Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
Corso di Laurea Magistrale in Matematica

ON THE LOG-CONCAVITY OF THE CHARACTERISTIC POLYNOMIAL OF A MATROID

Tesi di Laurea Magistrale in Matematica

Relatore:
Chiar.mo Prof.
Luca Moci

Correlatore:
Chiar.mo Prof.
Luca Migliorini

Presentata da:
Vecchi Lorenzo

Sessione unica
Anno Accademico 2019/2020

Summary

In this dissertation we address a long-standing conjecture, due to Heron, Rota and Welsh on the log-concavity of the characteristic polynomial of a matroid.

After decades of attempts and a series of partial results, the conjecture was fully solved in 2018 by Adiprasito, Huh and Katz, using combinatorial analogues of several results in Algebraic Geometry concerning a particular cohomology ring called Chow ring. In February 2020, a new, simpler proof was announced by Braden, Huh, Matherne, Proudfoot and Wang. This dissertation is conceived to be a self-contained guide to support the reader in understanding these two papers, providing also the necessary background, a wide horizon ranging from Hodge Theory to Combinatorics to Toric Geometry. Moreover, we provide concrete and nontrivial examples of computations of Chow rings, of which we feel current literature is still lacking.

Matroid Theory has its roots in the 1935 article *On the abstract properties of linear independence* by Whitney. Since then, matroids have been widely used in Graph Theory, Coding Theory and Optimization: for example, matroids can be used to solve some problems concerning duality in graphs; they also describe optimization problems on which greedy algorithms are proved to be optimal. Due to the surprisingly wide variety of situations that can be modelled using a matroid structure, this theory was considered for long time a branch of Applied Mathematics.
In the last years, a new generation of mathematicians revolutionized this perspective, discovering a deep and surprising interplay of Matroid Theory, Algebraic Topology and Algebraic Geometry. A milestone in this process was the paper *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs*, published in 2012 by June Huh, in which the Hodge-Riemann relations for the De Concini-Procesi wonderful model were used to obtain the log-concavity relations for matroids realizable on fields of characteristic zero. The following step was represented by the 2018 article *Hodge Theory for combinatorial geometries* in which Adiprasito, Huh and Katz gave a full proof of the conjecture based on an elaborate inductive argument that let them prove the combinatorial Hard Lefschetz Theorem and Hodge-Riemann bilinear relations. The above-mentioned 2020 preprint, *A semi-small decomposition of the Chow ring of a matroid*, mentioned above, gives a proof of the previous results inspired by a decomposition of the Chow ring induced by semi-small maps between projective varieties (as introduced by de Cataldo and Migliorini in 2002), proving one more time that Matroid Theory can claim its rightful place next to the other branches of Pure Mathematics.

The dissertation is organized in four chapters as follows. In the first chapter we define matroids using different equivalent sets of axioms and describe various invariants associated to them, such as the lattice of flats and the characteristic polynomial. We also show how to perform operations such as direct sum, truncation, restriction and contraction. Lastly, we make a quick digression on log-concave sequences and show famous classical examples from Combinatorics. In the second chapter we give an extensive overview of the necessary results in Hodge Theory for Riemannian, Hermitean and Kähler manifolds in order to demonstrate the so-called "Hodge Package" using the cohomology ring of (p, q)-forms and the intersection cohomology ring. We then describe the Kähler cone, the ample cone and semismall maps. In the third chapter we exploit different structures coming from Lattice Theory and Toric Geometry to define the Chow ring of a matroid, a graded algebra that plays the combinatorial counterpart of the cohomology ring of a variety. In the last chapter, we prove that the Chow ring satisfies a combinatorial version of Poincaré Duality, Hard Lefschetz Theorem and Hodge-Riemann bilinear relations, by exhibiting a decomposition that resembles the one induced by semi-small maps between projective varieties. Lastly, we prove the log-concavity conjecture using all the tools introduced in the previous chapters.

Introduzione

In questa tesi studiamo la celebre congettura, attribuita a Heron, Rota e Welsh, riguardante la log-concavità del polinomio caratteristico di un matroide.

La dimostrazione di tale congettura, data nel 2018 da Adiprasito, Huh e Katz, si basa sulla costruzione di una versione combinatoria di vari risultati di geometria algebrica riguardanti un particolare anello di coomologia detto anello di Chow. Nel febbraio 2020, è stata annunciata una nuova dimostrazione più semplice in una prepubblicazione di Braden, Huh, Matherne, Proudfoot e Wang. Questa tesi è stata concepita per essere una guida completa alla lettura di questi due articoli, e mira a fornire sia il background necessario, che spazia dalla teoria di Hodge, alla combinatoria alla geometria torica, sia esempi concreti e non banali di calcoli sul Chow ring, ancora quasi completamente assenti in letteratura.

La teoria dei matroidi ha le sue origini nell'articolo del 1935 *On the abstract properties of linear independence* di Whitney. Fin da subito, i matroidi sono stati utilizzati ampiamente in teoria dei grafi, teoria dei codici e ottimizzazione: ad esempio, i matroidi risolvono i problemi riguardanti la dualità dei grafi; inoltre, descrivono una classe di problemi di ottimizzazione su cui gli algoritmi *greedy* sono ottimali. Data la grande varietà di situazioni che possono essere modellizzate usando i matroidi, questa teoria è stata a lungo considerata un ramo della matematica applicata.
Negli ultimi anni, una nuova generazione di matematici ha rivoluzionato questa prospettiva, scoprendo un sorprendente e profondo legame tra la teoria dei matroidi, la topologia algebrica e la geometria algebrica. Il primo risultato in questa direzione è stato l'articolo *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs*, pubblicato nel 2012 da June Huh, in cui l'autore ottiene le relazioni di log-concavità su matroidi realizzabili in caratteristica zero usando le relazioni di Hodge-Riemann per il modello meraviglioso di De Concini-Procesi. Il passo successivo è stato l'articolo del 2018, *Hodge Theory for combinatorial geometries*, in cui Adiprasito, Huh e Katz forniscono una dimostrazione completa della congettura, basata su un elaborato argomento induttivo che permette di mostrare le versioni combinatorie del Teorema Hard Lefschetz e delle relazioni di Hodge-Riemann. L'articolo sopraccitato del 2020, *A semi-small decomposition of the Chow ring of a matroid*, fornisce una dimostrazione dei risultati precedenti ispirata a una decomposizione dell'anello di Chow indotta da mappe semi-piccole tra varietà proiettive (introdotte nel 2002 da de Cataldo e Migliorini). Tutti questi risultati mostrano in effetti come la teoria dei matroidi possa rivendicare a pieno titolo il suo posto a fianco delle altre branche della matematica pura.

La tesi è organizzata in quattro capitoli. Nel primo definiamo i matroidi usando diversi sistemi assiomatici equivalenti e descriviamo vari invarianti ad essi associati, tra cui il reticolo dei flats e il polinomio caratteristico. Mostriamo anche come compiere le operazioni di somma diretta, troncamento, restrizione e contrazione. Infine, compiamo una breve digressione sulle successioni log-concave e mostriamo alcuni esempi classici provenienti dalla combinatoria. Nel secondo capitolo, forniamo una panoramica dettagliata dei risultati di teoria di Hodge per varietà riemanniane, hermitiane e kahleriane per dimostrare il cosiddetto "pacchetto di Hodge" usando sia la coomologia delle (p, q)-forme, sia la coomologia di intersezione. Descriviamo poi il cono Kähler, il cono ampio e le mappe semipiccole. Nel terzo capitolo, utilizziamo strutture provenienti dalla teoria dei reticoli e dalla geometria torica per definire l'anello di Chow di un matroide, un'algebra graduata che gioca il ruolo della controparte combinatoria dell'anello di coomologia di una varietà. Nell'ultimo capitolo mostriamo che l'anello di Chow soddisfa le versioni combinatorie della Dualità di Poincaré, del Teorema Hard Lefschetz e delle relazioni bilineari di Hodge-Riemann, esibendo una decomposizione che richiama quella indotta da mappe semipiccole tra varietà proiettive. Infine, mostriamo la congettura di log-concavità utilizzando tutti gli strumenti introdotti in precedenza.

Contents

1 Preliminary notions on Matroid Theory **1**
1.1 Matroids 1
1.2 The lattice of flats 7
1.3 Submatroids 15
1.4 The characteristic polynomial 19
1.5 Unimodality and log-concavity 29

2 Hodge Theory background **35**
2.1 Linear Algebra tools 35
2.2 Compact, oriented, Riemannian manifolds 36
2.3 Hermitean manifolds 41
2.4 Kähler manifolds 48
2.5 Semismall maps 55
2.6 The axiomatic algebraic setup 57

3 The Chow ring of a matroid **61**
3.1 Basic notions of Toric Geometry 61
3.2 The Chow ring of a lattice 69
3.3 Useful properties of Σ_M and $CH(M)$ 72

4 The proof **81**
4.1 The decomposition of the Chow ring 81
4.2 The proof of log-concavity 90

Chapter 1

Preliminary notions on Matroid Theory

The purpose of this chapter is to set the notation and give the notions necessary to understand Theorem 1.4.21, which was stated in [2] and our dissertation aims to prove. The many examples are given, both to help the readers to familiarize with the topic and to let them appreciate the importance of the result. Our main reference for this chapter is [3].

1.1 Matroids

In this first section we aim to define the algebraic structure of *matroid* using different equivalent axiomatic structures. Informally speaking, a matroid is a finite set on which you can define the concept of *linear independence*.

Definition 1.1.1. A *matroid* M is a couple $(E, \mathscr{I})$, where

- E is a finite set (which can be identified with the set $\{1, 2, \ldots, n\}$ for $n = |E|$) called *ground set*;
- $\mathscr{I} \subseteq \mathscr{P}(E)$, called the *family of independent sets*, satisfies the following axioms (I)
  - (I1) $\emptyset \in \mathscr{I}$,
  - (I2) If $I_1 \in \mathscr{I}$ and $I_2 \subseteq I_1$, then $I_2 \in \mathscr{I}$,
  - (I3) If $I_1, I_2 \in \mathscr{I}$ and $|I_1| = |I_2| + 1$, then there exists $i \in I_1 \setminus I_2$ such that $I_2 \cup \{i\} \in \mathscr{I}$.

If $E = \emptyset$, M vacuously satisfies all the axioms (I); if not explicitly stated, we will always consider nonempty matroids. We will also say that $\mathscr{I}$ *is a matroid on* E, and we will identify M with $\mathscr{I}$ when it is clear which ground set E we are using.

Remark 1.1.2. When needed and if it does nor rise confusion, we will use the following notation: if $A = \{i_1, \ldots, i_k\} \in \mathscr{P}(E)$, we will write $A = i_1 \ldots i_k$

without brackets and commas, obviously implying the subset containing those elements. When we have to deal with sets of sets, writing

$$\mathscr{I} = \{\emptyset, 1, 2, 3, 12, 13, 23, 123\}$$

instead of

$$\mathscr{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\},$$

makes the writing much lighter and comprehensible. In particular, both the notations $i \in E$ and $i \subseteq E$ will then make sense and will be used depending if we are focusing on i as an element or as a singleton.

The starting point for axioms (I), as for many of the structures we are going to introduce, is the concept of linear independence between vectors of a vector space.

Definition 1.1.3. We say that $M = (E, \mathscr{I})$ is *representable on a field* $\mathbb{K}$ if there exist a $\mathbb{K}$-vector space V, and a map $\Phi : E \to V$ such that

$$I = \{i_j\}_j \in \mathscr{I} \Leftrightarrow \{\Phi(i_j)\}_j \text{ are linearly independent as vectors of } V.$$

We call the list of vectors $\Phi(E)$ a *realization* of M. If it is clear that we are working with a realization of M, we may use as ground set the list of vectors that satisfies this property.

Some matroids can be represented only on some fields; there exist matroids which are not representable on any field.

We give two other important definitions, inspired by vector spaces: the *rank function* and the concept of *basis*. Note that, in the case of representable matroids, the definitions coincide with the usual concepts of rank and basis in a vector space.

Definition 1.1.4. The *rank function* of a matroid M is the function

$$\rho : \mathscr{P}(E) \to \mathbb{Z}_{\geq 0}$$

such that

$$\rho(A) = \max \{|I| \text{ such that } I \subseteq A, I \in \mathscr{I}\}.$$

A *basis* of a matroid is any maximal independent set $B \in \mathscr{I}$. We will denote the family of bases of M with $\mathscr{B}$.
We also define the *rank of* M as

$$\rho(M) := \rho(E).$$

Note that $\rho(M) = \rho(B) = |B|$ for any basis $B \in \mathscr{B}$.

Example 1.1.5. The *Vámos matroid* V_8 is a rank 4 matroid on 8 elements which is not representable over any field. We describe it by giving its family of bases thanks to Theorem 1.1.10. We say that every subset of four elements is a basis except for the following

$$\{1234, 1256, 3456, 3478, 5678\}.$$

Remark 1.1.6. It is trivial to note that all bases of a matroid need have the same cardinality. In fact, let $B_1, B_2 \in \mathscr{B}$ and $|B_1| < |B_2|$; then from (I2) and (I3) there exists $i \in B_2 \setminus B_1$ such that $B_1 \cup i$ is still independent and therefore B_1 is not maximal and cannot be a basis.

Lemma 1.1.7. *The following property holds for ρ:*

$$\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B),$$

for any $A, B \in \mathscr{P}(E)$. This is also called sub-modularity.

Proof. We prove the result for $A = C \cup i$ e $B = C \cup j$ and $i, j \in E$; the rest is a simple generalization of this case, which can be proved using finite induction. The result is then equivalent to

$$\rho(C \cup i \cup j) + \rho(C) \leq \rho(C \cup i) + \rho(C \cup j).$$

If $\rho(C \cup i) = \rho(C)$, we can reduce ourselves to

$$\rho(C) \leq \rho(C \cup j),$$

which is trivially true from the monotonicity of max. If $\rho(C \cup i) = \rho(C \cup j) = \rho(C) + 1$, the result becomes

$$\rho(C \cup i \cup j) \leq \rho(C) + 2,$$

also trivially true. □

We can then give this theorem of characterization of matroids using the rank function

Theorem 1.1.8. *A function $\rho : \mathscr{P}(E) \to \mathbb{Z}_{\geq 0}$ is the rank function of a matroid on E if and only if it satisfies the following axioms (R)*

(R1) $0 \leq \rho(A) \leq |A|$;

(R2) If $A \subseteq B$, then $\rho(A) \leq \rho(B)$;

(R3) $\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B)$,

for any $A, B \in \mathscr{P}(E)$.

Proof. (R1) and (R2) are trivial. We proved (R3) in Lemma 1.1.7. Conversely, suppose axioms (R) hold. We define $\mathscr{I} = \{A | \rho(A) = A\}$ and prove that $\mathscr{I}$ satisfies (I). From (R1) we have

$$\rho(\emptyset) = 0 \Rightarrow \emptyset \in \mathscr{I}.$$

Using (R2), let $A \subseteq B$ and $\rho(B) = |B|$. Then, from (R3)

$$\begin{aligned} \rho(B) =& \rho(A \cup (B \setminus A)) \leq \rho(A) + \rho(B \setminus A) - \rho(A \cap (B \setminus A)) \\ =& \rho(A) + \rho(B \setminus A), \end{aligned}$$

therefore $|B| = \rho(A) + \rho(B \setminus A)$, and using (R1) on $B \setminus A$,

$$\rho(A) = |B| - \rho(B \setminus A) \geq |B| - (|B \setminus A|) = |A|.$$

Using (R1) again on A we obtain the desired equality. Lastly, let us consider A, B such that $|B| = |A| + 1$. Suppose that for any $i \in B \setminus A$

$$\rho(A \cup i) = \rho(A) = |A|.$$

Then,

$$\rho(A \cup i \cup j) \leq \rho(A \cup i) + \rho(A \cup j) - \rho((A \cup i) \cap (A \cup j)) = |A|,$$

for any $i \neq j \in B \setminus A$. Therefore, $\rho(A \cup B) = \rho(A \cup (B \setminus A)) = |A|$, and this is a contradiction because $B \subseteq A \cup B$, so

$$\rho(A \cup B) \geq \rho(B) = |B| = |A| + 1 > |A|.$$

□

Remark 1.1.9. We can then define $\mathscr{B}$ as the family of subsets of maximal rank.

Knowing the ground set and the family of bases $\mathscr{B}$ lets us reconstruct the matroid, if we define $\mathscr{I}$ as the family of all the subsets of all the elements of $\mathscr{B}$. Therefore, we could think that, instead of using $\mathscr{I}$, we can describe the matroid by giving a family $\mathscr{B}$. This can be done provided that the elements of $\mathscr{B}$ satisfy the following exchange property

Theorem 1.1.10. *A nonempty family $\mathscr{B} \subset \mathscr{P}(E)$ is the family of the bases of a matroid on E if and only if it satisfies (B)*

(B1) If $B_1, B_2 \in \mathscr{B}$ and $i \in B_1 \setminus B_2$, there exists $j \in B_2 \setminus B_1$ such that

$$(B_1 \setminus i) \cup j \in \mathscr{B}.$$

Proof. If $B_1 \in \mathscr{B} \subset \mathscr{I}$, then $B_1 \setminus i \in \mathscr{I}$ from (I2). Therefore, from (I3), there exists $j \in B_2 \setminus B_1$ such that $(B_1 \setminus i) \cup j \in \mathscr{B}$. Conversely, if (B1) holds, (I1) and (I2) are trivial because we defined a set to be independent if it is a subset of a basis. Lastly, we consider I_1, I_2 independent and $|I_1| < |I_2|$: let B_1, respectively B_2, a basis which contains I_1, respectively I_2. Surely $I_1 \subsetneq B_1$ (otherwise I_1 and I_2 would have the same cardinality), therefore there exists $i \in B_1 \setminus I_1$. From (B1), there exists $j \in B_2$ such that $(B_1 \setminus i) \cup j \in \mathscr{B} \subset \mathscr{I}$, and so $I_1 \cup j \in \mathscr{I}$. The last thing to check is that we can choose j in I_2. If by contradiction that could not be done, we could keep on exchanging elements in B_1 with elements in $B_2 \setminus I_2$ using (B1) until we obtain a new basis $\tilde{B}$ such that $B_2 \setminus \tilde{B} = I_2$. This is a contradiction because (B1) should hold for these particular bases. □

The following definitions come from Graph Theory, which will give us another nice class of matroids.

Definition 1.1.11. An element $i \in E$ is a *loop* if its rank is 0 or, equivalently if $i \notin \mathscr{I}$. Two elements $i_1, i_2 \in E$ are said to be *parallel* if they have rank 1 and $\rho(\{i_1, i_2\}) = 1$ or, equivalently, if they are independent but $\{i_1, i_2\}$ is not. A matroid is *simple* if it does not contain loops nor parallel elements.

Remark 1.1.12. In a representable matroid every loop is mapped to the 0 vector. Two parallel elements are mapped to parallel vectors.

Definition 1.1.13. A matroid M is said to be *graphic* if there exists a graph $G = (V, E(G))$ such that the ground set of M is in bijection with $E(G)$ and a set of edges is said to be independent if and only if it does not contain any cycle in G.
In a graphic matroid, $\mathscr{I}$ is the family of all subforests of G; $\mathscr{B}$ is the family of spanning forests of G.

Definition 1.1.14. An element $i \in E$ is a *coloop* if it belongs to any basis $B \in \mathscr{B}$.

In graphic matroids, an edge is a coloop if removing it from the graph increases the number of connected components of G (also called *bridge, isthmus* or *cut-edge*).

Notice that you could have more ways of representing a matroid, or it would be better to say that there exist different sets with the same underlying matroid structure.

Definition 1.1.15. We say that two matroids $M_1 = (E_1, \mathscr{I}_1)$ and $M_2 = (E_2, \mathscr{I}_2)$ are *isomorphic* if there exists a bijection

$$f : E_1 \to E_2,$$

such that

$$\{i_1, \ldots, i_k\} \in \mathscr{I}_1 \Leftrightarrow \{f(i_1), \ldots, f(i_k)\} \in \mathscr{I}_2;$$

in other words, f is a matroid isomorphism if it preserves the rank.

Example 1.1.16. For representable matroids, any automorphism f of V gives a matroid isomorphism between $\mathscr{I}$ and $f(\mathscr{I})$.

Remark 1.1.17. A graphic matroid is representable. Let us build a matroid isomorphism. Every loop can be mapped to the zero vector. Then, label each vertex of the graph and map the edge $e = (v_i, v_j)$ to vector $e_i - e_j$. This is indeed a matroid isomorphism since it maps dependent sets to all and only dependent sets. If we take a subgraph which is not a forest, it contains a cycle $\{v_{i_1}, \ldots, v_{i_k}\}$ and if we consider the corresponding vectors in its realization we have

$$\sum_{j=1}^{k-1}(e_{i_j} - e_{i_{j+1}}) + (e_{i_k} - e_{i_1}) = e_{i_1} - e_{i_k} + e_{i_k} - e_{i_1} = 0.$$

Remark 1.1.18. The converse is not true: there exist representable matroids which are not graphic, e.g.

$$E = \{1, 2, 3, 4\}$$
$$\mathscr{B} = \{12, 13, 14, 23, 24, 34\},$$

which we will call in a moment $U_{2,4}$, uniform matroid of rank 2 on 4 elements. A realization of M is given by four vectors in $\mathbb{K}^2$, no one multiple of the others; however, there does not exist a graph with four edges such that any three of them form a cycle (you could form a triangle with the first three, but then you have no possible choices for the fourth one).

Another interesting class of matroids is the above-mentioned family of uniform matroids:

Definition 1.1.19. We define *uniform matroid of rank k on n elements*, denoted $U_{k,n}$, as follows. The ground set is $E = \{1, 2, \dots, n\}$, while

- The independent set is $\mathscr{I} = \{A \in \mathscr{P}(E), |A| \leq k\}$;
- The family of bases is then $\mathscr{B} = \{B \in \mathscr{I}, |I| = k\}$;
- The rank function is defined as

$$\rho(A) = \begin{cases} |A|, & |A| \leq k \\ k, & |A| > k \end{cases}.$$

Definition 1.1.20. A particular case of uniform matroid is the *Boolean matroid on n elements*,

$$B_n := U_{n,n}.$$

In B_n, we define

- $\mathscr{I} = \mathscr{P}(E)$;
- $\mathscr{B} = \{E\}$; in particular, all elements in E are coloops;
- $\rho(A) = |A|$ for any $A \in \mathscr{P}(E)$.

Remark 1.1.21. A matroid M is boolean if and only if $\rho(M) = |M|$. In fact, the condition is equivalent to ask that the only element of $\mathscr{B}$ is E.

We give one last definition that, for our purposes, is going to be used to classify the (simple) matroids that we will study.

Definition 1.1.22. We say that a matroid M is a *direct sum* of two matroids

$$M = M_1 \oplus M_2,$$

if, equivalently,

- $\mathscr{I} \cong \mathscr{I}_1 \times \mathscr{I}_2$, that is an independent set of M is a disjoint union of an independent set of M_1 and an independent set of M_2;
- $\rho = \rho_1 + \rho_2$;
- $\mathscr{B} \cong \mathscr{B}_1 \times \mathscr{B}_2$, that is a basis of M is a disjoint union of a bases of M_1 and a bases of M_2.

It is also easy to check that all these operations yield the construction of a matroid that satisfies axioms (I), (R) or (B), respectively.

In representable matroids, the operation of direct sum is exactly equivalent to the one we define on vector spaces. The direct sum of two graphic matroids can be seen as simply considering the two graphs $G(M_1)$ and $G(M_2)$ as one graph with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. As a matroid, this is isomorphic to a graph built by identifying a vertex of $G(M_1)$ with a vertex of $G(M_2)$ (this should make sense, since gluing graphs on only a vertex does not create new cycles).

Lemma 1.1.23. *If M has m coloops (suppose they are labelled $1, \dots, m$ for simplicity), then*

$$M = B_m \oplus M_2.$$

Proof. By definition of coloops, $\{1, \dots, m\}$ is a subset of any basis. Define M_2 as the matroid with bases $B \setminus \{1, \dots, m\}$ for any basis $B \in \mathscr{B}$. □

Definition 1.1.24. A matroid M is said to be *connected* if we cannot write it as a direct sum of two proper matroid.

Corollary 1.1.25. *A connected matroid does not have any coloop.*

Proof. Straightforward consequence of Lemma 1.1.23. □

1.2 The lattice of flats

In this section we define the *flats* of a matroid, which will play a fundamental role in the rest of the dissertation, and we show that their family has the structure of lattice. As we will see, a flat can somehow be seen as a set "closed" under dependence relations.

Definition 1.2.1. Let M be a matroid. A subset $F \in \mathscr{P}(E)$ is a *flat* if for any $i \in E \setminus F$

$$\rho(F \cup i) = \rho(F) + 1.$$

In representable matroids, we can identify the family of flats with the set of all and only the subspaces of V generated by subsets $A \in \mathscr{P}(E)$.

Definition 1.2.2. If $\rho(A \cup i) = \rho(A)$, we say that i *depends on* A and we write $i \sim A$.
We also define the *closure operator* as the function

$$\sigma : \mathscr{P}(E) \to \mathscr{P}(E)$$

such that $\sigma(A)$ is the set of all elements in E that depend on A.

Theorem 1.2.3. *The following statements are equivalent and give a characterization of the flats of a matroid:*

1. *F is a flat;*
2. *$\sigma(F) = F$;*
3. *if $i \in E \setminus F$, then $i \nsim F$.*

Proof. The proofs are all trivial. □

Remark 1.2.4. We give a list of other statements about flats, the proof of which is also trivial.

- If i belongs to any flat, then it is a loop;
- if $i \sim A$ and j is parallel to i, then $j \sim A$;
- $\rho(A) = \rho(\sigma(A))$;

- if $i \in A$, then $i \sim A$;
- therefore, $A \subseteq \sigma(A)$ and if $B \subseteq A$, then $\sigma(B) \subseteq \sigma(A)$;
- $\emptyset$ is a flat if and only if M has no loops;
- M is simple if and only if $\rho(A) = 1 \Leftrightarrow |A| = 1$, for every A.

Theorem 1.2.5. *If F_1 and F_2 are flats in M, then $F_1 \cap F_2$ is too.*

Proof. From the previous remarks $F_1 \cap F_2 \subseteq \sigma(F_1 \cap F_2)$. Conversely, since $F_1 \cap F_2 \subseteq F_1, F_2$, we have that $\sigma(F_1 \cap F_2) \subseteq \sigma(F_1), \sigma(F_2)$ and so

$$\sigma(F_1 \cap F_2) \subseteq \sigma(F_1) \cap \sigma(F_2) = F_1 \cap F_2.$$

By Theorem 1.2.3, $F_1 \cap F_2$ is a flat. □

Definition 1.2.6. A flat of rank 1 is said to be an *atom* of M. A *hyperplane* of M is a flat of rank $\rho(M) - 1$. Furthermore, F is a hyperplane if and only if, equivalently:

- $F \subsetneq E$ is a maximal closed set;
- $\sigma(F) \neq E$ and $\sigma(F \cup i) = E$ for any $i \in E \setminus F$;
- No basis is contained in F, but for any $i \in E \setminus F$, there exists $B_i \in \mathscr{B}$ such that $B_i \in F \cup i$.

The family of flats of M is a lattice, that we will now describe.

Definition 1.2.7. Let M be a matroid. We define the following poset $\mathscr{L}(M)$, called *lattice of flats*, as the set of flats in M ordered by inclusion.

Remark 1.2.8. The following properties of $\mathscr{L}(M)$ hold:

- $\mathscr{L}(M)$ is finite with a minimum, $\sigma(\emptyset)$, and a maximum, E;
- The atoms of $\mathscr{L}(M)$ are the flats of rank 1;
- An element of $\mathscr{L}(M)$ is covered by E if and only if it is a hyperplane of M;
- A flat F_1 covers another flat F_2 in $\mathscr{L}(M)$ if and only if $F_2 \subseteq F_1$ and $\rho(F_1) = \rho(F_2) + 1$.

Theorem 1.2.9. *The lattice of flats $\mathscr{L}(M)$ is indeed a lattice, it satisfies the Jordan-Dedekind condition (maximal chains between two elements have all the same length) and it is semimodular, that is, if F_1 and F_2 cover $F_1 \wedge F_2$, then $F_1 \vee F_2$ covers F_1 and F_2.*

Proof. By Theorem 1.2.5, the meet of two elements exists and it is well defined,

$$F_1 \wedge F_2 := F_1 \cap F_2.$$

Furthermore, the join of two elements is

$$F_1 \vee F_2 = \bigcap_{\substack{F \in \mathscr{L}(M) \\ F_1 \cup F_2 \subseteq F}} F = \sigma(F_1 \cup F_2).$$

The Jordan-Dedekind condition is a direct consequence of Remark 1.2.8.
The semimodularity can be proved using 1.1.7 and 1.2.4 to get

$$\begin{aligned}\rho(F_1 \vee F_2) + \rho(F_1 \wedge F_2) =& \rho(\sigma(F_1 \cup F_2)) + \rho(F_1 \cap F_2)\\ =& \rho(F_1 \cup F_2) + \rho(F_1 \cap F_2)\\ \leq& \rho(F_1) + \rho(F_2),\end{aligned}$$

Hence, if $F_1 \wedge F_2 \lessdot F_1, F_2$ (and using the Jordan-Dedekind condition we have $\rho(F_1) = \rho(F_2)$ and $\rho(F_1 \wedge F_2) = \rho(F_1) - 1$),

$$\rho(F_1 \vee F_2) \leq \rho(F_1) + \rho(F_1) - (\rho(F_1) - 1) = \rho(F_1) + 1.$$

Using the fact that $F_1 \subsetneq F_1 \cup F_2 \subseteq \sigma(F_1 \cup F_2) = F_1 \vee F_2$, we observe that $\rho(F_1) < \rho(F_1 \vee F_2)$, which completes the proof. □

Definition 1.2.10. A finite lattice $\mathscr{L}$ is called *geometric* if it is semimodular and any element can be written as the join of atoms of the lattice,

$$A \in \mathscr{L} \Leftrightarrow A = \bigvee_{j=1}^{k} x_j.$$

Theorem 1.2.11. *A finite lattice $\mathscr{L}$ is isomorphic to the lattice of flats $\mathscr{L}(M)$ of a matroid M if and only if it is geometric.*

Proof. We have already proved that $\mathscr{L}(M)$ is semimodular. Let then F be a rank k flat. There exists an independent set $\{i_1, \ldots, i_k\} \in \mathscr{I}$ contained in F. Each of its elements is independent as a singleton for (I2) and $\rho(\{i_n, i_m\}) = 2$ for $n \neq m$, therefore $i_n \notin \sigma(i_m)$ for $n \neq m$. Hence, the atoms $\sigma(i_1), \ldots \sigma(i_k)$ are distinct and

$$F = \bigvee_{j=1}^{k} \sigma(i_j).$$

This proves that $\mathscr{L}(M)$ is geometric. Conversely, let $\mathscr{L}$ be a geometric lattice and let A be the set of its atoms. Define $\mathscr{I}$ as the family of subsets $X = \{x_i\}_i$ such that

$$X \in \mathscr{I} \Leftrightarrow h\left(\bigvee x_i\right) = |X|,$$

where h is the height function of the lattice $\mathscr{L}$. Moreover, h is submodular, so

$$h(X) \leq \sum_i h(x_i) = |X|.$$

It is then trivial to see that if $X \subseteq Y$,

$$h\left(\bigvee_i x_i\right) \leq h\left(\bigvee_j y_j\right).$$

Lastly, if $X, Y \in \mathscr{P}(E)$,

$$\begin{aligned}h\left(\bigvee_i x_i\right) + h\left(\bigvee_j y_j\right) \geq& h\left(\bigvee_i x_i \vee \bigvee_j y_j\right) + h\left(\bigvee_i x_j \wedge \bigvee_j y_j\right)\\ \geq& h\left(X \vee Y\right) + h\left(X \wedge Y\right).\end{aligned}$$

This proves that $\rho(X) = h\left(\bigvee x_i\right)$ satisfies the axioms (R). We can then define $M = M(\mathscr{L})$ as the matroid with such a rank function to complete the proof. $\square$

It is noteworthy to see that M is completely described by $\mathscr{L}(M)$, if we decide to overlook loops and parallel elements.

Theorem 1.2.12. *The correspondence between a geometric lattice $\mathscr{L}$ and the matroid $M(\mathscr{L})$ defined on the family of atoms of $\mathscr{L}$ is a bijection between the family of finite geometric lattices and the family of simple matroids.*

Proof. Let x, y be two distinct atoms of a geometric lattice $\mathscr{L}$. If ρ is the rank function of $M(\mathscr{L})$, clearly $\rho(x) = 1$, and $\rho(\{x, y\}) = 2$, therefore we can conclude that $M(\mathscr{L})$ is simple and

$$\mathscr{L}(M(\mathscr{L})) = \mathscr{L}$$

using 1.2.4. Conversely, if $\mathscr{L} = \mathscr{L}(M)$ is the geometric lattice of a simple matroid M, clearly

$$M(\mathscr{L}(M)) \cong M.$$

$\square$

Definition 1.2.13. Following the previous result we can introduce the term *combinatorial geometry* to refer indistinctly to a simple matroid or the geometric lattice associated to it.

Theorem 1.2.12 will be of great importance in the following sections because it lets us focus only on simple matroids. In general, we can always recognize in a matroid an underlying simple structure to which loops and parallel elements are added; this is very clear when, in Graph Theory, we extend the notion of simple graph to the one of multigraph. We claim that many properties of a matroid are not changed by considering the simplified version, or are easily reconstructed from it.

Remark 1.2.14. From this point, except if explicitly stated, we will always suppose to work with simple matroids; in particular we will only deal with lattices $\mathscr{L}$ in which

$$\min \mathscr{L} = \emptyset.$$

As said before, we will then try to reconstruct general results for non-simple matroid from the ones for simple matroids.

We now list all combinatorial geometries on four or less elements, describing them through $\mathscr{B}$, $\mathscr{L}(M)$ and vectorial and graphic realizations (except for $U_{2,4}$ which we already observed is not graphic).

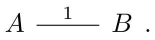
For $n = 1$, we only have the Boolean matroid B_1, $\mathscr{B}_{B_1} = \{1\}$,

$$\mathscr{L}(B_1) = \begin{array}{c} 1 \\ \uparrow \\ \emptyset \end{array}$$

It is realized by:

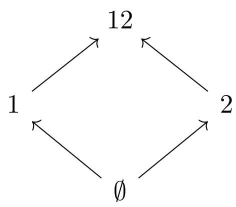
- $\{e_1\} \subset \mathbb{K}^1$,

- A connected graph with two vertices and one edge



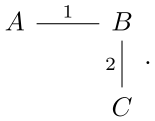
For $n = 2$, we only have the Boolean matroid $B_2 = B_1 \oplus B_1$, $\mathscr{B}_{B_2} = \{12\}$,

$\mathscr{L}(B_2) =$



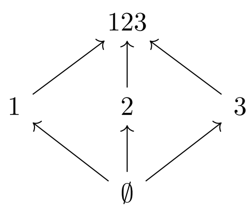
It is realized by:

- $\{e_1, e_2\} \subset \mathbb{K}^2$,
- A connected graph with three vertices and two edges



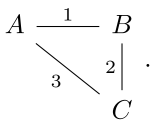
For $n = 3$:
Uniform matroid $U_{2,3}$, $\mathscr{B}_{U_{2,3}} = \{12, 13, 23\}$:

$\mathscr{L}(U_{2,3}) =$



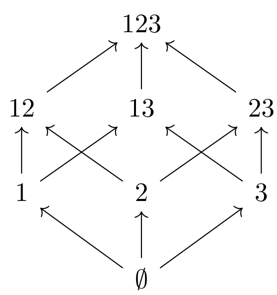
It is realized by:

- $\{e_1, e_2, e_1 + e_2\} \subset \mathbb{K}^2$,
- A three-cycle



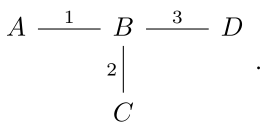
Boolean matroid $B_3 = B_1 \oplus B_1 \oplus B_1$, $\mathscr{B}_{B_3} = \{123\}$:

$\mathscr{L}(B_3) =$



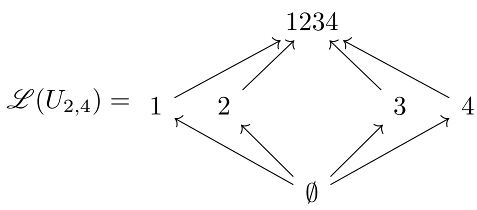
It is realized by:

- $\{e_1, e_2, e_3\} \subset \mathbb{K}^3$,
- A three-edge tree



.

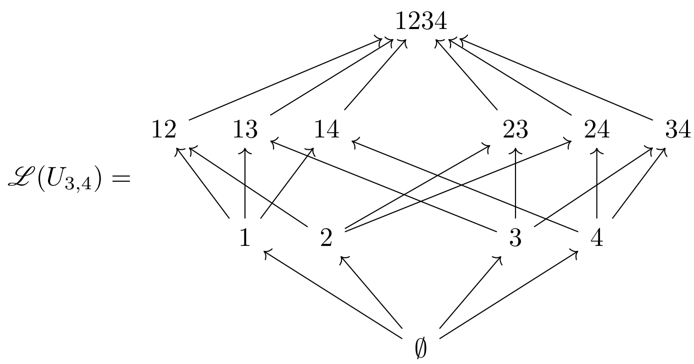
For $n = 4$:
Uniform matroid $U_{2,4}$, $\mathscr{B}_{U_{2,4}} = \{12, 13, 14, 23, 24, 34\}$:



It is realized by:

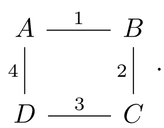
- $\{e_1, e_2, e_1 + e_2, e_1 - e_2\} \subset \mathbb{K}^2$
- No graphs.

Uniform matroid $U_{3,4}$, $\mathscr{B}_{U_{3,4}} = \{123, 124, 134, 234\}$:



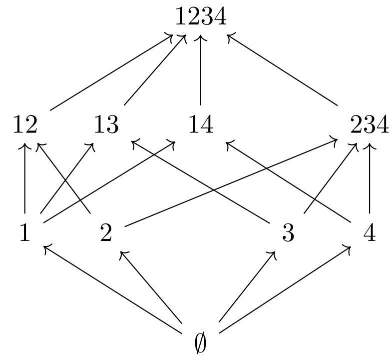
It is realized by:

- $\{e_1, e_2, e_3, e_1 + e_2 + e_3\} \subset \mathbb{K}^3$,
- A four-cycle



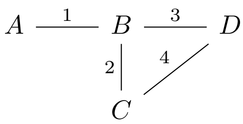
A matroid which we will denote $M_4 := B_1 \oplus U_{2,3}$, $\mathscr{B}_{M_4} = \{123, 124, 134\}$:

$\mathscr{L}(M_4) =$



It is realized by:

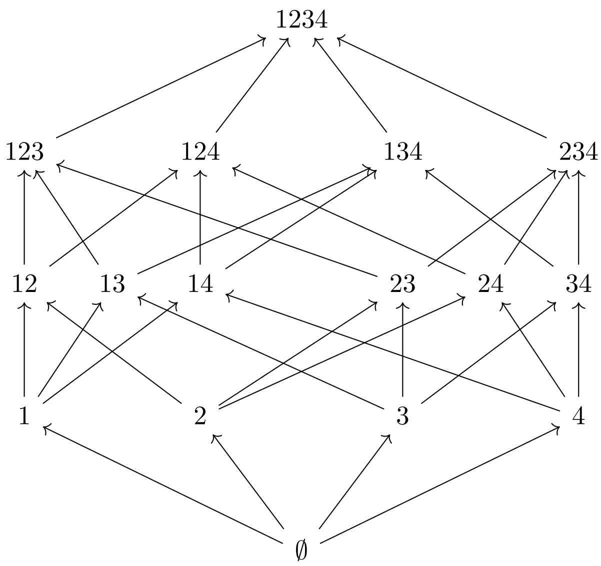
- $\{e_1, e_2, e_3, e_2 + e_3\} \subset \mathbb{K}^3$,
- A graph with one coloop and a three-cycle



.

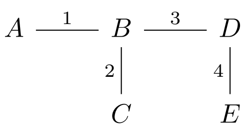
Boolean matroid $B_4 = B_1 \oplus B_1 \oplus B_1 \oplus B_1$, $\mathscr{B}_{B_4} = \{1234\}$:

$\mathscr{L}(B_4) =$



It is realized by:

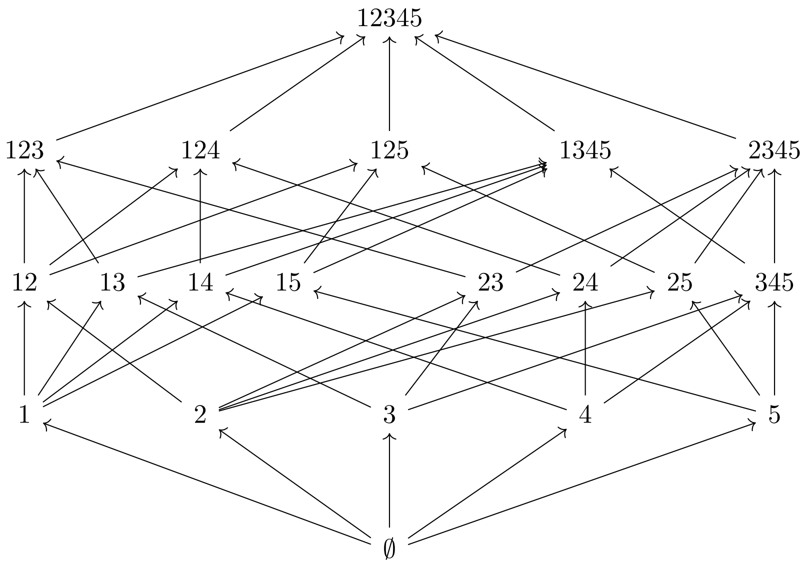
- $\{e_1, e_2, e_3, e_4\} \subset \mathbb{K}^4$
- Any four-edge tree



.

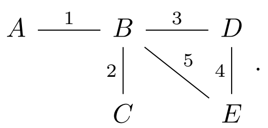
For the sake of having some slightly bigger examples, we will also work on the following matroids on five elements. The first one is $M_5 := B_2 \oplus U_{2,3}$, $\mathscr{B}_{M_5} = \{1234, 1235, 1245\}$:

$\mathscr{L}(M_5) =$



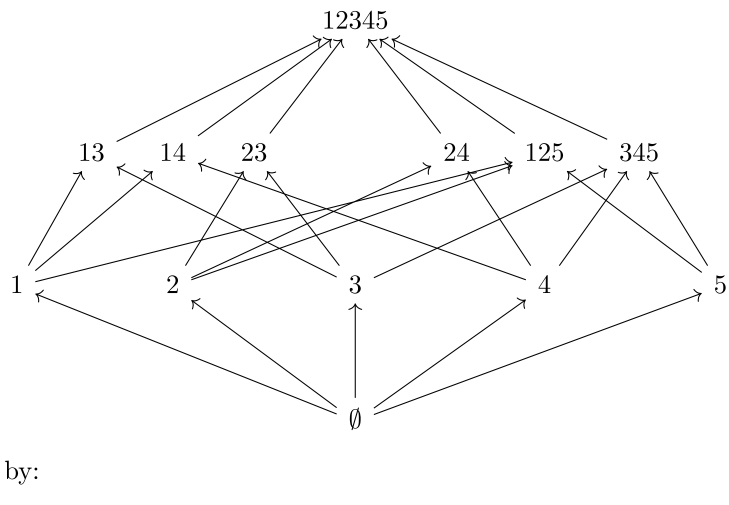
It is realized by:

- $\{e_1, e_2, e_3, e_4, e_3 + e_4\} \subset \mathbb{K}^4$
- Any graph with two coloops and a three-cycle



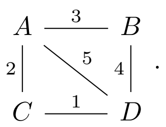
The second one is the connected matroid which we will call N_5, determined by $\mathscr{B}_{N_5} = \mathscr{P}(E) \setminus \{125, 345\}$:

$\mathscr{L}(N_5) =$



It is realized by:

- $\{e_1, e_2, e_3, e_1 + e_2 + e_3, e_1 + e_2\} \subset \mathbb{K}^3$
- The following graph



1.3 Submatroids

Something we can do with a matroid is building from it a "smaller" one, where smaller means with lower rank, on a smaller ground set or both. We call the results *submatroids*. In particular, we define the following three submatroids: the *truncation*, the *restriction* (or *localization*) and the *contraction*.

Definition 1.3.1. Let $M = (E, \mathscr{I})$ be a matroid on E and $0 < k \leq \rho(M)$. Define $t_k(\mathscr{I}) = \{I | I \in \mathscr{I}, |I| \leq k\}$. The matroid

$$t_k(M) = (E, t_k(\mathscr{I}))$$

is called the *truncation of M at k*.

Remark 1.3.2. If M is representable on V, then $t_k(M)$ is too. In fact, it can be realized by taking the projections of all the vectors on a suitable k-subspace. Intuitively, when we project on such a subspace the only subsets of vectors whose rank is affected are the ones with rank greater than k.

Remark 1.3.3. If $k' \leq k$, we have

$$t_{k'}(U_{k,n}) = U_{k',n}.$$

Example 1.3.4. Let us consider M_4:

$$t_2(M_4) = U_{2,4}.$$

In general, if M is simple, which means that the set of atoms is exactly E, we have that

$$t_2(M) = U_{2,n}.$$

Remark 1.3.5. The rank function of $t_k(M)$ is

$$\rho_k(A) = \begin{cases} \rho(A), \text{ if } \rho(A) < k \\ k, \text{ if } \rho(A) \geq k \end{cases}$$

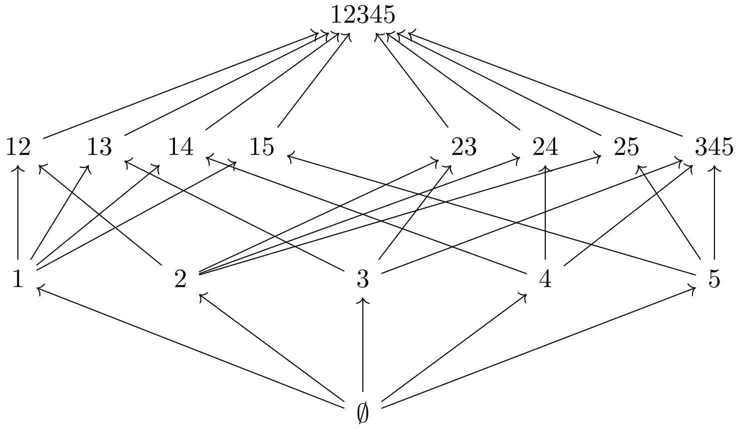
Remark 1.3.6. We prove that the lattice $\mathscr{L}(t_k(M))$ coincides with $\mathscr{L}(M)$ up to height $k-1$, and has E at height k. If F has rank $j < k$ in M, then $F \in \mathscr{L}(t_k(M))$ because if that were not true, there would exist $i \in E$ such that

$$\rho_k(F \cup i) = \rho_k(F) = \rho(F) = j,$$

hence $\rho(F \cup i) = \rho(F)$ which would be a contradiction. Lastly, let us consider a flat F of rank $\rho_k(F) = k - 1$; from the previous considerations, for any $i \notin F$, $F \cup i$ has rank $\rho_k(F \cup i) = k$. This means that $F \cup i$ contains an independent set of rank k, which is a basis in $t_k(M)$. From 1.2.6 and 1.2.8 we can conclude that E is the only rank k flat.

Example 1.3.7.

$\mathscr{L}(t_3(M_5)) =$



Definition 1.3.8. Let $M = (E, \mathscr{I})$ be a matroid and $A \in \mathscr{P}(E)$. We define $\mathscr{I}_{M^A} = \{I | I \subseteq A, I \in \mathscr{I}\}$. The matroid

$$M^A = (A, \mathscr{I}_{M^A})$$

is called *restriction (or localization) of* M *at* A. A particular case is given by

$$M \setminus i := M^{E \setminus i}, \text{ where } i \in E,$$

called *deletion of* i *from* M.

Remark 1.3.9. The rank of the restriction M^A is $\rho(M^A) = \rho(A)$.

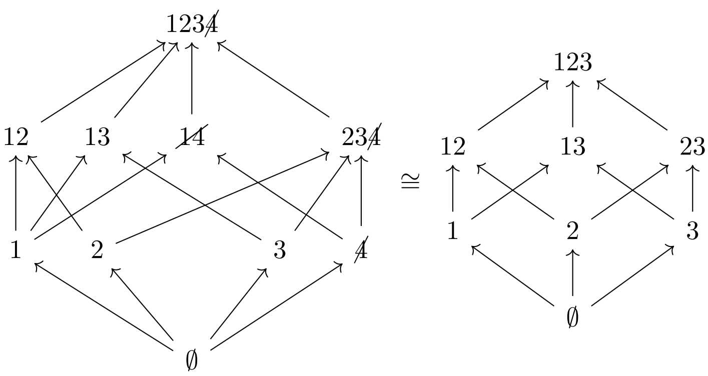
Remark 1.3.10. The deletion of an element is easy to visualize in representable and graphic matroids: we just delete the vector from our list (or the edge from our graph) and consider the resulting matroid.

Remark 1.3.11. The lattice $\mathscr{L}(M^A)$ is isomorphic to a sublattice of $\mathscr{L}(M)$ which can be constructed in the following way: starting from the atoms and going up, substitute each flat F with $F \cap A$ and delete it if $F \cap A$ is already in the lattice. In the particular case we are restricting to a flat F of M, the lattice is exactly the interval

$$\mathscr{L}(M^F) = [\emptyset, F] \subseteq \mathscr{L}(M).$$

Example 1.3.12. Consider $M = M_4$ and $A = 123$. Then,

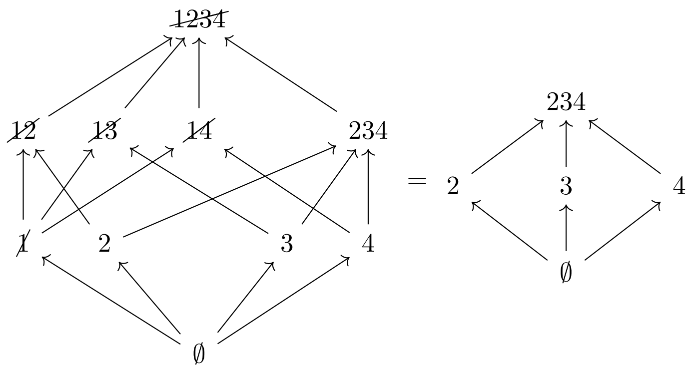
$\mathscr{L}(M\backslash 4) =$



$= \mathscr{L}(B_3)$.

If we then take $F = 234$,

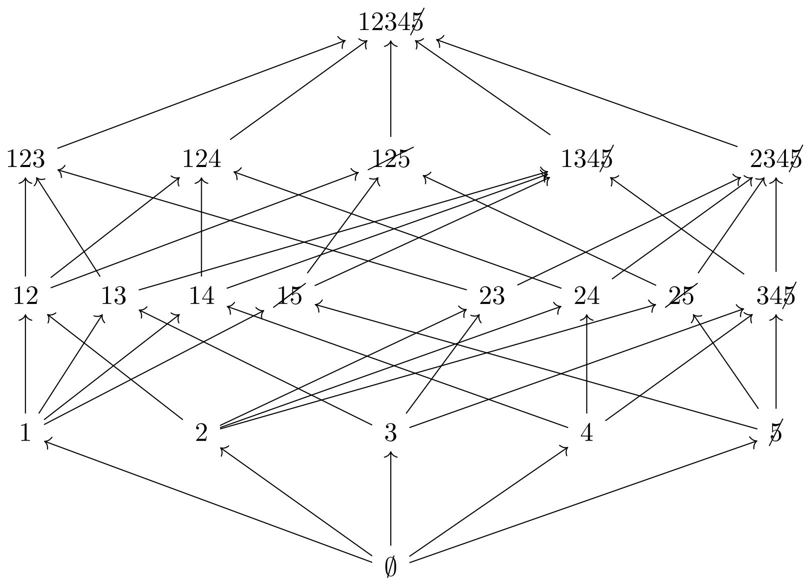
$\mathscr{L}(M\backslash 1) =$



$= \mathscr{L}(U_{2,3})$.

Remark 1.3.13. We can restrict any matroid M to a Boolean matroid of rank $\rho(M)$: we just restrict to M^B for any basis B.

$\mathscr{L}(M_5\backslash 5) =$



$= \mathscr{L}(B_4)$.

Example 1.3.14. Similar considerations show us, for example, that

$$\mathscr{L}(M_5 \setminus 1) = \mathscr{L}(M_4),$$
$$\mathscr{L}((M_5)^{12}) = \mathscr{L}(B_2),$$
$$\mathscr{L}((M_5)^{123}) = \mathscr{L}(B_3),$$
$$\mathscr{L}((M_5)^{345}) = \mathscr{L}(U_{2,3}).$$

Definition 1.3.15. Let $M = (E, \rho)$ be a matroid on E with rank function ρ, and let A be a subset of E. Define the following integer-valued function on $E \setminus A$,

$$\rho_{M_A}(T) = \rho(A \cup T) - \rho(A).$$

We call the matroid

$$M_A = (E \setminus A, \rho_{M_A})$$

the *contraction of M by A* (it is easy to see that ρ_{M_A} satisfies (R)).
We denote the contraction by a singleton $i \subseteq E$ with

$$M/i := M_i.$$

Remark 1.3.16. The rank of the contraction M_A is

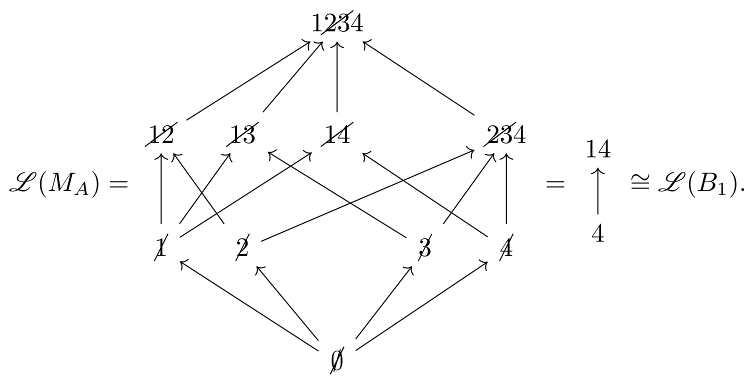
$$\rho_{M_A}(M_A) = \rho(E) - \rho(A) = \rho(M) - \rho(A).$$

Remark 1.3.17. As we did with the restriction, we can build the lattice $\mathscr{L}(M_A)$ starting from $\mathscr{L}(M)$. This can be done by taking the interval $[F, E] \subseteq \mathscr{L}(M)$ where F is the smallest flat containing A (which is well-defined because if A is contained in two incomparable flats then it is also contained in their intersection), and then intersecting all its elements with $E \setminus A$.

Example 1.3.18. Contraction by a basis gives us the lattice with only the flat $E \setminus B$, where every element is a loop.

Remark 1.3.19. Contraction by a vector v_i in a representable matroid is represented by considering the projection of $E \setminus v_i$ on the hyperplane orthogonal to v_i. Vectors who were parallel to v_i then become loops; vectors who were not parallel in M may become parallel in M/v_i. Contraction in a graph G by an edge e_i can be seen as a graph in which we have deleted the edge e_i and identified its endpoints. This construction should make even clearer that contraction does not preserve the simplicity of a matroid. Let us see a concrete example.

Example 1.3.20. Let us consider $M = M_4$ and $A = 23$. Then,

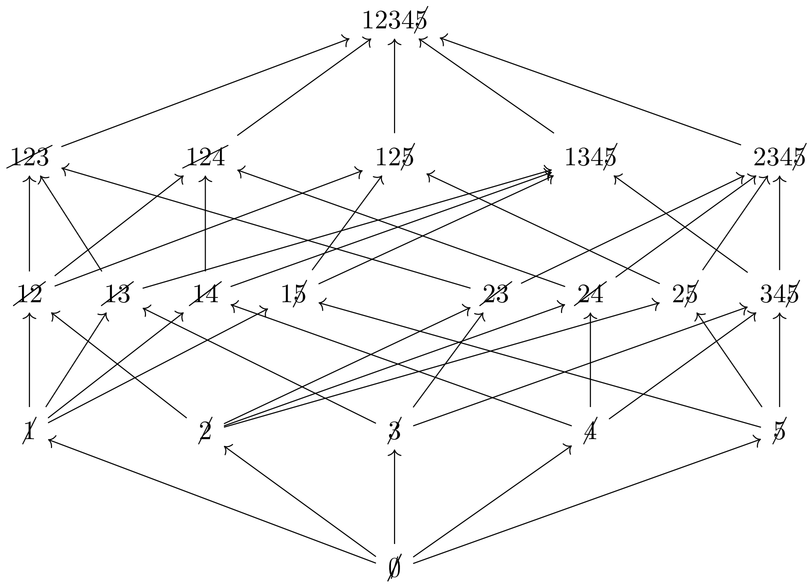


The element 4 has become a loop in the contraction. The above-mentioned procedure guarantees that we do not create loops when we contract by a flat F, since $\min \mathscr{L}(M_F) = \emptyset$.

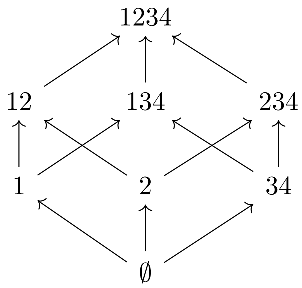
Unfortunately, as we will see in the next example, even contraction by a flat still does not guarantee that we obtain a combinatorial geometry.

Example 1.3.21. Let $M = M_5$ and $F = 5$. Then,

$\mathscr{L}(M_5/5) =$



$=$



$\cong \mathscr{L}(B_3).$

As we can see, in this case the contraction produced two parallel elements, namely 3 and 4.
Similarly, we can obtain

$$\mathscr{L}((M_5)/1) = \mathscr{L}(M_4),$$

$$\mathscr{L}((M_5)_{12}) = \mathscr{L}(U_{2,3}),$$
$$\mathscr{L}((M_5)_{13}) \cong \mathscr{L}(B_2),$$
$$\mathscr{L}((M_5)_{345}) = \mathscr{L}(B_2).$$

1.4 The characteristic polynomial

In this section we recall the definition of the *Moebius function* on a poset and use it to define the invariant $\chi_M(\lambda)$, the *characteristic polynomial* of a matroid.

Definition 1.4.1. Let P be a partially ordered set. The *Moebius function* μ *of* P is a map

$$\mu : P \times P \to \mathbb{Z},$$

defined recursively by

$$\mu(x,x) = 1, \text{ for all } x \in P,$$
$$\mu(x,y) = 0, \text{ for all } x \not\leq y,$$
$$\mu(x,y) = -\sum_{x \leq z < y} \mu(x,z), \text{ for all } x \leq y.$$

Definition 1.4.2. We define the *incidence algebra of* P to be the set of all real functions

$$f : P \times P \to \mathbb{R},$$

such that $f(x,y) = 0$, if $x \not\leq y$, with the following operations:

$$f(x,y) + g(x,y) = (f+g)(x,y),$$
$$(af)(x,y) = a(f(x,y)),$$
$$(f * g)(x,y) = \sum_{x \leq z \leq y} f(x,z)g(z,y).$$

The identity function is the *delta function*

$$\delta(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise.} \end{cases}$$

Remark 1.4.3. The *zeta function* $\zeta : P \times P \to \mathbb{R}$ defined as

$$\zeta(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{otherwise,} \end{cases}$$

is such that

$$(\zeta * \mu)(x,y) = \sum_{x \leq z \leq y} \zeta(x,z)\mu(z,y) = \sum_{x \leq z \leq y} \mu(z,y) = \delta(x,y).$$

Hence, ζ is the inverse of μ in the incidence algebra of P.

Theorem 1.4.4. *Let $f : P \to \mathbb{R}$ be a function on a poset. If $g : P \to \mathbb{R}$ is defined as*

$$g(x) = \sum_{y \leq x} f(y),$$

then we have

$$f(x) = \sum_{y \leq x} g(y)\mu(y,x).$$

Proof. The hypothesis implies that, for any $x \in P$

$$\begin{aligned}\sum_{y \leq x} g(y)\mu(y,x) = \sum_{y \leq x} \mu(y,x) \sum_{z \leq y} f(z) = \sum_{y \leq x} \sum_{z \in P} f(z)\zeta(z,y)\mu(y,x) = \\ = \sum_{z} f(z) \sum_{y \leq x} \zeta(z,y)\mu(y,x) = \sum_{z} f(z)\delta(z,x) = \\ = f(x).\end{aligned}$$

□

Example 1.4.5. The Moebius function of the lattice of subsets of a set E is

$$\mu(A, B) = (-1)^{|B|-|A|},$$

for $A \subseteq B$.

Next theorem will be stated using the notation of combinatorial geometries, but holds for any finite lattice (substitute $\emptyset$ with $\min \mathscr{L}$).

Theorem 1.4.6. *(Weisner) Let $\mathscr{L}$ be a combinatorial geometry and $G \neq \emptyset$. Then, for any $F \in \mathscr{L}$,*

$$\sum_{x|x \vee G=F} \mu(\emptyset, x) = 0.$$

Proof. If $F = G$,

$$\sum_{x|x \vee G=G} \mu(\emptyset, x) = \sum_{\emptyset \leq x \leq G} \mu(\emptyset, x) = 0.$$

Let now F be a minimal flat that does not satisfy the theorem. Then,

$$\sum_{x|x \vee G \leq F} \mu(\emptyset, x) = \sum_{x \vee G=F} \mu(\emptyset, x) + \sum_{x \vee G<F} \mu(\emptyset, x).$$

By minimality of F the last sum is zero, while the left-hand side of the equation is also zero, giving the required result. □

The next corollary does not seem to be very important now, but it will be used in one of the key passages in the proof of Theorem 1.4.21; since it is a general result for lattices we state it here and we will recall it in due time.

Corollary 1.4.7. *Let i be an atom of a combinatorial geometry. The following result holds:*

$$\mu(\emptyset, F) = -\sum_{i \notin G \lessdot F} \mu(\emptyset, G),$$

for any flat F.

Proof. We might as well consider the case where $F = E$, since the other cases can be seen as a restriction of this one to the interval $[\emptyset, F] \subseteq \mathscr{L}$. From the properties of rank functions on lattices we have that

$$\rho(G \wedge i) + \rho(G \vee i) \leq \rho(G) + \rho(i) = \rho(G) + 1.$$

Weisner's Theorem states that

$$\sum_{G|G \vee i=E} \mu(\emptyset, G) = 0.$$

Now, if $i \leq G$, then $G \wedge i = i$, therefore $\rho(E) \leq \rho(G)$ implies that $G = E$; if, instead, $i \nleq G$, $G \wedge i = \emptyset$, therefore G is a hyperplane, thus the result. □

Corollary 1.4.8. *If $G \lessdot F$, $\mu(\emptyset, G)$ and $\mu(\emptyset, F)$ have different signs.*

Proof. From Corollary 1.4.7 $\mu(\emptyset, F)$ is written as the opposite of a sum of values of μ on elements of same rank and therefore, from induction, they all have the same sign. □

We are finally ready to introduce the main object of our work: *the characteristic polynomial* $\chi_M(\lambda)$.

Definition 1.4.9. Let M be a matroid on E. We define its *characteristic polynomial* to be

$$\chi_M(\lambda) = \sum_{A \in \mathscr{P}(E)} (-1)^{|A|} \lambda^{\rho(M)-\rho(A)}.$$

We can immediately see that this is a polynomial matroid invariant, since two matroids with same rank functions (hence isomorphic) have the same characteristic polynomial.

Definition 1.4.10. We denote the absolute value of the coefficient of $\lambda^{\rho(M)-k}$ in $\chi_M(\lambda)$ with ω_k and call it *k-th Whitney number of the first kind.*

Remark 1.4.11. One can easily see that any matroid with at least one loop has characteristic polynomial $\chi_M(\lambda) \equiv 0$. In fact, if we let i be a loop,

$$\begin{aligned}
\chi_M(\lambda) &= \sum_{A \in \mathscr{P}(E)} (-1)^{|A|} \lambda^{\rho(M)-\rho(A)} = \\
&= \sum_{i \notin A} (-1)^{|A|} \lambda^{\rho(M)-\rho(A)} + \sum_{i \in A} (-1)^{|A|} \lambda^{\rho(M)-\rho(A)} = \\
&= \sum_{i \notin A} (-1)^{|A|} \lambda^{\rho(M)-\rho(A)} + \sum_{i \notin A} (-1)^{|A \cup i|} \lambda^{\rho(M)-\rho(A \cup i)} = \\
&= \sum_{i \notin A} (-1)^{|A|} \lambda^{\rho(M)-\rho(A)} + \sum_{i \notin A} (-1)^{|A|+1} \lambda^{\rho(M)-\rho(A)} = \\
&= \sum_{i \notin A} (-1)^{|A|} \lambda^{\rho(M)-\rho(A)} - \sum_{i \notin A} (-1)^{|A} \lambda^{\rho(M)-\rho(A)} = 0
\end{aligned}$$

With similar arguments, we can show that adding parallel elements to M does not change $\chi_M(\lambda)$. This also means that $\chi_M(\lambda)$ does not provide a classification.

These remarks lead us to believe that all we need to study is simple matroids, since characteristic polynomials for non-simple matroids can be found quite easily from simple ones. Theorems 1.4.12 and 1.4.16 give us powerful tools to compute the characteristic polynomials of simple matroids.

Theorem 1.4.12. *If M is a simple matroid on E and $\mathscr{L}(M)$ its lattice of flats,*

$$\chi_M(\lambda) = \sum_{F \in \mathscr{L}(M)} \mu(\emptyset, F) \lambda^{\rho(M)-\rho(F)},$$

where μ is the Moebius function of $\mathscr{L}(M)$ and ρ the rank function of M. Equivalently, we write

$$\omega_k = (-1)^k \sum_{F | \rho(F)=k} \mu(\emptyset, F).$$

Proof. We have to prove that

$$\sum_{\rho(A)=k} (-1)^{|A|} = \sum_{\rho(F)=k} \mu(\emptyset, F).$$

For $k = 0$ and $k = 1$ it is trivial since M is simple. Let us prove it for $k = \rho(M)$ supposing it holds for any $k < \rho(M)$.

$$\begin{aligned}\sum_{\rho(F)=\rho(M)} \mu(\emptyset, F) =& \mu(E) = -\sum_{G<E} \mu(\emptyset, G) = -\sum_{j=0}^{\rho(M)-1} \sum_{\rho(G)=j} \mu(\emptyset, G) = \\ &= -\sum_{j=0}^{\rho(M)-1} \sum_{\substack{A\in\mathscr{P}(E)\\ \rho(A)=j}} (-1)^{|A|} = -\sum_{\substack{A\in\mathscr{P}(E)\\ \rho(A)<\rho(M)}} (-1)^{|A|} = \\ &= \sum_{\rho(A)=\rho(M)} (-1)^{|A|} - \sum_{A\in\mathscr{P}(E)} (-1)^{|A|},\end{aligned}$$

and the latter sum is zero for the known combinatorial fact

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j = 0.$$

Hence, we have just proved that

$$\mu(\emptyset, F) = \sum_{\substack{A\subset F\\ \rho(A)=\rho(F)}} (-1)^{|A|}.$$

Adding on all flats of rank k we get

$$\sum_{\rho(F)=k} \mu(\emptyset, F) = \sum_{\rho(F)=k} \sum_{\substack{A\subset F\\ \rho(A)=\rho(F)}} (-1)^{|A|} = \sum_{\rho(A)=k} (-1)^{|A|},$$

where the last equality comes from the fact that if A is contained in two flats F_1 and F_2, then it is contained in $F_1 \cap F_2$, which is a flat of lower rank, hence we are counting every subset A exactly once. □

Therefore, we can easily compute the coefficients $(-1)^k\omega_k$ inspecting the lattice $\mathscr{L}(M)$, calculating recursively the values of μ and then adding together the values at the same height in the lattice.

Corollary 1.4.13. *The coefficients of $\chi_M(\lambda)$ alternate in sign. Hence, we can write*

$$\chi_M(\lambda) = \sum_{k=0}^{\rho(M)} (-1)^k \omega_k \lambda^{\rho(M)-k}.$$

Proof. Since the coefficients of $\chi_M(\lambda)$ are sums of evaluations of the Moebius function, we conclude directly from Corollary 1.4.8. □

Remark 1.4.14. Since $\mu(\emptyset, E) = -\sum_{F\neq E} \mu(\emptyset, F)$, we have that

$$\omega_{\rho(M)} = -\sum_{k=0}^{\rho(M)-1} (-1)^k \omega_k.$$

Hence, it is clear that $\chi_M(1) = 0$ for any matroid M.

Definition 1.4.15. We define the *reduced characteristic polynomial* $\bar{\chi}_M(\lambda)$ as

$$\bar{\chi}_M(\lambda) = \frac{\chi_M(\lambda)}{\lambda - 1};$$

its coefficient will be denoted $\bar{\omega}_k$.

Theorem 1.4.16. *Let M be a matroid and $i \in E$.*

- *If i is a coloop in M, then*

$$\chi_M(\lambda) = (\lambda - 1)\chi_{M \setminus i}(\lambda).$$

- *If i is not a coloop, then*

$$\chi_M(\lambda) = \chi_{M \setminus i}(\lambda) - \chi_{M/i}(\lambda).$$

Proof. The proof is based on computations on the coefficients similar to the ones performed before (See for example Lemma 7.13 in [17]). □

This technique is also called *deletion-contraction*, because it lets us compute the characteristic polynomial recursively, performing a series of these operations on the matroid.

Corollary 1.4.17. *The following result holds when i is not a coloop:*

$$\sum_{j=0}^{\rho(M)} \omega_j = \sum_{j=0}^{\rho(M \setminus i)} \omega(M \setminus i)_j + \sum_{j=0}^{\rho(M/i)} \omega(M/i)_j,$$

where $\omega(M \setminus i)$ and $\omega(M/i)$ denote the Whitney numbers of the first kind of the deletion and contraction matroid, respectively.

Proof. If i is not a coloop, $\rho(M \setminus i) = \rho(M)$, while $\rho(M/i) = \rho(M) - 1$. Hence, using Theorem 1.4.16, we can write

$$\begin{aligned}\chi_M(\lambda) = &\sum_{j=0}^{\rho(M)} (-1)^j \omega(M \setminus i)_j \lambda^{\rho(M)-j} - \sum_{j=0}^{\rho(M)-1} (-1)^j \omega(M/i)_j \lambda^{\rho(M)-1-j} = \\ =&\omega(M \setminus i)_0 \lambda^{\rho(M)} + \sum_{j=0}^{\rho(M)-1} (-1)^{j+1} (\omega(M \setminus i)_{j+1} + \omega(M/i)_j) \lambda^{\rho(M)-1-j},\end{aligned}$$

where we have isolated the 0-th term of the first sum and reordered the remaining terms to obtain the result as it is shown. We now use Corollary 1.4.8 to observe that we can obtain the sum of the Whitney numbers of the first kind of a matroid, simply computing $(-1)^{\rho(M)}\chi_M(-1)$. Therefore,

$$(-1)^{\rho(M)}\chi_M(-1) = \sum_{j=0}^{\rho(M)} \omega_j = \omega(M \setminus i)_0 + \sum_{j=0}^{\rho(M)-1} \left(\omega(M \setminus i)_{j+1} + \omega(M/i)_j\right),$$

and thus the result. □

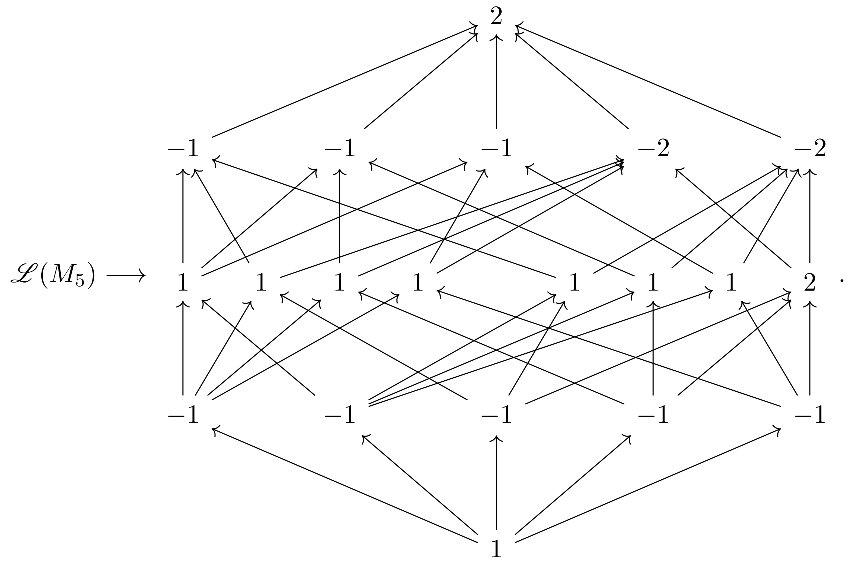
Corollary 1.4.18. *The Boolean matroid* B_n *has characteristic polynomial*

$$\chi_{B_n}(\lambda) = (\lambda - 1)^n.$$

Proof. We can iterate first part of Theorem 1.4.16 n times, since Boolean matroids are made of n coloops and $B_n \setminus i$ is isomorphic to B_{n-1}. □

We will now show how to use the result to find the characteristic polynomial of M_5 in two different ways.

Example 1.4.19. Let us compute the values of the Moebius function of $\mathscr{L}(M_5)$ and replace the flat F with the corresponding value $\mu(\emptyset, F)$ in the lattice.



Hence, the characteristic polynomial is

$$\chi_{M_5}(\lambda) = \lambda^4 - 5\lambda^3 + 9\lambda^2 - 7\lambda + 2.$$

Now let us verify the result using the deletion-contraction with $i = 5$:

$$\begin{aligned}\chi_{M_5}(\lambda) =&\chi_{M_5\setminus 5}(\lambda) - \chi_{M_5/5}(\lambda) = \\ =&\chi_{B_4}(\lambda) - \chi_{B_3}(\lambda) = (\lambda - 1)^4 - (\lambda - 1)^3 = \\ =&\lambda^4 - 5\lambda^3 + 9\lambda^2 - 7\lambda + 2.\end{aligned}$$

Or, similarly (always choosing $i = 1$ and then relabelling the elements accordingly),

$$\begin{aligned}\chi_{M_5}(\lambda) =&(\lambda - 1)\chi_{M_5\setminus 1}(\lambda) = (\lambda - 1)\chi_{M_4}(\lambda) = \\ =&(\lambda - 1)^2\chi_{M_4\setminus 1}(\lambda) = (\lambda - 1)^2\chi_{U_{2,3}}(\lambda) = \\ =&(\lambda - 1)^2[\chi_{U_{2,3}\setminus 1}(\lambda) - \chi_{U_{2,3}/1}(\lambda)] = (\lambda - 1)^2[\chi_{B_2}(\lambda) - \chi_{B_1}(\lambda)] = \\ =&(\lambda - 1)^2[(\lambda - 1)^2 - (\lambda - 1)] = \\ =&\lambda^4 - 5\lambda^3 + 9\lambda^2 - 7\lambda + 2.\end{aligned}$$

We now list the characteristic polynomials of all combinatorial geometries on at most four elements. One can find characteristic polynomials of bigger matroids taking their simplified version and operating deletions and contractions to reduce the problem to one of these cases, as we have shown above (note that we could have just given the value for B_1):

- $\chi_{B_1}(\lambda) = \lambda - 1$;
- $\chi_{B_2}(\lambda) = (\lambda - 1)^2$;
- $\chi_{U_{2,3}}(\lambda) = \lambda^2 - 3\lambda + 2$;
- $\chi_{B_3}(\lambda) = (\lambda - 1)^3$;
- $\chi_{U_{2,4}}(\lambda) = \lambda^2 - 4\lambda + 3$;
- $\chi_{U_{3,4}}(\lambda) = \lambda^3 - 4\lambda^2 + 6\lambda - 3$;
- $\chi_{M_4}(\lambda) = \lambda^3 - 4\lambda^2 + 5\lambda - 2$;
- $\chi_{B_4}(\lambda) = (\lambda - 1)^4$;

Observe that, since we are working with combinatorial geometries, the coefficient of $\lambda^{\rho(M)-1}$ is always equal to $|E|$, since the atoms of the lattice are all and only the singletons.

Remark 1.4.20. The characteristic polynomial of the uniform matroid $U_{k,n}$ is

$$\chi_{U_{k,n}}(\lambda) = \lambda^k - n\lambda^{k-1} + \binom{n}{2}\lambda^{n-2} - \ldots + (-1)^{k-1}\binom{n}{k-1}\lambda + (-1)^k\binom{n-1}{k-1}.$$

In fact, the Moebius function of $U_{k,n}$ agrees with the one of B_n for all terms except E, and

$$\omega_k = \mu(\emptyset, E) = -\sum_{F \subsetneq E} \mu(\emptyset, F) = -\sum_{i=0}^{k-1}(-1)^i\omega_i = -\sum_{i=0}^{k-1}(-1)^i\binom{n}{i}.$$

Now, it is easy to check that for any n,

$$\chi_{U_{2,n}} = \lambda^2 - n\lambda + n - 1.$$

To complete the proof, we suppose the formula is true for $U_{k,n}$ and show it holds for $U_{k+1,n}$.

$$\begin{aligned}
\omega_{k+1} = -\sum_{i=0}^{k}(-1)^i\binom{n}{i} &= -\left[\sum_{i=0}^{k-1}(-1)^i\binom{n}{i} + (-1)^k\binom{n}{k}\right] = \\
=&\omega_k - (-1)^k\binom{n}{k} = (-1)^k\binom{n-1}{k-1} - (-1)^k\binom{n}{k} = \\
=&(-1)^k\left[\binom{n-1}{k-1} - \binom{n}{k}\right] = \\
=&(-1)^{k+1}\binom{n-1}{k},
\end{aligned}$$

where last equality is given by Stifel's Formula.

At last, we can finally state the Heron-Rota-Welsh Conjecture, which was stated in the 1970s and it is now a theorem proved by Adiprasito-Huh-Katz.

Theorem 1.4.21. *(AHK). Let M be a matroid. The sequence of its Whitney numbers of the first kind ω_k form a sequence that is log-concave, that is*

$$\omega_{k-1}\omega_{k+1} \leq \omega_k^2,$$

for all $1 \leq k \leq \rho(M)$.
In particular, the sequence is unimodal.

One now may ask what kind of relevance this theorem has inside the theory of matroids. Firstly, log-concavity and unimodality are two properties shared by many sequences coming from every field of mathematics (we will see some examples coming from combinatorics in the next section). Secondly, the techniques used in the proof are very new and make great use of very different results in Lattice Theory, Toric Geometry and Algebraic Geometry (in particular, Hodge Theory); therefore, knowing how to use these new tools may lead the way in proving similar or stronger results. Lastly, the characteristic polynomial encodes useful information on the matroid and on several objects associated with it, as the next examples will show. Hence, any kind of information on the behaviour of its coefficients may be of great importance.

Example 1.4.22. Let G be a graph. We define as *q-colouring of G* any map

$$c : \mathbb{Z}_q \to V(G).$$

The colouring is said to be *proper* if

$$(v_i, v_j) \in E(G) \Rightarrow c(v_i) \neq c(v_j),$$

for any $v_i \neq v_j \in V(G)$. Informally, we can think of this as painting every vertex of the graph choosing from q different colours and saying it is a proper colouring if no two joined vertices have the same colour. We want to discover how many proper q-colouring a graph has, with q fixed. It is trivial to see that the result is a polynomial in the variable q; we call this polynomial $\chi_G(q)$, *chromatic polynomial of G*. The following result holds:

Theorem 1.4.23. *Let G be a graph.*

$$\chi_G(q) = q^n \chi_{M(G)}(q),$$

where n is the number of connected components of G and $M(G)$ is the graphic matroid associated to G.

Proof. We prove the result supposing G is connected, since the factor q^n, depends only on the fact that we have q possibilities of choosing the first colour for any connected component of G.
Firstly, we observe that if G has a loop, clearly $\chi_G(q) = 0$, since there cannot be a proper q-colouring for any q. Next, we observe that having parallel edges does not change the number of proper q-coloring, since we are not adding any extra condition on the connection of two vertices. We can now just prove that the formula holds for connected simple graphs. The way we do it is by showing

that we can find the same recursion for deletion-contraction. If G consists of just one coloop, clearly,

$$\chi_G(q) = q(q-1) = q\chi_{B_1}(q).$$

We are left to prove that

$$\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q),$$

for any edge $e \in E$. This is true because $\chi_{G\setminus e}(q)$ counts all the q-colourings of G that are proper everywhere except for maybe on edge e; $\chi_{G/e}(q)$ counts the q-colourings of G that are proper everywhere and are not proper in e (where we could then perform a contraction to obtain a proper q-colouring of G/e). In particular, if $e = (v_i, v_j)$ is a coloop, it can be shown that $\chi_{G\setminus e}(q) = q\chi_{G/e}(q)$. Hence,

$$\chi_G(q) = (q-1)\chi_{G/e}(q).$$

Since the two polynomials agree on the base cases and follow the same recursion, the theorem is proved. □

Example 1.4.24. Let M be a simple representable matroid on $\mathbb{R}$. Consider the family $\mathscr{H} = \{H_i\}$, where H_i is the hyperplane orthogonal to v_i in $V = \mathbb{R}^N$. Denote $A = V \setminus \mathscr{H}$, the complement of the union of said hyperplanes. Then, the following theorem holds

Theorem 1.4.25. *The number of connected components $r(\mathscr{H})$ of A is*

$$r(\mathscr{H}) = \sum_j \omega_j = |\chi_M(-1)|.$$

Proof. The second equivalence comes from Corollary 1.4.17. Let $M = B_1$ and $E = \{e_1\}$. $H_1 = 0$; $r(A) = r(\mathbb{R} \setminus \{0\}) = 2$ and

$$\sum_j \omega_j = 1 + 1 = 2.$$

We can conclude the proof by showing $r(\mathscr{H})$ follows the same recursion for deletion-contraction. Consider an element v_i: deletion of v_i results in the deletion of H_i from $\mathscr{H}$, denoted $\mathscr{H} \setminus H_i$; contraction by v_i gives us the family $\mathscr{H}/H_i := \{H_j \cap H_i | j \neq i\} \subseteq H_i$, (note that the contraction may give us less than $n-1$ hyperplanes in $\mathbb{R}^{N-1}$).
Let us consider the number of regions in the deletion, $r(\mathscr{H} \setminus H_i)$: inserting back H_i cuts some of the regions in two, which means that

$$r(\mathscr{H}) = r(\mathscr{H} \setminus H_i) + |\{\text{regions cut in two}\}|.$$

But each region that was cut in two intersects H_i in one of the regions of $\mathscr{H} \setminus H_i$, so the latter summand is exactly $r(\mathscr{H}/H_i)$. □

Similarly, if we make the same construction over a field $\mathbb{F}_q$, and set $p(\mathscr{H})$ to be the number of points in $A = (\mathbb{F}_q)^N \setminus \mathscr{H}$, the following theorem holds

Theorem 1.4.26.

$$p(\mathscr{H}) = \chi_M(q).$$

Proof. For $M = B_1$, we have in $\mathbb{F}_q$, the hyperplane $H_1 : \{x_1 = 0\}$, so

$$p(\mathscr{H}) = q - 1 = \chi_{B_1}(q).$$

In general, if we have $\mathscr{H} = \{H_1, \dots, H_n\}$, Let us consider $p(\mathscr{H} \setminus H_i)$. If we add back H_i, to compute $p(\mathscr{H})$ we only need to subtract the number of points in $H_i \setminus (\bigcup H_i \cap H_j)$. These are exactly the points in the complement of $\mathscr{H}/H_i$, hence the result. □

Example 1.4.27. Let us compute explicitly the result for B_n. Consider the canonical basis in $\mathbb{F}_q$, $e_i = (\delta_{i,j})$. Therefore,

$$\mathscr{H} = \{H_i : x_i = 0\}.$$

By the inclusion-esclusion principle,

$$\begin{aligned} \left|\bigcup H_i\right| =& \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{\mathscr{I}_k \subseteq \{1,\dots,n\} \\ |\mathscr{I}_k| = k}} \left| \bigcap_{i \in \mathscr{I}_k} H_i \right| = \\ =& nq^{n-1} - \binom{n}{2} q^{n-2} + \dots + (-1)^{n-1} = \\ =& \sum_{k=1}^{n} \binom{n}{k} (-1)^k q^{n-k}. \end{aligned}$$

Hence,

$$\left| \left(\bigcup H_i\right)^C \right| = q^n - \sum_{k=1}^{n} \binom{n}{k} (-1)^k q^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{n-k} = (q-1)^n.$$

Example 1.4.28. Lastly, if we construct the hyperplanes over $\mathbb{C}$ and consider $A = \mathbb{C}^N \setminus \mathscr{H}$, we have that

$$\omega_k = \beta_k(A),$$

where β_k denotes the k-th Betti number in the cohomology ring $H^*(A)$. While we will not prove the result (see, for example, [15]), we can show that it holds on the easiest example. Let $M = B_1$, $A = \mathbb{C}^* \cong \mathbb{S}^1$, so

$$H^0(A) = 1 = \omega_0,$$

$$H^1(A) = 1 = \omega_1.$$

1.5 Unimodality and log-concavity

In this section we state some properties and provide some known examples of log-concave sequences appearing in combinatorics. We will follow and comment on [12] and [16].

Definition 1.5.1. A real function f is said to be *log-concave* if its composition with the log function is concave. That is, for any $x \leq z \leq y$

$$\frac{(\log \circ f)(x) + (\log \circ f)(y)}{2} \leq (\log \circ f)(z).$$

Using know properties of logarithms this is equivalent to

$$f(x)f(y) \leq (f(z))^2 .$$

A sequence of real numbers $a_0 \ldots, a_n$ is log-concave if, for any $0 < i < n$,

$$a_{i-1}a_{i+1} \leq a_i^2.$$

Example 1.5.2. The Normal Gaussian function $f(x) = e^{-x^2}$ is not concave but it is log-concave, since

$$\log \circ f(x) = -x^2.$$

Definition 1.5.3. A sequence of real numbers $a_0, \ldots, a_n$ is said to be *unimodal* if there exists $0 \leq k \leq n$ such that

$$a_0 \leq a_1 \leq \ldots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \ldots \geq a_n.$$

Remark 1.5.4. It is easy to see that, if we do not have internal zeroes (that is, $a_i \neq 0$ for $0 < i < n$), log-concavity implies unimodality. In fact, suppose there is a j such that $a_{j-1} > a_j < a_{j+1}$, then

$$a_{j-1}a_{j+1} > a_j^2,$$

which is a contradiction.

We say a polynomial is log-concave if its coefficients form a log-concave sequence.

Theorem 1.5.5. *If $A(q)$ and $B(q)$ are log-concave polynomials, then so is $A(q)B(q)$.*

Proof. If $\deg A = m$ and $\deg B = n$, we can write

$$A(q) = \sum_{i=0}^{m+n} a_i q^i,$$

$$B(q) = \sum_{j=0}^{m+n} b_j q^j,$$

and consider the following matrices

$$X = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_{m+n} \\ 0 & a_0 & a_1 & \ldots & a_{m+n-1} \\ 0 & 0 & a_0 & \ldots & a_{m+n-2} \\ & & \ldots & & \\ 0 & \ldots & & 0 & a_0 \end{pmatrix}, Y = \begin{pmatrix} b_0 & b_1 & b_2 & \ldots & b_{m+n} \\ 0 & b_0 & b_1 & \ldots & b_{m+n-1} \\ 0 & 0 & b_0 & \ldots & b_{m+n-2} \\ & & \ldots & & \\ 0 & \ldots & & 0 & b_0 \end{pmatrix}.$$

From the log-concavity of the coefficients, we can see that every 2×2 minor of X and Y is non-negative. Using Cauchy-Binet identity for $k = 2$ we can then see that

$$\det XY = \sum_{1 \leq j_1 < j_2 \leq m+n} \det (A_{j_1,j_2}) \det (B_{j_1,j_2});$$

this also shows that any 2×2 minor of XY is non-negative. But the entries of XY are the coefficients of $(AB)(q)$, hence the result. ☐

For our purposes, an important consequence of this fact is the following

Corollary 1.5.6. *Theorem 1.4.21 holds if $\bar{\chi}_M(\lambda)$ is log-concave.*

Proof. The characteristic polynomial $\chi_M(\lambda)$ can be written as

$$\chi_M(\lambda) = (\lambda - 1)\bar{\chi}_M(\lambda),$$

which is a product of two log-concave polynomials. □

Example 1.5.7. One of the most famous examples of log-concave sequences is the n-th row of Pascal's triangle. In fact, for any $0 < k < n$,

$$\begin{aligned}\binom{n}{k-1}\binom{n}{k+1} =& \frac{n!n!}{(n-k+1)!(k-1)!(n-k-1)!(k+1)!} = \\ =& \frac{(n!)^2(n-k)k}{((n-k)!)^2(k!)^2(n-k+1)(k+1)} = \\ =& \binom{n}{k}^2 \frac{(n-k)k}{(n-k+1)(k+1)} \leq \binom{n}{k}^2.\end{aligned}$$

Remark 1.5.8. If our sequences have a combinatorial meaning, that is there exist sets $S_0, \ldots, S_n$ such that $a_i = |S_i|$, log-concavity can be proved by exhibiting explicit injections

$$\Phi_j = S_{j-1} \times S_{j+1} \to S_j \times S_j,$$

while unimodality can be proved by exhibiting explicit injections

$$\phi_j : S_j \to S_{j+1},$$

for $0 \leq j < k$ and explicit surjections

$$\psi_j : S_j \to S_{j+1},$$

for $k \leq j < n$.

We can then give a combinatorial proof for Example 1.5.7.

Proof. Let us define $\Phi_k : S_{k-1} \times S_{k+1} \to S_k \times S_k$ in the following way: define $X_j := X \cap \{1, \ldots, j\}$ and $X_j^C = X \setminus X_j$. Let $(A, B) \in S_{k-1} \times S_{k+1}$ and choose j to be the maximal element such that $|A_j| = |B_j| - 1$ (such an element exists because $|A_0| = |B_0| = 0$ and $|A_n| = |B_n| - 2$). Then,

$$\Phi_k(A, B) = (B_j \cup A_j^C, A_j \cup B_j^C).$$

This is indeed a map onto $S_k \times S_k$, since if we let $|A_j| = x$, we have that $|B_j \cup A_j^C| = (x+1) + [(k-1) - x] = k$ and $|A_j \cup B_j^C| = x + [(k+1) - (x+1)] = k$. To complete the proof we show that an inverse for Φ_k can be found. Let (C, D) be an element of $S_k \times S_k$ mapped onto by Φ_k. There must exist a maximal element i such that $|D_i| = |C_i| - 1$; then

$$\Psi_k(C, D) = (D_i \cup C_i^C, C_i \cup D_i^C)$$

is an inverse for Φ_k, hence the result. □

Example 1.5.9. The *Stirling numbers of the first kind*, denoted $s(n,k)$, are the coefficients of the polynomial

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k.$$

These coefficients alternate in sign and also have combinatorial meaning: the sequence $c_n(k) = |s(n,k)|$ counts the number of permutations of n elements having exactly k disjoint cycles. For fixed n, this is a log-concave sequence. For $n = 0$ the result is vacuously true. Let us prove it for n, supposing it holds for $n-1$. It is easy to prove that

$$c_n(k) = c_{n-1}(k-1) + (n-1)c_{n-1}(k).$$

Suppose you want to create a permutation of n objects with k cycles, starting from a permutation on $n-1$ objects. This can be done in two ways:

- We could insert the n-th element as a singleton cycle (i.e. a fixed point) in any permutation with $k-1$ cycles;
- We could take a k-cycle permutation and insert the n-th element in one of the existing cycles. Let us write the first permutation as

  $$(a_1 \cdots a_{j_1})(a_{j_1+1} \cdots a_{j_2}) \cdots (a_{j_{k-1}+1} \cdots a_{n-1}),$$

  where the elements of each cycle are in lexicographic order and the cycles are then ordered again lexicographically to avoid considering a permutation more than once. Insertion of a_n can then be made in $n-1$ different ways.

This recursion formula can be used to reduce the log-concavity inequality to

$$\begin{cases} c_{n-1}(k)c_{n-1}(k-2) \leq c_{n-1}(k-1)^2 \\ (n-1)^2 c_{n-1}(k-1)c_{n-1}(k+1) \leq +(n-1)^2 c_{n-1}(k)^2 \\ (n-1)\left[c_{n-1}(k-1)c_{n-1}(k) + c_{n-1}(k-2)c_{n-1}(k+1)\right] \leq 2(n-1)c_{n-1}(k-1)c_{n-1}(k) \end{cases}.$$

By induction hypothesis the first two inequalities are satisfied; the third one can be then reduced to

$$c_{n-1}(k-2)c_{n-1}(k+1) \leq c_{n-1}(k-1)c_{n-1}(k);$$

multiplying each side by $c_{n-1}(k)$ and applying repeatedly the induction hypothesis gives us the result.

Example 1.5.10. The *Stirling numbers of the second kind*, denoted $S(n,k)$, count the number of k-block partitions of a set of n elements. They satisfy the following recursion (its proof is very similar to the one for $c_n(k)$):

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

This can be used to show that, for fixed n, the sequence $S(n,k)$ is log-concave.

The next, and last, example should give a sense of how important a general theorem on matroids such as 1.4.21 can be.

Example 1.5.11. Consider the following conjectures:

- (Read-Hoggar). In any graph G the absolute value of the coefficients of the chromatic polynomial (see Example 1.4.22) form a log-concave sequence.
- (Welsh). Let V be a $\mathbb{K}$-vectore space, A a finite subset of V and $f_i(A)$ the number of linearly independent subsets of A of size i. These form a log-concave sequence.

Proof. The proofs of these conjectures were first given in [14]. Now they can actually be seen as easy corollaries of Theorem 1.4.21. □

Chapter 2

Hodge Theory background

As we have said in the first chapter, the theorem we are aiming to prove is purely combinatorial, but its proof relies on various results from Hodge Theory, also called the Kähler package. This set of theorems will be used "only" to build an axiomatic algebraic structure (which mimics the cohomology ring of a Kähler manifold), so the proof we give could be studied without all the background in Algebraic Geometry we are going to give; however, we think that it is more useful to get a sense of the geometric origin of said tools, especially if one wants to build new ones using similar techniques. While trying not to get too technical, we will mainly follow [10] and [11].

2.1 Linear Algebra tools

In this first section we introduce some notions for a vector space, which we will be needed to study manifolds through their tangent bundles.

Definition 2.1.1. Let $(V, < -, - >)$ be a Euclidean $\mathbb{R}$-vector space of dimension N and $\{e_1, \ldots, e_N\}$ a basis for V. Denote

$$e_I := e_{i_1} \wedge \ldots \wedge e_{i_p},$$

with $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\}$. We define the *exterior algebra of* V as

$$\Lambda(V) := \bigoplus_{0 \leq k \leq N} \Lambda^k(V),$$

where $\Lambda^k(V) := \mathbb{R} < e_I >_{|I|=k}$ is the *space of k-forms.* We can define a natural inner product $(-,-)$ on $\Lambda(V)$, by declaring each $\Lambda^k(V)$ orthogonal to the others and

$$(v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k) = \det || < v_i, w_j > ||.$$

Remark 2.1.2. The space $\Lambda^k(V)$ has dimension $\binom{N}{k}$. In particular,

$$\dim \Lambda^N(V) = 1.$$

Hence, $\Lambda^N(V)^* := \Lambda^N(V) \setminus \{0\}$ has two connected components.

Definition 2.1.3. Declaring one of the two connected components of $\Lambda^N(V)^*$ to be positive, is called an *orientation* of V. Such component will be denoted by $\Lambda^N(V)^+$.

Definition 2.1.4. Let $\{e_1, \ldots, e_N\}$ be a positive orthonormal basis for an oriented Euclidean space $(V, < -, - >, \Lambda^N(V)^+)$. We call *volume element* the N-form

$$dV := e_1^* \wedge \ldots \wedge e_N^* \in \Lambda^N(V^*)^+.$$

This is well-defined since, given another positive orthonormal basis $e_i' = Ae_j$,

$$e_1'^* \wedge \ldots \wedge e_N'^* = (\det A) e_1^* \wedge \ldots \wedge e_N^*$$

and $\det A = +1$ because A is a positive isometry.

Lastly, we define the *Hodge operator*:

Definition 2.1.5. Let $\big(V, < -, - >, \Lambda^N(V)^+\big)$ be an oriented Euclidean vector space. Define

$$\begin{aligned} * : \Lambda^k(V) &\longrightarrow \Lambda^{N-k}(V) \\ e_I &\longmapsto \varepsilon(I, c(I)) e_{C(I)} \end{aligned}$$

where $c(I)$ is the complement of I in the set $\{1, \ldots, N\}$ and $\varepsilon(I, c(I))$ is the sign of the permutation that reorders the indexes from $(I, c(I))$ to $(1, \ldots, n)$. This is equivalent to say that

$$e_I \wedge *e_I = dV.$$

Remark 2.1.6. The following properties of $*$ can be easily checked.

- $* : \Lambda^k(V) \cong \Lambda^{N-k}(V)$,
- $u \wedge *v = (u, v) dV$, for any $u, v \in \Lambda^k(V)$,
- $*(1) = dV$ and $*(dV) = 1$,
- $**_{|\Lambda^k(V)} = (-1)^{k(N-k)} Id_{|\Lambda^k(V)}$.

2.2 Compact, oriented, Riemannian manifolds

The main purpose of this section is to revise the main features of manifolds and state the Hodge decomposition theorem, the Hodge isomorphism theorem and the Poincaré duality theorem.

Let M be a manifold and $(U; x)$ a local chart centered around $m \in M$; M comes equipped with the following vector bundles:

- The *tangent bundle* $T(M; \mathbb{R})$, whose fiber at m is the *tangent space* $T_m(M; \mathbb{R})$, that is, the real span $T_m(M; \mathbb{R}) = \mathbb{R} < \partial_{x_1}, \ldots, \partial_{x_N} >$, where $\{\partial_{x_i}\}$ represent the directional derivatives;
- The *cotangent bundle* $T^*(M; \mathbb{R})$, whose fiber at m is the *cotangent space* $T_m^*(M; \mathbb{R})$, that is the real span $T_m^*(M; \mathbb{R}) = \mathbb{R} < dx_1, \ldots, dx_N >$, where $\{dx_i\}$ is the dual basis of $\{\partial_{x_i}\}$;

- The *exterior algebra bundle*

$$\Lambda(T^*(M;\mathbb{R})) = \bigoplus_{0\le k\le N} \Lambda^k(T^*(M;\mathbb{R})),$$

where the *k-th exterior bundle* $\Lambda^k(T^*(M;\mathbb{R}))$ has as fibers the real span $\Lambda^k(T^*_m(M;\mathbb{R})) = \mathbb{R} < dx_I >_{|I|=k}$.

Definition 2.2.1. The elements of the real vector space

$$E^k(M) := C^\infty(M, \Lambda^k(T^*(M;\mathbb{R})))$$

are called *k-forms* on M. If u is a k-form, this means that

$$u(m) \in \Lambda^k(T^*_m(M;\mathbb{R}));$$

on a local chart $(U;x)$, u can be written as

$$u = \sum_{|I|=k} u_I dx_I.$$

We denote by

$$d : E^k(M) \to E^{k+1}(M)$$

the exterior derivation of differential forms; locally, we can write

$$du = \sum_j \frac{\partial u_I}{\partial_{x_j}} dx_j \wedge dx_I.$$

Definition 2.2.2. We define the *k-th deRham cohomology group* $H^k_{dR}(M;\mathbb{R})$ as the quotient group

$$H^k_{dR}(M;\mathbb{R}) = \frac{\text{closed } k\text{-forms on } M}{\text{exact } k\text{-forms on } M}.$$

Let us now describe the three properties cited in the title of this section.

Definition 2.2.3. A manifold M is

- *Riemannian* if it is equipped with a *Riemannian metric* g, which is a smoothly-varying family of inner products $g = \{g_m\}_{m\in M}$, each defined on a fiber $T_m(M;\mathbb{R})$ of the tangent bundle.
- *Oriented* if $\Lambda^N(T^*(M;\mathbb{R})) \setminus M$ has two connected components; declaring one to be positive is an *orientation* of M.
- *Compact*, if it is compact as a topological space.

We now want to define the *volume* of a manifold.

Remark 2.2.4. Let (M,g) be a Riemannian manifold. Its metric g can be seen as a non-degenerate bilinear form, which can then be represented by a matrix G_U on a chart U. Consider the matrices

$$G^*_U := G_U^{-1};$$

if the matrices G_U are positive definite and symmetric, then so are the matrices G^*_U and, therefore, that lets us define a non-degenerate bilinear form on the fibers of the cotangent bundle, called *dual metric*.

Definition 2.2.5. Let M be an oriented, Riemannian manifold (orientation is needed for the integral to be well-defined). We call *Riemannian volume element* the only positive N-form dV on M, such that dV_m is the volume element for $(T^*_m(M;\mathbb{R}), g^*_m)$. If the integral

$$\int_M dV$$

converges it is called the *volume* of M. If M is a compact, oriented, Riemannian manifold the volume is always well-defined.

Theorem 2.2.6. *(Stokes) Let M be an oriented manifold of dimension N and let u be a $(N-1)$-form with compact support. Then,*

$$\int_M du = 0.$$

Note that this is a weaker version of the theorem, but it is enough for our purposes.

Theorem 2.2.7. *(Poincaré Duality) Let M be a compact, oriented manifold. The map*

$$(u, v) \longmapsto \int_M u \wedge v$$

induces isomorphisms

$$H^k_{dR}(M;\mathbb{R}) \cong H^{N-k}_{dR}(M;\mathbb{R})^\vee,$$

for every k.

The Hodge operator $*$, defined on the exterior algebra bundle, can be extended to another operator on the differential forms, which we will still denote using $*$. On a local chart $(U;x)$ around m, for a k-form u, we can write

$$*u(m) = * \left(\sum_{|I|=k} u_i(m) dx_I \right) = \sum_{|I|=k} u_i(m) * (dx_I) \in \Lambda^{N-k}(M),$$

hence

$$* : E^k(M) \to E^{N-k}(M).$$

This also lets us define an inner product $<< -, - >>$ on the k-forms with compact support

$$<< u, v >> := \int_M u \wedge *v = \int_M (u, v) dV,$$

which is an inner product because $(-,-)$ is an inner product and the integral is well-defined and positive. Note that the hypothesis of compactness for M guarantees that $<< -, - >>$ is defined on all forms.

Definition 2.2.8. Let us denote by $d^* : E^k(M) \to E^{k-1}(M)$ the adjoint operator of d, that is

$$<< du, v >> = << u, d^* v >>,$$

for suitable forms on M.

Remark 2.2.9. Let M be a compact, oriented, Riemannian manifold. Then,

$$d^* = (-1)^{N(k+1)+1} * d * .$$

Proof.

$$\begin{aligned} << du, v >>= & \int_M du \wedge *v = \int_M d(u \wedge *v) - (-1)^{k-1} u \wedge d * v = \\ &= \int_M d(u \wedge *v) + \int_M (-1)^k u \wedge *v = \\ &= \int_M (-1)^k u \wedge (-1)^{(N-k+1)(k-1)} * *d * v = \\ &= \int_M u \wedge * \Big((-1)^{N(k+1)+1} * d * v \Big) = \\ &= << u, (-1)^{N(k+1)+1} * d * v >>, \end{aligned}$$

where, in the fourth equality we simplified the first term using Stokes' Theorem 2.2.6 and we used one of the properties stated in 2.1.6 for the second term. □

Definition 2.2.10. We define the *Laplace operator* $\Delta : E^k(M) \to E^k(M)$ as

$$\Delta := d^* d + dd^*.$$

The previous remark shows us that we could describe Δ using only d and $*$. We denote by

$$\mathcal{H}^k(M; \mathbb{R}) := \ker \Delta,$$

and call $u \in \mathcal{H}^k(M; \mathbb{R})$ a *k-th harmonic form.*

Remark 2.2.11. Let us recall that, since $*$ depends on the metric of M, so do the spaces of harmonic forms $\mathcal{H}^k(M; \mathbb{R})$.

Lemma 2.2.12. *The following hold:*

- $\Delta * = * \Delta$*;*
- $\Delta^* = \Delta$*, that is, the Laplace operator is self-adjoint.*

Proof. For the first statement, let us compute

$$\begin{aligned} \Delta * - * \Delta = & d^* d * + dd^* * - * d^* d - * dd^* = \\ = & (-1)^{N(k+1)+1} * d * d * - * d (-1)^{N(k+1)+1} * d * + \\ & d (-1)^{N(k+1)+1} * d (-1)^{k(N-k)} - (-1)^{k(N-k)} (-1)^{N(k+1)+1} d * d = 0. \end{aligned}$$

For the second statement we observe that

$$\begin{aligned} & << \Delta u, v >> - << u, \Delta v >>= \\ = & << d^* du, v >> + << dd^* u, v >> - << u, d^* dv >> - << u, dd^* v >>= \\ = & << du, dv >> + << d^* u, d^* v >> - << du, dv >> - << d^* u, d^* v >>= 0. \end{aligned}$$

□

Lemma 2.2.13.

$$u \in \mathcal{H}^k(M;\mathbb{R}) \Leftrightarrow du = 0 \text{ and } d^*u = 0.$$

Proof. The result follows immediately by observing that

$$\begin{aligned} << \Delta u, u >>= << dd^*u, u >> + << u, dd^*u >>= \\ = ||du||^2 + ||d^*u||^2. \end{aligned}$$

□

We are finally ready to state the main theorems of this section.

Theorem 2.2.14. *(Hodge orthogonal decomposition) Let M be a compact, oriented, Riemannian manifold. Then, $\dim_{\mathbb{R}} \mathcal{H}^k(M;\mathbb{R}) < \infty$ for any k and we have the following $<< -, - >>$-orthogonal decomposition*

$$E^k(M) = \mathcal{H}^k(M;\mathbb{R}) \oplus d\left(E^{k-1}(M)\right) \oplus d^*\left(E^{k+1}(M)\right).$$

Proof. See Theorem 6.8 in [18] for a full proof. □

Theorem 2.2.15. *(Hodge isomorphism theorem) Let M be a compact, oriented, Riemannian manifold. Then,*

$$H^k_{dR}(M;\mathbb{R}) \cong \mathcal{H}^k(M;\mathbb{R}),$$

for every k.

Proof. Since d^* is the adjoint of d, if $u \in \ker d$ and $v = d^*w \in Imd^*$,

$$<< u, v >>=<< du, w >>= 0,$$

hence,

$$\ker d = \left(Imd^*\right)^{\perp}.$$

But from the Hodge decomposition, we know that

$$\ker d = (Imd^*)^{\perp} = \mathcal{H}^k \oplus Imd,$$

hence the result. □

Lemma 2.2.16. *Let $\alpha = [u + dv] \in H^k_{dR}(M;\mathbb{R})$, with u a k-closed form and $v \in E^{k-1}(M)$. Then u is harmonic if and only if it has minimal norm amongst the representatives of α.*

Proof. Supposing u is harmonic,

$$||u + dv||^2 = ||u||^2 + ||v||^2 + 2 << u, dv >>\geq ||u||^2 + 2 << d^*u, v >>= ||u||^2.$$

Conversely, we have that

$$0 = \frac{d}{dt}\left(||u + tdv||^2\right)_{|t=0} = 2 << u, dv >>= 2 << d^*u, v >>,$$

for any $v \in E^{k-1}(M)$. Hence, $d^*u = 0$. □

This means that, once fixed a Riemannian metric g, for each class in the cohomology groups of M we can use a special representative, that is the unique harmonic form. We can then give a proof of Theorem 2.2.7.

Proof. Fix a Riemannian metric g. We know from 2.2.12 that a form u is harmonic if and only if $*u$ is. In fact,

$$\Delta * u = *\Delta u = *0 = 0$$

and conversely,

$$\Delta u = \Delta * *(-1)^{(N-k)k} u = (-1)^{(N-k)k} * \Delta * u = 0.$$

Using the equality

$$\int_M u^* \wedge u = \int_M (u,u) dV = ||u||^2,$$

we see that if u is a non-zero harmonic k-form, there is a non-zero harmonic $(N-k)$-form, $*u$, that pairs non-trivially with u. This gives injective morphisms between spaces, that are finite-dimensional from 2.2.14, hence, isomorphisms. □

Remark 2.2.17. Since we are working with finite-dimensional spaces, we have non-canonical isomorphisms between a space and its dual. Therefore, Poincaré Duality gives us

$$H^k_{dR}(M) \cong H^{N-k}_{dR}(M)^\vee \cong H^{N-k}_{dR}(M).$$

We could actually find a direct isomorphism combining the induced isomorphism

$$* : \mathcal{H}^k(M) \cong \mathcal{H}^{N-k}(M),$$

found in the previous proof, with 2.2.15. However, this is still a non-canonical isomorphism because it depends on the chosen metric g.

2.3 Hermitean manifolds

In this section we define suitable vector bundles and (p,q)-forms on complex manifolds, and find Hodge-theoretic results for Hermitean manifolds.

Definition 2.3.1. Let M be a topological manifold of dimension $2n$. We say that M is a *complex manifold* if it has a maximal holomorphic atlas, that is a maximal collection of complex charts in which the local changes of coordinates are holomorphic.

Example 2.3.2. The projective space $\mathbb{P}^n(\mathbb{C})$ is a complex manifold. Recall we define

$$\mathbb{P}^n(\mathbb{C}) := \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\sim},$$

where $u \sim v$ if there exists $t \in \mathbb{C}^*$ such that $u = tv$. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n(\mathbb{C})$ be the quotient map. We equip $\mathbb{P}^n(\mathbb{C})$ with the quotient topology, that is

$U \subseteq \mathbb{P}^n(\mathbb{C})$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{C}^{n+1} \setminus \{0\}$. We state that the following is an holomorphic atlas for $\mathbb{P}^n(\mathbb{C})$. Let

$$U_0 = \{p = [u_0; \dots; u_n] | u_0 \neq 0\},$$

and let

$$\begin{aligned} z^{(0)} : U_0 &\longrightarrow \mathbb{C} \\ [z_0; \dots; z_n] &\longmapsto \left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0} \right). \end{aligned}$$

Similarly, we define U_i and $z^{(i)}$. Indeed $\mathbb{P}^n(\mathbb{C}) = \bigcup_{i=0}^n U_i$. Let us verify the compatibility on $U_0 \cap U_1 = \{u_0, u_1 \neq 0\}$.

$$z^{(1)} \circ (z^{(0)})^{-1} : (t_1, \dots, t_n) \mapsto z^{(1)}[1; t_1; \dots; t_n] \mapsto \left(\frac{1}{t_1}, \frac{t_2}{t_1}, \dots, \frac{t_n}{t_1} \right),$$

which is a holomorphic function.

Remark 2.3.3. A complex manifold of dimension $\dim_{\mathbb{C}} M = n$ also has a structure of differential manifold of dimension $\dim_{\mathbb{R}} M = 2n$, if we make the identification

$$z = (z_1, \dots, z_n) \leftrightarrow (x_1, \dots, x_n, y_1, \dots, y_n),$$

where $z_j = x_j + iy_j$.

Let us now define proper vector bundles for complex manifolds.

Definition 2.3.4. Let M be a complex manifold and $(U; z | z_j = x_j + iy_j)$ a chart centered around $m \in M$. We define

- The *tangent real bundle*, $T(M; \mathbb{R})$, whose fiber at m can be seen as the $2n$-dimensional real span

$$T_m(M; \mathbb{R}) = \mathbb{R} < \partial_{x_1} \dots, \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n} >;$$

- The *tangent complex bundle*, $T(M; \mathbb{C}) = T(M; \mathbb{R}) \otimes \mathbb{C}$, whose fiber at m can be seen as the $2n$-dimensional complex span

$$T_m(M; \mathbb{C}) = \mathbb{C} < \partial_{x_1} \dots, \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n} > .$$

Notice that neither of these bundles matches the dimension of our manifold. We would like to introduce n-dimensional bundles. First of all we notice that $T(M; \mathbb{R})$ can be embeddded in $T(M; \mathbb{C})$ by means of the map

$$\begin{aligned} T(M; \mathbb{R}) &\hookrightarrow T(M; \mathbb{C}) \\ v &\mapsto (v \otimes 1). \end{aligned}$$

Let us define new coordinates

$$\partial z_j := \frac{1}{2} \left(\partial_{x_j} - i \partial_{y_j} \right),$$

$$\partial \bar{z}_j := \frac{1}{2} \left(\partial_{x_j} + i \partial_{y_j} \right).$$

This is a suitable change of coordinates, hence

$$T_m(M; \mathbb{C}) = \mathbb{C} < \partial_{z_1}, \dots, \partial_{z_n}, \partial_{\bar{z}_1}, \dots, \partial_{\bar{z}_n} > .$$

Definition 2.3.5. We call $T^{(1,0)}(M;\mathbb{C})$, *holomorphic tangent bundle*, the bundle whose fiber at m is the n-dimensional complex span

$$T^{(1,0)}_m(M;\mathbb{C}) = \mathbb{C} < \partial_{z_1}, \ldots, \partial_{z_n} >;$$

this is what we will use as a substitute for the tangent bundle. Similarly, we call $T^{(0,1)}(M;\mathbb{C})$, *anti-holomorphic tangent bundle*, the bundle whose fiber at m is the n-dimensional complex span

$$T^{(0,1)}_m(M;\mathbb{C}) = \mathbb{C} < \partial_{\bar{z}_1}, \ldots, \partial_{\bar{z}_n} >;$$

it is easy to check that

$$T(M;\mathbb{C}) = T^{(1,0)}(M;\mathbb{C}) \oplus T^{(0,1)}(M;\mathbb{C}).$$

Remark 2.3.6. We have the following isomorphisms

$$\begin{gathered} T^{(1,0)}(M;\mathbb{C}) \cong T(M;\mathbb{R}) \cong T^{(0,1)}(M;\mathbb{C}) \\ \partial_{z_j} \leftarrow\!\shortmid \partial_{x_j} \mapsto \partial_{\bar{z}_j} \\ i\partial_{z_j} \leftarrow\!\shortmid \partial_{y_j} \mapsto -i\partial_{\bar{z}_j}. \end{gathered}$$

Notice also that the complex conjugation gives isomorphisms

$$T^{(0,1)}(M;\mathbb{C}) = \overline{T^{(1,0)}(M;\mathbb{C})}.$$

Definition 2.3.7. By considering $\{dz_j = dx_j + idy_j\}$, dual basis of $\{\partial_{z_j}\}$, we can define the *cotangent real bundle* $T^*(M;\mathbb{R})$, the *cotangent complex bundle* $T^*(M;\mathbb{C})$, the *cotangent holomorphic bundle* $T^{(1,0)*}(M;\mathbb{C})$ and the *cotangent anti-holomorphic bundle* $T^{(0,1)*}(M;\mathbb{C})$. The analogous relation

$$T^*(M;\mathbb{C}) = T^{(1,0)*}(M;\mathbb{C}) \oplus T^{(0,1)*}(M;\mathbb{C}),$$

holds.

Remark 2.3.8. We also give similar isomorphisms

$$\begin{gathered} T^{(1,0)*}(M;\mathbb{C}) \cong T^*(M;\mathbb{R}) \cong T^{(0,1)*}(M;\mathbb{C}) \\ \frac{1}{2}dz_j \leftarrow\!\shortmid dx_j \mapsto \frac{1}{2}d\bar{z}_j \\ \frac{1}{2i}dz_j \leftarrow\!\shortmid dy_j \mapsto -\frac{1}{2i}d\bar{z}_j \end{gathered}$$

and as before we notice that

$$T^{(0,1)*}(M;\mathbb{C}) = \overline{T^{(1,0)*}(M;\mathbb{C})}.$$

This is not the only way we can define these bundles on M. In fact, let us consider a $2n$-dimensional manifold. The only thing that we need to add to get a complex manifold is the multiplication by i, which is a complex linear automorphism of $T(M;\mathbb{C})$. Therefore, we can introduce on our manifold a real linear automorphism J of $T(M;\mathbb{R})$, such that $J^2 = -Id$. More specifically we have

$$J(\partial_{x_j}) = \partial_{y_j}$$
$$J(\partial_{y_j}) = -\partial_{x_j}.$$

This implies that $J(\partial_{z_j}) = J\left(\frac{1}{2}(\partial_{x_j} - i\partial_{y_j})\right) = \frac{1}{2}(\partial_{y_j} + i\partial_{x_j}) = i\partial_{z_j}$, which is exactly what we wanted (since J has the role of i in the complex tangent bundle); similarly, $J(\partial_{\bar{z}_j}) = -i\partial\bar{z}_j$. Therefore, we define $T^{(1,0)}(M;\mathbb{C})$ and $T^{(0,1)}(M;\mathbb{C})$ as the eigen-spaces of, respectively, $+i$ and $-i$ for J. A similar reasoning leads to an identical result for J^* and the cotangent bundles.

Definition 2.3.9. We define the *space of complex k-forms* as

$$A^k(M) := E^k(M) \otimes \mathbb{C}.$$

Notice that, locally, a k-form u can be written as

$$u = \sum_{\substack{|I|=p,|J|=q \\ p+q=k}} u_{IJ} dz_I \wedge d\bar{z}_J.$$

We call each of the terms of the sum a *(p,q)-component*. More generally, we define the *space of (p,q)-forms* as

$$A^{p,q}(M) := C^\infty(M;\Lambda^{p,q}(T^*(M;\mathbb{C}))),$$

where

$$\Lambda^{p,q}(T^*(M;\mathbb{C})) := \Lambda^p(T^{(1,0)*}(M;\mathbb{C})) \otimes \Lambda^q(T^{(0,1)*}(M;\mathbb{C}))$$

where we have the decomposition

$$\Lambda^k(T^*(M;\mathbb{C})) = \bigoplus_{p+q=k} \Lambda^{p,q}(T^*(M;\mathbb{C})).$$

Informally, a (p,q)-form is a $(p+q)$-form that is holomorphic on p coordinates and anti-holomorphic on q coordinates.

Remark 2.3.10. Every complex manifold comes equipped with a natural orientation. Consider

$$o_U := dx_1 \wedge \ldots \wedge dx_n \wedge dy_1 \wedge \ldots \wedge dy_n,$$

on every chart U centered around a point $m \in M$. This gives an orientation of the chart U. From straightforward substitutions we get that

$$o_U = \left(\frac{i}{2}\right)^n (-1)^{\frac{(n-1)n}{2}} dz_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_n.$$

Let us now consider another chart U' centered around m; then,

$$o_{U'} = |J(z(m))|^2 o_U,$$

where $J(z(m))$ is the determinant of the Jacobian of the change of coordinates from z to z' on $U \cap U'$. Therefore, if we cover M with holomorphic charts, we can glue together the forms o_U to obtain a nowhere vanishing real $2n$-form that orients the manifold independently from the chosen atlas.

If we ignore the additional complex structure we introduced, we can find the previous results (Poincaré Duality, Hodge orthogonal decomposition and Hodge isomorphism) on complex forms treating them as forms on a $2n$-dimensional manifold. We now aim to define new cohomology groups to find specific results for complex forms on complex manifolds.

Remark 2.3.11. Let us consider the differential operator d (defined on real forms and extended on complex forms) we observe that

$$d\left(A^{p,q}(M)\right) \subseteq A^{p+1,q}(M) \oplus A^{p,q+1}(M).$$

Hence, if we define the projections

$$\pi^{p,q} : A^k(M) \to A^{p,q}(M) \subset A^k(M),$$

we can define the holomorphic and anti-holomorphic differential operators

$$\partial := \pi^{p+1,q} \circ d,$$

$$\bar{\partial} := \pi^{p,q+1} \circ d,$$

with the following relations

$$d = \partial + \bar{\partial},$$

$$\partial^2 = \bar{\partial}^2 = 0,$$

$$\partial\bar{\partial} = -\bar{\partial}\partial.$$

We can visualize these operators working on a diagram

$$\begin{array}{ccccc}
A^{p-1,q-1}(M) & \xrightarrow{\bar{\partial}} & A^{p-1,q}(M) & \xrightarrow{\bar{\partial}} & A^{p-1,q+1}(M) \\
\downarrow \partial & \searrow^{\partial\bar{\partial}} & \downarrow \partial & \searrow^{\partial\bar{\partial}} & \downarrow \partial \\
A^{p,q-1}(M) & \xrightarrow{\bar{\partial}} & A^{p,q}(M) & \xrightarrow{\bar{\partial}} & A^{p,q+1}(M) \\
\downarrow \partial & \searrow^{\partial\bar{\partial}} & \downarrow \partial & \searrow^{\partial\bar{\partial}} & \downarrow \partial \\
A^{p+1,q-1}(M) & \xrightarrow{\bar{\partial}} & A^{p+1,q}(M) & \xrightarrow{\bar{\partial}} & A^{p+1,q+1}(M)
\end{array}$$

where summing on the bottom-left to top-right diagonals you can reconstruct the spaces $A^k(M)$.

In particular, those relations let us define the following cohomology groups

Definition 2.3.12. We define the *k-th Dolbeault's cohomology groups* as

$$H^{p,q}_{\partial}(M) := \frac{\ker \partial : A^{p,q}(M) \to A^{p+1,q}(M)}{Im\partial : A^{p-1,q}(M) \to A^{p,q}(M)}$$

$$H^{p,q}_{\bar\partial}(M) := \frac{\ker\bar\partial : A^{p,q}(M) \to A^{p,q+1}(M)}{Im\bar\partial : A^{p,q-1}(M) \to A^{p,q}(M)}.$$

These correspond to the cohomology complexes described by each row and column of the previous diagram. We also define the *k-th Bott-Chern cohomology group* as

$$H^{p,q}_{BC}(M) := \frac{A^{p,q}(M) \cap \ker\partial \cap \ker\bar\partial}{\partial\bar\partial(A^{p-1,q-1}(M))},$$

where we can easily see that the quotient is well-defined since if $v \in A^{p-1,q-1}$, then

$$\partial\left(\partial\bar\partial v\right) = \partial^2\bar\partial v = 0,$$
$$\bar\partial\left(\partial\bar\partial v\right) = -\partial\bar\partial^2 v = 0.$$

These correspond to the cohomology complexes described by each top-left to bottom-right diagonal of the previous diagram.

We have already observed that the property of orientability is always satisfied by complex manifolds. Let us extend the notion of Riemannian manifold with the following definition.

Definition 2.3.13. A *Hermitean form* on a complex vector space W is a $\mathbb{C}$-bilinear form

$$h : W \times \overline{W} \to \mathbb{C},$$

such that

$$h(u,v) = \overline{h(v,u)}.$$

A *Hermitean metric* is a positive definite Hermitean form, that is

$$h(u,u) > 0,$$

for all $u \neq 0$ in W.

Definition 2.3.14. A *Hermitean metric* on a manifold M is a family $h = (h_m)_{m\in M}$ of smoothly-varying Hermitean metrics each defined on $T^{(1,0)}_m(M;\mathbb{C})$. A manifold M is said to be a *Hermitean manifold* if it is equipped with a Hermitean metric.

Remark 2.3.15. A Riemmanian metric is Hermitean if and only if the automorphism J is an isometry for h, that is

$$h(u,v) = h(Ju,Jv).$$

Proof. Suppose $h(u,v) = \overline{h(v,u)}$. Then,

$$\begin{aligned} h(Ju,Jv) =& \overline{h(Jv,Ju)} = \overline{h(iv,-iu)} = \overline{h(v,u)} = \\ =& h(u,v). \end{aligned}$$

□

Locally, a Hermitean metric can be written as the tensor product

$$h = \sum_{j,k} h_{jk} dz_j \otimes d\bar z_k,$$

where $(h_{j,k})$ is a $n \times n$ Hermitean matrix, that is $h_{j,k} = \bar{h}_{k,j}$. Moreover, since a complex manifold M has a natural orientation, the volume element is well-defined and h can be easily extended to a Hermitean metric $(-,-)$ on $\Lambda(T^*(M;\mathbb{C}))$.

Definition 2.3.16. We extend the Hodge operator as

$$u \wedge \overline{*v} := (u,v)dV,$$

for $u, v \in \Lambda^k(T^*(M;\mathbb{C}))$. As before, this lets us define the metric $<< -,- >>$

$$<< u, v >>= \int_M (u,v)dV = \int_M u \wedge \overline{*v}$$

on all compact k-forms (which means on all k-forms if M is compact).

It immediately follows that $*$ gives a complex isometry

$$* : \Lambda^{p,q}(T^*(M;\mathbb{C})) \cong \Lambda^{n-q,n-p}(T^*(M;\mathbb{C}))$$

and the following equivalences hold

$$*\pi^{p,q} = \pi^{n-q,n-p}*,$$

$$**_{|\Lambda^{p,q}(T^*(M;\mathbb{C}))} = (-1)^{p+q} Id_{|\Lambda^{p,q}(T^*(M;\mathbb{C}))}.$$

Definition 2.3.17. On a Hermitean manifold M, we define the *Hermitean form associated to* h, $\omega = \omega_h$, as

$$\omega(u,v) = h(Ju,v).$$

We can immediately observe that we can reconstruct h from ω, in fact

$$h(u,v) = \omega(-Ju,v),$$

so we can indiscriminately talk about a Hermitean manifold giving h or ω. Moreover,

$$\omega(u,v) = h(Ju,v) = h(J^2u, Jv) = h(-u, Jv) = -h(Jv,u) = -\omega(v,u),$$

which means ω is an alternating 2-form.

Lemma 2.3.18. *The Hermitean form ω_h is of type $(1,1)$.*

Proof. We recall that

$$\Lambda^2(T^*(M;\mathbb{C})) = \Lambda^{2,0}(T^*(M;\mathbb{C})) \oplus \Lambda^{1,1}(T^*(M;\mathbb{C})) \oplus \Lambda^{0,2}(T^*(M;\mathbb{C})).$$

We also observe that

$$\begin{aligned}\omega(Ju,Jv) =& h(J^2u,Jv) = h(-u,Jv) = h(-Ju,J^2v) = \\ =& h(Ju,v) = \omega(u,v).\end{aligned}$$

To show that the result holds, we just observe that for any $v \in T(M;\mathbb{C})$,

$$v - iJv \in T^{(1,0)}(M;\mathbb{C}).$$

Then,

$$\omega(v - iJv, w - iJw) = \omega(v, w) - i\omega(Jv, w) - i\omega(v, Jw) - \omega(Jv, Jw) = \\ = - i\,[\omega(Jv, w) + \omega(v, Jw)] = 0.$$

Similarly for elements in $T^{(0,1)}(M; \mathbb{C})$. This means indeed that $\omega \in \Lambda^{1,1}(T^*(M; \mathbb{C}))$.
□

Locally, we can write

$$\omega = \sum_{j,k} h_{j,k} dz_j \wedge d\bar{z}_k.$$

Remark 2.3.19. Elements of $\Lambda^{p,p}(T^*(M; \mathbb{C}))$ are said to be *real*, in the sense that this subspace is preserved under the action of conjugacy:

$$\overline{\Lambda^{p,q}(T^*(M; \mathbb{C}))} = \Lambda^{q,p}(T^*(M; \mathbb{C}))$$

Defining the adjoints $\partial^* = - * \bar{\partial} *$ and $\bar{\partial}^* = - * \partial *$, the Laplace operators Δ_∂ and $\Delta_{\bar{\partial}}$, and the (p, q)-harmonic spaces $\mathcal{H}^{p,q}_{\Delta_\partial}(M)$ and $\mathcal{H}^{p,q}_{\Delta_{\bar{\partial}}}(M)$ (which again depend on the metric h), lets us give the following results.

Theorem 2.3.20. *(Hodge Theory for Hermitean manifolds) Let M be a compact Hermitean manifold.*

- *Hodge orthogonal decomposition:*

$$A^{p,q}(M) = \mathcal{H}^{p,q}_{\Delta_\partial}(M) \oplus \partial\left(A^{p-1,q}(M)\right) \oplus \partial^*\left(A^{p+1,q}(M)\right),$$

$$A^{p,q}(M) = \mathcal{H}^{p,q}_{\Delta_{\bar{\partial}}}(M) \oplus \bar{\partial}\left(A^{p,q-1}(M)\right) \oplus \bar{\partial}^*\left(A^{p,q+1}(M)\right);$$

- *Hodge isomorphisms:*

$$\mathcal{H}^{p,q}_{\Delta_\partial}(M) \cong H^{p,q}_{\partial}(M),$$
$$\mathcal{H}^{p,q}_{\Delta_{\bar{\partial}}}(M) \cong H^{p,q}_{\bar{\partial}}(M)$$

- *Kodaira-Serre Duality: The integral on M gives the following non-degenerate complex bilinear maps*

$$H^{p,q}_{\partial}(M) \times H^{n-p,n-q}_{\partial}(M) \to \mathbb{C},$$

$$H^{p,q}_{\bar{\partial}}(M) \times H^{n-p,n-q}_{\bar{\partial}}(M) \to \mathbb{C}.$$

2.4 Kähler manifolds

A special case of Hermitean manifold is given by Kähler manifolds. After discussing the Kähler condition, we will focus on complex projective manifolds and state the Hard Lefschetz Theorem and the Hodge-Riemann bilinear relations.

Definition 2.4.1. We say that a Hermitean metric is a *Kähler metric* if

$$d\omega = 0.$$

A manifold with a Kähler metric is a *Kähler manifold*.

Equivalently, a metric is Kähler if ω admits a *local Kähler potential*, which means that for any $m \in M$ there exists a real function ϕ on a neighbourhood U of m such that

$$\omega = i\partial\bar{\partial}\phi.$$

In fact, it is trivial to note that

$$d(i\partial\bar{\partial}\phi) = (\partial + \bar{\partial})(i\partial\bar{\partial}\phi) = i\partial^2\bar{\partial}\phi - i\partial\bar{\partial}^2\phi = 0.$$

Conversely, since ω is real, if it is also closed, we can write locally

$$\omega_{|U} = d\alpha,$$

where α is a real 1-form on a local chart U. Hence, $\alpha = \beta + \bar{\beta}$, where β is a $(1,0)$-form (so that $\alpha = \bar{\alpha}$). But since ω is a $(1,1)$-form, we also have that $\partial\beta = \bar{\partial}\bar{\beta} = 0$. This lets us write

$$\omega = \bar{\partial}\beta + \partial\bar{\beta}.$$

Then, locally, we can write $\beta = \partial\gamma$ and $\bar{\beta} = \bar{\partial}\bar{\gamma}$, so finally we have

$$\omega = \partial\bar{\partial}(\bar{\gamma} - \gamma) = i\partial\bar{\partial}\phi,$$

where $\phi = i(\gamma - \bar{\gamma})$.

Remark 2.4.2. Since ω is a closed real 2-form, it has a cohomology class $[\omega]$ in the real deRham cohomology, $H^2_{dR}(M;\mathbb{R})$; we call $[\omega]$ the *Kähler class of* h.

Lemma 2.4.3. *Let M be a compact Kähler manifold. Then, its even cohomology groups are non-zero:*

$$H^{2k}_{dR}(M;\mathbb{R}) \neq \{0\}.$$

Proof. Considering ω on a local chart, we get

$$\omega^n = n!(dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n) = n! dV_m,$$

where dV_m is the volume element at m. Therefore, since M is compact

$$[\omega]^n \cdot [M] = \int_M \omega^n = n!V > 0.$$

This means that $[\omega] \neq 0 \in H^2_{dR}(M;\mathbb{R})$ and so do all its powers in the respective cohomology groups. $\square$

Example 2.4.4. The projective space is a Kähler manifold. Let us exhibit a Kähler metric h for $\mathbb{P}^1(\mathbb{C})$, which can be easily generalized for any n.
Consider the covering and local coordinates defined as in 2.3.2. Define a Kähler potential by describing it on each chart as

$$\phi_i = \frac{1}{2\pi}\log\left(1 + |z_1^{(i)}|^2\right).$$

This is well-defined since they agree on the intersections, in fact on $U_0 \cap U_1$:

$$\partial\bar{\partial}\log\left(1 + \frac{1}{|z^{(0)}|^2}\right) = \partial\bar{\partial}\log\left(|z^{(0)}|^2 + 1\right) - \partial\bar{\partial}\log\left(|z^{(0)}|^2\right),$$

and we conclude recalling that $|z|^2 = z\bar{z}$. Hence, we define ω on the charts as

$$\omega = i\partial\bar{\partial}\phi.$$

For example, on U_0, we can write

$$\omega = \frac{i}{2\pi}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}.$$

The associated metric is called the *Fubini-Study metric*; we can see that in $\mathbb{P}^1(\mathbb{C})$ it is the form associated to the metric induced by the stereographic projection of a sphere, since

$$\frac{i}{2\pi}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} = \frac{dx^2+dy^2}{\pi(1+x^2+y^2)^2}$$

.

Remark 2.4.5. A submanifold of a Kähler manifold inherits a metric which is still Kähler. Hence, every complex projective manifold is Kähler.

Forms on Kähler manifolds are easier to study, thanks to the form ω. We can introduce

Definition 2.4.6. The *Lefschetz operator* L on a Hermitean manifold M is

$$\begin{aligned} L : A^k(M) \to& A^{k+2}(M) \\ u \mapsto& \omega \wedge u. \end{aligned}$$

We also define its adjoint as $L^* := *L^{-1}*$.

We now state very important identities the proof of which is not interesting and therefore omitted (See Theorem 6.2.2 in [10]).

Theorem 2.4.7. *(Fundamental identities of Kähler geometry) Let M be a compact Kähler manifold. Then,*

- $[L^*, \partial] = i\bar{\partial}^*$, $[L^*, \bar{\partial}] = -i\partial^*$, $[\bar{\partial}^*, L] = i\partial$, $[\partial^*, L] = -i\bar{\partial}$;
- $[\partial, \bar{\partial}^*] = [\bar{\partial}, \partial^*] = 0$;
- $\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$;

where $[A, B]$ denotes the commutator of operators A and B.

Important corollaries are

Corollary 2.4.8. *Let M be a compact Kähler manifold. Then,*

- *Δ, Δ_∂, $\Delta_{\bar{\partial}}$ commute with all the other operators that we have introduced;*
- *The harmonic spaces do not depend on the Laplace operator, that is*

$$\mathcal{H}^k_\Delta(M) = \mathcal{H}^k_{\Delta_\partial}(M) = \mathcal{H}^k_{\Delta_{\bar{\partial}}}(M);$$

- *If we define the space of (p, q)-harmonic forms,*

$$\mathcal{H}^{p,q}_\Delta(M) := \mathcal{H}^k_\Delta(M) \cap A^{p,q}(M),$$

  then

$$\mathcal{H}^{p,q}_\Delta(M) = \mathcal{H}^{p,q}_{\Delta_\partial}(M) = \mathcal{H}^{p,q}_{\Delta_{\bar{\partial}}}(M).$$

Theorem 2.4.9. *(Decomposition theorem on Kähler manifolds) Let M be a compact Kähler manifold. Then,*

$$\mathcal{H}^k_\Delta(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_\Delta(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\Delta_\partial}(M) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_{\Delta_{\bar\partial}}(M).$$

Remark 2.4.10. These isomorphisms, composed with the Hodge isomorphism, give an immediate proof of Kodaira-Serre Duality from Theorem 2.3.20.

Another important consequence of these identities is the following

Remark 2.4.11. Let $\beta^k(M) = \dim_{\mathbb{C}} H^k(M;\mathbb{C})$ denote the k-th Betti number of a Kähler manifold M. Define the *Hodge numbers of M* as

$$h^{p,q} = \dim_{\mathbb{C}} H^{p,q}_{\bar\partial}(M).$$

The previous decomposition tells us that

$$\beta^k(M) = \sum_{p+q=k} h^{p,q}(M).$$

By observing that $H^{p,q}(M) \cong \overline{H^{q,p}(M)}$ and using Kodaira-Serre Duality, we get that for odd cohomology groups

$$\beta^{2k+1}(M) = 2\sum_{j=0}^{k} h^{j,2k+1-j}(M).$$

We just proved

Theorem 2.4.12. *The odd Betti numbers of a Kähler manifold are even.*

Definition 2.4.13. The *Kähler cone* $\mathcal{K}$ of a compact complex manifold M is the set of all Kähler classes $[\omega]$ of Kähler metrics h on M.

Remark 2.4.14. We recall that we have a decomposition

$$H^2(M;\mathbb{C}) = H^{2,0}(M;\mathbb{C}) \oplus H^{1,1}(M;\mathbb{C}) \oplus H^{0,2}(M;\mathbb{C}),$$

where $H^{0,2}(M;\mathbb{C}) = \overline{H^{2,0}(M;\mathbb{C})}$ and $H^{1,1}(M;\mathbb{C})$ is real. We have also already observed that

$$[\omega] \in H^2_{dR}(M;\mathbb{R}) \cap H^{1,1}(M;\mathbb{C}),$$

for any Kähler metric defined on a compact complex manifold, hence $\mathcal{K} \subseteq H^{1,1}(M;\mathbb{R})$.

Remark 2.4.15. The Kähler cone $\mathcal{K}$ is open; in fact, if we let η be a closed real $(1,1)$-form with h-norm sufficiently small it can be shown that $\omega' = \omega + \eta$ still satisfies the Kähler condition and therefore is associated to some Kähler metric h', proving that $[\omega] + [\eta] \in \mathcal{K}$.

Remark 2.4.16. The Kähler cone $\mathcal{K}$ is convex; in fact if h and h' are Kähler metrics, it can be shown that, for $0 < s < 1$, $sh + (1-s)h'$ is still Kähler. Hence,

$$s[\omega] + (1-s)[\omega'] \in \mathcal{K}.$$

We lastly define the ample cone, using the following simple facts about intersection cohomology (since we are interested only in those cases were the intersection cohomology and the deRham cohomology coincide).

Definition 2.4.17. A *(Weil) divisor* D is a finite linear combination with integer coefficients of submanifolds of M of codimension 1,

$$D = \sum_{\alpha} n_\alpha D_\alpha.$$

We say D is *effective* if all coefficients are positive.

From the definition, we know we can consider the class of a divisor $[D] \in H^2(M; \mathbb{Z})$.

Definition 2.4.18. A class $[D] \in H^2(M; \mathbb{Z})$ is said to be *ample* if there exists an immersion $M \subset \mathbb{P}^N(\mathbb{C})$ and a hyperplane h such that $[M \cap H]$ is a positive multiple of $[D]$. The *ample cone of* M is the set of all ample classes of M.

Theorem 2.4.19. *(Kodaira Embedding Theorem) A Kähler class is the class of an ample divisor if and only if it belongs in $H^{1,1}(M; \mathbb{R}) \cap H^2(M; \mathbb{Z})$.*

The class of a divisor is always an integer class, since we build it gluing together subvarieties, hence the condition is trivially necessary. The important part of the theorem is that this condition is also sufficient.

Theorem 2.4.20. *(Nakai-Moishezon Criterion) A class $[D]$ is ample if and only if*

$$[D]^r \cdot [N] > 0,$$

for all r-dimensional submanifolds $N \subset M$, where $\cdot$ denotes the cup-product in the cohomology ring and $[D]^r$ can be seen as the intersection of r hyperplanes in the class of D.

This criterion lets us write the ample cone as the family

$$\{\alpha \in H^2(M; \mathbb{R}) \cap H^{1,1}(M; \mathbb{C}) | [\alpha] \cdot [H] > 0, H \text{ hyperplane}\}.$$

In fact, for an ample form ω and a complex 1-dimensional submanifold Σ, we have

$$[\omega] \cdot [\Sigma] = \int_\Sigma \omega,$$

where $[\Sigma] \in H_2(M; \mathbb{Z})$ is the homology class of Σ. The value of the integral is positive since it is the volume of Σ with respect to the metric h.

We also have some results on the closure of $\mathcal{K}$, called *NEF cone.*

Theorem 2.4.21. *(Kleiman) A class $[D]$ is in the closure of $\mathcal{K}$ if*

$$[D] \cdot [C] \geq 0,$$

for all complex 1-dimensional submanifolds $C \subset M$.

Remark 2.4.22. From everything we said so far, the ample cone is only contained in the Kähler cone. Moreover, if $H^{2,0}(M; \mathbb{C}) \neq \{0\}$, the ample cone could be very small and even the trivial subgroup. However, if $H^{2,0}(M; \mathbb{C}) = \{0\}$, this means that $H^2(M; \mathbb{C}) = H^{1,1}(M; \mathbb{C})$, hence the integer points of the ample cone are dense in the Kähler cone. If not explicitly stated, we will only treat this latter case (see for example Theorem 2.4.29).

We are finally ready to state the Hard-Lefschetz Theorem and the Hodge-Riemann bilinear relations.
We have already defined the Lefschetz operator L on complex forms. This works on cohomology too, because from 2.4.7, we see that

$$L(\ker \Delta) \subseteq \ker \Delta,$$

hence we can define

$$\begin{aligned} L : H^k_{dR}(M;\mathbb{C}) &\to H^{k+2}_{dR}(M;\mathbb{C}) \\ \alpha &\mapsto L(\alpha) = [\omega] \wedge \alpha \end{aligned}$$

by using the respective unique harmonic representative.

Theorem 2.4.23. *(Hard-Lefschetz) Let M be a compact Kähler manifold of complex dimension n. Then*

$$L^k : H^{n-k}(M;\mathbb{C}) \to H^{n+k}(M;\mathbb{C})$$

is an isomorphism for $0 \leq k \leq n$.

Proof. We first observe that

$$\Delta(\omega \wedge \alpha) = \omega \wedge \Delta(\alpha);$$

this also means that $\Delta(\omega^k \wedge \alpha) = \omega^k \wedge \Delta(\alpha)$. Thus, we can indeed define the map

$$\omega^k \wedge - : \ker \Delta \to \ker \Delta,$$

which is also an isomorphism. We can then complete the proof using the Hodge isomorphism

$$\begin{array}{ccc} \mathcal{H}^{n-k}(M) & \xrightarrow{\omega^k \wedge -} & \mathcal{H}^{n+k}(M) \\ \Big\downarrow * & & \Big\downarrow * \\ H^{n-k}(M;\mathbb{C}) & \xrightarrow{L^k} & H^{n+k}(M;\mathbb{C}) \end{array} .$$

□

The importance of this result is given by the fact that from a topological point of view, $H^{n-k}(M;\mathbb{C})$ and $H^{n+k}(M;\mathbb{C})$ are isomorphic to each others only as duals. (HL) gives a direct isomorphism. Moreover, we have the following corollaries.

Definition 2.4.24. We define the *primitive subspace* P^{n-k} as

$$P^{n-k} = \ker L^{k+1} : H^{n-k}(M;\mathbb{C}) \to H^{n+k+2}(M;\mathbb{C}).$$

Definition 2.4.25. We define the *Lefschetz form* to be the following bilinear form

$$\begin{aligned} Q^k : H^{n-k} \times H^{n-k} &\to \mathbb{C} \\ (\alpha,\beta) &\mapsto Q^k(\alpha,\beta) = (-1)^{n-k} \int_M \omega^k \wedge \alpha \wedge \beta. \end{aligned}$$

Then, we can also write $Q^k(\alpha,\beta) = (-1)^{n-k} << \alpha, L^k(\beta) >>$.

Corollary 2.4.26. *(Lefschetz Decomposition) The following decomposition holds*

$$H^{n-k}(M;\mathbb{C}) = \bigoplus_{j\geq 0} L^j P^{n-k-2j}(M;\mathbb{C}),$$

and each term is orthogonal to the others with respect to Q.

Proof. Take $H^{n-k}(M;\mathbb{C})$ and the action of L^{k+1} on it. Clearly,

$$H^{n-k}(M;\mathbb{C}) = P^{n-k} \oplus H^{n+k+2}(M;\mathbb{C}) \cong P^{n-k} \oplus LH^{n-k-2}(M;\mathbb{C}),$$

where the isomorphism is given by (HL). The proof is then completed by finite induction. For the orthogonality we observe that if $\alpha \in P^{n-k}$ and $\beta = \omega^j \wedge \gamma \in L^j P^{n-k-2j}$, then

$$<< \alpha, L^{k+j}\gamma >>=<< L^{k+j}\alpha, \gamma >>= 0,$$

and similarly for the other cases. □

Corollary 2.4.27. *The Betti numbers form two symmetric unimodal sequences*

$$\beta^0 \leq \beta^2 \leq \ldots \leq \beta^n \geq \ldots \geq \beta^{2n},$$

$$\beta^1 \leq \beta^3 \leq \ldots \leq \beta^{n-1} = \beta^{n+1} \geq \ldots \geq \beta^{2n-1}.$$

Proof. Straightforward consequence of (HL): L is injective for $0 \leq k < n$ and surjective for $n \leq k < 2n$. □

Remark 2.4.28. Theorem (HL) is equivalent to state that the bilinear form Q^k is non-degenerate, because the composition

$$\begin{array}{ccccc} H^{n-k}(M) & \xrightarrow{L^k} & H^{n+k}(M) & \xrightarrow{\wedge} & Hom_{\mathbb{R}}(H^{n-k}(M), H^{2n}(M)) \\ \alpha & \longmapsto & L^k(\alpha) & \longmapsto & - \wedge L^k(\alpha) \end{array}$$

is an isomorphism if and only if L^k is, since the second map is an isomorphism thanks to (PD).

Since the Lefschetz Decomposition is orthogonal with respect to this form, we then just need to study its signature on all the terms of the decomposition. In order to state the theorem, we restrict to the even-graded cohomology groups and to the case $p = q$, which is the one that we will need in the rest of the dissertation.

Theorem 2.4.29. *(Hodge-Riemann Bilinear Relations - simple version) The form Q^k is positive definite on the primitive subspace P^k.*

Proof. We already know that the form is non-degenerate, because (HL) holds. This means that $Q^k(\alpha, \alpha)$ does not change sign on P^k. Clearly, $\omega \in P^k$ and

$$<< \omega, L^k\omega >>= \int_M \omega^n > 0,$$

because it is the volume of M. Hence, the result. □

Remark 2.4.30. From now on we will say that, indiscriminately, a Lefschetz operator L, a complex 1-form ω or the class of a complex submanifold $[D]$ of codimension 1 has property $(\mathrm{HL})_k$, if the associated operators L^k, $\omega^k \wedge -$ and $[D]^k \cdot -$ are isomorphisms; if they have property $(\mathrm{HL})_k$ for every k, we simply say that they have property (HL).
Similarly, we will say that a bilinear form Q has property $(\mathrm{HR})_k$ if it is positive definite on P^k and it has property (HR) if it does for every k.

2.5 Semismall maps

In this section we give other results for the property (HL) using semismall maps. Our main references will be [7] and [8].

Definition 2.5.1. Let $f : M \to N$ be a proper holomorphic map between projective manifolds. We call *strata* the sets

$$Y^k = \{y \in Y \,|\, \dim f^{-1}(y) = k\},$$

for every k.

Definition 2.5.2. A proper holomorphic map $f : M \to N$ between projective manifolds is *semismall* if

$$\dim Y^k + 2k \leq \dim M,$$

for every k. A stratum is said to be *relevant* if the equality holds. Equivalently, f is semismall if there is no irreducible submanifold $T \subseteq M$ such that $2 \dim T - \dim f(T) > \dim M$.

We have introduced the ample cone $\mathcal{K}$, observed it is open and gave the property (HL) to all its classes. One may ask what happens on the NEF cone with regards to (HL). We have the following theorem

Theorem 2.5.3. *(de Cataldo, Migliorini) Let $f : M \to \mathbb{P}^m$ and $D := f^{-1}(H)$, where H is a projective hyperplane. We also say that $[D]$ is semiample. Then, $[D] \cdot -$ satisfies (HL) if and only if f is semismall.*

Remark 2.5.4. Let $f : M \to N$ a projective morphism, A ample on N and $[D] := [f^*A]$. If $[D]$ satisfies (HL) then f is semismall.

Proof. If f is not semismall then there exists an irreducible submanifold $T \subseteq M$ such that

$$2 \dim T - n > \dim f(T).$$

Let $[T] \in H^{2(n-\dim T)}(M; \mathbb{R}) = H^{n-(2\dim T - n)}(M)$. The class $[D]^{2\dim T - n}$ has a representative that does not intersect T and therefore

$$L^{2\dim T - n}[T] = [D]^{2\dim T - n} \cdot [T] = 0,$$

which means that $L^{2\dim T-n}$ is not an isomorphism. □

The converse is also true, even if it takes more work to be proved.

Theorem 2.5.5. *Using the same notation as before, if f is semismall, then $[D]$ satisfies (HL).*

To prove it, we have to state some classic preliminary results.

Theorem 2.5.6. *(Weak Lefschetz Theorem) If $i : D \hookrightarrow M$ is the tautological embedding of a divisor D in a complex n-dimensional manifold M, then the induced maps*

$$i^* : H^k(M;\mathbb{Z}) \to H^{k+1}(D;\mathbb{Z})$$

are isomorphisms for $k < n-1$ and injective for $k = n-1$.

Proof. See, for example, the proof of a more general version of this theorem in [9] □

Lemma 2.5.7. *(Limit Lemma) Suppose $(L_j)_{j\geq 0}$ is a continuous family of Lefschetz operator satisfying (HL). If there exists $\bar{j}$ such that $L_{\bar{j}}$ satisfies (HR), then every L_j satisfies (HR).*

Proof. If every L_j satisfies (HL) then the respective Lefschetz forms Q_{L_j} form a continuous family of non-degenerate bilinear forms, hence they all have the same signature. □

Let us now prove Theorem 2.5.5

Proof. We prove it by induction on $\dim M = 2n$. If $n = 1$, $[D]$ is trivially ample in M and therefore it satisfies (HL). Suppose it holds for $n-1$ and let us prove it for n. We know (WL) holds on D, hence we have the following decomposition

$$\begin{array}{ccc} H^{n-k}(M) & \xrightarrow{L^k} & H^{n+k}(M) \\ \downarrow\cong & & \cong\uparrow \\ H^{n-k+1}(D) & \xrightarrow{L^{k-1}} & H^{n+k-1}(D) \end{array},$$

which gives us (HL) for $k < n-1$. We only need to prove that the result holds for $k = n-1$. We observe that i^* restricts to a map

$$i^* : P^{n-1}(M) \to P^n(D);$$

hence, if $0 \neq \alpha \in P^{n-1}(M)$, by (WL) and (HR) we have

$$0 \neq Q(i^*\alpha, i^*\alpha) = Q(\alpha, \omega \wedge \alpha).$$

It follows that $L : H^{n-1} \to H^{n+1}$ is injective and hence an isomorphism. To prove (HR), we observe that $[D]$ is semiample and therefore on the boundary of the NEF cone. Since $\mathcal{K}$ is convex, this means that for any $[B] \in \mathcal{K}$,

$$[f^*A] + \varepsilon[B] \in \mathcal{K},$$

for $0 < \varepsilon << 1$. This puts us in the hypothesis of Lemma 2.5.7 and lets us conclude the proof. □

We give one last important result, the Decomposition Theorem for semismall maps. Even if we did not introduce all the necessary tools to fully understand it, we state it anyway because our proof of the main theorem of our dissertation, uses a similar decomposition. Since we are not interested in the general version of the theorem, we will assume that the fibers in our varieties are always irreducible

Theorem 2.5.8. *(Decomposition Theorem for intersection cohomology groups - simple version) Let $f : M \to N$ be a proper map. There exist finitely many locally closed irreducible submanifolds $Y_\alpha \subset N$ and integers d_α such that*

$$IH^{n-k}(M) = \bigoplus_\alpha IH^{n-k-d_\alpha}(\overline{Y}_\alpha),$$

where IH^k denotes the k-th intersection cohomology group, which we observe is opportunely shifted in its degree by d_α, depending on Y_α. In the case of f being semismall the direct sum is over all relevant strata Y^k.

The proof proceeds one stratum at a time: higher dimensional strata are dealt with inductively, by cutting transversally with a generic hyperplane section D on N, so that one is reconduced to the semismall map $f^{-1}(D) \to D$. The really significant case left is that of a zero-dimensional relevant stratum S (See Theorem 3.4.1 in [7] for the full proof of the general case).

Example 2.5.9. Let us provide an example of this decomposition. Consider a complex surface X, that is a smooth 2-dimensional complex manifold in a suitable projective space. Call $\tilde{X}$ the following manifold: remove a point P on X and replace it with its tangent space, so that now, for example, two curves intersecting at P on X, intersect also on $\tilde{X}$ if and only if they are tangent in P. The tangent space can be seen as a projective line $\mathbb{P}^1(\mathbb{C})$. We call $\tilde{X}$ the *blowup of X at P*. We now denote

$$f : \tilde{X} \to X.$$

This map is clearly semismall, since f is the identity on $X \setminus P$ and $f^{-1}(P) = \mathbb{P}^1(\mathbb{C})$. The condition of semismallness is then trivially satisfied everywhere on $X \setminus P$ and

$$\dim P + 2\dim f^{-1}(P) = 0 + 2 \leq \dim \tilde{X}.$$

Observe that the stratum Y^0 is then relevant. If we consider the pullback

$$f^* : H^\circ(X) \to H^\circ(\tilde{X})$$

anything that does not contain P can be trivially lifted, hence

$$f^* : H^k(X) \cong H^k(\tilde{X})$$

for $k \neq 2$. If we consider now the degree 2, the only thing that we need to add is the class of the projective line, which gives us the decomposition

$$H^2(\tilde{X}) = f^*\left(H^2(X)\right) \oplus [\mathbb{P}^1],$$

which is clearly orthogonal with respect to the intersection product.

2.6 The axiomatic algebraic setup

As we have said at the beginning of this chapter, we want to focus on the algebraic properties of the cohomology ring we have studied. We retrace everything we proved on a generic graded algebra A, which now should play the role of $H^*(M; \mathbb{R})$.

Definition 2.6.1. We say that a graded Artinian ring $A^* = \oplus_{k=0}^r A^k$ is a *Poincaré Duality Algebra of dimension* r if

- $A^0 \cong \mathbb{R}$ and $A^r \cong \mathbb{R}$,
- $A^k \cong 0$ for $k < 0$ or $k > r$,
- the multiplication in A^* gives isomorphisms
$$A^{r-k} \to Hom_{\mathbb{R}}(A^k, A^r).$$

Example 2.6.2. The graded polynomial ring $R^* = \mathbb{R}[x]/(x^{r+1})$ trivially satisfies all these properties.

Definition 2.6.3. We call *degree map* any isomorphism
$$\deg : A^r \to \mathbb{R}.$$

This plays the role of the integral of complex n-forms on a complex n-dimensional manifold.

Example 2.6.4. A degree map on R^* is given by the evaluation $x^r \mapsto 1$.

Definition 2.6.5. Let $\ell \in A^1$, $0 \leq k \leq \frac{r}{2}$

- The *Lefschetz operator* L_ℓ^k *associated to* ℓ *on* A^k is the linear map
$$\begin{aligned} L_\ell^k : A^k &\to A^{r-k} \\ u &\mapsto \ell^{r-2k} u; \end{aligned}$$
- The *Hodge-Riemann form* Q_ℓ^k *associated to* ℓ *on* A^k is the symmetric bilinear form
$$\begin{aligned} Q_\ell^k : A^k \times A^k &\to \mathbb{R} \\ (u_1, u_2) &\mapsto (-1)^k \deg(u_1 \cdot L_\ell^k u_2); \end{aligned}$$
- The *primitive subspace* P_ℓ^k *of* A^k *associated to* ℓ is
$$P_\ell^k = \{u \in A^k | \ell \cdot L_\ell^k(u) = 0\} \subset A^k.$$

In these definitions, which resemble very much the ones given for the cohomology ring of a Kähler manifold, we shifted the indices only for convenience.

Definition 2.6.6. Let A^* be a Poincaré Duality Algebra.

- A^* has property $(\mathrm{HL})_\ell$ if L_ℓ^k is an isomorphism, for every $k \leq \frac{r}{2}$.
- A^* has property $(\mathrm{HR})_\ell$ if Q_ℓ^k is positive definite on P_ℓ^k, for every $k \leq \frac{r}{2}$
- If L_ℓ^k is an isomorphism, then
$$A^{k+1} = P_\ell^{k+1} \oplus \ell A^k.$$

  If A^* has property $(\mathrm{HL})_\ell$ we have the *Lefschetz decomposition*
$$A^k = \bigoplus_{j=0}^k \ell^j P_\ell^{k-j},$$
  for every $k \leq \frac{r}{2}$. This decomposition is orthogonal with respect to Q_ℓ^k.

Example 2.6.7. In Example 2.6.2, let $\ell = x$. Then,

- The multiplication by x^{r-2k} is a Lefschetz operator that has property (HL);
- $P^0 = \mathbb{R}$ and $P_x^k = \{0\}$ for all $k \neq 0$; hence (HR) is trivially true for $k \neq 0$ and
$$(-1)^0 \deg(a \cdot x^{r-2\cdot 0} a) = a^2 > 0,$$
for every $a \neq 0$.
- It follows immediately that the Lefschetz decomposition is trivial for every k
$$P^k = x^k P^0 = x^k \mathbb{R}.$$

We have the following equivalent conditions

Theorem 2.6.8. *Let $\ell \in A^1$.*

- *A^* has property (HL)$_\ell$ if and only if Q_ℓ^k is non-degenerate for every $k \leq \frac{r}{2}$;*
- *A^* has property (HR)$_\ell$ if and only if Q_ℓ^k is non-degenerate and has signature*
$$n_+ - n_- = \sum_{j=0}^{k} (-1)^{k-j} \left(\dim_{\mathbb{R}} A^j - \dim_{\mathbb{R}} A^{j-1} \right),$$
for every $k \leq \frac{r}{2}$.

The proof of this theorem traces the one in Remark 2.4.28, hence we do not need to do it again in details (See Proposition 7.6 in [2] for the full proof).

Lastly, we observe that (HL) and (HR) are both preserved under the tensor product of Poincaré Duality Algebras. Let $(A_1^*, \deg_1)$ and $(A_2^*, \deg_2)$ be two Poincaré Duality Algebras of dimension r_1 and r_2, respectively. $R_1^* \otimes R_2^*$ is a Poincaré Duality Algebra of dimension $r_1 + r_2$, which can be equipped with the induced degree map
$$\deg_1 \otimes \deg_2 : A_1^{r_1} \otimes A_2^{r_2} \to \mathbb{R}.$$

Theorem 2.6.9. *If R_1^* has property (HR)$_{\ell_1}$ and R_2^* has property (HR)$_{\ell_2}$, then $R_1^* \otimes R_2^*$ has property (HR)$_{(\ell_1 \otimes 1 + 1 \otimes \ell_2)}$. The same holds for (HL).*

The full proof can be found in Proposition 7.7 in [2]. We show how this theorem works in a simple case.

Example 2.6.10. Let us consider $P_1^* = \mathbb{R}[x_1]/(x_1^3)$ and $P_2^* = \mathbb{R}[x_2]/(x_2^4)$, each equipped with a degree map $\deg(x_i^{r_i}) = 1$. Define
$$P^* := P_1^* \otimes P_2^*,$$
which can be naturally identified with
$$P^* = \mathbb{R}[x_1, x_2]/(x_1^3, x_2^4).$$

Set $\ell := x_1 + x_2$ and $\deg : P^5 \to \mathbb{R}$, $\deg(x_1^2 x_2^3) = 1$. A straightforward computation shows us that the primitive subspaces are

$$\begin{aligned}
P^0 =& \mathbb{R}, \\
P^1 =& \{p(x_1, x_2) = a_1 x_1 + a_2 x_2 | p(x_1, x_2)(x_1 + x_2)^4 = 0\} = \\
=& \mathbb{R} < 3x_1 - 2x_2 > := \mathbb{R} < p_1 >, \\
P^2 =& \{p(x_1, x_2) = a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 | p(x_1, x_2)(x_1 + x_2)^2 = 0\} = \\
=& \mathbb{R} < x_1^2 - x_1 x_2 + x_2^2 > := \mathbb{R} < p_2 > .
\end{aligned}$$

Then, to show that the property (HR) is satisfied we just need to compute

$$Q^0_{x_1+x_2}(1,1) = (-1)^0 \deg(x_1 + x_2)^5 = \deg(10 x_1^2 x_2^3) > 0,$$

$$\begin{aligned}
Q^1_{x_1+x_2}(p_1, p_1) =& (-1)^1 \deg(x_1 + x_2)^3 (3x_1 - 2x_2)^2 = \\
=& -\deg(-9 x_1^2 x_2^3 - 36 x_1^2 x_2^3 + 12 x_1^2 x_2^3) = -\deg(-33 x_1^2 x_2^3) > 0,
\end{aligned}$$

$$\begin{aligned}
Q^2_{x_1+x_2}(p_2, p_2) =& (-1)^2 \deg(x_1 + x_2)(x_1^2 - x_1 x_2 + x_2^2)^2 = \\
=& \deg(-2 x_1^2 x_2^3 + x_1^2 x_2^3 + 2 x_1^2 x_2^3) = \deg(x_1^2 x_2^3) > 0.
\end{aligned}$$

Chapter 3

The Chow ring of a matroid

In this chapter we introduce a new tool in our study of matroids: Toric Geometry. After revising the basic notions of fans and showing how we can construct a complex toric variety from them, we define a one-to-one correspondence between simple matroids and the family of polyhedral rational fans Σ_M called *Bergman fans*. We then quickly explain the main ideas contained in [4]: lattices that satisfy certain hypothesis can be associated to graded algebras $CH(\mathscr{L})$, called *Chow rings*. Even though it is defined on the lattice, since we also have a one-to-one correspondence between simple matroids and geometric lattices, we will denote the Chow ring with $CH(M)$. The name Chow ring comes again from Algebraic Geometry, and we can think of this algebra as a special cohomology ring for complex manifolds that arises from Intersection Theory. The lattice $\mathscr{L}(M)$, the fan Σ_M, the toric variety X_{Σ_M} and the graded algebra $CH(M)$ are all strictly related: for example, it is proved that we have isomorphisms

$$CH(X_{\Sigma_M}) \cong CH(M);$$

this will be useful when we have to compute things in $CH(M)$, because we can choose between many equivalent ways to see its elements.

3.1 Basic notions of Toric Geometry

Toric Geometry is a powerful link between Algebraic Geometry and Combinatorics: in fact, toric varieties are geometric objects that can be described only with combinatorial information. In this section we define rays, cones and fans, which are a collection of cones in some affine space, and we describe how to construct a toric variety, gluing together the affine toric variety associated to every cone of the fan. We will mainly follow [6].

Definition 3.1.1. A *toric variety* is an irreducible variety X such that

- $(\mathbb{C}^*)^n$ is a Zariski open subset of X;
- the action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on X.

Example 3.1.2. Consider the complex projective space $\mathbb{P}^1(\mathbb{C})$, defined as

$$\mathbb{P}^1(\mathbb{C}) := \mathbb{C}^2 \setminus \{(0,0)\}/\sim,$$

where $x \sim \lambda x$ for any $\lambda \in \mathbb{C}^*$. Then, the map

$$\begin{aligned} \mathbb{C}^* &\to \mathbb{P}^1(\mathbb{C}) \\ t &\mapsto [1;t] \end{aligned}$$

lets us identify $\mathbb{C}^*$ with the Zariski open subset $U_0 = \{[u_0;u_1] \in \mathbb{P}^1(\mathbb{C}) | u_0 \neq 0\}$. Moreover,

$$t_1 \cdot [a_0, a_1] := [a_0; t_1 a_1]$$

is an action of $\mathbb{C}^*$ on $\mathbb{P}^1(\mathbb{C})$. Similarly, we can also consider

$$U_1 = \{[u_0;u_1] \in \mathbb{P}^1(\mathbb{C}) | u_1 \neq 0\} \cong \mathbb{C}^*.$$

Definition 3.1.3. A *convex polyhedral cone* σ is a subset of $\mathbb{R}^n$ of the form

$$\sigma = Cone(S) = \left\{ \sum_{v \in S} \lambda_v v | \lambda_v \geq 0 \right\},$$

where S is a finite set of vectors of $\mathbb{R}^n$ called the *set of generators* of σ. We can think of a cone as the intersection of a finite number of closed half-spaces.
The dimension of a cone σ is the dimension of the smallest subspace of $\mathbb{R}^n$ that contains σ. We also say that a convex cone is *strictly convex* if it does not contain any positive-dimensional subspace of $\mathbb{R}^n$. Since we will only work with strictly convex cones, from now on we will call them just *cones* for simplicity.

Definition 3.1.4. Given a cone $\sigma \subset \mathbb{R}^n$, its *dual cone* is the set

$$\sigma^\vee = \{u \in (\mathbb{R}^n)^* \mid < u, v > \geq 0, \text{ for all } v \in \sigma\},$$

where $< u, v >= u(v)$ is the natural pairing defined on $V^* \times V$.

Lemma 3.1.5. *If a cone σ is generated by S, then*

$$\sigma^\vee = \bigcap_{v \in S} H_v^+,$$

where $H_v^+ := \{u \in \mathbb{R}^n | < u, v > \geq 0\}$.

Proof. Obviously, if $u \in \sigma^\vee$, then in particular $< u, v > \geq 0$, for every $v \in S$. Conversely, if $< u, v > \geq 0$ for every $v \in S$ and $w \in \sigma$, then

$$< u, w >=< u, \sum_{v \in S} \lambda_v v >= \sum_{v \in S} \lambda_v < u, v > \geq 0.$$

□

Definition 3.1.6. Let $u \in (\mathbb{R}^n)^*$ and $\sigma \subset H_u^+$. Then

$$\tau := \{v \in \sigma | < u, v >= 0\}$$

is called a *face* of the cone σ. The cone itself is a face and we call *proper face* a face $\tau \neq \sigma$. Since $\sigma \subset H_u^+$ if and only if $u \in \sigma^\vee$, this definition gives us all the faces as u varies in $\sigma^\vee \setminus \{0\}$.
We call τ a *facet* if it is a face of codimension 1; a *ray* ρ is a face of dimension 1.

Lemma 3.1.7. *Let σ be a cone with rays $\rho_1, \dots, \rho_s$. Then, if we fix $v_i \in \rho_i \setminus \{0\}$ and $S = \{v_1, \dots, v_s\}$,*

$$\sigma = Cone(S).$$

Definition 3.1.8. We call *lattice* a free Abelian group of finite rank. This is not to be confused with the structures introduced in the first Chapter. If we pick a basis for a lattice N, we get an isomorphism $N \cong \mathbb{Z}^n$. We can also define its *dual lattice* M as

$$M = Hom_{\mathbb{Z}}(N, \mathbb{Z}).$$

Lastly, we define the vector spaces $N_{\mathbb{R}} := N \otimes \mathbb{R} \cong \mathbb{R}^n$ and $M_{\mathbb{R}} := M \otimes \mathbb{R} = N_{\mathbb{R}}^*$. A cone σ is said to be *rational* if $\sigma \subset N_{\mathbb{R}}$ and $\sigma = Cone(S)$ where S is a finite subset of N.

Rational cones have unique minimal generating sets, which can be determined by considering the minimal generator for each ray ρ of σ.

Lemma 3.1.9. *If σ is a rational polyhedral cone, then $\sigma^\vee$ is too.*

Proof. If $\sigma = Span_{\mathbb{R}_{\geq 0}}(e_1, \dots, e_s)$, then the points in $\sigma^\vee$ are the solutions of a finite number of linear inequalities with integer coefficients ($< m, e_i > \geq 0$). Hence, $\sigma^\vee$ must have a finite number of generators in N. □

Lemma 3.1.10. *(Gordan's Lemma) If σ is a rational polyhedral cone, then*

$$S_\sigma := \sigma^\vee \cap M$$

is a finitely generated semigroup.

Proof. We can find integral vectors $u_1, \dots, u_r$ such that the cone σ can be written as

$$\sigma = \{v | < u_i, v > \geq 0 \text{ for every } i\}.$$

Then, these u_i generate the dual cone $\sigma^\vee$. If we now consider $x \in S_\sigma$, we can write it as

$$x = \sum_i n_i u_i + \sum_i r_i u_i,$$

where the n_i are non-negative integers and $0 \leq r_i < 1$. Since x and the first sum are integrals, then the second sum must be integral as well, which means that we have only a finite number of possible coefficients for the second sum. □

Definition 3.1.11. If σ is a rational polyhedral cone, we define its *semigroup algebra* $\mathbb{C}[S_\sigma]$ to be the $\mathbb{C}$-vector space with S_σ as a basis; more explicitly we write χ^m for the basis vector corresponding to $m \in S_\sigma$ and set $\chi^{e_i^*} := x_i$. Thus, the elements of $\mathbb{C}[S_\sigma]$ are finite formal linear combinations $\sum_{m \in S_\sigma} a_m \chi^m$ where, if $m = \sum_i a_i e_i^*$,

$$\chi^m = \chi^{\sum_i a_i e_i^*} := \prod_i x_i^{a_i}.$$

The semigroup algebra is an integral domain which is finitely generated as a $\mathbb{C}$-algebra. Hence,

Definition 3.1.12. We define the *affine toric variety associated to* σ to be

$$U_\sigma := Spec\left(\mathbb{C}[S_\sigma]\right),$$

where if $A = \mathbb{C}[x_1, \ldots, x_r]/I$, we identify $Spec(A)$ with

$$V(I) := \{p \in \mathbb{C}^r | f(p) = 0 \text{ for all } f \in I\}.$$

Example 3.1.13. Let $\sigma = Cone(e_1, \ldots, e_n)$. Then, $\sigma^\vee = \sigma$ and $\sigma^\vee \cap \mathbb{Z}^n = \mathbb{Z}^n_{\geq 0}$ and the semigroup algebra is $\mathbb{C}[S_\sigma] = \mathbb{C}[x_1, \ldots, x_n]$. Therefore, $U_\sigma = \mathbb{C}^n$

Example 3.1.14. If $0 < d < n$ and $\sigma = Cone(e_1, \ldots, e_d) \subset \mathbb{R}^n$, we see that $\sigma^\vee = Cone(e_1, \ldots, e_d, \pm e_{d+1}, \ldots, \pm e_n)$ and therefore

$$\mathbb{C}[S_\sigma] = \mathbb{C}[x_1, \ldots, x_d, x_{d+1}^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Since $Spec(\mathbb{C}[x]) = \mathbb{C}$ and $Spec(\mathbb{C}[x, x^{-1}]) = \mathbb{C}^*$, then

$$U_\sigma = \mathbb{C}^d \times (\mathbb{C}^*)^{n-d}.$$

In particular,

$$U_{\{0\}} = (\mathbb{C}^*)^n.$$

Theorem 3.1.15. *Let $\sigma \subset N_\mathbb{R} \cong \mathbb{R}^n$ be a rational cone and $U_\sigma = Spec(\mathbb{C}[S_\sigma])$. Then V_σ is a normal toric variety of dimension n. Conversely, if V is a normal affine toric variety, then there exists a rational cone σ such that V is isomorphic to $V_\sigma = Spec(\mathbb{C}[S_\sigma])$.*

Proof. See Theorem 1.3.5 in [6]. □

Remark 3.1.16. An affine variety X is normal if its coordinate ring

$$\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]/I(X)$$

is a integral domain. This is equivalent to saying that (Reid '85)

- X is non-singular in codimension 1;
- functions defined on $X \setminus Y$, where Y is a subvariety of codimension 2, can be extended to X.

The next step to create toric varieties is gluing together different U_σ that contain the same $(\mathbb{C}^*)^n$. The first thing to define is the structure of *fan*.

Definition 3.1.17. A *fan* Σ is a finite collection of rational cones in $N_\mathbb{R}$ such that

- If $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$,
- if $\sigma, \tau \in \Sigma$, $\sigma \cap \tau$ is a common face.

We also denote the set of ray generators of Σ, which is the set of 1-dimensional cones, as V_Σ.

Definition 3.1.18. Let τ be a cone in Σ. We define its *star fan* as

$$star(\tau, \Sigma) = \{\bar{\sigma} | \tau \text{ is a face of } \sigma\},$$

where

$$\bar{\sigma} = \sigma + Span(\tau)/Span(\tau) \subset N_{\mathbb{R}}/Span(\tau).$$

We define its *link* as

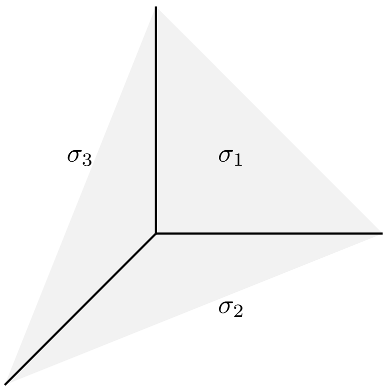
$$link(\tau, \Sigma) = \{\sigma' \in \Sigma | \sigma' \text{ is a face of a cone } \sigma, \tau \text{ is a face of } \sigma, \sigma' \cap \tau = \{0\}\}$$

Example 3.1.19. Consider a fan $\Sigma \subset \mathbb{R}^2$, $\Sigma = \{\sigma_i, \sigma_i \cap \sigma_j, \{0\}\}$, where

$$\sigma_1 = Cone(e_1, e_2),$$

$$\sigma_2 = Cone(e_1, -e_1 - e_2),$$

$$\sigma_3 = Cone(-e_1 - e_2, e_2).$$



If we now let $\tau = \sigma_1 \cap \sigma_2 = Cone(e_1)$, we obtain

$$star(\tau, \Sigma) = \{Cone(e_2), Cone(-e_2), \{0\}\} \subset N_{\mathbb{R}}/Span(e_1) \cong \mathbb{R}$$

and

$$link(\tau, \Sigma) = \{Cone(e_2), Cone(-e_1 - e_2), \{0\}\} \subset N_{\mathbb{R}}.$$

The fan Σ encodes the information needed to create an abstract variety X_Σ. Let us study how intersections and faces work before explaining how to use them to build X_Σ.

Lemma 3.1.20. *If τ is a face of σ, there exists $m_\tau \in S_\sigma$ such that*

$$\tau = \sigma \cap m_\tau^\perp = \{u | < m_\tau, u >= 0\}.$$

Proof. First, let us show that such a m exists in $\sigma^\vee$. In fact, if τ is a facet, then it is contained on a hyperplane with equation

$$\sum_j m_j x_j = 0,$$

hence, call $m_\tau = (m_1, \ldots, m_n)$. If τ is a face of codimension $k > 1$, it can be written as the intersection of k facets so

$$\tau = \bigcap_{j=1}^{k} \tau_j = \bigcap_{j=1}^{k} \sigma \cap m_{\tau_j}^{\perp} = \sigma \cap \left(\sum_{j=1}^{k} m_{\tau_j} \right)^{\perp}.$$

Now, we need to construct a m_τ that is also rational. Since $m_\tau \in \sigma^\vee$, it is contained in the interior of a face F of $\sigma^\vee$, which is rational. Thus, we can write $m_\tau = \sum_i a_i z_i$, where all the a_i are strictly positive, since τ is in the relative interior of F. Hence,

$$\sigma \cap m_\tau^{\perp} = \{u \in \sigma | < u, z_i >= 0 \text{ for all } i\},$$

which is what we wanted to prove. □

Remark 3.1.21. We can also prove that, if τ is a face of σ,

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-m_\tau),$$

where m_τ is the one from the previous lemma. We can observe directly from the definition that

$$\sigma^\vee + \mathbb{R}_{\geq 0}(-m_\tau) \subseteq \sigma^\vee + (m_\tau)^{\perp} \subseteq \tau^\vee.$$

Conversely, if $m \in S_\tau$, then $m + am_\tau \in S_\sigma$, because for any $v \in \tau = \sigma \cap m_\tau^{\perp}$

$$0 \leq < m + am_\tau, v >=< m, v > +a < m_\tau, v >=< m, v > .$$

In particular, $\mathbb{C}[S_\sigma] \subset \mathbb{C}[S_\tau]$ and so $U_\tau \hookrightarrow U_\sigma$ can be seen as a Zariski open subset of U_σ.

If we now consider $\sigma \cap \sigma'$ as a common face of σ and σ' in Σ, from the previous remark, we have immersions

$$U_{\sigma\cap\sigma'} \hookrightarrow U_\sigma$$

$$U_{\sigma\cap\sigma'} \hookrightarrow U_{\sigma'}.$$

Denote their images with $U_{\sigma\sigma'}$ and $U_{\sigma'\sigma}$, respectively. We then have an isomorphism

$$\varphi_{\sigma\sigma'} : U_{\sigma\sigma'} \cong U_{\sigma'\sigma}$$

which tells us how to glue the toric varieties on the common faces. Let us give an examples of this construction.

Example 3.1.22. We claim that the fan from 3.1.19 is such that $X_\Sigma = \mathbb{P}^2(\mathbb{C})$. We already know that $U_{\{0\}} = (\mathbb{C}^*)^2$. If we consider the 2-dimensional cones, we have

$$\sigma_1^\vee = \{m \in M | < m, u >\geq 0 \text{ for all } u \in \sigma_1\} = Span_{\mathbb{Z}}(e_1^*, e_2^*) \subset M,$$

and similarly,

$$\sigma_2^\vee = Span_{\mathbb{Z}}(-e_2^*, e_1^* - e_2^*) \subset M,$$

$$\sigma_3^\vee = \{m \in M | < m, u >\geq 0 \text{ for all } u \in \sigma_3\} = Span_{\mathbb{Z}}(-e_1^*, -e_1^* + e_2^*) \subset M.$$

We also observe that all the three pairs of generators are actually $\mathbb{Z}$-bases of M, which means there exist $A_2, A_3 \in SL(2,\mathbb{Z})$ such that $A_2(\sigma_2) = A_3(\sigma_3) = \sigma_1$. This means the three toric varieties are all isomorphic to U_{σ_1}. We now compute the semigroup algebra for σ_1,

$$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[\chi^{(1,0)}, \chi^{(0,1)}] = \mathbb{C}[x_1, x_2] = \mathbb{C}[x_1, x_2]/\{0\};$$

since $I = (0)$,

$$U_{\sigma_1} = Spec\mathbb{C}[x_1, x_2] = V(I) = \mathbb{C}^2.$$

Let us show now how these varieties behave on the intersections. Consider $\tau = \sigma_1 \cap \sigma_2 = \mathbb{R}_{\geq 0} e_1$. We have proved that, seeing τ as a face of σ_1 gives us

$$S_\tau = S_{\sigma_1} + \mathbb{Z}_{\geq 0}(-e_2^*) = Span_{\mathbb{Z}_{\geq 0}}(e_1^*, e_2^*, -e_2^*).$$

Hence,

$$\mathbb{C}[S_\tau] = \mathbb{C}[x_1, x_2, x_2^{-1}].$$

Similar computations give us the same result if we consider τ as a face of σ_2. Therefore the associated toric variety is,

$$U_\tau = \mathbb{C} \times \mathbb{C}^*$$

and, more specifically,

$$U_\tau = \{p \in U_{\sigma_1} | \chi^{m_1}(p) \neq 0\} \hookrightarrow U_{\sigma_1}$$

and

$$U_\tau = \{p \in U_{\sigma_2} | \chi^{m_2}(p) \neq 0\} \hookrightarrow U_{\sigma_2}.$$

This construction corresponds exactly to the usual construction of $\mathbb{P}^2(\mathbb{C})$, where we take three Zariski open subset $U_i = \{[u_1; u_2; u_3] | u_j \neq 0\}$ and glue them on the intersections with changes of coordinates (for example on $U_1 \cap U_2$)

$$\varphi_{12} : [u_1; 1; u_3] \mapsto \left[1; \frac{1}{u_1}; \frac{u_3}{u_1}\right].$$

Our next step in the study of fans is introducing piecewise linear functions.

Definition 3.1.23. Let us denote with $|\Sigma|$ the topological space made of the union of all cones of a fan $\Sigma \subset N_\mathbb{R}$. Consider a function $\ell : |\Sigma| \to \mathbb{R}$.

- We say ℓ is *piecewise linear* if it is continuous and its restriction to any cone is a linear function on $N_\mathbb{R}$;
- the function ℓ is said to be *integral* if

$$\ell(|\Sigma| \cap N) \subseteq \mathbb{Z};$$

- the function ℓ is said to be *positive* if

$$\ell(|\Sigma| \setminus \{0\}) \subseteq \mathbb{R}_{>0}.$$

Example 3.1.24. Consider a fan Σ and one of its rays $v_i \in V_\Sigma$. The *Courant function* x_{v_i} associated to v_i is defined as

$$x_{v_i}(v_j) = \delta_{ij}.$$

The Courant functions are piecewise linear and, since $\Sigma \subset N_\mathbb{R}$, they are also integral. Moreover, piecewise linear functions form a group, of which the Courant functions are a basis

$$PL(\Sigma) = \left\{ \sum_{v \in V_\Sigma} c_v x_v | c_v \in \mathbb{Z} \right\}.$$

Remark 3.1.25. If we have an integral linear function $m \in M_\mathbb{R}$ we can restrict it to a piecewise linear function on Σ using the map

$$\begin{aligned} res_\Sigma : M \to & PL(\Sigma) \\ m \mapsto & \sum_{v \in V_\Sigma} < m, v > x_v. \end{aligned}$$

Definition 3.1.26. We denote the following group

$$CH^1(\Sigma) := PL(\Sigma)/M.$$

This is the space of classes of piecewise linear functions on Σ where $\ell' \in [\ell]$ if and only if $\ell' - \ell$ is an integral linear function on $N_\mathbb{R}$.

An integral piecewise linear function on Σ is always equivalent to an integral piecewise linear function that is zero on a fixed cone $\sigma \in \Sigma$. This lets us give the following definitions

Definition 3.1.27. We say that ℓ is *convex*, respectively *strictly convex*, around $\sigma \in \Sigma$ if it is zero on σ and non-negative, respectively positive, on the rays of $link(\sigma, \Sigma)$.
We say ℓ is *(strictly) convex* if it is (strictly) convex on every cone in Σ.

Definition 3.1.28. The *ample cone* $\mathcal{K}_\Sigma \subset CH^1(\Sigma)$ of a fan Σ is the open convex cone of classes of strictly convex piecewise linear functions on Σ. The *NEF cone* $\mathcal{N}_\Sigma$ is the closure of the ample cone and contains the classes of convex piecewise linear functions on Σ.

Lastly, we give the definition of the *Chow ring of* Σ.

Definition 3.1.29. Let Σ be a fan. The *Chow ring of* Σ is a commutative graded algebra $CH(\Sigma)$

$$CH(\Sigma) := S_\Sigma/(I_\Sigma + J_\Sigma),$$

where

- $S_\Sigma = \mathbb{Q}[x_v | v \in V_\Sigma]$ and we denote $x_\sigma := \prod_{v \in \sigma} x_v$;
- I_Σ is the ideal generated by the linear forms

  $$\sum_{v \in V_\Sigma} < m, v > x_v,$$

  for $m \in M$;

- J_Σ is the ideal generated by the square-free monomials not in $Z^*(\Sigma) = \bigoplus Z^k(\Sigma)$, where
$$Z^k(\Sigma) := \bigoplus_{\dim \sigma = k} \mathbb{Q}x_\sigma.$$

Remark 3.1.30. If we identify the variables of S_Σ with the Courant functions on Σ, we observe that $CH^1(\Sigma)$ agrees with the Definition 3.1.26. This means that the Chow ring is the ring of polynomial piecewise linear maps on Σ with rational coefficients modulo the ideal generated by globally linear functions. As an algebra, $CH(M)$ is generated by $CH^1(M)$.

3.2 The Chow ring of a lattice

In this section we first define the Chow ring in a purely combinatorial way (See [5] and [4]) on the structure of semi-lattice, defining a graded algebra $D(\mathcal{L}, \mathcal{G})$, very similar to the one introduced for a fan, and use this definition on a combinatorial geometry $\mathscr{L}(M)$; then, we define a fan associated to a matroid Σ_M, called *Bergman fan*, and show that all these similar definitions actually coincide while dealing with matroids.

These definitions can be given for any *meet-semilattice* $\mathcal{L}$, which is a poset where any two elements $x, y \in \mathcal{L}$ have a greatest lower bound, which is called the *meet*, $x \wedge y$; in particular, we have a minimal element. Since we want to deal with combinatorial geometries, we will define everything on the semi-lattice $\mathcal{L} = \mathscr{L}(M) \setminus \{E\}$.

Definition 3.2.1. Let $\mathcal{G} \subseteq \mathcal{L}$ be a family of flats. We denote with $\mathcal{G}_{\leq F}$ the family
$$\mathcal{G}_{\leq F} = \{G \in \mathcal{G} | G \subseteq F\}.$$
The family $\max \mathcal{G}_{\leq F}$ is called the *set of factors of* F.

Definition 3.2.2. A set of flats $\mathcal{G} \subseteq \mathcal{L} \setminus \{\emptyset\}$ is called a *building set* if $\mathcal{G}$ generates $\mathcal{L}$ by $\vee$ and, for any $\{G, G_1, \ldots, G_t\}$ factors of F,

- $\mathcal{G}_{\leq G} \cap \mathcal{G}_{\leq G_1 \vee \ldots \vee G_t} = \emptyset$;
- if H is a flat such that $H < G$, then
$$H \vee G_1 \vee \ldots \vee G_t < G \vee G_1 \vee \ldots \vee G_t.$$

Remark 3.2.3. It is worth noting (and trivial to prove) that the set of atoms is the minimal building set, while $\mathcal{G} = \mathcal{L} \setminus \{\emptyset\}$ is the maximal building set for $\mathcal{L}$. For our purposes we will always work with the latter.

Definition 3.2.4. A subset N of a building set $\mathcal{G}$ is said to be *nested* if for any set of pairwise incomparable flats $G_1, \ldots, G_t \in N$ $(t \geq 2)$ the join $G_1 \vee \ldots \vee G_t \notin \mathcal{G}$. The nested sets of $\mathcal{G}$ form a simplicial complex denoted $\mathcal{N}(\mathcal{L}, \mathcal{G})$.

Definition 3.2.5. Let $\mathcal{L}$ be a finite meet-semilattice, $\mathcal{G}$ a building set for $\mathcal{L}$. The *algebra* $D(\mathcal{L}, \mathcal{G})$ *of* $\mathcal{L}$ *with respect to* $\mathcal{G}$ is defined as
$$D(\mathcal{L}, \mathcal{G}) := \mathbb{Q}[x_F | F \in \mathcal{G}]/(I + J),$$

where

- I is the ideal generated by

$$\{\sum_{G\geq i} x_G | i \text{ atom of } \mathcal{L}\},$$

- J is the ideal generated by

$$\left\{\prod_{i=1}^{t} x_{G_i} | \{G_1, \ldots, G_t\} \notin \mathcal{N}(\mathcal{L}, \mathcal{G})\right\}.$$

Remark 3.2.6. Working on $\mathcal{L} = \mathscr{L}(M) \setminus \{E\}$ with $\mathcal{G} = \mathcal{L} \setminus \{\emptyset\}$ gives us a really simple presentation for this algebra, that we will call *Chow ring of M* $CH(M) := D(\mathscr{L}(M) \setminus \{E\}, \mathcal{L} \setminus \{\emptyset\})$:

- $CH(M) = S_M/(I_M + J_M)$;
- $S_M = \mathbb{Q}[x_F | F \text{ non-empty proper flat of } M]$;
- (linear relations) I is the ideal generated by

$$\sum_{i_1\in F} x_F - \sum_{i_2\in F} x_F,$$

  for every couple of distint elements $i_1, i_2 \in E$;
- (incomparability relations) J is the ideal generated by

$$x_{F_1} x_{F_2},$$

  for every couple of incomparable flats F_1, F_2 in M.

The next step is defining a unimodular fan, called *Bergman fan*, for a simple matroid M.

Definition 3.2.7. Let $\mathcal{L}$ be a finite atomic semi-lattice with atoms $1, \ldots, n$. If $F \in \mathcal{L}$, then $F = i_1 \vee \ldots \vee i_k$. We define vectors in $\mathbb{R}^n$ for $F \in \mathcal{L}$ as

$$(v_F)_j = \begin{cases} 1, & \text{if } j \in F \\ 0, & \text{otherwise.} \end{cases}$$

We define the rational, polyhedral fan $\Sigma(\mathcal{L}, \mathcal{G})$ as

$$\Sigma(\mathcal{L}, \mathcal{G}) = \{Cone(N) | N \in \mathcal{N}(\mathcal{L}, \mathcal{G})\}.$$

This again has a very simple description for $\mathscr{L}(M) \setminus \{E\}, \mathcal{L} \setminus \{\emptyset\}$:

Definition 3.2.8. Let M be a matroid, $\mathbb{R}^E$ the vector space generated by $(e_i)_{i\in E}$. We denote, for any subset $S \subseteq E$,

$$e_S := \sum_{i\in S} e_i.$$

The *Bergman fan* Σ_M is the fan in $\mathbb{R}^E / < e_E >$ such that

- its family of rays is given by

$$\{\rho_F = Cone(e_F) | F \text{ non-empty flat of } M\};$$

- its family of cones is given by

$$\left\{\sigma_{\mathscr{F}} := Cone\,\{e_F\}_{F\in\mathscr{F}}\,|\mathscr{F} \text{ flag of non-empty proper flats of } M\right\}.$$

Lemma 3.2.9. *If $\Sigma = \Sigma_M$ is the Bergman fan of a matroid M, then*

$$CH(\Sigma_M) = CH(M).$$

Proof. We show that all the structures defining $CH(M)$ and $CH(\Sigma_M)$ coincide:

- We defined a ray for each non-empty proper flat, therefore $S_\Sigma = S_M$.
- Take the generator of I_M corresponding to a couple $i, j \in E$; this corresponds to a linear form $m \in M$ such that

$$< m, v >= \begin{cases} 1, \text{ if } v_i = 1 \\ -1, \text{ if } v_j = 1 \\ 0, \text{ otherwise} \end{cases}$$

  and therefore belongs to I_Σ. Conversely, if you consider $m \in M$, this determines uniquely a form in I_Σ, and it is uniquely determined by its values on the atoms $x_1, \dots, x_n$, denoted $< m, v_i >= m_i$. Hence,

$$\sum_{v\in V_\Sigma} < m, v > x_v = \sum_{i=1}^{n} m_i \sum_{i\in F} x_F.$$

  Since $\sum_{i=1}^{n} v_i = 0$, we also have $\sum_{i=1}^{n} m_i = 0$, hence the form is equal to

$$\sum_{i=1}^{n-1} m_i(\sum_{i\in F} x_F - \sum_{n\in F} x_F) \in I_M.$$

- Since we have a bijection between the family of flags of flats and cones, if F_1 and F_2 are incomparable, they do not form a cone in Σ_M and therefore $x_{F_1}x_{F_2} \in J_\Sigma$ if and only if it is in J_M.

□

Remark 3.2.10. At the beginning of the chapter, we said that the Chow ring is a structure that arises in the study of the intersection cohomology ring of a projective variety. As we know, the Bergman fan Σ_M is associated to a toric variety X_{Σ_M}; we claim that $CH(M) \cong CH(X_{\Sigma_M})$. Actually, we know that, in the representable case, the toric variety is the wonderful variety introduced in [13]. We quickly recall the ideas behind it.
Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential (i.e. $\bigcap H_i = \{0\}$) arrangement of complex linear hyperplanes in $\mathbb{C}^d$. Denote $\mathcal{M}(\mathcal{A}) := \mathbb{C}^d \setminus \mathcal{A}$; the topological information of this space is encoded in the lattice of the intersections $\mathscr{L}(\mathcal{A})$ (which, as we

observed in the first chapter, can be easily given a structure of matroid). We define an open embedding of $\mathcal{M}(\mathcal{A})$

$$\Psi : \mathcal{M}(\mathcal{A}) \to \mathbb{C}^d \times \prod_{X \in \mathscr{L}(\mathcal{A})} \mathbb{P}(\mathbb{C}^d/X);$$

the closure of its image

$$Y_{\mathcal{A}} := \overline{\Psi(\mathcal{M}(\mathcal{A}))}$$

is called the *wonderful model for* $\mathcal{A}$. The wonderful model $Y_{\mathcal{A}}$ has a natural projection map on the original ambient space

$$\pi : Y_{\mathcal{A}} \to \mathbb{C}^d$$

and is such that $\pi^{-1}(\mathcal{M}(\mathcal{A})) \cong \mathcal{M}(\mathcal{A})$ and $Y_{\mathcal{A}} \setminus \mathcal{M}(\mathcal{A})$ is a normal crossing divisor. The incredible result by De Concini-Procesi ([13]) is that

$$Y_{\mathcal{A}} \cong CH(\mathscr{L}(\mathcal{A})).$$

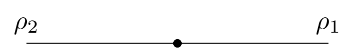
Feichtner and Yuzvinsky then extended the result on meet-semilattices using nested sets in building sets, to recover a similar result in the non-representable case; the resulting Chow ring $CH(M)$ is the one we studied throughout all this section.

3.3 Useful properties of Σ_M and $CH(M)$

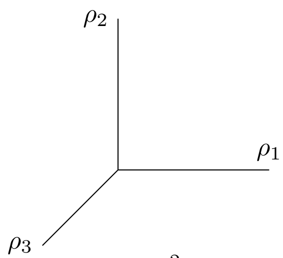
The Bergman fan becomes quite useful in proving some results on matroids, especially because they let us work on the Chow ring by means of piecewise linear functions. In this section we collect some more properties and remarks about it. Afterwards, we define specific maps that will help us describe the Chow Ring in terms of the deletion and contraction matroids. These, along with the *degree map* and the forms $\alpha_M, \beta_M \in CH^1(M)$ will be fundamental ingredients in the next chapter where we prove the main results.

Remark 3.3.1. If the ground set of a matroid has cardinality n, its Bergman fan Σ_M is defined on $\mathbb{R}^{n-1}$ and $\rho_n = -(\rho_1 + \ldots + \rho_{n-1})$. Moreover, the maximal dimension of its cones is given by $\rho(M) - 1$, since a cone is k-dimensional if and only if there exists a k-flag of proper flats.

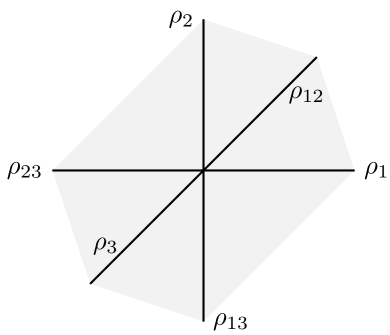
Example 3.3.2. Let us consider the Bergman fans of simple matroids with $|E| \leq 4$ (except for B_4, which is quite difficult to represent on paper). The fan for B_1 is just the origin $\Sigma_{B_1} = \{\{0\}\}$.



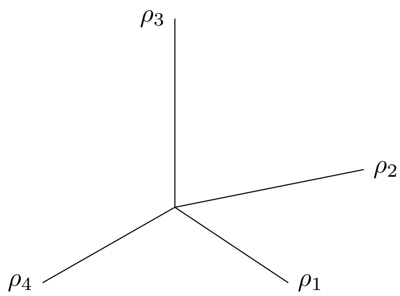
$\Sigma_{B_2} \subseteq \mathbb{R}^1$



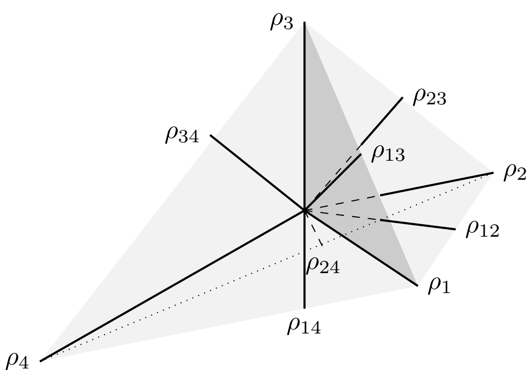
$\Sigma_{U_{2,3}} \subseteq \mathbb{R}^2$



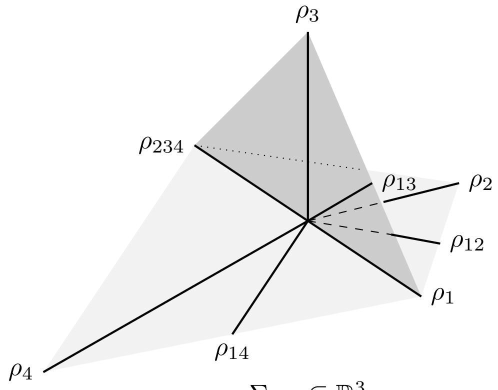
$\Sigma_{B_3} \subseteq \mathbb{R}^2$



$\Sigma_{U_{2,4}} \subseteq \mathbb{R}^3$



$\Sigma_{U_{3,4}} \subseteq \mathbb{R}^3$



$\Sigma_{M_4} \subseteq \mathbb{R}^3$

Definition 3.3.3. Let N and M be two matroids on the same ground set E. We say that M is a *quotient* of N if every flat of M is also a flat of N. This implies that $\Sigma_M \subseteq \Sigma_N$.

Example 3.3.4. A matroid on n elements is always a quotient of B_n. We can see for example that $\Sigma_{U_{2,3}} \subset \Sigma_{B_3}$. Since we are dealing with simple matroids, $U_{2,n}$ is a quotient of any matroid on n elements.

Lemma 3.3.5. *Let F be a non-empty proper flat of M. The linear isomorphism*

$$\begin{aligned} i_F : \mathbb{R}^{E\setminus F}/ < e_{E\setminus F} > \oplus \mathbb{R}^F / < e_F > \to & \mathbb{R}^E / < e_E, e_{E\setminus F} > \\ e_j \mapsto & e_j \end{aligned}$$

gives the following equivalence of fans

$$\Sigma_{M_F} \times \Sigma_{M^F} \cong star(\rho_F, \Sigma_M).$$

Proof. Directly from the definitions. □

Example 3.3.6. Take M_4 and $i = 1$. We know that $M_4/1 \cong U_{2,3}$ and $(M_4)^{\{1\}} \cong B_1$. Since $\Sigma_{B_1} \cong \{0\}$, we claim that $star(1, \Sigma_{M_4}) \cong \Sigma_{U_{2,3}}$. In fact,

$$\begin{aligned} star(1, \Sigma_{M_4}) = & \left\{ \frac{Cone(e_1 + e_2)}{< e_1 >}, \frac{Cone(e_1 + e_3)}{< e_1 >}, \frac{Cone(e_1 - e_1 - e_2 - e_3)}{< e_1 >} \right\} = \\ = & \{Cone(e_2), Cone(e_3), Cone(-e_2 - e_3)\} \cong \Sigma_{U_{2,3}}. \end{aligned}$$

Let us now make some considerations on $CH(M)$ as a graded algebra.

Remark 3.3.7. The Chow ring $CH(M)$ has the following structure of graded algebra

$$CH(M) = \bigoplus_{i=0}^{r} CH^i(M),$$

inherited by the algebra of polynomials $\mathbb{Q}[x_F]$. The maximum degree is r since any family of $r + 1$ flats surely has at least two incomparable flats which make the product equal to zero.

Example 3.3.8. The family of matroids of rank 2 $\{U_{2,n}\}_{n\in\mathbb{N}}$ all have the same Chow ring which is $CH(U_{2,n}) = \mathbb{Q}[x]/(x^2)$, since from the linear relations we get that $x_i = x_j$ for every $i, j \in E$.

Example 3.3.9. More generally, we can give a more explicit description of the Chow ring of matroids of rank less than or equal to 3. In fact, linear relations give us a system of linear equations of rank $n-1$, which can be used to find a basis for $CH^1(M)$. We will also see in Theorem 3.3.14 that $\dim CH^2(M) = 1$, and we can assume it is generated by any x_i^2. Let us show how to do it on $M_4 = U_{2,3} \oplus B_1$: the linear relations give us

$$\begin{cases} x_{13} = & x_2 - x_3 + x_{12} \\ x_{14} = & x_2 - x_4 + x_{12} \\ x_{234} = & x_1 + x_2 - x_3 - x_4 + 2x_{12} \end{cases}$$

This proves that every element of degree 1 has the form

$$p_1 = a + bx_1 + cx_2 + dx_3 + ex_4 + fx_{12};$$

this means that, using the incomparability relations, we now only have to describe the terms x_1x_{12} and x_2x_{12}. But using linear relations again,

$$x_1x_{12} = x_1\left(-x_1 + x_3 + x_{23}\right) = -x_1^2.$$

Example 3.3.10. Similar, yet more complicated, calculations, which we carried out using the software Singular, let us describe the Chow ring of the matroids on 5 elements we introduced in the first chapter ($t_3(M_5)$ and N_5 are rank 3 matroids and therefore can be studied as in the previous example). Matroid M_5 is a rank 4 matroid, with

$$\dim CH^0(M_5) = \dim CH^3(M_5) = 1,$$

$$\dim CH^1(M_5) = \dim CH^2(M_5) = 14.$$

This symmetry is no surprise and it foreshadows the fact that the Chow ring is a Poincaré Duality algebra.

Definition 3.3.11. Let i be an element in E. Define the linear forms

$$\alpha_{M,i} := \sum_{i\in F} x_F,$$

$$\beta_{M,i} := \sum_{i\notin F} x_F.$$

From the linear relations, their classes in $CH(M)$ do not depend on the element i and will be denoted by simply α_M e β_M.

Example 3.3.12. Consider $M = t_3(M_5)$. Then, for example,

$$\alpha_M = x_1 + x_{12} + x_{13} + x_{14} + x_{15},$$

$$\alpha_M^2 = -\frac{1}{3}x_1^2.$$

$$\beta_M = x_2 + x_3 + x_4 + x_5 + x_{23} + x_{24} + x_{25} + x_{345}.$$

Lemma 3.3.13. *Let $\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \ldots \subsetneq F_k\}$ be a flag of non-empty proper flats of M,*

1. *If there exists $m \leq k$ such that $\rho(F_m) \neq m$, then*
$$x_{F_1} \ldots x_{F_k} \alpha_M^{r-k} = 0 \in CH^r(M);$$
2. *If $\rho(F_m) = m$ for every $m \leq k$ (we will also call such a flag initial), then*
$$x_{F_1} \ldots x_{F_k} \alpha_M^{r-k} = \alpha_M^r \in CH^r(M);$$

Proof. If F is a non-empty proper flat and i does not belong in F, we have
$$x_F \alpha_M = x_F \sum_G x_G \in CH(M),$$
where the sum is on all the non-empty proper flats which contain both F and i (if they contain i they cannot be contained in F and if they are incomparable the product is zero, therefore the flats considered in the sum have to contain F). In particular, if $\rho(F) = r$, then the product is equal to zero for Remark 3.3.7. Let us prove the first statement by decreasing induction on k, which is necessarily less or equal to r. If $k = r$, then $\rho(F_k) = r$, and the product is zero due to the previous remark. If $k < r$, choose an element i that does not belong to F_k. Then,
$$x_{F_1} \ldots x_{F_k} \alpha_M^{r-k} = x_{F_1} \ldots x_{F_k} \left(\sum_G x_G \right) \alpha_M^{r-k-1} = \sum_G x_{F_1} \ldots x_{F_k} x_G \alpha_M^{r-k-1},$$

where the sum is made on all the flats containing i (and, following a similar reasoning as before, also containing F_k). Then, every member of the sum is zero since it is associated to a flag of $k+1$ flats multiplied by α_M^{k+1}.
We also prove the second statement by induction. For $k = 1$ we choose i in F_1 and write
$$\alpha_M^r = \left(\sum_G x_G \right) \alpha_M^{r-1},$$

where the sum is on all non-empty proper flats that contain i. From the first statement of this Lemma, if $k = 1$, there is only one non-zero term in the right-hand side, which is the one with x_i in it. This proves that
$$\alpha_M^r = x_i \alpha_M^{r-1}.$$

For $k \leq r+1$, suppose the result holds for $k-1$,
$$\alpha_M^r = x_{F_1} \ldots x_{F_{k-1}} \alpha_M^{r-(k-1)}.$$

The rank of F_{k-1} is less or equal to r, so there exists a flat $F_k \supsetneq F_{k-1}$; if we choose an element i in $F_k \setminus F_{k-1}$, then
$$\alpha_M^r = x_{F_1} \ldots x_{F_{k-1}} \left(\sum_G x_G \right) \alpha_M^{r-k},$$

where the sum is on all non-empty proper flats that contain F_{k-1} and i. From the first statement, there is only a non-zero term in the right-hand side, therefore

$$\alpha_M^r = x_{F_1} \dots x_{F_{k-1}} x_{F_k} \alpha_M^{r-k}.$$

□

Theorem 3.3.14. *If $\mathscr{F}$ and $\mathscr{G}$ are two maximal flags of non-empty proper flats, then*

$$x_{\mathscr{F}} = x_{\mathscr{G}} \in CH(M).$$

Proof. A maximal flag contains r flats. From Lemma 3.3.13,

$$x_{\mathscr{F}} = \alpha_M^r = x_{\mathscr{G}}.$$

□

Example 3.3.15. A smart way to compute the powers of α_M and β_M, if we have to do calculations by hand, is to exploit the incomparability relations, choosing a suitable representative as we show in a concrete example. Consider for example $M = N_5$ and $\alpha_M = x_1 + x_{13} + x_{14} + x_{125}$. Then,

$$\begin{aligned} \alpha_M^2 =& x_1\alpha_M + x_{13}\alpha_M + x_{14}\alpha_M + x_{125}\alpha_M = \\ =& x_1\alpha_{M,2} + x_{13}\alpha_{M,2} + x_{14}\alpha_{M,2} + x_{125}\alpha_{M,3} = \\ =& x_1 \left(x_{12} + x_{125}\right). \end{aligned}$$

Everything else has vanished because we multiplied each x_F by $\alpha_{M,i}$, where $i = \min(E \setminus F)$, resulting in many products between incomparable flats. Of course we can reduce everything to the form ax_1^2, using repeatedly the linear and incomparability relations, to get

$$\alpha_M^2 = -\frac{1}{2}x_1^2.$$

Definition 3.3.16. On the Chow ring $CH(M)$ we can define the following *degree map*

$$\deg_M : CH^r(M) \to \mathbb{Q}, x_{\mathscr{F}} \mapsto 1,$$

where $\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_r\}$ is a maximal flag of non-empty proper flats of M.

Remark 3.3.17. The degree of α_M^k is 1, since from Lemma 3.3.13,

$$\deg \alpha_M^r = \deg x_{\mathscr{F}} = 1.$$

Furthermore, in Theorem 3.3.14 we proved that $CH^r(M)$ is generated by α_M^r, therefore the degree map is an isomorphism from $CH^r(M)$ to $\mathbb{Q}$.

Example 3.3.18. The above computations for α_M^r show that the degree maps for matroids M_5, $t_3(M_5)$ and N_5 are, respectively,

$$\deg_{M_5} : \frac{1}{2}x_1^2 \mapsto 1$$

$$\deg_{t_3(M_5)} : -\frac{1}{3}x_1^2 \mapsto 1$$

$$\deg_{N_5} : -\frac{1}{2}x_1^2 \mapsto 1.$$

Let us now consider the following diagram, where F is a non-empty proper flat of M:

$$\begin{array}{ccc} CH(M) & \xrightarrow{\quad x_F \quad} & CH(M) \\ & \searrow^{\varphi_M^F} \quad {}^{\psi_M^F}\nearrow & \\ & CH(M_F) \otimes CH(M^F) & \end{array} .$$

We can describe the maps in two different ways: algebraically on the algebras $CH(M)$, $CH(M_F)$ and $CH(M^F)$, or geometrically, working on $CH(\Sigma)$ and knowing that $CH(\Sigma_{M_F}) \otimes CH(\Sigma_{M^F}) \cong CH(star(\sigma_F, \Sigma))$. Both options are valid and choosing one way of seeing things, lets us recover the other pretty easily.

Definition 3.3.19. The *pullback map* φ_M^F is the unique morphism of graded algebras

$$\varphi_M^F : CH(M) \to CH(M_F) \otimes CH(M^F)$$

such that

- If G is incomparable with F, $\varphi_M^F(x_G) = 0$;
- if G is non-empty and properly contained in F, $\varphi_M^F(x_G) = 1 \otimes x_G$;
- if G is a proper flat that properly contains F, $\varphi_M^F(x_G) = x_{G \setminus F} \otimes 1$.

Definition 3.3.20. The *pushforward map* ψ_M^F is the unique $CH(M)$-module morphism

$$\psi_M^F : CH(M_F) \otimes CH(M^F) \to CH(M)$$

such that

$$\psi_M^F \left(\prod_{F' \in \mathscr{S}'} x_{F' \setminus F} \otimes \prod_{F'' \in \mathscr{S}''} x_{F''} \right) = x_F \prod_{F' \in \mathscr{S}'} x_{F'} \prod_{F'' \in \mathscr{S}''} x_{F''},$$

for every collection $\mathscr{S}'$ of proper flats of M strictly containing F, and every collection $\mathscr{S}''$ of non-empty flats of M strictly contained in F.

Remark 3.3.21. The diagram is commutative and $\psi_M^F \circ \varphi_M^F = x_F$. Conversely, $\varphi_M^F \circ \psi_M^F = \varphi_M^F(x_F)$.

Lemma 3.3.22. *The pullback map satisfies these additional properties:*

- $\varphi_M^F(x_F) = -1 \otimes \alpha_{M^F} - \beta_{M_F} \otimes 1$.
- $\varphi_M^F(\alpha_M) = \alpha_{M_F} \otimes 1$;
- $\varphi_M^F(\beta_M) = 1 \otimes \beta_{M^F}$.

Proof. Let G be incomparable with F. Then, there must exist $i \in F \setminus G$ and $j \in G \setminus F$. From the linear relations in $CH(M)$,

$$x_F = x_G + \sum_{\substack{j \in H \\ H \neq G}} x_H - \sum_{\substack{i \in H \\ H \neq F}} x_H,$$

from which we obtain that

$$\begin{aligned}
\varphi_M^F(x_F) = \varphi_M^F &\left(x_G + \sum_{\substack{j\in H\\ H\neq G}} x_H - \sum_{\substack{i\in H\\ H\neq F}} x_H \right) = \\
= \varphi_M^F &\left(x_G + \sum_{\substack{j\in H\\ i\notin H\\ H\neq G}} x_H - \sum_{\substack{i\in H\\ j\notin H\\ H\neq F}} x_H \right),
\end{aligned}$$

where the second equivalence holds because if H contains both i and j, x_H would appear with different signs and could be cancelled out. Now, x_G and all the terms in the first sum are incomparable with F, hence their image through φ_M^F is zero. In the second sum we have

- all the flats containing i and contained in F,
- all the flats containing both i and F,
- some flats incomparable with F.

The result follows from the definition of φ_M^F, since

$$\begin{aligned}
\varphi_M^F(x_F) &= -\varphi_M^F \left(\sum_{\substack{i\in H\\ H\subsetneq F}} x_H \right) - \varphi_M^F \left(\sum_{\substack{i\in H\\ F\subsetneq H}} x_H \right) \\
&= -(1\otimes \alpha_{M^F}) - (\beta_{M_F}\otimes 1).
\end{aligned}$$

For the second statement, let us consider an element i that does not belong in F.

$$\alpha_M = \alpha_{M,i} = \sum_{\substack{i\in G\\ G\| F}} x_G + \sum_{\substack{i\in G\\ F\subset G}} x_G.$$

The first sum is on all flats incomparable with F and therefore is mapped to zero; the second sum is on all flats that contain both i and F, so their image is exactly $\alpha_{M_F,i}\otimes 1$.
The third statement comes from a similar reasoning, where we consider an element $i\in F$ and $\beta_M = \beta_{M.i}$. □

Let us see some concrete examples

Example 3.3.23. Consider $M = M_5$ and $F = 345$. We already studied in Chapter 1 the respective contraction by and restriction to F. Therefore,

$$\begin{array}{ccc}
CH(M_5) & \xrightarrow{\quad x_F \quad} & CH(M_5) \\
 & {}_{\varphi_M^F}\searrow \qquad \nearrow_{\psi_M^F} & \\
 & CH(B_2)\otimes CH(U_{2,3}) &
\end{array}.$$

We first compute the image of $\alpha_M = \alpha_{M,1}$.

$$\begin{aligned}\varphi(\alpha_{M,1}) =& \varphi(x_1 + x_{12} + x_{13} + x_{14} + x_{15} + x_{123} + x_{124} + x_{125} + x_{1345}) = \\ =& \varphi(x_{1345}) = x_{1345\setminus 345} \otimes 1 = \\ =& x_1 \otimes 1 = \alpha_{B_2,1} \otimes 1 \in CH(B_2) \otimes CH(U_{2,3}),\end{aligned}$$

where all the terms except one were cancelled out because they were incomparable with F and therefore their image is zero. To compute β_M we choose $i = 5$ and

$$\begin{aligned}\varphi(\beta_{M,5}) =& \varphi(x_1 + x_2 + x_3 + x_4 + x_{12} + x_{13} + x_{14} + x_{23} + x_{24} + x_{123} + x_{124}) = \\ =& \varphi(x_3 + x_4) = 1 \otimes (x_3 + x_4) = 1 \otimes \beta_{U_{2,3},5} \in CH(B_2) \otimes CH(U_{2,3}).\end{aligned}$$

Remark 3.3.24. The pushforward commutes with the degree map, that is

$$\deg_{M_F} \otimes \deg_{M^F} = \deg_M \circ \psi_M^F.$$

In fact, if $\rho(F) = k$,

$$\deg_M \left(x_F \prod_{F' \in \mathscr{S}'} x_{F'} \prod_{F'' \in \mathscr{S}''} x_{F''} \right) \neq 0$$

if and only if $\mathscr{S}'$ forms a flag of cardinality $r - k$ of flats containing F and $\mathscr{S}''$ forms a flag of cardinality $k - 1$ of flats contained in F; this is equivalent to the fact that the respective degrees in M_F and M^F are non-zero.

The technique we will use to prove that the Decomposition holds for $CH(M)$, along with Poincaré Duality, (HL) and (HR), is based on an inductive approach where we suppose that these results hold on matroids with smaller ground sets (such as, for example, M_F and M^F) and some maps are injective. We prove now in details one of these results, and will refer to this when similar proofs will be needed.

Theorem 3.3.25. *If $CH(M_F)$ and $CH(M^F)$ satisfy Poincaré Duality, then ψ_M^F is injective.*

Proof. We denote for simplicity $\deg_F = \deg_{M_F} \otimes \deg_{M^F}$; suppose there exists $\eta \neq 0$ such that $\psi_M^F(\eta) = 0$. If Poincaré Duality holds, there must exist an element ν such that $\deg_F(\nu\eta) = 1$. Moreover, φ_M^F is surjective, therefore there exist μ_1, μ_2 such that $\varphi_M^F(\mu_1) = \nu$ and $\varphi_M^F(\mu_2) = \eta$. Then,

$$\begin{aligned}1 &= \deg_F(\nu\eta) = \deg(\psi_M^F(\nu\eta)) = \deg(\psi_M^F(\varphi_M^F(\mu_1)\varphi_M^F(\mu_2)) = \deg(x_F\mu_1\mu_2) = \\ &= \deg(\mu_1 x_F \mu_2) = \deg(\mu_1\psi_M^F(\varphi_M^F(\mu_2)) = \deg(\mu_1\psi_M^F(\eta)) = \deg(0) = \\ &= 0\end{aligned}$$

where the second equivalence holds for Remark 3.3.24, while the fourth and the sixth hold for Remark 3.3.21. ∎

Chapter 4

The proof

We are finally ready to prove Theorem 1.4.21. We will always assume to work with a simple matroid M on $E = \{1, 2, \ldots, n\}$ of rank $r + 1$. We show that $CH(M)$ satisfies Poincaré Duality and property (HL) and (HR), using the decomposition described in [1] and then, in the last section, we conclude the proof of log-concavity as found in [2], by linking the Whitney numbers of the first kind to the Chow ring using initial descending flags of flats.

4.1 The decomposition of the Chow ring

Definition 4.1.1. Let i be an element of a matroid M. We define the following graded algebra morphism,

$$\theta_i = \theta_{M,i} : CH(M \setminus i) \to CH(M), x_F \mapsto x_F + x_{F \cup i},$$

where a variable x_G is set to zero if G is not a flat of M.
We also denote with $CH_{(i)} = Im\theta_i \subseteq CH(M)$ and with $\mathscr{S}_i$ the family

$$\mathscr{S}_i = \{F | F \subsetneq E \setminus i \text{ non-empty proper flat, such that } F, F \cup i \text{ are flats of } M\}.$$

Lemma 4.1.2. *We have the following compatibility results between θ_i and the degree map. Suppose $E \setminus i \neq \emptyset$; then,*

- *if i is not a coloop of M then θ_i commutes with the degree map,*

$$\deg_{M \setminus i} = \deg_M \circ \theta_i;$$

- *if i is a coloop of M then*

$$\deg_{M \setminus i} = \deg_M \circ x_{E \setminus i} \circ \theta_i = \deg_M \circ \alpha_M \circ \theta_i.$$

Proof. Let us prove the result when i is not a coloop. From 3.3.14 we have that $CH^r(M) =< \alpha_M^r >$ and $CH^r(M \setminus i) =< \alpha_{M \setminus i}^r >$ (we have the same maximal degree r because if i is not a coloop then M and $M \setminus i$ have the same rank). Since $\theta_i(\alpha_{M \setminus i}) = \alpha_M$, we have the result.
If i is a coloop, $E \setminus i$ is a flat of M, so

$$\varphi_M^{E \setminus i} \circ \theta_i = id$$

on $CH(M \setminus i)$. Using 3.3.24 we obtain

$$\deg_{M\setminus i} = \deg_M \circ \psi_M^{E\setminus i} = \deg_M \circ \psi_M^{E\setminus i} \circ \varphi_M^{E\setminus i} \circ \theta_i = \deg_M \circ x_{E\setminus i} \circ \theta_i.$$

Lastly, we observe that, if i is a coloop, from the definition of θ_i

$$\theta_i(\alpha_{M\setminus i}) = \alpha_M - x_{E\setminus i}.$$

Then,

$$\deg_M \circ x_{E\setminus i} \circ \theta_i = \deg_M \circ (\alpha_M - \theta_i(\alpha_{M\setminus i})) \circ \theta_i = \deg_M \circ \alpha_M \circ \theta_i.$$

□

We are finally ready to state the Decomposition Theorem and the Poincaré Duality theorem for the Chow Ring.

Theorem 4.1.3. *Let $CH(M)$ be the Chow Ring of a matroid M.*

- *If i is not a coloop of M we have the following decomposition*

$$CH(M) = CH_{(i)} \oplus \bigoplus_{F \in \mathscr{S}_i} x_{F\cup i} CH_{(i)}$$

and each term of the sum is orthogonal to the others with respect to the Poincaré Pairing on $CH(M)$.

- *If i is a coloop of M we have the following decomposition*

$$CH(M) = CH_{(i)} \oplus x_{E\setminus i} CH_{(i)} \oplus \bigoplus_{F \in \mathscr{S}_i} x_{F\cup i} CH_{(i)},$$

where each term is orthogonal to the others (except for the first two) with respect to the Poincaré Pairing on $CH(M)$.

- *(Poincaré Duality). For every $0 < k < \frac{r}{2}$, the bilinear form*

$$CH^k(M) \times CH^{r-k}(M) \to \mathbb{Q}, (\eta_1, \eta_2) \mapsto \deg(\eta_1 \eta_2)$$

is non-degenerate.

Before proving the theorem, we can give a better description of the terms of the sums, using the maps introduced in the previous section.

Lemma 4.1.4. *If $F \in \mathscr{S}_i$, then*

$$x_{F\cup i} CH_{(i)} = \psi_M^{F\cup i} \left(CH(M_{F\cup i}) \otimes \theta_i^{F\cup i} \left(CH(M^F) \right) \right),$$

where $M^{F\cup i} \setminus i = M^F$ and

$$\begin{aligned} \theta_i^{F\cup i} : CH(M^F) &\to CH(M^{F\cup i}) \\ x_G &\mapsto x_{G\cup i}. \end{aligned}$$

Proof. Consider $F \in \mathscr{S}_i$. The statement is equivalent to

$$\varphi_M^{F\cup i}\left(CH_{(i)}\right) = CH(M_{F\cup i}) \otimes \theta_i^{F\cup i}\left(CH(M^F)\right),$$

from 3.3.21. We have the following commutative diagram.

$$\begin{array}{ccccc}
CH(M \setminus i) & \xrightarrow{\theta_i} & CH(M) & & \\
 & & & \searrow^{\varphi_M^{F\cup i}} & \\
\downarrow{\scriptstyle \varphi_{M\setminus i}^F} & & & & CH(M_{F\cup i}) \otimes CH(M^{F\cup i}) \\
 & & & \nearrow_{1\otimes\theta_i^{F\cup i}} & \\
CH((M\setminus i)_F) \otimes CH((M\setminus i)^F) & \xrightarrow{q} & CH(M_{F\cup i}) \otimes CH(M^F) & &
\end{array}$$

where q is well-defined and surjective, because

- if $F \in \mathscr{S}_i$, i does not belong in F, hence $(M \setminus i)^F = M^F$;
- $M_{F\cup i} = (M/i)_F \subseteq (M \setminus i)_F$, where the inclusion comes from the inclusion of the respective fans.

Hence, the result. □

Example 4.1.5. Consider $M = N_5$, $F = 3 \in \mathscr{S}_1$. The diagram becomes as follows

$$\begin{array}{ccccc}
CH(M_4) & \xrightarrow{\theta_1} & CH(N_5) & & \\
 & & & \searrow^{\varphi_{N_5}^{13}} & \\
\downarrow{\scriptstyle \varphi_{M_4}^3} & & & & CH(B_1) \otimes CH(B_2) \\
 & & & \nearrow_{1\otimes\theta_1^{13}} & \\
CH(B_2) \otimes CH(B_1) & \xrightarrow{q} & CH(B_1) \otimes CH(B_1) & &
\end{array},$$

where in every Chow ring we have labelled in a suitable way the elements of the respective matroid. So, for example

$$\varphi_{M_4}^3\left(CH(M_4)\right) = \mathbb{Q}[x_2]/(x_2^2) \otimes \mathbb{Q}[x_3]/(x_3) \cong \mathbb{Q}[x_2]/(x_2^2),$$

$$q\left(\mathbb{Q}[x_2]/(x_2^2) \otimes \mathbb{Q}[x_3]/(x_3)\right) = \mathbb{Q}[x_2]/(x_2) \otimes \mathbb{Q}[x_3]/(x_3) \cong \mathbb{Q}$$

and finally,

$$\theta_1^{13}\left(CH(B_1)\right) = CH^0(B_2) \cong \mathbb{Q}.$$

If we consider instead $\left(\varphi_{N_5}^{13} \circ \theta_1\right)(CH(M_4))$, we easily see that the only elements in $CH(N_5)$ which are not mapped to zero by $\varphi_{N_5}^{13}$ are x_1, x_3 and x_{13}, and x_1 is not mapped onto by θ_1. So we just need to compute

$$\begin{aligned}
\left(\varphi_{N_5}^{13} \circ \theta_1\right)(x_3) =& \varphi_{N_5}^{13}(x_3 + x_{13}) = 1 \otimes x_3 + (-1 \otimes \alpha_{N_5^{13}} - \beta_{(N_5)_{13}}) = \\
=& 1 \otimes \alpha_{B_2} - 1 \otimes \alpha_{B_2} - 0 = 0.
\end{aligned}$$

Lemma 4.1.6. *If i is a coloop of M, then $x_{E\setminus i}CH_{(i)} = \psi_M^{E\setminus i}\left(CH(M\setminus i)\right).$*

Proof. We already observed that $\varphi_M^{E\setminus i}\circ\theta_i$ is the identity on $CH(M\setminus i)$, if i is a coloop. Then,

$$x_{E\setminus i}CH_{(i)} = x_{E\setminus i}\theta_i\left(CH(M\setminus i)\right) = \psi_M^{E\setminus i}\varphi_M^{E\setminus i}\theta_i\left(CH(M\setminus i)\right) = \psi_M^{E\setminus i}\left(CH(M\setminus i)\right).$$

□

Let us now prove Theorem 4.1.3.

Proof. The proof is by induction on the cardinality of E. If E is empty or is a singleton, the result is vacuously true. Let us then suppose that i is an element of E, that $E\setminus i$ is not empty and that the result holds on any matroid defined on any proper subset of E.
Suppose that i is not a coloop. We first prove the orthogonality of the terms of the sum. We observe that,

$$\begin{aligned} x_{F\cup i}CH_{(i)}\cdot CH_{(i)} =& x_{F\cup i}\theta_i\left(CH(M\setminus i)\right)\cdot\theta_i\left(CH(M\setminus i)\right) = \\ =& x_{F\cup i}\theta_i\left(CH(M\setminus i)\cdot CH(M\setminus i)\right)\subseteq x_{F\cup i}\theta_i\left(CH(M\setminus i)\right) = \\ =& x_{F\cup i}CH_{(i)}. \end{aligned}$$

Moreover, if $F\in\mathscr{S}_i$ then i is a coloop in $M^{F\cup i}$, so the maximal degree of $CH(M^F)$ is strictly less than $CH(M^{F\cup i})$ (which is $\rho(F)+1$); using Lemma 4.1.4 we observe that

$$x_{F\cup i}CH^{r-1}_{(i)} = \psi_M^{F\cup i}\left(CH^{r-\rho(F)-1}(M_{F\cup i})\otimes\left(\theta_i^{F\cup i}\left(CH(M^F)\right)^{\rho(F)}\right)\right) = \psi_M^{F\cup i}(0) = 0,$$

so $x_{F\cup i}CH^{r-1}_{(i)}$ is zero in degree r, hence the two terms are orthogonal with respect to the Poincaré pairing.
If we now consider $F_1, F_2\in\mathscr{S}_i$, we have two cases: if they are incomparable $x_{F_1\cup i}CH_{(i)}\cdot x_{F_2\cup i}CH_{(i)} = 0$, hence they are orthogonal; if, instead, $F_1 < F_2$, then

$$x_{F_1\cup i}CH_{(i)}\cdot x_{F_2\cup i}CH_{(i)}\subseteq x_{F_1\cup i}x_{F_2\cup i}CH_{(i)},$$

which is contained in $x_{F_1\cup i}CH_{(i)}$, since, being $F_1\cup i$ incomparable with F_2, we have

$$x_{F_1\cup i}x_{F_2\cup i} = x_{F_1\cup i}\left(x_{F_2}+x_{F_2\cup i}\right) = x_{F_1\cup i}\theta_i(x_{F_2})\in x_{F_1\cup i}CH_{(i)};$$

hence they also are orthogonal.
By inductive hypothesis and Lemma 4.1.2, the restriction of the Poincaré pairing of $CH(M)$ to $CH_{(i)}$ is non-degenerate; by Lemma 4.1.4 we can then prove that, for every $\nu_1,\nu_2\in CH(M_{F\cup i})\otimes CH(M^F)$ of complementary degrees,

$$\begin{aligned} &\deg_M\left(\psi_M^{F\cup i}\left(1\otimes\theta_i^{F\cup i}(\nu_1)\right)\cdot\psi_M^{F\cup i}\left(1\otimes\theta_i^{F\cup i}(\nu_2)\right)\right) = \\ =&\deg_M\left(\psi_M^{F\cup i}\left(\varphi_M^{F\cup i}\psi_M^{F\cup i}\left(1\otimes\theta_i^{F\cup i}(\nu_i)\right)\cdot\left(1\otimes\theta_i^{F\cup i}(\nu_2)\right)\right)\right) = \\ =&\deg_{M_{F\cup i}}\otimes\deg_{M^{F\cup i}}\left(\varphi_M^{F\cup i}\psi_M^{F\cup i}\left((1\otimes\theta_i^{F\cup i}(\nu_1))\cdot(1\otimes\theta_i^{F\cup i}(\nu_2))\right)\right) = \\ =&\deg_{M_{F\cup i}}\otimes\deg_{M^{F\cup i}}\left((-1\otimes\alpha_{M^{F\cup i}}-\beta_{M_{F\cup i}}\otimes 1)\cdot(1\otimes\theta_i^{F\cup i}(\nu_1))\cdot(1\otimes\theta_i^{F\cup i}(\nu_2))\right) = \\ =&-\deg_{M_{F\cup i}}\otimes\deg_{M^{F\cup i}}\left((1\otimes\alpha_{M^{F\cup i}})\cdot(1\otimes\theta_i^{F\cup i}(\nu_1))\cdot(1\otimes\theta_i^{F\cup i}(\nu_2))\right) = \\ =&-\deg_{M_{F\cup i}}\otimes\deg_{M^F}\left(\nu_1\nu_2\right), \end{aligned}$$

where the first equality comes from the fact that $\psi_M^{F\cup i}$ is a morphism of $CH(M)$-modules; the second comes from Lemma 3.3.24; the third from Remark 3.3.21; in the fourth we used that $\beta_{M_{F\cup i}} = 0$ since there are no flats that contain $F \cup i$ but not i; the fifth comes from Lemma 4.1.2. From inductive hypothesis, the restriction of the Poincaré pairing to $M_{F\cup i}$ and M^F is non-degenerate, so it is also non-degenerate on $x_{F\cup i}CH_{(i)}$.
Lastly we show that the sum spans all $CH(M)$. This is obvious in degree 0; in degree 1 we need to verify that x_G belongs to that sum for any non-empty proper flat G.
If $i \notin G$, then $G \setminus i$ is a flat of $M \setminus i$, so we can compute is image through θ_i and write

$$x_G = \theta_i(x_G) - x_{G\cup i}.$$

If $G \cup i$ is not a flat of M then

$$x_G = \theta_i(x_G) \in CH_{(i)};$$

if $G \cup i$ is a flat of M, then $G \in \mathscr{S}_i$, so

$$x_G \in CH_{(i)} \oplus x_{G\cup i}CH_{(i)}.$$

If, instead, $i \in G$, either $G \setminus i \in \mathscr{S}_i$, hence

$$x_G \in x_{(G\setminus i)\cup i}CH_{(i)},$$

or $G \setminus i \notin \mathscr{S}_i$ hence $G \setminus i$ is not a flat of M, so

$$\theta_i(x_{G\setminus i}) = x_{G\setminus i} + x_{(G\setminus i)\cup i} = x_G.$$

Since we have proved the result in degrees 0 and 1 and they generate $CH(M)$ as an algebra, we are left to prove that for every $k \geq 1$

$$CH_{(i)} \cdot CH^k(M) = CH^{k+1}(M).$$

For $k = 1$, since the decomposition holds in degree 1,

$$CH^1(M) = CH^1_{(i)} \oplus \bigoplus_{F\in\mathscr{S}_i} \mathbb{Q}x_{F\cup i},$$

hence,

$$CH^2(M) = \left(CH^1_{(i)} \oplus \bigoplus_{F\in\mathscr{S}_i} \mathbb{Q}x_{F\cup i}\right) \cdot \left(CH^1_{(i)} \oplus \bigoplus_{F\in\mathscr{S}_i} \mathbb{Q}x_{F\cup i}\right).$$

Then, we can just show that for any $F \in \mathscr{S}_i$, $x^2_{F\cup i} \in CH^1_{(i)} \cdot CH^1(M)$.

Consider

$$\begin{aligned} x_{F\cup i}^2 =& x_{F\cup i}\left(x_{F\cup i} + \sum_{\substack{i\in G\\ G\neq F\cup i}} x_G - \sum_{\substack{i\in G\\ G\neq F\cup i}} x_G \right) \\ =& x_{F\cup i}\left(\sum_{j\in G} x_G - \sum_{\substack{i\in G\\ G\neq F\cup i}} x_G \right) \\ =& x_{F\cup i}\left(\sum_{\substack{j\in G\\ i\notin G}} x_G - \sum_{\substack{i\in G\\ G\neq F\cup i\\ j\notin G}} x_G \right), \end{aligned}$$

where $j \notin F\cup i$. The product with the first sum is zero because the flats are all incomparable with $F\cup i$; in the second sum we have already proved the result for the flats that contain $F\cup i$. We then just need to check the products of the form $x_{F\cup i}x_G$ where $i\in G\subsetneq F\cup i$:

$$x_{F\cup i}x_G = \left(\theta_i(x_F) - x_F\right)x_G = \theta_i(x_F)\cdot x_G \in CH^1_{(i)}\cdot CH^1(M).$$

For $k>1$, we just observe that

$$\begin{aligned} CH^1_{(i)}\cdot CH^k(M) =& CH^1_{(i)}\cdot CH^1(M)\cdot CH^{k-1}(M) \\ =& CH^2(M)\cdot CH^{k-1}(M) = CH^{k+1}(M). \end{aligned}$$

This concludes the proof when i is not a coloop. If i is a coloop the proof is nearly identical: we just show the additional properties required.

The orthogonality of $x_{E\setminus i}CH_{(i)}$ with $x_{F\cup i}CH_{(i)}$ follows from the fact that $E\setminus i$ is incomparable with $F\cup i$. To show that the decomposition spans $CH(M)$, we have to verify that $x^2_{E\setminus i}\in CH^1_{(i)}\cdot CH^1(M)$, and

$$\begin{aligned} x^2_{E\setminus i} =& x_{E\setminus i}\left(x_{E\setminus i} + \sum_{\substack{j\in G\\ i\notin G\\ G\neq E\setminus i}} x_G - \sum_{\substack{j\in G\\ i\notin G\\ G\neq E\setminus i}} x_G \right) \\ =& x_{E\setminus i}\left(\sum_{i\in G} x_G - \sum_{\substack{j\in G\\ i\notin G\\ G\neq E\setminus i}} x_G \right). \end{aligned}$$

The product with the first sum is zero from the incomparability relations; the conditions on the second sum, using the fact that i is a coloop, are equivalent to ask that G belongs to $\mathscr{S}_i$, then

$$x_{E\setminus i}x_G = x_{E\setminus i}\left(\theta_i(x_G) - x_{G\cup i}\right) = x_{E\setminus i}\theta_i(x_G) \in CH^1_{(i)}\cdot CH^1(M).$$

Lastly, by inductive hypothesis, we know that $CH(M \setminus i)$ satisfies Poincarè duality; using Lemma 4.1.2 we observe that the Poincaré pairing on $CH(M)$ restricts to an isomorphism between $CH_{(i)}$ and $x_{E\setminus i}CH_{(i)}$. In fact, let $\theta_i(\eta_1) \in CH^k_{(i)}$, $\theta_i(\eta_2) \in CH^{r-k-1}_{(i)}(M)$, then

$$\deg_M \left(\theta_i(\eta_1) \cdot x_{E\setminus i}\theta_i(\eta_2)\right) = \deg_M \left(x_{E\setminus i}\theta_i(\eta_1\eta_2)\right) = \deg_{M\setminus i}(\eta_1\eta_2),$$

where we conclude by 4.1.2. Since $CH_{(i)}$ is a subring which is zero in degree r in $CH(M)$, the restriction of the Poincaré pairing to $CH_{(i)}$ is zero, thus $CH_{(i)}$ and $x_{E\setminus i}CH_{(i)}$ intersect only in $\{0\}$, and the restriction of the Poincaré pairing to $CH_{(i)} \oplus x_{E\setminus i}CH_{(i)}$ is non-degenerate. □

Let us compute explicitly some decompositions, using $t_3(M_5)$, N_5 and M_5.

Example 4.1.7. Consider $t_3(M_5)$ and $i = 1$. Then, $t_3(M_5) \setminus 1 \cong M_4$, which we recall has dimension 5 in degree 1. Therefore we can describe $CH_{(1)}$ in the following way:

$$\begin{aligned} \theta_1 : CH(M_4) &\to CH(t_3(M_5)), \\ x_2 &\mapsto x_2 + x_{12} \\ x_3 &\mapsto x_3 + x_{13} \\ x_4 &\mapsto x_4 + x_{14} \\ x_5 &\mapsto x_5 + x_{15} \\ x_{23} &\mapsto x_{23} \\ x_2^2 &\mapsto (x_2 + x_{12})^2 = \frac{2}{3}x_2^2 \end{aligned}$$

hence,

$$CH_{(1)} = \left\{a + b(x_2 + x_{12}) + c(x_3 + x_{13}) + d(x_4 + x_{14} + e(x_5 + x_{15}) + fx_{23} + gx_2^2\right\}.$$

Now, we see that $\dim CH_{(1)} = 7$, and it already contains $CH^0(t_3(M_5))$ and $CH^2(t_3(M_5))$; we also know that $\dim CH^1(t_3(M_5)) = 9$, thus we know that the remaining terms should have dimensions that add up to $9 - 5 = 4$.
Next, we observe that the family $\mathscr{S}_1$ is made of four elements, namely

$$\mathscr{S}_1 = \{2, 3, 4, 5\}.$$

We can now deduce that each of these four terms must be 1-dimensional. Computations on Singular indeed show that

$$x_{12}CH_{(1)} = \{ax_{12} + b(x_2 + x_{12})x_{12}\} = \{ax_{12}\}$$

and similarly for x_{13}, x_{14} and x_{15}. Therefore,

$$CH(t_3(M_5)) = CH_{(1)} \oplus \mathbb{Q}x_{12} \oplus \mathbb{Q}x_{13} \oplus \mathbb{Q}x_{14} \oplus \mathbb{Q}x_{15}.$$

Moreover, if we consider an element $p \in CH_{(1)}$ and an element of the form $ax_{12} \in x_{12}CH_{(1)}$, their product is

$$p \cdot ax_{12} = c(x_2 + x_{12}) \cdot ax_{12} = 0 \in CH^2(t_3(M_5)),$$

which shows the orthogonality with respect to the Poincaré pairing.

Example 4.1.8. Consider $M = N_5$ and $i = 5$. Then, $N_5 \setminus 5 \cong U_{3,4}$, which we recall has dimension 7 in degree 1. But $\dim CH^1(N_5) = 7$ as well, therefore we expect θ_5 to be an isomoprhism

$$\theta_5 : CH(U_{3,4}) \cong CH(N_5).$$

Indeed, if we write explicitly θ_5 we obtain

$$\begin{aligned}\theta_5 : CH(U_{3,4}) \to & CH(N_5)\\ x_1 \mapsto & x_1\\ x_2 \mapsto & x_2\\ x_3 \mapsto & x_3\\ x_4 \mapsto & x_4\\ x_{12} \mapsto & x_{12}\\ x_{13} \mapsto & x_{13}\\ x_{14} \mapsto & x_{14}\\ x_1^2 \mapsto & (x_1 + x_{15})^2\end{aligned}$$

and $\mathscr{S}_5 = \emptyset$. Therefore, the resulting decomposition is

$$CH(N_5) = CH_{(5)}.$$

Example 4.1.9. If we consider $i = 1 \in N_5$ and $N_5 \setminus 1 \cong M_4$, then we see that $\dim CH^1_{(1)} = 5$. Therefore, we should expect either $|\mathscr{S}_1| = 1$ or 2. Since, $\mathscr{S}_1 = \{3, 4\}$ we can conclude directly that

$$CH(N_5) = CH_{(1)} \oplus \mathbb{Q}x_{13} \oplus \mathbb{Q}x_{14}.$$

Example 4.1.10. Consider $M = M_5$ and $i = 1$. Then, $M_5 \setminus 1 \cong M_4$. Choosing an appropriate basis for $CH^1(M_4)$ and computing θ_1 as in the previous examples leads us to

$$CH_{(1)} =< 1, x_2+x_{12}, x_3+x_{13}, x_4+x_{14}, x_5+x_{15}, x_{23}+x_{123}, x_2^2+sx_2x_{12}+x_{12}^2 >,$$

which is 7-dimensional, and

$$\mathscr{S}_1 = \{2, 3, 4, 5, 23, 24, 25, 345\};$$

each of the terms $x_{F\cup 1}CH_{(1)}$ is proved to be 2-dimensional, for example,

$$\begin{aligned}x_{12}CH_{(1)} =& \{ax_{12} + b(x_2 + x_{12})x_{12} + fx_{12}x_{123} + g(x_2^2 + 2x_2x_{12} + x_{12}^2)x_{12}\} =\\ =& \{ax_{12} + b(x_2x_{12} + x_{12}^2) + fx_2x_{123}\}.\end{aligned}$$

Applying repeatedly the linear and incomparability relations,

$$\begin{aligned}x_{12}x_{123} =& x_{12}(-x_2 + x_5 - x_{12} - x_{23} - x_{24} + x_{345} - x_{124} + x_{1345}) =\\ =& x_{12}(-x_2 - x_{12} - x_{124}) =\\ =& x_{12}(-x_2 - x_{12} - x_{123} + x_{24} - x_{23} + x_{14} - x_{13} + x_4 - x_3) = -x_{12}(x_2 + x_{12}).\end{aligned}$$

Hence,

$$x_{12}CH_{(1)} = \{ax_{12} + (b-f)x_{12}(x_2 + x_{12})\}.$$

Moreover, since 1 is a coloop in M_5, we need to compute $x_{2345}CH_{(1)}$, which turns out to be 7-dimensional. Adding up all the resulting dimensions gives us a 30-dimensional space. We can check the result using Poincaré Duality: we are given as a fact that $\dim CH^0(M_5) = \dim CH^3(M_5) = 1$ and $\dim CH^1(M_5) = 14$ for previous considerations; Poincaré Duality implies that $CH(M_5)$ has dimension 14 in degree 2, since $CH^1(M_5) \cong CH^2(M_5)$. Thus, $\dim CH(M_5) = 1 + 14 + 14 + 1 = 30$, which is the result we found.

Theorem 4.1.11. *Let ℓ be a strictly convex piecewise linear function on Σ_{B_n}, seen as an element of $CH^1(M)$. Then $CH(M)$ has properties (HL)$_\ell$ and (HR)$_\ell$.*

Corollary 4.1.12. *Hence, the Lefschetz decomposition holds*

$$CH^k(M) = P_\ell^k \oplus \ell P_\ell^{k-1} \oplus \cdots \oplus \ell^k P_\ell^0.$$

Proof. The proof is by induction on the cardinality of E. The result is vacuously true for $n = 0, 1$, so we can suppose the cardinality is at least 2.
By inductive hypothesis, we know that for every non-empty proper flat F of M, the fans Σ_{M_F} and Σ_{M^F} satisfy (HL) and (HR) with respect to every strictly convex piecewise linear function on, respectively, Σ_{B_F} and Σ_{B^F}. From 2.6.9, $\Sigma_{M_F} \times \Sigma_{M^F}$ satisfies (HL) and (HR) for every strictly convex piecewise linear function on $\Sigma_{B_F} \times \Sigma_{B^F}$. In other words, the star of any ray in Σ_M satisfies (HR). This implies that Σ_M satisfies (HL), in fact if we take $\ell \in \mathcal{K}_{\Sigma_M}$ it can be written as

$$\ell = \sum_{e \in V_{\Sigma_M}} c_e x_e \in CH^1(\Sigma_M),$$

with positive coefficients c_e. We need to show that L_ℓ^k is injective for $k \leq \frac{r}{2}$. Let f be an element in the kernel of L_ℓ^k and consider

$$f_e := [f] \in CH^k(\Sigma_M)/ann(x_e).$$

Then,

- for every $e \in V_{\Sigma_M}$, $f_e \in P_{\ell_e}^k$;
- $\sum_{e \in V_{\Sigma_M}} c_e Q_{\ell_e}^k(f_e, f_e) = Q_\ell^k(f, f) = 0$,

but if the star of any ray satisfies (HR), this means that $f_e = 0$, so $x_e \cdot f = 0 \in CH(\Sigma_M)$. Since the elements $\{x_e\}$ generate the algebra, this means that $f = 0$. By convexity, we conclude that Σ_M satisfies (HL) for every strictly convex piecewise linear function on Σ_{B_n}. We are left to prove (HR).
Let ℓ be a piecewise linear function on Σ_{B_n}. By 2.6.8, we know that it satisfies (HR) if and only if for every $0 \leq k < \frac{r}{2}$, Q_l^k is non-degenerate with signature

$$\sum_{j=0}^{k} (-1)^{k-j} \left(\dim CH^j(M) - \dim CH^{j-1}(M)\right).$$

Since Σ_M satisfies (HL), we just need to prove that (HR) is satisfied by a strictly convex piecewise linear map ℓ_0 on Σ_B. In fact, given another such map ℓ_1, the maps of the form

$$\ell_t = (1-t)\ell_0 + t\ell_1,$$

are strictly convex for every $0 \leq t \leq 1$. Since (HL) is satisfied, Q_l^k is non-degenerate for every $0 \leq k < \frac{r}{2}$. This means that the form $Q_{\ell_t}^k$ all have the same signature. We conclude, since we know that (HR) holds for ℓ_0.
Because the condition for (HR) is open (being positive definite), we just need to verify that (HR) is verified by a (non strictly) convex piecewise linear function on Σ_B.
Let us suppose that M is not a Boolean matroid, for which the result can be proven with classical algebraic geometry, since $CH(B_n)$ can be identified with the cohomology ring of the associated toric variety $X_{\Sigma_{B_n}}$. We can then choose an element $i \in E$ which is not a coloop in M and consider the morphism of fans

$$\pi_i : \Sigma_M \to \Sigma_{M \setminus i}.$$

By induction, we know that $\Sigma_{M\setminus i}$ satisfies (HR) for every strictly convex piecewise linear function ℓ on $\Sigma_{B_n \setminus i}$. Let us consider the pullback $\ell_i = \ell \circ \pi_i$ and show that $CH(M)$ has property $(\text{HR})_{\ell_i}$, where we already know that ℓ_i is a (not necessarily strictly) convex piecewise linear map on Σ_{Bn}.
Since we have proved the decomposition

$$CH(M) = CH_{(i)} \oplus \bigoplus_{F \in \mathscr{S}_i} x_{F \cup i} CH_{(i)},$$

using the orthogonality of the terms, we need to prove that $(\text{HR})_{\ell_i}$ is satisfied on each term.

- The morphism θ_i is injective (see 3.3.25 for an analogous proof); in particular, θ_i is an isomorphism of $CH(M \setminus i)$-modules

  $$CH_{(i)} \cong CH(M \setminus i),$$

  hence if, by inductive hypothesis, (HR) holds on $CH(M \setminus i)$, then it holds on $CH_{(i)}$.

- Since Poincaré duality holds on $CH(M_{F\cup i})$ and on $CH(M^{F\cup i})$, $\psi_M^{F\cup i}$ and $\theta_i^{F\cup i}$ are injective and, in particular, we have the following isomorphisms of $CH(M \setminus i)$-modules

  $$CH(M_{F\cup i}) \otimes CH(M^{F\cup i}) \cong CH(M_{F\cup i}) \otimes \theta_i^{F\cup i}\left(CH(M^F)\right) \cong x_{F\cup i} CH_{(i)}.$$

  We conclude the proof by Lemmas 4.1.2 and 3.3.24.

□

4.2 The proof of log-concavity

This section is dedicated fully to the proof of Theorem 1.4.21, which relies on the properties (HL) and (HR) of the Chow ring $CH(M)$.

Definition 4.2.1. A flag of non-empty proper flats of M

$$\mathscr{F} = \{F_1 \subsetneq F_2 \subsetneq \ldots \subsetneq F_k\}$$

is said to be *initial* if $\rho(F_m) = m$ for every m. It is said to be *descending* if

$$\min(F_1) > \min(F_2) > \ldots > \min(F_k) > 1,$$

where $>$ is the natural order on $E = \{1, 2, \ldots, n\}$ We write $D_k(M)$ to denote the family of initial descending flags of non-empty proper flats of M.

Remark 4.2.2. We can observe directly from the definition that

$$D_k(M) \cong D_k(t_k(M)).$$

Lemma 4.2.3. *For every* $0 < k \leq r$,

$$\bar{\omega}_k(M) = |D_k(M)|,$$

where we recall $\bar{\omega}_k(M)$ *is the* k*-th reduced Whitney number of the first kind.*

Proof. We know the coefficients of the reduced characteristic polynomial are

$$\bar{\omega}_k(M) = (-1)^k \sum_{F|\rho(F)=k} \mu(\emptyset, F).$$

Let F_k be a rank k flat; then, by Weisner's Theorem (1.4.6),

$$\begin{aligned}
\mu(\emptyset, F_k) &= - \sum_{\substack{1 \notin F_{k-1} \\ \rho(F_{k-1})=k-1}} \mu(\emptyset, F_{k-1}) \\
&= + \sum_{\substack{1 \notin F_{k-1} \\ \rho(F_{k-1})=k-1}} \left(\sum_{\substack{\min(F_{k-1}) \notin F_{k-2} \\ \rho(F_{k-2})=k-2}} \mu(\emptyset, F_{k-2}) \right) \\
&= \ldots
\end{aligned}$$

What we are actually doing is counting the initial descending k-flags which terminate with F_k. Adding on all flats of rank k gives us the result. $\square$

Example 4.2.4. Let us see how these results are applied in our three examples, $t_3(M_5)$, N_5 and M_5.
In every matroid, all the 1-flags of the form $\mathscr{F} = \{i\}$ are initial and descending, except for $i = 1$; all the other 1-flags are not initial. This is coherent with the fact that $\omega_1 = |E|$ for every matroid and therefore, by definition and carrying out the division by $\lambda - 1$, $\bar{\omega}_1 = |E| - 1$. The family $D_2(t_3(M_5))$ is made by the following five 2-flags

$$D_2(t_3(M_5)) = \{\{3, 23\}, \{4, 24\}, \{4, 345\}, \{5, 25\}, \{5, 345\}\};$$

this is also the family $D_2(M_5)$, as observed in 4.2.2. To conclude with M_5, we just need to find $D_3(M_5)$, which is

$$D_3(M_5) = \{\{3, 23, 123\}, \{4, 345, 1345\}, \{4, 345, 2345\}, \{5, 345, 1345\}, \{5, 345, 2345\}\}.$$

As a last example, we compute $D_2(N_5)$ and find that

$$D_2(N_5) = \{\{3, 23\}, \{4, 24\}, \{4, 345\}, \{5, 345\}\}.$$

Lemma 4.2.5. *For every* $0 < k \leq r$,

$$\beta_M^k = \sum_{\mathscr{F}} x_{\mathscr{F}},$$

where we are summing on every descending k*-flag of non-empty proper flats of* M.

Proof. We prove it by induction on k. If $k = 1$, the result is trivial, since

$$\beta_M = \beta_{M,1} = \sum_{1 \notin F} x_F.$$

For the inductive step, we write

$$\beta_M^{k+1} = \sum_{\mathscr{F}} \beta_M x_{\mathscr{F}},$$

where we sum on all the descending k-flags of non-empty proper flats of M. For each flag we consider the first flat and its minimal element, $\min(F_1)$, to write

$$\beta_M x_{\mathscr{F}} = \beta_{M,\min(F_1)} x_{\mathscr{F}} = \left(\sum_{\min(F_1) \notin F} x_F \right) x_{\mathscr{F}}.$$

By the incomparability relations, this is equivalent to

$$\beta_M^{k+1} = \sum_{\mathscr{G}} x_{\mathscr{G}},$$

where the sum is on all descending flags of non-empty proper flats of M of the form

$$\mathscr{G} = \{F \subsetneq F_1 \subsetneq \ldots \subsetneq F_k\}.$$

This concludes the proof. □

Example 4.2.6. Consider $M = N_5$. Then,

$$\begin{aligned}\beta_M^2 =& \beta_M (x_2 + x_3 + x_4 + x_5 + x_{23} + x_{24} + x_{345}) = \\ =& \beta_{M,2} (x_2 + x_{23} + x_{24}) + \beta_{M,3} (x_3 + x_{345}) + \beta_{M,4} x_4 + \beta_{M,5} x_5 = \\ =& \beta_{M,2} x_{23} + \beta_{M,2} x_{24} + \beta_{M,3} x_{345} = \\ =& x_3 x_{23} + x_4 x_{24} + x_4 x_{345} + x_5 x_{345}.\end{aligned}$$

Naturally, they correspond to the initial descending 2-flags, because $r = 2$. If we consider the descending 2-flags in $M = M_5$,

$$\begin{aligned}\beta_M^2 =& \beta_M (x_2 + x_3 + x_4 + x_5 + x_{23} + x_{24} + x_{25} + x_{345} + x_{2345}) = \\ =& \beta_{M,2} (x_2 + x_{23} + x_{24} + x_{25} + x_{2345}) + \beta_{M,3} (x_3 + x_{345}) + \beta_{M,4} x_4 + \beta_{M,5} x_5 = \\ =& (x_3 x_{23} + x_4 x_{24} + x_5 x_{25} + x_4 x_{2345} + x_5 x_{2345} + x_{23} x_{2345} + x_{24} x_{2345} + x_{25} x_{2345} + x_{345} x_{2345}) + \\ &+ (x_4 x_{345} + x_5 x_{345}),\end{aligned}$$

where, for example, $x_{23} x_{2345}$ is descending but not initial.

Theorem 4.2.7. *For every* $0 < k \leq r$,

$$\bar{\omega}_k = \deg(\alpha_M^{r-k}\beta_M^k).$$

Proof. By Lemma 4.2.5 we can write

$$\alpha_M^{r-k}\beta_M^k = \left(\sum_{\mathscr{F}} \alpha_M^{r-k} x_{\mathscr{F}}\right),$$

where the sum is on all descending flags. By Lemma 3.3.13 we know

$$\sum_{\mathscr{F}} \alpha_M^{r-k} x_{\mathscr{F}} = \begin{cases} \alpha_M^k, & \text{if } \mathscr{F} \text{ is initial} \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\alpha_M^{r-k}\beta_M^k = |D_k(M)| \cdot \alpha_M^k.$$

The result follows from 3.3.17 and 4.2.3. □

Example 4.2.8. We have already worked out on Singular the degree maps of $t_3(M_5)$, N_5 and M_5; we can also compute the various monomials $\alpha_M^{r-k}\beta_M^k$ and find

$$\begin{aligned} \deg CH^2(t_3(M_5)) &\to \mathbb{Q}. \\ \alpha^2 = -\frac{1}{3}x_1^2 &\mapsto 1 \\ \alpha\beta = -\frac{4}{3}x_1^2 &\mapsto 4 \\ \beta^2 = -\frac{5}{3}x_1^2 &\mapsto 5 \end{aligned}$$

Similarly,

$$\begin{aligned} \deg CH^2(N_5) &\to \mathbb{Q}. \\ \alpha^2 = -\frac{1}{2}x_1^2 &\mapsto 1 \\ \alpha\beta = -2x_1^2 &\mapsto 4 \\ \beta^2 = -2x_1^2 &\mapsto 4 \end{aligned}$$

Lastly,

$$\begin{aligned} \deg CH^3(M_5) &\to \mathbb{Q}. \\ \alpha^3 = \frac{1}{2}x_1^2 &\mapsto 1 \\ \alpha^2\beta = 2x_1^2 &\mapsto 4 \\ \alpha\beta^2 = \frac{5}{2}x_1^2 &\mapsto 5 \\ \beta^3 = x_1^2 &\mapsto 2 \end{aligned}$$

Lemma 4.2.9. *Let* ℓ_1 *and* ℓ_2 *be two elements in* $CH^1(M)$. *If* ℓ_2 *is NEF, then*

$$\deg(\ell_1\ell_1\ell_2^{r-2})\deg(\ell_2\ell_2\ell_2^{r-2}) \leq \deg^2(\ell_1\ell_2\ell_2^{r-2}).$$

Proof. We first prove the result when ℓ_2 is ample. Consider the Hodge-Riemann form $Q^1_{\ell_2}$ on $CH^1(M)$. From Theorem 4.1.11 and Corollary 4.1.12, we know that

$$CH^1(M) =< \ell_2 > \oplus P^1_{\ell_2}(M),$$

and the decomposition is orthogonal with respect to $Q^1_{\ell_2}$. Since (HR) holds, $Q^1_{\ell_2}$ is negative definite on $< \ell_2 >$ and positive definite on $P^1_{\ell_2}(M)$. Restricting $Q^1_{\ell_2}$ on the subspace $< \ell_1, \ell_2 >\subseteq CH^1(M)$,

- if ℓ_1 is a multiple of ℓ_2, e.g. $\ell_1 = a\ell_2$,

$$Q^1_{\ell_2}(\ell_1, \ell_1) = a^2 Q^1_{\ell_2}(\ell_2, \ell_2),$$

  hence the equality

$$Q^1_{\ell_2}(\ell_1, \ell_1)Q^1_{\ell_2}(\ell_2, \ell_2) = a^2\cdot\big(Q^1_{\ell_2}(\ell_2, \ell_2)\big)^2 = \big(a \cdot Q^1_{\ell_2}(\ell_2, \ell_2)\big)^2 = \big(Q^1_{\ell_2}(\ell_1, \ell_2)\big)^2;$$

- if ℓ_1 is not a multiple of ℓ_2, it belongs to $P^1_{\ell_2}$, where $Q^1_{\ell_2}$ is positive definite. Thus,

$$Q^1_{\ell_2}(\ell_1, \ell_1)Q^1_{\ell_2}(\ell_2, \ell_2) < 0,$$

  hence the strict inequality of the statement.

We now extend the proof to the NEF case. Let ℓ_1 be ample and consider the family

$$\ell_2(t) = \ell_2 + t \cdot \ell_1,$$

$t \in [0, 1]$, which are ample for every t. From the previous result,

$$Q^1_{\ell_2(t)}(\ell_1, \ell_1)Q^1_{\ell_2(t)}(\ell_2(t), \ell_2(t)) \leq \Big(Q^1_{\ell_2(t)}(\ell_1, \ell_2(t))\Big)^2,$$

and considering the limit $t \to 0$ we get the result. □

Remark 4.2.10. The elements $\alpha_M, \beta_M \in CH^1(M)$ are NEF.

Proof. Consider a cone $\sigma_{\mathscr{F}}$ in Σ_M and an element i not in any of the flats in $\mathscr{F}$. Then, α_M, i is a convex piecewise linear function that is zero on $\sigma_{\mathscr{F}}$, therefore NEF. Similarly for β, for which we choose i in $\min \mathscr{F}$. □

Example 4.2.11. Consider $M = B_3$ and $\mathscr{F} = \{3\}$. Then we can consider

$$\alpha_{M,1} = x_{\rho_1} + x_{\rho_{12}} + x_{\rho_{13}},$$

which is clearly zero on ρ_3 and non-negative because it is a sum of Courant functions.
If we consider $\mathscr{F} = \{1, 12\}$, then we get

$$\beta_{M,1} = x_{\rho_2} + x_{\rho_3} + x_{\rho_{23}},$$

which is zero on the cone $\sigma = Cone(\rho_1, \rho_{12})$, since it is zero on its generating rays.

Theorem 4.2.12. *For every* $0 < k \leq r$

$$\bar{\omega}_{k-1}(M)\bar{\omega}_{k+1}(M) \leq \bar{\omega}^2_k(M).$$

Proof. If $r = 1$, there is nothing to prove. Now consider $k < r$, and apply the inductive hypothesis on the truncation $t_k(M)$. For $k = r - 1$, Theorem 4.2.7 shows that te result is equivalent to

$$\deg(\alpha_M^2 \beta_M^{r-2}) \deg(\beta_M^2 \beta_M^{r-2}) \leq \left(\deg(\alpha_M^1 \beta_M^{r-1})\right)^2,$$

which is true by Lemma 4.2.9 and because β_M is NEF as observed in Remark 4.2.10. □

Bibliography

[1] Tom Braden, June Huh, Jacob P. Matherne, Nicholas Proudfoot and Botong Wang. (2020). *A semi-small decomposition of the Chow ring of a matroid*, arXiv:2002.03341.

[2] Karim Adiprasito, June Huh and Eric Katz. (2015). *Hodge Theory for combinatorial geometries*, Annals of Mathematics. 188. 10.4007/annals.2018.188.2.1.

[3] Dominic Welsh. (1976). *Matroid Theory.* London Mathematical Society Monographs.

[4] Eva M. Feichtner and Sergey Yuzvinsky. (2004). *Chow ring of toric varieties defined by atomic lattices*, Inventiones Mathematicae. 155. 515-536. 10.1007/s00222-003-0327-2.

[5] Eva M. Feichtner and Dimitry N. Kozlov. (2003). *Incidence combinatorics of resolutions.* Selecta Mathematica. 10. 10.1007/s00029-004-0298-1.

[6] David A. Cox, John B. Little, Hal Schenck. (2011). *Toric varieties*, American Mathematical Society. 10.1365/s13291-012-0048-9.

[7] Mark A. De Cataldo, Luca Migliorini. (2002). *The Hard Lefschetz Theorem and the topology of semismall maps.* Annales Scientifiques de l'École Normale Supérieure. 35. 759-772. 10.1016/S0012-9593(02)01108-4.

[8] Mark A. De Cataldo, Luca Migliorini. (2002). *The Chow motive of semismall resolutions.* Mathematical Research Letters. 11. 10.4310/MRL.2004.v11.n2.a2.

[9] Luca Migliorini. (1995). *A smooth family of minimal surfaces of general type over a curve of genus at most one is trivial*, Journal of Algebraic Geometry. 4. 353-361.

[10] Mark A. De Cataldo. (2005). *Lectures on the Hodge Theory of projective manifolds.*

[11] Dominic Joyce. (2019). *Complex manifolds and Kähler geometry*, Graduate Summer School, Nairobi.

[12] Matthew Baker. (2017). *Hodge Theory in Combinatorics.* Bulletin of the American Mathematical Society. 55. 10.1090/bull/1599.

[13] Corrado De Concini and Claudio Procesi. (1996). *Wonderful models of subspace arrangements.* Selecta Mathematica. 1. 10/1007/BF01589496.

[14] June Huh. (2010). *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs.* Journal of the American Mathematical Society. 25. 10.2307/23265095.

[15] Peter Orlik, Hiroaki Terao, *Arrangements of Hyperplanes.* (1992). Springer.

[16] Richard Stanley. (2006). *Log-concave and Unimodal Sequences in Algebra, Combinatorics and Geometry.* Annals of the New York Academy of Sciences. 576. 500 - 535. 10.1111/j.1749-6632-1989.tb16434.x.

[17] Eric Katz. (2016). *Matroid theory for algebraic geometers.* In: Baker M., Payne S. (eds) Nonarchimedean and Tropical Geometry. Simons Symposia. Springer, Cham. 10.1007/978-3-319-30945-3_12.

[18] Frank W. Warner. (1971). *Foundations of differential manifolds and Lie groups*, Scott, Foresman and Company.