classes of *Generalized Hypergeometric functions*:

$$ {}_pF_q\left[\{a_1,...,a_p\},\{b_1,...,b_q\};z\right] \equiv \sum_{n=0}^{\infty} \frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n}\frac{z^n}{n!} \tag{B.13}$$

The Generalized Hypergeometric functions of any class are analytic in z on the entire complex plane, and symmetric in respect of any permutation of the a_p or b_q indices. Such functions appear in multiple fields of Mathematics.
In our discussion we encountered the class:

$$ {}_3F_2\left[\{a_1,a_2,a_3\},\{b_1,b_2\};z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n(a_3)_n}{(b_1)_n(b_2)_n}\frac{z^n}{n!} \tag{B.14}$$

As it, computed for $z=1$, can be used to expand the Clebsh-Gordan coefficients. There is plenty of identities involving the function ${}_3F_2\left[\{a_1,a_2,a_3\},\{b_1,b_2\};1\right]$, two of them have been crucial to obtain our proof in 5.2, and are:

$$\begin{aligned} {}_3F_2\left[\{a_1,a_2,a_3\},\{b_1,b_2\};1\right] = &\frac{\Gamma(b_2)\Gamma(a_1-b_1+1)\Gamma(a_2-b_1+1)\Gamma(a_3-b_1+1)}{\Gamma(a_3)\Gamma(1-b_1)\Gamma(a_1+a_2-b_1+1)\Gamma(1-b_1+b_2)} \\ &{}_3F_2\left[\{a_2-b_1+1,a_1-b_1+1,b_2-a_3\},\{a_1+a_2-b_1+1,b_2-b_1+1\};1\right] \\ &-\frac{\Gamma(b_2)\Gamma(a_1-b_1+1)\Gamma(a_2-b_1+1)\Gamma(a_3-b_1+1)\Gamma(b_1-1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(1-b_1)\Gamma(b_2-b_1+1)} \\ &{}_3F_2\left[\{a_1-b_1+1,a_2-b_1+1,a_3-b_1+1\},\{2-b_1,b_2-b_1+1\};1\right] \end{aligned} \tag{B.15}$$

With an existence condition given by:
$Re(a_3) > 0 \wedge Re(b_1+b_2-a_1-a_2-a_3) > 0$.
And:

$$\begin{aligned} {}_3F_2\left[\{a_1,a_2,a_3\},\{b_1,b_2\};1\right] = &\frac{\Gamma(a_1-b_1+1)\Gamma(a_3-b_1+1)}{\Gamma(1-b_1)\Gamma(a_1+a_3-b_1+1)} \\ &{}_3F_2\left[\{a_1,b_2-a_2,a_3\},\{a_1+a_3-b_1+1,b_2\};1\right] \\ &+\frac{\Gamma(a_1-b_1+1)\Gamma(a_2-b_1+1)\Gamma(a_3-b_1+1)\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(2-b_1)\Gamma(1-b_1+b_2)} \\ &{}_3F_2\left[\{a_1-b_1+1,a_2-b_1+1,a_3-b_1+1\},\{2-b_1,1-b_1+b_2\};1\right] \end{aligned} \tag{B.16}$$

With an existence condition given by:
$Re(a_2-b_1+1) > 0 \wedge Re(b_1+b_2-a_1-a_2-a_3) > 0$

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