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The Cauchy problem for the wave equation

Relatore:
Prof. Fausto Ferrari

Presentata da:
Francesco Benetti

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Abstract
In questo lavoro viene esposta la teoria del problema di Cauchy per l'equazione delle onde in un mezzo omogeneo e isotropo in dimensione qualunque.
I primi due capitoli sono incentrati sull'approcio classico alla soluzione del problema.
In particolare, nel primo capitolo si studia il problema in tutto lo spazio, mentre nel secondo in un dominio limitato, con condizioni al contorno.
Nel terzo capitolo viene esposta la teoria degli Spazi di Sobolev, che verrà poi applicata nel capitolo successivo, nella cosiddetta formulazione debole del problema.
L'ultimo capitolo è dedicato alle applicazioni fisiche: vengono studiate le onde elettromagnetiche e le onde gravitazionali, la cui recente scoperta ha aperto nuovi orizzonti nello studio del cosmo.

Contents

Introduction

In this work we give an introduction to the Cauchy problem for the wave equation in a homogeneous isotropic medium in an arbitrary number of dimensions.

Waves are ubiquitous in physics: let's think for instance to the electromagnetic waves, describing the propagation in space of the electromagnetic field, or sound waves. Recently, the existence of gravitational waves, predicted by the general theory of relativity, has been established experimentally by LIGO-VIRGO collaboration, opening new possibilities in the observation of the universe.
This fact confirms what it is well known since long time, about the fundamental importance of discovering and developing new mathematical tools for studying ondulatory phenomena. In particular, how waves originate from a source and how they evolve, starting from a given initial configuration. This is achieved by solving the Cauchy problem for the well known equation describing the propagation of waves. In this problem we look for functions of class C^2 which satisfy the wave equation in some open domain and agree with given initial data at $t = 0$. In the case in which the domain is bounded, we can also demand that the solution of the problem satisfies some given conditions on the boundary. In this case the problem is known as *boundary value problem*.
However, during the last century, mathematicians have discovered that the classical spaces in which they looked for solutions (functions which are two times continuously differentiable in space and time in the interior of the domain and continuously differentiable on the boundary), are not the natural spaces the solutions in most cases belong to. Indeed, in many situations, if the initial data do not possess very high regularity, solutions belonging to these spaces (the now so-called *classical solutions*) does not exist. Since we are interested in physical applications, in which initial data can exhibit high irregularities, it is important to find a way to solve this problem.
It turns out that the natural spaces in which looking for solutions are the so-called *Sobolev spaces*. They are functions spaces in which a weaker notion of differentiation is defined and the solution usually has to solve an integral equation. For this reason it is costumary to say that the wave equation is satisfied in the weak sense. The study of these spaces have been carried on by Sergej L'vovič Sobolev in the thirties using tools of functional analysis, but it is mainly based on the notion of integration by parts.

We now describe in more detail the content of this thesis.

In Chapter 1 we study the *global Cauchy problem* in an arbitrary numbers of dimensions, giving the definition of a classical solution and proving its existence and uniqueness. We then obtain the famous *d'Alambert formula* for the solution in one dimensions and we employ the *spherical means method*, *method of descent* and *Duhamel's method* to obtain from that the representation formula for solution in n dimensions.
Our main resources for this chapter were [8], [6] and [4]

In Chapter 2 we restrict the attention to bounded domains, giving the definition of the boundary value problem and looking at the most common ways to assign boundary conditions in literature. We then prove existence and uniqueness of the solution and we give an introduction to the *method of separation of variables*, studying explicitely the case of waves in a ball in three dimensions.
For references, see [1], [8] and [5]

The third Chapter is entirely devoted to the theory of Sobolev spaces. We give the definition of *weak derivative* and we study its main properties. We then show how functions in Sobolev spaces can be approximated by smooth functions and we introduce the notions of *trace* and *extension*, which are fundamental for the weak formulation of boundary value problems. In the final part of the Chapter we find embeddings between different Sobolev spaces and we define *functions mapping time into Banach spaces.*
For this chapter we have followed [4]

In Chapter 4 we give the definition of a *weak solution* and we discuss the weak formulation of boundary value problems, showing that, under some assumptions on initial data, the solution exists and it is unique. We give two different approaches: the first one makes use of the *Hille-Yosida's theory* for the homogeneous problem in a general domain. The second one uses the *Galerkin's method* to explicitely construct the solution in a bounded domain.
See [3] and [4].

Finally in Chapter 5 we present two phyisical situations in which the wave equation emerges from other equations in physics. In the first part, we show how *electromagnetic waves* can be found starting from *Maxwell equations* in a particular *gauge* and we solve the wave equation, finding the *scalar and vector potentials* for a point-like charged particle with arbitrary motion.
In the second part we show that *Einstein field equations* assume the form of a wave equation in a particular gauge, predicting the existence of *gravitational waves*, which are perturbations of space-time propagating with the speed of light. We study the

interaction of these waves with matter and the emission from a source, finding explicitely the expression for waves emitted by a binary system in circular orbit.
For the contents of this chapter, see [7] and [2].

Chapter 1

Classical global problem

We present the classical approach to the solution of the *Cauchy problem* for the *wave equation* in n dimensions. The definition of the problem is given and we prove existence and uniqueness of classical solutions, under certain assumptions on the regularity of the initial data. Initially, we focus the attention on the *homogeneous global Cauchy problem*, giving the solution in one dimension and then obtaining the one for general n from that, using the *spherical means method*. The cases with n odd and with n even are studied separately and the differences exhibited by the solutions in the two cases are analyzed. We then turn to the nonhomogeneous global problem, using *Duhamel's method* to reduce it to the homogeneous one and obtaining the general solution of the problem for all n.

1.1 Global Cauchy problem

In this section we introduce the global Cauchy problem for the wave equation and we define a classical solution.

Definition 1.1. We say that a function $u(x,t)$, defined in $\mathbb{R}^n \times [0,\infty[$, is a solution of the nonhomogeneous global Cauchy problem for the wave equation, provided $u \in C^2(\mathbb{R}^n \times [0,\infty[)$ and u satisfies

$$\begin{cases} \Box u = f(x,t) \;\; in\; \mathbb{R}^n \times [0,\infty[\\ u = g, \;\; u_t = h \;\; on\; \mathbb{R}^n \times \{0\} \end{cases} \tag{1.1}$$

where $\Box := \partial_t^2 - c^2\nabla^2$ is d'Alambert operator, c is a positive constant, $g, h \,:\, \mathbb{R}^n \to \mathbb{R}$ and $f \,:\, \mathbb{R}^n \times [0,\infty[\to \mathbb{R}$ are functions, whose regularity will be discussed later, representing initial data and external force acting on the system respectively.

1.2 Well posedness

We examine the well posedness of the problem (1.1), showing that there exists at most one solution. The existence will be proven in section 1.4., where we will find explicit formulas for the solution in all dimensions. We also show that the solution represents a signal that propagates in space with finite speed and we define its domain of dependence.

Definition 1.2.1. Let $u(x,t)$ be a solution of (1.1) and let (x_0,t_0) be a point in $\mathbb{R}^n \times [0,\infty[$. Define the *backward cone* of u with vertex at (x_0,t_0) as

$$C_{(x_0,t_0)} = \{(x,t) \in \mathbb{R}^n \times [0,\infty[: |x-x_0| \le c(t_0-t)\} = \bigcup_{t=0}^{t=t_0} B(x_0, c(t_0-t)) \times \{t\} \tag{1.2}$$

where $B(x,r)$ denotes the ball in R^n with center x and radius r.

Define also the *energy*

$$E(t) := \frac{1}{2}\int_{B(x_0,c(t_0-t))} \{u_t^2 + c^2|\nabla u|^2\}dx \tag{1.3}$$

Now, let $\tilde{u}(x,t)$ be a solution of the homogeneous Cauchy problem with zero initial data, i.e. $\tilde{u}$ satisfies

$$\begin{cases} \Box\tilde{u} = 0 \;\; in\ \mathbb{R}^n \times [0,\infty[\\ \tilde{u} = 0, \;\; \tilde{u}_t = 0 \;\; on\ \mathbb{R}^n \times \{0\} \end{cases} \tag{1.4}$$

We have the following:

Lemma 1.2.1. *The energy $E(t)$ associated with $\tilde{u}$ is a decreasing function of t.*

Proof. We may write

$$E(t) = \frac{1}{2}\int_0^{c(t_0-t)} dr \int_{\partial B(x_0,r)} \{\tilde{u}_t^2 + c^2|\nabla\tilde{u}|^2\}dS \tag{1.5}$$

so that:

$$\frac{dE}{dt} = -\frac{c}{2}\int_{\partial B(x_0,c(t_0-t))} \{\tilde{u}_t^2 + c^2|\nabla\tilde{u}|^2\}dS + \int_{B(x_0,c(t_0-t))} \{\tilde{u}_t\tilde{u}_{tt} + c^2\nabla\tilde{u}\cdot\nabla\tilde{u}_t\}dx \tag{1.6}$$

The last integral can be done by parts

$$\int_{B(x_0,c(t_0-t))} \nabla\tilde{u}\cdot\nabla\tilde{u}_t dx = \int_{\partial B(x_0,c(t_0-t))} \tilde{u}_t\tilde{u}_\nu dS - \int_{B(x_0,c(t_0-t))} \tilde{u}_t\nabla^2\tilde{u}dx \tag{1.7}$$

where ν is the outward normal to the sphere $\partial B(x_0, c(t_0 - t))$, and $\tilde{u}_\nu = \nabla \tilde{u} \cdot \nu$.

$$
\begin{aligned}
\frac{dE}{dt} &= \int_{B(x_0,c(t_0-t))} \tilde{u}_t \{\tilde{u}_{tt} - c^2 \nabla^2 \tilde{u}\} dx + \frac{c}{2} \int_{\partial B(x_0,c(t_0-t))} \{2c\tilde{u}_t \tilde{u}_\nu - \tilde{u}_t^2 - c^2 |\nabla \tilde{u}|^2\} dS \\
&= \frac{c}{2} \int_{\partial B(x_0,c(t_0-t))} \{2c\tilde{u}_t \tilde{u}_\nu - \tilde{u}_t^2 - c^2 |\nabla \tilde{u}|^2\} dS
\end{aligned}
\tag{1.8}
$$

since the first integral vanishes.
Now, by the Cauchy-Schwarz inequality, $|\tilde{u}_t \tilde{u}_\nu| \leq |\tilde{u}_t||\nabla \tilde{u}|$, so that

$$
2c\tilde{u}_t \tilde{u}_\nu - \tilde{u}_t^2 - c^2 |\nabla \tilde{u}|^2 \leq 2c|\tilde{u}_t||\nabla \tilde{u}| - \tilde{u}_t^2 - c^2 |\nabla \tilde{u}|^2 = -(\tilde{u}_t - c|\nabla \tilde{u}|)^2 \leq 0 \tag{1.9}
$$

and therefore $\frac{dE}{dt} \leq 0$, as claimed. □

We immediately have:

Theorem 1.2.1. *There exists at most one solution of* (1.1).

Proof. Let $u_1(x,t)$, $u_2(x,t)$ be two solutions with the same initial data.
Then $\tilde{u} := u_1 - u_2$ is a solution of (1.4). Since $E(0) = 0$ and $E(t)$ is a decreasing function, we have $E(t) = 0 \; \forall \; t > 0$. Therefore $\tilde{u}_t$ and $|\nabla \tilde{u}|$ vanish identically for each t. This implies that $\tilde{u}$ is a constant and, since $\tilde{u}(x,0) = 0$, we obtain $\tilde{u}(x,t) = 0 \; \forall \; (x,t) \in \mathbb{R}^n \times [0, \infty[$. □

Definition 1.2.2. Fix a point (x_0, t_0). We define the domain of dependence of a solution of (1.1) as the set of points (x,t) in $\mathbb{R}^n \times [0, \infty[$ on which the value of the solution at (x_0, t_0) depends.

We also have the following:

Corollary 1.2.1. *Suppose $u, u_t, f = 0$ on $B(x_0, ct_0)$. Then $u = 0$ within the cone $C_{(x_0,t_0)}$.*

In particular, the values of the initial data g, h outside $B(x_0, ct_0)$ have no effects on the solution within $C_{(x_0,t_0)}$. Therefore, the domain of dependence of the solution at the point (x_0, t_0) is a subset of $B(x_0, ct_0)$. This shows that the solution of the wave equation is a signal propagating with finite speed and that this speed is c.

From now and until section (1.5) we focus on homogeneous problem, i.e. we set $f = 0$ in (1.1).

1.3 One dimensional problem

We solve the one-dimensional homogeneous problem changing to characteristics coordinates and we derive the famous d'Alambert formula. This will also constitute the starting point for the solution of the problem in higher dimensions.

The one-dimensional homogeneous problem reads as

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \;\; in\ \mathbb{R} \times [0, \infty[\\ u = g, \quad u_t = h \;\; on\ \mathbb{R} \times \{0\} \end{cases} \tag{1.10}$$

We note that the differential operator $\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}$ can be splitted as $(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})$. This suggests that, changing to the coordinates $\xi = x - ct,\ \ \eta = x + ct$, the so called characteristics coordinates, the wave equation takes the simpler form:

$$u_{\xi\eta}(\xi, \eta) = 0 \tag{1.11}$$

The solution now follows easily, since there exists $f \in C(\mathbb{R})$, such that $u_\xi(\xi, \eta) = f(\xi)$ and there exists also $G \in C(\mathbb{R})$, such that $u(\xi, \eta) = F(\xi) + G(\eta)$, with F a primitive of f. Going back to the original coordinates, we then find that the solution to the wave equation is given by:

$$u(x, t) = F(x - ct) + G(x + ct) \tag{1.12}$$

whith F, G arbitrary differentiable functions. It is evident that F and G represent waves propagating in the positive and negative directions respectively, since an observer moving with speed $\pm c$ sees constant F, G respectively. Imposing the initial conditions, we find:

$$u(x, 0) = F(x) + G(x) = g(x) \tag{1.13}$$
$$u_t(x, 0) = c(G'(x) - F'(x)) = h(x) \tag{1.14}$$

solved by

$$F(x) = \frac{1}{2} g(x) - \frac{1}{2c} \int^x h(y)dy \tag{1.15}$$
$$G(x) = \frac{1}{2} g(x) + \frac{1}{2c} \int^x h(y)dy \tag{1.16}$$

Therefore, u is given by:

$$u(x, t) = \frac{1}{2}\{g(x + ct) + g(x - ct)\} + \frac{1}{2c} \int_{x-ct}^{x+ct} h(y)dy \tag{1.17}$$

(1.17) is the so-called *d'Alambert formula*. We need to check that it really represents a solution of the problem.

Theorem 1.3.1. *Assume $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$ and define u by d'Alambert formula. Then*
(i) $u \in C^2(\mathbb{R} \times [0, \infty[)$
(ii) u solves the wave equation in $\mathbb{R} \times [0, \infty[$
(iii) $\forall x_0 \in \mathbb{R}$, $\lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0)$, $\lim_{(x,t)\to(x_0,0)} u_t(x,t) = h(x_0)$.

The proof is a straightforward calculation.

Remark 1.3.1. We see from d'Alambert formula that the solution at the point (x, t) depends on g at the points $x - ct$ and $x + ct$ and on h in the whole interval $[x - ct, x + ct]$. This interval is therefore the domain of dependence of the solution in one dimension.

Remark 1.3.2. It is remarkable that d'Alambert formula makes sense also for g continuous and h bounded. In this case u is only continuous and, according to our definition, it is not a classical solution of the Cauchy problem. Nevertheless, we can find a precise sense in which u can be considered a valid solution even with these requirements on the initial data. In this case u is said to be a *weak solution* of the problem. The development of this concept will be the subject of later chapters.

1.4 Spherical means method

We employ the spherical means method for multivariable functions and we obtain the *Darboux equation.*

Definition 1.4.1. Let $f \in C(\mathbb{R}^n)$. For every fixed $x \in \mathbb{R}^n$, we define the spherical mean of f, denoted as $F(x, r)$, as:

$$F(x,r) = \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} f(y) dS_y \tag{1.18}$$

where ω_n is the n-dimensional measure of the unitary sphere.
The spherical mean may be rewritten as an average over the unitary sphere centered in the origin, setting $y = x + r\xi$.

$$F(x,r) = \frac{1}{\omega_n} \int_{|\xi|=1} f(x + r\xi) dS_\xi \tag{1.19}$$

With F written in this form, the following facts follow immediately:

(i) The map $r \to F(x, r)$ can be extended for $r < 0$ and becomes an even function of the variable r.
(ii) The function f, being continuous, can be recovered from F taking the limit $r \to 0$.

(iii) If $f \in C^k(\mathbb{R}^n)$, the same is true for F, since we can differentiate under the integral sign.

In particular, let $f \in C^2(\mathbb{R}^n)$. Then we have:

$$\frac{\partial}{\partial r}F = \frac{1}{\omega_n}\int_{|\xi|=1} \nabla_x f(x+r\xi)\cdot\xi dS_\xi = \frac{r}{\omega_n}\int_{|\xi|=1} \nabla_x^2 f(x+r\xi)d\xi \tag{1.20}$$

$$= \frac{1}{\omega_n r^{n-1}}\int_{|y-x|<r} \nabla_x^2 f(y)dy = \frac{1}{r^{n-1}}\int_0^r \rho^{n-1}d\rho\nabla_x^2[\frac{1}{\omega_n\rho^{n-1}}\int_{|y-x|=\rho} f(y)dS_y] \tag{1.21}$$

$$= \frac{1}{r^{n-1}}\int_0^r \rho^{n-1}d\rho\nabla_x^2 F \tag{1.22}$$

Multiplying by r^{n-1} and differentiating with respect to r, we find that F satisfies

$$\frac{\partial^2}{\partial r^2}F + \frac{n-1}{r}\frac{\partial}{\partial r}F - \nabla_x^2 F = 0 \tag{1.23}$$

Equation (1.23) is the so-called *Darboux equation*

Now, suppose the function $u(x,t)$ is a solution of the n-dimensional wave equation, that is $u_{tt} - c^2\nabla^2 u = 0$. Forming its time dependent spherical mean, we can compute

$$\nabla_x^2 U(x,r,t) = \nabla_x^2[\frac{1}{\omega_n}\int_{|\xi|=1} u(x+r\xi,t)dS_\xi] \tag{1.24}$$

$$= \frac{1}{\omega_n}\int_{|\xi|=1} \nabla_x^2 u(x+r\xi,t)dS_\xi = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}[\frac{1}{\omega_n}\int_{|\xi|=1} u(x+r\xi,t)dS_\xi] \tag{1.25}$$

$$= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}U(x,r,t) \tag{1.26}$$

Thus, by (1.23):

$$\frac{\partial^2}{\partial t^2}U - c^2\{\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\}U = 0 \tag{1.27}$$

Equation (1.27) is known as *Euler-Poisson-Darboux equation*, depending on the number of dimensions.

Remark 1.4.1. Fix $x \in \mathbb{R}^n$. Note that the operator $\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}$ is the radial part of the Laplace operator in n dimensions. We therefore see that the spherical mean of a solution of the wave equation is again a solution. This is not surprising because, since the wave equation is invariant under rotations, starting from a solution, rotating around a point and averaging over the unit sphere, we must reach another solution.

Remark 1.4.2. Suppose $u(x,t)$ is a solution of the Cauchy problem (1.1) with $f \equiv 0$. Then it is clear that, with x fixed, its spherical mean is a solution of the same problem, where g and h are replaced by their spherical means G and H around the point x.

1.4.1 Solution of the wave equation for odd n

We now solve the homogeneous wave equation in an odd number of dimensions, turning the Euler-Poisson-Darboux equation into the one dimensional equation and then using the d'Alambert formula.

We first need the following:

Lemma 1.4.1. *Let $\phi \in C^{k+1}(\mathbb{R},\mathbb{R})$. Then, for $k = 1, 2, ...$:*

(i) $(\frac{d^2}{dr^2})(\frac{1}{r}\frac{d}{dr})^{k-1}(r^{2k-1}\phi) = (\frac{1}{r}\frac{d}{dr})^{k}(r^{2k}\frac{d\phi}{dr})$

(ii) $(\frac{1}{r}\frac{d}{dr})^{k-1}(r^{2k-1}\phi) = \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{d^j\phi}{dr^j}$

where the constants β_j^k don't depend on ϕ.
Furthermore, $\beta_0^k = 1 \cdot 3 \cdot 5 \cdots (2k-1) = (2k-1)!!$.

The proof is by induction.

Now assume $n \geq 3$ is an odd integer and set $n = 2k+1$. Let also $u \in C^{k+1}(\mathbb{R}^n \times [0,\infty[)$ be a solution of (1.1) with $f \equiv 0$ for some functions h, g.

Definition 1.4.2. Define the *modified spherical means* as

$\tilde{U}(r,t) := (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U(x,r,t))$

$\tilde{G}(r) := (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}G(x,r))$

$\tilde{H}(r) := (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}H(x,r))$

We can now prove the following fundamental result:

Lemma 1.4.2. *$\tilde{U}$ solves the one dimensional Cauchy problem*

$$\begin{cases} \tilde{U}_{tt} - c^2\tilde{U}_{rr} = 0 \quad \text{in } \mathbb{R}\times[0,\infty[\\ \tilde{U} = \tilde{G}, \quad \tilde{U}_t = \tilde{H} \quad \text{on } \mathbb{R}\times\{0\} \\ \tilde{U} = 0 \quad \text{on } \{0\}\times[0,\infty[\end{cases} \tag{1.28}$$

Proof. Applying Lemma 1.4.1. (i) and using the fact that U satisfies the Darboux equation:

$$\tilde{U}_{rr} = (\frac{\partial^2}{\partial r^2})(\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U) = (\frac{1}{r}\frac{\partial}{\partial r})^{k}(r^{2k}U_r) \tag{1.29}$$

$$= (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}[r^{2k-1}U_{rr} + 2kr^{2k-2}U_r] \tag{1.30}$$

$$= (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}[r^{2k-1}(U_{rr} + \frac{n-1}{r}U_r)] \tag{1.31}$$

$$= (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}\frac{1}{c^2}U_{tt}) = \frac{1}{c^2}\tilde{U}_{tt} \tag{1.32}$$

The initial conditions follow by continuity. Finally, applying Lemma 1.4.1. (ii), we see that $\tilde{U} = 0$ on $\{r = 0\}$. □

D'Alambert formula thus gives, for $0 \leq r \leq ct$,

$$\tilde{U}(r,t) = \frac{1}{2}\{\tilde{G}(r+ct) - \tilde{G}(ct-r)\} + \frac{1}{2c}\int_{ct-r}^{ct+r}\tilde{H}(y)dy \tag{1.33}$$

Now, recall that $u(x,t) = lim_{r\to 0}U(x,r,t)$ and that lemma 1.4.1. (ii) asserts

$$\tilde{U}(r,t) = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}U(x,r,t)) = \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{\partial^j}{\partial r^j}U(x,r,t) \tag{1.34}$$

and so

$$u(x,t) = lim_{r\to 0}\frac{\tilde{U}(r,t)}{\beta_0^k r} \tag{1.35}$$

ì(1.33) and Lemma 1.4.1 imply:

$$u(x,t) = \frac{1}{(n-2)!!}lim_{r\to 0}[\frac{\tilde{G}(r+ct) - \tilde{G}(ct-r)}{2r} + \frac{1}{2cr}\int_{ct-r}^{ct+r}\tilde{H}(y)dy] \tag{1.36}$$

$$= \frac{1}{(n-2)!!}[\frac{1}{c}\frac{\partial}{\partial t}\tilde{G}(ct) + \tilde{H}(ct)] \tag{1.37}$$

We then finally obtain the representation formula

$$u(x,t) = \frac{1}{(n-2)!!}[(\frac{\partial}{\partial t})(\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-3}{2}}(\frac{1}{\omega_n c^{n-1}t}\int_{|y-x|=ct}g(y)dS_y) \tag{1.38}$$

$$+ (\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-3}{2}}(\frac{1}{\omega_n c^{n-1}t}\int_{|y-x|=ct}h(y)dS_y)] \tag{1.39}$$

We have to check that this expression really provides a solution of our homogeneous Cauchy problem.

Theorem 1.4.1. *Assume n is an odd integer, $n \geq 3$ and suppose also $g \in C^{m+1}(\mathbb{R}^n)$, $h \in C^m(\mathbb{R}^n)$, for $m = \frac{n+1}{2}$. Define u by (1.38). Then*
(i) $u \in C^2(\mathbb{R}^n \times [0, \infty[)$
(ii) $u_{tt} - \nabla^2 u = 0$ in $\mathbb{R}^n \times [0, \infty[$
(iii) $lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0)$, $lim_{(x,t)\to(x_0,0)} u_t(x,t) = h(x_0)$ $\forall\, x_0 \in \mathbb{R}^n$.

Proof. (i) follows directly from the regularity conditions on the initial data, (ii) and (iii) can be obtained easily using lemma 1.4.1. □

Remark 1.4.3. Comparing d'Alambert formula with (1.38), we see that the latter involves the derivatives of g. This suggests that, for $n > 1$, a solution of the Cauchy problem need not for time $t > 0$ be as smooth as initial value g. Thus, irregularities in g may focus at times $t > 0$, thereby causing u to be less regular.

1.4.2 Solution for even n

We apply the *method of descent* to obtain a representation formula in even dimensions, starting from the one in odd dimensions. We then analyze the differences exhibited by the solutions.

Assume n is an even integer and suppose u is a solution of the homogeneous Cauchy problem. The trick, known as *method of descent*, consists in noticing that the function $\bar{u}(x_1, ..., x_{n+1}, t) := u(x_1, ..., x_n, t)$ solves the wave equation in $\mathbb{R}^{n+1} \times [0, \infty[$, with the initial conditions $\bar{u} = \bar{g}$, $\bar{u}_t = \bar{h}$ on $\mathbb{R}^{n+1} \times \{0\}$, with

$$\begin{cases} \bar{g}(x_1, ..., x_{n+1}) := g(x_1, ..., x_n) \\ \bar{h}(x_1, ..., x_{n+1}) := h(x_1, ..., x_n) \end{cases} \tag{1.40}$$

Now, fix $x \in \mathbb{R}^n$ and write $\bar{x} = (x_1, ..., x_n, 0) \in \mathbb{R}^{n+1}$. Then, (1.38), with $n + 1$ replacing n, gives

$$u(x,t) = \frac{1}{(n-1)!!}[(\frac{\partial}{\partial t})(\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-2}{2}}(\frac{1}{\omega_{n+1}c^n t}\int_{|\bar{y}-\bar{x}|=ct} \bar{g}(\bar{y})d\bar{S}_{\bar{y}}) \tag{1.41}$$

$$+ (\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-2}{2}}(\frac{1}{\omega_{n+1}c^n t}\int_{|\bar{y}-\bar{x}|=ct} \bar{h}(\bar{y})d\bar{S}_{\bar{y}})] \tag{1.42}$$

Note that the intersection of the n+1-sphere of radius ct and the halfspace $\{y_{n+1} \geq 0\}$ is the graph of the function $\gamma(y) = \sqrt{(ct)^2 - |y-x|^2}$ for $y \in B(x, ct) \subset \mathbb{R}^n$. Likewise, the intersection of the n+1-sphere with the halfplane $\{y_{n+1} \leq 0\}$ is the graph of $-\gamma$.

Therefore, we have:

$$\frac{1}{\omega_{n+1}(ct)^n}\int_{|\bar{y}-\bar{x}|=ct}\bar{g}(\bar{y})d\bar{S}_{\bar{y}} = \frac{2}{(n+1)\omega_{n+1}(ct)^n}\int_{|y-x|\le ct} g(y)\sqrt{1+|\nabla\gamma(y)|^2}dy \tag{1.43}$$

$$= \frac{2}{(n+1)\omega_{n+1}(ct)^{n-1}}\int_{|y-x|\le ct}\frac{g(y)}{\sqrt{(ct)^2-|y-x|^2}}dy \tag{1.44}$$

$$\tag{1.45}$$

Inserting this expression and the similar one with h in place of g into (1.38), and recalling that $\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n+2}{2})}$, we find

$$u(x,t) = \frac{1}{n!!}[(\frac{\partial}{\partial t})(\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-2}{2}}(\frac{1}{\omega_n c^{n-1}}\int_{|y-x|\le ct}\frac{g(y)}{\sqrt{(ct)^2-|y-x|^2}}dy) \tag{1.46}$$

$$+ (\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-2}{2}}(\frac{1}{\omega_n c^{n-1}}\int_{|y-x|\le ct}\frac{h(y)}{\sqrt{(ct)^2-|y-x|^2}}dy)] \tag{1.47}$$

This is the representation formula for the solution in even dimensions.

Theorem 1.4.2. *Assume n is an even integer, $n \ge 2$ and suppose also $g \in C^{m+1}(\mathbb{R}^n)$, $h \in C^m(\mathbb{R}^n)$, for $m = \frac{n+2}{2}$. Define u by (1.46). Then*
(i) $u \in C^2(\mathbb{R}^n \times [0,\infty[)$
(ii) $u_{tt} - \nabla^2 u = 0$ in $\mathbb{R}^n \times [0,\infty[$
(iii) $lim_{(x,t)\to(x_0,0)}u(x,t) = g(x_0)$, $lim_{(x,t)\to(x_0,0)}u_t(x,t) = h(x_0)$ $\forall\, x_0 \in \mathbb{R}^n$.

Proof. The proof follows directly from that of Theorem 1.4.1. for odd n. □

Looking at the representation formulas (1.38) and (1.46), we see that the most important difference between the solutions in odd and even dimensions lies in the nature of their domains of dependence. In fact, to compute the solution at the point (x_0, t_0) in an odd dimensional space, we only need to have information on g, h and their derivatives on the sphere $\partial B(x_0, ct_0) = \{x : |x - x_0| = ct_0\}$, while, in an even dimensional space, we need the values of the data in all $B(x_0, ct_0) = \{x : |x - x_0| \le ct_0\}$.
Suppose g, h have their support in a bounded open set $\Omega \subset \mathbb{R}^n$ and n is odd. In order to have $u(x,t) \ne 0$, the point x has to lie on a sphere of radius ct centered at a point $y \in \Omega$. The union of such spheres contains the support of the solution u at the time t. Therefore, the support of u spreads in space with speed c and it is bounded by the spheres with radius ct and centers in $\partial\Omega$. For example, take $\rho > 0$ and suppose $\Omega = B(0,\rho) = \{x : |x| < \rho\}$. Then $\partial B(x, ct)\bigcap B(0,\rho) \ne 0$ only when x lies in the spherical shell bounded by the spheres $\partial B(0, ct+\rho)$ and $\partial B(0, ct-\rho)$. In particular, for each fixed x and all sufficiently large t (namely $t > \frac{|x|+\rho}{c}$) we have $u(x,t) = 0$. A disturbance originating in $B(0,\rho)$ is confined, at the time t, to a shell of thickness 2ρ expanding with speed c. On the

contrary, if n is even, the initial data starts affecting the solution at the point (x,t) after a time $t_{min} = \frac{d(x,\Omega)}{c}$, but then they continue forever, since $\Omega \bigcap B(x,ct) \neq 0$ for $t > t_{min}$. This discussion shows that sharp signals can propagate only in odd dimensions, a result known as *Huygens's strong principle*. It is worth mentioning that, while the support spreads out, the solution decays in time, so that the total energy is conserved.

Setting n=3 in (1.38) and n=2 in (1.46), we obtain the solutions in three and two dimensions respectively. After carrying out the derivative with respect to t, we obtain:

$$u(x,t) = \frac{1}{4\pi c^2 t^2} \int_{|y-x|=ct} \{g(y) + \nabla g(y) \cdot (y-x) + th(y)\} dS_y \quad n = 3 \tag{1.48}$$

$$u(x,t) = \frac{1}{2\pi ct} \int_{|y-x|\leq ct} \frac{g(y) + \nabla g(y) \cdot (y-x) + th(y)}{\sqrt{(ct)^2 - |x-y|^2}} dy \quad n = 2 \tag{1.49}$$

1.5 Nonhomogeneous problem

We make use of the *Duhamel's method* to solve the nonhomegeneous Cauchy problem in all dimensions, converting it to the homogeneous one.

The Duhamel's method consists in looking at the nonhomogeneous problem as a sequence of homogeneous ones for different values of a parameter s and then integrating over s to obtain the desidered solution. We can give an intuitive motivation of the method by a physical argument: the function f appearing on the right hand side of the nonhomogeneous wave equation represents an external force acting on the system. According to Newton's law, this force changes the velocity of the solution u between two constant time hyperplane $\{t = s\}$ and $\{t = s + ds\}$ by $f(x,s)ds$. Then, in order to get the solution at time $s + ds$ from the one at time s we must add to it a new solution of the homogeneous wave equation with initial data prescriped on $\{t = s\}$, $u(x,s) = 0,\ u_t(x,s) = f(x,s)ds$. The solution of the nonhomegneous problem is obtained by adding all this solutions integrating over s from 0 to t. We now give this argument a rigorous form and we prove that the function we obtain really provides a solution of (1.1). Since we already know the solution of the homogeneous problem and the wave equation is linear, it is sufficient to consider the problem

$$\begin{cases} \Box u = f(x,t) \ \ in\ \mathbb{R}^n \times [0,\infty[\\ u = 0, \ \ u_t = 0 \ \ on\ \mathbb{R}^n \times \{0\} \end{cases} \tag{1.50}$$

For $s \leq t$, define $\tilde{u}(x,t;s)$ to be the solution of the homogeneous problem

$$\begin{cases} \Box \tilde{u} = 0 \ \ in\ \mathbb{R}^n \times [s,\infty[\\ \tilde{u} = 0, \ \ \tilde{u}_t = f(x,s) \ \ on\ \mathbb{R}^n \times \{t = s\} \end{cases} \tag{1.51}$$

Now set

$$u(x,t) := \int_0^t \tilde{u}(x,t;s)ds \tag{1.52}$$

We have the following:

Theorem 1.5.1. *Assume $n \geq 2$ and $f \in C^{[\frac{n}{2}]+1}(\mathbb{R}^n \times [0,\infty[)$, where $[\ ]$ denotes the integer part. Define u by (1.52). Then:*
(i) $u \in C^2(\mathbb{R}^n \times [0,\infty[)$
(ii) $\Box u = f(x,t)$ in $\mathbb{R}^n \times [0,\infty[$
(iii) $lim_{(x,t)\to(x_0,0)} u(x,t) = 0, \quad lim_{(x,t)\to(x_0,0)} u_t(x,t) = 0 \quad \forall\, x_0 \in \mathbb{R}^n$.

Proof. If n is odd, $[\frac{n}{2}]+1 = \frac{n+1}{2}$. According to Theorem 1.4.1., $\tilde{u}(x,t;s) \in C^2(\mathbb{R}^n \times [0,\infty[)$ for each $s \geq 0$ and so $u \in C^2(\mathbb{R}^n \times [0,\infty[)$. If n is even, $[\frac{n}{2}] + 1 = \frac{n+2}{2}$. Hence $u \in C^2(\mathbb{R}^n \times [0,\infty[)$, according to Theorem 1.4.2. We now compute:

$$u_t(x,t) = \tilde{u}(x,t;t) + \int_0^t \tilde{u}_t(x,t;s)ds = \int_0^t \tilde{u}_t(x,t;s)ds \tag{1.53}$$

$$u_{tt}(x,t) = \tilde{u}_t(x,t;t) + \int_0^t \tilde{u}_{tt}(x,t;s)ds = f(x,t) + \int_0^t \tilde{u}_{tt}(x,t;s)ds \tag{1.54}$$

On the other hand:

$$\nabla^2 u(x,t) = \int_0^t \nabla^2 \tilde{u}(x,t;s)ds = \int_0^t \tilde{u}_{tt}(x,t;s)ds \tag{1.55}$$

This shows (ii). Clearly, we also have $u(x,0) = u_t(x,0) = 0$. $\Box$

Looking more closely at the three dimensional case, we have:

$$\tilde{u}(x,t;s) = \frac{1}{4\pi c^2(t-s)} \int_{|x-y|=c(t-s)} f(y,s)dS_y \tag{1.56}$$

so that

$$u(x,t) = \frac{1}{4\pi c^2} \int_0^t \frac{ds}{t-s} \int_{|x-y|=c(t-s)} f(y,s)dS_y \tag{1.57}$$

$$= \frac{1}{4\pi} \int_{|x-y|\leq ct} \frac{f(y, t - \frac{|y-x|}{c})}{|y-x|} dy \tag{1.58}$$

We see that the solution at the point x and time t depends on the value of the external force on the set $\{y : |y-x| \leq ct\}$ at the earlier time $t' = t - \frac{|x-y|}{c}$; this shows again that the solution propagates in space with speed c. For this reason formula (1.58) is known as *retarded potential*.

Clearly the complete solution of the nonhomogeneous Cauchy problem is obtained summing (1.52) with either (1.38) or (1.46).

Chapter 2

Boundary value problem

In this chapter we study the Cauchy problem for the wave equation in a domain that is a bounded open subset of $\mathbb{R}^n$. We see that, in order the problem to be well posed, we must specify some *boundary conditions*; that is, we must impose that the desidered solution of the problem satisfy certain conditions on the boundary of its space-time domain. Cauchy problems, together with these boundary conditions, are known as *initial boundary value* problems.
We start the chapter by giving the formal definition of these kind of problems and by exposing the most common ways to assign boundary conditions. We next prove *uniqueness* of the solution, for given initial and bounday conditions.
We continue by presenting the *separation of variables method* for problems with *homogeneous boundary conditions* and we give an example of application, solving explicitly the problem for waves in the unitary ball in $\mathbb{R}^3$ with *homogeneous Dirichlet boundary conditions*.

2.1 Generalities

We expose the main features of *initial-boundary value problems* for the wave equation and the most common ways to assign *boundary conditions* for such problems.

Notations: (i) Let $U \subset \mathbb{R}^n$ be open and bounded, $k \in \mathbb{N}$.
We say ∂U is of class C^k if for each point $x_0 \in \partial U$ there exist $r > 0$ and a function $\gamma \in C^k(\mathbb{R}^{n-1}, \mathbb{R})$, such that, upon relabeling and reorienting the coordinates axes if necessary, we have:

$$U \cap B(x_0, r) = \{x \in B(x_0, r) : x_n > \gamma(x_1, ..., x_{n-1})\} \tag{2.1}$$

Likewise, ∂U is of class C^∞ if it is of class C^k for every $k \in \mathbb{N}$.
(ii) If $T > 0$, we call space-time cylinder the set $U_T := U \times [0, T]$. We also denote the hypersurface $\partial U \times [0, T]$ as ∂U_T.

(iii) Suppose f is a function defined in U_T.- We say $f \in C^{l,k}(U_T)$ meaning that f is l-times differentiable with respect to space variables and k-times differentiable with respect to time.

In what follows U will indicate a bounded open subset of $\mathbb{R}^n$ with boundary of class C^1.

Definition 2.1. We say a function $u(x,t) : \bar{U}_T \to \mathbb{R}$ is a classical solution of a initial-boundary value problem for the wave equation, provided $u \in C^{2,2}(U_T) \cap C^{1,1}(\bar{U}_T)$ and u satisfies:

$$\begin{cases} \Box u = f(x,t) \;\; in\ U_T \\ u = g, \quad u_t = h \;\; on\ U \times \{0\} \\ +\ boundary\ conditions \;\; on\ \partial U_T \end{cases} \tag{2.2}$$

where $g, h : \bar{U} \to \mathbb{R}$, $f : \bar{U}_T \to \mathbb{R}$ represent initial conditions and external forces acting on the system respectively.

The following are the most common ways to assign boundary values to a function u:

(i) *Dirichlet conditions:* The value of the solution u is directly assigned along ∂U_T; that is, we set $u(x,t) = a(x,t)$ on ∂U_T, for a given $a \in C^{1,1}(\partial U_T)$.
An example may be given by a problem in which an electromagnetic wave is reflected by a metallic surface and the component of electric field parallel to the surface is required to vanish identically.

(ii) *Newmann conditions:* The *normal derivative* of the solution along ∂U_T is assigned; that is, we set $\partial_\nu u = a$ on ∂U_T, where ν is the outward normal to ∂U_T and $\partial_\nu u = \nabla u \cdot \nu$.
For example we can study the motion of a vibrating membrane and we can assign its velocity on the boundary.
Robin conditions: A linear combination of u and its normal derivative is given along ∂U_T; that, is, we set $\alpha u + \beta \partial_\nu u = a$ on ∂U_T.

Mixed conditions The value of the solution is given on $\partial \Omega_T$, a relatively open subset of ∂U_T and its normal derivative $\partial_\nu u$ is assigned on $\partial U_T - \partial \Omega_T$.

2.2 Uniqueness of the solution

We see that also in the case of bounded domain, with appropriate boundary conditions, the Cauchy problem is still well posed. In fact, the following result holds:

Theorem 2.2.1. *The problem* (2.2)*, together with one of the boundary conditions (i)-(iv), has at most one classical solution.*

The proof follows a reasoning similar to that of Theorem 1.2.1.

Proof. Define the energy function associated with u, as:

$$E(t) = \frac{1}{2}\int_U \{u_t^2 + c^2|\nabla u|^2\}dx \tag{2.3}$$

We have:

$$\frac{dE}{dt} = \int_U \{u_t u_{tt} + c^2|\nabla u| \cdot \nabla u_t\}dx = \int_U \{u_{tt} - c^2\nabla^2 u\}u_t dx + c^2\int_{\partial U} u_\nu u_t dS \tag{2.4}$$

$$= \int_U f u_t dx + c^2\int_{\partial U} u_\nu u_t dS \tag{2.5}$$

where we have integrated by parts.

Now, let $u_1,\ u_2$ be two classical solutions of (2.2) with the same initial and boundary data. Then, by linearity, $w = u_1 - u_2$ is a solution of the homogeneous wave equation with zero initial and boundary data. We are left with:

$$\frac{dE}{dt} = c^2\int_{\partial U} w_\nu w_t dS \tag{2.6}$$

For a problem with Dirichlet boundary conditions, we have $w = 0$ on ∂U_T and therefore $w_t = 0$ on ∂U. For a problem with Newmann boundary conditions, we have $w_\nu = 0$ on ∂U_T and for a problem with mixed conditions both w_t and w_ν vanish. In all these situations we see that energy is conserved in time and, since it vanished at $t = 0$, it vanishes at any time. This further implies $w = 0$ at any time, as in Theorem 1.2.1.
For a problem with Robin boundary conditions, we have $\alpha w + \beta w_\nu = 0$ and therefore:

$$\frac{dE}{dt} = -\frac{\alpha}{\beta}c^2\int_{\partial U} w w_t dS = -\frac{\alpha}{\beta}c^2\frac{d}{dt}\int_{\partial U} w^2 dS \tag{2.7}$$

This shows that the quantity

$$E + \frac{\alpha}{\beta}c^2\int_{\partial U} w^2 dS \tag{2.8}$$

is constant in time and, since it vanishes at $t = 0$, it vanishes at any time, implying again $w = 0$ at any time. $\square$

2.3 Separation of variables

We expose the *separation of variables method* for solving problems with *homogeneous boundary conditions* and we present some examples.

Definition 2.3.1. We say a boundary value problem have homogeneous boundary conditions if the solution is required to vanish at the boundary at any times, i.e. we demand

$$u(x,t) = 0 \ \ on \ \partial\Omega_T \tag{2.9}$$

The separation of variables method for solving boundary value problems for partial differential equations with homogeneous boundary conditions, consists in looking for solutions expressible as products of functions of a single variable. We therefore looking for solutions of the form:

$$u(x_1, ..., x_n) = \prod_{i=1}^{n} u_i(x_i) \tag{2.10}$$

Substituting (2.10) into the PDE we obtain a set of n ordinary differential equations with associated boundary conditions, that can be solved using well known methods for ordinary differential equations.

We now give an example of application for the wave equation in three dimensions.

2.3.1 Waves in a ball

Consider the following homogeneous initial-boundary value problem:

$$\begin{cases} \Box u = 0 \ \ in \ B^2 \times [0, \infty] \\ u = g, \ \ u_t = h \ \ on \ U \times \{0\} \\ u = 0 \ \ on \ S^2 \times [0, \infty] \end{cases} \tag{2.11}$$

where $B^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$, $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ are the unitary ball and the unitary sphere in $\mathbb{R}^3$. We also demand the function to be regular and bounded everywhere.
It is natural to work in spherical coordinates. The wave equation thus takes the form:

$$\frac{1}{r}\frac{\partial^2(ru)}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{2.12}$$

with $r \in [0,1], \ \theta \in [0,\pi], \ \phi \in [0, 2\pi]$, and the boundary condition becomes

$$u(1, \theta, \phi, t) = 0 \tag{2.13}$$

Looking for a solution of (2.12) of the form (2.10), we set

$$u(r,\theta,\phi,t) = R(r)\Theta(\theta)\Phi(\phi)T(t) \tag{2.14}$$

Substituting in (2.12) and dividing by u, we find:

$$\frac{1}{R}\frac{1}{r}\frac{d^2(rR)}{dr^2} + \frac{1}{\Theta}\frac{1}{r^2\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + \frac{1}{\Phi}\frac{1}{r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2} - \frac{1}{T}\frac{1}{c^2}\frac{d^2T}{dt^2} = 0 \tag{2.15}$$

If we move the last term to the right hand side, we note that we have an equation where a time-independent term is set equal to a time-dependent one and therefore, in order to the equation be valid, both its sides must be equal to a constant, say $-\lambda^2$

$$\frac{1}{R}\frac{1}{r}\frac{d^2(rR)}{dr^2} + \frac{1}{\Theta}\frac{1}{r^2\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + \frac{1}{\Phi}\frac{1}{r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2} + \lambda^2 = 0 \tag{2.16}$$

$$\frac{d^2T}{dt^2} + c^2\lambda^2 T = 0 \tag{2.17}$$

Multiplying by $r^2\sin^2\theta$ in (2.16), we can separate the ϕ-dependent term and, setting the two sides of the resulting equation equal to the constant $-m^2$, we find:

$$\frac{r\sin^2\theta}{R}\frac{d^2(rR)}{dr^2} + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + \lambda^2 - m^2 = 0 \tag{2.18}$$

$$\frac{d^2T}{dt^2} + c^2\lambda^2 T = 0 \tag{2.19}$$

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 \tag{2.20}$$

With other algebraic manipulations, we can further separate the equations in r and θ, and, setting both sides equal to a constant k, we finally obtain:

$$\begin{cases} r\frac{d^2(rR)}{dr^2} + (\lambda^2 r^2 - k)R = 0 \quad r \in [0,1] \\ R(1) = 0 \end{cases} \tag{2.21}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + (k - \frac{m^2}{\sin^2\theta})\Theta = 0 \quad \theta \in [0,\pi] \tag{2.22}$$

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 \quad \phi \in [0,2\pi] \tag{2.23}$$

$$\frac{d^2T}{dt^2} + c^2\lambda^2 T = 0 \quad t > 0 \tag{2.24}$$

Equations (2.23), (2.24) are straightforward:

$$T(t) = c_+e^{ic\lambda t} + c_-e^{-ic\lambda t} \tag{2.25}$$

$$\Phi(\phi) = c_1e^{im\phi} + c_2e^{-im\phi} \tag{2.26}$$

Since ϕ is an angle, Φ must posses 2π periodicity ant this forces m to be an integer.

We can turn equations (2.21), (2.22) into standard forms by changing variables: in (2.21) let $\chi = \frac{r}{R}$ and rescale $\rho = \lambda r$; in (2.22) define $x = \cos\theta$. In terms of these new variables (2.21), (2.22) become:

$$\frac{d^2\chi}{d\rho^2} + (1 - \frac{k}{\rho^2})\chi = 0 \tag{2.27}$$

$$\frac{d}{dx}[(1-x^2)\frac{d\Theta}{dx}] + (k - \frac{m^2}{1-x^2})\Theta = 0 \tag{2.28}$$

(2.27) is known as *spherical Bessel equation* and (2.28) as *associated Legendre equation*. It is a standard result, which do not prove here, that equation (2.28) admits nonzero solutions, regular in $x \in [-1, 1]$ only if $k = l(l+1)$, with l integer, and $-l \leq m \leq l$.

Although solutions to (2.27), (2.28) are well known, we show how (2.27) can be easily solved, using the so-called *Frobenius method*.
The idea lying behind this method is to look for a solution expressed in power series, substitute it back in the equation, finding in this way relations satisfied by coefficients of terms of equal power. This will give us a recorsive relation for the coefficients.

Therefore, we consider a solution of the form

$$\chi(\rho) = \sum_{i=0}^{\infty} a_i \rho^{i+\alpha} \tag{2.29}$$

where α must be determined. Substituting back in (2.27), we obtain:

$$\sum_{i=0}^{\infty}\{(i+\alpha)(i+\alpha-1) - l(l+1)\}a_i\rho^{i+\alpha-2} + \sum_{i=2}^{\infty} a_{i-2}\rho^{i+\alpha-2} = 0 \tag{2.30}$$

For $i = 0$, (2.30) gives $\alpha = -l$, or $\alpha = l+1$. Since we want the solution to be regular in $\rho = 0$, we must have $k = l+1$. For $i = 1$, (2.30) gives $a_i = 0$ and, for $i > 1$, we see that coefficents of odd order vanish and the ones of even order are related by the following recorsive relation:

$$a_{2i} = (-1)^i \frac{(i+l)!}{2^i i!(2i+2l+1)!} a_0 \tag{2.31}$$

Going back to the old variables, we find:

$$R(r) = a_0 \sum_{i=0}^{\infty} (-1)^i \frac{(i+l)!}{2^i i!(2i+2l+1)!} (\lambda r)^{2i+l} \tag{2.32}$$

If we set $a_0 = 2^l$ we obtain:

$$R(r) = j_l(\lambda r) \tag{2.33}$$

where j_l denotes the *spherical Bessel function* of order l.
Another expression for these functions is:

$$j_l(r) = (-1)^l (\frac{d}{dr})^l \frac{\sin r}{r} \tag{2.34}$$

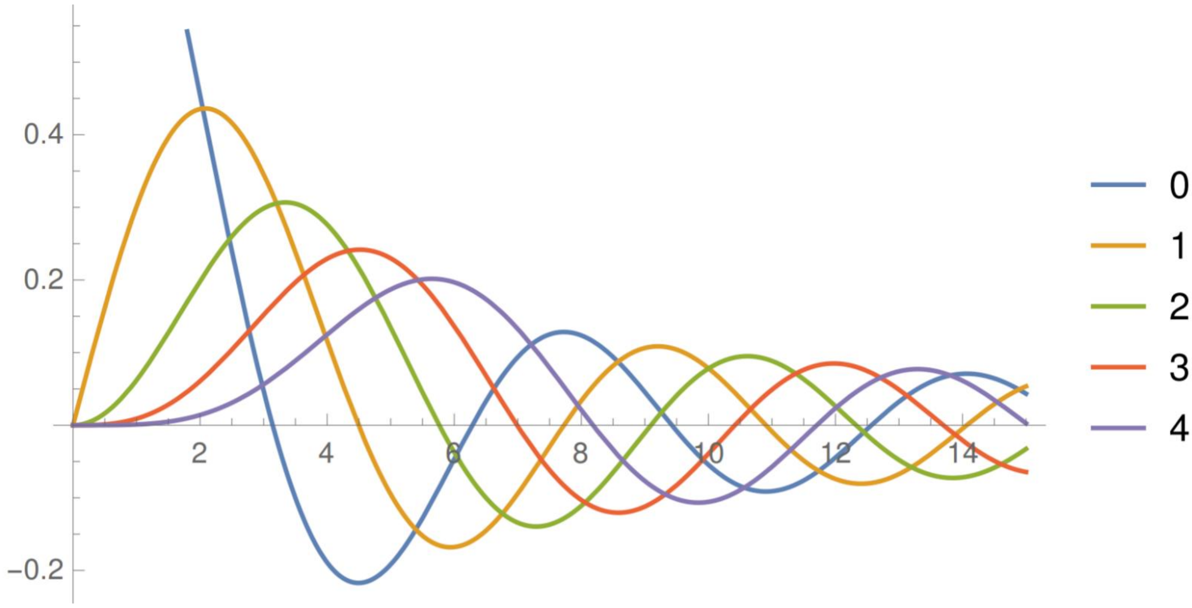

Figure 2.1: Plot of the spherical Bessel functions for lowest values of l

The condition $R(1) = 0$ limits the possible values λ can take; in fact, we must have $\lambda = z_{lk}$, where z_{lk} denotes the k-th zero of the spherical Bessel function of order l.

In a similar way, we can obtain the solutions of (2.28), with $k = l(l+1)$. These are the so-called *associated Legendre polynomials*. For $m \geq 0$ they are given by:

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{\frac{m}{2}} \frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \tag{2.35}$$

and, for $m \leq 0$,

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x) \tag{2.36}$$

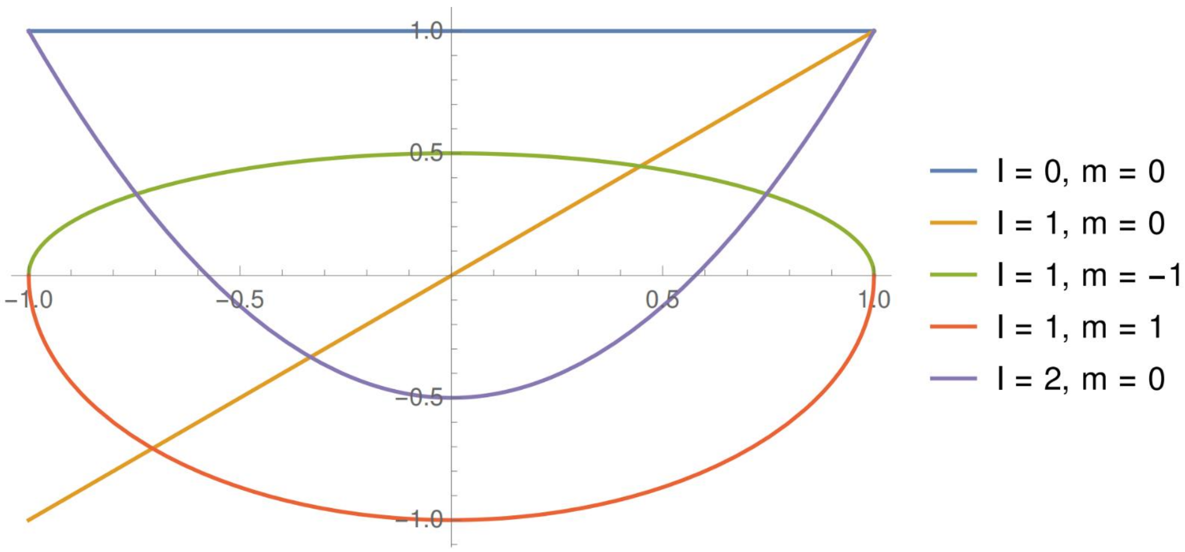

Figure 2.2: Plot of the Legendre polynomials of lower orders

Taking the product of (2.35) and (2.26), we see that the angular part of the solution, for given l, m is represented by a *spherical harmonic*.

$$Y_l^m(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi} \tag{2.37}$$

The following figure shows the squared modulus of the spherical harmonics for the lowest value of l and m.

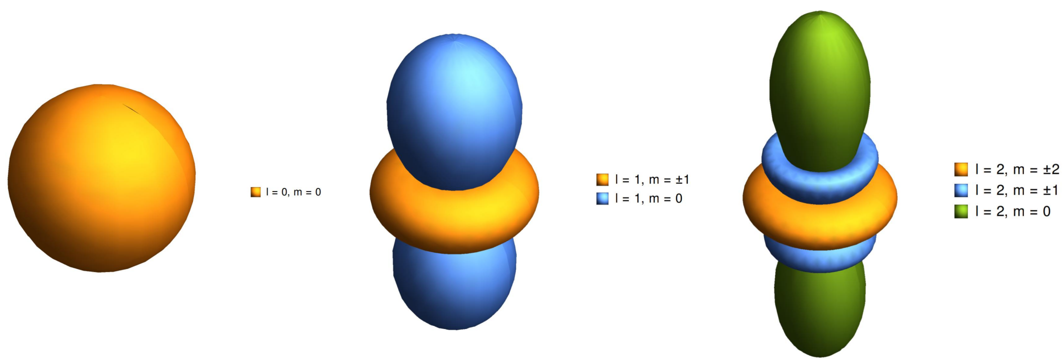

Figure 2.3: Plot of squared modulus of the spherical harmonics for the lowest value of l and m.

Therefore we see that our separable solution (2.14) can be written as:

$$u_{l,m,k}(r, \theta, \phi, t) = j_l(z_{l,k}r)Y_l^m(\theta, \phi)(A_{l,m,k}e^{icz_{l,k}t} + B_{l,m,k}e^{-icz_{l,k}t}) \tag{2.38}$$

Since the wave equation is linear, we would like to obtain the general solution to the problem superimposing solutions of the form (2.38).
The following theorem is fundamental for this purpose.

Theorem 2.3.1. *The following results hold:*

(i) The spherical harmonics are a complete set of orthogonal functions for the set $L^2(S^2)$. Therefore, they satisfy the orthogonal relation:

$$\int_{S^2} Y_l^{*m} Y_{l'}^{m'} dS = \delta_{l,l'}\delta_{m,m'} \tag{2.39}$$

Furthermore, every function $f \in L^2(S^2)$ can be written as an expansion of the form:

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \tilde{f}_l^m Y_l^m(\theta, \phi) \tag{2.40}$$

with the coefficients $\tilde{f}_l^m$ given by:

$$\tilde{f}_l^m = \int_{S^2} Y_m^{*l} f dS \tag{2.41}$$

We denote by V_l the $2l+1$-dimensional vector space spanned by the spherical harmonics with fixed l.

(ii) The spherical Bessel functions form a complete set of orthogonal functions for the set $L^2([0,1])$.
They satisfy the orthogonal relation:

$$\int_0^1 j_l(z_{l,k}r) j_l(z_{l,k'}r) r^2 dr = \frac{1}{2} j_{l+1}^2(z_{l,k})\delta_{k,k'} \tag{2.42}$$

Furthermore, every function $f \in L^2([0,1])$ can be written as an expansion of the form:

$$f(r) = \sum_{k=1}^{\infty} \tilde{f}_k j_l(z_{l,k}r) \tag{2.43}$$

with the coefficients $\tilde{f}^k$ given by:

$$\tilde{f}^k = \frac{2}{j_{l+1}^2(z_{l,k})} \int_0^1 j_l(z_{l,k}r) f r^2 dr \tag{2.44}$$

(iii) The space $L^2(B^2)$ can be decomposed as:

$$L^2(B^2) = \bigoplus_{l=0}^{\infty} L^2([0,1]) \otimes V_l \tag{2.45}$$

The time-dependent part can be treated in a similar way.
The reader can find discussions on Theorem 2.3.1. in [1] and in [5].

According to Theorem 2.3.1., the function

$$u(r,\theta,\phi,t)=\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}j_l(z_{l,k}r)Y_l^m(\theta,\phi)(A_{l,m,k}e^{icz_{l,k}t}+B_{l,m,k}e^{-icz_{l,k}t}) \tag{2.46}$$

is candidated to be the general solution of (2.11). The coefficients $A_{l,m,k}, B_{l,m,k}$ can be determined imposing that u satisfies the initial conditions

$$u=g, \quad u_t=h \quad on\ B^2\times\{0\} \tag{2.47}$$

For $t=0$, we have:

$$u(r,\theta,\phi,0)=\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}j_l(z_{l,k}r)Y_l^m(\theta,\phi)(A_{l,m,k}+B_{l,m,k}) \tag{2.48}$$

$$u_t(r,\theta,\phi,0)=\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}j_l(z_{l,k}r)Y_l^m(\theta,\phi)icz_{l,k}(A_{l,m,k}-B_{l,m,k}) \tag{2.49}$$

Since, according to Theorem 2.3.1., we can also expand the initial conditions as:

$$g(r,\theta,\phi)=\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\tilde{g}_{l,m,k}j_l(z_{l,k}r)Y_l^m(\theta,\phi) \tag{2.50}$$

$$h(r,\theta,\phi)=\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\tilde{h}_{l,m,k}j_l(z_{l,k}r)Y_l^m(\theta,\phi) \tag{2.51}$$

where

$$\tilde{g}_{l,m,k}=\frac{2}{j_{l+1}^2(z_{l,k})}\int_{B^2}j_l(z_{l,k}r)Y_l^{*m}(\theta,\phi)g(r,\theta,\phi)dS \tag{2.52}$$

and the same for h.
Comparing (2.48) and (2.50), we obtain:

$$A_{l,m,k}=\frac{1}{2}(\tilde{g}_{l,m,k}+\frac{\tilde{h}_{l,m,k}}{icz_{l,k}}) \tag{2.53}$$

$$B_{l,m,k}=\frac{1}{2}(\tilde{g}_{l,m,k}-\frac{\tilde{h}_{l,m,k}}{icz_{l,k}}) \tag{2.54}$$

Substituting in (2.46), we finally obtain:

$$u(r,\theta,\phi,t)=\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} j_l(z_{l,k}r)Y_l^m(\theta,\phi)(\tilde{g}_{l,m,k}\cos(z_{l,m,k}ct)+\frac{\tilde{h}_{l,m,k}}{cz_{l,k}}\sin(z_{l,m,k}ct)) \tag{2.55}$$

Now, the following questions emerge:

(1) Any finite linear combination of separable solutions $u_{k,l,m}$ is a solution of (2.11). Is the same true for (2.55)?. The answer would be positive if we could differentiate term by term the infinite sum. For this to be allowed, we have to prove that:
(i) The series of first and second derivates with respect to space and time converge uniformly to some functions.
(ii) For some $x_0 \in U_T$, the series (2.55) converge.
This is not an easy task and the answer depends strongly on initial data g, h.

(2) In which sense does (2.55) satisfy the initial data? For instance, is it true that

$$u(r,\theta,\phi,t) \to g(r',\theta',\phi') \quad u_t(r,\theta,\phi,t) \to h(r',\theta',\phi') \tag{2.56}$$

if $(r,\theta,\phi,t) \to (r',\theta',\phi',0)$?

(3) In which sense (2.55) satisfy the Dirichlet boundary condition? Is it true that $u(r,\theta,\phi,t) \to 0$ as $(r,\theta,\phi,t) \to (1,\theta,\phi,t)$ for each θ,ϕ,t?

Questions of this sort arises because we are looking for a classical solution, i.e. we require the solution to possess high regularity. It turns out that, when the initial data are not regular enough, a classical solution does not exist and, in general, it is a difficult task to prove that a candidate function is indeed a classical solution. This fact can be seen also in Theorem 1.4.1. and Theorem 1.4.2., where we had to require higher and higher regularity on the initial data as the number of dimensions increased, in order to obtain a classical solution to the problem.
Since we are interested in physical situations, in which the initial data are often even piecewise differentiable (think for examples at a string whose center has been pulled down in such a way that it assumes initially the form of the graph of absolute value), we need a formulation of the problem which requires a weaker notion of solution and can deal with less regular initial data.
This is achieved by demanding that initial data and solutions belong to *Sobolev spaces*, which are functions spaces in which a different notion of differentiation is defined. The solutions belonging to those spaces are called *weak solutions*.
Sobolev spaces and weak solutions will be the subjects of the two following chapters.

Chapter 3

Sobolev spaces

We recall here the basis of *Sobolev spaces* theory. We start by giving the definition of *weak derivative*, thought for dealing with functions not endowed by the classical notion of differentiation. We then define Sobolev spaces as spaces of functions whose weak derivatives have finite $L^p - norm$ up to a given order and we prove that they are Banach spaces with a very specific norm. Next we study how functions in Sobolev spaces can be approximated by smooth functions, which are dense. We then define two linear bounded operators acting on functions in the Sobolev space $W^{1,p}(U)$, where $U \subset \mathbb{R}^n$ is open and bounded: the *extension operator* extends functions in $W^{1,p}(U)$ to functions in $W^{1,p}(\mathbb{R}^n)$ and the *trace operator* solves the problem of assign values on the boundary of U to functions in $W^{1,p}(U)$.
Next we study how different Sobolev spaces are related and we establish embeddings between these spaces, finding inequalities involving their norms.
After we have defined the dual space $H^{-1}(U)$ and we have proven his basic properties, with the theory of partial differential equations in mind, we finally turn to Sobolev spaces with functions mapping time into Banach spaces and we see how these functions behaves under weak differentiation.

3.1 Weak derivative

Multiindex notation: We call *multiindex* a vector $\alpha = (\alpha_1, ..., \alpha_n)$, whose components are nonnegative integers. The sum of two multiindeces α, β is defined componentwise and we say that $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$, for every $i = 1, ..., n$. The order of a multiindex is the number $|\alpha| = \sum_{i=1}^n \alpha_i$ and the factorial is $\alpha! = \prod_{i=1}^n \alpha_i!$. We also associate with α the differential operator $D\alpha := \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} ... \partial^{\alpha_n}}$, so that the Taylor expansion of a function f can be written as $f(x) = \sum_{|\alpha|} \frac{D^\alpha f(x_0)}{\alpha!}(x - x_0)^\alpha$, where $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$.

In what follows U will indicate a bounded open subset of $\mathbb{R}^n$.

Definition 3.1.1. Let $C_c^\infty(U)$ denote the space of infinitely differentiable functions $\phi : U \to \mathbb{R}$ with compact support in U. A function ϕ belonging to $C_c^\infty(U)$ is called a *test function.*

Let $u \in C^k(U)$, with k a positive integer. Then, if $\phi \in C_c^\infty(U)$ and α is a multiindex with $|\alpha| = k$, we see, by integrating by parts:

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U D^\alpha u \phi dx \tag{3.1}$$

There are no boundary terms, since ϕ is compactly supported in U.

We next examine formula (3.1) and ask whether some variant of it might still be true even if u is not k times continuously differentiable. Now, the left hand side makes sense if u is only locally summable: the problem is rather that, if u is not C^k, then the expression $D^\alpha u$ on the right hand side has no obvious meaning. We solve this difficulty by asking if there exists a locally summable function v for which (3.1) is valid, with v formally replacing $D^\alpha u$. This leads to the definition of weak derivative.

Definition 3.1.2. Suppose $u, v \in L^1_{loc}(U)$ and α is a multiindex. We say that v is the α^{th}-weak partial derivative of u, written $D^\alpha u = v$, provided:

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx \tag{3.2}$$

for all test functions $\phi \in C_c^\infty(U)$.

Thus, the notion of weak derivative turns the definition of derivative into an integral equation that must be satisfied for all test functions. In this way the regularity requiriments a function must have to be differentiable are weaker. The following lemma shows that, if a weak derivatove exists, it is unique.

Lemma 3.1.1. *A weak α^{th} partial derivative of a function $u \in L^1_{loc}(U)$, if it exists, is uniquely defined up to a set of zero measure.*

Proof. Assume $v, \tilde{v} \in L^1_{loc}(U)$ satisfy

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx = (-1)^{|\alpha|} \int_U \tilde{v} \phi dx \tag{3.3}$$

for all $\phi \in C_c^\infty(U)$. Then $\int_U (v - \tilde{v}) \phi dx = 0$ for all $\phi \in C_c^\infty(U)$; whence $v - \tilde{v} = 0$ a.e. $\square$

We now consider spaces of functions that behave well under weak differentiation.

Fix $1 \le p \le \infty$ and let k be a nonnegative integer.

Definition 3.1.3. The Sobolev space $W^{k,p}(U)$ consists of all locally summable functions $u : U \to \mathbb{R}$ such that, for each multiindex α with $|\alpha| \le k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

Notation: If $p = 2$ we usually write $H^k(U) = W^{k,2}(U)$.

Next we verify some properties of the weak derivative.

Theorem 3.1.1. *Assume $u, v \in W^{k,p}(U)$, $|\alpha| \le k$. Then*

(i) $D^\alpha u \in W^{k-|\alpha|,p}(U)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u$
$\forall\ \alpha, \beta$, with $|\alpha| + |\beta| \le k$.
(ii) $\forall\ \lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(U)$ e $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$
(iii) If $V \subset U$, V open, then $u \in W^{k,p}(V)$.
(iv) If $\zeta \in C_c^\infty(U)$, then $\zeta u \in W^{k,p}(U)$ and

$$D^\alpha(\zeta u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u \tag{3.4}$$

where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$.

Proof. (i) Fix $\phi \in C_c^\infty(U)$. Then $D^\beta\phi \in C_c^\infty(U)$, and so

$$\begin{aligned}\int_U D^\alpha u D^\beta u \phi dx &= (-1)^{|\alpha|} \int_U u D^{\alpha+\beta}\phi dx \\ &= (-1)^{|\alpha|}(-1)^{|\alpha+\beta|} \int_U D^{\alpha+\beta} u \phi dx \\ &= (-1)^{|\beta|} \int_U D^{\alpha+\beta} u \phi dx \end{aligned} \tag{3.5}$$

Thus $D^\beta(D^\alpha u) = D^{\alpha+\beta}u$ in the weak sense.
(ii) and (iii) follows immediately.
We prove (iv) by induction on $|\alpha|$. Suppose first $|\alpha| = 1$ and choose any $\phi \in C_c^\infty(U)$. Then:

$$\begin{aligned}\int_U \zeta u D^\alpha \phi dx &= \int_U \{u D^\alpha(\zeta\phi) - u D^\alpha(\zeta)\phi\} dx \\ &= -\int_U (\zeta D^\alpha u + u D^\alpha \zeta)\phi dx \end{aligned} \tag{3.6}$$

Thus $D^{\alpha}(\zeta u) = \zeta D^{\alpha}u + uD^{\alpha}\zeta$, as required.
Next assume $l < k$ and formula (3.4) is valid for all $|\alpha| \leq l$ and all functions ζ. choose a multiindex α with $|\alpha| = l+1$. Then $\alpha = \beta + \gamma$ for some $|\beta| = l,\ |\gamma| = 1$. Then, for ϕ as above:

$$\begin{aligned}
\int_U \zeta u D^{\alpha}\phi dx = \int_U \zeta u D^{\beta}(D^{\gamma}\phi) dx & \qquad\qquad = \\
= (-1)^{|\beta|} \int_U \sum_{\sigma\leq\beta} \binom{\beta}{\sigma} D^{\sigma}\zeta D^{\beta-\sigma} u D^{\gamma}\phi dx \\
= (-1)^{|\beta|+|\gamma|} \int_U \sum_{\sigma\leq\beta} \binom{\beta}{\sigma} D^{\gamma}(D^{\sigma}\zeta D^{\beta-\sigma}u)\phi dx \\
= (-1)^{|\alpha|} \int_U \sum_{\sigma\leq\beta} \binom{\beta}{\sigma} [D^{\rho}\zeta D^{\alpha-\rho}u + D^{\sigma}\zeta D^{\alpha-\sigma}u]\phi dx \quad (\rho = \sigma + \gamma) \\
= (-1)^{|\alpha|} \int_U [\sum_{\sigma\leq\alpha} \binom{\alpha}{\sigma} D^{\sigma}\zeta D^{\alpha-\sigma}u]\phi dx
\end{aligned} \tag{3.7}$$

since $\binom{\beta}{\sigma-\gamma} + \binom{\beta}{\sigma} = \binom{\alpha}{\sigma}$.

This completes the proof. □

We now examine the properties of Sobolev spaces as function spaces, starting with the following:

Definition 3.1.4. (i) If $u \in W^{k,p}(U)$, the function

$$||u||_{W^{k,p}(U)} := \begin{cases} (\sum_{|\alpha|\leq k} \int_U |D^{\alpha}u|^p dx)^{\frac{1}{p}} & (1 \leq p < \infty) \\ \sum_{|\alpha|\leq k} ess\ sup_U |D^{\alpha}u| & (p = \infty) \end{cases} \tag{3.8}$$

is a norm
. (ii) let $\{u_m\}_{m=1}^{\infty}$ and $u \in W^{k,p}(U)$. We say u_m converges to u in $W^{k,p}(U)$ in $W^{k,p}(U)$, provided:

$$lim_{m\to\infty} ||u - u_m||_{W^{k,p}(U)} = 0 \tag{3.9}$$

We also write $u_m \to u$ in $W^{k,p}_{loc}(U)$, to mean $u_m \to u$ in $W^{k,p}(V)$ for each $V \subset\subset U$.
(iii) We denote by $W^{k,p}_0(U)$ the closure of $C^{\infty}_c(U)$ in $W^{k,p}(U)$.
Thus, $u \in W^{k,p}_0(U)$ if and only if there exists a sequence $\{u_m\}$ in $C^{\infty}_c(U)$ such that $u_m \to u$ in $W^{k,p}(U)$.

The following result shows that Sobolev spaces are complete with respect to the norm (3.8).

Theorem 3.1.2. *For each $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(U)$ is a Banach space.*

Proof. Let us first of all check that $||u||_{W^{k,p}(U)}$ is a norm.
Clearly $||\lambda u||_{W^{k,p}(U)} = |\lambda| ||u||_{W^{k,p}(U)}$ and $||u||_{W^{k,p}(U)} = 0$ if and only if $u = 0$ a.e.
Next assume $u, v \in W^{k,p}(U)$. Then, if $1 \leq p < \infty$, triangle inequality in $L^p(U)$ implies:

$$||u+v||_{W^{k,p}(U)} = (\sum_{|\alpha|\leq k} ||D^\alpha u + D^\alpha v||^p_{L^p(U)})^{\frac{1}{p}} \tag{3.10}$$

$$\leq (\sum_{|\alpha|\leq k} (||D^\alpha u||_{L^p(U)} + ||D^\alpha v||_{L^p(U)})^p)^{\frac{1}{p}} \tag{3.11}$$

$$\leq (\sum_{|\alpha|\leq k} ||D^\alpha u||^p_{L^p(U)})^{\frac{1}{p}} + (\sum_{|\alpha|\leq k} ||D^\alpha v||^p_{L^p(U)})^{\frac{1}{p}} \tag{3.12}$$

$$= ||u||_{W^{k,p}(U)} + ||v||_{W^{k,p}(U)} \tag{3.13}$$

It remains to show that $W^{k,p}(U)$ is complete. So assume $\{u_m\}_{m=1}^\infty$ is a Cauchy sequence in $W^{k,p}(U)$. Then, for each $|\alpha| \leq k$, $\{D^\alpha u_m\}_{m=1}^\infty$ is a Cauchy sequence in $L^p(U)$. Since $L^p(U)$ is complete, there exist functions $u_\alpha \in L^p(U)$ such that $D^\alpha u_m \to u_\alpha$ in $L^p(U)$ for each $|\alpha| \leq k$. In particular $u_m \to u_{(0,\dots,0)} := u$ in $L^p(U)$.
We now claim

$$u \in W^{k,p}(U), \quad D^\alpha u = u_\alpha, \quad |\alpha| \leq k. \tag{3.14}$$

For every fixed $\phi \in C_c^\infty(U)$, we have, invoking the Lebesgue dominated convergence theorem and the definition of weak derivative:

$$\int_U u D^\alpha \phi dx = lim_{m\to\infty} \int_U u_m D^\alpha \phi dx \tag{3.15}$$

$$= lim_{m\to\infty} (-1)^{|\alpha|} \int_U D^\alpha u_m \phi dx \tag{3.16}$$

$$= (-1)^{|\alpha|} \int_U u_\alpha \phi dx \tag{3.17}$$

Thus (3.14) is valid. Since therefore $D^\alpha u_m \to D^\alpha u$ in $L^p(U)$ for all $|\alpha| \leq k$, we see that $u_m \to u$ in $W^{k,p}(U)$, as required. □

3.2 Approximations

In this section we investigate under which assumptions a function $u \in W^{k,p}(U)$ can be approximated by smooth functions belonging to spaces which are dense in $W^{k,p}(U)$.

We start recalling some useful facts about *mollification* of functions.

Definition 3.2.1. Let $\eta :\ \mathbb{R}^n \to \mathbb{R}$ be the function

$$\eta(x) = \begin{cases} Ce^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.18}$$

where C is a normalization constant. Now define, for $\epsilon > 0$, the rescaled function

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta(\frac{x}{\epsilon}) \tag{3.19}$$

η_ϵ is called the standard mollifier. It can be proven that $\eta_\epsilon \in C^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \eta_\epsilon dx = 1$ and $spt(\eta_\epsilon) \subset B(0, \epsilon)$, where $spt(\eta)$ denote the support of η.

The reason for which η is called a mollifier becomes clear when we look at how it acts on functions by convolution.

Definition 3.2.2. Let $f \in L^1(U)$. Define $U_\epsilon = \{x \in U : dist(x, \partial U) > \epsilon\}$. We introduce the mollified function $f^\epsilon := \eta_\epsilon \star f$; that is, for $x \in U_\epsilon$,

$$f^\epsilon(x) := \int_{B(0,\epsilon)} \eta_\epsilon(y) f(x-y) dy \tag{3.20}$$

The properties of f^ϵ are summarized in the following result:

Theorem 3.2.1. *Let $f \in L^1(U)$. We have:*

(i) $f^\epsilon \in C^\infty(U_\epsilon)$
(ii) $f^\epsilon \to f$ a.e. as $\epsilon \to 0$
(iii) If $f \in C(U)$, then $f^\epsilon \to f$ uniformly on compact subsets of U.
(iv) If $1 \leq p < \infty$ and $f \in L^p_{loc}(U)$, then $f^\epsilon \to f$ in $L^p_{loc}(U)$.

The proof can be found in [4], Appendix C.

The previous theorem shows that, even if the original function f exhibits wild irregularities, the mollification f^ϵ is smooth on the set U_ϵ and f^ϵ approximates f better and better as ϵ approaches zero.
The mollification approach provides a way to locally approximate functions in Sobolev spaces by smooth functions.

Theorem 3.2.2. *Assume $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Set $u^\epsilon := \eta_\epsilon \star u$ in U_ϵ. Then $u^\epsilon \to u$ in $W^{k,p}_{loc}(U)$, as $\epsilon \to 0$.*

Proof. We start by proving that, if $|\alpha| \leq k$, then

$$D^\alpha u^\epsilon = \eta_\epsilon \star D^\alpha u \tag{3.21}$$

That is, the ordinary α^{th}-partial derivative of the smooth function u^ϵ is the mollification of the α^{th}-weak partial derivative of u. To confirm this, we compute, for $x \in U_\epsilon$,

$$D^\alpha u^\epsilon(x) = D^\alpha \int_U \eta_\epsilon(x-y)u(y)dy \tag{3.22}$$

$$= \int_U D_x^\alpha \eta_\epsilon(x-y)u(y)dy \tag{3.23}$$

$$= (-1)^{|\alpha|} \int_U D_y^\alpha \eta_\epsilon(x-y)u(y)dy \tag{3.24}$$

We don't have boundary terms since $spt(\eta) \subset U_\epsilon$.
Now, for fixed $x \in U_\epsilon$, the function $\phi(y) = \eta_\epsilon(x-y)$ belongs to $C_c^\infty(U)$. Consequently the definition of the α^{th}-weak partial derivative implies:

$$\int_U D_y^\alpha \eta_\epsilon(x-y)u(y)dy = (-1)^{|\alpha|} \int_U \eta_\epsilon(x-y) D_y^\alpha u(y)dy \tag{3.25}$$

Thus:

$$D^\alpha u^\epsilon(x) = (-1)^{2|\alpha|} \int_U \eta_\epsilon(x-y) D_y^\alpha u(y)dy = [\eta_\epsilon \star D^\alpha u](x) \tag{3.26}$$

This establishes (3.21).
Now choose an open set $V \subset\subset U$. In view of (3.21) and Theorem 3.2.1., for each $|\alpha| \leq k$, $D^\alpha u^\epsilon \to D^\alpha u$ in $L^p(V)$ as $\epsilon \to 0$. Consequently:

$$||u^\epsilon - u||_{W^{k,p}(V)} = \sum_{|\alpha| \leq k} ||D^\alpha u^\epsilon - D^\alpha u||^p_{L^p(V)} \to 0 \tag{3.27}$$

as $\epsilon \to 0$.
This proves the assertion. □

Next we show that we can find smooth functions which provides a global approximation of functions in Sobolev spaces. That is, we look for smooth functions that approximate u in $W^{k,p}(U)$ and not only in $W^{k,p}_{loc}(U)$. We have the following result:

Theorem 3.2.3. *Suppose $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then there exist functions $u_m \in C^\infty(U) \cap W^{k,p}(U)$ such that $u_m \to u$ in $W^{k,p}(U)$.*

Proof. We have $U = \cup_{i=1}^\infty U_i$, where

$$U_i := \{x \in U : d(x, \partial U) > \frac{1}{i}\} \tag{3.28}$$

Write $V_i := U_{i+3} - \bar{U}_{i+1}$.
Choose also any open set $V_0 \subset\subset U$, so that $U = \cup_{i=0}^{\infty} V_i$. Now, let $\{\zeta_i\}_{i=0}^{\infty}$ be a smooth partition of unity subordinate to the open sets $\{V_i\}_{i=0}^{\infty}$; that is, suppose

$$\begin{cases} 0 \leq \zeta_i \leq 1, \quad \zeta_i \in C_c^{\infty}(V_i) \\ \sum_{i=0}^{\infty} \zeta_i = 1 \ \ on\ U \end{cases} \tag{3.29}$$

According to Theorem 3.1.1, $\zeta_i u \in W^{k,p}(U)$ and $spt(\zeta_i u) \subset V_i$.
Now fix $\delta > 0$ and choose $\epsilon_i > 0$ so small that $u^i := \eta_{\epsilon_i} \star (\zeta_i u)$ satisfies

$$\begin{cases} ||u^i - \zeta_i u||_{W^{k,p}(U)} \leq \frac{\delta}{2^i+1} \\ spt(u^i) \subset W_i \end{cases} \tag{3.30}$$

where $W_i := U_{i+4} - \bar{U}_i$ for $i \in \mathbb{N}$. Write then $v := \sum_{i=0}^{\infty} u^i$. This function belongs to $C^{\infty}(U)$, since for each open set $V \subset\subset U$ there are at most finitely many nonzero terms in the sum.
Since $u = \sum_{i=0}^{\infty} \zeta_i u$, we have for each $V \subset\subset U$:

$$||u - v||_{W^{k,p}(V)} \leq \sum_{i=0}^{\infty} ||u^i - \zeta_i u||_{W^{k,p}(U)} \tag{3.31}$$

$$\leq \delta \sum_{i=0}^{\infty} \frac{1}{2^i + 1} = \delta \tag{3.32}$$

Take the supremum over sets $V \subset\subset U$, to conclude $||u - v||_{W^{k,p}(U)} \leq \delta$. □

Until now we haven't made any assumptions on the set U, since we were not interested in the values the functions take at the boundary ∂U. We now ask under which assumptions there exist smooth functions which approximate $u \in W^{k,p}(U)$ also on the closure of U. This requires some conditions to exclude ∂U being wild geometrically.

Theorem 3.2.4. *Assume ∂U is of class C^1 and suppose also $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then there exist functions $u_m \in C^{\infty}(\bar{U})$ such that $u_m \to u$ in $W^{k,p}(U)$.*

Proof. Fix any point $x^0 \in \partial U$ and let $r > 0$ as in definition of boundary of class C^1.
Set $V := U \cap B(x^0, \frac{r}{2})$ and define the shifted point $x^{\epsilon} := x + \lambda \epsilon e_n$, for $x \in V$ and $\epsilon > 0$. Observe that, for $\lambda > 0$ large enough, we have $B(x^{\epsilon}, \epsilon) \subset U \cap B(x^0, r)$ $\forall x \in V$ and $\forall \epsilon > 0$.
Now define the shifted function $u_{\epsilon}(x) := u(x^{\epsilon})$, for $x \in V$ and write $v^{\epsilon} = \eta_{\epsilon} \star u_{\epsilon}$; clearly $v^{\epsilon} \in C^{\infty}(\bar{V})$. We now claim

$$v^{\epsilon} \to u \quad in\ W^{k,p}(V) \tag{3.33}$$

To confirm this, take α to be any multiindex with $|\alpha| \leq k$. Then

$$||D^\alpha v^\epsilon - D^\alpha u||_{L^p(V)} \leq ||D^\alpha v^\epsilon - D^\alpha u_\epsilon||_{L^p(V)} + ||D^\alpha u_\epsilon - D^\alpha u||_{L^p(V)} \tag{3.34}$$

The first term vanishes in the limit $\epsilon \to 0$ by reasoning similar to that in the proof of Theorem 3.2.2. and the second term also vanishes since translation is continue in the L^p-norm.

Now select $\delta > 0$. Since ∂U is compact, we can find finitely many points $x_i^0 \in \partial U$, radii $r_i > 0$, sets $V_i = U \cap B(x_i^0, \frac{r_i}{2})$ and functions $v_i \in C^\infty(\bar{V}_i)$ such that $\partial U \subset \cup_{i=1}^N B^0(x_i^0, \frac{r_i}{2})$ and

$$||v_i - u||_{W^{k,p}(V_i)} \leq \delta \tag{3.35}$$

Take also an open set $V_0 \subset\subset U$, such that $U \subset \cup_{i=0}^N V_i$ and select a function $v_0 \in C^\infty(\bar{V}_0)$ satisfying

$$||v_0 - u||_{W^{k,p}(V_0)} \leq \delta \tag{3.36}$$

Now let $\{\zeta_i\}_{i=0}^N$ be a smooth partition of unity subordinate to the open sets $\{V_i\}_{i=0}^N$ in U. Define $v := \sum_{i=0}^N (\zeta_i v_i)$: then $v \in C^\infty(\bar{U})$. Furthermore, since $u = \sum_{i=0}^N (\zeta_i u)$, we see using Theorem 3.1.1., that for each $|\alpha| \leq k$:

$$||D^\alpha v - D^\alpha u||_{L^p(U)} \leq \sum_{i=0}^N ||D^\alpha(\zeta_i v_i) - D^\alpha(\zeta_i u)||_{L^p(V_i)} \tag{3.37}$$

$$\leq C \sum_{i=0}^N ||v_i - u||_{W^{k,p}(V_i)} = CN\delta \tag{3.38}$$

□

3.3 Extension and trace operators

We introduce two bounded linear operators (Appendix A) acting on the set $W^{1,p}(U)$: the *extension operator* and the *trace operator*. The former provides a way to extend functions in $W^{1,p}(U)$ to functions in the Sobolev space $W^{1,p}(\mathbb{R}^n)$, the latter permits us to assign boundary values along ∂U to a function in $W^{1,p}(U)$. This will be fundamental for our study of boundary value problems for the wave equation.

3.3.1 Extensions

Our goal is to extend functions in the Sobolev space $W^{1,p}(U)$ to functions in the Sobolev space $W^{1,p}(\mathbb{R}^n)$. Observe that extending $u \in W^{1,p}(U)$ to be zero in $\mathbb{R}^n - U$ does not work in general, as we may thereby create such a bad discontinuity along ∂U that the extended function no longer has a weak first partial derivative. We must instead consider a way to preserve weak derivatives across the boundary. The introduction of the *extension operator* solves this problem. In particular, we have the following:

Theorem 3.3.1. *Assume ∂U is C^1. Select a bounded open set V such that $U \subset\subset V$. Then there exists a bounded linear operator*

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n) \tag{3.39}$$

such that for each $u \in W^{1,p}(U)$:

(i) $Eu = u$ a.e. in U
(ii) spt$(Eu) \subset V$
(iii) $||Eu||_{W^{1,p}(\mathbb{R}^n)} \leq C||u||_{W^{1,p}(U)}$

the constant C depending only on p, U and V.

Remark 3.3.1. The absuntion on the regularity of the boundary is fundamental. It is well known that without this hypotesis it is possible to construct some counterexemples.

Definition 3.3.1.1. We call Eu an extension of the function u in $W^{1,p}(\mathbb{R}^n)$.

Proof. Fix $x^0 \in \partial U$ and suppose first that ∂U is flat near x^0, lying in the plane $\{x_n = 0\}$. Then we may assume there exists an open ball $B(x^0, r)$, such that

$$\begin{cases} B^+ := B \cap \{x_n \geq 0\} \subset \bar{U} \\ B^- := B \cap \{x_n \leq 0\} \subset \mathbb{R}^n - U \end{cases} \tag{3.40}$$

Temporarily suppose also $u \in C^\infty(\bar{U})$. Define then

$$\bar{u} := \begin{cases} u(x) & x \in B^+ \\ -3u(x_1, ..., x_{n-1}, -x_n) + 4u(x_1, ..., x_{n-1}, -\frac{x_n}{2}) & x \in B^- \end{cases} \tag{3.41}$$

We claim $\bar{u} \in C^1(B)$. To check this, let us write $u^- := \bar{u}|_{B^-}$, $u^+ := \bar{u}|_{B^+}$. Then

$$\frac{\partial u^-}{\partial x_n}(x) = 3\frac{\partial u}{\partial x_n}(x_1, ..., x_{n-1}, -x_n) - 2\frac{\partial u}{\partial x_n}(x_1, ..., x_{n-1}, -\frac{x_n}{2}) \tag{3.42}$$

and so $u^-_{x_n}|_{\{x_n=0\}} = u^+_{x_n}|_{\{x_n=0\}}$. Now, since $u^+ = u^-$ on $\{x_n = 0\}$, we see as well that $u^-_{x_i}|_{\{x_n=0\}} = u^+_{x_i}|_{\{x_n=0\}}$ for $i = 1, ..., n-1$. But then these two equalities together imply $D^\alpha u^-_{\{x_n=0\}} = D^\alpha u^+_{\{x_n=0\}}$ for each $|\alpha| \leq 1$ and so $\bar{u} \in C^1(B)$, as claimed.
Using this calculation, we immediately obtain:

$$||\bar{u}||_{W^{1,p}(B)} \leq C||u||_{W^{1,p}(B^+)} \tag{3.43}$$

for some constant C which does not depend on u.

Let us next consider the situation that ∂U is not necessarily flat near x^0. Let γ, r as in definition of boundary of class C^1. Define:

$$\begin{cases} y_i = x_i := \Phi^i(x) & i \in \{1, ..., n-1\} \\ y_n = x_n - \gamma(x_1, ..., x_{n-1}) := \Phi^n(x) \end{cases} \tag{3.44}$$

$$\begin{cases} x_i = y_i := \Psi^i(y) & i \in \{1, ..., n-1\} \\ x_n = y_n + \gamma(x_1, ..., x_{n-1}) := \Psi^n(y) \end{cases} \tag{3.45}$$

and write $y = \Phi(x)$, $x = \Psi(y)$.
Then $\Phi(x) = \Psi(y)^{-1}$ and the map $x \to \Phi(x) = y$ straightens out ∂U near x^0. Define then $u'(y) := u(\Psi(y))$. Repeating the calculation made above in the new coordinates, we find that the extension of the function $u'(y)$ on all of B, written $\bar{u}'$, is C^1 and we have the estimate

$$||\bar{u}'||_{W^{1,p}(B)} \leq C||u'||_{W^{1,p}(B^+)} \tag{3.46}$$

Let $W := \Psi(B)$. Then converting back to the x-variables, we obtain an extension $\bar{u}$ of u to W, with

$$||\bar{u}||_{W^{1,p}(W)} \leq C||u||_{W^{1,p}(U)} \tag{3.47}$$

Now, choose $W_0 \subset\subset U$. Since ∂U is compact, there exist finitely many points $x_i^0 \in \partial U$, open sets W_i, and extensions $\bar{u}_i$ of u to W_i such that $U \subset \cup_{i=0}^N W_i$. Suppose also $\{\zeta_i\}_{i=0}^N$ is an associated partition of unity and let $\bar{u} := \sum_{i=0}^N (\zeta_i \bar{u}_i)$, where $\bar{u}_0 = u$. Then, using (3.47), we obtain the bound

$$||\bar{u}||_{W^{1,p}(\mathbb{R}^n)} \leq C||u||_{W^{1,p}(U)} \tag{3.48}$$

for some constant C, depending on U, p, n, but not on u.

Furthermore, we can arrange for the support of $\bar{u}$ to lie within V, with $U \subset\subset V$.
We henceforth write $Eu := \bar{u}$ and observe that the mapping $u \to Eu$ is linear.
Suppose now that u is not necessarily $C^\infty(\bar{U})$, but belongs to $W^{1,p}(U)$. Choosing a sequence of functions $u_m \in C^\infty(\bar{U})$ converging to u in $W^{1,p}(U)$ and using estimate (5.41) and the linearity of E, we obtain:

$$||Eu_m - Eu_l||_{W^{1,p}(\mathbb{R}^n)} \leq C||u_m - u_l||_{W^{1,p}(U)} \tag{3.49}$$

Thus $\{Eu_m\}_{m=1}^\infty$ is a Cauchy sequence in $W^{1,p}(\mathbb{R}^n)$ and so converges to $\bar{u} := Eu$.
This extension, which does not depend on the particular choice of the approximating sequence, satisfies the conclusions of the theorem. $\square$

3.3.2 Traces

We discuss the possibility of assigning boundary values along ∂U to a function $u \in W^{1,p}(U)$, assuming that ∂U is C^1. Now, if $u \in C(\bar{U})$, then clearly u has values on ∂U in the usual sense. The problem is that a function in $W^{1,p}(U)$ is not in general continuous and, even worse, is only defined up to sets of measure zero in U. Since ∂U has n-dimensional Lebesgue measure zero, there is no direct meaning we can give to the expression "u restricted to the boundary". The notion of a *trace operator* solves this problem.

Theorem 3.3.2. *Let $1 \leq p < \infty$ and assume ∂U is C^1. Then there exists a bounded linear operator*

$$T : W^{1,p}(U) \to L^p(\partial U) \tag{3.50}$$

which satisfies the following properties:

(i) $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\bar{U})$.
(ii)

$$||Tu||_{L^p(\partial U)} \leq C||u||_{W^{1,p}(U)} \tag{3.51}$$

for each $u \in W^{1,p}(U)$, the constant C depending only on p and U.

Definition 3.3.2.1. We call Tu the trace of u on ∂U.

Proof. Assume first $u \in C^1(\bar{U})$, $x^0 \in \partial U$, ∂U is flat near x^0, lying in the plane $\{x_n = 0\}$. Choose a ball B as in the proof of Theorem 3.3.1.1. and let $\tilde{B}$ denote the concentric ball with radius $\frac{r}{2}$.
Select $\zeta \in C_c^\infty(B)$, with $\zeta \geq 0$ in B and $\zeta = 1$ in $\tilde{B}$. Denote also by $\Gamma := \partial U \cap \tilde{B}$. Set $x' = (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1} = \{x_n = 0\}$. Then:

$$\int_\Gamma |u|^p dx' \leq \int_{\{x_n=0\}} \zeta|u|^p dx' \tag{3.52}$$

$$= -\int_{B^+} (\zeta|u|^p)_{x_n} dx \tag{3.53}$$

$$= -\int_{B^+} \{|u|^p \zeta_{x_n} + p|u|^{p-1}(sgn\ u)u_{x_n}\zeta\} dx \tag{3.54}$$

$$\leq C\int_{B^+} (|u|^p + |Du|^p) dx \tag{3.55}$$

where we employed Young's inequality (Appendix A).

If ∂U is not flat near x^0, we can change coordinates near x^0 obtaining the setting above. Applying estimate (3.52), we obtain the bound

$$\int_{\Gamma'} |u|^p dx' \leq C\int_U (|u|^p + |Du|^p) dx \tag{3.56}$$

where Γ' is some open subset of ∂U containing x^0.

Now, since ∂U is compact, there exist finitely many points $x^0 \in \partial U$ and open subsets $\Gamma_i \subset \partial U$, $i \in \{1, ..., N\}$ such that $\partial U \subset \cup_{i=1}^{N} \Gamma_i$ and

$$||u||_{L^p(\Gamma_i)} \leq C||u||_{W^{1,p}(U)} \tag{3.57}$$

Consequently, if we write

$$Tu := u|_{\partial U} \tag{3.58}$$

then:

$$||Tu||_{L^p(\partial U)} \leq C|||u||_{W^{1,p}(U)} \tag{3.59}$$

for some constant C, which does not depend on u.

Assume now $u \in W^{1,p}(U)$. Then there exist functions $u_m \in C^\infty(\bar{U})$ converging to u in $W^{1,p}(U)$. Then, from (5.34), we have:

$$||Tu_m - Tu_l||_{L^p(\partial U)} \leq C||u_m - u_l||_{W^{1,p}(U)} \tag{3.60}$$

so that $\{Tu_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $L^p(\partial U)$. We define

$$Tu := \lim_{m\to\infty} Tu_m \tag{3.61}$$

the limit taken in $L^p(\partial U)$. This definition does not depend on the particular choice of functions approximating u.

Finally, if $\in W^{1,p}(U) \cap C(\bar{U})$, the functions $u_m \in C^\infty(\bar{U})$ constructed in the proof of theorem 3.2.4. converge uniformly to u in $\bar{U}$. Hence $Tu = u|_{\partial U}$. □

Now we look more closely what it means for a function to have zero trace. It turns out that the functions having zero trace are exactly those in the set $W_0^{1,p}$.

Theorem 3.3.3. *Suppose $u \in W^{1,p}(U)$, with ∂U of class C^1. Then $u \in W_0^{1,p}(U)$ if and only if $Tu = 0$ on ∂U.*

Proof. Suppose first $u \in W_0^{1,p}(U)$.
Then by definition there exists a sequence of functions $u_m \in C_c^\infty(U)$ converging to u in $W^{1,p}(U)$. Since $Tu_m = 0$ on ∂U for each m, and T is a linear bounded operator, $Tu = 0$ on ∂U.

Conversely assume $Tu = 0$ on ∂U.
Using a partition of unity and flattening out ∂U, we may as well assume

$$\begin{cases} u \in W^{1,p}(\mathbb{R}^n_+) & \textit{u has compact support in } \bar{\mathbb{R}}^n_+ \\ Tu = 0 & \textit{on } \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1} \end{cases} \tag{3.62}$$

then since $Tu = 0$ on $\mathbb{R}^{n-1}$, there exist functions $u_m \in C^1(\bar{\mathbb{R}}^n_+)$ such that $u_m \to u$ in $W^{1,p}(\mathbb{R}^n_+)$ and $Tu_m = u_m|_{\mathbb{R}^{n-1}} \to 0$ in $L^p(\mathbb{R}^{n-1})$.
Now, if $x' \in \mathbb{R}^{n-1}$, $x_n \geq 0$, we have:

$$|u_m(x', x_n)| \leq |u_m(x', 0)| + \int_0^{x_n} |u_{m,x_n}(x', t)| dt \tag{3.63}$$

Thus:

$$\int_{\mathbb{R}^{n-1}} |u_m(x', x_n)|^p dx' \leq C(\int_{\mathbb{R}^{n-1}} |u_m(x', 0)|^p dx' \tag{3.64}$$

$$+ x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |Du_m(x', t)|^p dx' dt) \tag{3.65}$$

Letting $m \to \infty$, we deduce:

$$\int_{\mathbb{R}^{n-1}} |u(x', x_n)|^p dx' \leq C x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |Du|^p dx' dt \tag{3.66}$$

for a.e. $x_n > 0$.

Next let $\zeta \in C^\infty(\mathbb{R})$, satisfy $0 \leq \zeta \leq 1$, $\zeta = 1$ on $[0,1]$ and $\zeta = 0$ on $\mathbb{R} - [0,2]$, and write

$$\begin{cases} \zeta_m(x) := \zeta(m x_n) \;\; x \in \mathbb{R}^n_+ \\ w_m := u(x)(1 - \zeta_m) \end{cases} \tag{3.67}$$

Then:

$$\begin{cases} w_{m,x_n} = u_{x_n}(1 - \zeta_m) - m u \zeta' \\ D_{x'} w_m = D_{x'} u (1 - \zeta_m) \end{cases} \tag{3.68}$$

From which we obtain:

$$\int_{\mathbb{R}^n_+} |Dw_m - Du|^p dx \leq C \int_{\mathbb{R}^n_+} |\zeta_m|^p |Du|^p dx \tag{3.69}$$

$$+ C m^p \int_0^{\frac{2}{m}} \int_{\mathbb{R}^{n-1}} |u|^p dx' dt := A + B \tag{3.70}$$

Now, since $\zeta \neq 0$ only if $0 \leq x_n \leq \frac{2}{m}$,

$$A \to 0 \;\; as \; m \to \infty \tag{3.71}$$

To estimate the term B, we utilize the inequality (3.66):

$$B \leq C m^p (\int_0^{\frac{2}{m}} t^{p-1} dt)(\int_0^{\frac{2}{m}} \int_{\mathbb{R}^{n-1}} |Du|^p dx' dx_n) \tag{3.72}$$

$$\leq C \int_0^{\frac{2}{m}} \int_{\mathbb{R}^{n-1}} |Du|^p dx' dx_n \to 0 \quad as \; m \to \infty \tag{3.73}$$

Employing (3.69), (3.71), (3.72), we deduce $Dw_m \to Du$ in $L^p(\mathbb{R}^n_+)$.
Since clearly $w_m \to u$ in $L^p(\mathbb{R}^n_+)$, we conclude

$$w_m \to u \;\; in \, W^{1,p}(\mathbb{R}^n_+) \tag{3.74}$$

But $w_m = 0$ if $0 < x_n < \frac{1}{m}$. We can therefore mollify the w_m to produce functions $u_m \in C_c^\infty(\mathbb{R}^n_+)$ such that $u_m \to u$ in $W^{1,p}(\mathbb{R}^n_+)$. Hence $u \in W_0^{1,p}(\mathbb{R}^n_+)$. □

3.4 Sobolev inequalities

In this section we study embeddings of various Sobolev spaces into others. We begin by proving the so-called *Sobolev inequalities*, establishing inequalities between norms in different spaces for smooth functions. This will then leads to estimates for arbitrary functions in the various Sobolev spaces, since smooth functions are dense. Next we concatenate these inequalities to obtain a theorem that summarises relations between the various Sobolev spaces.

It turns out that, for a given n, the nature of the embeddings depends upon whether

$$1 \le p < n \tag{3.75}$$

$$p = n \tag{3.76}$$

$$n < p \le \infty \tag{3.77}$$

3.4.1 Case $1 \le p < n$

In this section we assume $1 \le p < n$. We first ask whether we can establish an estimate of the form

$$||u||_{L^q(\mathbb{R}^n)} \le C||Du||_{L^p(\mathbb{R}^n)} \tag{3.78}$$

for certain constants $C > 0$, $1 \le q < \infty$ and all functions $u \in C_c^\infty(\mathbb{R}^n)$.
By a rescaling argument, we see that if any such inequality holds, then the number q cannot be arbitrary, but must in fact have a very specific form. To see this, choose a function $u \in C_c^\infty(\mathbb{R}^n)$, $u \neq 0$, and define for $\lambda > 0$ the rescaled function $u_\lambda(x) := u(\lambda x)$.
Applying the estimate (3.78) to $u_\lambda(x)$, we find:

$$||u_\lambda||_{L^q(\mathbb{R}^n)} \le C||Du_\lambda||_{L^p(\mathbb{R}^n)} \tag{3.79}$$

Now we have:

$$\int_{\mathbb{R}^n} |u_\lambda(x)|^q dx = \int_{\mathbb{R}^n} |u(\lambda x)|^q dx = \lambda^{-q} \int_{\mathbb{R}^n} |u(y)|^q dy \tag{3.80}$$

and

$$\int_{\mathbb{R}^n} |Du_\lambda(x)|^p dx = \lambda^p \int_{\mathbb{R}^n} |Du(\lambda x)|^p dx = \frac{\lambda^p}{\lambda^n} \int_{\mathbb{R}^n} |Du(y)|^p dy \tag{3.81}$$

Inserting these equalities into (3.79), we discover:

$$||u||_{L^q(\mathbb{R}^n)} \leq C\lambda^{1-\frac{n}{p}+\frac{n}{q}} ||Du||_{L^p(\mathbb{R}^n)} \tag{3.82}$$

But then if $1-\frac{n}{p}+\frac{n}{q} \neq 0$ we can upon sending λ to either 0 or ∞ obtain a contradiction. Thus if in fact the desidered inequality (3.78) holds, we must necessarily have $1-\frac{n}{p}+\frac{n}{q} = 0$, so that $q = \frac{np}{n-p}$.
This observation motivates the following:

Definition 3.4.1.1. If $1 \leq p < n$, the *Sobolev conjugate* of p is defined as

$$p* := \frac{np}{n-p} \tag{3.83}$$

Note that $\frac{1}{p*} = \frac{1}{p} - \frac{1}{n}$, $p* > p$.

The foregoing argument shows the estimate (3.78) can only possibly be true for $q = p*$. The following theorem proves this inequality is in fact valid.

Theorem 3.4.1. *Assume $1 \leq p < n$. There exists a constant C, depending only on p and n, such that*

$$||u||_{L^{p*}(\mathbb{R}^n)} \leq C||Du||_{L^p(\mathbb{R}^n)} \tag{3.84}$$

for each $u \in C^1_c(\mathbb{R}^n)$,

Proof. First assume $p = 1$.
Since u has compact support, for each $i \in \{1, ..., n\}$ and $x \in \mathbb{R}^n$, we have:

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, ..., y_i, ..., x_n) dy_i \tag{3.85}$$

and so:

$$|u(x)| \leq \int_{-\infty}^{\infty} |Du(x_1, ..., y_i, ..., x_n)| dy_i \tag{3.86}$$

from which we obtain:

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} (\int_{-\infty}^{\infty} |Du(x_1, ..., y_i, ..., x_n)| dy_i)^{\frac{1}{n-1}} \tag{3.87}$$

Integrating with respect to x_1:

$$\begin{aligned}
\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 &\leq \int_{-\infty}^{\infty} \prod_{i=1}^{n} (\int_{-\infty}^{\infty} |Du| dy_i)^{\frac{1}{n-1}} dx_1 \\
&= (\int_{-\infty}^{\infty} |Du| dy_i)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} (\int_{-\infty}^{\infty} |Du| dy_i)^{\frac{1}{n-1}} dx_1 \\
&\leq (\int_{-\infty}^{\infty} |Du| dy_i)^{\frac{1}{n-1}} (\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_i)^{\frac{1}{n-1}} \qquad (3.88)
\end{aligned}$$

where the last inequality follows from the general Hölder inequality (Appendix A). Now integrate (3.88) with respect to x_2.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 dx_2 \leq (\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_2)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=1\, i\neq 2}^{n} I_i^{\frac{1}{n-1}} dx_2 \qquad (3.89)$$

where we have defined

$$I_1 := \int_{-\infty}^{\infty} |Du| dy_1 \quad I_i := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_i \;\; i = 3, ..., n \qquad (3.90)$$

Applying once more the general Hölder inequality, we find:

$$\begin{aligned}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 dx_2 &\leq (\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dx_1 dy_2)^{\frac{1}{n-1}} (\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dy_1 dx_2)^{\frac{1}{n-1}} \\
&\prod_{i=3}^{n} (\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du| dy_i dx_1 dx_2)^{\frac{1}{n-1}} \qquad (3.91)
\end{aligned}$$

We continue by integrating with respect to x_3,...,x_n, to find:

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx \leq \prod_{i=1}^{n} (\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} |Du| dx_1 ... dy_i ... dx_n)^{\frac{1}{n-1}} = (\int_{\mathbb{R}^n} |Du| dx)^{\frac{n}{n-1}} \qquad (3.92)$$

This proves the theorem for $p = 1$.

Consider now the case that $1 < p < n$. We apply estimate (5.45) to the function $v := |u|^{\gamma}$, where $\gamma > 1$ is to be selected. Then:

$$\begin{aligned}
(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}} dx)^{\frac{n-1}{n}} &\leq \int_{\mathbb{R}^n} |D|u|^{\gamma}| dx = \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1} |Du| dx \\
&\leq \gamma (\int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}} dx)^{\frac{p-1}{p}} (\int_{\mathbb{R}^n} |Du|^{p})^{\frac{1}{p}} \qquad (3.93)
\end{aligned}$$

We choose γ so that $\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1} = p*$. That is, we set $\gamma = \frac{p(n-1)}{n-p}$. Thus, estimate (5.43) becomes:

$$(\int_{\mathbb{R}^n} |u|^{p*} dx)^{\frac{1}{p*}} \leq C(\int_{\mathbb{R}^n} |Du|^p dx)^{\frac{1}{p}} \tag{3.94}$$

This completes the proof. □

Now the following theorem follows easily.

Theorem 3.4.2. *Let ∂U be C^1, $1 \leq p < n$ and $u \in W^{1,p}(U)$. Then $u \in L^{p*}(U)$ and we have the estimate*

$$||u||_{L^{p*}(U)} \leq C||u||_{W^{1,p}(U)} \tag{3.95}$$

the constant C depending only on p, n and U.

Proof. Since ∂U is C^1, there exists according to theorem 3.3.1., an extension $Eu := \bar{u} \in W^{1,p}(\mathbb{R}^n)$, such that

$$\begin{cases} \bar{u} = u \;\; in \; U, \;\; \bar{u} \; has \; compact \; support \\ ||u||_{W^{1,p}(\mathbb{R}^n)} \leq C||u||_{W^{1,p}(U)} \end{cases} \tag{3.96}$$

Because $\bar{u}$ has compact support, there exist functions $u_m \in C_c^\infty(\mathbb{R}^n)$ converging to $\bar{u}$ in $W^{1,p}(\mathbb{R}^n)$.
Now, according to theorem 3.4.1.,

$$||u_m - u_l||_{L^{p*}(\mathbb{R}^n)} \leq C||Du_m - Du_l||_{L^p(\mathbb{R}^n)} \quad for \; all \; m \geq 1 \tag{3.97}$$

Thus $u_m \to \bar{u}$ in $L^{p*}(\mathbb{R}^n)$ as well. Since theorem 3.4.1. also implies $||u_m||_{L^{p*}(\mathbb{R}^n)} \leq C||Du_m||_{L^p(\mathbb{R}^n)}$, we have the bound

$$||\bar{u}||_{L^{p*}(\mathbb{R}^n)} \leq C||D\bar{u}||_{L^p(\mathbb{R}^n)} \tag{3.98}$$

This inequality and (3.96) completes the proof. □

If the function u belongs to $W_0^{1,p}(U)$ the estimate can be improved.

Theorem 3.4.3. *Suppose $u \in W_0^{1,p}(U)$ for some $1 \leq p < n$. Then we have the estimate:*

$$||u||_{L^q(U)} \leq C||Du||_{L^p(U)} \tag{3.99}$$

for each $q \in [1, p*]$, the constant C depending only on n, p, q and U.

Proof. Since $u \in W_0^{1,p}(U)$, we know there exists a sequence of functions $u_m \in C_c^\infty(U)$ converging to u in $W^{1,p}(U)$. We extend each function u_m to be 0 on $\mathbb{R}^n - \bar{U}$ and apply theorem 3.4.1. to discover $||u||_{L^{p*}(U)} \leq C||Du||_{L^p(U)}$. Since U is bounded, we furthermore have $||u||_{L^q(U)} \leq C||u||_{L^{p*}(U)}$ if $1 \leq q \leq p*$. □

Remark 3.4.1. The preceeding theorem shows that the norm $||Du||_{L^p(U)}$ is equivalent to $||u||_{W^{1,p}(U)}$ on $W_0^{1,p}(U)$, providing U is bounded.

3.4.2 Case $n < p < \infty$

We will show that if $u \in W^{1,p}(U)$, then u belongs to some other spaces.
We first need to define Hölder continuous functions, generalizing the well known Lipschitz continuous ones and Hölder spaces.

Definition 3.4.2.1. (i) If $u: U \to \mathbb{R}$ is bounded and continuous, we write:

$$||u||_{C(\bar{U})} := sup_{x \in U}|u(x)| \tag{3.100}$$

(ii) The γ^{th}-Hölder seminorm of $u: U \to \mathbb{R}$ is

$$[u]_{C^{0,\gamma}(\bar{U})} := sup_{x,y \in U,\, x \neq y}\{\frac{|u(x)-u(y)|}{|x-y|^{\gamma}}\} \tag{3.101}$$

and the γ^{th}-Hölder norm is

$$||u||_{C^{0,\gamma}(\bar{U})} := ||u||_{C(\bar{U})} + [u]_{C^{0,\gamma}(\bar{U})} \tag{3.102}$$

for $0 < \gamma \leq 1$.

The Hölder space $C^{k,\gamma}(\bar{U})$ consists of all functions $u \in C^k(\bar{U})$ for which the norm

$$||u||_{C^{k,\gamma}(\bar{U})} := \sum_{|\alpha| \leq k} ||D^{\alpha}u||_{C(\bar{U})} + \sum_{|\alpha|=k} [D^{\alpha}u]_{C^{0,\gamma}(\bar{U})} \tag{3.103}$$

is finite.

Now we are ready to prove that if $u \in W^{1,p}(U)$, then u is in fact Hölder continuous.

Theorem 3.4.4. *Assume $n < p \leq \infty$. Then there exists a constant C, depending only on p and n, such that:*

$$||u||_{C^{0,\gamma}(\mathbb{R}^n)} \leq C||u||_{W^{1,p}(\mathbb{R}^n)} \tag{3.104}$$

for all $u \in C^1(\mathbb{R}^n)$, where $\gamma := 1 - \frac{n}{p}$.

Proof. First choose any ball $B(x,r) \subset \mathbb{R}^n$. We claim there exists a constant C, depending only on n, such that:

$$\int_{B(x,r)}^{av} |u(y)-u(x)|dy \leq C\int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}dy \tag{3.105}$$

where $\int_{B(x,r)}^{av}$ denotes the average on $B(x,r)$.
To prove this, fix any point $w \in \partial B(0,1)$. Then, if $0 < s < r$,

$$\begin{aligned} |u(x+sw)-u(x)| &= |\int_0^s \frac{d}{dt}u(x+tw)dt| \\ &= |\int_0^s Du(x+tw)\cdot w dt| \leq \int_0^s |Du(x+tw)|dt \end{aligned} \tag{3.106}$$

hence:

$$\int_{\partial B(0,1)} |u(x+sw)-u(x)|dS \leq \int_0^s \int_{\partial B(0,1)} |Du(x+tw)|dSdt$$
$$= \int_0^s \int_{\partial B(0,1)} |Du(x+tw)|\frac{t^{n-1}}{t^{n-1}}dSdt \tag{3.107}$$

Let $y = x + tw$. Then, converting from polar coordinates, we have:

$$\int_{\partial B(0,1)} |u(x+sw)-u(x)|dS \leq \int_{B(x,s)} \frac{|Du(y)|}{|y-x|^{n-1}}dy$$
$$\leq \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}dy \tag{3.108}$$

Multiply by s^{n-1} and integrate from 0 to r with respect to s:

$$\int_{B(x,r)} |u(y)-u(x)|dy \leq \frac{r^{n-1}}{n}\int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}dy \tag{3.109}$$

This is (3.105).

Now fix $x \in \mathbb{R}^n$. We apply inequality (3.105) as follows:

$$\begin{aligned}|u(x)| &\leq \int^{av}_{B(x,1)} |u(y)-u(x)|dy + \int^{av}_{B(x,1)} |u(y)|dy \\ &\leq C\int_{B(x,1)} \frac{|Du(y)|}{|y-x|^{n-1}}dy + C||u||_{L^p(B(x,1))} \\ &\leq C(\int_{\mathbb{R}^n} |Du|^p dy)^{\frac{1}{p}}(\int_{B(x,1)} \frac{dy}{|x-y|^{(n-1)\frac{p}{p-1}}})^{\frac{p-1}{p}} + C||u||_{L^p(\mathbb{R}^n)} \\ &\leq C||u||_{W^{1,p}(\mathbb{R}^n)}\end{aligned} \tag{3.110}$$

The last estimate holds since $p > n$ implies $(n-1)\frac{p}{p-1} < n$; so that

$$\int_{B(x,1)} \frac{1}{|x-y|^{(n-1)\frac{p}{p-1}}} < \infty \tag{3.111}$$

Since $x \in \mathbb{R}^n$ is arbitrary, inequality (3.110) implies:

$$sup_{\mathbb{R}^n}|u| \leq C||u||_{W^{1,p}(\mathbb{R}^n)} \tag{3.112}$$

Next choose any two points $x, y \in \mathbb{R}^n$ and write $r = |x-y|$. Let $W := B(x,r) \cap B(y,r)$. Then:

$$|u(x)-u(y)| \leq \int_W^{av} |u(x)-u(z)|dz + \int_W^{av} |u(y)-u(z)|dz \tag{3.113}$$

But inequality (3.105) allows us to estimate:

$$\begin{aligned}
\int_W^{av} |u(x)-u(z)|dz &\leq C \int_{B(x,r)}^{av} |u(x)-u(z)|dz \\
&\leq C\Big(\int_{B(x,r)} |Du|^p dz\Big)^{\frac{1}{p}} \Big(\int_{B(x,r)} \frac{dz}{|x-z|^{(n-1)\frac{p}{p-1}}}\Big)^{\frac{p-1}{p}} \\
&\leq C\big(r^{n-(n-1)\frac{p}{p-1}}\big)^{\frac{p-1}{p}} ||Du||_{L^p(\mathbb{R}^n)} \\
&= Cr^{1-\frac{n}{p}} ||Du||_{L^p(\mathbb{R}^n)}
\end{aligned} \tag{3.114}$$

Likewise,

$$\int_W^{av} |u(y)-u(z)|dz \leq Cr^{1-\frac{n}{p}} ||Du||_{L^p(\mathbb{R}^n)} \tag{3.115}$$

Inserting (3.114) and (3.115) into (3.113) yelds:

$$|u(x)-u(y)| \leq C|x-y|^{1-\frac{n}{p}} ||Du||_{L^p(\mathbb{R}^n)} \tag{3.116}$$

Thus:

$$[u]_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} = sup_{x\neq y}\Big\{\frac{|u(x)-u(y)|}{|x-y|^{1-\frac{n}{p}}}\Big\} \leq C||Du||_{L^p(\mathbb{R}^n)} \tag{3.117}$$

This inequality and (5.52) together yeld:

$$||u||_{C^{0,\gamma}(\mathbb{R}^n)} = sup_{\mathbb{R}^n}|u| + sup_{x\neq y}\Big\{\frac{|u(x)-u(y)|}{|x-y|^{1-\frac{n}{p}}}\Big\} \leq C||u||_{W^{1,p}(\mathbb{R}^n)} \tag{3.118}$$

□

Remark 3.4.2.1. A slight variant of the proof above provides the estimate

$$|u(y)-u(x)| \leq Cr^{1-\frac{n}{p}} \Big(\int_{B(x,2r)} |Du(z)|^p dz\Big)^{\frac{1}{p}} \tag{3.119}$$

for all $u \in C^1(B(x,2r)),\ y \in B(x,r),\ n < p < \infty$. By an approximation the same bound is valid for $u \in W^{1,p}(B(x,2r)),\ n < p < \infty$.

Definition 3.4.2.2. We say a function v is a *version* of a given function u provided

$$u = v \quad a.e. \tag{3.120}$$

We have the following:

Theorem 3.4.5. *Suppose ∂U is C^1. Assume $n < p \leq \infty$ and $u \in W^{1,p}(U)$. Then u has a version $u^c \in C^{0,\gamma}(\bar{U})$, for $\gamma = 1 - \frac{n}{p}$ with the estimate*

$$||u^c||_{C^{0,\gamma}(\bar{U})} \leq C||u||_{W^{1,p}(U)} \tag{3.121}$$

The constant C depending only on n, p and U.

Proof. Since ∂U is C^1, there exists according to Theorem 3.3.1. an extension $Eu = \bar{u} \in W^{1,p}(\mathbb{R}^n)$ with the usual properties. Since $\bar{u}$ has compact support, we have, according to Theorem 3.2.2. the existence of a sequence of functions $u_m \in C_c^\infty(\mathbb{R}^n)$, such that:

$$u_m \to \bar{u} \quad in \;\; W^{1,p}(\mathbb{R}^n) \tag{3.122}$$

Now, according to Theorem 3.4.4., $||u_m - u_l||_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C||u_m - u_l||_{W^{1,p}(\mathbb{R}^n)}$ for all $m \geq 1$; whence there exists a function $u^c \in C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ such that:

$$u_m \to u^c \quad in \;\; C^{0,1-\frac{n}{p}}(\mathbb{R}^n) \tag{3.123}$$

From the properties of the extension and the uniquess of the limit, we see that $u^c = u$ a.e. in U; so that u^c is a version of u.
Since theorem 3.4.4. also implies $||u_m||_{C^{0,\gamma}(\bar{U})} \leq C||u_m||_{W^{1,p}(U)}$, assertions (3.122) and (5.57) yield:

$$||u^c||_{C^{0,\gamma}(\mathbb{R}^n)} \leq C||\bar{u}||_{W^{1,p}(\mathbb{R}^n)} \tag{3.124}$$

This inequality, together with the properties of extension, complete the proof. □

Remark 3.4.2.2. In view of Theorem 3.4.5., we will always identify a function $u \in W^{1,p}(U)$ with $p > n$ with its continuous version.

3.4.3 General Sobolev inequalities

We now concatenate the estimates established in subsections 3.4.1. and 3.4.2. to obtain the following theorem, which summarises all the embeddings between Sobolev spaces and other spaces of functions.

Theorem 3.4.6. *Let ∂U be C^1. Assume $u \in W^{k,p}(U)$.*

(i) If $k < \frac{n}{p}$, then $u \in L^q(U)$, where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$.
We have in addition the estimate

$$||u||_{L^q(U)} \leq C||u||_{W^{k,p}(U)} \tag{3.125}$$

The constant C depending only on k, p, n and U.

(ii) Se $k > \frac{n}{p}$, then $u \in C^{k-[\frac{n}{p}]-1,\gamma}(\bar{U})$, where $[\,]$ denote the integer part and

$$\gamma = \begin{cases} [\frac{n}{p}] + 1 - \frac{n}{p} \quad if \ \frac{n}{p} \ is\ not\ an\ integer \\ any\ positive\ number \ < 1, \quad if \ \frac{n}{p} \ is\ an\ integer \end{cases} \tag{3.126}$$

We have in addition the estimate

$$||u||_{C^{k-[\frac{n}{p}]-1,\gamma}(\bar{U})} \leq C||u||_{W^{k,p}(U)} \tag{3.127}$$

The constant C depending only on k, p, n, γ and U.

Proof. (i) Assume $k < \frac{n}{p}$. Then since $D^\alpha u \in L^p(U)$ for all $|\alpha| = k$, theorem 3.4.1. implies

$$||D^\beta u||_{L^{p*}(U)} \leq C||u||_{W^{k,p}(U)} \ \ if \ |\beta| = k - 1 \tag{3.128}$$

and so $u \in W^{k-1,p*}(U)$. Similarly we find $u \in W^{k-2,p**}(U)$, where $\frac{1}{p**} = \frac{1}{p*} - \frac{1}{n} = \frac{1}{p} - \frac{2}{n}$. Continuing, we find after k steps that $u \in W^{0,q}(U) = L^q(U)$, for $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. The estimate (3.125) follows immediately from multiplying the relevant estimates at each step of the above argument.

(ii) Suppose now $k < \frac{n}{p}$ and $\frac{n}{p}$ is not an integer. Then, as above, we see

$$u \in W^{k-l,r}(U) \tag{3.129}$$

for $\frac{1}{r} = \frac{1}{p} - \frac{l}{n}$ provided $lp < n$. We choose the integer l so that $l < \frac{n}{p} < l + 1$, that is we set $l = [\frac{n}{p}]$. Consequently $r = \frac{pn}{n-pl} > n$. Hence (3.129) and theorem 3.4.4. imply that $D^\alpha u \in C^{0,1-\frac{n}{r}}(\bar{U})$ for all $|\alpha| \leq k - l - 1$. Observe also that $1 - \frac{n}{r} = [\frac{n}{p}] + 1 - \frac{n}{p}$. Thus $u \in C^{k-[\frac{n}{p}]-1,[\frac{n}{p}]+1-\frac{n}{p}}(\bar{U})$ and the estimate (3.127) follows easily.
Finally, suppose (ii) holds with $\frac{n}{p}$ an integer. Set $l = [\frac{n}{p}] - 1 = \frac{n}{p} - 1$. We have, as above, $u \in W^{k-l,r}(U)$, for $r = \frac{pn}{n-pl} = n$. Hence, theorem 3.4.1. implies $D^\alpha u \in L^q(U)$ for all $|\alpha| \leq k - l - 1 = k - [\frac{n}{p}]$ and all $n \leq q < \infty$. Therefore theorem 3.4.4. further implies $D^\alpha u \in C^{0,1-\frac{n}{q}}(\bar{U})$ for all $n < q < \infty$ and all $|\alpha| \leq k - [\frac{n}{p}] - 1$. Consequently $u \in C^{k-[\frac{n}{p}]-1,\gamma}(\bar{U})$ for each $0 < \gamma < 1$. As before, estimate (3.127) follows easily. □

3.5 The space H^{-1}

We study the dual space of $H_0^1(U) = W_0^{1,2}(U)$, giving the definition of its norm and its characterization in terms of function in $L^2(U)$.

Definition 3.5.1. We denote by $H^{-1}(U)$ the dual space of $H_0^1(U)$.

That is, a function f belongs to $H^{-1}(U)$ provided f is a bounded linear functional on $H_0^1(U)$.

Notation: We denote by $<,>$ the pairing between $H^{-1}(U)$ and $H_0^1(U)$, that is $< f, u >= f(u)$ for $f \in H^{-1}(U)$ and $u \in H_0^1(U)$.

We equipe the space H^{-1} with the following norm:

$$||f||_{H^{-1}(U)} = sup\{< f, u > : u \in H_0^1,\ ||u||_{H_0^1(U)} \leq 1\} \tag{3.130}$$

It turns out that a function in $H^{-1}(U)$, where $U \in \mathbb{R}^n$, is completely determined by the assignment of n functions in $L^2(U)$.

Theorem 3.5.1. *(i) Assume $f \in H^{-1}(U)$. Then there exist functions $f^0, f^1, ..., f^n$ in $L^2(U)$ such that*

$$< f, v >= \int_U \{f^0 v + \sum_{i=1}^{n} f^i v_{x_i}\} dx \tag{3.131}$$

for each $v \in H_0^1(U)$.
(i) Furthermore

$$||f||_{H^{-1}}(U) = inf\{(\int_U \sum_{i=0}^{n} |f^i|^2 dx)^{\frac{1}{2}} : f\ satisfies\ (i)\ for\ f^0, f^1, ..., f^n\} \tag{3.132}$$

Notation: We write $f = f^0 - \sum_{i=1}^n f^i_{x_i}$ whenever (i) holds.

Proof. (i) Given $u, v \in H_0^1(U)$, we define the inner product

$$(u, v) := \int_U \{Du \cdot Dv + uv\} dx \tag{3.133}$$

It is easy to see that with this product $H_0^1(U)$ becomes a Hilbert space (Appendix A). Now, let $f \in H^{-1}(U)$. We apply the Riesz Representation Theorem (Appendix A) to deduce the existence of a unique function $u \in H_0^1(U)$ satisfying $(u, v) =< f, v >$ for all $v \in H_0^1(U)$; that is

$$< f, v >= \int_U \{Du \cdot Dv + uv\} dx \tag{3.134}$$

This establishes (i) for

$$\begin{cases} f^0 = u \\ f^i = u_{x_i} \quad i = 1, ..., n \end{cases} \tag{3.135}$$

(ii) Assume now $f \in H^{-1}(U)$, with

$$< f, v >= \int_U \{g^0 v + \sum_{i=1}^{n} g^i v_{x_i}\} dx \tag{3.136}$$

for $g^0, ..., g^n \in L^2(U)$. Setting $v = u$ in (3.134) and using (3.136), we deduce:

$$\int_U \{|Du|^2 + |u|^2\} dx \leq \int_U \sum_{i=0}^{n} |g^i|^2 dx \tag{3.137}$$

Thus (3.135) implies:

$$\int_U \sum_{i=0}^{n} |f^i|^2 dx \leq \int_U \sum_{i=0}^{n} |g^i|^2 dx \tag{3.138}$$

From (3.131) it follows that:

$$| < f, v > | \leq (\int_U \sum_{i=0}^{n} |f^i|^2 dx)^{\frac{1}{2}} \tag{3.139}$$

if $||v||_{H_0^1(U)} \leq 1$. Consequently:

$$||f||_{H^{-1}(U)} \leq (\int_U \sum_{i=0}^{n} |f^i|^2 dx)^{\frac{1}{2}} \tag{3.140}$$

Setting $v = \frac{u}{||u||_{H_0^1(U)}}$ in (3.134), we deduce that, in fact:

$$||f||_{H^{-1}(U)} = (\int_U \sum_{i=0}^{n} |f^i|^2 dx)^{\frac{1}{2}} \tag{3.141}$$

Assertion (ii) now follows easily. $\square$

3.6 Spaces involving time

In this section we turn the attention to other sorts of Sobolev spaces, these comprising *functions mapping time into Banach spaces.* These will be essential in the construction of weak solutions for the have equation.

We start by giving some basic definitions:

Definiton 3.6.1. Let X be a Banach Space, with norm $||\ \ ||$, and $T > 0$. The space $L^p(0,T;X)$ consists of all measurable functions $\mathbf{u} : [0,T] \to X$ with:

$$\begin{cases} ||\mathbf{u}||_{L^p(0,T,X)} := (\int_0^T ||\mathbf{u}(t)||^p dt)^{\frac{1}{p}} < \infty \quad 1 \leq p < \infty \\ ||\mathbf{u}||_{L^\infty(0,T,X)} := ess\ sup_{0\leq t\leq T}||\mathbf{u}(t)|| < \infty \end{cases} \tag{3.142}$$

Definition 3.6.2. The space $C([0,T];X)$ is the set of all continuous functions $\mathbf{u}$: $[0,T] \to X$ with

$$||\mathbf{u}||_{C([0,T];X)} := max_{0\leq t\leq T}||\mathbf{u}(t)|| < \infty \tag{3.143}$$

Definition 3.6.3. Let $\mathbf{u} \in L^1(0,T;X)$. We say $\mathbf{v} \in L^1(0,T;X)$ is the weak derivative of $\mathbf{u}$, written $\mathbf{u}' = \mathbf{v}$, provided:

$$\int_0^T \phi'(t)\mathbf{u}(t)dt = -\int_0^T \phi(t)\mathbf{v}(t)dt \tag{3.144}$$

for all scalar functions $\phi \in C_c^\infty(0,T)$, where integrals are taken componentwise.

Definition 3.6.4. (i) The Sobolev space $W^{1,p}(0,T;X)$ consists of all functions $\mathbf{u} \in L^p(0,T;X)$ such that $\mathbf{u}'$ exists in the weak sense and belongs to $L^p(0,T;X)$. Furthermore,

$$||\mathbf{u}||_{W^{1,p}(0,T;X)} := \begin{cases} (\int_0^T \{||\mathbf{u}(t)||^p + ||\mathbf{u}'(t)||^p\}dt)^{\frac{1}{p}} < \infty \quad 1 \leq p < \infty \\ ess\ sup_{0\leq t\leq T}(||\mathbf{u}(t)|| + ||\mathbf{u}'(t)||) \end{cases} \tag{3.145}$$

(ii) We write $H^1(0,T;X) = W^{1,2}(0,T;X)$.

We have the following result:

Theorem 3.6.1. *Let $\boldsymbol{u} \in W^{1,p}(0,T;X)$ for some $1 \leq p \leq \infty$. Then:*

(i) $\boldsymbol{u} \in C([0,T];X)$
(ii)$\boldsymbol{u}(t) = \boldsymbol{u}(s) + \int_s^t \boldsymbol{u}'(\tau)d\tau \quad for\ all\ 0 \leq s \leq t \leq T$
(iii) Furthermore, we have the estimate:

$$max_{0\leq t\leq T}||\boldsymbol{u}(t)|| \leq C||\boldsymbol{u}||_{W^{1,p}(0,T;X)} \tag{3.146}$$

the constant C depending only on T.

Proof. Extend $\mathbf{u}$ to be $\mathbf{0}$ on $]-\infty,0[$ and $]T,\infty[$, and set $\mathbf{u}^\epsilon := \eta_\epsilon \star \mathbf{u}$, η_ϵ denoting the usual mollifier on $\mathbb{R}$. Then $(\mathbf{u}^\epsilon)' = \eta_\epsilon \star \mathbf{u}'$ on $]\epsilon, T-\epsilon[$. Then, as $\epsilon \to 0$,

$$\begin{cases} \mathbf{u}^\epsilon \to \mathbf{u} \quad in\ L^p(0,T;X) \\ (\mathbf{u}^\epsilon)' \to \mathbf{u}' \quad in\ L^p(0,T;X) \end{cases} \tag{3.147}$$

Fixing $0 < s < t < T$, we compute

$$\mathbf{u}^\epsilon(t) = \mathbf{u}^\epsilon(s) + \int_s^t (\mathbf{u}^\epsilon)'(\tau) d\tau \tag{3.148}$$

Thus, in the limit,

$$\mathbf{u}(t) = \mathbf{u}(s) + \int_s^t \mathbf{u}'(\tau) d\tau \tag{3.149}$$

for a.e. $0 < s < t < T$. As the mapping $t \to \int_0^t \mathbf{u}'(\tau) d\tau$ is continuous, assertions (i) and (ii) follow. The estimate now follows easily. □

We are now interested in what happens when the functions $\mathbf{u}$ and $\mathbf{u}$' lie in different spaces.

Theorem 3.6.2. *Suppose $\boldsymbol{u} \in L^2(0, T; H_0^1(U))$, with $\boldsymbol{u}' \in L^2(0, T; H^{-1}(U))$. Then:*

(i) $\boldsymbol{u} \in C([0, T]; L^2(U))$
after possibly being redifined on a set of zero measure.
(ii) The mapping $t \to ||\boldsymbol{u}(t)||^2_{L^2(U)}$ is absolutely continuous, with

$$\frac{d}{dt} ||\boldsymbol{u}(t)||^2_{L^2(U)} = 2 < \boldsymbol{u}'(t), \boldsymbol{u}(t) > \tag{3.150}$$

for a.e. $0 \leq t \leq T$.
(iii) Furthermore, we have the estimate

$$max_{0 \leq t \leq T} ||\boldsymbol{u}(t)||_{L^2(U)} \leq C(||\boldsymbol{u}(t)||_{L^2(0,T;H_0^1(U))} + ||\boldsymbol{u}'||_{L^2(0,T;H^{-1}(U))}) \tag{3.151}$$

the constant C depending only on T.

Proof. Extend $\mathbf{u}$ to the larger interval $[-\sigma, T + \sigma]$ for $\sigma > 0$ ad consider again the mollification $\mathbf{u}^\epsilon = \eta_\epsilon \star \mathbf{u}$. Then, for $\epsilon, \delta > 0$,

$$\frac{d}{dt} ||\mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t)||^2_{L^2(U)} = 2((\mathbf{u}^\epsilon)'(t) - (\mathbf{u}^\delta)'(t), \mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t))_{L^2(U)} \tag{3.152}$$

Thus:

$$||\mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t)||^2_{L^2(U)} = ||\mathbf{u}^\epsilon(s) - \mathbf{u}^\delta(s)||^2_{L^2(U)} + 2 \int_s^t < (\mathbf{u}^\epsilon)'(\tau) - (\mathbf{u}^\delta)'(\tau), \mathbf{u}^\epsilon(\tau) - \mathbf{u}^\delta(\tau) > d\tau \tag{3.153}$$

for all $0 \leq s, t \leq T$. Fix any point $s \in (0, T)$ for which $\mathbf{u}^\epsilon(s) \to \mathbf{u}(s)$ in $L^2(U)$. Consequently, (3.153) implies:

$$\begin{aligned} lim\ sup_{\epsilon,\delta \to 0}\ \ sup_{\,0 \leq t \leq T}\ ||\mathbf{u}^\epsilon(t) - \mathbf{u}^\delta(t)||^2_{L^2(U)} \leq lim_{\epsilon,\delta \to 0} \int_0^T \{||(\mathbf{u}^\epsilon)'(\tau) - (\mathbf{u}^\delta)'(\tau)||^2_{H^1(U)} \\ + ||\mathbf{u}^\epsilon(\tau) - \mathbf{u}^\delta(\tau)||^2_{H_0^1(U)}||\} d\tau = 0 \end{aligned} \tag{3.154}$$

Thus the smoothed functions $\{\mathbf{u}^\epsilon\}_{0<\epsilon\leq 1}$ converge in $C([0,T];L^2(U))$ to a limit $\mathbf{v} \in C([0,T];L^2(U))$. Since we also know $\mathbf{u}^\epsilon(t) \to \mathbf{u}(t)$ for a.e. t, we deduce $\mathbf{u} = \mathbf{v}$ a.e.

We similarly have:

$$||\mathbf{u}^\epsilon(t)||^2_{L^2(U)} = ||\mathbf{u}^\epsilon(s)||^2_{L^2(U)} + 2\int_s^t < (\mathbf{u}^\epsilon)'(\tau), \mathbf{u}^\epsilon(\tau) > d\tau \tag{3.155}$$

and so, identifying $\mathbf{u}$ with $\mathbf{v}$ as above,

$$||\mathbf{u}(t)||^2_{L^2(U)} = ||\mathbf{u}(s)||^2_{L^2(U)} + 2\int_s^t < \mathbf{u}'(\tau), \mathbf{u}(\tau) > d\tau \tag{3.156}$$

for all $0 \leq s, t \leq T$.

To obtain (3.151) we integrate (3.156) with respect to s, recall the Cauchy-Schwarz inequality $| < \mathbf{u}', \mathbf{u} > | \leq ||\mathbf{u}'||_{H^{-1}(U)}||\mathbf{u}||_{H^1_0(U)}$ and make some simple estimates. □

When we will study the regularity of the weak solutions of the wave equation, we will also need the following:

Theorem 3.6.3. *Assume ∂U is smooth. Take m to be a nonnegative integer. Suppose $\boldsymbol{u} \in L^2(0,T;H^{m+2}(U))$, with $\boldsymbol{u}' \in L^2(0,T;H^m(U))$. Then:*

(i) $\boldsymbol{u} \in C([0,T];H^{m+1}(U))$
after possibly being redefined on a set of zero measure.
(ii) Furthermore, we have the estimate

$$max_{0\leq t\leq T}||\boldsymbol{u}(t)||_{H^{m+1}(U)} \leq C(||\boldsymbol{u}(t)||_{L^2(0,T;H^{m+2}_0(U))} + ||\boldsymbol{u}'||_{L^2(0,T;H^m(U))}) \tag{3.157}$$

the constant C depending only on T, U and m.

Proof. Suppose first that $m = 0$, in which case:

$$\mathbf{u} \in L^2(0,T;H^2(U)) \quad \mathbf{u}' \in L^2(0,T;L^2(U)) \tag{3.158}$$

We select a bounded open set V, with $U \subset\subset V$, and then construct a corresponding extension $\bar{u} = E\mathbf{u}$. Then $\bar{u} \in L^2(0,T;H^2(V))$ and

$$||\bar{u}||_{L^2(0,T;H^2(V))} \leq C||\mathbf{u}||_{L^2(0,T;H^2(U))} \tag{3.159}$$

for an appropriate constant C. It can be proven in addition that $\bar{u}' \in L^2(0,T;L^2(V))$, with the estimate

$$||\bar{u}'||_{L^2(0,T;L^2(V))} \leq C||\mathbf{u}'||_{L^2(0,T;L^2(U))} \tag{3.160}$$

Assume for the moment that $\bar{u}$ is smooth. We then compute:

$$\begin{aligned}|\frac{d}{dt}(\int_V |D\bar{u}|^2 dx)| = 2|\int_V D\bar{u}\cdot D\bar{u}' dx| = 2|\int_V \Delta\bar{u}\bar{u}' dx|\\ \leq C(||\bar{u}||^2_{H^2(V)} + ||\bar{u}'||^2_{L^2(V)})\end{aligned} \tag{3.161}$$

There is no boundary term when we integrate by parts, since the extension $\bar{u}$ has compact support within V. Integrating and recalling (3.159),(3.160), it follows that;

$$max_{0\leq t\leq T}||\mathbf{u}(t)||_{H^1(U)} \leq C(||\mathbf{u}||_{L^2(0,T;H^2(U))} + ||\mathbf{u}'||_{L^2(0,T;L^2(U))}) \tag{3.162}$$

We obtain the same estimate if $\mathbf{u}$ is not smooth, upon approximating by $\mathbf{u}^\epsilon := \eta_\epsilon \star \mathbf{u}$. As in the previous proof, it also follows that $\mathbf{u} \in C([0,T]; H^1(U))$.

In the general case that $m \geq 1$, we let α be a multiindex of order $|\alpha| \leq m$, and set $\mathbf{v} := D^\alpha \mathbf{u}$. Then:

$$\mathbf{v} \in L^2(0,T;H^2(U)) \quad \mathbf{v}' \in L^2(0,T;L^2(U)) \tag{3.163}$$

We apply estimate (3.162), with $\mathbf{v}$ replacing $\mathbf{u}$ and sum over all indices $|\alpha| \leq m$ to derive (3.157). $\square$

Chapter 4

Weak solutions

In this chapter we apply the theory of Sobolev Spaces to the study of initial boundary value problems for the wave equation in n dimensions. In particular, we look for *weak solutions* to these problems; that is, functions mapping time into Sobolev spaces which satisfy the wave equation in an appropriate weak sense.
We begin by defining weak solutions and outlining the main steps of the weak approach in the theory of partial differential equations. Next we focus the attention on the homogeneous problem, setting the external force equal to zero and we prove, using *Hille-Yosida's theory* for first order ordinary systems, that, under appropriate assumptions on the initial data, a weak solution exists and it is unique. We then study the regularity of this solution and we see that, if we allow the initial data to possess higher regularity, we can recover a classical solution to the problem.
Next we consider the nonhomogeneous problem restricted to bounded domains and we show that we can explicitly construct the solutions using the so-called *Galerkin's method.* This approach also permits us to prove existence and uniqueness of weak solution in bounded domains with more general initial data than those required for the application of Hille-Yosida's theory.

4.1 Basic definitions

In what follows U will indicate an open subset of $\mathbb{R}^n$ with C^∞ boundary. We will also set $c = 1$.

Fix $T > 0$ and consider the following *initial boundary value problem* for the wave equation, with homogeneous Dirichlet boundary conditions:

$$\begin{cases} \Box u = f \;\; in\ U_T \\ u = g, \;\; u_t = h \;\; on\ U \times \{0\} \\ u = 0 \;\; on\ \partial U_T \end{cases} \tag{4.1}$$

with $g \in H^2(U) \cap H_0^1(U)$, $h \in H_0^1(U)$.

We would like to find an appropriate weak formulation of the problem in order to look for solutions that satisfy it in the weak sense.
We begin by supposing $u = u(x,t)$ is a smooth solution to (4.1) and defining the associated map $\mathbf{u} : [0,\infty[\to H^2(U) \cap H_0^1(U)$, by $[\mathbf{u}(t)](x) = u(x,t)$. We similarly introduce the function $\mathbf{f} : [0,\infty[\to L^2(U)$ defined by $[\mathbf{f}(t)](x) = f(x,t)$.
Now fix any function $v \in H_0^1(U)$, multiply the wave equation by v and integrate over U.

$$\int_U \mathbf{u}'' v dx - \int_U \nabla^2 \mathbf{u} v dx = \int_U \mathbf{f} v dx \tag{4.2}$$

Integrating the first term by parts, we obtain:

$$\int_U \{\mathbf{u}'' v + \nabla \mathbf{u} \cdot \nabla v - \mathbf{f} v\} dx = 0 \tag{4.3}$$

This leads to the definition of *weak solutions*.

Definition 4.1.1 We say a function $\mathbf{u} : [0,\infty] \to H^2(U) \cap H_0^1(U)$ is a *weak solution* of the problem (4.1), provided:

$$\mathbf{u} \in C([0,T]; H^2(U) \cap H_0^1(U)), \quad \mathbf{u}' \in C([0,T]; H_0^1(U)), \quad \mathbf{u}'' \in C([0,T]; L^2(U)) \tag{4.4}$$
$$\mathbf{u}(0) = g, \quad \mathbf{u}'(0) = h \tag{4.5}$$

and furthermore $\mathbf{u}$ satisfies (4.3) for a.e. $t \in [0,T]$.

Our strategy in solving problems involving function in Sobolev spaces consists in the following steps:

Step A. Existence and uniqueness of a weak solution is established.

Step B. The weak solution is proved to posses higher regularity, under appropriate assumptions on initial data.

Step C. A classical solution is recovered by showing that any weak solution that is $C^{2,2}(U_T) \cap C^{1,1}(\bar{U}_T)$ is a classical solution.

4.2 Step A : Existence and uniqueness

We prove uniqueness of weak solution to the *homogeneous initial boundary value problem* (i.e, we set $f = 0$ in (4.1)), using *Hille-Yosida's theory* for *maximal monotone operators* on Hilbert spaces.

Definition 4.2.1. Let H be a Hilbert space, $A : D(A) \subset H \to H$ an unbounded operator on H.
(i) We say A is monotone if

$$< Av, v > \geq 0 \quad \forall v \in H \tag{4.6}$$

(ii) A is said to be maximal monotone if in addiction $A + I$ is surjective, where I denotes the identity operator.

Hille-Yosida theorem establishes existence and uniqueness of the solution to first order Cauchy problems involving maximal monotone operators.

Theorem 4.2.1. *Let A be a maximal monotone operator on a Hilbert space H. Then, given any $u_0 \in D(A)$, there exists a unique function*

$$u \in C^1([0,T]; H) \cap C([0,T]; D(A)) \tag{4.7}$$

satisfying

$$\begin{cases} \frac{du}{dt} + Au = 0 & on\ [0,T] \\ u(0) = u_0 \end{cases} \tag{4.8}$$

Moreover:

$$||u_t|| \leq ||u_0||, \quad ||Au(t)|| \leq ||Au_0|| \quad \forall\, t \in [0,T] \tag{4.9}$$

The proof is rather long and we omit it. The reader can find it in [3], Chapter 7.

Now we are ready to prove the following result:

Theorem 4.2.2. *Assume $g \in H^2(U) \cap H_0^1(U)$, $h \in H_0^1(U)$. Then there exists a unique weak solution to the problem* (4.1)*, with $f = 0$. Moreover:*

$$||\boldsymbol{u}'||^2_{L^2(U)} + ||\nabla \boldsymbol{u}||^2_{L^2(U)} = ||h||^2_{L^2(U)} + ||\nabla g||^2_{L^2(U)} \quad \forall t > 0 \tag{4.10}$$

Remark 4.2.1. Equation (4.10) is a *conservation law* asserting that the energy of the system is invariant in time.

Proof. We write the wave equation in the form of a system of first order equations:

$$\begin{cases} \partial u_t - v = 0 & in\ U_T \\ \partial v_t - \nabla^2 u = 0 & in\ U_T \end{cases} \tag{4.11}$$

and we set $V := \begin{pmatrix} u \\ v \end{pmatrix}$, so that (4.11) takes the form $\frac{dV}{dt} + AV = 0$, with

$$AV = \begin{pmatrix} 0 & -I \\ -\nabla^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ -\nabla^2 u \end{pmatrix} \tag{4.12}$$

We now apply the Hille-Yosida's theory in the space $H = H_0^1(U) \times L^2(U)$, equipped with the inner product

$$< V_1, V_2 >:= \int_U \{\nabla u_1 \cdot \nabla u_2 + u_1 u_2 + v_1 v_2\} dx \tag{4.13}$$

Now consider the unbounded operator $A : D(A) \subset H \to H$ defined by (4.12), with

$$D(A) = (H^2(U) \cap H_0^1(U)) \times H_0^1(U) \tag{4.14}$$

Note that the boundary condition $u = 0$ *on* ∂U_T has been incorporated in the definition of the space H.

Now we claim that $A + I$ is maximal monotone in H. Indeed, if $V \in D(A)$, we have:

$$< AV, V >_H + ||V||_H^2 = \int_U \{u^2 + v^2 + |\nabla u|^2 - \nabla u \cdot \nabla v - uv - \nabla^2 uv\} dx \tag{4.15}$$

$$= \int_U \{u^2 + v^2 + |\nabla u|^2 - uv\} dx \geq 0 \tag{4.16}$$

Next we have to show that $A + 2I$ is surjective. Given $F = \begin{pmatrix} f \\ g \end{pmatrix}$, we must solve the equation $(A + 2I)V = F$; that is, the system

$$\begin{cases} -v + 2u = f \\ -\nabla^2 u + 2v = g \end{cases} \tag{4.17}$$

with $u \in H^2(U) \cap H_0^1(U)$, $v \in H_0^1(U)$.
It follows from (4.17) that $-\nabla^2 u + 4u = 2f + g$. This equation has a unique solution $u \in H^2(U) \cap H_0^1(u)$, according to the teory of elliptic partial differential equations. Then we obtain $v \in H_0^1(U)$ by taking $v = 2u - f$. This proves that $A + I$ is maximal monotone in H.

We can therefore apply Hille-Yosida's theorem to conclude that there exists a unique solution of the problem

$$\begin{cases} \frac{dV}{dt} + AV = 0 \quad in\ U \times [0, \infty[\\ V(0) = V_0 \end{cases} \tag{4.18}$$

with $V \in C^1([0,T];H) \cap C([0,T];D(A))$.
Since $V_0 = \begin{pmatrix} g \\ h \end{pmatrix} \in D(A)$, we obtain the existence and uniqueness of the homogeneous problem.

In order to prove (4.10), we multiply the wave equation by $\frac{\partial u}{\partial t}$ and we integrate over U. □

4.3 Step B: improved regularity

We prove that the solution to (4.1) has a higher regularity than that obtained in the previous section, provided one makes additional assumptions on the initial data.

We use the same notations as in step A.

Let $k \in \mathbb{N}$, $k \geq 2$. Define by induction the set

$$D(A^k) = \{v \in D(A^{k-1}) : Av \in D(A^{k-1})\} \tag{4.19}$$

It can be easily seen that $D(A^k)$ is a Hilbert space with the scalar product

$$< u, v >_{D(A^k)} = \sum_{j=0}^{k} < A^j u, A^j v > \tag{4.20}$$

for $u, v \in D(A^k)$.
We have the following result:

Theorem 4.3.1. *Assume the initial data $u_0 \in D(A^k)$ for some integer $k \geq 2$. Then the solution u to problem* (4.8) *satisfies:*

$$u \in C^{k-j}([0,T]; D(A^j)) \quad \forall j = 0, \ldots k \tag{4.21}$$

The reader can find a proof in [3], chapter 7.

Now it is easy to prove a regularity theorem for the solutions to (4.1), with $f = 0$.

Theorem 4.3.2. *Assume that the initial data satisfy*

$$g \in H^k(U) \quad h \in H^k(U) \quad \forall k \tag{4.22}$$

and the compatibility conditions on the boundary

$$\nabla^2 g = 0 \quad \nabla^2 h = 0 \quad \text{on } \partial U \;\; \forall\, j \geq 0, \;\; j \in \mathbb{N} \tag{4.23}$$

Then the solution to (4.1) *with $f = 0$ belongs to $C^\infty(\bar{U}_T)$.*

Proof. Let A be the operator in Theorem 4.2.2. It is easy to see, by induction on k, that:

$$D(A^k) = \{\begin{pmatrix} u \\ v \end{pmatrix} : \begin{pmatrix} u \in H^{k+1}(U) \text{ and } \nabla^2 u = 0 \text{ on } \partial U \ \forall\, j,\ 0 \le j \le [\frac{k}{2}] \\ v \in H^{k+1}(U) \text{ and } \nabla^2 v = 0 \text{ on } \partial U \ \forall\, j,\ 0 \le j \le [\frac{k+1}{2}] - 1 \end{pmatrix}\} \tag{4.24}$$

In particular, $D(A^k) \subset H^{k+1}(U) \times H^k(U)$ with continuous injection. Applying Theorem 4.3.1., we see that if $V_0 = \begin{pmatrix} g \\ h \end{pmatrix} \in D(A^k)$, then the solution $V = \begin{pmatrix} u \\ v \end{pmatrix}$ to (4.11) satisfies:

$$V \in C^{k-j}([0,T]; D(A^j)) \quad \forall j = 0, ..., k \tag{4.25}$$

thus $u \in C^{k-j}([0,\infty[; H^{j+1}(U))\ \forall j = 0, ..., k$. We conclude with the help of Theorem 3.4.6., that $u \in C^k(\bar{U}_T)$ for each k. □

4.4 Step C: recovery of classical solutions

Theorem 4.4.1. *Let* $\boldsymbol{u} \in C^2([0,T]; H^2(U) \cap H_0^1(U))$ *be a weak solution of* (4.1)*. Then,* $\boldsymbol{u}$ *is indeed a classical solution.*

Proof. Since $u(x,t) \in C^{2,2}(U_T) \cap C^{1,1}(\bar{U}_T)$, $Tu = u|_{\partial U}$ according to Theorem 3.3.2. and, since $u \in H_0^1(U)$, $Tu = 0$, according to Theorem 3.3.3. Therefore $u = 0$ on ∂U_T. On the other hand, we have:

$$\int_{U_T} (\mathbf{u}'' - \nabla^2 \mathbf{u} - \mathbf{f}) v dx = 0 \tag{4.26}$$

for all $v \in C_c^1(U)$. This implies $\mathbf{u}'' - \nabla^2 \mathbf{u} - \mathbf{f}$ a.e. on U_T.
Since $u(x,t) \in C^{2,2}(U_T) \cap C^{1,1}(\bar{U}_T)$, the equality in fact holds everywhere and thus u is a classical solution. □

4.5 Bounded domain case

We now consider $U \in \mathbb{R}^n$ to be open and bounded and we see that, under weaker absuntions on initial data than those given in Theorem 4.2.2., we can construct explicitly weak solutions to (4.1), using the so-called *Galerkin's method*.

We start by slightly modify the definition of weak solutions, in order to deal with weaker conditions on initial data and reformulate the problem in terms of the dual space $H^{-1}(U)$.
Suppose the initial data satisfy $g \in H_0^1(U)$, $h \in L^2(U)$.
Since the Dirichlet condition requires $u = 0$ on the boundary, we see that the natural space in which looking for solutions is $L^2([0,T], H_0^1(U))$. Furthermore, since we want

$\mathbf{u}(0) = g$, recalling Theorem 3.6.2., it is natural to demand $\mathbf{u}' \in L^2([0,T]; L^2(U))$. Finally, looking at the wave equation, we see that u_{tt} has the form:

$$u_{tt} = f + \nabla^2 u = g^0 + \sum_{j=1}^{n} g^j_{x_j} \tag{4.27}$$

with $g^0 = f$, $g^j = u_{x_j}$.
This suggests , recalling Theorem 3.5.1., that we should look for a weak solution $\mathbf{u}$ with $\mathbf{u}'' \in H^{-1}(U)$ for a.e. t and then reinterpret the first term in (4.2) as $< \mathbf{u}'', v >_{H^{-1}(U),H^1_0(U)}$, where $<,>_{H^{-1}(U),H^1_0(U)}$ denotes the pairing between $H^{-1}(U)$ and $H^1_0(U)$.

Definition 4.5.1 We say a function

$$\mathbf{u} \in L^2([0,T]; H^1_0(U)) \quad \mathbf{u}' \in L^2([0,T]; L^2(U)) \quad \mathbf{u}'' \in L^2([0,T]; H^{-1}(U)) \tag{4.28}$$

is a weak solution to the problem (4.1), with $g \in H^1_0(U)$, $h \in L^2(U)$., provided:

(i) $< \mathbf{u}'', v >_{H^{-1}(U),H^1_0(U)} + \sum_{i=1}^{n} < \mathbf{u}_{x_i}, v_{x_i} >_{L^2(U)} = < \mathbf{f}, v >_{L^2(U)}$ for each $v \in H^1_0(U)$ and a.e. $t \in [0,T]$.
(ii) $\mathbf{u}(0) = g$, $\mathbf{u}'(0) = h$

We now construct the weak solution to the problem (4.1) employing Galerkin's mehod. We start by solving a finite dimensional approximation: select smooth functions $\{w_k\}_{k\in\mathbb{N}}$ such that:

(i) $\{w_k\}_{k=1}^{\infty}$ is an orthogonal complete system for the space $H^1_0(U)$
(ii) $\{w_k\}_{k=1}^{\infty}$ is an orthonormal complete system for the space $L^2(U)$.

Next, fix a positive integer m, and write

$$\mathbf{u}_m(t) := \sum_{k=1}^{m} d^k_m(t) w_k \tag{4.29}$$

where we intend to select the coefficients $d^k_m(t)$ in order to satisfy

$$d^k_m(0) = < g, w_k >_{H^1_0(U)} \tag{4.30}$$

$$d'^k_m(0) = < h, w_k >_{L^2(U)} \tag{4.31}$$

and

$$< \mathbf{u}''_m, w_k >_{H^{-1}(U),H^1_0(U)} + \sum_{i=1}^{n} < \mathbf{u}_{m_{x_i}}, w_{k_{x_i}} >_{L^2(U)} = < \mathbf{f}, w_k >_{L^2(U)} \tag{4.32}$$

Theorem 4.5.1. *For each integer m, there exists a unique function $\boldsymbol{u}_m$ of the form* (4.29)*, satisfying* (4.30)*,* (4.31) *and* (4.32)*.*

Proof. Assuming $\mathbf{u}_m$ to be given by (4.29), we project (4.32) on the w_ks, obtaining:

$$d_m''^k(t) + \sum_{i=1}^{n} e^{kl}(t) d_m^l(t) \ = \ f^k(t) \tag{4.33}$$

where $e^{kl} \ = < w_{l_{x_i}}, w_{k_{x_i}} >_{H_0^1(U)}$ and $f^k(t) \ = < \mathbf{f}(t), w_k >_{L^2(U)}$.
(4.33), together with the initial conditions (4.30) and (4.31), is a linear system of ordinary differential equations and, according to standard theory of ordinary differential equations, there exists a unique function $\mathbf{d}_m(t) = (d_m^1(t), ..., d_m^m(t))$, that is of class C^2 and solves the problem. □

Now we want to send $m \to \infty$ and so we need estimates that ensure uniform convergence in m. The following result fulfills this purpose.

Theorem 4.5.2. *There exists a constant C, depending only on U, such that:*

$$max_{0 \le t < \infty}(||\boldsymbol{u}_m||_{H_0^1(U)} + ||\boldsymbol{u}_m'||_{L^2(U)}) + ||\boldsymbol{u}_m''||_{L^2([0,T];H^{-1}(U))} \tag{4.34}$$

$$\le C(||\boldsymbol{f}||_{L^2([0,T];L^2(U))} + ||g||_{H_0^1(U)} + ||h||_{L^2(U)}) \tag{4.35}$$

for each m.

The reader can find the proof in [4], chapter 7.

Now we can prove uniqueness of the weak solution.

Theorem 4.5.3. *There exists a weak solution of* (4.1) *in the sense of Definition 4.5.1.*

Proof. According to (4.34), we see that the sequence $\{\mathbf{u}_m\}_{m=1}^\infty$ is bounded in $L^2([0,T]; H_0^1(U))$, $\{\mathbf{u}_m'\}_{m=1}^\infty$ is bounded in $L^2([0,T]; L^2(U))$, $\{\mathbf{u}_m''\}_{m=1}^\infty$ is bounded in $L^2([0,T]; H^{-1}(U))$.
As a consequence, there exists a subsequence $\{\mathbf{u}_{m_l}\}_{m=1}^\infty$ and functions $\mathbf{u} \in L^2([0,T]; H_0^1(U))$, $\mathbf{u}' \in L^2([0,T]; L^2(U))$, $\mathbf{u}'' \in L^2([0,T]; H^{-1}(U))$, such that:

$$\begin{cases} \mathbf{u}_{m_l} \to \mathbf{u} \ \ \textit{weakly in} \ \ L^2([0,T]; H_0^1(U)) \\ \mathbf{u}_{m_l}' \to \mathbf{u}' \ \ \textit{weakly in} \ \ L^2([0,T]; L^2(U)) \\ \mathbf{u}_{m_l}'' \to \mathbf{u}'' \ \ \textit{weakly in} \ \ L^2([0,T]; H^{-1}(U)) \end{cases} \tag{4.36}$$

(see Appendix A for the notion of weak convergence in Hilbert spaces).

Next, fix an integer N and choose a function $\mathbf{v} \in C^1([0,T]; H_0^1(U))$ of the form:

$$\mathbf{v}(t) = \sum_{k=1}^{N} d^k(t) w_k \tag{4.37}$$

where $d^k(t)$ are smooth functions. Select $m \geq N$, multiply (4.32) by $d^k(t)$, sum over $k = 1, ..., N$ and then integrate with respect to t, to discover:

$$\int_0^T (< \mathbf{u}_m'', \mathbf{v} >_{H^{-1}(U),H_0^1(U)} + \sum_{i=1}^n < \mathbf{u}_{m_{x_i}}, \mathbf{v}_{x_i} >_{L^2(U)})dt \ = \ \int_0^T < \mathbf{f}, \mathbf{v} >_{L^2(U)} dt \quad (4.38)$$

Now set $m = m_l$ and, taking the limit, we find:

$$\int_0^T (< \mathbf{u}'', \mathbf{v} >_{H^{-1}(U),H_0^1(U)} + \sum_{i=1}^n < \mathbf{u}_{x_i}, \mathbf{v}_{x_i} >_{L^2(U)})dt \ = \ \int_0^T < \mathbf{f}, \mathbf{v} >_{L^2(U)} dt \quad (4.39)$$

This equality then holds for all functions $\mathbf{v} \in C^1([0,T]; H_0^1(U))$, since functions of the form (4.37) are dense in this space. It follows further that:

$$< \mathbf{u}'', v >_{H^{-1}(U),H_0^1(U)} + \sum_{i=1}^n < \mathbf{u}_{x_i}, v_{x_i} >_{L^2(U)} = < \mathbf{f}, v >_{L^2(U)} \quad (4.40)$$

for all $v \in H_0^1$ and a.e. t.

We must now verify that the initial conditions are satisfied.
Choose any function $\mathbf{v} \in C^2([0,T]; H_0^1(U))$, that vanishes at $t = T$. Then, integrating by parts twice with respect to t in (4.39), we find:

$$\begin{aligned} &\int_0^T (< \mathbf{v}'', \mathbf{u} >_{H^{-1}(U),H_0^1(U)} + \sum_{i=1}^n < \mathbf{u}_{x_i}, \mathbf{v}_{x_i} >_{L^2(U)})dt \\ &= \int_0^T < \mathbf{f}, \mathbf{v} >_{L^2(U)} dt - < \mathbf{u}(0), \mathbf{v}'(0) >_{H_0^1(U)} + < \mathbf{u}'(0), \mathbf{v}(0) >_{L^2(U)} \end{aligned} \quad (4.41)$$

Similarly, from (4.38), we find:

$$\begin{aligned} &\int_0^T (< \mathbf{v}'', \mathbf{u}_m >_{H^{-1}(U),H_0^1(U)} + \sum_{i=1}^n < \mathbf{u}_{m_{x_i}}, \mathbf{v}_{x_i} >_{L^2(U)})dt \\ &= \int_0^T < \mathbf{f}, \mathbf{v} >_{L^2(U)} dt - < \mathbf{u}_m(0), \mathbf{v}'(0) >_{H_0^1(U)} + < \mathbf{u}_m'(0), \mathbf{v}(0) >_{L^2(U)} \end{aligned} \quad (4.42)$$

Setting $m = m_l$ and taking the limit:

$$\begin{aligned} &\int_0^T (< \mathbf{v}'', \mathbf{u} >_{H^{-1}(U),H_0^1(U)} + \sum_{i=1}^n < \mathbf{u}_{x_i}, \mathbf{v}_{x_i} >_{L^2(U)})dt \\ &= \int_0^T < \mathbf{f}, \mathbf{v} >_{L^2(U)} dt - < g, \mathbf{v}'(0) >_{H_0^1(U)} + < h, \mathbf{v}(0) >_{L^2(U)} \end{aligned} \quad (4.43)$$

Comparing (4.41) and (4.43), we conclude $\mathbf{u}(0) = g$, $\mathbf{u}'(0) = h$, since $\mathbf{v}$ is arbitrary. □

The next steps consist in proving uniqueness and higher regularity of weak solution, in order to recover from this the classical solution to the problem. We give only the statements of these results; the reader can find proofs in [4], chapter 7.

Theorem 4.5.4. *A weak solution of* (4.1)*, in the sense of Definition 4.5.1., is unique.*

Theorem 4.5.5. *Let m be a nonnegative integer and assume:*

$$\begin{cases} g \in H^{m+1}(U) \\ h \in H^m(U) \\ \frac{d^k \boldsymbol{f}}{dt^k} \in L^2([0,T]; H^{m-k}(U)) \quad for\ k = 0, ..., m \end{cases} \tag{4.44}$$

Suppose also the following m^{th}-order compatibility conditions hold:

$$\begin{cases} g_0 := g \in H_0^1(U) \\ h_1 := h \in H^m(U) \\ g_{2l} := \frac{d^{2l-2}\boldsymbol{f}}{dt^{2l-2}}(.,0) + \nabla^2 g_{2l-2} \in H_0^1(U) \quad if\ m = 2l \\ h_{2l+1} := \frac{d^{2l-1}\boldsymbol{f}}{dt^{2l-1}}(.,0) + \nabla^2 h_{2l-1} \in H_0^1(U) \quad if\ m = 2l+1 \end{cases} \tag{4.45}$$

Then:

$$\frac{d^k \boldsymbol{u}}{dt^k} \in L^\infty([0,T]; H^{m+1-k}(U)) \quad for\ k = 0, ..., m+1 \tag{4.46}$$

with the estimate

$$ess\ sup_{0 \le t < \infty} \sum_{k=0}^{m+1} ||\frac{d^k \boldsymbol{u}}{dt^k}||_{H^{m+1-k}(U)} \le C(\sum_{k=0}^{m} ||\frac{d^k \boldsymbol{f}}{dt^k}||_{H^{m-k}(U)} + ||g||_{H^{m+1}(U)} + ||h||_{H^m(U)}) \tag{4.47}$$

Corollary 4.5.1. *Assume $g, h \in C^\infty(\bar{U})$, $f \in C^\infty(\bar{U}_T)$ and the m^{th}-order compatibility conditions* (4.45) *hold for every m.*
Then the problem (4.1) *has a unique solution*

$$u \in C^\infty(\bar{U}_T) \tag{4.48}$$

Chapter 5

Physical applications

In this final chapter we discuss two relevant physical situations in which wave equation emerges in a natural way. We begin by study Maxwell equations and we show that, if we choose a particular gauge, equations for *scalar and vector potentials* turns into nonhomogeneous wave equations, depending on *charges and currents distributions* in space and time. In this way we see that Maxwell theory predicts the existence of *electromagnetic waves*, that are periodic oscillations of electric and magnetic fields propagating in space. We then explicitely solve the equations for the case of a point charged particle moving arbitrarily and we find *electric and magnetic fields* emitted by the particle.
In the second part of the chapter we study gravitational waves, starting by introducing linearized gravity theory and then showing that *Einstein equations* turns into wave equation in an appropriate gauge. We next study the effects of gravitational waves on test masses in different frame of references and the waves emission by a distribution of mass and energy, finding the explicit expression for the radiation emitted by a binary system in circular orbit.
Notions of differential geometry and general relativity are required in order to completely understand the topics discussed in the second part of this chapter. The reader can find an overview of these topics in [2].

5.1 Electromagnetic waves

We find equations describing electromagnetic waves from Maxwell equations.

5.1.1 Maxwell equations

The entire electromagnetic theory is encoded in Maxwell equations. They are a set of eight (two scalar and two vectorial) partial differential equations that must be satisfied by the components of electric and magnetic fields and relating them with charges and currents distributions. In SI units they read as:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{5.1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{5.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{5.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{5.4}$$

where $\mathbf{j}$, ρ are the charge and current density and μ_0, ϵ_0 are the vacuum permeability and the vacuum permittivity.
The fact that $\mathbf{B}$ is divergence-free implies the existence of a vector $\mathbf{A}$, called vector potential, such that:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{5.5}$$

Inserting (5.5) in (5.2), we find $\nabla \times (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) = 0$. Therefore, there exists a scalar function Φ, called scalar potential, satisfying:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi \tag{5.6}$$

It turns out that Maxwell equations are better exspressed using potentials. Inserting (5.5), (5.6) into (5.1), (5.2), (5.3), (5.4), we obtain the equations

$$\nabla^2 \Phi + \frac{\partial}{\partial t}\nabla \cdot \mathbf{A} = -\frac{\rho}{\epsilon_0} \tag{5.7}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \mathbf{A}}{\partial t}) = -\mu_0 \mathbf{j} \tag{5.8}$$

Since $\mathbf{B} = \nabla \times \mathbf{A}$, the value of the magnetic field is unaffected by a transformation of the form

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla f \tag{5.9}$$

where f is a scalar function. Under this transformation, the electric field changes as

$$\mathbf{E} \to \mathbf{E}' = -\frac{\partial \mathbf{A}}{\partial t} - \nabla(\Phi + \frac{\partial f}{\partial t}) \tag{5.10}$$

Thus, if Φ transforms as

$$\Phi \to \Phi' = \Phi - \frac{\partial f}{\partial t} \tag{5.11}$$

we see that also the electric field remains unchanged.
These potentials transformations that do not affect the values of the fields are called *gauge transformations*. Since the fields does not vary, gauge transformations do not

change the physical effects.

We can now use gauge freedom to choose a condition satisfied by the potentials that simplifies the Maxwell equations; indeed, looking at (5.7) and (5.8), we see that if we choose $\mathbf{A}$, Φ so that:

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0 \tag{5.12}$$

Maxwell equations decouple into two nonhomogeneous wave equations for the potentials.

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{5.13}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j} \tag{5.14}$$

The condition (5.12) is called *Lorentz condition* and the gauge in which it holds is called *Lorentz gauge.*

Remark 5.1.1.1. The Lorentz condition does not fix the potentials univocally. In fact, we can perform transformations of the form (5.9), (5.11) preserving the condition (5.12). Since the Lorentz condition transforms as

$$\nabla \cdot \mathbf{A}' + \frac{1}{c^2}\frac{\partial \Phi'}{\partial t} + \nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0 \tag{5.15}$$

we see that this condition is preserved exactly when the function f generating the gauge transformation satisfies the homogeneous wave equation.

The gauge freedom therefore implies that, among the four scalar functions defining the potentials (the function Φ and the three components of $\mathbf{A}$), only two are really indipendent (one can be determined using Lorentz condition and another one using the residual gauge freedom).

Using equation (1.58) in the first chapter, we find the solution to (5.13) and (5.14).

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0}\int_{|x-y|\leq ct} \frac{\rho(\mathbf{y}, t - \frac{|\mathbf{y}-\mathbf{x}|}{c})}{|\mathbf{y}-\mathbf{x}|} dy \tag{5.16}$$

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi}\int_{|x-y|\leq ct} \frac{\mathbf{j}(\mathbf{y}, t - \frac{|\mathbf{y}-\mathbf{x}|}{c})}{|\mathbf{y}-\mathbf{x}|} dy \tag{5.17}$$

5.1.2 Lienard-Wiechert potentials

We now use equations (5.16), (5.17) to find the potentials generated by a point charged particle with arbitrary motion. The charge and current densities are given by:

$$\rho(\mathbf{x}, t) = q\delta^3(\mathbf{x} - \mathbf{x}_0(t)) \tag{5.18}$$

$$\mathbf{j}(\mathbf{x}, t) = q\mathbf{v}(t)\delta^3(\mathbf{x} - \mathbf{x}_0(t)) \tag{5.19}$$

where $q, \mathbf{v}, \mathbf{x}_0$ are the charge, velocity and position of the particle respectively. Inserting (5.18), (5.19) into (5.16), (5.17), we find:

$$\Phi(\mathbf{x},t) = \frac{q}{4\pi\epsilon_0}\int_{\mathbb{R}^3}\frac{\delta^3(\mathbf{x}-\mathbf{x}_0(t'_{ret}))}{|\mathbf{y}-\mathbf{x}|}dy \tag{5.20}$$

$$\mathbf{A}(\mathbf{x},t) = \frac{q\mu_0}{4\pi}\int_{\mathbb{R}^3}\frac{\mathbf{v}(t)\delta^3(\mathbf{x}-\mathbf{x}_0(t'_{ret}))}{|\mathbf{y}-\mathbf{x}|}dy \tag{5.21}$$

where we have set $t'_{ret} = t - \frac{|\mathbf{y}-\mathbf{x}|}{c}$.

We focus on the scalar potential, since the derivation of the vector potential proceeds in the same way.
The trick consists in looking at the time as a coordinate, introducing a new delta function in time and integrating over a four-dimensional domain. We have:

$$\Phi(\mathbf{x},t) = \frac{q}{4\pi\epsilon_0}\int_{\mathbb{R}^4}\frac{\delta^3(\mathbf{x}-\mathbf{x}_0(t'))}{|\mathbf{y}-\mathbf{x}|}\delta(t'-t'_{ret})dt'dy \tag{5.22}$$

Changing the order of integration in space and time,

$$\Phi(\mathbf{x},t) = \frac{q}{4\pi\epsilon_0}\int_{\mathbb{R}^4}\frac{\delta(t'-t'_{ret})}{|\mathbf{y}-\mathbf{x}|}\delta^3(\mathbf{x}-\mathbf{x}_0(t'))dydt' \tag{5.23}$$

The inner integration in space coordinates can now be easily performed: the delta function picks out $\mathbf{y} = \mathbf{x}_0(t')$ and $t'_{ret} = t - \frac{|\mathbf{x}-\mathbf{x}_0(t')|}{c}$.

$$\Phi(\mathbf{x},t) = \frac{q}{4\pi\epsilon_0}\int_{\mathbb{R}}\frac{\delta(t'-t'_{ret})}{|\mathbf{x}-\mathbf{x}_0(t')|}dt' \tag{5.24}$$

In performing the integration in time, we have to keep in mind that t'_{ret} is a function of the point $(\mathbf{x},t)$ and the source trajectory. We therefore use the well known property of the delta function

$$\delta(f(t)) = \sum_i \frac{\delta(t-t_i)}{|\frac{df}{dt}(t_i)|} \tag{5.25}$$

where the sum runs over the zeros of f.
Since, for any given point $(\mathbf{x},t)$ and source trajectory there is only one retarded time t_{ret}, solution to the equation $t_{ret} = \frac{|\mathbf{x}-\mathbf{x}_0(t_{ret})|}{c}$, we have:

$$\delta(t'-t'_{ret}) = \frac{\delta(t'-t_{ret})}{\frac{\partial}{\partial t'}\{t'-(t-\frac{1}{c}|\mathbf{x}-\mathbf{x}_0(t')|)\}|_{t'=t_{ret}}} \tag{5.26}$$

$$= \frac{\delta(t'-t_{ret})}{\{1-\frac{1}{c}\frac{\mathbf{x}-\mathbf{x}_0(t')}{|\mathbf{x}-\mathbf{x}_0(t')|}\cdot\mathbf{v}_0(t')\}|_{t'=t_{ret}}} \tag{5.27}$$

$$= \frac{\delta(t'-t_{ret})}{1-\beta_0(t_{ret})\cdot\mathbf{n}(t_{ret})} \tag{5.28}$$

where $\beta = \frac{\mathbf{v}}{c}$ and $\mathbf{n}(t_{ret}) = \frac{\mathbf{x}-\mathbf{x}_0(t_{ret})}{|\mathbf{x}-\mathbf{x}_0(t_{ret})|}$ is the versor pointing from the particle position to the point in which the potential is calculated, evaluated at the retarded time.
Finally, the integration in time picks out $t' = t_{ret}$ and we find:

$$\Phi(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(1-\beta_0\cdot\mathbf{n})|\mathbf{x}-\mathbf{x}_0|}\right)_{t_{ret}} \tag{5.29}$$

$$\mathbf{A}(\mathbf{x}, t) = \frac{q\mu_0 c}{4\pi}\left(\frac{\beta_0}{(1-\beta_0\cdot\mathbf{n})|\mathbf{x}-\mathbf{x}_0|}\right)_{t_{ret}} = \frac{\beta_0}{c}\Phi(\mathbf{x}, t) \tag{5.30}$$

Formulas (5.29), (5.30) are the so-called *Lienard-Wiechert potentials*.
Frome these we can compute the electric and magnetic fields, using (5.5) and (5.6). The calculation is simple, but quite long and we omit it. The final result is:

$$\mathbf{E}(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_0}\left(\frac{(\mathbf{n}-\beta_0)(1-\beta_0^2)}{|\mathbf{x}-\mathbf{x}_0|^2(1-\beta_0\cdot\mathbf{n})^3}\right)_{t_{ret}} + \frac{q}{4\pi\epsilon_0}\left(\frac{\mathbf{n}\times\{(\mathbf{n}-\beta_0)\times\frac{\dot{\beta}_0}{c}\}}{|\mathbf{x}-\mathbf{x}_0|(1-\beta_0\cdot\mathbf{n})^3}\right)_{t_{ret}} \tag{5.31}$$

$$\mathbf{B}(\mathbf{x}, t) = \frac{\mathbf{n}}{c}\times\mathbf{E}(\mathbf{x}, t) \tag{5.32}$$

Looking at these expressions, we see that the fields generated by a moving particle are composed by two different terms. The first one is the *static field term* and we see that, if the velocity of the particle is constant, this term does not depend on the retarded time at all. This property is consistent with the principle of relativity: a charge moving with constant velocity must appear to a static observer in exactly the same way a static charge appears to a moving observer and, in the latter case, the direction of the field must change instantaneously. Thus, static field term points from the instantaneous position of the particle to the point in which the field is calculated, if the velocity does not change during the retarded time delay.
The second term is different from zero only if the particle is accelerating and it is proportional to the acceleration. It is called *radiation term*, since it describes the electromagnetic waves emitted by the particle and it contains all informations about the motion of the particle that can't be eliminated changing reference frame with a Lorentz transformation. Since the first term goes as $|\mathbf{x}-\mathbf{x}_0|^{-2}$ and the second one as $|\mathbf{x}-\mathbf{x}_0|^{-1}$, we see that far from the particle the static field is neglegible.

5.2 Gravitational waves

We present an introduction to the theory of *gravitational waves*. In particular, we show that, in the limit of weak gravitational field, we can find a gauge in which *Einstein equations* takes the form of a wave equation, describing perturbations in the space-time geometry. We describe interaction between waves and test masses in different frames and the production of waves by a given distribution of mass and energy.

5.2.1 Linearized Einstein equations

Suppose we are in presence of a very weak gravitational field, such that the space-time metric generated by this field can be written as an expansion around the flat Minkowski metric, in the form:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| << 1 \tag{5.33}$$

We can therefore perform all the calculations in the framework of *linearized approximation of general relativity*, keeping in the equations only terms linear in $h_{\mu\nu}$. We will therefore rise and lower indices with Minkowksi metric and we will taking derivations using the usual partial derivative instead of using covariant derivative.

Remark 5.2.1.1: Since tensors' components depend on the frame of reference, what we really mean in (5.33) is that there exists a reference frame in which that condition is satisfied in a sufficiently large region of space.

With a simple calculation, we find *Christoffel symbols, Riemann tensor, Ricci tensor* and *Ricci scalar* associated with metric (5.33). Keeping only the first order terms in $h_{\mu\nu}$, we obtain:

$$\Gamma^{\mu(1)}_{\nu\lambda} = \frac{1}{2}\eta^{\mu\rho}\{\partial_\lambda h_{\rho\nu} + \partial_\nu h_{\rho\lambda} - \partial_\rho h_{\nu\lambda}\} \tag{5.34}$$

$$R^{(1)}_{\mu\nu\rho\sigma} = \frac{1}{2}\{\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial\nu h_{\nu\rho}\} \tag{5.35}$$

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\{\partial_\lambda\partial_\nu h^\lambda_\mu + \partial_\lambda\partial_\mu h^\lambda_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\} \tag{5.36}$$

$$R^{(1)} = \eta^{\mu\nu}R^{(1)}_{\mu\nu} = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h \tag{5.37}$$

where $h = h^\lambda_\lambda = \eta^{\lambda\mu}h_{\mu\lambda}$ is the trace of $h_{\mu\nu}$ and $\Box = \partial^\mu\partial_\mu$ is d'Alambert operator associated with Minkowski metric.
The vacuum Einstein equations $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$, become:

$$\partial_\lambda\partial_\nu h^\lambda_\mu + \partial_\lambda\partial_\mu h^\lambda_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} = 0 \tag{5.38}$$

Lorenz gauge

Once we have chosen a reference frame in which (5.33) holds, we still have the freedom to choose a particular gauge. Consider a coordinate transformation of the form:

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x) \tag{5.39}$$

where ξ^μ is a vector field whose partial derivatives $\partial_\nu\xi^\mu$ are at most of the same order of magnitude of $h_{\mu\nu}$. Since the metric is a (0,2) rank tensor, under a generic coordinate transformation, it changes as

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \tag{5.40}$$

Neglecting higher order terms, we obtain:

$$h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_\nu \xi_\mu + \partial_\mu \xi_\nu) \tag{5.41}$$

Therefore, if $\partial_\nu \xi^\mu$ are small, the condition (5.33) still holds and the coordinate transformation (5.39) is a symmetry of linearized theory. Note also that the linearized Riemann tensor is invariant, and not only covariant, under (5.39) and so we can compute it in the most convenient gauge.

Remark 5.2.1.2: We see that if $\partial_\nu \xi_\mu + \partial_\mu \xi_\nu = 0$ the metric remains the same. The vector field ξ^μ is a Killing vector of the linearized metric.

Now, using the gauge freedom (5.41), we want to establish which property the vector field ξ^μ must obey in order to put linearized Einstein field equations in a simple form. Define

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \tag{5.42}$$

If we impose the *Lorenz gauge condition*

$$\partial^\nu \bar{h}_{\mu\nu} = 0 \tag{5.43}$$

we see that Einstein equations assume in this gauge the wave equation form.

$$\Box \bar{h}_{\mu\nu} = 0 \tag{5.44}$$

(5.41) becomes:

$$\bar{h}'_{\mu\nu}(x') = \bar{h}_{\mu\nu}(x) - (\partial_\nu \xi_\mu + \partial_\mu \xi_\nu - \eta_{\mu\nu}\partial_\alpha \xi^\alpha) \tag{5.45}$$

In order to find the property ξ^μ must obey to satisfy Lorenz condition, wetake derivative in (5.45), finding:

$$(\partial^\nu \bar{h}'_{\mu\nu})(x') = \partial^\nu \bar{h}_{\mu\nu}(x) - \Box \xi_\mu \tag{5.46}$$

Therefore, if we have initially $\partial^\nu \bar{h}_{\mu\nu}(x) = f_\mu(x)$, we can switch to Lorenz gauge, choosing a vector field which obeys $\Box \xi_\mu = f_\mu$. According to results of chapter 1 and 2, we know this equation always admits a solution. Since we have chosen the four components of ξ^μ, we are left with only 10-4=6 indipendent components of the symmetric tensor $\bar{h}_{\mu\nu}$. We have therefore seen that, if we choose a particular gauge, the linearized vacuum Einstein equations predict the existence of space-time perturbations propagating as waves at the speed of light.

TT gauge

Choosing the Lorenz gauge does not remove completely the gauge freedom. Infact, we see that, if $\bar{h}_{\mu\nu}(x)$ satisfies the Lorenz condition (5.43), the tranformed metric $\bar{h}'_{\mu\nu}(x')$, under $x'^{\mu}(x') = x^{\mu}(x) + \xi^{\mu}(x)$, also does, if the vector field ξ^{μ} obeys the homogeneous wave equation $\Box\xi^{\mu} = 0$. Therefore, equation (5.45) implies we can subtract from the metric's six indipendent components, the functions

$$\xi_{\mu\nu} = \partial_{\nu}\xi_{\mu} + \partial_{\mu}\xi_{\nu} - \eta_{\mu\nu}\partial_{\alpha}\xi^{\alpha} \tag{5.47}$$

which also satisfy $\Box\xi_{\mu\nu} = 0$. In this way we can choose ξ^{μ} in order to impose four further conditions on the metric and we are left with only two indipendent components.

One choice consists in selecting ξ^{0} in order to make the metric traceles $\bar{h} = 0$ (note this implies $\bar{h}_{\mu\nu} = h_{\mu\nu}$). The remaining three functions ξ^{i} can then be choosen in order to have $h_{0i} = 0$. Then, the condition (5.41), defining the Lorenz gauge, implies, with $\mu = 0$:

$$\partial^{0} h_{00} = 0 \tag{5.48}$$

That is, h_{00} is constant in time. Since we are dealing with perturbations that change in time, we can set, without loss of generality, $h_{00} = 0$. Therefore, only the pure spatial components of the metric are different from zero and the conditions we have imposed read:

$$h_{0\mu} = 0, \quad h^{i}_{i} = 0, \quad \partial^{j} h_{ij} = 0 \tag{5.49}$$

A metric satisfying (5.49) are said to be in the *TT gauge* (Transverse Traceless gauge) and we will write the metric as $h^{TT}_{\mu\nu}$.

We now analize the most simple solution of $\Box h^{TT}_{\mu\nu} = 0$: a plane wave. This has the following form:

$$h^{TT}_{ij} = \epsilon_{ij} e^{ik_{\mu}x^{\mu}} \tag{5.50}$$

where ϵ_{ij} is a symmetric tensor carrying informations about the wave polarization and $k^{\mu} = (\frac{\omega}{c}, k)$ is the wave vector. Since the wave travels with the speed of light, it is a light vector $k^{\mu}k_{\mu} = 0$.
If we write its spatial part as $\mathbf{k} = |k|\mathbf{n}$, where $\mathbf{n}$ is a versor, the condition $\partial^{j} h^{TT}_{ij} = 0$ translates to $n^{j} h^{TT}_{ij} = 0$ and we see that the only metric's components different from zero lie in the plane perpendicular to the wave's propagation direction.
Consider for example a wave propagating in the z direction. Then $\mathbf{n} = (0, 0, 1)$ and we have, taking the real part:

$$h^{TT}_{ij}(t, z) = \begin{pmatrix} h_1 & h_2 & 0 \\ h_2 & -h_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} cos[\omega(t - \frac{z}{c})] \tag{5.51}$$

The metric perturbed by this wave is then:

$$\begin{aligned} ds^2 = & - c^2dt^2 + \{1 + h_1cos[\omega(t - \frac{z}{c})]\}dx^2 + \{1 - h_1cos[\omega(t - \frac{z}{c})]\}dy^2 \\ & + 2h_2cos[\omega(t - \frac{z}{c})]dxdy + dz^2 \end{aligned} \tag{5.52}$$

5.2.2 Interaction with test masses

In the present section we study the interaction of gravitational waves with test masses in different frames of reference: the so-called *TT frame* and the usual *Laboratory frame.* The effects of gravitational waves will be studied looking at the *geodesic equation* of motion for the test masses.

TT frame

Choosing a gauge physically corresponds to select a particular frame of reference. We now want to understand which reference frame is associated with the TT gauge and how this frame can be physically realized.

The answer can be found looking at the geodesic equation for a mass in a region of space-time perturbed by a gravitational wave.
Let's parametrize the geodesic with the proper time and suppose a test mass is at rest at $\tau = 0$. We then have:

$$\frac{d^2x^i}{d\tau^2} = -\{\Gamma^i_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\}_{\tau=0} = -\{\Gamma^i_{00}(x)(\frac{dx^0}{d\tau})^2\}_{\tau=0} \tag{5.53}$$

By (5.34) we find $\Gamma^i_{00} = \frac{1}{2}(2\partial_0 h_{0i} - \partial_i h_{00})$. This quantity vanishes by the TT gauge condition. Therefore, if a mass was at rest before interacting with the wave, it remains at rest. In other words, the coordinates defining the TT reference frame also stretch in such a way that the position of the mass respect to them remains the same. This reference frame can then be realized using the masses themselves as coordinates. We can use a mass to define the origin, another one to define the point $(1, 0, 0)$, and so on. By definition, in this reference frame, the masses positions remain the same at any time. Note also that, in TT frame, the proper time measured by a clock travelling along a time-like trajectory $x^\mu(\tau) = (x^0(\tau), x^i(\tau))$ is given by:

$$c^2d\tau^2 = c^2dt^2(\tau) - (\delta_{ij} + h^{TT}_{ij})\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}d\tau^2 \tag{5.54}$$

where we have set $x^0(\tau) = ct(\tau)$. Since a particle initially at rest remains at rest, $\frac{dx^i}{d\tau} = 0$. Therefore, in the TT frame, the proper time measured by a clock on a mass initially at rest coincides with the coordinate time t.

The results obtained show that, in General relativity, physical effects are not described by the coordinates changes, since the theory is invariant under arbitrary coordinate transformations. In particular, the fact that the coordinates of test masses initially at rest does not change in the TT frame, does not mean that gravitational waves have not physical effects, but only that the coordinates have been chosen in such a way that they not change under the action of the waves.
Physical effects must be studied looking at *proper distances* and *proper times*, that are indipendent from the observer.
Let's consider for example two events with TT coordinates $(t, x_1, 0, 0)$, $(t, x_2, 0, 0)$ respectively. The *coordinate separation* it is not changed by the passage of a wave propagating along the z direction. On the other hand, looking at (5.52), we see that the proper distance between these events oscillates in time:

$$s = (x_2 - x_1)\{1 + h_+ cos(\omega t)\}^{\frac{1}{2}} \sim L\{1 + \frac{1}{2}h_+ cos(\omega t)\} \tag{5.55}$$

In general, if the coordinate separation vector between two events is $\mathbf{L}$, the proper distance changes as $s^2 = L^2 + h_{ij}L^iL^j$ and, neglecting higher order terms, we have:

$$\ddot{s} \sim \frac{1}{2}\ddot{h_{ij}}\frac{L^i}{L}L^j \tag{5.56}$$

where dot indicates derivation with respect to proper time. If we write $\frac{L^i}{L} = n^i$, $s = n^i s_i$:

$$\ddot{s}_i \sim \frac{1}{2}\ddot{h_{ij}}L^j \sim \frac{1}{2}\ddot{h_{ij}}s^j \tag{5.57}$$

where we have set, at the lowest order in h, $L^j = s^j$.
Equation (5.57) is the geodesic equation expressed in terms of proper distance and proper time.
If the two test masses are mirrors between which a light ray is reflected, the proper distance between the mirrors determines the time interval between two reflections and, since gravitational waves change the proper distance, they can be detected measuring the frequency of reflections.

Laboratory frame

Although gravitational waves assume their simplest form in the TT frame, their effects are studied on Earth in the so-called *laboratory frame*, in which coordinates axis are defined using rigid rulers. In this system, we expect that masses initially at rest change their position when they meeet a travelling wave.
The simplest laboratory frame which can be studied is the one inside an orbiting satellite,

in which the measure apparatus is in free fall with respect to the gravitational field generated by the Earth and the wave. In this situation, using *Fermi-Walker parallel transport*, we can find normal coordinates in which the metric reduces to the Minkowski metric along the geodesic followed by the apparatus and approximates it in a neighborhood of space-time.

$$ds^2 \sim -c^2dt^2 + \delta_{ij}dx^i dx^j \tag{5.58}$$

We don't have first order corrections to the metric in $|x^i|$, since first order partial derivatives of the metric vanishes along the geodesic around which we expand. At the second order, we find:

$$ds^2 \sim -\{1 + R_{0i0j}x^i x^j\}c^2dt^2 - (\frac{4}{3}R_{0jik}x^j x^k)cdtdx^i + \{\delta_{ij} - \frac{1}{3}R_{ikjl}x^k x^l\}dx^i dx^j \tag{5.59}$$

where the Riemann tensor is evaluated along the geodesic.
The situation is more complicated for a laboratory on Earth, since we have to take into account the acceleration -**g** felt by an object with respect to a local inertial frame and the Earth rotation with respect to a local system of gyroscopes. The metric can be found explicitely changing the coordinates; the calculation is rather long and we omit it. The result, up to order $O(x^i x^j)$, is:

$$\begin{aligned} ds^2 \sim &-\{1 - \frac{2}{c^2}\mathbf{g}\cdot\mathbf{x} + \frac{(\mathbf{g}\cdot\mathbf{x})^2}{c^4} - \frac{(\Omega\times\mathbf{x})^2}{c^2} + R_{0i0j}x^i x^j\}c^2dt^2 \\ &+ 2c\{\frac{\epsilon_{ijk}}{c}\Omega^j x^k - \frac{2}{3}R_{0jik}x^j x^k\}dtdx^i \\ &+ \{\delta_{ij} - \frac{1}{3}R_{ikjl}x^k x^l\}dx^i dx^j \end{aligned} \tag{5.60}$$

where Ω is the angular velocity of the Earth rotation.
Gravitational waves effects are of second order and they compete with higher magnitude effects, such as Earth gravity and Coriolis acceleration. This makes their study difficult. However, since waves can have very high frequencies, they vary considerably within a scale of time in which other effects are almost constant and they can be isolated and studied separately. If also the Earth gravity is compensated by a system of suspensions, we can use the metric (5.59) also in the laboratory frame on Earth to study the free motion of test masses in the plane z = constant.
The action of the gravitational waves on test masses can be conveniently studied looking at the *geodesic deviation equation*, which gives the separation between two close geodesics; the equation can be simplified noticing that the Christoffel symbols vanish along the geodesic around which we expand and that we can take only the zero-zero component, since the measure apparatus moves at a non relativistic speed. We have:

$$\frac{d^2\xi^i}{d\tau^2} = -R^i_{0j0}\xi^j(\frac{dx^0}{d\tau})^2 \tag{5.61}$$

where ξ is the separation vector between two geodesics.
If a mass is initially at rest, after the interaction with the wave, it acquires a velocity $\frac{dx^i}{d\tau} = cO(h)$; therefore:

$$dt^2 = d\tau^2\{1 + \frac{1}{c^2}\frac{dx^i}{d\tau}\frac{dx_i}{d\tau}\} = d\tau^2\{1 + O(h^2)\} \tag{5.62}$$

Since, in (5.61), the Riemann tensor is already of order $O(h)$, at the first order, $t = \tau$ and the geodesic deviation equation becomes:

$$\ddot{\xi}^i = -c^2 R^i_{0j0}\xi^j \tag{5.63}$$

where dot denotes derivation with respect to coordinate time in the laboratory frame. Since, as we pointed out earlier, Riemann tensor is invariant under linearized coordinate transformations, we can compute it in the TT frame, where it takes a particularly simple form: $R^i_{0j0} = -\frac{1}{2c^2}\ddot{h}^{TT}_{ij}$. Therefore, we obtain:

$$\ddot{\xi}_i = \frac{1}{2}\ddot{h}^{TT}_{ij}\xi^j \tag{5.64}$$

Remark 5.2.2.1: According to (5.64), in the laboratory frame, the effects of gravitational waves on a test mass m can be described, using Newtonian mechanics, by the force

$$F_i = \frac{m}{2}\ddot{h}^{TT}_{ij}\xi^j \tag{5.65}$$

+ and × polarizations

We show explicitely the effects of a plane wave propagating in the z direction on a circular ring of test masses initially at rest on the x-y plane in the laboratory frame. Since for a wave propagating in the z direction we have $h^{TT}_{ij} = 0$ for $i = 3$ or $j = 3$, we see from (5.64) that the masses remains on the plane x-y.
We say the wave has polarization + if it has the form:

$$h^{TT}_{ab} = h_+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sin(\omega t) \tag{5.66}$$

where $a, b = 1, 2$ are indices in the x-y plane.
The position of a test mass after the interaction with the wave is $\xi_a(t) = (x_0 + \delta x(t), y_0 + \delta y(t))$, where (x_0, y_0) is the initial position and the perturbations $\delta x, \delta y$ are small. The equation (5.64) takes the form:

$$\delta\ddot{x} = -\frac{h_+}{2}(x_0 + \delta x)\omega^2 \sin(\omega t) \tag{5.67}$$

$$\delta\ddot{y} = +\frac{h_+}{2}(y_0 + \delta y)\omega^2 \sin(\omega t) \tag{5.68}$$

with first order solution

$$\delta x(t) = +\frac{h_+}{2} x_0 \sin(\omega t) \tag{5.69}$$

$$\delta y(t) = -\frac{h_+}{2} y_0 \sin(\omega t) \tag{5.70}$$

Likewise, we say the wave is $\times$ polarized if it has the form:

$$h_{ab}^{TT} = h_\times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin(\omega t) \tag{5.71}$$

with first order solution

$$\delta x(t) = +\frac{h_\times}{2} y_0 \sin(\omega t) \tag{5.72}$$

$$\delta y(t) = +\frac{h_\times}{2} x_0 \sin(\omega t) \tag{5.73}$$

The following figure shows the periodic deformations of a ring of test particles, initially at rest in the laboratory frame, due to interaction with a + and a $\times$ polarized wave respectively.

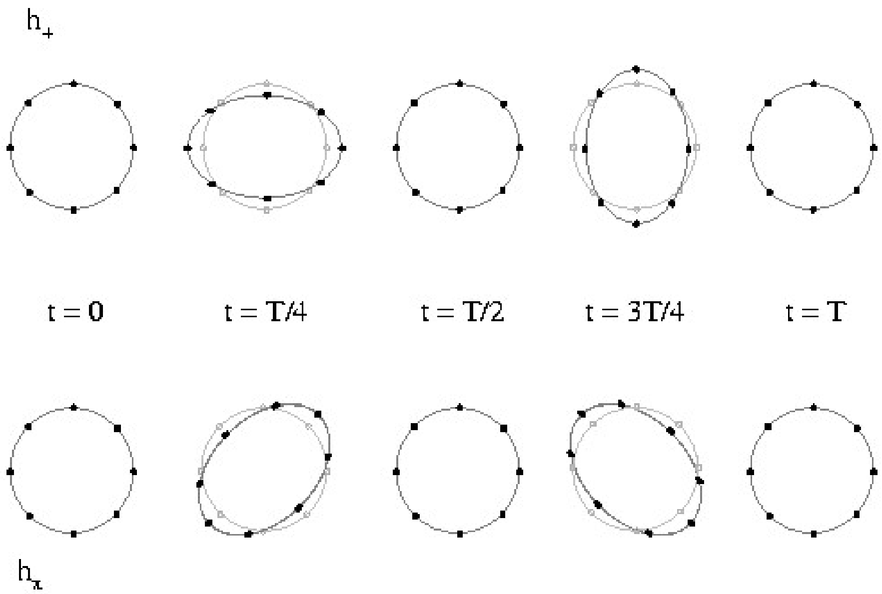

Figure 5.1: Deformations caused by + and $\times$ polarized waves on a ring of test masses.

5.2.3 Gravitational waves emission

We now study the emission of gravitazional waves by a mass distribution. We see that the source, in order to emit gravitational waves, must posses a certain degree of asymmetry and we obtain the expression for the waves emitted by a time-varying source. Finally, we look at binary systems in circular orbit and we find the waves emitted by these kind of sources.

In the presence of matter, the linearized Einstein equations, in the Lorenz gauge, take the form of a nonhomogeneous wave equation.

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T^{(0)}_{\mu\nu} \tag{5.74}$$

Note that only the zeroth order term of the source's *stress-energy tensor* appears in the right hand side of the equation. This is due to the fact that $T_{\mu\nu}$ must itself already be small in order for the linearized approximation to be valid, i.e. $T_{\mu\nu}$ should be of order $h_{\mu\nu}$. Therefore, any terms in $T_{\mu\nu}$ depending on $h_{\mu\nu}$ would already be of order $O(h^2_{\mu\nu})$ and can be dropped.
According to equation (1.57) in chapter 1, the solution is given by:

$$\bar{h}_{\mu\nu}(t,x) = \frac{4G}{c^4} \int_V \frac{T^{(0)}_{\mu\nu}(t - \frac{|x-x'|}{c}, x')}{|x-x'|} d^3x' \tag{5.75}$$

where we integrate over the space volume occupied by the source and the stress-energy tensor is calculated at the earlier time $t - \frac{|x-x'|}{c}$, since the waves travel with the speed of light.
Since we look fot solutions expressed in the TT gauge, we can focus only on the spatial components $\bar{h}_{ij}$, with $i, j = 1, 2, 3$.
Far from the source, if the wave length of the emitted radiation is much bigger that the source's dimension, we can write equation (5.75) as:

$$\bar{h}_{ij}(t,x) = \frac{4G}{c^4}\frac{1}{r} \int_V T^{(0)}_{ij}(t - \frac{r}{c}, x') d^3x' \tag{5.76}$$

where we have set $r = |x|$.
(5.76) can be simplified making use of the Virial theorem. We obtain, after some calculations:

$$\int_V T^{(0)}_{ij} d^3x' = \frac{1}{2c^2}\frac{d^2}{dt^2} \int_V T^{(0)}_{00} x'_i x'_j d^3x' \tag{5.77}$$

The tensor

$$q_{ik}(t) = \frac{1}{c^2} \int_V T^{(0)}_{00} x'_i x'_j d^3x' \tag{5.78}$$

is called source *quadrupole moment.*
Equation (5.76) can be expressed in terms of q_{ij}, as:

$$\bar{h}_{ij}(t,x) = \frac{2G}{c^4}\frac{1}{r} \ddot{q}_{ij}(t - \frac{r}{c}) \tag{5.79}$$

(5.79) shows that, for an isolated system, the first non vanishing term in the multipole expansion is the quadrupole term. This can be explained noticing that the derivative of the gravitational dipole $d_G^i = \int_V T_{00} x'^i d^3x'$ vanishes by the conservation of momentum. Therefore, a distribution of masses, in order to emit gravitational waves, must have a non vanishing quadrupole term and so a certain degree of asymetry. In particular, a spherically symmetric star that pulses changing periodically its radius can't emit gravitational waves: a result known as Birkhoff's theorem.

The most convenient choice for studying waves emission is, as usual, the TT gauge. Let's consider a plane wave propagating in the direction given by the versor n^i. The projector

$$P_{ij} = \delta_{ij} - n_i n_j \tag{5.80}$$

gives the components of a symmetric tensor transverse to n^i. Using P_{ij} we can construct another projector:

$$\Lambda_{ijkl} = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl} \tag{5.81}$$

Λ_{ijkl} is called the TT projector and it is clear that it projects a symmetric tensor to the TT gauge. Therefore, we can apply it to (5.79), finding:

$$\bar{h}_{ij}^{TT}(t,x) = \frac{2G}{c^4} \frac{1}{r} \ddot{q}_{ij}^{TT}(t - \frac{r}{c}) \tag{5.82}$$

where $q_{ij}^{TT} = \Lambda_{ijkl} q^{kl}$.

Waves emitted by a binary system in circular orbit.

Studying the wave emitted by these kind of systems is very important, since they are very common in universe: think of binary stars, for example and they usually involve huge masses, making the detection of waves possible. It is remarkable that in the first experimental observation of gravitational waves, made in 2015 by LIGO and Virgo Scientific Collaboration, the waves detected were emitted by a system of this kind, in particular by two merging black holes.
Let m_1, m_2 be the mass of the two stars, r_1, r_2 their distance from their center of mass, $M = m_1 + m_2$, $l = r_1 + r_2$, μ the reduced mass. In the center of mass frame, if we take the plane $z = 0$ as that of the orbital motion, the coordinates of the stars are:

$$\begin{cases} x_1(t) = \frac{m_2}{M} l \cos(\omega t) \\ y_1(t) = \frac{m_2}{M} l \sin(\omega t) \\ z_1(t) = 0 \end{cases} \quad \begin{cases} x_2(t) = -\frac{m_1}{M} l \cos(\omega t) \\ y_1(t) = -\frac{m_1}{M} l \sin(\omega t) \\ z_2(t) = 0 \end{cases} \tag{5.83}$$

where $\omega = \sqrt{\frac{GM}{l^3}}$ is the orbital angular velocity, obtained by the third Kepler law.
The zero-zero component of the stress-energy tensor of the system is given by:

$$T^{00} = \sum_{i=1}^{2} m_i c^2 \delta(x - x_i(t))\delta(y - y_i(t))\delta(z) \tag{5.84}$$

and the quadrupole moment's components are easily obtained:

$$q_{xx} = m_1 \int_V x^2 \delta(x - x_1)dx\delta(y - y_1)dy\delta(z)dz + \tag{5.85}$$

$$m_2 \int_V x^2 \delta(x - x_2)dx\delta(y - y_2)dy\delta(z)dz \tag{5.86}$$

$$= m_1 x_1^2 + m_2 x_2^2 = \frac{\mu}{2} l^2 \cos(2\omega t) + C_1 \tag{5.87}$$

$$q_{yy} = -\frac{\mu}{2} l^2 \cos(2\omega t) + C_2 \tag{5.88}$$

$$q_{xy} = \frac{\mu}{2} l^2 \sin(2\omega t) + C_3 \tag{5.89}$$

The other components vanish.
Consider a plane wave propagating in the z direction, perpendicular to the orbital plane. The propagation direction is $n = (0, 0, 1)$ and we obtain:

$$P_{ij} = \delta_{ij} - n_i n_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.90}$$

$$q_{xx}^{TT} = (P_{xm}P_{xn} - \frac{1}{2}P_{xx}P_{mn})q^{mn} = \frac{1}{2}(q_{xx} - q_{yy}) \tag{5.91}$$

$$q_{yy}^{TT} = (P_{ym}P_{yn} - \frac{1}{2}P_{yy}P_{mn})q^{mn} = -\frac{1}{2}(q_{xx} - q_{yy}) \tag{5.92}$$

$$q_{xy}^{TT} = (P_{xm}P_{yn} - \frac{1}{2}P_{xy}P_{mn})q^{mn} = q_{xy} \tag{5.93}$$

Therefore, the radiation emitted in the z direction is given by:

$$h_{ab}^{TT}(t) = -\frac{4G}{c^4}\frac{1}{z}\mu l^2 \omega^2 \begin{pmatrix} \cos(2\omega(t - \frac{z}{c})) & \sin(2\omega(t - \frac{z}{c})) \\ \sin(2\omega(t - \frac{z}{c})) & -\cos(2\omega(t - \frac{z}{c})) \end{pmatrix} \tag{5.94}$$

Note that:
(i) The wave has both + and × polarization.
(ii) Since $h_{yy}^{TT} = ih_{xy}^{TT}$, the wave is circularly polarized.
(ii) Radiation is emitted at a double frequency compared to that of orbital motion.

Appendix A

Banach, Hilbert and L^p spaces

A.1 Banach spaces

Definition A.1.1. Let V be a vector space over $\mathbb{C}$. We say a function $|| \, || : V \to \mathbb{C}$ is a norm if the following properties hold $\forall v, w \in V, \ \lambda \in \mathbb{C}$:

(i) $||v|| \geq 0$ and $||v|| = 0$ iff $v = \mathbf{0}$
(ii) $||\lambda v|| = |\lambda| ||v||$
(iii) $||v + w|| \leq ||v|| + ||w||$

We denote by $V_{|| \, ||}$ the space V with the norm $|| \, ||$.

Definition A.1.2. Let $\{v_n\}_{n \in \mathbb{N}}$ be a sequence in $V_{|| \, ||}$.

(i) We say the sequence converges to $v \in V_{|| \, ||}$ with respect to $|| \, ||$ if:

$$\lim_{n \to \infty} ||v_n - v|| \to 0 \tag{A.1}$$

(ii) We say the sequence is a Cauchy sequence if:

$$\forall \, \epsilon > 0 \, \exists \, \bar{n} \in \mathbb{N} : ||v_m - v_n|| < \epsilon \, \forall \, m \geq n > \bar{n} \tag{A.2}$$

Definition A.1.3. We say the space $V_{|| \, ||}$ is complete with respect to the norm $|| \, ||$ if every Cauchy sequence in $V_{|| \, ||}$ converges to some $v \in V_{|| \, ||}$.

We will call a complete space a *Banach space*

Definition A.1.4. Let X, Y be Banach spaces. A mapping $A : X \to Y$ is a linear operator, provided

$$A(\lambda u + \mu v) = \lambda A(u) + \mu A(v) \tag{A.3}$$

for all $\lambda, \mu \in \mathbb{C}$ and $u, v \in X$.
If X is a real Banach space, a linear mapping $u^* : X \to \mathbb{R}$ is called a linear functional on X.

Definition A.1.5. We define the norm of a linear operator $A : X \to Y$ as:

$$||A|| := sup\{||Au||_Y : ||u||_X \leq 1\} \tag{A.4}$$

We say that A is *bounded* if $||A||$ is finite.
It is easy to prove that A is bounded if and only if it is continuous.

Definition A.1.6. Let X be a real Banach space. We define the *topological dual space* X^* as the set of all continuous functionals $F : X \to \mathbb{R}$.

Definition A.1.7. Let X be a real Banach space. We say a sequence $\{u_k\}_{k=1}^{\infty} \subset X$ converges weakly to a function $u \in X$ if

$$< u^*, u_k > \to < u^*, u > \tag{A.5}$$

for each bounded linear functional $u^* \in X$.

A.2 Hilbert spaces

Definition A.2.1. Let V be a vector space over $\mathbb{C}$. We call *inner product* on V a function $<,> : V \times V \to \mathbb{C}$, satisfying the following properties $\forall\, u, v, w\ \in V,\ \lambda \in \mathbb{C}$:

(i) $< v, w > = < w, v >^*$
(ii) $< \lambda v, w > =\ \lambda < v, w >$
(iii) $< u + v, w >=< u, w > + < v, w >$
(iv) $< v, v > \in \mathbb{R},\ < v, v > \geq 0$ and $< v, v > =\ 0$ iff $v = 0$.

Theorem A.2.1. *(Cauchy-Schwarz inequality) Let V be a vector space over $\mathbb{C}$, with a inner product $<,>$. Then the following inequality holds $\forall u, v \in V$:*

$$| < u, v > |^2 \leq < u, u >< v, v > \tag{A.6}$$

Starting from a inner product we can obtain a norm on V, setting $||v|| = \sqrt{< v, v >}$. This leads to the definition of *Hilbert spaces*:

Definition A.2.2. Let V be a vector space over $\mathbb{C}$, with a inner product $<,>$. We say V is a Hilbert space if it is complete with respect to the norm induced by $<,>$.

Definition A.2.3. Let V be a vector space over $\mathbb{C}$ with inner product $<,>$, $u, v \in V$. We say that u and v are orthogonal with respect to $<,>$ if $< u, v >= 0$.

Definition A.2.4. Let V be a Hilbert space over $\mathbb{C}$ with inner product $<,>$ and let A be a family of indices. A set $\{v_\alpha\}_{\alpha \in A}$ of vectors in V is called an *orthonormal system* if

$$< v_\alpha, v_\beta > = \ \delta_{\alpha,\beta} \ \forall \ \alpha, \beta \ \in A \tag{A.7}$$

Theorem A.2.2. *(Parseval inequality) Let $\{v_\alpha\}_{\alpha \in A}$ be an orthonormal system on a Hilbert space V with inner product $<,>$, $v \in V$. Then the set of indices α for which the numbers $< v_\alpha, v > \neq \ 0$ is at most numerable.*
Moreover, the following inequality holds:

$$\sum_\alpha | < v_\alpha, v > |^2 \leq ||v||^2 \tag{A.8}$$

where the sum runs over the set of indices α such that $< v_\alpha, v > \neq \ 0$ and where $|| \ ||$ is the norm induced by $<,>$.

Definition A.2.5. An orthonormal system in a Hilbert space V is said to be *complete* if it maximal, i.e. we cannot add any vector to it obtaining again an orthonormal system.

Theorem A.2.3. *Let $\{v_\alpha\}_{\alpha \in A}$ be an orthonormal system on a Hilbert space V with inner product $<,>$. The following are equivalent:*

(i) $\{v_\alpha\}_{\alpha \in A}$ is an orthonormal system
(ii) The closure of the space generated by the $v_\alpha s$ coincides with V.
(iii) Every vector $v \in V$ can be written as

$$v = \sum_\alpha < v_\alpha, v > v_\alpha \tag{A.9}$$

where the sum runs over the, at most numerable, set of indices α such that $< v_\alpha, v > \neq \ 0$.
For every $v \in V$

$$||v||^2 \ = \ \sum_\alpha | < v_\alpha, v > |^2 \tag{A.10}$$

where $|| \ ||$ is the norm induced by $<,>$.

Theorem A.2.4. ***Riesz Representation Theorem:*** *V^* can be canonically identified with V. More precisely, for each $u^* \in V^*$, there exists a unique $u \in V$ such that:*

$$< u^*, v >_{V,V^*} = < u, v > \tag{A.11}$$

for all $v \in V$, where $< u^*, v >_{V,V^*}$ denotes the pairing between V and V^*.
The mapping $u^* \to u$ is a linear isomorphism of V^* onto V.

A.3 L^p spaces

Definition A.3.1: Let $\Omega \in R^n$ be open, $p \in \mathbb{R}$, $p < \infty$. We define $L^p(\Omega)$ to be the space of all measurable functions $f : \Omega \to \mathbb{C}$ for which the p-th power of absolute value is Lebesgue integrable, that is:

$$\int_\Omega |f(x)|^p dx \ < \ \infty \tag{A.12}$$

if $p = \infty$, we define $L^\infty(\Omega)$ as the set of all measurable functions for which the essential supremum of absolute value in Ω is finite; that is:

$$inf\{M \geq 0 \ : |f(x)| \leq M \ a.e. \ in \ \Omega\} < \infty \tag{A.13}$$

The following facts can be proven:

Theorem A.3.1. *Let $1 \leq p \leq \infty$. The space $L^p(\Omega)$ is a vector space, with sum and scalar multiplication defined pointwise.*

Theorem A.3.2. *Let $1 \leq p \leq \infty$. The function $|| \, ||_p : L^p(\Omega) \to \mathbb{C}$*

$$||f||_p = \begin{cases} (\int_\Omega |f(x)|^p dx)^{\frac{1}{p}} \ \ if \ p < \infty \\ ess \ sup_\Omega |f(x)| < \infty \ \ if \ p = \infty \end{cases} \tag{A.14}$$

is a norm.

Remark A.3.1. Strictly speaking the function (A.14) is not a norm, since $||f|_p = 0$ if f is zero a.e. and not only if f is the zero function. We can solve the problem taking quotient in L^p, identifying functions that are different only on a set of measure zero.

Theorem A.3.3. ***Young inequality:*** *Let $1 < p, q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$. Then, $\forall \ a, b > 0$:*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{A.15}$$

Theorem A.3.4. ***Hölder inequality:*** *Let $1 \leq p, q \leq \infty$, satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Suppose also $f \in L^p(\Omega)$, $g \in L^q(\Omega)$. Then $fg \in L^1(\Omega)$ and*

$$||fg||_1 \leq ||f||_p ||g||_q \tag{A.16}$$

Theorem A.3.5. ***General Hölder inequality:*** *Let $p_1 \leq p_2 \leq ... \leq p_m \leq \infty$, with $\sum_{j=1}^m \frac{1}{p_j} = 1$, and assume $f_j \in L^{p_j}(\Omega)$. Then:*

$$|| \prod_{j=1}^m f_j ||_1 \leq \prod_{j=1}^m ||f_j||_{p_j} \tag{A.17}$$

Theorem A.3.6. *Let* $1 \leq p \leq \infty$, $\Omega \in \mathbb{R}^n$. *The space* $L^p(\Omega)$, *with the norm* $|| \, ||_p$, *is a Banach space.*

Theorem A.3.7. *The space* $L^2(\Omega)$, *with the following inner product, for* $f, g \in L^2(\Omega)$*:*

$$< f, g >_2 := \int_\Omega f\bar{g}dx \tag{A.18}$$

is a Hilbert space.

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