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Emerging models for the slope of a curve: a reinvention activity for upper secondary school

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To Madam Mary, her children
and all the pupils of Talek Primary School;
may all your wishes come true
...
e a Fede e alla sua vita

Introduction

In a standard school lesson of mathematics, or in standard mathematics books, every topic is usually presented starting from what in its historical development represents the end of the discovery process. Would it be a good idea to invert this teaching trend and start designing lessons and activities able to push students to "reinvent" a mathematical notion? The research described in this thesis partially answers to this question, focusing on the concept of "slope of a curve" as taught in upper-secondary school. The whole research investigates if an already designed and progressively improved task has the potential to make relevant informal models of the concept of slope of a curve in one point, emerge from the students' solution; the consequent question inquired is whether the teacher could effectively build a rigorous lesson and present the formal knowledge about the topic, starting from these models.

This task was actually conceived in the context of MERIA Project, a Project funded by the European Union aimed at providing teachers in 4 European countries with teaching materials designed for implementing an inquiry-based mathematics lesson. The so called "Slide task" (students are required to design a playground slide, by providing concrete equations of a line and a curve that represent respectively the straight and the bended part of the slide, so that they meet "smoothly") had already been tested in different schools in Croatia, Slovenia and the Netherlands; its outcomes showed how students effectively came up with meaningful ideas and suggested how the concept of slope of a curve, and consequently of the derivative, could be naturally built by their minds. From the results of these pilot lessons, the interest in trying to analyse pattern or similar approaches in solving the task as well as finding and describing possible original ways in which the concept of slope of a curve could be formally and rigorously presented, has grown.

The idea that the standard definition of the derivative as the limit of the rate of change of the function doesn't come naturally to the students' mind and involves a notion, namely the limit, that is already hard to be understood, has been confirmed by several previous researches. On the other side, some alternative ways of defining and applying the derivative have already been rigorously designed. In particular in this thesis we will describe teaching strategies that use the concepts of multiplicity of the point of intersection of curve and line (the intended tangent line), local linearity and "transition points", concepts whose traces have been noticed in the students' solutions of the slide task.

The task has been later tested in three further lessons in the Netherlands, where the students' action has been monitored by observers. The purpose of these latest lessons, besides collecting ulterior solutions to the task, was examining whether a different setting of the lesson (differing mainly in the duration of the lesson) or a higher level in mathematics of the class could increase the chance of having more meaningful models emerging.

In Chapter 1 we will describe the theories that stand behind the designing of the task and the conception of the whole MERIA project: IBMT (Inquiry-based mathematics teaching), TDS (theory of didactical situations) and RME (realistic mathematics education).
In Chapter 2 we will describe the general features of MERIA project, the slide-task and the reasons that stand behind the choice of this specific task: conditions limiting the standard approach to teaching the derivative and the potential of principles of reinvention and emergent models.
The alternative formal approaches in which the notion of slope of a curve (and

consequently of the derivative) could be taught, are presented in Chapter 3. In Chapter 4 we will describe in details the implementing of the slide-task in the three latest lessons and in the Conclusions chapter the outcomes of all the lessons studied are analysed in order to answer to the research questions raised at the end of Chapter 2.

Introduzione

Può un concetto matematico essere "reinventato" dagli studenti, se facilitati dalla progettazione di una lezione inquiry-based?
Questa tesi cerca di rispondere parzialmente a questa domanda, indagando come una precisa attività a gruppi possa favorire l'emergere di idee e modelli informali significativi riguardanti il concetto di "pendenza di una curva", che possono poi essere formalizzati fino ad introdurre propriamente la nozione di derivata.
Durante l'attività viene richiesto di "progettare" uno scivolo per bambini, fornendo le equazioni di una retta e una curva che si uniscano "senza salti". Gli studenti lavorano autonomamente, senza alcun intervento da parte del docente; gli approcci da loro scelti per risolvere il problema, che matematicamente si traduce nel problema di trovare la retta tangente alla curva in un punto, svelano come la formulazione matematica di tale concetto possa essere costruita dagli studenti in autonomia e successivamente formalizzata con l'intervento dell' insegnante.

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Chapter 1

Theoretical framework

In this chapter we will describe the theories that stand behind MERIA Project and the designing of the task that will be discussed in this thesis.
These theories are: IBMT (Inquiry-Based Mathematics Teaching), TDS (Theory of Didactical Situations) and RME (Realistic Mathematics Education).

1.1 Inquiry Based Mathematics Teaching

The alternative to the traditional teaching attitude proposed and pursued in the MERIA project can be roughly characterized as Inquiry Based Mathematics Teaching. Inquiry Based Mathematics Teaching -abbreviated as IBMT- is an approach to teaching that emphasizes the student's role in the learning process. IBMT was born as a general science teaching approach, later developed in mathematics education with its specific features. The main idea is that the teacher should present to the students activities which conduct them to expand their actual mathematical knowledge, or even construct new knowledge.

The idea which is at the basis of inquiry based teaching was formulated for the first time more than one century ago. The educational researcher and philosofer John Dewey was the first to indirectly encourage teachers to use inquiry as a teaching strategy. Dewey is indeed often associated with the phrase “learning by doing”. Throughout the 20th century, the idea as well as different approaches to this method have been developed. Explicit guidelines on how a teacher could adapt her/his teaching attitude in an inquiry-based perspective were formulated. In recent times IBMT has been widely promoted by researchers in mathematics education. As a matter of fact, besides MERIA in the last decade the European Union has financed several projects geared to promote and implement inquiry-based science teaching, such as PRIMAS or FIBONACCI projects.
But how can IBMT be practically implemented in classroom? First the teacher has to find or create problems which require the students to develop new knowledge. In IBMT a problem is more than a certain task, exercise or an activity. A problem is open in the sense that it requires students to engage in experimenting, hypothesizing about possible solutions, communicating hypotheses and possible solution strategies, and maybe pose further questions to be studied as a part of a process of its solving [Winsløw et al., 2017]. This means that the students' use of their own knowledge, intuition and ideas should guide the study of the unknown situation present in the problem and thus lead them to acquire new knowledge. Moreover these problems can rise from purely mathematical issues as well as from experiments or situations in the real word. The important thing is that students' creativity and curiosity drive the problem solving process and that these abilities can be further developed by engaging in the process itself.

In addition to the problem design, the scenario of the lesson plays a very important role. The students can work on their own, or ideally in small groups so that also their ability to collaborate with peers can be fostered. The teacher in this phase only acts as a facilitator in guiding the students in the problem solving process. This guiding role must be played not by providing the student with answers or solutions but by posing questions and thus driving the inquiry process. To scaffold the students' work is then really challenging: if the teacher provides too much direction there is no real and efficient inquiry and the "learning by doing" potential is ruined; on the other side if the teacher gives too little direction, students can get stuck and are discouraged from solving the problem autonomously [Winsløw et al., 2017]. Surely the Teacher's action should have characteristics such as responsiveness, which means that scaffolding must be adapted to each student's need, and fading, in the sense that it must gradually decrease as the students progress with their inquiry. Another significant factor is the need for a classroom environment where students feel safe to speak out and to make mistakes.

1.1.1 Conditions for successful implementation of IBMT

Although inquiry offers compelling opportunities for mathematics learning, due to its specific nature there are many obstacles that limit a successful implementation of IBMT [Winsløw et al., 2017]. Some of the most significant challenges are:

- *The practical constraints of the learning context.* The activities of inquiry-based learning must fit within the practical constraints of the learning environment, such as the restrictions imposed by available resources and fixed schedules. Here the biggest enemies to a successful implementation of IBMT are time and the traditional textbooks whose tasks and problems are mostly structured in such a way that students hardly need to think about the solving procedure.

- *The concern about assessing student performance* Examinations in schools and standardized knowledge-based tests mainly focus on students' capacity for reproduction skills. Therefore, teachers are in conflict to prepare their students for the exams or to implement IBMT within class.

- *Background knowledge.* If the students, or some of the students, lack the pre-knowledge required by the problem the risk is that they lose the opportunity to develop it, since they will be unable to complete meaningful investigations.
- *Behavior of students* In traditional educational activities students are not typically asked to manage extended complex processes or groups activities without the explicit guidance of the teacher. Such a lesson could be then problematic in term of noise and disorder and mainly students with a lack of self-discipline and motivation may fail in achieving the potential of inquiry-based learning.

As we can see these challenges appear in different forms and an effective response to them requires high didactic and pedagogical skills as well as the use of curricular design strategies.
MERIA design of modules relies on two frameworks that can be seen as different approaches to IBMT, namely RME (Realistic Mathematics Education) and TDS (Theory of Didactical Situations).

1.2 Realistic Mathematics Education

Realistic Mathematics Education -hereafter abbreviated as RME- is an approach to mathematics teaching which was developed in the Netherlands. The main characteristic of this theory is that rich, “realistic” situations are given a prominent role in the learning process. Here “real” must not be interpreted as shaped on reality. Although “real-world” situation are important in RME, “realistic” in this specific context means that the situations offered in the problems must be experientially real in the students’ mind. In other words students must be offered problem situations which they can imagine. The contextualized meaning of “rich” is that a problem or situation enhances students’ cognitive achievements if it connects the student’s common sense, it leads to mathematical models which can be useful for different situations and it allows different and original approaches or solutions.

The German mathematician Hans Freudenthal (1905-1990) contributed significantly to the development of RME theory. Freudenthal considered mathematics as a human activity. That's why mathematics should not be taught as a closed system, but as a set of problems which are relevant to students' experiences.

RME is then oriented on empowering the activity of mathematising reality and if it's possible even that of mathematising mathematics. Mathematising can be seen as an equivalent to axiomatising, formalising, schematising, modelling, ecc.. Freudenthal himself insisted on "including in this one term the organizing activity of the mathematician, whether it affects mathematical content and expression, or more naive, intuitive, say lived experience, expressed in everyday language" [Freudenthal, 1991].

Adri Treffers[1] introduced the distinction between horizontal mathematising and vertical mathematising.

Horizontal mathematising leads from the world of life to the world of symbols; it's the mathematical treatment of a problem or a situation.

Vertical mathematising relates to the more or less sophisticated mathematical processing; it is the process of reorganization within the mathematics discipline [Freudenthal, 1991]. For RME these two dimensions of mathematising are both essential to the learning process. It should start indeed with the transposition of real-world problems into a mathematical domain, with its models and symbols. Then the students with vertical mathematising should manipulate these models and symbols in order to draw a more general conclusion.

[1] Adri Treffers has been a mathematics teacher in the Netherlands from 1959 to 1969 and since then has worked as a developer, researcher and as a professor at the Freudenthal Institute and at the Department of Educational Science at Utrecht University.

The principles that regulate RME were formulated, and reformulated over the years, mainly by Treffers. Some of the main design principles are:

- *The reality principle.* As already written, this refers to the importance of posing problems that can help students grant meaning to the mathematical construct they use to solve them.
- *The activity principle.* RME promotes mathematics as a human activity, and this is achieved if the student becomes an active participant in the learning process. It is by doing actively that she/he learns.
- *The level principle.* The students pass gradually from an informal model, specifically related to the problem context, to the more general and precise next one. Then also the teaching operating must be progressive so that it can proceed in parallel with the students' learning activity.
- *The guidance principle.* The teaching-learning activity must be structured as a guided-reinvention of the mathematical concepts. Freudenthal stressed the importance for the teacher to guide towards reinvention: "Guiding means striking a delicate balance between the force of teaching and the freedom of learning." [Freudenthal, 1991].
- *The interactivity principle.* This principle refers to the value of the learning activity as a social activity. The learning and reinventing process can be also carried out individually, but it becomes meaningful only if the learner's own invention can be compared with the other learner's and the teacher's ideas.

The idea that in a process of knowledge construction the students and their ideas can play the central role and the teacher in these defined situations acts just as a guide or a facilitator sets up a natural connection between RME and another well established theory: The Theory of Didactical Situations.

1.3 Theory of Didactical situations

The Theory of Didactical situations (TDS) is a line of research in the field of mathematics education that was started by Guy Brousseau in the late 60's. In this theory, the focus is on the situations, which represent a set of explicitly or implicitly established relations between the teacher, the learner and the learning context. A key component of this theory is indeed what is called the "milieu"; it represents the whole environment and circumstances that interact with the student. When the milieu is properly designed by the teacher, the learner can interact with it autonomously. This occurs in those situations that in TDS are called "a-didactical situations". As a matter of fact we can distinguish between "didactical situations" and "a-didactical situations":

- *didactical situations* are defined by the teacher's explicit intent to teach. The teacher actively shows passages, gives clear directions or suggestions in order to let the students reach the cognitive achievement aimed.

- *a-didactical situations* in which the teacher lets the students work autonomously. The students' action is not guided by the teacher's explicit directions, but by the use of their existing knowledge, intuitions and interest in solving a problem or task that she/he was given.

The learning potential lies in the dialectic between a-didactical and didactical situations: a-didactical situations carry the potential for building new personal knowledge, but this knowledge assumes meaningful value only if validated by the teacher through the didactical situation. It's important indeed to differentiate the personal knowledge and the institutional knowledge. The first one is that knowledge that is built up by the students in their interaction with a mathematical problem. The institutional knowledge is validated and made public; let's say it is the knowledge we find in books or which is stated by educational institutions. It's possible to find a beneficial balance between a-didactical and didactical situations if the teacher manages to create a milieu which leads the students to expand their personal knowledge and that this is later validated either by the teacher her/himself or by comparison with other students' formulations. The students in this way can see their own knowledge being formalized and becoming closer to what can be regarded as institutional knowledge.

The interaction between the two types of knowledge is rarely performed by the students without a mastered and systematic organization of the lessons by the teacher. According to Brousseau, the lesson, or the learning module, should be organized in 5 phases:

- *Devolution*
  The first phase is purely didactical; the teacher submits the problem and the purpose of it to the students. The milieu is in this way well defined by the teacher and by the activity's rules. The students just have to listen to the instructions and, once they clearly understand the activity, they involve themselves in it.
- *Action*
  The student engages in the problem. He reflects and then productively acts. It's in this phase that she/he creates her/his own personal knowledge.
- *Formulation*
  After having worked alone or with a small group of classmates, the student presents and motivates his solution and ideas to the whole class. This happens as an open discussion. The teacher is not meant to intervene, her/his role is just to make sure that any student has the chance to present her/his hypothesis and ideas.
- *Validation*
  The students are asked to compare their results and perhaps find com-

mon features or differences between them. The teacher can investigate more deeply the process of ideas construction by posing detailed questions or more explicitly show the effectiveness or either limits of the students' solutions. In some cases, if the milieu is properly designed, the students can find an autonomous way to verify their hypothesis (using material tools, or geogebra..ecc).

- *Institutionalisation*
  The teacher sums up the ideas presented by the students, shows their effectiveness or either limits and shows how these informal solutions and ideas relate to the mathematical learning goals of the lesson. It's very important that in this last phase, the teacher doesn't turn the lesson into a typical lecture in which the students' presented ideas become useless. Indeed, it's starting from those that the institutional knowledge should be presented, as a combined outcome of the classroom discussion refined by the teacher's knowledge.

By following these phases in the presented order the classical lesson texture is reversed. In the standard teaching practice, institusionalisation is placed at the beginning of the lesson. The teacher provides from the beginning of the lesson or module the students with the concepts and rules to apply them. In such an approach, the teacher clearly has explicit expectations and the students are aware of their responsibility in satisfy those expectations. At the same time the student her/himself knows the teacher's habits and attitudes and tries to solve the proposed problems focusing on what the teacher might desire as an answer. These mutual expectations about each other's specific roles and responsibilities is known as *didactical contract*, since this system of reciprocal obligations resembles indeed a contract. Obviously also in the TDS approach a shadow of the didactical contract remains as an obstacle to the students involvement in the knowledge construction. But the more they're trained to this type of activity, the more fearless they become to try to find a solution and speak out to present their ideas. And thanks to the last phase of the process, the institutional knowledge will finally appear to them as more consistent and meaningful. As some researches in this field show [Strømskag Måsøval, 2011], they will also more easily transfer a specific knowledge acquired to other contests or problems also when they are not strictly didactical. This adequately responds to what for Brousseau should be the aim of a teacher:

> "The objects of teaching and the knowledge communicated must allow the student to engage in all non-didactical situations and social practices as a responsible subject and not as a student" [Brousseau, 1990, p- 322-323]

.

Clearly it's not feasible nor necessary to address each mathematical topic in class by using a-didactical situations and the phases previously described. It's a good practice for the teacher though, when designing and programming a unit, to identify those that can be regarded as the core principles of the subject or topic, and that she/he desires to be constructed autonomously by the students.

Chapter 2

The MERIA Project and the task on the slide

A teaching approach which combines ideas from RME and TDS to design IBMT based modules, characterises the Erasmus+ project MERIA.

In this chapter we will give a short presentation of the Project and will describe how the specific research discussed in this thesis fits the context.

2.1 The MERIA Project

The MERIA Project was started in September 2016 and will last until the Summer 2019. The Project is a cooperation between higher education institutions and schools from four European countries: Croatia, Denmark, the Netherlands and Slovenia.

"MERIA" stands for "Mathematics Education, Relevant, Interesting, Applicable". The main goal of the Project is indeed to enhance quality of mathematics education in secondary schools and to show to students that the subject is significant, attractive and useful.

The RME principle of mathematics seen as a human activity leads to the concept of *relevant* mathematics; if the education is student-centred and importance is given to the student's autonomous discovery of mathematics, the students will be able to recognize their own potential for mathematical reasoning and how it can be relevant to every day life problems.

The use of realistic problems as well as the TDS teaching approach supports the subject to be seen as *interesting*. It promotes a positive attitude towards mathematics and it enhances students' self-confidence. As a matter of fact they will feel encouraged by succeeding or even just getting close to the right result, they will be fostered to take responsibilities, will have the chance to explore various ways of solving problems and mainly, by engaging in group activities, they will amuse themselves.

Finally, the use of the aforementioned strategies will allow students to use the learned mathematical tools in everyday situations. Mathematics becomes *applicable* to the students' eyes. They will get used to notice patterns in different aspects of real world, such as nature, economics or society, to use models and modelling, to summarise ideas and formulate conclusions and to communicate in a scientific and rigorous way these conclusions.

Essentially the combination of TDS and RME to design inquiry-based mathematics lessons, has the potential to make the lesson -and the subject itself-appear relevant, interesting and applicable.

MERIA aims to reach these results by focusing on the development of teachers' competences. That's why its purpose is to provide secondary school teachers in all Europe with new teaching materials that can be helpful to establish such inquiry situations. These materials, in order to be shareable, must be based on systematic didactical design and explicit teaching strategies. The main result of the Project is indeed a set of new display teaching scenarios and modules which will be based on the theoretical framework presented in the first chapter.

A *scenario* describes the teaching methods for a lesson: it describes the curriculum area of the lesson and its specific goals, the target knowledge and competences and gives clear instructions on how to perform a lesson based on TDS.

A *module* is a set of both written and digital materials (if needed) accompanying a teaching scenario. It includes explicit motivations for the choice of the specific problem, experiments with students' assignment, teaching methods, results and digital worksheets.

In Appendix A the reader can find templates, provided in the MERIA Project website [Winsløw and Jessen, 2017], describing the structure of a scenario and a module.

2.2 The slide task

The main task for the program participants is to design such modules, test them in practice in schools in the four involved countries and later make them available to teachers in other countries.
The task we are going to discuss in this thesis was conceived in this context. The aim of the task is to support students in the reinvention of the notion of slope of a curve in a point.

2.2.1 The task

Students, who still haven't learned about derivative, are asked to design a playground slide consisting of a bent and a straight part joining without bumps. They collaborate in groups of three and should give as an outcome concrete equations of a curve and a line which meet smoothly.

2.2.2 Reasons behind the choice of the task

The problem of understanding the concept of derivative, and the consequent problem of how to teach it, is still one of the biggest challenges of mathematics education at the secondary school and university level [Byerley and Thompson, 2017, Fuentealba et al., 2019, Gravemeijer and Doorman, 1999, Roundy et al., 2014, Weber et al., 2012, Winsløw et al., 2017].

The derivative of a function at one point is almost always introduced in calculus courses, as well as in standard mathematics and calculus books for high school or university, as the rate of change of the function at that point, which geometrically represents the slope of the tangent line to the graph of the function at that point.

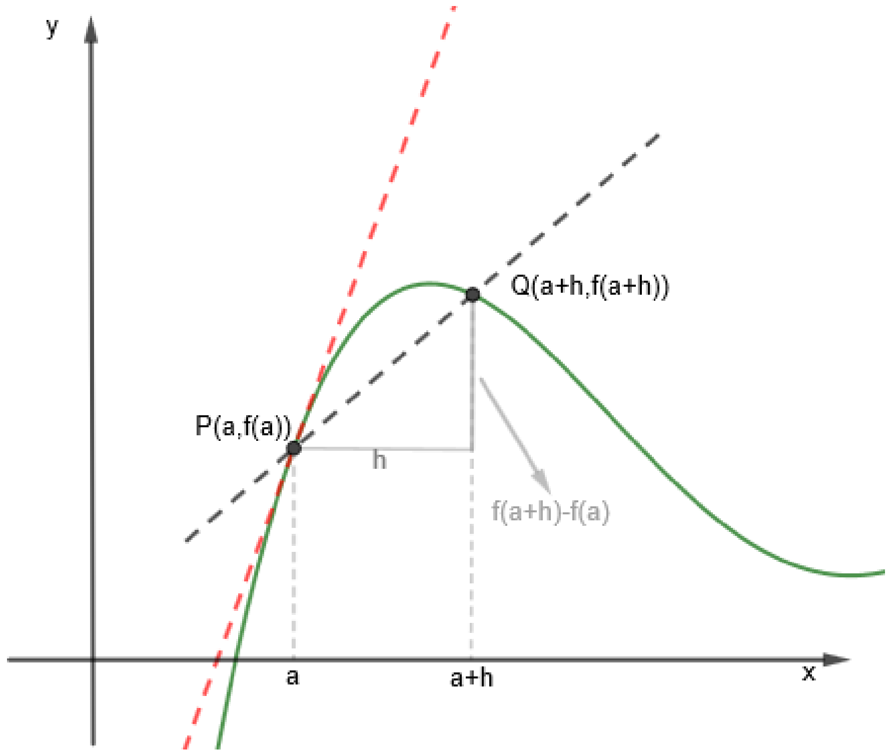


Figure 2.1: Standard approach to derivative.

A typical teaching approach would be for example starting with drawing the plot of a function (Figure 2.1) and a secant line passing through the point of the intended tangency and another point and write the difference quotient:

$$\frac{f\left(a+h\right)-f\left(a\right)}{h}$$

which corresponds to the slope of the secant line.

Then the teacher would show that by moving the second point closer and closer to the first point the secant line tends to coincide with the tangent line at the first point. At that point the formal definition of the derivative of a function at one point is introduced :

$$f'\left(a\right)=\lim_{h\to 0}\frac{f\left(a+h\right)-f\left(a\right)}{h}$$

In this approach the concept of the limit has to precede the derivative.
As showed by many previous researches [Bressoud et al., 2016, Kidron and Tall, 2014, Roundy et al., 2014, Tall, 2013, Van den Heuvel-Panhuizen and Drijvers, 2014], the concept and the use of limit and difference quotient in this context form serious obstacles for students. Especially Tall [Tall, 2013] points out how the limit concept, although it is an excellent foundation for mathematical analysis at the highest level, has proved to be a source of cognitive difficulties for students. Students often struggle to envision and make sense of a sliding secant line and its relationship to rate of change on a small interval [Weber et al., 2012] and also to interpret the different quotient as an amount of change in one quantity in relation to a change in another and not just as “how fast the function is changing” [Thompson, 1994]. In addition to the cognitive problems that the limit concept may bring, its introduction suddenly appears for no reason [Gravemeijer and Doorman, 1999].

In 1983 Orton (cited in [Bressoud et al., 2016]) provided one of the earliest descriptions of students’ difficulties with derivatives: he showed that while the students he had studied were generally proficient at computing derivatives, they carried significant misunderstandings in the concept of the derivative itself. Orton hypothesis was confirmed in 1994 by Ferrini-Mundy and Graham (cited in [Bressoud et al., 2016]) who, using interview methods, discovered that students could compute derivatives while being unable to connect those results to such tasks as giving the equation of a tangent line.

According to Tall, as cited in [Gravemeijer and Doorman, 1999], in such a sequence for presenting the derivative, the fallacy stays in trying to simplify a complex mathematical topic by breaking it up in smaller parts and order these in a sequence that is logical for mathematicians but not easy to be followed and understood by students. He suggests then to look for situa-

tions and problems that can work as informal starting points, from which a cognitive growth of the mathematical concept is possible. This means that the teacher should design lessons that can offer the students an opportunity to construct their understanding, starting from their own informal knowledge. This is also what RME aims at [Gravemeijer and Doorman, 1999].
On this issue it is also interesting to observe the development of the derivative concept from an historical point of view. In the article"The Changing Concept of Change: The Derivative from Fermat to Weierstrass" [Grabiner, 1983], Judith V. Grabiner remarks the difference between the standard book exposition of the topic and its historical development:

> "Fermat implicitly used it; Newton and Liebniz discovered it; Taylor, Euler, Maclaurin developed it; Lagrange named and characterized it; and only at the end of this long period of development did Cauchy and Weierstrass define it". [Grabiner, 1983, p. 205]

This is indeed the reverse of the order which is given in the standard explication of the concept by school books and teachers. As Grabiner suggests, the teacher should allow the students to discover the concept not as mere learners but as real mathematicians, showing them the rigorous definition not as the starting point of the lesson but as the result of their research. From this point of view students should have the chance to explore the different uses of the derivative before having any formal concept of it, when they still are "where mathematicians were before Fermat" [Grabiner, 1983, p. 206].

The above presented reasons stand behind the choice of the subject of the task, while its designing is inspired by the idea of guided reinvention. The context problem invites students to use their own informal knowledge and what is expected is that models for the definition of the slope of a curve at a point will emerge in a natural way. The emergent modelling design principle explains how an adequately guided students' activity can be the foundation for a reinvention process [Gravemeijer and Doorman, 1999,Doorman and Ea,

2009].
In recent years the reinvention and emergent models principles have been tested with successful outcomes in upper secondary school and bachelor university level [Gravemeijer and Doorman, 1999, Doorman and Ea, 2009, Oehrtman et al., 2014, Herbert and Pierce, 2008].

A balanced use of didactical and a-didactical situations can confer to the activity a proper extent of guidance. Mainly in the action phase if the students involve themselves in an autonomous quest to the solution, they have the chance to use methods and previous knowledge that they consider meaningful. Emergent models will be authentic and not altered by the teacher's intervention. As a consequence, once the concept has been formalised in the institutionalisation phase starting from their own models, students can experience ownership of the reinvention.

The aim of this research is indeed to investigate whether combining RME and TDS to design such a task for an inquiry based mathematics lesson can lead to a more meaningful and natural understanding of the notion of slope of a curve.
We intend to investigate:

1. What strategies do students tend to use and how do these relate to their level?

2. How do the student strategies can be linked to the teaching approaches that we are formulating in the next chapter?

3. Is it feasible and easy for teachers to use students' infomal models emerging from their outcomes to institutionalise the concept of slope of the tangent line of a curve at a point?

Results of this research could also partly answer to the question:
Is the reinvention principle suitable and feasible at secondary school level?

Chapter 3

Pilot studies and hypothetical approaches

Pilot studies have been conducted in schools in the Netherlands, Croatia and Slovenia with students who still hadn't learned about the derivative (In particular the results which will be discussed in this section have been produced in pilot lessons in 4 classes in the Netherlands).

In this chapter we will briefly discuss the results obtained from these pilot lessons and how they lead to formulate the hypothetical alternative ways of institutionalisation of the derivative concept.

Students of age around 15 were asked to design the slide collaborating in groups of three. The data collected from these experiences consist in the students' outcomes and in the teachers' and -in some cases- observers' self reports about what they noticed during the activity. Students were asked to give as an outcome concrete equations describing the line and the curve which form the slide. The choice of the specific curve was up to the students; the majority of the groups used a parabola, some opted for hyperbola or the circle and no groups ventured to use functions such as the logarithm or the sine. Some of the groups just tried to compute the equations using sketches of the line and the curve while others used digital graphing tools like GeoGebra or a graphic calculator.

3.1 Approaches observed

The very open formulation of the task allowed the students to come up with very different ideas and approaches to solve it. The main distinction between the approaches is in the way students tried to achieve smoothness, in the tools and previous knowledge they used and in their means of evaluation of the work done.

In this phase just an a-posteriori analysis of the students works was done. This made very hard to interpret which idea came up to the students' mind and lead them to the found solution. In some cases the presence of more details, more sketches and computations made more clear in what way the students were trying to achieve smoothness.

The approaches observed in some of the groups -taken from different classes- were for example:

1. Students in a group drew a parabola, but without giving its equation. Then they drew a straight line. They kept the slope of the line fixed and tried to vary its intersection point with the y-axis in order to find "when it is fluent". It seems -but this is only an hypothesis- that they were trying to connect the two figures using the local straightness of the curve in the point of connection.

2. Other students drew an hyperbole $y = \frac{1}{x}$. The drawing was very precise, so that they could see visually what was the line that was tangent to the curve. By using the intersection points of the line with the curve and the axis, they determined the equation of the line. It is possible that students here (Fig. 3.1) used unconsciously the symmetry of the

hyperbole, which made easier to find the line $y = -x + 2$ which is also symmetric with respect to the line $y = x$. It seems then that they were looking for a line which has just one intersection point with the curve.

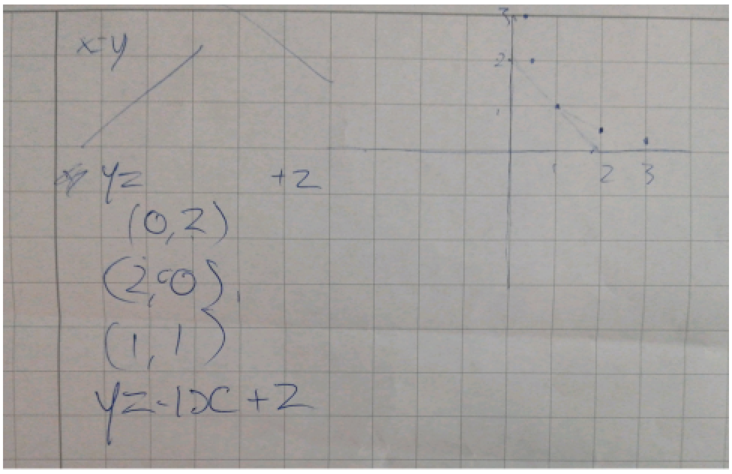


Figure 3.1: Example of solution with the hyperbole.

3. Students of this group started with the parabola $y = \frac{1}{4}(x-6)^2 + 2$. They then chose for the line: $y = -x+b$. In order to find the parameter b they set an equality between the two functions and found b such as the final equation became $\frac{1}{4}x^2 - 2x + 4 = 0$.
   They did so probably because they realised that this equation has a root of molteplicity 2. In this case the students' computation clearly shows that they were looking for a line which intersects the curve in one unique point.

4. In one class all the students used Geogebra to find their solution. The analysis of the only resulting equations with no observations or comments made almost impossible to hypothesise in which way students tried to have the right connection between the two curves. They could have taken one secant line passing through two points of the curve and then moved one point closer and closer to the other; or they could have

taken a point on the curve and a line passing through that point to then rotate this line until there seems to be just one intersection point. Or also they could have taken a curve and zoomed in until the part of the curve they were focusing in became visually straight and so drawn a straight line so that in the focus area it would coincide with the curve. The only cases in which it is easier to guess what has been done is when the curve and the line visually seem to intersect in just one point, but by precise verification there isn't even an intersection point. This means that students didn't start by taking one or two points on the curve, but probably took a curve and a random line and tried to translate or rotate this -with no constictions to the curve- unless it seemed to be tangent to the curve. But this is also just another hypothesis.

During the validation phase, implemented with the whole class, students had to check if their designs would fit with their idea of smoothness. This was the explicit request of that phase of the lesson, but obviously students would check their work also during the action phase to be sure they were working correctly.

It is possible to classify students' validation approaches in three categories:

1. *Visual*: Some students just checked visually if their drawing "looked good". With the help of GeoGebra, if they were using it, they zoomed in on the intended point of tangency in order to check the smoothness on a smaller scale.

2. *Algebraic*: Some students set a system with the equations of the curve and line found, and tried to compute whether the system had the desired unique intersection point.

3. *Numerical*: Some students validated their designs by construction, if they had used numerical datas. It means that they would take the intersection point and another point on the curve very close to the

the previous one and then compute the slope of the imaginary line connecting the two points; they would then check if this "average slope" corresponds to the slope of the straight line they used for the slide.

By observing each group's work, including both their action and the means of evaluation, it is possible to notice the recurrence of some designing strategies.

An idea that emerged in many solutions, even if in many different forms, is to choose the line and the curve in such a way that they meet in just one point. For convenience we can label the ensemble of approaches with this property as "**Bounding line approach**", because they can be linked to the definition of tangent line as the unique local bounding line.

Students who focused on adjusting the slope of the line or the position of the curve so that they could fit best against each other, or that zoomed in in the focus area until they could visualise a piece of the curve as straight, used an informal idea which is close to the definition of the tangent line as the best linear approximation of the curve; that's why we bring these approach together under the name of "**Linear approximation approach**".

In the Appendix B the reader can find a table in which we summarized the description of the students' work in each groups, specifying the equations given as an outcome -when they were actually found- and the evaluation method chosen (or that the teacher could suggest to either prove the correctness or show flaws in the result).

3.2 Possible institutionalisation methods

The presence of traces of the aformentioned approaches were used to hypothesise what informal mathematical models emerged from the students' work. An interesting fact is that barely any groups of students used the secant lines approach. This is a clear sign of the fact that this approach, which is the approach found in standard school books, does not come naturally to learners. On the other hand, most of the students came up with other ideas that are essential to the notion of slope of a curve in a point, such as those that we called "Bounding line" and "Linear approximation" approach.

As we said in the First Chapter, according to TDS, in the Institutionalisation phase the teacher should present the formal knowledge starting right from the students' informal models. In the context that we are presenting then, the teacher must be prepared to present the institutionalised knowledge by transforming students' models into models for mathematical formal reasoning.

Here we present 3 ways in which it is possible to connect the potential observed in the analysed students' work to the mathematical formal knowledge regarding the notion of slope of a curve in a point.

3.2.1 Algebraic approach

As we said, many students chose the curve and the line so that they would meet in just one intersection point. The crucial point is that as a matter of fact that unique intersection point is a "double point". R. Michael Range in his works "Using high school algebra for a natural approach to derivatives and continuity" and "Where Are Limits Needed in Calculus? " [Michael Range, 2018, Michael Range, 2011] shows a rigorous approach to teach the derivative which essentially uses this idea which involves multiplicity.

It is simple to convince the students that a point of tangency (they should be familiar to this concept in the context of circles) is at least a double point. If the teacher draws a curve and a tangent line at one point she/he can show that by slightly perturbing the line, the tangency point suddenly splits into two points (fig. 3.2). How can this method be computed algebraically? We

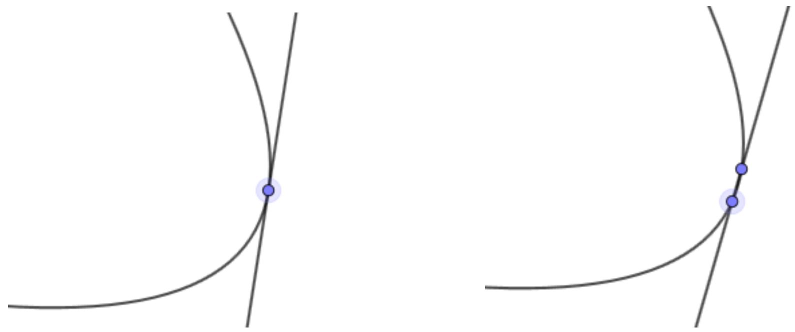


Figure 3.2: Perturbation of the tangent line reveals two points of intersection.

begin with showing how it works with a simple parabola, the function that was chosen by most of the students.

Example 3.1. Let us consider the tangent to $y = x^2$ at the point (a, a^2). Let us take te generic line passing through this point; its equation is given by

$$y - a^2 = m(x - a)$$

that we can rewrite as

$$x^2 - a^2 = m(x - a)$$

and again as

$$x^2 - a^2 - m(x - a) = 0$$

Let us now factorize it as

$$(x - a)(x + a) - m(x - a) = 0$$

$$(x - a)[(x + a) - m] = 0$$

$$(x - a)[x - (m - a)] = 0$$

Since we want a to be a solution with multiplicity 2, the expression $m - a$ must be equal to a

$$m - a = a$$

and so we have

$$m = 2a$$

Example 3.2. We can give also an example of a rational function such as $y = \frac{1}{x^2}$.

We want to compute the slope of its tangent line at the generic point $\left(a, \frac{1}{a^2}\right)$. As before, we consider the generic line passing through that point:

$$y - \frac{1}{a^2} = m(x - a)$$

which we rewrite as

$$\frac{1}{x^2} - \frac{1}{a^2} = m(x - a)$$

Let us solve the equation found:

$$a^2 - x^2 = m(x - a)a^2x^2$$

$$-(x - a)(a + x) = m(x - a)a^2x^2$$

$$(x - a)[ma^2x^2 + (x - a)] = 0$$

As before, we want a to be a root of multiplicity 2 of this function. This means that $ma^2x^2 + x - a$ must be a multiple of $x - a$.
By computing the division we find that the rest is equal to $2a + ma^4$.
We want it to vanish, so it must be $m = -\frac{2}{a^3}$.

One could argue that we are taking into account as examples only everywhere convex or concave functions, so that when we compute their intersection points with a line we can obtain 0, 1 or a maximum of 2 of such points. This isn't a real issue in cases in which we have a tangent line to the curve at a point (what we know being a double point), and this line intersect the curve somewhere else.
Cases which are relevant are those in which, by taking a secant line to a curve at a point and slightly rotating this, it reveals 3 points of intersection.

Let us show an example of this situation.

Example 3.3. Let's consider the function $y = x^3$ and let's study it in the point $(0, 0)$. We can easily see that by slightly rotating the line, the point

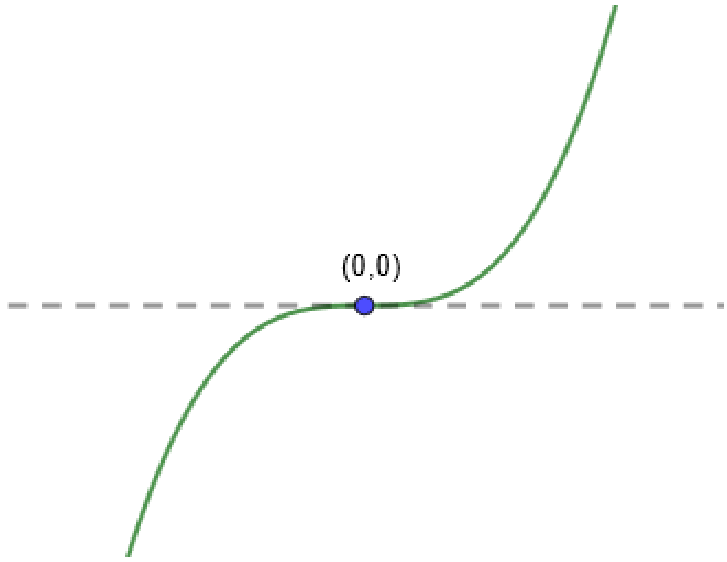


Figure 3.3: Tangent line to the curve $y = x^3$ at the point $(0, 0)$.

separates into 3 distinct points.
That's why we say that the intersection point of the curve and the line (in this example the specific line is the x-axis $y = 0$) is a triple point, or a point of multiplicity three.

We can now compute the intersection between the curve $y = x^3$ and the generic line passing through $(0, 0)$ to verify that the tangency line is $y = 0$.

We have:

$$y = x^3$$

and the generic line:

$$y - 0 = m(x - 0)$$

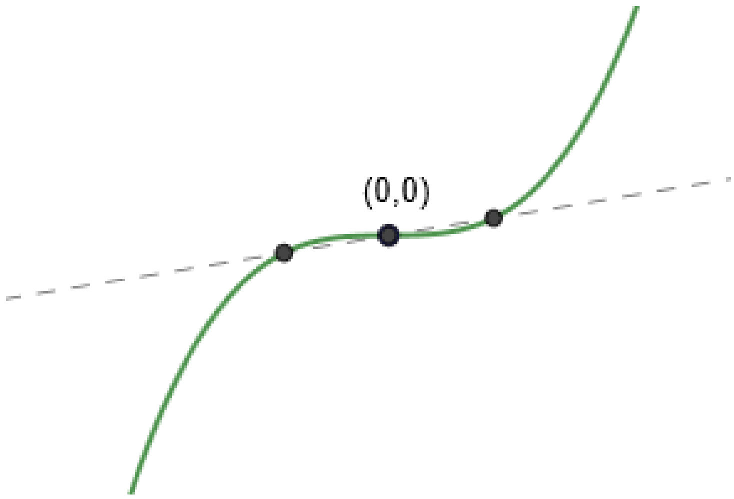


Figure 3.4: Slightly rotated line shows three points of intersection.

Together they lead to the equation:

$$x^3 - 0 = m(x - 0)$$

$$x^3 - mx = 0$$

$$x(x^2 - m) = 0$$

$$x(x - \sqrt{m})(x + \sqrt{m}) = 0$$

Now we want the point $(0, 0)$ to be a solution of the equation with multiplicity three.
This means that m must be equal to zero.
So we have that the tangent line is $y = 0$.

Based on these and more examples Michael Range [Michael Range, 2018, p. 439] gives a reasonably informal geometric definition of a tangent[1]:

[1] This fundamental idea to relate tangents to double roots is not new at all; already Descartes, Leibniz and Newton used it for different functions and curves(see [Grabiner, 1983, Michael Range, 2018, Michael Range, 2011]).

Definition 3.4. *A tangent to a curve at a point on the curve is a line that intersects the curve in such a way that some suitable arbitrarily small displacement of the line will split the point in two (or possibly more) distinct points of intersection. We say that the point of tangency is a point of multiplicity two (or greater).*

It is possible to generalise the method used for the previous functions for a generic polynomial and its graph. We can take the generic formula for a polynomial with degree n:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$

We want to determine the tangent to the graph of P at the generic point $(a, P(a))$. As in all the previous examples we take the generic line passing through this point:

$$y = P(a) + m(x - a)$$

and by computing its intersection with the curve, we obtain:

$$P(x) = P(a) + m(x - a)$$

We don't need to find the roots of this polynomial; we are only interested in the trivial root $x = a$ and in finding whether this has multiplicity 2 or greater. We rearrange the above equation in the form:

$$P(x) - P(a) = m(x - a)$$

As we know - and the students should know - from the theory related to polynomials, if one polynomial of degree $n \geq 1$ $Q(x)$ has a as a root, i.e. $Q(a) = 0$, then there exists a polynomial $q(x)$ of degree $n - 1$ such that $Q(x) = q(x)(x - a)$.

In our case $P(x) - P(a)$ has indeed a zero at the point a, so we can rewrite it as:

$$P(x) - P(a) = q(x)(x - a)$$

The previous equation becomes:

$$q(x)(x-a) = m(x-a)$$

$$[q(x) - m]\,(x-a) = 0$$

Since we want a to be a root of $[q(x) - m]$, we must have $m = q(a)$.

We can formalise this result in the following theorem that solves the tangent problem for any polynomial [Michael Range, 2018]:

Theorem. *Let P be a polynomial of degree $n \geq 2$ and let $(a, P(a))$ be a point on its graph. Then there exists a unique line through $(a, P(a))$ that intersects the graph at that point with muliplicity greater than one. The slope m of that line is given by $q(a)$, where q is the polynomial of degree $n-1$ in the factorisation $P(x) - P(a) = q(x)(x-a)$.*

We can then call this unique line *the tangent to the graph of P at the point* $(a, P(a))$ and call $m = q(a)$ the *derivative of P at the point a.*

We managed to give such definitions without using the limits. In the two cited articles ([Michael Range, 2018, Michael Range, 2011]) Michael Range shows how it is possible to extend this method based on multiplicity to more general algebraic functions like rational functions, roots and more.
Moreover it is possible to easily obtain the standard rules for differentiation using this approach.

Again, limits are not needed at all; all that is needed are the basic elementary algebraic tools that we used in case of polynomials, suitably extended to more general or complicated functions.

About how to extend the approach to other functions and how to obtain differentiation rules and the reader can find full details in Michael Range's papers [Michael Range, 2018, Michael Range, 2011].

Obviously, the multiplicity method reaches its limits when taking into account functions that cannot be defined by some algebraic expressions. As a matter of fact we could state that as long as one wants to compute tangent lines to polynomials or other general algebraic functions the classical limit-based approach can be easily substituted by this algebraic approach; but since this become really elaborate when working with transcendent functions, we eventually need to introduce a more sophisticated approach that allows us to handle even more general functions than algebraic ones.

3.2.2 Local approximation

Analysing the students' work, another designing strategy that was often noticed is that which we called "locally linear approach". Even though when using this strategy it was difficult for the students to come up with concrete equations, many times this informal idea of the tangent line as the best linear approximation of the curve lead them to design the slide accurately.

The teacher could then start from this designing strategy and show how it is possible to precisely compute the tangent line to a curve at a point.

Let us see some examples.

Example 3.5. We start again with the most chosen curve: the parabola.

Let's consider the function $y = x^2$. We aim to find the tangent line to this curve at the generic point (a, a^2). It's useful here to use a small trick:

if we translate the point (a, a^2) to the origin, computing the tangent line becomes much easier.

Our function becomes then:

$$y = (x + a)^2 - a^2$$

and we can rewrite it as:

$$y = x^2 + 2ax$$

We are interested now in what happens specifically in the point $(0, 0)$.

This is the moment in which to use the students' idea. If we take a point which is very close to the intended point of tangency -namely zooming in- locally the curve is very close to a line. Indeed when the value of x is very close to 0, the value of x^2 is even more close to zero. We can therefore ignore the term x^2 and the previous term becomes:

$$y = 2ax$$

.

We obtained then $y = 2ax$ as a linear approximation of the function $y = (x + a)^2 - a^2$ around the origin, which is the linear approximation of the function $y = x^2$ around the point (a, a^2).

Hence $2a$ is the slope of the tangent line to the parabola $y = x^2$ at the point (a, a^2).

Example 3.6. As a second example we can again take the rational function $y = \frac{1}{x^2}$ and compute its tangent line at the generic point $(a, \frac{1}{a^2})$.

We use the same trick as before and translate the point $(a, \frac{1}{a^2})$ to the origin. The function becomes:

$$y + \frac{1}{a^2} = \frac{1}{(x+a)^2}$$

and we rewrite it as:

$$y = \frac{a^2 - (x+a)^2}{a^2(x+a)^2}$$

$$y = \frac{-x^2 - 2ax}{a^2(x+a)^2}$$

As in the previous example, the intended point of tangency is now $(0,0)$ and when the value of x is very close to 0, then $x^2 \approx 0$ and $(x+a)^2 \approx a^2$.
We obtain then:

$$y = \frac{-2ax}{a^4} = \frac{-2x}{a^3}$$

Therefore $\frac{-2x}{a^3}$ is the slope of the tangent line to the function $y = \frac{1}{x^2}$ at the point $(a, \frac{1}{a^2})$.

Example 3.7. For this approach we can give also an example of a trascendent function.
Let us consider the function $y = \sin x$ and let us find its tangent line at the point $(a, \sin a)$.
We start by translating this point to the origin and we obtain the function:

$$y + \sin a = \sin(x+a)$$

Using the trigonometric sum rule we can rewrite it as:

$$y + \sin a = \sin x \cos a + \sin a \cos x$$

As in the previous examples our focus is in the surroundings of the point $(0, 0)$.
When the value of x is very close to 0 we have that $\sin x \approx x$ and $\cos x \approx 1$.
We obtain:

$$y + \sin a = x \cos a + \sin a$$

$$y = x \cos a$$

and so we have that the slope of the tangent line at the point $(a, \sin a)$ is $\cos a$.

What has been done in the examples in this section is substantially using the concept of limit without explicitly saying it. But the way in which the concept was used here, without using formal and elaborate formulas or definitions, can be much more natural for the students.

We have given some examples of how to rigorously compute the slope of the tangent line with the "locally linear approach". Furthermore it is possible to formally introduce and define the derivative using this idea.
Such a characterisation has been done by Jerrold Marsden and Alan Weinstein in their book "Calculus Unlimited" [Marsden and Weinstein, 1981, p. 31-44].
This text presents an alternative treatment of calculus without the use of limits. The prerequisites to the understanding of the methods presented are a knowledge of functions, graphs, high school algebra and trigonometry.
For example a definition of the derivative is given by using the concept of "rapidly vanishing points". The notion which supports this method is essentially the local linearity.
Let us show more in detail this concept of "rapidly vanishing point".

Definition 3.8. *We say that a function f vanishes at x_0 if $f(x_0) = 0$, namely if x_0 is a root of f.*

We can notice that some functions vanish "more rapidly" than others. If we consider the two functions $f(x) = x - 1$ and $g(x) = 5(x-1)^2$ we can notice that both vanish at $x = 1$ but g vanishes "more rapidly" than f (fig. 3.5):

$$\begin{array}{cc} f(2) = 1 & g(2) = 5 \\ f(0) = -1 & g(0) = 5 \\ f(1,1) = 0,1 & g(1,1) = 0,05 \\ f(0,9) = -0,1 & g(0,9) = 0,05 \\ f(1,01) = 0,01 & g(1,01) = 0,0005 \\ f(0,99) = -0,01 & g(0,99) = 0,0005 \\ f(1,001) = 0,001 & g(1,001) = 0,000005 \\ f(0,999) = -0,001 & g(1,0001) = 0,000005 \end{array}$$

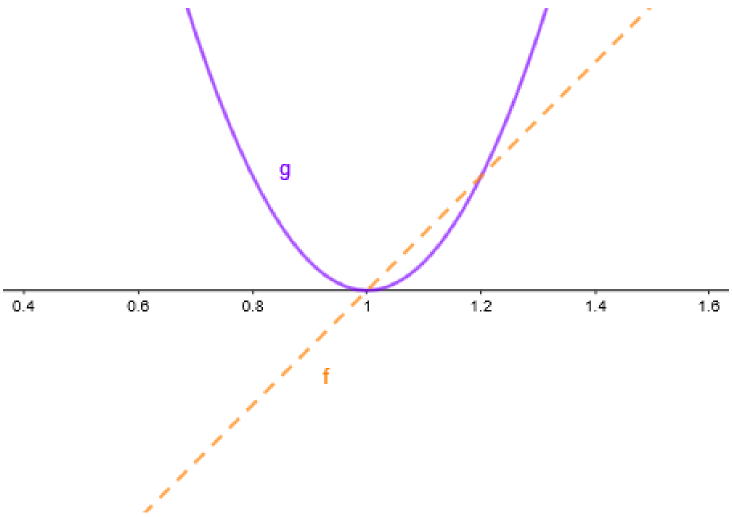


Figure 3.5: Function g vanishes more rapidly than function f.

How can we express this concept of "rapidly vanishing" more formally? Marsden and Weinstein give this definition.

Definition 3.9. *Let r be a function such that $r(x_0) = 0$. Then $r(x)$ vanishes rapidly at $x = x_0$ if and only if, for every positive number ϵ, there is an open interval I about x_0 such that, for all $x \neq x_0$ in I, $\mid r(x) \mid < \epsilon \mid x - x_0 \mid$.*

We can at this point give a definition of “derivative” which uses this concept of rapidly vanishing functions:

Definition 3.10. *We say that a function f is differentiable at the point x_0 and that m is the derivative of f at the point x_0, if the function $r(x)$ defined by*

$$r(x) = f(x) - [f(x_0) + m(x - x_0)]$$

vanishes rapidly at x_0.

In Marsden and Weinstein's work, most of the standard rules for differentiation -sum rule, constant multiple rule, product rule, quotient rule, ...- are introduced and proved by using this definition. Again, limits are never explicitely used in these definitions and proofs.

The reader can read about this approach in details in [Marsden and Weinstein, 1981].

3.2.3 Sign change

We want here to present another alternative way to introduce and define the derivative, which is also outlined in "Calculus unlimited" [Marsden and Weinstein, 1981, p 1-13].
We haven't found in the pilot studies an explicit use of the idea which stands behind this approach, but it appears so natural and easy to us that we do not preclude that when conducting a lesson using the slide task some students might come up with such an intuition.

This approach uses the concept of change of sign.
Let us start with giving the following definition:

Definition 3.11. *Let f be a function and x_0 a real number. We say that f* ***changes sign from negative to positive*** *at x_0 if there is an open interval (a, b) containing x_0 such that f is defined on (a, b) (except possibly at x_0) and*

$$f(x) < 0 \quad if\, a < x < x_0$$

and

$$f(x) > 0 \quad if\, x_0 < x < b$$

Similarly, we say that f ***changes sign from positive to negative*** *at x_0 if there is an open interval (a, b) containing x_0 such that f is defined on (a, b) (except possibly at x_0) and*

$$f(x) > 0 \quad if\, a < x < x_0$$

and

$$f(x) < 0 \quad if\, x_0 < x < b$$

Basically, a function is said to change sign when its graph crosses from one side of the x-axis to the other.

It is possible to find the slope of the tangent line to a curve at a point using this concept of "changing sign".
We can indeed define the derivative as follows:

Definition 3.12. *Let f be a function whose domain contains an open interval about x_0. We say that the number m_0 is the* ***derivative of f*** *at x_0 if:*

1. *For every $m < m_0$, the function*

$$f(x) - [f(x_0) + m(x - x_0)]$$

changes sign from negative to positive at x_0;

2. *For every $m > m_0$, the function*

$$f(x) - [f(x_0) + m(x - x_0)]$$

changes sign from positive to negative at x_0.

If such a number m_0 exists, we say that f is **differentiable** at x_0, and we write $m_0 = f'(x_0)$. If f is differentiable at each point of its domain, we just say that f is differentiable. The process of finding the derivative of a function is called differentiation.

Geometrically, the definition says that every line through $(x_0, f(x_0))$ with slope less than m_0 crosses the graph of f from above to below, while each line with slope greater than m_0 crosses f from below to above (see fig 3.6).

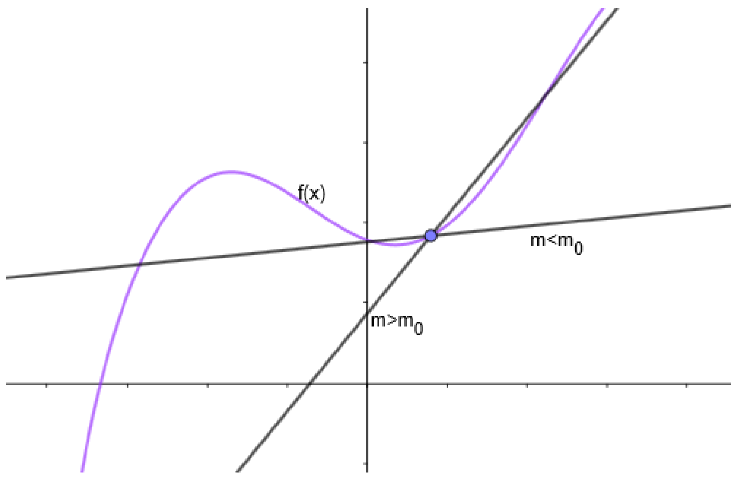


Figure 3.6: Lines with slope different from m_0 cross the curve.

We can now present some concrete examples, using the same functions we have used in the previous sections:

Example 3.13. Let us consider the function $y = x^2$ and find its derivative at the point (a, a^2).

We must consider the difference

$$f(x) - [f(x_0) + m(x - x_0)]$$

which in our case is:

$$x^2 - a^2 - m(x - a)$$

$$= (x - a)(x + a) - m(x - a)$$

$$= (x - a)(x + a - m)$$

This term changes sign at $x = a$ $\forall m$, except when $m = 2a$.
More precisely $\forall m < 2a$ and $\forall m > 2a$ the function $x^2 - a^2 - m(x - a)$ changes sign.
Hence $m = 2a$ is the derivative of $y = x^2$ at the point (a, a^2), namely it is the slope of the tangent line to $y = x^2$ at the point (a, a^2).

Example 3.14. As a second example we consider $y = \frac{1}{x^2}$ at the point $(a, \frac{1}{a^2})$.

In this case the difference$f(x) - [f(x_0) + m(x - x_0)]$ is:

$$\frac{1}{x^2} - \frac{1}{a^2} - m(x - a)$$

$$= \frac{a^2 - x^2}{a^2x^2} - m(x - a)$$

$$= \frac{(a - x)(a + x)}{a^2x^2} + m(a - x)$$

$$= (a - x)\left(\frac{a + x}{a^2x^2} + m\right)$$

This last term changes sign when $x = a$ $\forall m$ except when $m = -\frac{2a}{a^4} = -\frac{2}{a^3}$.

Hence $m = -\frac{2}{a^3}$ is the derivative of $y = \frac{1}{x^2}$ at the point $(a, \frac{1}{a^2})$.

The preceding examples show how derivatives may be calculated directly from the definition. Exactly as it is done in standard calculus courses, in Marsden and Weinstein's approach, functions are not differentiated using the definition; instead general differentiation rules are introduced. These, once derived, enable to differentiate many functions quite simply.

As an example we will derive the rule to find the tangent line to any parabola at any point:

Theorem 3.2.1. *Let $f(x) = ax^2 + bx + c$, where a, b, and c are constants, and let x_0 be any real number. Then f is differentiable at x_o, and $f'(x_o) = 2ax_o + b$.*

Proof. We must investigate the sign change of the function $f(x) - [f(x_0) + m(x - x_0)]$ at the point x_0. We have:

$$\begin{aligned} f(x) - [f(x_0) + m(x - x_0)] \\ = ax^2 + bx + c - [ax_0^2 + bx_0 + c + m(x - x_0)] \\ = a(x^2 - x_0^2) + b(x - x_0) + c - c - m(x - x_0) \\ = (x - x_0)[a(x + x_0) + b - m] \end{aligned}$$

The factor $[a(x + x_0) + b - m]$ is a (possibly constant) linear function whose value at x_0 is $2ax_0 + b - m$.

- if $m < 2ax_0 + b$, this factor is positive at $x = x_0$ and, since it is a linear function, it is positive also when x is near x_0.
  Thus the product of $[a(x + x_0) + b - m]$ with $(x - x_0)$ changes sign from negative to positive at x_0.

- if $m > 2ax_0 + b$, this factor is negative at $x = x_0$ and, since it is a linear function, it is negative also when x is near x_0.
  Thus the product of $[a(x + x_0) + b - m]$ with $(x - x_0)$ changes sign from positive to negative at x_0.

The number $m_o = 2ax_o + b$ satisfies the definition of the derivative, and so $f'(x_o) = 2ax_o + b$. $\square$

In "Calculus Unlimited" Marsden and Weinstein also reformulate the derivative in terms of the concept of "transition points". Other concepts, such as that of "sign change", can also be expressed in terms of transitions. We will not go in further details with it, but the reader can find these in [Marsden and Weinstein, 1981, p. 16].

Chapter 4

Experimental lessons

The results briefly described in the previous chapter were obtained during pilot studies. After those pilot lessons the slide task was tested in three further lessons. These lessons were given after the previous results had been analysed. As a consequence, the teacher who conducted the lesson and the observers were more aware of the informal ideas that could quite possibly come to the students' mind; thus they were more focused in seeing if such ideas, as well as different ones, emerged in the students' work. Moreover, in each of these lessons there were at least two observers in addition to the teacher who conducted the lesson; this means that it was possible to monitor the whole action of the observed groups. This happened to be really worthwhile because it made easier to comprehend the students' ideas that stand behind their approaches, or also to understand which obstacles arose.

In this chapter we are going to describe these experimental lessons. We will describe the set-up of the lesson, the students' action in details and briefly the formulation, validation and institutionalisation phases.

4.1 Lesson 1

The first of these last lessons was given in a “Mathematics A”, 4th grade (level 10) class[1]. Students were in total 21 and had been previously divided by their teacher in 7 groups, each made of 3 students. There were in total 5 observers: the class’ usual teacher, one colleague who teaches in the same school, two researchers of the Freudenthal Institute for science and mathematics education who are both involved in the whole MERIA project -one of those conducted the lesson- and me.
The lesson lasted in total 45 minutes.

4.1.1 Devolution phase

This phase lasted 5 minutes. The teacher who conducted the lesson presented

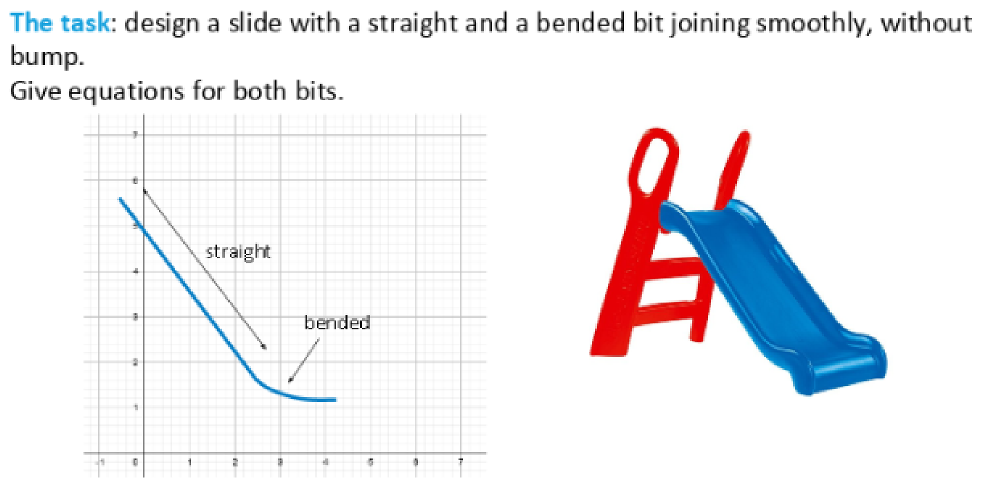


Figure 4.1: The teacher presents the task.

the task by projecting images of a playground slide and of a plot made of a straight part and a bended part joining smoothly (fig. 4.1).

[1]In appendix C the reader who is not familiar with the Dutch school system can find a description of this, as well as the description of the mathematics curriculum in secondary school in the Netherlands.

Then he refreshed the students' memory by revising the general line equation and the meaning of the coefficients. For the bended part he didn't do the same, in order to leave completely free and unaffected the choice of the curve.
Students were then provided with working sheets. The structure of these was revised after the pilot studies. It should be noted that in a TDS-based lesson the teacher doesn't intervene in the action phase and for this reason all that constitutes the "milieau" must be designed properly.

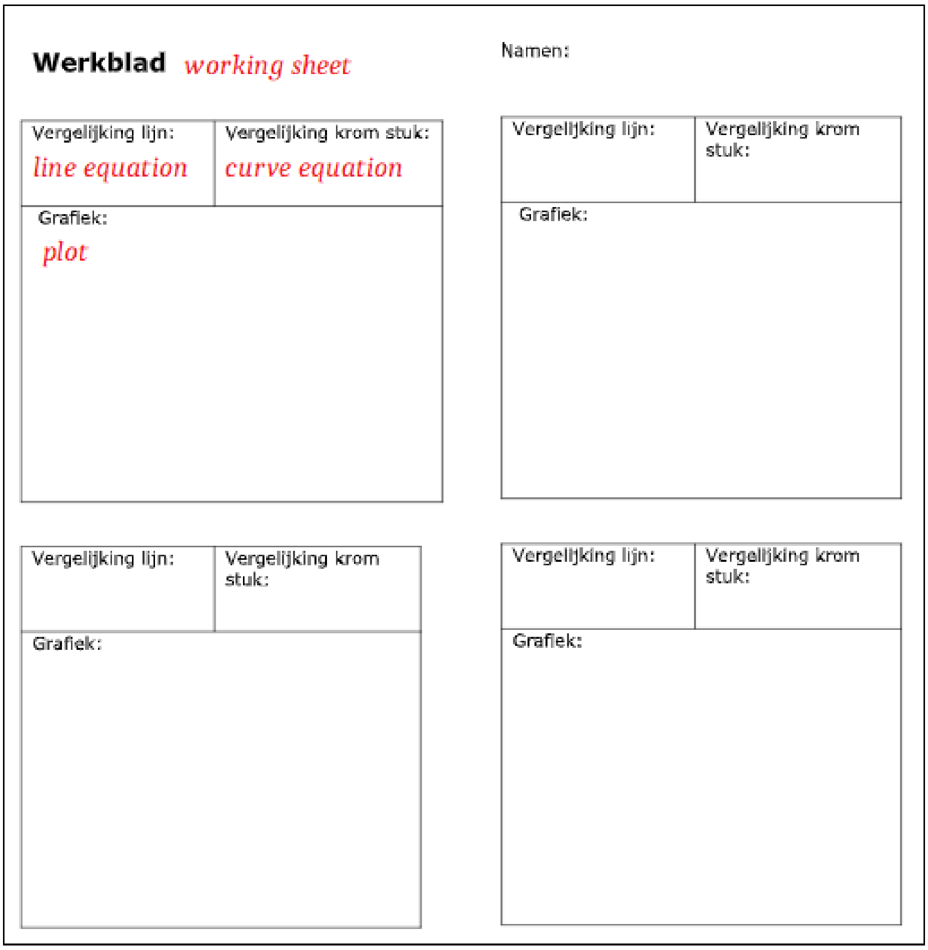


Figure 4.2: Working sheet used in previous pilots.

In the pilot studies the working sheet consisted in a sequence of squares in which the students had to fill in their improving drawings and equations (see fig.4.2).

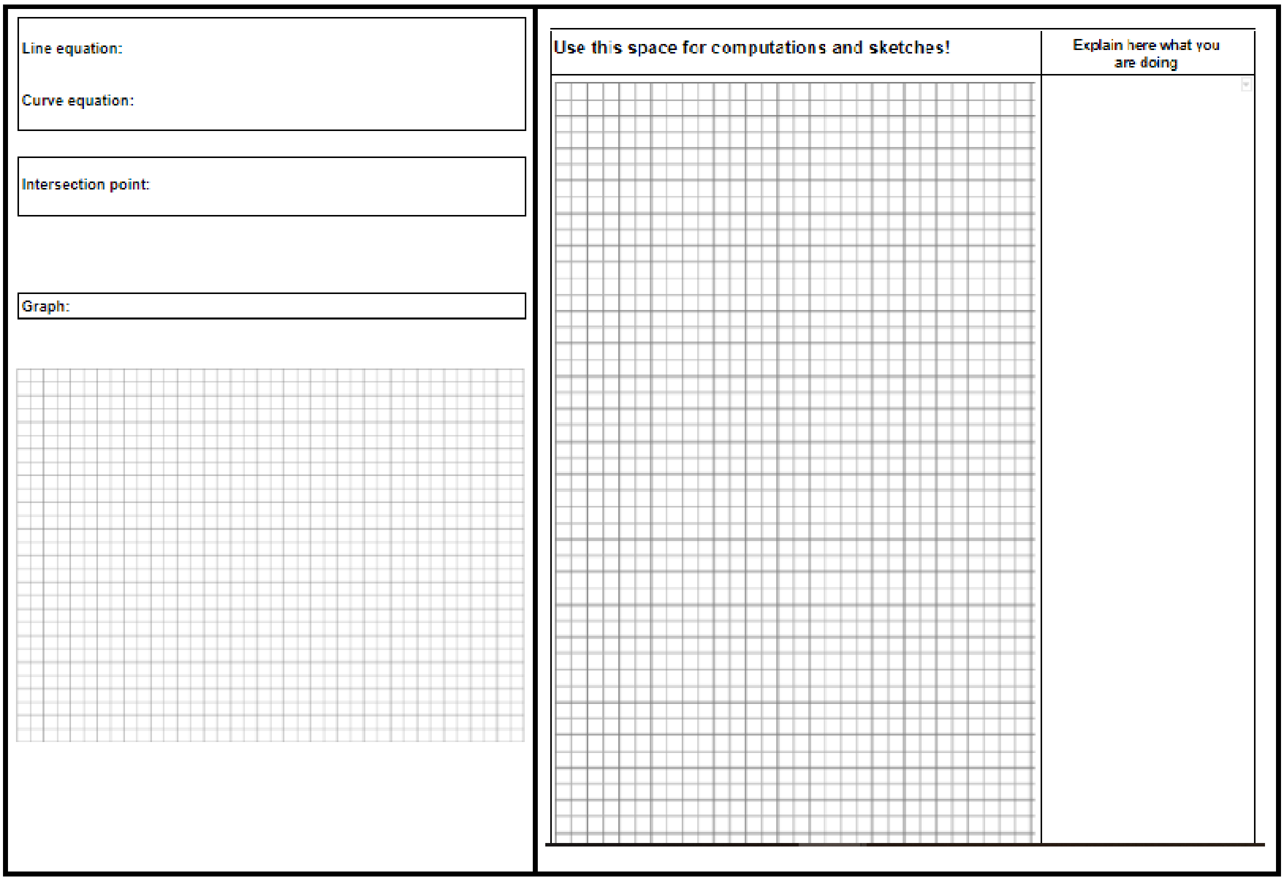


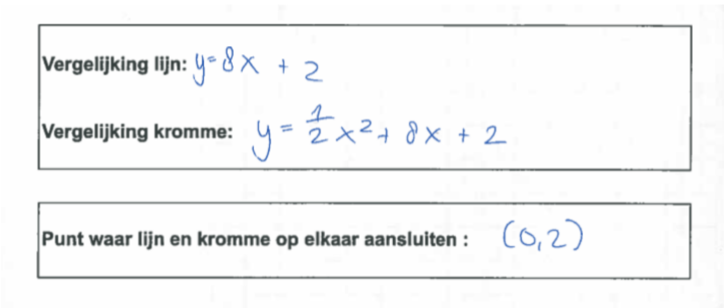
Figure 4.3: New working sheet (English version).

While analysing the results, we noticed that most of the times students, even when they found concrete equations, didn't specify the point at which the line and the curve met. The first adjustment made to the working sheet was explicitly asking to write the coordinates of this points. (see fig. 4.3). Moreover we noticed that students tended to make not very precise drawings of the functions. That's why we decided to provide them with chequered sheets, so as to encourage accuracy. We also decided to remove the blocks and leave the students the discretion to arrange as they wanted all the space on the sheet (again, see fig. 4.3).

4.1.2 Action phase

The action phase started. The groups of students started working on the task. In 4 of the 7 groups there was an observant, monitoring the students' action and taking accurate notes of it. The teacher who conducted the lesson was walking around the class, glancing over their work to see if he could notice some significant approach or ideas.
This phase lasted 20 minutes in total.
Of the 7 groups, 3 came up with some relevant ideas. We will present here in details their action.

- **Group 22**[2] This group was the only one to find a complete and correct solution. Also their intuition can be very fascinating.
  After a few attempts they found as the final solution (see fig. 4.4):
  - **Line equation:** $y = 8x + 2$
  - **Curve equation:** $y = \frac{1}{2}x^2 + 8x + 2$
  - **Intersection point:** $(0, 2)$



Vergelijking lijn: $y = 8x + 2$

Vergelijking kromme: $y = \frac{1}{2}x^2 + 8x + 2$

Punt waar lijn en kromme op elkaar aansluiten : $(0, 2)$

Figure 4.4: Solution found by Group 22.

They haven't provided a drawing for these equations, but their solution, computed with GeoGebra, looks like this:

[2] The numbering of the groups follows the table of all the results that the reader can find in Appendix B.

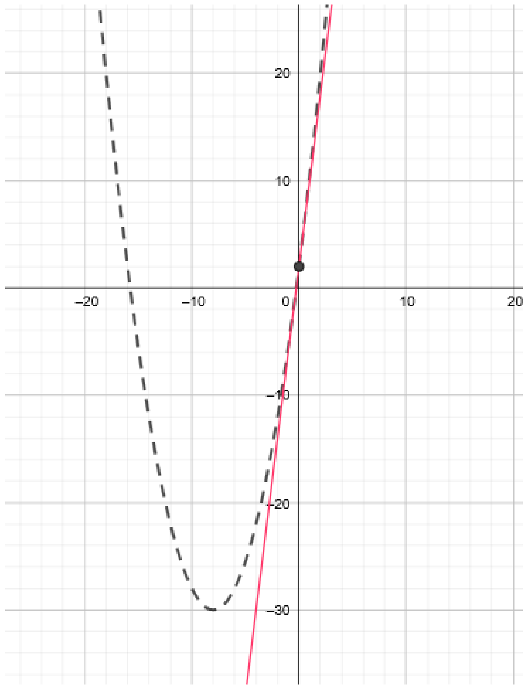


Figure 4.5: Equations given by Group 22 plotted with GeoGebra.

As we said, the solution is correct, in the sense that the chosen line is exactly the tangent line to the curve at the point $(0, 2)$.

It is curious, though, that according to the observer's notes, the group had found another solution before the definitive one:

- **Line equation:** $y = 2x + 2$
- **Curve equation:** $y = \frac{1}{2}x^2 + 2x + 2$
- **Intersection point:** $(0, 2)$

Not only the tangency is achieved here, but the figure would likely represent a playground slide too (we can notice that the other one was way too steep to be a real slide). Anyway, it is curious that the students didn't leave any trace of this solution on their worksheet and chose the other solution as the final one.

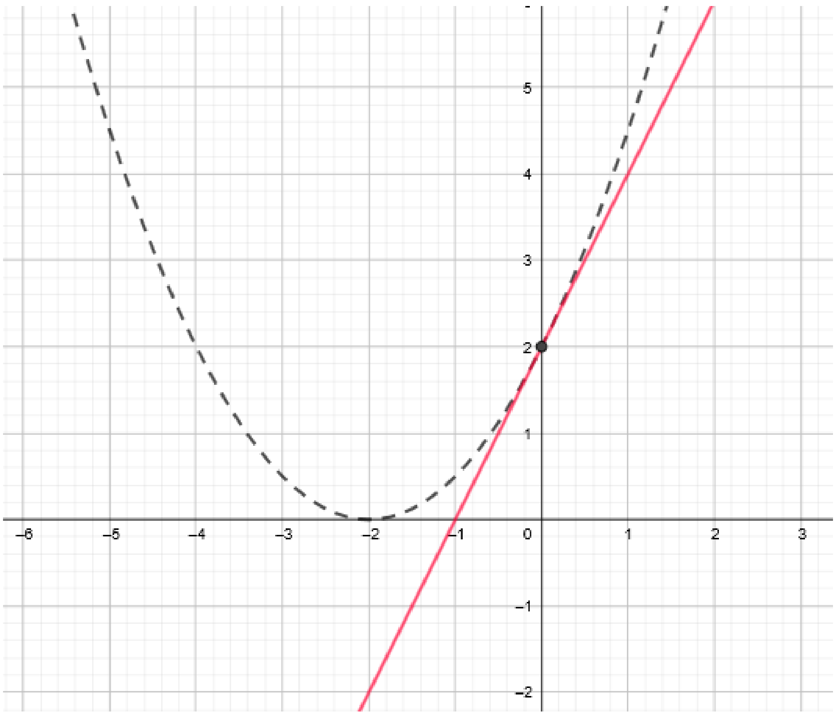


Figure 4.6: Other solution found by Group 22.

Let us show now how this solution can be intriguing.
On the work sheet students explicitely wrote:

> "b in the formula of the parabola and a in the formula of the line must be the same."

specifying that they mean that "the direction coefficients of the line and the curve must be the same".
Furthermore, they had the remarkable idea of choosing (in both their solution) the intersection point with the y-axis of both the line and the curve, as the intersection point of these two.
In this way such point, $(0, 2)$ has $x = 0$.

The fact that they stated that "b in the formula of the parabola and a in the formula of the line must be the same", shows that probably they realised that, around $x = 0$, the term "x^2" can be ignored.

This is exactly the technique that we used in Chapter 3 in the section "Local approximation" (3.2.2).

- **Group 20** Students in this group didn't even find an equation for the curve. However they had an interesting approach that could be used by the teacher and connected to the formal institutionalisation.

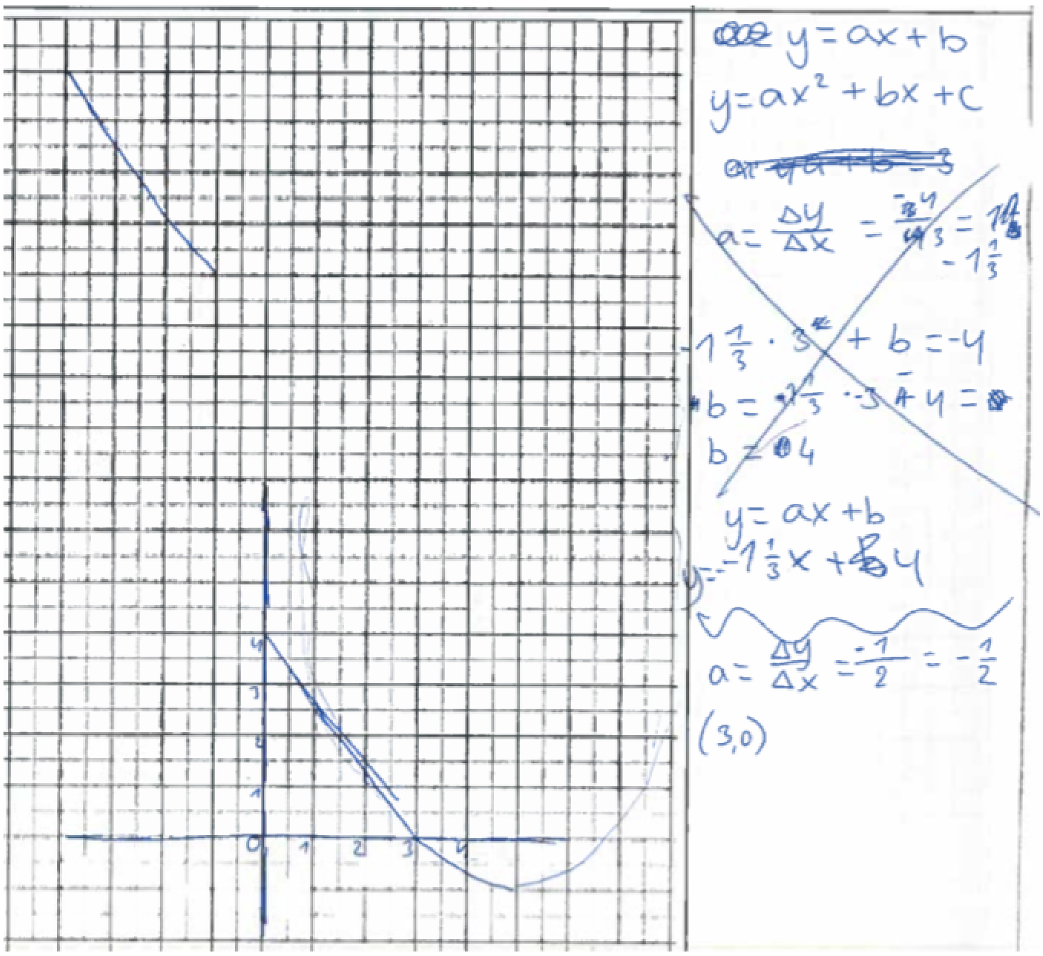


Figure 4.7: Action of Group 20.

They started drawing a straight line and a parabola. They computed the equation of the line using two points on it: $y = -\frac{4}{3}x + 4$.
Then they used an interesting method to try to find the parameters for

the parabola equation: $y = ax^2 + bx + c$ (but eventually they didn't find them):
they took the intersection point and another point on the parabola (quite close to the first one) and they calculated their difference quotient.
Taking the average slope of one piece of the parabola was an interesting idea; though, they chose a too big "piece" and used this idea only to compute the equation of the parabola and not in order to have a good fit.
Nevertheless the teacher could show how their idea of taking the average slope of the parabola could be meaningful and appropriate, if correctly applied.

- **Group 24** This group's action was also observed; thus we have a complete overview of all the dynamic of the group's action.
  Also in this case they didn't come up with some relevant approach, but the development of their solution's pursuit is interesting in some way.

  They started drawing a line and finding its equation; they made a mistake with the sign of the slope coefficient. Then they had to decide which curve to use. In this respect two girls in the group had a curious exchange of words:

  - *Student A:* We could use a parabola;
  - *Student B:* No, because a parabola is never straight at any point. We have to use a curve which consists itself of a straight and a bended part. The parabola does not;
  - *Student A:* But such a curve doesn't exist. Any curve has a straight part;
  - *Student B:* That's true;
  - *Student A:* We should use the parabola and connect it with the straight line we have already chosen.

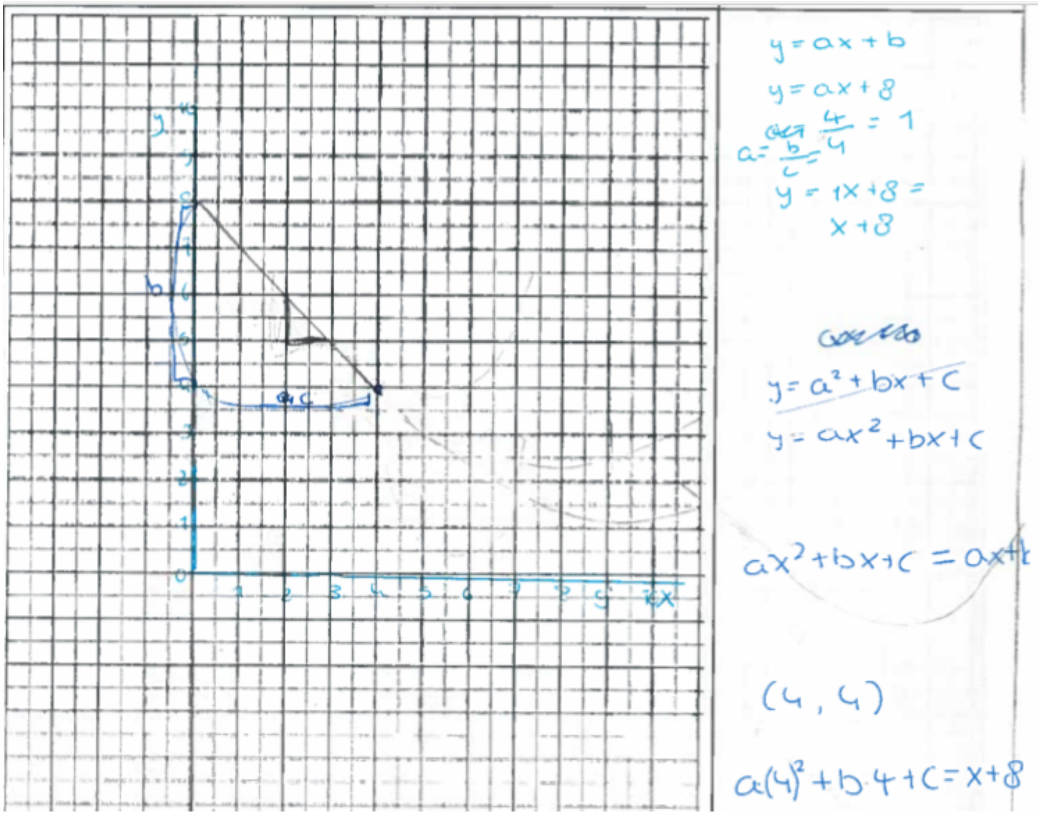


Figure 4.8: Action of Group 24.

This dialogue was interesting because it could have preceded some reasoning about if the last piece of the curve linking with the line has to be "somehow" straight; and this could have revealed their idea of how the connection can be smooth.
Unluckily in the remaining few minutes they focused mainly on how to determine the coefficients for the parabola that they had drawn; so they didn't get to investigate how to achieve the smoothness algebraically.

The only conclusion that we can make is that it seems that they tried to link graphically the parabola to the line in such a way that in the point of intersection the curve has the same direction of the line.

As we wrote at the beginning of this section, only these 3 groups came up with the relevant ideas we have presented. For the sake of completeness, we will briefly here show what the other groups have done.

- **Group 19:** These students also chose the parabola. They didn't give a definitive solution nor a drawing, but there are two pairs of line-parabola equations written in their worksheet; unfortunately there is no way to understand which technique they used. The first solution pair is:

  – **Line equation:** $y = -4x^2 + 1$

  – **Curve equation:** $y = -4x + 50$

  which is in any way wrong. The second one is:

  – **Line equation:** $y = -6x - 59$

  – **Curve equation:** $y = 0.15x^2$

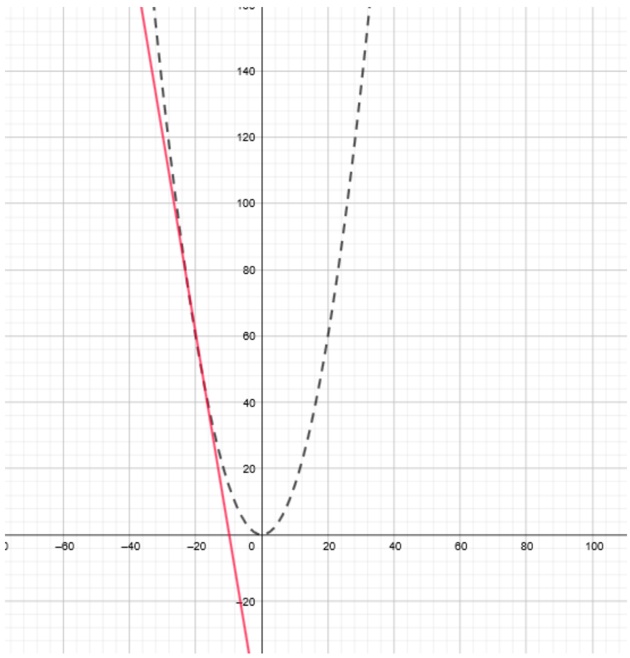


Figure 4.9: Equations given by Group 19 plotted with GeoGebra.

When computing this last couple of equations with GeoGebra, the solution could look correct at first glance (see fig. 4.9). However, by magnifying the picture or computing the intersection points, it is possibile to see that these are 2 instead of only one.
The reader can notice that the students chose high numerical parameters. As the observer has also noted, they gave importance to the fact that their equations could represent a realistic slide (in particular they were thinking about a ski-jump slide).

- **Group 21:** Students of this group found a solution that is correct, in the sense that the line chosen is tangent to the parabola chosen:

  - **Line equation:** $y = -x$
  - **Curve equation:** $y = x^2 + 0.25$

  Strangely though, they gave as intersection point $(0.49, 0.49)$ instead of $(-0.5, 0.5)$. There was no observer for this group and that is un-

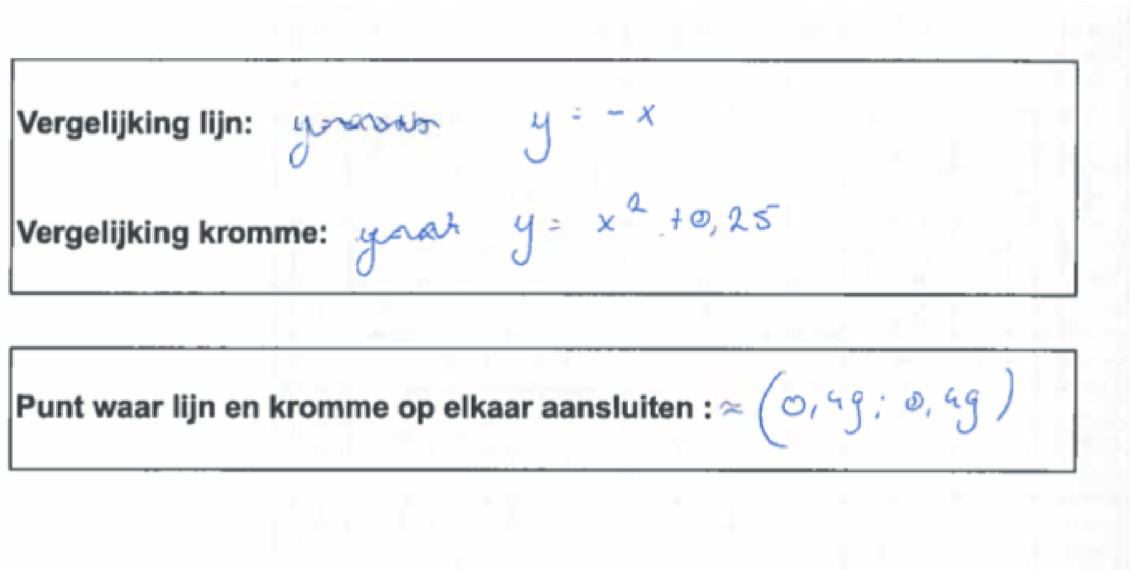


Figure 4.10: Solution given by Group 21.

fortunate because the students found a solution which is -besides the intersection point- correct and with a realistic steepness (see fig. 4.11), but without leaving any traces of their reasoning or accurate sketches.

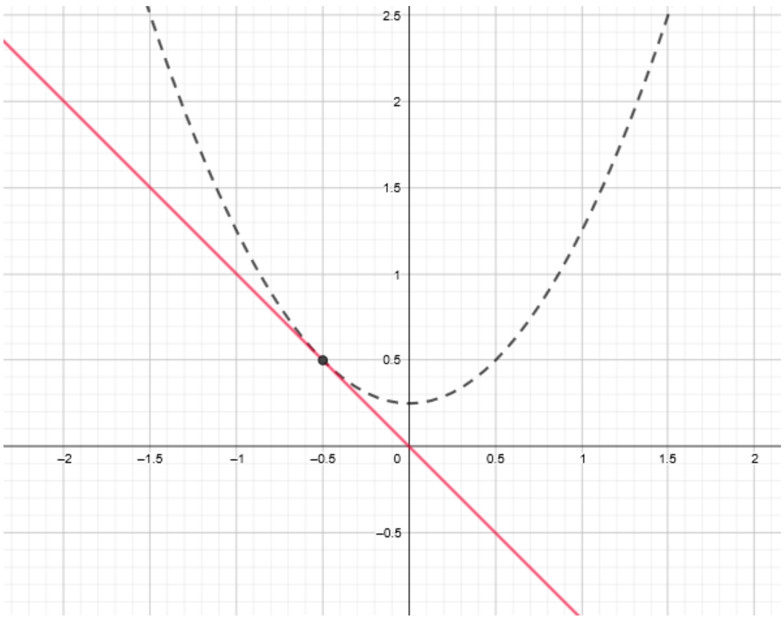


Figure 4.11: Equations given by Group 21 plotted with GeoGebra.

- **Group 23:**

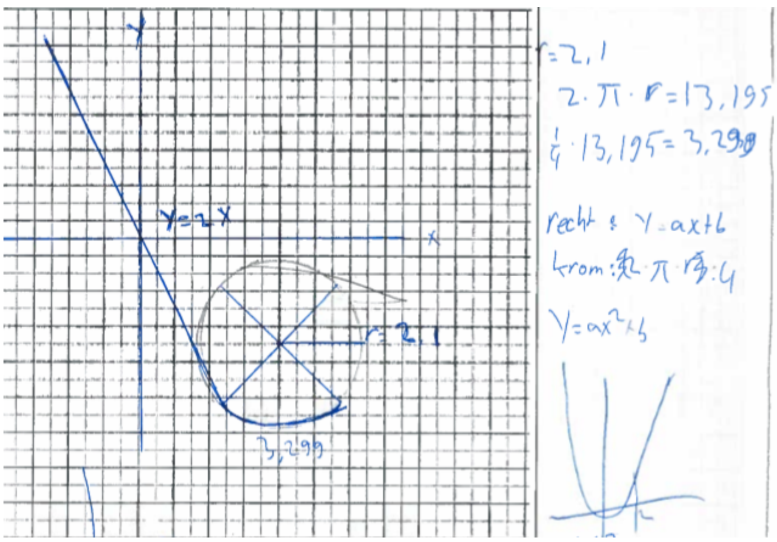


Figure 4.12: Drawing and computation made by Group 23.

Students in this group didn't come up with any equations for the bended part. They appeared to be particularly weak and lost in solving the task, because after having drawn a line (giving a wrong equation for it) and a circle, instead of computing the equation for this one, they tried to compute its length (see fig. 4.12).

- **Group 25:** This group just wrote the equation for a line and made some inaccurate drawings.

4.1.3 Formulation, validation and institutionalisation phase

When the time for the action phase was over, the teacher asked if any group was willing to present their work. Students appeared shy and didn't want to speak out; unfortunately this happened also to the students of Group 22, so the teacher missed the chance to use their result for the institutionalisation.

Another group presented their work. The solution given was not correct and the teacher started asking questions about how they could verify their solution. The ideas of zooming in or computing the intersection points came out, but then the time for the lesson was over and so the teacher didn't have the chance to analyse more deeply these ideas.

4.2 Lesson 2

The second lesson was conducted in Christelijk Gymnasium in Utrecht in a 4th grade, Mathematics B class.
Students were divided in 9 groups with 3 students each. During the lesson, besides the teacher who conducted the lesson, there were two observers: the class' usual teacher and me.
Differently from the pilot lessons and Lesson 1, in this occasion the research group had two houres at its disposal. The first hour was dedicated to the devoultion and action phase. Then after two days the teacher came back in the class for the formulation, validation and institutionalisation phase.

4.2.1 Devolution phase

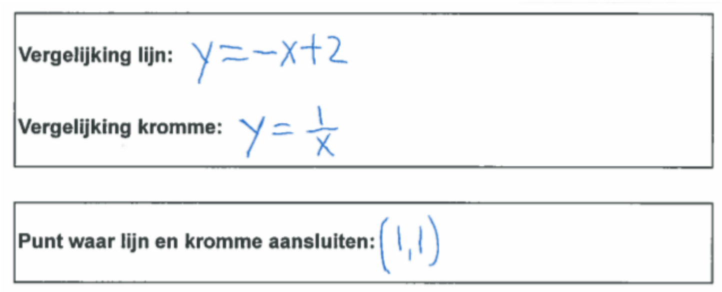
This phase lasted 5 minutes and was conducted in the same way as in the lesson described in the previous section. Students were provided with the same instructions and materials. This time, though, some of the students had at their disposal their laptop. We will see in the "action phase" section how they used this tool.

4.2.2 Action phase

As in the other lesson, the students worked on the task with their group mates. Meanwhile the teacher who conducted the lesson walked around the class and stopped in each group for some minutes to observe their general strategy. Two of the groups were observed during the whole "action phase" by the observers. The main difference from the other lessons was the duration of the phase: students had the chance to work for 45 minutes on the task.

In this class, of the 9 groups, 3 gave a correct solution and 2 gave a nearly correct solution. As before we start by presenting the works of the groups that came up with relevant ideas.

- **Group 33:** This group has been observed for the whole action phase. At the beginning they started with giving a solution with the circle. Then they changed and opted for the hyperbola. They gave the correct equations and intersection point:


Vergelijking lijn: $y = -x + 2$

Vergelijking kromme: $y = \frac{1}{x}$

Punt waar lijn en kromme aansluiten: (1,1)

Figure 4.13: Solution given by group 33.

  - **Line equation:** $y = -x + 2$
  - **Curve equation:** $y = \frac{1}{x}$
  - **Intersection point:** $(1, 1)$

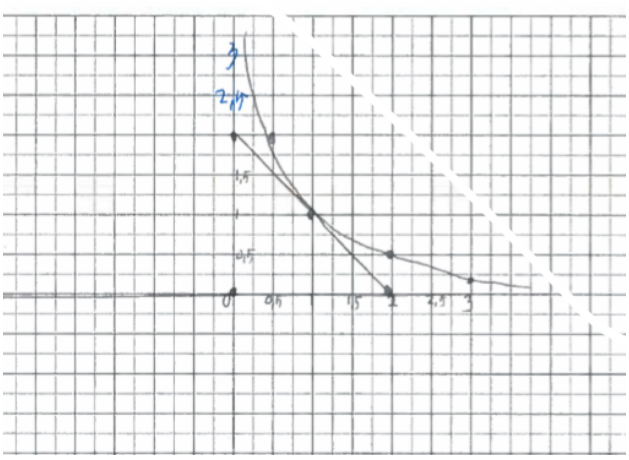


Figure 4.14: Drawing made by group 33.

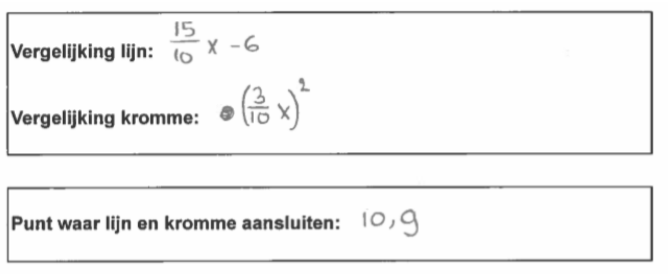
They also provided a precise drawing of it (fig. 4.14).
They didn't write a lot about why they made this choice, but the observer reported that they expressly said that they were using the symmetry of the hyperbola and line chosen. We had already seen the exact same approach in the pilot lessons. This shows that this idea could come quite naturally to the students' mind.
How could then the teacher institutionalise the concept of slope of a curve in a point starting from such a result given by students? This approach doesn't clearly link to one of the approaches hypothesised; Nevertheless, we think that the occuring of this approach could be a good opportunity to present the formal concept using the teaching approach that in the former chapter we have called "Sign changed approach". Using the simmetry of both the line and the curve with respect to the line $y = x$, it is easy to show that by slightly rotating the line, either clockwise or anticlockwise, around the intersection point, the line overtakes the curve.

Another significant aspect of this group's work is that, after the teacher asked the students how they could verify that their solution is effectively correct, they answered that they could *"Zoom in forever"*.

- **Group 31:** This group worked with the Graphic Calculator and it has also been observed. They haven't found a correct solution but during their attempt to solve the task they had a significant intuition.

  They started with choosing a line and giving a random equation for a parabola and then repeatedly changed its parameters to make it lean better against the line.
  After having tried a few times, they changed their strategy and decided to fix the line and a point on it in which they wanted the line to connect and then changed the parameters of the line. At the end they gave as the final result:

- **Line equation:** $y = \frac{15}{10}x - 6$
- **Curve equation:**$y = \left(\frac{3}{10}x\right)^2$
- **Intersection point:** $(10, 9)$



Vergelijking lijn: $\frac{15}{10}$ x −6

Vergelijking kromme: $\left(\frac{3}{10}x\right)^2$

Punt waar lijn en kromme aansluiten: 10, 9

Figure 4.15: Solution (not correct) given by group 31.

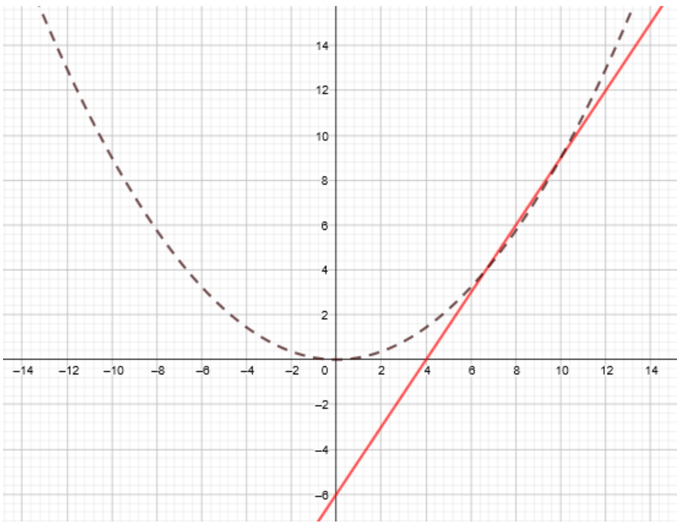


Figure 4.16: Equations given by group 31 plotted with GeoGebra.

On the Graphic Calculator this solution did look very good, but the students themselves started inquiring themselves whether they could find a rigorous way to verify their solution. At that point one student said:

> "I think that when the line touches the curve, in that small part the equation of the parabola must be the same of the line".

They didn't go further with this idea because they weren't able to convert it to proper mathematical formulation. However this intuition could be easily formalised by using the "local approximation" approach during the institutionalisation phase; the idea of the line's equation being the same as the curve's equation in the "small part" where the two connect, is the idea which stands behind the cited approach.

These were the groups whose work is clearly linkable to some of the teaching approaches we have proposed. Other groups gave a correct solution; even if they didn't use ideas which were distinctly expression of one of the approaches hypothesised, the teacher could anyway start from their result to formalise the concept of the slope of the curve, using the strategy that she/he prefers. We will show here such results.

- **Group 27:**

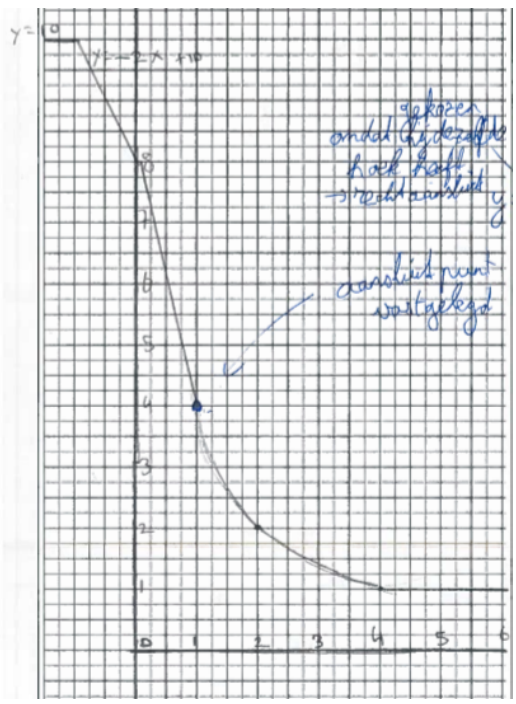


Figure 4.17: Drawing produced by group 27.

This group worked with GeoGebra. They just fixed the hyperbola and then found a line that they thought fitted nicely the curve. They just relied on their visual evaluation to both produce and verify their result which is:

   - **Line equation:** $y = -4x + 8$
   - **Curve equation:** $y = \frac{4}{x}$

- **Group 32:** This group also gave a correct final solution:
   - **Line equation:** $y = -2x + 7$
   - **Curve equation:** $y = \left(x - \frac{7}{2}\right)^2 + 1$
   - **Intersection point:** $(2.5, 2)$

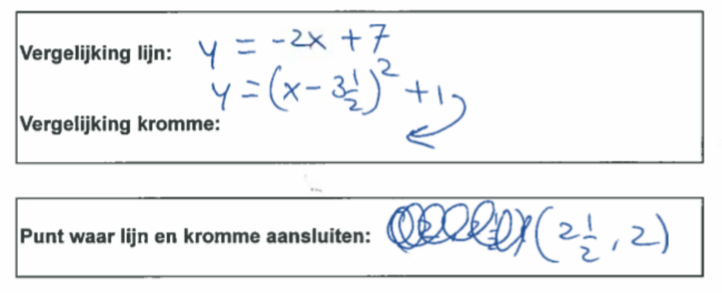


Figure 4.18: Solution given by group 32.

They started with fixing the line and then chose the parabola. They didn't give an explanation about how they chose it. Though, while the teacher was walking from desk to desk he stopped by this group and challenged them to find a rigorous way to verify their result. Then the students computed the intersection points of the parabola and the line and verified that there was one unique intersection point (precisely, two coincident points).

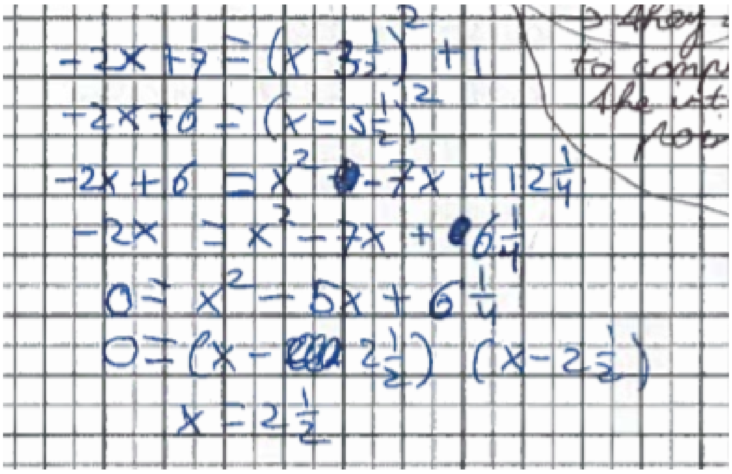


Figure 4.19: Computation made by group 32 to verify their result.

As we did for the other lesson we have described, for the sake of completeness we will briefly show also the action of the groups that went close to a correct result or failed.

- **Group 26:** Students of this group used a strategy that we have already seen in other groups' work: they made a drawing of the line and the curve that they wanted to use and then tried to find their equations using data extracted from the drawing. In this specific case they drew a line and a parabola and used the intersection point of the line with the y-axis, the point of intersection of the line and the parabola and the top of the parabola.
  Their final solution is:
  - **Line equation:** $y = -2x + 8$
  - **Curve equation:**$y = \frac{5}{4}\left(x - \frac{11}{2}\right)^2$

  They made mistakes in their computation, but anyway also if they had not made such mistakes, their solution would have had two intersection points instead of only one.

- **Group 28:** This group worked with GeoGebra. They started with fixing a line and then chose an arbitrary parabola and changed its parameters in order to get a better fit. Their final solution is:

  - **Line equation:** $y = -2x + 8$
  - **Curve equation:**$y = \frac{5}{4}\left(x - \frac{11}{2}\right)^2$

  This solution is not correct, in the sense that there are 2 intersection points, even if when computing these equations with GeoGebra there seems to be just one intersection point.

- **Group 29:** This group just gave the equation:

  - **Curve equation:** $y = (x - 5)^4$

  There are no signs in their work sheets of an attempt to find a solution. Probably they were not working seriously.

- **Group 30:** They decided to use the hyperbola: $y = \frac{1}{x}$. They just set

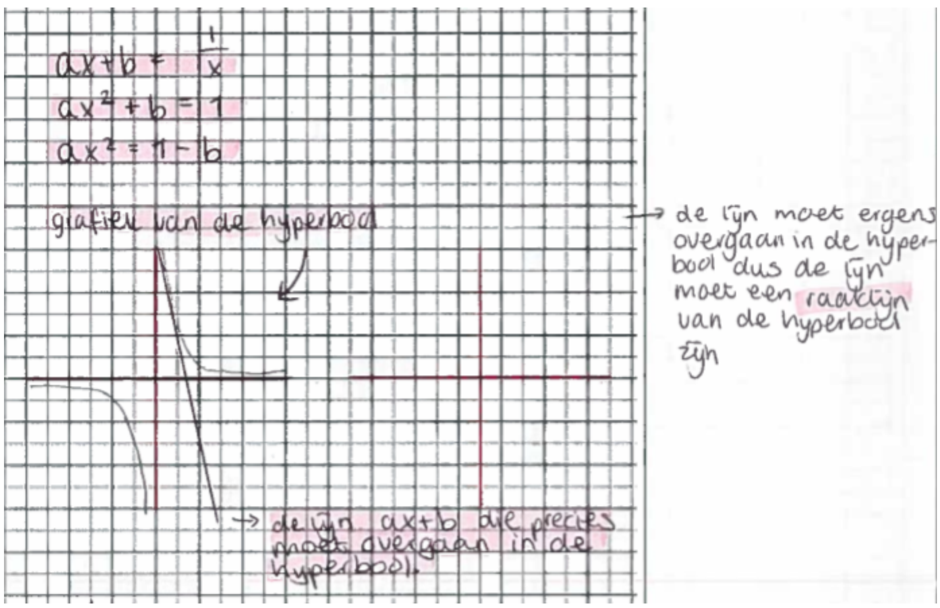


Figure 4.20: Action of Group 30.

an equality between this equation and the general equation of a line: $y = ax + b$. They started with an algebraic approach, but they didn't go further probably because they didn't have a clear idea of how to manipulate the equation found in order to find the correct parameters for the line. However they explicitely wrote that the line had to be the "tangent line" to the curve.

- **Group 34:** Students in this group only gave an equation for the line and decided to use a parabola, but they didn't even provide an equation for this one.

4.2.3 Formulation, validation and institutionalisation phase

After two days the teacher who conducted the lesson came back to the school to give the second part of this lesson. As already mentioned, having more time at disposal for the whole lesson about the task was very favourable to its success. Nevertheless, this had its disadvantages: students weren't divided in groups anymore and they likely hadn't perfect memory of the work they had produced two days before. For these reasons the teacher decided to skip the formulation phase and presented himself the most remarkable results. Then he encouraged the whole class to find means of evaluation of the work presented. Again, the idea of zooming in came out, but that was also explicitly stated by Group 33.
The teacher then gave a few examples (one using a circle, one using an hyperbola and one using a parabola) of how to find the tangent line to a curve at a point, stressing the idea of local linearity and zooming in.

At this point we must make clear that during these experimental lessons the teacher didn't have the chance to really present the whole concept of the tangent line, with computation rules and different examples, during the institutionalisation phase. The time at his disposal was still too short and

mainly the task was not carried out at the time of the year, or at the point in the curriculum, in which the topic had to be introduced. These are obstacles that appeared because of the experimental and out of the ordinary nature of the lesson.
On the other hand, when a teacher is presenting the task in her/his own class, at the right moment of the curriculum, at this time of the lesson she/he could actually use the theory presented in Chapter 3 to introduce the topic of the tangent line, and so of the derivative.

4.3 Lesson 3

Also the third and last lesson was conducted in Christelijk Gymnasium in Utrecht in a 4th grade, Mathematics B class.
Students were divided in 10 groups with 3 students each. Also in this case during the lesson, besides the teacher who conducted the lesson, there were two observers: the class' usual teacher and me.
An entire lesson lasting 1 hour and 50 minutes was dedicated to the task.

4.3.1 Devolution phase

This phase lasted 5 minutes and was conducted in the same way as in the lessons described in the previous sections. Also in this class some of the students used their laptop to work with GeoGebra.

4.3.2 Action phase

Outcomes of this lesson were very successful. Of the 10 groups, 7 gave a correct solution. As we did for the other lessons, we start presenting the solutions that clearly show intuitions that can be easily linked to the approaches presented in the third chapter.

- **Group 35:**

  Students of this group relied on an algebraic method to find the solution. It is clear that they were aware since the beginning of the fact that their goal was to find a curve and a line which intersect in only one point. Indeed they didn't use graphic tools nor tried to sketch the line and the curve before finding suitable equations for them. They gave the equation for a hyperbola: $y = \frac{6}{x}$. Then at the beginning they hazarded a guess over the equation for the line $y = -x+5$. They set an equality between these two formulas and they got two different results:

  $x = 2$
  $x = 3$

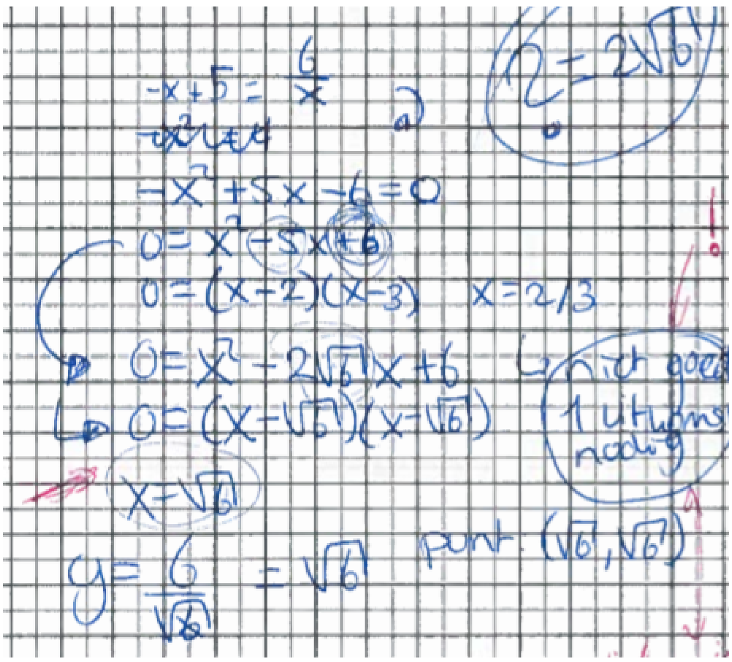


Figure 4.21: Computation made by group 35.

As clearly shown by their computation (see fig.4.21), they tried to adjust the final form of the equation $-x^2 + 5x - 6 = 0$ in order to find

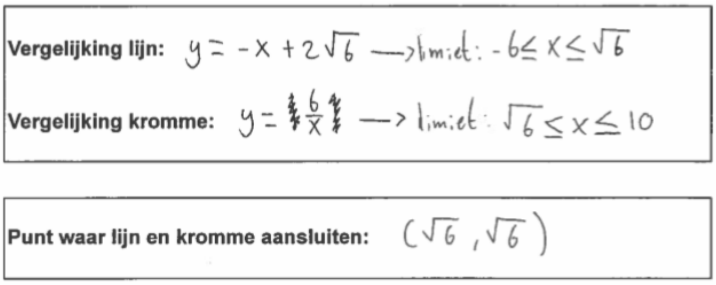
a formula which represents a square. They only changed the second parameter into $2\sqrt{6}x$.
They obtained so:

$$-x^2 + 2\sqrt{6}x - 6 = 0$$

$$(x - \sqrt{6})(x - \sqrt{6}) = 0$$

which has a double root $x = \sqrt{6}$.
their final solution was then:

- **Line equation:** $y = -x + 2\sqrt{6}$
- **Curve equation:** $y = \frac{6}{x}$
- **Intersection point:** $(\sqrt{6}, \sqrt{6})$



Vergelijking lijn: $y = -x + 2\sqrt{6} \longrightarrow$ limiet: $-6 \leq x \leq \sqrt{6}$

Vergelijking kromme: $y = \frac{6}{x} \longrightarrow$ limiet: $\sqrt{6} \leq x \leq 10$

Punt waar lijn en kromme aansluiten: $(\sqrt{6}, \sqrt{6})$

Figure 4.22: Solution given by group 35.

- **Group 41:** This group tried out different strategies to solve the task. First they chose which curve to use and opted for the parabola: $y = \frac{1}{2}x^2$.
  Then they chose an arbitrary line and repeatedly changed its parameters in order to make it better connect to the parabola. Once they had found one line which they considered appropriate, they started focusing on how to evaluate the correctness of the found equations. These were:
  - **Line equation:** $y = 3x - \frac{40}{9}$
  - **Curve equation:** $y = \frac{1}{2}x^2$

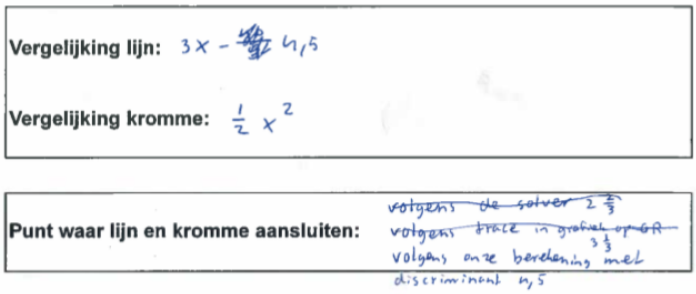


Figure 4.23: Solution given by group 41.

They also opted for an algebraic approach, set an equality between the two formulas and manipulated the resulting equation. At that point one student said:

> "I think that the discriminant should be equal to zero so that we have just one solution".

His group-mates agreed, but they soon got stuck because anyone could remember the formula for the discriminant. They switched to different evaluation methods: one student started using values found with the Graphic Calculator (although it is not clear how he used them to verify the result) while the other students kept manipulating the previous equation in order to discover if it could be the expression of a square or not. Unfortunately, when they got to:

$$9x^2 - 54x + 80$$

they didn't realise that by just changing 80 into 81 they would have obtained:

$$(3x - 9)^2$$

.

We can disclose that later, during the formulation phase, the student of this group who was presenting their work, was suggested the formula for the discriminant and could find, while presenting, a correct solution:

- **Line equation:** $y = 3x - 4,5$
- **Curve equation:** $y = \frac{1}{2}x^2$

- **Group 43:** Students started with making a drawing of a line and a parabola which meet without bumps. They then found their equations

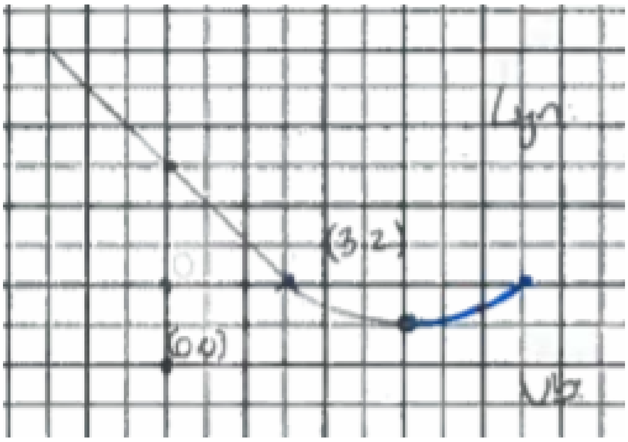


Figure 4.24: Drawing made by group 43.

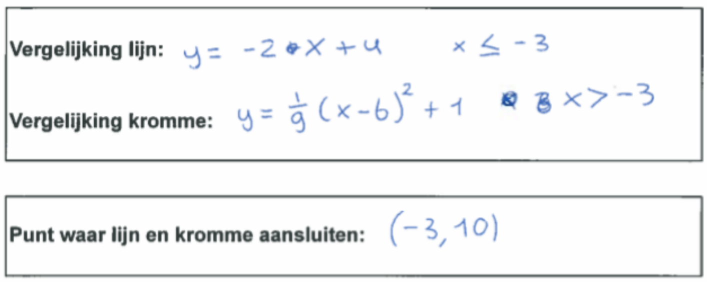
using values pulled by the drawing: for the line they used the slope and the intersection with the y-axis while for the parabola they used its top and the point of junction with the line $(3, 2)$.

They obtained so the solution:

- **Line equation:** $y = -x + 5$
- **Curve equation:** $y = \frac{1}{9}(x - 6)^2 + 1$

They typed these equation in GeoGebra but they were not satisfied because, as they said, "there is still a bump". Again using GeoGebra they chose a different line: $y = -2x + 4$. They tried to validate their result by zooming in the point of junction and this evaluation method convinced them of their choice. At that point the teacher challenged them to find a an additional method that could precisely confirm the correctness of their result. They then decided to compute the intersection points of the line and the curve and stated: "we want to have only one intersection point, otherwise we would have a bump". As wished, they found an unique intersection point: $(-3, 10)$. Their final result was then:

- **Line equation:** $y = -2x + 4$ $\qquad x \leq -3$
- **Curve equation:** $y = \frac{1}{9}(x-6)^2 + 1$ $\qquad x > -3$
- **Intersection point:** $(-3, 10)$



Vergelijking lijn: $y = -2x + 4$ $\quad x \leq -3$

Vergelijking kromme: $y = \frac{1}{9}(x-6)^2 + 1$ $\quad x > -3$

Punt waar lijn en kromme aansluiten: $(-3, 10)$

Figure 4.25: Solution given by group 43.

- **Group 37:** This group, as the previous one, opted for a graphic approach. They firstly made a precise drawing of a line and a parabola meeting smoothly.

  Using values from the picture they found the line equation: $y = 9 - 2x$. The junction point had $x = 3$ so they set the equality:

  $$9 - 2x = (x-6)^2 a$$

  They used the wrong equation for the parabola: they considered its top to be $(6, 0)$, while in their drawing the top was $(6, 1)$ (see fig. 4.26). Appereantly they were lucky, because the line they chose is not tangent to the parabola they chose, while the data used to set the equality lead to a correct result.

  Their final equations and point of intersection, which are correct, are then:

  - **Line equation:** $y = 9 - 2x$

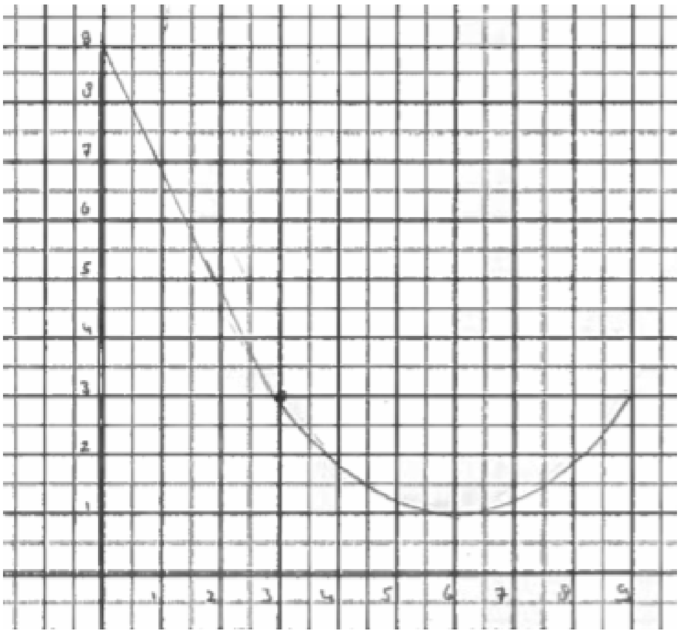


Figure 4.26: Drawing made by group 37.

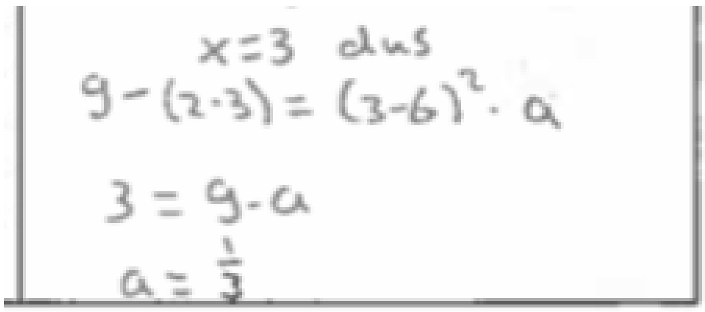


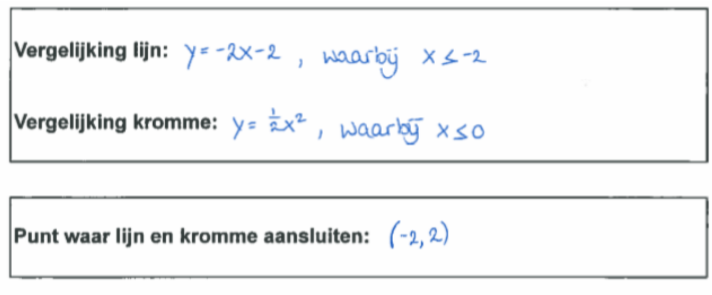
Figure 4.27: Computation made by group 37.

- **Curve equation:** $y = (x-6)^2\frac{1}{3}$
- **Intersection point:** $(3, 3)$

- **Group 40:**

  They first found a solution with a parabola and its tangent line at the top and then tried to see if there was a relation between the parameters for the line and the parabola. This approach appeared not to be helpful to solve the task. They then just chose a different line, which appeared to fit nicely with curve in GeoGebra. Their definitive solution is:

- **Line equation:** $y = -2x - 2$
- **Curve equation:** $y = \frac{1}{2}x^2$
- **Intersection point:** $(-2, 2)$


Vergelijking lijn: $y = -2x - 2$, waarbij $x \leq -2$

Vergelijking kromme: $y = \frac{1}{2}x^2$, waarbij $x \leq 0$

Punt waar lijn en kromme aansluiten: $(-2, 2)$

Figure 4.28: Solution given by group 40.

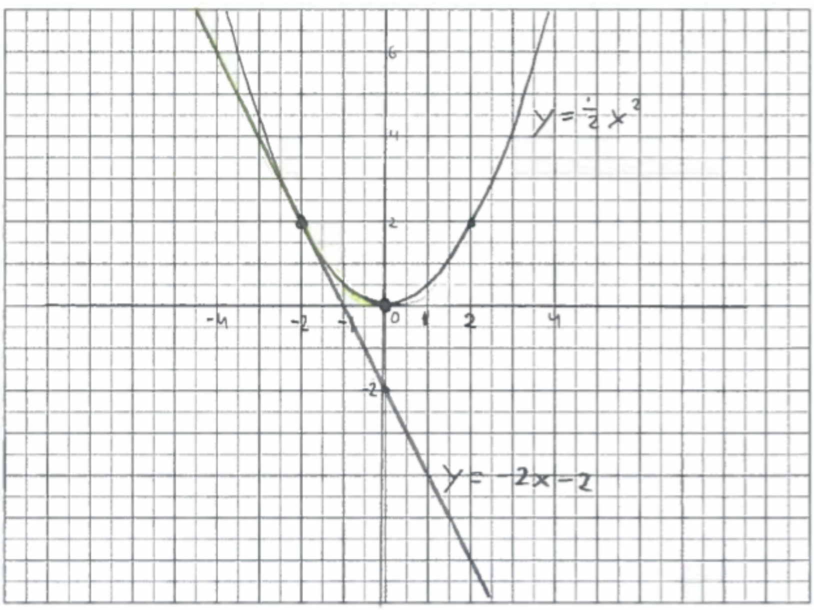
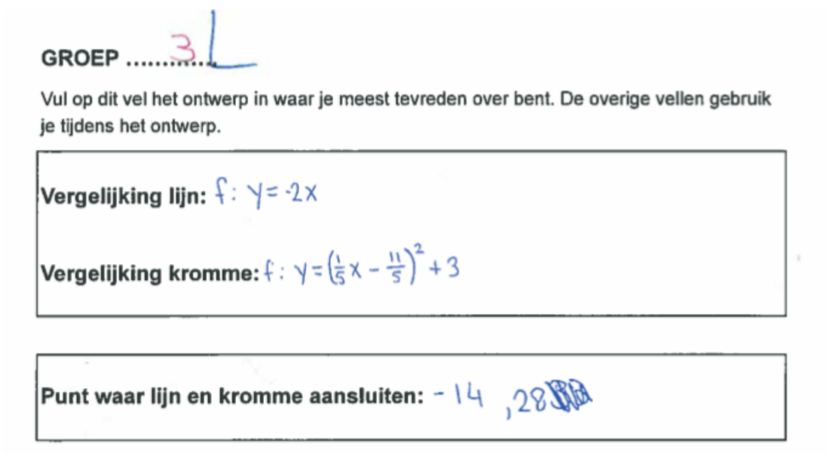


Figure 4.29: Drawing made by group 40.

- **Group 42:** This group gave a correct result but it is not really relevant because they used a circle and knew from euclidean geometry that the tangent line should be perpendicular to the radius.

- **Group 44:** This group worked with GeoGebra. They first fixed a line: $y = -2x$. Then they chose a parabola and modified it by making it wider or translating it horizontaly. Their final and correct result is:

  – **Line equation:** $y = -2x$
  – **Curve equation:** $y = \left(\frac{1}{5}x - \frac{11}{5}\right)^2 + 3$
  – **Intersection point:** $(-14, 28)$


GROEP 3L

Vul op dit vel het ontwerp in waar je meest tevreden over bent. De overige vellen gebruik je tijdens het ontwerp.

Vergelijking lijn: f: $y = -2x$

Vergelijking kromme: f: $y = \left(\frac{1}{5}x - \frac{11}{5}\right)^2 + 3$

Punt waar lijn en kromme aansluiten: -14 ,28

Figure 4.30: Result given by group 44.

To verify it, they started computing the intersection points, but didn't finish, probably because the time for the action phase was over.

- **Group 36:** This group worked with the circle, but just gave an equation for a circle, which doesn't even correspond to the circle they have drawn.

- **Group 38:** The circle was the choice made also by this group. At the beginning they fixed the line and tried to change the radius of the circle. Then they made the useful discovery, unluckily not useful for

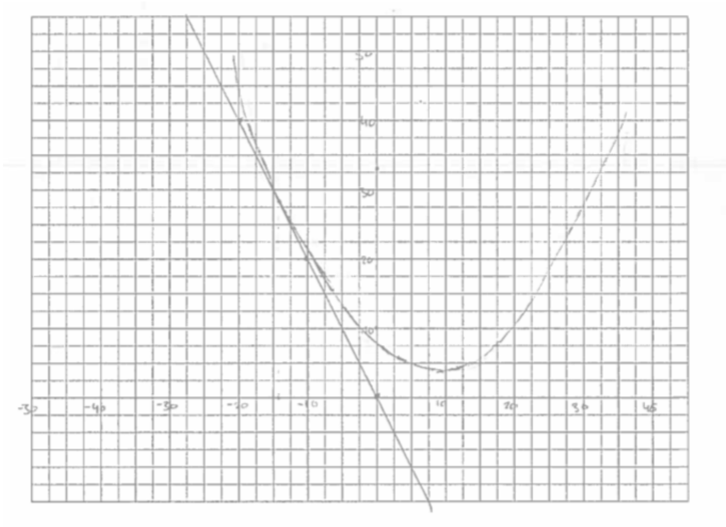


Figure 4.31: Drawing made by group 44.

the researchers, of the button in GeoGebra which automatically draws the tangent line to the chosen curve and this made the solving process far too easy.

- **Group 39:** Students in this group made a drawing of a parabola ($y = \frac{1}{2}x^2$). Then they set an equation between the parabola $y = \frac{1}{2}x^2$ and the formula for the general line $y = ax + b$, but they didn't go further with this approach.

4.3.3 Formulation, validation and institutionalisation phase

After 45 minutes, the teacher declared the action phase finished. He had already chosen a few groups that used an interesting approach; so he invited students from groups 35 and 41 to present their work. As already written, the student from group 41 was able to correct his group's solution during the formulation phase, thanks to the help of the rest of the class.
Both these groups used an algebraic strategy. The teacher then, also for the validation, kept with this approach. Students appeared really interested,

probably because many of them had that same intuition to solve the task. About the institutionalisation, we refer the reader to what has been written about it at the end of the section “Lesson 2”.

Conclusions

Together with the three lessons described in the previous chapter, we decided to take into account for the final analysis also three of the classes in which pilot lessons were performed and that we partly discussed in the third chapter. The three chosen classes are those for which we have a quantitatively complete overview; even if, as already said, observations were not as accurate as in the last lessons, for these classes we were able to analyse what all the groups had written in their work-sheets.
In total in our final analysis we can count the performances of 6 classes, for a total of 44 groups.

The data was analysed by one researcher of the Freudenthal Institute -who is also the teacher who performed the lessons- and me and then compared.

We started the analysis with looking for designing patterns in the groups' action.
First we used the students' results and processes and, when present, the observer's notes to agree on a description for the group's strategy. Then we discerned ten categories to classify such strategies; they are described in table 4.1.

Label	Description
D	**D**raw a line and curve and find the equations from data (like points, slope of a line, or top of a parabola) taken from the drawing
PS	Choose a line and a curve (equation); then vary the **P**arameter for the **S**lope of the line
PT	Choose a line and a curve (equation); then vary the **P**arameter to **T**ranslate the line
PC	Choose a line and a curve (equation); then vary the **P**arameter(s) of the **C**urve
A	Use **A**lgebraic means to find a good design: e.g. computing intersection points
HS	Use the tangent line perpendicular to the **S**ymmetry axis of a **H**yperbola
C	Use the tangent line perpendicular to the radius of a **C**ircle
R	The **R**est: strategies not mentioned above
O	**O**ther: not traceable strategy
N	**N**o serious attempt registered

Table 4.1: Classification labels for students' design strategies.

As visible in the complete descriptions provided in the previous chapter, many groups used different strategies. They often started with one approach to later change it, or used more than one strategy to find their solution. So the labels are not exclusive: each group could be labelled with more than one category. In Table 4.2 we reported in the second column the lists of approaches noticed in each group[3].

After having systematically classified each group's work, we decided for each of them if they could be helpful as model or starting points to present the

[3]In Table 4.1 we placed the lessons in chronological order, but kept the numbers 1, 2 and 3 for those that in the previous chapter we have called "Lesson 1", "Lesson 2" and "Lesson 3"; that's why we called "1*","2*" and "3*" the previous three classes.

formal knowledge in the institutionalisation. In particular for every group we determined if one specific approach (**S** for "Secant Line" namely the traditional approach, **L** for "Local approximation", **A** for Algebraic and **T** for "Transition point/Sign change") emerged, if their work could be more vaguely connected to different approaches (**V** for "Various") or if none of the teaching approaches could be presented starting from their action (**N**).

Always in table 4.2, in the third column, we reported the occurrence of the cited possible institutionalisation methods for each class.

Class label	Strategies registered in each class	Teaching approach
1*	PC,PC,O,PT,HS,D/PS,O,N	V,V,N,T,N,T,N
2*	PC/PT,O/PT,O,O,O,PS/PC,O	V,V,V,V,V,V,V
3*	A,C,C	A,T,N
1	PT/PC,D,O,C/R,O,R,N	V,V,V,L,N,L,N
2	D,R,PC,O,R,PC/PS/PT,PC,HS,O	V,V,V,N,A,L,A,T,N
3	A,C,D,PT/PC/C,D,R,PC/PT/PS/A,C,D,PC	A,N,L,V,N,V,A,V,V,A

Table 4.2: Strategies and connecting teaching approach per group.

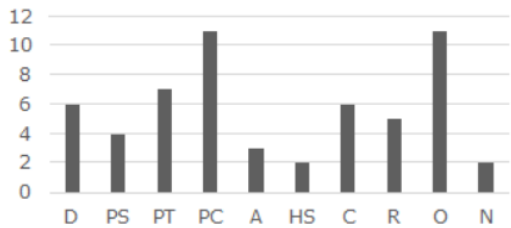


Figure 4.32: Frequency of students' strategies.

To determine in which category each group's work would fit better, we haven't only considered the designing strategy, but also the evaluation methods used (if used). About these, we make some remarks.
Firstly, what in Chapter 2 we have called "numerical validation approach"

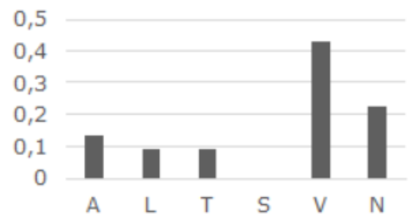


Figure 4.33: Relative frequency of approaches.

never appeared in the students' work; they mainly used an algebraic approach (namely computing the intersection points) or a visual approach (namely magnifying the plot or stating that "it looks good"). About this last method, it is important to mention that it can be used to verify that one result is not correct and not to verify that it is correct. Moreover, students' usage of this method doesn't show necessarily a conscious application of the concept of local linearity; only when this was more or less explicit we labelled it with L.

We can mention another validation method that we noticed quite often; this occurred in those cases in which previous knowledge about euclidean geometry was used. Students who used the fact that in the circle the tangent line is perpendicular to the radius or explicitly used the symmetry of the hyperbola $y = \frac{1}{x}$, felt confident about their solution because they were sure of these geometrical properties. For these cases we decided to suggest a link with the "Sign change/Transition points approach" (**T**), because using these symmetry properties that are familiar to students, it is easy to show that by slightly rotating the line around the point of tangency the line overtakes the curve[4].

Also the use of designing strategy **PS** (varying the slope of the line) appeared to us an indication of the attempt to find a line which does not overtake the curve; also in these cases we opted for the label **T**[5].

However in both cases the link to approach **T** is only a suggestion and rep-

[4]This concept is better explained in the description of the solution given by *Group 33*, in Chapter 4.

[5]This happened only in *Group 7*.

resents then a weaker connection with respect to cases that we labelled with **L** or **A**.

After this qualitative analysis we quantified the presence of correct results, nearly correct results and recognisable approaches per group, in order to investigate whether there could be a connection between the occurrence of significant solutions and some aspects of the lesson structure such as the mathematics level of the students, the length of the lesson or the use of some tools. This is summarised in Table 4.3:

Class label	Grade	Mathematics level	Length of action phase in minutes	Number of Groups	% of groups with a correct (or nearly correct) solution
1*	3 VWO	Maths A	25	8	12.5
2*	3 VWO	Maths A	25	7	100
3*	3 VWO	Maths B	25	3	66.7
1	4 VWO	Maths A	25	7	42.9
2	4 VWO	Maths B	35	9	55.6
3	4 VWO	Maths B	50	10	80

Table 4.3: Overview of results per class.

What can we conclude from these quantitative data, together with data in Table 4.2?

Firstly we can make some observations about what designing strategies have been used by students.

- Strategy PC (varying parameters of the curve) was often used (fig. 4.32). This surprised us, because it would be much easier to first fix

the curve and then modifying parameters for the line to get a better fit. However there could be some explanations for the students' approach: they prefer starting with something that is easier to find, as the line equation, and then focus on the more difficult challenge of finding the curve equations. It could also be possible that they unconsciously think of the descent on the slide: this dynamic perspective would explain why they first think about the line and then about the bended part.
If this last hypothesis was confirmed, this would represent a weakness of the task; adjustments to the task and to the way it is presented could be needed.

- We have noticed that the use of an algebraic approach is more frequent in classes with a higher mathematics level (see Table 4.2). The algebraic knowledge they need for this task is not that advanced but they probably feel more confident than other students in using it.

- Some comments about the absence of traces of the secant lines approach (fig. 4.33) should be included: the process of taking a secant line and then moving one of the intersection points towards the other is a dynamic process which couldn't be used naturally when working with pen and paper. That would be easier to implement if using GeoGebra. Anyway we haven't seen this approach even in groups which worked with GeoGebra. There could be an explanation also for this: the secant line approach is a process which starts from a "wrong" situation to then change it making it better and better (the secant line which gradually become closer and closer to the wished tangent line). In such a task, it is unnatural for the student to start intentionally with a wrong construction to then improve it.

- The linear approximation approach, although used more or less as often as the algebraic approach, has never been used properly to find a concrete solution, but only to validate the result or cited as a general idea around the solution of the task. The same arguments given for approach **S** could explain this: zooming in and taking two points on the curve (once that it looks locally straight) to design the desired line, would also mean giving a wrong (nearly correct, but still wrong) result.

- In lessons 3* and 3 we haven't registered cases in which the group's strategy is unclear (**O**) (Tab. 4.2). This could be explained by the fact that during those lessons the teacher pushed and even helped the students to leave a written explanation of what they were doing or trying to do.

Then, we can draw conclusions comparing the relative frequency of correct solutions and other aspects concerning the lesson.
We must mention that the high frequency of correct or nearly correct solutions in classes 2* or 3* need some considerations:

- In class 2* all the groups used GeoGebra. If we consider only correct solutions, these represent the 14% of the total: of the 7 groups, only one found a curve and a line which is really tangent to that curve. The other groups, helped by the graphic tool, found a line which intersected the curve in 2 or no points, but that plotted with GeoGebra looked tangent.

- In lesson 3*, the teacher previously selected only some students, choosing those who were good at mathematics and motivated to keep with the more advanced maths class the following year. As a consequence the chosen students are not very representative of their whole class; this would explain the high relative frequency of correct answers in this lesson.

Taken into account the specificity of these two classes, we can notice that better results (in terms of correctness of the results and presence of approaches and ideas that could be useful for the institutionalisation) are registered in classes with a higher level in mathematics and in which students had more time to solve the task (see Tab. 4.3)

In the lessons in which the time at disposal for the action phase was around 25 minutes students tended to lose a lot of time in trying to find the equations for the line and the curve and they didn't get to really investigate how to achieve smoothness, which is the main goal of the task. With more time, not only they could engage in this last aspect, but they could try different designing and evaluating strategies (as noticeable in Table 4.2).

A good solution to time issues suggested by our results is that when the time that can be dedicated to the task is short, the teacher could make available to all the students laptops so that they could work with GeoGebra (with the aformentioned benefits).

All these final considerations answer to the research questions formulated at the end of Chapter 2. In particular the activity could be easily replicated by an upper secondary school teacher; if the class has a suitable level in mathematics, and an adequate amount of time is dedicated to the task, meaningful models can most likely emerge. Moreover, if the activity is carried out at the proper stage of the curriculum a teacher would have the chance to immediately institutionalise the formal knowledge. She/He should only be aware of the fact that with no observers, monitoring relevant approaches during the action phase could be complicated; a suggestion is to use an entire lesson for the devolution and action phase and then keep with the rest of the phases in a following lesson so that all groups' actions can be accurately studied, as has been done in Lesson 2.

In conclusion, the realisation of such activity can support the reinvention of the concept of slope of a curve at a point and lead to a more meaningful and natural understanding of it.

Appendix A

MERIA templates

In this appendix the reader can find templates describing the structure of a scenario and a module, as provided from the MERIA team in a booklet downloadable from the website of the Project [Winsløw and Jessen, 2017].

Target knowledge	A precise mathematical formulation of the goal.		
Broader goals	Broader achievements such as competences, possible applications, reasoning etc.		
Prerequisite mathematical knowledge	Precise formulation of what mathematical knowledge, skills and competences the students are expected to possess (before engaging with this situation).		
Grade	Grade number and age of students.		
Time	Estimated time and number of lessons. (45-60 minutes entities).		
Required material	All sorts of needed artefacts.		
Problem: The exact formulation of the main problem, which the teacher devolves to the students (possibly after some preparatory activity).			
	Teacher's actions incl. instructions	**Students' actions and reactions**	**Observations from implementation**
Devolution (didactical) Time estimate			
Action (adidactical) Time estimate			
Formulation (didactical/adidactical) Time estimate			
Validation (didactical/adidactical) Time estimate			
Institutionalisation (didactical) Time estimate			
Possible ways for students to realize target knowledge	- Be mathematically explicit about the strategies that students might follow. Remember to emphasize when a strategy can split into a scenario with ICT or without ICT using only pen and paper, as well as if the strategy requires to look at special cases.		
Further study	- What are possible applications / generalization of the notion or the concept studied?		
List of additional materials	- Students' productions (snapshots of boards, reports, assignments, posters etc.) - Formulations of students' assignments, reports or other productions required from students based on the lesson - Table for recording students' strategies - Video		

Figure A.1: Template for MERIA Scenario.

2. Template for MERIA module

The scenario

A short description of the scenario is written in table format. Remember to indicate time, instructions and possible different student strategies together with the goals of the scenario. Column for observations will be omitted in the module.

Explanation of materials

In this section a short explanation on how to use additional materials should be provided presenting each possible material or resource (hand out, (electronic) worksheets, dice, building blocks and other artefacts) and suggesting how to use it in a certain phase. Some of the resources might be for the teacher only.

Variations based on didactic variables

In this section, possible variations are described together with recommendations regarding what could be changed and what should not be changed. The variations can be based on changes of the milieu (changing the use of ICT, adding or removing mathematical objects or artefacts), the didactic time (a particular phase being shorter/longer) or the organization of the class work individually, in pairs or in groups. Further, the impact and consequences of variations should be discussed based on the context they are to be realized in – and draw on experiences from the testing of the scenario.

Observations from practice

Here the main observations and reflections from observing the test implementations of the scenario should be shared. This means a description of less successful strategies and suggestions on when and how to intervene, description of strategies with different level of success supported by data (photo of students' work, assignments etc.)

Evaluation tools

It can be an advantage to deliver an evaluation tool in the form of an immediate task (or more simple tasks) the students are supposed to be able to solve if they have reached the learning goal of the scenario. Also, to provide suggestions for further problems regarding the target knowledge. The possible strategies suggested in the scenario can be used as a "checking tool" during teaching. If the teacher creates a sheet with the list of strategies, (s)he can note which groups follow which strategy and whether any important ones are left out. This can be addressed during the scenario, but also used after the lesson for reflection.

Rationale and RME perspectives on the scenario

An elaborated presentation of how the target knowledge can be reached with the scenario. Further, RME elements can be addressed, such as: choice of mathematical target knowledge, relevance and applicability, inquiry skills and potential for additional lessons (providing a storyline for a whole chapter or presenting "a bigger picture").

Figure A.2: Template for MERIA Module.

Appendix B

Overview of the action of analysed groups

In this appendix the reader can find a table for each lesson, in which for every group are specified:

- the strategy/strategies chosen;
- the solution given (when correct);
- the suggested institutionalisation approach.

We remind that, as described in the Conclusions, the labels that we have given to the different approaches to distinguish them are:

Label	Description
D	**D**raw a line and curve and find the equations from data (like points, slope of a line, or top of a parabola) taken from the drawing
PS	Choose a line and a curve (equation); then vary the **P**arameter for the **S**lope of the line
PT	Choose a line and a curve (equation); then vary the **P**arameter to **T**ranslate the line
PC	Choose a line and a curve (equation); then vary the **P**arameter(s) of the **C**urve
A	Use **A**lgebraic means to find a good design: e.g. computing intersection points
HS	Use the tangent line perpendicular to the **S**ymmetry axis of a **H**yperbola
C	Use the tangent line perpendicular to the radius of a **C**ircle
R	The **R**est: strategies not mentioned above
O	**O**ther: not traceable strategy
N	**N**o serious attempt registered

About the possible institutionalisation, we write **A** for “Algebraic”, **L** for “Local linearity”, **T** for “Transition points”, **S** for “Secant lines”, **V** for “Various approaches” and **N** if none of the approaches could be easily linked to the students’ action.

Group	Used Strategies	Equations given (when correct)	Suggested institution-alisation
1	PC	wrong	V
2	PC	wrong	V
3	O	wrong	V
4	PT	wrong	N
5	HS	$y = \frac{1}{x}, \quad y = -x + 2$	T
6	N	wrong	N
7	D/PS	wrong	T
8	O	wrong	N

Table B.1: Overview of results in class 1*.

Group	Used Strategies	Equations given (when correct)	Suggested institution-alisation
9	PC/PT	close to correct solution	V
10	O/PT	close to correct solution	V
11	O	close to correct solution	V
12	O	close to correct solution	N
13	O	close to correct solution	T
14	PS/PC	close to correct solution	N
15	O	$y = -2.5 + x, \quad y = 0.1x^2$	T

Table B.2: Overview of results in class 2*.

Group	Used Strategies	Equations given (when correct)	Suggested institution-alisation
16	A	$y = \frac{1}{4}(x-6)^2 + 2, \quad y = -x + 7$	V
17	C	$y = -x - 2, \quad x^2 + y^2 = 2$	V
18	C	wrong	V

Table B.3: Overview of results in class 3*.

Group	Used Strategies	Equations given (when correct)	Suggested institution-alisation
19	PT/PC	close to correct solution	V
20	D	wrong	V
21	O	close to correct solution	V
22	C/R	close to correct solution	N
23	O	wrong	T
24	R	wrong	N
25	N	wrong	T

Table B.4: Overview of results in class 1.

Group	Used Strategies	Equations given (when correct)	Suggested institution-alisation
26	D	wrong	V
27	R	$y = \frac{4}{x}, \quad y = -4x + 8$	V
28	PC	close to correct solution	V
29	O	wrong	N
30	R	wrong	A
31	PC/PS/PT	close to correct solution	L
32	PC	$y = -2x + 7, \quad y = \left(x - \frac{7}{2}\right)^2 + 1$	A
33	HS	$y = -x + 2, \quad y = \frac{1}{x}$	T
34	O	wrong	N

Table B.5: Overview of results in class 2.

Group	Used Strategies	Equations given (when correct)	Suggested institution-alisation
35	A	$y = \frac{6}{x}, \quad y = -x + 2\sqrt{6}$	A
36	C	wrong	N
37	D	$y = 9 - 2x, \quad y = (x-6)^2 \, \frac{1}{3}$	L
38	PT/PC/C	close to correct solution	V
39	D	wrong	N
40	R	$y = -2x - 2, \quad y = \frac{1}{2}x^2$	V
41	OPC/PT/PS/A	$y = \frac{1}{2}x^2, \quad y = 3x - 4.5$	A
42	C	$y = -x + 6, \quad x^2 + y^2 = \left(3\sqrt{2}\right)^2$	V
43	D	$y = -2.5 + x, \quad y = \frac{1}{9}(x-6)^2 + 1$	V
44	PC	$y = -2x, \quad y = \left(\frac{1}{5}x - \frac{11}{5}\right)^2 + 3$	A

Table B.6: Overview of results in class 3.

Appendix C

Education system and Mathematics curriculum in the Netherlands

This appendix provides information about the education system of the Netherlands and the mathematics curriculum in Secondary school[1].

The Dutch education system is made up of primary education, secondary education and higher education (see fig. C.1).
Education is compulsory for pupils aged 5 to 16. Students between the ages of 16-18 are subject to basic qualification requirement; this means that they must attend school until they reach the age of 18, unless they obtain a diploma before.

Primary education lasts 8 years. In class 8, the final year of primary education, pupils choose a secondary education pathway. Usually teachers recommend the school type for each student.

[1] All informations are retrieved from
https://www.epnuffic.nl/en/publications/education-system-the-netherlands.pdf
and
https://www.nro.nl/wp-content/uploads/2015/05/Netherlands.pdf

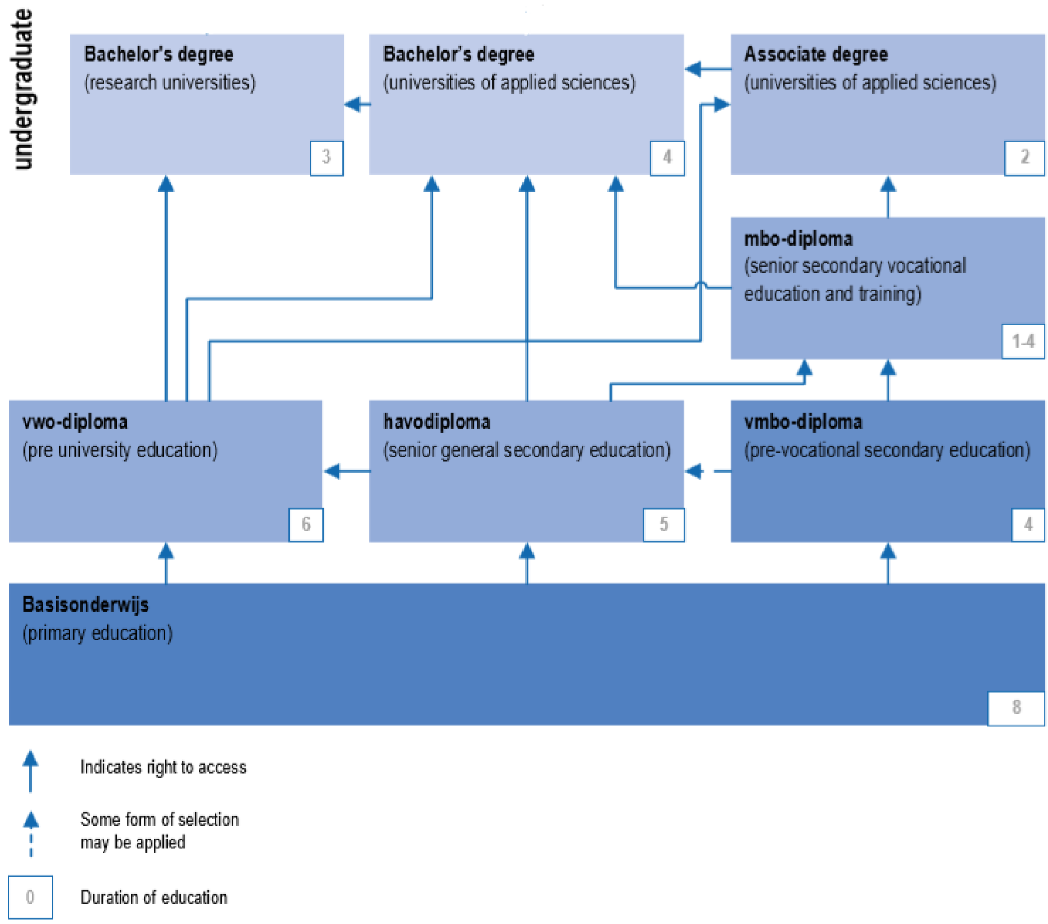


Figure C.1: Dutch education system until undergraduate level.

The two basic pathways of **secondary education** are

- general education (VMBO-T, and HAVO or VWO);
- pre-vocational secondary education (VMBO-bb/kb/gl).

General secondary education (HAVO/VWO)

Students can choose between two types of education:

- senior general secondary education (HAVO);
- pre university education (VWO).

There are different types of VWO schools: *gymnasium, atheneum, VWO+, technasium.*
In *gymnasium* students follow all regular atheneum subjects supplemented with Greek and Latin. In VWO+ they follow regular atheneum curricula with Latin. *Technasium* emphasises research and design and focuses on the development of science-related skills.

In both HAVO and VWO taught contents are a broad range of subjects during the initial years, followed by a subject cluster for deeper specialisation. The 4 available subject clusters are:

- culture and society
- economics and society
- nature and health
- nature and technology

Pre-vocational secondary education (VMBO)
Like general secondary education, VMBO offers in the first years a broad range of subjects. By the end of the second year students choose 1 of the following learning pathways:

- basic vocational programme;
- advanced vocational programme;
- combined programme;
- theoretical programme.

Higher education
Higher education is based around a binary system, which distinguishes between research-oriented higher education (WO) and higher professional education (HBO).

Admission to research-oriented higher education (WO) requires a VWO diploma or HBO first-year certificate.
Admission to higher professional education (HBO) requires a HAVO or VWO diploma.

Mathematics curriculum in Secondary school

For the first two years of secondary school, the mathematics curriculum comprises nine core objectives. By the end of the first two years of secondary education, students should have been taught how to do the following:

- Use appropriate mathematical language to organize mathematical thinking, explain things to others, and understand explanations in the context of mathematics;

- Recognize and use mathematics to solve problems in practical situations, both individually and in collaboration with others;

- Establish a mathematical argument and distinguish it from opinion, learning to give and receive mathematical criticism and to respect other ways of thinking;

- Recognize the structure and coherence of the systems of positive and negative numbers, decimal numbers, fractions, percentages, and proportions, and learn to work with these systems meaningfully in practical situations;

- Make exact calculations, provide estimates, and demonstrate an understanding of accuracy, order of magnitude, and margin of error appropriate to a given situation;

- Make measurements, recognize the structure and coherence of the metric system, and calculate with measurements in common applications;

- Use informal notations, schematic representations, tables, diagrams, and formulas to understand connections between quantities and variables;
- Work with two- and three-dimensional shapes and solids, make and interpret representations of these objects, and calculate and reason using their properties;
- Learn to describe, order, and visualize data systematically, and to judge data, representations, and conclusions critically.

From the third year students can choose at what level they want to keep with the study of the subject.
In VWO they can choose between:

- Wiskunde C *Mathematics "ultra light"*
- Wiskunde A *Mathematics "light"*
- Wiskunde B *Mathematics*
- Wiskunde D *Mathematics extra*

Wiskunde D is an optional subject, not required for any universitary courses.
In HAVO students can choose between Wiskunde A, Wiskunde B or Wiskunde D.

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