Alma Mater Studiorum · Università di Bologna

SCUOLA DI SCIENZE
Dipartimento di Fisica e Astronomia
Corso di Laurea Magistrale in Fisica

Non-local Gravity and Cosmology from Noether Symmetries

Relatore:

Prof. Roberto Balbinot

Correlatore:

Prof. Salvatore Capozziello

Presentata da:

Mattia Migliozzi

Anno Accademico 2018/2019

Abstract

L'argomento principale della mia tesi di laurea è lo studio della teoria della gravità non locale e in particolare una generalizzazione, nel formalismo di Ricci, di studi precedenti. Problemi come la materia e l'energia oscura, l'orizzonte e la piattezza, sottolineano che il Modello Cosmologico Standard, basato sul Modello Standard delle Particelle e sulla Relatività Generale, fallisce quando si vuole descrivere l'intero Universo, specialmente a regimi estremi di scale ultraviolette. Inoltre, la relatività generale non funziona come una teoria fondamentale in grado di fornire una descrizione quantistica dello spazio-tempo. I risultati di queste carenze e, prima di tutto, l'assenza di una teoria soddisfacente della gravità quantistica spingono verso una modifica della relatività generale necessaria per avvicinarsi, almeno, a una descrizione semi-classica.

La non località risulta essere una proprietà esclusiva del mondo quantistico, quindi è plausibile che l'inclusione di tale caratteristica nella relatività generale possa essere un buon modo per andare verso una teoria quantistica della gravità.

Questa caratteristica è inclusa nella formulazione lagrangiana, con un termine che contiene l'inverso dell'operatore di d'Alambert, chiamato quindi operatore non locale. Allo scopo di ottenere una teoria molto generale è possibile scrivere la lagrangiana nella seguente forma: $f(R, \Box^{-1}R)$, dove per la necessità di gestire questa teoria con un metodo variazionale si introduce un campo scalare $\phi = \Box^{-1}R$. È possibile quindi ottenere la relativa equazione di campo e l'equazioni cosmologiche. In particolare è possibile selezionare la forma funzionale della $f(R, \phi)$ usando il primo prolungamento del teorema di Noether.

Si ottengono diverse espressioni funzionali le quali ognuna contiene delle costanti legate al teorema di Noether. In particolare tutte le soluzioni sono rinormalizzabili e riproducono la Relatività Generale. Concludendo è importante sottolineare che le equazioni cosmologiche riproducono, in maniera naturale, non solo gli effetti della energia oscura ma anche l'inflazione cosmologica.

"Un tempo si pensava che le leggi scientifiche fossero ben stabilite e irrevocabili. Lo scienziato scopre fatti e leggi e aumenta costantemente la quantità delle conoscenze sicure e indubitabili. Oggi abbiamo riconosciuto che la scienza non può dare alcuna garanzia del genere. Le leggi scientifiche possono essere rivedute, spesso risulta che esse sono non solo localmente scorrette ma interamente sbagliate, facendo asserzioni su entità che non sono mai esistite."

Paul K. Feyerabend (Contro il metodo. Abbozzo di una teoria anarchica della conoscenza).

Contents

INTRODUCTION

The history of the gravitational interaction

Gravity is probably the first fundamental interaction experienced by mankind, as a matter of fact, it is related to the phenomena of everyday life. For such motivations, gravity has inspired and it is still fascinating the minds of scientists.

During the fourth century b.C. the Greek philosopher Aristotle, (384-322 b.C) developed some theories for describing natural phenomena. He believed that the "four elements", water-fire-air-earth, were the fundamental constituents of the Earth. Aristotle also considered skies and every particle made by a fifth element, the ether or quintessence. In particular, in order to explain motions, he elaborated the "natural places" theory, according to which the bodies move towards the place closest to them. For example, smoke and the other light bodies went up because they were similar to air and on contrary, stones and heavy bodies fell down because they were similar to earth. Gravity was not studied only in the the occidental philosophy, in fact, in ancient India Aryabhata first identified the force to explain the reason why objects do not fall when the earth rotates, Brahmagupta described gravity as an attractive force and used the term "gruhtvaakarshan" for gravity.

Galileo Galilei, at the end of the 16th century, introduced the scientific method which is the philosophical base of modern science. With the help of pendula and inclined planes, he studied terrestrial gravity.

However, it was not until 1665, when Isaac Newton published the *Principia* where he introduced the "inverse-square law" or "universal law of gravitation", that terrestrial gravity was linked with the celestial gravity in

a single theory. Using his own words:
"...I deduced that the forces which keep the planets in their orbs must be reciprocally as the squares of their distances from the centres about which they revolve: and thereby compared the force required to keep the Moon in her Orb with the force of gravity at the surface of the Earth; and found them answer pretty nearly..."

Newton's theory is able to correctly predict different phenomena both at terrestrial scale and planetary scale. Newton founded the conceptual basis of his theory on two key ideas:

1. the idea of absolute space, that is a rigid and imperturbable arena where phenomena take place;
2. the Weak Equivalence Principle, which states that inertial and gravitational mass coincide.

Newton's theory reached its greatest success when it was used to predict the existence of Neptune based on motions of Uranus. We can legitimately wonder in which case or sense a theory could be right or not. To be more specific we have to focus on how large the portion of the physical world well described by such a theory is.

First doubt took place in 1855 when the astronomer Urban Le Verrier observed a 35 arc-second excess precession of Mercury's orbit and then, in 1882, Simon measured accurately to be 43 arc-second. Le Verrier tried to explain such precession, supposing the existence of a yet not observed planet: Vulcan.

Conceptually, in 1893, Ernst Mach stated what was later called by Albert Einstein as "Mach's Principle". This one is the first important attack to Newton's theory. Such as Einstein said, we can summarize Mach's idea in the following line:

"...inertia originates in a kind of interaction between bodies...".

This is clearly in opposition with Newton's idea of inertia always relative to absolute space. There exists an alternative formulation of Mach's principle, which was given by Dicke:

"...The gravitational constant should be a function of the mass distribution in the Universe...".

Now Newton's basic axioms of the gravitational constant as being universal and unchanged, have to be reconsidered.

Only in 1905, when Albert Einstein completed Special Relativity, Newtonian gravity was seriously challenged. This new theory of Einstein's appeared to be incompatible with Newtonian ideas relative motion and all linked concepts had to be generalised to non-inertial frames. Finally, in 1905, Einstein published the theory of General Relativity(GR), a generalization of Special Relativity that admits gravity and any type of accelerated frames.

Albert Einstein suggested some test, the so-called "three classical test of General Relativity" which were: the perihelion precession of Mercury's orbit, the deflection of light by the Sun, the gravitational redshift of light. Remarkably, the theory matched perfectly the experimental results as showed by Lense-Thirring, in 1918, for Mercury's precession and by Arthur Eddington, during a solar eclipse in 1919, for deflection of light. Einstein theory overcomes Newtonian gravity and is still successful and well-accepted. However, Newton's theory is still used for some application and it is important to stress out that every good theory of gravitation has to reproduce it in weak field limit.

This brief historical introduction, besides its own interest, it has been proposed for a practical reason. Physicists are facing now some similar problems about how space and time are made and moreover, there are not well-understood issues like dark matter and dark energy.

About a Good Theory of Gravity

Every relativistic theory of gravity has to satisfy some minimal phenomenological requirements. First of all, it has to match with the astrophysical observation like the orbits and self-gravitating structures. Hence, as we have already said, it has to reproduce Newtonian dynamic in the weak-field limit. Furthermore, it has to pass the well experimented Solar system tests. Secondly, it has to be consistent with Galactic dynamics considering the now observed baryonic constituent, as the luminous components like stars. Last but not least, at the cosmological scale, any theory should reproduce parameters like expansion rate, Hubble constant, density parameter in a self-consistent way. General Relativity is the best accepted theory capable of explaining the observation. It is based on the idea that space-time is a synolon which, in the case of absence of gravity, leads to Minkowski's structure. Any good Theory of gravity, GR of course included, has to fulfill the following assumption:

- "The Principle of Relativity", that states there is not a preferred inertial frame that should be chosen *a priori* (if any exist).

- "The Principle of Equivalence", that requires inertial effects to be locally indistinguishable from gravitational effects, in other words, the equivalence between inertial and gravitational mass.
- "The Principle of General Covariance" which requires field equations to be covariant in form and invariant under diffeomorphisms.
- "The principle of Causality" affirms that each point of space-time should admit a universally valid notion of past, present and future.

In his work Einstein postulated the equivalence between the gravitational potential and the metric tensor $g_{\mu\nu}$, necessary to measure the distance between the events of the space-time. He introduced, the squared infinitesimal line element which is not dependent on the coordinates system

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \,.$$

Therefore, the metric coefficients are the gravitational potential and the space-time is curved by the distribution of the matter-energy sources. Einstein and Hilbert, using different approaches[1], obtained the relation between metric and the matter-stress-energy tensor $T_{\mu\nu}$, called the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \,.$$

Their choice was totally arbitrary (they chose a torsion-less connection) motivated by a matter of simplicity, both from the mathematical and the physical point of view. As Levi-Civita showed, olso in 1919, the curvature of a manifold is not only a metric notion but it depends on the connection Γ. While g fixes the causal structure, light cones, the connection Γ fixes the free-fall.

Some alternative of gravity, which tries to extend GR, start by this point of view. In fact, there are "Purely Metric Theory" or "Purely Affine Theory" that works on connection and the "Metric-Affine Theory" or "First Order Formalism Theories" which contemporary work on g and Γ. Einstein himself and many others (Eddington, Weyl, Schrödinger) tried to extend the Einstein-Hilbert action, wondering to unify gravity and electromagnetism but soon they had to deal with a non-linear theory. The idea to extend Einstein's gravity has been taken again to obtain a theoretical consistent description of observation. These classes of theories focus their attention on the geometric part of the field equation and they do not take into account exotic components in the source side, such as dark matter or dark energy.

[1] The Hilbert's approach will be useful later in this work.

Beyond Einstein

The physical world could be divided into microcosm and macrocosm respectively described by the Standard Model, based on Quantum Field Theory and the General Relativity. In Quantum Field Theory, space-time is Minkowskian, even if it exist a generalization, on curved-spacetime, which supports the idea of quantum field flowing on space-time. That is a first attempt to make this opposite world coherent with each other, it is known as the semi-classical model, in the framework in which gravity is still classic and the remaining are quantum-field.

It is clear that it still represent an open problem, where no one theories are able to give a satisfying description of quantum-gravity. Anyway a final proof that gravity should have quantum feature does not exist at all and consequently there is no proof of the graviton existence.

The Plank scale, $10^{-35}\ m$, seems currently experimentally inaccessible and it is doubtful that any experiment in a near future could investigate these lengths. There are, anyhow, different reasons that lead scientific resource beyond the curiosity, one above all is the Big Bang. In this scenario, indeed, the Universe goes necessarily through the scale Plank era, where interactions are unified.

CHAPTER 1

THE LAGRANGIAN FORMULATION

1.1 The Einstein-Hilbert Action

Einstein, between 1905 and 1915, was not the only one who was working on the problem, indeed, also the German mathematician David Hilbert was collaborating. In particular, using a totally different approach, he obtained the field equation 5 days before Einstein, who despite this, is considered the father of General Relativity because he achieved this equations on the base of physics principles and not only by a mathematical point of view. However, having developed the action describing the gravitational field, Hilbert's approach was as well useful; such an action read as:

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{2k}R + \mathcal{L}_{\mathcal{M}}\Big], \tag{1.1}$$

where $g = det(g_{\mu\nu})$ is the determinant of the metric tensor, R is the Ricci scalar while $\mathcal{L}_{\mathcal{M}}$ represents any kind of matter Lagrangian. Before starting with theoretical calculations it is worth mentioning that $\mathcal{L}_{\mathcal{M}}$ leads to the stress-energy tensor on the right side of Einstein field equations:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{\mathcal{M}})}{\delta g^{\mu\nu}}. \tag{1.2}$$

Let us now focus on the first part of the action (1.1). We state that its variation with respect to the inverse metric tensor $g^{\mu\nu}$, vanishes.

$$\begin{aligned} 0 = \delta S = \delta(\sqrt{-g})R + \sqrt{-g}\delta g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \\ = \Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\Big)\sqrt{-g}\delta g^{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}. \end{aligned} \tag{1.3}$$

Where in the last equivalence we use the following relations:

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu}, \tag{1.4}$$

$$\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}, \tag{1.5}$$

It is now clear where in relation (1.3) Einstein's tensor arise from:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\,. \tag{1.6}$$

In order to finally achieve the field equations, it remains only to evaluate the last term of the last equivalence in relation (1.3). In fact in the case it vanishes, the action principle leads us to well known Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}. \tag{1.7}$$

1.1.1 The Palatini Relation

In this subsection, we want to exactly determine the contribution of the previously mentioned term and it is important to remark its vanishing is not a really trivial question. In particular to show that, it is quite its utility in the determination of Palatini Identity[4].

First of all, we have to take into account the Riemann tensor:

$$R^{\rho}_{\sigma\mu\nu} = \Gamma^{\rho}_{\sigma\nu,\mu} - \Gamma^{\rho}_{\sigma\mu,\nu} + \Gamma^{n}_{\sigma\nu}\Gamma^{\rho}_{\mu n} - \Gamma^{n}_{\sigma\mu}\Gamma^{\rho}_{\nu n}, \tag{1.8}$$

labelling comma the partial derivatives. Now we have to vary the last relation with respect to the inverse metric tensor.

$$\delta R^{\rho}_{\sigma\mu\nu} = \delta\Gamma^{\rho}_{\sigma\nu,\mu} - \delta\Gamma^{\rho}_{\sigma\mu,\nu} + \delta\Gamma^{n}_{\sigma\nu}\Gamma^{\rho}_{\mu n} + \Gamma^{n}_{\sigma\nu}\delta\Gamma^{\rho}_{\mu n} - \delta\Gamma^{n}_{\sigma\mu}\Gamma^{\rho}_{\nu n} - \Gamma^{n}_{\sigma\mu}\delta\Gamma^{\rho}_{\nu n}, \tag{1.9}$$

Moreover, reminding how the covariant derivative works, it is possible to rewrite the previous relation as:

$$\delta R^{\rho}_{\sigma\mu\nu} = \nabla_{\mu}\delta\Gamma^{\rho}_{\nu\sigma} - \nabla_{\nu}\delta\Gamma^{\rho}_{\mu\sigma}. \tag{1.10}$$

We may now obtain the variation of the Ricci curvature tensor simply by contracting two indices of the Riemann tensor variation getting the Palatini relation:

$$\delta R_{\sigma\nu} = \nabla_{\rho}(\delta\Gamma^{\rho}_{\nu\sigma}) - \nabla_{\nu}(\delta\Gamma^{\rho}_{\rho\sigma}). \tag{1.11}$$

Finally it is possible to get the variation of the Ricci scalar:

$$\delta R \;=\; R_{\sigma\nu}\delta g^{\sigma\nu} \,+\, g^{\sigma\nu}\delta R_{\sigma\nu} \;=\; R_{\sigma\nu}\delta g^{\sigma\nu} \,+\, \nabla_\rho\Big(g^{\sigma\nu}\delta\Gamma^\rho_{\nu\sigma} \,-\, g^{\sigma\rho}\delta\Gamma^\mu_{\mu\sigma}\Big), \quad (1.12)$$

It is now evident that the last term leads to a negligible surface term, even if is important pointing out some consideration about that.

In fact, considering the Gauss Ostrogradsky theorem[53]:

$$\int d^4x\nabla_\rho\Big(g^{\sigma\nu}\delta\Gamma^\rho_{\nu\sigma} - g^{\sigma\rho}\delta\Gamma^\mu_{\mu\sigma}\Big) = \lim_{R\to\infty} R^2\int_{t_i}^{t_f}\int d\Omega\, n_\rho\Big(g^{\sigma\nu}\delta\Gamma^\rho_{\nu\sigma} - g^{\sigma\rho}\delta\Gamma^\mu_{\mu\sigma}\Big), \quad (1.13)$$

we assume an asymptotic radial behaviour:

$$\lim_{R\to\infty} R^2\Big(g^{\sigma\nu}\delta\Gamma^\rho_{\nu\sigma} - g^{\sigma\rho}\delta\Gamma^\mu_{\mu\sigma}\Big) = 0 \quad \forall t \in [t_i, t_f]\,. \quad (1.14)$$

The global term will finally vanish and allow us to obtain the Einstein field equations.

CHAPTER 2
THE EXTENDED THEORIES OF GRAVITY

Different issues arisen from Cosmology, Astrophysics and Quantum Field Theory suggest to extend the General Relativity in order to overcome several shortcomings emerging at conceptual and experimental level. The standard Einstein theory fails when one wants to give a quantum description of space-time. In particular, the approach based on corrections and extensions of the Einstein scheme, it is paradigmatic in the study of gravitational interaction. Otherwise, such theories have acquired great interest in cosmology since they "naturally" exhibit inflationary behaviours which can go beyond the issues of standard cosmology. From an astrophysical point of view, Extended Theories of Gravity do not require to find candidates for dark energy and dark matter at fundamental level; the approach starts from considering only the observed ingredients as gravity, radiation and baryonic matter. This approach fully agree with the early geometrical description of General Relativity. Scalar-tensor theories (as Brans-Dicke theory or Non-Local theories) and $f(R)$-models agree with observed cosmology, extragalactic and galactic observations and Solar System tests, and they are capable to explain the accelerated expansion of the universe and the missing matter effect of self-gravitating structures[7]. Despite these preliminary results,there is no final model which satisfies all the open issues, however the paradigm seems promising in order to achieve a self-consistent theory.[21][22][23].

2.1 Metric $f(R)$ gravity in general

Let us examine now the variational principle and the field equations of an extended theory of gravity, the $f(R)$ theory in the metric formalism. The salient feature of these theories is given by the fourth order of field equations which lets the theory more complicated then Einstein's one (which is recovered as particular limit of $f(R)$). Due to their higher order, these field equations admit a much richer variety of solutions than the Einstein equations[8]. A consequence of introducing an arbitrary function, is related to the necessity of explaining the accelerated expansion and structure formation of the Universe without adding unknown forms of dark energy or dark matter[21]. Indeed, we will see how is possible to reproduce the cosmological constant from geometrical consideration.

We consider the space-time as a pair (M, g) with M a four-dimensional manifold and g a metric on M. The Lagrangian is an arbitrary function of the Ricci scalar $L[g] = f(R)$, the relation between the Ricci scalar and the metric tensor is given taking a Levi-Civita connection on the manifold[22]. The general action can be written as:

$$S = \frac{1}{2k} S_{met} + S_{matter}, \tag{2.1}$$

where, as we still said in the previous chapter, the second term is the usual matter term and the first one is the bulk term:

$$S_{met} = \int d^4x \sqrt{-g} f(R), \tag{2.2}$$

with the convention:

$$\begin{aligned} 0 = \delta S = \int d^4x \Big[\delta(\sqrt{-g}) f(R) + \sqrt{-g} f'(R) \delta R\Big] \\ = \int d^4x \sqrt{-g} \delta g^{\mu\nu} \Big[f'(R) R_{\nu\mu} - \frac{1}{2} g_{\mu\nu} f(R)\Big] \\ + \int d^4x \sqrt{-g} \delta g^{\mu\nu} f'(R) \delta R_{\mu\nu}, \end{aligned} \tag{2.3}$$

standing $f'(R) = \frac{\delta f}{\delta R}$. For the purpose of evaluating the last term is important to recover the (A.5) relation and discarding a global term from integration by part, so that we finally get:

$$\int d^4x \sqrt{-g} \delta g^{\mu\nu} \Big[- \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \Box f'(R)\Big]. \tag{2.4}$$

So recollecting all the terms we obtain the field equations:

$$f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} = \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \Box f'(R)\,. \tag{2.5}$$

It is easy to show that for $f(R) = R$ we recover the General Relativity. These equations can be re-arranged in the Einstein-like form:

$$G_{\mu\nu} = \frac{1}{f'(R)}\left[-\nabla_\mu\nabla_\nu f'(R) + g_{\mu\nu}\Box f'(R) - g_{\mu\nu}\frac{f(R) - f'(R)R}{2}\right]. \quad (2.6)$$

The right-hand side of Eq. (2.6) is then regarded as an effective stress-energy tensor, which we call curvature fluid energy-momentum tensor $T^{curv}_{\mu\nu}$ sourcing the effective Einstein equations.

It is easy to show that this extension of the Einstein-Hilbert's Action gives rise to a model naturally capable to take into account the cosmological constant. As a matter of a fact, by considering a first order Taylor expansion of the $f(R)$ function we obtain a linear Lagrangian with a constant term, namely

$$f(R) = f_0 + f_1 R. \quad (2.7)$$

It is always possible to set $f_0 = -2\Lambda$ and $f_1 = 1$, which give rise to

$$S = \frac{1}{2k}\int d^4x\sqrt{-g}\,(R - 2\Lambda) + \int d^4x\sqrt{-g}\mathcal{L}_{\mathcal{M}}, \quad (2.8)$$

that yields

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (2.9)$$

As it is well-know this equation is capable to reproduce the de-Sitter cosmic accelerated expansion.

An important fourth-order subcase of $f(R)$ theory, is the Starobinsky quadratic action [41][42]

$$S_{Sta} = \frac{M_P^2}{2}\int d^4x\sqrt{-g}\Big(R + \alpha R^2\Big), \quad (2.10)$$

which leads to the following field equations:

$$G_{\mu\nu} + \alpha\Big[2R(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R) + 2(g_{\mu\nu}\Box R - \nabla_\mu\nabla_\nu R)\Big] = kT_{\mu\nu}. \quad (2.11)$$

With trace:

$$\Box R - m^2(R + kT) = 0. \quad (2.12)$$

being $m^2 = \frac{1}{6\alpha}$. Equation (2.8) can be seen as an effective Klein-Gordon equation for the effective scalar field degree of freedom R (sometimes called *Scalaron*). Starobinsky model is the one that is best capable to fit with the 2018 Cosmic Microwave Background (CMB) data from Plank satellite [48].

2.2 The $f(R)$ cosmological equation

In the above discussion we use the Friedmann-Lemaitre-Robertson-Walker(FLRW) line element:

$$ds^2 = dt^2 - a^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\Big], \tag{2.13}$$

Where $a(t)$ is the so called scale factor and $k = -1, 0, 1$ according respectively to a spherical, flat or hyperbolic universe. The standard approach is to use a perfect fluid description for matter with stress-energy tensor:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}. \tag{2.14}$$

With these elements it is possible to obtain the cosmological equations for a $f(R)$ theory:

$$3f'H^2 = k\Big[\rho + \frac{Rf' - f}{2} - 3H\dot{R}f''\Big]. \tag{2.15}$$

$$2H^2 f' + 3\dot{H}f' = -k\Big[p + \dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf')\Big]. \tag{2.16}$$

These are the cosmological equation for the $f(R)$ of gravity.

2.3 The Brans-Dicke Theory

The f(R) gravity is not the only way to extend General Relativity, as will be shown in this chapter and forthcoming. The Brans-Dicke action is an example of a scalar-tensor theory, a gravitational theory in which the interaction is mediated by a scalar field. The gravitational constant G is no longer assumed to be constant but instead, 1/G is replaced by a coordinates dependent scalar field ϕ. The following action contains the complete description of the Brans-Dicke theory:

$$S_{BD} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\Big[\phi R - \frac{\omega}{\phi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu - V(\phi)\Big] + S^m. \tag{2.17}$$

The variation of this action with respect to $g^{\mu\nu}$ yields to the field equation:

$$G_{\mu\nu} = \frac{8\pi}{\phi}T_{\mu\nu} + \frac{\omega}{\phi}\Big(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi\Big) + \frac{1}{\phi}\Big(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\Big) - \frac{V}{2\phi}g_{\mu\nu}. \tag{2.18}$$

The last equation describes how the stress-energy tensor and scalar field ϕ together affect spacetime curvature.

By varying the action with respect to ϕ we obtain:

$$\frac{2\omega}{\phi}\Box\phi + R - \frac{\omega}{\phi^2}\nabla^\alpha\phi\nabla_\alpha\phi + 3\frac{\Box\phi}{\phi} + \frac{2V}{\phi} = 0. \tag{2.19}$$

Taking the trace of (3.2),

$$R = -\frac{8\pi T}{\phi} + \frac{\omega}{\phi^2}\nabla^\alpha\phi\nabla_\alpha\phi + 3\frac{\Box\phi}{\phi} + \frac{2V}{\phi} \tag{2.20}$$

and using the last two equation, we get,

$$\Box\phi = \frac{1}{2\omega + 3}\Big(8\pi T + \phi\frac{dV}{d\phi} - 2V\Big). \tag{2.21}$$

This equation allow us to say that the trace of the stress-energy tensor acts as the source for the scalar field ϕ.

The term proportional to $\phi\frac{dV}{d\phi} - 2V$ on the right hand-side of previous equation vanishes if the potential has the form $V(\phi) = m^2\frac{\phi^2}{2}$ familiar from the Klein-Gordon equation and from particle physics[12][22].

As we mentioned in the introduction of this chapter, from the equation (3.1) we can see that the effective coupling constant depends on ϕ:

$$G_{eff} = \frac{1}{\phi}. \tag{2.22}$$

By this consideration it is possible to observe that ϕ must be positive if we want to recover an attractive gravity. It is important to stress that Barns-Dicke theory is a attempt to include the Mach Principle where G is considered to be not constant. Also it is generally agreed that the convergence of Brans-Dicke gravity to general relativity can occur during the matter-dominated era, or even during the inflationary phase of the early universe[37].

2.3.1 Equivalence between scalar-tensor and $f(R)$ gravity

For the sake of completeness, in this sub-section we want to show how the scalar-tensor theories and the $f(R)$ are related. In metric $f(R)$ models if we set $\phi \equiv R$ the action

$$S = \frac{1}{2k}\int d^4x\sqrt{-g}f(R) \tag{2.23}$$

is rewritable as

$$S = \frac{1}{2k}\int d^4x\sqrt{-g}\left[\psi(\phi)R - V(\phi)\right] \tag{2.24}$$

where

$$\psi = f'(\phi), \quad V(\phi) = \phi f'(\phi) - f(\phi), \quad f''(\phi) \neq 0. \tag{2.25}$$

It is trivial that by setting $R = \phi$, from the equation (2.24) we recover the $f(R)$ action (2.23). *Vice-versa*, varying the action (2.24) with the respect to ϕ

$$R\frac{d\psi}{d\phi} - \frac{dV}{d\phi} = (R - \phi)\, f''(R) = 0. \tag{2.26}$$

The equation (2.26) leads us to $R = \phi$ if $f''(R) \neq 0$. The action (2.24) has the Brans-Dicke form when $\omega = 0$. As a matter of fact, by recovering the Brans-Dicke action (2.17)

$$S_{BD} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\Big[\phi R - \frac{\omega}{\phi} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu - V(\phi)\Big] + S^m. \tag{2.27}$$

and considering $\omega = 0$ we obtain the action (2.24). An $\omega = 0$ Brans-Dicke theory was originally proposed for obtaining a Yukawa correction to the Newtonian potential in the weak-field limit [33] and called "O'Hanlon theory" or "massive dilaton gravity".

CHAPTER 3

NON LOCAL THEORY OF GRAVITY

3.1 The issues about a good quantum theory of gravity

The physical universe is now well described on one side by Quantum Mechanics, in particular, Quantum Field Theory and on the other side by General Relativity. The Standard Model is successfully able to treat at a fundamental level Electromagnetism, Weak and Strong interaction, therefore, is questionable whether it is possible to include also gravity in this model. How could be possible to unify the quantum states of Hilbert space and observable like Hermitian operator, with General Relativity or in general with gravity?

The conceptual problem is related to the difference between the probabilistic description of micro-cosmos and the classical so deterministic framework of Einstein theory. Quantum gravity forces us to modify the idea of space and time, along the direction opened by Einstein's GR, in order to make them fully compatible with quantum theory. In Einstein's theory, space and time lose their properties of being a fixed framework in which the dynamical world is immersed. They are identified with the gravitational field and acquire dynamical properties. When we take quantum mechanics into account, we realize that space-time should be described like a quantum field, and therefore, it should have a microscopic granular structure (like the photons or other quantum mediators) and a probabilistic dynamics. Building the mathematical language and the conceptual structure for making sense of such notions of quantum space and quantum time is the challenge for a quantum theory of gravity.

As well as Schrodinger equation is able to describe the time evolution of a state, a quantum equation of gravity is supposed to be able to provide us with the probability to have a certain evolution of our spacetime.

It is also interesting to investigate what kind of phenomena need to be described by a quantum theory of gravity and so which are the scales where that theory is required. The scale where General Relativity does not work at all is the so-called *Plank Scale*, defined by using the natural constant $\hbar$, c and G whose characteristic length turns out to be:

$$l_p \equiv \sqrt{\frac{G\hbar}{c^3}}; \tag{3.1}$$

likewise it is possible to define other quantities, like plank energy, time and so on. The order of such length is $l_p \sim 10^{-35}\, m$ many order smaller than subatomic scale, far away inaccessible by our experiment.

Besides these problems, there are other conceptual problems that arise when we classical consider the metric tensor $g_{\mu\nu}$ as a background in Quantum Field Theory, in fact,indeed, in this case space-time is not a simple arena where fields act on but is at the same time one of these fields. That becomes more evident when in Einstein theory if we consider that in GR the background is not given "*a priori*" but is a solution of field equations.[1]

Anyway is possible to follow a classical procedure of quantization, treating gravity like the electromagnetic fields and using covariant and canonical quantization. In the first case, the *covariant quantization*, the metric is split in flat term and a perturbation part:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{3.2}$$

This approach leads us to two problems the first one is related to renormalization, in fact, the theory has a one-loop divergence for graviton-matter and two loops for interaction graviton-graviton that cannot be controllable. The second problem is related to the fact that in General Relativity geometry and connection are determined by the full metric tensor $g_{\mu\nu}$.[2]

The *canonical quantization* does not split up the metric in two parts but in order to be applicable needs a Hamiltonian formulation of Einstein theory.

3.2 Non-Locality induced by Acceleration

When referring to the everyday world we are used to giving a position to an object, so locality can be interpreted as the capability to determine the position of given particles or fields. To be more clear, in modern physic the principle of locality states that, according to special relativity, an object is directly influenced only by its immediate surroundings. A theory which includes the principle of locality it is called a "local theory".

Before Einstein's theory, Newtonian physics admitted an instantaneous "action at distance" and the paradigmatic example of this is the universal law of gravitation. According to Newton, if it was possible to switch off the source of gravitational interaction, a particle moving inside that field (like the Earth in relation to the Sun) will instantaneously escape out in rectilinear trajectory.

Locality hence goes beyond the Newtonian physics, in fact, something in the space has to mediate the action and so, in simple words, something may be the information carrier.

In General Relativity, there is not a mediator of this interaction and interaction at distance is explained in terms of space-time curvature; which, according to special relativity is supposed to be non-instantaneous.

The non-locality property, is a particular feature which only belongs to the quantum world and is a kind of instantaneous interaction at distance and it is coherent with Einstein theory as long as there is no information transmission.

For example, if we consider a pair of electrons created together, in order to preserve Pauli's principle and to let the total angular momentum to be conserved, they must have opposite spin direction. In quantum theory, a superposition is also possible, so that the two electrons can be considered to simultaneously have both clockwise and anti-clockwise spin. If they are separated by any distance and later observed, the second particle instantaneously takes the opposite spin with the respect to the first, so that the pair maintains its zero total spin, no matter how far they are.

The idea is to give a non-local description of gravity, property which is induced, as we are going to show, by acceleration.

The standard theory of relativity is based on the assumption that an accelerated observer is point-wise inertial. Thus, the association between actual accelerated systems and ideal inertial systems is purely local in the theories of special and general relativity. It is intuitively clear that an accelerated observer could be considered inertial, during an experiment, if the observer's acceleration is such that its motion is uniform and rectilinear during the experimented process. Lorentz invariance can then be employed to predict the result of such experiment [16].

Let $\frac{\lambda}{c}$ be the intrinsic timescale for the process under consideration, and let $\frac{L}{c}$ be the acceleration timescale over which the velocity of the observer changes appreciably; then, the condition for the validity of the connection between theory and experiment is:

$$\lambda \ll L \tag{3.3}$$

Let us now consider the measurement of the frequency of an incident plane monochromatic wave of frequency ω by an accelerated observer. Until the observer with velocity $v(t)$ is considered inertial, then the local inertial frame of the observer can be related to the background global inertial frame by a Lorentz transformation. Subsequently, the Doppler effect may be employed to give:

$$\omega'(t) = \gamma[\omega - v(t)k]. \tag{3.4}$$

It is necessary to register various oscillations of the incident wave for obtaining an adequate determination of its frequency; on the other hand, Eq. (3.4) holds only if during this time the velocity does not highly change during the time considered. We can express this condition from the standpoint of the fundamental inertial observers as:

$$nT|\dot{\mathbf{v}}(t)| \ll v(t), \tag{3.5}$$

where T is the period and n is the number of cycles and in this case, v(t) is the magnitude of **v**. Then, considering the relation between period and the wavelength, it is possible to rewrite the last equation as:

$$\lambda \ll \frac{c^2}{a}. \tag{3.6}$$

with a being the a is the magnitude of the acceleration. Finally we want to show how the acceleration is related to the other wave characteristic and how Lorentz invariance could be extended to the non-inertial system only when $\lambda \ll L$.

As far as we are concerned, the paradigmatic example of the accelerating system is represented by the universe and therefore, being the relation (3.3) no longer valid, an induced Non-Local Theory of gravity has been developed. These theories are history-dependent and the usual partial differential equations are replaced by integral-differential equations, like the Non-Local Poisson's equation[28]. What we expect is to obtain a Lagrangian formulation of these theories, that allows us to reproduce Dark Matter and Dark Energy effects.

3.3 Non-Local Poisson's Equation

In this theory of non-local gravity, gravitation is described by a local field that satisfies integrodifferential equations. Thus gravity is a non-local extension of Newtonian gravity where non-locality is introduced through a "constitutive" integral kernel. As we are going to show this theory is capable to reproduce the Dark Matter effect on the source of gravitational field[28].

The first step is to recover the classical Poisson's Equation:

$$\nabla^2 \Phi_N(t, x) = 4\pi G \rho(t, x)\,, \tag{3.7}$$

which can be written as:

$$\nabla^2 \Phi(x) + \sum_i \int d^3y \frac{\partial k(x,y)}{\partial x^i} \frac{\partial \Phi(y)}{\partial y^i} = 4\pi G \rho(x). \tag{3.8}$$

Here Φ_N stands for the Newtonian gravitational potential and for simplicity, any kind of temporal dependence of the gravitational potential and matter density has been neglected. Moreover, the non-local kernel k is a smooth function of u and v, so that $k(x, y) = K(u, v)$ where:

$$u = x - y \quad v = \frac{|\nabla_y \Phi(y)|}{|\nabla_x \Phi(x)|}. \tag{3.9}$$

For the sake of simplicity is possible to consider a linear $k(x, y) = K(u)$,so that we obtain $\frac{\partial k}{\partial x^i} = -\frac{\partial k}{\partial y^i}$. Furthermore, let us assume that in the limit $|y| \to \infty$, the quantity $|k(x, y)\nabla_y \Phi(y)|$ goes to zero faster than $1/y^2$; then, using integration by parts and Gauss's theorem, Non-Local Poisson's Equation can be written as:

$$\nabla^2 \Phi_l(x) + \int d^3y\, k(x - y) \nabla^2 \Phi_l(y) = 4\pi G \rho(x). \tag{3.10}$$

That is a Fredholm integral equation of the second kind that has a unique solution, and which can be expressed in terms of the reciprocal convolution kernel $q(u)$ as:

$$\nabla^2 \Phi_l(x) = 4\pi G \rho(x) + 4\pi G \int d^3y\, q(x - y) \rho(y). \tag{3.11}$$

The non-locality gives rise to an additive source term for the gravitational field that could be interpreted as the Dark Matter term.

3.4 A first case of Non-Local Theory

In this section, we want to evaluate the consequence of an alteration of the classical Einstein-Hilbert action, including a non-local term. The non-locality resides in D'Alambert inverse operator[30]. The new action has the following form:

$$S_{NL} = \frac{1}{2k}\int d^4x\sqrt{-g(x)}R(x)\Big[1 + f(\Box^{-1}R)(x)\Big] + \int d^4x\sqrt{-g(x)}\mathcal{L}_m. \quad (3.12)$$

where $k = 8\pi G$, R is the Ricci scalar, f is any function of the non-local-operator, L_m it is the matter Lagrangian and $\Box$ is the usual D'Alambert operator. The non-local term can be written in this way:

$$(\Box^{-1}f)(x) \equiv \mathcal{G}[f(x)] = \int d^4x'\sqrt{-g(x')}f(x,x')G(x,x'), \quad (3.13)$$

where $G(x,x')$ is f is the retarded Green function evaluated at the Ricci scalar. It is clear that by setting $f(\Box^{-1}R) = 0$ we get the Einstein theory.

3.4.1 The field equations

In order to obtain the new field equation let us focus on the geometric part of the action, since the matter Lagrangian could be included later.

$$S = \frac{1}{2k}\int d^4x\sqrt{-g}R\Big[1 + f(\Box^{-1}R)\Big]. \quad (3.14)$$

It is clear that the first term lead us to Einstein-Hilbert tensor.

1)

$$S = \frac{1}{2k}\int d^4x\sqrt{-g}R \implies G_{\mu\nu}. \quad (3.15)$$

2)

$$\begin{aligned}
&\delta\Big[\sqrt{-g}Rf(\Box^{-1}R)\Big] = \\
&\quad = \delta(\sqrt{-g})Rf(\Box^{-1}R + \sqrt{-g}(\delta R)f(\Box^{-1}R) + \sqrt{-g}R(\delta f(\Box^{-1}R)) = \\
&= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}Rf(\Box^{-1}R)\delta g^{\mu\nu} + \sqrt{-g}(\delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu})f(\Box^{-1}R) + \\
&\qquad + \sqrt{-g}R\delta f(\Box^{-1}R) = \sqrt{-g}\delta g^{\mu\nu}G_{\mu\nu}f(\Box^{-1}R) + \\
&+ \sqrt{-g}(-\nabla_\mu\nabla_\nu\delta g^{\mu\nu} + \Box(g_{\mu\nu}\delta g^{\mu\nu}))f(\Box^{-1}R) + \sqrt{-g}R\delta f(\Box^{-1}R). \quad (3.16)
\end{aligned}$$

Let us focus on the last term of the previous equation and without going through the details of the calculation, after some algebra we get[25]:

$$\begin{aligned}\sqrt{-g}R\delta f(\Box^{-1}R) &= \sqrt{-g}Rf'(\Box^{-1}R)\Big[\Box^{-1}\delta R - \Box^{-1}(\delta\Box)\Box^{-1}R\Big] = \\ &= \sqrt{-g}\Box^{-1}(Rf'(\Box^{-1}R))[\delta g^{\mu\nu}R_{\mu\nu} + (-\nabla_\mu\nabla_\nu\delta g^{\mu\nu} + \Box(g_{\mu\nu}\delta g^{\mu\nu}))] + \\ &\quad - \sqrt{-g}\Box^{-1}(Rf'(\Box^{-1}R))\Big[\delta(\Box)R\Big]. \end{aligned} \tag{3.17}$$

By using $\Box^{-1} \times \Box = \mathbb{1}$, neglecting the boundary terms and using (3.13) we can rewrite the last term of the equation (3.17):

$$R\mathcal{G}\Big[Rf'(\mathcal{G}[R])\Big]\delta(\sqrt{-g}) + \partial_\mu\Big(\mathcal{G}\Big[Rf'(\mathcal{G}[R])\Big]\Big)\partial_\nu(\mathcal{G}[R])(g^{\mu\nu}\delta(\sqrt{-g}) + \sqrt{-g}\delta g^{\mu\nu}). \tag{3.18}$$

All terms are been determined so is finally possible to obtain the field equations. These showed results are useful to determinate the new field equations, indeed by using the results from (3.16) to (3.18) we obtain:

$$\begin{aligned} & G_{\mu\nu} + (-\nabla_\mu\nabla_\nu f(\mathcal{G}[R]) + g_{\mu\nu}\Box f(\mathcal{G}[R])) + G_{\mu\nu}f(\mathcal{G}[R]) + R_{\mu\nu}\mathcal{G}[Rf'(\mathcal{G}[R])] + \\ & - \nabla_\mu\nabla_\nu\mathcal{G}[Rf'(\mathcal{G}[R])] + g_{\mu\nu}\Box(\mathcal{G}[Rf'(\mathcal{G}[R])]) - \frac{1}{2}Rg_{\mu\nu}\mathcal{G}[Rf'(\mathcal{G}[R])] + \\ & + \partial_\rho(\mathcal{G}[Rf'(\mathcal{G}[R])])\partial_\sigma(\mathcal{G}[R])(\delta^\rho_\mu\delta^\sigma_\nu - \frac{1}{2}g^{\rho\sigma}g_{\mu\nu}). \end{aligned} \tag{3.19}$$

Finally, we have the field equations:

$$\begin{aligned} & G_{\mu\nu} + \Big(G_{\mu\nu} + g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\Big)\Big\{f(\mathcal{G}[R]) + \mathcal{G}[Rf'(\mathcal{G}[R])]\Big\} + \\ & + \partial_\rho(\mathcal{G}[Rf'(\mathcal{G}[R])])\partial_\sigma(\mathcal{G}[R])(\delta^\rho_\mu\delta^\sigma_\nu - \frac{1}{2}g^{\rho\sigma}g_{\mu\nu}) = 8\pi G T_{\mu\nu}. \end{aligned} \tag{3.20}$$

It is evident now how non-local term affects the Einstein field equations and what are the additional terms.

3.5 The state of the art

In the figure 4.1 we want to present the state of the art about Non Local theories[19]. The diagram shows how to recover the different theories of gravity starting from the scalar-field representation of the general theory where has been included also the teleparlallel theories.

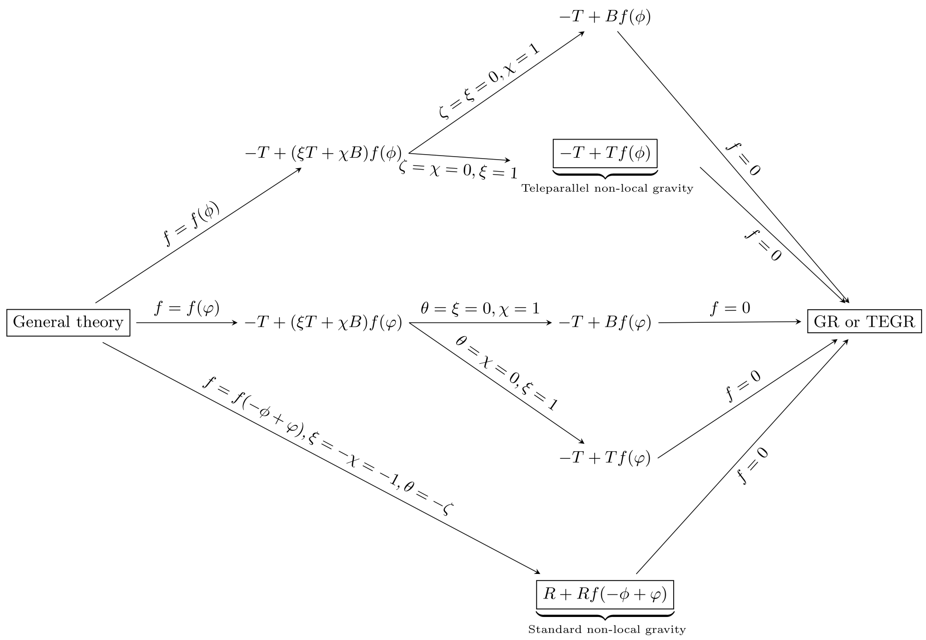

Figure 3.1: Diagram of the Non-Local Theories

In this diagram $\phi = \Box^{-1}T$, $\varphi = \Box^{-1}B$ and ξ and χ are coupling constants. The Ricci scalar R and the torsion scalar T may be linked thanks to a boundary term, in fact:

$$R = -T + \frac{2}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}T^\mu\right) = -T + B. \tag{3.21}$$

and so

$$\Box^{-1}R = -\Box^{-1}T + \Box^{-1}B = -\phi + \varphi. \tag{3.22}$$

Clearly, the curvature and torsion representations "converge" only for the linear theories in R, GR, and in T, Teleparallel Equivalent of General Relativity(TERG). The TEGR is a gauge description of the gravitational interactions and torsion defined through the Weitzenbock connection

$$\tilde{\Gamma}^\alpha_{\mu\nu} = E^\alpha_a \partial_\mu e^a_\nu, \tag{3.23}$$

instead of the Levi-Civita connection, used by GR. Where has been used e^a_μ for tetrads and E^α_a for their inverse[10][19]. Hence, in this theory, the manifold is flat but endorsed with torsion. The dynamical fields of the theory are the four linearly independent vierbeins and they are related to the metric tensor and its inverse by

$$g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu, \quad g^{\mu\nu} = \eta^{ab}E^\mu_a E^\nu_b, \tag{3.24}$$

where η_{ab} is the Minkowskian flat metric. Anyway, the TERG action is given by

$$S_{TEGR} = -\frac{1}{2k}\int d^4xeT + \int d^4xe\mathcal{L}_{\mathcal{M}} \tag{3.25}$$

with e begin $e = det(e^a_\mu) = \sqrt{-g}$ and T is the torsion scalar. It is clear that by recovering the relation (3.21) the sclars R and T differ only by a boundary term and so the Einstein-Hilbert action and the TEGR (3.25) are equivalent. In the teleparallel framework, recently it was proposed a similar kind of non-local gravity based on the torsion scalar T. In this theory, the action reads as follows[25]

$$S_{teleparlallel-NL} = -\frac{1}{2k}\int d^4xe(x)T(x) + \frac{1}{2k}\int d^4xe(x)T(x)f(\Box^{-1}T) + S_{\mathcal{M}}, \tag{3.26}$$

where the function f depends on the non-local term. The TERG is recovered by setting $f(\Box^{-1}T) = 0$.

Let us now present a generalization, which we call Generalized Non-local Teleparallel Gravity (GNTG). Its action is given by

$$S_{GNTG} = -\frac{1}{2k}\int d^4xe(x)T(x) + \frac{1}{2k}\int d^4xe(x)\left(\xi T(x) + \chi B(x)\right) f(\Box^{-1}T, \Box^{-1}B). \tag{3.27}$$

Here, T is the torsion scalar, B is the boundary term previously mentioned and $f(\Box^{-1}T, \Box^{-1}B)$ is a arbitrary function of the non local term and of the non local boundary term. Instead, the Greek letters ξ and χ are coupling constant. By setting $\xi = -\chi = -1$ one obtain the Ricci Scalar R. Directly from the equation (3.21), if $f(\Box^{-1}T, \Box^{-1}B) = f(-\Box^{-1}T + \Box^{-1}B)$, the action takes the well-known form $Rf(\Box^{-1}R)$ of the relation (3.12). Least but not last if we set $\chi = 0$ and $f(\Box^{-1}T, \Box^{-1}B) = f(\Box^{-1}T)$, the non local teleparallel action (3.26) is recovered. The other parameters θ and ζ which appear in the figure 3:1 are Lagrange multipliers required for including, respectively, the constrains

$$\Box\phi - T = 0, \quad \Box\varphi - B = 0. \tag{3.28}$$

This aspect, related to constrain, will be better investigated in the fourth and fifth chapter. Anyhow, the purpose of this section was to show the state of the art of non-local theories and to briefly present also the teleparallel theories.

3.6 A general case of Non-Local Gravity

In this section we set out to describe the main feature of a more general theory, using what learned in the previous section. As we have just shown, in order to obtain a new more general field equation, it is important to select a good Lagrangian term. In the equation (3.12) the Lagrangian is a product of R and a function of $\Box^{-1}R$. Therefore, we want to consider a more general Lagrangian which depends arbitrary on the variables R and $\Box^{-1}R$, namely:

$$S = \frac{1}{2k}\int d^4x\sqrt{-g}f(R,\Box^{-1}R) + \int d^4x\sqrt{-g}\mathcal{L}_m. \tag{3.29}$$

Before starting with the variational method lets focus us on this notation:

$$f' = \frac{\delta f}{\delta R}, \qquad f^\star = \frac{\delta f}{\delta(\Box^{-1}R)}, \tag{3.30}$$

Now we can use the variational method on the geometric term:

$$\begin{aligned}
&\delta(\sqrt{-g})f(R,\Box^{-1}R) + \sqrt{-g}\delta f(R,\Box^{-1}R) = \\
&\quad = \delta(\sqrt{-g})f(R,\Box^{-1}R) + \sqrt{-g}\Big[f'\delta R + f^\star\delta(\Box^{-1}R)\Big] = \\
&= \delta(\sqrt{-g})f(R,\Box^{-1}R) + \sqrt{-g}f'(R,\Box^{-1}R)\delta R + \sqrt{-g}f^\star\delta(\Box^{-1}R).
\end{aligned} \tag{3.31}$$

It's clear that the first two terms of the last equivalence are similar to $f(R)$ in form while the last one, is the non-local term already treated in the last section. Therefore, using these concepts and studying them separately, it is possible to get:

1) The first term:

$$f'(R,\Box^{-1}R)R_{\mu\nu} - \frac{f(R,\Box^{-1}R)}{2}g_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R,\Box^{-1}R) + g_{\mu\nu}\Box f'(R,\Box^{-1}R). \tag{3.32}$$

2) The second term[30]:

$$\sqrt{-g}f^\star\delta(\Box^{-1}R) = \sqrt{-g}f^\star\Big[\Box^{-1}\delta R - \Box^{-1}(\delta\Box)\Box^{-1}R\Big] =$$
$$= \sqrt{-g}f^\star\Big\{\Box^{-1}(\delta\Box)\Box^{-1}R + \Box^{-1}[\delta g^{\mu\nu}R_{\mu\nu} + (-\nabla_\mu\nabla_\nu\delta g^{\mu\nu} + \Box(g_{\mu\nu}\delta g^{\mu\nu}))]\Big\} =$$
$$= \Big(G_{\mu\nu} + g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\Big)\mathcal{G}[f^\star] + \partial_\rho(\mathcal{G}[f^\star])\partial_\sigma(\mathcal{G}[R])\Big[\delta^\rho_\mu\delta^\sigma_\nu - \frac{1}{2}g^{\rho\sigma}g_{\mu\nu}\Big]. \tag{3.33}$$

Recollecting these two terms finally we get the **the field equations**:

$$R_{\mu\nu}f' - \frac{1}{2}g_{\mu\nu}f - \nabla_\mu\nabla_\nu f' + g_{\mu\nu}\Box f' + \Big(G_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\Big)\mathcal{G}[f^\star] +$$
$$+ \partial_\rho(\mathcal{G}[f^\star])\partial_\sigma(\mathcal{G}[R])\Big[\delta^\rho_\mu\delta^\sigma_\nu - \frac{1}{2}g^{\rho\sigma}g_{\mu\nu}\Big] = 8\pi G T_{\mu\nu}. \tag{3.34}$$

We could say in a simple way, that the equation takes the aspected form, in fact as soon as we wrote the action, it was clear that it would have led us to something similar to $f(R)$ and non-local theory.

In order to test the last equation we can prove that is possible to obtain the old equation (3.20). We consider :

$$f(R, \Box^{-1}R) = R\Big[1 + g(\Box^{-1}R)\Big], \tag{3.35}$$

$$f' = \frac{\delta f}{\delta R} = 1 + g(\Box^{-1}R), \qquad f^\star = \frac{\delta f}{\delta(\Box^{-1}R)} = R\frac{\delta g(\Box^{-1}R)}{\delta(\Box^{-1}R)} \equiv Rg'. \tag{3.36}$$

Now replacing these relations into (3.34) we get the less general case (3.20).

CHAPTER 4

COINSTRAINED NON-LOCAL GRAVITY BY S2 STAR ORBITS

In this chapter, we consider the non-local theory (3.12) proposed by Deser and Woodard but in its local representation. We apply the first prolongation of Noether Symmetry Approach, discussed in appendix A, in a spherically symmetric spacetime and find those functional forms of the distortion function, that leave the point-like Lagrangian invariant. Moreover, we find the weak field limit of the theory with the exponential coupling and we also calculate the Post-Newtonian (PN) limit.

The local representation of this non-local model can be formulated as a biscalar-tensor theory, even if one of the two scalar fields is not dynamical. The PN analysis gives rise to two new length scales, where only one of them is physically relevant; the other one is related to the auxiliary degree of freedom introduced for localizing the original action. Finally, we consider the orbits of S2 star around the Galactic center and by comparing the PN terms of our theory with observations, we are able to set some bounds on the above dynamical length scale. S-stars are the bright stars which move around the centre of our Galaxy, where the compact radio source Sagittarius A*(or Sgr A*) is located. For one of them, called S2, a deviation from its Keplerian orbit was observed [43].

It is clear that the non-localities are not expected to be significantly relevant at astrophysical and galactic scales, because otherwise they would have been observed. However, what we see, is that our approach is consistent with the orbits of S2 star around Sgr A*, in order to show its range of validity at astrophysical and cosmological scales.

4.1 The point-like lagrangian and Noether symmetries

In order to obtain a point-like Lagrangian we need to recover the Deser and Woodard non-local modification of the Einstein-Hilbert action (3.12), which has the following form:

$$S_{NL} = \frac{1}{2k}\int d^4x\sqrt{-g(x)}R(x)\Big[1+f(\Box^{-1}R)(x)\Big].$$

In order to localize the action we introduce two auxiliary scalar fields ϕ and ξ, where the first one satisfies a Klein-Gordon equation:

$$\Box\phi - R = 0 \tag{4.1}$$

and the second one is introduced as Lagrangian multiplier of the constrain related to ϕ, namely

$$S_{NL} = \frac{1}{2k}\int d^4x\sqrt{-g}\left[R\left(1+f(\phi)\right)+\xi\left(\Box\phi - R\right)\right], \tag{4.2}$$

where integrating out the total derivatives we obtain

$$S_{NL} = \frac{1}{2k}\int d^4x\sqrt{-g}\left[R\left(1+f(\phi)-\xi\right)-\nabla^\mu\xi\nabla_\mu\phi\right]. \tag{4.3}$$

By varying the action with respect to ξ and ϕ respectively, we get

$$\Box\phi - R = 0, \tag{4.4}$$

$$\Box\xi = -R\frac{df}{d\phi}. \tag{4.5}$$

Moreover, the variation of the action with respect to the metric we yields

$$\begin{aligned}(1+f(\phi)-\xi)G_{\mu\nu}+\frac{1}{2}g_{\mu\nu}\nabla^\alpha\xi\nabla_\alpha\phi &= \\ = k^2T_{\mu\nu}+\nabla_\mu\xi\nabla_\nu\phi+\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)&\left(f(\phi)-\xi\right). \end{aligned}\tag{4.6}$$

The method consist in selecting a symmetry for the background spacetime which, in our case, is spherically symmetric. The metric is given by the following line element

$$ds^2 = e^{\nu(r,t)}dt^2 - e^{\lambda(r,t)}dr^2 - r^2d\Omega^2, \tag{4.7}$$

where $\nu(r,t)$ and $\lambda(r,t)$ are two arbitrary function depending on time t and on the radial variable r since we do not know *a priori* if Birkhoff's theorem holds in non-local gravity.

Then, substituting this metric in the Lagrangian density in (4.3):

$$\begin{aligned}\mathcal{L} = e^{-\frac{1}{2}(\lambda+\nu)}\Big(& -e^{\nu}r^2\nu_r\phi_r f'(\phi) + e^{\nu}r^2\lambda_t\phi_t f'(\phi) - \\ & -2e^{\nu}f(\phi)\left(e^{\lambda}+r\lambda_r-1\right) - 2e^{\lambda+\nu} + 2e^{\nu} + e^{\nu}r^2\xi_r\phi_r + e^{\nu}r^2 + \\ & + e^{\nu}r^2\nu_r\xi_r - e^{\lambda} - e^{\lambda}r^2\xi_t\phi_t - e^{\lambda}r^2\lambda_t\xi_t + 2e^{\nu}\xi\left(e^{\lambda}+r\lambda_r-1\right) - 2e^{\nu}r\lambda_r\Big)\end{aligned} \tag{4.8}$$

where the subscript denotes differentiation with respect to the variable.

Now taking into account the equation (A.50) related to the first prolongation of the Noether theorem:

$$X^{[1]}\mathcal{L} + \mathcal{L}\left(\frac{d\xi^t}{dt} + \frac{d\xi^r}{dr}\right) = \frac{dh^t}{dt} + \frac{dh^r}{dr}, \tag{4.9}$$

where h^t and h^r are two arbitrary functions depending on $(t, r, \nu, \lambda, \phi, \xi)$. Expanding the above condition, we find a system of 75 equations with 9 unknown variables, 6 coefficients of the Noether vector $(\xi^t, \xi^r, \eta^{\nu}, \eta^{\lambda}, \eta^{\phi}, \eta^{\phi})$, 2 unknown functions h^t, h^r and the form of the distortion function $f(\phi)$[43]. Solving the system we find two possible models that are invariant under point transformations, that are

$$f(\phi) = c_4 + c_3\phi \ \text{ and } \ f(\phi) = c_4 + \frac{c_4}{c_1}e^{c_1\phi}. \tag{4.10}$$

This approach allowed us to select the distortion function and the Lagrangians in agreement with the symmetries of the astrophysical system.

4.2 The weak field approximation

To obtain the weak field approximation we take into account the exponential form of the distortion function and then we compare the result with the S2 star orbits. As in General Relativity, in order to recover the Newtonian potential for time-like particles we have to expand the g_{00} component of the metric to $\Phi \sim v^2 \sim O(2)$, where Φ is the Newtonian potential and v is the 3-velocity of a fluid element. If we want to study the Post Newtonian limit we have to expand the components of the metric as:

$$g_{00} \sim O(6), \ \ g_{0i} \sim O(5), \ \text{ and } g_{ij} \sim O(4). \tag{4.11}$$

We also assume that in the weak field approximation the Birkhoff's Theoreme holds and so a static and spherically symmetric metric works as well. With this position, the metric is

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\Omega^2. \tag{4.12}$$

Since the metric depends only on the radial coordinate, it is reasonable to consider that the scalar fields ϕ and ξ depends only on the radial coordinate. The expansion of the metric components, as well as the scalar fields, is:

$$A(r) = 1 + \frac{1}{c^2}\Phi^{(2)} + \frac{1}{c^4}\Phi^{(4)} + \frac{1}{c^6}\Phi^{(6)} + O(8) \tag{4.13}$$

$$B(r) = 1 + \frac{1}{c^2}\Psi^{(2)} + \frac{1}{c^4}\Psi^{(4)} + O(6) \tag{4.14}$$

$$\phi(r) = \phi_0 + \frac{1}{c^2}\phi^{(2)} + \frac{1}{c^4}\phi^{(4)} + \frac{1}{c^6}\phi^{(6)} + O(8) \tag{4.15}$$

$$\xi(r) = \xi_0 + \frac{1}{c^2}\xi^{(2)} + \frac{1}{c^4}\xi^{(4)} + \frac{1}{c^6}\xi^{(6)} + O(8) \tag{4.16}$$

If we substitute the exponential form f(ϕ), namely $f(\phi) = 1 + e^{\phi}$, into the the equations (4.4)-(4.6) and if we consider the previously perturbation expansion we obtain:

$$\begin{aligned} A(r) = 1 - \frac{2G_N M\phi_c}{c^2 r} &+ \frac{G_N^2 M^2}{c^4 r^2}\left[\frac{14}{9}\phi_c^2 + \frac{18r_\xi - 11r_\phi}{6r_\xi r_\phi} r\right] + \\ &- \frac{G_N^3 M^3}{c^6 r^3}\left[\frac{50r_\xi - 7r_\phi}{12r_\xi r_\phi}\phi_c + \frac{16\phi_c^3}{27} - \frac{r^2(2r_\xi^2 - r_\phi^2)}{r_\xi^2 r_\phi^2}\right] \end{aligned} \tag{4.17}$$

$$B(r) = 1 + \frac{2G_N M\phi_c}{3c^2 r} + \frac{G_N^2 M^2}{c^4 r^2}\left[\frac{2}{9}\phi_c^2 + \left(\frac{3}{2r_\xi} - \frac{1}{r_\phi}\right) r\right] \tag{4.18}$$

Similar relations can be obtained for the two scalar field, $\phi(r)$ and $\xi(r)$, showed in [43]. We see that two length scales arise, r_ϕ and r_ξ, related to the two scalar fields ϕ and ξ. Is finally possible to show how the non locality affects the Newtonian potential, which turns out to be

$$\begin{aligned} V(r) = U_{NL} - U_N = \frac{G_N M\phi_c}{r} &+ \frac{G_N^2 M^2}{c^4 r^2}\left[\frac{14}{9}\phi_c^2 + \frac{18r_\xi - 11r_\phi}{6r_\xi r_\phi} r\right] + \\ &- \frac{G_N^3 M^3}{c^6 r^3}\left[\frac{50r_\xi - 7r_\phi}{12r_\xi r_\phi}\phi_c + \frac{16\phi_c^3}{27} - \frac{r^2(2r_\xi^2 - r_\phi^2)}{r_\xi^2 r_\phi^2}\right] \end{aligned} \tag{4.19}$$

To compare our results with real trajectories we have to find some constraints on the two new length scales. This is why, the results of the simulations are now compared with the orbits of S2 star around the Galactic Center[43]. The results obtained for r_ϕ and r_ξ parameters shows that the S2 star orbit in non-local gravity fits the astrometric data better than Keplerian orbit. The most probable value for the scale parameter r_ϕ it varies from 0.1 to 2.5 AU, otherwise it is not possible to obtain a constrains on the parameter r_ξ of non-local gravity using only observed astrometric data since this it is associated with one of the scalar fields which is not dynamical.

The obtained orbital precession of the S2 star in non-local gravity is of the same order of magnitude as in General Relativity; in the future, more precise astronomical data will better constrain the non-local gravity parameters. However, it is reasonable to think that non-local effects do not have a significant role at scales comparable to the S2 astrophysical scales, but only at cosmological ones. Finally, as we are going to see in the next chapters, that the forms of the distortion function $f(\phi)$, that leave the action invariant, are the same as the ones of cosmological minisuperspace.

CHAPTER 5

NON LOCAL COSMOLOGY FROM NOETHER SYMMETRIES

In the previous chapter, we discussed some methods able to extend General Relativity using the Lagrangian formulation through the variational approach. In every case, we need a procedure that could be capable to select a functional form of the functions involved in these theories whether they are $f(R)$ or for modified non-local theories $f(R, \Box^{-1}R)$.

Clearly these functions need to be suitable for the symmetries of the system and the methodology that allows us to select that compatibility is exactly the Noether Theorem and in particular, for our aim, its first prolongation. As said before this necessity is related to the fact that in the Friedman-Lemaitre-Robertson-Walker metric, the action will be point-like.

With a view to studying the Non-Local Theories we need to include the following constraint in the action:

$$\phi = \Box^{-1}R. \tag{5.1}$$

where ϕ is a scalar field and its inclusion is mediated by using another scalar field χ which plays the role of Lagrange-multiplier.

Following the procedure showed in the Appendix A we define a configuration space $\mathcal{Q} \equiv \{a, R, \phi, \chi\}$ and much more important its tangent space $\mathcal{TQ} \equiv \{a, R, \phi, \chi . \dot{a}, \dot{R}, \dot{\phi}, \dot{\chi}\}$ where the generator of the infinitesimal transformation and the Lagrangian $\mathcal{L}$ are defined.

5.1 The point-like Lagrangian

As anticipated we need to recover the Non-Local Action (3.29) and for the sake simplicity we use the physical units $G = c = \hbar = 1$ and also we do not consider the matter term. So the action becomes:

$$S = \int d^4x\sqrt{-g}F(R,\phi). \tag{5.2}$$

where already it has been said:

$$\phi = \Box^{-1}R \qquad \text{or} \qquad \Box\phi = R \tag{5.3}$$

this suggests that the scalar field ϕ must satisfy a Klein-Gordon equation. However, in particular, is possible to include these constrains into the action by using the Lagrange multiplier χ, because in any case the term $\Box\phi - R$ is evidentially zero:

$$S = \int d^4x\sqrt{-g}\Big[F(R,\phi) + \chi(\Box\phi - R)\Big]. \tag{5.4}$$

Integrating by parts and setting to zero the boundary term it is possible to rewrite the term which involves the box operator:

$$S = \int d^4x\sqrt{-g}\Big[F(R,\phi) - \chi R - \nabla_\mu\chi\nabla^\mu\phi\Big]. \tag{5.5}$$

Now it is possible to take into account the flat FRLW metric in the flat case $k = 0$, namely:

$$ds^2 = dt^2 - a^2(t)\Big[dx_1^2 + dx_2^2 + dx_3^2\Big], \tag{5.6}$$

then assuming that the scalar field only depends on the time the covariant derivatives become time derivatives and also the invariant measure gives rise to $2\pi^2 dta^3$ where t now is the cosmological time. The action therefore is

$$S = 2\pi^2\int dt\, a^3\Big(F(R,\phi) - \chi R - \dot{\phi}\dot{\chi}\Big). \tag{5.7}$$

To obtain the canonical point-like Lagrangian which depends on all the variable of the tangent space, above mentioned, we need to set R like a dynamical constraint by using another Lagrange multiplier λ. In that metric the Ricci scalar is:

$$R = -6\Big(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\Big), \tag{5.8}$$

and so we obtain:

$$S = 2\pi^2\int dt\, a^3\Big\{F(R,\phi) - \chi R - \dot{\phi}\dot{\chi} - \lambda\Big[R + 6\Big(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\Big)\Big]\Big\}. \tag{5.9}$$

The Lagrange multiplier λ could be obtained by varying the action with respect to R, that leads us to:

$$\lambda = \frac{\partial F(R,\phi)}{\partial R} - \chi \tag{5.10}$$

and the the action becomes:

$$S = 2\pi^2 \int dt\, a^3 \Big\{ F(R,\phi) - \chi R - \dot{\phi}\dot{\chi} - \Big(\frac{\partial F(R,\phi)}{\partial R} - \chi \Big) \Big[R + 6\Big(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \Big) \Big] \Big\}. \tag{5.11}$$

After integrating by parts, it is finally possible to achieve the get point-like Lagrangian:

$$\mathcal{L}(a, R, \phi, \chi.\dot{a}, \dot{R}, \dot{\phi}, \dot{\chi}) = 6a^2\dot{a}\dot{R}\frac{\partial^2 F(R,\phi)}{\partial R^2} + 6a^2\dot{a}\dot{\phi}\frac{\partial^2 F(R,\phi)}{\partial R \partial \phi} + \\ + 6a\dot{a}^2\frac{\partial F(R,\phi)}{\partial R} + a^3 F(R,\phi) - a^3 R\frac{\partial F(R,\phi)}{\partial R} - 6a^2\dot{a}\dot{\chi} - 6a\chi\dot{a}^2 - a^3\dot{\chi}\dot{\phi}. \tag{5.12}$$

This result will be important in the next section where we will determinate the Noether symmetries of these theories and so the form relative of the function $f(R,\phi)$.

5.2 The Noether Symmetries

As it has been said in the Theorem 4 (First prolongation of Noether Theorem) a Noether symmetry for the point-like Lagrangian (5.12) exists if it satisfies the condition (A.50):

$$X^{[1]}\mathcal{L} + \mathcal{L}\dot{\xi} = \dot{g}(t, q^i). \tag{5.13}$$

By explaining each term and taking into account that our Lagrangian is not dissipative $\frac{\partial \mathcal{L}}{\partial t} = 0$, the previous condition gives rise to:

$$\eta^i(t,q)\frac{\partial \mathcal{L}}{\partial q^i} + (\dot{\eta}^i - \dot{q}^i\dot{\xi})\frac{\partial \mathcal{L}}{\partial \dot{q}^i} + \mathcal{L}\dot{\xi} = \dot{g}(t, q^i). \tag{5.14}$$

To be clearer we can write each term of the first prolongation of the Noether vector:

$$X^{[1]} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial R} + \gamma\frac{\partial}{\partial \phi} + \delta\frac{\partial}{\partial \chi} + \\ + (\dot{\alpha} - \dot{a}\dot{\xi})\frac{\partial}{\partial \dot{a}} + (\dot{\beta} - \dot{R}\dot{\xi})\frac{\partial}{\partial \dot{R}} + (\dot{\gamma} - \dot{\phi}\dot{\xi})\frac{\partial}{\partial \dot{\phi}} + (\dot{\delta} - \dot{\chi}\dot{\xi})\frac{\partial}{\partial \dot{\chi}} \tag{5.15}$$

where the functions α, β, γ, δ, ξ depend on t, a, R, ϕ, χ and their derivatives:

$$\dot{\alpha} = \frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial a}\dot{a} + \frac{\partial \alpha}{\partial R}\dot{R} + \frac{\partial \alpha}{\partial \phi}\dot{\phi} + \frac{\partial \alpha}{\partial \chi}\dot{\chi} \tag{5.16}$$

$$\dot{\beta} = \frac{\partial \beta}{\partial t} + \frac{\partial \beta}{\partial a}\dot{a} + \frac{\partial \beta}{\partial R}\dot{R} + \frac{\partial \beta}{\partial \phi}\dot{\phi} + \frac{\partial \beta}{\partial \chi}\dot{\chi} \tag{5.17}$$

$$\dot{\gamma} = \frac{\partial \gamma}{\partial t} + \frac{\partial \gamma}{\partial a}\dot{a} + \frac{\partial \gamma}{\partial R}\dot{R} + \frac{\partial \gamma}{\partial \phi}\dot{\phi} + \frac{\partial \gamma}{\partial \chi}\dot{\chi} \tag{5.18}$$

$$\dot{\delta} = \frac{\partial \delta}{\partial t} + \frac{\partial \delta}{\partial a}\dot{a} + \frac{\partial \delta}{\partial R}\dot{R} + \frac{\partial \delta}{\partial \phi}\dot{\phi} + \frac{\partial \delta}{\partial \chi}\dot{\chi} \tag{5.19}$$

$$\dot{\xi} = \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial a}\dot{a} + \frac{\partial \xi}{\partial R}\dot{R} + \frac{\partial \xi}{\partial \phi}\dot{\phi} + \frac{\partial \xi}{\partial \chi}\dot{\chi} \tag{5.20}$$

with the *gauge function* $g(t, q^i)$:

$$\dot{g} = \frac{\partial g}{\partial t} + \frac{\partial g}{\partial a}\dot{a} + \frac{\partial g}{\partial R}\dot{R} + \frac{\partial g}{\partial \phi}\dot{\phi} + \frac{\partial g}{\partial \chi}\dot{\chi}. \tag{5.21}$$

With these prescriptions the equation (4.14) leads us to a system of twenty-eight partial differential equations, related to the collected terms in $\dot{a}^2$, $\dot{a}\dot{\chi}$, $\dot{\chi}\dot{\phi}$, $\dot{a}\dot{R}$, $\dot{R}^2$ and many others. For entirety i will show the complete system in the Appendix B.

It is straightforward to show that by setting to zero each term of the time derivative of the gauge term, namely:

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial a} = \frac{\partial g}{\partial R} = \frac{\partial g}{\partial \phi} = \frac{\partial g}{\partial \chi} = 0, \tag{5.22}$$

we obtain that α, β, γ, δ, do not depend on time and also and on the contrary ξ depends only on time and not on the other variables.

After various algebraic manipulation is finally possible to show that these equations give rise to the most important and fundamental system composed by only six partial differential equations, which will allow us to determine the Noether symmetries and the functional form the $F(R, \phi)$. In particular, we first obtain that:

$$\alpha = \alpha(a);\ \ \beta = \beta(a, R, \phi, \chi);\ \ \gamma = \gamma(a, \phi);\ \ \delta = \delta(a, \chi); \tag{5.23}$$

$$\xi = \xi(t). \tag{5.24}$$

So the final system is:

$$
\begin{cases}
-3R\frac{\partial F}{\partial R}\alpha - aR\frac{\partial^2 F}{\partial R^2}\beta + a\frac{\partial F}{\partial \phi}\gamma - aR\frac{\partial^2 F}{\partial R \partial \phi}\gamma + 3F\alpha - aR\frac{\partial \xi}{\partial t}\frac{\partial F}{\partial R} + a\frac{\partial \xi}{\partial t}F = 0 \\
\\
-a\delta - \alpha\chi + a\gamma\frac{\partial^2 F}{\partial R\partial\phi} + \alpha\frac{\partial F}{\partial R} + a\beta\frac{\partial^2 F}{\partial R^2} + a^2\frac{\partial^2 F}{\partial R\partial\phi}\frac{\partial\gamma}{\partial a} - a^2\frac{\partial\delta}{\partial a} + a^2\frac{\partial^2 F}{\partial R^2}\frac{\partial\beta}{\partial a} - 2a\chi\frac{\partial\alpha}{\partial a} + \\
+2a\frac{\partial F}{\partial R}\frac{\partial\alpha}{\partial a} - a\frac{\partial\xi}{\partial t}\frac{\partial F}{\partial R} + a\chi\frac{\partial\xi}{\partial t} = 0 \\
\\
6a^2\frac{\partial^2 F}{\partial R^2}\frac{\partial\beta}{\partial\chi} - 12a\alpha - a^3\frac{\partial\gamma}{\partial a} - 6a^2\frac{\partial\alpha}{\partial a} - 6a^2\frac{\partial\delta}{\partial\chi} + 6a^2\frac{\partial\xi}{\partial t} = 0 \\
\\
-3a^2\alpha - a^3\frac{\partial\gamma}{\partial\phi} - a^3\frac{\partial\delta}{\partial\chi} + a^3\frac{\partial\xi}{\partial t} = 0 \\
\\
6a^2\frac{\partial^2 F}{\partial R\partial\phi}\frac{\partial\alpha}{\partial a} + 12a\frac{\partial^2 F}{\partial R\partial\phi}\alpha + 6a^2\frac{\partial^2 F}{\partial R^2}\frac{\partial\beta}{\partial\phi} + 6a^2\frac{\partial^3 F}{\partial R^2\partial\phi}\beta + 6a^2\frac{\partial^3 F}{\partial R\partial^2\phi}\gamma + 6a^2\frac{\partial^2 F}{\partial R\partial\phi}\frac{\partial\gamma}{\partial\phi} + \\
-a^3\frac{\partial\delta}{\partial a} - 6a^2\frac{\partial\xi}{\partial t}\frac{\partial^2 F}{\partial R\partial\phi} = 0 \\
\\
6a^2\frac{\partial^2 F}{\partial R^2}\frac{\partial\alpha}{\partial a} + 12a\frac{\partial^2 F}{\partial R^2}\alpha + 6a^2\frac{\partial^2 F}{\partial R^2}\frac{\partial\beta}{\partial R} + 6a^2\frac{\partial^3 F}{\partial R^3}\beta + 6a^2\frac{\partial^3 F}{\partial R^2\partial\phi}\gamma - 6a^2\frac{\partial\xi}{\partial t}\frac{\partial^2 F}{\partial R^2} = 0
\end{cases}
\tag{5.25}
$$

After long manipulations it is possible to determine how α, β, γ, δ depend on the variables of the configuration space:

$$\alpha(a) = k_1 a; \quad \beta(R) = -R(6k_1 + 2c_2); \quad \gamma(\phi) = c_3; \quad \delta(\chi) = c_2\chi; \tag{5.26}$$

$$\xi(t) = (3k_1 + c_2)t + c_4 \tag{5.27}$$

and the fundamental partial differential equation for $F(R, \phi)$:

$$F(R,\phi)(c_2 + 6k_1) - R\frac{\partial F(R,\phi)}{\partial R}(6k_1 + 2c_2) + c_3\frac{\partial F(R,\phi)}{\partial\phi} = 0, \tag{5.28}$$

where the constants that appear in previous relations come from the resolution of differential equations of the system (5.25).

5.3 The model selection

In the previous section, we have obtained a partial differential equation (5.28) for the function $F(R, \phi)$ that characterizes the Lagrangian of our theory. By using this equation we want to determinate the classes of solution compatible with the Noether theory in the cosmological framework.

Let us start recovering the partial differential equation (PDE):

$$F(R,\phi)(c_2+6k_1)-R\frac{\partial F(R,\phi)}{\partial R}(6k_1+2c_2)+c_3\frac{\partial F(R,\phi)}{\partial \phi}=0.$$

First of all, we suppose that the function is writeable like a product of two functions, one of the variable R and the second of the variable ϕ, namely:

$$F(R,\phi)=A(R)B(\phi). \tag{5.29}$$

that yields to

$$A(R)B(\phi)(c_2+6k_1)-R\frac{\partial A(R)}{\partial R}B(\phi)(6k_1+2c_2)+c_3A(R)\frac{\partial B(\phi)}{\partial \phi}=0. \tag{5.30}$$

First Case: Every coefficients are not null

We obtain:

$$\Big[A(R)(c_2+6k_1)-R\frac{\partial A(R)}{\partial R}(6k_1+2c_2)\Big]B(\phi)=-c_3A(R)\frac{\partial B(\phi)}{\partial \phi}. \tag{5.31}$$

By dividing both sides by $A(R)B(\phi)$ we obtain that the left term depends only on the variable R and the right term only on ϕ, so we can equal both members to a constant m, as follow:

$$\begin{cases} A(R)(c_2+6k_1)-R\frac{\partial A(R)}{\partial R}(6k_1+2c_2)=mA(R) \\ \frac{\partial B(\phi)}{\partial \phi}=-\frac{m}{c_3}B(\phi) \end{cases} \tag{5.32}$$

that has the following solution:

$$\begin{cases} A(R)=A_0\,R^{\frac{6k_1+c_2-m}{6k_1+2c_2}} \\ B(\phi)=B_0\,e^{-\frac{m}{c_3}\phi} \end{cases} \tag{5.33}$$

Summing up we finally obtain the first solution of the system (5.25):

$$\alpha(a)=k_1a;\quad \beta(R)=-R(6k_1+2c_2);\quad \gamma(\phi)=c_3;\quad \delta(\chi)=c_2\chi;$$

$$\xi(t)=(3k_1+c_2)t+c_4,$$

$$F(R,\phi)=F_0R^{\frac{6k_1+c_2-m}{6k_1+2c_2}}\,e^{-\frac{m}{c_3}\phi}. \tag{5.34}$$

It is straightforward and also important to stress that for $c_2 = 0$ and $m = 0$, we recover as a limit the General Relativity:

$$F(R) = F_0\, R. \tag{5.35}$$

In fact the action becomes:

$$S = \int d^4x\sqrt{-g}R. \tag{5.36}$$

that leads, as it has been shown in the first chapter, to the Einstein field equations.

Second Case: $c_2 + 6k_1 = 0$

We obtain:

$$c_2 R\frac{\partial A(R)}{\partial R}B(\phi) = c_3 A(R)\frac{\partial B(\phi)}{\partial \phi}. \tag{5.37}$$

By dividing both sides by $A(R)B(\phi)$ we obtain that each sides depends on a different variable, that is R for the left part and ϕ for the right part, so we can equal both to a constant m, as follow:

$$\begin{cases} R\frac{\partial A(R)}{\partial R}c_2 = mA(R) \\ \frac{\partial B(\phi)}{\partial \phi} = \frac{m}{c_3}B(\phi) \end{cases} \tag{5.38}$$

that has the following solution:

$$\begin{cases} A(R) = A_0\, R^{\frac{m}{c_2}} \\ B(\phi) = B_0\, e^{\frac{m}{c_3}\phi} \end{cases} \tag{5.39}$$

This allow us to determine the second solution of the system (5.25):

$$\alpha(a) = -\frac{c_2}{6}a; \quad \beta(R) = -Rc_2; \quad \gamma(\phi) = c_3; \quad \delta(\chi) = c_2\chi;$$

$$\xi(t) = \frac{c_2}{2}t + c_4;$$

$$F(R,\phi) = F_0 R^{\frac{m}{c_2}} e^{\frac{m}{c_3}\phi}. \tag{5.40}$$

This second case gives rise a less general case of the previous one and clearly in the same way we get General Relativity as a limit. In fact stating $m = c_2$ and $c_3 \gg m$ we recover $F(R) = F_0 R$ that is the Einstein-Hilbert Lagrangian.

Third case: $6k_1 + 2c_2 = 0$

It easily could be rewritten as: $3k_1 + c_2 = 0$. This condition leads to an important consequence:

$$\xi(t) = c_4 \Longrightarrow \frac{\partial \xi(t)}{\partial t} = 0 \tag{5.41}$$

Such condition, as is possible to see form the equation 5.14, leads us to the classical Noether Theorem without the prolongation, namely:

$$L_X \mathcal{L} = 0 \quad or \quad X\mathcal{L} = 0. \tag{5.42}$$

However we get this equation

$$-A(R)B(\phi)c_2 = c_3 A(R)\frac{\partial B(\phi)}{\partial \phi}. \tag{5.43}$$

By dividing both sides by $A(R)$ we obtain an equation which depends only on the function $B(\phi)$, as follow

$$\frac{\partial B(\phi)}{\partial \phi} = -\frac{c_2}{c_3} B(\phi) \tag{5.44}$$

and the related solution is:

$$B(\phi) = B_0 e^{-\frac{c_2}{c_3}\phi}. \tag{5.45}$$

By merging with the previous results, we finally obtain the third solution of the system (5.25):

$$\alpha(a) = -\frac{c_2}{3}a; \quad \beta(R) = 0; \quad \gamma(\phi) = c_3; \quad \delta(\chi) = c_2\chi;$$

$$\xi(t) = c_4;$$
$$F(R, \phi) = F_0 A(R) e^{-\frac{c_2}{c_3}\phi}. \tag{5.46}$$

Furthermore, even in this case, we can obtain the Einstein Theory as a limit, in fact, by imposing $c_3 \gg c_2$ and expanding at first order $A(R) = A_0 + A_1 R$, we obtain the General Relativity with a constant term that can be interpreted as the cosmological term:

$$S = \int d^4x\sqrt{-g}\Big(R - 2\Lambda\Big), \tag{5.47}$$

that is exactly the Hilbert-Einstein Action in presence of the cosmological constant.

Fourth case: $c_3 = 0$

In this case, the term related to the ϕ derivative vanishes and thus we get:

$$\left[A(R)(c_2+6k_1) - R\frac{\partial A(R)}{\partial R}(6k_1+2c_2)\right]B(\phi) = 0. \tag{5.48}$$

The term $B(\phi)$ clearly is not zero, thus we can get a relation only for $A(R)$:

$$A(R)(c_2+6k_1) = R\frac{\partial A(R)}{\partial R}(6k_1+2c_2), \tag{5.49}$$

that has the following solution:

$$A(R) = A_0 R^{\frac{6k_1+c_2}{6k_1+2c_2}}. \tag{5.50}$$

The fourth solution of the system (5.25) so becomes:

$$\begin{gathered}\alpha(a) = k_1 a; \quad \beta(R) = -R(6k_1+2c_2); \quad \gamma(\phi) = 0; \quad \delta(\chi) = c_2\chi; \\ \xi(t) = (3k_1+c_2)t + c_4; \\ F(R,\phi) = F_0 R^{\frac{6k_1+c_2}{6k_1+2c_2}} B(\phi). \end{gathered} \tag{5.51}$$

Even in this case we obtain the Einstein Theory if we consider $B(\phi) = B_0$ and $c_2 = 0$.

Now we want to analyse the situation in which the function $F(R,\phi)$ is writeable like a sum of two functions, one of the variable R and the second one of the variable ϕ, namely

$$F(R,\phi) = A(R) + B(\phi) \tag{5.52}$$

that, by means of the equation (5.28), yields

$$A(R)(c_2+6k_1) + B(c_2+6k_1) - R\frac{\partial A(R)}{\partial R}(2c_2+6k_1) + c_3\frac{\partial B(\phi)}{\partial \phi} = 0. \tag{5.53}$$

First Case: Every coefficients are not null

We obtain:

$$A(R)(c_2+6k_1) - R\frac{\partial A(R)}{\partial R}(2c_2+6k_1) = -B(c_2+6k_1) - c_3\frac{\partial B(\phi)}{\partial \phi}, \tag{5.54}$$

where the left term depends only on the variable R and the right term only on ϕ, so we can equal both sides to a constant m, as follow:

$$\begin{cases} A(R)(c_2+6k_1) - R\frac{\partial A(R)}{\partial R}(2c_2+6k_1) = m \\ -B(c_2+6k_1) - c_3\frac{\partial B(\phi)}{\partial \phi} = m \end{cases} . \tag{5.55}$$

It is possible to solve the first one recovering that:

$$y' + a(x)y = b(x) \tag{5.56}$$

has the following solution:

$$y(x) = e^{-K(x)}\Big[C + \int dx\, b(x)e^{K(x)}\Big], \tag{5.57}$$

where y is a function of x, $a(x)$ and $b(x)$ are generic functions of the x variable and $K(x) = \int dx\, a(x)$.

The previous system has the following solution:

$$\begin{cases} A(R) = A_0 R^{\frac{c_2+6k_1}{2c_2+6K_1}} + \frac{m}{c_2+6k_1} \\ B(\phi) = B_0 e^{-\frac{c_2+6k_1}{c_3}\phi} - \frac{m}{c_2+6k_1} \end{cases} . \tag{5.58}$$

We can finally obtain the fifth solution of the system (5.25):

$$\alpha(a) = k_1 a; \quad \beta(R) = -R(6k_1+2c_2); \quad \gamma(\phi) = c_3; \quad \delta(\chi) = c_2\chi;$$

$$\xi(t) = (3k_1+c_2)t + c_4;$$

$$F(R,\phi) = A_0 R^{\frac{6k_1+c_2}{6k_1+2c_2}} + B_0 e^{-\frac{c_2+6k_1}{c_3}\phi}. \tag{5.59}$$

Also, in this case, it is possible to recover the General Relativity, in fact, if $c_3 \gg c_2 + 6k_1$ and $c_2 = 0$ as it has already been shown, we recover the Einstein Theory.

Second Case: $c_2 + 6k_1 = 0$

It corresponds to:

$$-R\frac{\partial A(R)}{\partial R}c_2 = -c_3\frac{\partial B(\phi)}{\partial \phi}, \tag{5.60}$$

where the variables are clearly divided, so we can apply the well-known procedure, equalling both sidedes to a constant m:

$$\begin{cases} R\frac{\partial A(R)}{\partial R}c_2 = m \\ c_3\frac{\partial B(\phi)}{\partial \phi} = m \end{cases} \implies \begin{cases} A(R) = \frac{m}{c_2}\ln\Big(\frac{R}{R_0}\Big) + A_0 \\ B(\phi) = \frac{m}{c_3}\phi + B_0 \end{cases} \tag{5.61}$$

This leads us to the sixth solution of the system (5.25):

$$\alpha(a) = -\frac{c_2}{6}a; \quad \beta(R) = -Rc_2; \quad \gamma(\phi) = c_3; \quad \delta(\chi) = c_2\chi; \quad \xi(t) = \frac{c_2}{2}t + c_4;$$

$$F(R,\phi) = \frac{m}{c_2}\ln\left(\frac{R}{R_0}\right) + \frac{m}{c_3}\phi + F_0. \tag{5.62}$$

As well as in the previous cases the General Relativity could be obtained if we consider $c_3 \gg m$ and if we expand the natural logarithm at the first order around R_0. The constant term also, in that case, can be interpreted as the cosmological constant term.

Third case: $6k_1 + 2c_2 = 0$

By considering: $3k_1 + c_2 = 0$. A condition that leads to an important consequence:

$$\xi(t) = c_4 \Longrightarrow \frac{\partial\xi(t)}{\partial t} = 0 \tag{5.63}$$

which, as it is possible to see from the equation (5.14),let us to recover the classical Noether Theorem without the prolongation, namely:

$$L_X\mathcal{L} = 0 \quad or \quad X\mathcal{L} = 0 \tag{5.64}$$

However we get the following equation:

$$A(R)c_2 = -Bc_2 + c_3\frac{\partial B(\phi)}{\partial\phi}, \tag{5.65}$$

where is clear the dependence on ϕ of the right term and the the dependence on R of the left term. So it is possible to equal both to a constant m:

$$\begin{cases} A(R)c_2 = m \\ \frac{\partial B(\phi)}{\partial\phi}c_3 - B(\phi)c_2 = m \end{cases} \tag{5.66}$$

which admits the following solution:

$$\begin{cases} A(R) = \frac{m}{c_2} \\ B(\phi) = B_0 e^{\frac{c_3}{c_2}} - \frac{m}{c_2} \end{cases} \tag{5.67}$$

Summing up, we obtain the seventh solution of the system (5.25):

$$\alpha(a) = -\frac{c_2}{3}a; \quad \beta(R) = 0; \quad \gamma(\phi) = c_3; \quad \delta(\chi) = c_2\chi;$$

$$\xi(t) = c_4;$$
$$F(R,\phi) = F_0 e^{\frac{c_2}{c_3}\phi}. \tag{5.68}$$

Fourth case: $c_3 = 0$

Let us analyse the last solution with equation:

$$A(R)(c_2+6k_1) - R\frac{\partial A(R)}{\partial R}(2c_2+6k_1) = -B(\phi)(c_2+6k_1). \tag{5.69}$$

We can equal each term to a constant m because as before, the right term depends only on ϕ and the left term only on R:

$$\begin{cases} A(R)(c_2+6k_1) - R\frac{\partial A(R)}{\partial R}(2c_2+6k_1) = m \\ -B(\phi)(c_2+6k_1) = m \end{cases} \tag{5.70}$$

The first one has already been studied in the first case of function made of sum of $f(\phi)$ and $f(R)$ and so we get the following solution:

$$\begin{cases} A(R) = A_0 R^{\frac{c_2+6k_1}{2c_2+6k_1}} + \frac{m}{c_2+6k_1} \\ B(\phi) = -\frac{m}{c_2+6k_1} \end{cases} \tag{5.71}$$

Finally, we get the eighth and the last solution of the system (5.25):

$$\alpha(a) = k_1 a; \quad \beta(R) = -R(6k_1+2c_2); \quad \gamma(\phi) = 0; \quad \delta(\chi) = c_2\chi;$$
$$\xi(t) = (3k_1+c_2)t + c_4;$$
$$F(R,\phi) = F_0 R^{\frac{6k_1+c_2}{6k_1+2c_2}}. \tag{5.72}$$

It is straightforward to verify for $c_2 = 0$ we recover the General Relativity.

CHAPTER 6

THE COSMOLOGICAL EQUATIONS

In this last chapter, we want to analyse the cosmological equations related to our model of gravity and also to specialize the obtained equations, on the functions previously determined. In particular as, we are going to show, these models are capable to reproduce the de-Sitter solution without having to introduce the cosmological constant and also two scalar fields that in this sense they cover the role of inflation field.

6.1 Non-Local Cosmology from Noether Symmetries

It is now possible to determine the cosmological equation in general without the necessity of focusing on the function related to Noether Symmetries Approach.

First of all, we have to recover the point-like Lagrangian (5.12):

$$\mathcal{L}(a, R, \phi, \chi . \dot{a}, \dot{R}, \dot{\phi}, \dot{\chi}) = 6a^2\dot{a}\dot{R}\frac{\partial^2 F(R,\phi)}{\partial R^2} + 6a^2\dot{a}\dot{\phi}\frac{\partial^2 F(R,\phi)}{\partial R \partial \phi} + \\ + 6a\dot{a}^2\frac{\partial F(R,\phi)}{\partial R} + a^3 F(R,\phi) - a^3 R\frac{\partial F(R,\phi)}{\partial R} - 6a^2\dot{a}\dot{\chi} - 6a\chi\dot{a}^2 - a^3\dot{\chi}\dot{\phi},$$

and the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right) - \left(\frac{\partial \mathcal{L}}{\partial q^i}\right) = 0, \tag{6.1}$$

where $q^i = \{a, R, \phi, \chi\}$ and also the energy condition is:

$$E_{\mathcal{L}} = \dot{q}^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \mathcal{L} = 0, \tag{6.2}$$

Now with these relations, we are capable to determine the cosmological equation, that are:

$$\frac{1}{2}F(R,\phi) + \frac{1}{2}\dot{\chi}\dot{\phi} + \left[3H^2 + 3H\frac{d}{dt}\right]\chi + \left[3\left(\dot{H} + H^2\right) - 3H\frac{d}{dt}\right]\frac{\partial F(R,\phi)}{\partial R} = 0, \tag{6.3}$$

$$\frac{1}{2}F(R,\phi) - \frac{1}{2}\dot{\chi}\dot{\phi} + \left(2\dot{H} + 3H^2\right)\chi + \left(\dot{H} + 3H^2\right)\frac{\partial F(R,\phi)}{\partial R} + \\ + \left[\frac{d^2}{dt^2} + 3H\frac{d}{dt}\right]\left(\chi - \frac{\partial F(R,\phi)}{\partial R}\right) = 0, \tag{6.4}$$

$$\ddot{\phi} + 3H\dot{\phi} + 6\dot{H} + 12H^2 = 0, \tag{6.5}$$

$$\ddot{\chi} + 3H\dot{\chi} + \frac{\partial F(R,\phi)}{\partial \phi} = 0, \tag{6.6}$$

$$R = -6\left(\dot{H} + 2H^2\right). \tag{6.7}$$

The first one is related to the Energy condition and the others come from the Euler-Lagrange equation. In all of them, the following relations have been used :

$$H(t) = \frac{\dot{a}}{a}, \tag{6.8}$$

$$\frac{\ddot{a}}{a} = \dot{H} + H^2, \tag{6.9}$$

$$\Box = \frac{d^2}{dt^2} + 3H\frac{d}{dt}. \tag{6.10}$$

As it is possible to see, the equation for the scalar field ϕ (6.5) it only depends on the choice of $H(t)$ and it does not depends on what kind of function $F(R,\phi)$ we are considering. Such equation can be compared with the Guth Model[52] of inflation:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\phi,T)}{\partial \phi} = 0, \tag{6.11}$$

where the potential $V(\phi, T)$ is, in our case:

$$V(\phi, T) = \phi\left(6\dot{H} + 12H^2\right), \tag{6.12}$$

Similar considerations hold also for the scalar field χ that exactly satisfies a Guth Model of inflation.

Beyond this brief parenthesis, we want to show that these models admit a de-Sitter solution. In fact, we can consider an exponential expansion rate, namely:

$$a(t) = a_0 e^{\Lambda t}, \tag{6.13}$$

so

$$H(t) = \Lambda \tag{6.14}$$

and

$$R(t) = -12\Lambda^2. \tag{6.15}$$

The equation (6.5), hence becomes:

$$\ddot{\phi} + 3\Lambda\dot{\phi} + 12\Lambda^2 = 0, \tag{6.16}$$

that has this general solution:

$$\phi(t) = -\frac{c_1 e^{-3\Lambda t}}{3\Lambda} + c_2 - 4\Lambda t. \tag{6.17}$$

Let us recover the most general $F(R, \phi)$ in the first product case (5.34) that could be rewritten as:

$$F(R, \phi) = F_0 R^K e^{-C\phi}, \tag{6.18}$$

where K and C are arbitrary constant. This choice lead us to the following form of the equation (6.5):

$$\ddot{\chi} + 3\Lambda\dot{\chi} - Qe^{-C\phi} = 0, \tag{6.19}$$

where Q is only a constant which derive from derivation. The system formed by the Euler-Lagrangian equations and the energy condition, has an analytic solution only for certain coefficient values, so we select

$$\phi(t) = c_2 - 4\Lambda t, \tag{6.20}$$

where $c_1 = 0$. This particular case allows us to obtain a simple solution for the scalar field χ:

$$\chi(t) = \frac{3^K 4^{K-1}(-\Lambda)^K \Lambda^{K-2} e^{4\Lambda t - Cc_2}}{4C + 3} - \frac{c_3 e^{-3\Lambda t}}{3\Lambda} + c_4, \tag{6.21}$$

We are now capable to write the final system:

$$\begin{cases} 36c_4\Lambda^2 e^{Cc_2+(3-4C)\Lambda t} + (4C+3)12^K((4C-1)K+3)\left(-\Lambda^2\right)^K e^{3\Lambda t} = 0 \\ 12c_4\Lambda^2 e^{Cc_2+(3-4C)\Lambda t} + 12^K((4C-1)\Lambda+3)\left(-\Lambda^2\right)^K e^{3\Lambda t} = 0, \end{cases} \tag{6.22}$$

whose solution rises behind the condition:

$$\begin{cases} c_4 = 0 \\ 3 - K + 4CK = 0 \end{cases} . \tag{6.23}$$

and finally leads us to:

$$K = \frac{3}{1-4C}. \tag{6.24}$$

Summing up we can reproduce a de-Sitter solution, considering:

$$F(R,\phi) = R^K e^{-C\phi}, \tag{6.25}$$

$$\phi(t) = c_2 - 4\Lambda t, \tag{6.26}$$

$$\chi(t) = \chi_0 e^{4\Lambda t} - \frac{c_3 e^{-3\Lambda t}}{3\Lambda}, \tag{6.27}$$

where K and C are not independent, as the relation (6.24) shows. The constant χ_0 sums up the other constant in χ.

We can obtain a solution only for de-Sitter in the product case, in fact, in the sum case we do not have a solution for the cosmological equation. Following the same procedure that we have shown, is also possible to investigate a power low solution:

$$a(t) = a_0 t^m \tag{6.28}$$

but it has not a solution both in the product case and in the sum case.

Concluding we can stress that the existence of Noether's symmetries is a selection criterion for physically motivated models. In-fact, up to now, scalar fields, dark matter and dark energy contributions, were choose by hand in order to solve the cosmological and astrophysical issues. According to non-local theories, above involved, such quantities comes naturally out.

CHAPTER 7

CONCLUSIONS AND PERSPECTIVES

In this thesis, we have showed the theoretical foundations of the Extended Theories of Gravity, developed in order to overcome the shortcomings and the inconsistencies of General Relativity. These issues appear at the infrared and ultraviolet scales, respectively, on one side in the Cosmological and Astrophysical sectors and on the other side at the quantum level. Anyway, these theories need to satisfy two requirements:

1. recover the well-know results of the General Relativity, in other words they have to reproduce GR as a limit.

2. tfollow the geometrical approach, developed by Hilbert, without including the dark sector (Dark Matter and Dark Energy) of the cosmic pie; in-fact there is not a final proof of their existence.

Recently, data coming from astrophysical and cosmological observations result of extremely high quality and lead to the so-called Precision Cosmology. This new era of research is bringing to extend accurate methods of investigations, appropriate for experimental physics, also to cosmology and, generally, to astrophysics. From Precision Cosmology, a picture emerges in some way surprising, according to which also the relatively near (at small redshifts) universe results very different to the representation we had of it until nineties.

In synthesis, the universe can be represented as a spatially flat manifold with an ordinary matter-energy content (baryonic matter and radiation) well below the critical value necessary to obtain flatness from the Einstein-Friedmann equations. Furthermore, cosmological standard candles, used as distance indicators, suggest an accelerated expansion phase, hardly obtainable once we consider ordinary fluids as the source of cosmological equations. Specifically, discrepancy between the observed luminous matter and the critical density, needed to obtain a spatially flat universe and then to give rise to the accelerated expansion, can be only filled if one admits the existence of a cosmic fluid, with negative pressure, which does not result clustered as in the large scale structure. In the simplest scenario, this mysterious ingredient, known as dark energy, can be represented as the Einstein cosmological constant Λ and would account for about 70% to the global energy of the Universe. The remaining 30%, instead located in galaxies and clusters of galaxies, should be constituted for about 4% by baryons and for the rest by cold dark matter (CDM), theoretically describable through WIMPs (Weak Interacting Massive Particles) or axions. This cosmological model, the so-called ΛCDM represents a first step toward a new cosmology.

Beside the concordance with the observations, this model present some shortcoming in particular relative to the cosmological constant. If the cosmological constant constitutes the vacuum state of the gravitational field, we have to explain the 120 orders of magnitude between the observed value at cosmological level and the one predicted by any quantum gravity theory[11].

Moreover, there is the un-solved problem of the cosmic coincidence, for which the dark energy becomes relevant and comparable to the matter term (dark and baryonic) in too recent epochs, in fact they are of the same order of magnitude, being for the cosmological evolution, 30% and 70% very similar numbers.

A first attempt to generalize GR was the Brans-Dicke model, a scalar-tensor theory of gravity, developed in order to include the Mach's Principle in which the gravitational constant G is non longer supposes to be constant but it depends on ϕ scalar field.

The Einstein-Hilbert approach, that is linear in R, was extended including high order term in the Ricci scalar, like the Starobinsky model that is quadratic in R. It is possible to naturally generalize the EH action using a generic function of the Ricci scalar, the so-called $f(R)$ gravity. This model, as has been showed, do not require to find candidates for dark energy and dark matter and it exhibit inflationary behaviour.

An alternative approach to gravity based on torsion was presented, called the teleparallel approach that lead to the same conclusion of General Relativity without the necessity of introducing the equivalence principle among inertial masses and gravitational masses. Moreover, as in $f(R)$ models, extended theory of teleparallel gravity has been developed.

The main subject of my master thesis is the study of the non-local theory of gravity and in particular it is a generalization, in Ricci formalism, of Woodard-Deser action. In these theories the non locality is reproduced by the inverse of the box operator of the Ricci scalar, $\phi = \Box^{-1}R$. This modification of Einstein gravity was developed, not only in order to solve the astrophysical and cosmological shortcomings but since the non-locality is a intrinsic and exclusive feature of the quantum world, also as a probable link to a quantum theory of gravity.

In particular in this work we have studied a Lagrangian which involve a generic function of Ricci scalar and of the non-local operator, namely $f(R, \Box^{-}R)$. This choice is done because the Woodard-Deser theory is linear in R and also for overcoming the product relation between the R and the distortion function $f(\phi)$.

The aim was to select effective theories of gravity showing Noether symmetries with physical meaning capable of allowing the exact solution of related dynamical system. The approach, already developed in other contests, is particularly relevant in view of regularization and renormalization of theories of gravity. The so called Noether Symmetries Approach in this perspective allows us to consider reliable astrophysical and cosmological models whose point-like Lagrangians give rise to integrable dynamical system. Selecting first integrals of motion was the second part of the project so that exact cosmological solution can be achieved.

It is important to once more stress that this approach allow us to reproduced both the dark sector of the cosmic pie, i.e the dark matter and dark energy contribution and two scalar field compatible with the Guth inflation model, only *via* geometrical modification of the Einstein-Hilbert lagrangian. The hope is to find a physical interpretation of the obtained results in view of selecting reliable models capable, in principle, of addressing the behaviours of gravitational field at ultraviolet (quantum) and infrared(astrophysical, cosmological) scales.

APPENDIX A THE NOETHER SYMMETRIES APPROACH

A.1 Intrinsic formulation of the Euler-Lagrange Equations

The Euler-Lagrange equations are widely described in any book of Analytic Mechanics and for such motivation, we will not give a complete treatment about the argument but we will present an uncommon formulation. First of all, we need to introduce the Lagrangian function:

$$\mathcal{L} = T - U, \tag{A.1}$$

where U is the potential term that is at most a linear function of the velocity:

$$U(q, \dot{q}) = V(q) + A_i(q)\dot{q}^i, \tag{A.2}$$

such requirement comes from the fact that the generalized forces are not supposed to be dependent on the accelerations. Then the T term is the kinetic energy:

$$T = \frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j \tag{A.3}$$

a Lagrangian it said to be regular if:

$$det||a_{ij}|| = det\left|\left|\frac{\partial^2\mathcal{L}}{\partial\dot{q}^i\partial\dot{q}^j}\right|\right| \neq 0. \tag{A.4}$$

Finally, we introduce the Euler-Lagrangian Equations in the classical formalism:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^i}\right) - \left(\frac{\partial\mathcal{L}}{\partial q^i}\right) = 0. \tag{A.5}$$

In order to get the intrinsic formulation let us consider a point transformation on $\mathcal{TQ} = \{q^i, v^i\}$:

$$\begin{cases} q^i \mapsto Q^i(q) \\ v^i \mapsto V^i(q) = \frac{\partial Q^i}{\partial q^k} v^k \end{cases} \tag{A.6}$$

that transforms the Lagrangian:

$$\mathcal{L}(q, v) \mapsto \tilde{\mathcal{L}}(Q, V). \tag{A.7}$$

Now we want to demonstrate that the Euler Lagrange equations transform like a covector under point-transformation:

$$\frac{\partial \tilde{\mathcal{L}}}{\partial Q^i} = \frac{\partial \mathcal{L}}{\partial q^k}\frac{\partial q^k}{\partial Q^i} + \frac{\partial \mathcal{L}}{\partial v^k}\frac{\partial v^k}{\partial q^j}\frac{\partial q^j}{\partial Q^i}, \tag{A.8}$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial V^i} = \frac{\partial \mathcal{L}}{\partial v^k}\frac{\partial v^k}{\partial V^i}. \tag{A.9}$$

The (A.6) implies that:

$$\frac{\partial v^k}{\partial V^i} = \frac{\partial q}{\partial Q^i} \tag{A.10}$$

and so:

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial V^i} =& \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v^k}\frac{\partial v^k}{\partial V^i}\right) \\ &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v^k}\frac{\partial q^k}{\partial Q^i}\right) \\ &= \frac{d}{dt}\left(\frac{\partial\ \mathcal{L}}{\partial v^k}\right)\frac{\partial v^k}{\partial Q^i} + \frac{\partial \mathcal{L}}{\partial v^k}\frac{d}{dt}\frac{\partial v^k}{\partial Q^i} \\ &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v^k}\right)\frac{\partial v^k}{\partial Q^i} + \frac{\partial \mathcal{L}}{\partial v^k}\frac{\partial v^k}{\partial q^j}\frac{\partial q^j}{\partial Q^i} \end{aligned} \tag{A.11}$$

By considering the difference between (A.11) and (A.9) we obtain:

$$\frac{d}{dt}\left(\frac{\partial \tilde{\mathcal{L}}}{\partial V^i}\right) - \left(\frac{\partial \mathcal{L}}{\partial Q^i}\right) = \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v^i} - \frac{\partial \mathcal{L}}{\partial q^i}\right)\frac{\partial q^k}{\partial Q^i} \tag{A.12}$$

This suggests that the Euler-Lagrange can be multiplied by dq^i and considering the vanishing on the trajectories Γ of the one-form[1]:

$$\left(L_\Gamma\left(\frac{\partial \mathcal{L}}{\partial v^i}\right) - \frac{\partial \mathcal{L}}{\partial q^i}\right)dq^i = 0, \tag{A.13}$$

[1] where has been used L for Lie derivative and so it does not stand for the Lagrangian Function

as a condition equivalent to Euler-Lagrange thinking to associate the Lie derivative respect to Γ with the time derivative. A little algebra shows that:

$$\begin{aligned}\left(L_\Gamma\left(\frac{\partial\mathcal{L}}{\partial v^i}\right)-\frac{\partial\mathcal{L}}{\partial q^i}\right)dq^i =& L_\Gamma\left(\frac{\partial\mathcal{L}}{\partial v^i}\right)dq^i-\frac{\partial L}{\partial q^i}dq^i\\ &= L_\Gamma\left(\frac{\partial\mathcal{L}}{\partial v^i}dq^i\right)-\frac{\partial\mathcal{L}}{\partial v^i}L_\Gamma dq^i-\frac{\partial\mathcal{L}}{\partial q^i}dq^i\\ &= L_\Gamma\left(\frac{\partial\mathcal{L}}{\partial v^i}dq^i\right)-\frac{\partial\mathcal{L}}{\partial v^i}d(L_\Gamma q^i)-\frac{\partial\mathcal{L}}{\partial q^i}dq^i\\ &= L_\Gamma\left(\frac{\partial\mathcal{L}}{\partial v^i}dq^i\right)-d\mathcal{L}=0\end{aligned} \tag{A.14}$$

Now we can define the one form:

$$\theta_{\mathcal{L}}\equiv\frac{\partial\mathcal{L}}{\partial v^i}dq^i \tag{A.15}$$

and finally write the **intrinsic formulation of Euler-Lagrange equations**:

$$L_\Gamma\theta_{\mathcal{L}}-d\mathcal{L}=0 \tag{A.16}$$

A.2 Noether Theorem: Coordinate Formulation

In this and in the following section we discuss the connection between symmetries and constant of motion within the Lagrangian Formalism. This connection is provided by a fundamental theorem known as Noether Theorem.

We start defying the Cyclic Variables as a variable q^i such that the Lagrangian $\mathcal{L}$ does not depend on it:

$$\frac{\partial\mathcal{L}}{\partial q^i}=0. \tag{A.17}$$

This is equivalent to say that:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial v^i}=0. \tag{A.18}$$

By defining the **conjugate momentum**:

$$p_i\equiv\frac{\partial\mathcal{L}}{\partial v^i}. \tag{A.19}$$

The equation (A.18) tells us that the conjugate momenta relative to the cyclic variable are conserved quantities along with the motion. The generalization of the idea to find conserved quantities is provided by the Noether Theorem that joins the symmetries of the Lagrangian with constant of motion.

Theorem 1. ***(Noether Theorem):*** *Let*

$$X = a^i \frac{\partial}{\partial q^i} + \dot{a}^i \frac{\partial}{\partial v^i} \tag{A.20}$$

be the infinitesimal generator of the transformation on TQ

$$\begin{cases} q^i \mapsto q^i + \epsilon a^i \\ v^i \mapsto v^i + \epsilon \dot{a}^i \end{cases} \tag{A.21}$$

If the previous (A.21) *transformation is a symmetry for the Lagrangian ($X\mathcal{L} = 0$) then the quantity:*

$$\Sigma_0 = a^i \frac{\partial \mathcal{L}}{\partial v^i} \tag{A.22}$$

is a constant of motion for the dynamical system

Proof. Considering the Euler-Lagrange equations and multiplying them by a^i:

$$\left(\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v^i}\right) - \frac{\partial \mathcal{L}}{\partial q^i}\right) a^i = 0 \tag{A.23}$$

By using the Leibniz rule on the time derivatives, we get:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v^i} a^i\right) - \dot{a}^i \frac{\partial \mathcal{L}}{\partial v^i} - a^i \frac{\partial \mathcal{L}}{\partial q^i} = 0. \tag{A.24}$$

The second and the third term is the application of the generator X of the transformation and so:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v^i} a^i\right) = X\mathcal{L} \tag{A.25}$$

but for hypothesis the second part vanishes because $\mathcal{L}$ is invariant under the previous mentioned transformation and finally:

$$\frac{d}{dt}\Sigma_0 = 0 \longrightarrow \Sigma_0 = \mathit{constant}. \tag{A.26}$$

This theorem could be generalized to the **Quasi-Invariant Lagrangians** or rather when the application of the generator X of the transformation gives rise a total time derivative of a function $F = F(t)$, namely:

$$X\mathcal{L} = \frac{dF}{dt}. \tag{A.27}$$

Theorem 2. ***(Generalized Noether Theorem):*** *Let consider a dynamical system whose Lagrangian $\mathcal{L}$ is quasi-invariant under the group transformation* (A.21)*:*

$$\Sigma_0 = a^i \frac{\partial \mathcal{L}}{\partial v^i} - F \tag{A.28}$$

is a constant of the motion.

*Proof.*To prove this it is enough to go over again the previous proof.

A paradigmatic example is when the function $F(t)$ is the Lagrangian $\mathcal{L}$ itself and in the case that $a^i = v^i$ the conserved quantities is:

$$E_{\mathcal{L}} = v^i \frac{\partial \mathcal{L}}{\partial v^i} - \mathcal{L} \tag{A.29}$$

that represents the energy conservation.

A.3 Noether Theorem: Intrinsic Formalism

In the previous section, we have discussed the relation between symmetries and constant of motion and we have proved that Noether theorem into the coordinate picture. Let us elaborate it more, to obtain a coordinate independent formulation.

Let us consider a vector field $X \in \mathcal{X}(\mathcal{Q})$ that is the infinitesimal generator of a flow ϕ_t on $\mathcal{Q}$, then the tangent lift X^T of X is the infinitesimal generator of the tangent map $T\phi_t$. In local coordinates if $X = a^i(q)\frac{\partial}{\partial q^i}$, then

$$X^T = a^i(q)\frac{\partial}{\partial q^i} + \frac{\partial a^i(q)}{\partial q^j} v^j \frac{\partial}{\partial q^i} + \frac{da^i}{dt}\frac{\partial}{\partial v^i}. \tag{A.30}$$

Therefore now let go back to the intrinsic formulation of Euler-Lagrange equations (A.16) in intrinsic form and contract them with a generic vector field $X \in \mathcal{X}(\mathcal{TQ})$ that is practically the same procedure done for the coordinate formalism when we have multiplied by a^i.

$$i_X\Big(L_\Gamma \theta_{\mathcal{L}} - d\mathcal{L}\Big) = 0, \tag{A.31}$$

where considering the identity $L_X = i_x d$ on functions, we obtain:

$$i_X L_\Gamma \theta_{\mathcal{L}} - L_X \mathcal{L} = 0. \tag{A.32}$$

By using a well know proprieties of Lie derivative and inner product for any 1-form α:

$$[L_\Gamma, i_X]\alpha = i_{[\Gamma, X]}\alpha \tag{A.33}$$

By adapting this relation in our case, we get the following equation:

$$L_\Gamma i_X \theta_\mathcal{L} - i_{[\Gamma,X]}\theta_\mathcal{L} - L_X \mathcal{L} = 0, \tag{A.34}$$

Let now put our notion of a constant of the motion (A.28) into the intrinsic formulation. We have:

$$\frac{d}{dt}\Big(a^i \frac{\partial \mathcal{L}}{\partial v^i} - F\Big) = 0 \tag{A.35}$$

By considering that the derivative of a function along the dynamics is the same of its total derivative ($\frac{d}{dt} \mapsto L_\Gamma$) and writing $a^i \frac{\partial \mathcal{L}}{\partial v^i}$ like the contraction $i_X \theta_\mathcal{L}$, the relation (A.35) becomes:

$$L_\Gamma\Big(i_X \theta_\mathcal{L} - F\Big) = 0, \tag{A.36}$$

The quasi-invariant requirement in this formalism leads us to:

$$L_X \mathcal{L} = L_\Gamma F, \tag{A.37}$$

So equations (A.34) and (A.37) imply that:

$$i_{[\Gamma,X]}\theta_\mathcal{L} = 0, \tag{A.38}$$

These are the conditions that allow us to rewrite the Noether Theorem in intrinsic formalism. The condition (A.38) is automatically satisfied every time X is the tangent lift of a vector field on $\mathcal{Q}$ as is possible to see by direct calculation using the following expression:

$$X^T = a^i \frac{\partial}{\partial q^j} + (L_\Gamma a^i)\frac{\partial}{\partial v^i}, \tag{A.39}$$

$$\Gamma = v^k \frac{\partial}{\partial q^k} + F^k \frac{\partial}{\partial v^k}, \tag{A.40}$$

Finally, we can express the intrinsic formulation of Noether Theorem:

Theorem 3. ***(Noether Theorem):*** *Let X^T be a tangent lift generating an infinitesimal transformation such that:*

$$L_{X^T}\mathcal{L} = L_\Gamma F, \quad F \in \mathcal{F}(\mathcal{TQ}), \tag{A.41}$$

then to the transformation, there is associated with the constant of the motion:

$$I_{X^T} = i_{X^T}\theta_\mathcal{L} - F, \tag{A.42}$$

A.4 Prolongation of a point transformation

In this alternative approach to Noether Theorem, we consider a general *one parameter group of point transformation* involving time as well as the Lagrangian coordinates, that because in the following chapter we will discuss the Lagrangian in Friedman-Robertson-Walker metric that is precisely point-like. The invariance under that point-transformation of the Lagrangian will allow us to find *a posteriori* the form of the generator that gives rise these symmetries.

In general is possible to consider a Lagrangian that depends either on time from the Lagrangian coordinates and velocities, namely $\mathcal{L} = \mathcal{L}(t, q^i, \dot{q}^i)$, even if as we will see our Lagrangian does not explicitly depend on time. Considering an infinitesimal transformation which depends on ϵ parameter:

$$\begin{cases} t' = t + \epsilon\xi(t, q) + ... = t + \epsilon X t \\ q'^i = q^i + \epsilon\eta^i(t, q) + ... = q^i + \epsilon X q^i \end{cases} \tag{A.43}$$

where:

$$\xi(t, q) = \left.\frac{\partial t'}{\partial \epsilon}\right|_{\epsilon=0} \qquad \eta^i(t, q) = \left.\frac{\partial q'^i}{\partial \epsilon}\right|_{\epsilon=0} \tag{A.44}$$

and the generator of the infinitesimal transformation is X defined as:

$$X = \xi(t, q)\frac{\partial}{\partial t} + \eta^i(t, q)\frac{\partial}{\partial q^i}. \tag{A.45}$$

Now is possible to understand how the point transformation works on the derivatives of the Lagrangian coordinates and how it gives rise to the first, the second prolongation and so on, even if we only need of the first one prolongation.

$$\dot{q}'^i = \frac{dq'^i}{dt'} = \frac{dq^i + \epsilon d\eta^i}{dt + \epsilon d\xi} = \frac{\dot{q}^i + \epsilon\dot{\eta}^i}{1 + \epsilon\dot{\xi}} \tag{A.46}$$

Using the McLaurin series for $\frac{1}{1+\epsilon\dot{\xi}}$ and stopping it on the first order:

$$\dot{q}'^i = \dot{q}^i + \epsilon(\dot{\eta}^i - \dot{q}^i\dot{\xi}). \tag{A.47}$$

Theorem 4. ***(First Prolongation of Noether Theorem):*** *Let*

$$X = \xi(t, q)\frac{\partial}{\partial t} + \eta^i(t, q)\frac{\partial}{\partial q^i} \tag{A.48}$$

be the infinitesimal generator of the transformation (A.43) *and*

$$\mathcal{L} = \mathcal{L}(t, q^i, \dot{q}^i) \tag{A.49}$$

be a Lagrangian which describes the dynamical system. The action of this transformation leaves the Euler-Lagrangian equation invariant, if and only if there exists a function $g = g(t, q^i)$ that holds the following condition:

$$X^{[1]}\mathcal{L} + \mathcal{L}\dot{\xi} = \dot{g}(t, q^i), \tag{A.50}$$

where $X^{[1]}$ is the first prolongation of X in (A.48)*, namely:*

$$X^{[1]} = X + (\dot{\eta}^i - \dot{q}^i\dot{\xi})\frac{\partial}{\partial \dot{q}^i}. \tag{A.51}$$

So is possible to re-write the equation (A.50)*:*

$$\xi(t,q)\frac{\partial \mathcal{L}}{\partial t} + \eta^i(t,q)\frac{\partial \mathcal{L}}{\partial q^i} + (\dot{\eta}^i - \dot{q}^i\dot{\xi})\frac{\partial \mathcal{L}}{\partial \dot{q}^i} + \mathcal{L}\dot{\xi} = \dot{g}(t, q^i) \tag{A.52}$$

For any Noether symmetry there exist a function Σ_0 that is a first integral:

$$\Sigma_0 = \xi\left(\dot{q}^i\frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \mathcal{L}\right) - \eta^i\frac{\partial \mathcal{L}}{\partial q} + g \tag{A.53}$$

Poof. Let consider a transformation of the action that leaves the equation of motion invariant:

$$\int dt'\mathcal{L}'(t', q'^i, \dot{q}'^i) = \int dt[\mathcal{L}(t, q^i, \dot{q}^i) + \epsilon\dot{g}(t, q^i)], \tag{A.54}$$

where the extremes $q(t_{in})$ and $q(t_{fin})$ are fixed and the variation of *gauge function* g is zero, namely:

$$\delta g(t, q^i)\Big|_{ext} = 0 \tag{A.55}$$

that lead us to:

$$\delta\int dt'\mathcal{L}'(t', q'^i, \dot{q}'^i) = \delta\int dt\mathcal{L}(t, q^i, \dot{q}^i) \tag{A.56}$$

So we can perform the variation by using the transformation:

$$\int dt(1+\epsilon\dot{\xi})\mathcal{L}'(t', q'^i, \dot{q}'^i) = \int dt[\mathcal{L}(t, q^i, \dot{q}^i) + \epsilon\dot{g}(t, q^i)] \tag{A.57}$$

and so taking only the first order in ϵ we can replace $\mathcal{L}'$ with $\mathcal{L}$:

$$\int dt\Big[\delta\mathcal{L}(t, q^i, \dot{q}^i) + \epsilon\dot{\xi}\mathcal{L}(t, q^i, \dot{q}^i)\Big] = \epsilon\int dt\dot{g}(t, \dot{q}^i) \tag{A.58}$$

and then operating the variation of the Lagrangian inside the last equation we prove the theorem:

$$\xi(t,q)\frac{\partial \mathcal{L}}{\partial t} + \eta^i(t,q)\frac{\partial \mathcal{L}}{\partial q^i} + (\dot{\eta}^i - \dot{q}^i\dot{\xi})\frac{\partial \mathcal{L}}{\partial \dot{q}^i} + \mathcal{L}\dot{\xi} = \dot{g}(t, q^i). \tag{A.59}$$

This result will be fundamental in order to study the the Non-Local cosmological model.

APPENDIX B

THE NOETHER SYMMETRIES SYSTEM

In this Appendix we want to show for entirety the complete system related to the first prolongation of Noether Theorem:

$$
\left\{
\begin{array}{l}
-3R\frac{\partial F}{\partial R}\alpha - aR\frac{\partial^2 F}{\partial R^2}\beta + a\frac{\partial F}{\partial \phi}\gamma - aR\frac{\partial^2 F}{\partial R\partial \phi}\gamma + 3F\alpha - aR\frac{\partial \xi}{\partial t}\frac{\partial F}{\partial R} + a\frac{\partial \xi}{\partial t}F = 0 \\[3ex]
-a\delta - \alpha\chi + a\gamma\frac{\partial^2 F}{\partial R\partial \phi} + \alpha\frac{\partial F}{\partial R} + a\beta\frac{\partial^2 F}{\partial R^2} + a^2\frac{\partial^2 F}{\partial R\partial \phi}\frac{\partial \gamma}{\partial a} - a^2\frac{\partial \delta}{\partial a} + a^2\frac{\partial^2 F}{\partial R^2}\frac{\partial \beta}{\partial a} - 2a\chi\frac{\partial \alpha}{\partial a} + \\
+2a\frac{\partial F}{\partial R}\frac{\partial \alpha}{\partial a} - a\frac{\partial \xi}{\partial t}\frac{\partial F}{\partial R} + a\chi\frac{\partial \xi}{\partial t} = 0 \\[3ex]
12a\frac{\partial F}{\partial R}\frac{\partial \alpha}{\partial \chi} + 6a^2\frac{\partial^2 F}{\partial R^2}\frac{\partial \beta}{\partial \chi} + 6a^2\frac{\partial^2 F}{\partial R\partial \phi}\frac{\partial \gamma}{\partial \chi} - 12a\chi\frac{\partial \alpha}{\partial \chi} - 12a\alpha - a^3\frac{\partial \gamma}{\partial a} - 6a^2\frac{\partial \alpha}{\partial a} - 6a^2\frac{\partial \delta}{\partial \chi} + \\
+6a^2\frac{\partial \xi}{\partial t} = 0 \\[3ex]
6a^2\frac{\partial^2 F}{\partial R\partial \phi}\frac{\partial \alpha}{\partial \chi} - 6a^2\frac{\partial \alpha}{\partial \phi} - 3a^2\alpha - a^3\frac{\partial \gamma}{\partial \phi} - a^3\frac{\partial \delta}{\partial \chi} + a^3\frac{\partial \xi}{\partial t} = 0 \\[3ex]
6a^2\frac{\partial^2 F}{\partial R\partial \phi}\frac{\partial \alpha}{\partial a} + 12a\frac{\partial F}{\partial R}\frac{\partial \alpha}{\partial \phi} + 12a\frac{\partial^2 F}{\partial R\partial \phi}\alpha + 6a^2\frac{\partial^2 F}{\partial R^2}\frac{\partial \beta}{\partial \phi} + 6a^2\frac{\partial^3 F}{\partial R^2\partial \phi}\beta + 6a^2\frac{\partial^3 F}{\partial R\partial \phi^2}\gamma \\
+6a^2\frac{\partial^2 F}{\partial R\partial \phi}\frac{\partial \gamma}{\partial \phi} - 12a\chi\frac{\partial \alpha}{\partial \phi} - a^3\frac{\partial \delta}{\partial a} - 6a^2\frac{\partial \delta}{\partial \phi} - 6a^2\frac{\partial \xi}{\partial t}\frac{\partial^2 F}{\partial R\partial \phi} = 0 \\[3ex]
6a^2\frac{\partial^2 F}{\partial R^2}\frac{\partial \alpha}{\partial \chi} - 6a^2\frac{\partial \alpha}{\partial R} - a^3\frac{\partial \gamma}{\partial R} = 0 \\[3ex]
6a^2\frac{\partial^2 F}{\partial R^2}\frac{\partial \alpha}{\partial a} + 12a\frac{\partial F}{\partial R}\frac{\partial \alpha}{\partial R} + 12a\frac{\partial^2 F}{\partial R^2}\alpha + 6a^2\frac{\partial^2 F}{\partial R^2}\frac{\partial \beta}{\partial R} + 6a^2\frac{\partial^3 F}{\partial R^3}\beta + 6a^2\frac{\partial^2 F}{\partial R\partial \phi}\frac{\partial \gamma}{\partial R} + \\
+6a^2\frac{\partial^3 F}{\partial R^2\partial \phi}\gamma - 12a\chi\frac{\partial \alpha}{\partial R} - 6a^2\frac{\partial \delta}{\partial R} - 6a^2\frac{\partial \xi}{\partial t}\frac{\partial^2 F}{\partial R^2} = 0 \\[3ex]
-a^3\frac{\partial \gamma}{\partial \chi} - 6a^2\frac{\partial \alpha}{\partial \chi} = 0 \\[3ex]
6a^2\frac{\partial^2 F}{\partial R\partial \phi}\frac{\partial \alpha}{\partial \phi} - a^3\frac{\partial \delta}{\partial \phi} = 0 \\[3ex]
6a^2\frac{\partial^2 F}{\partial R^2}\frac{\partial \alpha}{\partial \phi} + 6a^2\frac{\partial^2 F}{\partial R\partial \phi}\frac{\partial \alpha}{\partial R} - a^3\frac{\partial \delta}{\partial R} = 0 \\[3ex]
6a^2\frac{\partial^2 F}{\partial R^2}\frac{\partial \alpha}{\partial R} = 0
\end{array}
\right.
$$

$$
\begin{cases}
-6a\frac{\partial F}{\partial R}\frac{\partial \xi}{\partial R}+6a\chi\frac{\partial \xi}{\partial R}-6a^2\frac{\partial \xi}{\partial a}\frac{\partial^2 F}{\partial R^2}=0\\
-6a\frac{\partial F}{\partial R}\frac{\partial \xi}{\partial \phi}+6a\chi\frac{\partial \xi}{\partial \phi}-6a^2\frac{\partial \xi}{\partial a}\frac{\partial^2 F}{\partial R\partial \phi}=0\\
-6a\frac{\partial F}{\partial R}\frac{\partial \xi}{\partial a}+6a\frac{\partial \xi}{\partial a}\chi=0\\
6a^2\frac{\partial \xi}{\partial a}-6\frac{\partial \xi}{\partial \chi}a\frac{\partial F}{\partial R}+6a\frac{\partial \xi}{\partial \chi}\chi=0\\
-6a^2\frac{\partial \xi}{\partial R}\frac{\partial^2 F}{\partial R^2}=0\\
-6a^2\frac{\partial \xi}{\partial R}\frac{\partial^2 F}{\partial R\partial \phi}-6a^2\frac{\partial \xi}{\partial \phi}\frac{\partial^2 F}{\partial R^2}=0\\
6\frac{\partial \xi}{\partial R}a^2-6a^2\frac{\partial \xi}{\partial \chi}\frac{\partial^2 F}{\partial R^2}=0\\
-6a^2\frac{\partial \xi}{\partial \phi}\frac{\partial^2 F}{\partial R\partial \phi}=0\\
-6a^2\frac{\partial \xi}{\partial \phi}-6a^2-6a^2\frac{\partial \xi}{\partial \chi}\frac{\partial^2 F}{\partial R\partial \phi}-a^3\frac{\partial \xi}{\partial a}=0\\
6a^2\frac{\partial \xi}{\partial \chi}=0\\
a^3\frac{\partial \xi}{\partial R}=0\\
a^3\frac{\partial \xi}{\partial \phi}=0\\
a^3\frac{\partial \xi}{\partial \chi}=0\\
-a^3\frac{\partial \xi}{\partial a}R\frac{\partial F}{\partial R}+a^3\frac{\partial \xi}{\partial a}F+12a\frac{\partial F}{\partial R}\frac{\partial \alpha}{\partial t}-12a\chi\frac{\partial \alpha}{\partial t}+6a^2\frac{\partial \beta}{\partial t}\frac{\partial^2 F}{\partial R^2}+6a^2\frac{\partial \gamma}{\partial t}\frac{\partial^2 F}{\partial R\partial \phi}+\\
-6a^2\frac{\partial \delta}{\partial t}=\frac{\partial g}{\partial a}\\
-a^3\frac{\partial \xi}{\partial R}R\frac{\partial F}{\partial R}+a^3\frac{\partial \xi}{\partial R}F+6a^2\frac{\partial^2 F}{\partial R^2}\frac{\partial \alpha}{\partial t}=\frac{\partial g}{\partial R}\\
-a^3\frac{\partial \xi}{\partial \phi}R\frac{\partial F}{\partial R}+a^3\frac{\partial \xi}{\partial \phi}F+6a^2\frac{\partial^2 F}{\partial R\partial \phi}\frac{\partial \alpha}{\partial t}-a^3\frac{\partial \delta}{\partial t}=\frac{\partial g}{\partial \phi}\\
-a^3\frac{\partial \xi}{\partial \chi}R\frac{\partial F}{\partial R}+a^3\frac{\partial \xi}{\partial \chi}F-6a^2\frac{\partial \alpha}{\partial t}-a^3\frac{\partial \gamma}{\partial t}=\frac{\partial g}{\partial \chi}
\end{cases}
\tag{B.1}
$$

APPENDIX C

USEFUL RELATION

In this chapter, we want to recollect some useful relation, often used in this work, about the variation of Christoffel symbol and Ricci scalar. First of all, we have to remind the simplest expression for anholonomic and torsionless connection:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda k} \Big[g_{\mu k,\nu} + g_{\nu k,\mu} - g_{\mu\nu,k} \Big] \tag{C.1}$$

where has been used the usual symbol "," for partial derivatives. Then is possible to determinate the variation of this connection.

$$\begin{aligned} \delta\Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2} g^{\lambda k} \Big[g_{\mu k,\nu} + g_{\nu k,\mu} - g_{\mu\nu,k} \Big] = \\ &= \frac{1}{2} \delta g^{\lambda k} \Big[g_{\mu k,\nu} + g_{\nu k,\mu} - g_{\mu\nu,k} \Big] + \frac{1}{2} g^{\lambda k} \Big[\delta g_{\mu k,\nu} + \delta g_{\nu k,\mu} - \delta g_{\mu\nu,k} \Big] = \\ &= \frac{1}{2} g^{\lambda k} \Big[\nabla_{\nu} \delta g_{\mu k} + \nabla_{\mu} \delta g_{\nu k} - \nabla_{k} g_{\mu\nu} \Big] \end{aligned} \tag{C.2}$$

It is also possible to rewrite it in another good form for our purpose.

$$\delta\Gamma^{\lambda}_{\mu\nu} = -\frac{1}{2} \Big[g_{\mu\sigma} \nabla_{\nu} \delta g^{\lambda\sigma} + g_{\nu\sigma} \nabla_{\mu} \delta g^{\lambda\sigma} - g_{\mu\sigma} g_{\nu\tau} \nabla^{\lambda} g^{\sigma\tau} \Big] \tag{C.3}$$

and also:

$$\delta\Gamma^{\lambda}_{\mu\lambda} = -\frac{1}{2} g_{\lambda\alpha} \nabla_{\mu} (\delta g^{\lambda\alpha}) \tag{C.4}$$

These relations are useful for the main result:

$$g^{\mu\nu} \delta R_{\mu\nu} = -\nabla_{\mu} \nabla_{\nu} (\delta g^{\mu\nu}) + \Box (g_{\mu\nu} \delta g^{\mu\nu}) \tag{C.5}$$

This relation is really important into extended theories of general relativity, in particular, his first application for $f(R)$ theory.

ACKNOWLEDGEMENTS

Ringrazio i miei genitori, Renato e Carolina, per il sostegno datomi negli anni. Mio fratello Andrea, per la compagnia di tutti i giorni e i miei amici per il sostegno, l'allegria e la serenità donatami. Non posso non ringraziare il mio relatore e mio correlatore, il professore Roberto Balbinot e il professore Salvatore Capozziello: persone squisite, pazienti, sempre disponibili, che mi hanno guidato, ispirato professionalmente e moralmente nella realizzazione di questo lavoro di tesi. Infine vorrei ringraziare me stesso per averci sempre creduto e per essere stato tenace, cercando di fare il meglio possibile.

DEDICATION

A Coloro Che Verranno

Davvero, vivo in tempi bui!
La parola innocente è stolta. Una fronte distesa
vuol dire insensibilità. Chi ride,
la notizia atroce
non l'ha saputa ancora.

Quali tempi sono questi, quando
discorrere d'alberi è quasi un delitto,
perché su troppe stragi comporta silenzio!
E l'uomo che ora traversa tranquillo la via
mai più potranno raggiungerlo dunque gli amici
che sono nell'affanno?

È vero: ancora mi guadagno da vivere.
Ma, credetemi, è appena un caso. Nulla
di quel che faccio m'autorizza a sfamarmi.
Per caso mi risparmiano. (Basta che il vento giri,
e sono perduto).
"Mangia e bevi!", mi dicono: "E sii contento di averne".
Ma come posso io mangiare e bere, quando
quel che mangio, a chi ha fame lo strappo, e
manca a chi ha sete il mio bicchiere d'acqua?
Eppure mangio e bevo.

Vorrei anche essere un saggio.
Nei libri antichi è scritta la saggezza:
lasciar le contese del mondo e il tempo breve
senza tema trascorrere.
Spogliarsi di violenza,
render bene per male,
non soddisfare i desideri, anzi
dimenticarli, dicono, è saggezza.
Tutto questo io non posso: davvero, vivo in tempi bui!
Nelle città venni al tempo del disordine,
quando la fame regnava.
Tra gli uomini venni al tempo delle rivolte,
e mi ribellai insieme a loro.
Così il tempo passò
che sulla terra m'era stato dato.

Il mio pane, lo mangiai tra le battaglie.
Per dormire mi stesi in mezzo agli assassini.
Feci all'amore senza badarci
e la natura la guardai con impazienza.
Così il tempo passò
che sulla terra m'era stato dato.

Al mio tempo le strade si perdevano nella palude.
La parola mi tradiva al carnefice.
Poco era in mio potere. Ma i potenti
posavano più sicuri senza di me; o lo speravo.
Così il tempo passò
che sulla terra m'era stato dato.

Le forze erano misere. La meta era molto remota.
La si poteva scorgere chiaramente, seppure anche per me
quasi inattingibile.
Così il tempo passò
che sulla terra m'era stato dato.
Voi che sarete emersi dai gorghi
dove fummo travolti
pensate

quando parlate delle nostre debolezze
anche ai tempi bui
cui voi siete scampati.

Andammo noi, più spesso cambiando paese che scarpe,
attraverso le guerre di classe, disperati
quando solo ingiustizia c'era, e nessuna rivolta.

Eppure lo sappiamo: anche l'odio contro la bassezza
stravolge il viso.
Anche l'ira per l'ingiustizia
fa roca la voce. Oh, noi
che abbiamo voluto apprestare il terreno alla gentilezza, noi non
si poté essere gentili.

Ma voi, quando sarà venuta l'ora
che all'uomo un aiuto sia l'uomo,
pensate a noi
con indulgenza.

(Bertolt Brecht)

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